Spacecraft Dynamics and Control

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Lecture 10: Rendezvous and Targeting - Lambert's Problem

Introduction

In this Lecture, you will learn:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Numerical Problem: Suppose we are in an equatorial parking orbit or radius r. Given a target with position $\vec{r}$ and velocity $\vec{v}$, calculate the Δv required to intercept the target before it reaches the surface of the earth.

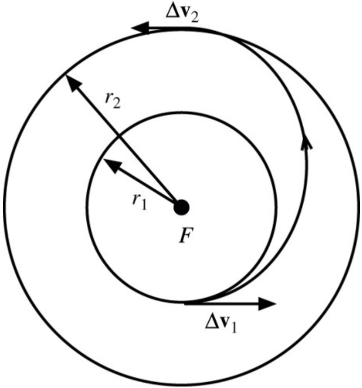
Problems we Have Solved

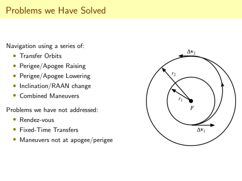
Navigation using a series of:

- Transfer Orbits
- Perigee/Apogee Raising
- Perigee/Apogee Lowering
- Inclination/RAAN change
- Combined Maneuvers

Problems we have not addressed:

- Rendez-vous
- Fixed-Time Transfers
- Maneuvers not at apogee/perigee





A Brief Note on Rendez-vous using Hohman transfer between circular orbits

- The transfer orbit can begin at any point in a circular orbit
- Need to calculate the relative phase between the vehicle and target at which to begin the transfer orbit.
- Let θ_0 denote the initial angle between the position vectors of the vehicle and target at the beginning of the Hohman transfer.
- The target moves at angular velocity $\dot{f} = n_t = \frac{2\pi}{T_{target}}$, which is the mean motion.
- The vehicle moves through an angle of $\Delta f = \pi$ radians during the transfer orbit
- The transfer orbit takes an amount of time $T_{hohmann}/2$.
- The relative angle between vehicle and target at arrival is $\theta_0 + n_t \cdot \frac{T_{hohmann}}{2} - \pi$.
- Thus the required relative phase at which to begin the transfer is

$$\theta_0 + n_t \cdot \frac{T_{hohmann}}{2} - \pi = 0 \qquad \text{or} \qquad \theta_0 = \pi - n_t \cdot \frac{T_{hohmann}}{2} = \pi - \pi \frac{T_{hohman}}{T_{target}}$$

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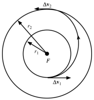
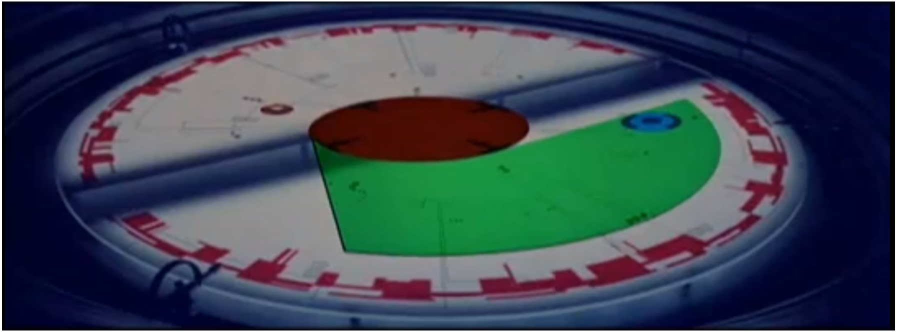


The Problem with phasing

Problem: We have to wait.



Remember what happened to the Death star?

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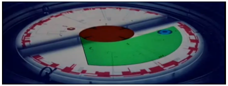
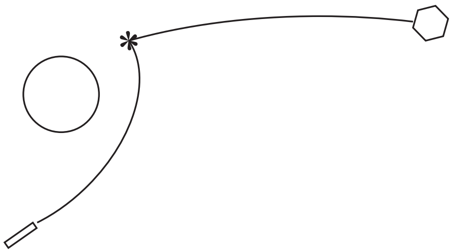


Remember what happened to the Death star?

- The death star had to wait for about 100° of phase (or $\Delta t = \frac{100}{360} T_{ds}$) before it was in range of the rebel base.
- The rebels solved Lambert's problem and calculated an intercept trajectory with TOF$< \Delta T = \frac{100}{360} T_{ds}$.

Asteroid Interception



Suppose that:

- Our time to intercept is limited.
- The target trajectory is known.

Problem: Design an orbit starting from $\vec{r}_0$ which intersects the orbit of the asteroid at the same time as the asteroid.

