

Spacecraft Dynamics and Control

Matthew M. Peet
Arizona State University

Lecture 13: The Effect of a Non-Spherical Earth

Introduction

In this Lecture, you will learn:

The Non-Spherical Earth

- The gravitational potential
- Expression in the R-T-N frame
- Perturbations
 - ▶ Periodic
 - ▶ Secular

Mission Planning

- Sun-Synchronous Orbits
- Frozen Orbits
- Critical Inclination

Recall The Perturbation Equations

$$\vec{F}_{disturbance} = R\hat{e}_R + T\hat{e}_T + N\hat{e}_N$$

Semi-major Axis

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}} [eR \sin f + T(1+e \cos f)] \quad \tan \frac{E_{ecc}}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2}$$

Inclination:

$$\frac{d}{dt} i = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \cos(\omega + f)}{1+e \cos f}$$

Eccentricity:

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i (1+e \cos f)}$$

RAAN:

$$\dot{\omega} = -\dot{\Omega} \cos i + \sqrt{\frac{a(1-e^2)}{e^2 \mu}} \left(-R \cos f + T \frac{(2+e \cos f) \sin f}{1+e \cos f} \right)$$

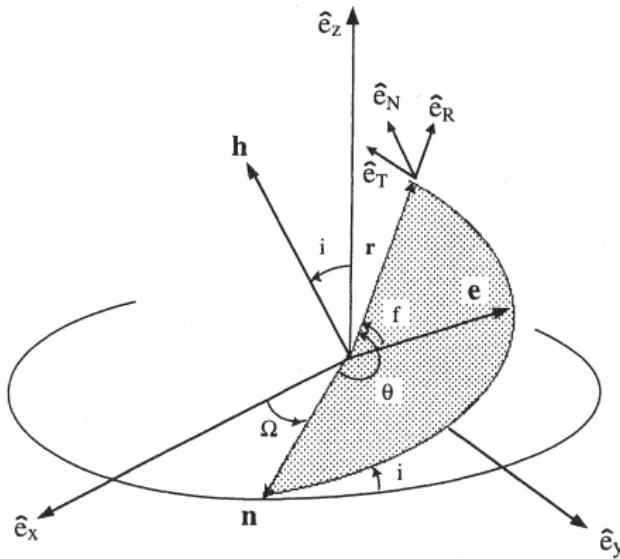
Recall

Satellite-Normal Coordinate System

$$\vec{F} = N\hat{e}_N + R\hat{e}_R + T\hat{e}_T$$

Satellite-Normal CS (R-T-N):

- \hat{e}_R points along the earth → satellite vector.
- \hat{e}_N points in the direction of \vec{h}
- \hat{e}_T is defined by the RHR
 - ▶ $\hat{e}_T \cdot v > 0$.



The Non-spherical Earth

The Spherical Earth

Recall that gravity for a point mass is

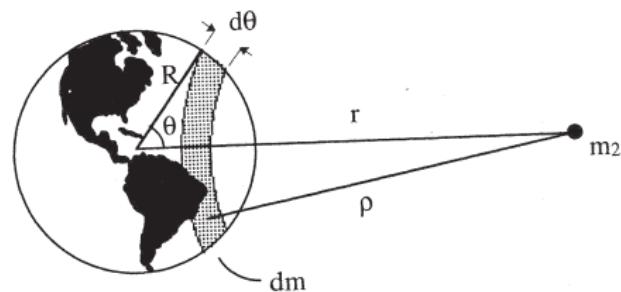
$$\vec{F} = -\mu \frac{\vec{r}}{\|\vec{r}\|^2}$$

Gravity force derives from the potential field.

$$\vec{F} = \nabla U$$

To find U , we integrate

$$dU = -\frac{2\pi R^2 G \sigma m_2 \sin \theta}{\rho} d\theta$$



For a uniform spherical mass,

- There is symmetry about the line \vec{r}_{12} .
- The point-mass approximation holds.

The Non-spherical Earth

A Distorted Potential Field

For a spherical earth, dU is symmetric

$$dU = -\frac{2\pi R^2 G \sigma m_2 \sin \theta}{\rho} d\theta$$

The actual gravity field

- Is not precisely spherical.
- density varies throughout the earth.

The result is a distorted potential field.

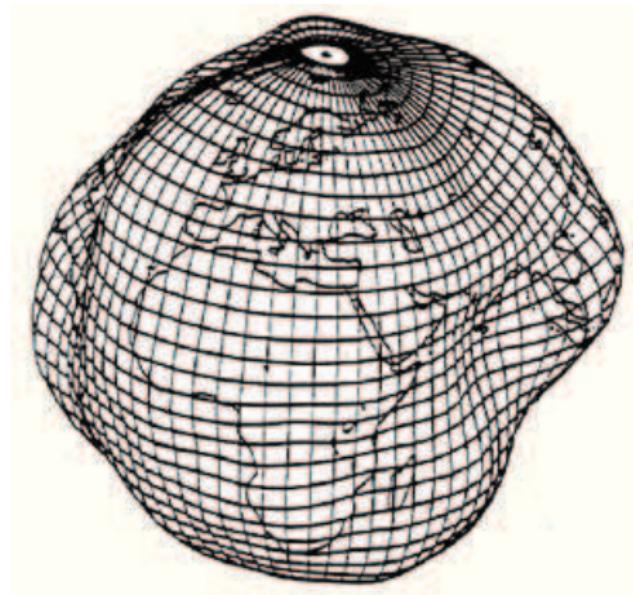


Figure: The geoid, 15000:1 scale

The Non-spherical Earth

A Distorted Potential Field

Socrates: So how do we derive the potential field?

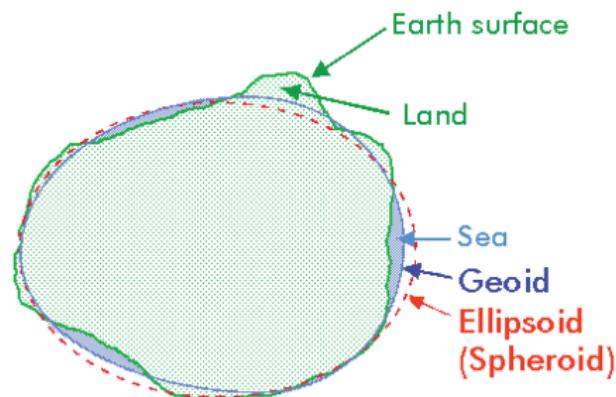
Tycho Brahe: We measure it!!!

Definition 1.

Physical Geodesy is the study of the gravitational potential field of the earth.

Definition 2.

The **Geoid** is equipotential surface which coincides with the surface of the ocean.



NASA's Geodesy Video

Development of Geodesy

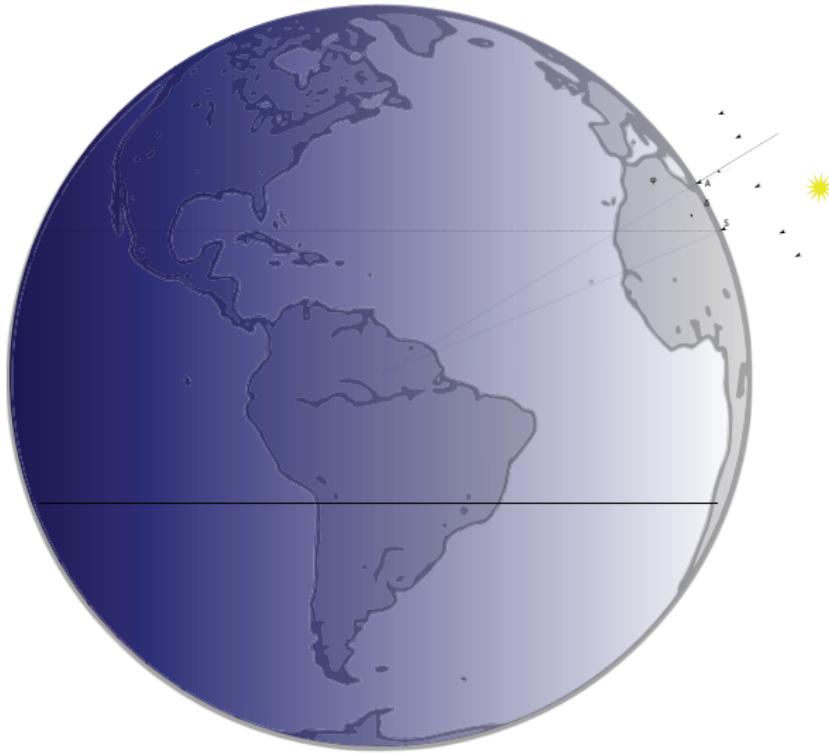
Eratosthenes of Cyrene(276-195 BC)

The first measurements of the earth were made by Eratosthenes

- Third Librarian of Library of Alexandria (240BC).
- Invented “Geography”
- Invented Latitude and Longitude
- Measured the circumference of the earth.
 - ▶ The difference in angle between high noon at two points on the earth.
 - ▶ Measured using deep wells
- May have starved to death.



Geometry of Eratosthenes



The Non-spherical Earth

A Distorted Potential Field

Question: So how do we measure the potential field of the earth?

LAGEOS: Laser Geodynamics Satellites

1. Precisely measure the orbit of a satellite as it orbits the earth
2. Account for drag, third-body dynamics, etc.
3. Remaining perturbation must be caused by gravitational potential

The orbits of the LAGEOS satellites are measured precisely by laser reflection.

Note: Only measures potential along path of the orbit.

- We must observe for a long time to get comprehensive data.

$$a = 12,278\text{ km}, \quad i = 109.8^\circ, 52.6^\circ,$$

Launch dates: 1976, 1992



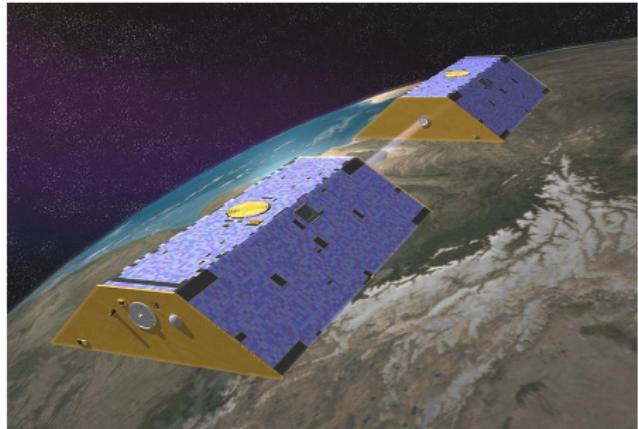
The Non-spherical Earth

Measuring satellite positions from earth is inaccurate.

- Atmospheric Distortion

GRACE (2002):

1. Measure the relative position of two adjacent satellites
2. Relative motion yields gradient of the potential field



Less fancy methods:

- Survey markers
- Altimetry
- Ocean level variation

Data from GRACE

The Non-spherical Earth

Question: So what is $U(\vec{r})$?

Response: Too much data to write as a function.

In order to be useful, we match the data to a few basis functions.

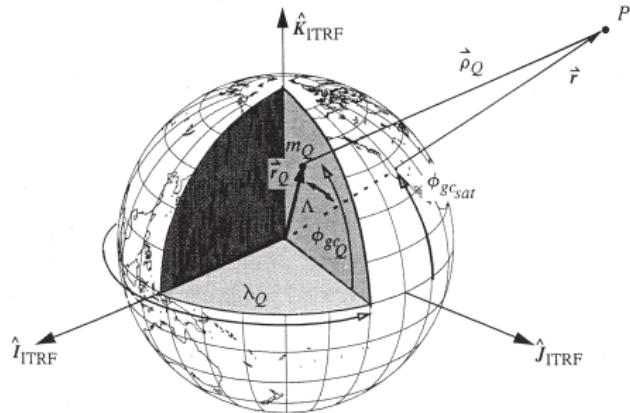
Coordinates: Express position using

ϕ_{gc} , λ , r .

- ϕ_{gc} is declination from equatorial plane.
- r is radius
- λ is right ascension, measured from Greenwich meridian.

We will have a function of form

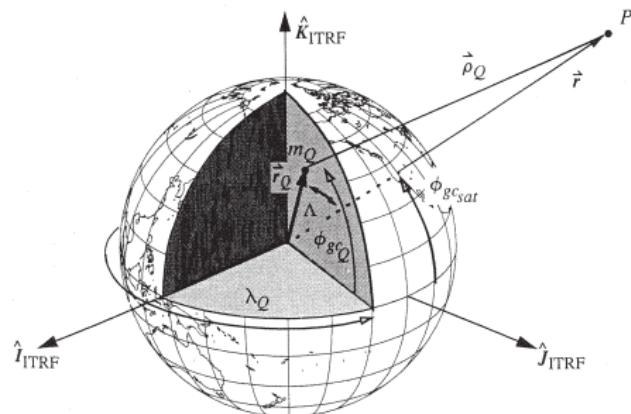
$$U(\phi_{gc}, \lambda, r)$$



The Harmonics

The potential has the form

$$U(\phi_{gc}, \lambda, r) = \frac{\mu}{r} + U_{zonal}(r, \phi_{gc}) \\ + U_{sectorial}(r, \lambda) \\ + U_{tesseral}(r, \phi_{gc}, \lambda)$$

