Chapter 4

Boolean Algebra and Logic Simplification

（布尔代数和逻辑简化）

4-1 BOOLEAN OPERATIONS AND EXPRESSIONS
（布尔运算和表达式）

- Boolean algebra is the mathematics of digital systems.
  (布尔代数是数字系统的数学工具)

- *A* variable is a symbol (usually an italic uppercase letter) used to represent a logic quantity. Any single variable can have a 1 or 0 value.
  （通常用一个斜体大写字母表示一个变量。一个单变量的取值为0或者1）

4-1 BOOLEAN OPERATIONS AND EXPRESSIONS
（布尔运算和表达式）

- The complement is the inverse of a variable and is indicated by a bar over the variable (overbar). The complement of the variable *A* is read as "not *A*" or "*A* bar". Sometimes a prime symbol rather than an overbar is used to denote the complement of a variable.

- 一个变量的反码也称之为其补码，通常用符号上面的一横表示。读作"not *A*" or "*A* bar"。有时候也用一撇代替一横杠，用来表示一个变量的反码/补码

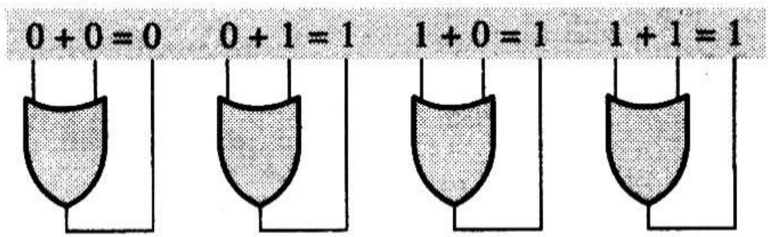
$$A \qquad \overline{A}$$

$$A+B, \quad A+B+\overline{C}, \quad \overline{A}+B+C+\overline{D}$$

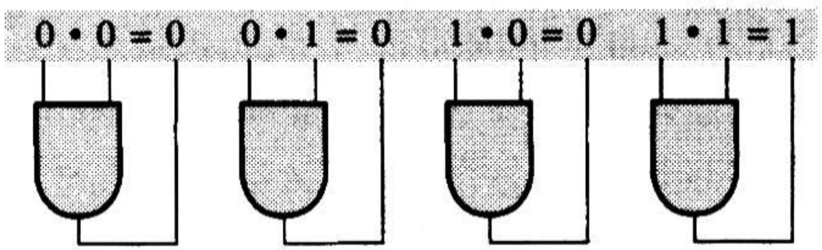
$$AB, \quad AB\overline{C}, \quad \overline{A}BC\overline{D}$$

Boolean Operations and Expressions

- Boolean addition is equivalent to the OR operation and the basic rules are illustrated with their relation to the OR gates as follows:



- Boolean multiplication is equivalent to the AND operation and the basic rules are illustrated with their relation to the AND gates as follows:



AND (Boolean Multiplication) : $\cdot$

e.g. A AND B = A $\cdot$B = AB

OR (Boolean Addition): $+$

e.g. A OR B = A+B

Inverter(NOT): $\bar{\ }$

e.g. $\overline{A}$

sum term

sum term: sum of literals. (Literal（因子）：Variable or complement of a variable)

e.g. $A + B,\ A + \overline{B} + \overline{C},\ \overline{A} + B + \overline{C} + D$

Example:

Determine the value of A, B, C, and D which make the sum term $A + \overline{B} + C + \overline{D} = 0$

Solution:

$A + \overline{B} + C + \overline{D} = 0 \Rightarrow A = 0, \overline{B} = 0, C = 0, \overline{D} = 0$

$\Rightarrow A = 0, B = 1, C = 0, D = 1$

product term

product term: product of literals.

e.g. $AB, \; A\overline{B}C, \; A\overline{B}CD$

Example:
Determine the value of A, B, C, and D which make the product term $A\overline{B}C\overline{D} = 1$

Solution:

$$A\overline{B}C\overline{D} = 1 \Rightarrow A = 1, \overline{B} = 1, C = 1, \overline{D} = 1$$

$$\Rightarrow A = 1, B = 0, C = 1, D = 0$$

4-2 LAWS AND RULES OF BOOLEAN ALGEBRA
布尔代数的定律和法则

	Laws	Boolean Expression
1	Commutative law of addition Commutative law of multiplication (交换律)	$A+B=B+A$ $AB=BA$
2	Associative law of addition Associative law of multiplication (结合律)	$A+(B+C)=(A+B)+C$ $A(BC)=(AB)C$
3	Distributive law (分配率)	$A(B+C)=AB+AC$

Laws and Rules of Boolean Algebra (I)

Rule Number	**Boolean Expression**
1	$A + 0 = A$
2	$A + 1 = 1$
3	$A \cdot 0 = 0$
4	$A \cdot 1 = A$
5	$A + A = A$
6	$A + \overline{A} = 1$

Laws and Rules of Boolean Algebra (II)

Rule Number	**Boolean Expression**
7	$A \cdot A = A$
8	$A \cdot \overline{A} = 0$
9	$\overline{\overline{A}} = A$
10	$A + AB = A$
11	$A + \overline{A}B = A + B$
12	$(A+B)(A+C) = A + BC$
13	$AB + \overline{A}C + BC = AB + \overline{A}C$

Laws and Rules of Boolean Algebra _PROOF (I)

- This law is similar to absorption in that it can be employed to eliminate extra elements from a Boolean expression:

$$A + \overline{A}B = A + B$$

[PROOF]

$$A+\overline{A}B=(A+AB)+\overline{A}B=A+(AB+\overline{A}B)$$

$$=A+(A+\overline{A})B=A+1\cdot B=A+B$$

Laws and Rules of Boolean Algebra _PROOF (II)

$$(A+B)(A+C) = A+BC$$

[PROOF]

$$\begin{aligned}(A+B)(A+C) &= (A+B)A+(A+B)C \\ &= AA+AB+AC+BC \\ &= A+AB+AC+BC \\ &= A(1+B+C)+BC \\ &= A+BC\end{aligned}$$

Laws and Rules of Boolean Algebra (III)

- Consensus theorem

$$AB + \overline{A}C + BC = AB + \overline{A}C$$

[PROOF]

$$AB + \overline{A}C + BC = AB + \overline{A}C + (A + \overline{A})BC$$
$$= AB + \overline{A}C + ABC + \overline{A}BC$$
$$= (AB + ABC) + (\overline{A}C + \overline{A}CB) = AB + \overline{A}C$$

The key to using this theorem is to find a variable and its complement, note the associated terms, and eliminate the included term (the consensus term), which is composed of the associated terms.

4-3 DEMORGAN'S THEOREMS 狄摩根定理

- ***DeMorgan's first theorem***:
- *The complement of a product of variables is equal to the sum of the complements of the variables.*

$$\overline{XY} = \overline{X} + \overline{Y}$$

(equivalency of the NAND and negative-OR gates)

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

- ***DeMorgan's second theorem***:
  The complement of a sum of variables is equal to the product of the complements of the variables.

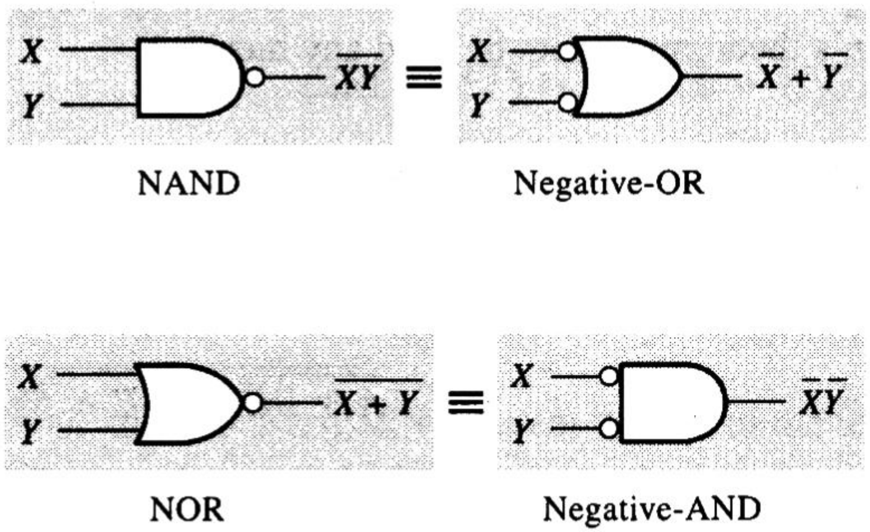
$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

(equivalency of the NOR and negative-AND gates.)

Inputs		Output	
X	Y	$\overline{X+Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

DeMorgan's Theorems

- The gate equivalencies and truth tables for DeMorgan's theorems are shown below:



Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X}+\overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Inputs		Output	
X	Y	$\overline{X+Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

FIGURE 4–15

Gate equivalencies and the corresponding truth tables that illustrate DeMorgan's theorems. Notice the equality of the two output columns in each table. This shows that the equivalent gates perform the same logic function.

Example: Apply DeMorgan's theorem to the expressions:

$$\overline{XYZ},\ \overline{X + Y + Z}$$

Solution:

$$\overline{XYZ} = \overline{X} + \overline{YZ} = \overline{X} + (\overline{Y} + \overline{Z})$$
$$= \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{X + Y + Z} = \overline{X} \cdot \overline{Y + Z} = \overline{X}\,\overline{Y}\,\overline{Z}$$

Example: Apply DeMorgan's theorems to each of the following expressions:

(a) $\overline{(A+B+C)D}$ (b) $\overline{ABC+DEF}$

Solution:

(a) Let $A+B+C=X$, $D=Y$:

$$\overline{(A+B+C)D} \Rightarrow \overline{XY}=\overline{X}+\overline{Y}=\overline{A+B+C}+\overline{D}$$

$$\overline{A+B+C} \Rightarrow \overline{A+B+C}=\overline{A}\,\overline{B}\,\overline{C}$$

$$\overline{(A+B+C)D} \Rightarrow \overline{A}\,\overline{B}\,\overline{C}+\overline{D}$$

(b) $$\overline{ABC+DEF} = (\overline{ABC})(\overline{DEF})$$
$$= (\overline{A}+\overline{B}+\overline{C})(\overline{D}+\overline{E}+\overline{F})$$

Example: Apply DeMorgan's theorem to the following expression:

$$\overline{\overline{A+B\overline{C}}+D(\overline{E+\overline{F}})}$$

Solution:

$$\overline{\overline{A+B\overline{C}}+D(\overline{E+\overline{F}})}$$

$$\overline{X+Y}=\overline{X}\,\overline{Y}$$

$$=(\overline{\overline{A+B\overline{C}}})(\overline{D(\overline{E+\overline{F}})})$$

$$\overline{XY}=\overline{X}+\overline{Y}$$

$$=(A+B\overline{C})(\overline{D}+(\overline{\overline{E+\overline{F}}}))$$

$$=(A+B\overline{C})(\overline{D}+E+\overline{F})$$

4.4 Logic function Simplification

Objective:

To reduce a particular expression in its ***simplest form*** or ***change its form to a more convenient one*** to implement the expression most efficiently. The approach taken in this section is to use the ***basic laws, rules, and theorems*** of Boolean algebra to simplify an expression.

- Example 4-5 Using Boolean algebra, simplify this expression:

$$X = AB + A(B + C) + B(B + C)$$

- Solution:

$$X = AB + A(B + C) + B(B + C)$$

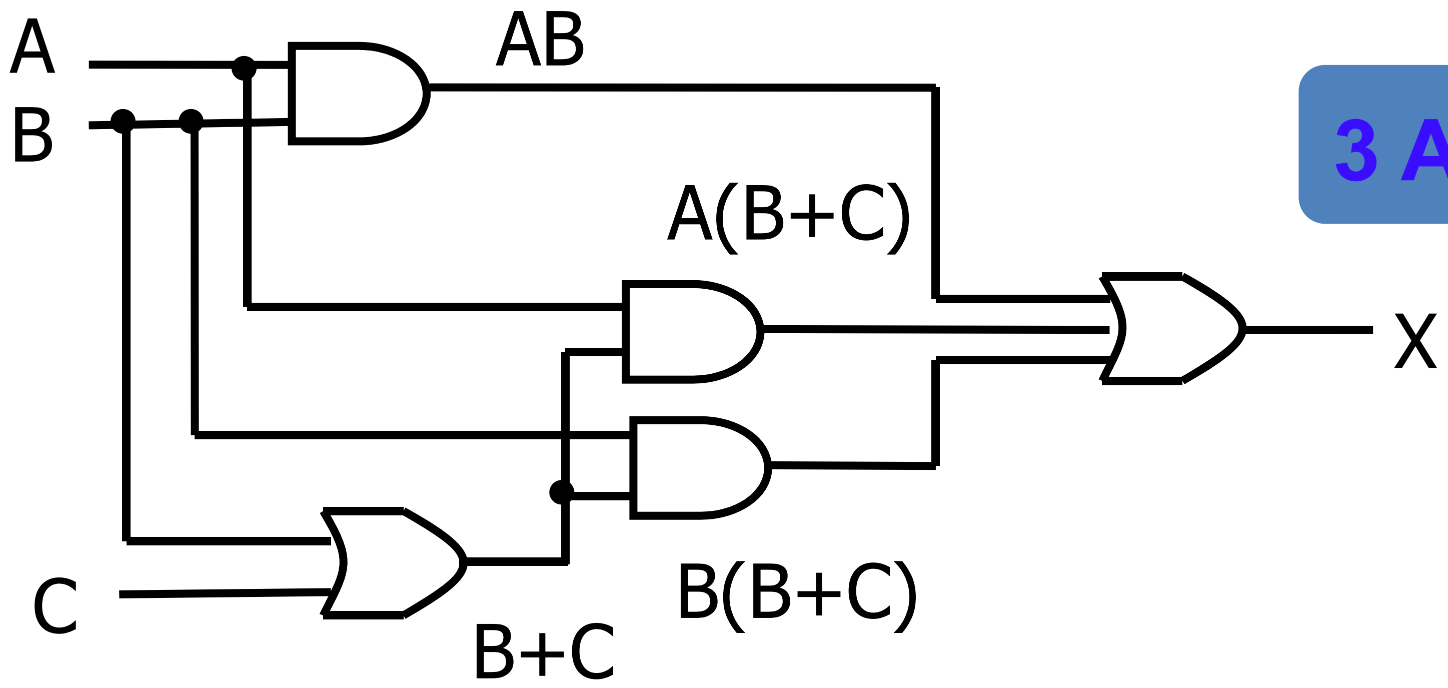
$$= AB + AB + AC + BB + BC$$

$$= AB + AB + AC + B + BC$$

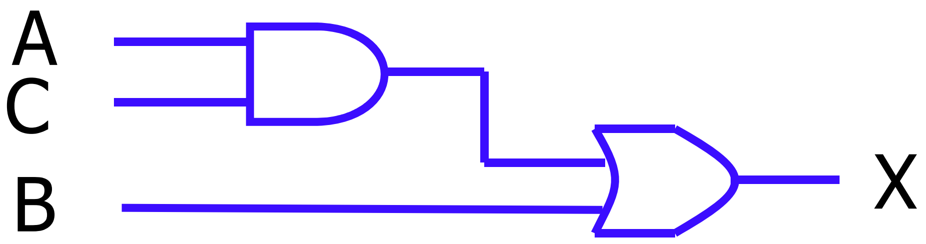
$$= AB + AC + B + BC$$

$$= AB + AC + B$$

$$= AC + B$$



3 AND,2 OR



1 AND,1 OR

Example:

$$(A\overline{B}(C + BD) + \overline{A}\overline{B})C$$

Example:

$$
\begin{aligned}
&(A\overline{B}(C+BD)+\overline{A}\,\overline{B})C \\
&=(A\overline{B}C+A\overline{B}BD+\overline{A}\,\overline{B})C \\
&=(A\overline{B}C+A\cdot 0\cdot D+\overline{A}\,\overline{B})C \\
&=(A\overline{B}C+\overline{A}\,\overline{B})C \\
&=A\overline{B}CC+\overline{A}\,\overline{B}C \\
&=A\overline{B}C+\overline{A}\,\overline{B}C \\
&=(A+\overline{A})\overline{B}C \\
&=1\cdot\overline{B}C \\
&=\overline{B}C
\end{aligned}
$$

4 AND,2 OR

1 AND

Review: 常用的化简方法

$$1.A+\overline{A}=1 \Rightarrow AB+A\overline{B}=A$$

Example

$$F=A\overline{\overline{B}CD}+A\overline{B}CD \quad =A$$

$$AB+A\overline{B}=A$$

$2.\ A + AB = A$

Example:

$$F = A + \overline{\overline{A} \cdot \overline{BC}}\left(\overline{A} + \overline{\overline{B}\,\overline{C} + D}\right) + BC$$

$$\overline{X \cdot Y} = \overline{X} + \overline{Y}$$

$$= A + BC + (A + BC)\left(\overline{A} + \overline{\overline{B}\,\overline{C} + D}\right)$$

$$A + AB = A$$

$$= A + BC$$

$$3.\ A+\overline{A}B=A+B$$

Example:

$$F=AC+\overline{A}D+\overline{C}D$$

$$=AC+(\underline{\overline{A}+\overline{C}})D$$

$$\overline{X\cdot Y}=\overline{X}+\overline{Y}$$

$$=AC+\overline{AC}D \Leftarrow A+\overline{A}B=A+B$$

$$=AC+D$$

$$4.\ A+\overline{A}=1,\ \ A+A=A$$

$$\Rightarrow \textit{using repeated terms}$$

Example:

$$F=\overline{A}B\overline{C}+\overline{A}BC+ABC \quad \Leftarrow A+A=A$$

$$=\overline{A}B\overline{C}+\underline{\overline{A}BC+\overline{A}BC}+ABC$$

$$=(\overline{A}B\overline{C}+\overline{A}BC)+(\overline{A}BC+ABC)$$

$$=\overline{A}B(C+\overline{C})+(\overline{A}+A)BC \quad \Leftarrow A+\overline{A}=1$$

$$=\overline{A}B+BC$$

Simplification Using Boolean Algebra

- Examples:

$$Y_1 = A\bar{B} + AC + BC$$

$$Y_2 = ABD + A\bar{B}C\bar{D} + A\bar{C}DE + A$$

$$Y_3 = \bar{A}BC + (A + \bar{B})C$$

$$Y_4 = \overline{\bar{A}BC} + \overline{A\bar{B}}$$

$$Y_5 = A\bar{B}(\bar{A}CD + \overline{AD + \bar{B}\bar{C}})(\bar{A} + B)$$

$$Y_6 = AC(\bar{C}D + \bar{A}B) + BC\overline{\overline{\bar{B} + AD} + CE}$$

$$Y_7 = \overline{A\bar{B}\bar{C}D + A\bar{C}DE + \bar{B}D\bar{E} + A\bar{C}\bar{D}E}$$

Simplification Using Boolean Algebra

- Solutions: $Y_1 = A\overline{B} + AC + BC$

$$= BC + \overline{B}A + CA = BC + \overline{B}A = A\overline{B} + BC$$

$$Y_2 = ABD + A\overline{B}C\overline{D} + A\overline{C}DE + A$$

$$= A(BD + \overline{B}C\overline{D} + \overline{C}DE + 1) = A \cdot 1 = A$$

$$Y_3 = \overline{A}BC + (A + \overline{B})C$$

$$= \overline{A}BC + \overline{\overline{A}B}C = (\overline{A}B + \overline{\overline{A}B})C = 1 \cdot C = C$$

Simplification Using Boolean Algebra

- Solutions:

$$Y_4 = \overline{\overline{A}BC} + \overline{A\overline{B}} = A + \overline{B} + \overline{C} + \overline{A} + B = 1$$

$$Y_5 = A\overline{B}(\overline{A}CD + \overline{AD + \overline{B}\,\overline{C}})(\overline{A} + B)$$

$$= A\overline{B}(\overline{A}CD + \overline{AD + \overline{B}\,\overline{C}})\overline{A\overline{B}} = 0$$

$$Y_6 = AC(\overline{C}D + \overline{A}B) + BC\,\overline{\overline{\overline{B} + AD} + CE}$$

$$= AC\overline{C}D + AC\overline{A}B + BC(\overline{B} + AD)\overline{CE}$$

$$= BC\overline{B}\,\overline{CE} + BCAD\overline{CE} = BCAD(\overline{C} + \overline{E}) = BCAD\overline{E} = ABCD\overline{E}$$

Simplification Using Boolean Algebra

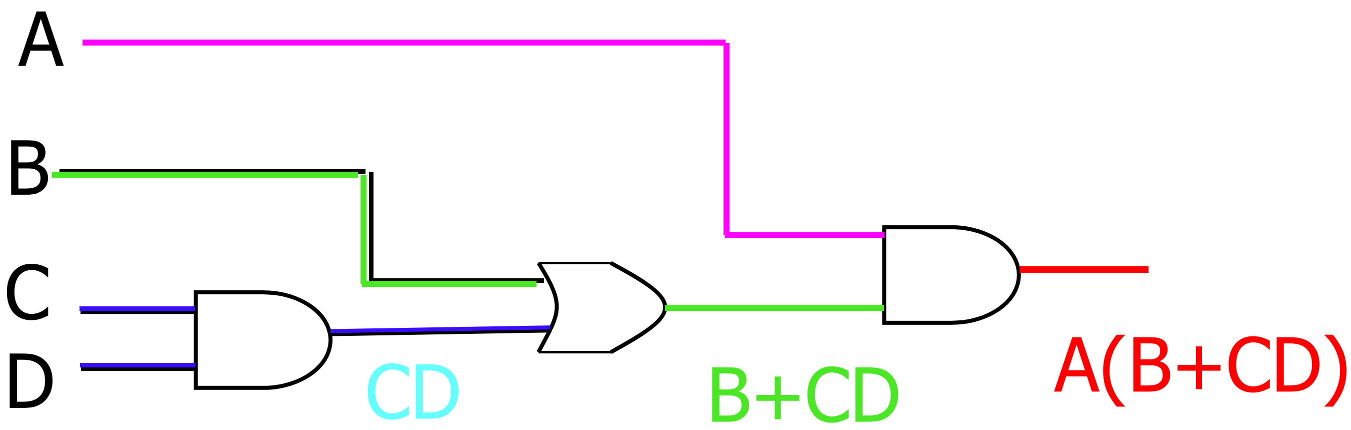
- Solutions:

$$
\begin{aligned}
Y_7 &= \overline{A\bar{B}\bar{C}D + A\bar{C}DE + \bar{B}D\bar{E} + A\bar{C}\bar{D}E} \\
&= \overline{A\bar{B}\bar{C}D + A\bar{C}(D + \bar{D})E + \bar{B}D\bar{E}} \\
&= \overline{A\bar{B}\bar{C}D + A\bar{C}E + \bar{B}D\bar{E}} = \overline{EA\bar{C} + \bar{E}\bar{B}D + A\bar{C}\bar{B}D} \\
&= \overline{EA\bar{C} + \bar{E}\bar{B}D} \\
&= (\bar{E} + \bar{A} + C)(E + B + \bar{D}) \\
&= B\bar{E} + \bar{D}\bar{E} + \bar{A}E + \bar{A}B + \bar{A}\bar{D} + CE + BC + C\bar{D} \\
&= (EC + \bar{E}B + CB) + (E\bar{A} + \bar{E}\bar{D} + \bar{A}\bar{D}) + \bar{A}B + C\bar{D} \\
&= EC + \bar{E}B + E\bar{A} + \bar{E}\bar{D} + \bar{A}B + C\bar{D} = (EC + \bar{E}\bar{D} + C\bar{D}) + (E\bar{A} + \bar{E}B + \bar{A}B) \\
&= EC + \bar{E}\bar{D} + E\bar{A} + \bar{E}B = \bar{A}E + B\bar{E} + CE + \bar{D}\bar{E}
\end{aligned}
$$

4-5 BOOLEAN ANALYSIS OF LOGIC CIRCUITS
逻辑电路的布尔分析

- *Logic functions*
- *truth table*

Example:



Truth Table

Inputs				Output
A	B	C	D	A(B+CD)
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	0
0	1	1	0	0
0	1	1	1	0
1	0	0	0	0
1	0	0	1	0
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	1
1	1	1	1	1

4.6 Standard Forms of Boolean Expression（布尔表达式的标准形式）

Definitions (SOP)

- ***Literal（因子）:*** Variable or complement of a variable
- ***Product Term:*** A single literal or a product (AND) of 2 or more literals
- ***Sum of Products***: Logical sum (OR) of product terms
- ***Domain（域）:*** the set of variables contained in the expression in either complemented or uncomplemented form
- ***Minterm (最小项):***: Normal product term with all variables in the domain appearing
- ***Standard (Canonical) Sum Of Products:*** Sum of minterms

Minterms and Maxterms for Three Binary Variables

Input			Minterms			Maxterms	
A	B	C	Terms	Designation		Terms	Designation
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0		$A+B+C$	M_0
0	0	1	$\bar{A}\bar{B}C$	m_1		$A+B+\bar{C}$	M_1
0	1	0	$\bar{A}B\bar{C}$	m_2		$A+\bar{B}+C$	M_2
0	1	1	$\bar{A}BC$	m_3		$A+\bar{B}+\bar{C}$	M_3
1	0	0	$A\bar{B}\bar{C}$	m_4		$\bar{A}+B+C$	M_4
1	0	1	$A\bar{B}C$	m_5		$\bar{A}+B+\bar{C}$	M_5
1	1	0	$AB\bar{C}$	m_6		$\bar{A}+\bar{B}+C$	M_6
1	1	1	ABC	m_7		$\bar{A}+\bar{B}+\bar{C}$	M_7

Minterm和它的编号

- 在n变量逻辑函数中，每个最小项均包含了这n个变量的原变量形式或反变量形式，且在每个最小项中仅出现一次，那么可以推导出一共有2^n个最小（例如：ABC三个变量，最小项有8个）
- 输入变量的每一组取值都使一个对应的最小项的值等于1
- A=1，B=1，C=1 时ABC=1，把111看成二进制值就是7，为了使用方便将ABC=1最小项记为m7

SOP Form（积之和形式）

***The Sum-of-product (SOP) Form*:**

Terms consisting of the product of literals (variables or their complements) are summed by Boolean addition.

Example

$$AB + CD + EF$$

$$ABC + \overline{A}\,\overline{B}C + A\overline{B}C$$

Conversion of a general expression to SOP form

Any logic expression can be changed into SOP form by applying Boolean algebra techniques .

Conversion methods:

Applying laws and rules, e.g. distributive law

$$A(B + CD) = AB + ACD$$

Example: Convert each of the following Boolean expression to SOP form:

$(a)\ AB + B(CD + EF)$ $\quad$ $(b)\ (A + B)(B + C + D)$

$(c)\ \overline{\overline{A + B} + C}$

Solution

$$(a)\ AB + B(CD + EF) = AB + BCD + BEF$$

$$(b)\ (A + B)(B + C + D) = AB + AC + AD + BB + BC + BD$$

$$= AB + AC + AD + B + BC + BD$$

$$(c)\ \overline{\overline{A + B} + C} = \overline{\overline{A + B}}\,\overline{C} = (A + B)\overline{C} = A\overline{C} + B\overline{C}$$

Standard SOP Form (Sum of Minterms Form)最小项之和形式

A ***standard SOP*** expression is one in which ***all the variables*** in the domain appear in each product term in the expression.

Example

$$ABC + \overline{A}BC + A\overline{B}\overline{C}$$

$$ABCD + \overline{A}\overline{B}CD + A\overline{B}\overline{C}D$$

The standard SOP form is important in constructing truth table or for Karnaugh map simplification which we will discuss later.

