A Fuzzy-PIE Representation of T-S Fuzzy Systems with Delays and Stability Analysis via LPI method

Shuangshuang Wu, * **Fuchun Sun,** * **Matthew M. Peet,** ** **Changchun Hua** ***

* *Department of Computer Science and Technology, Tsinghua University, Beijing, 100085, China (e-mail: shuangshuangwu@mail.tsinghua.edu.cn).*
** *School of Matter, Transport and Energy, Arizona State University, Tempe, AZ 85231 USA (e-mail: mpeet@asu.edu).*
*** *Electrical Engineering Department, Yanshan University, Qinhuangdao, 066004, China.*

Abstract: Inspired by the recently proposed Partial Integral Equality(PIE) representation for linear delay systems, this paper proposes a fuzzy-PIE representation for T-S fuzzy systems with delays for the first time. Inspired by the free-weighting matrix technique, this paper introduces the free-weighting Partial Integral (PI) operators. Based on the novel representation and free-weighting PI operators, the stability issue is investigated for the T-S fuzzy systems with delays. The corresponding conditions are given as Linear Partial Inequality (LPI) and can be solved by the MATLAB toolbox PIETOOLS. Compared with the existing results, our method has no need of the bounding technique and a large amount of matrix operation. The numerical examples show the superiority of our method. This paper adds to the expanding field of LPI approach to fuzzy systems with delays.

Keywords: T-S fuzzy systems with delays, stability, fuzzy -PIE, free-weighting PI operator

1. INTRODUCTION

T–S fuzzy model with time delays is a vital and efficient tool for solving the nonlinear systems with delays. It allows us to project the advances of linear delayed systems analysis to nonlinear delayed systems. The appearance of the commonly used techniques for solving linear delay systems has led to an increase in research on T-S fuzzy systems with delays in these years (An and Wen, 2011; Kwon et al., 2016; Hua et al., 2020; Datta et al., 2020). Among the researches, stability assessment is one major issue aiming at finding a maximum allowable upper bound for assuring the stability of the system. There are two directions to improve the results.

One is to utilize the continuously developing techniques used in studies on linear delay systems, mainly about choosing a novel LKF with less restriction, and getting a tighter lower bound for the LKF derivative. Combing the augmented LKF and Wirtinger-based integral inequality, some results are presented for T-S fuzzy systems with delays in Zeng et al. (2014); Kwon et al. (2016); Tan et al. (2018); Lian et al. (2020); Qiu et al. (2021). The delay-decomposition augmented LKF and the LKF combing delay-product-type functional were proposed in Zeng et al. (2014) and Lian et al. (2020), respectively. Sheng et al. (2021) presented less conservative stability and stabilization conditions with lower computational complexity, where an symmetric membership-function-dependent LKF was used. By using Bessel-Legendre integral inequality, some enhanced results can be found in Datta et al. (2020); Sadek et al. (2022); Li et al. (2021).

Another is to make more use of the properties of fuzzy system itself, called the membership-function-dependent approach. It introduces the membership functions into the LKFs and adds more decision variables. However, the derivatives of membership function come into the LKF derivatives subsequently. To deal with this problem, analyzing the time derivative of membership function is necessary. For the cases where the bounds of membership function derivatives exist and are known, less conservative global and local stability conditions of T-S fuzzy system with constant delay are given in Wang and Liu (2018). Recently, a linear switching method was presented in Wang and Lam (2018) where a switched LKF was employed assuming that the switching is in finite time and the switching rule is dependent on the membership function derivative. The linear switching method has been improved in Wang et al. (2022) as polynomial matrix switching. Recently, combining membership-function-dependent approach and the improved inequality technique, many results have been presented in Lian et al. (2020); Zhi et al. (2021); Sheng et al. (2021); Wang et al. (2022).

Unfortunately, all the papers above exploit the integral inequality and thus the conservatism of inequality technique itself limits the superiority of results. Nowadays, one delay-free compact representation called Partial Integral Equation (PIE) was proposed for linear delay systems in Peet (2020). Using this representation, the LPI-based conditions for stability and control issue of linear delay systems are obtained in Peet and Gu (2019); Wu et al. (2019). Since there are no delay terms on the outside, commonly used inequality technique and complex matrix operation were no longer necessary. However, such results have not been extended to T-S fuzzy systems with delays.

This paper aims to apply the recently proposed PIE representation and LPI-based method to T-S fuzzy systems with delays. Firstly, a class of fuzzy-PIE system is proposed, and its stability is studied using LPI method. Then we prove this fuzzy-PIE system can be used to equivalently represent the solutions of a set T-S fuzzy systems with delays. Based on the representation and novel free-weighting PI operators, LPI-form stability condition for T-S fuzzy systems with delays is given, which can be solved efficiently using the MATLAB toolbox PIETOOLS Shivakumar et al. (2020a). Numerical examples are given to illustrate the proposed method.