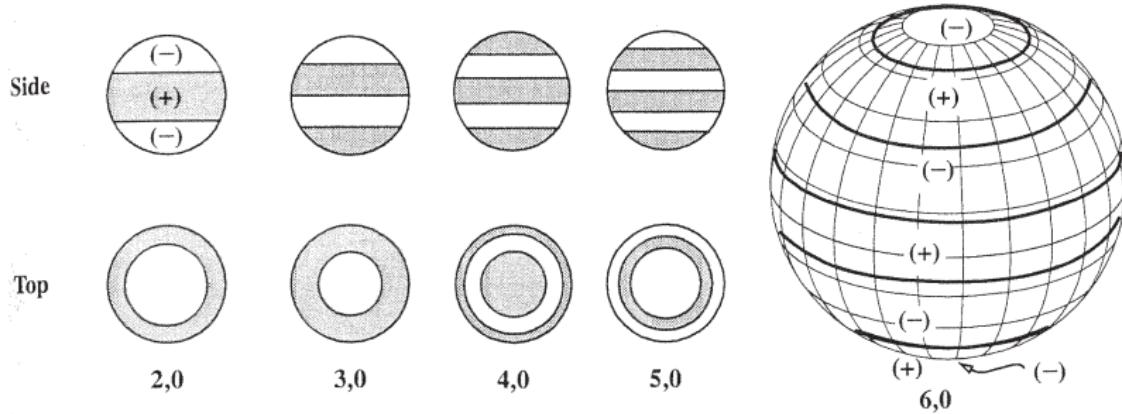


Actually, $U_{sectorial}$ varies with ϕ_{gc} , but not very actively.

The Zonal Harmonics

Zonal Harmonics: These have the form

$$U_{zonal}(r, \phi_{gc}) = \frac{\mu}{r} \sum_{i=2}^{\infty} J_i \left(\frac{R_e}{r} \right)^i P_i(\phi_{gc})$$



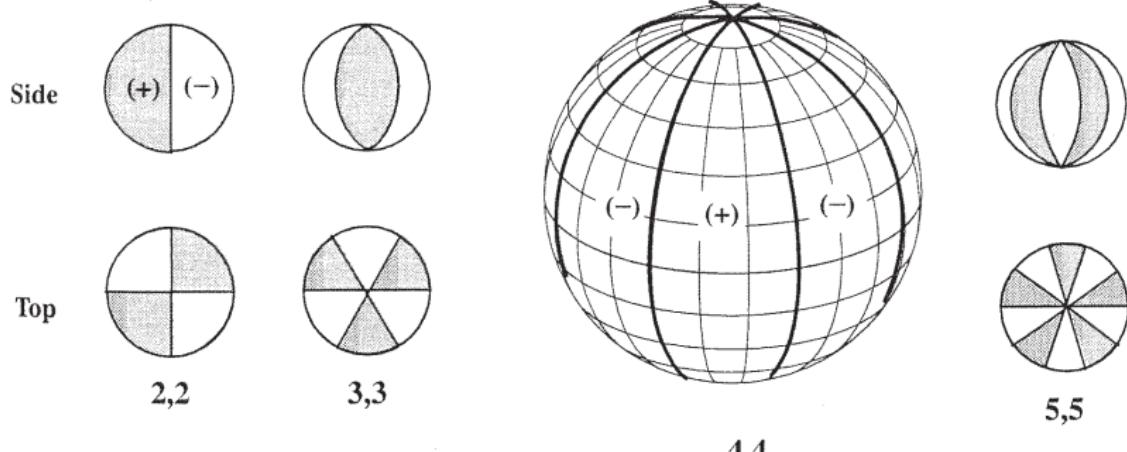
- R_e is the earth radius
- P_i are the Legendre Polynomials
- The J_i are determined by the Geodesy data!

Zonal harmonics vary only with latitude.

The Sectorial Harmonics

Sectorial Harmonics: These have the form

$$U_{sect}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i=2}^{\infty} (C_{i,sect} \cos(i\lambda) + S_{i,sect} \sin(i\lambda)) \left(\frac{R_e}{r}\right)^i \cos(\phi_{gc})^i$$



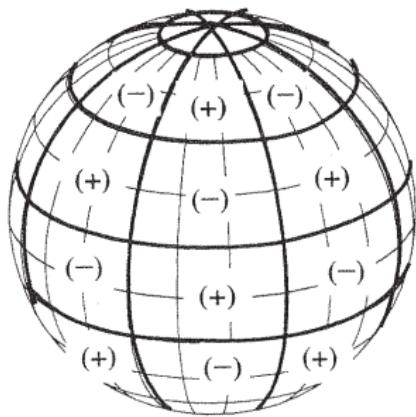
- Divides globe into slices by longitude.
- Varies with ϕ_{gc} , but $P_{i,i}(\phi_{gc})$ is uniformly positive.
- The $C_{i,sectorial}$ and $S_{i,sectorial}$ are also determined by the Geodesy data!

Sectorial harmonics vary only with longitude.

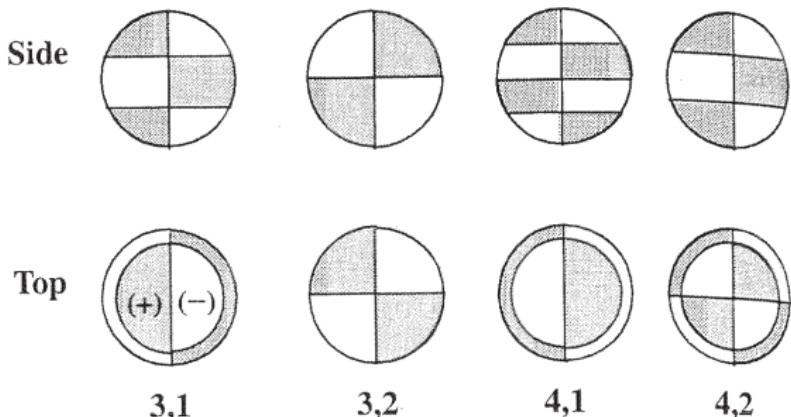
The Tesseral Harmonics

These have the form

$$U_{tesseral}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i,j=2}^{\infty} (C_{i,j} \cos(i\lambda) + S_{i,j} \sin(i\lambda)) \left(\frac{R_e}{r}\right)^i P_{i,j}(\phi_{gc})$$



4,3



- Divides globe into slices by longitude and latitude.
- The $C_{i,j}$ and $S_{i,j}$ are also determined by the Geodesy data!

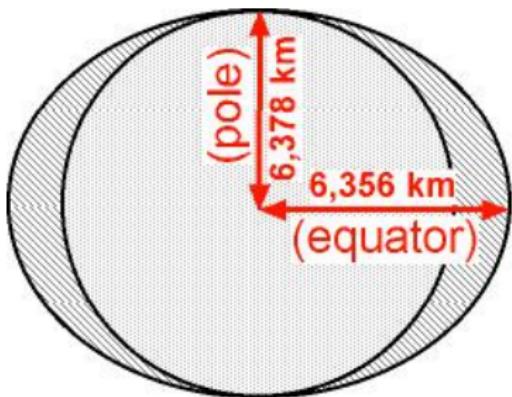
The J2 Perturbation

For simplicity, we ignore all harmonics except the first zonal harmonic.

$$\Delta U_{J2}(r, \phi_{gc}) = \frac{\mu}{r} \left(J_2 \left(\frac{R_e}{r} \right)^2 \left[\frac{3}{2} \sin^2(\phi_{gc}) - \frac{1}{2} \right] \right)$$

This corresponds to a single band about the equator.

- The earth is 21 km wider than it is tall.
- A flattening ratio of $\frac{1}{300}$.
- $J_2 = .0010826$
- $J_3 = .000002532$
- $J_4 = .000001620$



The J2 Perturbation

Defined in the Wrong Coordinate System

$$\Delta U_{J2}(r, \phi_{gc}) = \frac{\mu}{r} \left(J_2 \left(\frac{R_e}{r} \right)^2 \left[\frac{3}{2} \sin^2(\phi_{gc}) - \frac{1}{2} \right] \right)$$

- Obviously Expressed in the ECI Frame
- Since $\sin \phi_{gc} = \frac{z}{r}$,

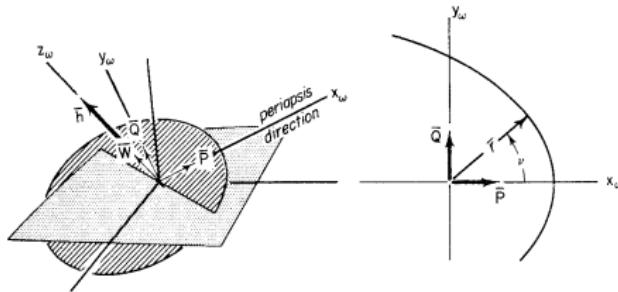
$$U_{J2}(r, \phi_{gc}) = \frac{\mu}{r} \frac{J_2}{2} \left(\frac{R_e}{r} \right)^2 \left[\frac{3z^2}{r^2} - 1 \right]$$

We now calculate the perturbation force as

$$\begin{aligned}\vec{F} &= \frac{\partial U_{J2}}{\partial r} \hat{e}_R + \frac{\partial U_{J2}}{\partial z} \hat{e}_z \\ &= \mu J_2 R_e^2 \left[\frac{3z}{r^5} \hat{e}_z + \left(\frac{3}{2r^4} - \frac{15z^2}{2r^6} \right) \hat{e}_R \right]\end{aligned}$$

But to use our perturbation equations, we need a force expressed in the R-T-N frame.

Recall: Perifocal to ECI Transformation



To convert a PQW vector to ECI, we use

$$\vec{r}_{ECI} = R_3(\Omega)R_1(i)R_3(\omega)\vec{r}_{PQW} = R_{PQW \rightarrow ECI}\vec{r}_{PQW}$$

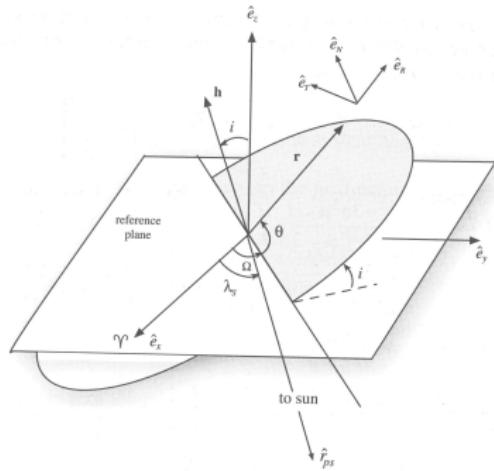
$$R_{PQW \rightarrow ECI} = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i & -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i & \sin \Omega \sin i \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i & -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i & -\cos \Omega \sin i \\ \sin \omega \sin i & \cos \omega \sin i & \cos i \end{bmatrix}$$

The R-T-N to ECI Transformation

An additional rotation gives us the R-T-N frame.

$$R_{RTN \rightarrow ECI}$$

$$= \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos(\omega + f) & -\sin(\omega + f) & 0 \\ \sin(\omega + f) & \cos(\omega + f) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_{RTN \rightarrow ECI} =$$

$$\begin{bmatrix} \cos \Omega \cos \theta - \sin \Omega \sin \theta \cos i & -\cos \Omega \sin \theta - \sin \Omega \cos \theta \cos i & \sin \Omega \sin i \\ \sin \Omega \cos \theta + \cos \Omega \sin \theta \cos i & -\sin \Omega \sin \theta + \cos \Omega \cos \theta \cos i & -\cos \Omega \sin i \\ \sin \theta \sin i & \cos \theta \sin i & \cos i \end{bmatrix}$$

Where for brevity, we define $\theta = \omega + f$. This gives us the expression

$$\hat{e}_z = \sin i \sin(\omega + f) \hat{e}_R + \sin i \cos(\omega + f) \hat{e}_T + \cos i \hat{e}_N$$

Forces in the R-T-N Frame

From the rotation matrices, we have that

$$\hat{e}_z = \sin i \sin(\omega + f) \hat{e}_R + \sin i \cos(\omega + f) \hat{e}_T + \cos i \hat{e}_N$$

and since

$$z = r \sin \phi_{gc} = r \sin i \sin(\omega + f),$$

this yields the disturbing force in the R-T-N frame:

$$\begin{aligned}\vec{F} = & \frac{-3\mu J_2 R_e^2}{r^4} \left[\left(\frac{1}{2} - \frac{3 \sin^2 i \sin^2 \theta}{2} \right) \hat{e}_R \right. \\ & + \sin^2 i \sin \theta \cos \theta \hat{e}_T \\ & \left. + \sin i \sin \theta \cos i \hat{e}_N \right]\end{aligned}$$

where again, for brevity, we use $\theta = \omega + f$

The J2 Perturbation

The primary effect of J_2 is on Ω and ω .

We plug the force equations into the expressions for $\dot{\Omega}$ and $\dot{\omega}$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i (1 + e \cos f)}$$

to get

$$\dot{\Omega} = -\frac{3\mu J_2 R_e^2}{hp^3} \cos i \sin^2 \theta [1 + e \cos f]^3$$

This is the instantaneous rate of change.

