Modern Control Systems

Matthew M. Peet
Illinois Institute of Technology

Lecture 17: Operators on Signal Space

Now we are ready to define the Laplace transform.

Recall: $L_2(-\infty,\infty)$ is the space of functions with inner product given by

$$\langle u, y\rangle_{L_2} = \int_{-\infty}^{\infty} u(t)^* y(t) dt$$

Now, we propose a new signal space in the frequency domain: $\hat{L}_2$

Definition 1.

$\hat{L}_2$ is the inner-product space of functions $\hat{f} : \mathbb{R} \to \mathbb{C}^n$ with form $\hat{f}(\imath\omega)$ and inner product

$$\langle \hat{u}, \hat{y}\rangle_{\hat{L}_2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\imath\omega)^* \hat{v}(\imath\omega) d\omega$$

$\hat{L}_2$ inherits the norm

$$\|\hat{u}\|_{\hat{L}_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|^2 d\omega$$

Note: The use of "$\hat{\cdot}$" notation will refer to frequency-domain spaces, signals and operators.

The Fourier Transform

Definition 2.

For any function $u : \imath\mathbb{R} \to \mathbb{R}^n$, we define the **Fourier Transform** of u: ϕu by

$$\hat{u} = (\phi u)(\imath\omega) = \int_{-\infty}^{\infty} u(t) e^{-\imath\omega t} dt$$

- Note that we neglected the signal space
- On L_2, we have $\phi u(\imath\omega) = \langle u, e^{-\imath\omega t} \rangle_{L_2}$

Theorem 3.

- *If $u \in L_1$, the $(\phi u)(\imath\omega)$ is well defined for all $\omega \in \mathbb{R}$*
- *If $u \in L_2$, then*

$$\lim_{T \to \infty} \int_{-T}^{T} u(t) e^{-\imath\omega t} dt$$

  exists for almost all ω.

  - *When limit does not exist, define $(\phi u)(\imath\omega) = 0$*

The "Inverse Fourier Transform"

Note: We have not shown that ϕ has an inverse (or any other properties).

Definition 4.

Given a function $\hat{u} : \mathbb{R} \to \mathbb{C}^n$, we propose the operator ϕ^{-1}

$$(\phi^{-1}\hat{u})(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(\imath\omega)e^{\imath\omega t}d\omega$$

- In this case $(\phi^{-1}\hat{u})(t) = \langle \hat{u}, e^{\imath\omega t}\rangle_{\hat{L}_2}$
- If $\hat{u} \in \hat{L}_2(\imath\mathbb{R})$, then $(\phi^{-1}u)(t)$ exists for almost all t.

Invertible?

- If $(\phi u)(\imath\omega)$ exists for almost all ω, then
$$(\phi^{-1}\phi u)(t) = u(t)$$
for almost all t.
- If $(\phi^{-1}\hat{u})(t)$ exists for almost all t, then
$$(\phi\phi^{-1}\hat{u})(\imath\omega) = u(\imath\omega)$$
for almost all ω.

The Plancherel Theorem

The Fourier Transform and its inverse are Unitary.

Theorem 5.

1. $\phi : L_2(-\infty, \infty) \to \hat{L}_2(\imath\mathbb{R})$ *and*
$$\langle u, v \rangle_{L_2} = \langle \phi u, \phi v \rangle_{\hat{L}_2} \qquad \textit{for all} \quad u, v \in L_2.$$
2. $\phi^{-1} : \hat{L}_2(\imath\mathbb{R}) \to L_2(-\infty, \infty)$ *and*
$$\langle \hat{u}, \hat{v} \rangle_{\hat{L}_2} = \langle \phi^{-1} \hat{u}, \phi^{-1} \hat{v} \rangle_{L_2} \qquad \textit{for all} \quad \hat{u}, \hat{v} \in \hat{L}_2.$$

Unitary because
$$\langle u, \phi^* \phi v \rangle = \langle \phi u, \phi v \rangle = \langle u, v \rangle$$
for all u, v, which implies $\phi^* \phi = I$.

