SOS for Nonlinear Delayed Models in Biology and Networking

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Summary. In this chapter we illustrate how the Sum of Squares approach can be used for understanding the stability properties of models of networks of biological and communication systems. The models we consider are in the form of nonlinear delay differential equations with multiple, incommensurate delays . Using the sum of squares approach, appropriate Lyapunov-Krasovskii functionals can be constructed, both for testing delay-independent and delay-dependent stability. We illustrate the methodology using examples from congestion control for the Internet and gene regulatory networks.

1 Introduction

Delay differential equations can be used to model the behaviour of dynamical systems in which aftereffect is important. Examples include communication and computer networks (e.g., Internet congestion control), population dynamics of species interaction (e.g., predator-prey models with maturation), biological systems (e.g., gene regulatory networks with auto-regulation) etc. In all these examples, the delays inherent in the systems are important and cannot be ignored in modelling or analysis as this can lead to incorrect or misleading conclusions about the system under study.

A major problem in trying to understand the properties of communication and biological networks is that many times, multiple, incommensurate time-delays are required to accurately represent them. For example, when routers attempt to control Internet congestion by feeding back price signals to the users, there is an intrinsic propagation delay whose size is a function of the distance of the router from the user. Moreover, because users typically use many links in order to transmit their data, the feedback signals will be outdated by multiple incommensurate delays. As another example, consider predator-prey interactions - in this case, multiple time-delays would be needed to adequately model the system, due to the difference in the maturation rates of different species.

Nonlinear delay differential equations with multiple time delays are, however, very difficult to analyze. Even in the case of *linear* system descriptions, it is known that for multiple time-delays lying in the interval $\tau_i \in [0, \overline{\tau_i}]$ the question of stability is NP-hard [1]. Many results that have been developed in the past concentrate on linear systems with multiple time-delays and the use of frequency domain tools for performing the stability test [2, 3, 4, 5]. Other results use Lyapunov arguments [6, 7, 8, 9], but these tests are in general more conservative.

For the case of nonlinear time-delay systems, very few approaches have been proposed for algorithmic analysis of systems with multiple, incommensurate time-delays, despite the fact that such models appear naturally in many practical systems. In this chapter, we will show how the sum of squares decomposition of multivariable polynomials can be used to construct appropriate Lyapunov-Krasovskii functionals for polynomial time-delay systems in order to verify stability. This chapter concentrates more on the applications than the theory, for which the reader is referred to [10, 11, 12, 13].

We first present two examples where multiple time-delays appear naturally – communication networks and biology – and in Section 3 briefly outline the tools that can be used to answer the stability question. We return to these examples in Section 4 before concluding the chapter.

1.1 Notation

The notation we will be using is standard. $\mathbb{R}$ denotes the reals and $\mathbb{R}^n$ the $n-$dimensional Euclidean space. We denote by $\mathbb{R}[x]$ the ring of real polynomials in variables x and by $Z_d[x]$ we denote the vector of monomials in x of degree d or less. We consider delays $\tau_i \in \mathbb{R}_+$, $i = 0, 1, \ldots, K$ with $0 = \tau_0 < \tau_1 < \ldots < \tau_K = \tau$ and define $H = \{-\tau_0, \ldots, -\tau_k\}$ and $H^c = [-\tau, 0] \backslash H$. We will also denote the intervals $I_i = [-\tau_i, -\tau_{i-1}]$. Further, denote $C([-\tau, 0], \mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the norm on C is defined as $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$. Suppose $\sigma \in \mathbb{R}$, $\rho \geq 0$ and $x \in C([\sigma - \tau, \sigma + \rho], \mathbb{R}^n)$; then for any $t \in [\sigma, \sigma + \rho]$, define $x_t \in C$ by $x_t(\theta) = x(t + \theta), \theta \in [-\tau, 0]$. We will also be using symbolic independent variables to reference state and delayed state variables. To facilitate understanding: $\hat{x}_k^{(i)}$ will reference $x_i(t - \tau_k)$ where $l = 1, \ldots, K$. We will use $\hat{x}_k$ to denote the vector whose elements are $\hat{x}_k^{(i)}$, $i = 1, \ldots, n$. In a similar fashion, $\hat{y}_k^{(i)}$ will be used for $x_i(t + \theta - \tau_k)$ an $\hat{z}_k^{(i)}$ for $x_i(t + \xi - \tau_k)$.

2 An Introduction to the Network Models

2.1 Network Congestion Control for the Internet

Internet congestion control [14] is a distributed algorithm to allocate available bandwidth to competing sources so as to avoid congestion collapse by ensuring that link capacities are not exceeded. The need for congestion control for the Internet emerged in the mid-1980s, when congestion collapse resulted in unreliable file transfer. In 1988 Jacobson [15] proposed an admittedly ingenious scheme for congestion control. The shortcomings of this scheme and its successors such as TCP Reno and Vegas have only recently become apparent: they are not scalable to arbitrary networks with very large capacities and multiple, non-commensurate time-delays. New designs of Active Queue Management (AQM) and/or Transmission Control Protocol (TCP) schemes have been proposed that provide scalable stability in the presence of heterogeneous delays, which can be verified at least for the linearization about a steady-state.

The simplest adequate modeling framework for network congestion control is in the form of nonlinear deterministic delay-differential equations [16, 17, 18]. Some work has been done on the analysis of such systems usually for the single-bottleneck link case, using either Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals, IQC methods, passivity etc.

Let us consider a simple network instantiation of what is known a 'primal' congestion control scheme, shown in Figure 1. It consists of $L = 3$ links, labeled $l = 1, 2, 3$ and $S = 2$ sources, $i = 1, 2$. For this network, we define an

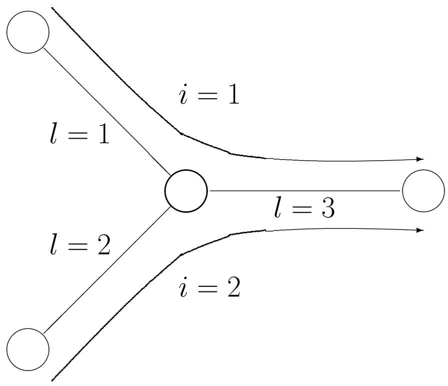

Fig. 1. A network topology under consideration.

$L \times S$ routing matrix R which is 1 if source i uses link l and 0 otherwise:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

The architecture of network congestion control is shown in Figure 2. To each source i we associate a transmission rate x_i. All sources whose flow passes through resource l contribute to the *aggregate rate* y_l for resource l, the rates being added with some forward time delay $\tau^f_{i,l}$. Hence we have:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}) \tag{1}$$

The resources l react to the aggregate rate y_l by setting congestion information p_l, the price at resource l. This is the Active Queue Management part of the picture. The prices of all the links that source i uses are aggregated to form q_i, the *aggregate price* for source i, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}) \tag{2}$$

The prices q_i can then be used to set the rate of source i, x_i, which completes the loop. The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l}, \quad \forall\, l \tag{3}$$

The capacity of link l is given by c_l. For a general network, the interconnection is shown in Figure 2. In this section we choose the control laws for TCP and AQM as follows:

$$\begin{aligned} p_l(t) &= \left(\frac{y_l(t)}{c_l}\right)^B \\ \dot{x}_i(t) &= 1 - x_i(t - \tau_i) q_i(t) \end{aligned}$$

Here $p_l(t)$ corresponds to the probability that the queue length exceeds B in a $M/M/1$ queue with arrival rate $y_l(t)$ and capacity c_l, and the source law corresponds to a queue length with proportionally fair source dynamics.

In the particular case of the network shown in Figure 1, the closed loop dynamics become:

$$\begin{aligned} \dot{x}_1 &= 1 - x_1(t-\tau_1)\left[\left(\frac{x_1(t-\tau_1)}{c_1}\right)^B + \left(\frac{x_1(t-\tau_1) + x_2(t - \tau^b_{1,3} - \tau^f_{2,3})}{c_3}\right)^B\right] \\ \dot{x}_2 &= 1 - x_2(t-\tau_2)\left[\left(\frac{x_2(t-\tau_2)}{c_2}\right)^B + \left(\frac{x_2(t-\tau_2) + x_1(t - \tau^f_{1,3} - \tau^b_{2,3})}{c_3}\right)^B\right] \end{aligned}$$

We will show in the examples section how a model like this can be analyzed.

2.2 Autoregulation in gene regulatory networks

Cells live in complex environments and need to respond to a range of different signals and parameter changes, such as temperature, pressure, the presence of

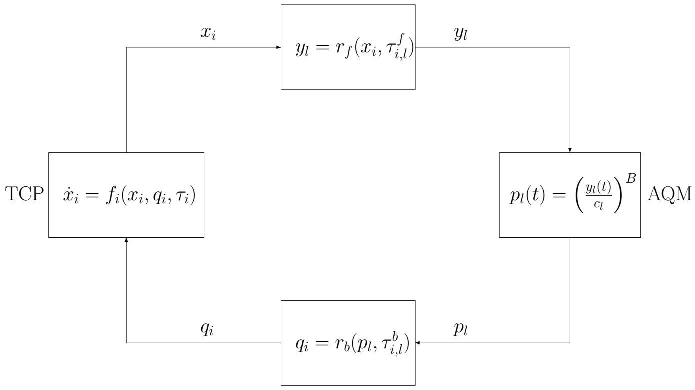

Fig. 2. The internet as an interconnection of sources and links through delays.

nutrients etc. At the same time, the cell needs to respond to several internal signals. The way cells react to such changes is by producing appropriate proteins that act upon the internal and external environments, countering their effect.

Upon an environmental change, special proteins, called *transcription factors* become activated and bind to DNA to regulate the rate at which target genes are read. This is a two-step process: first, genes are read (transcribed) into mRNA, which is then translated into protein. This means, of course, that there are two time-delays from the time RNA polymerase and transcription factors bind to DNA and the time the protein is produced: translation cannot happen before transcription has been completed.

Another important aspect is that the resulting proteins can act on the environment but may also be transcription factors for other genes, in a feedback fashion. In particular, transcription factors can act as *activators*, in which case they increase the transcription rate of a gene, or *repressors*, in which case they reduce it. The strength of the effect of a transcription factor on the transcription rate of its target gene is described by an *input function*:

- In the case of activation, denoted by $X \to Y$, i.e., where the production of protein Y is controlled by a transcription factor X that acts as an activator, the number of molecules of Y is a function of the concentration of X in its active form, X^*. This is a monotonically increasing, S-shaped function, which is usually captured using a so-called Hill function:

$$f(x^*) = \frac{\beta_X \, (x^*)^n}{K^n + (x^*)^n}$$

  where K is the activation coefficient, n is the Hill coefficient and β is the maximal expression level.
- In the case of repression, denoted by $X \dashv Y$, the Hill function is a decreasing S-shaped function of the form

$$f(x^*) = \frac{\beta_X K^n}{K^n + (x^*)^n}$$

The latter means that the production of protein Y is controlled by X which acts as an inhibitor.

The simplest gene regulatory network we can consider is a negative auto-regulation motif, in which the gene product acts as a transcription factor which represses its own production. We can model this as follows:

$$\begin{aligned}\dot{x}_1 &= -\lambda_1 x_1 + c_1 x_2(t-\tau_1)\\ \dot{x}_2 &= -\lambda_2 x_2 + \frac{c_2}{1+(x_1(t-\tau_2))^n}\end{aligned}$$

where x_1 denotes the concentration of the protein while x_2 denotes the concentration of the mRNA produced by transcription of the gene encoding the protein. In the above model, λ_1 and λ_2 are natural degradation/dilution rates, while c_1 and c_2 are translation and transcription rates respectively. Also, τ_1 is the time required for the mRNA to be produced and transported to the ribosomes in order to initiate the transcription process, while τ_2 is the time-delay required before the production of the protein can actually regulate its own production. The parameter n is a Hill coefficient. We will analyze this model in Section 4.