- The angles θ and f will cycle from 0° to 360° over each orbit.
- We would like to know how much of that perturbation is **secular**?
- What is the average over θ ?

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{\dot{\theta}} = \frac{\dot{\Omega}}{h/r^2}$$

Averaging the J2 Perturbation

Starting with

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{h/r^2} = -3J_2 \left(\frac{R_e}{p}\right)^2 \cos i \sin^2 \theta [1 + e \cos(\theta - \omega)]$$

Then the average change over an orbit is

$$\frac{d\Omega}{d\theta} \Big|_{AV} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Omega}{d\theta} d\theta = -\frac{3J_2}{2\pi} \left(\frac{R_e}{p}\right)^2 \cos i \int_0^{2\pi} \sin^2 \theta [1 + e \cos(\theta - \omega)] d\theta$$

Now we use $\cos(\theta - \omega) = \cos \omega \cos \theta + \sin \omega \sin \theta$ to get

$$\begin{aligned} \int_0^{2\pi} \sin^2 \theta [1 + e \cos(\theta - \omega)] d\theta &= \int_0^{2\pi} \sin^2 \theta d\theta + e \int_0^{2\pi} \sin^2 \theta \cos(\theta - \omega) d\theta \\ &= \pi + e \cos \omega \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta + e \sin \omega \int_0^{2\pi} \sin^3 \theta d\theta \\ &= \pi + 0 + 0 = \pi \end{aligned}$$

Thus, we have

$$\frac{d\Omega}{d\theta} \Big|_{AV} = -\frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

Averaging the J2 Perturbation

Given

$$\frac{d\Omega}{d\theta} \Big|_{AV} = -\frac{3}{2} J_2 \left(\frac{R_e}{p} \right)^2 \cos i$$

we can use the fact that

$$n = \frac{d\theta}{dt} \Big|_{AV}$$

to get the final expression

$$\dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left(\frac{R_e}{p} \right)^2 \cos i$$

J2 Nodal Regression

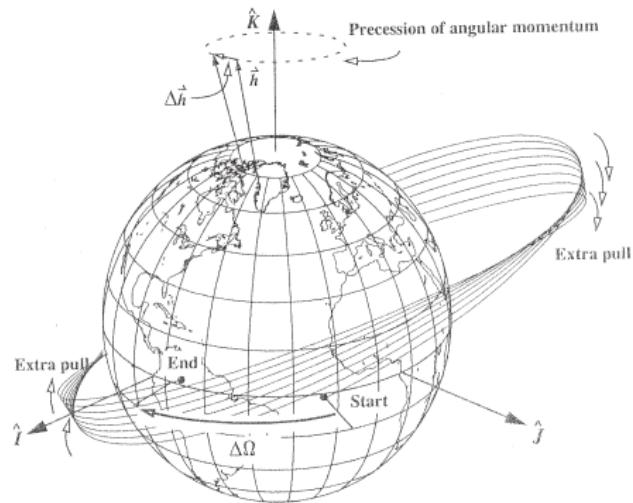
Physical Explanation

The ascending node migrates opposite the direction of flight

$$\dot{\Omega}_{J2,av} = -\frac{3}{2}nJ_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

The equatorial bulge produces extra pull
in the equatorial plane

- Creates a torque on the angular momentum vector
- Like gravity, the torque causes \vec{h} to precess.
- Only depends on inclination



J2 Nodal Regression

Magnitude

The nodal regression rate is often large. **Cannot Be Neglected!!!.**

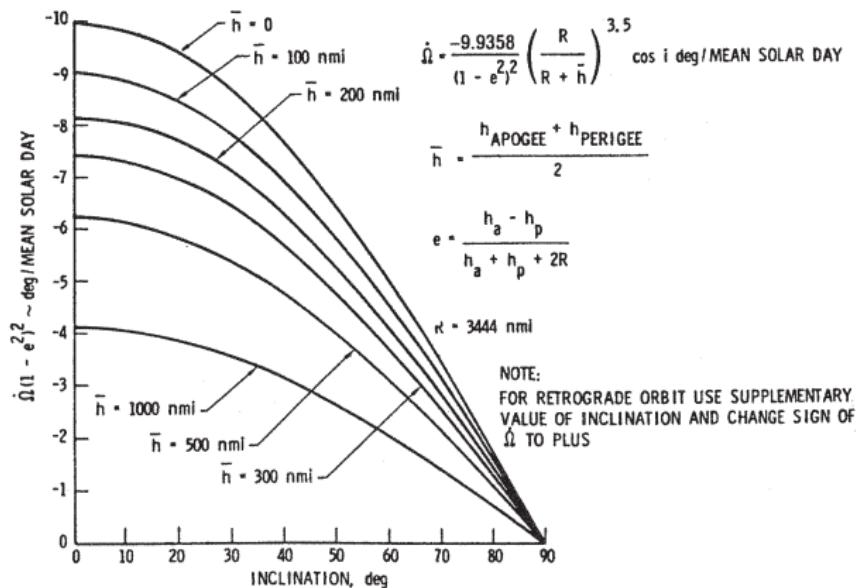


Fig. 10.2 Regression rate due to oblateness vs inclination for various values of average altitude.

Figure: Magnitude of Regression Rate vs. inclination and altitude

J2 Apsidal Rotation

Recall the Argument of Perigee Equation:

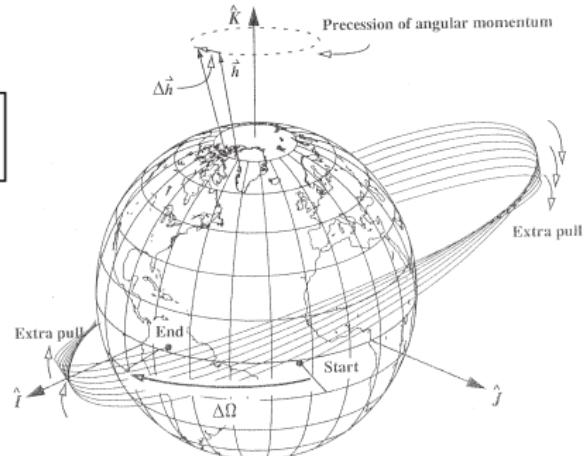
$$\dot{\omega} = -\dot{\Omega} \cos i + \sqrt{\frac{a(1-e^2)}{e^2 \mu}} \left(-R \cos f + T \frac{(2+e \cos f) \sin f}{1+e \cos f} \right)$$