最小项的性质

- 1. 在输入变量的任何取值下必有一个最小项，而且仅有一个最小项的值为1
- 2. 全体最小项之和为1
- 3. 任何两个最小项的乘积为0
- 4. 具有相邻的两个最小项之和可以合并成一项并消去一对因子
  - 若两个最小项只有一个因子不同，则称这两个最小项具有相邻性
  - 例如

$$\overline{A}B\overline{C} + AB\overline{C} = (\overline{A} + A)B\overline{C} = B\overline{C}$$

Examples: Convert the following Boolean expression into standard SOP form.

$$A\overline{B}C + \overline{A}\,\overline{B} + AB\overline{C}D$$

Solution

$$= A\overline{B}C\underline{(D+\overline{D})} + \overline{A}\,\overline{B}\underline{(C+\overline{C})}\underline{(D+\overline{D})} + AB\overline{C}D$$

$$= \underline{A\overline{B}CD + A\overline{B}C\overline{D}}$$

$$+ \underline{\overline{A}\,\overline{B}CD + \overline{A}\,\overline{B}\,\overline{C}D + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}} + AB\overline{C}D$$

$$A\overline{B}CD = 1\cdot\overline{0}\cdot 1\cdot 1 = 1 \qquad m_{11}$$

Definitions (POS)

- ***Sum Term:*** A single literal or a sum (OR) of 2 or more literals
- ***Product of Sums:*** Logical product of sum terms
- ***Maxterm (最大项)*** : Normal sum term with all variables in the domain appearing
- ***Standard (Canonical) Product of Sums:*** Product of Maxterms

Minterms and **Maxterms for Three Binary Variables**

Input			Minterms		Maxterms	
A	B	C	Terms	Designation	Terms	Designation
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0	$A+B+C$	M_0
0	0	1	$\bar{A}\bar{B}C$	m_1	$A+B+\bar{C}$	M_1
0	1	0	$\bar{A}B\bar{C}$	m_2	$A+\bar{B}+C$	M_2
0	1	1	$\bar{A}BC$	m_3	$A+\bar{B}+\bar{C}$	M_3
1	0	0	$A\bar{B}\bar{C}$	m_4	$\bar{A}+B+C$	M_4
1	0	1	$A\bar{B}C$	m_5	$\bar{A}+B+\bar{C}$	M_5
1	1	0	$AB\bar{C}$	m_6	$\bar{A}+\bar{B}+C$	M_6
1	1	1	ABC	m_7	$\bar{A}+\bar{B}+\bar{C}$	M_7

POS Form（和之积形式）

***The Product-of-sum (POS) Form*:**

Terms consisting of the sum of literals (variables or their complements) are multiplied by Boolean multipication.

Example

$$(A+B+C)(A+\bar{B}+C)(\bar{A}+B+\bar{C})$$

$$(A+B)(A+C+D)(\bar{A}+\bar{B}+C+D)$$

Standard POS Form (Product of Maxterms Form)最大项之积形式

A ***standard POS*** expression is one in which ***all the variables*** in the domain appear in each sum term in the expression.

Example

$$(A + B + C)(\overline{A} + B + C)(A + \overline{B} + \overline{C})$$

$$(A + B + C + D)(\overline{A} + \overline{B} + C + D)(A + \overline{B} + \overline{C} + D)$$

Conversion of POS to Standard POS form

***Steps*:**

1. Add to each nonstandard product term a term made up of a missing variables and its complement. This results in two sum terms.
2. Apply rule $A + BC = (A + B)(A + C)$
3. Repeat step 1 until all resulting sum terms contain all variables in the domain in either complemented or un-complemented form.

Example: Convert following Boolean algebra expression into standard POS form.

$$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

Solution

$$\underbrace{(A+\overline{B}+C)}_{1}\underbrace{(\overline{B}+C+\overline{D})}_{2}(A+\overline{B}+\overline{C}+D)$$

$$1.\ A+\overline{B}+C=(A+\overline{B}+C+D\overline{D})=(A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})$$

$$2.\ \overline{B}+C+\overline{D}=A\overline{A}+\overline{B}+C+\overline{D}=(A+\overline{B}+C+\overline{D})(\overline{A}+\overline{B}+C+\overline{D})$$

$$(A+\overline{B}+C)(\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

$$=(A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})(A+\overline{B}+C+\overline{D})$$

$$(\overline{A}+\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$

4.7 Conversion between Truth table and logic functions with SOP or POS form

The combination of input variable values in truth table can be converted to ***minterms*** (product terms) or a ***maxterms*** (sum terms).

- For minterm: 1 is variable, 0 is complement
- For maxterm: 1 is complement, 0 is variable

Example:

m_{10}

1010

$A\bar{B}C\bar{D}$ ***Minterm***

1010

$\bar{A}+B+\bar{C}+D$ ***Maxterm***

1010

M_{10}

Minterms and Maxterms for Three Binary Variables

Input			Minterms			Maxterms	
A	B	C	Terms	Designation		Terms	Designation
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0		$A+B+C$	M_0
0	0	1	$\bar{A}\bar{B}C$	m_1		$A+B+\bar{C}$	M_1
0	1	0	$\bar{A}B\bar{C}$	m_2		$A+\bar{B}+C$	M_2
0	1	1	$\bar{A}BC$	m_3		$A+\bar{B}+\bar{C}$	M_3
1	0	0	$A\bar{B}\bar{C}$	m_4		$\bar{A}+B+C$	M_4
1	0	1	$A\bar{B}C$	m_5		$\bar{A}+B+\bar{C}$	M_5
1	1	0	$AB\bar{C}$	m_6		$\bar{A}+\bar{B}+C$	M_6
1	1	1	ABC	m_7		$\bar{A}+\bar{B}+\bar{C}$	M_7

Example: From the truth table, determine the standard SOP expression and the equivalent standard POS expression.

Input			Output
A	B	C	x
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Solution:

Sum of minterms expression:

$$X = \overline{A}BC + A\overline{B}\,\overline{C} + AB\overline{C} + ABC$$

$$= m_3 + m_4 + m_6 + m_7$$

$$= \sum(3,4,6,7)$$

Product of maxterms expression:

$$X = (A+B+C)(A+B+\overline{C})$$

$$(A+\overline{B}+C)(\overline{A}+B+\overline{C})$$

$$= M_0 \times M_1 \times M_2 \times M_5$$

$$= \prod(0,1,2,5)$$

$$X = \sum(3,4,6,7) = \prod(0,1,2,5)$$

logic functions

- Using m_i to represent minterms.

  Sum of minterms expression:

$$f = \sum_i m_i$$

- Using M_i to represent maxterms.

  Product of maxterms experssion:

$$f = \prod_i M_i$$

Exercise: From the truth table, determine the standard SOP expression and the equivalent standard POS expression.

Input			Output
A	B	C	x
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	1

Conversion of SOP/ POS form to Truth Table

Example: Determine the truth table for following expression:

$$(A+B+C)(A+\overline{B}+C)(A+\overline{B}+\overline{C})(\overline{A}+B+\overline{C})(\overline{A}+\overline{B}+C)$$

Solution:

$$\underbrace{(A+B+C)}_{000}\underbrace{(A+\overline{B}+C)}_{010}\underbrace{(A+\overline{B}+\overline{C})}_{011}\underbrace{(\overline{A}+B+\overline{C})}_{101}\underbrace{(\overline{A}+\overline{B}+C)}_{110}$$

Input			Output
A	B	C	x
0	0	0	0
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Conversion Standard SOP form to Standard to POS form

Exercise: Convert the SOP expression to an equivalent POS expression:

$$\overline{A}\overline{B}\overline{C}+\overline{A}B\overline{C}+\overline{A}BC+A\overline{B}C+ABC$$

000+010+011+101+111

Input			Output
A	B	C	x
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Equivalent POS:

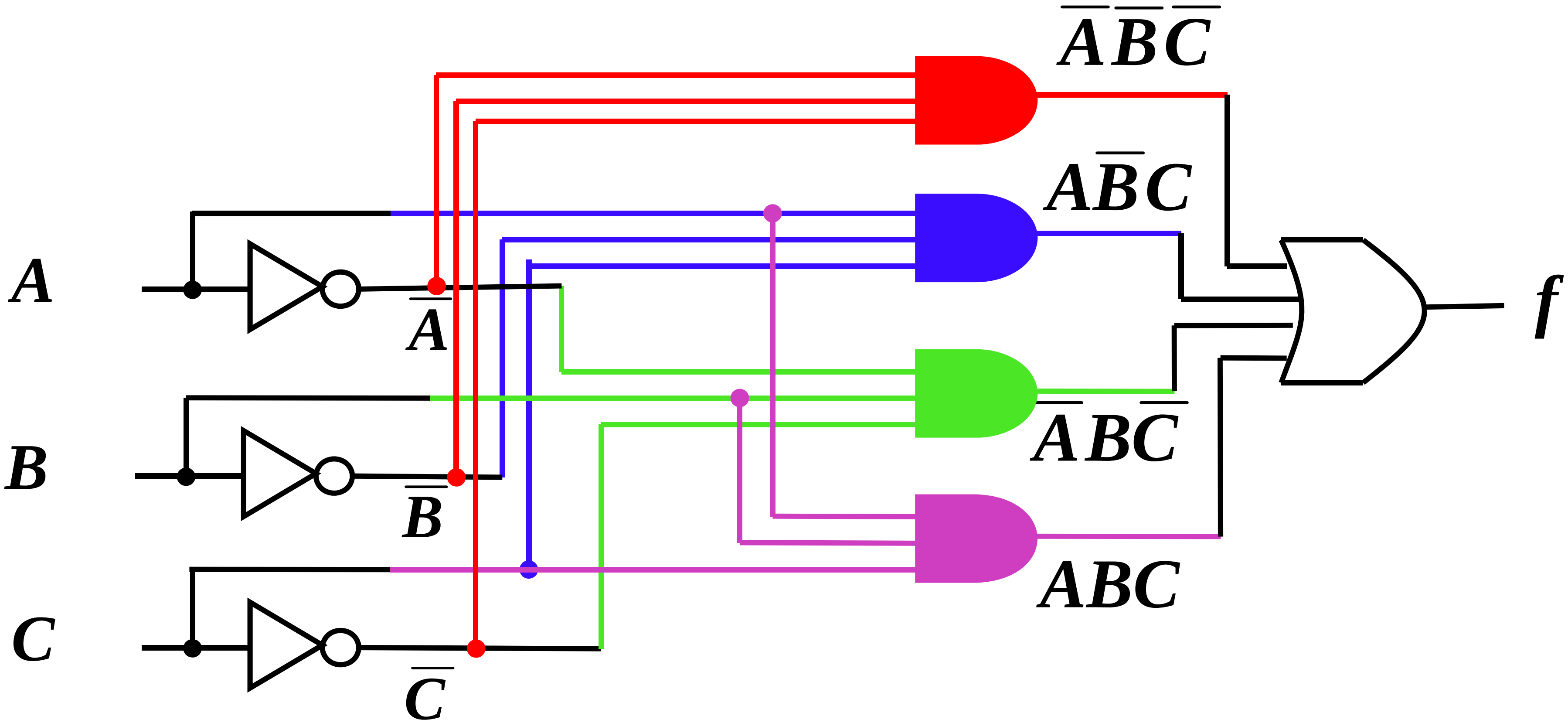
$$(A+B+\overline{C})(\overline{A}+B+C)(\overline{A}+\overline{B}+C)$$

4.8 Conversion of Logic Functions to Logic Diagrams

Converting from logic functions to logic diagrams involves using the symbols assigned to logic functions and replacing the terms of a logic equation with the appropriate logic symbols.

Example: Convert the following function to a logic diagram, use AND, OR, and NOT gates.

$$f = \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + \overline{A}B\overline{C} + ABC$$

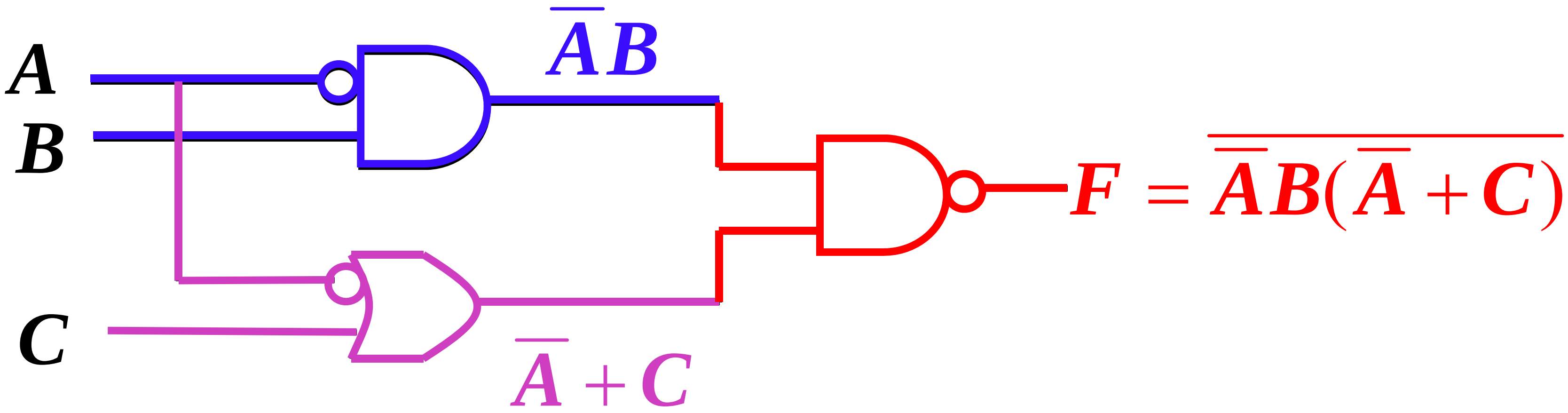


Converting Logic Diagrams to Logic Functions

In case we only get ***logic diagram*** of the circuit, and we want to analyze the circuit or understand the ***purpose*** of the circuit, we need to ***convert the logic diagram into logic function***. It is just the opposite process from the previous one—from function to diagram.

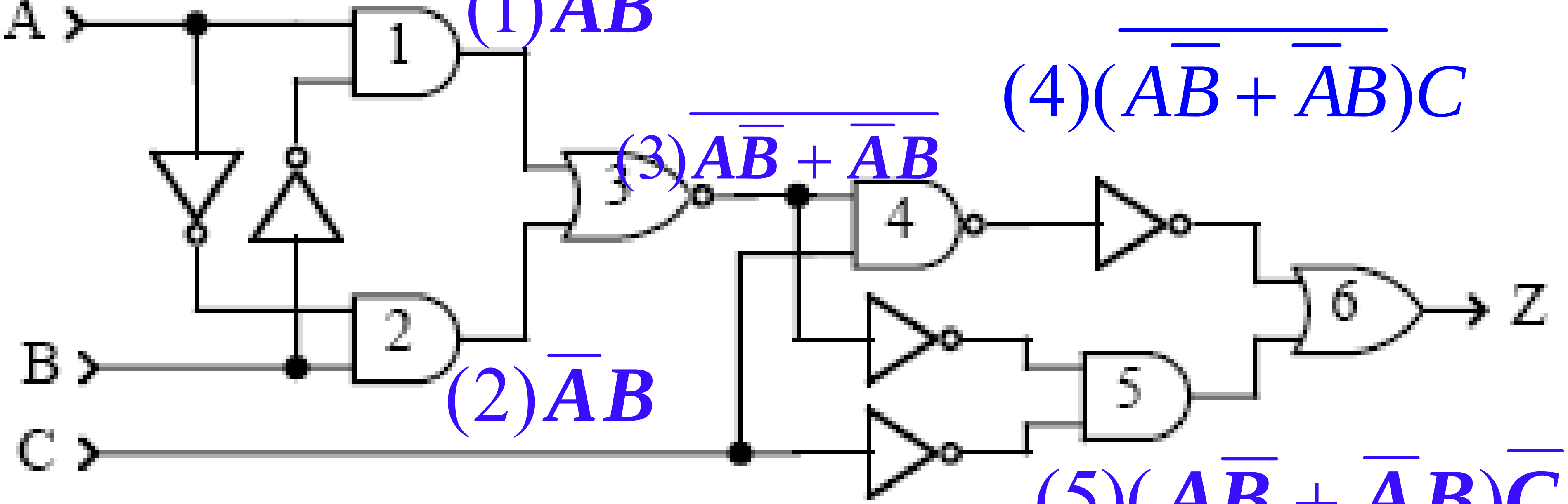
Determine output functions for each gate, step by step from left to right, then get the result.

Example:



A
B
C
$\overline{A}B$
$\overline{A}+C$
$F=\overline{\overline{A}B(\overline{A}+C)}$

Exercise



(1) $A\bar{B}$

(2) $\bar{A}B$

(3) $\overline{A\bar{B} + \bar{A}B}$

(4) $(\overline{A\bar{B} + \bar{A}B})C$

(5) $(A\bar{B} + \bar{A}B)\bar{C}$

(6) $\overline{A\bar{B} + \bar{A}B}C + (A\bar{B} + \bar{A}B)\bar{C}$

$$= (A \oplus B) \oplus C$$

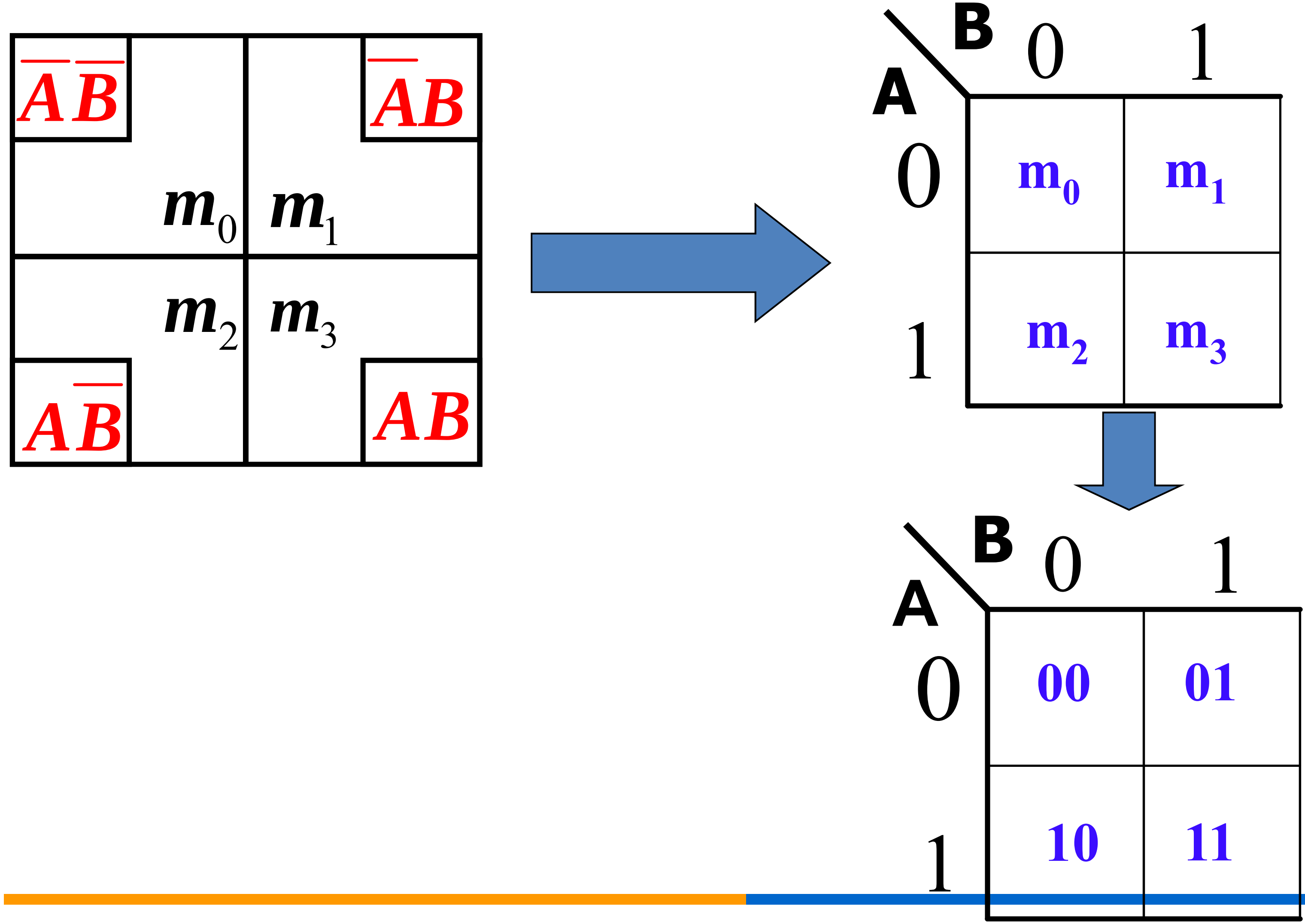
4.9 Karnaugh Map (卡诺图)

- The Karnaugh map provides a ***systematic method (系统性方法)*** for ***simplifying Boolean expressions(简化布尔表达式)***, will produce the simplest SOP or POS expression, known as minimum expression (最简表达式).
- A Karnaugh map is ***an array(矩阵/阵列) of cells*** in which ***each cell(单元) represents a binary value of the input variables（一组输入变量的二进制值）.***
- 将n变量的全部最小项各用一个小方块表示，并使具有逻辑相邻性的最小项在几何位置上也相邻的排列起来。所得到的图形叫n变量的卡诺图。是美国工程师卡诺提出来的。

1 Karnaugh Map Expression

- 2-variable Karnaugh map
- 3-variable Karnaugh map
- 4-variable Karnaugh map
- 5-variable Karnaugh map

2-variable Karnaugh map



Cell Adjacency

The cell in Karnaugh map are arranged so that there is only ***a single-variable change*** between adjacent cells.

在任何一行或一列两端的最小项仅有一个变量不同。因此从几何位置上应该把卡诺图看成是上下，左右闭合的图形。

3-variable Karnaugh Map

A \ BC	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

保证图中几何位置相邻的最小项在逻辑上具有相邻性，这些数码不能按照自然二进制数从小到大的顺序排列，而必须按照图中的方式排列，以确保相邻的两个最小项仅有一个变量是不同的。这样的数值也形成了一组Gray码

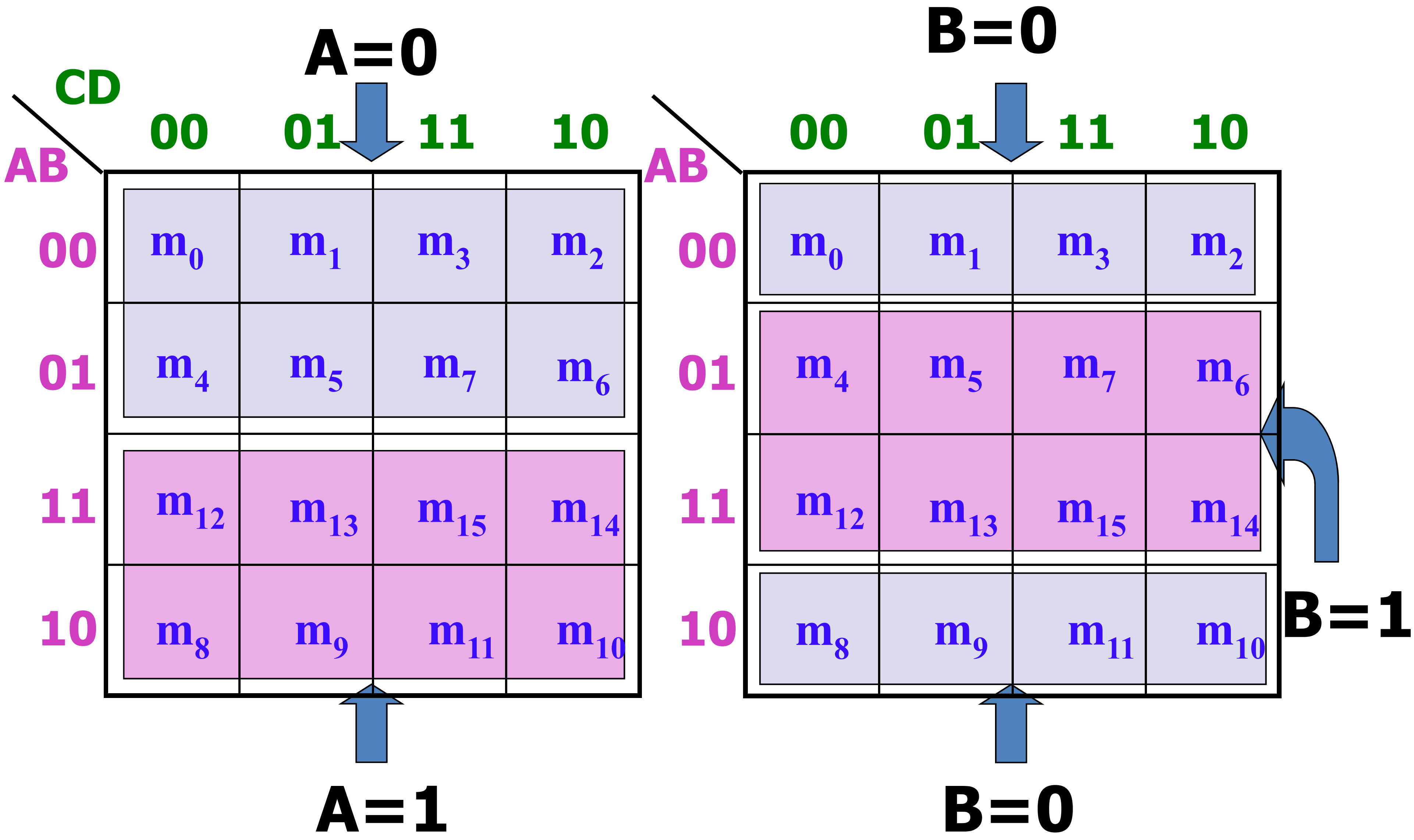
Note: The combination of BC is **00—01—11—10** instead of **00—01—10—11**.

