Stability Analysis of Linear Time-Delay Systems using Semidefinite Programming

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Abstract

This paper presents an algorithm for stability analysis of linear time-delay systems. We describe a general methodology for proving stability of linear time-delay systems by computing solutions to an operator-theoretic version of the Lyapunov inequality via semidefinite programming. The result is stated in terms of a nested sequence of sufficient conditions which are of increasing accuracy. This approach is generalized to the case of parametric uncertainty by considering parameter-dependent Lyapunov functionals. Numerical examples are given to demonstrate convergence of the algorithm.

Key words: Time-delay, Infinite dimensional systems, Stability, Semidefinite programming, Parametric uncertainty

1 Introduction

The study of stability of systems of delayed linear systems has been an active area of research for some time. A complete summary of the results in this field is beyond the scope of this paper. However, an overview of these results can be obtained from various survey papers and books on the subject, see for example [1–4]. Previous results on this subject can usually be grouped into analysis either in the frequency-domain or in the time-domain. Frequency-domain techniques typically attempt to determine whether all roots of the characteristic equation of the system lie in the left half-plane. This approach is complicated by the transcendental nature of the characteristic equation, which imply the existence of a possibly infinite number of roots. Time-domain techniques generally use the potential energy methods of Lyapunov, an approach which was extended to systems of functional differential equations by Krasovskii in [5]. Stability results in this area are grouped into delay-dependent and delay-independent conditions. If a delay-dependent condition holds, then stability is guaranteed for a specific value or range of values of the delay. If a delay-independent condition holds, then the system is stable for all finite values of the delay. A particularly interesting result on the linear delay-independent case appears in [6].

In this paper, we address the question of stability of systems of linear differential equations which contain delay. Although the general question of stability of systems with linear dynamics and uncertain delays has been shown to be NP-hard [7], we obtain a nested sequence of sufficient conditions which are of non-decreasing accuracy and which approaches the analytical limit. Stability of the class of linear time-delay systems we consider has been shown to be equivalent to existence of a positive definite quadratic functional whose derivative is negative definite along trajectories of the system. Furthermore, it has been shown that one can assume such functionals to be defined by the combination of positive multiplier and integral operators. Therefore, stability of a linear time-delay system can be equivalently posed as the existence of a positive solution to a certain operator-theoretic Lyapunov inequality. This operator inequality, defined by the derivative of the complete quadratic functional, can be posed as a convex feasibility problem over certain infinite-dimensional convex cones defined by positive operators. Our approach to solving this optimization problem is to consider subsets of the convex cones mentioned which can be represented using polynomial functions of bounded degree. In this case, we show that the convex feasibility problem can be solved using semidefinite programming. Naturally, as we allow the degree bound to increase, the accuracy of our algorithm will increase, as

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will the size of the associated semidefinite program.

This paper is organized as follows. In the sections 2-4, we discuss background on linear time-delay systems and give necessary and sufficient conditions for stability in terms of existence of a positive quadratic functional and a negative Lie derivative functional which are defined by multiplier and integral operators. We then present a parametrization of the set of positive polynomials of bounded degree using positive semidefinite matrices. We also present a generalization to matrices of polynomials. Following this, we demonstrate how to represent certain positive multiplier and integral operators, defined by polynomials of bounded degree, using semidefinite programming constraints. In Sections 5-7, we construct the semidefinite programs associated with three classes of linear time-delay systems. We will also present numerical examples which demonstrate the convergence of the algorithm for specific linear systems as a function of the degree bound. Finally, in section 8, we will conclude and discuss topics of ongoing research.

2 Notation

$\mathcal{C}$ denotes the space of continuous vector-valued functions. $\mathcal{C}_\tau$ denotes the space of continuous vector-valued functions defined on the interval $[-\tau, 0]$. For $x \in \mathcal{C}_\tau$, $\|x\|$ denotes the supremum norm. L_2 will be used to denote the Hilbert space of square Lebesgue-integrable vector-valued functions defined on an interval. $\|\cdot\|_2$ denotes the L_2 norm. We will occasionally also associate with $\mathcal{C}_\tau$ an inner product space equipped with the inner product associated with L_2. Thus for $x, y \in \mathcal{C}_\tau$, $\langle x, y\rangle$ denotes the L_2 inner product. An operator A on an inner-product space X is defined to be positive if $\langle x, Ax\rangle \geq 0$ for all $x \in X$.

Let $\mathbb{S}^n$ denote the space of $n \times n$ symmetric matrices. Denote by $\mathbb{R}[x]$ the ring of scalar polynomials in variables x. Denote by $\mathbb{R}^{n\times m}[x]$ the set of n by m matrices with scalar elements in $\mathbb{R}[x]$. Let $\mathbb{S}^n[x]$ denote the set of symmetric n by n matrices with elements in $\mathbb{R}[x]$ and define $\mathbb{S}^n_d[x]$ to be the element of $\mathbb{S}^n[x]$ of degree d or less. Let $\mathcal{P}^Y \subset \mathbb{R}[x]$ denote the convex cone of scalar polynomials which are non-negative on Y. Let $\mathcal{P}^+ \subset \mathbb{R}[x]$ denote the convex cone of globally non-negative scalar polynomials. Let $\mathcal{S}^Y_n \subset \mathbb{S}^n[x]$ denote the convex cone of elements $M \in \mathbb{S}^n[x]$ such that $M(x) \geq 0$ for all $x \in Y$. Let $\mathcal{S}^+_n$ denote the convex cone of elements $M \in \mathbb{S}^n[x]$ such that $M(x) \geq 0$ for all x. We let $Z_d[x]$ denote the $\binom{n+d}{d}$-dimensional vector of monomials in n variables x of degree d or less. Define $\bar{Z}^n_d[x] := I_n \otimes Z_d[x]$, where I_n is the identity matrix in $\mathbb{S}^n$. Finally, we note that for the sake of notational convenience, we will often use the expression $M(x) \in X$ to indicate that the function $M \in X$ for some set of functions X. This will hopefully not cause substantial amounts of confusion.

3 Background

In this paper, we consider time-delay systems which can be expressed in the following form, where we assume for convenience that $\tau_i > \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$.

$$\dot{x}(t) = \sum_{i=0}^{K} A_i x(t-\tau_i) + \int_{-\tau_K}^{0} A(\theta)x(t+\theta)d\theta \qquad (1)$$

Here $A_i \in \mathbb{R}^{n\times n}$ and $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is bounded on $[-\tau_K, 0]$. We say that $x \in \mathcal{C}$ is a ***solution*** of Equation (1) with initial condition $x_0 \in \mathcal{C}_{\tau_K}$ if $x(t) = x_0(t)$ for $t \in [-\tau_K, 0]$ and Equation (1) holds for all $t \geq 0$. It can be shown that elements of this class of system admit a unique solution for every initial condition $x_0 \in \mathcal{C}_{\tau_K}$. We can associate with systems of this form a solution map $G : \mathcal{C}_{\tau_K} \to \mathcal{C}$, where $x = Gx_0$ if x is a solution of Equation (1) with initial condition x_0. Stability is defined in terms of this solution map.

Definition 1 *The solution map G defined by equation (1) is* ***stable*** *if G is continuous at 0 with respect to the supremum norms on $\mathcal{C}$ and $\mathcal{C}_{\tau_K}$.*

Definition 2 *The solution map G defined by equation (1) is* ***asymptotically stable*** *if $x = Gx_0$ implies $\lim_{t\to\infty} x(t) = 0$ for any $x_0 \in \mathcal{C}_{\tau_K}$*

With any solution x and time $t \geq 0$, one can denote the element $x_t \in \mathcal{C}_{\tau_K}$, where $x_t(\theta) = x(t+\theta)$ for all $\theta \in [-\tau_K, 0]$. This element is considered to be the ***state*** of the system at time t. For a given solution map, we define the ***flow map*** $\Gamma : \mathcal{C}_{\tau_K} \times \mathbb{R}_+ \to \mathcal{C}_{\tau_K}$ by

$$y = \Gamma(x_t, \Delta t) = (Gx_t)_{\Delta t} = x_{t+\Delta t},$$

which maps the state at time t to the state at time $t+\Delta t$.

3.1 Stability Theorem

Lyapunov Theory can be extended to time-delay systems through the use of Lyapunov-Krasovskii functionals. One such extension is defined by the following general theorem [2] which can be applied to the nonlinear as well as the linear case.

Theorem 3 *Consider a solution map G defined by Equation (1). Suppose $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, and that $u(s) > 0, v(s) > 0$ for $s > 0$ and $u(0) = v(0) = 0$. Suppose there exists a continuous functional $V : \mathcal{C} \to \mathbb{R}$ such that both of the following hold for all $\phi \in \mathcal{C}_\tau$.*

$$u(|\phi(0)|) \leq V(\phi) \leq v(\|\phi\|) \qquad (2)$$

$$\dot{V}(\phi) = \limsup_{\Delta t\to 0^+} \frac{1}{\Delta t}\left(V(\Gamma(\phi, \Delta t)) - V(\phi)\right) \leq -w(|\phi(0)|)$$

Here Γ is the flow map defined by G. Then the solution map G is stable. If $w(s) > 0$ for $s > 0$, then the solution map is asymptotically stable.