The argument of perigee (ω) is linked to RAAN (Ω). The average value is

$$\frac{d\omega}{d\theta} = -\frac{d\Omega}{d\theta} \cos i + \frac{3J_2 R^2}{2p^2} \left[1 - \frac{3}{2} \sin^2 i \right]$$

where

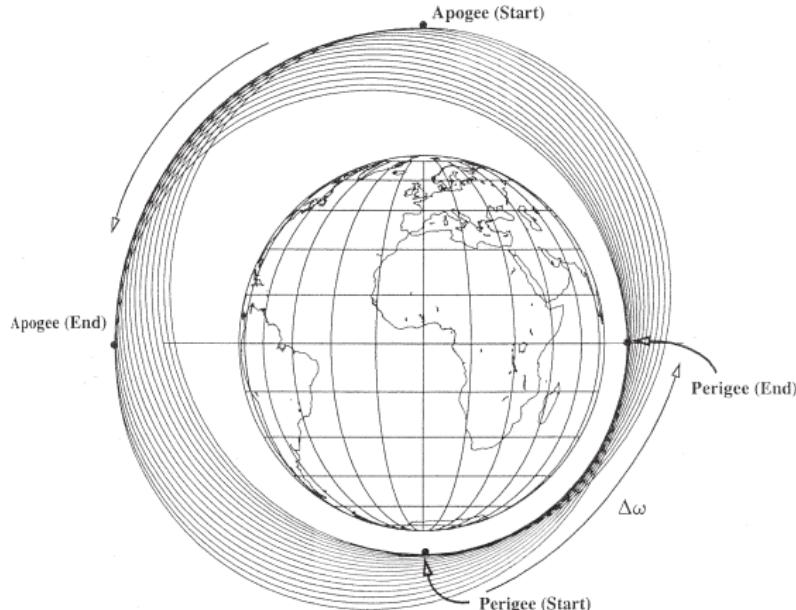
$$\begin{aligned}\frac{d\Omega}{d\theta} \cos i &= -\frac{3}{2} J_2 \left(\frac{R_e}{p} \right)^2 \cos^2 i \\ &= -\frac{3}{2} J_2 \left(\frac{R_e}{p} \right)^2 (1 - \sin^2 i)\end{aligned}$$



J2 Apsidal Rotation

Similar to nodal regression, but perigee moves forward or backward, depending on inclination.

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p} \right)^2 \left[2 - \frac{5}{2} \sin^2 i \right]$$



J2 Apsidal Rotation

Magnitude

The apsidal rotation rate is often large.

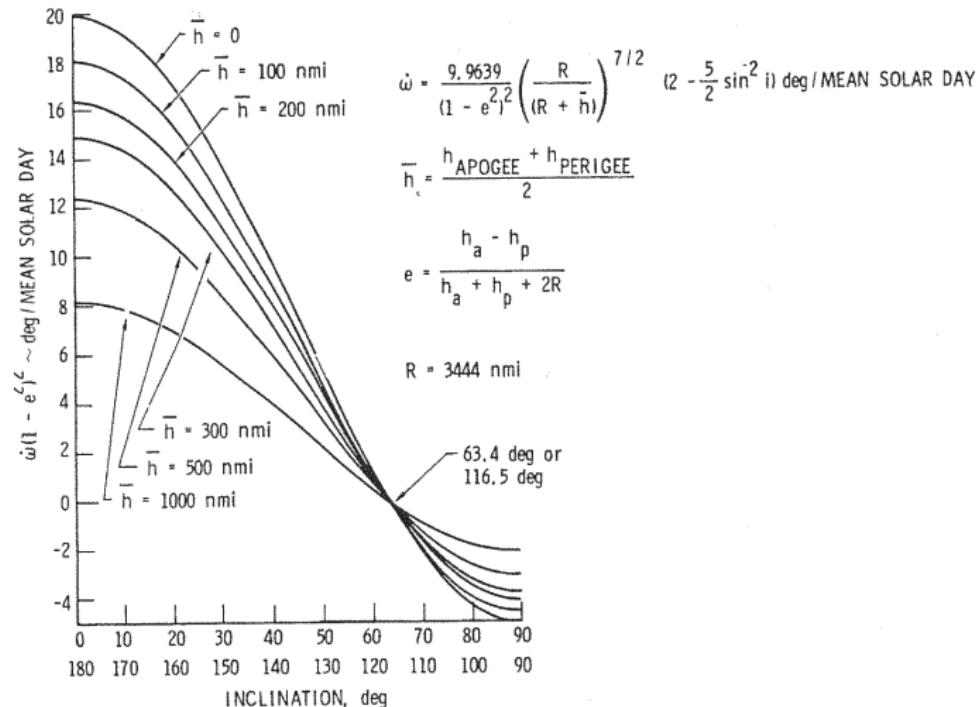


Figure: Magnitude of Regression Rate vs. inclination and altitude

J2 Effect

Other Elements: Eccentricity

The J_2 effect on other elements is usually minor. $\dot{a} \cong 0$.

