A Result on Global Stability of Internet Congestion Control

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Abstract— **In this paper, we address the question of stability of certain TCP/AQM congestion control protocols described by nonlinear, discontinuous differential equations with delay. We analyze a well-known model, whose dynamics were previously shown to be locally stable for arbitrary delay via analysis of its linearization. We use a generalization of passivity theory to show that the nonlinear, discontinuous model is both input-output and asymptotically stable for arbitrary delay given a restriction on a certain parameter. These results apply to the case of a single link with sources of identical fixed delay and demonstrate that in certain cases, stability of the nonlinear model holds under the same conditions as the linearized model.**

I. Introduction and Prior Work

The analysis of Internet congestion control protocols has received much attention recently. Explicit mathematical modeling of the Internet has allowed analysis of existing protocols from a number of different theoretical perspectives and has generated suggestions for improvement to current protocols. This work has been motivated by concern about the ability of current protocols to ensure stability and performance of the Internet as the number of users and amount of bandwidth continues to increase. Although the protocols that have been used in the past have performed well as the Internet has increased in size, analysis [1] indicates that as capacities and delays increase, instability will become a problem.

Many algorithms have been proposed for Internet congestion control, some of which have been shown to be globally stable in the presence of delays, nonlinearities and discontinuities. These proofs can be grouped into several categories according to methodology. In particular, Lyapunov-Razumhikin theory has been used to show global stability in [2], [3], [4], [5], [6], Lyapunov-Krasovskii functionals have been used to show global stability in [7], [8], [9], [10] and an input-output approach was taken in [5], [11]. In all of these cases, stability has been proven with varying degrees of conservatism with respect to restrictions on system parameters or delays. In this paper, we address stability of a dual implementation of the protocols suggested by Low and Lapsley [12] based on the work of Kelly et al. [13]. We consider a particular implementation of these protocols for which a linear stability result was obtained for arbitrary network topology and delays [14] subject to a bound on a certain gain parameter, α. This paper shows in certain cases that this same bound on the gain parameter, α, also implies global stability. The importance of this result lies in the fact that the value of the gain parameter is inversely proportional to the equilibrium queue length.

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The system considered in this paper is described by differential equations with delay and a non-static, discontinuous nonlinearity. Although frequency domain techniques are effective when applied to linear systems with delay, these tools fail in the presence of nonlinearity. In addition, although time-domain analysis of nonlinear finite dimensional systems has had some success, analysis of the infinite dimensional systems associated with delay has been more problematic. In this paper, we obtain improved results by decomposing the nonlinear, discontinuous, delayed system into the interconnection of a linear system with delay and a nonlinear, discontinuous system without delay. We analyze the subsystems separately and prove a passivity result for each. One benefit of such an approach is that it allows us to use frequency-domain arguments in addressing the infinite dimensional linear system. We can then use time-domain arguments in the analysis of the single state nonlinear system. If the nonlinearity were static, then these results could be combined using the Nyquist criterion. However, since the the nonlinear component varies with time, we are obliged to adopt the generalized passivity framework of Rantzer and Megretski [15].

This paper is organized as follows. In Section II, we discuss background including the TCP model, stability of differential equations and the theory of Integral Quadratic Constraints. In Section III, we decompose the system dynamics into linear and nonlinear components and use the generalized passivity framework to prove input-output stability of the system. We also show that this implies asymptotic stability. Finally, in Section IV, we conclude the paper.

II. Background Material

A. Notation

Let $\mathbb{R}_+ = \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$, and let $\mathcal{C}$ denote the set of continuous functions. $\mathcal{C}_\tau$ denotes the Banach space of continuous functions $u : [-\tau, 0] \to \mathbb{R}^n$ with norm $\|u\| = \sup_{t\in[-\tau,0]} \|u(t)\|_2$. A function $x : \mathbb{R} \to \mathbb{R}$ is said to be ***absolutely continuous*** if for any integer N and any sequence $t_1, \dots, t_N$, we have $\sum_{k=1}^{N-1} |x(t_k) - x(t_{k+1})| \to 0$ whenever $\sum_{k=1}^{N-1} |t_k - t_{k+1}| \to 0$.

L_2 is the Hilbert space of Lebesgue measurable real vector-valued functions $x : \mathbb{R}^+ \to \mathbb{R}^n$ with inner-product $\langle u, v\rangle_2 = \int_0^\infty u(t)^T v(t) dt$. P_T is the truncation operator such that if $y = P_T z$, then $y(t) = z(t)$ for all $t \leq T$ and $y(t) = 0$ otherwise. L_{2e} denotes the space of functions such that for any $T > 0$ and $y \in L_2 e$, we have $P_T y \in L_2$. We also make use of the space $W_2 = \{y : y, \dot{y} \in L_2\}$ with inner product $\langle x, y\rangle_{W_2} = \langle x, y\rangle_{L_2} + \langle \dot{x}, \dot{y}\rangle_{L_2}$ and extended space $W_{2e} = \{y : y, \dot{y} \in L_{2e}\}$. The dimensions of the various L_2 and W_2 spaces used should be clear from context and are not explicitly stated.

A causal operator $H : L_{2e} \to L_{2e}$ is bounded if $H(0) = 0$ and if it has finite gain, defined as

$$\|H\| = \sup_{u\in L_2 \neq 0} \frac{\|Hu\|}{\|u\|}$$

$\hat{u}$ denotes the either the Fourier or Laplace transform of u, depending on u. We will also make use of the following specialized set of transfer functions which define bounded linear operators on L_2. $\mathcal{A}$ is defined to be those transfer functions which are the Laplace transform of functions of the form

$$g(t) = \begin{cases} h(t) + \sum_{i=1}^{N} g_i \delta(t - t_i) & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $h \in L_1$, $g_i \in \mathbb{R}$ and $t_i \geq 0$.

B. Stability Framework

Two concepts of stability will be used in this paper. The first, input-output stability, is used to define stability of an operator and describes the relationship between inputs and outputs. The second, internal stability, defines stability of a differential equation and is a property of the behavior of the state given initial conditions.

Input-output stability: Consider an operator, Ψ.

Definition 1: For normed spaces X, Y, the operator Ψ is Y stable on X if it defines a single-valued map from X to Y and there exists some β such that $\|\Psi u\|_Y \leq \beta \|u\|_X$ for all $u \in X$.

Internal stability: Now consider a delay-differential equation of the following form

$$\dot{x}(t) = \begin{cases} f\big(x(t), x(t-\tau)\big) & \text{for all } t \geq \tau \\ f\big(x(t), x_0(t-\tau)\big) & \text{for all } t \in [0, \tau) \end{cases} \quad (1)$$

where $f(0,0) = 0$. The system is well-posed if for each $x_0 \in \mathcal{C}_\tau$ there exists a unique $x \in \mathcal{C}$ such that x satisfies (1) for $t \geq 0$ and $x(0) = x_0(0)$. In this case, the function f defines a map

$$\Phi_f : \mathcal{C}_\tau \to \mathcal{C}$$

Definition 2: The solution map Φ defined by f is ***globally stable*** on $X \subset \mathcal{C}_\tau$ if

(i) Φx is bounded for any $x \in X$

(ii) Φ is continuous at 0 with respect to the supremum norm on $\mathcal{C}$ and $\mathcal{C}_\tau$.

Definition 3: The solution map Φ defined by f is ***globally asymptotically stable*** on $X \subset \mathcal{C}_\tau$ if $y = \Phi x_0$ implies $\lim_{t\to\infty} y(t) = 0$ for any $x_0 \in X$.

C. Theory of Integral-Quadratic Constraints

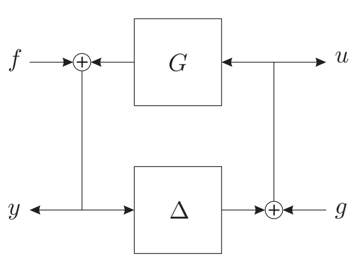

Fig. 1. Interconnection of systems

Let G be a linear operator with transfer function $\hat{G} \in \mathcal{A}$ and let the operator $\Delta : L_2 \to L_2$ be causal and bounded. Define inputs $f \in W_2$ and $g \in L_2$. The ***interconnection*** of G and Δ is defined by the following equations.

$$y = Gu + f \qquad u = \Delta y + g$$

Definition 4 (Jönsson [16], p71): The interconnection of G and Δ is ***well-posed*** if for every pair (f, g) with $f \in W_2$ and $g \in L_2$, there exists a solution $u \in L_{2e}$, $y \in W_{2e}$ and the map $(f, g) \to (y, u)$ is causal.

If the interconnection of Δ and G is well posed, then the interconnection defines an operator $\Phi : W_2 \times L_2 \to W_{2e} \times L_{2e}$. In this paper we use the following generalization of the work by Rantzer and Megretski [15] as presented in the thesis work by Jönsson [16].

Definition 5: Let $\Pi_B : j\mathbb{R} \to \mathbb{C}^{n\times n}$ be a bounded and measurable function that takes Hermitian values and $\lambda \in \mathbb{R}$. We say that Δ ***satisfies the IQC*** defined by Π_B, λ, if there exists a positive constant γ such that for all $y \in W_2$ and $v = \Delta y \in L_2$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \Pi_B(j\omega) \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} d\omega + 2\langle v, \lambda \dot{y} \rangle \geq -\gamma |y(0)|^2$$

Theorem 6: Assume that

1) G is a linear causal bounded operator with $s\hat{G}(s), \hat{G}(s) \in \mathcal{A}$
2) For all $\kappa \in [0,1]$, the interconnection of $\kappa\Delta$ and G is well-posed
3) For all $\kappa \in [0,1]$, $\kappa\Delta$ satisfies the IQC defined by Π_B, λ
4) There exists $\eta > 0$ such that for all $\omega \in \mathbb{R}$

$$\begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix}^* \left(\Pi_B(j\omega) + \begin{bmatrix} 0 & \lambda j\omega^* \\ \lambda j\omega & 0 \end{bmatrix} \right) \begin{bmatrix} \hat{G}(j\omega) \\ I \end{bmatrix} \leq -\eta I$$

Then the interconnection of G and Δ is $W_2 \times L_2$ stable on $W_2 \times L_2$.

