Systems Analysis and Control

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Lecture 17: Compensator Design

Overview

In this Lecture, you will learn:

Lead-Lag Compensation

- Designing Leads
- Designing Lags
- Combining Leads and Lags

Notch Filters

- Providing extra zeros
- Eliminates annoying frequency components.

Recall: Pole-Zero Compensation

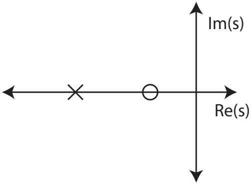
Definition 1.

A **Pole-Zero Compensator** is of the form

$$K(s) = \frac{s+z}{s+p}$$

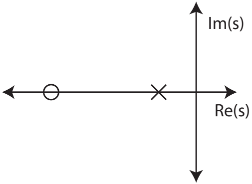
Lead Compensation

- $p < z$
- Replaces Pure Zero



Lag Compensation

- $z < p$
- Replaces Integrator



Lead Compensation

Example

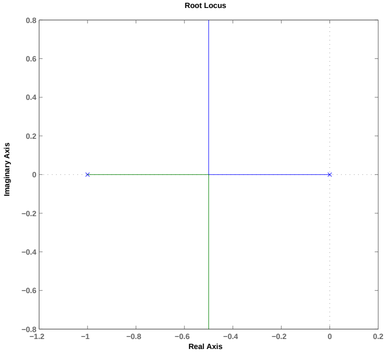
$$G(s) = \frac{1}{s(s+1)}$$

Asymptotes: $\pm 90°$
Intercept: $\alpha = -.5$

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} = \frac{-1-0}{2-0} = -.5$$

Break Point: $s = -.5$

$$n'd + d'n = 2s + 1 = 0$$



Conclusion: At high gain, we get

- High Frequency Oscillation
- Lots of overshoot
- Fixed Settling Time

$$G_c(s) = \frac{s+4}{s+5}$$

Asymptotes: $\pm 90°$
Intercept: $\alpha = -1$

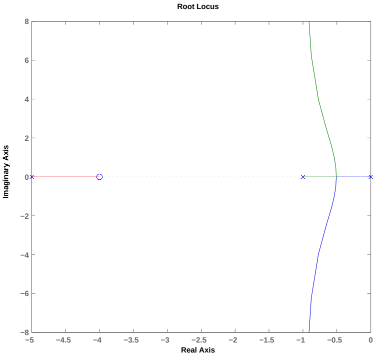
$$\alpha = \frac{\sum p_i - \sum z_i}{n-m}$$
$$= \frac{-5+4}{2-0} + \frac{-1-0}{2-0} = -.5 - .5 = -1$$

Break Point: $s = -.438$

$$n'd + d'n$$
$$= (s+5)(s^2+s) + (3s^2+12s+5)(s+4)$$

Conclusion: At high gain, we get

- Improved Settling time
- Slightly less overshoot



Lead Compensation

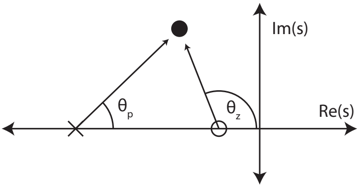
Phase Interpretation

The effect of a **Lead Compensator**

- Add Phase at every point

$$\angle K(s)G(s) = \angle K(s) + \angle G(s)$$

Points compensate by moving left.



Lead Compensation
Pole Placement

Lead-Lag can be used to do pole-placement

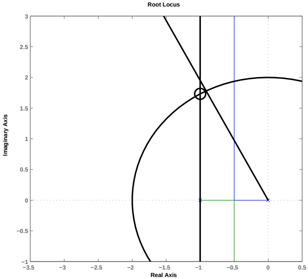
$$G(s) = \frac{1}{s(s+1)}$$

Now suppose we want:

- 20% Overshoot
- $\omega_n = 2$
- $T_s < 4$

We choose a desired point on the root locus:

- The intersection of
  - $\omega_n = 2$
  - $\sigma < -1$



$$s_{1,2} = -1 \pm \sqrt{3}\imath$$

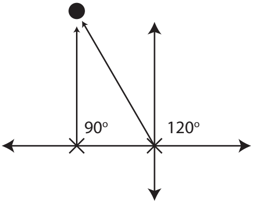
Question: Can we achieve this point exactly using Pole-Zero compensation?

Lets start with a basic question:

- Is s already on the root locus?

Lets check:

$$\angle G(s) = \sum \angle(s - z_i) - \sum \angle(s - p_i)$$



Working out the geometry:

$$\angle G(s) = -90^\circ - 120^\circ = -210^\circ$$

Not on the Root Locus!