Notations: I denotes the identity matrix with dimension clear from context. A block-diagonal matrix is denoted by diag$\{\cdots\}$. We use $L_2^n[T]$ to denote the vector-valued Lesbesque square integrable functions which map $T \to \mathbb{R}^n$. The space $Z_{m,n}$:=$\mathbb{R}^m\times L_2^n[-1,0]$ is an inner-product space with the inner product defined as

$$\left\langle \begin{bmatrix} y \\ \psi \end{bmatrix}, \begin{bmatrix} x \\ \phi \end{bmatrix} \right\rangle = y^T x + \int_{-1}^{0} \psi(s)^T \phi(s) ds,$$

where $x, y \in \mathbb{R}^m$ and $\psi, \phi \in L_2^n[-1, 0]$. The inner product $\langle\cdot,\cdot\rangle$ is in $Z_{m,n}$ space without any special notation and $\|\mathbf{x}\| = \sqrt{<\mathbf{x},\mathbf{x}>}, \forall \mathbf{x} \in Z_{m,n}$. There are actually two kinds of PI operators given in Shivakumar et al. (2020a), in this paper, "PI operator" specifically refers to the 4-PI operator.

2. PRELIMINARIES

The definition of PI operators, PIE system and LPI together with the properties of PI operators are introduced. A commonly used lemma in the stability research on T-S fuzzy system with delays is also given here.

2.1 (4-)PI operator

Partial Integral (PI) operators are an extension of matrices to infinite-dimensional spaces. The class of PI operators form an algebra of bounded linear multiplier and integral operators defined jointly on $\mathbb{R}^n$ and L_2.

A (4-)PI operator $\mathcal{P} : \mathbb{R}^m \times L_2^n[a, b] \to \mathbb{R}^p \times L_2^q[a, b]$ is in the form of

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}_{i=0}^2 \end{bmatrix} \begin{bmatrix} x \\ \Phi \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^{0} Q_1(s)\Phi(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}_{i=0}^2}\right)\Phi(s) \end{bmatrix}$$

and

$$\mathcal{P}_{\{R_i\}_{i=0}^2}\phi(s) := R_0(s)\phi(s) + \int_{-1}^{s} R_1(s,\theta)\phi(\theta)d\theta + \int_{s}^{0} R_2(s,\theta)\phi(\theta)d\theta$$

where $P : \mathbb{R}^m \to \mathbb{R}^q$, $Q_1 : [a, b] \to \mathbb{R}^{p\times n}$, $Q_2 : [a, b] \to \mathbb{R}^{q\times m}$, $R_0 : [a, b] \to \mathbb{R}^{n\times m}$, $R_i : [a, b] \times [a, b] \to \mathbb{R}^{m\times n}$ for $i = 1, 2$.

Remark 1. This PI operator can represent the most relations that appears in the time delay systems and partial differential equations with boundary conditions. It contributes to the creation of PIE representation.

2.2 PIE system

A PIE system is a class of system described by a set of differential equations that are parameterized by PI operators. Specifically, we say $\mathbf{z} \in Z_{m,n}$ solves the PIE for initial condition $z_0 \in Z_{m,n}$ if

$$\mathcal{T}\dot{\mathbf{z}}(t) = \mathcal{A}\mathbf{z}(t), \quad \mathbf{z}(0) = \mathbf{z}_0 \in Z_{m,n} \tag{1}$$

where $z \in \mathbb{R}^q$, $\mathcal{T}$ and $\mathcal{A} : Z_{m,n} \to Z_{m,n}$ are PI operators. The PIE formulation provides a new alternative representation to a large class of linear infinite dimensional systems including delay differential formulation (Peet, 2020; Shivakumar et al., 2020b). The stability of a PIE system is defined as follows.

Definition 1. The PIE system (1) defined by $\{\mathcal{T}, \mathcal{A}\}$ is said to be stable if any solution to the PIE system (1) satisfies $\lim_{t\to\infty} \|\mathcal{T}\mathbf{z}\| \to 0$.

2.3 LPI

Linear Partial Integral Inequality (LPI) is an equality constraint involved with PI operator variables, which is used to solve a convex feasibility or optimization problem.

For example, if $\mathcal{A}$ and $\mathcal{P}$ are PI operators, then

$$\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} \prec 0,$$

is an LPI. The LPI constraint involves the operations between PI operator variables, such as addition, adjoint, composition, and concatenation. A simple declaration is given here.

For any two PI operators, $\mathcal{P}_1$ and $\mathcal{P}_2$, we have

a. $\mathcal{P}_1 + \mathcal{P}_2$ is also a PI operator.
b. $\mathcal{P}_1^*$ stands for the adjoint of $\mathcal{P}_1$ and is also a PI operator.
c. $\mathcal{P}_1\mathcal{P}_2$ represents the composition of $\mathcal{P}_1$ and $\mathcal{P}_2$ is also a PI operator.
d. $\mathcal{P}_1(: X \to Z) \succ 0$ if $\forall\sigma \in X$, $\langle\sigma, \mathcal{P}_1\sigma\rangle \succ 0$.

Remark 2. LPI's definition is similar to that for Linear Matrix Inequality (LMI). The operation among the PI operators such as the addition, adjoint, composition and verification of the positivity of a LPI do not need any manual involvement and can be solved directly by MATLAB package PIETOOLS. More calculation details on how the PIETOOLS works can be found in Shivakumar et al. (2020a).

