# Symmetric neural networks

May 2, 2019

## 1 Convolutions have translational equivariance

The following is quoted from Wikipedia:

The convolution commutes with translations, meaning that

$$\tau_x(f * g) = (\tau_x f) * g = f * (\tau_x g) \tag{1}$$

where $\tau_x f$ is the translation of the function $f$ by $x$ defined by:

$$(\tau_x f)(y) = f(y - x). \tag{2}$$

If $f$ is a **Schwartz function** [a] , then $\tau_x f$ is the convolution with a translated Dirac delta function $\tau_x f = f * \tau_x \delta$. So translation invariance of the convolution of Schwartz functions is a consequence of the associativity of convolution.
Furthermore, under certain conditions, convolution is the most general translation invariant operation. Informally speaking, the following holds:

> Suppose that $S$ is a bounded linear operator acting on functions which commutes with translations: $S(\tau_x f) = \tau_x(Sf)$ for all $x$. Then $S$ is given as convolution with a function (or distribution) $gS$; that is $Sf = gS * f$.

Thus some translation invariant operations can be represented as convolution. Convolutions play an important role in the study of time-invariant systems, and especially LTI system theory. The representing function $gS$ is the impulse response of the transformation $S$.
A more precise version of the theorem quoted above requires specifying the class of functions on which the convolution is defined, and also requires assuming in addition that $S$ must be a continuous linear operator with respect to the appropriate topology. It is known, for instance, that every continuous translation invariant continuous linear operator on $L1$ is the convolution with a finite Borel measure. More generally, every continuous translation invariant continuous linear operator on $L_p$ for $1 \leq p < \infty$ is the convolution with a tempered distribution whose Fourier transform is bounded. To wit, they are all given by bounded Fourier multipliers.

[a] Schwartz space is the function space of all functions whose derivatives are rapidly decreasing. This space has the important property that the Fourier transform is an automorphism on this space.

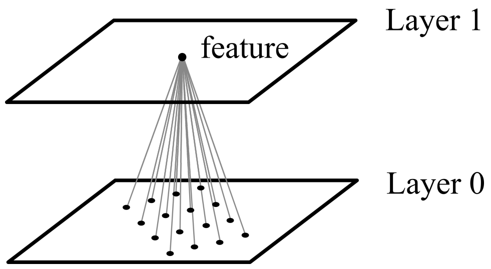
In 1989, Yann LeCun invented **convolutional** neural networks (ConvNets) which established the importance of this type of neural networks for **computer vision** (it seems to remain the dominant approach to this day).

For this he won the Turing Award recently.

In computer vision, translational invariance is obviously desirable. The structure of ConvNets has translational invariance built-in, so it does not need to be learned, thus making learning more efficient. This enabled the breakthrough in machine vision to surpass human level, around 2011.

## 2 Structure of a traditional neural network

Recall that a neural network is constructed from **features**:



(3)

A **feature** = **neuron**, is a **filter** applied to a *local* part of the **input signal**, usually implemented as a **dot product** followed by a **non-linearity** ∮:

$$\boxed{\text{neuron}} = \boxed{\text{feature}} = ∮ \, \langle \boldsymbol{v}, \boldsymbol{w} \rangle. \tag{4}$$

Such a function is also called a **ridge function**. $\boldsymbol{v}$ is the **input** vector, $\boldsymbol{w}$ is the **weight** vector.

Last time I forgot to mention that the non-linearity ∮ being a **sigmoid** function is not essential to a neural network. In fact, any non-linearity will do, as long as it is *continuous* and has first-order derivatives. That even includes **piecewise-linear** functions.

The "universal" approximation ability of neural networks comes from the **Weierstrass approximation theorem** (1885) that says that every continuous function can be uniformly approximated by polynomials. This is later generalized to the **Stone-Weierstrass** theorem (1937). For neural networks, the universal approximation property can be proven with just a single layer — which means it does not depend on the "deep" property.

If we have a **pool** of features or neurons, they form a **layer**:

$$\boxed{\text{pool of neurons}} = \boxed{\text{layer}} = ∮( \, W\boldsymbol{v} \, ) \tag{5}$$

where $W$ is a **matrix** of weights. ∮ is applied *component-wise* to the resulting vector.

A deep network is just the **composition** of many layers:

$$\boxed{\text{deep network}} = [∮ \, W]^L \;\; \boldsymbol{v}. \tag{6}$$

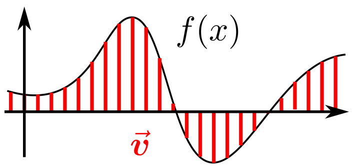
## 3 Structure of a ConvNet

In the convolutional network, each **feature** is replaced by a convolutional feature:

$$\boxed{\text{convolutional feature}} = \oint(\ f * g\ ) \tag{7}$$

where $f, g$ are functions, $f$ is a **filter** applied to the **input** $g$.

We can regard the input signal as a **function** or as a discretized **vector** of its graph:



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so there is no big difference between a function and a vector, in this context.

When implemented on a computer, the discrete, **full convolution** is defined as [*]:

$$\boxed{\text{full convolution}} \quad \boldsymbol{x} * \boldsymbol{k} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \boldsymbol{x}_{ij}\, \boldsymbol{k}_{u-i,v-j}. \tag{9}$$

A **valid** convolution is a restriction of the full convolution to a **window**:

$$\boxed{\text{valid convolution}} \quad \boldsymbol{x} * \boldsymbol{k} = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \boldsymbol{x}_{i+u,j+v}\, \boldsymbol{k}_{\text{rot }i,j}\, \chi(i,j) \tag{10}$$

$$\chi(i,j) = \begin{cases} 1, & 0 \le i, j \le n \\ 0, & \text{otherwise} \end{cases}$$

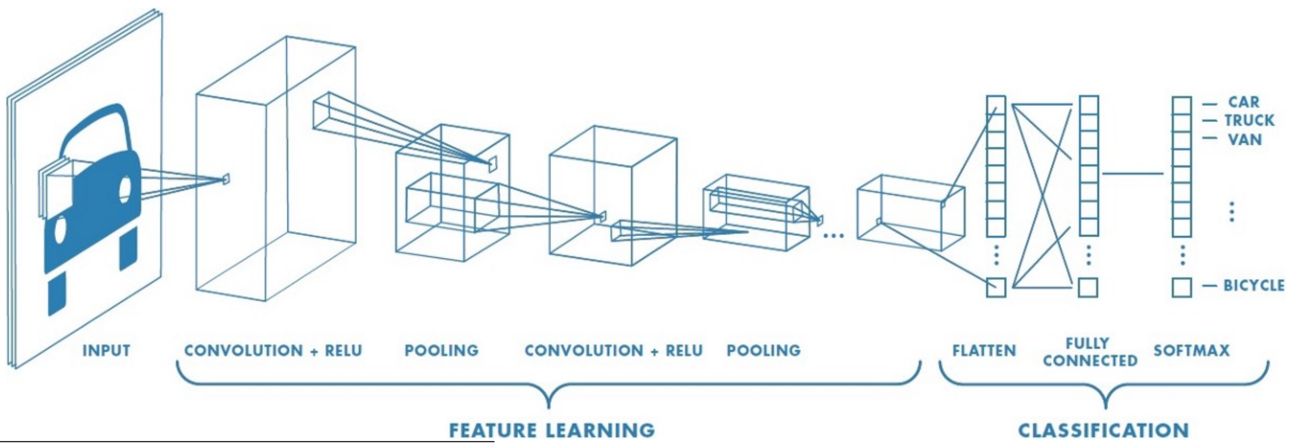
where $\boldsymbol{k}_{\text{rot}}$ is $\boldsymbol{k}$ rotated by 180°.

The pool of local features has a lot of **redundancy**, which can be reduced by **sub-sampling** them:

$$\boxed{\text{sub-sampling}} \quad \boldsymbol{x}_{i,j}^{(\ell+1)} = \oint(\ \beta \sum_{u=ir}^{(i+1)r-1} \sum_{v=jr}^{(j+1)r-1} \boldsymbol{x}_{u,v}^{(\ell)}\ ) \tag{11}$$

where $\beta$ is a scalar weight. The output is a matrix of *reduced* size: $\left( \dfrac{m-n-1}{r} \times \dfrac{m-n+1}{r} \right)$.

Repeating this structure gives rise to the following typical architecture (which I won't explain in detail):



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[*]Thanks to 孙小也 at 知乎 (Zhihu.com) for providing these formulas.

## 4 Symmetric neural networks