- Show that $\phi^{-1} = \phi^*$ is the Inverse Fourier Transform

Now we know L_2 and $\hat{L}_2$ are isomorphic.

The Fourier System

Let $y = Gu$.

- Then $y = \phi^{-1}\phi G\phi^{-1}\phi u$
- OR $\hat{y} = (\phi G\phi^{-1})\hat{u}$
- We'll return to $\phi G\phi^{-1}$ shortly

Because L_2 and $\hat{L}_2$ are isomorphic, $\hat{L}_2$ are the coordinates of u in the Fourier basis.

- $\hat{L}_2(\imath\omega)$ is the coordinate of basis $e^{-\imath\omega t}$.

The problem with operators on $\hat{L}_2$ is they are not always **Causal**.

Laplace Transform

Now consider the space $L_2[0,\infty)$.

Definition 6.

Given $u \in L_2[0,\infty)$, the Laplace Transform of u is $\hat{u} = \Lambda u$, where

$$\hat{u}(s) = (\Lambda u)(s) = \lim_{T\to\infty} \int_0^T u(t)e^{-st}dt$$

if this limit exists.

Note that for $u \in L_2[0,\infty)$, $\Lambda u = \phi u$.

- Laplace transform acts on a subspace of $L_2(-\infty,\infty)$
- Laplace and Fourier transforms coincide on the imaginary axis.
- Thus the image of the Laplace transform is "smaller" than the image of ϕ.
  - Speaking of which: What is the image?
  - Its a bit more complicated.....

Analytic Functions

Let $u \in L_2[0, \infty)$. Suppose $\text{Re}(s) > 0$.

- Then $e^{-st} \in L_2$ - a basis function
- Then
$$\hat{u}(s) = (\Lambda u)(s) = \int_0^\infty e^{-st} u(t) dt = \langle e^{-st}, u \rangle_{L_2} < \infty$$
- $(\Lambda u)(s)$ is well-defined everywhere in the right-half-plane (RHP).

Definition 7.

In complex analysis, a function is **analytic** if it is continuous and bounded.

- More generally, a function is analytic if the Taylor series converges everywhere in the domain.
- $\text{image}\,\Lambda$ is a subset of analytic functions bounded on the right-half-plane.
  - Note we didn't prove continuous.

To the point

Definition 8.

A function $\hat{u} : \bar{\mathbb{C}}^+ \to \mathbb{C}^n$ is in H_2 if

1. $\hat{u}(s)$ is analytic on the Open RHP (denoted $\mathbb{C}^+$)
2. For almost every real ω,

$$\lim_{\sigma \to 0^+} \hat{u}(\sigma + \imath\omega) = \hat{u}(\imath\omega)$$

   - Which means continuous on the imaginary axis

3. $$\sup_{\sigma \geq 0} \int_{-\infty}^{\infty} \|\hat{u}(\sigma + \imath\omega)\|_2^2 < \infty$$

   - Which means bounded on every vertical line.

Maximum Modulus Principle

Theorem 9 (Maximum Modulus).

An analytic function cannot obtain its extrema in the interior of the domain.

Hence if $\hat{u}$ satisfies 1) and 2), then

$$\begin{aligned}\sup_{\sigma \geq 0} \int_{-\infty}^{\infty} \|\hat{u}(\sigma + \imath\omega)\|_2^2 &= \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|_2^2 d\omega \\ &= \|\hat{u}\|_{\hat{L}_2} = \|\phi u\|_{\hat{L}_2} \\ &= \|u\|_{L_2}\end{aligned}$$

Thus we equip H_2 with the $\hat{L}_2$ norm and inner product

$$\|\hat{u}\|_{H_2} = \int_{-\infty}^{\infty} \|\hat{u}(\imath\omega)\|_2^2 d\omega, \qquad \langle \hat{u}, \hat{y} \rangle_{H_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(\imath\omega)^* \hat{v}(\imath\omega) d\omega$$

- This is a valid inner product because H_2 is isomorphic to the image $\phi L_2[0,\infty)$, which is a subspace of $\hat{L}_2$.
- Paley-Wiener