4-variable Karnaugh Map

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Cell Adjacency

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}


A=0
CD
AB
00 01 11 10
00 m_0 m_1 m_3 m_2
01 m_4 m_5 m_7 m_6
11 m_{12} m_{13} m_{15} m_{14}
10 m_8 m_9 m_{11} m_{10}
A=1
B=0
AB
00 01 11 10
00 m_0 m_1 m_3 m_2
01 m_4 m_5 m_7 m_6
11 m_{12} m_{13} m_{15} m_{14}
10 m_8 m_9 m_{11} m_{10}
B=1
B=0

C=1

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

D=1

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

5-variable Karnaugh Map (1)

C=0

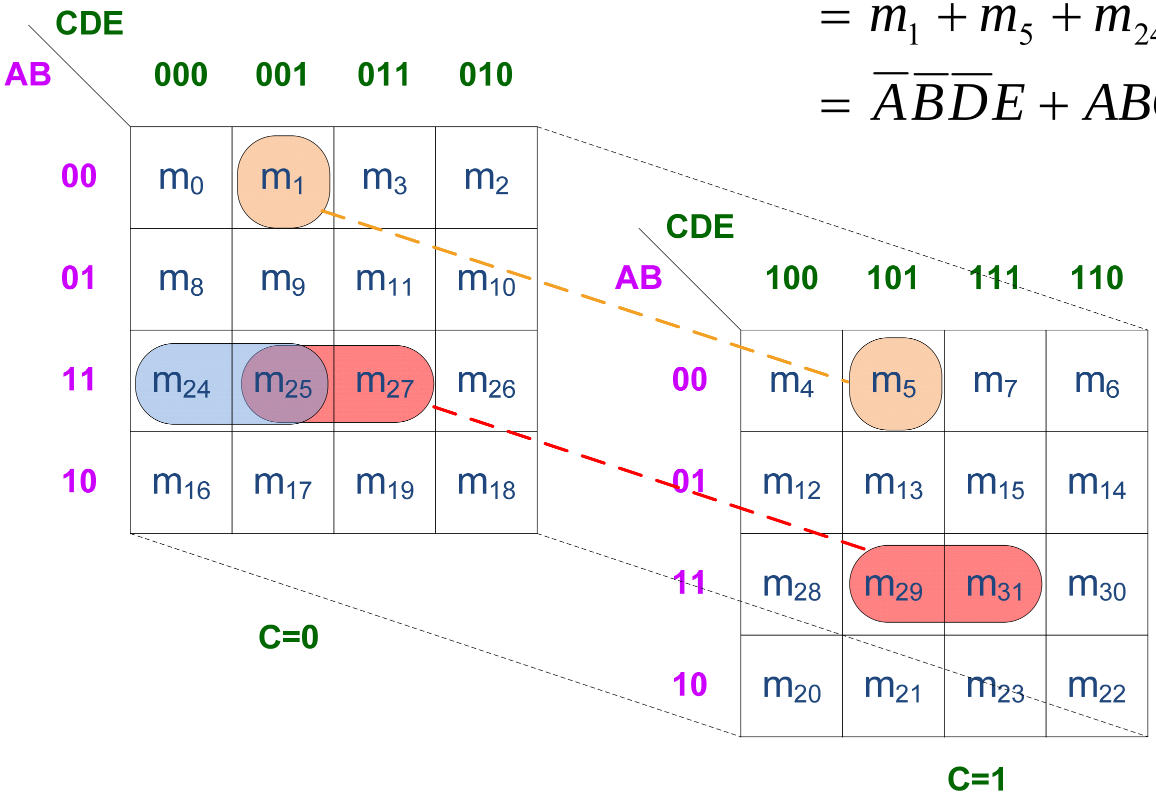
AB \ CDE	000	001	011	010
00	m_0	m_1	m_3	m_2
01	m_8	m_9	m_{11}	m_{10}
11	m_{24}	m_{25}	m_{27}	m_{26}
10	m_{16}	m_{17}	m_{19}	m_{18}

C=1

AB \ CDE	100	101	111	110
00	m_4	m_5	m_7	m_6
01	m_{12}	m_{13}	m_{15}	m_{14}
11	m_{28}	m_{29}	m_{31}	m_{30}
10	m_{20}	m_{21}	m_{23}	m_{22}

5-variable Karnaugh Map (2)

$$F = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D}E + \overline{A}\,\overline{B}C\overline{D}E + AB\overline{C}\,\overline{D}\,\overline{E} + ABC\overline{D}\,\overline{E}$$
$$+ AB\overline{C}DE + ABC\overline{D}E + ABCDE$$
$$= m_1 + m_5 + m_{24} + m_{25} + m_{27} + m_{29} + m_{31}$$
$$= \overline{A}\,\overline{B}\,\overline{D}E + AB\overline{C}\,\overline{D} + AB\overline{E}$$



2 Karnaugh Map SOP Minimization

Mapping a Standard SOP expression
映射标准的最小项之和的函数到卡诺图

Steps:

1. 一个逻辑函数都表示为若干最小项之和的形式。那么自然也就可以设法用卡诺图来表示任意一个逻辑函数。

2.步骤：把逻辑函数化为最小项之和的形式。然后再卡诺图上与这些最小项对应的位置上填入1

任何一个逻辑函数都等于他的卡诺图中填入1的那些最小项之和

$$X = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}\,C + A\,B\,\overline{C} + A\,\overline{B}\,\overline{C}$$

000 001 110 100

A \ BC	00	01	11	10
0	1	1	m_3	m_2
1	1	m_5	m_7	1

Example

$$F = \overset{0011}{\bar{A}\bar{B}CD} + \overset{0101}{\bar{A}B\bar{C}D} + \overset{1101}{AB\bar{C}D} + \overset{1111}{ABCD}$$

$$+ \underset{1100}{AB\bar{C}\bar{D}} + \underset{0001}{\bar{A}\bar{B}\bar{C}D} + \underset{1010}{A\bar{B}C\bar{D}}$$

AB \ CD	00	01	11	10
00		1	1	
01		1		
11	1	1	1	
10				1

Mapping a Nonstandard SOP expression

Example

$$X = \overline{A} + A\overline{B} + AB\overline{C}$$

映射非标准的最小项之和的函数到卡诺图

0××	10×	110
000 001 010 011	100 101	

A \ BC	00	01	11	10
0	1	1	1	1
1	1	1	m_7	1

3 Karnaugh Map Simplification

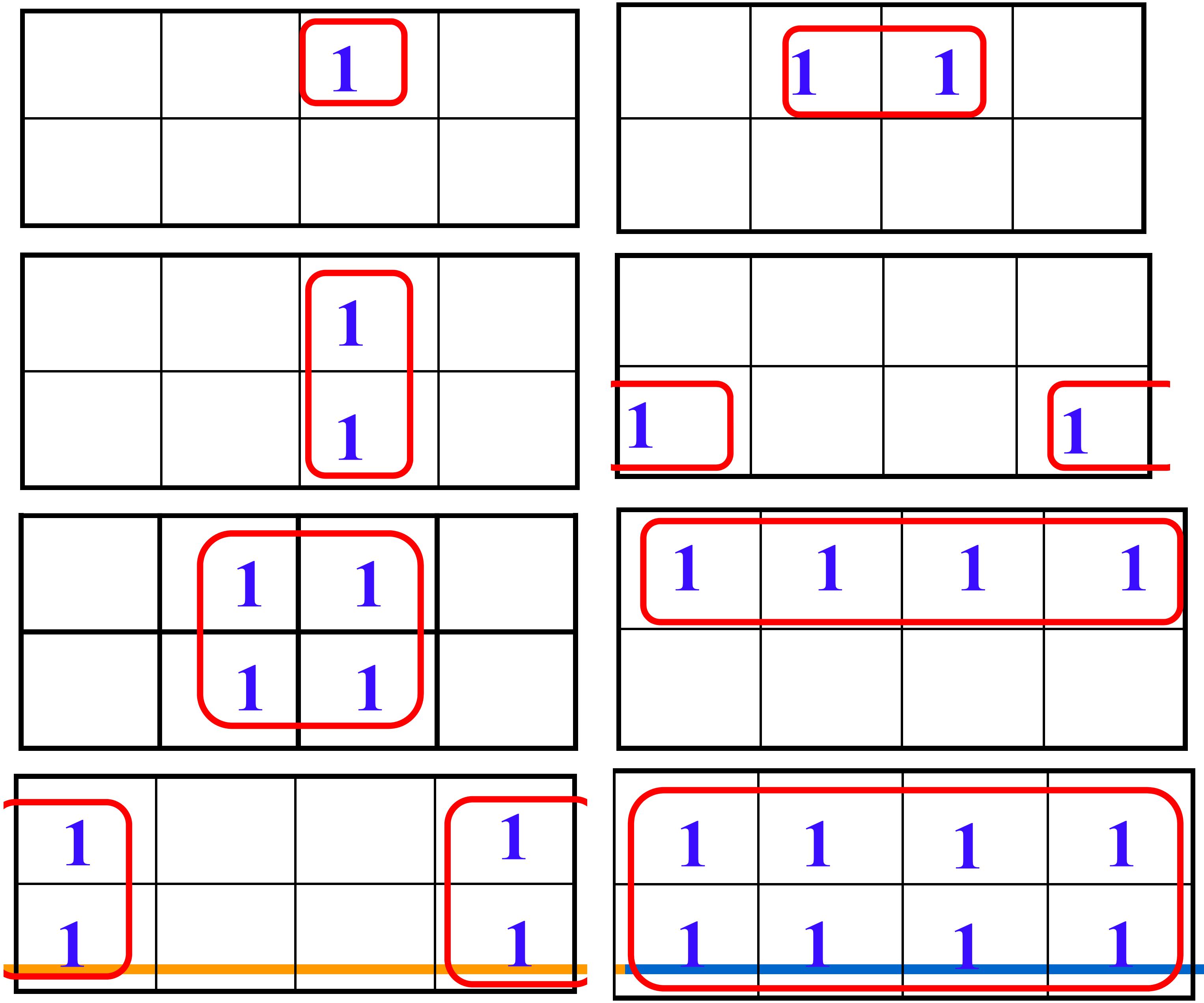
Steps

1. **Grouping the *1*s. Rules:**
   a) **A group must contain either 1,2,4,8,16 cells.**
   b) **Each cell in a group must be *adjacent* to one or more cells in the same group**
   c) **Always include the largest possible of 1s in a group in accordance with rule a)**
   d) ***Each 1* on the map must be included in at least one group**

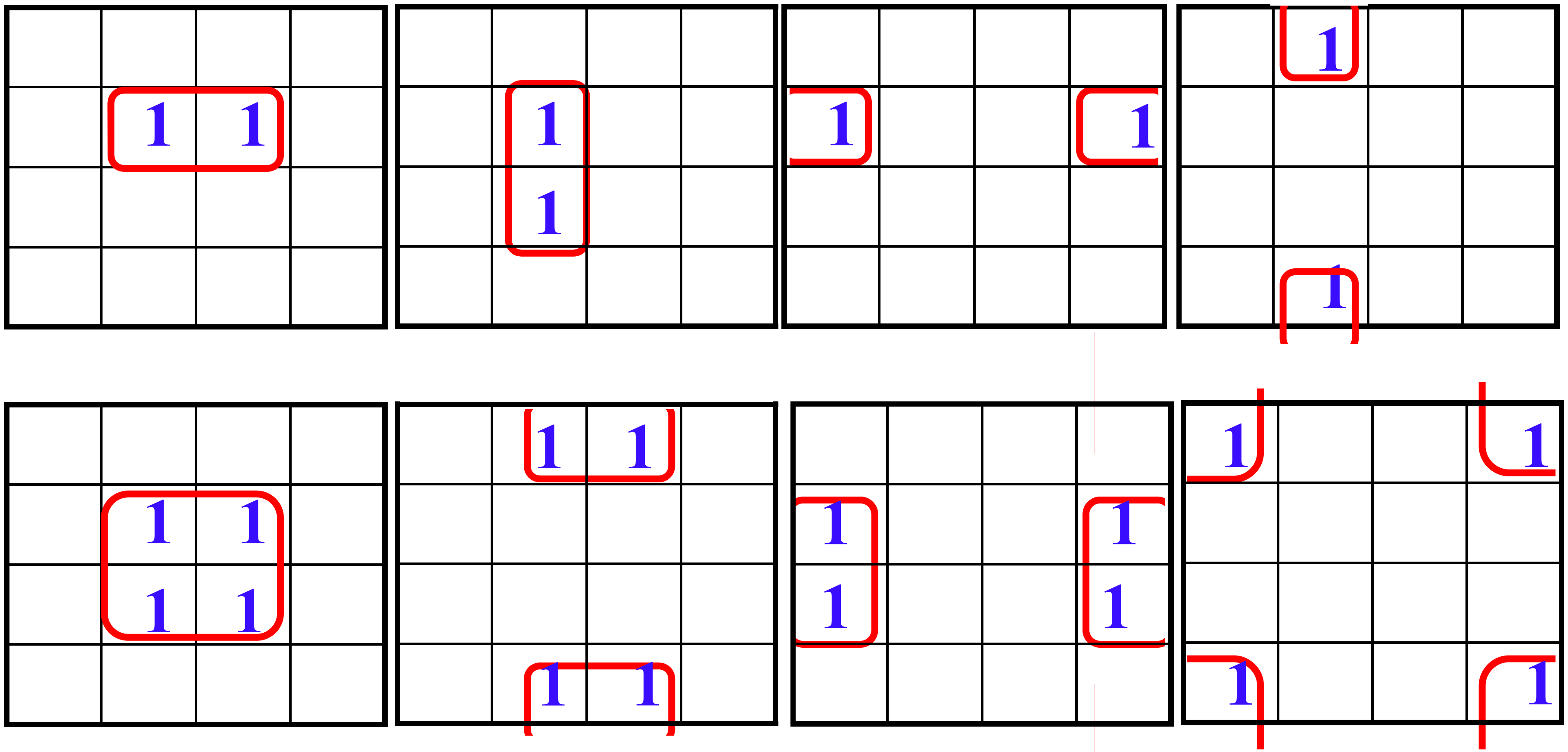
3 Karnaugh Map Simplification

- 化简的依据:具有相邻性的最小项可以合并，并消去不同的因子。卡诺图的几何相邻位置与逻辑相邻位置上一致的，因而能直观的找出具有相邻性的最小项并将其合并化简。
- 合并最小项的规则：合并1的原则
  - A：一组必须包含1，2，4，8，16
  - B:每组中的CELL必须相邻
  - C：符合规则A的含有最多的1的组
  - D：在图中的每个1必须被一个组包含

3-variable Karnaugh map



4-variable Karnaugh map



	1		
	1		
	1		
	1		

1	1	1	1

1	1	1	1
1	1	1	1

	1	1	
	1	1	
	1	1	
	1	1	

1	1	1	1
1	1	1	1

1			1
1			1
1			1
1			1

Karnaugh Map Simplification(II)

2. **Determining the minimum SOP expression from the map. Rules:**
   a) **Variables that occur both un-complemented or complemented within the group are *eliminated*.**

从图中决定最小的SOP，规则：
a) 在同一组中那些互补与非互补发生的变量可以消去

Karnaugh Map Simplification(III)

b) **Determine the minimum product for each group. For example, a 3-variable map:**