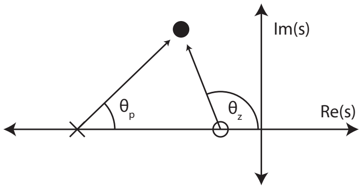
The point s lacks 30° of phase.

To place the point s on the root locus:

- we need to add 60° of phase at this point.

Phase is sum of zeros minus poles

- Zeros add phase
- Poles subtract phase.



We can add 30° if we use a pole-zero combo:

- Add a zero at 60°
- Add a pole at 30°

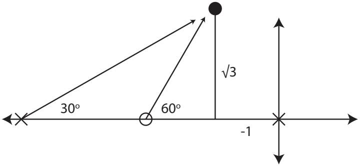
$$K(s) = \frac{s - z}{s - p}$$

Use reverse geometry to find p and z.

Zero:

$$\tan 60^\circ = \frac{\sqrt{3}}{x}$$

$$x = \frac{\sqrt{3}}{\tan 60^\circ} = 1$$



Pole:

$$\tan 30^\circ = \frac{\sqrt{3}}{x}$$

$$x = \frac{\sqrt{3}}{\tan 30^\circ} = 3$$

$$p = -1 - x = -4$$

$$z = -1 - x = -2$$

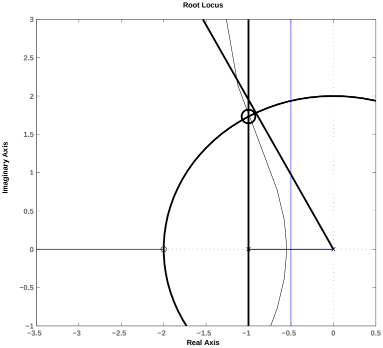
Now, the root locus passes through s.

To find the gain at this point

- Use `rlocfind`
- Use $k = \frac{|d(s)|}{|n(s)|}$.

For this example,

$$k = 6.00$$



Potential Problem: May adversely affect other poles.

Root Locus Demo 2

Pole Placement

Wiley+ Root Locus Demo 2

Make the phase 180°.

The other big use of root locus is to change **Departure Angles**.
Recall the Suspension system problem with integral feedback:

$$G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}\frac{1}{s}$$

The poles are

- $p_{1,2} = -.957 \pm 1.23\imath$
- $p_{3,4} = -.0433 \pm .641\imath$

At pole $p_{3,4} = -.0433 + .641\imath$, the phase is $-156°$.

Departure Angle:

$$\angle_{dep} = \angle G(s) + 180° = 24°$$

Goal: Increase the departure angle to $90°$ or more.

Suppose we want a departure angle of $\angle_{dep} = 100^\circ$.

- Recall
$$\angle_{dep} = \angle G(s) + 180^\circ$$
- Required Phase $\angle G(s)$
$$\angle_{req} G(s) = \angle_{dep} - 180^\circ = -80^\circ$$
- Required Phase Change:
$$\Delta \angle G(s) = \angle_{req} G(s) - \angle G(s) = -80 + 156^\circ = 76^\circ$$

Lead Compensation

Departure Angles

We need to add $76°$.

- Zero at $90°$.
- Pole at $14°$.

Recall departure point is $p_3 = -.0433 + .641\imath$

Zero:

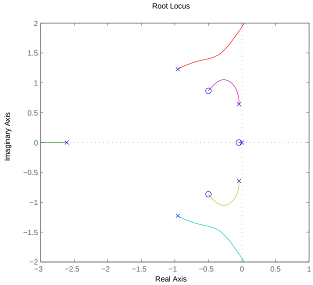
- $\theta = 90°$,
- $\Delta x = 0$
- $z = -.0433$

Pole:

$$\tan 14° = \frac{.641}{\Delta x}$$

$$\Delta x = \frac{.641}{\tan 14°} = 2.57$$

So $p = -.0433 - \Delta x = -2.61$.



Controller:

$$K(s) = \frac{s + .0433}{s + 2.61}$$

Predict the Steady-State Error.

$$G(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+1s+1} \qquad\qquad K(s) = \frac{s-z}{s-p} = \frac{n_K(s)}{d_K(s)}$$

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} \frac{1}{1+G(s)K(s)} \\ &= \frac{d_K(0)}{d_K(0)+kn_K(0)G(0)} \\ &= \frac{-p}{-p+-zkG(0)} \end{aligned}$$

If

- p is small
- z is large

Then

$$e_{ss} \cong \frac{p}{kz}\frac{1}{G(0)}$$

$$G(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+1s+1}$$

Say we want steady-state error less than $.01$.