3.2 Complete Quadratic Functionals

There have been a number of results concerning necessary and sufficient conditions for stability of linear time-delay systems in terms of the existence of quadratic functionals. These results are significant in that they allow us to restrict our search for a Lyapunov-Krasovskii functional to a specific class without introducing any conservatism. Recall that we consider differential equations of the following form.

$$\dot{x}(t) = \sum_{i=0}^{K} A_i x(t-\tau_i) + \int_{-\tau_K}^{0} A(\theta)x(t+\theta)d\theta \qquad (3)$$

We now make the additional assumption that A is continuous on $[-\tau_K, 0]$. The following comes from Gu et al. [1].

Definition 4 *We say that a functional $V : \mathcal{C}_\tau \to \mathbb{R}$ is of the complete quadratic type if there exists a matrix $P \in \mathbb{S}^n$ and matrix-valued functions $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S : \mathbb{R} \to \mathbb{S}^n$ and $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta,\eta) = R(\eta,\theta)^T$ such that the following holds.*

$$V(\phi) = \phi(0)^T P\phi(0) + 2\phi(0)^T \int_{-\tau}^{0} Q(\theta)\phi(\theta)d\theta$$
$$+ \int_{-\tau}^{0} \phi(\theta)^T S(\theta)\phi(\theta)d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T R(\theta,\eta)\phi(\eta)d\theta d\eta$$

Theorem 5 *Suppose the system described by Equation (3) is asymptotically stable. Then there exists a complete quadratic functional V and $\eta > 0$ such that the following holds for all $\phi \in \mathcal{C}_\tau$.*

$$V(\phi) \geq \eta\|\phi(0)\|^2 \qquad \text{and} \qquad \dot{V}(\phi) \leq -\eta\|\phi(0)\|^2$$

Furthermore, the matrix-valued functions which define V can be taken to be continuous everywhere except possibly at points $\theta, \eta = -\tau_i$, for $i = 1, \dots, K-1$.

3.3 Sum-of-Squares and Convex Optimization

Many difficult problems in analysis and control can be reformulated as convex optimization problems of the following form.

$$\max \gamma : \qquad f_0(x) - \gamma \in \mathcal{P}^Y$$

Although the formulation of the problem is convex, it is not tractable since there exists no efficient test for membership in the set $\mathcal{P}^Y$. Indeed, the question of whether $f \in \mathcal{P}^+$, that is, $f(x) \geq 0$ for all $x \in \mathbb{R}^n$, is NP hard for polynomials of degree 4 or more. Thus there is unlikely to exist a computationally tractable set membership test for $\mathcal{P}^+$ unless $P = NP$. However, there may exist some convex cone $\Sigma \subset \mathcal{P}^+$ for which the set membership test is computationally tractable. One such cone is defined as follows.

Definition 6 *A polynomial $s \in \mathbb{R}[x]$ satisfies $s \in \Sigma_s \subset \mathcal{P}^+$ if it can be represented in the following form for some finite set of polynomials $g_i \in \mathbb{R}[x]$, $i = 1, \dots, m$.*

$$s(x) = \sum_{i=1}^{m} g_i(x)^2$$

*An element $s \in \Sigma_s$ is referred to as a **sum-of-squares polynomial**. The set Σ_s^d is defined as the elements of Σ_s of degree d or less.*

In [8], it was shown that the set membership $f \in \Sigma_s$ can be represented as a semidefinite programming constraint.

Lemma 7 *A degree $2d$ polynomial $f \in \mathbb{R}[x]$ satisfies $f \in \Sigma_s$ if and only if there exists some matrix $Q \geq 0$ such that*

$$f(x) = Z_d[x]^T Q Z_d[x].$$

The equality constraint in Lemma 7 is affine in the monomial coefficients of f. Therefore, set membership in Σ_s can be tested using semidefinite programming. The question of how well Σ_s represents $\mathcal{P}^+$ has been a topic on ongoing research for some time. We refer the reader to the survey paper by Reznick [9] for an overview of this subject.

Membership in the cone $\mathcal{P}^K$ is harder to test than $\mathcal{P}^+$. Consider a region K_f defined by $K_f := \{x \in \mathbb{R}^n : f_i(x) \geq 0,\ i = 1, \dots, n_K\}$ for $f_i \in \mathbb{R}[x]$. A condition for membership in $\mathcal{P}^{K_f}$ comes from the following simplified version of a Positivstellensatz result from Putinar [10] for sets which satisfy a condition we call ***P-compact***.

Definition 8 *We say that $f_i \in \mathbb{R}[x]$ for $i = 1, \dots, n_K$ define a **P-compact** set K_f, if there exist $h \in \mathbb{R}[x]$ and $s_i \in \Sigma_s$ for $i = 0, \dots, n_K$ such that the level set $\{x \in \mathbb{R}^n : h(x) \geq 0\}$ is compact and such that the following holds.*

$$h(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

The condition that a region be P-compact may be difficult to verify. However, some important special cases include any region K_f such that all the f_i are linear and any region K_f defined by f_i such that there exists some i for which the level set $\{x : f_i(x) \geq 0\}$ is compact.

Theorem 9 *Suppose K_f, as defined above, is P-compact. Suppose p lies in the interior of $\mathcal{P}^{K_f}$. Then there exist $s_i \in \Sigma_s$ for $i = 1, \dots, n_K$, such that the following holds.*

$$p(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

Theorem 9 does not represent a tractable test for membership in $\mathcal{P}^{K_f}$ since no bounds exist for the degree of the s_i. However, it does provide a basis for a sequence of tractable sufficient conditions.

Definition 10 *For a given degree bound, d, and K_f of the form given above, we define the convex cone $\Upsilon_d^{K_f}$ of functions $p \in \mathbb{R}[x]$ such that there exist $s_i \in \Sigma_s^d$ for $i = 1, \ldots, n_K$ such that the following holds.*

$$p(x) - \sum_{i=1}^{n_K} s_i(x) f_i(x) \in \Sigma_s$$

3.4 SOS Constraints for Matrix-Valued Functions

We now present an extension of the above results to the convex cones $\mathcal{S}^+$ and $\mathcal{S}^K$ of non-negative matrices of polynomials. Of course, one can always test $M \in \mathcal{S}^+$ by testing whether $y^T M(x) y \in \Sigma_s$. However, the resulting introduction of auxiliary variables y dramatically increases the complexity. An alternative condition is given as follows.

Definition 11 *For $M \in \mathbb{S}^n[x]$, we denote $M \in \bar{\Sigma}_s$ if there exist $G_i \in \mathbb{R}^{n\times n}[x]$ such that $M(x) = \sum_{i=1}^m G_i(x)^T G_i(x)$. An element $M \in \bar{\Sigma}_s$ is referred to as a **sum-of-squares matrix function**. $\bar{\Sigma}_s^d$ are the elements of $\bar{\Sigma}_s$ of degree d or less.*

The following lemma shows that membership in $\bar{\Sigma}_s$ can be represented as a semidefinite programming constraint.

Lemma 12 *Suppose $M \in \mathbb{S}_{2d}^n[x]$. $M \in \bar{\Sigma}_s$ if and only if there exists some matrix $Q \geq 0$ such that the following holds.*

$$M(x) = \left(\bar{Z}_d^n[x]\right)^T Q \bar{Z}_d^n[x]$$

The complexity of the membership test associated with Lemma 12 is considerably lower than that associated with the introduction of auxiliary variables. Specifically, the variables associated with the first test are of order $n\binom{n_2+d}{d}$ as opposed to order $\binom{n+n_2+d+1}{d+1}$. Furthermore, the use of the test associated with Lemma 12 does not increase the conservativity since it can be shown that $M(x) \in \bar{\Sigma}_s$ if and only if $y^T M(x) y \in \Sigma_s$. As was the case for Σ_s, the question of how well $\bar{\Sigma}_s$ approximates $\mathcal{S}_n^+$ is an open question. However, in the case of a single variable, we have the following result due to Choi et al. [11].

Lemma 13 *For $x \in \mathbb{R}$, $M \in \mathbb{S}^n[x]$, $M \in \mathcal{S}_n^+$ if and only if $M \in \bar{\Sigma}_s$.*

Lemma 13 will find considerable application in Section 4.

Now consider the set $\mathcal{S}_n^{K_f}$ for some semi-algebraic set, K_f, defined by polynomials f_i. Putinar's Positivstellensatz was extended to SOS matrices in [12].

Theorem 14 *Suppose that K_f is P-compact and $M \in \mathcal{S}_n^{K_f}$. Then there exist $\eta > 0$ and $S_i \in \bar{\Sigma}_s$ for $i = 1, \ldots, n_f$ such that the following holds.*

$$M(x) - \sum_{i=1}^{n_f} f_i(x) S_i(x) - \epsilon I \in \bar{\Sigma}_s$$

This result leads to the following sequence of tractable conditions for membership in $\mathcal{S}^{K_f}$.

Definition 15 *For $d \geq 0$ and K_f as defined previously, we denote $M \in \bar{\Upsilon}_{n,d}^{K_f} \subset \mathcal{S}_n^{K_f}$ if there exist $S_i \in \bar{\Sigma}_s^d$ such that the following holds.*

$$M(x) - \sum_{i=1}^{n_f} f_i(x) S_i(x) \in \bar{\Sigma}_s$$

For any $d \geq 0$, we can express membership in $\bar{\Upsilon}_{n,d}^{K_f} \subset \mathcal{S}_n^{K_f}$ as a semidefinite programming constraint.

4 Positive Operators

In this section, we present the convex cones of positive multiplier and integral operators which define the complete quadratic functional and some its derivative forms. We then use polynomials of bounded degree to parametrization certain convex subsets of these cones in terms of positive semidefinite matrices. These results will allow us to express the Lyapunov stability conditions in terms of semidefinite programming problems. To begin, consider the complete quadratic functional.

$$V(x) = x(0)^T P x(0) + 2\int_{-\tau}^{0} x(0)^T Q(\theta) x(\theta) d\theta$$
$$+ \int_{-\tau}^{0} x(\theta)^T S(\theta) x(\theta) d\theta + \int_{-\tau}^{0}\int_{-\tau}^{0} x(\theta)^T R(\theta,\omega) x(\omega) d\theta d\omega$$

We can associate with V and $\epsilon > 0$ an operator $A : \mathcal{C}_\tau \to \mathcal{C}_\tau$ such that $V(x) = \langle x, Ax\rangle$, where A is defined as follows.

$$(Ay)(\theta) = \begin{bmatrix} P - \epsilon I & Q(\theta) \\ Q(\theta)^T & S(\theta) \end{bmatrix} y(\theta) + \int_{-\tau}^{0} \begin{bmatrix} 0 & 0 \\ 0 & R(\theta,\omega) \end{bmatrix} y(\omega) d\omega$$
$$= (A_1 y)(\theta) + (A_2 y)(\theta),$$

The operator A is a combination of multiplier operator A_1 and integral operator A_2. The complete quadratic functional, V satisfies the positivity condition of the stability theorem if the integral operator, A_2, is positive on $\mathcal{C}_\tau$ and there exists some $\epsilon > 0$ such that the multiplication operator, A_1, is positive on the subspace $X \subset \mathcal{C}_\tau$ where

$$X := \{x \in \mathcal{C}_\tau \mid, x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } x_1(\theta) = x_2(0) \text{ for all } \theta\}.$$

We now define the specific sets of operators which define the complete quadratic functional and its derivative forms.

Definition 16 *For a matrix-valued function $M : \mathbb{R} \to \mathbb{S}^n$, we denote by A_M the multiplication operator such that*

$$(A_M x)(\theta) = M(\theta)x(\theta)$$

Definition 17 *For a matrix-valued function $R : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $R(\theta,\omega) = R(\omega,\theta)^T$, we denote by B_R the integral operator such that*

$$(B_R x)(\theta) = \int_{-\tau}^{0} R(\theta,\omega)x(\omega)d\omega$$

Definition 18 *For a continuous matrix-valued function M, we say $M \in H_1^+$ if $\langle x, A_M x\rangle \ge 0$ for all $x \in X$.*

Definition 19 *For a continuous matrix-valued function $R : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ such that $R(\theta,\omega) = R(\omega,\theta)^T$, we say $R \in H_2^+$ if $\langle x, B_R x\rangle \ge 0$ for all $x \in \mathcal{C}_\tau$.*

Definition 20 *For a piecewise-continuous matrix-valued function M, we say $M \in \tilde{H}_1^+$ if $\langle x, A_M x\rangle \ge 0$ for all $x \in X$.*

Definition 21 *For a piecewise-continuous matrix-valued function $R : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{n\times n}$ such that $R(\theta,\omega) = R(\omega,\theta)^T$, we say $R \in \tilde{H}_2^+$ if $\langle x, B_R x\rangle \ge 0$ for all $x \in \mathcal{C}_\tau$.*

Definition 22 *For a continuous matrix-valued function M, we say $M \in H_3^+$ if $\langle x, A_M x\rangle \ge 0$ for all $x \in X_3$, where*

$$X_3 := \{x \in \mathcal{C}_\tau \mid , x = \begin{bmatrix} x_1^T & x_2^T & x_3^T \end{bmatrix}^T \text{ and } x_1(\theta) = x_3(0) \text{ and } x_2(\theta) = x_3(-\tau) \text{ for all } \theta\}.$$

Definition 23 *For a piecewise-continuous matrix-valued function M, we say $M \in \tilde{H}_3^+$ if $\langle x, A_M x\rangle \ge 0$ for all $x \in \tilde{X}_3$, where*

$$\tilde{X}_3 := \Big\{x \in \mathcal{C}_\tau \mid , x = \begin{bmatrix} x_1^T & \dots & x_{K+2}^T \end{bmatrix}^T \text{ and } x_i(\theta) = x_{K+2}(-\tau_{i-1}) \text{ for all } \theta,\ i = 1,\dots,K+1\Big\}.$$

4.1 Multiplier Operators and Spacing Functions

In this subsection, we consider the convex cones H_1^+ and H_3^+ which define positive multiplication operators on X_1 and X_3 respectively. We use the following theorem.

Theorem 24 *Suppose $M : \mathbb{R} \to \mathbb{S}^{2n}$ is a continuous matrix-valued function. Then the following are equivalent*

(1) There exists some $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_\tau$.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \ge \epsilon \|x\|_2^2$$

(2) There exists some $\epsilon' > 0$ and some continuous matrix-valued function $T : \mathbb{R} \to \mathbb{S}^n$ such that the following holds.

$$\int_{-\tau}^{0} T(\theta)d\theta = 0$$

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \ge 0 \qquad \text{for all } \theta \in [-\tau, 0]$$

PROOF. $(2 \Rightarrow 1)$ Suppose statement 2 is true, then

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon' \|x\|_2^2$$
$$= \int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \ge 0.$$

$(1 \Rightarrow 2)$ Suppose that statement 1 holds for some M. Write M as

$$M(\theta) = \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) \end{bmatrix}.$$

We first prove that $M_{22}(\theta) \ge \epsilon I$ for all $\theta \in [-\tau, 0]$. By statement 1, we have that

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & M_{22}(\theta) - \epsilon I \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \ge 0.$$

Now suppose that $M_{22}(\theta) - \epsilon I$ is not positive semidefinite for all $\theta \in [-\tau, 0]$. Then there exists some $x_0 \in \mathbb{R}^n$ and $\theta_1 \in [-\tau, 0]$ such that $x_0^T(M_{22}(\theta_1) - \epsilon I)x_0 < 0$. By continuity of M_{22}, if $\theta_1 = 0$ or $\theta_1 = -\tau$, then there exists some $\theta_1' \in (-\tau, 0)$ such that $x_0^T(M_{22}(\theta_1') - \epsilon I)x_0 < 0$. Thus assume $\theta_1 \in (-\tau, 0)$. Now, since M_{22} is continuous, there exists some x_1 and $\delta > 0$ where $\theta_1 + \delta < 0$, $\theta_1 - \delta > -\tau$ and such that $x_1^T(M_{22}(\theta) - \epsilon I)x_1 \le -1$ for all $\theta \in [\theta_1 - \delta, \theta_1 + \delta]$. Then for $\beta > \max\{1/(-\theta_1 - \delta), 1/(\tau + \theta_1 - \delta)\}$, let

$$x(\theta) = \begin{cases} (1 + \beta(\theta - (\theta_1 - \delta)))x_1 & \theta \in [\theta_1 - \delta - 1/\beta, \theta_1 - \delta] \\ x_1 & \theta \in [\theta_1 - \delta, \theta_1 + \delta] \\ (1 - \beta(\theta - (\theta_1 + \delta)))x_1 & \theta \in [\theta_1 + \delta, \theta_1 + \delta + 1/\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}_\tau$, $x(0) = 0$ and $\|x(\theta)\| \le \|x_1\|$ for all $\theta \in [-\tau, 0]$. Now, since M_{22} is continuous, it is bounded on $[-\tau, 0]$. Therefore, there exists some $\epsilon_2 > 0$ such that $M_{22}(\theta) - \epsilon I \le \epsilon_2 I$. Then let $\beta \ge 2\epsilon_2\|x_1\|^2/\delta$ and we have the following.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta - \epsilon \|x\|^2$$
$$= \int_{\theta_1-\delta}^{\theta_1+\delta} x_1^T (M_{22}(\theta) - \epsilon I) x_1 d\theta$$
$$+ \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} x(\theta)^T (M_{22}(\theta) - \epsilon I) x(\theta) d\theta$$
$$+ \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} x(\theta)^T (M_{22}(\theta) - \epsilon I) x(\theta) d\theta$$
$$\le -2\delta + \epsilon_2 \int_{\theta_1-\delta-1/\beta}^{\theta_1-\delta} \|x(\theta)\|^2 d\theta + \epsilon_2 \int_{\theta_1+\delta}^{\theta_1+\delta+1/\beta} \|x(\theta)\|^2 d\theta$$
$$\le -2\delta + 2\epsilon_2 \|x_1\|^2/\beta \le -\delta$$

Therefore, by contradiction, we have that $M_{22}(\theta) \ge \epsilon I$ for all $\theta \in [-\tau, 0]$. Now we define $\epsilon' = \epsilon/2$ and $\tilde{M}_{22}(\theta) = M_{22}(\theta) - \epsilon' I \ge \epsilon' I$. We now note that since $\tilde{M}_{22}(\theta)$ is upper and lower bounded and is continuous, it can be shown that $\tilde{M}_{22}(\theta)^{-1}$ is bounded and continuous. We now prove statement 2 by construction. Suppose M satisfies statement 1. Let

$$T(\theta) = T_0 - (M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T).$$

Here

$$T_0 = \frac{1}{\tau} \int_{-\tau}^{0} (M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T) d\theta.$$

Then we have that T is continuous and

$$\int_{-\tau}^{0} T(\theta) d\theta = \tau T_0 - \tau T_0 = 0.$$

This implies that $T \in \Omega$. We now prove that $T_0 \ge 0$. For any vector $z_0 \in \mathbb{R}^n$, suppose z is a continuous function such that $z(0) = (I + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T) z_0$. Then let $x(\theta) = z(\theta) - \tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T z_0$. Then x is continuous, $x(0) = z_0$ and by statement 1 we have the following.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T \begin{bmatrix} M_{11}(\theta) & M_{12}(\theta) \\ M_{12}(\theta)^T & \tilde{M}_{22}(\theta) \end{bmatrix} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta$$
$$= z_0^T \left(\int_{-\tau}^{0} M_{11}(\theta) - M_{12}(\theta)\tilde{M}_{22}^{-1}(\theta)M_{12}(\theta)^T d\theta \right) z_0$$
$$+ \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta$$
$$= z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T \tilde{M}_{22}(\theta) z(\theta) d\theta \ge \epsilon' \|x\|_2^2$$

We now show that this implies that $T_0 \ge 0$. Suppose there exists some y such that $y^T T_0 y < 0$. Then there exists some z_0 such that $z_0 T_0 z_0 = -1$. Now let $\alpha > 1/\tau$ and

$$z(\theta) = \begin{cases} (I + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T) z_0 (1 + \alpha\theta) & \theta \in [-1/\alpha, 0] \\ 0 & \text{otherwise.} \end{cases}$$

Then z is continuous, $z(0) = (I + \tilde{M}_{22}(0)^{-1} M_{12}(0)^T) z_0$ and $\|z(\theta)\|^2 \le \|z(0)\|^2$ for all $\theta \in [-\tau, 0]$. Recall that $\tilde{M}_{22}(\theta) \le (\epsilon_2 + \epsilon') I = \epsilon_3 I$. Let $\alpha > 2\epsilon_3 \|z(0)^2\|$ and then we have the following.

$$z_0^T T_0 z_0 + \int_{-\tau}^{0} z(\theta)^T M_{22}(\theta) z(\theta) d\theta \le -1 + \epsilon_3 \int_{-1/\alpha}^{0} \|z(\theta)\|^2$$
$$\le -1 + \epsilon_3 \|z(0)\|^2/\alpha < -\frac{1}{2}$$

However, this contradicts the previous relationship. Therefore, we have by contradiction that $T_0 \ge 0$. Now by using the invertibility of $\tilde{M}_{22}(\theta) = M_{22}(\theta) - \epsilon' I \ge \epsilon' I$ and the Schur complement transformation, we have that statement 2 is equivalent to the following for all $\theta \in [-\tau, 0]$.

$$\tilde{M}_{22}(\theta) \ge 0$$
$$M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T \ge 0$$

We have already proven the first condition. Finally, we have the following.

$$M_{11}(\theta) + T(\theta) - M_{12}(\theta)\tilde{M}_{22}(\theta)^{-1} M_{12}(\theta)^T = T_0 \ge 0$$

Thus we have shown that statement 2 is true.

Definition 25 *Given n and τ, we refer to any piecewise-continuous, matrix-valued function $T : \mathbb{R} \to \mathbb{R}^n$ such that $\int_{-\tau}^{0} T(\theta) d\theta = 0$ as a* ***spacing function****, denoted $T \in \Omega$.*

Theorem 24 states that any element of H_1^+ which satisfies a certain strict positivity condition can be represented by the combination of a positive semidefinite matrix-valued function and a spacing function. This allows us to search for elements of H_1^+ by simultaneously searching over the set of positive semidefinite matrix-valued functions and the set of spacing functions. For any $d \in \mathbb{Z}^+$, the following gives a tractable condition for membership in H_1^+.

Definition 26 *Given n and τ, define Ω_p^d to be the set of functions $T \in \mathbb{S}_d^n[\theta]$ such that $\int_{-\tau}^{0} T(\theta) d\theta = 0$.*

Definition 27 *For a given $d \ge 0$, $M \in \mathbb{S}^{2n}[\theta]$ satisfies $M \in G_1^d$ if there exists a $T \in \Omega_p^d$ such that*

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+.$$

Membership in the convex cone G_1^d can be implemented as a semidefinite programming constraint by noting that the constraint that a polynomial integrate to 0 is affine in the coefficients and that for the single variable case, $\bar{\Sigma}_s = \mathcal{P}^+$. We now consider H_3^+. We can use a simple extension of the previous theorem.

Lemma 28 *Let $M : \mathbb{R} \to \mathbb{S}^{3n}$ be a continuous matrix-valued function. Then the following are equivalent.*

(1) There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_\tau$.

$$\int_{-\tau}^{0} \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} x(0) \\ x(-\tau) \\ x(\theta) \end{bmatrix} \geq \epsilon \|x\|_2^2$$

(2) There exists some $\epsilon' > 0$ and a continuous matrix-valued function $T : \mathbb{R} \to \mathbb{S}^{2n}$ such that the following holds.

$$\int_{-\tau}^{0} T(\theta) d\theta = 0$$

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for all } \theta \in [-\tau, 0]$$

Proof omitted.

Definition 29 *For a given $d \geq 0$, $M \in \mathbb{S}^{3n}[\theta]$ satisfies $M \in G_3^d$ if there exists a $T \in \mathbb{S}^{2n}[\theta]$ such that $T \in \Omega_p^d$ and*

$$M(\theta) + \begin{bmatrix} T(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+.$$

Similar to Theorem 24, Lemma 28 allows us to use $G_3^d \subset H_3^+$ to replace $M \in H_3^+$ with a semidefinite programming constraint.

4.1.1 Piecewise-Continuous Spacing Functions

We now consider the sets $\tilde{H}_1^+$ and $\tilde{H}_3^+$ of piecewise continuous functions which define positive multiplication operators on X_1 and $\tilde{X}_3$ respectively. The following lemma is a generalization of Theorem 24.

Lemma 30 *Suppose $S_i : \mathbb{R} \to \mathbb{S}^{2n}$, $i = 1, \dots, K$ are continuous symmetric matrix-valued functions with domains $[-\tau_i, -\tau_{i-1}]$ where $\tau_i > \tau_{i-1}$ for $i = 1, \dots, K$ and $\tau_0 = 0$. Then the following are equivalent.*

(1) There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_{\tau_K}$.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(0) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

(2) There exists an $\epsilon' > 0$ and continuous symmetric matrix valued functions, $T_i : \mathbb{R} \to \mathbb{S}^n$, such that

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for } \theta \in [-\tau_i, -\tau_{i-1}], \; i = 1, \dots, K,$$

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Proof omitted.

Lemma 30 allows us to represent the constrain $S \in \tilde{H}_1^+$ by considering a piecewise-continuous function S to be defined by $S(\theta) := S_i(\theta)$ for $\theta \in [-\tau_i, -\tau_{i-1}]$.