D. A Congestion Control Protocol

The following model for optimizing flow rates in a network was proposed by Kelly et al. [13]. We denote the set of sources which utilize link j by I_j and the set of links used by source i by J_i.

$$\text{maximize} \quad \sum_{i=1}^{N} U_i(x_i) \quad \text{subject to} \quad x_i \geq 0, \quad \sum_{i\in I_j} x_i \leq c_j$$

Here c_j is the capacity of link j and N is the number of sources. We consider the dual problem with dual variable $p \in \mathbb{R}^M$ which is given by

$$\text{minimize} \qquad h(p) \qquad \text{subject to} \qquad p \geq 0,$$

where M is the number of links and the dual function h is given by

$$h(p) = \sum_{i=1}^{N} \Big(U_i\big(x_{\text{opt},i}(p)\big) \Big) - p^T (R x_{\text{opt}}(p) - c)$$

$$x_{\text{opt},i}(p) = \max\{0, U_i'^{-1}(q_i(p))\} \qquad q(p) = R^T p.$$

Here R is the routing matrix and is defined by $R_{ji} = 1$ if $j \in J_i$ and is 0 otherwise. To construct a dynamical system which converges to the solution of the dual problem, we consider the gradient projection algorithm. A continuous-time implementation of this is as follows.

$$\dot{p}_j(t) = \begin{cases} \gamma_j(y_j(t) - c_j) & p_j(t) > 0 \\ \max\{0, \gamma_j(y_j(t) - c_j)\} & p_j(t) \le 0 \end{cases}$$
$$x_i(t) = \max\{0, U_i'^{-1}(q_i(t))\}$$
$$y(t) = Rx(t), \qquad q(t) = R^T p(t)$$

γ_j denotes a gain parameter, corresponding to step-size in discrete time. This algorithm has the property that it is decentralized, corresponding to the separable structure of the constraints. To ensure that the algorithm will converge in the current internet framework, we also consider the delay in transmitting packets from the source to the link and then receiving acknowledgements at the source. The delay from source i to link j is denoted τ_{ij}^f and the delay from link j to source i is denoted τ_{ij}^b. For any source i, $\tau_i = \tau_{ij}^f + \tau_{ij}^b$ for all $j \in J_i$. Thus we have that $y_j(t) = \sum_{i \in I_j} x_i(t - \tau_{ij}^f)$ and $q_i(t) = \sum_{j \in J_i} p_j(t - \tau_{ij}^b)$.

The work by Paganini et al. [14] introduced a class of utility functions under which this delayed system was shown to have a stable linearization about its positive equilibrium point for a fixed gain parameter $\gamma_j = 1/c_j$. This class was given by the set of U_i such that

$$\frac{d}{dq_i} U_i'^{-1}(q_i) = -\frac{\alpha_i}{M_i \tau_i} U_i'^{-1}(q_i),$$

where M_i is a bound on the number of links in the path of source i and $\alpha_i < \pi/2$. In particular, the choice of

$$U_i(x) = \frac{M_i \tau_i}{\alpha_i} x \left(1 - \ln \frac{x}{x_{\max,i}}\right),$$

with restricted domain $x \le x_{\max,i}$ was suggested in [14] as a strictly concave utility function such that the function $U_i'^{-1}(q) = x_{\max,i} e^{-\frac{\alpha_i}{M_i \tau_i} q} \ge 0$ has the necessary derivative. In this paper we prove that under certain conditions global stability of this protocol holds under the same conditions as stability of its linearization. Previous attempts to analyze global stability of this algorithm can be found in, for example [6], [9].

III. Results

In this section we reformulate the proposed congestion control algorithm as the interconnection of a linear system with delay and a nonlinear system without delay. This approach was motivated by the work of Wang[5] and Jönsson[17]. We then use the IQC defined by Π_B, $\lambda = \frac{2}{\pi}$, where $\Pi_B = \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix}$, $\beta = \alpha/(\alpha_{\max}\tau)$ and $\alpha_{\max} = \ln(x_{\max}/c)/\big((x_{\max}/c) - 1\big)$ to establish $W_2 \times L_2$ stability on W_2 of the interconnection for any $\tau \ge 0$, $0 < \alpha < \pi/2\alpha_{\max}$. It is assumed that $x_{\max} > c$. We also show that $W_2 \times L_2$ stability on W_2 of the interconnection implies asymptotic stability of the original formulation of the congestion control protocol.

