Spacecraft and Aircraft Dynamics

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Lecture 5: Hyperbolic Orbits

Introduction

In this Lecture, you will learn:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Given t, find r and v

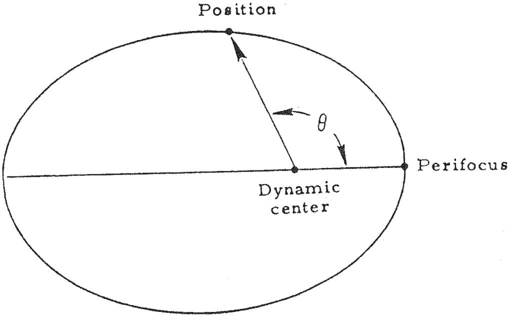
For elliptic orbits:

1. Given time, t, solve for Mean Anomaly
$$M(t) = nt$$
2. Given Mean Anomaly, solve for Eccentric Anomaly
$$M(t) = E - e \sin E$$
3. Given eccentric anomaly, solve for true anomaly
$$\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$$
4. Given true anomaly, solve for r
$$r(t) = \frac{a(1-e^2)}{1 + e \cos f(t)}$$

Does this work for Hyperbolic Orbits? Lets recall the angles.

What are these Angles?

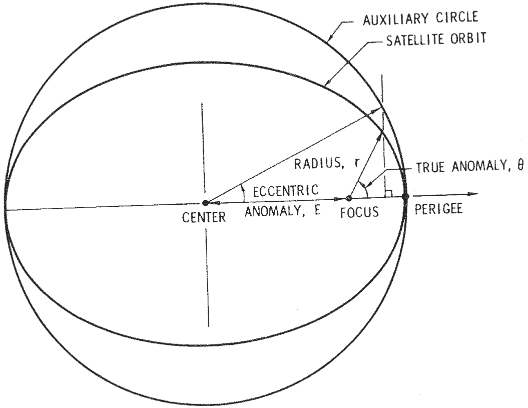
True Anomaly, f (θ)



- The angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$.
- The angle the position vector makes with periapse.

What are these Angles?

Eccentric Anomaly, E

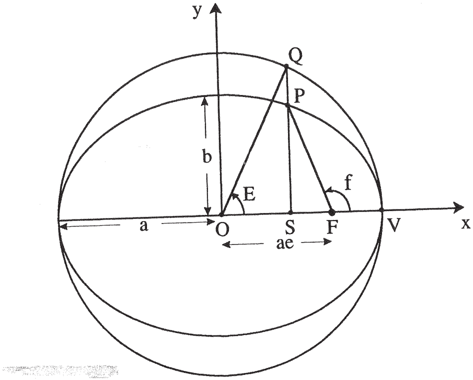


- Measured from center of ellipse to a auxiliary reference circle.

What are these Angles?

Mean Anomaly

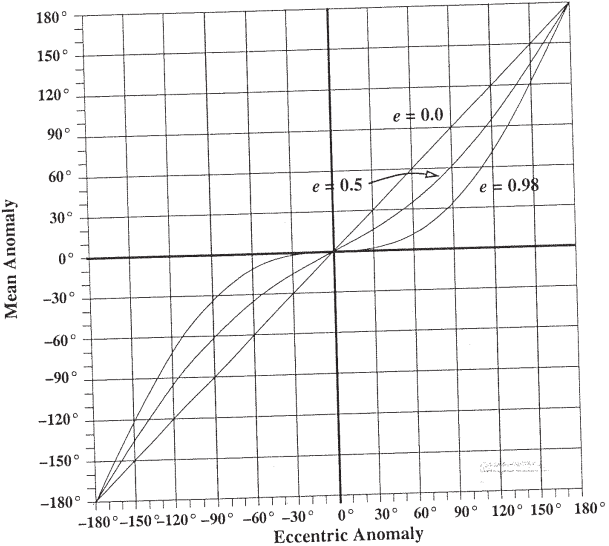
$$M(t) = 2\pi \frac{t}{T} = 2\pi \frac{A_{PFV}}{A_{Ellipse}}$$

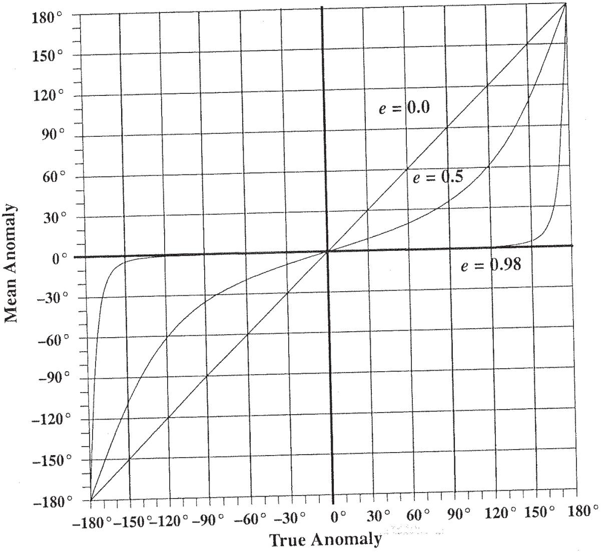


- The fraction of area of the ellipse which has been swept out, in radians.

Relationships between M, E, and f

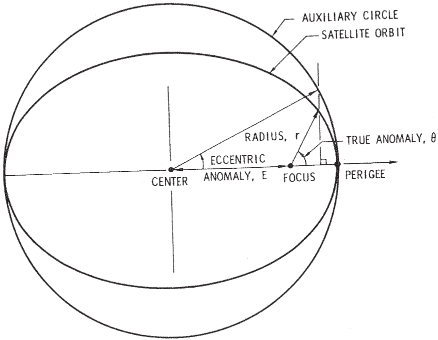
M vs. E





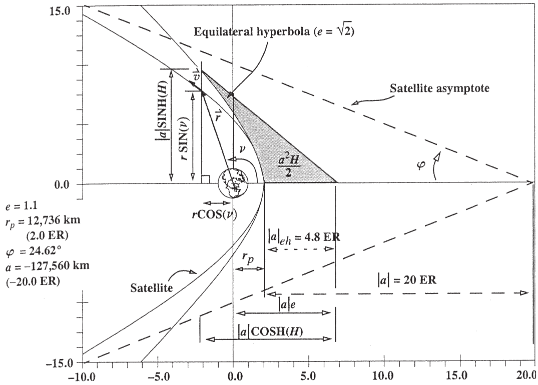
180°
150°
120°
90°
60°
30°
0°
−30°
−60°
−90°
−120°
−150°
−180°
Mean Anomaly
e = 0.0
e = 0.5
e = 0.98
−180° −150° −120° −90° −60° −30° 0° 30° 60° 90° 120° 150° 180°
True Anomaly

- The orbit does not repeat (no period, T)
  - We can't use
    $$T = 2\pi\sqrt{\frac{\mu}{a^3}}$$
  - What is mean motion, n?
- No reference circle
  - Eccentric Anomaly is Undefined



Note: In our treatment of hyperbolae, we do **NOT** use the *Universal Variable* approach of Prussing/Conway and others.

We will not get into details!

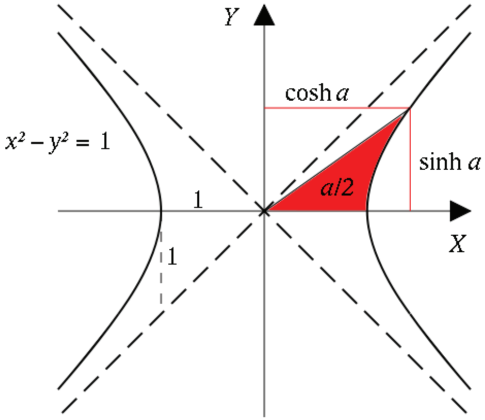


- defined using the reference hyperbola, tangent at perigee

$$x^2 - y^2 = 1$$

Recall your Hyperbolic Trig.

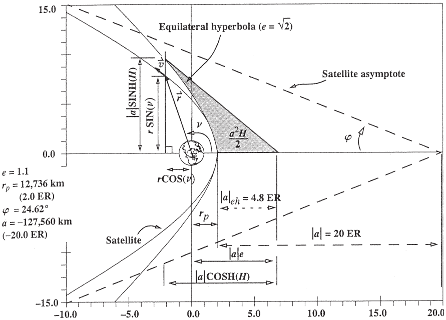
Cosh and Sinh



Relate area of reference hyperbola to lengths.

- Yet another branch of mathematics developed for solving orbits (Lambert).

Hyperbolic Anomaly



- Hyperbolic Anomaly, H is a measure of Area.
- Hyperbolic Trig gives a relationship to true anomaly, which is

$$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f}{2}\right)$$

- Alternatively,

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

Hyperbolic Kepler's Equation

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

Definition 1 (Hyperbolic Kepler's Equation).

