Assessing a set of additive utility functions for multicriteria decision-making, the UTA method

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The purpose of the method presented in this paper is to assess additive utility functions which aggregate multiple criteria in a composite criterion, using the information given by a subjective ranking on a set of stimuli or actions (weak-order comparison judgments) and the multicriteria evaluations of these actions. It is an ordinal regression method using linear programming to estimate the parameters of the utility function.

Stability and sensitivity analysis leads to the assessment of a set of utility functions by means of post-optimality analysis techniques in linear programming.

Finally, a simple illustrative example is presented and some extensions of the method are proposed.

Introduction

A considerable amount of work has been published on multiattribute utility theory (MAUT) and its use in Decision Analysis. A theoretical framework based on axiomatic considerations enables us to know in which conditions one can use more simple models such as an additive utility function. Nevertheless, as far as assessment procedures are concerned, and despite of the work and procedures developed by Keeney and Raïffa [9], a lot of researchers think there is a lack of realism in the approach, when confronted to many managerial decision process (see, for instance, von Winterfeldt [26]). In Europe relatively few applications of MAUT exist, and those developed in England (cf. [1]), for instance, are rather pragmatic assesssment procedures which do not follow the whole theoretical procedure developed by Keeney and Raïffa.

In France and Belgium, there has been a lot of work done in French, on preference modelling and assessment procedures used in private firms and public administration. Most of the approaches use a partial comparison of the alternatives instead of the assessment of a unique utility function (see Roy [17]).

We present in this paper an assessment procedure of a set of utility functions (UTA [1] method). We assume the axiomatic basis that underlies MAUT since we assume the existence of an additive utility function. But we keep in mind some of the realistic properties of the French approach; the model assessed by UTA is not a single utility function, but is a set of utility functions, all of them being models consistent with the decision-maker's a priori preferences. In order to assess such a set of utility functions, we use an ordinal regression method. Using linear programming, it adjusts optimally additive non-linear utility functions so that they fit data which consist of multicriteria evaluations of some alternatives and a subjective ranking of these alternatives given by the decision-maker.

Sensitivity analyses, by means of post-optimality analysis techniques, leads to a set of utility functions.

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[1] UTilité Additive.

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Each of them gives the same quality adjustment, i.e. the same number of inconsistencies in the estimation procedure.

Used interactively, the UTA method can reduce the set of assessed utility functions to a single utility function. But it is also possible to use the set of utility functions to assess a so-called outranking relation which gives only a partial order on the alternatives. This paper does not develop these last points.

Section 1 is a brief overview on additive utilities and their assessment procedures. Section 2 contains the mathematical model. An illustrative example is presented in Section 3 and extensions of the model are discussed in Section 4.

1. Additive utility functions and assessment procedures

1.1. Additive utility functions

In multicriteria decision-making, one usually consider a set of actions (alternatives, stimuli,...), called A, which is valued by a family of criteria $\boldsymbol{g} = (g_1, g_2, \ldots, g_n)$. A classical operational attitude of assessing a model of overall preference of an individual (decision-maker,...) leads to the aggregation of all criteria into a unique criterion called a utility function (Roy [16], Keeney and Raïffa [9]),

$$U(\boldsymbol{g}) = U(g_1, g_2, \ldots, g_n). \tag{1}$$

Let us call P the strict preference relation and I the indifference relation. If $\boldsymbol{g}(a) = [g_1(a), g_2(a), \ldots, g_n(a)]$ is the multicriteria evaluation of an action a, then the following properties generally hold for the utility function U:

$$U[\boldsymbol{g}(a)] > U[\boldsymbol{g}(b)] \Leftrightarrow aPb, \tag{2}$$

$$U[\boldsymbol{g}(a)] = U[\boldsymbol{g}(b)] \Leftrightarrow aIb \tag{3}$$

and the relation $R = P \cup I$ is a weak order.

The utility function is additive if it is of the form

$$U(\boldsymbol{g}) = \sum_{i=1}^{n} u_i(g_i). \tag{4}$$

It is the most used form in practice, especially in its linear form of a weighted sum of criteria

$$U(\boldsymbol{g}) = \sum_{i=1}^{n} p_i g_i, \tag{5}$$

where each marginal utility $u_i(g_i)$ is entirely determined by the criterion g_i and a weight p_i. A fundamental restrictive hypothesis when using a unique additive function (4), and then a fortiori (5), is the mutual preferential independence condition (Keeney and Raïffa [9]). Other conditions of additivity have been proposed by Fishburn [4,5].

It is common, in current applications, to normalize the utility function. Suppose that

$$U(\boldsymbol{g}) = \sum_{i=1}^{n} p_i w_i(g_i), \quad \text{where } w_i(g_i) = \frac{1}{p_i} u_i(g_i) \quad \text{for all } i, \tag{6}$$

and where the factors p_i are the weights of the criteria. Let g_i^* and g_{i*} be respectively the most and less preferred value (grade) on the criterion i. The most common normalization constraints are the following (Keeney and Raïffa [9]):

$$\begin{cases} \sum_{i=1}^{n} p_i = 1, \\ w_i(g_{i*}) = 0 \quad \text{for all } i, \\ w_i(g_i^*) = 1 \quad \text{for all } i. \end{cases} \tag{7}$$

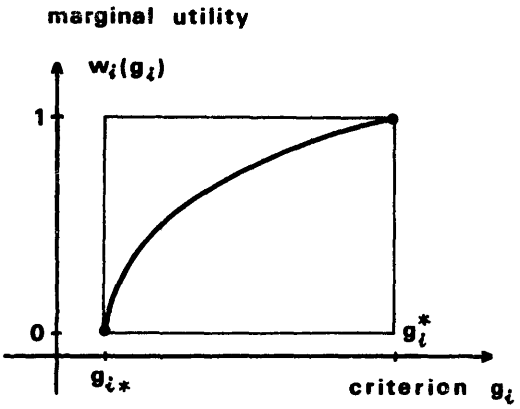

Fig. 1. The normalized marginal utility function.

If we assume the usual hypothesis of a non-decreasing preference on each criterion, the marginal utilities u_i or w_i are then monotone non-decreasing functions of the g_i (Fig. 1). Furthermore, the weights p_i can be interpreted as the relative importance of the criteria. When using the additive form (4), the normalization constraints (7) become

$$\begin{cases} \sum_{i=1}^{n} u_i(g_i^*) = 1, \\ u_i(g_{i*}) = 0 \text{ for all } i. \end{cases} \tag{8}$$

Form (4) with constraints (8) is equivalent to form (6) with constraints (7).

1.2. Assessment procedures of additive utility functions

It is possible to consider two classes of assessing methods. Fishburn [6] gives an excellent review of the first class; he distinguishes direct methods where the utilities $u_i(g_i)$ are estimated from judgments (rating, ranking, preferences on lotteries) for each individual criterion, and compensation methods where these are estimated from "investigation" in terms of compensation and so involve tradeoffs between at least two criteria usually with the help of indifference curves.

The second class of methods is based, either upon the observation of choices made in A by one or several individuals (choice of products, modal choice in transportation, acceptance of candidates, etc.), or upon the knowledge of subjective preferences on A (total or partial weak-orders on products, projects, well-known reference alternatives, etc.). The problem is then to estimate an additive utility function, that is as "consistent" as possible with the observed choices or known subjective preferences.

Table 1 presents some assessing methods of the second class. Methods 3, 4 and 5 estimate the weights of

Table 1
Some assessing methods; (5): estimation of p_i, (4) and (6): estimation of $u_i(g_i)$

No.	Method	Nature of the analysed preference	Type of additive utility
1	Discriminant analysis	Nominal	(5)
2	Multiple Linear Regression	Cardinal	(5)
3	Wagner [25]	Cardinal	(5)
4	ORDREG (Srinivasan and Shocker [22])	Ordinal	(5)
5	Pekelman and Sen [14]	Ordinal	(5)
6	DISQUAL (Saporta [18])	Nominal	(4)
7	MORALS (Young, De Leeuw and Takane [27])	Ordinal	(6)
8	UTA (method presented in section 2).	Ordinal	(4) or (6)

the criteria in an optimal way using linear programming techniques ("goal programming", Charnes and Cooper [3]). These approaches sometimes "force" the criteria to be quantitative variables without dealing with qualitative information.

The purpose of the UTA method, presented in Section 2, is to estimate utility functions of type (4) or (6) also using linear programming.

2. Development of the UTA method

2.1. The input data

Let $G_i = [g_{i*}, g_i^*]$, $i = 1, 2, \ldots, n$ be the intervals in which the values of each criterion are found, then we call product consequence space the set $G = \times_{i=1}^{n} G_i$.

The subjective preference is a weak order $R = (P, I)$ on a set (let us call it A') of real or imaginary actions with multicriteria evaluations in G. The data then consists of the multicriteria evaluation information and the weak order R defined on A'. The method described below proceeds in two steps, i.e. the assessment of an optimal utility and its sensitivity analysis using special linear programming techniques.

2.2. Step 1: Assessment of an "optimal" utility function $U^*(g)$

When one or several evaluation scales G_i are continuous or when they contain too large a number of grades g_i^j, it is possible to estimate the corresponding marginal utility functions in a piecewise linear fashion. We suppose that the extreme values g_{i*}, g_i^* on each criterion are finite and we cut the interval $[g_{i*}, g_i^*]$ into $(\alpha_i - 1)$ equal intervals $[g_i^j, g_i^{j+1}]$. α_i is given by the analyst who indicates in that way the number of estimated points of each marginal utility u_i. The end points g_i^j are then given by the formula

$$g_i^j = g_{i*} + \frac{j-1}{\alpha_i - 1}(g_i^* - g_{i*}).$$

The variables to estimate are $u_i(g_i^j)$. The marginal utility of an action a is approximated by a linear interpolation. Thus, for $g_i(a) \in [g_i^j, g_i^{j+1}]$, we have

$$u_i[g_i(a)] = u_i(g_i^j) + \frac{g_i(a) - g_i^j}{g_i^{j+1} - g_i^j}\left[u_i(g_i^{j+1}) - u_i(g_i^j)\right].$$

For example, if $[g_i^j, g_i^{j+1}] = [3,4]$ and $g_i(a) = 3.8$, then $u_i[g_i(a)] = 0.2u_i(3) + 0.8u_i(4)$.

When the interval G_i is discrete with few grades, we can choose α_i equal to the number of grades. If, for instance, $G_i = [5, 6, 7, 8, 9, 10]$, we put $\alpha_i = 6$ in order to find the utilities $u_i(5), u_i(6), \ldots, u_i(10)$.

Considering the relations (2), (3) and (4), let us call

$$U'[g(a)] = \sum_{i=1}^{n} u_i[g_i(a)] + \sigma(a) \quad \text{for all } a \in A', \tag{9}$$

$\sigma(a)$ being a potential error relative to the utility

$$U[g(a)] = \sum_{i=1}^{n} u_i[g_i(a)]$$

The introduction of potential variables $\sigma(a) \geq 0$, $a \in A'$, instead of variables of type z_{ab} for all pairs $(a, b) \in R$ (as in methods 3–5 of Table 1; for the definition of such variables z_{ab}, see formula (13)) is possible because of the transitivity of R.

Indeed, it is useless to write all the equalities and inequalities of type (2) and (3) (cf. Fig. 2). Writing

$$U'[g(a)] - U'[g(b)] \geq \delta \Leftrightarrow aPb \tag{10}$$

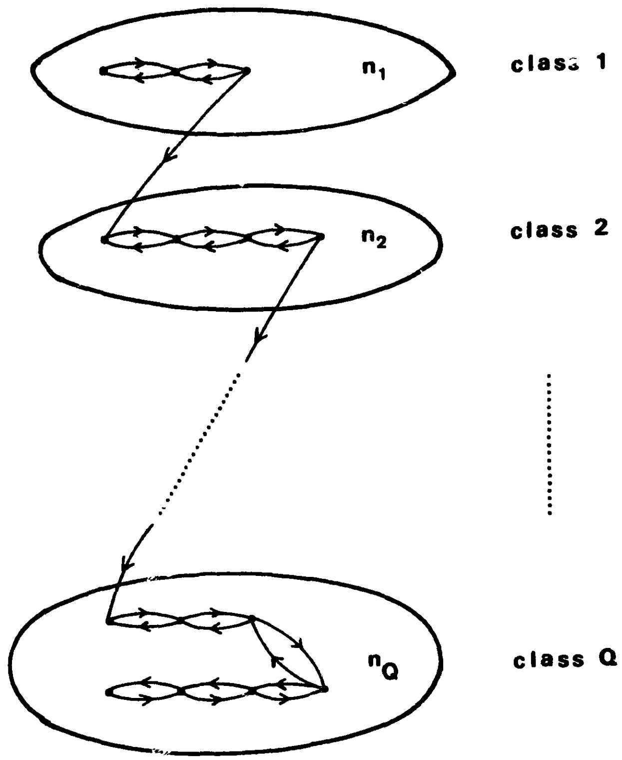

Fig. 2. The R^* relation deduced from the weak order containing Q indifference classes

with $\delta > 0$, a small real number[2] depending on $|A'|$, the number Q of indifference classes in R and on the values of indifference thresholds s_i defined below (see formula (14)). Relations (9) and (10) give

$$\sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + \sigma(a) - \sigma(b) \geq \delta \Leftrightarrow aPb, \tag{11}$$

and for indifferent pairs of actions, we write

$$\sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + \sigma(a) - \sigma(b) = 0 \Leftrightarrow aIb. \tag{12}$$

Taking three actions a, a', a'' belonging to the following three different indifference classes of R (aPa' and $a'Pa''$) we have from (11)

$$\sum_{i=1}^{n} u_i[g_i(a)] - \sum_{i=1}^{n} u_i[g_i(a')] + \sigma(a) - \sigma(a') \geq \delta,$$

$$\sum_{i=1}^{n} u_i[g_i(a')] - \sum_{i=1}^{n} u_i[g_i(a'')] + \sigma(a') - \sigma(a'') \geq \delta$$

[2] δ must be choosen in order to separate significantly two classes of the R weak order, but it should never be greater than $1/Q$. It is recommended to use different values for δ to select the one giving the best adjustment. These values must lie for instance in the interval $[\frac{1}{10}Q, 1/Q]$.

so, summing the two inequalities

$$\sum_{i=1}^{n} u_i[g_i(a)] - \sum_{i=1}^{n} u_i[g_i(a'')] + \sigma(a) - \sigma(a'') \geqslant 2\delta \Rightarrow aPa''.$$

Thus the condition of type (11) for the pair (a, a'') carries no further information once preferences between (a, a') and (a', a'') are known. Similarly, the following consequences hold:

aPa' and $a'Ia'' \Rightarrow aPa''$,
aIa' and $a'Ia'' \Rightarrow aIa''$.

Let us consider a weak-order R with Q indifference classes (Fig. 2), each of them having n_q actions $(q = 1, 2, \ldots, Q)$. Only the relations represented by the arrows have to be taken into account in order to satisfy the system (11)–(12), the other being redundant. Let us call $R^* = P^* \cup I^*$ such a sub-relation of $R = P \cup I$.

The property of transitivity used in our model forbids us to analyse non-transitive preferences. Otherwise, one would have to assess a model in which each pair is associated with either one or two variables according to the following formulae:

$$\begin{cases} \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} \geqslant \delta \Leftrightarrow aPb, \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} - z_{ba} = 0 \Leftrightarrow aIb \\ \text{with } z \geqslant 0. \end{cases} \tag{13}$$

This extension is proposed in Section 4.1.