A. Preliminary Results

If we consider the problem of a single link and a single source, then from the development in Section II-D we have that $y(t) = x(t - \tau^f)$ and $q(t) = p(t - \tau^b)$ where $\tau^f + \tau^b = \tau$. Given an initial condition $x_0 \in \mathcal{C}_\tau$, the dynamics can now be summarized as $p(t) = x_0(t)$ for $t \in [-\tau, 0]$ and the following for $t \ge 0$.

$$\dot{p}(t) = \begin{cases} \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1 & p(t) > 0 \\ \max\{0, \frac{x_{\max}}{c} e^{-\frac{\alpha}{\tau} p(t-\tau)} - 1\} & p(t) \le 0 \end{cases} \tag{2}$$
$$x(t) = x_{\max} e^{-\frac{\alpha}{\tau} p(t - \tau^b)} \tag{3}$$

Since the dynamics of Equation (2) are decoupled from those of (3) and stability of x follows from that of p, we need only consider stability of Equation (2). Now consider the equilibrium point of Equation (2), $p_0 = \frac{\tau}{\alpha} \ln \frac{x_{\max}}{c}$. As is customary, we change to variable z, where $z(t) = p(t) - p_0$ so that the origin is an equilibrium point. Now we have $z(t) = x_0(t) - p_0$ for $t \in [-\tau, 0]$ and the following for $t \ge 0$

$$\dot{z}(t) = \begin{cases} e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1 & z(t) > -p_0 \\ \max\{0, e^{-\frac{\alpha}{\tau} z(t-\tau)} - 1\} & z(t) \le -p_0 \end{cases} \tag{4}$$

For convenience and efficiency of presentation, we will refer to the solution map defined by Equation (4) as $A : \mathcal{C}_\tau \to \mathcal{C}$. Implicit in these dynamics is the constraint $z(t) \ge -p_0$. Assuming that any initial condition satisfies this constraint, we can include the constraint in the dynamics without altering the solution map. For convenience, we define the following bounded continuous functions.

$$f_1(y) = \min\{e^{\frac{\alpha}{\tau} y} - 1, e^{\frac{\alpha}{\tau} p_0} - 1\}, \quad f_2(y) = \max\{0, f_1(y)\},$$
$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \\ f_2(y) & \text{otherwise} \end{cases}$$

We now have the following for $t \ge 0$.

$$\dot{z}(t) = f_c(z(t), -z(t - \tau)) \tag{5}$$

We briefly note that it can be shown that the solution map $A : \mathcal{C}_\tau \to \mathcal{C}$ defined as above is well-posed.

1) Separation into subsystems: Equation (5) is a delay-differential equation defined by a nonlinear, discontinuous function. To aid in the analysis, we will reformulate the problem as the interconnection of two subsystems where the $W_2 \times L_2$ stability on W_2 of this interconnection implies asymptotic stability on X of the original formulation for some set X. Define the map G by $w = Gu$ if

$$w(t) = \int_{t-\tau}^{t} u(\theta) d\theta$$

Note that G is a linear operator which can be represented by the convolution with $g(t) = \text{step}(t) - \text{step}(t - \tau) \in L_1$. This implies that $\hat{G} \in \mathcal{A}$. Moreover, G can be represented in the frequency domain by $\hat{G}(s) = \frac{1 - e^{-\tau s}}{s}$ which implies G is a bounded operator on L_2 since $\|\hat{G}(j\omega)\|_\infty = \tau$. In addition, $s\hat{G}(s) \in \mathcal{A}$ since it can be represented by convolution with $\delta(t) - \delta(t - \tau)$. Define the map Δ_z by $z = \Delta_z y$ if $z(0) = 0$ and $\dot{z}(t) = f_c\big(z(t), y(t) - z(t)\big)$ for all $t \ge 0$. We define the map Δ by $v = \Delta y$ if $v(t) = \dot{z}(t)$ where $z = \Delta_z y$. Again, we note that it can be shown that the maps Δ and Δ_z are well-posed.

If we now form the interconnection of G and $\kappa\Delta$ as defined above with a single input $f \in W_2$, we can construct the map from input f to outputs y, u. For convenience and efficiency of presentation, we will denote the interconnection map for $\kappa = 1$ by $B : W_2 \to W_{2e} \times L_{2e}$. Furthermore, for $\kappa = 1$, we denote the map from input f to internal variable z by B_z. For $t \leq 0$, $u(t) = y(t) = z(t) = f(t) = 0$ and for $t \geq 0$, the interconnection dynamics combine as follows.

$$\begin{aligned} u(t) &= \kappa \dot{z}(t), \\ y(t) &= \int_{t-\tau}^{t} u(t)dt + f(t) = \kappa\big(z(t) - z(t-\tau)\big) + f(t), \\ \dot{z}(t) &= f_c(z(t), y(t) - z(t)) \\ &= f_c(z(t), f(t) - \kappa z(t-\tau) - (1-\kappa)z(t)) \end{aligned}$$

Again, we note that it can be shown that the interconnection is well-posed for any $\kappa \in [0, 1]$.