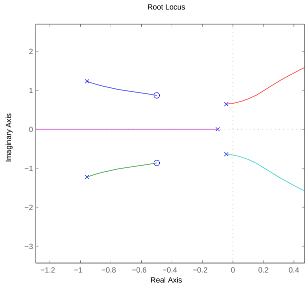
$$e_{ss} \cong \frac{p}{kz}\frac{1}{G(0)} = \frac{p}{kz} =\leq .01$$

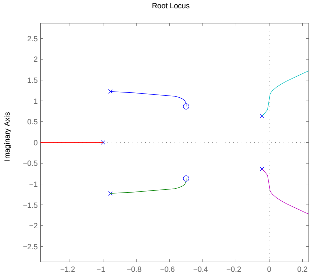
or $p \leq .01kz$

- Assume $k > 10$
- Choose $p = .1$
- Result: $z = 100$

Alternatively, $p = 1,\ z = 1000$.

- But there are dangers!





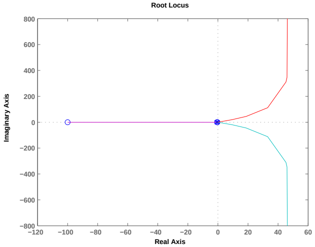
Lag Compensation

Notice some negative effects of Lag

- Asymptotes still at $\pm 90°$

Center of Asymptotes:

$$\begin{aligned}\alpha &= \frac{\sum p_i - \sum z_i}{n-m} \\ &= \frac{\sum p_{i,old} - \sum z_{i,old}}{n-m} + \frac{\sum p_{i,new} - \sum z_{i,new}}{n-m} \\ &= \alpha_{old} + \frac{p-z}{2}\end{aligned}$$



Creates a Shift in Asymptotes by

$$\Delta\alpha \cong \frac{z}{2} = -50$$

for the suspension problem.

Mitigate the effect of lag compensation

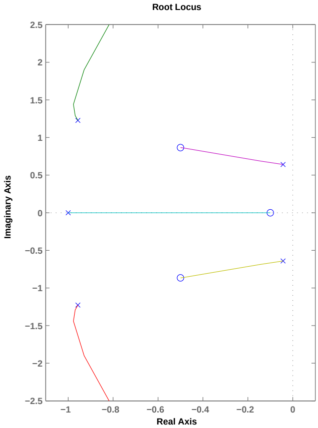
- Add some Lead Compensation
  - Zero at $z = .01$
  - Pole at $p = 20$

Phase at s_1

$$\angle G(s) = -25.8^\circ$$

Departure Angle:

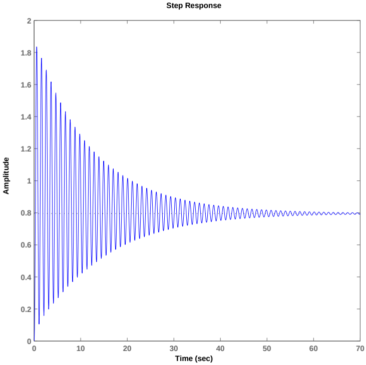
$\angle_{dep} = 180 + \angle G(s) = 154.25^\circ$

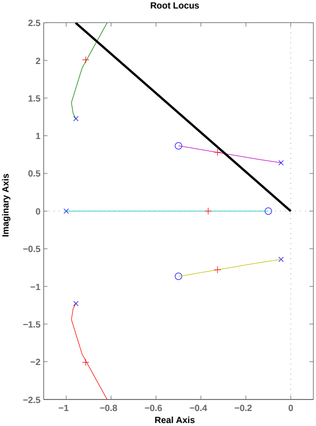


Use `rlocfind` to pick off

- Maximum stable gain

$$k = .7768$$



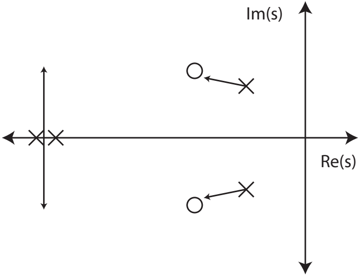


Conclusion:

- Lead-Lag Improves Performance
- Can't do everything.

Problem Can't Stabilize those poles at

$$s = -.0433 \pm .641\imath$$



One solution is to use a *Notch Filter*.

Definition 2.

