

Systems Analysis and Control

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Lecture 18: The Frequency Response

Overview

In this Lecture, you will learn:

Introduction to the Frequency Domain

- Life without Newton
 - ▶ “Who needs a model, anyway?”
- Black Boxes.

Frequency Response

- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots

Root Locus Demo 1

Wiley+ Root Locus Demo 1

Root Locus Demo 2

Wiley+ Root Locus Demo 2

The Frequency Response

Definition 1.

The **Frequency Response** is the *steady-state* output of a system with sinusoidal input.

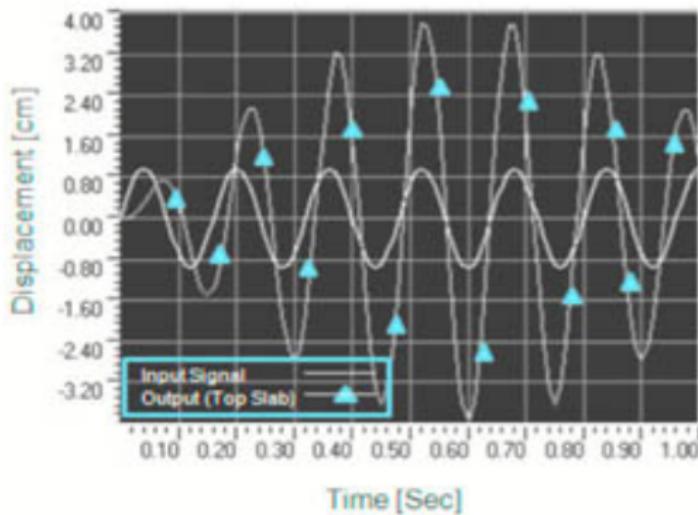


Figure: Response of Concrete Slabs to Soil Excitation (FEM)

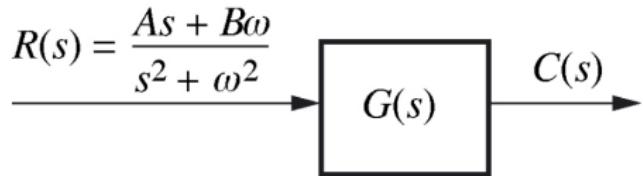
The Frequency Response

A Sinusoidal Input:

$$\begin{aligned} u(t) &= A \sin(\omega t) + B \cos(\omega t) \\ &= \sqrt{A^2 + B^2} \sin\left(\omega t - \tan^{-1}\left(\frac{B}{A}\right)\right) \\ &= M \sin(\omega t + \phi) \end{aligned}$$

Laplace Transform:

$$\hat{u}(s) = \frac{Bs + A\omega}{s^2 + \omega^2}$$

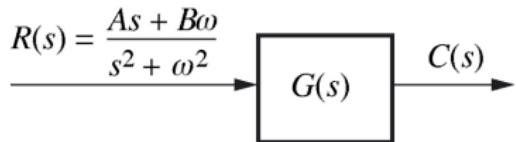


- $M = \sqrt{A^2 + B^2}$
- $\phi = -\tan^{-1}\left(\frac{B}{A}\right)$

The Frequency Response

For now, set $B = 0$, then $u(t) = A \sin \omega t$.

$$\hat{u}(s) = \frac{A\omega}{s^2 + \omega^2}$$



For a given stable transfer function,

$$G(s) = \frac{n(s)}{(s + p_1) \cdots (s + p_n)},$$

then by partial-fraction expansion

$$\begin{aligned}\hat{y}(s) &= G(s)\hat{u}(s) \\ &= G(s) \frac{A\omega}{(s + i\omega)(s - i\omega)} \\ &= \frac{r_1}{s + p_1} + \cdots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + i\omega} + \frac{\beta}{s - i\omega}.\end{aligned}$$

The Frequency Response

Partial Fraction Expansion:

$$\hat{y}(s) = \frac{r_1}{s + p_1} + \cdots + \frac{r_n}{s + p_n} + \frac{\alpha}{s + i\omega} + \frac{\beta}{s - i\omega}$$

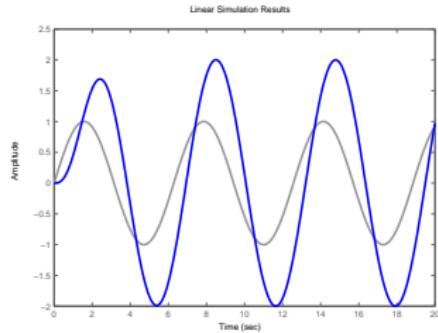
Inverse Laplace Transform:

$$y(t) = r_1 e^{-p_1 t} + \cdots + r_n e^{-p_n t} + \alpha e^{-i\omega t} + \beta e^{i\omega t}$$

But we want the **Steady-State Response**.