B. Δ satisfies the IQC

In this section we show that if $\alpha > 0$, then Δ and consequently $\kappa\Delta$ are bounded and satisfy the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0, 1]$. The methods used in this section were motivated by those in Jönnson [17] and Wang [5]. For $\gamma = 4\beta/\pi > 0$, we prove the following for all $y \in W_2$, $v = \Delta y$.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix}^* \begin{bmatrix} 0 & \beta \\ \beta & -\frac{4}{\pi} - 2 \end{bmatrix} \begin{bmatrix} \hat{y}(j\omega) \\ \hat{v}(j\omega) \end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \geq -\gamma |y(0)|^2$$

By Parseval's equality, this is equivalent to $\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle \geq -\frac{\gamma}{2}|y(0)|^2$. A critical result used in the analysis of this section is the existence of a sector bound on the nonlinearity f_1 and consequently on f_2, i.e. $0 \leq f_i(x)x \leq \beta x^2$ where $\beta = \frac{e^{\frac{\alpha}{\tau}p_0} - 1}{p_0}$, denoted $f_i \in \text{sector}[0, \beta]$. Also notice that

$$f_c(x, y) = \begin{cases} f_1(y) & \text{if } x > -p_0 \text{ or } y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 7: If $v = \Delta y$ with $y \in W_2$, then

1) $v \in L_2$ with norm bound $\beta\|y\|$,
2) $\langle v, \beta y - v\rangle \geq 0$

Proof: As a consequence of the above sector bounds, we have that $f_c(x, y)^2 \leq \beta y f_c(x, y)$. Let $z = \Delta_z y$, then this implies

$$\dot{z}(t)^2 = f_c\big(z(t), y(t) - z(t)\big)\dot{z}(t) \leq \beta\big(y(t) - z(t)\big)\dot{z}(t)$$

Now for any $T \geq 0$, we have

$$\begin{aligned} \|P_T v\|^2 &= \int_0^T v(t)^2 dt = \int_0^T \dot{z}(t)^2 dt \leq \beta\int_0^T \dot{z}(t)(y(t) - z(t))dt \\ &= \beta\int_0^T \dot{z}(t)y(t)dt - \frac{\beta}{2}(z(T)^2 - z(0)^2) \leq \beta\langle P_T\dot{z}, y\rangle \qquad (6) \\ &\leq \beta\|P_T\dot{z}\|\|y\| = \beta\|P_T v\|\|y\| \qquad (7) \end{aligned}$$

Therefore, $\|P_T v\| \leq \beta\|y\|$ for all $T \geq 0$. Thus $v \in L_2$ with norm bounded by $\beta\|y\|$. Statement 2 follows from line 7 by letting $T \to \infty$. ■

Lemma 8: Let $z = \Delta_z y$ with $y \in W_2$, then $\lim_{t\to\infty} z(t) = 0$.

Proof: Let $v = \dot{z} = \Delta y$. Suppose that $T_2 > T_1 > 0$ and let $H = P_{T_2} - P_{T_1}$. Then

$$\begin{aligned} \|Hv\|_2^2 &= \int_{T_1}^{T_2} \dot{z}(t)^2 dt \leq \beta\int_{T_1}^{T_2} \dot{z}(t)y(t)\,dt - \beta\int_{T_1}^{T_2} \dot{z}(t)z(t)\,dt \\ &= \beta\langle Hv, Hy\rangle - \frac{\beta}{2}(z(T_2)^2 - z(T_1)^2) \\ &\leq \beta\|Hv\|_2\|Hy\|_2 - \frac{\beta}{2}\big(z(T_2)^2 - z(T_1)^2\big) \end{aligned}$$

Hence

$$z(T_2)^2 - z(T_1)^2 \leq 2\|Hv\|_2\|Hy\|_2 - \frac{2}{\beta}\|Hv\|_2^2 \leq 2\|Hv\|_2\|Hy\|_2$$

By Lemma 7, $v \in L_2$. Since $\|v\|$ and $\|y\|$ exist, we can use the Cauchy criterion and the above inequality to establish that for any $\delta > 0$, there exists a T_δ such that $T_2 > T_1 > T_\delta$ implies $(z(T_2)^2 - z(T_1)^2) < \delta$. It can be shown that this implies that for any infinite increasing sequence $\{T_i\}$, $\{z(T_i)^2\}$ is a Cauchy sequence and therefore $z(t)^2$ converges to a limit. Since z is continuous, this implies that $z(t)$ also converges to a limit, z_∞. Since $y \in W_2$, we have $\lim_{t\to\infty} y(t) = y_\infty = 0$. Recall $f_c(a, b) = f_1(b)$ at points such that $a > -p_0$ or $b > 0$. Suppose $z_\infty \neq 0$. If $z_\infty < 0$, then $y_\infty - z_\infty > 0$ and $f_c(a, b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. If $z_\infty > 0$, then $z_\infty > 0 > -p_0$ and $f_c(a, b) = f_1(b)$ in some neighborhood of $(z_\infty, y_\infty - z_\infty)$. Since f_1 is continuous, we have $\lim_{t\to\infty} \dot{z}(t) = \lim_{t\to\infty} f_c(z(t), y(t) - z(t)) = f_1(y_\infty - z_\infty) = f_1(-z_\infty)$. Thus if $z_\infty \neq 0$, then $\dot{z}$ has a nonzero limit. However, since $\dot{z} \in L_2$, it cannot have a nonzero limit, thus we conclude by condradiction that $z_\infty = 0$. ■

Lemma 9: If $v = \Delta y$ with $y \in W_2$, then $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$.