- *Before* the asteroid intersects the earth (when $r(t) = 6378$)

Missile Defense

Problem: ICBM's have re-entry speeds in excess of 8km/s (Mach 26).

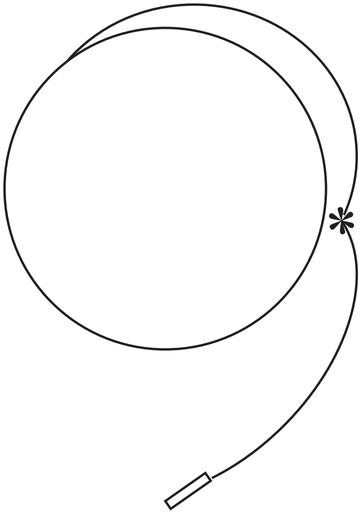
- Patriot missiles can achieve max of Mach 5.

Objective: Intercept ballistic trajectory before missile re-entry

- *Before* the missile intersects the atmosphere
- When $r(t) = 6378km + \cong 200km$

Complications:

- Plane changes may be required.
- The required time-to-intercept may be small.
  - Hohman transfer is not possible



The Targeting Problem

Step 1: Determine the orbit of the **Target**

Step 1 can be accomplished one of two ways:

Method 1:

1. Given $\vec{r}(t_1)$ and $\vec{v}(t_1)$, find $a, e, i, \omega_p, \Omega$ and $f(t_1)$
   - we have covered this approach in Lecture 6.
2. Unfortunately, it is difficult to measure $\vec{v}$

Method 2:

1. Given two observations $\vec{r}(t_1)$ and $\vec{r}(t_2)$, find $a, e, i, \omega_p, \Omega$ and $f(t_0)$.
   - Alternatively, find $\vec{v}(t_1)$ and $\vec{v}(t_2)$
2. This is referred to as Lambert's problem (the topic of this lecture)

Note: This is a *boundary-value* problem:

- We know some states at two points.
- In contrast to the *initial value* problem, where we know all states at the initial time.
- Unlike initial-value problems, boundary-value problems cannot always be solved.

Carl Friedrich Gauss (1777-1855)

The problem of orbit determination was originally solved by C. F. Gauss in 1801

Boring/Conservative/Grumpy (Monarchist).

One of the greatest mathematicians

- Professor of Astronomy in Göttingen
- Motto: "pauca sed matura" (few but ripe)

Discovered

- Gaussian Distributions
- Gauss' Law (collaboration with Weber)
- Non-Euclidean Geometry (maybe)
- Least Squares (maybe)

Legendre published the first solution to the Least Squares problem in 1805

- In typical fashion, Carl Friedrich Gauss claimed to have solved the problem in 1795 and published a more rigorous solution in 1809.
- This more rigorous solution first introduced the normal probability distribution (or Gaussian distribution)

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- Gauss focused on simplification/distillation/perfection of existing ideas.
- Least-squares is also claimed by Legendre.
- Non-Euclidean geometries discovered in 1829 by Bolyai. Problem of parallel lines. No hard evidence to support Gauss' claim (1932). "To praise it would amount to praising myself. For the entire content of the work ... coincides almost exactly with my own meditations which have occupied my mind for the past thirty or thirty-five years"
- Gauss (at 7) is the source of that story about the student who summed up the numbers from 1 to 100.
- Wanted a heptadecagon inscribed on his tombstone (17-sided equilateral polygon)
- pauca sed matura
- Disliked teaching, believing students robbed him of his time. He especially hated when students took notes in class, saying they should listen instead.
- Kept a playlist of his favorite songs in a notebook.

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The story of orbit determination is a bit complicated.

- Lambert's problem relies on the numerical solution of Lambert's equation.
- Lambert proposed Lambert's equation in 1761, but the proof was purely geometric. He also proposed a series expansion for this equation.
- Lagrange actually proved Lambert's equation.
- Gauss initally solved the 3 observation problem where we don't have range, only declination and right ascension. We won't actually cover the solution to this problem, as it is rather involved. However, this was the original basis of the story of Gauss and Piazzi.
- Gauss's method for solving Lambert's equation followed shortly thereafter in Theoria Motus.
- The first modern algorithms for solving Lambert's problem only became available in the late 1950's.
- These algorithms are very touchy and ill-tempered, especially for multi-revolution orbits and at the transition between elliptic and hyperbolic orbits.