In the next section, we discuss briefly the tools that we will develop for analyzing nonlinear time-delay systems with multiple, incommensurate time-delays.

3 Methodology

Consider a time-delay system with multiple delays of the form

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K)) \tag{4}$$

where $f : \mathbb{R}^{n\times(K+1)} \to \mathbb{R}^n$ with $f(0,\ldots,0) = 0$ is such that a unique solution to the above delay-differential equation exists from an appropriate initial condition close to 0. In the sequel, we assume that the vector field $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_k]$. One type of Lyapunov functional that can be used to verify the delay-dependent (DD) stability of the zero steady-state takes the form

$$V(\phi) = \int_{-\tau}^{0} v_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} v_2(\phi(\xi), \phi(\theta), \theta, \xi)\, d\theta\, d\xi. \tag{5}$$

This functional has a derivative of the general form

$$\begin{aligned}\dot{V}(\phi) = &\int_{-\tau}^{0} v_3(\phi(0), \phi(-\tau_1), \ldots, \phi(-\tau_K), \phi(\theta), \theta) d\theta\\ &- \int_{-\tau}^{0}\int_{-\tau}^{0} \left(\frac{\partial v_2}{\partial \theta} + \frac{\partial v_2}{\partial \xi}\right) d\theta\, d\xi,\end{aligned} \tag{6}$$

where the map v_1, v_2 to v_3 will be presented in later sections.

In this chapter we will consider Lyapunov-Krasovskii functionals for studying the delay-independent and delay-dependent stability of the zero equilibrium of (4). This approach will transform these notions of stability into polynomial non-negativity conditions which are in general difficult to test. A sufficient condition for polynomial non-negativity which is worst-case polynomial-time verifiable is the existence of a sum of squares (SOS) decomposition. More details on positive polynomials and the SOS decomposition can be found in [19, 20, 21, 22]. Software, SOSTOOLS to support the theory is also available [23].

Denote by Σ the SOS cone and by Σ_d the subset of Σ of polynomials of degree d of less. First note that if $a \in \mathbb{R}[x]$ is an SOS, then it is globally nonnegative. In order to ensure that it is positive *definite* and radially unbounded we can use a polynomial 'shaping' function $\varphi(x)$. For example, given a polynomial $a(x)$ of degree $2d$, if we let

$$\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^{d} \epsilon_{ij} \geq \gamma \quad \forall\ i = 1, \ldots, n, \tag{7}$$

with γ a positive number, and $\epsilon_{ij} \geq 0$ for all i and j, then the condition $a(x) - \varphi(x) \in \Sigma$ guarantees the positive definiteness of $a(x)$, i.e., $a(x) > 0, x \neq 0$. Moreover, $a(x)$ is radially unbounded.

Another issue is related to testing non-negativity of a polynomial over a bounded domain instead of globally. Conditional satisfiability conditions can be tested using a generalization to the S-procedure, which is based on Putinar's representation [24] in Real Algebraic Geometry.

Given $p \in \mathbb{R}[x]$, suppose we want to ensure that $p(x) > 0$ on the set $D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, N_1\}$. Then one can search for Lagrange-type multipliers $\lambda_i \in \Sigma_k$ so that $p(x) + \sum_{i=1}^{N_1} \lambda_i(x) g_i(x) \in \Sigma$. Searching for $\lambda_i(x)$ of a fixed degree k so that the above expression is SOS is a semidefinite programme. Note that if p and g_i are quadratic forms and λ_i are constants, the above test is indeed the S-procedure, which can fail if $i \geq 2$. However, it has been shown in [24] that if D is compact and another mild condition holds on the $g_i(x)$ (the highest degree homogeneous parts of the g_i's have no common zeros in $\mathbb{R}^n$ except at 0), then there is a k for which the above test will succeed – it is indeed a necessary and sufficient condition. Other tests can also be formulated, which may have different properties. See e.g. [25].

3.1 Independent of Delay stability, Multiple-Delay case

The equilibrium of a time-delay system is said to be Independent Of Delay (IOD) stable if it is stable for all positive values of the time-delays. For simplicity, here we only develop the case in which there is a single steady-state and we seek to show global stability. Simple sufficient IOD stability conditions for global stability can be found in the following proposition:

Proposition 1. *Consider the system given by (4) with* 0 *a steady state. Suppose there exist functions* $v_0, v_{1i} : \mathbb{R}^n \to \mathbb{R}$ *for* $i = 1, \ldots, K$*, a positive definite radially unbounded function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *and a non-negative function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *such that for all* $\hat{x}_0, \ldots, \hat{x}_K, \hat{y}_0 \in \mathbb{R}^n$*:*

1) $v_0(\hat{x}_0) - \varphi(\hat{x}_0) \geq 0$,
2) $v_{1i}(\hat{y}_0) \geq 0$ *for all* $i = 1, \ldots, K$,
3) $\nabla_{\hat{x}_0} v_0(\hat{x}_0)^T f(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K) + \sum_{i=1}^K (v_{1i}(\hat{x}_0) - v_{1i}(\hat{x}_i)) + \psi(\hat{x}_0) \leq 0.$

Then the steady-state is globally stable. If $\psi(\hat{x}_0) > 0$*, then the steady-state is globally asymptotically stable.*

Proof. Consider the following Lyapunov-Krasovskii functional:

$$V(\phi) = v_0(\phi(0)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} v_{1i}(\phi(\theta)) d\theta.() \tag{8}$$

The first two conditions guarantee that this functional is positive definite and radially unbounded, by construction of φ. Differentiating $V(\phi)$ along the system trajectories we get:

$$\dot{V} = \nabla_{\hat{x}_0} v_0(\hat{x}_0)^T f(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K) + \sum_{i=1}^{K} (v_{1i}(\hat{x}_0) - v_{1i}(\hat{x}_i))$$

which is non-positive by condition 3) in the above proposition. Therefore, the steady-state is globally stable using the Lyapunov Krasovskii Theorem [7]. Moreover, if $\psi(\hat{x}_0) > 0$ then we have $\dot{V} < 0$ and therefore the steady-state is globally asymptotically stable.

The cases of local stability, robustness analysis etc., can be dealt with in a unified manner. See [10] for more details.

Delay-Dependent, Multiple-Delay Case

The Lyapunov functions that one needs to use in order to show delay-dependent stability in the presence of multiple, incommensurate delays have piecewise continuous kernels, and take the form (5). Sufficient conditions for DD stability can then be written as follows:

Proposition 2. *Consider the system given by (4) with* 0 *a steady state and* f *a polynomial vector field. Given a vector of monomials* $Z_d[\hat{y}_0, \theta]$ *consider* $Q \succeq 0$ *and let* Q_{ij} *be the i,jth block of* Q *and*

$$v_{2ij}(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d^T[\hat{y}_0, \theta] Q_{ij} Z_d[\hat{z}_0, \xi], \ \text{for } i = 1, \ldots, K, \quad j = 1, \ldots, K.$$

Furthermore, suppose there exist polynomials $v_{1i}(\hat{x}_0, \hat{y}_0, \theta)$*,* $r_{1i}(\hat{x}_0, \theta)$ *and* $r_{2i}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K, \theta)$ *for* $i = 1, \ldots, K$*, a positive definite radially unbounded polynomial function* $\varphi(\hat{x}_0) > 0$ *and a non-negative polynomial function* $\psi(\hat{x}_0)$ *such that for all* $\hat{y}_0, \hat{x}_0, \ldots, \hat{x}_K \in \mathbb{R}^n$*:*

1) $v_{1i}(\hat{x}_0, \hat{y}_0, \theta) + r_{1i}(\hat{x}_0, \theta) - \varphi(\hat{x}_0) \geq 0$, *for all* $\theta \in [-\tau_i, -\tau_{i-1}]$ *for each* $i = 1, \ldots, K$.