Definition 31 *Given n and τ, define $\tilde{\Omega}_p^d$ to be the set of piecewise-continuous functions $T : \mathbb{R} \to \mathbb{S}^n$ such that $T(\theta) = T_i(\theta)$ for $T_i \in \mathbb{S}_d^n[\theta]$ and $\theta \in [-\tau_i, -\tau_{i-1}]$ where $i = 1, \dots, K$ and such that*

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0$$

Definition 32 *For a given $d \geq 0$, we say that $M : \mathbb{R} \to \mathbb{S}^{2n}$ satisfies $M \in \tilde{G}_1^d$ if there exists $M_i \in \mathbb{S}_d^{2n}[\theta]$ such that $M(\theta) = M_i(\theta)$ for $\theta \in [-\tau_i, -\tau_{i-1}]$ and there exists a $T \in \tilde{\Omega}_p^d$, defined by $T_i \in \mathbb{S}_d^n[\theta]$ on the interval $[-\tau_i, -\tau_{i-1}]$, and such that*

$$M_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+ \qquad i = 1, \dots, K.$$

Similar to Theorem 24, Lemma 30 allows us to use $\tilde{G}_1^d \subset \tilde{H}_1^+$ to replace $M \in \tilde{H}_1^+$ with a semidefinite programming constraint. Now for $\tilde{H}_3^+$, we have the following lemma.

Lemma 33 *Suppose $S_i : \mathbb{R} \to \mathbb{S}^{n(K+2)}$ are continuous matrix valued functions with domains $[-\tau_i, -\tau_{i-1}]$ for $i = 1, \dots, K$ where $\tau_i > \tau_{i-1}$ for $i = 1, \dots, K$ and $\tau_0 = 0$. Then the following are equivalent.*

(1) There exists an $\epsilon > 0$ such that the following holds for all $x \in \mathcal{C}_{\tau_K}$.

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix}^T S_i(\theta) \begin{bmatrix} x(-\tau_0) \\ \vdots \\ x(-\tau_K) \\ x(\theta) \end{bmatrix} d\theta \geq \epsilon \|x\|_2^2$$

(2) There exists an $\epsilon' > 0$ and continuous matrix valued functions $T_i : \mathbb{R} \to \mathbb{S}^{n(K+1)}$ such that

$$S_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & -\epsilon' I \end{bmatrix} \geq 0 \qquad \text{for } \theta \in [-\tau_i, -\tau_{i-1}], \; i = 1, \dots, K,$$

$$\sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} T_i(\theta) = 0.$$

Proof omitted.

Definition 34 *For a given $d \geq 0$, we say that $M : \mathbb{R} \to \mathbb{S}^{n(K+2)}$ satisfies $M \in \tilde{G}_3^d$ if there exist $M_i \in \mathbb{S}_d^{n(K+2)}[\theta]$ such that $M(\theta) = M_i(\theta)$ on $[-\tau_i, -\tau_{i-1}]$ and*

$T_i \in \mathbb{S}_d^{n(K+1)}[\theta]$ such that $T(\theta) = T_i(\theta)$ on $[-\tau_i, -\tau_{i-1}]$ implies $T \in \tilde{\Omega}_p^d$ and such that

$$M_i(\theta) + \begin{bmatrix} T_i(\theta) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{P}^+ \qquad i = 1, \ldots, K.$$

Similar to Theorem 24, Lemma 33 allows us to use $\tilde{G}_3^d \subset \tilde{H}_3^+$ to replace $M \in \tilde{H}_3^+$ with a semidefinite programming constraint.

4.1.2 Parameter-Dependent Spacing Functions

In this subsection, we consider the case which arises when the dynamics of the system depend on some uncertain time-invariant vector of parameters.

Lemma 35 *For a given $d \in \mathbb{Z}^+$ and $S \in \mathbb{S}_d^{2n}[\theta, y]$, suppose there exists some $T \in \mathbb{S}_d^n[\theta, y]$ such that the following holds.*

$$\int_{-\tau}^{0} T(\theta, y) d\theta = 0$$

$$S(\theta, y) + \begin{bmatrix} T(\theta, y) & 0 \\ 0 & 0 \end{bmatrix} \in \bar{\Sigma}_s$$

Then $S(\cdot, y) \in H_1^+$ for all $y \in \mathbb{R}^p$.

Now suppose we wish to impose the condition that $S(\cdot, y) \in H_1^+$ for all $y \in \{y : p_i(y) \geq 0, i = 1, \ldots, N\}$ for scalar polynomials p_i.

Lemma 36 *Suppose there exist functions S_i such that $S_i(\cdot, y) \in H_1^+$ for $i = 0, \ldots, N$ for all y and such that the following holds.*

$$S(\theta, y) = S_0(\theta, y) + \sum_{i=1}^{N} p_i(y) S_i(\theta, y)$$

Then $S(\cdot, y) \in H_1^+$ for all $y \in \{y : p_i(y) \geq 0, i = 1, \ldots, N\}$.

The purpose of these lemmas is to show how our representation of H_1^+ can be used to prove stability in the presence of parametric uncertainty. Clearly, these results are motivated by the construction $\bar{\Upsilon}_d^K$ presented in subsection 3.4. The generalization of Lemmas 35 and 36 to H_3^+, $\tilde{H}_1^+$ and $\tilde{H}_3^+$ is similar and therefore not explicitly discussed.

4.2 Positive Finite-Rank Integral Operators

We now consider the set H_2^+ of continuous matrix-valued kernel functions which define positive integral operators on $\mathcal{C}_\tau$. These operators are compact and satisfy the following inequality for all $x \in \mathcal{C}_\tau$.

$$\int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T k(\theta, \omega) x(\omega) d\theta d\omega \geq 0$$

Because $\mathcal{C}_\tau$ is an infinite dimensional space, we cannot fully parameterize the operators which are positive on this space using a finite-dimensional set of positive semidefinite matrices. Instead, we will consider the subset of finite-rank operators.

Theorem 37 *Let A be a compact Hermitian operator which is positive on $Y_m \subset \mathcal{C}_\tau$, where*

$$Y_d := \{p \in \mathcal{C}_\tau : p \text{ is an } n\text{-dimensional vector of univariate polynomials of degree } d \text{ or less.}\}$$

Then there exists a $Q \geq 0$ such that

$$\langle x, Ax \rangle = \int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T k(\theta, \omega) x(\omega) d\theta d\omega \qquad \forall\, x \in Y_m$$

$$k(\theta, \omega) := \bar{Z}_d^n(\theta)^T Q \bar{Z}_d^n(\omega).$$

Before proving the theorem, we quote the following lemma which follows directly from the Spectral Theorem [13].

Lemma 38 *Let $\{e_i\}_{i=1}^p$ be a basis for some finite dimensional subspace Y of an inner product space Z. Let A be some compact Hermitian operator which is positive on Y. Then there exists some $K \geq 0$ such that the following holds for all $x \in Y$.*

$$\langle x, Ax \rangle = \sum_{i,j=1}^{p} K_{ij} \langle e_i, x \rangle \langle e_j, x \rangle$$

PROOF. Define e_i to be the transpose of the i^{th} row of $\bar{Z}_d^n$, then $\{e_i\}_{i=1}^{n(d+1)}$ forms a basis for the finite dimensional subspace $Y_m \subset \mathcal{C}_\tau$. By Lemma 38, there exists some $Q \geq 0$ such that the following holds where $p = n(d+1)$.

$$\begin{aligned}
\langle x, Ax \rangle &= \sum_{i,j=1}^{p} Q_{ij} \langle e_i, x \rangle \langle e_j, x \rangle \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} \sum_{i,j=1}^{p} Q_{ij} e_i(\theta)^T x(\theta) e_j(\omega)^T x(\omega) d\theta d\omega \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T \sum_{i,j=1}^{p} \left(e_i(\theta) Q_{ij} e_j(\omega)^T \right) x(\omega) d\theta d\omega \\
&= \int_{-\tau}^{0} \int_{-\tau}^{0} x(\theta)^T \bar{Z}_d^n[\theta]^T Q \bar{Z}_d^n[\omega] x(\omega) d\theta d\omega
\end{aligned}$$

Definition 39 *For a given integer $d \geq 0$, we denote the set of compact Hermitian operators which are positive on Y_d by G_2^{2d}.*

This theorem shows how we can use semidefinite programming to represent the subset $G_2^{2d} \subset H_2^+$ which consists of finite rank elements of H_2^+ with polynomial eigenvectors of bounded degree. Naturally, as we increase the degree bound, d, the rank of the operators in G_2^{2d} will increase, as will the computational complexity of the problem.

4.2.1 Piecewise-Continuous Positive Finite-Rank Operators

In this subsection, we consider the set $\tilde{H}_2^+$ of matrix-valued kernel functions which are discontinuous only at points $\theta, \omega = \{-\tau_i\}_{i=1}^{K-1}$ and which define positive integral operators on $\mathcal{C}_\tau$. Although the results of the previous section can be applied directly, the kernel functions defined in such a manner will necessarily be continuous since they are constructed using a finite number of continuous functions. In order to allow for the construction of operators defined by piecewise-continuous matrix valued functions, we use the following lemma.

Lemma 40 *Let M be a matrix valued function $M : \mathbb{R}^2 \to \mathbb{S}^n$ which is discontinuous only at points $\theta, \omega = -\tau_i$ for $i = 1, \ldots, K-1$ where the τ_i are increasing and $\tau_0 = 0$. Then $M \in \tilde{H}_2^+$ if and only if there exists some continuous matrix valued function $R : \mathbb{R}^2 \to \mathbb{R}^{nK\times nK}$ such that $R \in H_2^+$ and the following holds where $I_i = [-\tau_i, -\tau_{i-1}]$, $\Delta_i = \tau_i - \tau_{i-1}$.*

$$M(\theta,\omega) = M_{ij}(\theta,\omega) \qquad \text{for all } \theta \in I_i, \quad \omega \in I_j$$

$$M_{ij}(\theta,\omega) = R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}, \frac{\tau_K}{\Delta_j}\omega + \tau_{j-1}\frac{\tau_K}{\Delta_j}\right)$$

$$R(\theta,\omega) = \begin{bmatrix} R_{11}(\theta,\omega) & \ldots & R_{1K}(\theta,\omega) \\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \ldots & R_{KK}(\theta,\omega) \end{bmatrix}$$

PROOF. ($\Leftarrow$) Define $\theta_i(\theta) = \frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}$ and $\omega_i(\omega) = \frac{\tau_K}{\Delta_i}\omega + \tau_{i-1}\frac{\tau_K}{\Delta_i}$. Then $\theta_i(-\tau_i) = -\tau_K$, $\theta_i(-\tau_{i-1}) = 0$, $\theta(\theta_i) = \frac{\Delta_i}{\tau_K}\theta_i - \tau_{i-1}$ and $\frac{d\theta}{d\theta_i} = \frac{\Delta_i}{\tau_K}$. The same relation holds between ω and ω_i. Because M is piecewise continuous, M_{ij} as defined above are continuous. Then for any $x \in \mathcal{C}_{\tau_K}$, let $x_i(\theta) = \frac{\Delta_i}{\tau_K}x\left(\frac{\Delta_i}{\tau_K}\theta - \tau_{i-1}\right) \in \mathcal{C}_{\tau_K}$ and the following holds.

$$\begin{aligned}
&\int_{-\tau_K}^0\int_{-\tau_K}^0 x(\theta)^T M(\theta,\omega)x(\omega) \\
&= \sum_{i,j=1}^K \int_{I_i}\int_{I_j} x(\theta)^T M_{ij}(\theta,\omega)x(\omega)d\theta d\omega \\
&= \sum_{i,j=1}^K \int_{-\tau_K}^0\int_{-\tau_K}^0 x_i(\theta_i)^T R_{ij}(\theta_i,\omega_j)x_j(\omega_j)d\theta_i d\omega_j \\
&= \int_{-\tau_K}^0\int_{-\tau_K}^0 \begin{bmatrix} x_1(\theta) \\ \vdots \\ x_K(\theta)\end{bmatrix}^T R(\theta,\omega)\begin{bmatrix} x_1(\omega) \\ \vdots \\ x_K(\omega)\end{bmatrix} d\theta d\omega \ge 0
\end{aligned}$$

Thus $R \in H_2^+$ implies $M \in H_2^+$.

($\Rightarrow$) Now suppose $M \in H_2^+$. Then for any nK dimensional vector valued function in $\mathcal{C}_{\tau_K}$, $x(\theta) = \left[x_1(\theta)^T \cdots x_K(\theta)^T\right]^T$, define $\tilde{x}(\theta) = \frac{\tau_K}{\Delta_i}x_i\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}\right)$ for $\theta \in I_i$. Then $\tilde{x}$ is continuous except possibly at points $\{-\tau_i\}$ and we have the following.

$$\begin{aligned}
&\int_{-\tau_K}^0\int_{-\tau_K}^0 x(\theta)^T R(\theta,\omega)x(\omega)d\theta d\omega \\
&= \sum_{i,j=1}^K \int_{-\tau_K}^0\int_{-\tau_K}^0 x_i(\theta_i)^T R_{ij}(\theta_i,\omega_j)x_j(\omega_j)d\theta_i d\omega_j \\
&= \sum_{i,j=1}^K \int_{I_i}\int_{I_j} \tilde{x}(\theta)^T M_{ij}(\theta,\omega)\tilde{x}(\omega)d\theta d\omega \\
&= \int_{-\tau_K}^0\int_{-\tau_K}^0 \tilde{x}(\theta)^T M(\theta,\omega)\tilde{x}(\omega)d\theta d\omega
\end{aligned}$$

where M is as defined above. Since M is bounded and positive on $x \in \mathcal{C}_{\tau_K}$, it can be shown that M is also positive on the space of functions which are continuous except possibly at points $\{-\tau_i\}_{i=1}^{K-1}$. Therefore, we have that $R \in H_2^+$.

Definition 41 *For an integer $d \ge 0$, we denote by $\tilde{G}_2^d$ the set of piecewise continuous matrix valued functions $M : \mathbb{R}^2 \to \mathbb{S}^n$ such that there exist some $R : \mathbb{R}^2 \to \mathbb{R}^{nK\times nK}$ such that $R \in G_2^d$ and the following holds where I_i, Δ_i are as defined above.*

$$M(\theta,\omega) = M_{ij}(\theta,\omega) \qquad \text{for all } \theta \in I_i, \quad \omega \in I_j$$

$$M_{ij}(\theta,\omega) = R_{ij}\left(\frac{\tau_K}{\Delta_i}\theta + \tau_{i-1}\frac{\tau_K}{\Delta_i}, \frac{\tau_K}{\Delta_j}\omega + \tau_{j-1}\frac{\tau_K}{\Delta_j}\right)$$

$$R(\theta,\omega) = \begin{bmatrix} R_{11}(\theta,\omega) & \ldots & R_{1K}(\theta,\omega) \\ \vdots & & \vdots \\ R_{K1}(\theta,\omega) & \ldots & R_{KK}(\theta,\omega) \end{bmatrix}$$

Lemma 40 allows us to use elements of the set G_2^d to represent the subset of $\tilde{H}_2^+$ of finite-rank operators with piecewise-continuous eigenvectors. Since the transformation from G_2^d to $\tilde{G}_2^d$ is affine, this allows us to represent the constraint $R \in \tilde{H}_3$ using semidefinite programming.

4.3 Parameter Dependent Positive Finite-Rank Operators

We now briefly discuss the extension of the results of the previous two subsections when the dynamics contain parametric uncertainty.

Lemma 42 *For a given $d \in \mathbb{Z}^+$, suppose there exists a $P \in \bar{\Sigma}_s^d$ such that*

$$M(\theta,\omega,\alpha) = \bar{Z}_n^d[\theta]^T P(\alpha)\bar{Z}_n^d[\omega].$$

Then $M(\cdot,\cdot,\alpha) \in H_2^+$ for all $\alpha \in \mathbb{R}^n$.

Define $K_p := \{\alpha : p_i(\alpha) \ge 0, i = 1, \ldots, l\}$.

Lemma 43 *For a given $d \in \mathbb{Z}^+$, suppose there exist $P \in \bar{\Upsilon}^{K_p}_{n(d+1),d}$ such that*

$$M(\theta,\omega,\alpha) = \bar{Z}^d_n[\theta]^T P(\alpha) \bar{Z}^d_n[\omega].$$

Then $M(\cdot,\cdot,\alpha) \in H_2^+$ for all $\alpha \in K_p$.

These lemmas allow us to search for parameter dependent positive integral operators using the set of positive semidefinite matrix functions. The extension of these lemmas to the set $\tilde{H}_2^+$ should be clear and is not explicitly discussed.

5 Results

We now turn our attention to the following three classes of time-delay systems which we will explicitly consider.

(1) $\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$
(2) $\dot{x}(t) = A_0 x(t) + \sum_{i=1}^n A_i x(t-\tau_i)$
(3) $\dot{x}(t) = A_0 x(t) + \int_{-\tau}^0 A(\theta) x(t+\theta) d\theta$

Here $x(t) \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}[\theta]$. These classes contain constant, finite aftereffect. We refer to system 1) as the case of single delay, system 2) as the case of multiple delays and system 3) as the case of distributed delay. In this section, we show that in each of these cases, the derivative of the complete quadratic functional along trajectories of the system can be represented by a quadratic functional defined by the integral and multiplier operators parameterized in the previous section. Moreover, the transformation from the coefficients of the polynomials defining the complete quadratic functional to those defining its derivative is affine.

5.1 The Single Delay Case

In this section, we consider the first subclass of time-delay systems which are given in the following form for matrices $A, B \in \mathbb{R}^{n\times n}$.

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{4}$$

The following theorem gives a condition for stability of the system.

Theorem 44 *For integer $d \geq 0$, the solution map, G, defined by Equation (4) is asymptotically stable if there exists a constant $\epsilon > 0$, matrix $P \in \mathbb{S}^n$, and continuous matrix-valued functions $S : \mathbb{R} \to \mathbb{S}^n$, $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta,\eta) = R(\eta,\theta)^T$ and such that the following holds.*

$$\begin{bmatrix} P-\epsilon I & \tau Q \\ \tau Q^T & \tau S \end{bmatrix} \in G_1^d \qquad R \in G_2^d$$

$$-D_1 \in G_3^d \qquad M \in G_2^d$$

$$D_1(\theta) = \begin{bmatrix} D_{11} & PB - Q(-\tau) & \tau(A^T Q(\theta) - \dot{Q}(\theta) + R(0,\theta)) \\ *^T & -S(-\tau) & \tau(B^T Q(\theta) - R(-\tau,\theta)) \\ *^T & *^T & -\tau \dot{S}(\theta) \end{bmatrix}$$

$$M(\theta,\omega) = \frac{d}{d\theta} R(\theta,\omega) + \frac{d}{d\omega} R(\theta,\omega)$$

$$D_{11} = PA + A^T P + Q(0) + Q(0)^T + S(0) + \epsilon I$$

PROOF. We use the following complete quadratic functional.

$$\begin{aligned} V(\phi) &= \phi(0)^T P \phi(0) + 2\phi(0)^T \int_{-\tau}^0 Q(\theta)\phi(\theta) d\theta \\ &+ \int_{-\tau}^0 \phi(\theta)^T S(\theta) \phi(\theta) d\theta + \int_{-\tau}^0 \int_{-\tau}^0 \phi(\theta)^T R(\theta,\omega) \phi(\omega) d\theta d\omega \\ &= \frac{1}{\tau} \int_{-\tau}^0 \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T \begin{bmatrix} P-\epsilon I & \tau Q(\theta) \\ \tau Q(\theta)^T & \tau S(\theta) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix} d\theta \\ &+ \int_{-\tau}^0 \int_{-\tau}^0 \phi(\theta)^T R(\theta,\omega)\phi(\omega) d\theta d\omega + \epsilon \|\phi(0)\|^2 \geq \epsilon \|\phi(0)\|^2 \end{aligned}$$

All that is required for asymptotic stability is strict negativity of the the derivative. The Lie derivative of this functional along trajectories of the system is given by the following.

$$\begin{aligned} \dot{V}(\Gamma(\phi,0)) &= \phi(0)^T P(A\phi(0) + B\phi(-\tau)) \\ &+ (\phi(0)^T A^T + \phi(-\tau)^T B^T) P \phi(0) \\ &+ 2\int_{-\tau}^0 \phi(\theta)^T Q(\theta)^T (A\phi(0) + B\phi(-\tau)) d\theta \\ &+ 2\phi(0)^T \left(Q(0)\phi(0) - Q(-\tau)\phi(-\tau) - \int_{-\tau}^0 \dot{Q}(\theta)\phi(\theta) d\theta \right) \\ &+ \phi(0)^T S(0)\phi(0) - \phi(-\tau)^T S(-\tau)\phi(-\tau) \\ &- \int_{-\tau}^0 \phi(\theta)^T \dot{S}(\theta)\phi(\theta) d\theta \\ &+ \int_{-\tau}^0 \left(\phi(0)^T R(0,\theta)\phi(\theta) - \phi(-\tau)^T R(-\tau,\theta)\phi(\theta) \right) d\theta \\ &+ \int_{-\tau}^0 \left(\phi(\theta)^T R(\theta,0)\phi(0) - \phi(\theta)^T R(\theta,-\tau)\phi(-\tau) \right) d\theta \\ &- \int_{-\tau}^0 \int_{-\tau}^0 \phi(\theta) \left(\frac{d}{d\theta} R(\theta,\omega) + \frac{d}{d\omega} R(\theta,\omega) \right) \phi(\omega) d\theta d\omega \\ &= \frac{1}{\tau} \int_{-\tau}^0 \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix}^T D_1(\theta) \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix} \\ &- \int_{-\tau}^0 \int_{-\tau}^0 \phi(\theta) M(\theta,\omega)\phi(\omega) d\theta d\omega - \epsilon\|\phi(0)\|^2 \leq -\epsilon \|\phi(0)\|^2 \end{aligned}$$

Therefore, if the conditions of the theorem hold, then the functional is strictly decreasing along trajectories of the system, proving asymptotic stability.

Example 1: In this example, we compare our results with the discretized Lyapunov functional approach used by Gu et al. in [1] in the case of a system with a single delay. Although numerous other papers have also given sufficient conditions for stability of time-delay systems, e.g. [14–16], we use the approach introduced by Gu since it has demonstrated a particularly high level of precision. When we are comparing with the piecewise linear approach here and throughout this chapter, we will only consider examples which have been presented in the work [1] and we will compare our results with the numbers that are cited therein. We use SOSTools [17] and SeDuMi [18] for solution of all semidefinite programming problems. Now consider the following system of delay differential equations.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

The problem is to estimate the range of τ for which the differential equation remains stable. Using the method presented in this paper and by sweeping τ in increments of .1, we estimate the range of stability to be an interval. We then use a bisection method to find the minimum and maximum stable delay. Our results are also summarized in Table 1 and are compared to the analytical limit and the results obtained in [1].

Our results			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.10017	1.6249	1	.1006	1.4272
2	.10017	1.7172	2	.1003	1.6921
3	.10017	1.71785	3	.1003	1.7161
Analytic	.10017	1.71785			

Table 1
τ_{max} and τ_{min} for discretization level N_2 using the piecewise-linear Lyapunov functional and for degree d using our approach and compared to the analytical limit

Clearly, the results for this test case illustrate a high rate of convergence to the analytical limit. However, although the results presented here give reasonable estimates for the interval of stability, they do not prove stability over any interval, but rather only at the specific values of τ for which the algorithm was tested. To provide a more rigorous analysis, we now include τ as an uncertain parameter and search for parameter dependent Lyapunov functionals which prove stability over an interval. The results from this test are given in Table 3.

Example 2: In this example, we illustrate the flexibility of our algorithm through a simplistic control design and analysis problem. Suppose we wish to control a simple inertial mass remotely using a PD controller. Now suppose that the derivative control is half of the proportional control. Then we have the following dynamical system.

d in τ	d in θ	τ_{min}	τ_{max}
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 2
Stability on the interval $[\tau_{min}, \tau_{max}]$ vs. degree using a parameter-dependent functional

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

It is easy to show that this system is stable for all positive values of a. However, because we are controlling the mass remotely, some delay may be introduced due to, for example, the fixed speed of light. We assume that this delay is known and changes sufficiently slowly so that for the purposes of analysis, it may be taken to be fixed. Now we have the following delay-differential equation with uncertain, time-invariant parameters a and τ.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Whereas before the system was stable for all positive values of a, now, for any fixed value of a, there exists a τ for which the system will be unstable. In order to determine which values of a are stable for any fixed value of τ, we divide the parameter space into regions of the form $a \in [a_{\min}, a_{\max}]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. This type of region is compact and can be represented as a semi-algebraic set using the polynomials $p_1(a) = (a - a_{\min})(a - a_{\max})$ and $p_2(\tau) = (\tau - \tau_{\min})(\tau - \tau_{\max})$. By using these polynomials, we are able to construct parameter dependent Lyapunov functionals which prove stability over a number of parameter regions. These regions are illustrated in Figure 1.

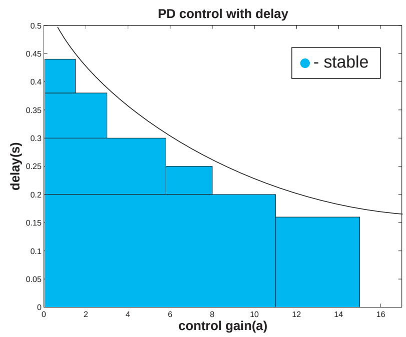

Fig. 1. Regions of Stability for Example 2

5.2 *Multiple Delay Case*

We now consider the case of multiple delays. The system is now defined by the following where $A_i \in \mathbb{R}^{n \times n}$, $\tau_i > \tau_{i-1}$ for $i = 1, \ldots, K$ and $\tau_0 = 0$.

$$\dot{x}(t) = \sum_{i=0}^{K} A_i x(t - \tau_i) \tag{5}$$

The difference between the analysis of the case of a single delay and that of multiple delays is that the matrix-valued functions defining the complete quadratic functional necessary for stability may now contain discontinuities at discrete points given by the values of the delay. In this case, we express the the complete quadratic functional in the following form where $Q_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S_i : \mathbb{R} \to \mathbb{S}^n$ and $R_{ij} : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ for $i, j = 1, \dots, K$ are continuous matrix-valued functions, $P \in \mathbb{S}^n$ and $R_{ij}(\theta,\omega) = R_{ji}(\omega,\theta)^T$.

$$V(\phi) = \phi(0)^T P\phi(0) + 2\sum_{j=1}^{K} \phi(0)^T \int_{-\tau_i}^{-\tau_{i-1}} Q_i(\theta)\phi(\theta)d\theta$$
$$+ \sum_{j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \phi(\theta)^T S_i(\theta)\phi(\theta)d\theta$$
$$+ \sum_{i=1}^{K}\sum_{j=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} \phi(\theta)^T R_{ij}(\theta,\omega)\phi(\omega)d\theta d\omega$$

Theorem 45 *For integer $d \geq 0$, suppose $P \in \mathbb{S}^n$, $\eta > 0$, $Q_i : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S_i : \mathbb{R} \to \mathbb{S}^n$ and $R_{ij} : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R_{ij}(\theta,\omega) = R_{ji}(\omega,\theta)^T$ are continuous for $i, j = 1, \dots, K$. Let $R(\theta,\omega) = R_{ij}(\theta,\omega)$ for $\theta \in I_i$, $\omega \in I_j$ where $I_i = [-\tau_i, -\tau_{i-1}]$. Then the solution map defined by equation (5) is asymptotically stable if the following holds.*

$$M \in \tilde{G}_1^d \qquad R \in \tilde{G}_2^d$$
$$-D \in \tilde{G}_3^d \qquad L \in \tilde{G}_2^d$$

$$M(\theta) = \begin{bmatrix} P - \eta I & \tau Q_i(\theta) \\ \tau Q_i(\theta)^T & \tau S_i(\theta) \end{bmatrix} \qquad \theta \in I_i$$

$$L(\theta,\omega) = \frac{\delta}{\delta\theta} R_{ij}(\theta,\omega) + \frac{\delta}{\delta\theta} R_{ij}(\theta,\omega) \qquad \theta \in I_i, \omega \in I_j$$

$$D(\theta) = \begin{bmatrix} D11 & \tau D12_i(\theta) \\ \tau D12_i(\theta)^T & \tau D22_i(\theta) \end{bmatrix} \qquad \text{for } \theta \in I_i$$

$$D11_{11} = PA_0 + A_0^T P + Q_1(0) + Q_1(0)^T + S_1(0) - \eta I$$

$$D11_{ij} = \begin{cases} PA_{i-1} - Q_{i-1}(-\tau_{i-1}) + Q_i(-\tau_{i-1}) & i, j = 1, 2 \dots K \\ S_i(-\tau_{i-1}) - S_{i-1}(-\tau_{i-1}) & i = j = 2 \dots K \\ PA_K - Q_K(-\tau_K) & i, j = 1, K+1 \\ -S_K(-\tau_K) & i = j = K+1 \\ 0 & \text{otherwise} \end{cases}$$

$$D12_j(\theta) = \begin{bmatrix} R_{1j}(0,\theta) + A_0^T Q_j(\theta) - \dot{Q}_j(\theta) \\ R_{2j}(-\tau_1,\theta) - R_{1j}(-\tau_1,\theta) + A_1^T Q_j(\theta) \\ \vdots \\ R_{(K)j}(-\tau_{K-1},\theta) - R_{(K-1)j}(-\tau_{K-1},\theta) + A_1^T Q_j(\theta) \\ A_K^T Q_j(\theta) - R_{Kj}(-\tau_K,\theta) \end{bmatrix}$$

$$D22_i(\theta) = -\dot{S}_i(\theta)$$

PROOF. We use the following complete quadratic functional.

$$V(\phi) = \frac{1}{\tau}\int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T M(\theta) \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}$$
$$+ \int_{-\tau_K}^{0}\int_{-\tau_K}^{0} \phi(\theta)^T R(\theta,\omega)\phi(\omega)d\theta d\omega + \eta\|\phi(0)\|^2 \geq \eta\|\phi(0)\|^2$$

The system is asymptotically stable if the derivative of the functional is strictly negative. The Lie derivative of the functional along trajectories of the systems is given by the following.

$$\dot{V}(\Gamma(\phi,0)) = \frac{1}{\tau}\int_{-\tau_K}^{0} \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(0) \\ \vdots \\ \phi(-\tau_K) \\ \phi(\theta) \end{bmatrix} d\theta$$
$$+ \int_{-\tau_K}^{0} \phi(\theta)^T G(\theta,\omega)\phi(\omega)d\theta d\omega - \eta\|\phi(0)\|^2 \leq -\eta\|\phi(0)\|^2$$

Thus if the conditions of the theorem hold, then the derivative of the functional is strictly negative, which implies asymptotic stability.

5.3 *Examples of Multiple Delay*

Example 3: Consider the following system of delay-differential equations.

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -\frac{9}{10} \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} \left[\frac{1}{20}x(t - \frac{\tau}{2}) + \frac{19}{20}x(t-\tau)\right]$$

Again, the system is stable when τ lies on some interval. The problem is to search for the minimum and maximum value of τ for which the system remains stable. In applying the methods of this paper, we again use a bisection method to find the minimum and maximum value of τ for which the system remains stable. Our results are summarized in Table 3 and are compared to the analytical limit as well as piecewise-linear functional method. For the piecewise functional method, N_2 is the level of both discretization and subdiscretization.

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 3
τ_{max} and τ_{min} using the piecewise-linear Lyapunov functional of Gu et al. and our approach and compared to the analytical limit

6 Distributed Delay Case

We now consider the case of distributed delay where the dynamics are given by the following functional differential equation where $A_0 \in \mathbb{R}^{n\times n}$ and $A \in \mathbb{R}^{n\times n}[\theta]$.

$$\dot{x}(t) = A_0 x(t) + \int_{-\tau}^{0} A(\theta)x(t+\theta)d\theta \tag{6}$$

We can again assume that the matrix-valued functions defining the complete quadratic functional are continuous. This leads to the following theorem.

Theorem 46 *For integer $d \geq 0$, the solution map, G, defined by equation (6) is asymptotically stable if there exists a constant $\eta > 0$, a matrix $P \in \mathbb{S}^n$ and matrix functions $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S : \mathbb{R} \to \mathbb{S}^n$, $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ where $R(\theta, \omega) = R(\omega, \theta)^T$ and such that the following holds.*

$$\begin{bmatrix} P-\eta I & \tau Q \\ \tau Q^T & \tau S \end{bmatrix} \in G_1^d \qquad -D \in G_3^d$$
$$R \in G_2^d \qquad M \in G_2^d$$

where

$$D(\theta) = \begin{bmatrix} D_{11}+\eta I & \tau D_{12}(\theta) \\ \tau D_{12}(\theta)^T & -\tau \dot{S}(\theta) \end{bmatrix}$$
$$D_{11} = \begin{bmatrix} PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & -Q(-\tau) \\ -Q(-\tau)^T & -S(-\tau) \end{bmatrix}$$
$$D_{12}(\theta) = \begin{bmatrix} A_0^T Q(\theta) + PA(\theta) - \frac{d}{d\theta}Q(\theta) + R(0,\theta) \\ -R(-\tau,\theta) \end{bmatrix}$$
$$M(\theta,\omega) = \frac{d}{d\theta}R(\theta,\omega) + \frac{d}{d\omega}R(\theta,\omega) - A(\theta)^T Q(\omega) - Q(\theta)^T A(\omega)$$

PROOF. We consider the complete quadratic functional

$$V(\phi) = \frac{1}{\tau}\int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T \begin{bmatrix} P-\eta I & \tau Q(\theta) \\ \tau Q(\theta)^T & \tau S(\theta) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(\theta) \end{bmatrix}^T d\theta$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T R(\theta,\omega)\phi(\omega)d\theta d\omega + \eta\|\phi(0)\|^2 \geq \eta\|\phi(0)\|^2$$

The system is asymptotically stable if the Lie derivative is strictly negative. The Lie derivative of the functional along trajectories of the system is given by

$$\dot{V}(\Gamma(\phi,0)) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix}^T D(\theta) \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(\theta) \end{bmatrix} d\theta$$
$$- \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(\theta)^T M(\theta,\omega)\phi(\omega)d\omega d\theta - \eta\|\phi(0)\|^2 \leq -\eta\|\phi(0)\|^2$$

Thus, if the conditions of the theorem hold, then the system is asymptotically stable.

7 Conclusion

In this paper, we have shown how to compute solutions to an operator-theoretic version of the Lyapunov inequality. Our approach is to combine results from real algebraic geometry with functional analysis in order to parameterize certain convex cones of positive operators using the convex cone of positive semidefinite matrices. We then show that the operator inequalities can be expressed using affine constraints on these matrices. This allows us to compute solutions using semidefinite programming, for which there exist efficient numerical algorithms. We have further extended our results to the case when the dynamics contain parametric uncertainty using a construction based on certain results of Putinar and others. The numerical examples given in this Chapter demonstrate a quick convergence to the analytic limit of stability.

We conclude by mentioning that the methods of this paper can be used to construct full-rank solutions of the Lyapunov inequality. Furthermore, we have developed methods for constructing the inverses of these full-rank operators. By computing solutions to the Lyapunov inequality for the adjoint system constructed by Delfour and Mitter [19], this invertibility result seems to imply that one can construct stabilizing controllers for linear time-delay systems. This work is ongoing, however, and is therefore not detailed in this paper.

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