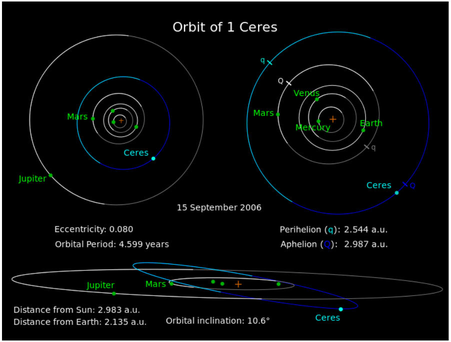
Discovery and Rediscovery of Ceres

The pseudo-planet Ceres was discovered by G. Piazzi

- Observed 12 times between Jan. 1 and Feb. 11, 1801
- Planet was then lost.

Complication:

- Observation was only declination and right-ascension.
- Observations were only spread over 1% of the orbit.
  - No ranging info.
- For this case, three observations are needed.



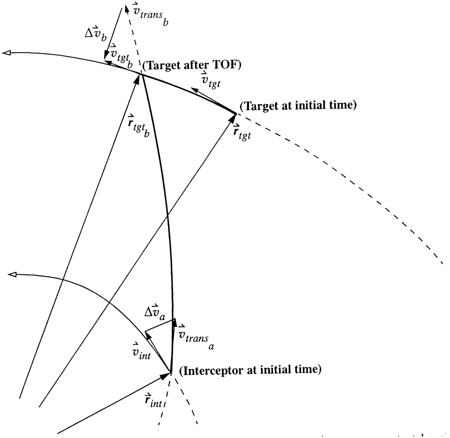
C. F. Gauss solved the orbit determination problem and correctly predicted the location. Gauss' last rushed publication!

- Planet was re-found on Dec 31, 1801 in the correct location.

Once we have found the orbit of the target, we can determine where the target will be at the desired time of impact, t_2.

Procedure:

- The difference $t_2 - t_1$ is the Time of Flight (TOF)
- Calculate $M(t_2) = M(t_1) + n(t_2 - t_1)$
- Use $M(t_2)$ to find $E(t_2)$.
- Use $E(t_2)$ to find $f(t_2)$.
- Use $f(t_2)$ to find $\vec{r}(t_2)$.



For a given

- Initial Position, $\vec{r}_1$
- Final Position, $\vec{r}_2$
- Time of Flight, TOF

the transfer orbit is uniquely (not really) determined.

Challenge: Find that orbit!!!

Difficulties:

- Where is the second focus?
- May require initial plane-change.
- May use LOTS of fuel.

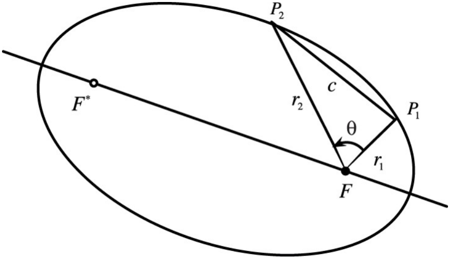


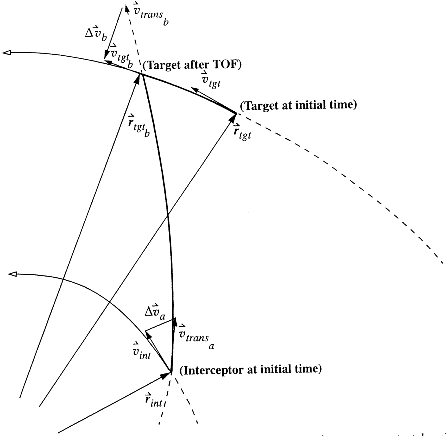
Figure: For given P_1 and P_2 and TOF, the transfer ellipse is uniquely determined.

On the Plus Side:

- We know the change in true anomaly, Δf...
- For this geometry, TOF only depends on a.

Once we have found the transfer orbit,

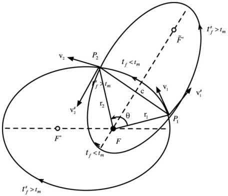
- Calculate $\vec{v}_{tr}(t_1)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_1)$
- Calculate $\vec{\Delta v} = \vec{v}_{tr}(t_1) - \vec{v}(t_1)$

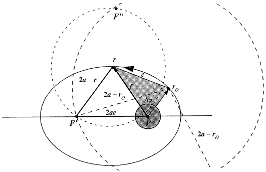


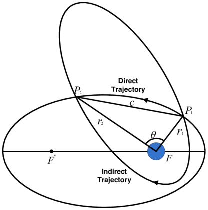
What do we know?

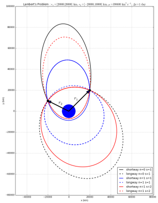
- Location of focus, F (Earth or Sun)
- Point $\vec{r}(t_1)$ on the orbit.
- Point $\vec{r}(t_2)$ on the orbit.

