Spacecraft Dynamics and Control

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Lecture 17: Stability of Torque-Free Motion

Attitude Dynamics

In this Lecture we will cover:
Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation

Review: Euler Equations

$$\dot{\omega}_x = -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t)$$
$$\dot{\omega}_y = -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t)$$
$$\dot{\omega}_z = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t)$$

Axisymmetric Case: $I_x = I_y$

- $\dot{\omega}_z = 0$ - ω_z is fixed
- Allows us to linearize the equations
- Allows us to solve the equations explicitly

Non-axisymmetric Case $I_x \neq I_y$.

- We will have to rely on linearization

Linearization of the Euler Equations

Linearization allows us to consider small deviations about an equilibrium.

- We need to define the equilibrium

CASE: Stability of Spin about a principle axis.

- Nominal motion is

$$\omega_0(t) = \begin{bmatrix} \omega_{x,0}(t) \\ \omega_{y,0}(t) \\ \omega_{z,0}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n \end{bmatrix}$$

- This is an equilibrium because

$$\dot{\omega}_{x,0}(t) = -\frac{I_z - I_y}{I_x}\omega_{y,0}(t)\omega_{z,0}(t) = 0$$
$$\dot{\omega}_{y,0}(t) = -\frac{I_x - I_z}{I_y}\omega_{x,0}(t)\omega_{z,0}(t) = 0$$
$$\dot{\omega}_{z,0}(t) = -\frac{I_y - I_x}{I_z}\omega_{x,0}(t)\omega_{y,0}(t) = 0$$

Linearization of the Euler Equations

Now consider small disturbances to this equilibrium

$$\omega(t) = \omega_0 + \Delta\omega(t)$$

Then $\Delta\omega(t) = \omega(t) - \omega_0$ and

$$\begin{aligned}
\Delta\dot{\omega}(t) = \dot{\omega}(t) - 0 &= \begin{bmatrix} -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t) \\ -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t) \\ -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{I_z - I_y}{I_x}(\omega_{y,0} + \Delta\omega_y(t))(\omega_{z,0} + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y}(\omega_{x,0} + \Delta\omega_x(t))(\omega_{z,0} + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z}(\omega_{x,0} + \Delta\omega_x(t))(\omega_{y,0} + \Delta\omega_y(t)) \end{bmatrix} \\
&= \begin{bmatrix} -\frac{I_z - I_y}{I_x}\Delta\omega_y(t)(n + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y}\Delta\omega_x(t)(n + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z}\Delta\omega_x(t)\Delta\omega_y(t) \end{bmatrix}
\end{aligned}$$

Now because we have assumed that $\Delta\omega$ is small, products of the form $\Delta\omega_x \Delta\omega_y$ are very small indeed. Using this observation, we make the following **Approximations:**

$$\Delta\omega_x \Delta\omega_y = 0, \quad \Delta\omega_x \Delta\omega_z = 0, \quad \Delta\omega_z \Delta\omega_y = 0$$

This yields the following set of linearized equations:

$$\Delta\dot{\omega}(t) = \begin{bmatrix} -\frac{I_z - I_y}{I_x} \Delta\omega_y(t)(n + \Delta\omega_z(t)) \\ -\frac{I_x - I_z}{I_y} \Delta\omega_x(t)(n + \Delta\omega_z(t)) \\ -\frac{I_y - I_x}{I_z} \Delta\omega_x(t) \Delta\omega_y(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{I_z - I_y}{I_x} n \Delta\omega_y(t) \\ -\frac{I_x - I_z}{I_y} n \Delta\omega_x(t) \\ 0 \end{bmatrix}$$

Linearization of the Euler Equations

Thus the evolution of small disturbances is governed by a set of linear equations.

$$\begin{bmatrix}\Delta\dot{\omega}_x(t)\\ \Delta\dot{\omega}_y(t)\end{bmatrix} = \begin{bmatrix} 0 & -\frac{I_z-I_y}{I_x}n \\ -\frac{I_x-I_z}{I_y}n & 0\end{bmatrix}\begin{bmatrix}\Delta\omega_x(t)\\ \Delta\omega_y(t)\end{bmatrix}$$

$$\Delta\dot{\omega}_z(t) = 0$$

- The third equation $\Delta\dot{\omega}_z = 0$ implies $\Delta\omega_z = constant$.
- The first two equations combine to yield

$$\begin{aligned}\Delta\ddot{\omega}_x(t) &= -\frac{I_z-I_y}{I_x}n\Delta\dot{\omega}_y(t)\\ &= \frac{I_z-I_y}{I_x}\frac{I_x-I_z}{I_y}n^2\Delta\omega_x(t)\end{aligned}$$

If we take the Laplace transform of this equation, we get

$$s^2\Delta\hat{\omega}_x(s) = \frac{(I_z-I_y)(I_x-I_z)}{I_xI_y}n^2\Delta\hat{\omega}_x(s)$$

Stability Analysis

From

$$s^2 \Delta\hat{\omega}_x(s) = \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2 \Delta\hat{\omega}_x(s)$$

we get the transfer function

$$\hat{G}(s) = \frac{1}{s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}$$

or if you prefer, the characteristic equation

$$\lambda(s) = s^2 - \frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2$$

The roots of this characteristic equation are

$$s = \pm\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}$$

Review Stability Analysis

Consider a differential equation with characteristic equation, $\lambda(s)$:

Recall that the roots of the characteristic equation tell us about the behaviour of the variable $\Delta\omega_x$.

