A Sum-of-Squares Approach to the Analysis of Zeno Behavior in Hybrid Dynamical Systems

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***Abstract*— In this paper, conditions for the stability of Zeno executions in hybrid dynamical systems with nonlinear dynamics and nonlinear transitions are presented. Zeno executions are non-equilibrium trajectories which experience an infinite number of transitions within a bounded time interval. The main result of the paper shows how to use the Sum-of-Squares optimization method to construct Lyapunov functions proving that all trajectories with initial conditions within a subset of the state-space are Zeno executions. Examples illustrating the use of the proposed technique are also provided. Finally, we use Sum-of-Squares to show convergence to Zeno equilibria for systems with parametric uncertainty in the vector field and transition map, along with an illustrative example.**

I. Introduction

In recent years, hybrid dynamical systems have garnered significant interest. Hybrid systems are dynamical systems with trajectories that exhibit both continuous flows and discrete transitions. As such, hybrid systems provide a framework for the study of a variety of man-made systems, including electrical systems with switching [1], communication networks [2], embedded computing [3], and air traffic control [4].

Recent research into hybrid systems has yielded results on stability of equilibria [5] and observability and controllability [6]. Several Lyapunov-based techniques for the analysis of hybrid systems, including the use of multiple Lyapunov functions [7], the construction of piecewise-quadratic Lyapunov functions [8], and the utilization of Lyapunov techniques for robust stability analysis [9] have also been presented. More recently, a means to assess stability of hybrid systems by constructing higher-order polynomial Lyapunov functions using sum-of-squares techniques was presented in [10], and a method to perform robust stability analysis using sum of squares techniques was provided in [11]. However, there are still behaviors of hybrid systems that require further study, such as the Zeno phenomenon.

Zeno behavior occurs when infinite transitions between discrete states occur in a finite period of time. Trajectories exhibiting this behavior are called Zeno executions, and converge to a set of points known as a Zeno equilibrium. Zeno hybrid systems are described in detail in, for example, [12]. Zeno behavior can cause simulations to halt or fail, since infinitely many transitions would need to be simulated, as noted in, e.g., [12]. This problem was addressed in [13] and [14], which describe methods to regularize hybrid systems to ensure that trajectories continue after the Zeno equilibrium. Sufficient conditions for Zeno behavior in a first quadrant hybrid systems were given in [15], and further sufficient conditions for systems with nonlinear vector fields based on constant approximations were given in [16]. More recently, necessary and sufficient Lyapunov conditions for the existence of Zeno equilibria were first given in [17]. These results were extended in [18], where the concept of Zeno stability was described as an extension of finite-time asymptotic stability. Moreover, [18] provided Lyapunov conditions for Zeno stability of compact sets.

In this paper, we use sum-of-squares optimization to construct the Lyapunov functions described by the Lyapunov theorem presented in [17]. This method allows us to use semidefinite programming to construct Lyapunov functions which prove the stability of Zeno equilibria. Moreover, the method presented in this paper allows for the verification of Zeno stability for polynomial hybrid systems with nonlinear vector fields and transitions. Last, we present a method to verify Zeno stability for systems with paramatric uncertainties.

The outline of the paper is as follows: in Section II, definitions of sum-of-squares polynomials, the positivstellensatz, hybrid systems and their executions, and Zeno executions and equilibria are presented. Section II also details Lyapunov conditions for the existence of Zeno behavior as described in [17]. In Section III, we present a method to construct Lyapunov functions to prove Zeno stability using Sum-of-Squares optimization, and in Section IV, illustrative examples are provided. In Section V, a method to determine the stability of Zeno equilibria for systems with parametric uncertainties is given.

II. Preliminaries

In this section, we provide a brief introduction to Sum-of-Squares polynomials and Positivstellensatz results, as well as definitions for hybrid systems, their executions, and Zeno behavior. We use $\mathbf{R}[x]$ to denote the ring of polynomials generated by variables $x = (x_1, .., x_n)$.

A. Sum of Squares Polynomials

Definition 1. (Sum of Squares Polynomial) A polynomial $f(x) : \mathbb{R}^n \to \mathbb{R}$ is said to be Sum of Squares (SOS) if there exist polynomials $p_i(x) : \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \sum_i (p_i(x))^2$$

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We use $f \in \Sigma_x \subset \mathbf{R}[x]$ to denote that f is SOS.

The following result gives a polynomial-time complexity test to determine whether a polynomial is SOS.

Theorem 1. *For a polynomial, p of degree $2d$, $p \in \Sigma_x$ if and only if there exists a positive semidefinite matrix Q, such that*

$$p(x) = Z(x)^T QZ(x)$$

where $Z(x)$ is the vector of monomials of degree d or less

Therefore, checking whether a polynomial is SOS is equivalent to checking the existence of a positive-semidefinite matrix Q, which is a convex feasibility problem. Thus, while checking polynomial positivity is NP-hard, checking whether a polynomial is SOS is decidable in polynomial time.

