Systems Analysis and Control

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Lecture 20: Drawing Bode Plots, Part 2

Overview

In this Lecture, you will learn:

Simple Plots

- Real Zeros
- Real Poles
- Complex Zeros
- Complex Poles

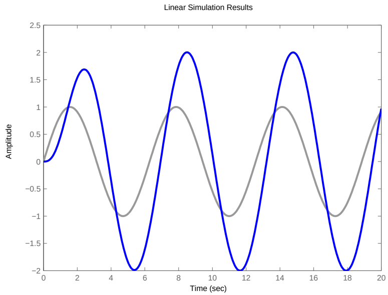
Recall: Frequency Response

Input:

$$u(t) = M\sin(\omega t + \phi)$$

Output: Magnitude and Phase Shift

$$y(t) = |G(\imath\omega)| M \sin(\omega t + \phi + \angle G(\imath\omega))$$



Frequency Response to $\sin\omega t$ is given by $G(\imath\omega)$

Review

Recall: Bode Plot

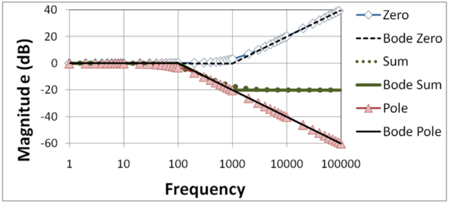
Definition 1.

The Bode Plot is a pair of log-log and semi-log plots:

1. Magnitude Plot: $20\log_{10}|G(\imath\omega)|$ vs. $\log_{10}\omega$
2. Phase Plot: $\angle G(\imath\omega)$ vs. $\log_{10}\omega$

Bite-Size Chucks:

$$\angle G(\imath\omega) = \sum_i \angle(\imath\omega\tau_{zi}+1) - \sum_i \angle(\imath\omega\tau_{pi}+1)$$



$$20\log|G(\imath\omega)| = \sum_i 20\log|\imath\omega\tau_{zi}+1| - \sum_i 20\log|\imath\omega\tau_{pi}+1|$$

Plotting Simple Terms

Plotting Normal Zeros

$$G_1(s) = (\tau s + 1)$$

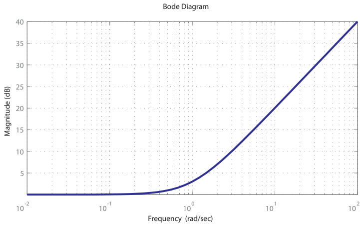
$$|\imath\omega\tau + 1| \cong \begin{cases} 1 & \omega << \frac{1}{\tau} \\ |\imath\omega\tau| & \omega >> \frac{1}{\tau} \end{cases}$$

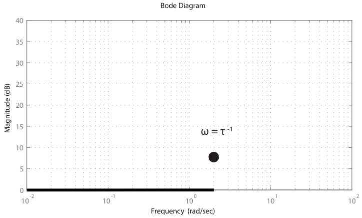
Behaves as

- A constant at low ω.
- A zero at high ω.

Break Point at $\omega = \frac{1}{\tau}$.
Magnitude:

$$20\log|\imath\omega\tau + 1| = \begin{cases} 0 \text{ dB} & \omega << \frac{1}{\tau} \\ 3.01 \text{ dB} & \omega = \frac{1}{\tau} \\ 20\log\omega & \omega >> \frac{1}{\tau} \end{cases}$$





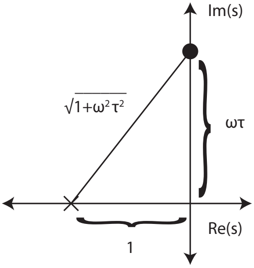
Plotting Real Zeros

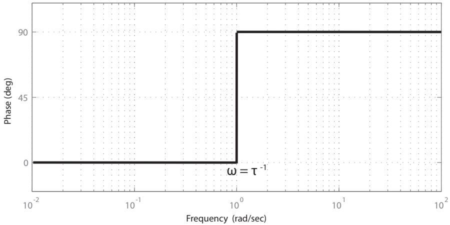
Phase

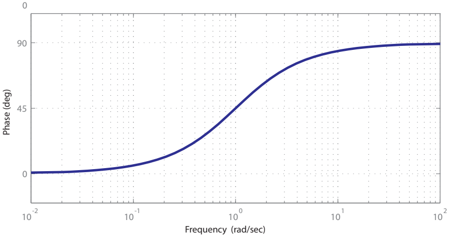
Geometry:

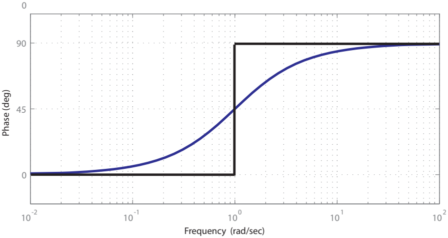
$$\angle(\imath\omega\tau + 1) = \tan^{-1}\left(\frac{\omega\tau}{1}\right)$$

$$\angle(\imath\omega\tau + 1) = \begin{cases} 0^\circ & \omega << \frac{1}{\tau} \\ 45^\circ & \omega = \frac{1}{\tau} \\ 90^\circ & \omega >> \frac{1}{\tau} \end{cases}$$



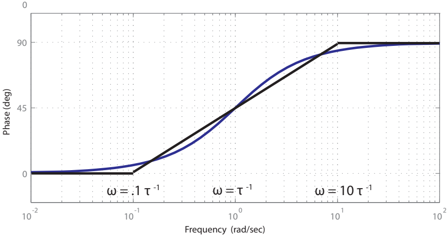






We can improve our sketch using a line

$$\angle(\imath\omega\tau + 1) = \begin{cases} 0^\circ & \omega < \frac{.1}{\tau} \\ 45^\circ\left(1 + \log \omega\tau\right) & \frac{.1}{\tau} < \omega < \frac{10}{\tau} \\ 90^\circ & \omega > \frac{10}{\tau} \end{cases}$$



Plotting Real Poles
Magnitude

We can plot real zeros

- Now lets do the poles.

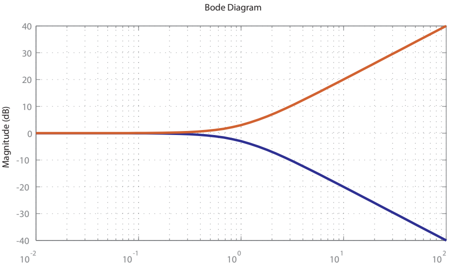
Poles are the opposite of zeros

$$G_{zero} = G_{pole}^{-1}$$
$$G_{pole}(\imath\omega) = \frac{1}{\imath\omega\tau + 1}$$
$$G_{zero}(\imath\omega) = \imath\omega\tau + 1$$

Magnitude:

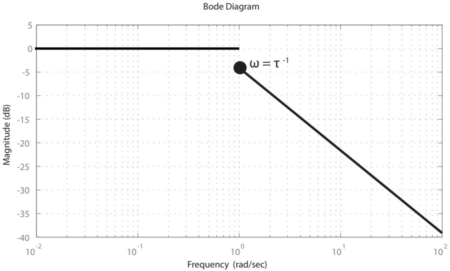
$$|G_{pole}(\imath\omega)| = \frac{1}{|\imath\omega\tau + 1|}$$

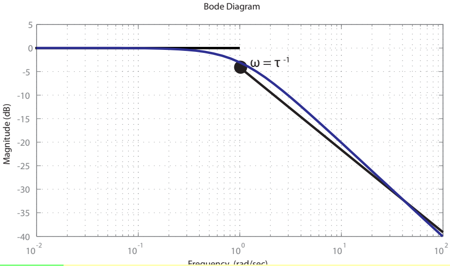
$$20\log|G_{pole}(\imath\omega)| = -20\log|\imath\omega\tau + 1|$$



Our approximation is also a reflection of the zero

$$20\log\frac{1}{|\imath\omega\tau+1|}=\begin{cases}0\text{ dB} & \omega << \frac{1}{\tau}\\ -3.01\text{ dB} & \omega = \frac{1}{\tau}\\ -20\log\omega & \omega >> \frac{1}{\tau}\end{cases}$$





We can plot real zeros

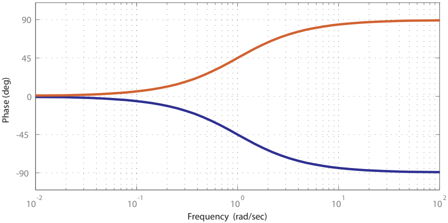
- Now lets do the poles.

Poles are the opposite of zeros

$$G_{pole}(\imath\omega) = \frac{1}{\imath\omega\tau + 1} = G_{zero}^{-1}(\imath\omega)$$

Phase:

$$\begin{aligned}\angle G_{pole}(\imath\omega) &= -\tan^{-1}\omega\tau \\ &= -\angle G_{zero}(\imath\omega)\end{aligned}$$



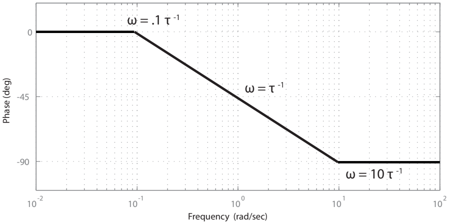
Plotting Real Poles

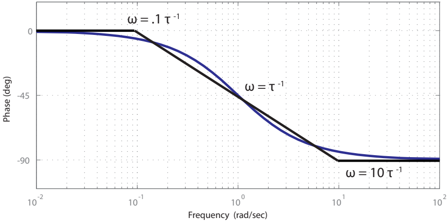
Phase

Our approximation is also a reflection of the zero

$$\angle(\imath\omega\tau + 1)^{-1}$$

$$= \begin{cases} 0^\circ & \omega < \frac{.1}{\tau} \\ -45^\circ\,(1 + \log \omega\tau) & \frac{.1}{\tau} < \omega < \frac{10}{\tau} \\ -90^\circ & \omega > \frac{10}{\tau} \end{cases}$$





We now have the pieces for a Lead-Lag Compensator.

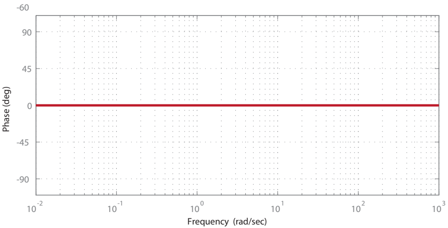
Example: Lead Compensator

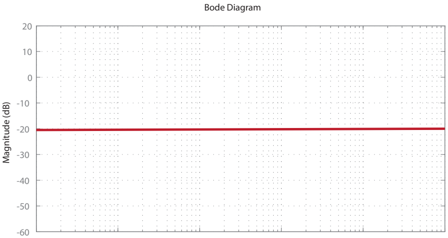
$$G(s) = \frac{s+1}{s+10}$$

First Step: Standard Form

$$G(s) = \frac{s+1}{s+10} = \frac{1}{10}\frac{s+1}{\frac{1}{10}s+1}$$

Second Step: Plot the Constant $c = .1 = -20\text{dB}$.

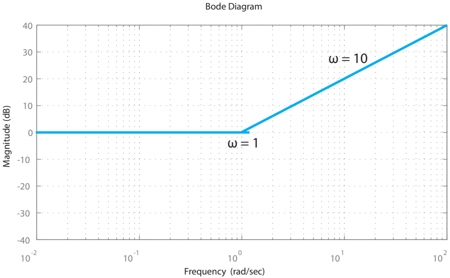




Third Step: Plot the Zero at $\omega = 1$.

Magnitude:

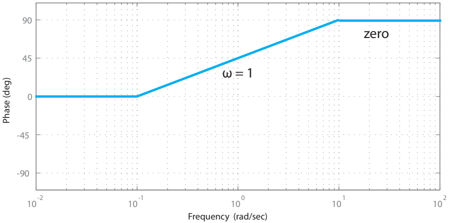
$$20\log|\imath\omega + 1| = \begin{cases} 0 \text{ dB} & \omega << 1 \\ 3.01 \text{ dB} & \omega = 1 \\ 20\log\omega & \omega >> 1 \end{cases}$$



Phase:

$$\angle(\imath\omega + 1)$$

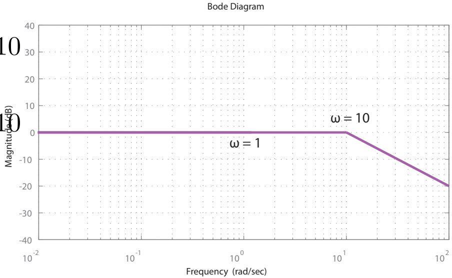
$$= \begin{cases} 0° & \omega < .1 \\ 45°\,(1 + \log\omega) & .1 < \omega < 10 \\ 90° & \omega > 10 \end{cases}$$



Fourth Step: Plot the Pole at $\omega = 10$.

