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LAMINAR BOUNDARY LAYERS

9.1 INTRODUCTION

The boundary layer of a flowing fluid is the thin layer close to the wall. In a flow field, viscous stresses are very prominent within this layer. Although the layer is thin, it is very important to know the details of flow within it. The main-flow velocity within this layer tends to zero while approaching the wall. Also the gradient of this velocity component in a direction normal to the surface is large as compared to the gradient of this component in the streamwise direction.

9.2 BOUNDARY LAYER EQUATIONS

In 1904, Ludwig Prandtl, the well-known German scientist, introduced the concept of boundary layer [1] and derived the equations for boundary layer flow by correct reduction of the Navier–Stokes equations. He hypothesised that for fluids having a relatively small viscosity, the effect of internal fictitious in the fluid is significant only in a narrow region surrounding the solid boundaries or bodies over which the fluid flows. Thus, close to the body is the boundary layer where shear stresses exert an increasingly larger effect on the fluid as one moves from free stream towards the solid boundary. However, outside the boundary layer where the effect of the shear stresses on the flow is small compared to values inside the boundary layer (since the velocity gradient $\partial u/\partial y$ is negligible), the fluid particles experience no vorticity, and therefore, the flow is similar to a potential flow. Hence, the *surface* at the boundary layer interface is a rather fictitious one dividing rotational and irrotational flow. Prandtl's model regarding the boundary layer flow is shown in Fig. 9.1. Hence with the exception of the immediate vicinity of the surface, the flow is frictionless (inviscid) and the velocity is U. In the region very near to the surface (in the thin layer), there is friction in the flow which signifies that the fluid is retarded until it adheres to the surface. The transition of the mainstream velocity from zero at the surface to full magnitude takes place across the boundary layer. Its thickness is δ which is a function of the coordinate direction x. The thickness is considered to be very small compared to the characteristic length L of the domain. In the normal direction, within the thin layer, the gradient $\partial u/\partial y$ is very large compared to the gradient in the flow direction $\partial u/\partial x$. The next step is to simplify the Navier–Stokes equations for steady two-dimensional laminar incompressible flows. Considering the Navier–Stokes equations together with the equation of continuity, the following dimensional form is obtained:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right] \tag{9.1}$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{\mu}{\rho}\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right] \tag{9.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{9.3}$$

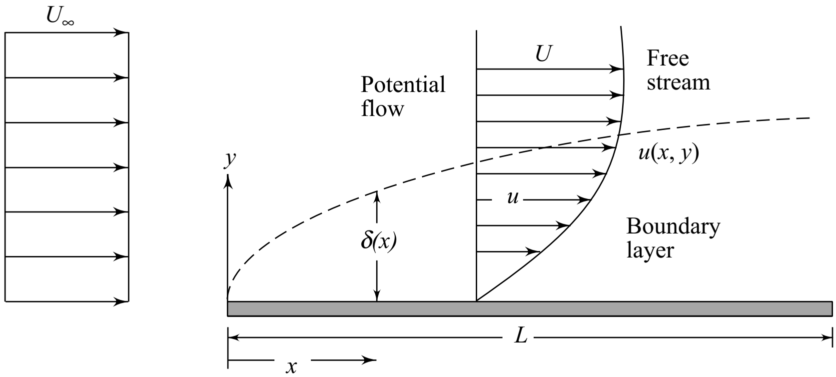

Fig. 9.1 Boundary layer on a flat plate

Here the velocity components u and v are acting along the streamwise x and normal y directions respectively. The static pressure is p, while ρ is the density and μ is the dynamic viscosity of the fluid.

The equations are now non-dimensionalised. The length and the velocity scales are chosen as L and U_∞ respectively. The non-dimensional variables are

$$u^* = \frac{u}{U_\infty},\ v^* = \frac{v}{U_\infty},\ p^* = \frac{p}{\rho U_\infty^2}$$

$$x^* = \frac{x}{L},\ y^* = \frac{y}{L}$$

where U_∞ is the dimensional free stream velocity and the pressure is non-dimensionalised by twice the dynamic pressure $p_d = (1/2)\ \rho\ U_\infty^2$. Using these non-dimensional variables, Eqs (9.1) to (9.3) become

$$u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{\partial p^*}{\partial x^*}+\frac{1}{\text{Re}}\left[\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right] \tag{9.4}$$

$$u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}=-\frac{\partial p^*}{\partial y^*}+\frac{1}{\text{Re}}\left[\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right] \tag{9.5}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{9.6}$$

where the Reynolds number,

$$\mathrm{Re} = \frac{\rho U_\infty L}{\mu}$$

Let us now examine what happens to the u velocity as we go across the boundary layer. At the wall the u velocity is zero. The value of u on the inviscid side, i.e., on the free stream side beyond the boundary layer. is U. For the case of external flow over a flat plate, this U is equal to U_∞.

Based on the above, we can identify the following scales for the boundary layer variables:

Variable	*Dimensional scale*	*Non-dimensional scale*
u	U_∞	1
x	L	1
y	δ	$\varepsilon\ (=\delta/L)$

The symbol ε describes a value much smaller than 1. Now let us look at the order of magnitude of each individual terms involved in Eqs (9.4), (9.5) and (9.6). We start with the continuity Eq. (9.6). One general rule of incompressible fluid mechanics is that we are not allowed to drop any term from the continuity equation. From the scales of boundary layer variables, the derivative $\partial u^*/\partial x^*$ is of the order 1. The second term in the continuity equation $\partial v^*/\partial y^*$ should also be of the order 1. Now, what makes $\partial v^*/\partial y^*$ to have the order 1? Admittedly v^* has to be of the order ε because y^* becomes ε $(= \delta/L)$ at its maximum. Next, consider Eq. (9.4). Inertia terms are of the order 1. Among the second order derivatives, $\partial^2 u^*/\partial x^{*2}$ is of the order 1 and $\partial^2 u^*/\partial y^{*2}$ contains a large estimate of $(1/\varepsilon^2)$. However, after multiplication with 1/Re, the sum of these two second order derivatives should produce at least one term which is of the same order of magnitude as the inertia terms. This is possible only if the Reynolds number (Re) is of the order of $1/\varepsilon^2$.

The order of Reynolds number can be determined in the following manner. While deriving the boundary layer equation, the basic assumption is that outside the boundary layer, the inertia force is >> viscous force. However, within the boundary layer, the inertia force and viscous forces are comparable. If inertia force and the viscous force components can be represented by $\rho u \frac{\partial u}{\partial x}$ and $\mu \frac{\partial^2 u}{\partial y^2}$ respectively, we can say that within the boundary layer

$$\rho u \frac{\partial u}{\partial x} \sim \mu \frac{\partial^2 u}{\partial y^2}$$

or

$$\rho U \frac{U}{L} \sim \mu \frac{U}{\delta^2}$$