A **Notch Filter** consists of

- Two Complex Zeros
  - Used to Capture Troublesome Poles
- Two Real Poles far out in the LHP

Lag Compensation

Notch-Filter Example

To attack the poles at

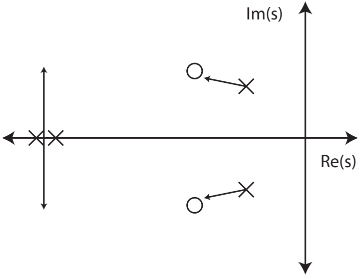
$$s = -.0433 \pm .641\imath$$

Lets use a notch filter at

$$z_{1,2} = -.5 \pm .641\imath$$

Poles at

$$p_{1,2} = -20$$



$$K(s) = \frac{(s + .5 + .641\imath)(s + .5 - .641\imath)}{(s + 20)^2}$$
$$= \frac{s^2 + s + .66}{s^2 + 40s + 400}$$

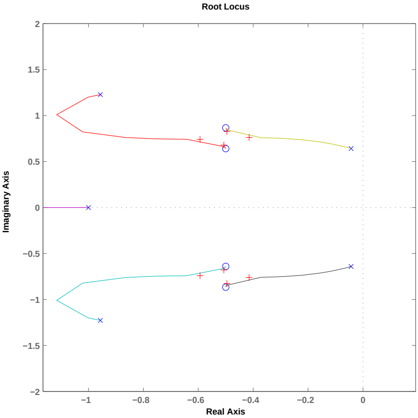
Lag Compensation

Notch-Filter Example

Combining the Notch-Filter with the Lag filter.

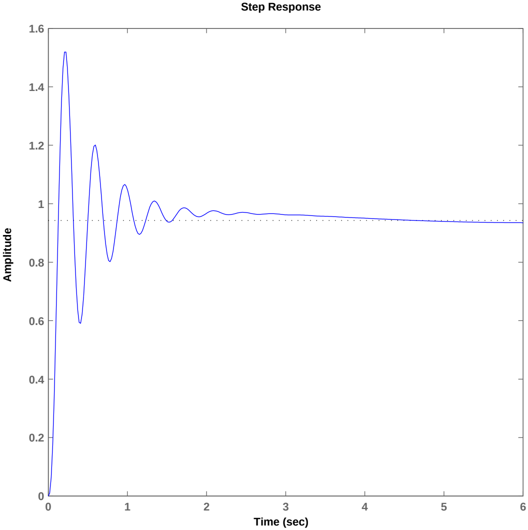
- Using `rlocfind`, we pick off the point

$$k = 10$$



Lag Compensation

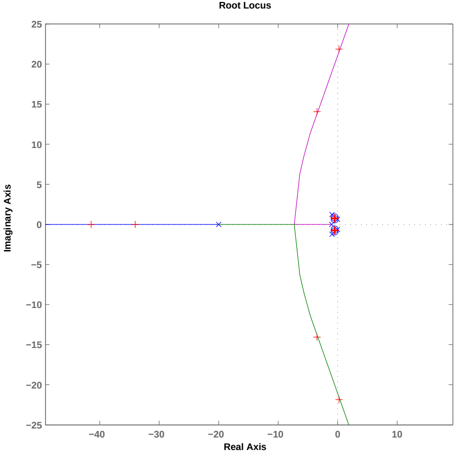
Notch-Filter Example



Lag Compensation

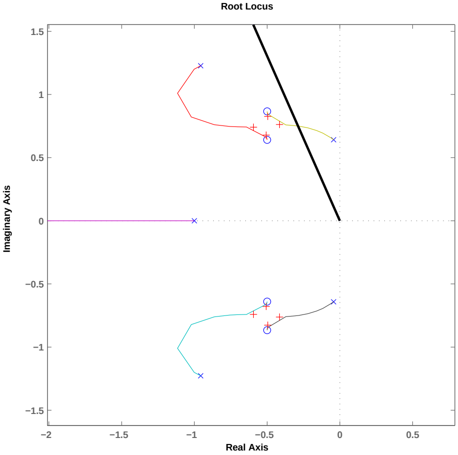
Notch-Filter Example

Don't Forget about the other poles!



$$k_{max} = 19.4$$

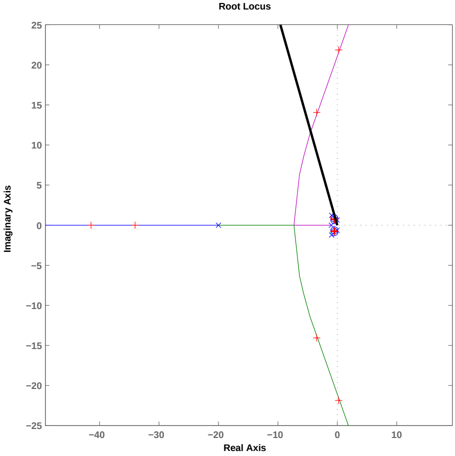
What about 30% overhoot?



Lag Compensation

Notch-Filter Example

What about 30% overhoot?



Don't forget those other poles!

Summary

What have we learned today?

Lead-Lag Compensation

- Designing Leads
- Designing Lags
- Combining Leads and Lags

Notch Filters

- Providing extra zeros
- Eliminates annoying frequency components.

Next Lecture: The Frequency Domain