This is enough to determine the orbital plane









We also know the change in True Anomaly,

- $f(t_2) - f(t_1) = \theta$ or $f(t_2) - f(t_1) = 360° - \theta$

Q: Is this enough to determine the orbit?

A: No. Also need semi-major axis, a, or distant focus, F'.

Lambert's Conjecture

Semi-major axis, a only depends on Δt (TOF)

Recall, we are **given** Δt

First: Calculate some lengths

- $c = \|\vec{r}_2 - \vec{r}_1\|$ is the *chord*.
- $s = \frac{c+r_1+r_2}{2}$ is the *semi-perimeter*.
  - NOT semiparameter.

Modern Formulation of **Lambert's Equation:**

$$\Delta t = \sqrt{\frac{a^3}{\mu}} \left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

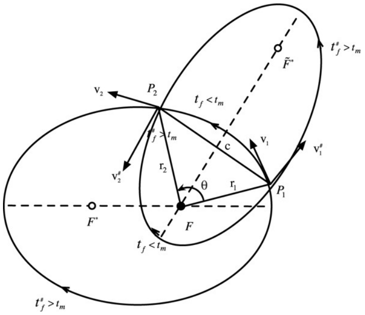
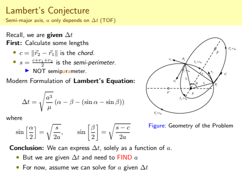


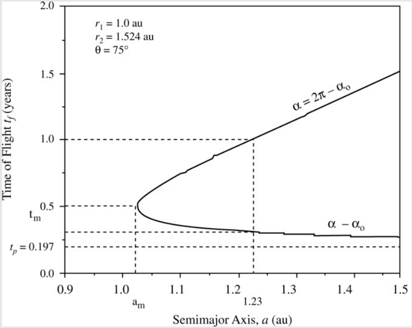
Figure: Geometry of the Problem

Conclusion: We can express Δt, solely as a function of a.

- But we are given Δt and need to FIND a
- For now, assume we can solve for a given Δt



- That TOF only depends on a was Lambert's conjecture! Only proved rigorously the year (1776) before he died (1777, Tuberculosis?).
- Was originally a clerk is the iron mines. Then a tutor.
- Another problem: There are 2 solutions to the equation!



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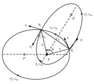

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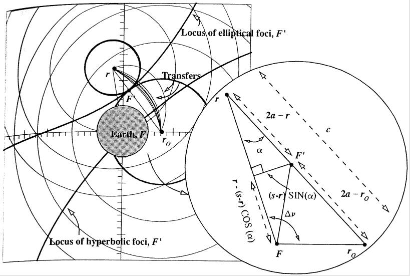
- The semi-perimeter is half the perimeter of the triangle shown in the figure.



How to use a to define the transfer orbit

Lets find the hidden Focus, F'

Recall the defining property of an ellipse:

- At any point on an ellipse, the sum of the distances to the foci is always $2a$

The value of a restricts possible locations of the hidden focus.

- **Focal Circle 1:**
  Circle of radius $2a - r_1$ about $\vec{r}_1$.
- **Focal Circle 2:**
  Circle of radius $2a - r_2$ about $\vec{r}_2$.

The **Intersection** of these circles gives the two possible locations of the hidden focus, F' or F''.

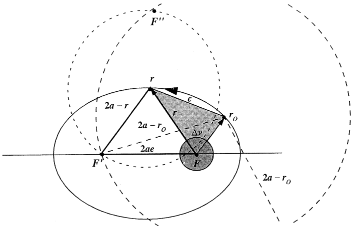


Figure: Potential Locations of Second Focus

The Long Way and the Short Way

For a given a, the two potential foci F' and F'' correspond to the two solutions to Lambert's equation

- The **Short Way** Corresponds to the small Δt solution.
- The **Long Way** Corresponds to the large Δt solution.

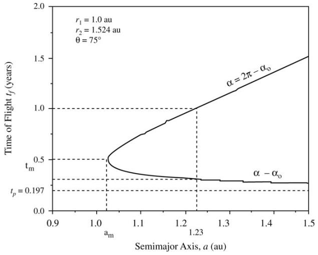


Figure: First arc of transfer times

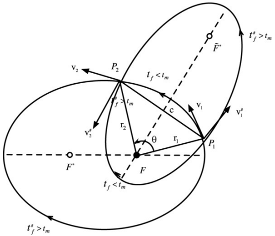


Figure: Long way and short way

In addition, the direction of travel along each ellipse can be reversed to obtain the **Retrograde Path**

- A total of Four (4) possible transfers for a given value of semi-major axis, a.
- If we include multi-revolution orbits, an infinite number of transfers can be obtained.