- The roots may be real, imaginary, or a mixture: $s = a + b\imath$

There are **Three Cases**:

1. **[Instability:]** If Real$(s) = a > 0$ for any root of $\lambda(s)$, then small disturbances will grow over time.
2. **[Stability:]** If Real$(s) = a < 0$ for all roots of $\lambda(s)$, then small disturbances will vanish over time.
3. **[Neutral Stability:]** If Real$(s) = a = 0$ for any root of $\lambda(s)$, then small disturbances will persist, but will not grow.

Stability of Torque-free Spin

Now recall the roots of $\lambda(s)$ for the torque-free spaceraft spinning about the $\hat{z}$ axis with angular velocity n.

- The roots of $\lambda(s)$ are

$$s = \pm\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}$$

We can break down our stability analysis into three cases:

CASE 1: Spin about the major axis ($I_z > I_x$ and $I_z > I_y$).

1. In this case $(I_z - I_y) > 0$ and $(I_x - I_z) < 0$.
2. Then the roots are purely imaginary
3. $s = \pm \imath b$ where $\sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y} n^2}$ is real.

Stability of Torque-free Spin

$$s = \pm\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2}$$

CASE 2: Spin about the minor axis ($I_z < I_x$ and $I_z < I_y$).

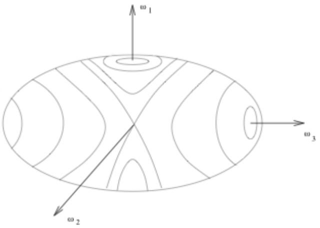
1. In this case $(I_z - I_y) < 0$ and $(I_x - I_z) > 0$.
2. The roots are also purely imaginary
3. $s = \pm \imath b$ where $\sqrt{\frac{(I_z - I_y)(I_z - I_x)}{I_x I_y} n^2}$ is real.

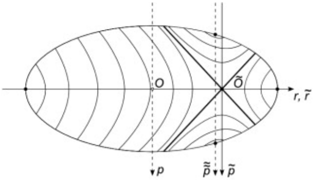
CASE 3: Spin about the intermediate axis ($I_y < I_z < I_x$ or $I_x < I_z < I_y$).

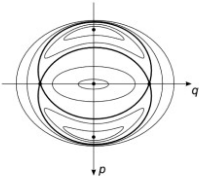
1. In this case $(I_z - I_y)(I_x - I_z) > 0$.
2. The roots are real
3. $s = \pm a$ where $\sqrt{\frac{(I_z - I_y)(I_x - I_z)}{I_x I_y} n^2} > 0$ is real.
4. One of the roots has positive real part - **UNSTABLE**

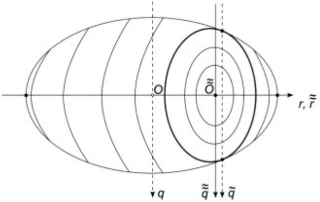
Thus spin about an intermediate axis is always unstable (small deviations will eventually get big!)

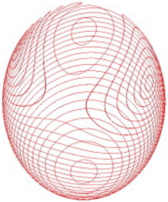
Polhodes











This effect be visualized using Polhodes.

- Positions of the axis of rotation, $\vec{\omega}$
- For fixed energy, lines are of constant angular momentum $\vec{h}$.

Instability of the intermediate axis

Figure: A Deck of Cards on the ISS (Garriott)

Figure: Simulated Ellipsoid

Figure: A Textbook on the ISS (Petit)

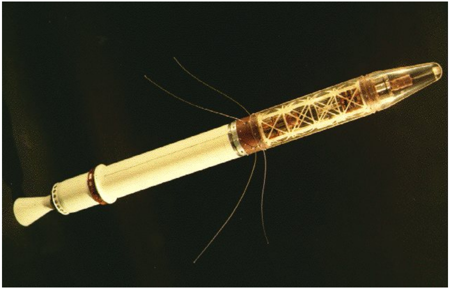
Destabilization caused by Energy Dissipation

Summary:

- Spin about intermediate axis - **Unstable**
- Spin about major or minor axis - **Neutral Stability**

What about Disturbances?

- Fuel Sloshing
- Flexible Structures
- Heat dissipation



Problem:

- Newton's Second Law predicts Conservation of **Momentum**
- It says nothing about **Kinetic Energy**!!!

Question: What is the effect of losses in Kinetic Energy?

Destabilization caused by Energy Dissipation

Question: How to relate energy drain $\dot{T} < 0$ to changes in $\vec{\omega}$?

Consider the expression for Kinetic Energy:

$$2T = \omega_x^2 I_x + \omega_y^2 I_y + \omega_z^2 I_z$$

Meanwhile, the total angular momentum is

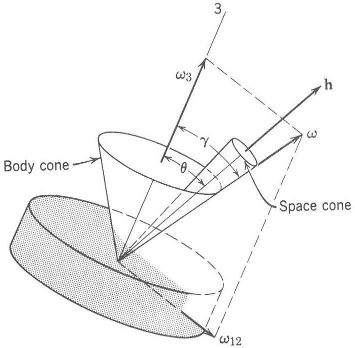
$$h^2 = I_x^2 \omega_x^2 + I_y^2 \omega_y^2 + I_z^2 \omega_z^2$$

Consider the Axisymmetric Case: $I_x = I_y$. Then

$$2T = I_x(\omega_x^2 + \omega_y^2) + \omega_z^2 I_z$$

$$h^2 = I_x^2(\omega_x^2 + \omega_y^2) + I_z^2 \omega_z^2$$

where we know $\omega_{xy}^2 = \omega_x^2 + \omega_y^2$ is fixed.



We substitute $\omega_x^2 + \omega_y^2 = \frac{h^2 - I_z^2 \omega_z^2}{I_x^2}$ into the expression for T to get

$$2T = \frac{h^2 - I_z^2 \omega_z^2}{I_x} + \omega_z^2 I_z = \frac{h^2}{I_x} + \omega_z^2 I_z \left(1 - \frac{I_z}{I_x}\right)$$

Destabilization caused by Energy Dissipation

Now consider the angle (θ) by which $\vec{h}$ differs from $\hat{z}$.

$$\cos\theta = \frac{h_z}{h} = \frac{I_z \omega_z}{h}$$

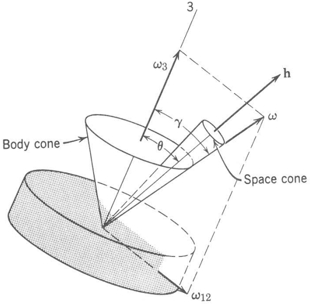
We would like to express T in terms of θ.

Noting that $\omega_z = \frac{h}{I_z}\cos\theta$, we find

$$2T = \frac{h^2}{I_x} + \frac{h^2}{I_z}\cos^2\theta(1 - \frac{I_z}{I_x})$$

Taking the time-derivative, we find

$$\begin{aligned}\dot{T} &= -\frac{h^2}{2I_z} 2\cos\theta\sin\theta\left(1 - \frac{I_z}{I_x}\right)\dot{\theta}\\ &= \frac{h^2}{2I_z} 2\cos\theta\sin\theta\left(\frac{I_z}{I_x} - 1\right)\dot{\theta}\end{aligned}$$



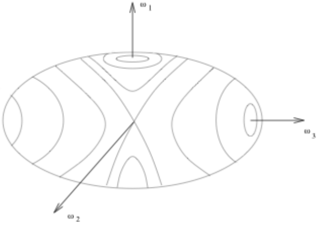
Destabilization caused by Energy Dissipation

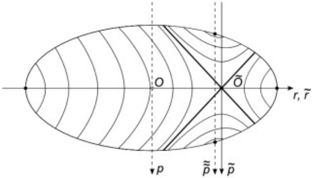
$$\dot{T} = \frac{h^2}{2I_z} 2\cos\theta \sin\theta \left(\frac{I_z}{I_x} - 1 \right) \dot{\theta}$$

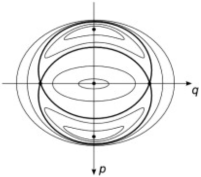
There are two cases

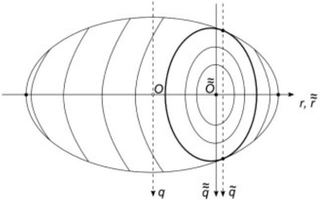
1. **CASE 1:** If $I_x > I_z$, then $\dot{T} < 0$ implies that $\dot{\theta} > 0$.
   - Spin Axis $\hat{z}$ is **UNSTABLE**
2. **CASE 2:** If $I_x < I_z$, then $\dot{T} < 0$ implies that $\dot{\theta} < 0$
   - Spin Axis $\hat{z}$ is **STABLE**

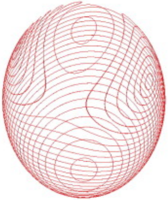
Polhodes











The effect of energy dissipation can also be visualized using Polhodes.

- Each line has constant $\frac{h^2}{T}$.
- Rotation proceeds from large T to small T.

Major Axis Rule

Theorem 1 (Major Axis Rule).

1. *Spin about the major axis is stable*
2. *Spin about any other axis is unstable*

Attitude Dynamics

In this Lecture we have covered:
Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation