Logic in Hilbert space

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Summary

- It seems possible to construct a model of the **untyped** λ-calculus in Hilbert space, with function application $f(g)$ implemented as $[\![g]\!] \circ [\![f]\!]$.

- Doing so allows **self-application** of logic predicates (Curry's paradox can be avoided by the fuzzy truth value "don't know")

- The use of **continuous maps** may be advantageous in machine-learning because **generalization** seems to work best with "continuous" domains (as opposed to maps acting on symbolic logic representations which may be discontinuous).

- Elements in the infinite-dimensional $\mathcal{H}$ can be realized on a computer as **neural networks** (which are functions $\mathbb{R}^n \to \mathbb{R}^n$ **finitely** generated from sets of weights).

0 Inspiration

In the 1960's Dana Scott constructed a model for untyped λ-calculus, using a domain D_∞ with the property $D_\infty^{D_\infty} \cong D_\infty$. This started off the field known as **domain theory**.

Scott initially believed that such models cannot exist, but later discovered that they can be constructed. In retrospect, this is not surprising because the Church-Rosser theorem demonstrated that the untyped λ-calculus is consistent.

Scott's idea is to begin with an initial domain D_0 and define D_{n+1} to be the function space $D_n \to D_n$.

Thus it is guaranteed, for any domain $d \in D_\infty$, one can always find a function space $d \to d$. Therefore the space D_∞ is isomorphic to $D_\infty \to D_\infty$.

That this claim does not contradict Cantor's theorem (a set X cannot be isomorphic to X^X) is because the functions are restricted to **continuous** maps, sending open sets to open sets, also known as Scott-continuous (these are not *all* the functions in $X \to X$).

The detailed definition of D_∞ involves building a cumulative hierarchy of infinite sequences, with pairs of operators ψ_n, Ψ_n going up and down levels. For a gentle exposition one may read (Stenlund 1972), Ch.1 §6.

1 Requirements

In order to define a logical calculus in $\mathcal{H}$, what we need are:

- a family of functions dense in $\mathcal{H}$
- self-application: maps can act on other maps; how to define $f(g)$?
- define the $\mathbf{S}, \mathbf{K}, \mathbf{I}$ combinators in combinatory logic
- how does an element $e \in \mathcal{H}$ translate to and from a (syntactic) logic formula?

2 Elements in $\mathcal{H}$

The structure of D_∞ is reminescent of Cantor's ordinal number ε_0:

$$\varepsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}} = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \dots\} \tag{2.1}$$

but this number is still "smaller" than the **continuum**, ie. the real line $\mathbb{R}$. This led me to think that models of λ-calculus may be found in the Hilbert space of continuous functions.

But such a Hilbert space would be ∞-dimensional. Next I consider the **neural network** as a function f:

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n \\ x & \mapsto & y \end{array} \tag{2.2}$$

and notice that f and x, y are "unequal" because f can **act on** x, y but not the other way round. This is partly because f is ∞-dimensional whereas x, y are finite-dimensional. So, what if we increase n to ∞, then perhaps x, y would become the same kind of objects as f ? In an informal sense $\mathcal{H}$ can be regarded as $\mathbb{R}^\infty$.

The Universal Approximation theorem of (Cybenko 1989) (with later extensions by others) states that the family of neural networks is **dense** in the space of continuous functions, with respect to the supremum norm. This means that we have a nice way of generating elements in $\mathcal{H}$ that can be finitely represented in a computer.

3 Function application

Now we lack the notion of a function **applying** to another function, such as $f(g)$. Since we only need the functions as elements of $\mathcal{H}$, the domains $\mathbb{R}^n$ is somewhat "immaterial". We might as well assume $\mathbb{R}^n$ to be common among all functions (the dimension n can be fixed for implementation considerations), and thus function **composition** such as $f \circ g$ always exists. So we define:

$$[\![f(g)]\!] = [\![g]\!] \circ [\![f]\!] \tag{3.1}$$

where $[\![\bullet]\!]$ denotes "model of". (The reversed order $g \circ f$ is to prepare for dealing with tuples; see below.)

4 Combinatory logic in $\mathcal{H}$

We need to implement the combinators defined by:

- $\mathbf{I}x = x$
- $\mathbf{K}xy = x$
- $\mathbf{S}xyz = xz(yz)$

which obviously requires that some functions take on > 1 arguments. So we need the notion of **tuples** and of functions applying to tuples.

