Chapter 4

Boolean Algebra and Logic Simplification

（布尔代数和逻辑简化）

4-1 BOOLEAN OPERATIONS AND EXPRESSIONS
（布尔运算和表达式）

- Boolean algebra is the mathematics of digital systems.
  (布尔代数是数字系统的数学工具)

- *A* variable is a symbol (usually an italic uppercase letter) used to represent a logic quantity. Any single variable can have a 1 or 0 value.
  （通常用一个斜体大写字母表示一个变量。一个单变量的取值为0或者1）

4-1 BOOLEAN OPERATIONS AND EXPRESSIONS
（布尔运算和表达式）

- The complement is the inverse of a variable and is indicated by a bar over the variable (overbar). The complement of the variable *A* is read as "not *A*" or "*A* bar". Sometimes a prime symbol rather than an overbar is used to denote the complement of a variable.

- 一个变量的反码也称之为其补码，通常用符号上面的一横表示。读作"not *A*" or "*A* bar"。有时候也用一撇代替一横杠，用来表示一个变量的反码/补码

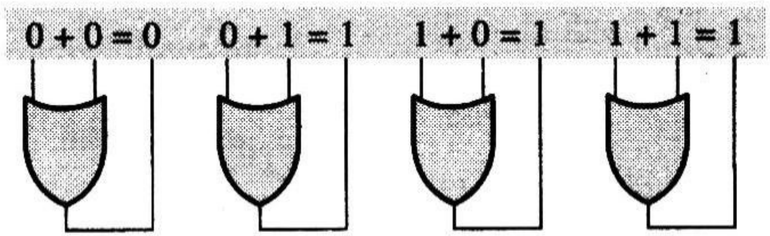
$$A \qquad \overline{A}$$

$$A+B, \qquad A+B+\overline{C}, \qquad \overline{A}+B+C+\overline{D}$$

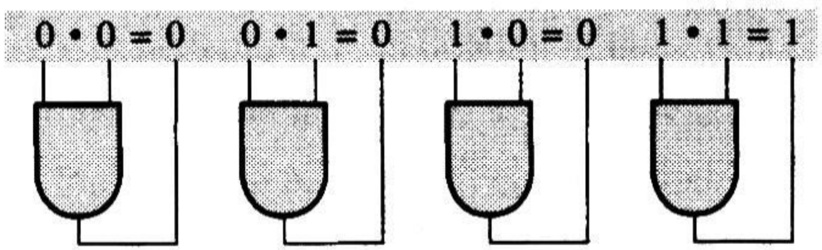
$$AB, \qquad AB\overline{C}, \qquad \overline{A}BC\overline{D}$$

Boolean Operations and Expressions

- Boolean addition is equivalent to the OR operation and the basic rules are illustrated with their relation to the OR gates as follows:



- Boolean multiplication is equivalent to the AND operation and the basic rules are illustrated with their relation to the AND gates as follows:



AND (Boolean Multiplication) : $\cdot$

e.g. A AND B = $A \cdot B$ = AB

OR (Boolean Addition): $+$

e.g. A OR B = A+B

Inverter(NOT): $\bar{\ }$

e.g. $\overline{A}$

sum term

sum term: sum of literals.

e.g. $A+B,\ A+\overline{B}+\overline{C},\ \overline{A}+B+\overline{C}+D$

Example:

Determine the value of A, B, C, and D which make the sum term $A+\overline{B}+C+\overline{D}=0$

Solution:

$$A+\overline{B}+C+\overline{D}=0 \Rightarrow A=0, \overline{B}=0, C=0, \overline{D}=0$$

$$\Rightarrow A=0, B=1, C=0, D=1$$

product term

product term: product of literals.

e.g. $AB,\ A\overline{B}C,\ A\overline{B}CD$

Example:
Determine the value of A, B, C, and D which make the product term $A\overline{B}C\overline{D} = 1$

Solution:

$$A\overline{B}C\overline{D} = 1 \Rightarrow A = 1, \overline{B} = 1, C = 1, \overline{D} = 1$$

$$\Rightarrow A = 1, B = 0, C = 1, D = 0$$

4-2 LAWS AND RULES OF BOOLEAN ALGEBRA
布尔代数的定律和法则

	Laws	Boolean Expression
1	Commutative law of addition Commutative law of multiplication (交换律)	$A+B=B+A$ $AB=BA$
2	Associative law of addition Associative law of multiplication (结合律)	$A+(B+C)=(A+B)+C$ $A(BC)=(AB)C$
3	Distributive law (分配率)	$A(B+C)=AB+AC$

Laws and Rules of Boolean Algebra (I)

Rule Number	**Boolean Expression**
1	$A + 0 = A$
2	$A + 1 = 1$
3	$A \cdot 0 = 0$
4	$A \cdot 1 = A$
5	$A + A = A$
6	$A + \overline{A} = 1$

Laws and Rules of Boolean Algebra (II)

Rule Number	**Boolean Expression**
7	$A \cdot A = A$
8	$A \cdot \overline{A} = 0$
9	$\overline{\overline{A}} = A$
10	$A + AB = A$
11	$A + \overline{A}B = A + B$
12	$(A+B)(A+C) = A + BC$
13	$AB + \overline{A}C + BC = AB + \overline{A}C$

Laws and Rules of Boolean Algebra _PROOF (I)

- This law is similar to absorption in that it can be employed to eliminate extra elements from a Boolean expression:

$$A + \overline{A}B = A + B$$

[PROOF]

$$A + \overline{A}B = (A + AB) + \overline{A}B = A + (AB + \overline{A}B)$$

$$= A + (A + \overline{A})B = A + 1 \cdot B = A + B$$

Laws and Rules of Boolean Algebra _PROOF (II)

$$(A+B)(A+C) = A+BC$$

[PROOF]

$$\begin{aligned}(A+B)(A+C) &= (A+B)A+(A+B)C \\ &= AA+AB+AC+BC \\ &= A+AB+AC+BC \\ &= A(1+B+C)+BC \\ &= A+BC\end{aligned}$$

Laws and Rules of Boolean Algebra (III)

- Consensus theorem

$$AB + \overline{A}C + BC = AB + \overline{A}C$$

[PROOF]

$$AB + \overline{A}C + BC = AB + \overline{A}C + (A + \overline{A})BC$$

$$= AB + \overline{A}C + ABC + \overline{A}BC$$

$$= (AB + ABC) + (\overline{A}C + \overline{A}CB) = AB + \overline{A}C$$

The key to using this theorem is to find a variable and its complement, note the associated terms, and eliminate the included term (the consensus term), which is composed of the associated terms.

4-3 DEMORGAN'S THEOREMS
狄摩根定理

- ***DeMorgan's first theorem***:
- *The complement of a product of variables is equal to the sum of the complements of the variables.*

$$\overline{XY} = \overline{X} + \overline{Y}$$

(equivalency of the NAND and negative-OR gates)

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

- ***DeMorgan's second theorem***:
  The complement of a sum of variables is equal to the product of the complements of the variables.

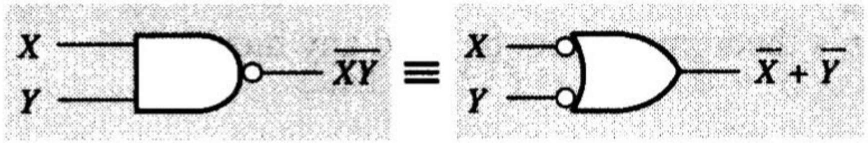
$$\overline{X + Y} = \overline{X}\,\overline{Y}$$

(equivalency of the NOR and negative-AND gates.)

Inputs		Output	
X	Y	$\overline{X+Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

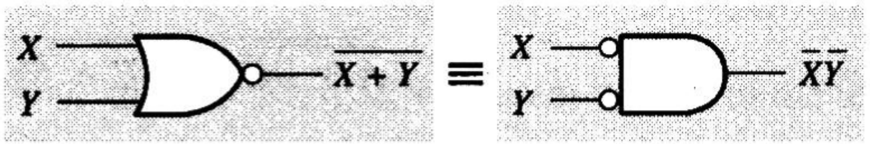
DeMorgan's Theorems

- The gate equivalencies and truth tables for DeMorgan's theorems are shown below:



NAND Negative-OR

Inputs		Output	
X	Y	$\overline{XY}$	$\overline{X} + \overline{Y}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0



NOR Negative-AND

Inputs		Output	
X	Y	$\overline{X + Y}$	$\overline{X}\,\overline{Y}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

FIGURE 4–15

Gate equivalencies and the corresponding truth tables that illustrate DeMorgan's theorems. Notice the equality of the two output columns in each table. This shows that the equivalent gates perform the same logic function.

Example: Apply DeMorgan's theorem to the expressions:

$$\overline{XYZ},\ \overline{X + Y + Z}$$

Solution:

$$\overline{XYZ} = \overline{X} + \overline{YZ} = \overline{X} + (\overline{Y} + \overline{Z})$$
$$= \overline{X} + \overline{Y} + \overline{Z}$$

$$\overline{X + Y + Z} = \overline{X} \cdot \overline{Y + Z} = \overline{X}\,\overline{Y}\overline{Z}$$

Example: Apply DeMorgan's theorems to each of the following expressions:

(a) $\overline{(A+B+C)D}$ (b) $\overline{ABC+DEF}$

Solution:

(a) Let $A+B+C=X,\quad D=Y$:

$\overline{(A+B+C)D} \Rightarrow \overline{XY}=\overline{X}+\overline{Y}=\overline{A+B+C}+\overline{D}$

$\overline{A+B+C} \Rightarrow \overline{A+B+C}=\overline{A}\,\overline{B}\,\overline{C}$

$\overline{(A+B+C)D} \Rightarrow \overline{A}\,\overline{B}\,\overline{C}+\overline{D}$

$$(b)\quad \overline{ABC+DEF} = \overline{(ABC)}\,\overline{(DEF)}$$
$$= (\overline{A}+\overline{B}+\overline{C})(\overline{D}+\overline{E}+\overline{F})$$

Example: Apply DeMorgan's theorem to the following expression:

$$\overline{\overline{A+B\overline{C}}+D(\overline{E+\overline{F}})}$$

Solution:

$$\overline{\overline{A+B\overline{C}}+D(\overline{E+\overline{F}})}$$

$$\overline{X+Y}=\overline{X}\,\overline{Y}$$

$$\Rightarrow\ =(\overline{\overline{A+B\overline{C}}})(\overline{D(\overline{E+\overline{F}})})$$

$$\overline{XY}=\overline{X}+\overline{Y}$$

$$\Rightarrow\ =(A+B\overline{C})(\overline{D}+\overline{\overline{(E+\overline{F})}})$$

$$\Rightarrow\ =(A+B\overline{C})(\overline{D}+E+\overline{F})$$

4.4 Logic function Simplification

Objective:

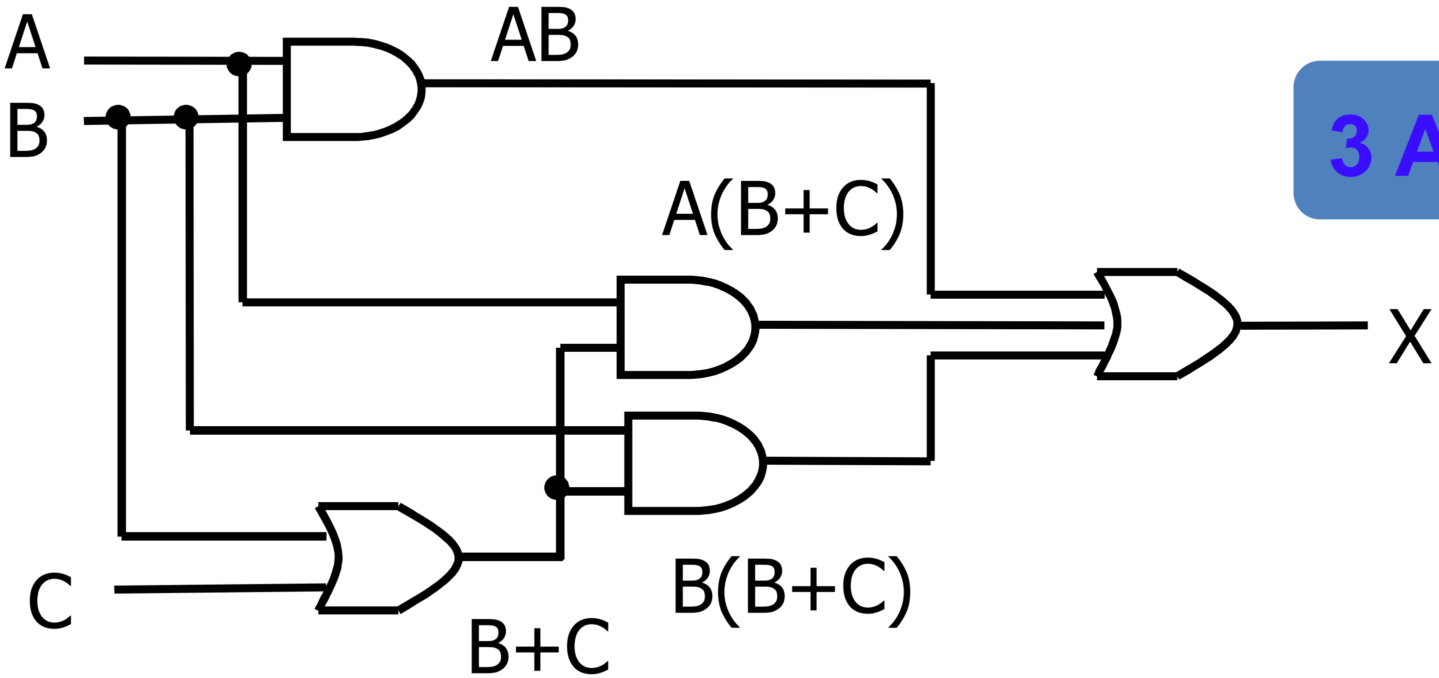
To reduce a particular expression in its ***simplest form*** or ***change its form to a more convenient one*** to implement the expression most efficiently. The approach taken in this section is to use the ***basic laws, rules, and theorems*** of Boolean algebra to simplify an expression.

- Example 4-5 Using Boolean algebra, simplify this expression:

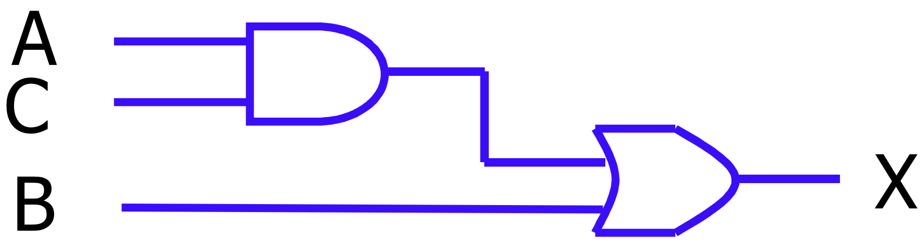
$$X = AB + A(B+C) + B(B+C)$$

- Solution:

$$X = AB + A(B+C) + B(B+C)$$
$$= AB + AB + AC + BB + BC$$
$$= AB + AB + AC + B + BC$$
$$= AB + AC + B + BC$$
$$= AB + AC + B$$
$$= AC + B$$



3 AND,2 OR



1 AND,1 OR

Example:

$$(A\overline{B}(C+BD)+\overline{A}\overline{B})C$$

Example:

$$(A\overline{B}(C+BD)+\overline{A}\,\overline{B})C$$

$$=(A\overline{B}C+A\overline{B}BD+\overline{A}\,\overline{B})C$$

$$=(A\overline{B}C+A\cdot 0\cdot D+\overline{A}\,\overline{B})C$$

$$=(A\overline{B}C+\overline{A}\,\overline{B})C$$

$$=A\overline{B}CC+\overline{A}\,\overline{B}C$$

$$=A\overline{B}C+\overline{A}\,\overline{B}C$$

$$=(A+\overline{A})\overline{B}C$$

$$=1\cdot\overline{B}C$$

$$=\overline{B}C$$

4 AND,2 OR

↓

1 AND

Review: 常用的化简方法

$$1.\ A+\overline{A}=1 \Rightarrow AB+A\overline{B}=A$$

Example

$$F=A\overline{\overline{B}CD}+A\overline{B}CD \quad =A$$

$$AB+A\overline{B}=A$$

$2. A + AB = A$

Example:

$$F = A + \overline{\overline{A} \cdot \overline{BC}}(\overline{A} + \overline{\overline{B}\,\overline{C} + D}) + BC$$

$$\overline{X \cdot Y} = \overline{X} + \overline{Y}$$

$$= A + BC + (A + BC)(\overline{A} + \overline{\overline{B}\,\overline{C} + D})$$

$$A + AB = A$$

$$= A + BC$$

$$3. A + \overline{A}B = A + B$$

Example:

$$F = AC + \overline{A}D + \overline{C}D$$
$$= AC + (\underline{\overline{A} + \overline{C}})D$$

$$\overline{X \cdot Y} = \overline{X} + \overline{Y}$$

$$= AC + \overline{AC}D$$

$$A + \overline{A}B = A + B$$

$$= AC + D$$

$$4. A+\overline{A}=1,\ \ A+A=A$$

$\Rightarrow$ ***using repeated terms***

Example:

$$F=\overline{A}B\overline{C}+\overline{A}BC+ABC \quad \Leftarrow A+A=A$$

$$=\overline{A}B\overline{C}+\underline{\overline{A}BC+\overline{A}BC}+ABC$$

$$=(\overline{A}B\overline{C}+\overline{A}BC)+(\overline{A}BC+ABC)$$

$$=\overline{A}B(C+\overline{C})+(\overline{A}+A)BC \quad \Leftarrow A+\overline{A}=1$$

$$=\overline{A}B+BC$$

Simplification Using Boolean Algebra

- Examples:

$$Y_1 = A\overline{B} + AC + BC$$

$$Y_2 = ABD + A\overline{B}C\overline{D} + A\overline{C}DE + A$$

$$Y_3 = \overline{A}BC + (A + \overline{B})C$$

$$Y_4 = \overline{\overline{A}BC} + \overline{A\overline{B}}$$

$$Y_5 = A\overline{B}(\overline{A}CD + \overline{AD + \overline{B}\,\overline{C}})(\overline{A} + B)$$

$$Y_6 = AC(\overline{C}D + \overline{A}B) + BC\overline{\overline{\overline{B} + AD} + CE}$$

$$Y_7 = \overline{A\overline{B}\,\overline{C}D + A\overline{C}DE + \overline{B}D\overline{E} + A\overline{C}\,\overline{D}E}$$