B. The Positivstellensatz

A Positivstellensatz is a result from real algebraic geometry which provides a means to verify the positivity of a polynomial over a semialgebraic set. In this section, we provide some necessary mathematical background, as well as Stengle's Positivstellensatz [19].

Definition 2. (Semialgebraic Set) A semialgebraic set is a set of the form

$$S := \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1,\ldots,n_1,\ h_i(x) = 0, i = 1,\ldots,n_3\}$$

where each $f_i \in \mathbf{R}[x]$, $g_i \in \mathbf{R}[x]$, and $h_i \in \mathbf{R}[x]$

Definition 3. (Multiplicative Monoid) A multiplicative monoid $\mathscr{M}$ generated by elements $\{f_1,\ldots,f_n\} \in \mathbf{R}[x]$ is the set

$$\mathscr{M} := \left\{ p \in \mathbf{R}[x] : p = \prod_{i=1}^{n} f_i^{k_i},\ k_i \in \mathbb{N} \right\}$$

Thus, $\mathscr{M}$ is the set of of finite products of $\{f_1,\ldots,f_n\}$.

Definition 4. (Cone) For given elements $\{f_1,\ldots,f_n\} \in \mathbf{R}[x]$, let $\bar{\mathscr{M}}$ be the set of products defined by

$$\bar{\mathscr{M}} := \left\{ p \in \mathbf{R}[x] : p = \prod_{i=1}^{n} f_i^{k_i}, k_i \in \{0,1\} \right\}$$

and let M denote the cardinality of $\bar{\mathscr{M}}$. The cone $\mathscr{P}$ generated by $\{f_1,\ldots,f_n\} \in \mathbf{R}[x]$ is the subset of $\mathbf{R}[x]$ defined as

$$\mathscr{P} := \left\{ p \in \mathbf{R}[x] : p = s_0 + \sum_{i=1}^{M} s_i m_i,\ m_i \in \bar{\mathscr{M}},\ s_i \in \Sigma_x \right\}.$$

$\mathscr{P}$ satisfies the following properties:

1) $a,b \in \mathscr{P}$ implies $a+b \in \mathscr{P}$
2) $a,b \in \mathscr{P}$ implies $a \cdot b \in \mathscr{P}$
3) $a \in \mathbf{R}[x_1,\ldots,x_n]$ implies $a^2 \in \mathscr{P}$

Definition 5. (Ideal) The Ideal $\mathscr{I}$ generated by $\{f_1,\ldots,f_n\} \in \mathbf{R}[x]$ is defined as

$$\mathscr{I} := \left\{ p \in \mathbf{R}[x] : p = \sum_{i=1}^{n} q_i f_i,\ q_i \in \mathbf{R}[x] \right\}$$

Note that $\mathscr{I}$ must satisfy

1) $a,b \in \mathscr{I}$ implies $a+b \in \mathscr{I}$
2) $a \in \mathscr{I}; b \in \mathbf{R}[x]$ implies $ab \in \mathscr{I}$

Theorem 2. (Stengle's Positivstellensatz) *Consider the polynomials $\{f_1, f_2,\ldots,f_{n_1}\} \in \mathbf{R}[x]$, $\{g_1, g_2,\ldots,g_{n_2}\} \in \mathbf{R}[x]$, and $\{h_1,h_2,\ldots,h_{n_3}\} \in \mathbf{R}[x]$. Let $\mathscr{P}$ be the cone generated by $\{f_i\}_{i=1,2,\ldots,n_1}$, $\mathscr{M}$ be the multiplicative monoid generated by $\{g_j\}_{j=1,2,\ldots,n_2}$, and $\mathscr{I}$ be the Ideal generated by $\{h_k\}_{k=1,2,\ldots,n_3}$. Then, the following statements are equivalent:*

1) $\{x \in \mathbb{R}^n : f_i(x) \geq 0, g_j(x) \neq 0, h_k(x) = 0, i = 1,\ldots n_1, j = 1,\ldots,n_2, k = 1,\ldots,n_3\} = \emptyset$
2) $\exists F \in \mathscr{P}$, $\exists G \in \mathscr{M}$, $\exists H \in \mathscr{I}$ *s.t.*

$$F + G^2 + H \equiv 0$$

The Positivstellensatz has been described as a generalization of the S-procedure [20]. However, while the S-procedure provides information regarding the positivity of quadratic forms such that other quadratic forms are also positive, the Positivstellensatz can be used to obtain certificates of positivity for polynomials of arbitrary degree over semialgebraic sets. We use the Positivstellensatz extensively in this paper to construct Lyapunov functions which are positive on bounded sets (see sections III and V).

For further details and proofs, we refer to [19] and [21].

C. Hybrid Systems

In this section, we define hybrid systems and their executions. We use similar notation to that given in [22] and, more recently, [17].