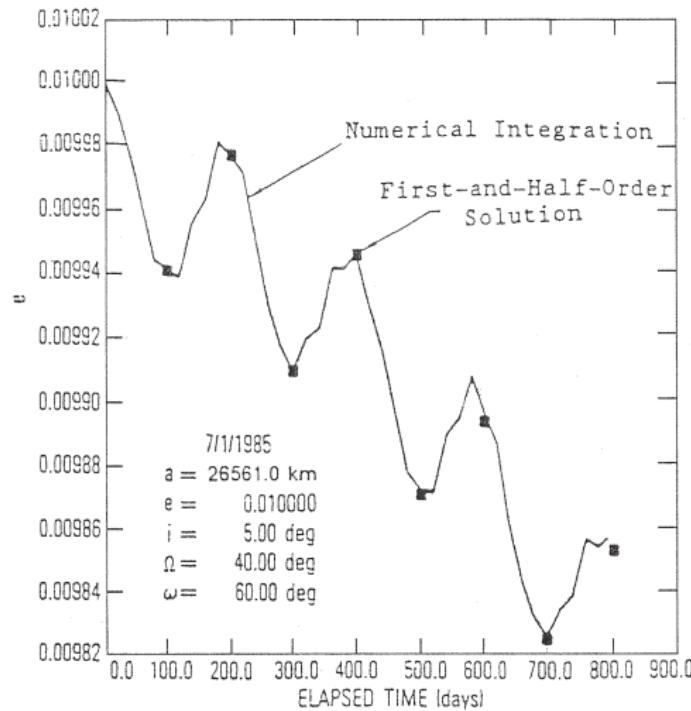


Figure: Eccentricity Change for Low-Inclination Orbit

J2 Effect

Other Elements: Eccentricity

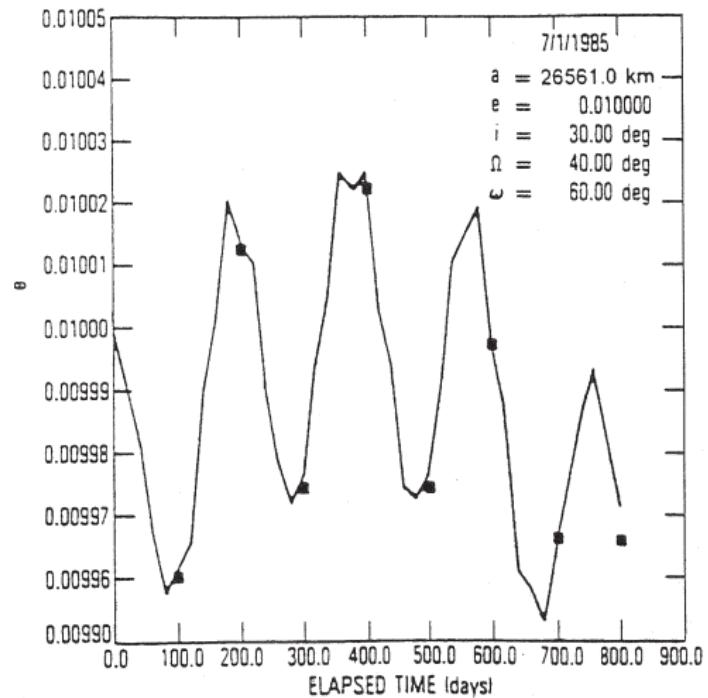


Figure: Eccentricity Change for Moderate-Inclination Orbit

J2 Effect

Other Elements: Eccentricity

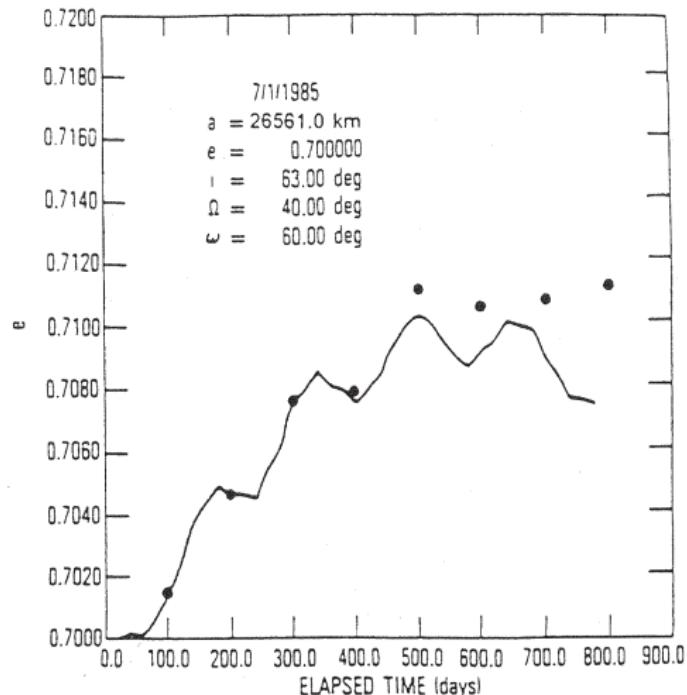


Figure: Eccentricity Change for High-Inclination Orbit

J2 Effect

Other Elements: Inclination

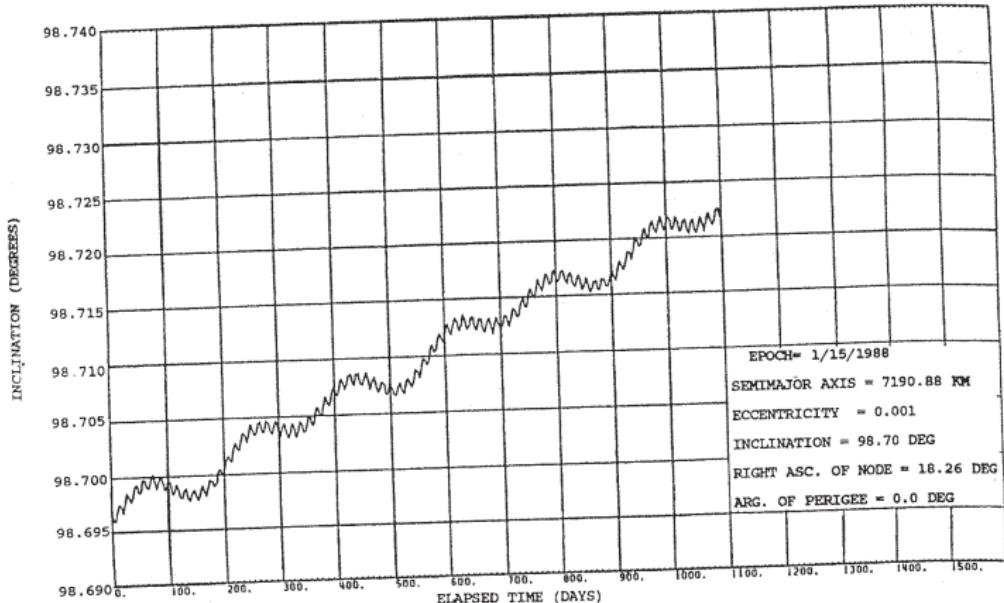


Fig. 10.6 Inclination variation without correction (5:30 orbit).

Figure: Inclination Change for Eccentric and Circular Orbits

J_2 Special Orbits

Critical Inclination

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p} \right)^2 \left[2 - \frac{5}{2} \sin^2 i \right]$$

Definition 3.

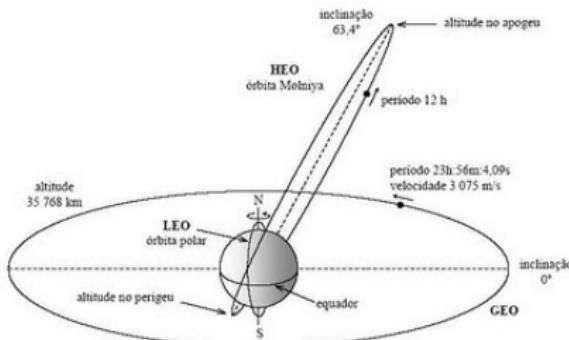
A **Critically Inclined Orbit** is one where $\dot{\omega} = 0$

For a critically inclined orbit,

$$4 - 5 \sin^2 i = 0$$

which means

$$\begin{aligned} i &= \sin^{-1} \sqrt{4/5} \\ &= 63.43^\circ \quad \text{or} \quad 116.57^\circ \end{aligned}$$



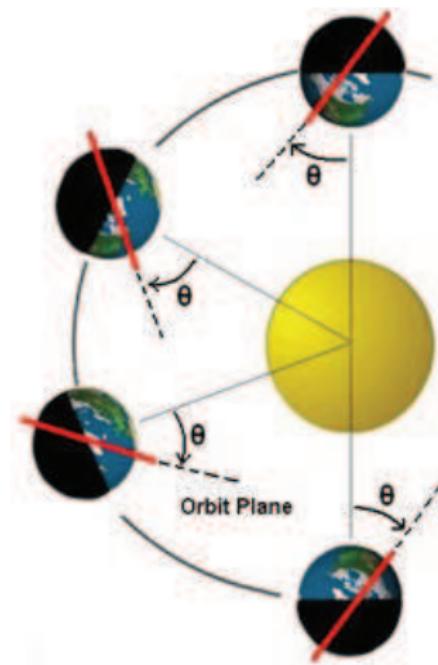
J_2 Special Orbits

Sun-Synchronous Orbit

Sun-Synchronous orbits maintain the same orientation of the orbital plane with respect to the sun.