- Poles p_i are all stable.
 - ▶ $\lim_{t \rightarrow \infty} e^{-p_i t} = 0$
- These are called **Transient Responses**
- only left with

$$y_{ss}(t) = \alpha e^{-i\omega t} + \beta e^{i\omega t}$$



The Frequency Response

Since $\pm i\omega$ are isolated poles, by the remainder theorem:

$$\begin{aligned}\alpha &= G(s) \frac{A\omega}{(s + i\omega)(s - i\omega)} (s + i\omega)|_{s=-i\omega} \\&= G(-i\omega) \frac{A\omega}{-2i\omega} \\&= G(-i\omega) \frac{A}{-2i}\end{aligned}$$

Likewise,

$$\beta = G(i\omega) \frac{A}{2i}$$

Then

$$\begin{aligned}y_{ss}(t) &= \alpha e^{-i\omega t} + \beta e^{i\omega t} \\&= A \frac{G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}}{2i}\end{aligned}$$

Complex Numbers

Complex Conjugates

Issue: $G(-\omega)$ is the complex conjugate of $G(\omega)$.

Definition 2.

For a complex number $s = a + bi$, the **Complex Conjugate** of s is

$$s^* = a - bi$$

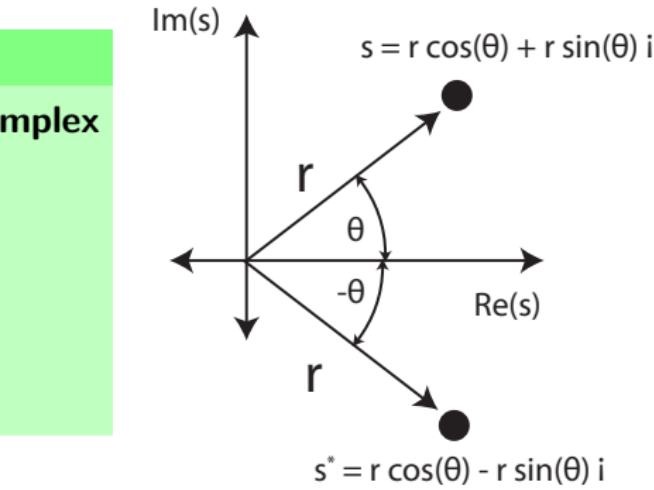
- Just replace $i \rightarrow -i$.
- $re^{i\theta} \rightarrow re^{-i\theta}$

Magnitude is unchanged. Phase is reversed

For $s = re^{i\theta}$,

Phase: $\angle s = \theta$

- $\angle s^* = -\theta = -\angle s$



Magnitude: $|s| = r$

- $|s^*| = r = |s|$

The Frequency Response

Complex Conjugate: $G(-\omega) = G(\omega)^*$

$$y_{ss}(t) = A \frac{G(\omega)e^{\imath\omega t} - G(-\omega)e^{-\imath\omega t}}{2\imath}$$

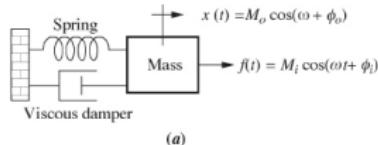
Recall that we can express $G(\omega)$ as

$$G(\omega) = |G(\omega)|e^{\angle G(\omega)\imath}$$

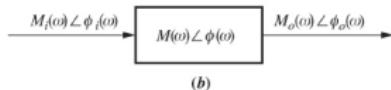
and $|G(\omega)| = |G(\omega)^*| = |G(-\omega)|$, $\angle G(-\omega) = \angle G(-\omega)^* = -\angle G(\omega)$

$$\begin{aligned} y_{ss}(t) &= A \frac{G(\omega)e^{\imath\omega t} - G(-\omega)e^{-\imath\omega t}}{2\imath} \\ &= |G(\omega)|A \frac{e^{\angle G(\omega)\imath}e^{\imath\omega t} - e^{-\angle G(\omega)\imath}e^{-\imath\omega t}}{2\imath} \\ &= |G(\omega)|A \frac{e^{(\omega t + \angle G(\omega))\imath} - e^{-(\omega t + \angle G(\omega))\imath}}{2\imath} \\ &= |G(\omega)|A \sin(\omega t + \angle G(\omega)) \end{aligned}$$