Proof: Let $z = \Delta_z v$ and define the variable $r(t) = y(t) - z(t)$ and the set $M = \{t : z(t) > -p_0 \;\text{ or }\; r(t) \geq 0\}$, then

$$\begin{aligned} \langle v, \dot{y} - v\rangle &= \langle \dot{z}, \dot{y} - \dot{z}\rangle = \int_0^\infty \dot{z}(t)\dot{r}(t)dt = \int_M f_1(r(t))\dot{r}(t)dt \\ &\leq \beta\|y\|\|\dot{y}\| + \beta^2\|v\|^2. \end{aligned}$$

Since $y \in W_2$, we have that y is absolutely continuous and thus r is absolutely continuous. Since r, z are absolutely continuous functions and since by Lemma 8, we have $z(t) \to 0$, we can partition the set M into the countable union of sequential disjoint intervals $\bigcup_i I_i \bigcup I_f$ where $I_i = [T_{a,i}, T_{b,i})$ with $\{T_{a,i}\}, \{T_{b,i}\} \subset \mathbb{R}^+$ and $I_f = [T_{a,f}, \infty)$. To see that the intervals are closed on the left, suppose I_i were open on the left. Then, since $T_{a,i} \notin M$, $z(T_{a,i}) = -p_0$ and $r(T_{a,i}) < 0$. However, since r is continuous, $r(T_{a,i} + \eta) < 0$ for η sufficiently small. Since $r(t) < 0$ implies $\dot{z}(t) \leq 0$, we have that $z(T_{a,i} + \eta) < 0$ and thus $T_{a,i} + \eta \notin M$ for η sufficiently small, which is a contradiction. Thus all the intervals are closed on the left. Similarly, one can show that all the intervals are open on the right. Now, consider time

$T_a > 0$, where $T_a \in M$ defines the start of one of the intervals described above. If $z(T_a) > -p_0$, then since z is continuous, $z(T_a - \eta) > -p_0$ for all η sufficiently small. Therefore $T_a - \eta \in M$ for all η sufficiently small. This contradicts the statement that the intervals are disjoint. We thus conclude $z(T_a) = -p_0$ and consequently $r(T_a) \geq 0$ by definition of M. Now suppose $r(T_a) > 0$. Since r is continuous, $r(T_a - \epsilon) > 0$ and consequently $T_a - \epsilon \in M$ for all ϵ sufficiently small, which contradicts the statement that the intervals are disjoint. Therefore we conclude $r(T_a) = 0$ if $T_a \neq 0$. Then

$$\langle v, \dot{y} - v\rangle = \sum_i \int_{I_i} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^{\infty} f_1(r(t))\dot{r}(t)dt$$
$$= \sum_i \int_{T_{a,i}}^{T_{b,i}} f_1(r(t))\dot{r}(t)dt + \int_{T_{a,f}}^{\infty} f_1(r(t))\dot{r}(t)dt$$

We will assume that $T_{a,1} = 0$. If $T_{a,1} \neq 0$, we have $r(T_{a,1}) = 0$ and the proof becomes simpler. Since $f_1(r)$ is continuous in r and $r(t)$ is absolutely continuous in time, by the substitution rule we have

$$\langle v, \dot{y} - v\rangle = \sum_i \int_{r(T_{a,i})}^{r(T_{b,i})} f_1(r)dr$$
$$= \int_{r(0)}^{r(T_{b,1})} f_1(r)dr + \sum_{i\neq 1} \int_0^{r(T_{b,i})} f_1(r)dr$$
$$= \int_{r(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr$$

Since $f_1 \in \text{sector}[0,\beta]$, $\int_0^{r(T_{b,i})} f_1(r)dr \geq 0$ for any $r(T_{b,i}) \in \mathbb{R}$. The summation converges since it is bounded, increasing. Furthermore, since $r(0) = y(0) - z(0) = y(0)$ and $|\int_0^y f_1(r)dr| \leq f_1(y)y \leq \beta y^2$ for any y, we have

$$\langle v, \dot{y} - v\rangle = \int_{y(0)}^{0} f_1(r)dr + \sum_i \int_0^{r(T_{b,i})} f_1(r)dr \geq -\beta|y(0)|^2$$

■

To summarize this section, we have shown that Δ is bounded and that for any $y \in W_2$ and $v = \Delta y$, we have $\langle v, \beta y - v\rangle \geq 0$ and $\langle v, \dot{y} - v\rangle \geq -\beta|y(0)|^2$. Therefore, we conclude that Δ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$, since