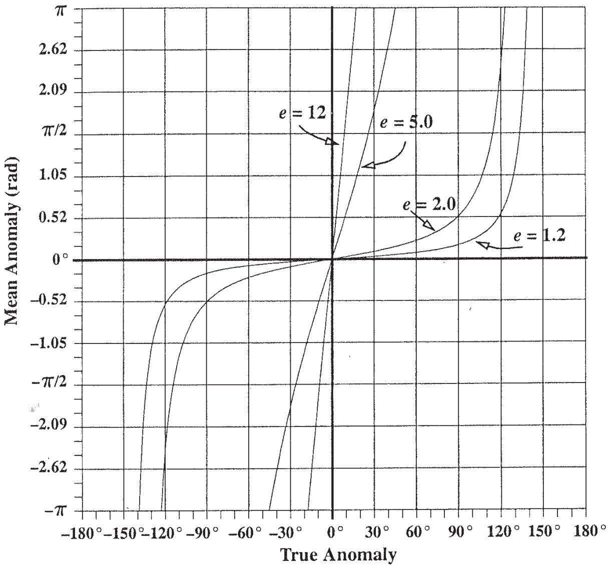
$$M = \sqrt{\frac{\mu}{-a^3}}t = e\sinh(H) - H$$

Newton Iteration for Hyperbolic Anomaly:

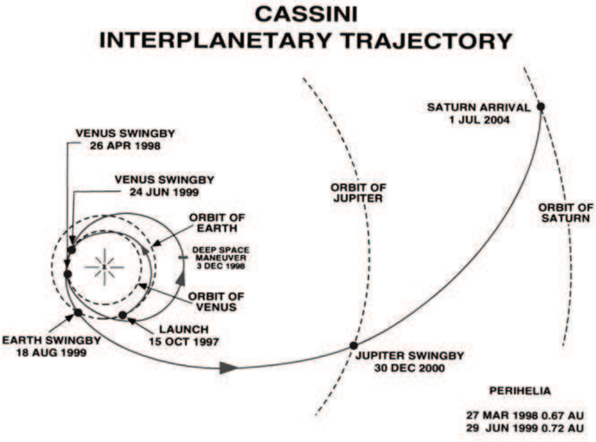
$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

with $H_1 = M$.

Relationship between M and f for Hyperbolic Orbits



Example: Jupiter Flyby



Problem: Suppose we want to make a flyby of Jupiter. The relative velocity at approach is $v_\infty = 10km/s$. To achieve the proper turning angle, we need an eccentricity of $e = 1.07$. Radiation limits our time within radius $r = 100,000km$ to 1 hour (radius of Jupiter is $71,000km$). Will the spacecraft survive the flyby?

Example: Jupiter Flyby

Example Continued

Solution: First solve for a and p. $\mu = 1.267E8$.

- The total energy of the orbit is given by

$$E_{tot} = \frac{1}{2} v_\infty^2$$

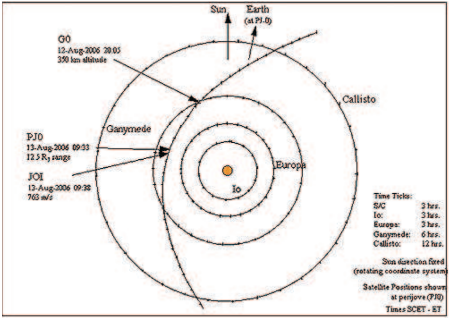
- The total energy is expressed as

$$E = -\frac{\mu}{2a} = \frac{1}{2} v_\infty^2$$

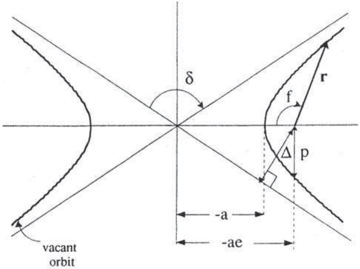
which yields

$$a = -\frac{\mu}{v_\infty^2} = -1.267E6$$

- The parameter is $p = a(1 - e^2) = 1.8359E5$



Example Continued



We need to find the time between $r_1 = 100,000km$ and $r_2 = 100,000km$. Find f at each of these points.

- Start with the conic equation:

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

- Solving for f,

$$f_{1,2} = \cos^{-1}\left(\frac{1}{e} - \frac{r}{ep}\right) = \pm 64.8\,\mathrm{deg}$$

Example Continued

Given the true anomalies, $f_{1,2}$, we want to find the associated times, $t_{1,2}$.