Definition 6. (Hybrid System) A hybrid system $\mathbf{H}$ is a tuple:

$$\mathbf{H} = (Q,E,D,F,G,R)$$

where

- $Q \subset \mathbb{Z}$ is a finite collection of discrete states or indices.
- $E \subset Q \times Q$ is a collection of edges. For any edge $e = (q,q')$ we use the functions s and t to denote the start and end, so that for $e = (q,q')$, $s(e) = q$ and $t(e) = q'$.
- $D = \{D_q\}_{q \in Q}$ is a collection of Domains, where for each $q \in Q$, $D_q \subseteq \mathbb{R}^n$.
- $F = \{f_q\}_{q \in Q}$ is a collection of vector fields, where for each $q \in Q$, $f_q : D_q \to \mathbb{R}^n$.
- $G = \{G_e\}_{e \in E}$ is a collection of guard sets, where for each $e = (q,q') \in E$, $G_e \subset D_q$
- $R = \{\phi_e\}_{e \in E}$ is a collection of Reset Maps, where for each $e = (q,q') \in E$, $\phi_e : G_e \to D_{q'}$.

Definition 7. A cyclic hybrid system $\mathbf{H}_c$ is a hybrid system where for each domain $q \in Q$, we can associate a unique edge $e(q) = (q,q_i) \in E$ such that $s(e(q)) = q$ and such that for any $q \in Q$, $q = t(e(t(e(\cdots t(e(t(e(q))))))))$. That is, the set of edges forms a directed graph.

Assumption 1:

For the purposes of this paper, we consider hybrid systems with polynomial vector fields and resets, and semialgebraic domains and guard sets. We implicitly assume every hybrid system is of this form and that associated with every hybrid system is a set of polynomials g_{qi}, $h_{e,k}$ for $q \in Q$, $e \in E$, $i = k = 1, \cdots, K_q$ and $k = 1, \cdots, N_q$ for some $K_q, N_q > 0$. In this framework, the domains of the hybrid system $\mathbf{H}$ are defined as

$$D_q = \{x \in \mathbb{R}^n \,:\, g_{qk}(x) \geq 0,\, k = 1, 2, \cdots, K_q\} \tag{1}$$

where $g_{qk} \in \mathbf{R}[x]$, and $K_q \in \mathbb{N}$. Similarly, the guard sets are defined as

$$G_e = \{x \in \mathbb{R}^n : h_{e,0}(x) = 0,\, h_{e,k}(x) \geq 0,\, k = 1, 2, \cdots, N_q\} \tag{2}$$

where each $h_{ek} \in \mathbf{R}[x]$, and $N_q \in \mathbb{N}$. Lastly, for each $e = (q, q') \in E$, the reset map ϕ_e is given by the vector-valued polynomial function

$$\phi_e = [\phi_{e,1}, \cdots, \phi_{e,n}]^T \tag{3}$$

where $\phi_{e,j} \in \mathbf{R}[x]$ for $j = 1, \cdots, n$.

Definition 8. (Hybrid System Execution) We say that the tuple

$$\chi = (I, T, p, C)$$

is an *execution* of the hybrid system $\mathbf{H}$ if: where

- $I \subseteq \mathbb{N}$ is index of intervals.
- $T = \{T_i\}_{i \in I}$ are a set of open time intervals associated with points in time τ_i as $T_i = (\tau_i, \tau_{i+1}) \subset \mathbb{R}^{n+}$ where $T_{i+1} = (\tau_{i+1}, \tau_{i+2})$.
- $p : I \to Q$ maps each interval to a domain.
- $C = \{c_i(t)\}_{i \in I}$ is a set of continuously differentiable functions satisfying
  1) $\dot{c}_i(t) = f_{p(i)}(c_i(t))$ for $t \in T_i$ and for all $i \in I$.
  2) $c_i(t) \in D_{p(i)}$ for $t \in T_i$ and for all $i \in I$.
  3) $c_i(\tau_{i+1}) \in G_{(p(i), p(i+1))}$ for all $i \in I$.
  4) $c_{i+1}(\tau_{t+1}) = \phi_{(p(i), p(i+1))}(c_i(\tau_i))$ for all $i \in I$.

D. Zeno Behavior in Hybrid Dynamical Systems

We now present definitions of Zeno executions, equilibria, and stability, along with necessary and sufficient conditions for Zeno stability as presented in [17] and [18].

Definition 9. (Zeno Execution) We say an execution $\chi = (I, T, p, C)$ of a hybrid System $\mathbf{H} = (Q, E, D, F, G, R)$ is Zeno if

1) $I = \mathbb{N}$
2) $\lim_{i \to \infty} \tau_i < \infty$

Definition 10. (Zeno Equilibrium) A set $z = \{z_q\}_{q \in Q}$ is a Zeno equilibrium of a Hybrid System $\mathbf{H} = (Q, E, D, F, G, R)$ if it satisfies

1) For each edge $e = (q, q') \in E$, $z_q \in G_e$ and $\phi_e(z_q) = z_{q'}$.
2) $f_q(z_q) \neq 0$ for all $q \in Q$.