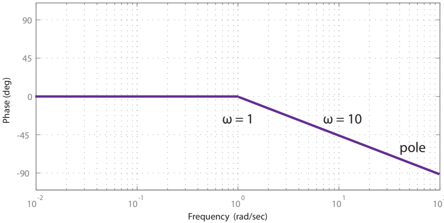
Magnitude:

$$20 \log \left| \frac{\imath\omega}{10} + 1 \right|^{-1} = \begin{cases} 0 \text{ dB} & \omega << 10 \\ -3.01 \text{ dB} & \omega = 10 \\ -20 \log \omega & \omega >> 10 \end{cases}$$

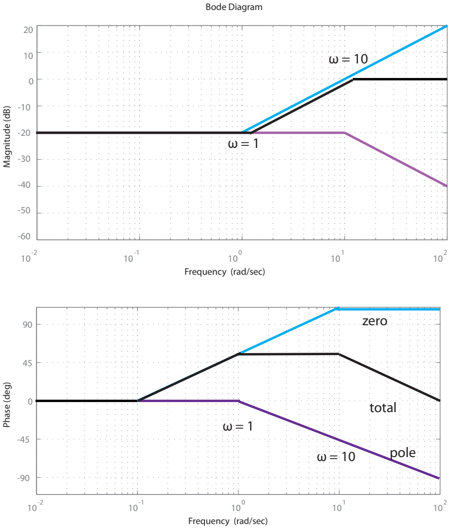


Phase:

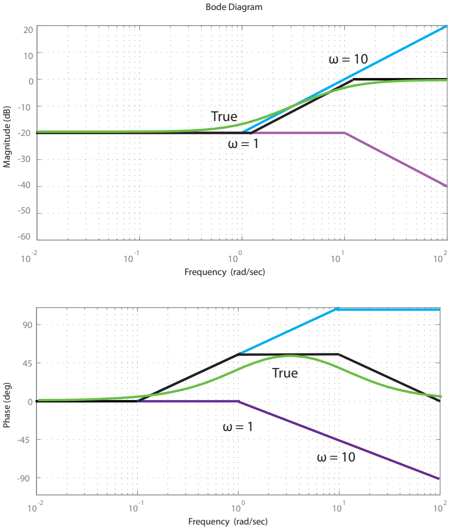
$$\angle \left(\frac{\imath\omega}{10} + 1 \right)^{-1} = \begin{cases} 0^\circ & \omega < 1 \\ -45^\circ \left(1 + \log \frac{\omega}{10}\right) & 1 < \omega < 100 \\ -90^\circ & \omega > 100 \end{cases}$$



Add them all together

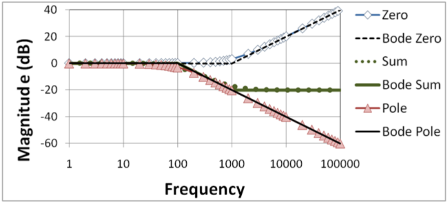


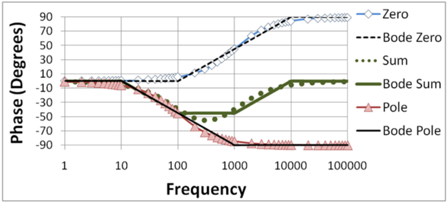
Compare with the True Plot



Example: Lag Compensator

$$G(s) = \frac{1}{10}\frac{s+1000}{s+100} = \frac{\frac{s}{1000}+1}{\frac{s}{100}+1}$$





- Pole at $\omega = 100$
- Zero at $\omega = 1000$

Output Phase "Lags" behind the input.

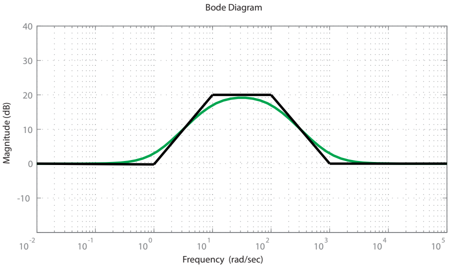
Put them together to get a Lead-Lag Compensator.

$$G(s) = \frac{s+1}{s+10}\frac{s+1000}{s+100} = \frac{s+1}{\frac{s}{10}+1}\frac{\frac{s}{1000}+1}{\frac{s}{100}+1}$$

- Zero at $\omega = 1$
- Pole at $\omega = 10$
- Pole at $\omega = 100$
- Zero at $\omega = 1000$

Magnitude: Changes in Slope.

- $\omega = 1$: $+20$
- $\omega = 10$: -20
- $\omega = 100$: -20
- $\omega = 1000$: $+20$



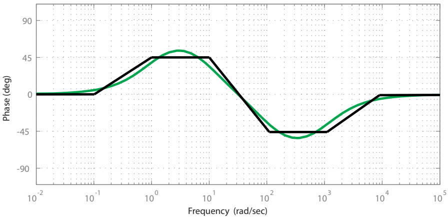
Plotting Real Poles and Zeros

Lead-Lag

$$G(s) = \frac{s+1}{s+10}\frac{s+1000}{s+100}$$

Phase: Changes in Slope.

- $\omega = .1$: $+45°$
- $\omega = 1$: $-45°$
- $\omega = 10$: $-90°$
- $\omega = 100$: $+90°$
- $\omega = 1000$: $+45°$
- $\omega = 10000$: $-45°$



Complex Poles and Zeros

Now we move on to the final topic: **Complex Poles and Zeros**
Complex Zeros:

$$G(s) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

First Step: Put in standard form

$$G(s) = \omega_n^2 \left(\left(\frac{s}{\omega_n} \right)^2 + 2\zeta \frac{s}{\omega_n} + 1 \right)$$

- So $y_{ss} = 1$

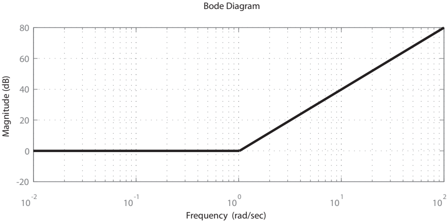
Ignoring the constant

$$G_1(\imath\omega) = \left(-\frac{\omega}{\omega_n} \right)^2 + 2\zeta \frac{\omega}{\omega_n}\imath + 1 = \begin{cases} 1 & \frac{\omega}{\omega_n} << 1 \\ 2\zeta\imath & \frac{\omega}{\omega_n} = 1 \\ -\left(\frac{\omega}{\omega_n}\right)^2 & \frac{\omega}{\omega_n} >> 1 \end{cases}$$

Magnitude:

$$20\log|G_1(\imath\omega)| = 20\log\left|-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{s\omega}{\omega_n}\imath + 1\right|$$

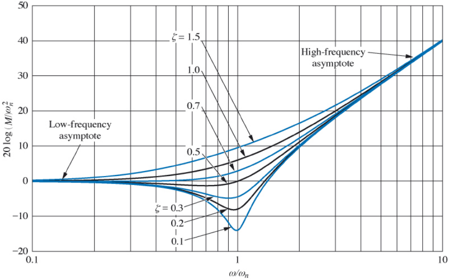
$$= \begin{cases} 0dB & \frac{\omega}{\omega_n} << 1 \\ 20\log(2\zeta) & \frac{\omega}{\omega_n} = 1 \\ 40(\log\omega - \log\omega_n) & \frac{\omega}{\omega_n} >> 1 \end{cases}$$



Complex Poles and Zeros

The Behavior near $\omega = \omega_n$ depends on ζ.

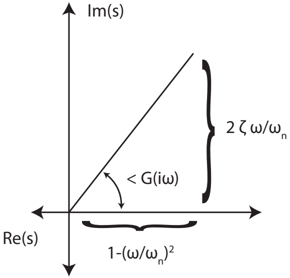
- When $\zeta = 0$,
$$20 \log |G_1(\imath\omega)| = 20 \log(2\zeta) = -\infty$$
- When $\zeta = .5$,
$$20 \log |G_1(\imath\omega)| = 20 \log 1 = 0$$
- When $\zeta = 1$,
$$20 \log |G_1(\imath\omega)| = 20 \log 2 = 6.02dB$$

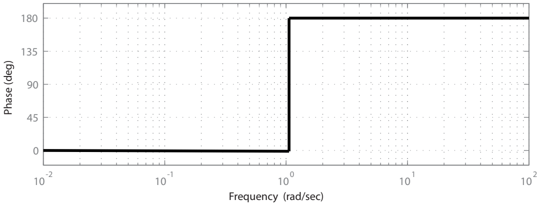


Complex Poles and Zeros

Phase:

$$\angle G_1(\imath\omega) = \angle\left(1-\left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}\imath\right)$$
$$= \tan^{-1}\left(\frac{2\zeta\omega\omega_n}{\omega_n^2-\omega^2}\right)$$
$$= \begin{cases} 0^\circ & \frac{\omega}{\omega_n} << 1 \\ 90^\circ & \frac{\omega}{\omega_n} = 1 \\ 180^\circ & \frac{\omega}{\omega_n} >> 1 \end{cases}$$

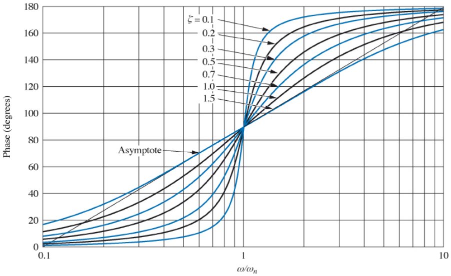




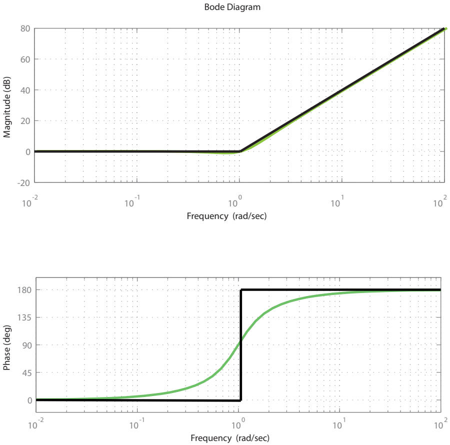
Complex Poles and Zeros

The Behavior near $\omega = \omega_n$ depends on ζ. For $\omega = \omega_n = 1$,

- When $\zeta = 0$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = \infty$
- When $\zeta = .1$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = 10$
- When $\zeta = .5$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = 2$
- When $\zeta = 1$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = 1$



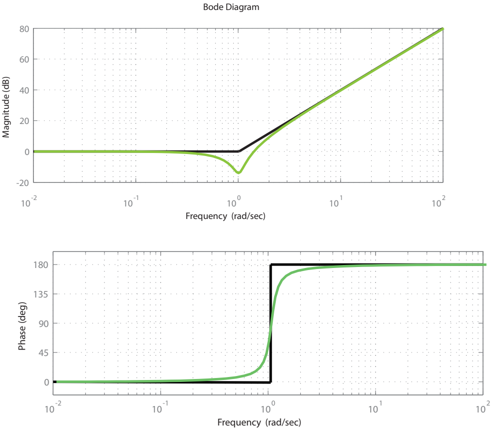
Comparison vs. True for $\omega_n = 1$, $\zeta = .5$.



Good for magnitude, bad for phase

Complex Poles and Zeros

Comparison vs. True for $\omega_n = 1$, $\zeta = .1$.

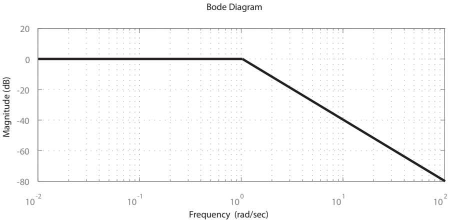


Complex Poles

Treatment of Complex Poles is similar

Complex Poles:

$$G(s) = \frac{1}{\left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right)}$$



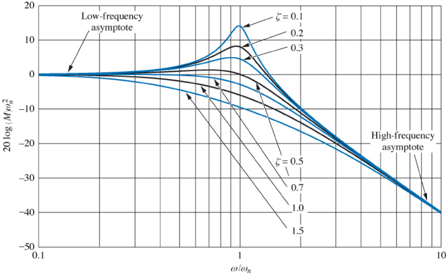
Magnitude:

$$20\log|G_1(\imath\omega)| = -20\log\left|1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{s\omega}{\omega_n}\imath\right|$$

$$= \begin{cases} 0dB & \frac{\omega}{\omega_n} << 1 \\ -20\log(2\zeta) & \frac{\omega}{\omega_n} = 1 \\ -40(\log\omega - \log\omega_n) & \frac{\omega}{\omega_n} >> 1 \end{cases}$$

The Behavior near $\omega = \omega_n$ depends on ζ.