- Only solve for t_2, get t_1 by symmetry.
- First find Hyperbolic Anomaly,

$$H_2 = \tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f_2}{2}\right)\right) = .1173$$

- Now use Hyperbolic anomaly to find mean anomaly

$$M_2 = e\sinh(H_2) - H_2 = .0085$$

  - This is the "easy" direction.
  - No Newton iteration required.
- t_2 is now easy to find

$$t_2 = M_2\sqrt{\frac{-a^3}{\mu}} = 1076.6$$

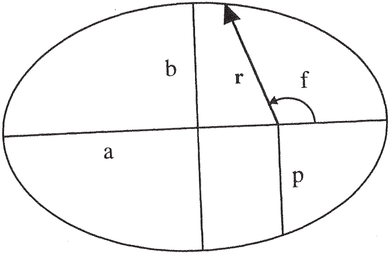
Finally, we conclude $\Delta t = 2 * t_2 = 2153s = 35min$.

So the spacecraft survives.

The Orbital Elements

So far, all orbits are parameterized by 3 parameters

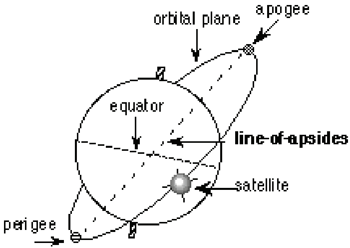
- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

The Orbital Elements

Note: We have shown how to use a, e and f to find the scalars r and v.

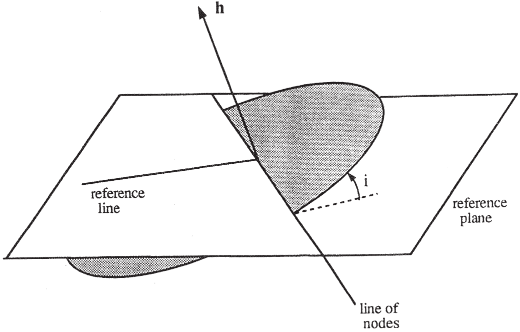


Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

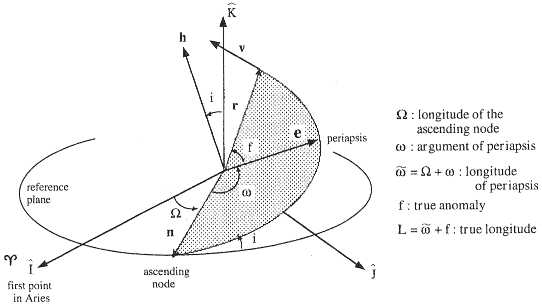
- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.

Inclination, i



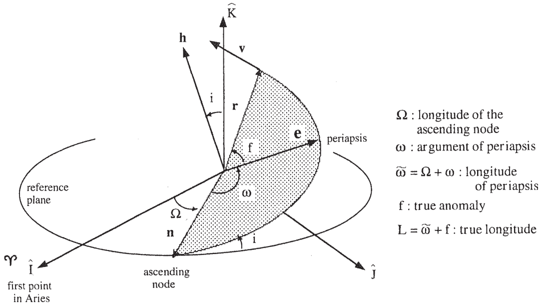
Angle the orbital plane makes with the reference plane.

Right Ascension of Ascending Node, Ω



Angle measured from reference direction in the reference plane to intersection with orbital plane.

Argument of Periapse, ω



Angle measured from reference plane to point of periapse.

Summary

This Lecture you have learned:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Properties of Keplerian Orbits

Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis $a < 0$ b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}\big/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 + b^2/a^2} \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a\,(1 - e)$	$q = p/2$	$q = a\,(1 - e)$
Velocity, V, at distance, r, from Focus	$V^2 = \mu/r$	$V^2 = \mu\,(2/r - 1/a)$	$V^2 = 2\mu/r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, $\mathcal{E}$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = 0$	$\mathcal{E} = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu}$	$n = \sqrt{\mu/(-a)^3}$
Period, P	$P = 2\pi/n$	$P = 2\pi/n$	$P = \infty$	$P = \infty$
Anomaly	$v = M = E$	Eccentric anomaly, E $\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{v}{2} = D\big/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e\sin E$	$M = qD + (D^3/6)$	$M = (e\sinh F) - F$
Distance from Focus, $r = q\,(1+e)/(1+e\cos v)$	$r = a$	$r = a\,(1 - e\cos E)$	$r = q + (D^2/2)$	$r = a\,(1 - e\cosh F)$
$r\,dr/dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\,\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\,\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{dv}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$	$\frac{dA}{dt} = \sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$

$\mu = GM$ is the gravitational constant of the central body; v is the true anomaly, and $M \equiv n\,(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion.