Systems Analysis and Control

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Lecture 22: The Nyquist Criterion

Overview

In this Lecture, you will learn:

Complex Analysis

- The Argument Principle
- The Contour Mapping Principle

The Nyquist Diagram

- The Nyquist Contour
- Mapping the Nyquist Contour
- The closed Loop
- Interpreting the Nyquist Diagram

Review

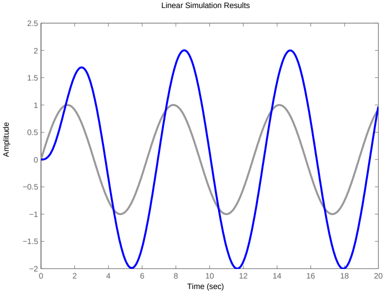
Recall: Frequency Response

Input:

$$u(t) = M \sin(\omega t + \phi)$$

Output: Magnitude and Phase Shift

$$y(t) = |G(\imath\omega)| M \sin(\omega t + \phi + \angle G(\imath\omega))$$



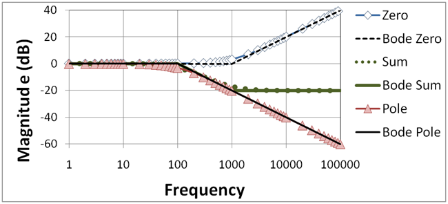
Frequency Response to $\sin \omega t$ is given by $G(\imath\omega)$

Review

Recall: Bode Plot

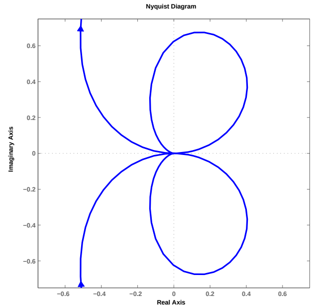
The Bode Plot is a way to visualize $G(\imath\omega)$:

1. Magnitude Plot: $20\log_{10}|G(\imath\omega)|$ vs. $\log_{10}\omega$
2. Phase Plot: $\angle G(\imath\omega)$ vs. $\log_{10}\omega$



Bode Plots

If we only want a single plot we can use ω as a *parameter*.



A plot of $Re(G(\imath\omega))$ vs. $Im(G(\imath\omega))$ as a function of ω.

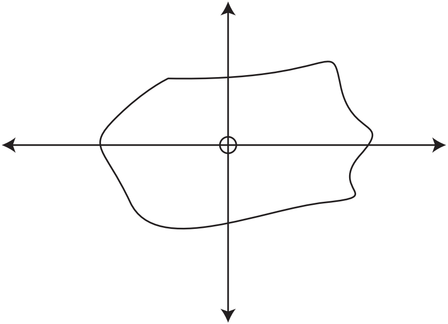
- Advantage: All Information in a single plot.
- AKA: Nyquist Plot

Question: How is this useful?

The Nyquist Plot

To Understand Nyquist:

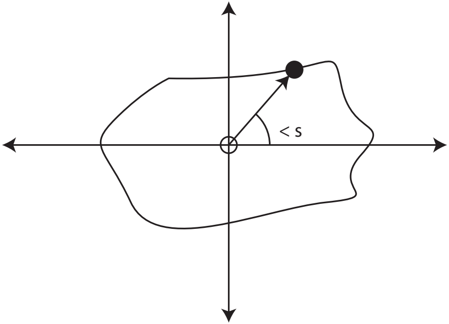
- Go back to Root Locus
- Consider a single zero: $G(s) = s$.



Draw a curve around the pole

What is the phase at a point on the curve?

$$\angle G(s) = \angle s$$



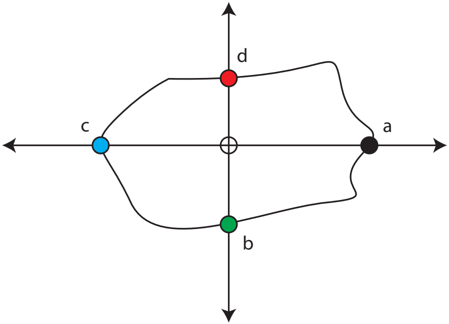
The Nyquist Plot

Consider the phase at four points

1. $\angle G(a) = \angle 1 = 0°$
2. $\angle G(b) = \angle - \imath = -90°$
3. $\angle G(c) = \angle - 1 = -180°$
4. $\angle G(d) = \angle \imath = -270°$

The phase decreases along the curve until we arrive back at a.

- The phase *resets* at a by $+360°$



The reset is **Important**!

- There would be a reset for *any* closed curve containing z or *any* starting point.
- We went around the curve *Clockwise* (CW).
  - If we had gone Counter-Clockwise (CCW), the reset would have been $-360°$.

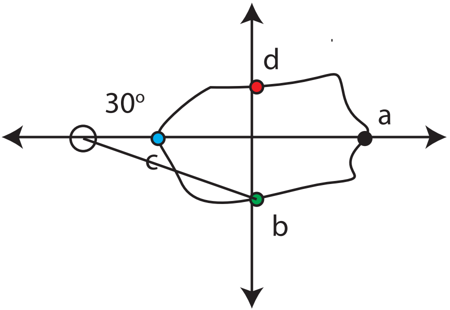
The Nyquist Plot

Now consider the **Same Curve** with

$$G(s) = s + 2$$

Phase at the same four points

1. $\angle G(a) = \angle 3 = 0°$
2. $\angle G(b) = \angle 2 - \imath \cong -30°$
3. $\angle G(c) = \angle 1 = 0°$
4. $\angle G(d) = \angle 2 + \imath \cong 30°$



In this case the transition back to $0°$ is smooth.

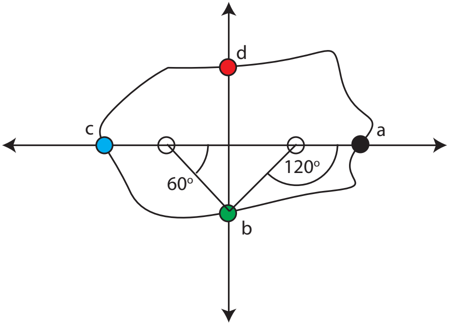
- No reset is required!

Question What if we had encircled 2 zeros?

Phase at the same four points

1. $\angle G(a) = 0^\circ$
2. $\angle G(b) = -180^\circ$
3. $\angle G(c) = -360^\circ$
4. $\angle G(d) = -540^\circ$

- The phase *resets* at a by $+720^\circ$



Rule: The reset is $+360 \cdot \#_{zeros}$.

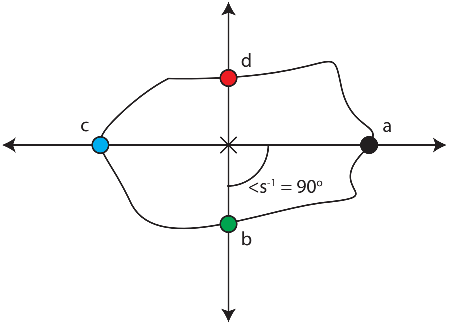
The Nyquist Plot

Question What about encircling a pole?

Consider the phase at four points

1. $\angle G(a) = \angle 1 = 0^\circ$
2. $\angle G(b) = \angle \frac{1}{-\imath} = \angle \imath = 90^\circ$
3. $\angle G(c) = \angle -1 = 180^\circ$
4. $\angle G(d) = \angle \frac{1}{\imath} = \angle -\imath = 270^\circ$

- The phase *resets* at a by -360°



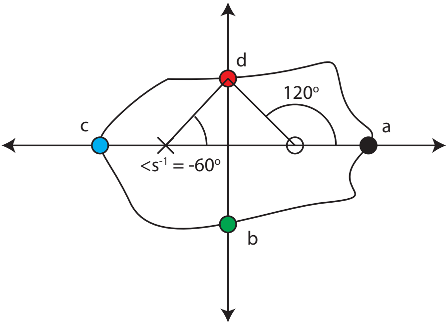
Rule: The reset is $-360 \cdot \#_{poles}$.

Question: What if we combine a pole and a zero?

Consider the phase at four points

1. $\angle G(a) = 0°$
2. $\angle G(b) = -60°$
3. $\angle G(c) = 0°$
4. $\angle G(d) = 60°$

- There is *no reset* at a.



Rule: Going CW, the reset is $+360 \cdot (\#_{zeros} - \#_{poles})$.

A consequence of the *Argument Principle* from Complex Analysis.

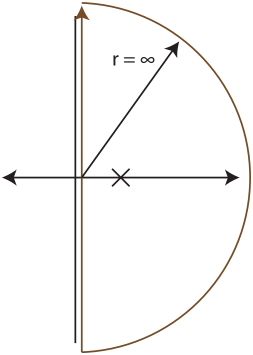
How can this observation be used?
Consider Stability.

- $G(s)$ is stable if it has no poles in the right half-plane

Question: How to tell if any poles are in the RHP?
Solution: Draw a curve around the RHP and count the resets.

Define the Curve:

- Starts at the origin.
- Travels along imaginary axis till $r = \infty$.
- At $r = \infty$, loops around clockwise.
- Returns to the origin along imaginary axis.



A Clockwise Curve

The reset is $+360 \cdot (\#_{zeros} - \#_{poles})$.

If there is a negative reset, there is a pole in the RHP

If we encircle the right half-plane,

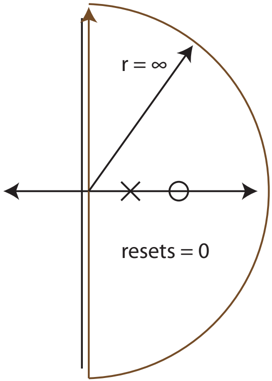
$$\text{The reset is } +360 \cdot (\#_{zeros} - \#_{poles}).$$

Question 1:

- How to determine the number of resets along this curve?

Question 2:

- Zeros can hide the poles!
- What to do?



Contour Mapping

Lets answer the more basic question first:

- How to determine the number of resets along this curve?

Definition 1.

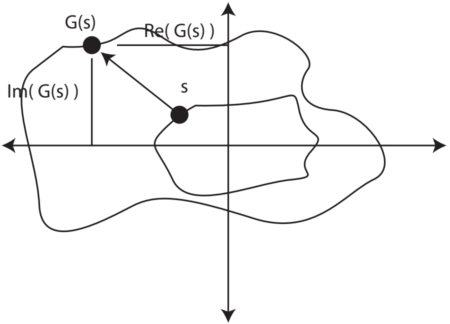
Given a contour, $\mathcal{C} \subset X$, and a function $G : X \to X$, the **contour mapping** $G(\mathcal{C})$ is the curve $\{G(s) : s \in \mathcal{C}\}$.

In the complex plane, we plot

$$Im(G(s)) \text{ vs. } Re(G(s))$$

along the curve $\mathcal{C}$

- Yields a new curve, $\mathcal{C}_G$.



Contour Mapping

Key Point: For a point on the mapped contour, $s^* = G(s)$,

$$\angle s^* = \angle G(s)$$

- We measure θ, not phase.

To measure the 360° resets in $\angle G(s)$

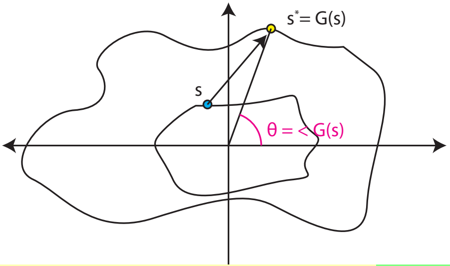
- We count the number of $+360^\circ$ resets in θ!
- We count the number of times $\mathcal{C}_G$ encircles the origin **Clockwise**.

A reset occurs every time $\mathcal{C}_G$ encircles the origin clockwise.

- Makes the resets much easier to count!

Assumes the contour doesn't hit any poles or zeros, otherwise

- $G(s) \to \infty$ and we lose count.
- $G(s) \to 0$ and we lose count.



Contour Mapping

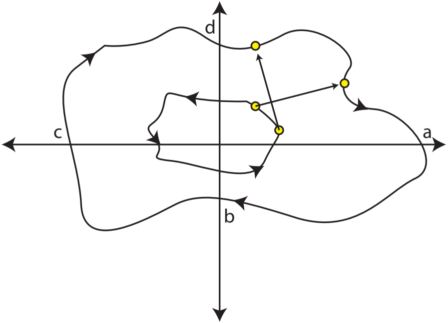
Example 1

Remember Direction is important
If the original Contour was counter-clockwise

$$\text{The reset is } +360 \cdot (\#_{poles} - \#_{zeros}).$$

In this example we see $\mathcal{C}_G$

- encircles the origin once going Clockwise
  - $\theta_a = 0$
  - $\theta_b = -90^\circ$
  - $\theta_c = -180^\circ$
  - $\theta_d = -270^\circ$
- A *Positive Reset* of $+360^\circ$.



Thus $+360 \cdot (\#_{poles} - \#_{zeros}) = +360$

$$(\#_{poles} - \#_{zeros}) = 1$$

At least one pole in the region.

Contour Mapping

Assume the original Contour was clockwise

The reset is $+360 \cdot (\#_{zeros} - \#_{poles})$.

There are 5 counter-clockwise encirclements of the origin.

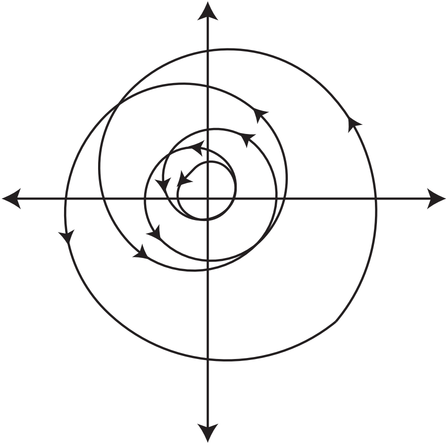
- A *Negative Reset* of $-360° \cdot 5$.

Thus

$$+360 \cdot (\#_{zeros} - \#_{poles}) = -360 \cdot 5$$

$$(\#_{zeros} - \#_{poles}) = -5$$

At least 5 poles in the region.



The Nyquist Contour

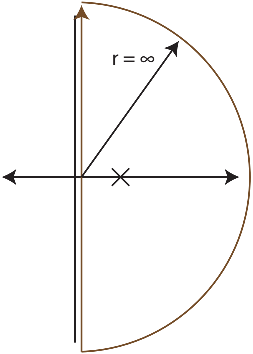
Conclusion: If we can plot the contour mapping, we can find the relative # of poles and zeros.

Definition 2.

The **Nyquist Contour**, $\mathcal{C}_N$ is a contour which contains the imaginary axis and encloses the right half-place. The Nyquist contour is clockwise.

A Clockwise Curve

- Starts at the origin.
- Travels along imaginary axis till $r = \infty$.
- At $r = \infty$, loops around clockwise.
- Returns to the origin along imaginary axis.



The Nyquist Contour

To map the Nyquist Contour, we deal with two parts

- The imaginary Axis.
- The loop at ∞.

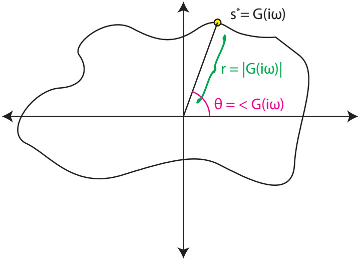
The Imaginary Axis

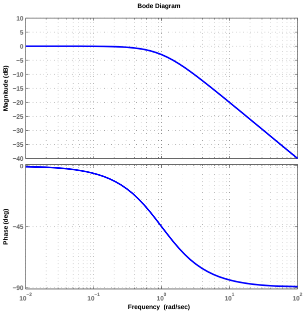
- Contour Map is $G(\imath\omega)$
- Plot $Re(G(\imath\omega))$ vs. $Im(G(\imath\omega))$

Data Comes from Bode plot

- Plot $Re(G(\imath\omega))$ vs. $Im(G(\imath\omega))$

Map each point on Bode to a point on Nyquist





The Nyquist Contour

The Loop at ∞: 2 Cases

$$G(s) = \frac{n(s)}{d(s)} = \frac{a_0 s^m + \cdots a_m}{b_0 s^n + \cdots b_n}$$

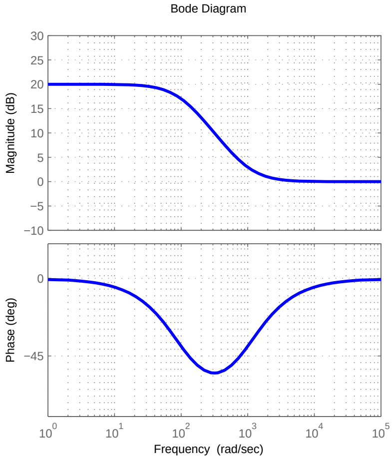
Case 1: $G(s)$ is Proper, but not strictly

- Degree of $d(s)$ same as $n(s)$
- As $\omega \to \infty$, $G(s)$ becomes constant
  - Magnitude becomes fixed

$$\lim_{s \to \infty} \frac{n(s)}{d(s)} = \frac{n(s)}{d(s)} = \frac{a_0}{b_0}$$

  - Phase varies (More on this later)

We can use the Nyquist Plot



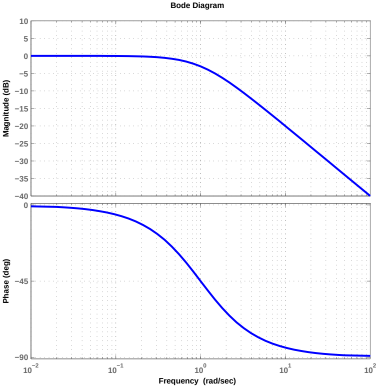
The Loop at ∞:

$$G(s) = \frac{n(s)}{d(s)} = \frac{a_0 s^m + \cdots a_m}{b_0 s^n + \cdots b_n}$$

Case 1: $G(s)$ is Strictly Proper

- Degree of $d(s)$ greater than $n(s)$
- As $\omega \to \infty$, $|G(\imath\omega)| \to 0$

$$\lim_{s \to \infty} G(s) = \lim_{\omega \to \infty} \frac{n(s)}{d(s)} = 0$$



Can't tell what goes on at ∞!

Nyquist Plot is useless

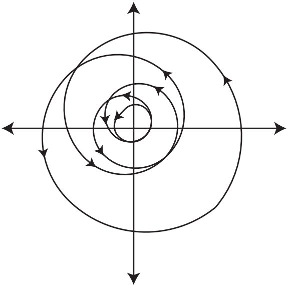
The Nyquist Contour

If we limit ourself to proper, but not strictly proper.

Because the Nyquist Contour is clockwise,

The number of clockwise encirclements of 0 is

- The $\#_{poles} - \#_{zeros}$ in the RHP



Conclusion: Although we can map the RHP onto the Nyquist Plot, we have two problems.

- Can only determine $\#_{poles} - \#_{zeros}$
- Doesn't work for strictly proper systems.

The Nyquist Contour

Our solution to all problems is to consider Systems in Feedback

- Assume we can plot the Nyquist plot for the open loop.
- What happens when we close the loop?

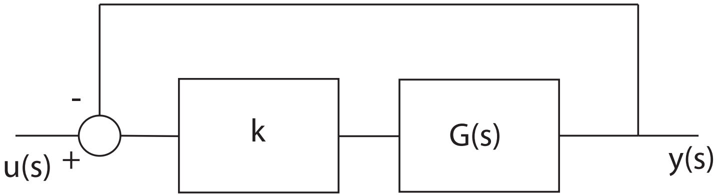
The closed loop is

$$\frac{kG(s)}{1 + kG(s)}$$

We want to know when

$$1 + kG(s) = 0$$

Question: Does $\frac{1}{k} + G(s)$ have any zeros in the RHP?



This is a better question.
$\frac{1}{k} + G(s)$ **is Proper, but not Strictly**

$$\frac{1}{k} + G(s) = \frac{d(s) + kn(s)}{kd(s)}$$

- Degree of $d(s)$ greater than or equals $n(s)$
- $degree(d(s) + kn(s)) = degree(d(s))$

Numerator and denominator have same degree!

We know about the poles of $\frac{1}{k} + G(s)$

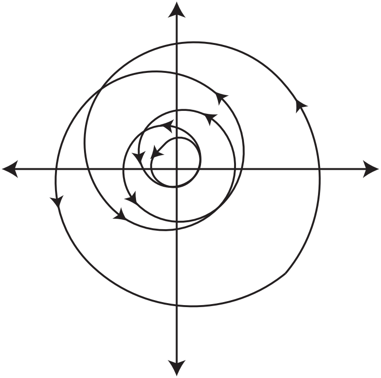
- poles are the poles of the open loop
- We know if the open loop is stable!
- we know if any poles are in RHP.

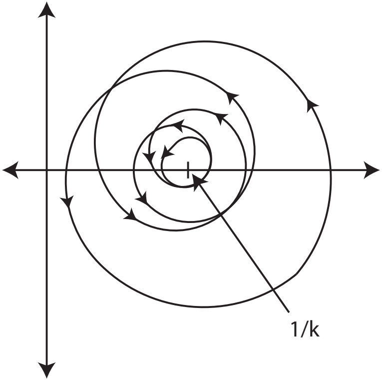
The Nyquist Contour

Closed Loop

Mapping the Nyquist contour of $\frac{1}{k} + G(s)$ is easy!

1. Map the Contour for $G(s)$
2. Add $\frac{1}{k}$ to every point





Shifts the plot by factor $\frac{1}{k}$

Conclusion: If we map the Nyquist Contour for $\frac{1}{k}+G(s)$

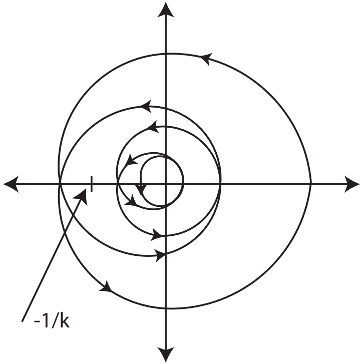
- The # of clockwise encirclements of 0 is $\#_{poles}-\#_{zeros}$ of closed loop in the RHP.
- The # of zeros of $\frac{1}{k}+G(s)$ in RHP is # of clockwise encirclements plus # of open-loop poles of $G(s)$ in RHP.

Instead of shifting the plot, we can shift the origin to point $-\frac{1}{k}$

The number of unstable closed-loop poles is $N+P$, where

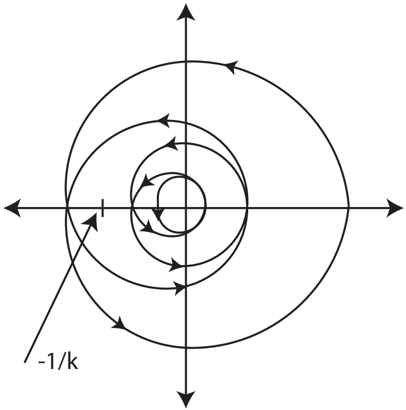
- N is the number of clockwise encirclements of $\frac{-1}{k}$.
- P is the number of unstable open-loop poles.

If we get our data from Bode, typically $P=0$



The Nyquist Contour

Example



Two CCW encirclements of $-\frac{1}{k}$

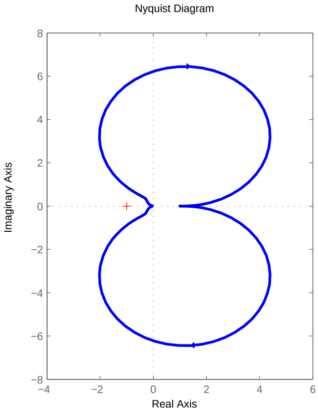
- Assume 1 unstable Open Loop pole $P = 1$
- Encirclements are CCW: $N = -2$
- $N + P = -1$: No unstable Closed-Loop Poles

Nyquist lets us quickly determine the regions of stability

The Suspension Problem

- Open Loop is Stable: $P = 0$
- No encirclement of $-1/k$
    - Holds for any $k > 0$

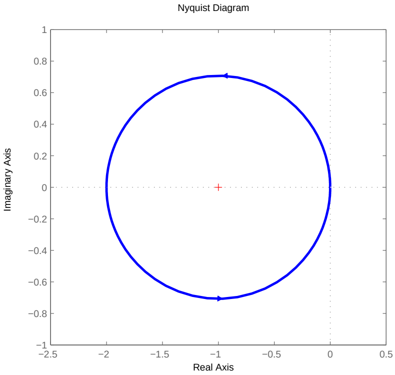
Closed Loop is *stable* for any $k > 0$.



The Inverted Pendulum with Derivative Feedback

- Open Loop is Unstable: $P = 1$
- CCW encirclement of $-1/k$
  - Holds for any $-2 < \frac{-1}{k} < 0$
  - Holds for any $k > \frac{1}{2}$
- When $k \geq \frac{1}{2}$, $N = -1$

Closed Loop is *stable* for $k > \frac{1}{2}$.



Summary

What have we learned today?

Complex Analysis

- The Argument Principle
- The Contour Mapping Principle

The Nyquist Diagram

- The Nyquist Contour
- Mapping the Nyquist Contour
- The closed Loop
- Interpreting the Nyquist Diagram

Next Lecture: Drawing the Nyquist Plot