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix}^*\begin{bmatrix}0 & \beta\\ \beta & -\frac{4}{\pi}-2\end{bmatrix}\begin{bmatrix}\hat{y}(j\omega)\\ \hat{v}(j\omega)\end{bmatrix} + \frac{4}{\pi}\langle v, \dot{y}\rangle \geq -\frac{4\beta}{\pi}|y(0)|^2.$$

We conclude that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for any $\kappa \in [0,1]$, since

$$\frac{2}{\pi}\langle \kappa v, \dot{y} - \kappa v\rangle + \langle \kappa v, \beta y - \kappa v\rangle \geq \kappa\big(\frac{2}{\pi}\langle v, \dot{y} - v\rangle + \langle v, \beta y - v\rangle\big)$$
$$\geq -\kappa\frac{2\beta}{\pi}|y(0)|^2 \geq -\frac{2\beta}{\pi}|y(0)|^2.$$

C. Properties of G

Recall that we define the map G as follows. $w = Gu$ if $w(t) = \int_{t-\tau}^{t} u(\theta)d\theta$. Furthermore recall that G can be represented as a transfer function in the frequency domain by $\hat{G} = \frac{1-e^{-j\omega\tau}}{j\omega}$. Now, examine the term

$$\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}^*\left(\Pi_B + \begin{bmatrix}0 & \lambda j\omega^*\\ \lambda j\omega & 0\end{bmatrix}\right)\begin{bmatrix}\hat{G}(j\omega)\\ I\end{bmatrix}$$
$$= \begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix}^*\begin{bmatrix}0 & \beta + \frac{2}{\pi}j\omega^*\\ \beta + \frac{2}{\pi}j\omega & -\frac{4}{\pi}-2\end{bmatrix}\begin{bmatrix}\frac{1-e^{-j\omega\tau}}{j\omega}\\ 1\end{bmatrix}$$
$$= 2\cdot\text{Real}\left(\beta\frac{1-e^{-j\omega\tau}}{j\omega} - \frac{2}{\pi}e^{-j\omega\tau} - 1\right)$$
$$= 2\left(\beta\tau\frac{\sin(\omega\tau)}{\omega\tau} - \frac{2}{\pi}\cos(\omega\tau) - 1\right) = 2p(\omega\tau)$$

It can be shown that for any $\beta\tau < \pi/2$, there exists an $\eta > 0$ such that $p(\omega) < -\eta$ for all ω. Since $\beta\tau = \tau(e^{\frac{\alpha}{\tau}p_0} - 1)/p_0 = \alpha/\alpha_{\text{max}}$, if $\alpha < \pi\alpha_{\text{max}}/2$, we have that $\beta\tau < \pi/2$, and hence if $0 < \alpha < \pi\alpha_{\text{max}}/2$, condition 4 of Theorem 6 is satisfied.

D. Stability of the Interconnection

We conclude our discussion of input-output stability with the following Theorem concerning the stability of the the interconnection of Δ and G.

Theorem 10: Suppose $\alpha \in (0, \pi\alpha_{\text{max}}/2)$. Then the map B, defining the interconnection of Δ and G is $W_2 \times L_2$ stable on W_2.

Proof: We have shown that G is a linear causal bounded operator with $\hat{G}(s), s\hat{G}(s) \in \mathcal{A}$. We have also shown that the interconnection of G and $\kappa\Delta$ is well-posed for all $\kappa \in [0,1]$ and that $\kappa\Delta$ satisfies the IQC defined by $\Pi_B, \lambda = \frac{2}{\pi}$ for all $\kappa \in [0,1]$. Finally, we have that for all $\alpha \in (0, \pi\alpha_{\text{max}}/2)$, condition 4 of Theorem 6 is satisfied. We can therefore use Theorem 6 to prove $W_2 \times L_2$ stability on W_2 of the interconnection for any $\alpha \in (0, \pi\alpha_{\text{max}}/2)$. ■

E. Asymptotic Stability

Recall that the original solution map A is defined by the following.

$$\dot{z}(t) = f_c(z(t), -z(t-\tau))\ \forall t \geq 0, \quad z(t) = x_0(t)\ \forall t \in [-\tau, 0] \tag{8}$$

The interconnection maps B and B_z, however, are defined by the following differential equation.

$$\dot{z}(t) = f_c(z(t), f(t) - z(t-\tau)) \quad \forall t \geq 0, \quad z(t) = 0 \quad \forall t \leq 0 \tag{9}$$

In the previous section, we have proven $W_2 \times L_2$ stability on W_2 of the map B where B represents a reformulation of the problem in the input-output framework. We would like to show, however, that for some $X \subset \mathcal{C}_\tau$ this result also implies asymptotic stability on X of the solution map A, where A represents the original formulation of the problem. This is done in the following Theorem.

Theorem 11: Suppose $\alpha \in (0, \pi\alpha_{\text{max}}/2)$. Then the delay-differential equation (2) describing the algorithm proposed by Paganini et al. [14] is asymptotically stable on $X = \{x : x \in W_2 \cap \mathcal{C}_\tau, x(t) \geq -p_0, x(-\tau) > -p_0\}$.

Proof: We have already shown $W_2 \times L_2$ stability on W_2 of the map B. Let $x_0 \in X$ be an arbitrary initial condition. It can be shown that for any initial condition $x_0 \in X$, there exists some $f \in W_2$ and $T > 0$ such that $(Ax_0)(t) = (B_z f)(t+T)$ for all $t \geq 0$. Let $(y, u) = Bf$, then $y \in W_2$. Furthermore, recall $B_z f = \Delta_z y$ where B_z is the map to internal variable z. By Lemma 8, if $y \in W_2$, then $\lim_{t\to\infty}(\Delta_z y)(t) = 0$. Therefore, $\lim_{t\to\infty}(Ax_0)(t) = \lim_{t\to\infty}(B_z f)(t+T) = \lim_{t\to\infty}(\Delta_z y)(t+T) = 0$. ■

IV. Conclusion

To summarize, for the case of a single source with a single link, we have demonstrated both input-output and asymptotic stability. We have used a generalized passivity framework to decompose a difficult nonlinear, discontinuous, infinite-dimensional problem into separate linear, infinite-dimensional and nonlinear, finite-dimensional subproblems, each of which is amenable to existing analysis techniques. A key feature of the analysis of the nonlinear subsystem was the existence of a sector bound on the nonlinearity.

In addition to the case of a single source with a single link, the result presented in this paper applies directly to the case of multiple sources with identical fixed delay. The proof may also be easily adapted to alternate implementations of the proposed linearized protocols so long as they admit a similar sector bound on the nonlinearity. In addition to analysis of multiple heterogeneous delays, directions for future research include analysis of other proposed congestion control protocols with an aim toward reducing conservative of existing nonlinear stability results.

References

[1] S. Low, F. Paganini, J. Wang, and J. Doyle, "Linear stability of TCP/RED and a scalable control," *Computer Networks Journal*, vol. 43, no. 5, pp. 633–647, Dec. 2003.

[2] L. Ying, G. Dullerud, and R. Srikant, "Global stability of internet congestion controllers with heterogeneous delays," in *Proceedings of the American Control Conference*, 2004.

[3] S. Deb and R. Srikant, "Global stability of congestion controllers for the internet," *IEEE Transactions on Automatic Control*, vol. 48, no. 6, p. 1055, 2003.

[4] C. Hollot and Y. Chait, "Nonlinear stability analysis for a class of TCP/AQM networks," in *Proceedings of the IEEE Conference on Decision and Control*, Dec. 2001, Orlando, FL.

[5] Z. Wang and F. Paganini, "Global stability with time-delay in network congestion control," in *Proceedings of the IEEE Conference on Decision and Control*, Dec. 2002, Las Vegas, NV.

[6] ——, "Global stability with time-delay of a primal-dual congestion control," in *Proceedings of the IEEE Conference on Decision and Control*, 2003.

[7] T. Alpcan and T. Basar, "A utility-based congestion control scheme for internet-style networks with delay," in *Proceedings of IEEE INFOCOM*, 2003.

[8] F. Mazenc and S.-I. Niculescu, "Remarks on the stability of a class of TCP-like congestion control models," in *Proceedings of the IEEE Conference on Decision and Control*, 2003.

[9] A. Papachristodoulou, "Global stability analysis of a TCP/AQM protocol for arbitrary networks with delay," in *Proceedings of the IEEE Conference on Decision and Control*, 2004.

[10] ——, "Analysis of nonlinear delay differential equation models of TCP/AQM protocols using sums of squares," in *Proceedings of the IEEE Conference on Decision and Control*, 2004.

[11] X. Fan, M. Arcak, and J. Wen, "L_p stability and delay robustness of network flow control," in *Proceedings of the IEEE Conference on Decision and Control*, 2003.

[12] S. H. Low and D. E. Lapsley, "Optimization flow control-I: Basic algorithm and convergence," *IEEE/ACM Transactions on Networking*, vol. 7, no. 6, pp. 861–874, Dec. 1999.

[13] F. P. Kelly, A. Maulloo, and D. Tan, "Rate control for communication networks: Shadow prices, proportional fairness, and stability," *Journal of the Operations Research Society*, vol. 49, no. 3, pp. 237–252, 1998.

[14] F. Paganini, J. Doyle, and S. Low, "Scalable laws for stable network congestion control," in *Proceedings of the IEEE Conference on Decision and Control*, 2001, Orlando, FL.

[15] A. Rantzer and A. Megretski, "System analysis via integral quadratic constraints part II," Department of Automatic Control, Lund Institute of Technology, Tech. Rep., Sept. 1997, ISSN 0280-5316.

[16] U. Jonsson, "Robustness analysis of uncertain and nonlinear systems," Ph.D. dissertation, Department of Automatic Control, Lund Institute of Technology, 1996.

[17] U. Jonsson and A. Megretski, "The Zames-Falb IQC for systems with integrators," *IEEE Transactions on Automatic Control*, vol. 45, no. 3, pp. 560–565, Mar. 2000.