Spacecraft and Aircraft Dynamics

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Lecture 11: The Effect of a Non-Spherical Earth

Introduction

In this Lecture, you will learn:

The Non-Spherical Earth

- The gravitational potential
- Expression in the R-T-N frame
- Perturbations
  - Periodic
  - Secular

Mission Planning

- Sun-Synchronous Orbits
- Frozen Orbits
- Critical Inclination

Recall The Perturbation Equations

$$\vec{F}_{disturbance} = R\hat{e}_R + T\hat{e}_T + N\hat{e}_N$$

Semi-major Axis

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[eR\sin f + T(1+e\cos f)\right]$$

Eccentricity:

$$\tan\frac{E_{ecc}}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}$$

Inclination:

$$\frac{d}{dt}i = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\cos(\omega+f)}{1+e\cos f}$$

RAAN:

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\sin(\omega+f)}{\sin i(1+e\cos f)}$$

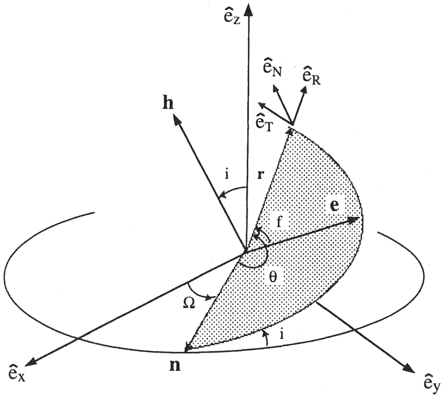
Argument of Perigee:

$$\dot{\omega} = -\dot{\Omega}\cos i + \sqrt{\frac{a(1-e^2)}{e^2\mu}}\left(-R\cos f + T\frac{(2+e\cos f)\sin f}{1+e\cos f}\right)$$

$$\vec{F} = N\hat{e}_N + R\hat{e}_R + T\hat{e}_T$$

Satellite-Normal CS (R-T-N):

- $\hat{e}_R$ points along the earth $\to$ satellite vector.
- $\hat{e}_N$ points in the direction of $\vec{h}$
- $\hat{e}_T$ is defined by the RHR
  - $\hat{e}_T \cdot v > 0.$



Recall that gravity for a point mass is

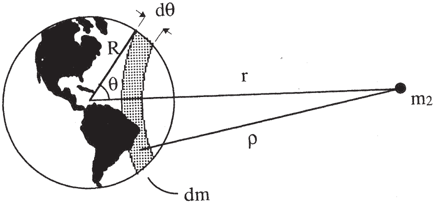
$$\vec{F} = -\mu \frac{\vec{r}}{\|\vec{r}\|^2}$$

Gravity force derives from the potential field.

$$\vec{F} = \nabla U$$

To find U, we integrate

$$dU = -\frac{2\pi R^2 G\sigma m_2 \sin\theta}{\rho} d\theta$$



For a uniform spherical mass,

- There is symmetry about the line $\vec{r}_{12}$.
- The point-mass approximation holds.

The Non-spherical Earth

A Distorted Potential Field

For a spherical earth, dU is symmetric

$$dU = -\frac{2\pi R^2 G\sigma m_2 \sin\theta}{\rho} d\theta$$

The actual gravity field

- Is not precisely spherical.
- ρ varies throughout the object.

The result is a distorted potential field.

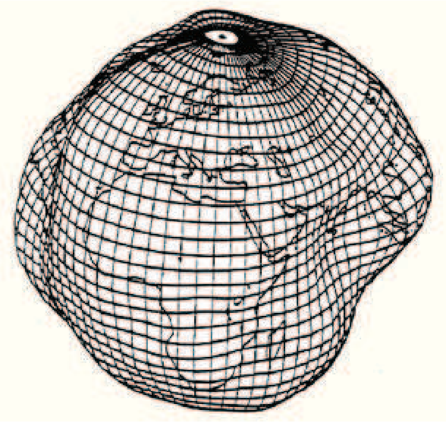


Figure: The geoid, 15000:1 scale

Socrates: So how do we derive the potential field?