Tuples

We can implement tuples like (g, h) simply by stacking them vertically:

$$(g, h) = \begin{bmatrix} g \\ h \end{bmatrix} \tag{4.1}$$

which is a $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ function. This can be extended to dimension kn. Function application can be implemented as:

$$[\![f(g, h)]\!] = \begin{bmatrix} [\![g]\!] \\ [\![h]\!] \end{bmatrix} \circ [\![f]\!] \tag{4.2}$$

where f is of type $\mathbb{R}^{2n} \to \mathbb{R}^{n}$. We can visualize the right hand side like this:

$$\begin{array}{cccc} g & \diagdown & & \\ & & f & — \\ h & \diagup & & \end{array} \tag{4.3}$$

Interpreting logic formulas

It is helpful to bear in mind that a functional form

$$f(g, h) = e \tag{4.4}$$

in Hilbert space corresponds to a logic formula

$$g \wedge h \overset{f}{\Longrightarrow} e \tag{4.5}$$

where f plays the role of an **implication**, or logical **consequence**, $g \wedge h \vdash e$. In fact f encodes the *entire* formula $g \wedge h \Rightarrow e$, but even more than that, f can be applied to other arguments. (Readers may recognize that the function f realizes an implication statement in logic via the **Curry-Howard isomorphism**.)

With tuples, we can easily implement the combinators $\mathbf{S}, \mathbf{K}, \mathbf{I}$. The treatment for λ-calculus would be analogous, and I would add that to a later version of this paper when I have time.

Arities

There is a remaining problem with arities. Because we allow tuples, compositions with tuples leave us with functions with arity > 1. For example, the following two applications of f yield the same result c, and we would like to make these definitions of f consistent:

$$\begin{tikzcd}[row sep=small] a \arrow[dr, dash] & & \\ & f \arrow[r, dash] & c \\ b \arrow[ur, dash] & & \end{tikzcd} \qquad f(a,b) = c \tag{4.6}$$

$$d \;—\; f \;—\; c \qquad f(d) = c$$

but the **arity** of f appears different in the equations. My solution is to adjoin a **null** input to the single input, that is:

$$\begin{tikzcd}[row sep=small] d \arrow[dr, dash] & & \\ & f \arrow[r] & c \\ \emptyset \arrow[ur, dash] & & \end{tikzcd} \qquad f(d, \emptyset) = c \tag{4.7}$$

In implementation, this can be done "on demand", as the number of conjunctions in the premise of a logic rule is usually a small number.

5 Usage as AI logic

Usually there would be a "big" neural network, representing a special function f, that plays the role of a **knowledge base**, consisting of (the conjunction of) a huge number of logic rules of the form $\Gamma \vdash \Delta$. This f would send various logic formulas to their conclusions, for a single inference step. In this sense f embodies all the *knowledge* in the AI system.

What is special with this Hilbert-space model is that we can now construct arbitrary logic formulas as elements in $\mathcal{H}$, which can be acted upon by f.

For example, "if we touch fire we may get burnt" is a logic formula of the form $A \Rightarrow B$. In a primitive AI (or animal), this formula is *implicitly* stored in the transition function f, which performs the logic inference $A \vdash B$. The animal or AI has no conscious reckoning of $A \Rightarrow B$. Now with Hilbert space we can create an element $A \Rightarrow B$, ie. an object that can **participate** in logic inference. We can take arbitrarily complex logic formulas and have them represented as Hilbert space elements.

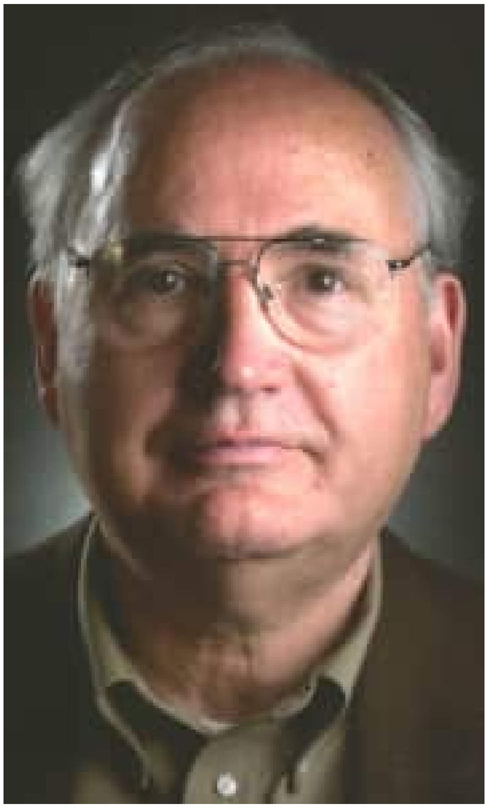
A simple logic formula such as $A \Rightarrow B$ is a **section** of the big map f restricted to the domain A. In this case, the restricted function is **degenerate** and is in effect just a pair (A, B). Also, a single atomic proposition, eg. A, can be implemented as $\top \Rightarrow A$, a practice commonly used in Prolog.

6 Conclusion

I am not highly confident of this invention, as its technical details are a bit complicated (We could build a simpler AI using a big neural network to act on *symbolic* logic formulas; indeed the BERT model is a case in point). If human-level intelligence turns out to be very hard to learn by an AGI, at least we can turn to this, more powerful approach.

Questions, comments welcome ☺

Dana Scott (1932-) and Sören Stenlund (1943-2019):



References

Cybenko (1989). "Approximation by superposition of a sigmoidal function". In: *Mathematics of control, signals and systems.*

Stenlund (1972). *Combinators, λ-terms and proof theory.*