Taking into account the hypothesis of Section 1.1 on monotonicity of preferences, the marginal utilities $u_i(g_i)$ must satisfy the set of constraints (14):

$$u_i(g_i^{j+1}) - u_i(g_i^j) \geqslant s_i, \quad j = 1, 2, \ldots, \alpha_i - 1, \; i = 1, 2, \ldots, n, \tag{14}$$

$s_i \geqslant 0$ being indifference thresholds defined on each criterion g_i. It is not necessary to use indifference thresholds in this model $(s_i = 0)$. Nevertheless, when the evaluation g_i and the subjective preference R are given by the same individual, it can be useful to introduce such a threshold in order to avoid phenomena such as: $u_i(g_i^{j+1}) = u_i(g_i^j)$ when $g_i^{j+1} P g_i^j$. One can first study the case where $s_i = 0$ for every i, and then choose the thresholds for some criteria when we know the first set of weights $u_i(g_i^*)$.

The utilities $u_i(g_i^j)$ are estimated by means of a linear programme with (8), (11), (12), (14) as constraints and with an objective function depending on the $\sigma(a)$, $a \in A'$, as described below.

Table 2
Dimensions of the (PL1) linear programme

Constraints			Variables	
Formula	Sign	Number	Nature	Number
(11)	$\geqslant$	$Q-1$	$u_i(g_i^j)$	$\sum_{i=1}^{n} (\alpha_i - 1)$
(12)	$\geqslant$	$2\sum_{i=1}^{Q} (n_i - 1)$		
(14)	$\geqslant$	$\sum_{i=1}^{n} (\alpha_i - 1)$	$\sigma(a)$	$\lvert A' \rvert$
$\sum_{i=1}^{n} u_i[g_i^*] = 1$	$\geqslant$	2		

For simplicity, we use a linear objective function in order to minimize the amount of total deviation (15):

$$F = \sum_{a \in A'} \sigma(a). \tag{15}$$

It is possible to weigh the potential errors in order to take into account a different degree of confidence in each ranked action

$$F = \sum_{a \in A'} p(a)\,\sigma(a). \tag{15'}$$

The linear programme is the following one (PL1):

$$\left\{\begin{array}{ll} [\min]\ F = \sum_{a \in A'} \sigma(a) & \\ \text{under the constraints:} & \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + \sigma(a) - \sigma(b) \geqslant \delta & \text{if } aP^*b, \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + \sigma(a) - \sigma(b) = 0 & \text{if } aI^*b, \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geqslant s_i & \text{for all } i \text{ and } j, \\ \sum_{i=1}^{n} u_i(g_i^*) = 1 & \\ u_i(g_{i*}) = 0,\ u_i(g_i^j) \geqslant 0,\ \sigma(a) \geqslant 0 & \text{for all } i \text{ and } j \text{ and} \\ & \text{for all } a \in A'. \end{array}\right. \tag{PL1}$$

Table 2 gives the dimensions of (PL1) taking into account the transitivity of the preferences (Fig. 2).

The structure of (PL1) is such that it is more useful to solve the dual linear programme. Doing so avoids introducing artificial variables and saves computer time and memory.

Let (PL1′) be the associated canonical form of (PL1). Let $\mathscr{A}$ be the matrix of (PL1′), $\boldsymbol{x}$ the column vector of the marginal utilities u_i and the deviations $\sigma(a)$, $\boldsymbol{b}$ the second member of the linear programme, and $\boldsymbol{c}$ the row vector in which the first $(\sum_{i=1}^{n}(\alpha_i - 1))$ elements are 0 and the $|A'|$ following elements are 1. Then (PL1′) can be written

$$\left\{\begin{array}{l} [\min]\ F = \boldsymbol{cx} \\ \text{under the constraints:} \\ \mathscr{A}\boldsymbol{x} \geqslant \boldsymbol{b}, \\ \boldsymbol{x} \geqslant 0 \end{array}\right. \tag{PL1'}$$

where the dimensions of the matrix $\mathscr{A}$ are given in Table 2.

Let (PL2) be the dual linear programme of (PL1′). Solving PL1 (PL1′ or PL2) leads to an optimal utility function $U^*(\boldsymbol{g})$ and to a corresponding set of potential errors $\{\sigma(a), a \in A\}$.

2.3. Step 2: Assessment of a set $\mathscr{U}$ of utility functions by means of post-optimality analysis

Until now we have assessed one optimal utility function $U^*(\boldsymbol{g})$ which is an "optimal" numerical representation of the preference relation R. However, if the optimum F^* of (PL1) or G^* of (PL2) is 0, this means that the polyhedron of admissible solutions for $u_i(g_i)$ is not empty and that many utility functions lead to a perfect representation of the relation R.

Even when the optimal value F^* is strictly positive (in case of the empty polyhedron) the discussion about the optimality objective investigated in Section 4.1 shows that other solutions, less good for F, can nevertheless improve another satisfactory criterion namely Kendall's τ. The experience with the model

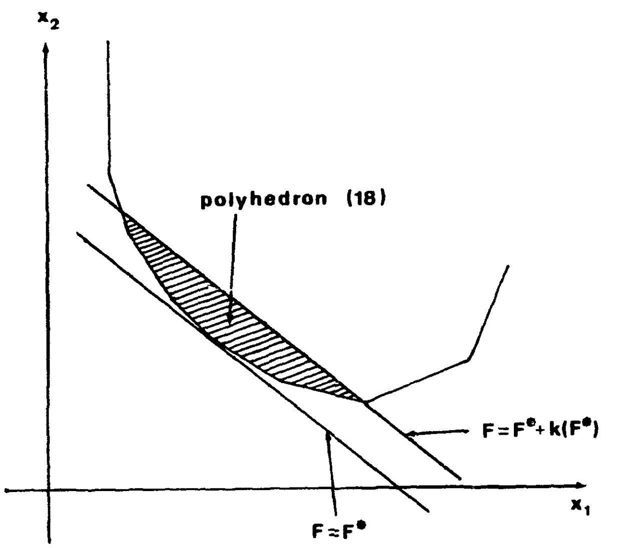

Fig. 3 Post-optimality analysis: exploration of the hatched polyhedron.

confirms that non-optimal utility functions $U(\boldsymbol{g})$ (for which $F > F^*$) give weak-orders R', which are closer to R (in the sense of Kendall or Spearman distances) than the weak-order deduced from the so-called optimal utility $U^*(\boldsymbol{g})$.

Some classical phenomena in mathematical programming such as primal or dual degeneracy, and as criteria correlation phenomena in statistics, are not taken into account in search for an optimal solution. It is then absolutely necessary to explore solutions around the obtained optimum. This analysis, called post-optimality analysis (cf. Van de Panne, [24]), is formalized in the following way.

Let $F^* = G^*$ be the optimal value of the objective function of (PL1) or (PL2) and let us consider a real positive threshold $k(F^*)$ which is a very small proportion of F^*. The k-optimality analysis consists in exploring the vertices of a new polyhedron which is obtained adding the constraint (16):

$$F \leqslant F^* + k(F^*) \tag{16}$$

which can also be written

$$-\sum_{a \in A'} \sigma(a) \geqslant -[F^* + k(F^*)] \tag{17}$$

to the linear programming (PL1) or (PL1′) constraints.

The set $\mathcal{U}$ of admissible utility functions as numerical representation of the preference relation R is defined by the following polyhedron:

$$\begin{cases} \mathcal{A}'\boldsymbol{x} \geqslant \boldsymbol{b}', \\ \boldsymbol{x} \geqslant 0 \end{cases} \tag{18}$$

with $\mathcal{A}'$ and $\boldsymbol{b}'$ being respectively the matrices $\mathcal{A}$ and $\boldsymbol{b}$ of (PL1′) increased by the supplementary line (17). This polyhedron is represented in 2-dimensional space by the hatched area in fig. 3.

The algorithms which could be used to explore $\mathcal{U}$ (the polyhedron (18)) are branch and bound methods (reverse simplex methods, cf. Van de Panne [24, p. 202–231]) or techniques dealing with the notion of the labyrinth in graph theory such as Tarry's method (cf. Charnes, Cooper [3]), or the method of Manas and Nedoma [12], etc. Through the variation of the parameter $k(F^*)$, and, in consequence, of the set $\mathcal{U}$ one can get a very clear idea of the stability of $U^*(\boldsymbol{g})$. From this sensitivity analysis we can deduce from the $u_i(g_i)$ intervals or average values, or even base the modelling of individual preferences on multiple utility functions consistent with R via deterministic or fuzzy outranking relations (see Jacquet-Lagrèze [8] or Siskos [20]).

Table 3
A subjective preference system for ten reference cars (source of criteria informations: Spécial Salon, L'action automobile et touristique, No. 238, octobre 1980)

Reference cars	Subjective ranking	Criteria					
		Maximal speed (km)	Consumption in town (lt/100 km)	Consumption at 120 km/h (lt/100 km)	Horse power (CV)	Space (m^2)	Price (francs)
Peugeot 505 GR	1	173	11.4	10.01	10	7.88	49500
Opel Record 2000 LS	2	176	12.3	10.48	11	7.96	46700
Citroën Visa Super "E"	3	142	8.2	7.30	5	5.65	32100
VW Golf 1300 GLS	4	148	10.5	9.61	7	6.15	39150
Citroën CX 2400 Pallas	5	178	14.5	11.05	13	8.06	64700
Mercedes 230	6	180	13.6	10.40	13	8.47	75700
BMW 520	7	182	12.7	12.26	11	7.81	68593
Volvo 244 DL	8	145	14.3	12.95	11	8.38	55000
Peugeot 104 ZS	9	161	8.6	8.42	7	5.11	35200
Citroën Dyane	10	117	7.2	6.75	3	5.81	24800

The UTA computer program under its present form gives a subset of $\mathfrak{A}$ which is very characteristic of polyhedron (18). Each vertex corresponds to a utility function for which one or more criteria get an extreme weight (maximum or minimum). This partial exploration of $\mathfrak{A}$ (the polyhedron (18)) is obtained by the solution of the following linear programmes (19):

$$\begin{cases} [\min] \sum_{i=1}^{n} \rho_i u_i(g_i^*) \\ \text{in:} \\ \text{polyhedron (18);} \end{cases} \qquad \begin{cases} [\max] \sum_{i=1}^{n} \rho_i u_i(g_i^*) \\ \text{in:} \\ \text{polyhedron (18)} \end{cases} \tag{19}$$

with $\rho_i = 0$ or 1 for all i. If, for example, one chooses $\rho_1 = 1$ and $\rho_i = 0$, $i \neq 1$, the solution of the (19) programmes gives the interval variation of the weight of criterion g_1 and consequently gives an idea of the importance of this criterion in the decision-maker's preferences.

3. A numerical illustration

The following example comes from a real experiment where subjects were asked to rank-order some cars they knew from a file of twenty eight cars evaluated through six quantitative criteria. The data we analyze here consist in a subjective weak order of ten of the proposed cars (see Table 3) for a given individual.

In order to assess the utility functions with the UTA method, one has to choose the criteria variation intervals and the α_i, s_i and δ parameters. The value choosen for δ is 0.01, the other values are given in Table 4.

In step 1, the optimal utility function is obtained in 30 pivots[3] by solving the (PL2) programme for which there are 31 constraints and 32 variables. As a result $F^* = 0.0$, consequently the estimated utility is

[3] The computer program contains about 1000 FORTRAN IV instructions and executes (PL2) using high performance techniques known under the name of "LU methods" (cf. Bartels [2], Tolla [23]). UTA data is a weak order on a set of reference actions (the subjective preference) and the multicriteria evaluations of these actions. A copy of the program is available at: LAMSADE, Université de Paris-Dauphine, Place du Maréchal De Lattre de Tassigny, 75775 Paris Cedex 16, France.

Table 4
Technical data used for the analysis of one subjective ranking

Criteria	g_{i*}	g_i^*	α_i	s_i
g_1	110	190	5	0
g_2	15	7	4	0
g_3	13	6	4	0
g_4	3	13	5	0
g_5	5	9	4	0
g_6	80000	20000	5	0

perfectly consistent with the subjective ranking and the multicriteria evaluation of Table 3. Such a consistency leads to a monotone increasing function in a "subjective ranking" versus "estimated utilities or predicted ranking" diagram (see Fig. 4).

In step 2, sensitivity analysis was performed with the choice $k(F^*) = 0.009$ (so just smaller than δ). We have solved twelve linear programmes taking as objective functions

$$[\min] u_i(g_i^*) \quad \text{and} \quad [\max] u_i(g_i^*), \quad i = 1, 2, \ldots, 6.$$

The twelve assessed utility functions are also perfectly consistent with the data, since Kendall's τ is always equal to 1.0. In order to summarize these results, we computed a utility function which is the mean of the twelve post-optimal utility functions, i.e. the mean value of each component $u_i(g_i^j)$. This new utility is also perfectly consistent.

The mean utility and the optimal utility estimated in step 1 are visualized in Fig. 5. To give an idea of the utility's stability, we present below the range of the weights.

$$0.000 \leq u_1(g_1^*) \leq 0.752, \qquad 0.000 \leq u_4(g_4^*) \leq 0.516,$$
$$0.000 \leq u_2(g_2^*) \leq 0.382, \qquad 0.064 \leq u_5(g_5^*) \leq 0.450,$$
$$0.000 \leq u_3(g_3^*) \leq 0.471, \qquad 0.029 \leq u_6(g_6^*) \leq 0.475.$$

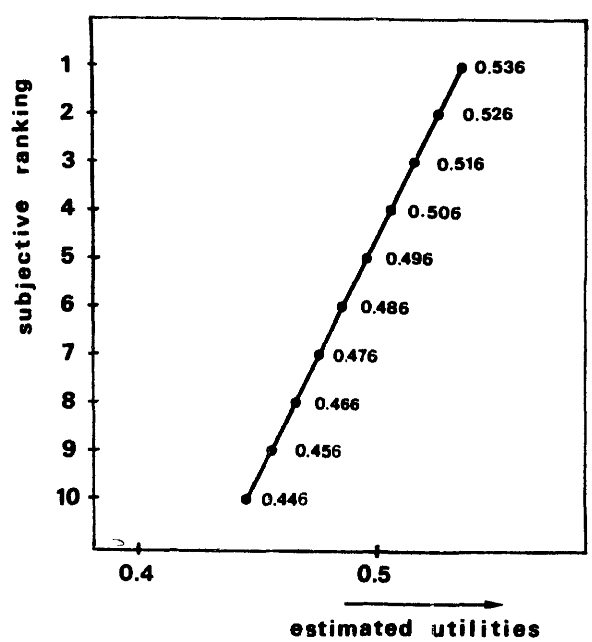

Fig. 4. Estimated optimal utility versus subjective preference.

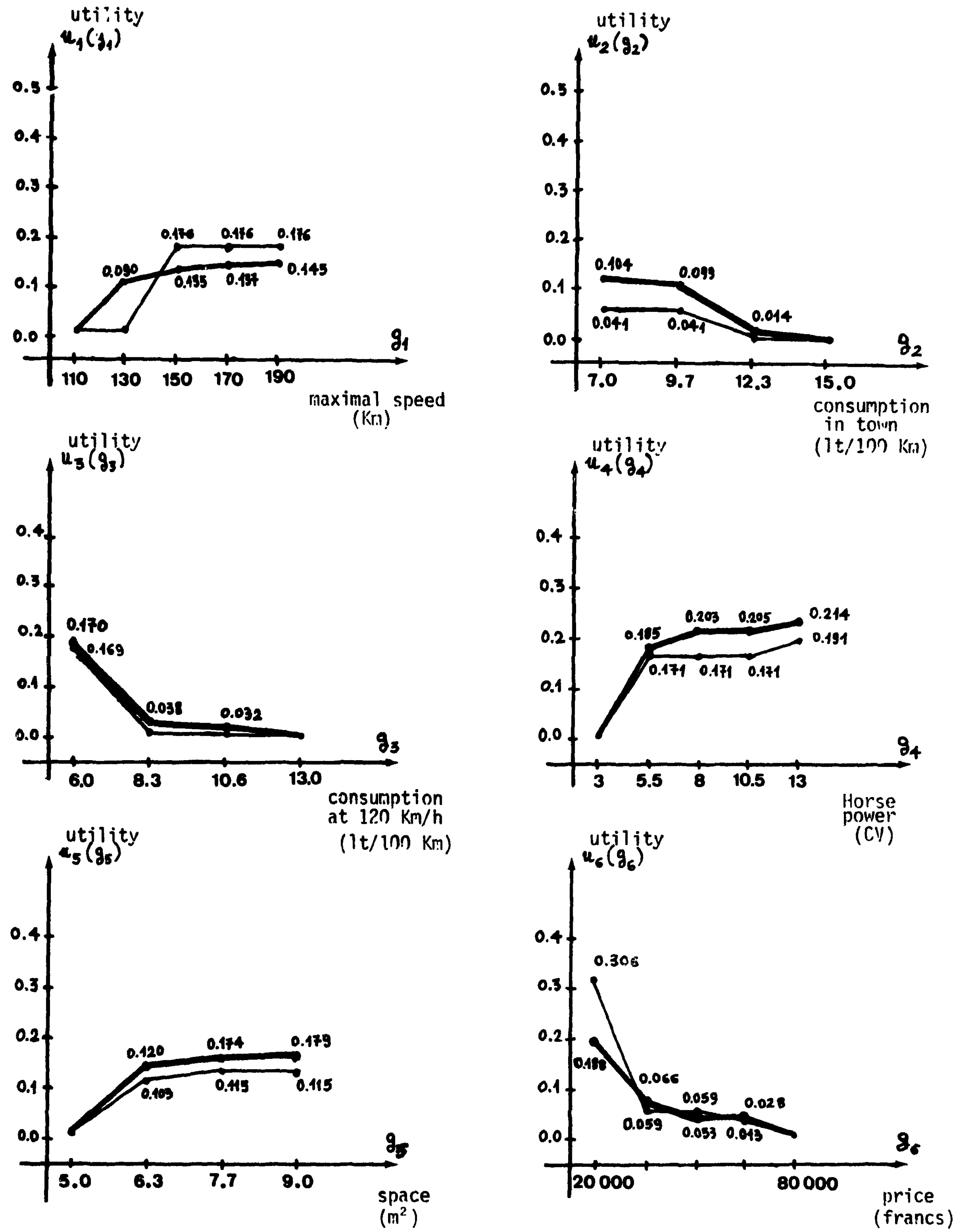

Fig. 5. Two assessed utility functions; ———: optimal utility, ▬▬▬: mean utility.

So far the UTA method has been applied in several domains of managerial decision-making problems an environment study for the choice of a highway plan between two French cities [10]; a preference analysis of a decision committee in Architecture ([8, Chapter 8]); a study of staff management [20]; an evaluation study of research and development projects for Guatemala [13]; an analysis of the market of beach games for the launching of a new game in France [19]; a general study for strategic decision-making in organizations [15]; and, finally, an evaluation study of a large number of furniture retail outlets for choosing a new retail-trading system [21].

4. Some extensions of the model

In this section, we present some extensions of this type of method to assess additive utility functions. We propose some other optimality objectives and an extension of the model when the subjective preference is nominal (a discriminant analysis problem).

4.1. Other optimality objectives

A subjective preference obtained from pairwise judgments is most often not transitive. An extension of the model is given then by (PL3).

$$\begin{cases} [\min]\ F = \sum_{(a,b)\in P} \lambda_{ab} z_{ab} + \sum_{(a,b)\in I} \lambda_{ab} z_{ab} & \\ \text{under the constraints:} & \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} \geqslant \delta & \text{if } aPb, \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + z_{ab} - z_{ba} = 0 & \text{if } aIb\ (\Rightarrow bIa), \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geqslant s_i & \text{for all } i \text{ and } j, \\ \sum_{i=1}^{n} u_i(g_i^*) = 1, & \\ u_i(g_{i*}) = 0,\ u_i(g_i^j) \geqslant 0,\ z_{ab} \geqslant 0 & \text{for all } i, j \text{ and } (a,b) \in R, \end{cases} \qquad \text{(PL3)}$$

λ_{ab} being a non negative weight reflecting a degree of confidence in the judgment aRb.