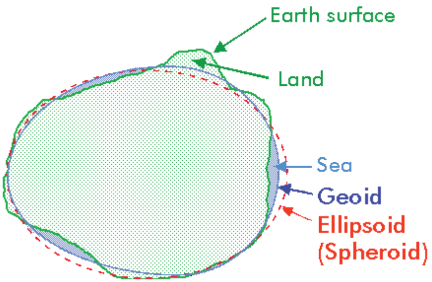
Tycho Brahe: We measure it!!!

Definition 1.

Physical Geodesy is the study of the gravitational potential field of the earth.

Definition 2.

The **Geoid** is equipotential surface which coincides with the surface of the ocean.



Question: So how do we measure the potential field of the earth?

LAGEOS: Laser Geodynamics Satellites

1. Precisely measure the orbit of a satellite as it orbits the earth
2. Account for drag, third-body dynamics, etc.
3. Remaining perturbation must be causes by gravitational potential

The orbits of the LAGEOS satellites are measured precisely by laser reflection.

Note: Only measures potential along path of the orbit.

- We must observe for a long time to get comprehensive data.

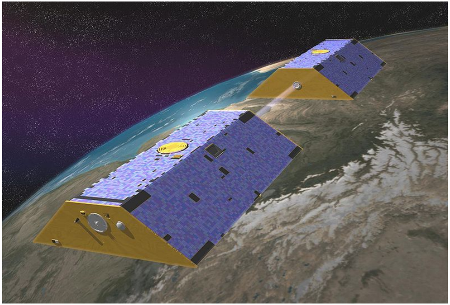
The Non-spherical Earth

Measuring satellite positions from earth is inaccurate.

- Atmospheric Distortion

GRACE:

1. Measure the relative position of two adjacent satellites
2. Relative motion yields gradient of the potential field



Less fancy methods:

- Survey markers
- Altimetry
- Ocean level variation

The Non-spherical Earth

Question: So what is $U(\vec{r})$?
Response: Too much data to write as a function.

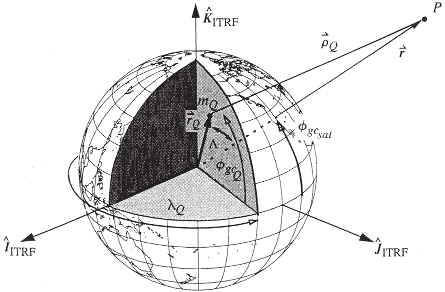
In order to be useful, we match the data to a few basis functions.

Coordinates: Express position using ϕ_{gc}, λ, r.

- ϕ_{gc} is declination from equatorial plane.
- r is radius
- λ is right ascension, measured from Greenwich meridian.

We will have a function of form

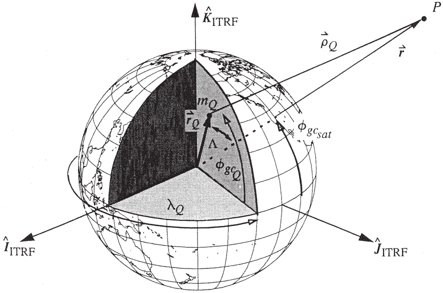
$$U(\phi_{gc}, \lambda, r)$$



The Harmonics

The potential has the form

$$U(\phi_{gc}, \lambda, r) = \frac{\mu}{r} + U_{zonal}(r, \phi_{gc}) + U_{sectorial}(r, \lambda) + U_{tesseral}(r, \phi_{gc}, \lambda)$$

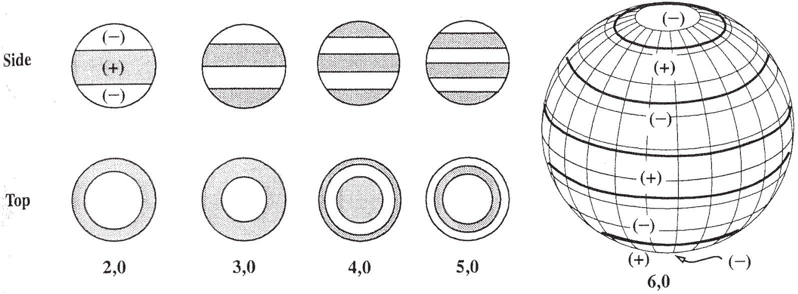


Actually, $U_{sectorial}$ varies with ϕ_{gc}, but not very actively.

The Zonal Harmonics

Zonal Harmonics: These have the form

$$U_{zonal}(r, \phi_{gc}) = \frac{\mu}{r} \sum_{i=2}^{\infty} J_i \left(\frac{R_e}{r} \right)^i P_i(\phi_{gc})$$



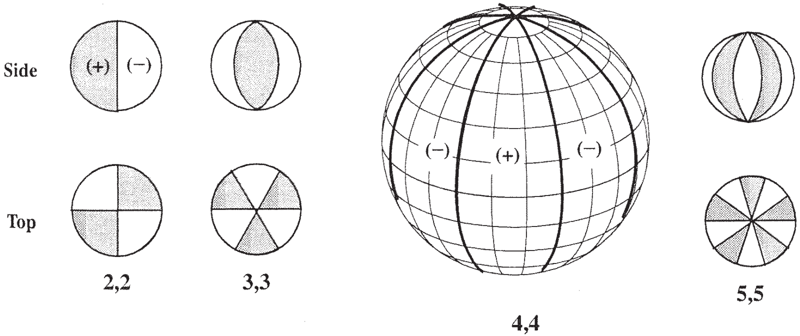
- R_e is the earth radius
- P_i are the Legendre Polynomials
- The J_i are determined by the Geodesy data!

Zonal harmonics vary only with latitude.

The Sectorial Harmonics

Sectorial Harmonics: These have the form

$$U_{sect}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i=2}^{\infty} \left(C_{i,sect} \cos(i\lambda) + S_{i,sect} \sin(i\lambda)\right) \left(\frac{R_e}{r}\right)^i \cos(\phi_{gc})^i$$



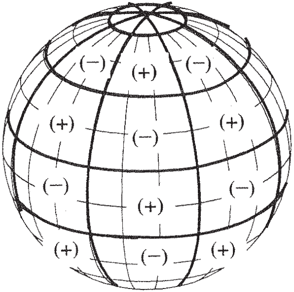
- Divides globe into slices by longitude.
- Varies with ϕ_{gc}, but $P_{i,i}(\phi_{gc})$ is uniformly positive.
- The $C_{i,sectorial}$ and $S_{i,sectorial}$ are also determined by the Geodesy data!

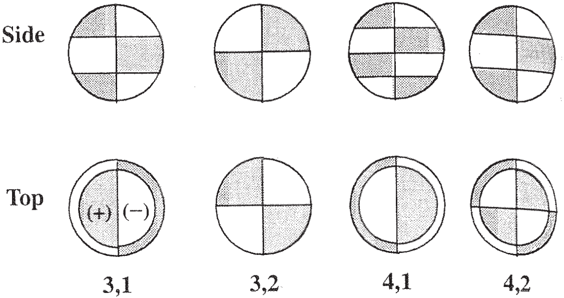
Sectorial harmonics vary only with longitude.

The Tesseral Harmonics

These have the form

$$U_{tesseral}(r,\phi_{gc},\lambda) = \frac{\mu}{r}\sum_{i,j=2}^{\infty}\left(C_{i,j}\cos(i\lambda) + S_{i,j}\sin(i\lambda)\right)\left(\frac{R_e}{r}\right)^i P_{i,j}(\phi_{gc})$$





- Divides globe into slices by longitude and latitude.
- The $C_{i,j}$ and $S_{i,j}$ are also determined by the Geodesy data!

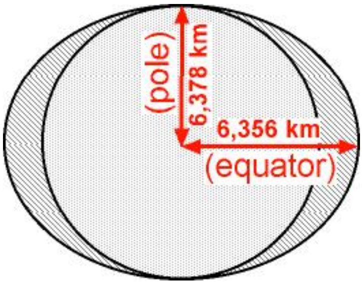
The J2 Perturbation

For simplicity, we ignore all harmonics except the first zonal harmonic.

$$\Delta U_{J2}(r, \phi_{gc}) = \frac{\mu}{r}(J_2 \left(\frac{R_e}{r}\right)^2 \left[\frac{3}{2}\sin^2(\phi_{gc}) - \frac{1}{2}\right]$$

This corresponds to a single band about the equator.

- The earth is 21 km wider than it is tall.
- A flattening ratio of $\frac{1}{300}$.
- $J_2 = .0010826$
- $J_3 = .000002532$
- $J_4 = .000001620$



The J2 Perturbation

Since $\sin\phi_{gc} = \frac{z}{r}$,

$$U_{J2}(r,\phi_{gc}) = \frac{\mu}{r}\frac{J_2}{2}\left(\frac{R_e}{r}\right)^2\left[\frac{3z^2}{r^2} - 1\right]$$

We now calculate the perturbation force as

$$\begin{aligned}\vec{F} &= \frac{\partial U_{J2}}{\partial r}\hat{e}_R + \frac{\partial U_{J2}}{\partial z}\hat{e}_z \\ &= \mu J_2 R_e^2\left[\frac{3z}{r^5}\hat{e}_z + \left(\frac{3}{2r^4} - \frac{15z^2}{2r^6}\right)\hat{e}_R\right]\end{aligned}$$

From the rotation matrices, we have that

$$\hat{e}_z = \sin i \sin(\omega + f)\hat{e}_R + \sin i \cos(\omega + f)\hat{e}_T + \cos i \hat{e}_N$$

and since

$$z = r\sin\phi_{gc} = r\sin i \sin(\omega + f)$$

Using $\theta = \omega + f$, this yields the disturbing force in the R-T-N frame:

$$\vec{F} = \frac{-3\mu J_2 R_e^2}{r^4}\left[\left(\frac{1}{2} - \frac{3\sin^2 i \sin^2\theta}{2}\right)\hat{e}_R + \sin^2 i \sin\theta\cos\theta\hat{e}_T + \sin i \sin\theta \cos i \hat{e}_N\right]$$

The J2 Perturbation

The primary effect of J_2 is on Ω and ω.
We plug the force equations into the expressions for $\dot{\Omega}$ and $\dot{\omega}$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i(1 + e\cos f)}$$

to get

$$\dot{\Omega} = -\frac{3\mu J_2 R_e^2}{hp^3} \cos i \sin^2\theta \left[1 + e\cos f\right]^3$$

If we average $\dot{\Omega}$ over an orbit, then we get the final expression

$$\dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

Likewise, we can find an expression for $\dot{\omega}$:

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$

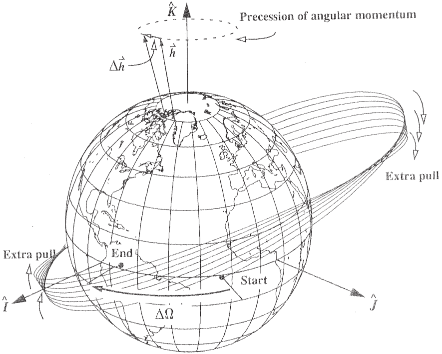
J2 Nodal Regression
Physical Explanation

The ascending node migrates opposite the direction of flight

$$\dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

The equatorial bulge produces extra pull in the equatorial plane

- Creates a torque on the angular momentum vector
- Like gravity, the torque causes $\vec{h}$ to precess.
- Only depends on inclination



