Spacecraft Dynamics and Control

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Lecture 9: Bi-elliptics and Out-of-Plane Maneuvers

Introduction

In this Lecture, you will learn:

Bi-elliptic Maneuvers

- 3-burn Maneuvers
- Comparison with Hohmann
- Numerical Example
  - Elliptic
  - Circular

Out-of-Plane Maneuvers

- Inclination Change
- Right Ascension Change

Numerical Problem: Suppose we are in a circular parking orbit at altitude $191km$. We desire a final altitude of $376,310km$. Design the energy optimal orbital maneuvers necessary to reach our desired orbit.

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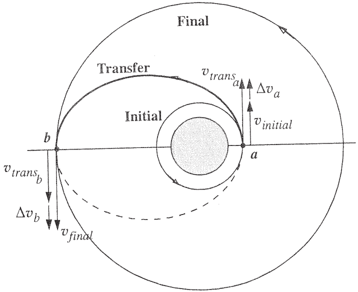
Numerical Problem: Suppose we are in a circular parking orbit at altitude $191km$. We desire a final altitude of $376,310km$. Design the energy optimal orbital maneuvers necessary to reach our desired orbit.

- For low earth orbit, RA has a linear progression due to earth's equatorial bulge. This effect is less significant at outer orbits and can be ignored or corrected.

The Oberth Effect

Generally it is better to make the initial burn at perigee.



For a burn at velocity v, the change in kinetic energy is

$$\Delta T = \frac{1}{2}(v + \Delta v)^2 - \frac{1}{2}v^2 = \frac{1}{2}\Delta v^2 + v \cdot \Delta v$$

For a fixed Δv, the second term $v \cdot \Delta v$ is much greater when v is large.

- For an elliptic orbit, maximum velocity is at perigee
- Lower orbits move faster
- It is much easier to achieve escape velocity when in low earth orbit

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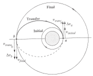

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- Of course, potential energy (normalized by mass) is the same before and after the burn!

This is also, why we burn at 0° Flight path angle.

- Otherwise, e.g. for $\angle FPA = 90^\circ$

$$\vec{v_f} = \begin{bmatrix} v \\ \Delta v \end{bmatrix}$$

- In this case, we have

$$T = \frac{1}{2}\|\vec{v}\|^2 = \frac{1}{2}(v^2 + \Delta v^2)$$

- So

$$\Delta T = \frac{1}{2}(v^2 + \Delta v^2) - \frac{1}{2}v^2 = \frac{1}{2}\Delta v^2$$

- You lose out on all of $v \cdot \Delta v$
- Since typically $v >> \Delta v$, the energy added is a fraction of the energy added for $\angle FPA = 0^\circ$, where $\Delta T = \frac{1}{2}\Delta v^2 + v \cdot \Delta v$

The Oberth Effect: Energy Explanation

Propulsive force results from expulsion of particles at high velocity.

Kinetic Energy of Propellant

- Suppose craft moving at velocity v_s.
- Particles are ejected with relative velocity $\Delta v_p > v_s$
- Absolute velocity of particles is $v_s - \Delta v_p$.
- Kinetic Energy of particles is

$$T_p \cong (v_s - \Delta v_p)^2$$

- The closer v_s is to Δv, the lower the kinetic energy.

Potential Energy of Propellant

- The potential energy of the propellant is

$$V = -\sqrt{\frac{\mu}{r}}$$

- the lower the propellant is ejected, the lower the potential energy

Conclusion: Propellant used at perigee has much less energy.

The energy not spent on propellant is retained by the spacecraft.

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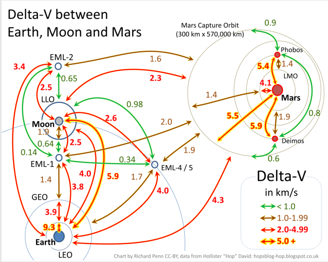
The energy not spent on propellant is retained by the spacecraft.

- For this reason, all significant Δv maneuvers in interplanetary missions are done as close to the gravity well of a planet as possible.
- This is entirely separate from the slingshot effect, but in both cases, a low periapse radius is desirable.

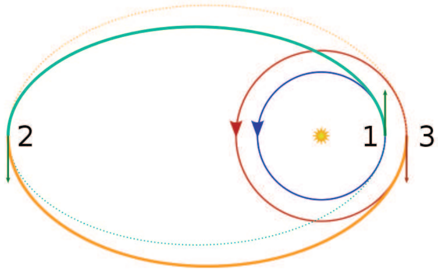
- Note this strategy is only effective when you are trying to increase the energy of the orbit.
- Doesn't apply to plane-change maneuvers.
- For apogee lowering, we want to dump as much energy as possible.



The Bi-Elliptic Transfer

The Hohmann transfer is the energy-optimal *2-impulse* transfer.