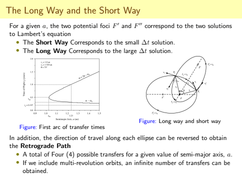


Illustration of the multiple solutions of Lambert's problem for multiple revolutions.

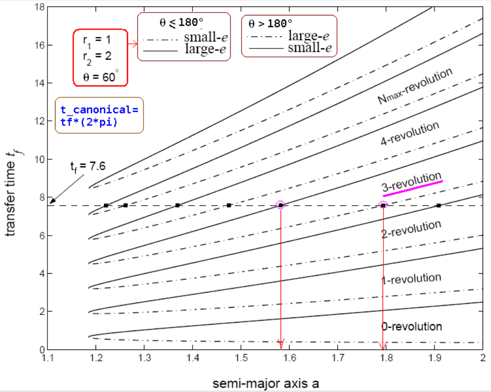


Figure: Sito di Astronomia Teorica by Giuseppe Matarazzo

To get retrograde transfers, we adjust by each arc by the period: $T(a) - \Delta t(a)$.

Note the focal locations vary continuously as we change a.

- As we vary a, the set of foci points F' and F'' form a *hyperbola*.
- As a increases, the two possible foci get farther apart.
- At a_{min}, The foci F' and F'' coincide.

  This is the minimum energy transfer ellipse

Conclusions

- $a_{\min} = \frac{r_1+r_2+c}{4}$ where $c = \|\vec{r}_2 - \vec{r}_1\|$.
- The minimum energy transfer yields the smallest a for which it is possible to have the two points on the same orbit.
- Hidden focus for the minimum energy transfer lies on the line between $\vec{r}_1$ and $\vec{r}_2$.
- This is NOT the Hohman transfer.

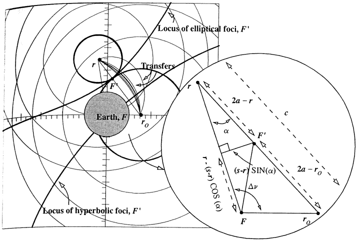


Figure: Potential Locations of Second Focus for a given a

The “minimum energy” transfer

The smallest achievable a

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- Minimum Energy means the orbit has minimum energy as per $E = -\frac{\mu}{2a}$. The Δv required is not necessarily minimized.
- That means you probably don’t want to use this transfer.
- At minimum energy orbit, $F' = F''$! Δt the long way is the same as Δt the short way.

How to Determine a given Δt?

Recall **Lambert's Equation:**

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

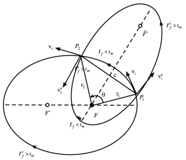
where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

- $c = \|\vec{r}_2 - \vec{r}_1\|$ is the *chord*.
- $s = \frac{c+r_1+r_2}{2}$ is the *semi-perimeter*.

Note the Similarity to using Kepler's Equation:

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left(E(t_2) - E(t_1) - e(\sin E(t_2) - \sin E(t_1))\right)$$



But the similarity is superficial

- Recall $M(t_2) - M(t_1) = n\Delta t$.
- No clear relationship between α and $E(t_1)$ or β and $E(t_2)$.
- Lambert's Equation is much harder to solve.

Given Δt, find a:

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right), \quad \sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \quad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

There are several ways to solve Lambert's Equation

- Newton Iteration
  - More Complicated than Kepler's Equation
- Series Expansion
  - Probably the easiest...
- Bisection
  - Relatively Slow, but easy to understand
  - Only works for *monotone* functions.

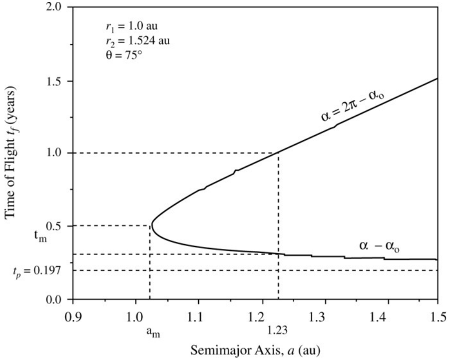
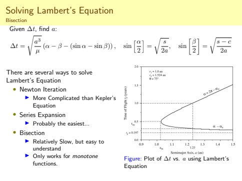


Figure: Plot of Δt vs. a using Lambert's Equation



- In the figure, a_m is the minimum energy transfer orbit. (Recall NOT minimum Δv).
- t_m is the transfer time (TOF) obtained by plugging a_m into Lambert's equation.
- t_p is the flight time of a parabolic orbit (corresponding to $a = \infty$)
- The function is *monotone* in the interval $TOF \in [t_p, t_m]$
- The other branch of the plot $(TOF > t_m)$ corresponds to use of the distant focus F''.

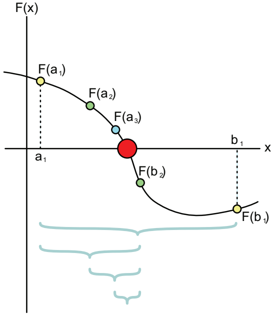
Solving Lambert's Equation via Bisection

Define $g(a) = \sqrt{\frac{a^3}{\mu}}\left(\alpha(a) - \beta(a) - (\sin\alpha(a) - \sin\beta(a))\right)$.

Root-Finding Problem:

Find a :

such that $g(a) = \Delta t$

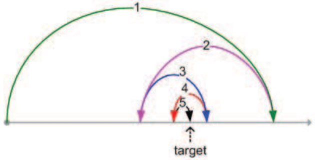


Bisection Algorithm:

1. Choose $a_{\min} = \frac{s}{2} = \frac{r_1 + r_2 + c}{4}$
2. Choose $a_{\max} >> a_{\min}$
3. Set $a = \frac{a_{\max} + a_{\min}}{2}$
4. If $g(a) > \Delta t$, set $a_{\min} = a$
5. If $g(a) < \Delta t$, set $a_{\max} = a$
6. Goto 3

This is guaranteed to converge to the unique solution (if it exists).

- We assume *Elliptic* transfers.



Lecture 10

Spacecraft Dynamics

2023-03-21

Solving Lambert's Equation via Bisection

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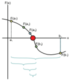

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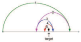

- By Elliptic solutions, we mean that we assume that the transfer orbit is elliptic.
- Parabolic solutions are possible, but not covered by Lambert's equations.
- We must check to make sure the solution is not parabolic before starting.

Make Sure a Solution Exists!!

- First calculate the minimum TOF.
- This corresponds to a parabolic trajectory

$$\Delta t_p = \frac{\sqrt{2}}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right)$$

  - Can get there even faster by using a hyperbolic approach (Not Covered).
- We should also calculate the Maximum TOF

$$\Delta t_{\max} = \sqrt{\frac{a_{\min}^3}{\mu}}\left(\alpha_{\max} - \beta_{\max} - (\sin\alpha_{\max} - \sin\beta_{\max})\right)$$

where

$$\sin\left[\frac{\alpha_{\max}}{2}\right] = \sqrt{\frac{s}{2a_{\min}}}, \qquad \sin\left[\frac{\beta_{\max}}{2}\right] = \sqrt{\frac{s-c}{2a_{\min}}}$$

- One can exceed this by going the long way around (Not Covered Here)

Bisection

Some Implementation Notes

Make Sure a Solution Exists!!

- First calculate the minimum TOF.
- This corresponds to a parabolic trajectory

$$\Delta t_p = \frac{\sqrt{2}}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right)$$

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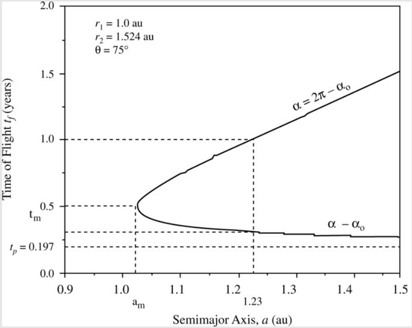


Figure: Note a_{min} and t_p restrict the arc of solutions considered and ensure the function is monotonic.

The bisection method would need to be reversed to search for the upper arc (long way solution).

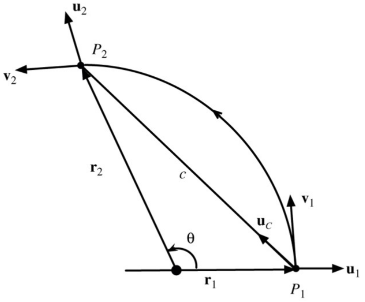
Calculating $\vec{v}(t_0)$ and $\vec{v}(t_f)$

Once we have a, calculating $\vec{v}$ is not difficult.

$$\vec{v}(t_1) = (B + A)\vec{u}_c + (B - A)\vec{u}_1, \qquad \vec{v}(t_2) = (B + A)\vec{u}_c - (B - A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\alpha}{2}\right), \qquad B = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\beta}{2}\right)$$



and the unit vectors

- $\vec{u}_1$ and $\vec{u}_2$ point to positions 1 and 2.

$$\vec{u}_1 = \frac{\vec{r}(t_1)}{r_1}, \qquad \vec{u}_2 = \frac{\vec{r}(t_2)}{r_2}$$

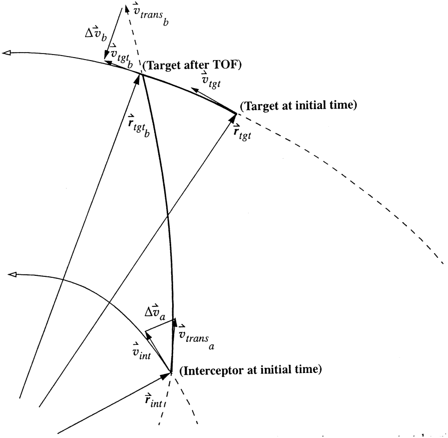
- $\vec{u}_c$ points from position 1 to 2.

$$\vec{u}_c = \frac{\vec{r}(t_2) - \vec{r}(t_1)}{c}$$

Calculating Δv

Once we have found the transfer orbit,

- Calculate $\vec{v}_{tr}(t_1)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_1)$
  - If a ground-launch, use rotation of the earth.
  - If in orbit, use orbital elements.
- Calculate $\vec{\Delta v} = \vec{v}_{tr}(t_1) - \vec{v}(t_1)$



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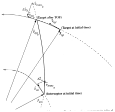


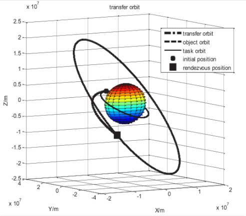


Fig 4: The optimal transfer orbit of the first example

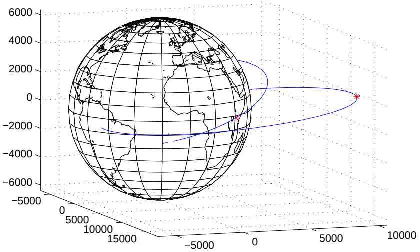
Problem: Suppose that Brasil launches an ICBM at Bangkok, Thailand.

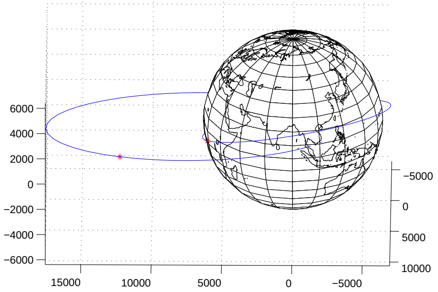
- We have an interceptor in the air with position and velocity

$$\vec{r}(t_1) = \begin{bmatrix}6045 & 3490 & 0\end{bmatrix} km \qquad \vec{v}(t_1) = \begin{bmatrix}-2.457 & 6.618 & 2.533\end{bmatrix} km/s.$$

- We have tracked the missile at
$r_t(t_1) = \begin{bmatrix}12214.839 & 10249.467 & 2000\end{bmatrix} km$ heading
$\vec{v}_t(t_1) = \begin{bmatrix}-3.448 & .924 & 0\end{bmatrix} km/s$.

Question: Determine the Δv required to intercept the missile before re-entry, which occurs in 30 minutes.





Numerical Example of Missile Targeting

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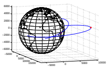
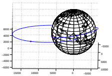




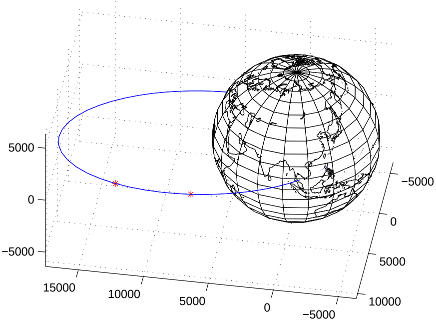
- The figure shows both the path of the ICBM and the current (temporary) orbit of the interceptor.
- The $*$ indicates the current positions of the ICBM and interceptor in their respective orbits.

Numerical Example of Missile Targeting

The first step is to determine the position of the ICBM in $t + 30$min.

Recall: To propagate an orbit in time:

1. Use $\vec{r}_{t1}$ and $\vec{v}_{t1}$ to find the orbital elements, including $M(t_1)$.
2. Propagate Mean anomaly $M(t_2) = M(t_1) + n\Delta t$ where $\Delta t = 1800s$.
3. Use $M(t_2)$ to find true anomaly, $f(t_2)$.
   - Requires iteration to solve Kepler's Equation.
4. Use the orbital elements, including $f(t_2)$ to find $\vec{r}(t_2)$



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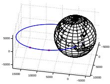

- This figure shows the position of the ICBM at the initial point and the desired point of interception.

Numerical Example of Missile Targeting

The next step is to determine whether an intercept orbit is feasible using TOF$=30$min.

Geometry of the Problem:

$$r_1 = \|\vec{r}_1\| = 6,980km, \quad r_2 = \|\vec{r}_t(t_2)\| = 12,282km,$$

$$c = \|\vec{r}_1 - \vec{r}_t(t_2)\| = 7,080km, \quad s = \frac{c + r_2 + r_1}{2} = 13,171km$$

Minimum Flight Time: Using the formula, the minimum (parabolic) flight time is

$$t_{\min} = t_p = \frac{\sqrt{2}}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right) = 12.8min$$

Thus we have more than enough time.

Maximum Flight Time: Geometry yields a minimum semi-major axis of

$$a_{\min} = \frac{s}{2} = 6,586km$$

Plugging this into Lambert's equation yields a maximum flight time of $t_{\max} = 37.3$min.

Numerical Example of Missile Targeting

What remains is to solve Lambert's equation:

$$\Delta t = \sqrt{\frac{a^3}{\mu}} \left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

Initialize our search parameters using $a \in [a_l, a_h] = [a_{\min}, 2s]$.

1. $a_1 = \frac{a_l + a_h}{2} = 8,232$ - $TOF = 21.14$min - too low, decrease a
   1.1 Set $a_h = a_1$
2. $a_2 = \frac{a_l + a_h}{2} = 7,409$ - $TOF = 24$min - too low, decrease a
   2.1 Set $a_h = a_2$
3. $a_3 = \frac{a_l + a_h}{2} = 6,997$ - $TOF = 26.76$min - too low, decrease a
   3.1 Set $a_h = a_3$
4. $\cdots$

K. $a_k = \frac{a_l + a_h}{2} = 6,744$ - $TOF = 29.99$
   K.1 Close Enough!

Numerical Example of Missile Targeting

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- In this example, $a_{\max}$ was chosen as $2s$. However, this was just a guess and if the TOF is near the parabolic flight time, a larger value should be chosen.

Numerical Example of Missile Targeting

Now we need to calculate Δv.

$$\vec{v}(t_1) = (B + A)\vec{u}_c + (B - A)\vec{u}_1, \qquad \vec{v}(t_2) = (B + A)\vec{u}_c - (B - A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\alpha}{2}\right) = .597, \qquad B = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\beta}{2}\right) = 4.2363$$

and the unit vectors

$$\vec{u}_1 = \begin{bmatrix} .866 \\ .5 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} .52 \\ .8414 \\ .1451 \end{bmatrix}, \quad \vec{u}_c = \begin{bmatrix} .0493 \\ .9666 \\ .2516 \end{bmatrix}$$

which yields

$$\vec{v}_t(t_1) = \begin{bmatrix} 3.3901 & 6.4913 & 1.2163 \end{bmatrix} km/s$$

Calculating Δv

$$\Delta v = \vec{v}_t(t_1) - \vec{v} = \begin{bmatrix} 5.847 & -.1267 & -1.3167 \end{bmatrix} km/s$$

For a total impulse of 6km/s.

Numerical Example of Missile Targeting

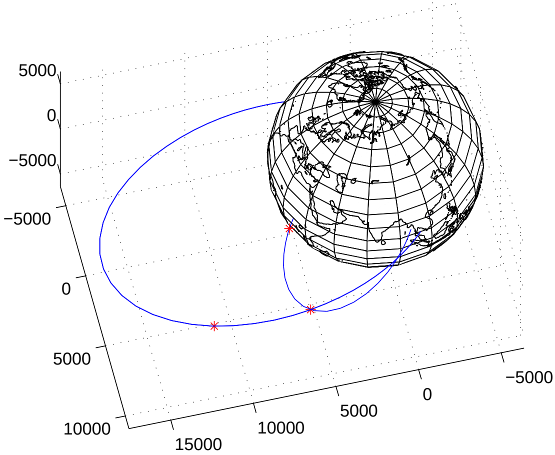

Figure: Intercept Trajectory

Numerical Example of Missile Targeting

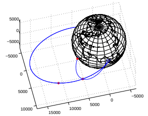

Figure: Intercept Trajectory

- This figure shows the ICBM and the path of the intercept trajectory.

Summary

This Lecture you have learned:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Next Lecture: Rocketry.