Systems Analysis and Control

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Lecture 24: Compensation in the Frequency Domain

Overview

In this Lecture, you will learn:

Lead Compensators

- Performance Specs
- Altering Phase Margin

Lag Compensators

- Change in steady-state error

Phase and Gain Margin

Recall: 3 indicators

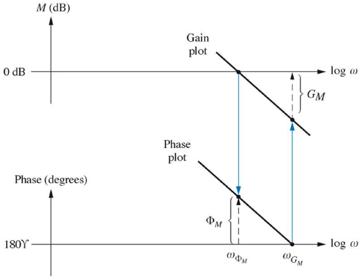
- Gain Margin
- Phase Margin
- Bandwidth

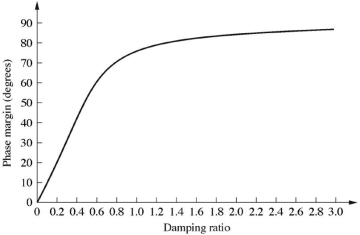
From Phase Margin and Closed-Loop Bandwidth:

- Percent Overshoot
- Peak Time
- Rise Time
- Settling Time

Percent Overshoot: ζ is from Φ_M only

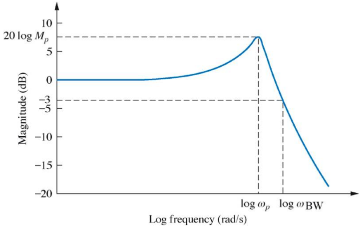
$$\%OS = e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

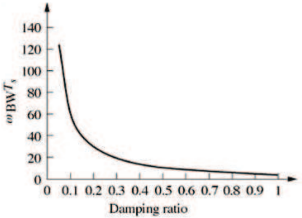


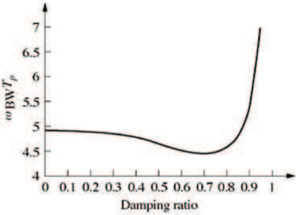


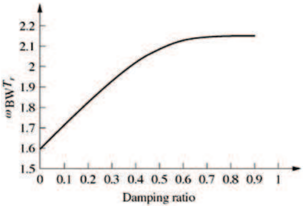
Phase and Gain Margin

Given ζ, we need ω_{BW} to find T_r, T_s and T_p









$$T_s = \frac{f_1(\zeta)}{\omega_{BW}} \qquad T_p = \frac{f_2(\zeta)}{\omega_{BW}} \qquad T_r = \frac{f_3(\zeta)}{\omega_{BW}}$$

Phase and Gain Margin

This is all analysis.
Design Problem: Achieve

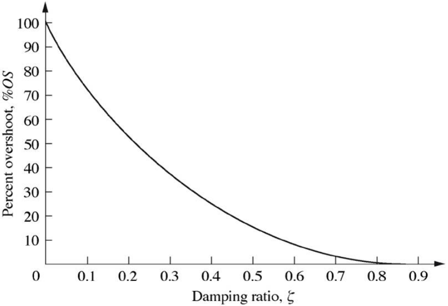
- 10% Overshoot
- $T_r = 2s$
- $T_s = 10s$

Step 1: Translate into Φ_M and ω_{BW} constraints.

Get desired ζ from 10% Overshoot

$$\zeta = \frac{-\ln\left(\frac{\%OS}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}}$$

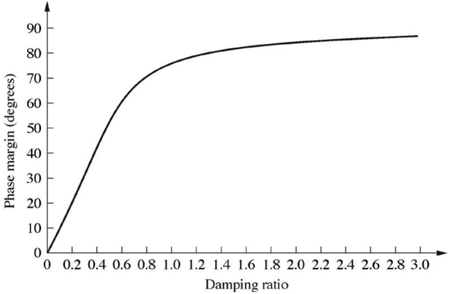
$$= .57$$



Phase and Gain Margin

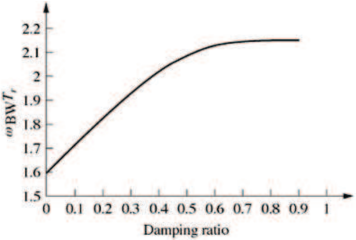
The desired ζ yields a desired phase margin.

$$\Phi_{M,desired} \cong \zeta_{des} \cdot 100$$
$$= 57^\circ$$



The toughest constraint will be Rise Time: $T_r < 2s$

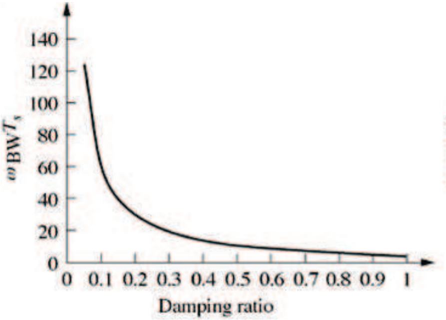
$$\omega_{BW} = \frac{f_3(\zeta)}{T_r}$$
$$> \frac{2.12}{2} = 1.06$$



Phase and Gain Margin

We can also look at settling time:
$T_s < 10s$

$$\omega_{BW} = \frac{f_1(\zeta)}{T_s}$$
$$> \frac{8}{10} = .8$$



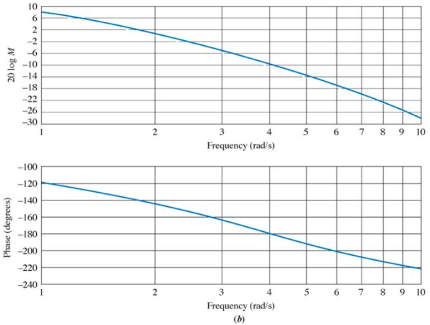
Therefore: We want

- Phase margin of $\Phi_M = 57°$
- Bandwidth of $\omega_{BW} > 1$

So far we only know how to *find* Φ_M and ω_{BW}, but not influence them.

- $\Phi_M = 35°$ at $\omega_c = 2.3$
- $\omega_{BW} \cong 3.7$

Question: How can we increase Φ_M and decrease ω_{BW}?



Answer:

- Increase gain at $\omega > \omega_{BW,desired}$
  - So that $|G(\imath\omega_{BW,desired})| = -7dB$.
- Increase phase by $22°$ at crossover frequency.

Lead Compensation

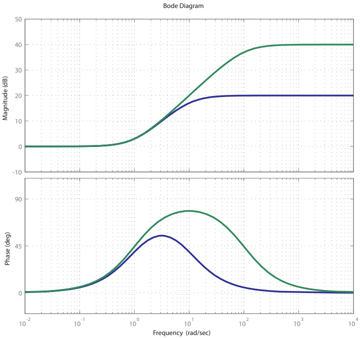
Lead Compensators

Question: How do we design our lead compensator?

- In root locus, we had pole placement.

Look at the Two Lead Compensators:

$$D(s) = k\frac{s+1}{\frac{s}{10}+1} \qquad \text{and} \qquad D_2(s) = k\frac{s+1}{\frac{s}{100}+1}$$



What are the differences?

Lead Compensators

Consider the generalized form of lead compensation

$$D(s) = k\frac{Ts+1}{\alpha Ts+1}$$

α determines how much phase is added

- $\alpha < 1$ for lead compensation
- $\alpha > 1$ for lag compensation

T determines where the phase is added.

- We want extra phase at the crossover frequency

Note that magnitude is also added at high frequency.

- Could increase the crossover frequency.
- Changes the phase margin.

Lead Compensators

Question: Where is the phase added?
Find the point of Maximum Phase.
The Phase contribution of the lead compensator is

$$\begin{aligned}\Phi(\omega) &= \angle D(\imath\omega)\\ &= \angle T(\imath\omega+1) - \angle(\alpha T\imath\omega+1)\\ &= \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)\end{aligned}$$

To find peak phase contribution, set $\frac{\partial \angle D(\imath\omega)}{\partial\omega} = 0$.

$$\frac{\partial\Phi}{\partial\omega} = \frac{1}{1+(T\omega)^2}T - \frac{1}{1+(\alpha T\omega)^2}\alpha T = 0$$

Which means

$$\begin{aligned}&1+(\alpha T\omega)^2 - (1+(T\omega)^2)\alpha\\ &= 1-\alpha+(\alpha-1)(\alpha T^2\omega^2) = 0\end{aligned}$$

Dividing by $1-\alpha$, we get

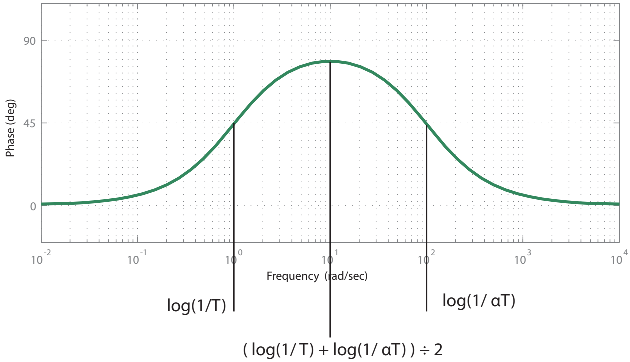
$$1-\alpha T^2\omega^2 = 0 \qquad \text{or} \qquad \omega_{max} = \frac{1}{T\sqrt{\alpha}}$$

So we get that the maximum phase contribution is at $\omega_{max} = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{T\alpha}}$.

Convert to a $\log \omega$ Bode plot:

$$\log \omega_{max} = \frac{1}{2} \left[\log \frac{1}{T} + \log \frac{1}{\alpha T} \right]$$

The maximum occurs at the average of $\log \frac{1}{T}$ and $\log \frac{1}{\alpha T}$.

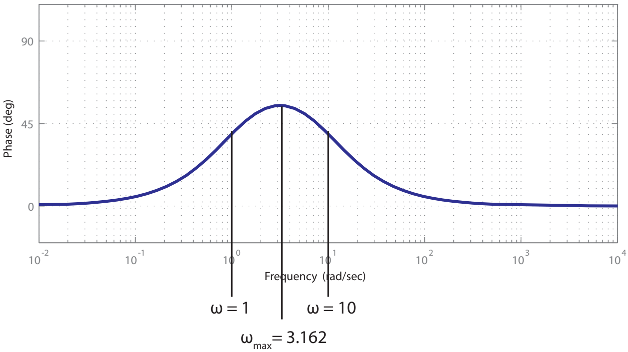


Lead Compensators

Reconsider our example:

$$D(s) = k\frac{s+1}{\frac{s}{10}+1}$$

- $T = 1$
- $\alpha = \frac{1}{10}$



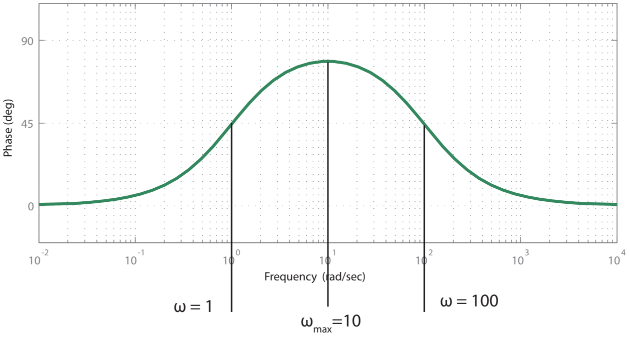
Maximum Phase occurs at $\omega = \frac{1}{T\sqrt{\alpha}} = 3.162$

Lead Compensators

Reconsider our example:

$$D(s) = k\frac{s+1}{\frac{s}{100}+1}$$

- $T = 1$
- $\alpha = \frac{1}{100}$



Maximum Phase occurs at $\omega = \frac{1}{T\sqrt{\alpha}} = 10$

Lead Compensators

So if we want to add phase at ω_c, then we need

$$T\sqrt{\alpha} = \frac{1}{\omega_c}$$

This is not definitive

- Depends on how much phase we want to add.

Now consider the case where we want to add 30° of phase margin.
Question: How much phase does a lead compensator add?

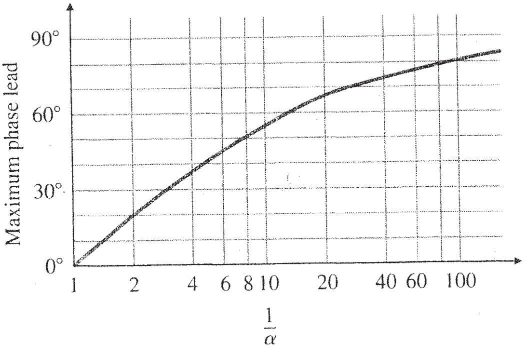
- We already know the frequency of peak phase.
- Make this the crossover frequency

$$\begin{aligned}\Phi_{\max} &= \tan^{-1}\frac{1}{\sqrt{\alpha}} - \tan^{-1}(\sqrt{\alpha}\omega)\\ &= \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right)\end{aligned}$$

Independent of T!

Lead Compensators

We can plot $\Phi_{\max}$ vs. $\frac{1}{\alpha}$.



For 30° phase, we want $\frac{1}{\alpha} = 3$.

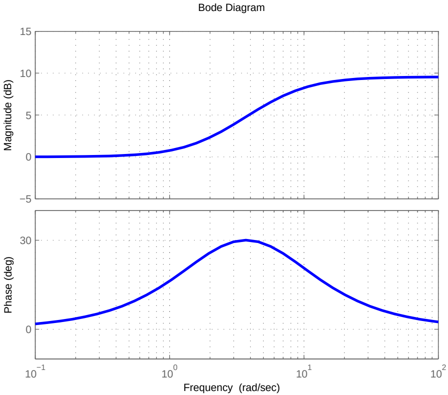
- Lets add 30° of phase at $\omega = 3.7$, we want
  - Lead will move crossover frequency.

$$T = \frac{1}{\omega_c \sqrt{\alpha}} = \frac{\sqrt{3}}{3.7} = .468$$

Lead Compensators

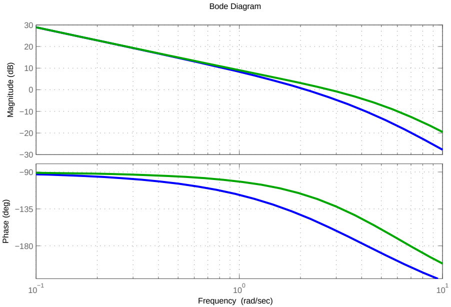
This gives us a lead compensator:

$$D(s) = \frac{Ts+1}{\alpha T s + 1} = \frac{.468s+1}{.156s+1}$$



Lead Compensators

Add this to the original plot to get



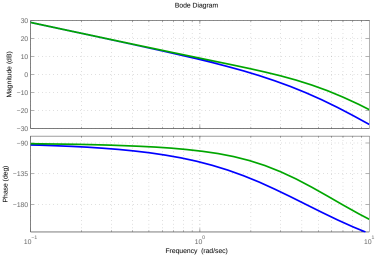
New Phase margin is $51.3°$. New $\omega_{BW} = 4.79$

- Crossover frequency is increased, which reduces the phase margin.

Lag Compensators

What about steady-state error?

- Want to increase $|G(0)|$
- No effect on Φ_M or ω_{BW}



$$e_{ss} = \frac{1}{1 + |G(0)|}$$

- Ignore the lead compensator.
- Increase $|G(0)|$ by 15dB.
- No change at ω_c or ω_{BW}

Lag Compensators

Question: How do we increase $|G(0)|$ without changing ω_{BW}?

- We want to reduce gain at $\omega = 0$.

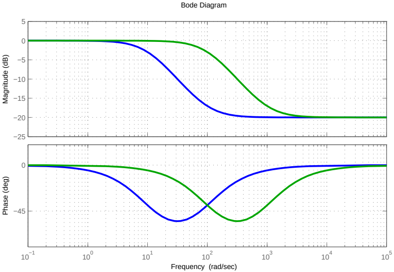
$$20\log|D(0)| \cong 15$$

- Want no effect on Φ_M or $|G(0)|$

$$20\log|D(\imath\omega_{BW})| \cong 0$$

Look at the Two Lag Compensators:

$$D(s) = \frac{\frac{s}{100}+1}{10\frac{s}{100}+1} \qquad \text{and} \qquad D_2(s) = \frac{\frac{s}{1000}+1}{10\frac{s}{1000}+1}$$



Lag Compensators

Lets use the form:

$$D(s) = \frac{Ts+1}{\alpha Ts+1}$$

α determines how much phase is added

- $\alpha > 1$ for lag compensation
- As before, min phase is given by $\omega = \frac{1}{T\sqrt{\alpha}}$
  - Center this point at low frequency

Change in magnitude at high frequency is

$$20\log\alpha(Ts+1) - 20\log(\alpha Ts+1)$$

If T is large, this is just 0. At low frequency, gain is

$$20\log\alpha$$

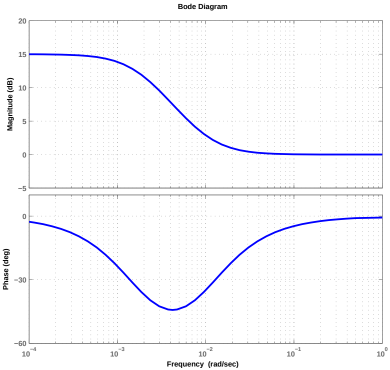
Lag Compensators

For our problem, set $T = .01$ and then we want

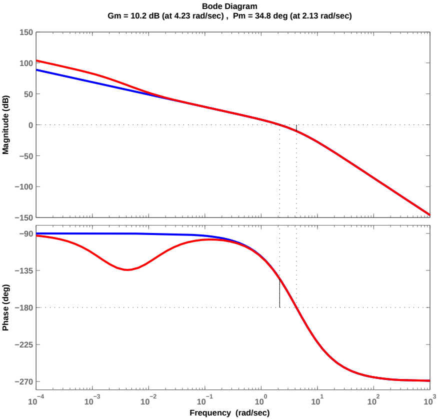
$$20 \log \alpha = 15$$

so $\alpha = 10^{.75} = 5.62$

$$D(s) = 5.62 \frac{.01s + 1}{.0562s + 1}$$



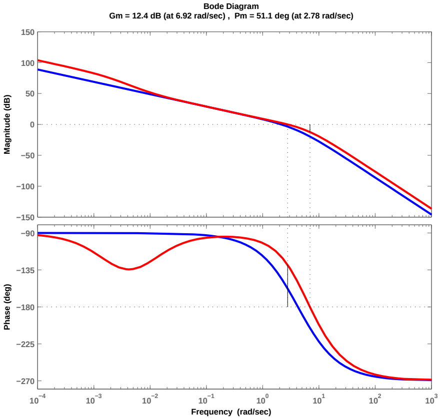
The system with lag compensation:



Bandwidth: $\omega_{BW} = 3.7$
Phase Margin: $\Phi_M = 35°$

Lag Compensators

Combine the lead and lag compensators:



- $\Phi_M = 51^\circ$
- $\omega_{BW} = 4.78$

You Have Learned: Classical Control Systems

Exam Material:
Root Locus

- Drawing
  - Asymptotes, Break Points, etc.
- Compensation
  - Gain, Lead-Lag, Notch Filters

Bode and Frequency Response

- What is frequency response?
- Drawing Bode Plot
- Closed Loop Dynamics
- Compensation

Nyquist Plot

- Drawing and Concepts
- Stability Margins

Next Course is MMAE543: Rise of the Machines