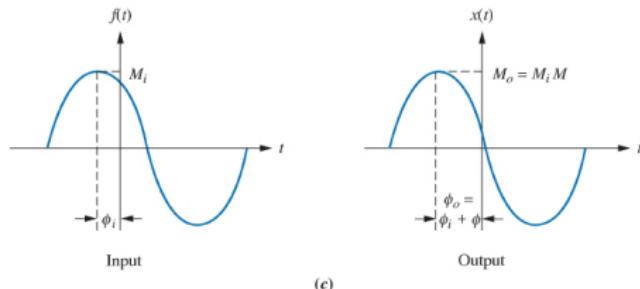
The Frequency Response



(a)



(b)



(c)

If the input is shifted:

Input:

$$u(t) = M \sin(\omega t + \phi)$$

Output:

$$y(t) = M|G(\omega)| \sin(\omega t + \phi + \angle G(\omega))$$

The Frequency Response

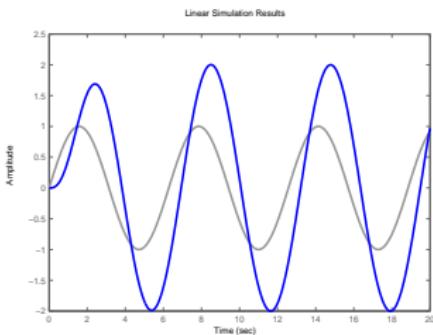
Conclusion: The response to a sinusoidal input $\sin \omega t$:

- A sinusoid with the same frequency.
- Phase is shifted by $\angle G(\omega)$.
- Magnitude is changed $|G(\omega)|$.

We refer to

- $|G(\omega)|$ is the Magnitude of Frequency Response
- $\angle G(\omega)$ is the Phase of Frequency Response

These depend only on ω and $G(\omega)$.



Frequency Response Planning

Applications

Application: Crane Oscillation

- Sinusoidal Input from Hanging load.
- Avoid Spillage.
- Avoid Tipping.

A Form of **Motion Control**.



Frequency Response Planning

Applications

[Figure: Simple Crane Sway Control](#)

[Figure: Industrial Crane Sway Control](#)

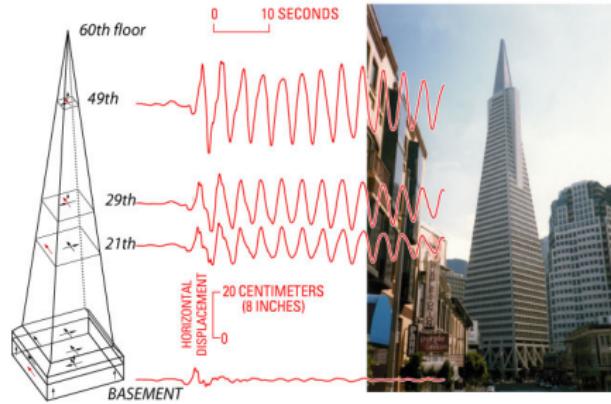
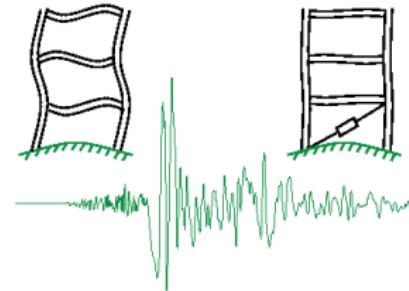
[Figure: Failure of Crane Control](#)

Frequency Response Planning

Modeling Structural Dynamics

Application: Building Response to Earthquakes

- Sinusoidal input from ground.
- Reduce peak output.