When the subjective preference is antisymmetric (without indifference judgments), a more suitable optimality objective would be: minimize the number of violated pairs of R in the weak order R' given by $U(\mathbf{g})$, which is equivalent to the objective: maximize Kendall's τ $\tau(R, R')$. The solution is given by a mixed linear programme (PL4) where the discrete variables are 0/1.

$$\begin{cases} [\min]\ F = \sum_{(a,b)\in R} \gamma_{ab} \Leftrightarrow [\max]\ \tau(R, R') & \\ \text{under the constraints:} & \\ \sum_{i=1}^{n} \{u_i[g_i(a)] - u_i[g_i(b)]\} + M \cdot \gamma_{ab} \geqslant \delta & \text{for all } (a,b) \in R, \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geqslant s_i & \text{for all } i \text{ and } j, \\ \sum_{i=1}^{n} u_i(g_i^*) = 1 & \\ u_i(g_{i*}) = 0,\ u_i(g_i^j) \geqslant 0 & \text{for all } i \text{ and } j, \\ \gamma_{ab} = 0 \text{ or } 1 \quad \text{for all } (a,b) \in R, & M, \text{ a large number.} \end{cases} \qquad \text{(PL4)}$$

If the difference $U[\mathbf{g}(a)] - U[\mathbf{g}(b)]$ for a pair $(a,b) \in R$ is greater than δ, then $\gamma_{ab} = 0$ and the judgment is respected, otherwise $\gamma_{ab} = 1$ and it is violated. So F represents the number of violated pairs in the overall preference aggregated by $U(\mathbf{g})$.

Because of computation difficulties, (PL4) could be used only for small problems with $|R|$ not too high.

4.2. A discriminant analysis method

In order to assess additive utilities when we know real individual past choices instead of a weak-order preference relation, we introduce into the model nominal variables instead of ordinal ones. As an example we present a model when the nominal variables can take two values. Let $U(g)$ be the utility function to estimate. U_0 is the level of acceptance/rejection which must be found in order to distinguish the set of accepted actions called A_1, and the set of the rejected actions called A_2. The model would be without errors if we had the inequalities (20):

$$\begin{cases} a \in A_1 \Leftrightarrow U(g(a)) \geq U_0, \\ a \in A_2 \Leftrightarrow U(g(a)) < U_0. \end{cases} \tag{20}$$

Introducing errors variables $\sigma(a)$, $U(g)$ can be estimated by means of linear programme (PL5):

$$\begin{cases} [\min] \ F = \sum_{a \in A'} \sigma(a) & \\ \text{under the constraints:} & \\ \sum_{i=1}^{n} u_i[g_i(a)] - U_0 + \sigma(a) \geq 0 & \text{for all } a \in A_1, \\ \sum_{i=1}^{n} u_i[g_i(a)] - U_0 - \sigma(a) \leq 0 & \text{for all } a \in A_2, \\ u_i(g_i^{j+1}) - u_i(g_i^j) \geq s_i & \text{for all } i \text{ and } j, \\ \sum_{i=1}^{n} u_i(g_i^*) = 1 & \\ u_i(g_{i*}) = 0, \ u_i(g_i^j) \geq 0 & \text{for all } i \text{ and } j, \\ U_0 \geq 0, \ \sigma(a) \geq 0 & \text{for all } a \in A'. \end{cases} \tag{PL5}$$

The objective consists in minimizing the sum of deviations from the threshold U_0 for the ill classified actions (see Fig. 6). Of course, it would be very easy in such a model to penalize differently the actions in the sets A_1 and A_2, using an objective of the type $F = \lambda \Sigma_{a \in A_1} \sigma(a) + \mu \Sigma_{a \in A_2} \sigma(a)$.

5. Concluding remarks

The method developed in this paper belongs to the family of ordinal regression analysis methods. Its main features are, first, to assess additive utility models instead of the traditional linear one and, second, to get a set of such utility functions as a model of the preferences of a decision-maker.

Another extension of the model have been developed in the case where the evaluations of a are under the form of probability distributions $\delta_i^a(g_i)$ (experts' evaluations, importance of the population involved,...).

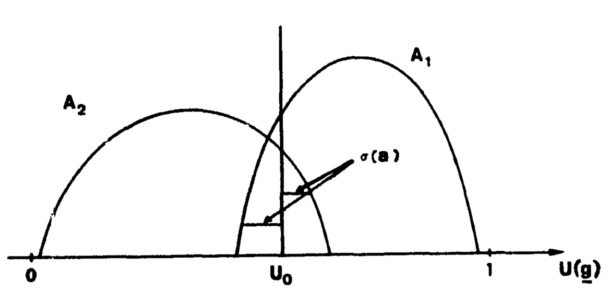

Fig. 6. Discriminant analysis: distributions of the actions of A_1 and A_2 on the assessed utility $U(g)$.

In the linear programme (PL1), $u_i[g_i(a)]$ would no longer be a single value or a linear interpolation between two values, but it would be a weighted sum of values which have to be estimated, $u_i[g_i(a)] = \Sigma_j \delta_i^a(g_i^j)\, u(g_i^j)$, where j represents the grade index. A particular case has been studied by Marchet [11] for a highway plan choice problem (cf. [10]).

The way for using UTA method in an O.R. interactive process is discussed by Jacquet-Lagrèze [7] and Siskos [20,21]. Such applications for real decision-making processes are developed in [15] and [21].

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