Symmetric neural networks

YKY

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1 General case for $y = Ax$

$$\boxed{\text{original}} \quad y_j = \sum_i a_{ij} x_i. \tag{1}$$

Equivariance implies:

$$\boxed{\text{swapped}} \quad y_j \cdot \sigma(x_j \ x_k) \quad = \quad y_k \quad \boxed{\text{original}} \tag{2}$$

$$\sum_{i \neq j,k} a_{ij} x_i + a_{kj} x_j + a_{jj} x_k \quad = \quad \sum_{i \neq j,k} a_{ik} x_i + a_{jk} x_j + a_{kk} x_k.$$

Comparing coefficients yields:

$$
\begin{aligned}
a_{ij} &= a_{ik} & \forall j, k, (i \neq j, k) \\
a_{kj} &= a_{jk} & \forall j, k \\
a_{jj} &= a_{kk} & \forall j, k.
\end{aligned}
\tag{3}
$$

In other words, the matrix A is of the form:

$$A = \alpha I + \beta 11^T. \tag{4}$$

2 Case for $y_k = A_k x \cdot x$

The general form of a "quadratic" vector function is:

$$y = (Ax) \cdot x + Bx + C. \tag{5}$$

We just focus on the quadratic term $(Ax) \cdot x$:

$$\boxed{\text{original}} \quad y_k = \sum_j \left[\sum_i a_{ij}^k x_i \right] x_j. \tag{6}$$

Equivariance implies:

$$\boxed{\text{swapped}} \quad y_k \cdot \sigma(x_k \ x_h) \;=\; y_h \quad \boxed{\text{original}} \tag{7}$$

$$\sum_{j \neq h,k} \sum_{i \neq h,k} a_{ij}^k x_i x_j + a_{hh}^k x_k^2 + a_{kh}^k x_h x_k + a_{hk}^k x_k x_h + a_{kk}^k x_h^2 \;=\; \sum_{j \neq h,k} \sum_{i \neq h,k} a_{ij}^h x_i x_j + a_{hh}^h x_h^2 + a_{kh}^h x_k x_h + a_{hk}^h x_h x_k + a_{kk}^h x_k^2$$

which yields:

$$
\begin{aligned}
a_{ij}^h &= a_{ij}^k & \forall h,k, (i,j) \neq (h,k) \\
a_{hh}^h &= a_{kk}^k & \forall h,k \\
a_{hh}^k &= a_{kk}^h & \forall h,k \\
a_{kh}^k + a_{hk}^k &= a_{kh}^h + a_{hk}^h & \forall h,k.
\end{aligned}
\tag{8}
$$