Basic categorical logic

YKY

1 Mar 2011

Basic category theory

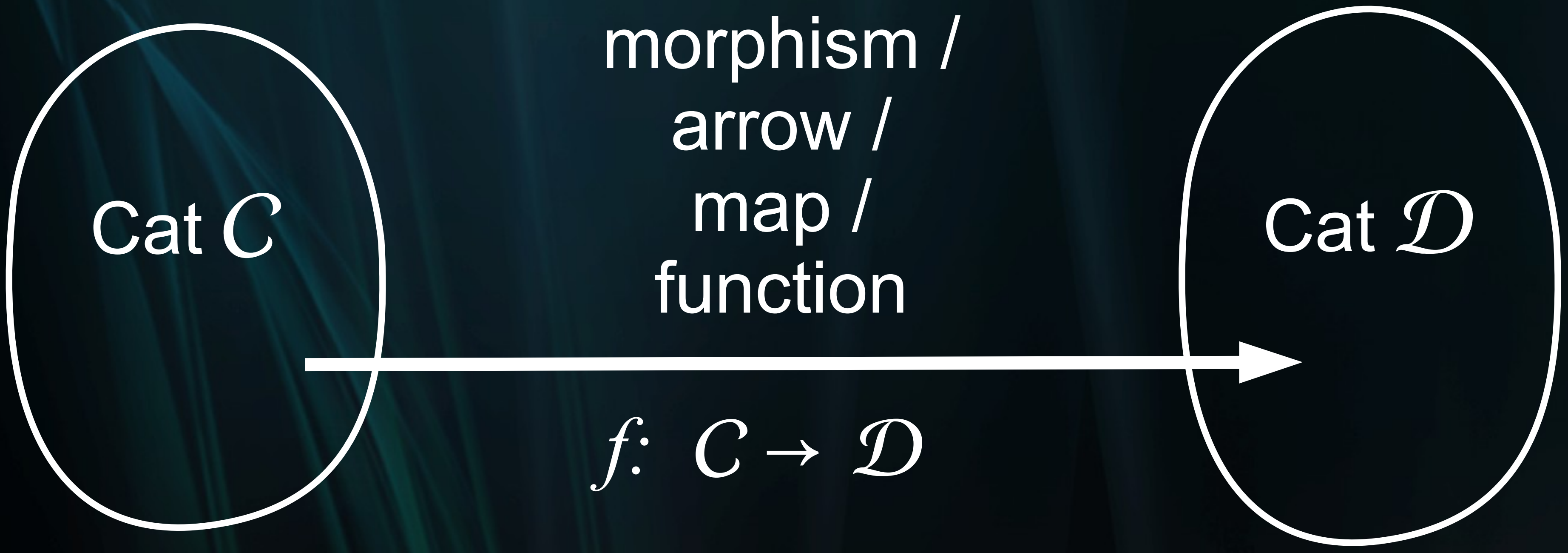
Category theory is poised to replace **set theory** as a *foundation* of mathematics.

Categorical logic is the application of category theory to logic.

Set theory

In set theory one talks about the **elements** of a set and how they are *mapped* to other elements by **functions**.

Category theory



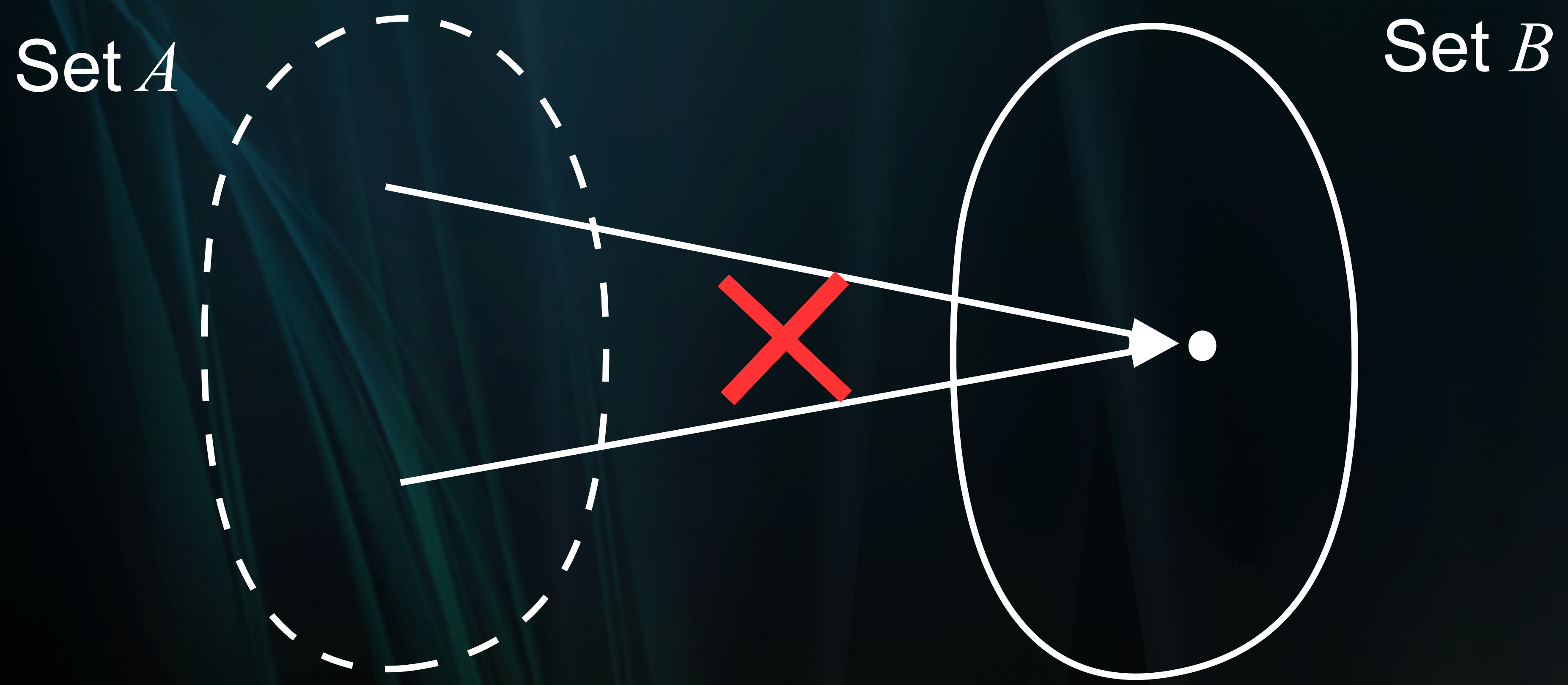
In category theory one does not “see” the elements inside a set and instead talks about single **morphisms** between 2 categories.