Note that for any $z \in \{z_q\}_{q \in Q}$, where $\{z_q\}_{q \in Q}$ is a Zeno equilibrium of a cyclic hybrid system $\mathbf{H}_c$,

$$\phi_{i-1} \circ \cdots \circ \phi_0 \cdots \phi_i(z) = z$$

Next, we define Zeno stability:

Definition 11. (Zeno Stability) Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a hybrid system with Zeno equilibrium $z = \{z_q\}_{q \in Q}$. The set z is Zeno stable if, for each $q \in Q$, there exist neighborhoods Z_q, where $z_q \in Z_q$, such that for any initial condition $x_0 \in \bigcup_{q \in Q} Z_q$, the execution $\chi = (I, T, p, C)$, with $c_o(t_0) = x_0$ is Zeno, and converges to z.

Note that this definition of Zeno stability is consistent with the stability definitions provided in [18]. We now reiterate the Lyapunov conditions for the stability of Zeno equilibria in cyclic hybrid systems presented in [17], which are as follows:

Theorem 3. (Lamperski and Ames) *Consider a hybrid system $\mathbf{H} = (Q, E, D, F, G, R)$, with an isolated Zeno equilibrium $\{z_q\}_{q \in Q}$. Let $\{W_q\}_{q \in Q}$ be a collection of open neighborhoods of $\{z_q\}_{q \in Q}$. Suppose there exist continuously differentiable functions $V_q : \mathbb{R}^n \to \mathbb{R}$ and $B_q : \mathbb{R}^n \to \mathbb{R}$, and non-negative constants $\{r_q\}_{q \in Q}$, γ_a, and γ_b, where $r_q \in [0, 1]$, and $r_q < 1$ for some q and such that*

$$
\begin{aligned}
&V_q(x) && > 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q && (4)\\
&V_q(z_q) && = 0, \quad \text{for all } q \in Q && (5)\\
&\nabla V_q^T(x) f_q(x) && \leq 0 \quad \text{for all } x \in W_q, q \in Q && (6)\\
&B_q(x) && \geq 0 \quad \text{for all } x \in W_q, q \in Q && (7)\\
&\nabla B_q^T(x) f_q(x) && < 0 \quad \text{for all } x \in W_q, q \in Q && (8)\\
&V_{q'}(R_{(q,q')}(x)) && \leq r_q V_q(x), && (9)\\
& && \quad \text{for all } e = (q, q') \in E \text{ and } x \in G_e \cap W_q &&\\
&B_q(R_{(q',q)}(x)) && \leq \gamma_b \left(V_q(R_{(q,q')}(x))\right)^{\gamma_a} && (10)\\
& && \quad \text{for all } e = (q, q') \in E \text{ and } x \in G_e \cap W_q. &&
\end{aligned}
$$

Then $\{z_q\}_{q \in Q}$ is Zeno stable.

As noted in [18], satisfying EC1-C2 guarantees asymptotic Zeno stability. To simplify notation, we will use the sufficient conditions of Theorem 4 as follows. Note that our subsequent analysis can be easily applied to directly to the conditions of Theorem 3 and in our numerical examples we have tested both sets of conditions and they yield similar results.

Theorem 4. *Let $\mathbf{H} = (Q, E, D, F, G, R)$ be a cyclic hybrid system with Zeno equilibrium $z = \{z_q\}_{q \in Q}$. Let $\{W_q \subset D_q\}_{q \in Q}$, be a collection of neighborhoods of the $\{z_q\}_{q \in Q}$. Suppose that there exist continuously differentiable functions $V_q : W_q \to \mathbb{R}$, and positive constants $\{r_q\}_{q \in Q}$ and γ, where $r_q \in (0, 1]$, and $r_q < 1$ for some q and such that*

$$
\begin{aligned}
&V_q(x) && > 0 \quad \text{for all } x \in W_q \backslash z_q, q \in Q && (11)\\
&V_q(z_q) && = 0, \quad \text{for all } q \in Q && (12)\\
&\nabla V_q^T(x) f_q(x) && \leq -\gamma \quad \text{for all } x \in W_q, q \in Q && (13)\\
&r_q V_q(x) && \geq V_{q'}(\phi_e(x)) && (14)\\
& && \quad \text{for all } e = (q, q') \in E \text{ and } x \in G_e \cap W_q. &&
\end{aligned}
$$

then z is a stable Zeno equilibrium.

Proof:

We show that if for each $q \in Q$, we can find a V_q such that (11)-(14) are satisfied, then the same V_q also satisfies (4)-(10). From inspection, it is clear that if V_q satisfies (11)-(14), then (4)-(6) and (9) are satisfied. Second, choose $B_q = V_q$ for each $q \in Q$. From inspection, it is clear that V_q also satisfies (7) and (8). Last, if $\gamma_a = \gamma_b = 1$, we get $V_q \le V_q$, where the equality holds. From this, we see that for each $q \in Q$, V_q also satisfies (10). Thus, the theorem is proved. $\square$

III. Using Sum-of-Squares Optimization to Determine Zeno Stability

Theorem 4 provides sufficient conditions for Zeno stability in cyclic hybrid systems. We now show that these conditions can be enforced using SOS

Let $\mathbf{H} = (Q,E,D,F,G,R)$ be a hybrid system with a Zeno equilibrium $\{z_q\}_{q\in Q}$. Let $\{W_q\}_{q\in Q}$ be a collection of neighborhoods of $\{z_q\}_{q\in Q}$. Moreover, suppose that each W_q is a semialgebraic set defined as

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1,2,...,K_{qw}\}$$

where $w_{qk} \in \mathbf{R}[x]$.
We define feasibility problem 1:

Feasibility Problem 1:

For hybrid system $\mathbf{H} = (Q,E,D,F,G,R)$, find

- a_{qk}, c_{qk}, i_{qk}, $\in \Sigma_x$, for $k = 1,2,...,K_{qw}$ and $q \in Q$;
- b_{qk}, d_{qk}, $j_{qk} \in \Sigma_x$, for $k = 1,2,...,K_q$ and $q \in Q$.
- $m_{e,l} \in \Sigma_x$ for $e \in E$ and $l = 1,2,...,N_q$
- V_q, $m_{e,0} \in \mathbf{R}[x]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q\in Q} \in (0,1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} \in \Sigma_x \quad \text{for all } q \in Q \tag{15}$$

$$V_q(z_q) = 0 \quad \text{for all } q \in Q \tag{16}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} \in \Sigma_x \quad \text{for all } q \in Q \tag{17}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} \in \Sigma_x \quad \text{for all } e = (q,q') \in E \tag{18}$$

Theorem 5. *Let $z = \{z_q\}_{q\in Q}$ be an isolated Zeno equilibrium of a hybrid system $\mathbf{H} = (Q,E,D,F,G,R)$. If Feasibility Problem 1 has a solution, then z is Zeno stable.*

Proof:

To prove the theorem we show that if V_q, $q \in Q$ are elements of a solution of Feasibility Problem 1, then for each $q \in Q$, the same V_q also satisfy (4)-(9) of Theorem 4. That is, we show that if the V_q satisfy (15)-(18), then the same V_q also satisfies (11)-(14).

First, we observe that (16) directly implies (12). Next, from (15), we know that

$$V_q(x) \ge \sum_{k=1}^{K_{qw}} a_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} b_{qk}(x) g_{qk}(x) + \alpha x^T x$$

Since $a_{qk}(x)$ and $b_{qk}(x)$ are SOS, and thus, always nonnegative, by the Positivstellensatz and the definitions of W_q and D_q, we have that $V_q(x) \ge \alpha x^T x$ for all $x \in W_q \subset D_q$. Thus, (15) implies (11) is satisfied. Similarly, from (17),

$$-\nabla V_q^T(x) f_q(x) - \gamma \ge \sum_{k=1}^{K_{qw}} c_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} d_{qk}(x) g_{qk}(x).$$

Since $c_{qk}(x)$ and $d_{qk}(x)$ are always nonnegative, by the definition of D_q and W_q, $\nabla V_q(x)^T f_q(x) \le -\gamma$ for $x \in \{x \in \mathbb{R}^n : g_{qk}(x) \ge 0,\ w_{qk}(x) \ge 0\} = D_q \cap W_q$ which implies (13) is satisfied. Next, from (18) we have that for all $e = (q,q') \in Q$,

$$r_q V_q(x) - V_{q'}(\phi_e(x)) \ge m_{e,0}(x) h_{e,0}(x) + \sum_{l=1}^{N_q} m_{e,l}(x) h_{e,l}(x) + \sum_{k=1}^{K_q} i_{qk}(x) w_{qk}(x) + \sum_{k=1}^{K_q} j_{qk}(x) g_{qk}(x).$$

First note that $h_{e,0}(x) = 0$ and hence $m_{e,0}(x) h_{e,0}(x) = 0$ on G_e. Since $m_{e,l} \in \Sigma_x$, we have $m_{e,l}(x) h_{e,l}(x) \ge 0$ on G_e. Similarly $j_{qk}(x) g_{qk}(x) \ge 0$ on D_q and $i_{qk}(x) w_{qk}(x) \ge 0$ on W_q. It follows that $r_q V_q(x) - V_{q'}(\phi_e(x)) \ge 0$ when $x \in G_e \cap W_q \cap D_q$ for all $e = (q,q') \in E$. Thus, we have shown that (18) implies (14). Thus we conclude that the solution elements V_q of Feasibility Problem 1 satisfy the conditions (11)-(14) of Theorem 4. Thus by Theorem 4 we conclude stability of the Zeno equilibrium. $\square$

IV. Examples

A. The Bouncing Ball

We first consider a very simple model of the bouncing ball:

Example 1.

A bouncing ball $\mathbf{B}$ is modeled by a tuple:

$$\mathbf{B} = (Q,E,D,F,G,R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0,q_0)\}$
- $D := \{x \in \mathbb{R}^2 : x_1 \ge 0\}$
- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \le 0\}$
- $R = \phi(x) = [0, -c x_2]^T$. Here, c is a coefficient of restitution.

Results: The Zeno equilibrium is $z = (0,0)^T$. We consider stability on the unit ball $W_q := \{x \in \mathbb{R}^n : x_1 \ge 0, 1 - x_1^2 - x_2^2 < 0\}$ for feasibility problem 1.

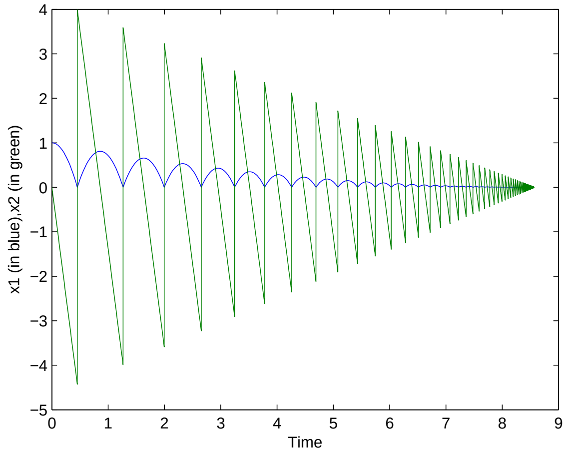

Fig. 1. Bouncing Ball with $c = 0.9$

Using SOSTOOLS to implement the conditions given in Feasibility Problem 1, for values of $c \in [0, .999]$ we were able to find a 4th–order $V(x)$ which verifies stability of Zeno executions using SOS polynomial multipliers of second order.

B. Water Tank Problem

A classic example of a hybrid system exhibiting Zeno behavior is the dual water tank, as described in [23]. The system consists of two water tanks sharing a single supply pipe, which pumps water at the constant rate W. Moreover, each tank leaks at a constant rate (v_1 and v_2). The supply pipe switches to tank i when the water level falls below a prescribed level r_i.

Example 2.

The two-tank system can be modeled by a hybrid system **T**, which is the tuple

$$\mathbf{T} = (Q, E, D, F, G, R)$$

where

- $Q = q_1, q_2$
- $E = \{(q_1, q_2), (q_2, q_1)\}$
- $D = \{D_{q_1}, D_{q_2}\}$, where

$$D_{q_1} := \{x \in \mathbb{R}^2 : x_2 - r_2 \geq 0, x_1 \geq 0\}$$

and

$$D_{q_2} := \{x \in \mathbb{R}^2 : x_1 - r_1 \geq 0, x_2 \geq 0\}$$

- $F = \{f_1, f_2\}$, where

$$f_1 = \begin{pmatrix} W - v_1 \\ -v_2 \end{pmatrix}$$

and

$$f_2 = \begin{pmatrix} -v_1 \\ W - v_2 \end{pmatrix}$$

- $G = \{G_{q_1 q_2}, G_{q_2 q_1}\}$, where

$$G_{q_1 q_2} := \{x \in \mathbb{R}^2 : r_2 - x_2 \geq 0\}$$

and

$$G_{q_2 q_1} := \{x \in \mathbb{R}^2 : r_1 - x_1 \geq 0\}$$

- $R = \{R_{q_1 q_2}, R_{q_2 q_1}\}$, where

$$R_{q_1 q_2} = R_{q_2 q_1} = x$$

Results:

The Zeno equilibrium is $z = [r_1, r_2]^T$. For solving Feasibility problem 1, we again consider the unit ball in both domains.

$$W_q := \{x \in \mathbb{R}^2 : ||x - z||^2 \leq 1\}$$

We then obtain fourth order $V_1(x)$ and $V_2(x)$ by solving Feasibility Problem 1. Exploring values of the parameter space, we find our algorithm is able to prove stability when $v_1 + v_2 < W$.

C. System with nonlinear resets and vector field

We now consider a more difficult model similar to that of the bouncing ball, but with a nonlinear vector field and a nonlinear reset.

Example 3.

The nonlinear hybrid system can be represented by **N**, which is the tuple:

$$\mathbf{N} = (Q, E, D, F, G, R)$$

where

- $Q = \{q_0\}$
- $E = \{(q_0, q_0)\}$
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$
- $G := \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$
- $F = \{f\}$, where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g + c_1 x_2^2 \end{pmatrix}$$

- $R = \phi(x) = [0, -c_2 x_2 (1 - c_3 x_2^2)]^T$. Here, c_1, c_2, and c_3 are positive constants satisfying $c_i < 1$.

Results The Zeno equilibrium is $z = (0, 0)$. We searched

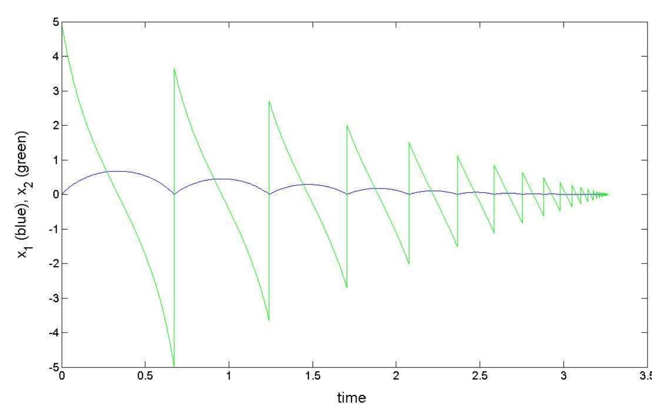

Fig. 2. Nonlinear Hybrid System with $c_1 = 0.5$, $c_2 = 0.8$, $c_3 = 0.001$

for a 4th-order $V(x)$ and multipliers that solves Feasibility Problem 1 using SOSTOOLS. While the range of Zeno-stable parameters was complicated, we were able to show Zeno-stability on the unit ball for a range of values.

We fix each c_i at certain values, and plot the other constants. In Figure 3, we set $c_1 = 0.99, 0.50$, and 0.001, and plot corresponding values of c_1 and c_2 such that $\mathbf{N}$ was stable. Similarly, in Figure 4, we set c_2 to some constant values,

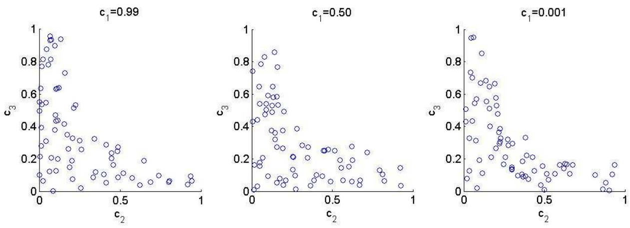

Fig. 3. Values of c_2 and c_3 such that $\mathbf{N}$ is stable for fixed c_1

and plot values of c_1 and c_3 such that $\mathbf{N}$ is stable. Last, in

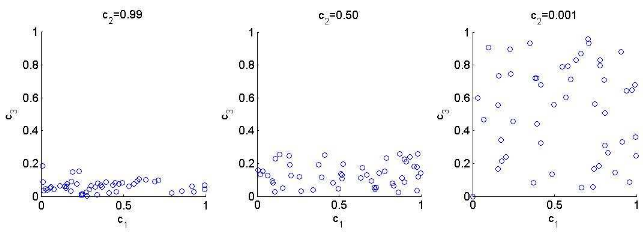

Fig. 4. Values of c_1 and c_3 such that $\mathbf{N}$ is stable for fixed c_2

Figure 5, we set c_3 equal to some constant values, and plot values of c_1 and c_2 such that $\mathbf{N}$ is stable.

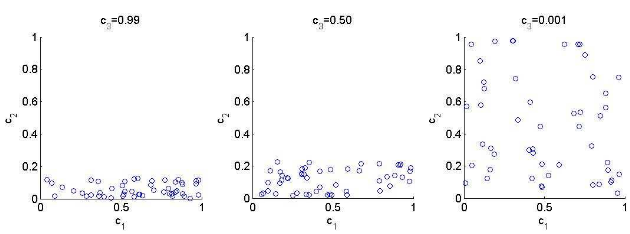

Fig. 5. Values of c_1 and c_2 such that $\mathbf{N}$ is stable for fixed c_3

V. Zeno Stability in Systems with Uncertainties

We now present a method to verify Zeno stability in cyclic hybrid systems with time-invariant uncertainties in the guards, vector fields, and resets.

Define the uncertain vector p to lie within a semialgebraic set $P := \{p \in \mathbb{R} : \tilde{p}_k(p) \geq 0, k = 1,2,...,K_1\}$. Let $\mathbf{H} = (Q,E,D,F,G,R)$ be a hybrid system with Zeno equilibrium $\{z_q\}_{q\in Q}$. Let $\{W_q\}_{q\in Q}$ be a collection of neighborhoods of $\{z_q\}_{q\in Q}$. We consider W_q of the form

$$W_q := \{x \in \mathbb{R}^n : w_{qk}(x) > 0, k = 1,2,...,K_q\}$$

where each $w_{qk}(x) \in \mathbf{R}[x]$.
Consider feasibility problem 2:
Feasibility Problem 2:

For hybrid system $\mathbf{H} = (Q,E,D,F,G,R)$, find

- a_{qk}, c_{qk}, i_{qk}, $\in \Sigma_{x,p}$, for $k = 1,2,...,K_{qw}$ and $q \in Q$;
- b_{qk}, d_{qk}, $j_{qk} \in \Sigma_{x,p}$, for $k = 1,2,...,K_q$ and $q \in Q$.
- η_{qk}, β_{qk}, $\zeta_{qk} \in \Sigma_{x,p}$, for $k = 1,2,...,K_1$ and $q \in Q$.
- $m_{e,l} \in \Sigma_{x,p}$ for $e \in E$ and $l = 1,2,...,N_q$
- V_q, $m_{e,0} \in \mathbf{R}[x,p]$ for $e \in E$ and $q \in Q$.
- Constants $\alpha, \gamma > 0$, $\{r_q\}_{q\in Q} \in (0,1]$ such that $r_q < 1$ for some $q \in Q$.

such that

$$V_q - \alpha x^T x - \sum_{k=1}^{K_{qw}} a_{qk} w_{qk} - \sum_{k=1}^{K_q} b_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \eta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{19}$$

$$V_q(z_q, p) = 0 \quad \text{for all } q \in Q \tag{20}$$

$$-\nabla V_q^T f_q - \gamma - \sum_{k=1}^{K_{qw}} c_{qk} w_{qk} - \sum_{k=1}^{K_q} d_{qk} g_{qk} - \sum_{k_1=1}^{K_1} \beta_{qk_1} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } q \in Q \tag{21}$$

$$r_q V_q - V_{q'}(\phi_e) - m_{e,0} h_{e,0} - \sum_{l=1}^{N_q} m_{e,l} h_{e,l} - \sum_{k=1}^{K_{qw}} i_{qk} w_{qk} - \sum_{k=1}^{K_q} j_{qk} g_{qk} - \sum_{k=1}^{K_1} \zeta_{qk} \tilde{p}_{qk} \in \Sigma_{x,p} \quad \text{for all } e = (q,q') \in E. \tag{22}$$

Theorem 6. *Let $z = \{z_q\}_{q\in Q}$ be a Zeno equilibrium of a hybrid system $\mathbf{H} = (Q,E,D,F,G,R)$. If there is a solution to Feasibility Problem 2, then z is Zeno stable for all $p \in P$.*

Proof:
Suppose the problem is feasible. If p is in P, $\tilde{p}_k(p) \geq 0$. Thus, by similar logic to that employed in the proof of Theorem III, we can show that V_q satisfies Conditions (11)-(14) for all $p \in \mathscr{P}$. By Theorem 4, this implies that the Zeno equilibrium is stable for all $p \in P$ $\square$

A. Illustrative Example: Bouncing Ball with Uncertainy

We use a variant of the bouncing ball model to illustrate computational analysis of robust Zeno stability using SOS as described above. Here, the coefficient of restitution is a time-invariant uncertain parameter.

Example 4.
A bouncing ball with parametric uncertainties in the reset map can be described by $\mathbf{B}_p$ which is the tuple:

$$\mathbf{B}_p = (Q,E,D,F,G,R)$$

where

- $Q = \{q_0\}$, which provides the discrete state
- $E = \{(q_0,q_0)\}$, which is the single edge from q_0 to itself
- $D := \{x \in \mathbb{R}^2 : x_1 \geq 0\}$ provides the domain. Thus, $g_{q_0} = x_1$.
- $G = \{x \in \mathbb{R}^2 : x_1 = 0,\ x_2 \leq 0\}$ provides the guard. Thus, $h_{(q_0,q_0),0} = x_1$, and $h_{(q_0,q_0),1} = -x_2$.

- $R = \phi(x) = [0, -px_2]^T$ provides the reset map.
- $F = f(x)$ provides a vector field mapping D to itself, and where

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -g \end{pmatrix}$$

We would like to prove stability of the Zeno equilibrium for $p \in (0, C)$ where $C \in [0, 1)$. We define $P = \{p \in \mathbb{R} : \tilde{p}(p) := p(p - C) \leq 0\}$ to describe the set of possible values of the uncertainty p.

Results:
From previous analysis, we see that the system exhibits Zeno behavior when $C < 1$. The Zeno equilibrium z is (0,0). Thus we search for a parameter-dependent variables which establish this property. Specifically, we choose a maximum values of C search for a 4th degree $V(x)$ along with SOS and polynomial multipliers. As a result, we were able to verify stability for $C = 0.99$, which agrees with the known analytical result to a high degree of accuracy.

VI. Conclusions

In this paper, we present a Lyapunov based method for determining the stability of Zeno equilibria in hybrid dynamical systems. The method presented makes use of the sum-of-squares decomposition, thus enabling the construction of higher-order Lyapunov functions. As such, the theorem presented can be used to verify the existence of Zeno behavior in systems with nonlinear vector fields and reset maps. Examples illustrating the use of the proposed method are also provided, including an example of a system with nonlinear reset maps. Lastly, a method to determine the stability of Zeno equilibria in systems with time-invariant uncertainties is also given. A bouncing ball with an uncertain coefficient of restitution is provided as an illustrative example, and the Zeno stability of the bouncing ball is verified for an uncertain parameter $p \in (0, 1]$.

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