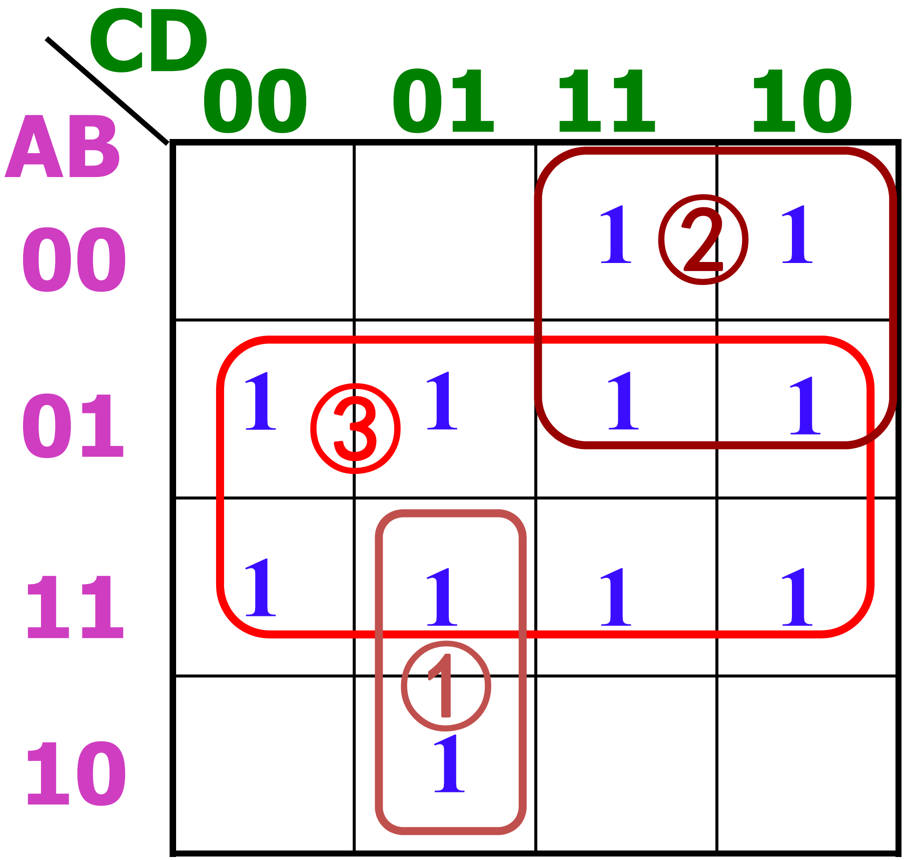
--A 1-cell group yield a 3-variable product term

--A 2-cell group yield a 2-variable product term

--A 4-cell group yield a 1-variable product term

--A 8-cell group yield a value of 1

Example



①Two 1s: $A\overline{C}D$

②Four 1s: $\overline{A}C$

③Eight 1s: B

$$F = A\overline{C}D + \overline{A}C + B$$

Example: Use a Karnaugh map to minimize the following expression:

$$F = A\overline{B}\,\overline{C} + ABC + AB\overline{C} + \overline{A}BC$$

100 **111** **110** **011**

$$F = \sum(m_3, m_4, m_6, m_7) = \sum(3,4,6,7)$$

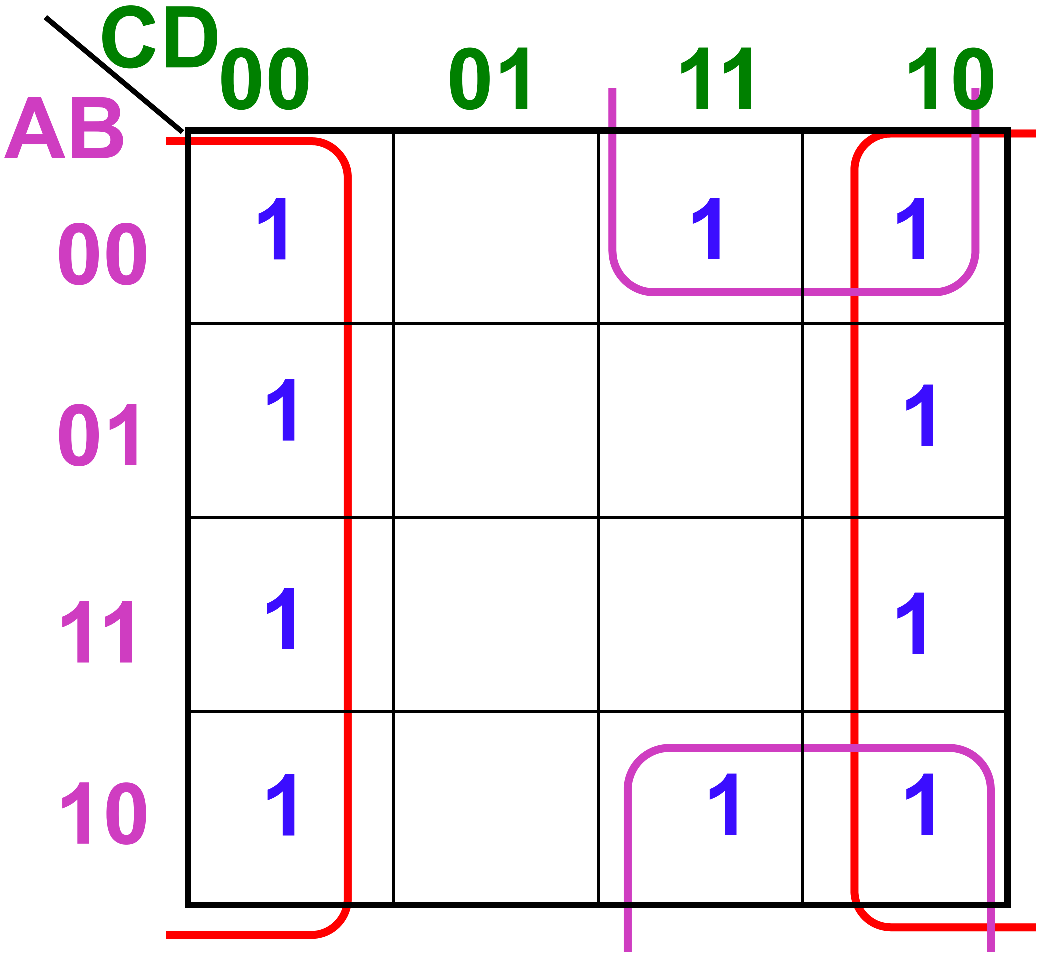
A \ BC	00	01	11	10
0			1	
1	1		1	1

BC $A\overline{C}$

$$F = BC + A\overline{C}$$

Example

$$F = \sum(0,2,3,4,6,8,10,11,12,14)$$



$\overline{D}$

$\overline{B}C$

$$F = \overline{D} + \overline{B}C$$

4 Minimization with "don't care" Terms

"don't care" terms:

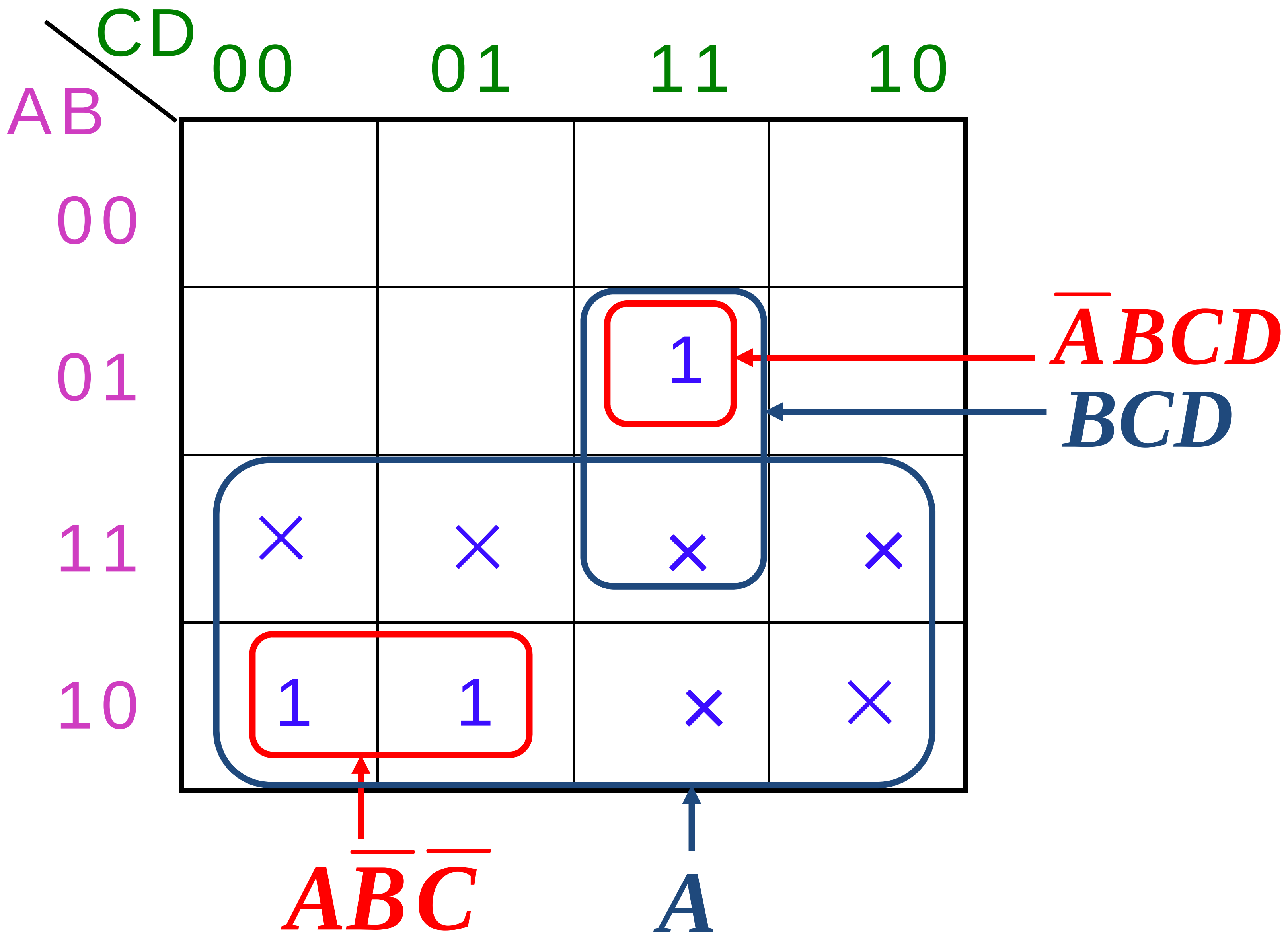
A combination of input literals that can be used as ***a 1 or a 0*** on Karnaugh map.

***Two special cases*:**

- Can not occur
- Can occur but we don't care the value

Example

BCD code	Inputs				Output
	A	B	C	D	Y
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	1
8	1	0	0	0	1
9	1	0	0	1	1
×	1	0	1	0	×
×	1	0	1	1	×
×	1	1	0	0	×
×	1	1	0	1	×
×	1	1	1	0	×
×	1	1	1	1	×



CD
AB
00
01
11
10
00
01
11
10
1
1
1
×
×
×
×
×
×
$\overline{A}BCD$
BCD
$A\overline{B}\overline{C}$
A

Results:

- **Without "don't care" terms**

$$Y = A\overline{B}\,\overline{C} + \overline{A}BCD$$

- **With "don't care" terms**

$$Y = A + BCD$$

Example: For decimal numbers, if x>=5， F=1； else F=0。

1) Truth table

A	B	C	D	F
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	0
0	1	0	1	1
0	1	1	0	1
0	1	1	1	1

A	B	C	D	F
1	0	0	0	1
1	0	0	1	1
1	0	1	0	×
1	0	1	1	×
1	1	0	0	×
1	1	0	1	×
1	1	1	0	×
1	1	1	1	×

2) Karnaugh map $F = A + BC + BD$

AB \ CD	00	01	11	10
00				
01		1	1	1
11	×	×	×	×
10	1	1	×	×

BC

BD

A

5 Mapping a Standard POS Expression

Steps:

- Determine the binary value of each sum term in the standard POS expression. This is the binary value that makes the terms equal to 0.
- As each sum term is evaluated, place a **0** on the Karnaugh map in the corresponding cell.

$$Y=(A+B+C)(A+B+\overline{C})(\overline{A}+\overline{B}+C)(\overline{A}+B+C)$$

$$Y=\prod(0,1,4,6)$$

000 001 110 100

A \ BC	00	01	11	10
0	0	0	m_3	m_2
1	0	m_5	m_7	0

Example

$$F = \prod(1,3,5,10,12,13,15)$$

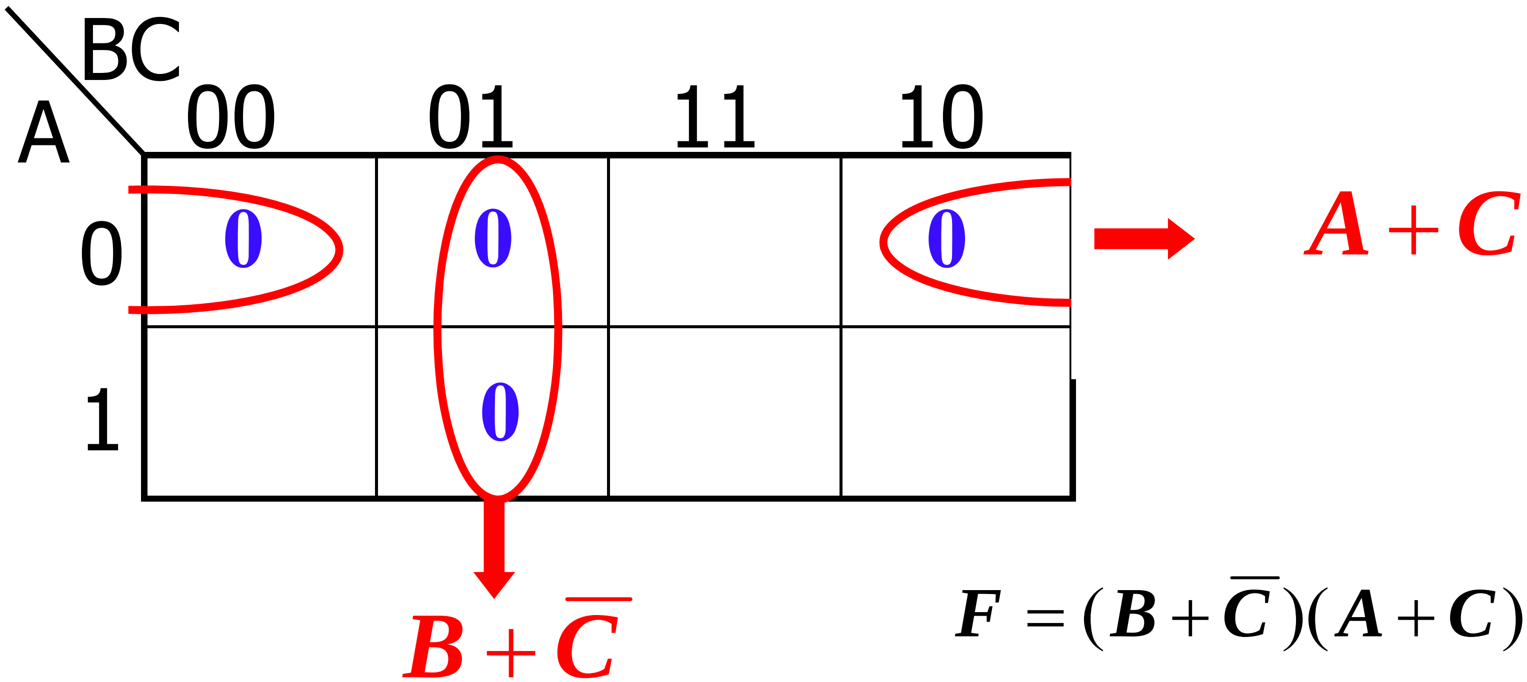
AB \ CD	00	01	11	10
00		0	0	
01		0		
11	0	0	0	
10				0

6 Karnaugh Map Minimization for POS Form

Example: Use a Karnaugh map to minimize the following expression:

$$F = (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(\overline{A} + B + \overline{C})$$

000 **001** **010** **101**



$$F = (B + \overline{C})(A + C)$$

Example: Use a Karnaugh map to minimize the following expression:

$$F = (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(A+\overline{B}+\overline{C})(\overline{A}+\overline{B}+C)$$
$$= \prod(0,1,2,3,6)$$

A \ BC	00	01	11	10
0	0	0	0	0
1	1	1	1	0

A

$A\overline{B}$ $\quad$ AC $\quad$ $\overline{B}+C$

$$F = A\overline{B} + AC$$

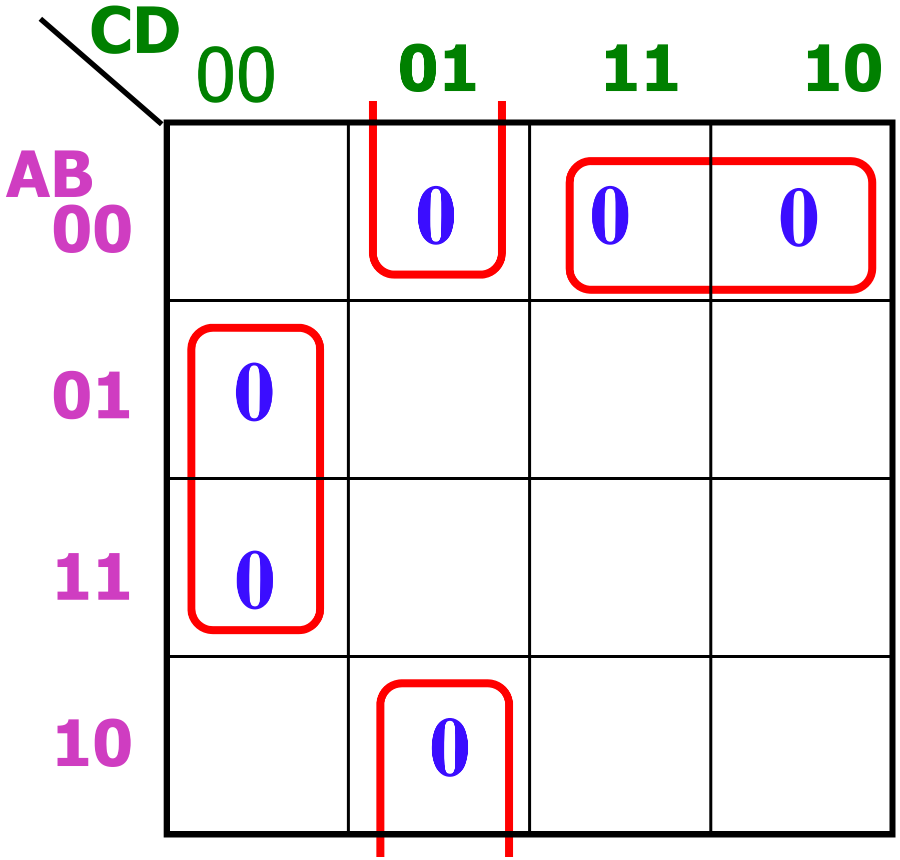
2AND,1 OR

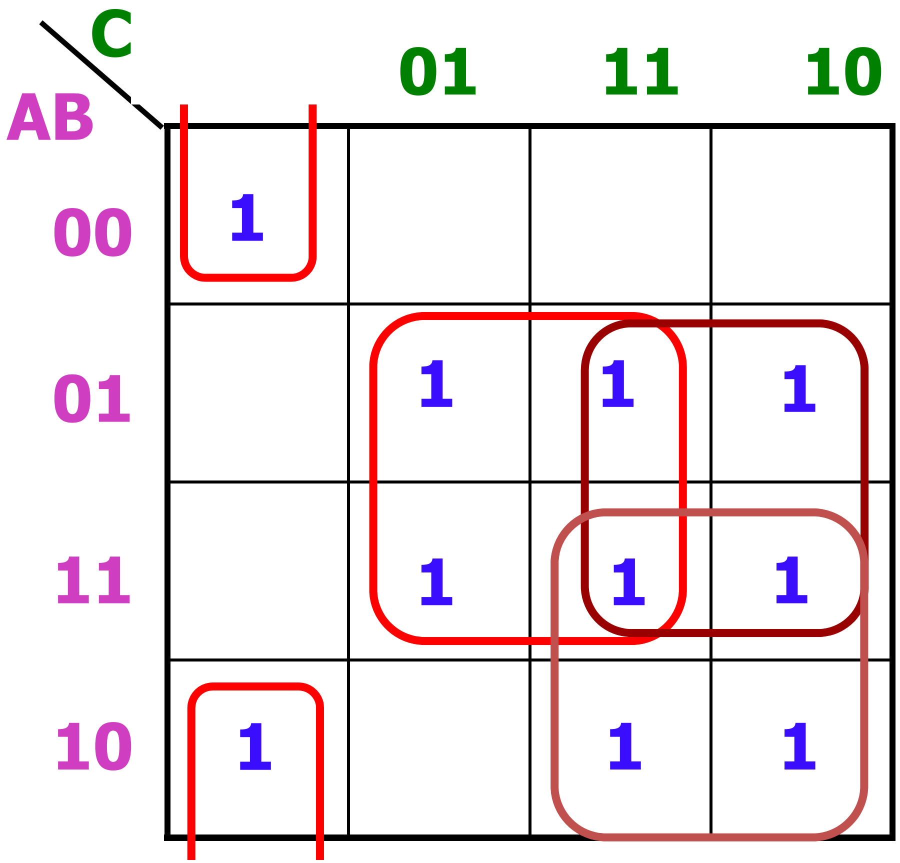
1 AND , 1 OR

$$F = A(\overline{B}+C)$$

$$F=\prod(1,2,3,4,9,12)$$

$$=\sum(0,5,6,7,8,10,11,13,14,15)$$





$$(A+B+\overline{C})(\overline{B}+C+D)(B+C+\overline{D}) \qquad AC+BC+BD+\overline{B}\,\overline{C}\,\overline{D}$$

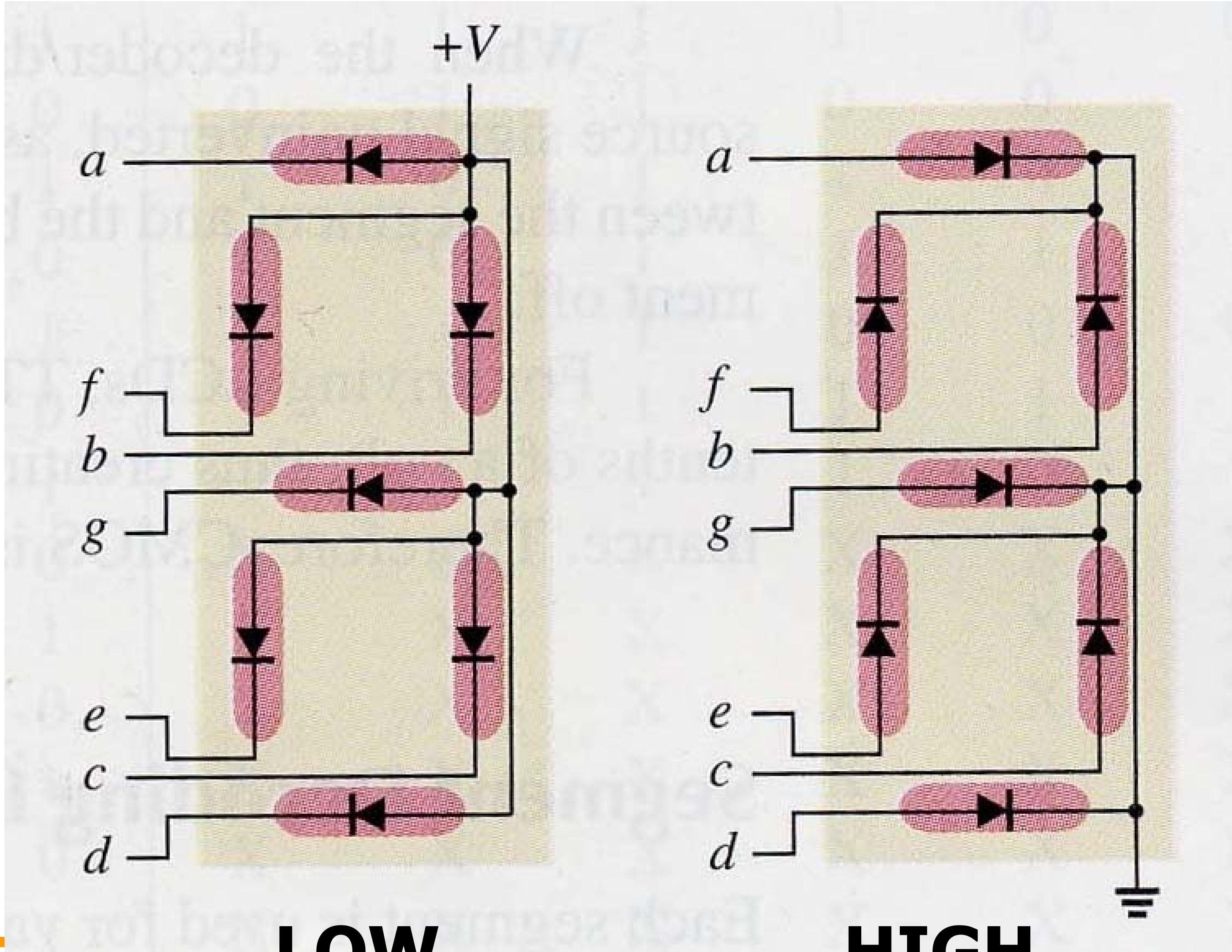
Application I : Seven-segment Display

Seven-segment display segments arrangement

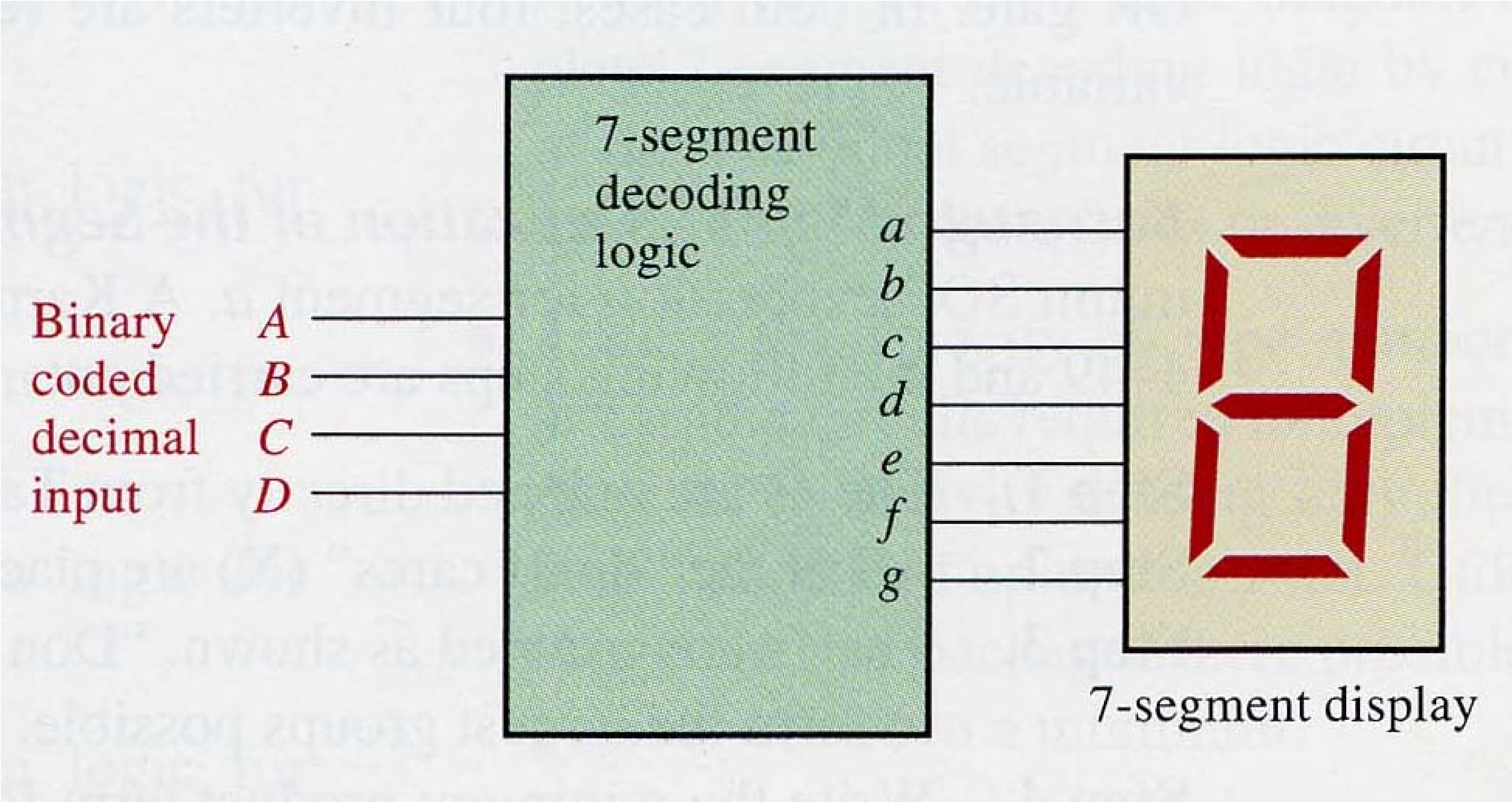
(a) Segment arrangement

(b) Formation of the ten digits and certain letters

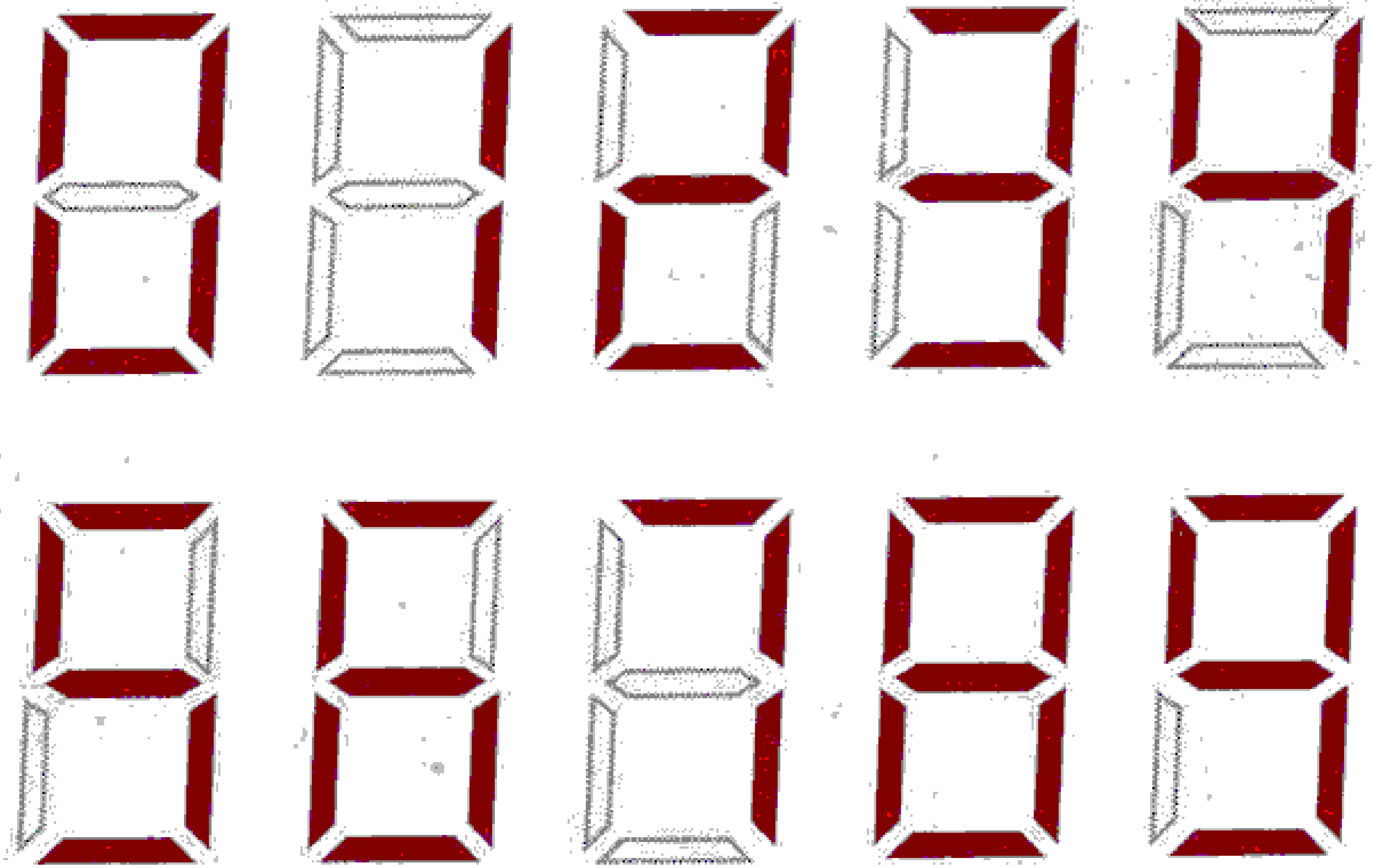
LED: light-emitting diodes



LOW HIGH



7-segment
decoding
logic
Binary
coded
decimal
input
A
B
C
D
a
b
c
d
e
f
g
7-segment display



Digit	Segments Activated
0	*a*, *b*, *c*, *d*, *e*, *f*
1	*b*, *c*
2	*a*, *b*, *d*, *e*, *g*
3	*a*, *b*, *c*, *d*, *g*
4	*b*, *c*, *f*, *g*
5	*a*, *c*, *d*, *f*, *g*
6	*a*, *c*, *d*, *e*, *f*, *g*
7	*a*, *b*, *c*
8	*a*, *b*, *c*, *d*, *e*, *f*, *g*
9	*a*, *b*, *c*, *d*, *f*, *g*

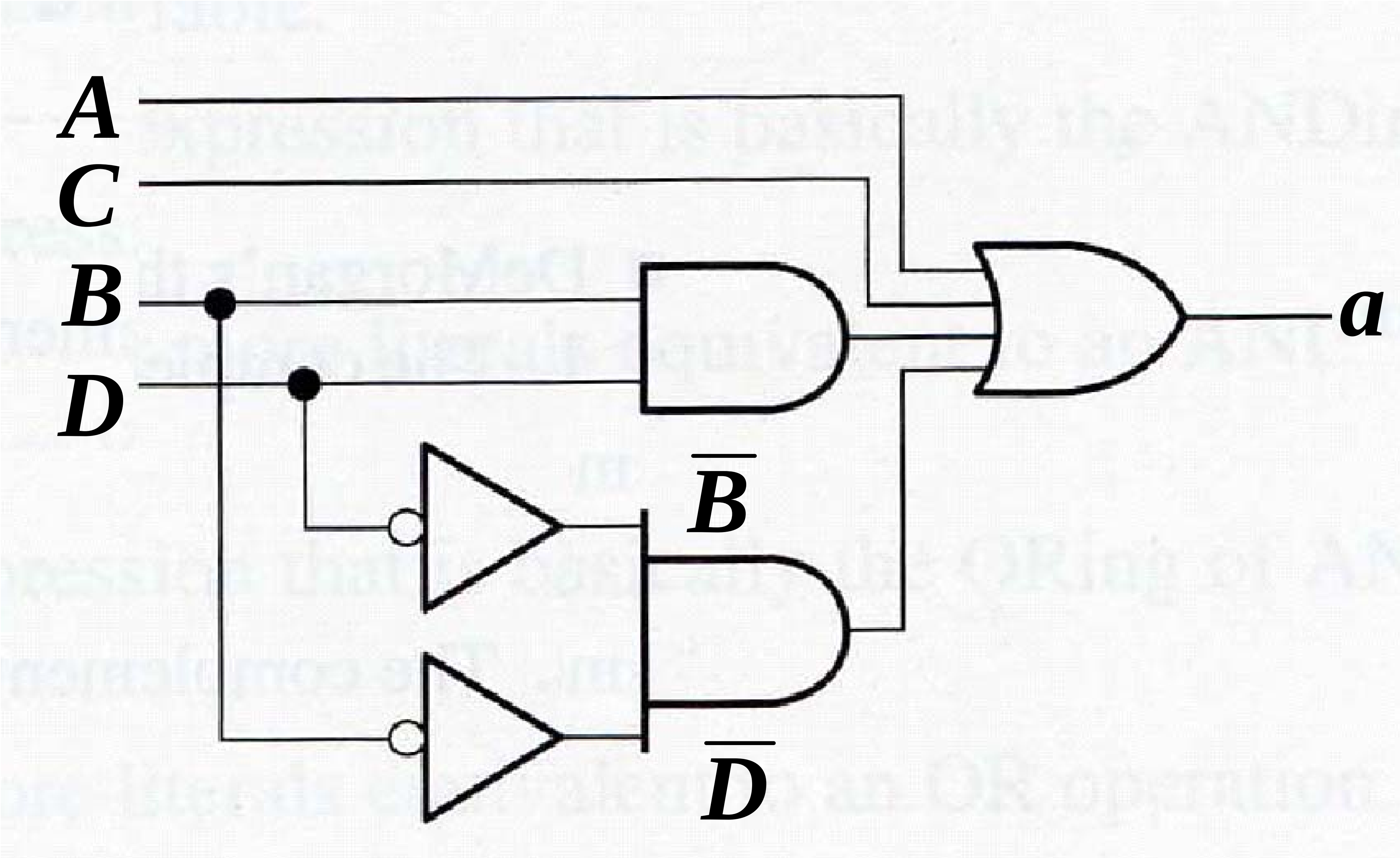
Decimal digit	Inputs				Segment Outputs						
	A	B	C	D	a	b	c	d	e	f	g
0	0	0	0	0	1	1	1	1	1	1	0
1	0	0	0	1	0	1	1	0	0	0	0
2	0	0	1	0	1	1	0	1	1	0	1
3	0	0	1	1	1	1	1	1	0	0	1
4	0	1	0	0	0	1	1	0	0	1	1
5	0	1	0	1	1	0	1	1	0	1	1
6	0	1	1	0	1	0	1	1	1	1	1
7	0	1	1	1	1	1	1	0	0	0	0
8	1	0	0	0	1	1	1	1	1	1	1
9	1	0	0	1	1	1	1	1	0	1	1
×	1	0	1	0	×	×	×	×	×	×	×
×	1	0	1	1	×	×	×	×	×	×	×
×	1	1	0	0	×	×	×	×	×	×	×
×	1	1	0	1	×	×	×	×	×	×	×
×	1	1	1	0	×	×	×	×	×	×	×
×	1	1	1	1	×	×	×	×	×	×	×

$$a = \overline{A}\,\overline{B}\,\overline{C}\,\overline{D} + \overline{A}\,\overline{B}C\overline{D} + \overline{A}\,\overline{B}CD + \overline{A}B\overline{C}D + \overline{A}BC\overline{D}$$

$$+\overline{A}BCD + A\overline{B}\,\overline{C}\,\overline{D} + A\overline{B}\,\overline{C}D$$

AB \ CD	00	01	11	10
00	1		1	1
01		1	1	1
11	×	×	×	×
10	1	1	×	×

$$a = A + C + BD + \overline{B}\,\overline{D}$$



A
C
B
D
a
$\overline{B}$
$\overline{D}$

Quiz

1. The associative law for addition is normally written as

   a. $A + B = B + A$

   b. $(A + B) + C = A + (B + C)$

   c. $AB = BA$

   d. $A + AB = A$

Quiz

2. The Boolean equation $AB + AC = A(B + C)$ illustrates

   a. the distribution law

   b. the commutative law

   c. the associative law

   d. DeMorgan's theorem

Quiz

3. The Boolean expression $A \cdot 1$ is equal to

   a. A

   b. B

   c. 0

   d. 1

Quiz

4. The Boolean expression $A + 1$ is equal to

   a. A

   b. B

   c. 0

   d. 1

Quiz

6. A Boolean expression that is in standard SOP form is

a. the minimum logic expression

b. contains only one product term

c. has every variable in the domain in every term

d. none of the above

Quiz

7. Adjacent cells on a Karnaugh map differ from each other by

   a. one variable

   b. two variables

   c. three variables

   d. answer depends on the size of the map

Quiz

8. The minimum expression that can be read from the Karnaugh map shown is

   a. $X = A$

   b. $X = \overline{A}$

   c. $X = B$

   d. $X = \overline{B}$

	$\overline{C}$	C
$\overline{A}\,\overline{B}$		
$\overline{A}B$		
AB	1	1
$A\overline{B}$	1	1

Quiz

9. The minimum expression that can be read from the Karnaugh map shown is

   a. $X = A$

   b. $X = \overline{A}$

   c. $X = B$

   d. $X = \overline{B}$

	$\overline{C}$	C
$\overline{A}\,\overline{B}$	1	1
$\overline{A}B$		
AB		
$A\overline{B}$	1	1

Quiz

10. In VHDL code, the two main parts are called the

    a. I/O and the module

    b. entity and the architecture

    c. port and the module

    d. port and the architecture

Homework

- Problems: 3, 6e, 10bd, 15b, 21, 22, 33a, 35a, 36a,40c, 44c, 46
- Give the simplest SOP form for segment *f* (exercise in PPT)