The nodal regression rate is often large. **Cannot Be Neglected!!!.**

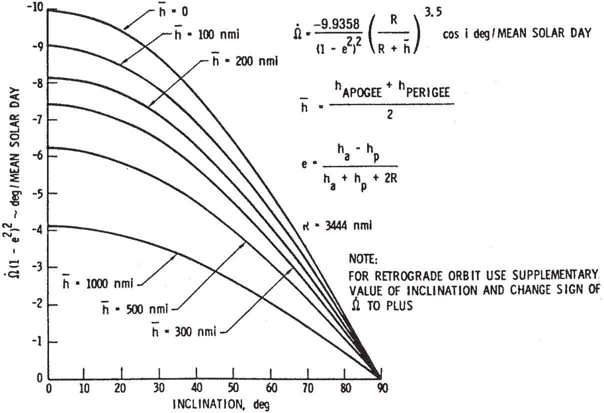


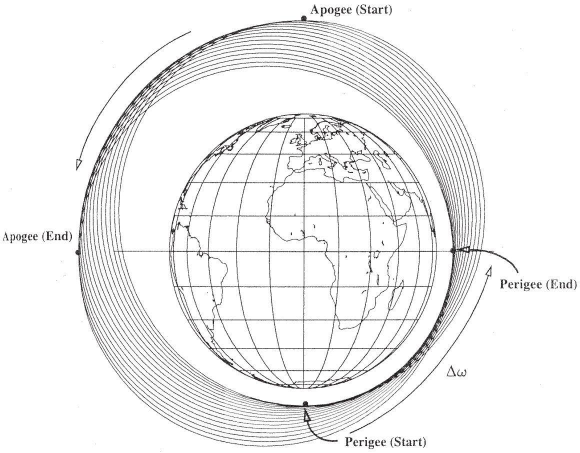
Fig. 10.2 Regression rate due to oblateness vs inclination for various values of average altitude.

Figure: Magnitude of Regression Rate vs. inclination and altitude

J2 Apsidal Rotation

Similar to nodal regression, but perigee moves forward or backward, depending on inclination.

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$



The apsidal rotation rate is often large.

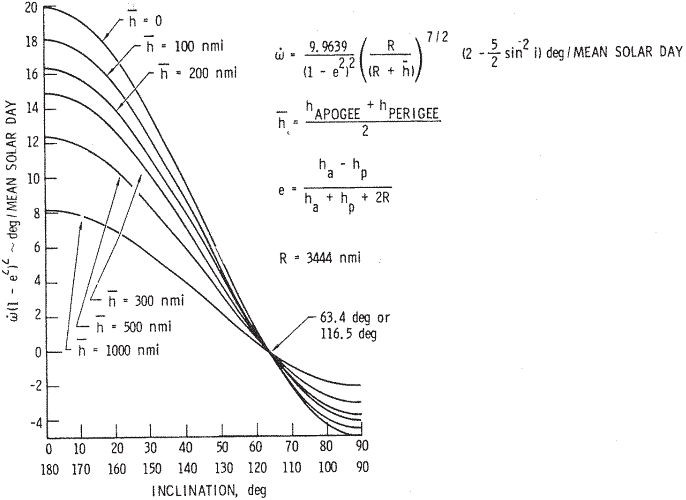


Figure: Magnitude of Regression Rate vs. inclination and altitude

The J_2 effect on other elements is usually minor. $\dot{a} \cong 0$.

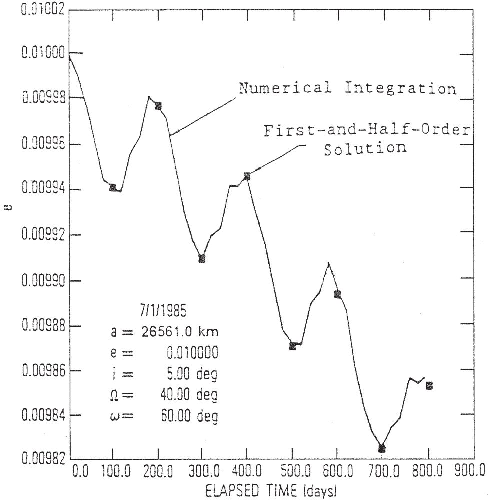


Figure: Eccentricity Change for Low-Inclination Orbit

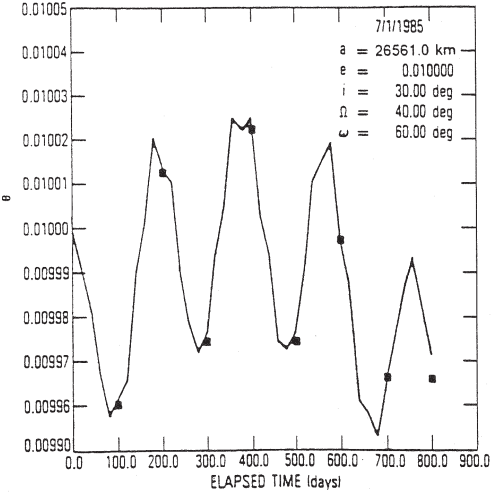


Figure: Eccentricity Change for Moderate-Inclination Orbit

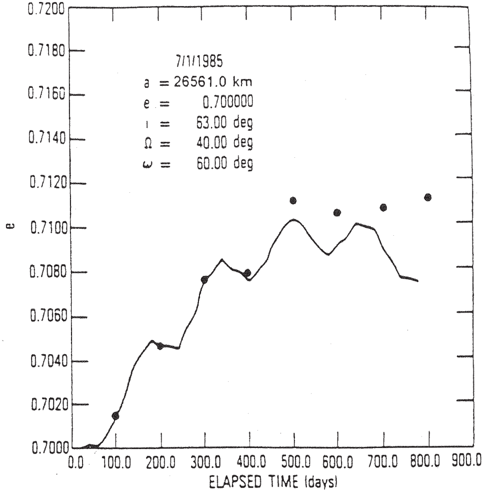


Figure: Eccentricity Change for High-Inclination Orbit

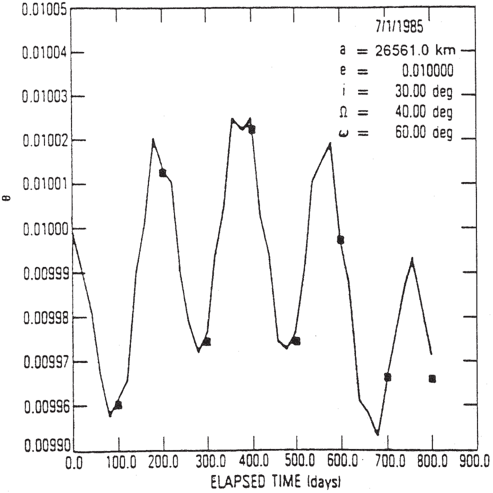


Figure: Inclination Change for Eccentric and Circular Orbits

J_2 Special Orbits

Critical Inclination

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$

Definition 3.

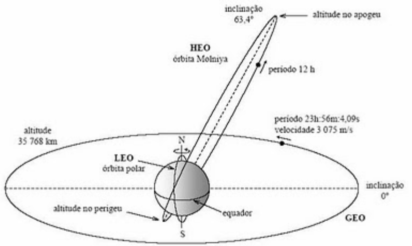
A **Critically Inclined Orbit** is one where $\dot{\omega} = 0$

For a critically inclined orbit,

$$4 - 5\sin^2 i = 0$$

which means

$$\begin{aligned} i &= \sin^{-1}\sqrt{4/5} \\ &= 63.43^\circ \quad \text{or} \quad 116.57^\circ \end{aligned}$$

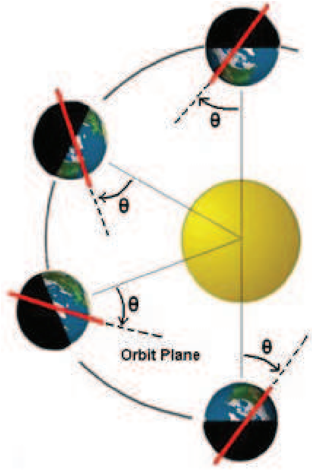


J_2 Special Orbits
Sun-Synchronous Orbits

Sun-Synchronous orbits maintain the same orientation of the orbital plane with respect to the sun.

Applications:

- Mapping
- Solar-Powered
- Shadow-evading
- Time-of-Day Apps



The earth rotates $360°$ about the sun every 365.25 days.

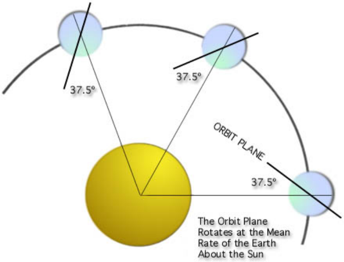
Definition 4.

A **Sun-Synchronous Orbit** is one where $\dot{\Omega} = .9855°/day = 1.992 \cdot 10^{-7} rad/s$.

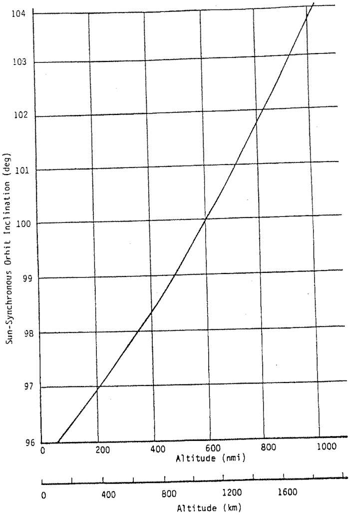
Thus

$$\cos i = -1.992 \cdot 10^{-7} \left(\frac{p}{R_e}\right)^2 \frac{2}{3nJ_2}$$

- The orbital plane rotates once every year.



Unlike critically inclined orbits, sun-synchronous orbits depend on altitude.



Numerical Example

Landsat

Problem: Design a sun-synchronous orbit with $r_p = R_e + 695km$ and $r_a = R_e + 705km$.

Solution: The desired inclination for a sun-synchronous orbit is given by

$$i = \cos^{-1}\left(-4.778 \cdot 10^{-6}\left(\frac{p}{R_e}\right)^2 \frac{2}{3nJ_2}\right)$$

For this orbit $a = R_e + 700km = 7078km$. The eccentricity is

$$e = 1 - \frac{r_p}{a} \cong 0$$

$J_2 = .0010826$. Thus the required inclination is

$$i = 1.716rad = 98.33^\circ$$

Numerical Example

Molnaya Orbit

Problem: Molnaya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 12 hours and which precesses at $\dot{\Omega} = -.2^\circ/day$.

Solution: We can first use the period to solve for a. From

$$n = \sqrt{\frac{\mu}{a^3}}$$

and $n = 2\pi/T = 2rad/day$ we have

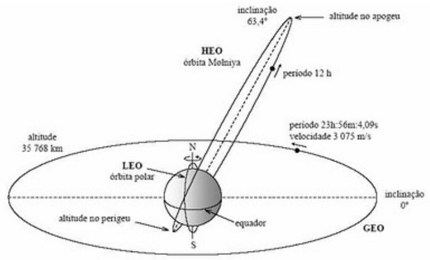
$$a =^3 \sqrt{\frac{\mu}{n^2}}$$

Now the critical inclination for $\dot{\omega} = 0$ is $i = 63.4^\circ$ or $i = 116.6^\circ$. We choose $i = 63.4^\circ$. To achieve $\dot{\Omega} = -.2^\circ/day$, we use

$$\dot{\Omega} = -\frac{3nJ_2R_e^2}{2a^2(1-e^2)^2}\cos i$$

Since a is already fixed, we must use e. We can solve for e as

$$e = \sqrt{1 + \sqrt{-\frac{3nJ_2R_e^2}{2\dot{\Omega}a^2}\cos i}} = .7459.$$



Note: Make sure the units of a and n match those of R_e and $\dot{\Omega}$, respectively.

Summary

This Lecture you have learned:

How to account for perturbations to Earth gravity

- Gravity Mapping
- Harmonic Functions
- J_2 Perturbation
  - Effect on Ω
  - Effect on ω
  - Minor effect (e, i)

How to design specialized orbits

- Critically - Inclined Orbit.
- Sun-Synchronous Orbit.
- Applications

Next Lecture: Final Review.