- Addition Energy savings can be bought at the expense of additional time.
- a 3-impulse trajectory



The bi-elliptic transfer uses the Oberth effect

1. initial impulse close to escape velocity.
2. perigee-raising maneuver at apogee.
3. apogee-*lowering* maneuver at perigee.

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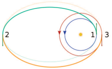


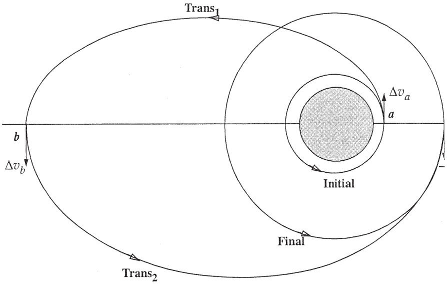
The bi-elliptic transfer uses the Oberth effect

1. initial impulse close to escape velocity.
2. perigee-raising maneuver at apogee.
3. apogee-*lowering* maneuver at perigee.

- Perigee raising gets easier the farther out you are.
- less Δv required.
- Less wasted propellant energy

The Bi-Elliptic Transfer

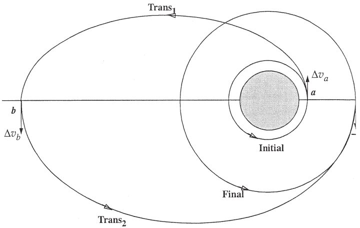
Suppose we want to raise a circular orbit of radius r_1 to radius r_2.



3 burns are required. Given r_1 and r_2, choose transfer radius $r_* >> r_f$.

1. Convert circular initial orbit at radius r_1 to elliptic transfer orbit 1 with perigee $r_p = r_1$ and apogee $r_a = r_*$.
2. At apogee, raise perigee of elliptic transfer orbit 2 to r_2.
3. At Perigee, circularize the final orbit by lowering perigee to r_2.

Suppose we want to raise a circular initial orbit of radius r_i to final circular orbit of radius r_f.



There are 2 transfer ellipses (both with apogee at r_*)(perigees at r_i and r_f)

Ellipse 1:

$$a_1 = \frac{r_i + r_*}{2}$$
$$e_1 = \frac{r_* - r_i}{r_* + r_i}$$

Ellipse 2:

$$a_2 = \frac{r_f + r_*}{2}$$
$$e_2 = \frac{r_* - r_f}{r_* + r_f}$$

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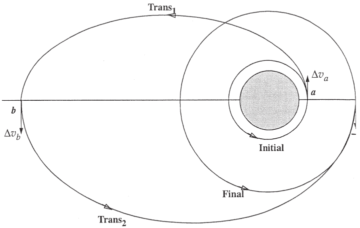


We can calculate the 3 burns as:

Burn 1:

$$\Delta v_1 = v_{1,p} - v_i = \sqrt{2\mu \frac{r_*}{r_i(r_i + r_*)}} - \sqrt{\frac{\mu}{r_i}}$$

Burn 2:

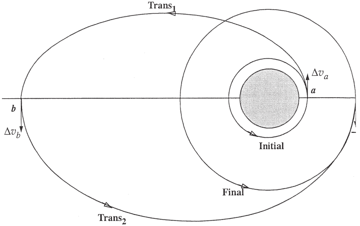
$$\Delta v_2 = v_{2,a} - v_{1,a} = \sqrt{2\mu \frac{r_f}{r_*(r_f + r_*)}} - \sqrt{2\mu \frac{r_i}{r_*(r_i + r_*)}}$$

Burn 3:

$$\Delta v_3 = v_f - v_{2,p} = \sqrt{\frac{\mu}{r_f}} - \sqrt{2\mu \frac{r_*}{r_f(r_f + r_*)}}$$

Notes on the Bi-Elliptic Transfer

Suppose we want to raise a circular initial orbit of radius r_i to final circular orbit of radius r_f.



Note that the third burn is retrograde.

- Δv_3 is clearly wasted energy.
- For this reason, bielliptics only work when $r_f >> r_i$ ($R := \frac{r_f}{r_i} \cong 11.94$).
  - $v_f << v_i$

Note that r_* is a free parameter.

- As $r_* \to \infty$, the bielliptic gets *more* efficient.
  - Escape and reinsertion.
- As $r_* \to \infty$, $\Delta t \to \infty$.
  - A tradeoff between time and efficiency.

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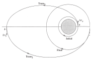

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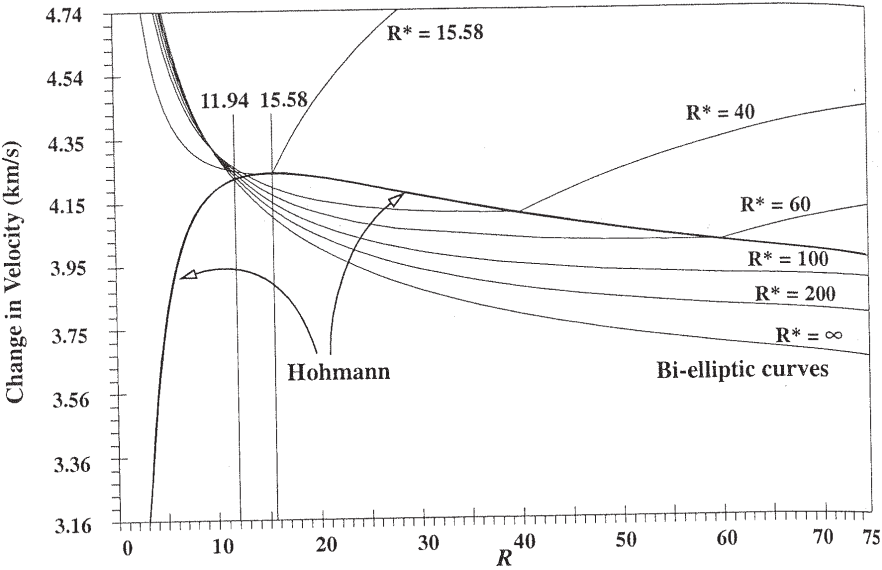
  - A tradeoff between time and efficiency.

- Recall the energy of an orbit is $E = -\frac{\mu}{2a}$.
- The difference in energy between target and initial orbit is partly a product of the kinetic energy change.
- The Oberth effect only becomes important when the energy difference between the orbits is large.

Notes on the Bi-Elliptic Transfer ($R = \frac{r_f}{r_i}$)



Numerical Example

Problem: Suppose we are in a circular parking orbit at altitude $191km$. We desire a final altitude of $376,310km$. Design the energy optimal orbital maneuvers necessary to reach our desired orbit.

Solution: First we choose between Hohmann and bi-elliptic. Note

$$r_i = 191km + 1ER = 1.03ER \qquad \text{and} \qquad r_f = 376,310km + 1ER = 60ER$$

Thus our ratio $R \cong 60$. In this case, it is clear that the bi-elliptic is better.

We choose a transfer radius of $r_* = 80ER$.

Ellipse 1: Our first transfer ellipse will have $a_1 = \frac{r_i + R_*}{2} = 40.5ER$. We have the following data

$$v_i = .985ER/TU$$
$$v_{1,p} = 1.385ER/TU$$
$$v_{1,a} = .0178ER/TU$$

Thus our initial velocity change is

$$\Delta v_1 = v_{1,p} - v_i = 1.385 - .985 = .4ER/TU$$

Ellipse 2: Our second transfer ellipse will have $a_2 = \frac{r_f + R_*}{2} = 70ER$. We have the following data

$$v_f = .129ER/TU$$
$$v_{2,p} = .138ER/TU$$
$$v_{2,a} = .103ER/TU$$

Our change from ellipse 1 to ellipse 2 requires

$$\Delta v_2 = v_{2,a} - v_{1,a} = .103 - .0178 = .0857ER/TU$$

Our final circularization requires

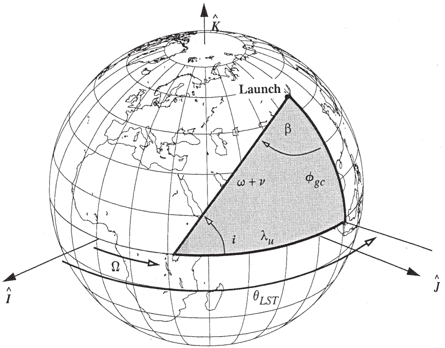
$$\Delta v_3 = v_f - v_{2,p} = .129 - .138 = -.009ER/TU$$

Conclusion:

- Our total Δv budget is $.4938ER/TU = 3.9km/s$.
- Budget for Hohmann is $4.0km/s$.
- The total duration of transit is 2650 TU = 593.9hr = 24.75 days.

Most satellites are launched from the surface of the earth.

- Launch Geometry restricts the initial orbital plane.

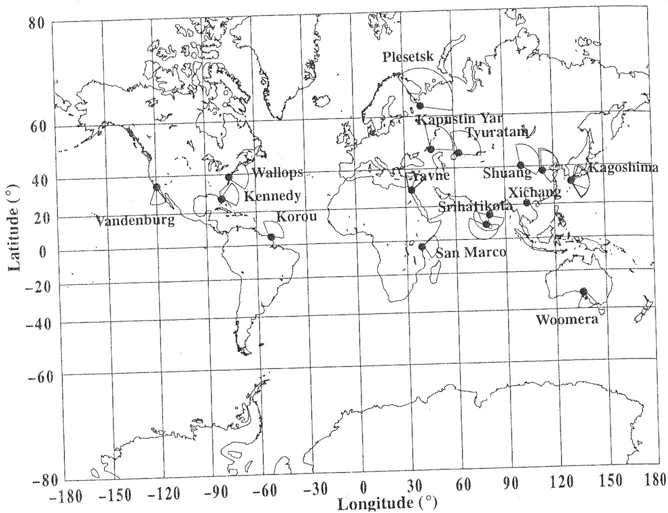


The two geometric features/constraints of launch are:

- latitude of the launch site, ϕ_{gc} (fixed).
- launch azimuth (direction), β (range of values).

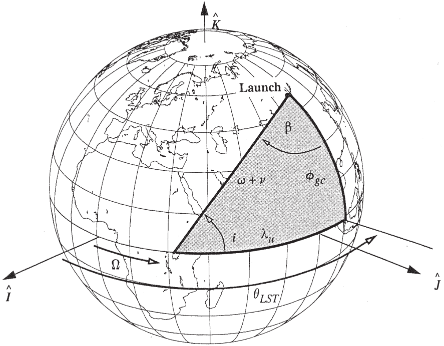
Launch Geometry

Site Restrictions

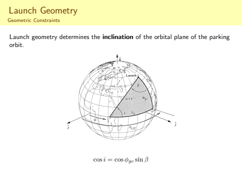


- The set of launch sites is restricted
- The range of launch azimuth is restricted

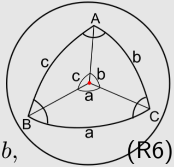
Launch geometry determines the **inclination** of the orbital plane of the parking orbit.



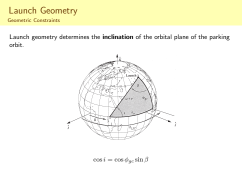
$$\cos i = \cos \phi_{gc} \sin \beta$$



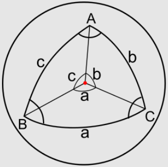
- Right ascension is also restricted, but can be modified by time-of-launch. To be discussed shortly.
- If $\beta = \pm 90^\circ$, then the minimum i is restricted by $i \geq \phi_{gc}$.
- Equation derived from Napier's rules for spherical right triangles ($C = 90^\circ$)
- $\beta = A$ (as defined on the previous figure) or $\beta = 180^\circ + A$ (using the tables)



(R1)	$\cos c = \cos a \, \cos b,$	(R6)	$\tan a = \cos B \, \tan c,$
(R2)	$\sin b = \sin B \, \sin c,$	(R7)	$\tan b = \cos A \, \tan c,$
(R3)	$\sin a = \sin A \, \sin c,$	(R8)	$\cos B = \sin A \, \cos b,$
(R4)	$\tan b = \tan B \, \sin a,$	(R9)	$\cos A = \sin B \, \cos a,$
(R5)	$\tan a = \tan A \, \sin b,$	(R10)	$\cos c = \cot B \, \cot A.$



- Alternative Napier's rules for spherical right triangles ($B = 90°$)
- In this case, $\beta = -A$ (as defined on the previous figure) or $\beta = 180° - A$ (using the tables)



(R1)	$\cos b = \cos a \, \cos c,$	(R6)	$\tan a = \cos C \, \tan b,$
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Site	Latitude (°)	Longitude (°)	Azimuth Min (°)	Azimuth Max (°)
Vandenberg	34.600 000	–120.600 000	147	201
Cape Kennedy	28.500 000	–80.550 000	37	112
Wallops	37.850 000	–75.466 67	30	125
Kourou	5.200 000	–52.800 000	340	100
San Marco	–2.933 333	40.200 000	50	150
Plesetsk	62.800 000	40.600 000	330	90
Kapustin Yar	48.400 000	45.800 000	350	90
Tyuratam	45.600 000	63.400 000	340	90
Sriharikota	13.700 000	80.250 000	100	290
Shuang-Ch'Eng-Tzu	40.416 667	99.833 333	350	120
Xichang	28.250 000	102.200 000	94	105
Tai-yuan	37.766 667	112.500 000	90	190
Kagoshima	31.233 333	131.083 333	20	150
Woomera	–30.950 000	136.500 000	350	15
Yavne	31.516 667	34.450 000	350	120

Typically, different sites are used for different purposes.

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Example: Vandenburg has

$$\phi_{gc} = 34.6^\circ \qquad \text{and} \qquad \beta + 180^\circ \in [147^\circ, 201^\circ]$$

Therefore

$$-.4483 < \cos i < .295$$

So the inclination is restricted as

$$72.84^\circ < i < 116.63^\circ$$

Launch Geometry

Site Restrictions

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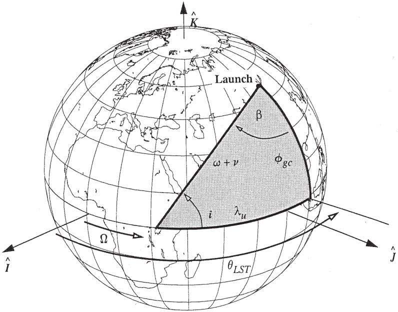
Azimuth in these tables is measured clockwise from due north $(\beta + 180^\circ)$

- There is a small correction to account for velocity gained from rotation of the Earth.
- I am unsure whether these tables include this correction.

Unlike inclination, the Right Ascension of the orbital plane can be chosen by **Launch Window**.



Referring to the triangle, our desired launch time (in Local Sidereal Time) is given by

$$\theta_{LST} = \Omega + \lambda_u$$

where λ_u can be found from β and i as

$$\cos \lambda_u = \frac{\cos \beta}{\sin i}$$

Launch Window
RAAN

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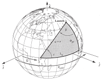

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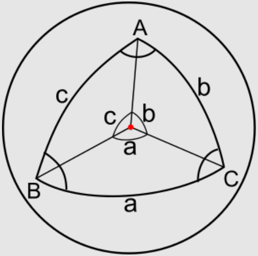

Law of Cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

Law of Sines:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

Here $a = \lambda_u$, $A = \beta$, $C = 90^\circ$, $B = i$.

Formula actually comes from (R9): $\cos A = \sin B \, \cos a$

Example: Launching into the Ecliptic Plane

For interplanetary missions, it is often desirable to establish an initial parking orbit aligned with the ecliptic plane.

- Desired RAAN: $\Omega = 0$
- Desired inclination: $i = 23.5°$
- Launch Site: Kourou/GSC ($\phi_{gc} = 5.2°$, $\theta_K = -52.8°$, $\beta \in [160°, 280°]$)

Challenge: Find θ_{LST} and β! Is it in the range of launch azimuths?

First, we note that since $\Omega = 0$, $\theta_{LST} = \lambda_u$

$$\cos i = \cos \phi_{gc} \sin \beta, \qquad \cos \theta_{LST} = \frac{\cos \beta}{\sin i}$$

Solving the first equation for β, we have

$$\beta = \sin^{-1}\left(\frac{\cos 23.5°}{\cos 5.2°}\right) = 67.05°, 112.95°$$

$$\theta_{LST} = cos^{-1}\left(\frac{\cos \beta}{\sin i}\right) = cos^{-1}\left(\frac{\cos 67.05°}{\sin 23.5°}\right) = 12.074°, 167.92°$$

We typically look for a posigrade orbit, so $\beta' = 180° + \beta$. Choosing $\beta = 67.05°$, we have

$$180° + \beta' = \beta = 247.05° \in [160°, 280°]$$

Note that $112° + 180° = 292° \notin [160°, 280°]$, so this is not a viable launch

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- Desired inclination: $i = 23.5^\circ$
- Launch Site: Kourou/GSC ($\phi_{gc} = 5.2^\circ$, $\theta_K = -52.8^\circ$, $\beta \in [160^\circ, 280^\circ]$)

Challenge: Find θ_{LST} and β! Is it in the range of launch azimuths?

First, we note that since $\Omega = 0$, $\theta_{LST} = \lambda_u$

$$\cos i = \cos\phi_{gc}\sin\beta, \qquad \cos\theta_{LST} = \frac{\cos\beta}{\sin i}$$

Solving the first equation for β, we have

$$\beta = \sin^{-1}\left(\frac{\cos 23.5^\circ}{\cos 5.2^\circ}\right) = 67.05^\circ, 112.95^\circ$$

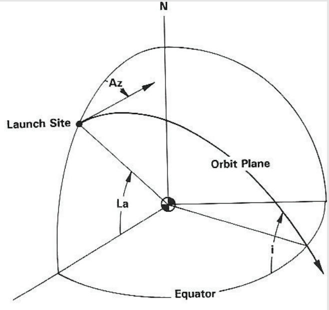
$$\theta_{LST} = cos^{-1}\left(\frac{\cos\beta}{\sin i}\right) = cos^{-1}\left(\frac{\cos 67.05^\circ}{\sin 23.5^\circ}\right) = 12.074^\circ, 167.92^\circ$$

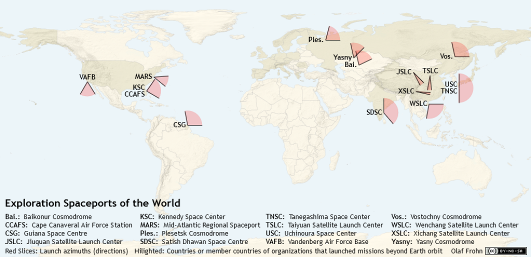
We typically look for a posigrade orbit, so $\beta' = 180^\circ + \beta$. Choosing $\beta = 67.05^\circ$, we have

$$180^\circ + \beta' = \beta = 247.05^\circ \in [160^\circ, 280^\circ]$$

Note that $112^\circ + 180^\circ = 292^\circ \notin [160^\circ, 280^\circ]$, so this is not a viable launch window!

- We have adjusted the table data from $\beta' \in [-20^\circ, 100^\circ]$ to $\beta \in [160^\circ, 280^\circ]$

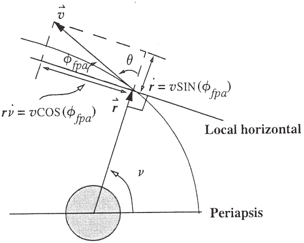


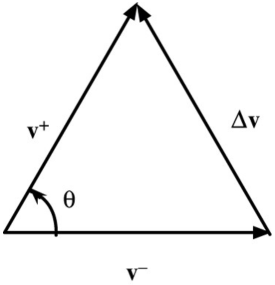


France owns French Guiana, and Kourou/CSG is a major spaceport for the ESA.

To change the inclination of an orbit requires Δv

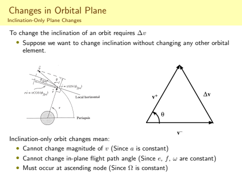
- Suppose we want to change inclination without changing any other orbital element.





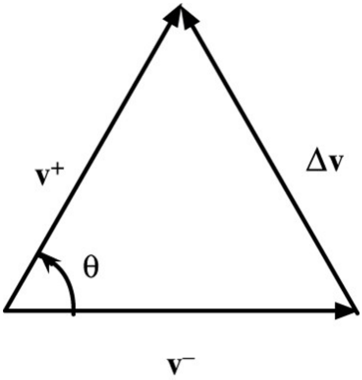
Inclination-only orbit changes mean:

- Cannot change magnitude of v (Since a is constant)
- Cannot change in-plane flight path angle (Since e, f, ω are constant)
- Must occur at ascending node (Since Ω is constant)



- Of course, we usually combine inclination changes with other orbit changes. We will address this in a later slide.
- Unlike changes in a, it is always better to change inclination when the velocity is *smallest*.
  - Oberth effect is not relevant because we are not adding energy to the orbit.

Inclination Only Plane Changes

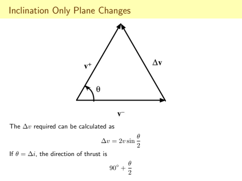


The Δv required can be calculated as

$$\Delta v = 2v \sin\frac{\theta}{2}$$

If $\theta = \Delta i$, the direction of thrust is

$$90^\circ + \frac{\theta}{2}$$

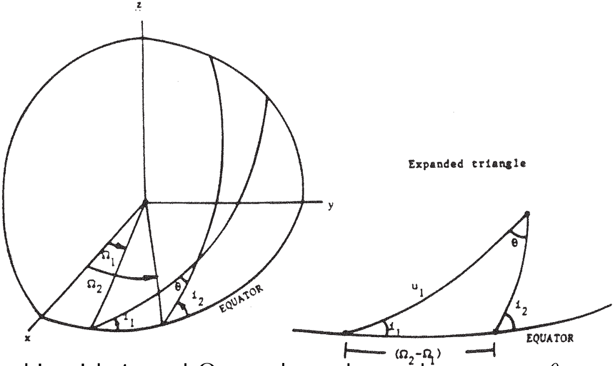


- For direction of thrust: Initial velocity is along the vector v^-. The direction of thrust is then measured by a $90^\circ + \frac{\Delta i}{2}$ counterclockwise rotation from the current velocity vector.
- The formula is derived by bisecting the triangle along the θ angle and calculating $\Delta v/2$.

Changes in Orbital Plane

General Rotations

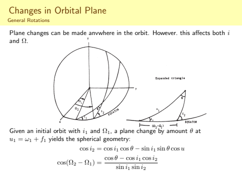
Plane changes can be made anywhere in the orbit. However, this affects both i and Ω.



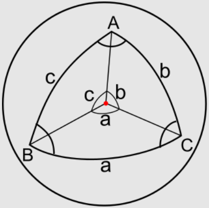
Given an initial orbit with i_1 and Ω_1, a plane change by amount θ at $u_1 = \omega_1 + f_1$ yields the spherical geometry:

$$\cos i_2 = \cos i_1 \cos\theta - \sin i_1 \sin\theta \cos u$$

$$\cos(\Omega_2 - \Omega_1) = \frac{\cos\theta - \cos i_1 \cos i_2}{\sin i_1 \sin i_2}$$



- $u_1 = \omega_1 + f_1$ is the arc measured from the ascending node (in the orbital plane). ω_1 is the argument of perigee of the initial orbit. f_1 is the true anomaly of the initial orbit at the time of the Δv.
- We are given Ω_1, ω_1, i_1 along with desired Ω_2, i_2
- We want to determine f_1 and θ. That is, when in the orbit to burn (f_1) and how big to make the angle change (θ).



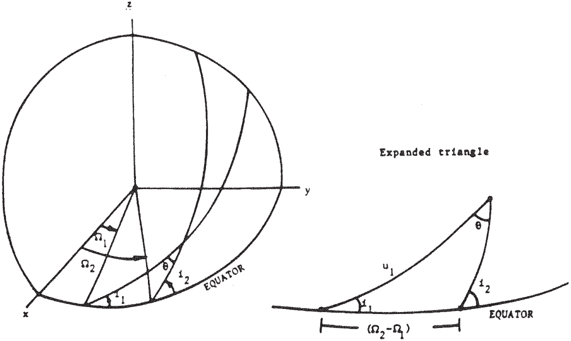
Law of Cosines:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

2nd Law of Cosines:

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a$$

Changing both Ω and i simultaneously is always more efficient than changing them separately.



If we are given an initial orbit with i_1 and Ω_1, along with desired elements i_2 and Ω_2, then required plane change (θ) and position (f_1) are given by:

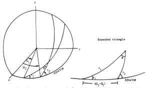
$$\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1)$$

$$\cos(u_1) = \cos(\omega_1 + f_1) = \frac{\cos i_1 \cos\theta - \cos i_2}{\sin i_1 \sin\theta}$$

Changes in Orbital Plane

Dual Purpose Plane Changes

Changing both Ω and i simultaneously is always more efficient than changing them separately.



If we are given an initial orbit with i_1 and Ω_1, along with desired elements i_2 and Ω_2, then required plane change (θ) and position (f_1) are given by:

$$\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1)$$

$$\cos(u_1) = \cos(\omega_1 + f_1) = \frac{\cos i_1 \cos\theta - \cos i_2}{\sin i_1 \sin\theta}$$

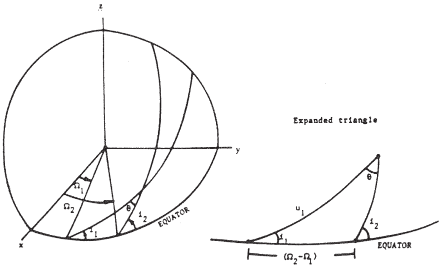
- First, we solve for θ (amount of plane change), then we solve for $u_1 = \omega_1 + f_1$. Then burn occurs at $f_1 = u_1 - \omega_1$
- Finding the new argument of periapse (ω_2) is a little complicated.

Combined Maneuvers

Inclination changes are by definition inefficient

$$\Delta v = 2v \sin \frac{\theta}{2}$$

- Up to 200% of total energy.
- Changes become more efficient as $\lim v \to 0$.
  - $v \to 0$ as $r \to \infty$.



It is often worth boosting the orbit to improve the efficiency of a plane change (See Homework.)

A typical strategy is to combine a plane change with a bi-elliptic transfer

Numerical Example: Combined Change

Problem: Suppose we are in an orbit with inclination $i = 55°$, $\Omega = 0°$ and $a = 1.8ER$. Determine the timing and Δv required to change the inclination to $i = 40°$ and RAAN to $\Omega = 45°$.

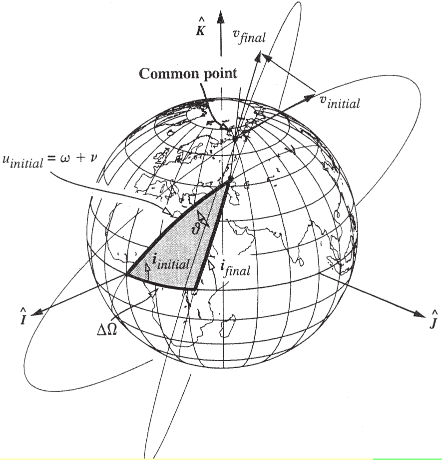
Solution: First find the plane change required

Using our formula,

$$\begin{aligned}\cos\theta &= \cos 55° \cos 40° \\ &\quad + \sin 55° \sin 40° \cos 45° \\ &= .8117\end{aligned}$$

Thus $\theta = 35.74°$. The timing for the Δv can be calculated from $u = \omega + f$ as

$$\begin{aligned}\cos u &= \frac{\cos 55° \cos 35.74° - \cos 40°}{\sin 35.74° \sin 55°} \\ &= -.628\end{aligned}$$



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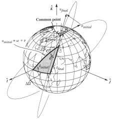


- In the image, a calligraphic form of θ is being used.

Numerical Example: Combined Change

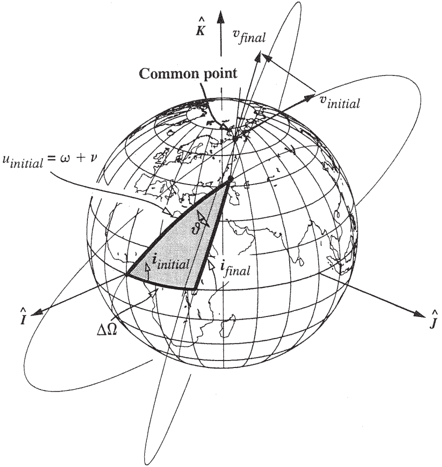
Since $\cos u = -.628$, we have that $u = 128.9°$.

Since the orbit is circular, $\omega = 0$ (or neglected). Thus the burn occurs at

$$f = 128.9°.$$

To calculate the Δv, we must first find the v at the desired point in the orbit. Since the orbit is circular, this is not difficult.

$$v = \sqrt{\frac{\mu}{r}} = .745 ER/TU$$

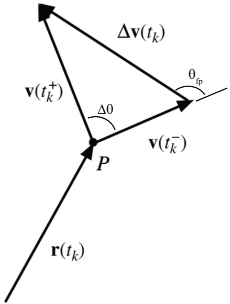


Then the required Δv can be calculated as

$$\Delta v = 2 \cdot v \cdot \sin\frac{\theta}{2} = .457 ER/TU$$

Suppose we want to combine a plane change $\Delta\theta$, with a velocity change

- For a perigee or apogee raising maneuver



- Initial velocity $v(t_k^-)$
  - Determined from initial or transfer orbit
- Final velocity $v(t_k^+)$
  - Determined from target or 2nd transfer orbit
- θ_{fp} is the direction of burn w/r to the current velocity vector.

Applies to both Hohmann and bi-elliptic transfers.

Law of Cosines: (To find magnitude of Δv)

$$\Delta v^2 = v(t_k^-)^2 + v(t_k^+)^2 - 2v(t_k^-)v(t_k^+)\cos\Delta\theta$$

Law of Sines: (To find direction of burn - θ_{fp}) AMBIGUITY!!!

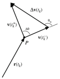
$$\frac{v(t_k^+)}{\sin(180^\circ - \theta_{fp})} = \frac{\Delta v}{\sin(\Delta\theta)} \quad \Rightarrow \quad \theta_{fp} = 180^\circ - \sin^{-1}\left(\frac{v(t_k^+)\sin(\Delta\theta)}{\Delta v}\right)$$

Combined Maneuvers
Change in Both Velocity and Plane

Suppose we want to combine a plane change $\Delta\theta$, with a velocity change
- For a perigee or apogee raising maneuver

- Initial velocity $v(t_k^-)$
  - Determined from initial or transfer orbit
- Final velocity $v(t_k^+)$
  - Determined from target or 2nd transfer orbit
- θ_{fp} is the direction of burn w/r to the current velocity vector.

Applies to both Hohmann and bi-elliptic transfers.



Law of Cosines: (To find magnitude of Δv)

$$\Delta v^2 = v(t_k^-)^2 + v(t_k^+)^2 - 2v(t_k^-)v(t_k^+)\cos\Delta\theta$$

Law of Sines: (To find direction of burn - θ_{fp}) AMBIGUITY!!!

$$\frac{v(t_k^+)}{\sin(180^\circ - \theta_{fp})} = \frac{\Delta v}{\sin(\Delta\theta)} \quad \Rightarrow \quad \theta_{fp} = 180^\circ - \sin^{-1}\left(\frac{v(t_k^+)\sin(\Delta\theta)}{\Delta v}\right)$$

- This logic leads to the common scenario where we boost the apogee/perigee while also performing an inclination change.
- In this case, the Δv magnitude and direction geometry is more complicated.
- Referring to the triangle, v^- is the initial magnitude of velocity. v^+ is the magnitude of the desired new orbit with altered r_a, r_p, etc.
- The angle between the two vectors is still $\theta = \Delta i$.
- Now, because the magnitudes of v^- and v^+ are different, we have to use the law of cosines to calculate the size of the Δv.
- Law of sines can then be used to compute the direction of burn.
- Note the law of sines has ambiguity which often affects the calculation of θ_{fp}. Probably better to use law of cosines for both parts!

$$\Delta v_{\mathrm{I}} = \sqrt{(v_2 - v_1)^2 + 4 v_1 v_2 \sin^2 \frac{\delta}{2}} \quad \text{(Rotation about the common apse line)} \tag{6.22}$$

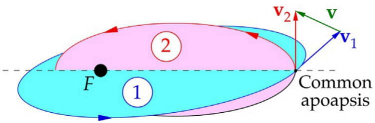


FIG. 6.23

Impulsive plane change maneuver at apoapsis.

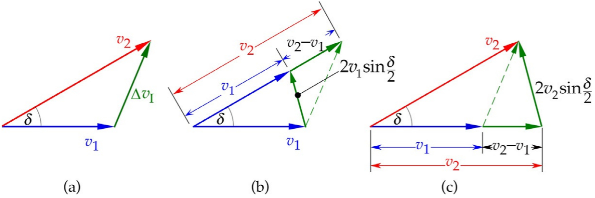


FIG. 6.24

The orbital plane rotates about the common apse line. (a) Speed change accompanied by plane change. (b) Plane change followed by speed change. (c) Speed change followed by plane change.

Summary

This Lecture you have learned:

Bi-elliptic Maneuvers

- 3-burn Maneuvers
- Comparison with Hohmann
- Numerical Example
  - Elliptic
  - Circular

Out-of-Plane Maneuvers

- Inclination Change
- Right Ascension Change

Next Lecture: Lambert's Problem.