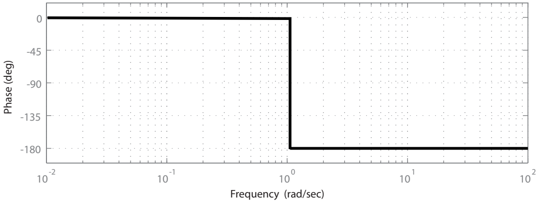
- When $\zeta = 0$,
$$20 \log |G_1(\imath\omega)| = 20 \log(2\zeta) = +\infty$$
- When $\zeta = .5$,
$$20 \log |G_1(\imath\omega)| = 20 \log 1 = 0$$
- When $\zeta = 1$,
$$20 \log |G_1(\imath\omega)| = 20 \log 2 = -6.02dB$$



Complex Poles

Phase:

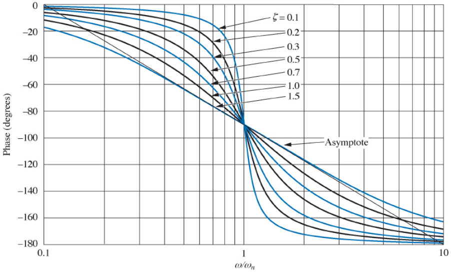
$$\angle G_1(\imath\omega) = -\angle\left(1 - \left(\frac{\omega}{\omega_n}\right)^2 + 2\zeta\frac{\omega}{\omega_n}\imath\right)$$
$$= \begin{cases} 0^\circ & \frac{\omega}{\omega_n} << 1 \\ -90^\circ & \frac{\omega}{\omega_n} = 1 \\ -180^\circ & \frac{\omega}{\omega_n} >> 1 \end{cases}$$



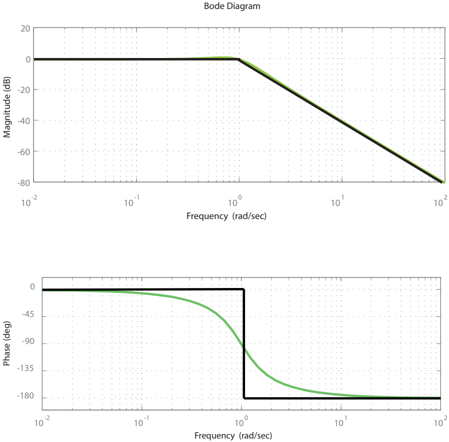
Complex Poles

The Behavior near $\omega = \omega_n$ depends on ζ. For $\omega = \omega_n = 1$,

- When $\zeta = 0$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = -\infty$
- When $\zeta = .1$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = -10$
- When $\zeta = .5$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = -2$
- When $\zeta = 1$, $\dfrac{d\angle G_1(\imath\omega)}{d\omega} = -1$



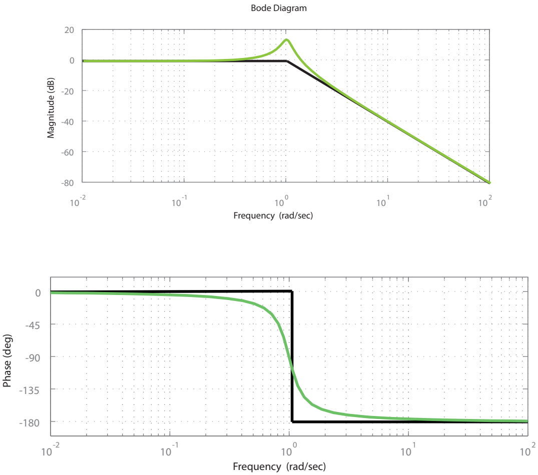
Comparison vs. True for $\omega_n = 1$, $\zeta = .5$.



Good for magnitude, bad for phase

Complex Poles

Comparison vs. True for $\omega_n = 1$, $\zeta = .1$.



Phase improves, Magnitude gets worse.

Summary

What have we learned today?

Simple Plots

- Real Zeros
- Real Poles
- Complex Zeros
- Complex Poles

Next Lecture: Compensation in the Frequency Domain