2) $$\nabla_{\hat{x}_0} v_{1i}(\hat{x}_0, \hat{y}_0, \theta)^T f(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K) - \frac{\partial v_{1i}}{\partial \theta}(\hat{x}_0, \hat{y}_0, \theta) + \frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\hat{x}_0, \hat{x}_{i-1}, -\tau_{i-1})$$
$$- \frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\hat{x}_0, \hat{x}_i, -\tau_i) + 2 \sum_{j=1}^{K} v_{2ij}(\hat{x}_{j-1}, \hat{y}_0, -\tau_{j-1}, \theta)$$
$$-2 \sum_{j=1}^{K} v_{2ij}(\hat{x}_j, \hat{y}_0, -\tau_j, \theta) + r_{2i}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K, \theta) + \psi(\hat{x}_0) \leq 0$$
for all $\theta \in [-\tau_i, -\tau_{i-1}]$ *and for each* $i = 1, \ldots, K$,

3) $$\sum_{i=1}^{K} \int_{I_i} r_{1i}(\hat{x}_0, \theta) d\theta = 0,$$

4) $$\sum_{i=1}^{K} \int_{I_i} r_{2i}(\hat{x}_0, \hat{x}_1, \ldots, \hat{x}_K, \theta) d\theta = 0.$$

5) There exist $R \succeq 0$ *and let* R_{ij}*, the i,jth block of* R *satisfy*

$$\frac{\partial v_{2ij}}{\partial \theta}(\hat{y}_0, \hat{z}_0, \theta, \xi) + \frac{\partial v_{2ij}}{\partial \xi}(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d[\hat{y}_0, \theta]^T R_{ij} Z_d[\hat{z}_0, \xi], \quad i, j = 1, \ldots, K.$$

Then the steady-state is globally stable. If $\psi(\hat{x}_0) > 0$*, then the steady-state is globally asymptotically stable.*

Proof. The proof of this proposition is based on ensuring that the following functional is a Lyapunov-Krasovksii functional:

$$V(\phi) = \sum_{i=1}^{K} \int_{I_i} v_{1i}(\phi(0), \phi(\theta), \theta) d\theta + \sum_{i=1}^{K} \sum_{j=1}^{K} \int_{I_i} \int_{I_j} v_{2ij}(\phi(\theta), \phi(\xi), \theta, \xi) d\theta d\xi. \tag{9}$$

Conditions (1) and (3) in the proposition ensure that $V(\phi) > 0$, so the first Lyapunov condition is satisfied and moreover V is radially unbounded.

The derivative of this functional along the trajectories of (4) is:

$$\dot{V}(\phi) = \sum_{i=1}^{K} \int_{I_i} \begin{pmatrix} \nabla_{\phi(0)} v_{1i}(\phi(0), \phi(\theta), \theta)^T f(\phi(0), \phi(-\tau_1), \ldots, \phi(-\tau_K)) \\ -\frac{\partial v_{1i}}{\partial \theta}(\phi(0), \phi(\theta), \theta) + \frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\phi(0), \phi(-\tau_{i-1}), -\tau_{i-1}) \\ -\frac{1}{\tau_i - \tau_{i-1}} v_{1i}(\phi(0), \phi(-\tau_i), -\tau_i) \\ +2 \sum_{j=1}^{K} v_{2ij}(\phi(-\tau_{j-1}), \phi(\theta), -\tau_{j-1}, \theta) \\ -2 \sum_{j=1}^{K} v_{2ij}(\phi(-\tau_j), \phi(\xi), -\tau_j, \theta) \end{pmatrix} d\theta$$
$$- \sum_{i=1}^{K} \sum_{j=i}^{K} \int_{I_i} \int_{I_j} \left(\frac{\partial v_{2ij}}{\partial \theta}(\phi(\theta), \phi(\xi), \theta, \xi) + \frac{\partial v_{2ij}}{\partial \xi}(\phi(\theta), \phi(\xi), \theta, \xi) \right) d\theta d\xi$$

The non-positivity of this time derivative is ensured by condition (2) when condition (4) is taken into account, as well as the decomposition given by (5). Therefore $V(\phi)$ is a Lyapunov-Krasovskii functional that proves global stability of the steady-state. If $\psi(\phi) > 0$, then $\dot{V}(\phi) < 0$ and the steady-state is globally asymptotically stable.

It is sometimes beneficial to consider other criteria for the stability analysis, such as combining all the above derivative conditions into one in more variables. For example, we can consider other Lyapunov functions, e.g., of the form:

$$\begin{aligned} V(x_t) &= V_0(x(t)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} V_{1i}(\theta_i, x(t), x(t+\theta_i))d\theta_i \\ &+ \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta_i}^{t} V_{2i}(x(\zeta))d\zeta d\theta_i \end{aligned} \tag{10}$$

Positivity tests similar to the ones shown earlier can be given in this case too.

We now turn to the examples we have considered in Section 2.

4 Examples

4.1 Network Congestion Control for the Internet

Consider the example developed in Section 2.1. Let $B = 2$, $c_1 = c_2 = 3$, and $c_3 = 1$. Then the steady-state of this system is $(x_1^*, x_2^*) = (0.6242, 0.6242)$. We consider delay sizes such that $\tau_{1,3}^b = 63$ms, $\tau_{1,3}^f = 93$ms, $\tau_{2,3}^f = 49$ms, $\tau_{2,3}^b = 77$ms. The system equations about the new steady-state become:

$$\begin{aligned} \dot{x}_1' &= 1 - [x_1'(t-0.154) + x_1^*] \\ &\left[\left(\frac{x_1'(t-0.154) + x_1^*}{3}\right)^2 + (x_1'(t-0.154) + x_2'(t-0.14) + x_1^* + x_2^*)^2\right] \\ \dot{x}_2' &= 1 - [x_2'(t-0.126) + x_2^*] \\ &\left[\left(\frac{x_2'(t-0.126) + x_2^*}{3}\right)^2 + (x_2'(t-0.126) + x_1'(t-0.14) + x_1^* + x_2^*)^2\right] \end{aligned}$$

where $x_i(t) = x_i'(t) + x^*$. The linearization of this system about the steady-state is stable and a Lyapunov function of the form (9) can be constructed. The same Lyapunov function can be constructed in a region of the state-space satisfying $\|x_{1_t}'\| \le 0.8x_1^*$ and $\|x_{2_t}'\| \le 0.8x_2^*$, thus showing that the steady-state is locally stable. The Lyapunov functional has the following form: all v_{1i} are 4th order, quadratic in $(\phi(0), \phi(\theta))$ and at most second order in θ and Z_d to construct v_{2ij} are third order: at most first order in $\phi(\theta)$ and second order in θ.

4.2 Stability of a negative gene autoregulation model with delays

Consider now the model developed in Section 2.2. We choose the following parameters: $\lambda_1 = \lambda_2 = 0.03$, $c_1 = c_2 = 1$ and investigate the stability bounds for the delays τ_i. The model we consider is very similar to the one developed in [26].

When the Hill coefficient is equal to 1, i.e., $n = 1$, the system is delay-independent stable and this can be verified through the construction of a Lyapunov Krasovskii functional. In this case we need to perform a local stability analysis as it does not make sense to consider negative initial conditions for concentration profiles. The resulting Lyapunov functional is quartic in all variables and is of the form (8).

5 Conclusions

In this chapter we have studied application examples from Biology and Communication networks and shown how the Sum of Squares decomposition of multivariable polynomials can be used to analyze the system behaviour.

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Index

delay
  incommensurate delays, 1
  multiple delays, 1

Gene Regulatory Networks, 4

Internet Congestion Control, 3

Lyapunov-Krasovskii, 1

Nonlinear time-delay system, 2

SOSTOOLS, 7
stability
  delay-dependent, 1
  delay-independent, 1
Sum of Squares, 1