Optimal Estimation (and Control) of Dynamic Systems with State Delay

Matthew M. Peet (with K. Gu)
Arizona State University
Tempe, AZ USA

IFAC 2018
Budapest, Hungary

June 30, 2018

Control of Differential Equations with State Delay

Consider a MIMO Linear Differential-Difference Equation.

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t),$$
$$y(t) = Cx(t) + D_1 w(t) + D_2 u(t),$$

Stability Analysis is a CLOSED PROBLEM.

- SOS analysis is accurate to 6 decimal places

However,

- H_∞-Optimal Controller Design is OPEN
- H_∞-Optimal Estimator Design is OPEN

In this talk, we will CLOSE the estimation problem.

- Also the control problem (ACC 2018)

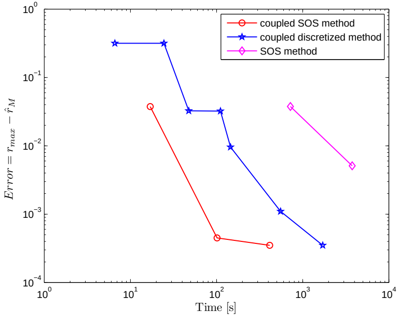


Figure: Comparison of asymptotic algorithms for maximum stable delay

H_∞-Optimal Observer Synthesis Problem to be Solved

Consider solutions of

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bw(t)$$
$$y(t) = C_2 x(t)$$

With a PDE observer (observed errors)(nominal dynamics)(corrective gains)

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{\phi}(t,-\tau) + L_1\left(C_2\hat{x}(t) - y(t)\right) + L_2\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$
$$+ \int_{-\tau}^{0} L_3(\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$
$$\partial_t\hat{\phi}(t,s) = \partial_s\hat{\phi}(t,s) + L_4(s)\left(C_2\hat{x}(t) - y(t)\right) + L_5(s)\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$
$$+ L_6(s)\left(C_2\hat{\phi}(t,s) - y(t+s)\right) + \int_{-\tau}^{0} L_7(s,\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$
$$\hat{\phi}(t,0) = \hat{x}(t)$$

Problem Definition:
Minimize γ such that there exist L_i such that if $z_e(t) = C_1(x(t) - \hat{x}(t))$, then for any $w \in L_2$, $\|z_e\|_{L_2} \le \gamma\|w\|_{L_2}$.

Roadmap of the Talk

Find $\mathcal{L} : Z \to Z$ such that

$$\dot{\mathbf{e}}(t) = \mathcal{A}\mathbf{e}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \quad u(t) = \mathcal{K}\mathbf{x}, \quad y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 u(t)$$

implies $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$

Step 1: Solve the problem as a abstract but convex Linear Operator Inequality.

Step 2: Parameterize All Operators using Matrices.

- Synthesis conditions now linear matrix constraints and operator positivity constraints
- $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ framework

Step 3: Enforce Operator Positivity using LMIs.

Step 4: Solve the LMI and Reconstruct the controller gains.

- Invert the operator using matrix manipulations.

An LMI for Optimal *Estimation* of **ODE**s

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1 w(t), \qquad y(t) = C_2 x(t) + Dw(t)$$

Observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)), \quad z_e(t) = C_1(\hat{x}(t) - x(t))$$

Lemma 1 (H_∞-Optimal Observer Synthesis).

Define the map $w \mapsto z_e$:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + LC_2 & -(B + LD) \\ \hline C_1 & 0 \end{array}\right].$$

The following are equivalent.

- *There exists a L such that $\|\hat{G}\|_{H_\infty} \le \gamma$.*
- *There exists a $P > 0$ and Z such that*

$$\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

The Observer Gain is recovered as $L = P^{-1}Z$.

Make a DDE look like an ODE: Put it in 1st-Order Form

Write the DDE as

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t), \qquad y(t) = \mathcal{C}x(t) + \mathcal{D}_1 w(t) + \mathcal{D}_2 u(t).$$

where

$$\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^K A_i \phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}, \quad \left(\mathcal{C}\begin{bmatrix} \psi \\ \phi_i \end{bmatrix}\right) := \left[C_0 \psi + \sum_i C_i \phi_i(-\tau_i)\right]$$

$$(\mathcal{B}_1 w)(s) := \begin{bmatrix} B_1 w \\ 0 \end{bmatrix}, \qquad (\mathcal{B}_2 u)(s) := \begin{bmatrix} B_2 u \\ 0 \end{bmatrix},$$

$$(\mathcal{D}_1 w)(s) := \left[D_1 w\right], \qquad (\mathcal{D}_2 u)(s) := \left[D_2 u\right]$$

Details: $\mathcal{A} : X \to Z_{n,K}$, $\mathcal{B}_1 : \mathbb{R}^m \to Z_{n,K}$, $\mathcal{B}_2 : \mathbb{R}^p \to Z_{n,n,K}$, $\mathcal{D}_1 : \mathbb{R}^m \to \mathbb{R}^q$, $\mathcal{D}_2 : \mathbb{R}^p \to \mathbb{R}^q$, and $\mathcal{C} : Z_{n,n,K} \to \mathbb{R}^p$ where

$$Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$$

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} := \tau_K y^T x + \sum_{i=1}^K \int_{-\tau_i}^0 \psi_i(s)^T \phi_i(s) ds$$

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \;:\; \begin{smallmatrix} \phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K] \end{smallmatrix} \right\}.$$

The DPS/DDE Equivalent of the Observer LMI

LMI Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p\times n}$ such that

$$\begin{bmatrix} e \\ w \end{bmatrix}^T \left[\begin{bmatrix} A^TP + C_2^TZ^T + PA + ZC_2 & -(PB+ZD) \\ -(PB+ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^TC_1 & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{bmatrix} e \\ w \end{bmatrix}$$

$$= (PAe)^Te + (PAe)^Te + (ZCe)^Te + (ZC_2e)^Te$$

$$-e^TPBw - (PBw)^Te - \gamma w^Tw + \frac{1}{\gamma}(C_1e)^T(C_1e) < 0$$

for all $e \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

DPS Version of Observer Synthesis: Minimize γ such that $\exists \mathcal{P} > 0$ and $\mathcal{Z}$ such that

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$-\langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^Tw + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m$$

An LMI for Optimal Control of **ODE**s

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t), \qquad y(t) = Cx(t) + D_1 w(t) + D_2 u(t).$$

Lemma 2 (Full-State Feedback Controller Synthesis).

Define:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C + D_2 K & D_1 \end{array}\right].$$

The following are equivalent.

- *There exists a* K *such that* $\|\hat{G}\|_{H_\infty} \le \gamma$.
- *There exists a* $P > 0$ *and* Z *such that*

$$\begin{bmatrix} PA^T + AP + Z^T B_2^T + B_2 Z & B_1 & PC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 P + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The Controller is recovered as $K = ZP^{-1}$.

- $P > 0$ ensures P is invertible.

The DPS/DDE Equivalent of the Synthesis LMI

LMI Version of Controller Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p\times n}$ such that

$$\begin{bmatrix} z \\ w \\ v \end{bmatrix}^T \begin{bmatrix} YA^T + AY + Z^T B_2^T + B_2 Z & B_1 & YC_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \begin{bmatrix} z \\ w \\ v \end{bmatrix}$$

$$= z^T P A^T z + z^T A P z + z^T Z^T B_2^T z + z^T B_2 Z z + z^T B_1 w + w^T B_1^T z - \gamma w^T w$$

$$+ v^T (CPz) + (CPz)^T v + v^T (D_2 Z z) + (D_2 Z z)^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v$$

$$\le 0$$

for all $z \in \mathbb{R}^n$, $w \in \mathbb{R}^m$, $v \in \mathbb{R}^q$

DPS Version of Controller Synthesis: Minimize γ such that $\exists \mathcal{P} : X \to X$ (coercive, $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P}(X) = X$) and $\mathcal{Z}$ such that

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w$$

$$+ v^T (\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T (\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T (D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon \|z\|_Z^2$$

for all $\mathbf{z} \in Z$, $w \in \mathbb{R}^m$, $v \in \mathbb{R}^q$

How to Solve these LOIs?

Enforce $\quad \mathcal{P} \geq 0 \quad$ or equivalently $\quad V(x) = \langle x, \mathcal{P}x \rangle \geq 0$

The Wrong Way: Project onto $\mathbb{R}^n$

1. **Model Transformations:** $V = z^T M z$ where $z(t) = x(t-\tau) + \int_{t-\tau}^{t} A_0 x(s) + A_1 x(s-\tau) ds.$
2. **Jensen's Inequality:** $V = z^T M z$ where $z(t) = \int_{-\tau}^{0} \phi(t,s) ds$.
3. **Wirtinger/Legendre:** $V = z^T M z$ where $z_i(t) = \int_{-\tau}^{0} L_i(s)\phi(t,s) ds$.

The Right Way: Lift LMIs to $\mathbb{R}^n \times L_2$. Let $V = \langle \mathbf{z}, M\mathbf{z} \rangle$ where $M > 0$ and

$$\mathbf{z}(s) = \begin{bmatrix} x \\ Z(s)\phi(s) \\ \int_{-\tau}^{0} Z(s,\theta)\phi(\theta)d\theta. \end{bmatrix} \quad \text{Then} \quad V(\mathbf{x}) := \int_{-\tau}^{0} \begin{bmatrix} x \\ \phi(s) \end{bmatrix} \left(\mathcal{P} \begin{bmatrix} x \\ \phi \end{bmatrix} \right)(s) ds$$

where

$$\left(\mathcal{P} \begin{bmatrix} x \\ \phi \end{bmatrix} \right)(s) = \begin{bmatrix} Px + \int_{-\tau}^{0} Q(\theta)\phi(\theta)d\theta \\ Q(s)^T x + S(s)\phi(s) + \int_{-\tau}^{0} R(s,\theta)\phi(\theta)d\theta \end{bmatrix}$$

$$P = M_{11} \cdot \frac{1}{\tau}\int_{-\tau}^{0} ds, \qquad Q(s) = \frac{1}{\tau}\left(M_{12} Z(s) + \int_{-\tau_K}^{0} M_{13} Z(\eta, s) d\eta \right), \qquad S(s) = \frac{1}{\tau} Z(s)^T M_{22} Z(s),$$

$$R(s,\theta) = Z(s)^T M_{23} Z(s,\theta) + Z(\theta,s)^T M_{32} Z(\theta) + \int_{-\tau}^{0} Z(\eta,s)^T M_{33} Z(\eta,\theta) d\eta$$

The PQRS Framework - Parametrization and Positivity

Parameterize all operators as

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta \end{bmatrix}$$

Positivity: To Constrain $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq 0$: Define $a_i = \frac{\tau_i}{\tau_K}$ and

$$\hat{Q}(s) := \begin{bmatrix}\sqrt{a_1}Q_1(a_1 s) & \cdots & \sqrt{a_K}Q_K(a_K s)\end{bmatrix}, \qquad \hat{S}(s) := \begin{bmatrix} S_1(a_1 s) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & S_K(a_K s)\end{bmatrix},$$

$$\hat{R}(s,\theta) := \begin{bmatrix} \sqrt{a_1 a_1}R_{11}(sa_1,\theta a_1) & \cdots & \sqrt{a_1 a_K}R_{1K}(sa_1,\theta a_K) \\ \vdots & \dots & \vdots \\ \sqrt{a_K a_1}R_{K1}(sa_K,\theta a_1) & \cdots & \sqrt{a_K a_K}R_{KK}(sa_K,\theta a_K)\end{bmatrix}.$$

Now constrain (using $g = 1$ and $g = -s(s+\tau_K)$)

$$P = M_{11}\cdot\frac{1}{\tau_K}\int_{-\tau_K}^0 g(s)ds, \quad \hat{S}(s) = \frac{1}{\tau_K}g(s)Z(s)^T M_{22}Z(s)$$

$$\hat{Q}(s) = \frac{1}{\tau_K}\left(g(s)M_{12}Z(s) + \int_{-\tau_K}^0 g(\eta)M_{13}Z(\eta,s)d\eta\right)$$

$$\hat{R}(s,\theta) = g(s)Z(s)^T M_{23}Z(s,\theta) + g(\theta)Z(\theta,s)^T M_{32}Z(\theta) + \int_{-\tau_K}^0 g(\eta)Z(\eta,s)^T M_{33}Z(\eta,\theta)d\eta$$

Matlab Command: `[P,Q,R,S]=sosjointpos_mat_ker_ndelay_PQRS`

How to work in the PQRS framework?

Take each term in the LOI and make it look like a PQRS operator

$$\langle \mathbf{z}, \mathcal{A}\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\mathbf{z}\rangle = \textbf{ bunch of terms } = \langle \tilde{\mathbf{z}}, \mathcal{P}_{\{D,E_i,F_i,G_{ij}\}}\tilde{\mathbf{z}}\rangle$$

What does a PQRS operator look like?

$$\left\langle \underbrace{\begin{bmatrix} h \\ \phi_i \end{bmatrix}}_{\tilde{\mathbf{z}}}, \mathcal{P}_{\{D,E_i,F_i,G_{ij}\}} \underbrace{\begin{bmatrix} h \\ \phi_i \end{bmatrix}}_{\tilde{\mathbf{z}}} \right\rangle_{Z_{r,n,K}}$$

$$= \tau_K h^T D h + \tau_K \sum_{i=1}^{K} \int_{-\tau_i}^{0} h^T E_i(s)\phi_i(s) ds + \tau_K \sum_i \int_{-\tau_i}^{0} \phi_i(s)^T E_i(s)^T h ds$$

$$+ \tau_K \sum_i \int_{-\tau_i}^{0} \phi_i(s)^T F_i(s)\phi_i(s) ds + \sum_{ij} \int_{-\tau_i}^{0}\int_{-\tau_j}^{0} \phi_i(s)^T G_{ij}(s,\theta)\phi_i(\theta)\, d\theta ds.$$

Take each term in $\langle \mathbf{z}, \mathcal{A}\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\mathbf{z}\rangle$ and associate it to a D, E_i, F_i or G_{ij}.

- Illustrated on the next few slides

Observer: Put Each Term in the PQRS Framework

Define $\mathbf{z} = \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{bmatrix}$ and $h = \begin{bmatrix} w^T & e_1^T & e_2(-\tau)^T \end{bmatrix}^T$.

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m,$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_{Z_n} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_{Z_{n,K}}$$

$$= \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix}^T \begin{bmatrix} D_1(s) & \tau E_1(s) \\ \tau E_1(s)^T & -\tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \\ e_2(s) \end{bmatrix} ds + \tau \int_{-\tau}^{0}\int_{-\tau}^{0} e_2(s)^T G(s,\theta) e_2(\theta) d\theta$$

$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_1, E_1, -\dot{S}, G\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

where

$$D_1(s) = \begin{bmatrix} 0 & * & * \\ 0 & PA_0 + A_0^T P + Q(0) + Q(0)^T + S(0) & * \\ 0 & A_1^T P - Q(-\tau)^T & -S(-\tau) \end{bmatrix}$$

$$E(s) = \begin{bmatrix} 0 \\ A_0^T Q(s) + R(s,0)^T - \dot{Q}(s) \\ A_1^T Q(s) - R(s,-\tau)^T \end{bmatrix} \qquad G(s,\theta) = -R_\theta(s,\theta) - R_s(s,\theta).$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e},\mathbf{e}\rangle_{L_2}+\langle \mathbf{e},\mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2}+\langle \mathcal{Z}\mathcal{C}_2\mathbf{e},\mathbf{e}\rangle_{L_2}+\langle \mathbf{e},\mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$-\langle \mathbf{e},\mathcal{P}\mathcal{B}w\rangle_{L_2}-\langle \mathcal{B}w,\mathcal{P}\mathbf{e}\rangle_{L_2}-\gamma w^T w+\frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e})<-\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e}\in X,\ w\in\mathbb{R}^m,$$

$$\langle \mathcal{Z}\mathcal{C}_2\mathbf{e},\mathbf{e}\rangle_{L_2}+\langle \mathbf{e},\mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}=2\tau e_1^T\left(Z_1C_2e_1+Z_2C_2e_2(-\tau)+\int_{-\tau}^0 Z_3(\theta)C_2e_2(\theta)d\theta\right)$$

$$+2\tau\int_{-\tau}^0 e_2(s)^T\left(Z_4(s)C_2e_1+Z_5(s)C_2e_2(-\tau)+Z_6(s)C_2e_2(s)+\int_{-\tau}^0 Z_7(s,\theta)C_2e_2(\theta)d\theta\right)$$

$$=\tau\begin{bmatrix} w\\ e_1\\ e_2(-\tau)\end{bmatrix}^T\underbrace{\begin{bmatrix}0&0&0\\0&Z_1C_2&Z_2C_2\\0&C_2^TZ_2&0\end{bmatrix}}_{D_2}\begin{bmatrix} w\\ e_1\\ e_2(-\tau)\end{bmatrix}+2\tau\int_{-\tau}^0\begin{bmatrix} w\\ e_1\\ e_2(-\tau)\end{bmatrix}^T\underbrace{\begin{bmatrix}0\\ C_2^TZ_4(s)^T+Z_3(s)C_2\\ C_2^TZ_5(s)^T\end{bmatrix}}_{E_2}e_2(s)ds$$

$$+\tau\int_{-\tau}^0 e_2(s)^T\underbrace{\left(Z_6(s)C_2+C_2^TZ_6(s)^T\right)}_{F_2}e_2(s)ds$$

$$+\tau\int_{-\tau}^0\int_{-\tau}^0 e_2(s)^T\underbrace{\left(Z_7(s,\theta)C_2+C_2^TZ_7(\theta,s)^T\right)}_{G_2}e_2(\theta)d\theta$$

$$=\left\langle \begin{bmatrix} h\\ \mathbf{e}_2\end{bmatrix},\mathcal{P}_{\{D_2,E_2,F_2,G_2\}}\begin{bmatrix} h\\ \mathbf{e}_2\end{bmatrix}\right\rangle_{L_2}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$-\langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m,$$

$$-\langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} = 2\int_{-\tau}^{0} e_1^T P B w ds + 2\int_{-\tau}^{0} e_2(s)^T \tau Q(s)^T B w ds$$

$$= \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & -B^T P & 0 \\ -PB & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_3} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix} + 2\tau \int_{-\tau}^{0} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} -B^T Q(s) \\ 0 \\ 0 \end{bmatrix}}_{E_3} e_2(s) ds$$

$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_3, E_3, 0, 0\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

Observer: Put Each Term in the PQRS Framework

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) + \epsilon\|\mathbf{e}\|^2 \le 0 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m$$

$$-\gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) + \epsilon\|\mathbf{e}\|^2$$

$$= \tau \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}^T \underbrace{\begin{bmatrix} -\frac{\gamma}{\tau} & 0 & 0 \\ 0 & \frac{1}{\gamma\tau}C_1^T C_1 + \epsilon I & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} w \\ e_1 \\ e_2(-\tau) \end{bmatrix}$$

$$= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D_4,0,0,0\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2}$$

Combine Terms and enforce Constraint

Suppose there exist P, Q, S, R, Z_i such that

$$\begin{aligned}
&\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2} \\
&- \langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) + \epsilon\|\mathbf{e}\|^2 \\
&= \left\langle \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix}, \mathcal{P}_{\{D,E,F,G\}} \begin{bmatrix} h \\ \mathbf{e}_2 \end{bmatrix} \right\rangle_{L_2} \le 0,
\end{aligned}$$

where $D = \sum_{i=1}^{5} D_i$, $E(s) = \sum_{j=1}^{3} E_i(s)$ and $G(s,\theta) = \sum_{j=1}^{2} G_i(s,\theta)$. Then if $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ and

$$\begin{aligned}
&\dot{\hat{\mathbf{x}}}(t) = \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right), \qquad \mathbf{y}(t)(s) = \begin{bmatrix} C_2 x(t) \\ C_2 x(t+s) \end{bmatrix} \\
&\hat{z}(t) = \mathcal{C}_1\mathbf{x}(t), \qquad z_e(t) = \hat{z}(t) - z(t), \qquad \mathbf{x}(t)(s) = \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix} \qquad (1)
\end{aligned}$$

and $z_e(t) = \hat{z}(t) - z(t)$, we have $\|z_e\|_{L_2} \le \gamma\|w\|_{L_2}$

Observer Gains Reconstruction

Let $\mathcal{P}_{\{\hat{P},\hat{Q},\hat{S},\hat{R}\}} = \mathcal{P}^{-1}_{\{P,Q,S,R\}}$. Then the observer dynamics are given by

$$\dot{\hat{x}}(t) = A_0\hat{x}(t) + A_1\hat{\phi}(t,-\tau) + L_1\left(C_2\hat{x}(t) - y(t)\right) + L_2\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$

$$+ \int_{-\tau}^{0} L_3(\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta, \qquad \hat{\phi}(t,0) = \hat{x}(t)$$

$$\partial_t\hat{\phi}(t,s) = \partial_s\hat{\phi}(t,s) + L_4(s)\left(C_2\hat{x}(t) - y(t)\right) + L_5(s)\left(C_2\hat{\phi}(t,-\tau) - y(t-\tau)\right)$$

$$+ L_6(s)\left(C_2\hat{\phi}(t,s) - y(t+s)\right) + \int_{-\tau}^{0} L_7(s,\theta)\left(C_2\hat{\phi}(t,\theta) - y(t+\theta)\right)d\theta$$

where

$$L_1 = \hat{P}Z_1 + \int_{-\tau}^{0}\hat{Q}(\theta)Z_4(\theta)d\theta, \quad L_2 = \hat{P}Z_2 + \int_{-\tau}^{0}\hat{Q}(\theta)Z_5(\theta)d\theta$$

$$L_3(\theta) = \hat{P}Z_3(\theta) + \hat{Q}(\theta)Z_6(\theta) + \int_{-\tau}^{0}\hat{Q}(s)Z_7(s,\theta)ds$$

$$L_4(s) = \hat{Q}(s)^T Z_1 + \hat{S}(s)Z_4(s) + \int_{-\tau}^{0}\hat{R}(s,\theta)Z_4(\theta)d\theta$$

$$L_5(s) = \hat{Q}(s)^T Z_2 + \hat{S}(s)Z_5(s) + \int_{-\tau}^{0}\hat{R}(s,\theta)Z_5(\theta)d\theta, \qquad L_6(s) = \hat{S}(s)Z_6(s)$$

$$L_7(s,\theta) = \hat{Q}(s)^T Z_3(\theta) + \hat{S}(s)Z_7(s,\theta) + \hat{R}(s,\theta)Z_6(\theta) + \int_{-\tau}^{0}\hat{R}(s,\xi)Z_7(\xi,\theta)d\xi.$$

Boring Numerical Examples

Numerical Example 1 In this example, we consider the unstable system

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 7 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

Applying the Ricatti approach in [Fattouh 1998] with $\epsilon = .001$ we obtain a L_2-gain of $\gamma = .580$. Applying the LOI, we obtain an L_2-gain of .236. Of all the systems we tested, this one showed the least improvement in performance.

Numerical Example 2 A modified form of [Fridman 2001].

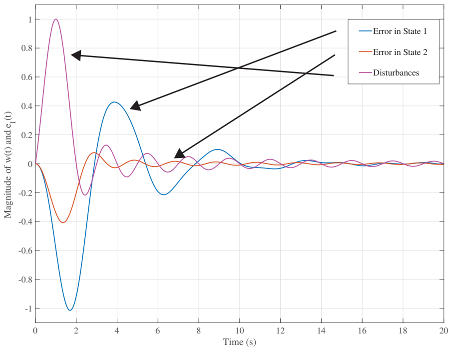
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

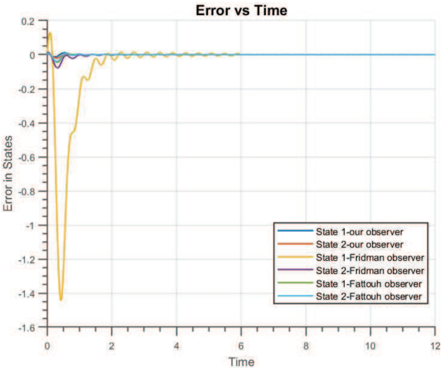
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

Using the original system with $\tau = 1$, a closed-loop gain of 22.8 was obtained in [Fridman 2001]. For this problem, [Fattouh 1998] was infeasible for any value of gain. Applying the LOI, we obtained a closed-loop gain of 2.33 using polynomials of degree 4.

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 7 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$





Conclusions:

- Extends the LMI Framework to DPS
  - Relies on a new Duality result
  - Other LK-based approaches can be used (But why?).
  - Most LMIs can be converted.
  - But be careful...
- Practical Implications
  - Solved the H_∞-**optimal Full-State Feedback Synthesis Problem** for multi-state multi-delay systems.
  - Solved the H_∞-**optimal Estimator Synthesis Problem** for multi-state single-delay systems.
  - Analytic Inverse allows controller and observer reconstruction.

Numerical Code Produced:

- LOI Toolbox
  - Packaged as DelayTools
  - Duality Test Now on CodeOcean
  - Both Papers on arXiv

Available for download at
`http://control.asu.edu`

- Next Talk:
  - Input Delay (Special Case of Observer Synthesis)
  - H_∞ optimal Dynamic Output Feedback Controller Synthesis

An LMI for Optimal Estimation of **ODE**s

Get rid of the delays and we have

$$\dot{x}(t) = Ax(t) + B_1 w(t), \qquad y(t) = C_2 x(t) + Dw(t)$$

Observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + L(C_2\hat{x}(t) - y(t)), \quad z_e(t) = C_1(\hat{x}(t) - x(t))$$

Lemma 3 (H_∞-Optimal Observer Synthesis).

Define the map $w \mapsto z_e$:

$$\hat{G}(s) = \left[\begin{array}{c|c} A + LC_2 & -(B + LD) \\ \hline C_1 & 0 \end{array}\right].$$

The following are equivalent.

- *There exists a L such that $\|\hat{G}\|_{H_\infty} \le \gamma$.*
- *There exists a $P > 0$ and Z such that*

$$\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB + ZD) \\ -(PB + ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix} < 0.$$

The Observer Gain is recovered as $L = P^{-1}Z$.

The DPS/DDE Equivalent of the Observer LMI

LMI Version of Observer Synthesis: Minimize γ such that $\exists P > 0$ and $Z \in \mathbb{R}^{p\times n}$ such that

$$\begin{bmatrix} e \\ w \end{bmatrix}^T \left[\begin{bmatrix} A^T P + C_2^T Z^T + PA + ZC_2 & -(PB+ZD) \\ -(PB+ZD)^T & -\gamma I \end{bmatrix} + \frac{1}{\gamma}\begin{bmatrix} C_1^T C_1 & 0 \\ 0 & 0 \end{bmatrix}\right] \begin{bmatrix} e \\ w \end{bmatrix}$$

$$= (PAe)^T e + (PAe)^T e + (ZCe)^T e + (ZC_2 e)^T e$$

$$-e^T PBw - (PBw)^T e - \gamma w^T w + \frac{1}{\gamma}(C_1 e)^T (C_1 e) < 0$$

for all $e \in \mathbb{R}^n$, $w \in \mathbb{R}^m$

DPS Version of Observer Synthesis: Minimize γ such that $\exists \mathcal{P} > 0$ and $\mathcal{Z}$ such that

$$\langle \mathcal{P}\mathcal{A}\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{L_2} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{L_2} + \langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{L_2}$$

$$-\langle \mathbf{e}, \mathcal{P}\mathcal{B}w\rangle_{L_2} - \langle \mathcal{B}w, \mathcal{P}\mathbf{e}\rangle_{L_2} - \gamma w^T w + \frac{1}{\gamma}(\mathcal{C}_1\mathbf{e})^T(\mathcal{C}_1\mathbf{e}) < -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^m$$

H_∞-Optimal Controller Synthesis Problem to be Solved

Consider solutions of

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t),$$
$$y(t) = Cx(t) + D_1 w(t) + D_2 u(t).$$

Problem Definition:
Minimize γ such that there exist K_0, K_{1i} and $K_{2i}(s)$ such that if

$$u(t) = K_0 x(t) + \sum_i K_{1i} x(t-\tau_i) + \sum_i \int_{-\tau_i}^{0} K_{2i}(s) x(t+s) ds$$

then for any $w \in L_2$, $\|y\|_{L_2} \le \gamma \|w\|_{L_2}$.

Put Each Term in the PQRS Framework

Define $\mathbf{z} = \begin{bmatrix} x \\ \phi_i \end{bmatrix}$ and $h = \begin{bmatrix} v^T & w^T & x^T & \phi_1(-\tau_1)^T & \cdots & \phi_K(-\tau_K)^T \end{bmatrix}^T$.

H_∞-optimal Controller Synthesis Condition: Let $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z} \rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w \rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z} \rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon \|z\|$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z} \rangle_{Z_{n,K}} + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z} \rangle_{Z_{n,K}} = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_1, E_{1i}, \dot{S}_i, G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}$$

where

$$D_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & C_0 + C_0^T & C_1 & \cdots & C_K \\ 0 & 0 & C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & C_K^T & 0 & 0 & -S_K(-\tau_K) \end{bmatrix}, \quad \begin{array}{c} C_0 := A_0 P + \tau_K \sum_{i=1}^{K} (A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0)), \\ C_i := \tau_K A_i S_i(-\tau_i), \end{array}$$

$$E_{1i}(s) := \begin{bmatrix} 0 & 0 & B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \quad B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} A_j R_{ji}(-\tau_j, s),$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T.$$

Put Each Term in the PQRS Framework

$$\left(\mathcal{Z}\begin{bmatrix}\psi\\ \phi_i\end{bmatrix}\right) := \left[Z_0\psi + \sum_i Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 Z_{2i}(s)\phi_i(s)ds\right]$$

$$\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$\langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z + \langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z = 2\tau_K x^T\left[B_2 Z_0 x + \sum_i B_2 Z_{1i}\phi_i(-\tau_i) + \sum_i \int_{-\tau_i}^0 B_2 Z_{2i}(s)\phi_i(s)ds\right]$$

$$= \tau_K \underbrace{\begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}^T}_{h^T} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0\\ *^T & 0 & 0 & 0 & \dots & 0\\ *^T & *^T & B_2 Z_0 + Z_0^T B_2^T & B_2 Z_{11} & \dots & B_2 Z_{1K}\\ *^T & *^T & *^T & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ *^T & *^T & *^T & *^T & \dots & 0\end{bmatrix}}_{D_2} \underbrace{\begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}}_{h}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v\\ w\\ x\\ \phi_1(-\tau_1)\\ \vdots\\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0\\ 0\\ B_2 Z_{2i}(s)\\ 0\\ \vdots\\ 0\end{bmatrix}}_{E_{2i}(s)} \phi_i(s)ds = \left\langle \begin{bmatrix} h\\ \phi_i\end{bmatrix}, \mathcal{P}_{\{D_2,E_{2i},0,0\}}\begin{bmatrix} h\\ \phi_i\end{bmatrix}\right\rangle_{Z_{r,n,K}}.$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w$$

$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon \|z\|$$

$$\begin{aligned}
&\langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \\
&= \tau_K x^T B_1 w + \tau_K (B_1 w)^T x - \gamma w^T w + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \\
&= \tau_K \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}^T}_{h^T} \underbrace{\frac{1}{\tau_K}\begin{bmatrix} -\gamma I & D_1 & 0 & 0 & \dots & 0 \\ D_1^T & -\gamma I & \tau_K B_1^T & 0 & \dots & 0 \\ 0 & \tau_K B_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}}_{D_3} \underbrace{\begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K) \end{bmatrix}}_{h} \\
&= \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_3,0,0,0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}
\end{aligned}$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v \le -\epsilon\|z\|$$

$$v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v = 2v^T\Bigg[\left(C_0 P + \sum_i \tau_K C_i Q_i(-\tau_i)^T\right)x + \tau_K \sum_i C_i S_i(-\tau_i)\phi_i(-\tau_i)$$
$$+ \sum_{i=1}^K \int_{-\tau_i}^0 \left(C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s)\right)\phi_i(s)ds\Bigg]$$

$$= \tau_K \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\begin{bmatrix} 0 & 0 & \frac{C_0 P}{\tau_K} + \sum_i C_i Q_i(-\tau_i)^T & C_1 S_1(-\tau_1) & \dots & C_K S_K(-\tau_K) \\ *^T & 0 & 0 & 0 & \dots & 0 \\ *^T & *^T & 0 & 0 & \dots & 0 \\ *^T & *^T & *^T & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & 0 \end{bmatrix}}_{D_4} \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}$$

$$+ 2\tau_K \sum_{i=1}^K \int_{-\tau_i}^0 \begin{bmatrix} v \\ w \\ x \\ \phi_1(-\tau_1) \\ \vdots \\ \phi_K(-\tau_K)\end{bmatrix}^T \underbrace{\frac{1}{\tau_K}\begin{bmatrix} C_0 Q_i(s) + \sum_j C_j R_{ji}(-\tau_j, s) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_{4i}(s)} \phi_i(s)ds = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D_4, E_{4i}, 0, 0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}}.$$

Put Each Term in the PQRS Framework

$$\langle \mathcal{A}\mathcal{P}\mathbf{z},\mathbf{z}\rangle_Z+\langle \mathbf{z},\mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z+\langle \mathcal{B}_2\mathcal{Z}\mathbf{z},\mathbf{z}\rangle_Z+\langle \mathbf{z},\mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z+\langle \mathbf{z},\mathcal{B}_1 w\rangle_Z+\langle \mathcal{B}_1 w,\mathbf{z}\rangle_Z-\gamma-\gamma w^T w$$
$$+v^T(\mathcal{C}\mathcal{P}\mathbf{z})+(\mathcal{C}\mathcal{P}\mathbf{z})^T v+v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z})+(\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v+v^T(D_1 w)+(D_1 w)^T v-\gamma v^T v\le-\epsilon\|z\|$$

$$v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z})+(\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v=2v^T\left[D_2Z_0x+\sum_i D_2Z_{1i}\phi_i(-\tau_i)+\sum_i\int_{-\tau_i}^0 D_2Z_{2i}(s)\phi_i(s)ds\right]$$

$$=\tau_K\begin{bmatrix}v\\w\\x\\\phi_1(-\tau_1)\\\vdots\\\phi_K(-\tau_K)\end{bmatrix}^T\underbrace{\frac{1}{\tau_K}\begin{bmatrix}0&0&D_2Z_0&D_2Z_{11}&\dots&D_2Z_{1K}\\ *^T&0&0&0&\dots&0\\ *^T&*^T&0&0&\dots&0\\ *^T&*^T&*^T&0&\dots&0\\ \vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ *^T&*^T&*^T&*^T&\dots&0\end{bmatrix}}_{D_5}\begin{bmatrix}v\\w\\x\\\phi_1(-\tau_1)\\\vdots\\\phi_K(-\tau_K)\end{bmatrix}$$

$$+2\tau_K\sum_{i=1}^K\int_{-\tau_i}^0\begin{bmatrix}v\\w\\x\\\phi_1(-\tau_1)\\\vdots\\\phi_K(-\tau_K)\end{bmatrix}^T\underbrace{\frac{1}{\tau_K}\begin{bmatrix}D_2Z_{2i}(s)\\0\\0\\0\\\vdots\\0\end{bmatrix}}_{E_{5i}(s)}\phi_i(s)ds=\left\langle\begin{bmatrix}h\\\phi_i\end{bmatrix},\mathcal{P}_{\{D_5,E_{5i},0,0\}}\begin{bmatrix}h\\\phi_i\end{bmatrix}\right\rangle_{Z_{r,n,K}}.$$

Combine Terms and enforce Constraint

And, finally,

$$\epsilon\|z\|_Z^2 = \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{\hat{I},0,I,0\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \qquad \text{where} \quad \hat{I} = \mathrm{diag}(0_{q+m}, I_n, 0_{nK})$$

Suppose there exist P, Q_i, S_i, R_{ij}, Z_0, Z_{1i}, and Z_{2i} such that

$$\langle \mathcal{A}\mathcal{P}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{A}\mathcal{P}\mathbf{z}\rangle_Z + \langle \mathcal{B}_2\mathcal{Z}\mathbf{z}, \mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_2\mathcal{Z}\mathbf{z}\rangle_Z + \langle \mathbf{z}, \mathcal{B}_1 w\rangle_Z + \langle \mathcal{B}_1 w, \mathbf{z}\rangle_Z - \gamma - \gamma w^T w$$
$$+ v^T(\mathcal{C}\mathcal{P}\mathbf{z}) + (\mathcal{C}\mathcal{P}\mathbf{z})^T v + v^T(\mathcal{D}_2\mathcal{Z}\mathbf{z}) + (\mathcal{D}_2\mathcal{Z}\mathbf{z})^T v + v^T(D_1 w) + (D_1 w)^T v - \gamma v^T v + \epsilon\|z\|_Z^2$$
$$= \left\langle \begin{bmatrix} h \\ \phi_i \end{bmatrix}, \mathcal{P}_{\{D+\hat{I}, E_i, \dot{S}_i + I, G_{ij}\}} \begin{bmatrix} h \\ \phi_i \end{bmatrix} \right\rangle_{Z_{r,n,K}} \le 0,$$

where $D = \sum_{i=1}^5 D_i$,and $E_i(s) = \sum_{j=1}^5 E_{ij}(s)$. Then there exists a feedback controller $u(t) = \mathcal{Z}\mathcal{P}^{-1}\mathbf{x}(t)$ which achieves CL H_∞ norm γ.

Matlab Code:

```
[P,Q,R,S] = sosjointpos_mat_ker_ndelay_PQRS_vZ
...
[P2,Q2,R2,S2] = sosjointpos_mat_ker_ndelay_PQRS_vZ
sosmateq(prog,D+P2); sosmateq(prog,Q2{i}+E{i});
sosmateq(prog,S2{i}+F{i}); sosmateq(prog,R2{i,j}+G{i,j});
```

How to ensure $\mathcal{P}(X) = X$

Not Needed for Optimal Estimator Synthesis

Recall PQRS Operators have the form

$$\begin{bmatrix} x' \\ \phi_i' \end{bmatrix}(s) = \left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s)$$
$$= \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta \end{bmatrix}$$

So to achieve $x' = \phi_i'(0)$, we need

$$Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds = \tau_K Q_i(0)^T x + \tau_K S_i(0)\phi_i(0) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(0,\theta)\phi_j(\theta)\, d\theta$$

or equivalently

$$P = \tau_K (Q_i(0)^T + S_i(0)), \qquad Q_j(s) = R_{ij}(0,s) \qquad \forall i,j$$

These are linear constraints on P and the coefficients of the polynomials Q_i, S_i, R_{ij}.

Complexity and Accuracy of Dual Stability ($\mathcal{AP} < 0$)

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	4	analytic
$\tau_{\max}$	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.6581	1.716	1.7178	1.7178	1.7178
$\tau_{\min}$	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t)$$
$$+ \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

$$\dot{x}(t) = -\sum_{i=1}^{K} \frac{x(t-i/K)}{K}$$

$K \downarrow\ n \rightarrow$	1	2	3	5	10
1	.366	.094	.158	.686	12.8
2	.112	.295	1.260	10.83	61.05
3	.177	1.311	6.86	96.85	5223
5	.895	13.05	124.7	2014	200950
10	13.09	59.5	5077	200231	NA

Table: CPU sec indexed by # of states (n) and # of delays (K)

Complexity Scaling Results:

- Viable when $nK < 50$

Significant reduction possible using Differential-Difference Formulation.

Roadmap of the Talk

The goal is to find $K \in \mathbb{R}^{m\times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: Solve the problem as a abstract but convex Linear Operator Inequality.

Step 2: Parameterize All Operators using Matrices.

- Synthesis conditions now linear matrix constraints and operator positivity constraints
- $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ framework

Step 3: Enforce Operator Positivity using LMIs.

Step 4: Reconstruct the controller gains.

- Invert the operator using matrix manipulations.

Analytic Formula for Operator Inversion

Suppose $\mathcal{P} := \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$, $Q_i(s) = H_i Z(s)$ and $R_{ij}(s,\theta) = Z(s)^T \Gamma_{ij} Z(\theta)$. Then $\mathcal{P}^{-1} = \mathcal{P}_{\left\{\hat{P},\hat{Q}_i,\hat{S}_i,\hat{R}_{ij}\right\}}$ where if we define

$$H = \begin{bmatrix} H_1 & \dots & H_K \end{bmatrix} \qquad \text{and} \qquad \Gamma = \begin{bmatrix} \Gamma_{11} & \dots & \Gamma_{1K} \\ \vdots & & \vdots \\ \Gamma_{K,1} & \dots & \Gamma_{K,K} \end{bmatrix},$$

then

$$\hat{P} = \left(I - \hat{H} V H^T\right) P^{-1}, \quad \hat{Q}_i(s) = \frac{1}{\tau_K} \hat{H}_i Z(s) S_i(s)^{-1}$$

$$\hat{S}_i(s) = \frac{1}{\tau_K^2} S_i(s)^{-1} \qquad \hat{R}_{ij}(s,\theta) = \frac{1}{\tau_K} S_i(s)^{-1} Z(s)^T \hat{\Gamma}_{ij} Z(\theta) S_i(\theta)^{-1},$$

where

$$\begin{bmatrix} \hat{H}_1 & \dots & \hat{H}_K \end{bmatrix} = \hat{H} = P^{-1} H \left(V H^T P^{-1} H - I - V\Gamma\right)^{-1}$$

$$\begin{bmatrix} \hat{\Gamma}_{11} & \dots & \hat{\Gamma}_{1K} \\ \vdots & & \vdots \\ \hat{\Gamma}_{K,1} & \dots & \hat{\Gamma}_{K,K} \end{bmatrix} = \hat{\Gamma} = -(\hat{H}^T H + \Gamma)(I + V\Gamma)^{-1}, \quad V = \begin{bmatrix} V_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & V_K \end{bmatrix}$$

$$V_i = \int_{-\tau_i}^{0} Z(s) S_i(s)^{-1} Z(s)^T ds$$

Reconstructing the Full-State Feedback Controller Gains

Finally, we recover the controller as

$$u(t) = K_0 x(t) + \frac{1}{\tau_K}\sum_i K_{1i} x(t-\tau_i) + \frac{1}{\tau_K}\sum_i \int_{-\tau_i}^{0} K_{2i}(s) x(t+s) ds$$

where (Z_0, Z_{1i}, Z_{2i} are variables, Z is a vector of monomials)

$$K_0 = Z_0 \hat{P} + \sum_j \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{H}_j^T$$

$$K_{1i} = Z_{1i} S_i(-\tau_i)^{-1}, \qquad O_i = \int_{-\tau_j}^{0} Z_{2j}(s) S_j(s)^{-1} Z(s)^T ds$$

$$K_{2i}(s) = \left(Z_0 \hat{H}_i Z(s) + Z_{2i}(s) + \sum_{j=1}^{K} \left(Z_{1j} S_j(-\tau_j)^{-1} Z(-\tau_j)^T + O_j \right) \hat{\Gamma}_{ji} Z(s) \right) S_i(s)^{-1}$$

Note: This is *Full-State* Feedback.

- Contrast with output feedback: $u(t) = Kx(t)$ or $u(t) = Ky(t-r)$.

Boring Numerical Examples

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ .1 \end{bmatrix} u(t)$$

d	1	2	3	Padé	Fridman 2003	Li 1997
$\gamma_{\min}(\tau = .999)$	.10001	.10001	.10001	.1000	.22844	1.8822
$\gamma_{\min}(\tau = 2)$	1.43	1.36	1.341	1.340	∞	∞
CPU sec	.478	.879	2.48	2.78	N/A	N/A

$$\dot{x}(t) = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} -.5 \\ 1 \end{bmatrix} w(t) + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & -.5 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

d	1	2	3	Padé
$\gamma_{\min}(\tau = .3)$	.3953	.3953	.3953	.3953
CPU sec	.655	1.248	2.72	2.91

Interesting Numerical Example

Fixing the Athenaeum Showers [Peet Thesis, 2006]

Tracking Control with integral feedback

- T_{2i} is the water temperature
- T_{1i} is the tap position
- τ_i is the time for water to move from tap to showerhead
- w_i is the desired water temperature (Not available to controller!)
- Opening the tap by user i decreases the water temperature of users $j \neq i$
- Minimize tap action and controller interference.

$$\dot{T}_{1i}(t) = T_{2i}(t) - w_i(t)$$

$$\dot{T}_{2i}(t) = -\alpha_i \left(T_{2i}(t-\tau_i) - w_i(t)\right) + \sum_{j \neq i} \gamma_{ij}\alpha_j \left(T_j(t-\tau_j) - w_j(t)\right) + u_i(t)$$

$$y_i(t) = \begin{bmatrix} T_{1i}(t) \\ .1u_i(t) \end{bmatrix}.$$

Fixing the Athenaeum Showers

$$\dot{x}(t) = A_0 x(t) + \sum_i A_i x(t-\tau_i) + B_1 w(t) + B_2 u(t), \quad y(t) = Cx(t) + D_1 w(t) + D_2 u(t)$$

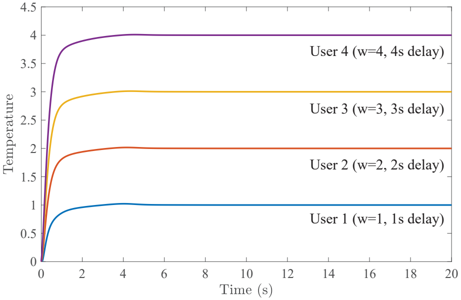
where

$$A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B_1 = \begin{bmatrix} -I \\ -\hat{\Gamma} + \mathrm{diag}(\alpha_1 \dots \alpha_K) \end{bmatrix}$$

$$\hat{A}_i(:,i) = \alpha_i \begin{bmatrix} \gamma_{i,1} & \dots & \gamma_{i,i-1} & -1 & \gamma_{i,i-1} & \dots & \gamma_{i,K} \end{bmatrix}^T$$

$$\hat{\Gamma}_{ij} = \alpha_j \gamma_{ij} = \begin{bmatrix} q_1 & \dots & q_K \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$C_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \; C_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \; D_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \; D_2 = \begin{bmatrix} 0 \\ .1I \end{bmatrix}$$



Complexity: 8 states, 4 delays, 4 inputs, 4 disturbances, 8 regulated outputs

Results: A Matlab simulation of the step response of the closed-loop temperature dynamics ($T_{2i}(t)$) with 4 users (w_i and τ_i as indicated) coupled with the controller with closed-loop gain of $.48$