2.4 Lemma

Lemma 2. (Wang and Lam, 2018) Suppose there exist a positive integer v and functions r_i satisfying $\sum_{i=1}^{v} r_i(t) = 1$. For matrices $X_i, i = 1, 2, \cdots, v$, define a matrix function $X_h = \sum_{i=1}^{v} r_i(t)X_i$. Then

$$\dot{X}_h \le 0$$

is satisfied if the switched situations are true:

$$\begin{cases} \text{if } \dot{r}_i(t) \ge 0, \text{then } X_i - X_v \le 0, \\ \text{if } \dot{r}_i(t) < 0, \text{then } X_i - X_v < 0, \end{cases} \tag{2}$$

for $i = 1, \cdots, v - 1$.

3. FUZZY-PIE SYSTEM AND ITS STABILITY

This section first defines a class of fuzzy-PIE system, and investigate the stability issue of the fuzzy-PIE system via the LPI method.

3.1 Fuzzy-PIE system

Inspired by the T-S fuzzy system model, the fuzzy-PIE system is put forward as

$$\mathcal{T}\dot{\mathbf{x}}(t) = \sum_{i=1}^{v} r_i(t)\mathcal{A}_i\mathbf{x}(t) \tag{3}$$
$$\mathbf{x}(0) = \mathbf{x}_0 \in Z_{m,n}$$

where $z \in \mathbb{R}^q$, $\mathcal{T}, \mathcal{A}_i : Z_{m,n} \to Z_{m,n}, \forall i = 1, \cdots, v$ are PI operators. r_i are fuzzy membership functions with respect to t satisfying

$$r_i(t) \geq 0, \sum_{i=1}^{v} r_i(t) = 1, \forall t. \tag{4}$$

Before we proceed, inspired by the Definition 1, we first present the definition of stability for fuzzy-PIE system. Moreover, Lemma 4 is given here, which will be used in the following analysis.

Definition 3. The fuzzy-PIE system (3) is defined to be quadratically stable if from any initial condition $\mathbf{x}_0$, the solution $\mathbf{x}(t)$ satisfies

$$\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\| \to 0,$$

for any r_i satisfying Eqn (4).

Lemma 4. Suppose there exist PI operators $\mathcal{X}_i : Z_{m,n} \to Z_{m,n}$ for $i = 1, 2, \cdots, v$, functions r_i are with respect to t satisfying Eqn (4). Define a PI operator $\mathcal{X}_h = \sum_{i=1}^{v} r_i(t)\mathcal{X}_i$. Then

$$\dot{\mathcal{X}}_h \preceq 0$$

is satisfied if the following situations are true:

$$\begin{cases} \text{if } \dot{r}_i(t) \leq 0, \text{ then } \mathcal{X}_i - \mathcal{X}_v \succeq 0, \\ \text{if } \dot{r}_i(t) > 0, \text{ then } \mathcal{X}_i - \mathcal{X}_v \prec 0, \end{cases} \tag{5}$$

for $i = 1, \cdots, v-1$.

Proof. Suppose the conditions (5) are satisfied for $i = 1, \cdots, v-1$ and $\sum_{i=1}^{v} r_i(t) = 1$. Then $r_i(t)$ and $\mathcal{X}_i$ satisfy

$$\sum_{i=1}^{v-1} \dot{r}_i(t)(\mathcal{X}_i - \mathcal{X}_v) \preceq 0,\ \sum_{i=1}^{v} \dot{r}_i(t) = 0.$$

Further, we get

$$\begin{aligned}\dot{\mathcal{X}}_h &= \sum_{i=1}^{v-1} \dot{r}_i(t)\mathcal{X}_i + \dot{r}_v(t)\mathcal{X}_v \\ &= \sum_{i=1}^{v-1} \dot{r}_i(t)\mathcal{X}_i - \sum_{i=1}^{v-1} \dot{r}_i(t)\mathcal{X}_v \\ &= \sum_{i=1}^{v-1} \dot{r}_i(t)(\mathcal{X}_i - \mathcal{X}_v) \preceq 0\end{aligned}$$

This completes the proof.

Remark 3. Lemma 4 is one PI-form version of the Lemma 2. The switching rules are dependent on the derivative of the function $r_i(t)$ and there exist 2^{v-1} cases in total. Instead of using the simple condition that $\mathcal{X}_i \leq 0$ for $i = 1, 2, \cdots, v$ to ensure $\mathcal{X}_h \leq 0$, this lemma makes use of the property of the multiplier function $r_i(t)$ and gets a less conservative condition without changing the number of the PI operator variables. For the cases that functions $r_i(t)$ are monotone changing, i.e., $\dot{r}_i(t) \geq 0$ or $\dot{r}_i(t) < 0$ always hold, this lemma becomes the case without switching.

3.2 LPI-based stability analysis of fuzzy-PIE system

In this subsection, a sufficient LPI-based robust condition of fuzzy-PIE system (3) is obtained. Inspired by the free-weighting matrix, the novel free-weighting PI operators are first introduced into the LPI-form stability condition. The free-weighting matrix was proposed for linear delay system in Wu et al. (2004). It is usually used to express the relationships between terms in Leibniz–Newton formula and proposed to overcome the conservatism of methods involving a fixed model transformation. The common use of free-weighting matrix in the research proves its effectiveness.