Obtaining Frequency Response Data

Controlling Structural Dynamics

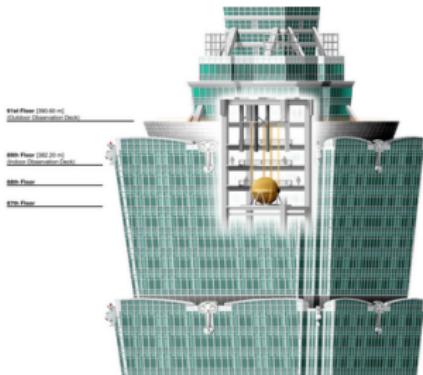


Figure: Earthquake Damping

The Frequency Response

This can work the other way too:

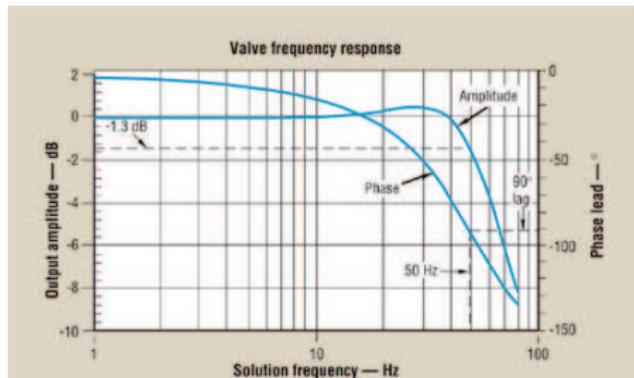
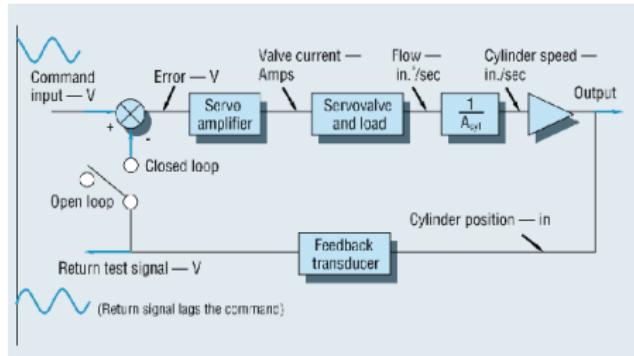
- **Input** $u(t) \sin \omega t$
- **Output:** $y(t) = M \sin(\omega t + \phi)$
- **Measure** M and ϕ
 - ▶ Relative Phase $\phi = \angle G(\omega)$
 - ▶ Magnitude: $M = |G(\omega)|$

Frequency Sweeping: Measure M and ϕ at every frequency

- Get functions $M(\omega)$ and $\phi(\omega)$

Reconstruct

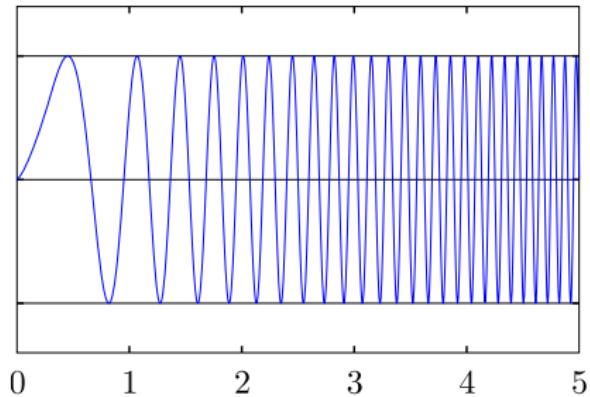
$$G(s) \cong M(s)e^{\phi(s)i}$$



The Frequency Response

Input: A Sinusoid of Increasing Frequency.

$$u(t) = \sin((\omega_0 + kt)t)$$



The Frequency Response

Figure: A Frequency Sweep in Circuit Analysis

Frequency Sweeping

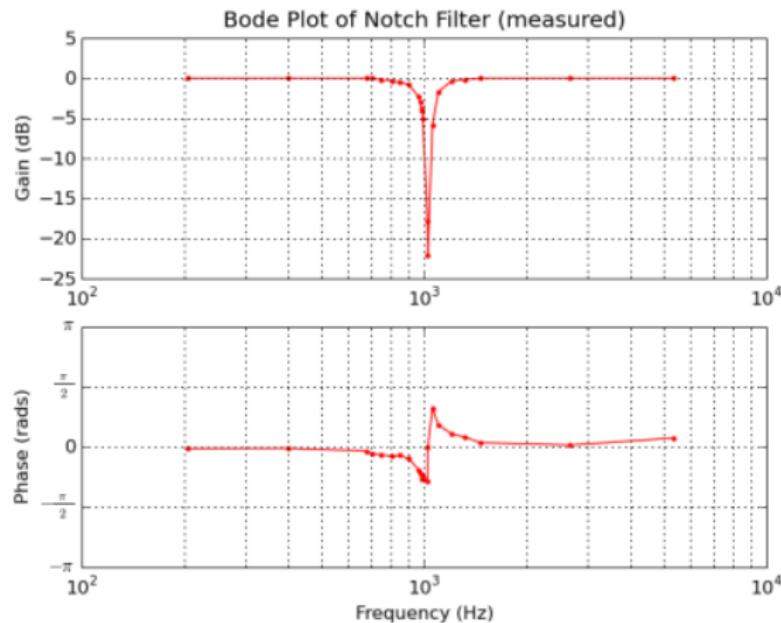
Magnitude and Phase Data



Frequency Sweeping

Magnitude and Phase Data

Magnitude and Phase Data for a Notch Filter



This type of Magnitude-Phase graph is called a Bode Plot

Frequency Sweeping

Magnitude and Phase Data

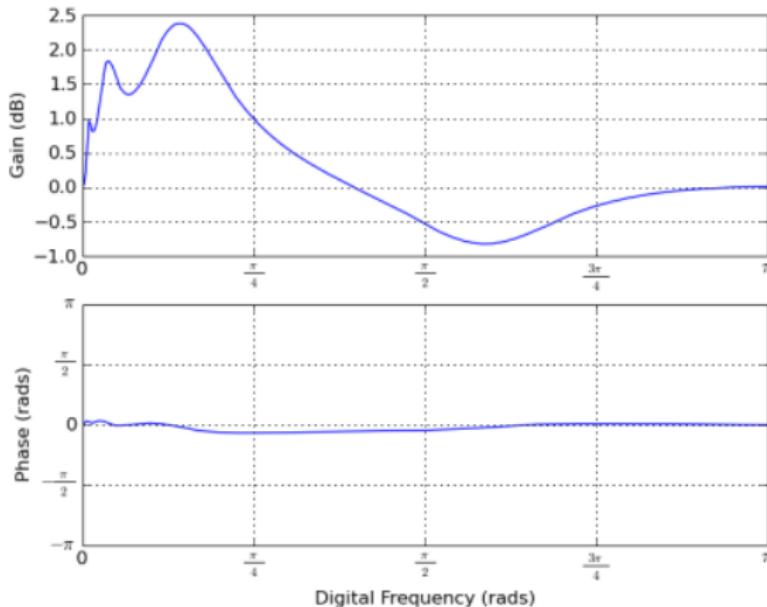


Figure: Data From a Graphic Equalizer

No Model is Required to understand the system.

Obtaining Frequency Response Data

Experimental Methods: Shake Tables

Figure: Shake Table

Figure: Shake Table with Linear Driver

Obtaining Frequency Response Data

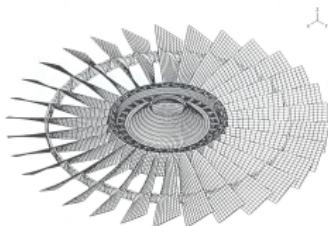
Finite-Element Modeling

For structures and rigid bodies.

- Dynamics are Partial-Differential Equations
 - ▶ Elasticity
- We can derive the model, but it would be too complicated.

We must rely on **Simulation**.

- Simulate a sinusoidal input
 - ▶ Record output displacement
- Resulting model is only an approximation.



Obtaining Frequency Response Data

Finite-Element Modeling

Figure: Satellite Frequency Response Analysis using NASTRAN

Summary

What have we learned today?

Introduction to the Frequency Domain

- Life without Newton
 - ▶ “Who needs a model, anyway?”
- Black Boxes.

Frequency Response

- Predicting Frequency Response
- Using Frequency Response Data
- Bode Plots

Next Lecture: The Bode Plot

Obtaining Frequency Response Data

Experimental Methods: Circuit Sweeping

Figure: Frequency Response Analysis in the Power Industry (Ad)