Monic and epic

These are 2 important concepts:

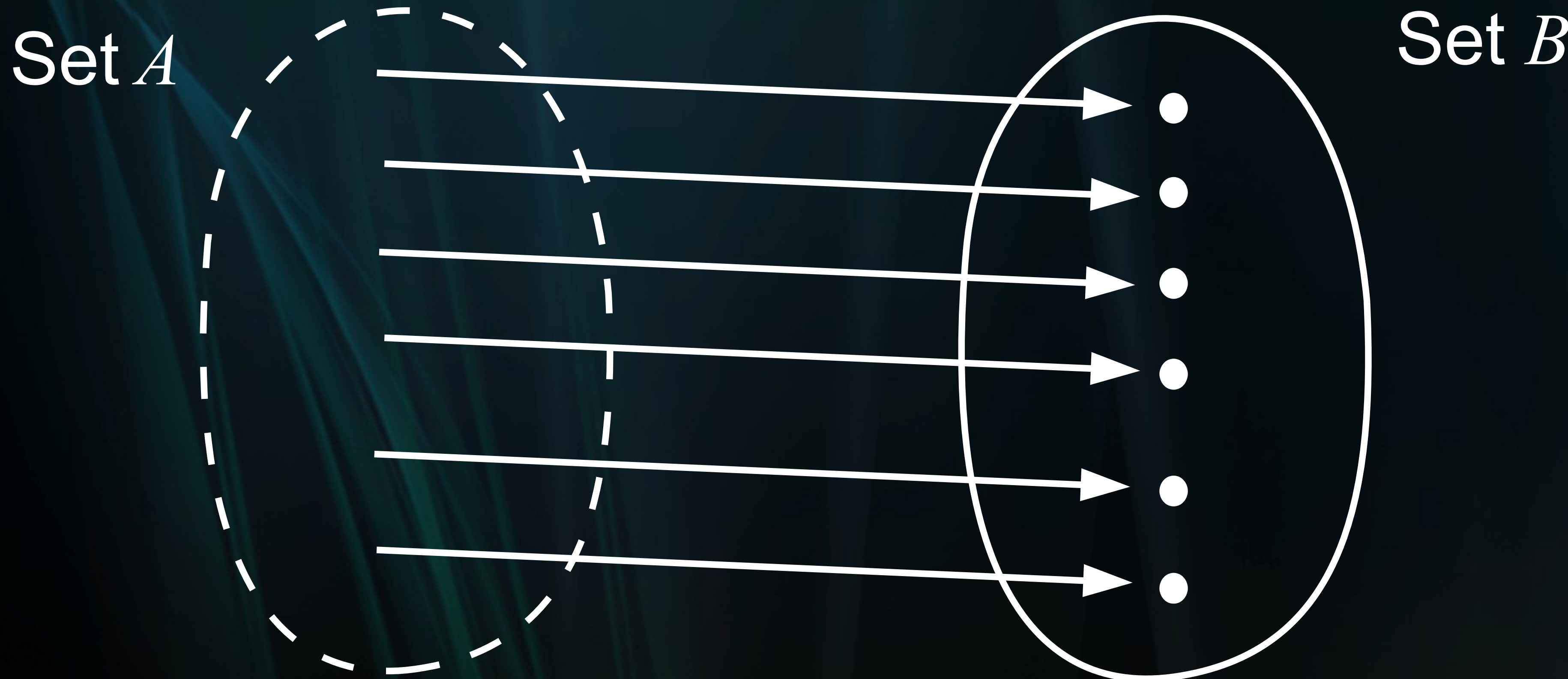
in cat theory		in set theory
monic	$\leftrightarrow$	**injection** = one-to-one
epic	$\leftrightarrow$	**surjection** = onto

Injection (单射)



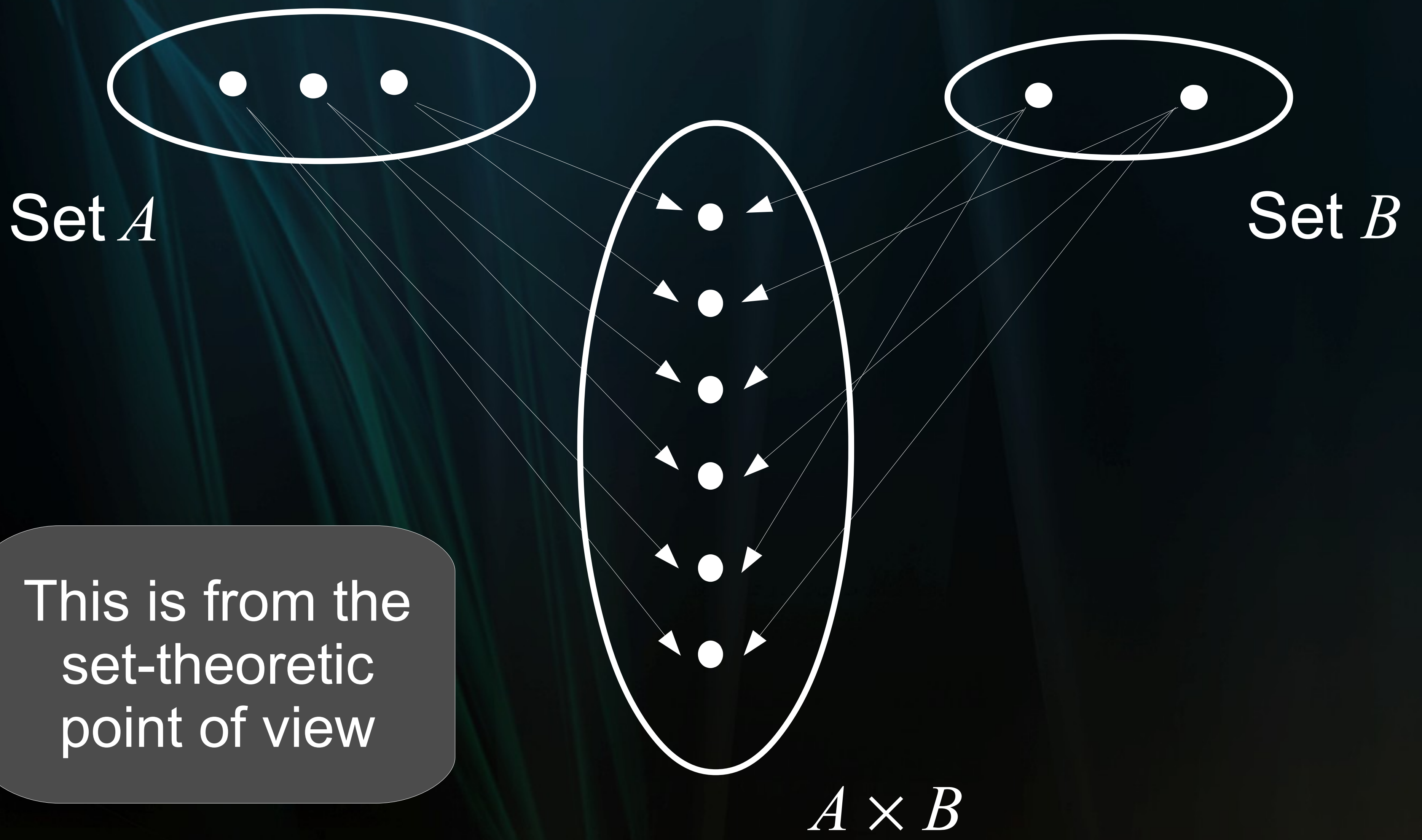
In an injection any source element can only map to a single target element.