Theorem 5. Suppose there exist bounded PI operators $\mathcal{P}_i$ satisfying $\mathcal{P}_i = \mathcal{P}_i^* \succ 0$ and any PI operators $\mathcal{M}_1, \mathcal{M}_2$ with the appropriate dimensions satisfying

$$\mathcal{P}_h = \sum_{i=1}^{v} r_i(t)\mathcal{P}_i, \quad \dot{\mathcal{P}}_h \prec 0,$$
$$\Phi_i = \begin{bmatrix} -\mathcal{M}_1^*\mathcal{A}_i - \mathcal{A}_i^*\mathcal{M}_1 & \mathcal{M}^* \\ \mathcal{M} & \mathcal{T}^*\mathcal{M}_2^*\mathcal{T} + \mathcal{T}^*\mathcal{M}_2\mathcal{T} \end{bmatrix} \prec 0 \tag{6}$$

where $\mathcal{M} = \mathcal{T}^*\mathcal{P}_i\mathcal{T} - \mathcal{T}^*\mathcal{M}_2^*\mathcal{A}_i + \mathcal{T}^*\mathcal{M}_1$. $\mathcal{T}, \mathcal{A}_i$ and $r_i(t)$ as defined in system (3) and (4). Then system (3) is stable.

Proof. Define the Lyapunov function as

$$V(\mathbf{x}) = \langle \mathcal{T}\mathbf{x}, \mathcal{P}_h\mathcal{T}\mathbf{x} \rangle \tag{7}$$

Since $\mathcal{P}_i$ are bounded, coercive and $r_i(t) \in [0, 1]$, then $\mathcal{P}_h$ is bounded, coercive and there exist positive scalars α, β satisfying $\alpha \|\mathbf{x}\|^2 \leq V(\mathbf{x}) \leq \beta \|\mathbf{x}\|^2$.

Since the solution $\mathbf{x}$ must satisfy Eqn (3), for the free-weighting-PI operators $\mathcal{M}_1$ and $\mathcal{M}_2$, we get

$$\begin{aligned}\mathcal{H} &= \left\langle \mathcal{M}_1\mathbf{x}(t) + \mathcal{M}_2\mathcal{T}\dot{\mathbf{x}}(t), \mathcal{T}\dot{\mathbf{x}}(t) - \sum_{i=1}^{v} r_i(\theta(t))\mathcal{A}_i\mathbf{x}(t) \right\rangle \\ &+ \left\langle \mathcal{T}\dot{\mathbf{x}}(t) - \sum_{i=1}^{v} r_i(\theta(t))\mathcal{A}_i\mathbf{x}(t), \mathcal{M}_1\mathbf{x}(t) + \mathcal{M}_2\mathcal{T}\dot{\mathbf{x}}(t) \right\rangle \\ &= 0\end{aligned} \tag{8}$$

Then combined with Eqn (8), differentiating $V(\mathbf{x}(t))$ gets

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) &= \left\langle \mathcal{T}\mathbf{x}(t), \dot{\mathcal{P}}_h\mathcal{T}\mathbf{x}(t) \right\rangle + \sum_{i=1}^{v} r_i(t) \langle \mathcal{T}\dot{\mathbf{x}}(t), \mathcal{P}_i\mathcal{T}\mathbf{x}(t) \rangle \\ &\quad + \sum_{i=1}^{v} r_i(t) \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}_i\mathcal{T}\dot{\mathbf{x}}(t) \rangle \\ &\leq \sum_{i=1}^{v} r_i(t) \{\langle \mathcal{T}\dot{\mathbf{x}}(t), \mathcal{P}_i\mathcal{T}\mathbf{x}(t) \rangle + \langle \mathcal{T}\mathbf{x}(t), \mathcal{P}_i\mathcal{T}\dot{\mathbf{x}}(t) \rangle\} + \mathcal{H} \\ &= \sum_{i=1}^{v} r_i(t) \left\langle \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}, \Phi_i \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} \right\rangle\end{aligned}$$

Then if Eqn (6) is satisfied, $\dot{V}(t) < 0$ comes true. With the assumption that the switching times is finite, from the

Lyapunov theory and Definition 3, we get the system is stable. This proof is complete.

4. APPLICATION TO THE T-S FUZZY SYSTEMS WITH DELAYS

This section investigates the stability of a class of T-S fuzzy systems with constant delays. To apply the LPI-condition obtained for the fuzzy-PIE system to our case, we will first represent the T-S fuzzy system with delays in the form fuzzy-PIE representation (3). Then the LPI-form stability condition of the T-S fuzzy system with delays is given.

4.1 Problem formulation

Consider the fuzzy system described as the following rules.

Plant rules i: IF θ_1 is M_{i1} and $\cdots$ and θ_w is M_{iw}, THEN

$$\dot{x}(t) = A_{i0}x(t) + \sum_{l=1}^{k} A_{il}x(t-\tau_l) \tag{9}$$

where i is a positive integer satisfying $1 \le i \le v$ and v is the number of the IF-THEN rules. $M_{i1}, M_{i2,}, \cdots, M_{iw}$ are the fuzzy sets, $\theta_1(t), \theta_2(t), \cdots, \theta_w(t)$ are the previous variables. $x(t) \in \mathbb{R}^n$ is the state vector. A_{i0} and A_{il} denote constant matrices of proper dimensions. τ_l are constant delays for $l = 0, 1, \cdots, k$ and $\tau = \max\{\tau_1, \cdots, \tau_k\}$.

Define the fuzzy membership function as

$$r_i(\theta(t)) = \frac{\prod_{j=1}^{q} \epsilon_{ij}(\theta(t))}{\sum_{i=1}^{v} \prod_{j=1}^{q} \epsilon_{ij}(\theta_j(t))} \tag{10}$$

where $\theta(t) = [\theta_1(t), \theta_2(t), \cdots, \theta_p(t)]$ and $\epsilon_{ij}(\theta_j(t)) \ge 0$ represent the grade of membership of $\theta_j(t)$ in fuzzy sets M_{ij}. Then we get

$$r_i(\theta(t)) \ge 0, \sum_{i=1}^{v} r_i(\theta(t)) = 1, \forall t. \tag{11}$$

To simplify, we use $r_i(t)$ as an shorthand of $r_i(\theta(t))$ in the following analysis.

Using a standard fuzzy inference method (using a singleton fuzzifier, product fuzzy inference, and weighted average defuzzifier), the system is expressed as follows

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{v} r_i(t) \left(A_{i0}x(t) + \sum_{l=1}^{k} A_{il}x(t-\tau_l) \right) \\ x(t) &= x_0, t \in [-\tau, 0] \end{aligned} \tag{12}$$

4.2 An Equivalent Fuzzy-PIE Representation

This part will give the conversion and additionally, prove the solutions of the system (12) are equivalent to the converted system (3) with

$$\begin{aligned} \mathcal{A}_i &:= \mathcal{P}\begin{bmatrix} A_{i0} + \sum_{l=1}^{k} A_{il}, & -\begin{bmatrix} A_{i1}, \cdots, A_{ik} \end{bmatrix} \\ 0, & \{H, 0, 0\} \end{bmatrix}, \\ \mathcal{T} &:= \mathcal{P}\begin{bmatrix} I, & 0 \\ \tilde{I}, & \{0, 0, -I\} \end{bmatrix}, \\ \tilde{I} &:= [\underbrace{I_n, I_n, \cdots, I_n}_{k}]^T, \\ H &:= \operatorname{diag}\{\frac{1}{\tau_1}, \cdots, \frac{1}{\tau_k}\}. \end{aligned} \tag{13}$$

In Peet (2020), combining Lemma 1 and Lemma 4 proves non-conservative conversion between linear DDEs and a PIE form. We extend to the fuzzy system with delay case as follows.

Lemma 6. Given function θ, positive constants $\tau_l, l = 1, 2, \cdots, k$, and function $x_0 \in L_2^n[-\tau, 0]$ where $\tau = \max\{\tau_1, \cdots, \tau_k\}$, the function x satisfies the T-S fuzzy delay system (12) defined by $\{A_{i0}, A_{il}, \tau_l, r_i\}$ where r_i are as defined in Eqn (10) if and only if $\mathbf{x}$ satisfies the fuzzy-PIE (3) defined by $\{\mathcal{T}, \mathcal{A}_i, r_i\}$ parameterized by Eqn (13), and $\mathbf{x}(t), \mathbf{x}_0$ are defined as $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ \partial_s \phi(t,s) \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} x_0 \\ \partial_s \phi_0(s) \end{bmatrix}$ where $\phi(t,s) = \left[x(t+\tau_1 s)^T, \cdots, x(t+\tau_k s)^T\right]^T, s \in [-1, 0]$ and $\phi_0(s) = \left[x_0^T, \cdots, x_0^T\right]^T$.

Proof. Let us to prove the sufficiency firstly. Suppose $\mathbf{x}, \mathbf{x}_0, \phi, \phi_0, \mathcal{T}, \mathcal{A}_i$ and r_i are as defined in this Lemma and $\mathbf{x}$ satisfies the fuzzy-PIE (3). From the definition of PI operators and Fundamental Theorem of Calculus and boundary conditions, we get

$$\begin{aligned} \mathcal{T}\dot{\mathbf{x}}(t) &= \begin{bmatrix} \dot{x}(t) \\ \partial_t \phi(t,s) \end{bmatrix}, \\ \mathcal{A}_i \mathbf{x}(t) &= \begin{bmatrix} A_{i0}x(t) + \sum_{l=1}^{k} + A_{il}x(t-\tau_l) \\ H\partial_s \phi(t,s) \end{bmatrix}, \\ \partial_t \phi(t,s) &= H\partial_s \phi(t,s), \\ \phi(0,s) &= \left[x(\tau_1 s)^T, \cdots, x(\tau_k s)^T\right]^T. \end{aligned} \tag{14}$$

Then if $\mathbf{x}$ satisfies the fuzzy-PIE (3), the function x satisfies Eqn (12). The sufficiency is proved. The necessity comes as the sufficiency and is omitted here.

4.3 LPI-based stability condition of the fuzzy delay system

Theorem 7. Given positive constants k, v, τ_l for $l = 1, 2, \cdots, k$, function $x_0 \in L_2^n[-\tau, 0]$, and function θ, suppose there exist matrix $P_i \in \mathbb{R}^{n\times n}$, matrix $P_{m1} \in \mathbb{R}^{n\times n}$, matrix $P_{m2} \in \mathbb{R}^{n\times n}$, matrix-valued polynomials $Q_i : [-1, 0] \to \mathbb{R}^{n\times m}, R_{i0} : [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{i1} \in [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{i2} : [-1, 0] \to \mathbb{R}^{m\times m}$, $Q_{m1} : [-1, 0] \to \mathbb{R}^{n\times m}, R_{m10} : [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{m11} : [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{m12} : [-1, 0] \times [-1, 0] \to \mathbb{R}^{m\times m}, Q_{m2} : [-1, 0] \to \mathbb{R}^{n\times m}, R_{m20} : [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{m21} : [-1, 0] \to \mathbb{R}^{m\times m}$, $R_{m22} : [-1, 0] \times [-1, 0] \to \mathbb{R}^{m\times m}$ such that the operator $\mathcal{P}_i = \mathcal{P}\begin{bmatrix} P_i, & Q_i \\ Q_i^T, & \{R_{i0}, R_{i1}, R_{i2}\} \end{bmatrix}$ satisfying $\mathcal{P}_i^* = \mathcal{P}_i \succ 0$ for $i = 1, 2, \cdots, v$ and $\mathcal{M}_j = \mathcal{P}\begin{bmatrix} P_{mj}, & Q_{mj} \\ Q_{mj}^T, & \{R_{mj0}, R_{mj1}, R_{mj2}\} \end{bmatrix}, j = 1, 2,$ such that

$$\begin{aligned} \mathcal{P}_h &= \sum_{i=1}^{v} r_i(t)\mathcal{P}_i, \quad \dot{\mathcal{P}}_h \prec 0, \\ \Phi_i &= \begin{bmatrix} -\mathcal{M}_1^*\mathcal{A}_i - \mathcal{A}_i^*\mathcal{M}_1 & \mathcal{M}^* \\ \mathcal{M} & \mathcal{T}^*\mathcal{M}_2^*\mathcal{T} + \mathcal{T}^*\mathcal{M}_2\mathcal{T} \end{bmatrix} \prec 0 \end{aligned} \tag{15}$$

for $i = 1, \cdots, v$, where $\mathcal{M} = \mathcal{T}^*\mathcal{P}_i\mathcal{T} - \mathcal{T}^*\mathcal{M}_2^*\mathcal{A}_i + \mathcal{T}^*\mathcal{M}_1$, $m = n \cdot k$. $\mathcal{T}, \mathcal{A}_i$ and $r_i(t)$ are as defined in Eqn (10) and (13). $r_i(t)$ satisfies (11). Then the system (12) is asymptotically stable.

Proof. For any solution $x(t)$ of the system (12), we define $\mathbf{x}(t) = \begin{bmatrix} x(t) \\ \partial_s \phi(t,s) \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} x_0 \\ \partial_s \phi_0(s) \end{bmatrix}$, $\phi(t,s) = \left[x(t+\tau_1 s)^T, \cdots, x(t+\tau_k s)^T\right]^T$ and $\phi_0(s) = \left[x_0^T, \cdots, x_0^T\right]^T$. $\mathcal{T}, \mathcal{A}_i$ are as defined in Eqn (13). Then from Lemma 6, $\mathbf{x}(t)$ satisfies the system (3). Suppose constraints (15) are satisfied, from Theorem 5 and Definition 3, the parameterized system (3) is stable and

$$\lim_{t\to\infty} \|\mathcal{T}\mathbf{x}(t)\| \to 0.$$

A simple calculation gives $\mathcal{T}\mathbf{x}(t) = \begin{bmatrix} x(t) \\ \phi(t,s) \end{bmatrix}$ and $\|x(t)\| \leq \left\| \begin{bmatrix} x(t) \\ \phi(t,s) \end{bmatrix} \right\|$. Hence we get for any $x(t)$ that satisfies system (12),

$$\lim_{t\to\infty} \|x(t)\| \to 0.$$

This implies that the system (12) is asymptotically stable. The proof is complete.

5. NUMERICAL IMPLEMENTATION

In this section, two numerical examples are presented to prove the competitive performance of our method. The MATLAB PIETOOLs are used to solve the derived LPI stability condition.

Example 1 Consider the following nonlinear delayed system (Wang and Lam, 2019):

$$\begin{aligned} \dot{x}_1(t) &= 0.5\left(1-\sin^2(\theta(t))\right)x_2(t) \\ &\quad - x_1\left(t-\tau\right) - \left(1+\sin^2(\theta(t))\right)x_1(t) \\ \dot{x}_2(t) &= \mathbf{sgn}\left(|\theta(t)| - \frac{\pi}{2}\right)\left(0.9\cos^2(\theta(t))-1\right)x_1(t-\tau) \\ &\quad - x_2(t-\tau) - \left(0.9+0.1\cos^2(\theta(t))\right)x_2(t). \end{aligned} \tag{16}$$

where τ is a constant delay. This system can be described by the following two-rule fuzzy model

Rule 1: *IF* $\theta(t)$ *is* $\pm\frac{\pi}{2}$, *THEN* $\dot{x}(t) = A_1 x(t) + A_{d1} x(t-\tau)$,
Rule 2: *IF* $\theta(t)$ *is* 0, *THEN* $\dot{x}(t) = A_2 x(t) + A_{d2} x(t-\tau)$
with the membership functions

$$r_1(t) = \frac{1}{1+e^{-2\theta_1(t)}}, \quad r_2(t) = \frac{e^{-2\theta_1(t)}}{1+e^{-2\theta_1(t)}},$$

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_{d1} = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} -1 & 0 \\ 0.1 & -1 \end{bmatrix}.$$

The stability bound of this system has been widely studied in Kwon et al. (2016); Zhao et al. (2018); Wang and Lam (2018) and so on. Using Theorem 7, the maximum delay bound τ_1 is 4.1495 under the condition $\dot{r}_1(t) \leq 0$, and the maximum delay bound τ_2 is 4.690 under the condition $\dot{r}_1(t) > 0$. For the case that only one condition exists, for example, $\theta(t) = t$ and then $\dot{r}_1(t) > 0$ always holds, we get $\tau_{max} = \tau_2 = 4.690$.

Suppose both conditions possibly exist for this system, for example, $\theta(t) = x_1(t)$ and the sign of $\dot{r}_1(t)$ depend on the state, then we get the maximum delay bound $\tau_M = \min\{\tau_1, \tau_2\} = 4.1495$. For this case, different maximum delay obtained in the existing papers and by Theorem are listed in Table I. Apparently, using the method in this

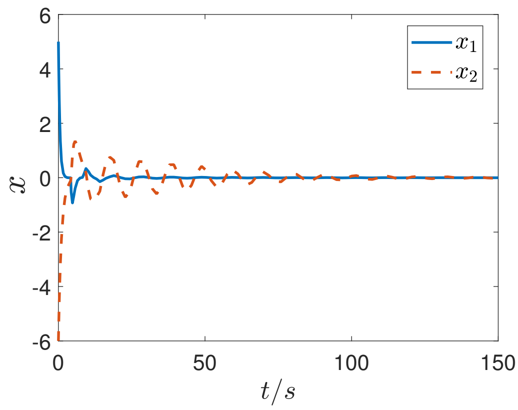

Fig. 1. State response for Example 1

paper yields a larger maximum bound. When $\tau = 4.1495$, the state response is given as Fig.1, and it shows that the system is asymptotically stable.

Table 1. The maximum admissible bound τ_M for Example 1

Methods	Delay interval
Kwon et al. (2016)	2.5932
Zhao et al. (2018)	3.3116
Wang and Lam (2018)	3.4859
Sheng et al. (2021)	3.6167
Wang et al. (2022) Corollary (q=3)	3.6947
Wang et al. (2022) Theorem* (q=3)	3.6928
Theorem 7	4.1495

Example 2 Consider the T-S fuzzy system (12) with

$$A_1 = \begin{bmatrix} -2.1 & 0.1 \\ -0.2 & -0.9 \end{bmatrix}, A_{d1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, A_{d2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}.$$

For this example, we get by applying Theorem 7, $\tau_1 = 5.9999$ for $\dot{r}_1 < 0$ and $\tau_2 = 7.1376$ for $\dot{r}_1 \geq 0$. Since the derivatives are unknown, the final maximum delay bound which can assure the stability is $\tau_{max} = \min\{\tau_1, \tau_2\}$. Table 2 lists the computed upper bounds by our method and different approaches in An and Wen (2011); Zeng et al. (2014); Kwon et al. (2016); Zhi et al. (2021); Tan et al. (2018); Zhang et al. (2021). It shows that our result is less conservative.

Table 2. The maximum admissible bound τ_M for Example 2

Methods	Delay interval
An and Wen (2011)(m=10)	4.41
Zeng et al. (2014)(m=3)	4.37
Kwon et al. (2016)	5.5826
Zhi et al. (2021)	5.5973
Tan et al. (2018)	5.73
Zhang et al. (2021)	5.92
Theorem 7	5.9999

6. CONCLUSION

This paper provides a new approach to stability analysis of a class of T-S fuzzy systems with delays. A fuzzy-PIE equation is first proposed, which provides an alternative, compact and delay-free representation for T-S fuzzy systems with delays. By means of the LPI method, a less conservative LPI-form stability condition for the fuzzy-PIE system is presented and then applied to the T-S fuzzy systems with delays. The effectiveness has been shown through two numerical examples. Future work will extend the problems of stability and constant delay to control and time-varying delay case.

REFERENCES

An, J. and Wen, G. (2011). Improved stability criteria for time-varying delayed T-S fuzzy systems via delay partitioning approach. 185(1), 83–94. doi: 10.1016/j.fss.2011.06.016.

Datta, R., Dey, R., Bhattacharya, B., Saravanakumar, R., and Kwon, O.M. (2020). Stability and stabilization of T-S fuzzy systems with variable delays via new bessel–legendre polynomial based relaxed integral inequality. *Information Sciences*, 522, 99–123.

Gu, K., Chen, J., and Kharitonov, V.L. (2003). *Stability of time-delay systems*. Springer Science and Business Media.

Hua, C., Wu, S., and Guan, X. (2020). Stabilization of T-S fuzzy system with time delay under sampled-data control using a new looped-functional. *IEEE Transactions on Fuzzy Systems*, 400–407.

Jafarov, E.M. (2001). Comparative analysis of simple improved delay-dependent stability criterions for linear time-delay systems: an augmented functional approach. In *Proceedings of the 2001 American Control Conference*, volume 5, 3389–3394. IEEE.