Applications:

- Mapping
- Solar-Powered
- Shadow-evading
- Time-of-Day Apps



J_2 Special Orbits

Sun-Synchronous Orbit

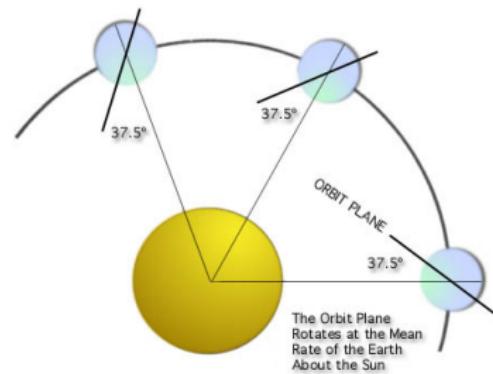
The earth rotates 360° about the sun every 365.25 days.

Definition 4.

A **Sun-Synchronous Orbit** is one where $\dot{\Omega} = .9855^\circ/day = 1.992 \cdot 10^{-7} rad/s$.

Thus

$$\cos i = -\frac{1}{1.992 \cdot 10^{-7}} \left(\frac{p}{R_e} \right)^2 \frac{2}{3nJ_2}$$

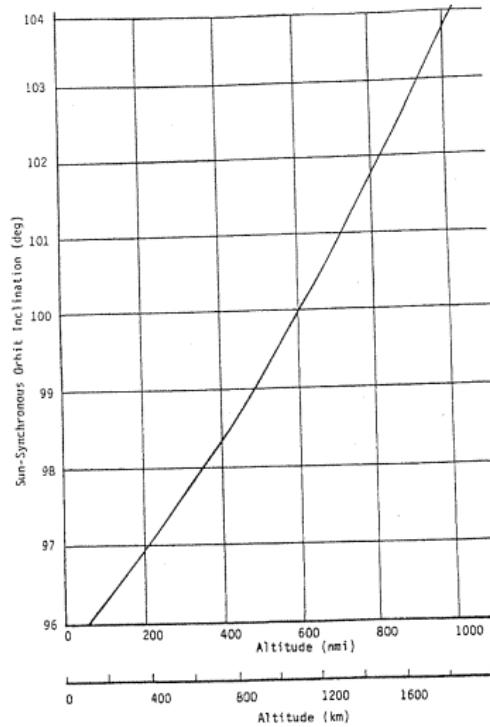


- The orbital plane rotates once every year.

J_2 Special Orbits

Sun-Synchronous Orbit

Unlike critically inclined orbits, sun-synchronous orbits depend on altitude.



Numerical Example

LANDSAT

Problem: Design a sun-synchronous orbit with $r_p = R_e + 695\text{km}$ and $r_a = R_e + 705\text{km}$.

Solution: The desired inclination for a sun-synchronous orbit is given by

$$i = \cos^{-1} \left(5.02 \cdot 10^6 \left(\frac{p}{R_e} \right)^2 \frac{2}{3nJ_2} \right)$$

For this orbit $a = R_e + 700\text{km} = 7078\text{km}$. The eccentricity is

$$e = 1 - \frac{r_p}{a} = .00071$$

Thus $p = a(1 - e^2) = 6999.65\text{km}$. $n = \sqrt{\frac{\mu}{a^3}} = .0011$. Finally, $J_2 = .0010826$. Thus the required inclination is

$$i = 1.716\text{rad} = 98.33^\circ$$

Numerical Example

Molnaya Orbit

Problem: Molnaya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 12 hours and which precesses at $\dot{\Omega} = -0.2^\circ/day$.

Solution: We can first use the period to solve for a . From

$$n = \sqrt{\frac{\mu}{a^3}} = 7.27 \cdot 10^{-5}$$

and $n = 2\pi/T = 2\text{rad}/\text{day}$ we have

$$a = \sqrt[3]{\frac{\mu}{n^2}} = 42,241\text{km}$$

Now the critical inclination for $\dot{\omega} = 0$ is $i = 63.4^\circ$ or $i = 116.6^\circ$. We choose $i = 63.4^\circ$. To achieve $\dot{\Omega} = -0.2^\circ/day$, we use

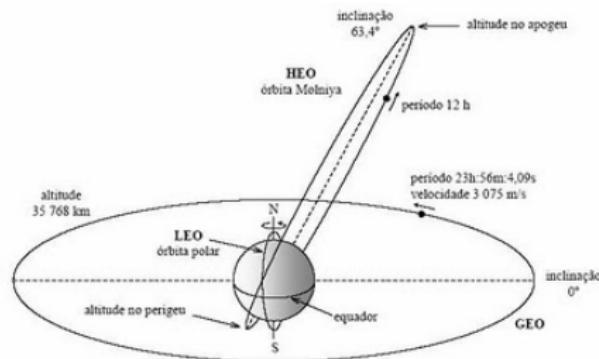
$$\dot{\Omega} = -\frac{3nJ_2R_e^2}{2a^2(1-e^2)^2} \cos i$$

Numerical Example

Molnaya Orbit, continued

Since a is already fixed, we must use e . We can solve for e as

$$e = \sqrt{1 - \sqrt{-\frac{3nJ_2R_e^2}{2\dot{\Omega}a^2} \cos i}} = .7459.$$



Note: Make sure the units of a and n match those of R_e and $\dot{\Omega}$, respectively.

Summary

This Lecture you have learned:

How to account for perturbations to Earth gravity

- Gravity Mapping
- Harmonic Functions
- J_2 Perturbation
 - ▶ Effect on Ω
 - ▶ Effect on ω
 - ▶ Minor effect (e, i)

How to design specialized orbits

- Critically - Inclined Orbit.
- Sun-Synchronous Orbit.
- Applications

Next Lecture: Interplanetary Mission Planning.