Surjection (满射)



In a surjection the map must cover the target domain <u>fully</u>.

Example: Cartesian product



This is from the set-theoretic point of view

Cartesian product

How can products be defined *categorically*?

First, note that there exist 2 maps that project the product object to each of the original objects.

projections

$$A \xleftarrow{proj_1} U \xrightarrow{proj_2} B$$

the product

Cartesian product

Given any set V and two arrows q1, q2, there exists a **unique arrow**, usually denoted by ! or a dotted line:

unique arrow

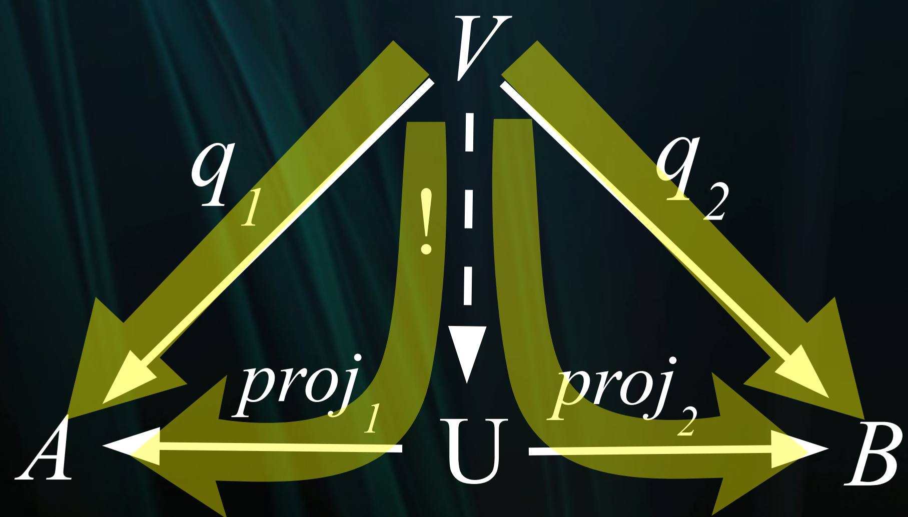
$$
\begin{tikzcd}
 & V \arrow[dl, "q_1"'] \arrow[d, dashed, "!"'] \arrow[dr, "q_2"] & \\
A & U \arrow[l, "proj_1"'] \arrow[r, "proj_2"] & B
\end{tikzcd}
$$

!

such that ...

Cartesian product

… the diagram **commutes**, that is to say, the different paths of composite mappings are always **equivalent**.



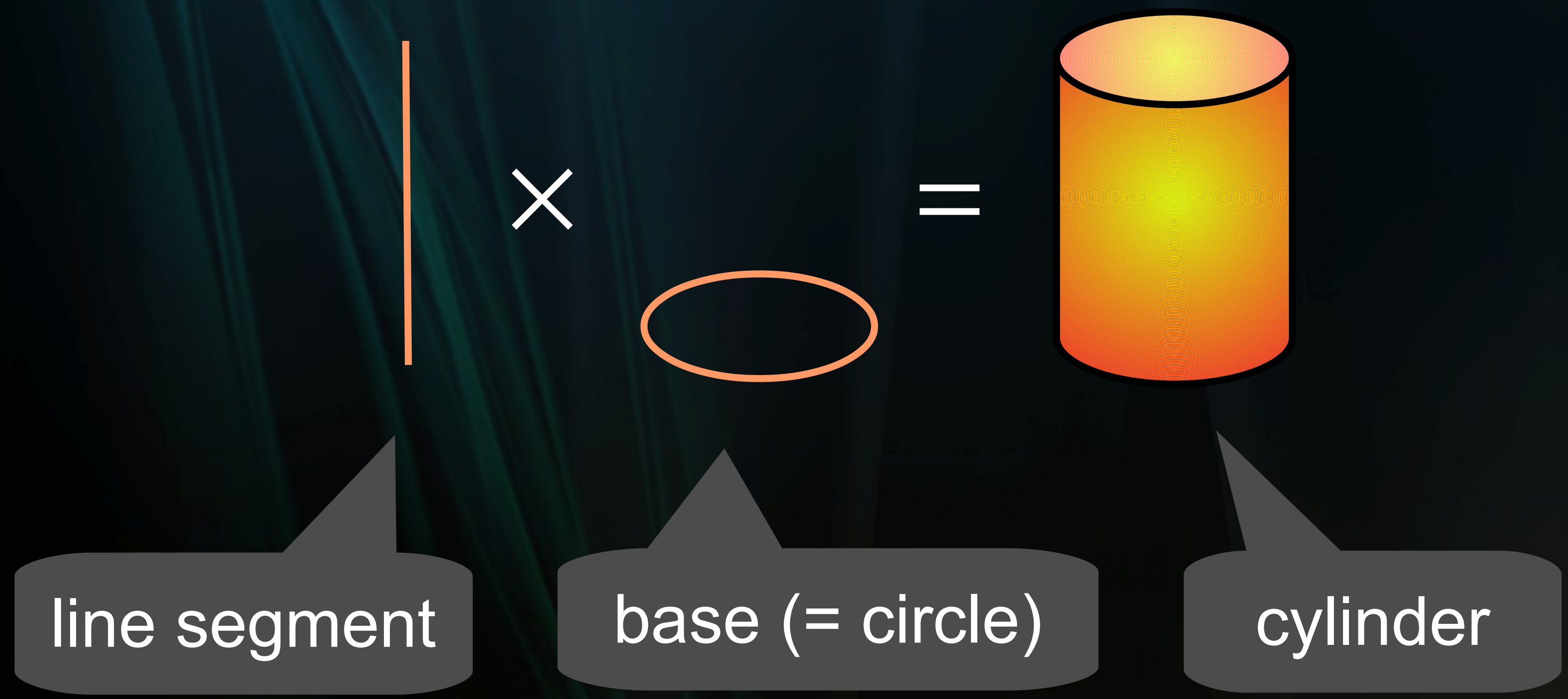
∘ = function composition

$proj_1 \circ ! = q_1$ $\qquad$ $proj_2 \circ ! = q_2$

Section and retraction

Fiber bundles

This is a topological product:

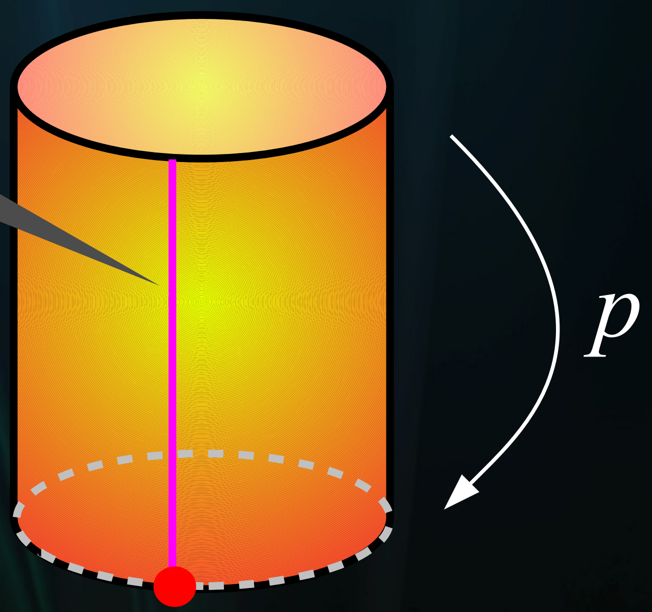


Fiber bundles

The Möbius strip is globally *not* a product of the midline and the segment (which would be a cylinder), but it is locally around each neighborhood the product of an arc and the segment.

Fiber bundles

a fiber "grows" on the base point x



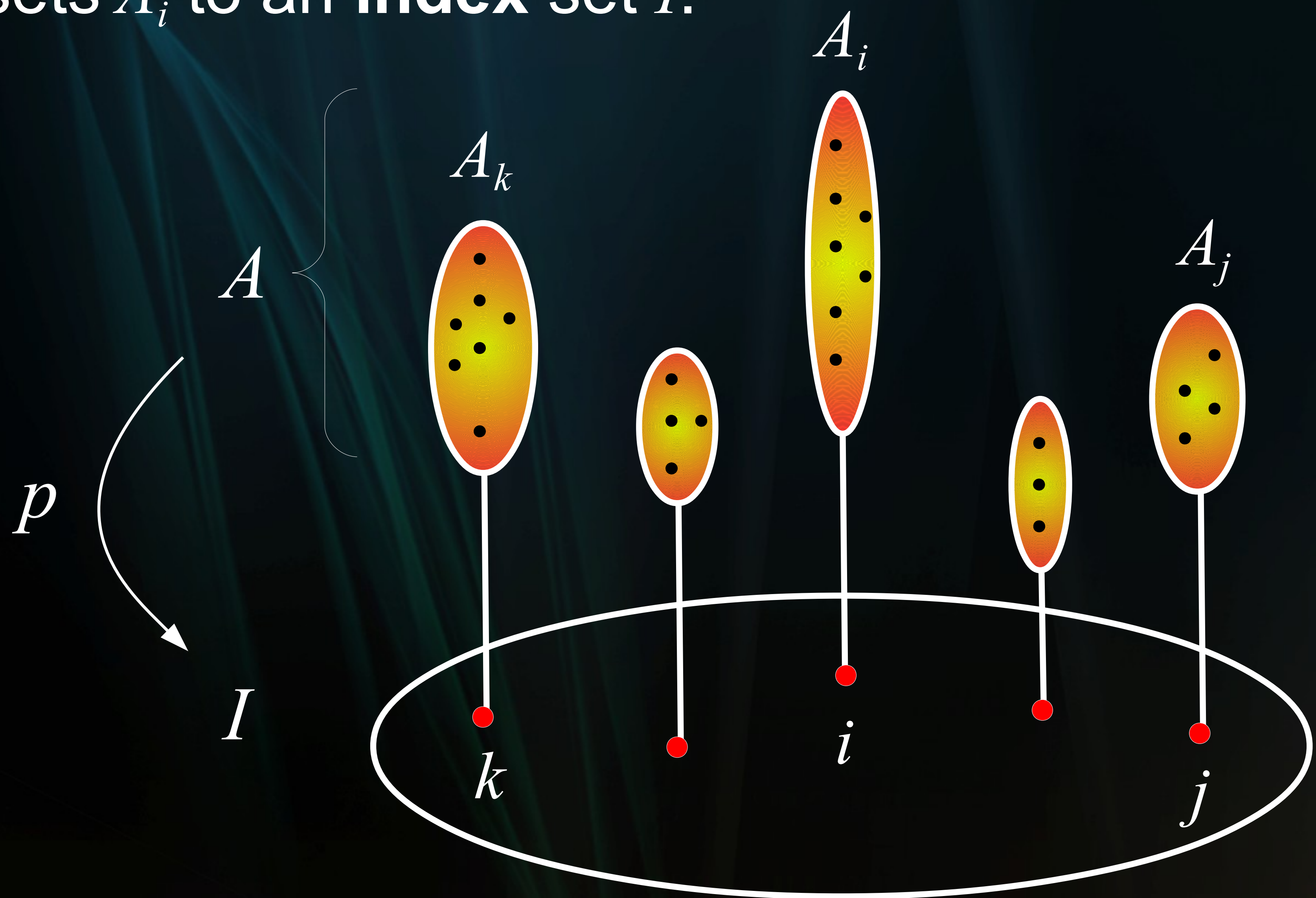
The map p **projects** the fiber to x.

The fiber is the **pre-image** $p^{-1}(x)$.

x

Fiber bundles

Abstractly, a bundle is a map p from a collection of sets A_i to an **index** set I.



Subobjects

subobject in cat theory	=	**subset** in set theory

To say $A \subseteq D$ ("A is a subset of D") is equivalent to saying "there exists an *injective* map $A \to D$", denoted as:

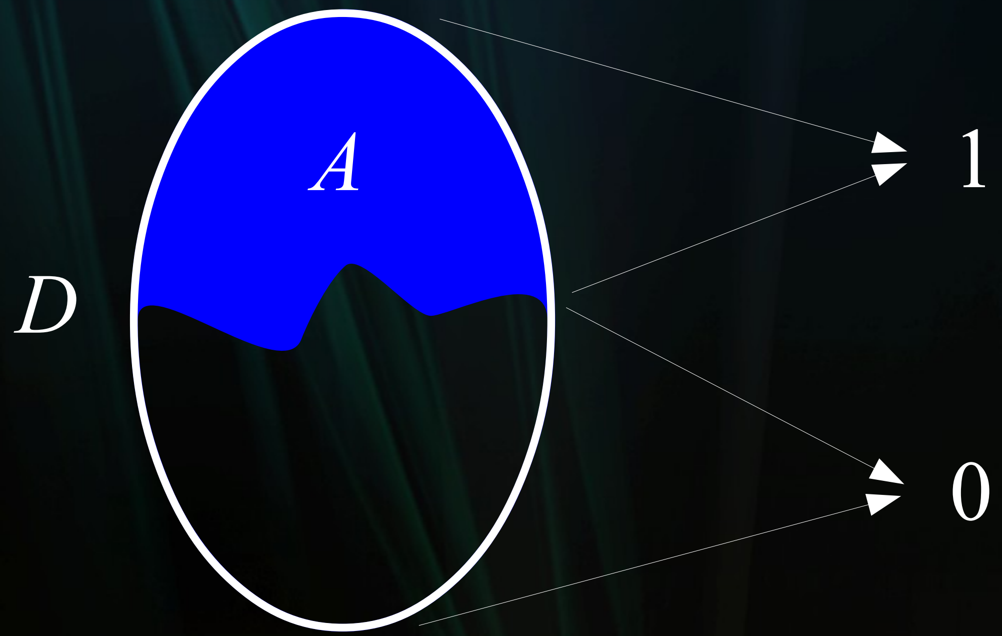
$$A \hookrightarrow D$$

"injective" ↔ "inclusion"

Characteristic function

The characteristic function χ_A of a set A is defined as:

$$\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \notin A \end{cases}$$



Subobject classifier

- Ω is called the **subobject classifier** or the **truth value object**
- Ω has 2 elements $\{0, 1\}$ corresponding to T and F
- When we map a domain D to Ω via $f\colon D \to \Omega$ we *sort the domain into various classes*, thus giving it structure. This is known as **fibration.**
- The set inclusion relation $\subseteq$ induces a pre-order and thus an equivalence class in the domain D. This means that in some domains, Ω may have more than 2 values, ie, *varying shades of truth*, suggestive of fuzzy logic.

Subobject classifier

Commutative diagram:

domain

monic arrow
(= inclusion)

characteristic
function

$$
\begin{tikzcd}
A \arrow[r, tail, "f"] \arrow[d, dashed, "!"'] & D \arrow[d, "\chi_f"] \\
1 \arrow[r, "true"] & \Omega
\end{tikzcd}
$$

unique
arrow

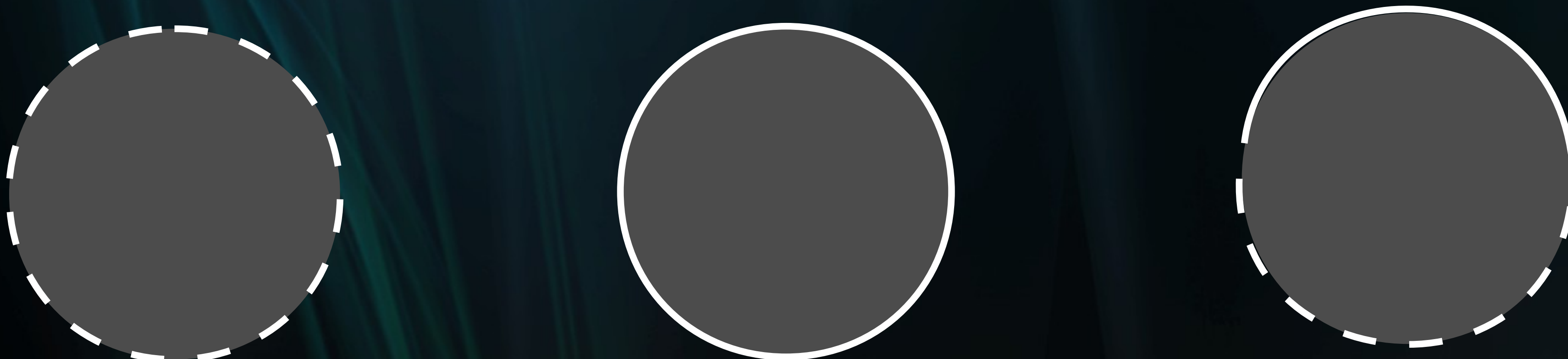
truth
value
object

object with only
one element

This function merely
picks out 1 from {0,1}

What is topology?

Topology is really the study of **open sets**.
(Open set = one that contains all of its **boundary points**)



open

closed

clopen

(neither open nor closed)

What is topology?

The collection of all open subsets of a space is called the **topology** of that space.

Topological concepts are precisely those that can be defined purely in terms of open sets, eg:

- limit points / convergence
- continuity
- path-connectedness
- compactness

whereas **distance** is not topological.

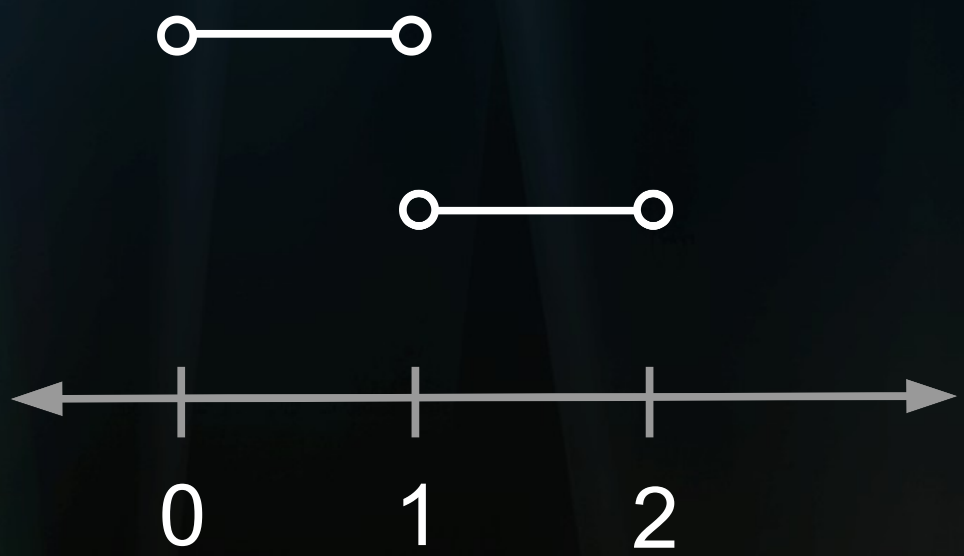
Topological semantics

In **topological semantics**, the truth value $[\ ]_V$ of a proposition is an **open set** on $\mathbb{R}$.

Eg:

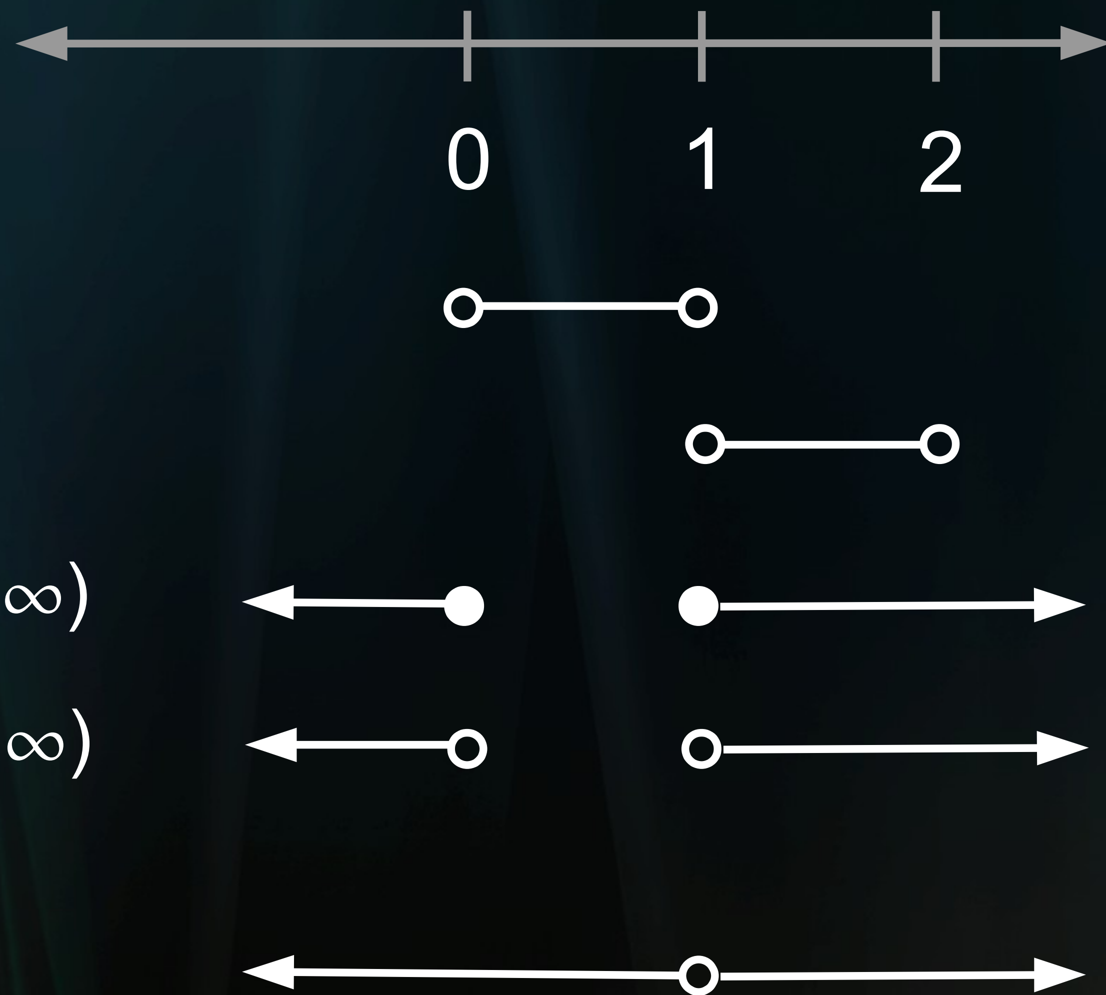
$[p]_V = (0,1)$

$[q]_V = (1,2)$



Topological semantics

More examples:



$[p]_V = (0,1)$

$[q]_V = (1,2)$

$[\neg p]_V = (-\infty,0] \cup [1,\infty)$

$[p \rightarrow q]_V = (-\infty,0) \cup (1,\infty)$

$[(p \rightarrow q) \vee (q \rightarrow p)]_V =$

$(-\infty,1) \cup (1,\infty) = \mathbb{R} \setminus \{1\}$

Geometric logic

= logic of topology

Comparison

Orders	Logic	Sets
$\leq$	$\Rightarrow$	$\subseteq$
$=$	$\Leftrightarrow$	$=$
top $\top$	T	universe
bottom $\bot$	F	Ø
meet $\sqcap$	$\wedge$	$\cap$
join $\sqcup$	$\vee$	$\cup$

Quantifiers

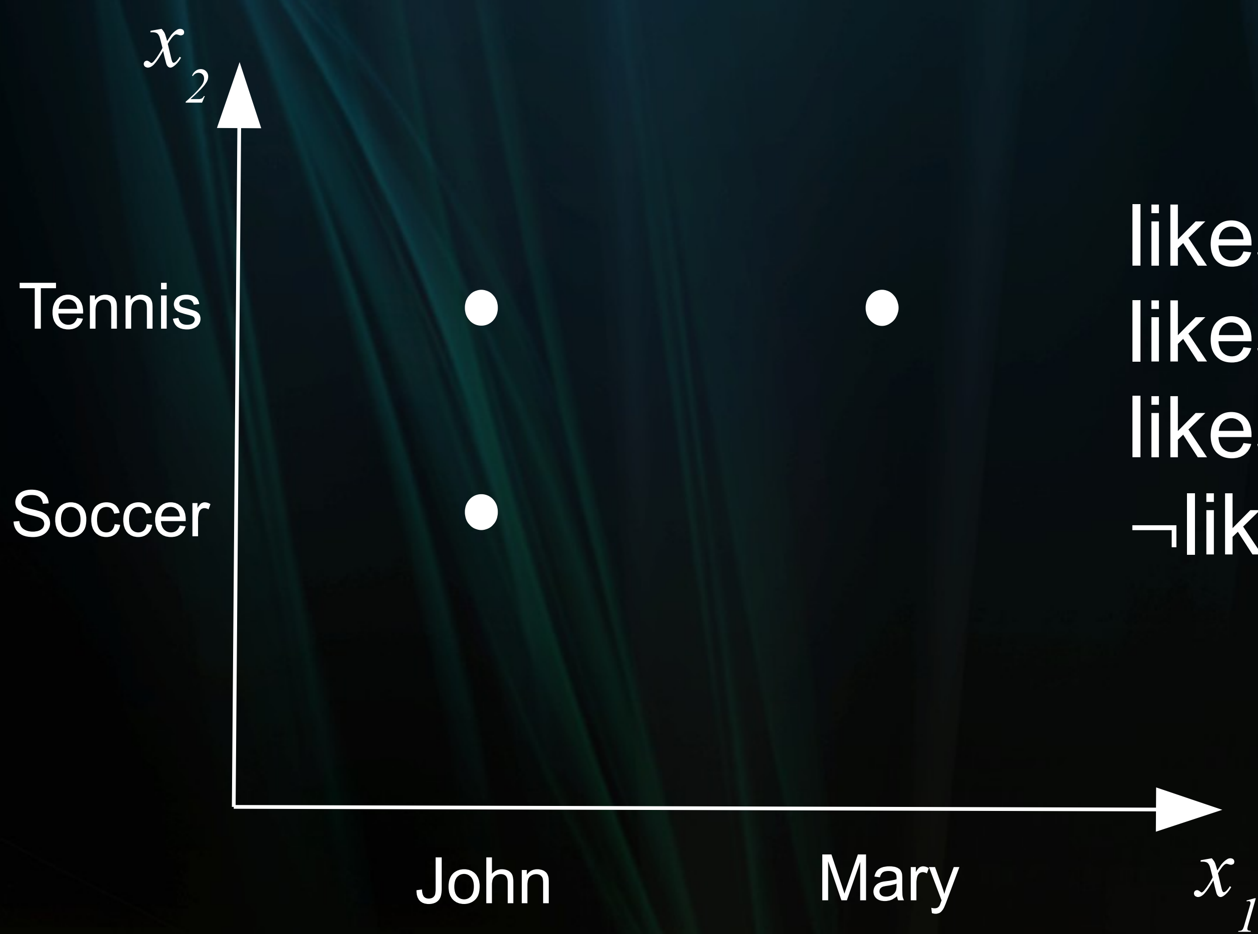
$\exists$ and $\forall$ are respectively the left and right **adjuncts** to a substitution map f:

$$\exists \dashv f \dashv \forall$$

This discovery is due to Lawvere in the 1960's

Relations

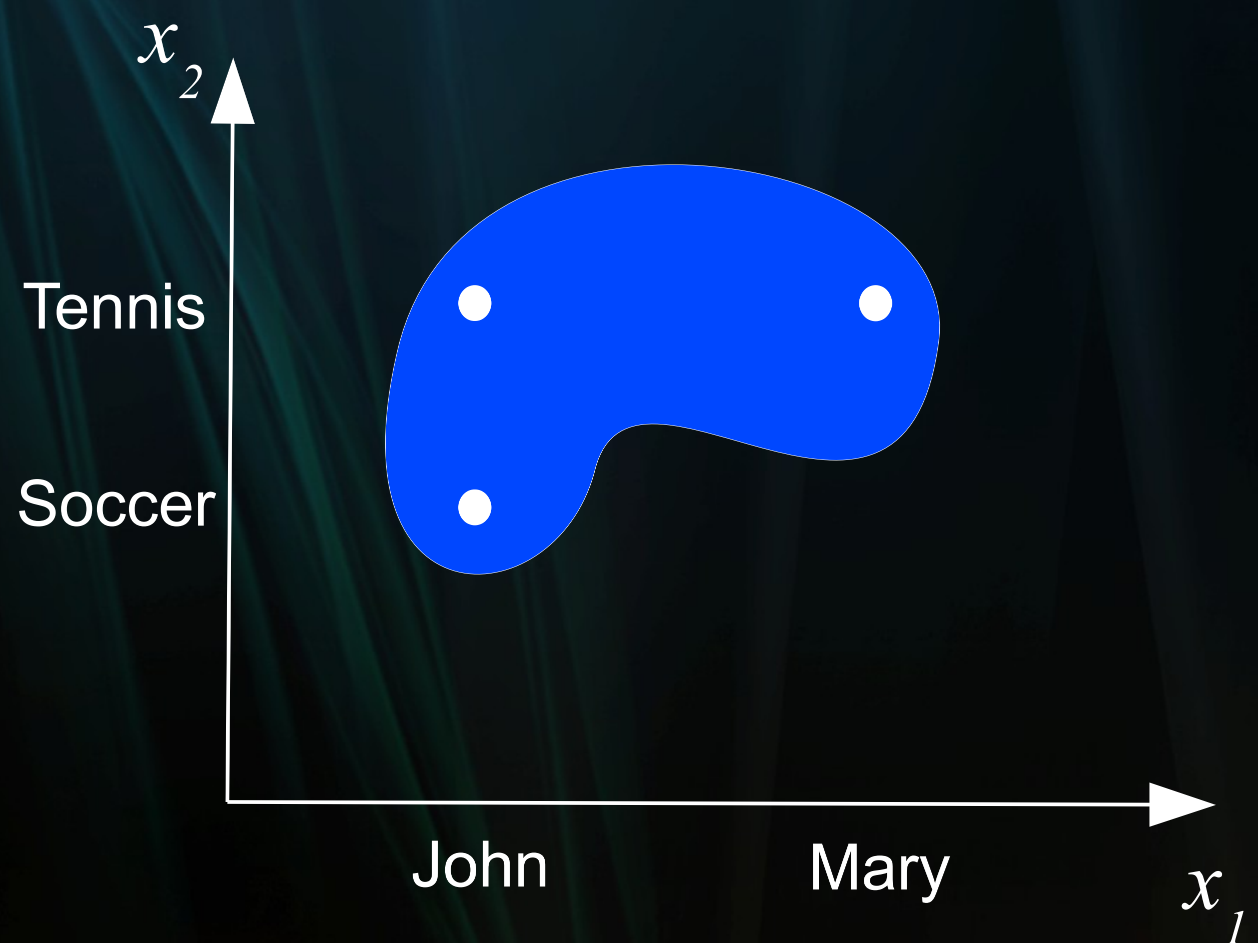
A **relation** is defined by its **graph** and can be represented spatially, eg:



likes(john, tennis)
likes(john, soccer)
likes(mary, tennis)
¬likes(mary, soccer)

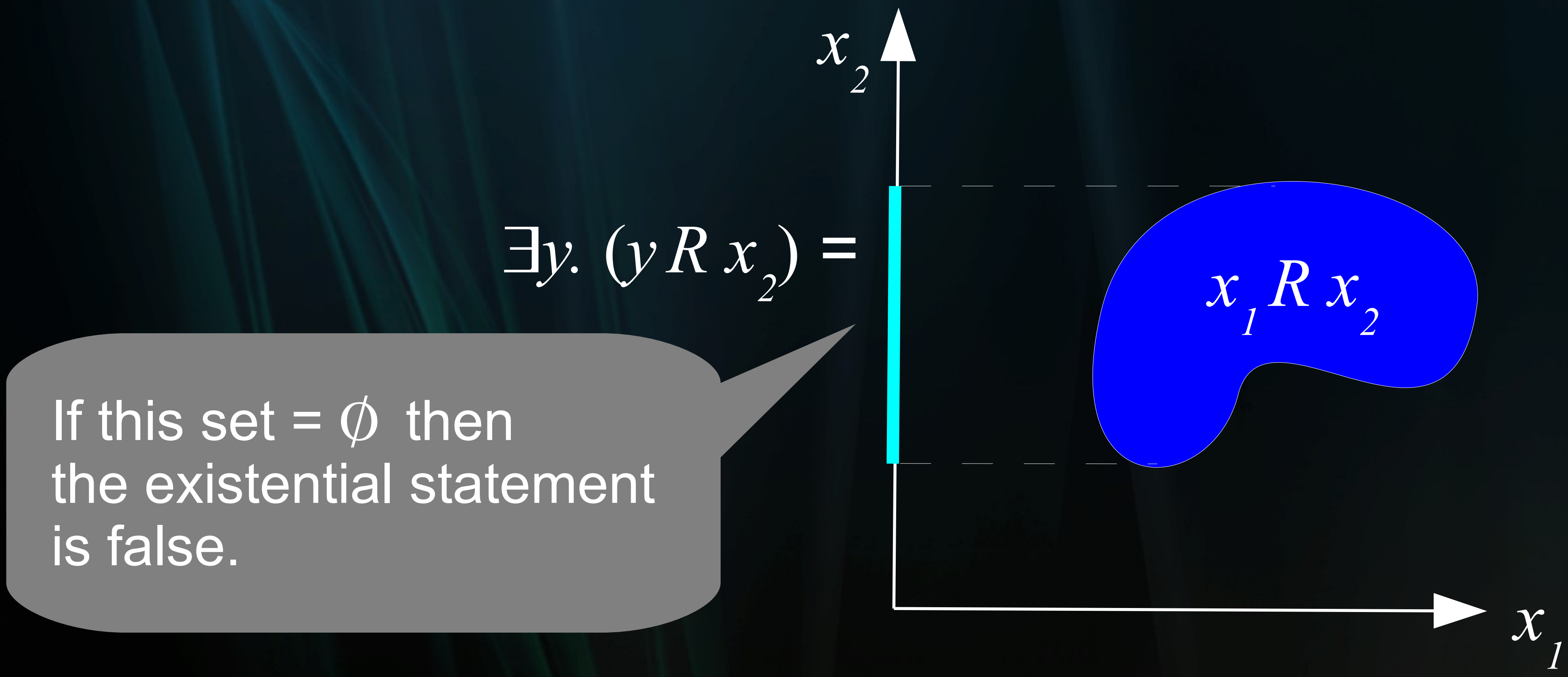
Relations

For illustration's sake assume that the domains are continuous, so the graph looks like this:



Existential quantification

∃ is equivalent to **projection** onto R's **range** or **domain**.



Due to the shape of this projection, this kind of algebra is called **cylindrical algebra**.

Adjunction

An **isomorphism** between 2 categories

$$\mathcal{C} \cong \mathcal{D}$$

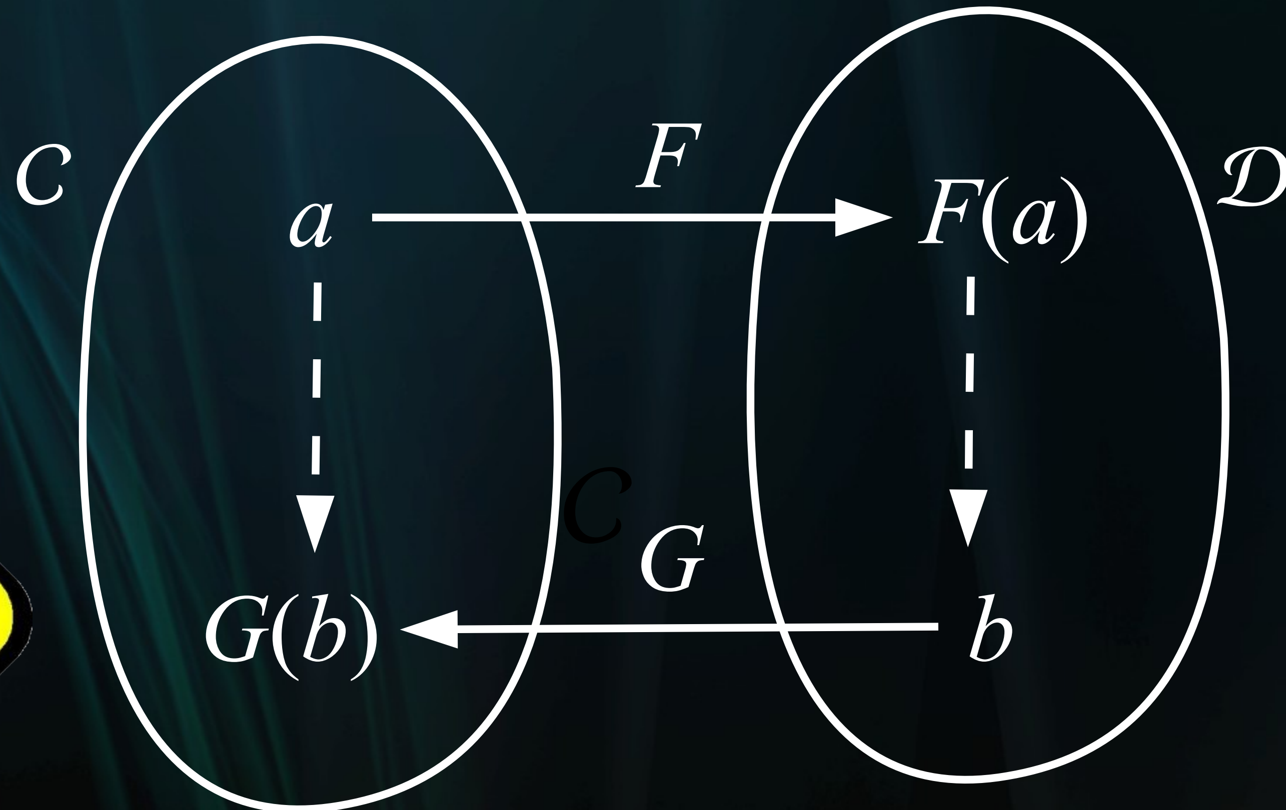
means they have the same mathematical structure (ie, essentially the same).

The **adjunction**

$$\mathcal{C} \dashv \mathcal{D}$$

is a *relaxation* of the idea of isomorphism.

Adjunction



This is the *internal* diagram of 2 categories

Example: monoid

Free monoid

… is an example of an adjunct situation

To-do

- Sheaves, locales, Heyting algebra
- Topoi and variable sets
- Awodey's idea “HOL is the logic of continuous variation”
- Probabilistic / fuzzy logic in categorical setting?

References

- Arbib & Manes 1975. *Arrows, structures, and functors – the categorical imperative*
- Awodey 2006. *Category theory*
- Barr & Wells 1990. *Category theory for computing science*
- Boltyanskii & Efremovich 2001. *Intuitive combinatorial topology*
- Catster, Warrick Univ. on YouTube: *Adjunction*
- Goldblatt 1984. *Topoi – the categorial analysis of logic*
- Halmos & Givant 1988. *Logic as algebra*
- Lawvere & Schanuel 1991. *Conceptual mathematics – a first introduction to categories*
- Pierce 1991. *Basic category theory for computer scientists*
- Plotkin 1994. *Universal algebra, algebraic logic, and databases*
- Schechter 2005. *Classical and nonclassical logics*
- Vickers 1989. *Topology via logic*