Kwon, O., Park, M., Park, J.H., and Lee, S. (2016). Stability and stabilization of T-S fuzzy systems with time-varying delays via augmented lyapunov-krasovskii functionals. *Information Sciences*, 372, 1–15.

Li, G., Peng, C., Xie, X., and Xie, S. (2021). On stability and stabilization of T-S fuzzy systems with time-varying delays via quadratic fuzzy lyapunov matrix. *IEEE Transactions on Fuzzy Systems*. doi: 10.1109/TFUZZ.2021.3128062.

Lian, Z., He, Y., Zhang, C.K., and Wu, M. (2020). Stability and stabilization of T-S fuzzy systems with time-varying delays via delay-product-type functional method. *IEEE transactions on cybernetics*, 50(6), 2580–2589.

Peet, M.M. (2020). Representation of networks and systems with delay: DDEs, DDFs, ODE–PDEs and PIEs. *Automatica*.

Peet, M.M. and Gu, K. (2019). SOS for systems with multiple delays: Part 1. H_∞ -optimal control. In *2019 American Control Conference (ACC)*.

Qiu, Y., Hua, C., and Wang, Y. (2021). Nonfragile sampled-data control of T-S fuzzy systems with time delay. *IEEE Transactions on Fuzzy Systems*. doi: 10.1109/TFUZZ.2021.3107748.

Sadek, B., Houssaine, T.E., and Noreddine, C. (2022). On designing lyapunov-krasovskii functional for time-varying delay t–s fuzzy systems. *Journal of the Franklin Institute*, 359, 2192–2205.

Seuret, A. and Gouaisbaut, F. (2013). Wirtinger-based integral inequality: Application to time-delay systems. *Automatica*, 49(9), 2860–2866.

Seuret, A. and Gouaisbaut, F. (2014). Complete quadratic lyapunov functionals using bessel-legendre inequality. In *2014 European Control Conference (ECC)*, 448–453. IEEE.

Sheng, Z., Lin, C., Chen, B., and Wang, Q.G. (2021). An asymmetric lyapunov-krasovskii functional method on stability and stabilization for T-S fuzzy systems with time delay. *IEEE Transactions on Fuzzy Systems*. doi: 10.1109/TFUZZ.2021.3076512.

Shivakumar, S., Das, A., and Peet, M.M. (2020a). PIETOOLS: A MATLAB toolbox for manipulation and optimization of partial integral operators. *American Control Conference (ACC)*.

Shivakumar, S., Das, A., Weiland, S., and Peet, M. (2020b). Duality and H_∞-optimal control of coupled ode-pde systems. *IEEE Conference on Decision and Control*.

Souza, F.O., Campos, V.C.S., and Palhares, R.M. (2014). On delay-dependent stability conditions for takagi-sugeno fuzzy systems. *Journal of the Franklin Institute*, 351(7), 3707–3718.

Tan, J., Dian, S., Zhao, T., and Chen, L. (2018). Stability and stabilization of T-S fuzzy systems with time delay via wirtinger-based double integral inequality. *Neurocomputing*, 275, 1063–1071.

Wang, L.K. and Lam, H.K. (2018). A new approach to stability and stabilization analysis for continuous-time takagi–sugeno fuzzy systems with time delay. *IEEE Transactions on Fuzzy Systems*, 26(4), 2460–2465.

Wang, L.K. and Lam, H.K. (2019). New stability criterion for continuous-time takagi-sugeno fuzzy systems with time-varying delay. *IEEE Transactions on Cybernetics*, 49(4), 1551–1556.

Wang, L.K., Lam, H.K., and Gu, J.H. (2022). Stability and stabilization for fuzzy systems with time delay by applying polynomial membership function and iteration algorithm. *IEEE Transactions on Cybernetics*. doi: 10.1109/LCSYS.2020.3038758.

Wang, L. and Liu, J. (2018). Local stability analysis for continuous-time takagi–sugeno fuzzy systems with time delay. *Neurocomputing*, 273, 152–158.

Wu, M., He, Y., She, J.H., and Liu, G.P. (2004). Delay-dependent criteria for robust stability of time-varying delay systems. *Automatica*, 40(8), 1435–1439.

Wu, S., Peet, M.M., and Hua, C. (2019). Estimator-based output-feedback stabilization of linear multi-delay systems using SOS. *IEEE*.

Zeng, H., Park, J.H., Xia, J.W., and Xiao, S.P. (2014). Improved delay-dependent stability criteria for t–s fuzzy systems with time-varying delay. *Applied Mathematics and Computation*, 235, 492–501.

Zhang, Z., Wang, D., and Gao, X.Z. (2021). Stability and stabilization condition for ts fuzzy systems with time-delay under imperfect premise matching via an integral inequality. *Int. J. Comput. Intell. Syst.*, 14(1), 11–22.

Zhao, X., Lin, C., Chen, B., and Wang, Q.G. (2018). A novel lyapunov-krasovskii functional approach to stability and stabilization for T-S fuzzy systems with time delay. *Neurocomputing*, 313, 288–294.

Zhi, L., He, Y., and Wu, M. (2021). Stability and stabilization for delayed fuzzy systems via reciprocally convex matrix inequality. *Fuzzy Sets and Systems*, 402, 124–141. Theme: Control Engineering.