

On Positive Forms and the Stability of Linear Time-Delay Systems

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Abstract—This paper presents a time-domain approach to stability analysis of linear time-delay systems using positive quadratic forms. We show that positivity of these forms is equivalent to certain convex constraints on the functions which define them. These results are then combined with recent developments in polynomial optimization to construct a nested sequence of sufficient conditions for stability which are expressible as semidefinite programs and which are of non-decreasing accuracy. We also address the case of parametric uncertainty and uncertain delay.

I. INTRODUCTION

The study of stability of delayed linear systems has been an active area of research for some time. An overview of this field of research can be obtained from the brief outline in Section III and from various survey papers and books on the subject, see for example [1], [2], [3], [4]. In this paper, we consider numerical algorithms for proving the stability of linear time-delay systems of the form

$$\dot{x}(t) = \sum_{i=1}^k A_i x(t - \tau_i),$$

where the τ_i are increasing and $A_i \in \mathbb{R}^{n \times n}$. Throughout Sections I and II, we will assume without loss of generality that $\tau_k = 1$ and $\tau_0 = 0$. Lyapunov stability analysis can be extended to such systems through the use of “Lyapunov functionals” which map *segments* of trajectory into the positive real numbers. Indeed, converse Lyapunov theorems [1] show that the linear time-delay system given above is stable if and only if there exists a positive Lyapunov functional $V : \mathcal{C}[0, 1] \rightarrow \mathbb{R}$ such that the functional decreases along trajectories of the system. Here $\mathcal{C}[0, 1]$ denotes the space of continuous functions on the interval $[0, 1]$. Furthermore, these theorems state that one can assume that such functionals are defined by some matrix functions M and R in the following manner.

$$V(\phi) = \int_0^1 \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds + \int_0^1 \int_0^1 \phi(s)^T R(s, \omega) \phi(\omega) ds d\omega. \quad (1)$$

The derivative of V has a similar structure to V and is also defined by matrix functions which are an affine transformation of M and R . Thus the question of stability of linear time-delay systems can be reduced to the question of

existence of functions M and R such that the functional V is positive and \dot{V} is negative. Since the set of M and R such that V is positive is a convex cone, the question of stability of a linear time-delay system is a convex feasibility problem. Of course, without an efficient parametrization of this cone, such equivalence is entirely academic. Indeed, it is far from clear what conditions one can impose on M and R which will result in positivity of the functional. For example, while uniform positivity of $M(s)$ is sufficient for positivity of the first component of V , it is highly conservative. Meanwhile, uniform positivity of $R(s, \omega)$ is not even sufficient for positivity of the second component.

The goal of this paper is to develop affine constraints on the functions M and R which are equivalent to positivity of the Lyapunov functional and which are amenable to testing through the use of semidefinite programming. Roughly speaking, the condition on M is given in terms of the existence of a slack variable T such that $M + T$ is uniformly positive and the condition on R implies the existence of a function G such that $R(s, \omega) = G(s)^T G(\omega)$. These conditions are described in Section II. The method of implementation of these conditions as affine semidefinite programming constraints is outlined in Section IV. In Section V, the results are combined into an algorithm for stability analysis. Finally, the paper concludes with a series of numerical examples.

II. A CHARACTERIZATION OF POSITIVE FUNCTIONALS

To begin the paper, we present results which allow us to characterize the set of matrix-valued functions which define positive Lyapunov functionals. Specifically, we consider matrix-valued functions M and R which define Lyapunov functionals, V , of the form given by Equation 1. In most cases, we will assume that the functions $M(s)$ and $R(s, \omega)$ are piecewise-continuous, by which we mean they are continuous except possibly at points $s, \omega = \tau_i$ where $\tau_i \in [0, 1]$.

The following theorem allows us to express positivity of the first component of V directly in terms of conditions on the function M through the use of a slack variable T . Specifically the theorem states that the first component is strictly positive if and only if there exists a slack variable T such that the combination of the function M and the slack variable is uniformly positive on the interval.

Theorem 1: Suppose $M : \mathbb{R} \rightarrow \mathbb{S}^{n+m}$ is piecewise-continuous. Then the following are equivalent.

- 1) There exists $\nu > 0$ such that for all $x \in \mathcal{C}[0, 1]$, $c \in \mathbb{R}^m$,

$$\int_0^1 \begin{bmatrix} c \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} ds \geq \nu \|x\|_2^2.$$

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- 2) There exists $\delta > 0$ and piecewise-continuous $T : \mathbb{R} \rightarrow \mathbb{S}^m$, such that $\int_0^1 T(s) ds = 0$ and

$$M(s) + \begin{bmatrix} T(s) & 0 \\ 0 & -\delta I \end{bmatrix} \geq 0 \quad \text{for all } s \in [0, 1].$$

Proof: Clearly 2 implies 1 with $\nu = \delta$.

(1 \Rightarrow 2) Assume that M is continuous except at the monotone increasing sequence of points $\tau_i \in [0, 1]$ where $\tau_1 = 0$ and $\tau_k = 1$. Suppose that statement 1 holds. Write M as

$$M(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) \end{bmatrix},$$

where $M_{22} : \mathbb{R} \mapsto \mathbb{S}^n$. To start the proof, we show that that $M_{22}(s) \geq \nu I$ for all $s \in [0, 1]$. This is necessary so that the following infimum exists at some finite value of z .

$$\inf_z \begin{bmatrix} c \\ z \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) - \frac{\nu}{2} I \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix}$$

By statement 1, we have that for all $x \in \mathcal{C}[0, 1]$, $c \in \mathbb{R}^m$,

$$\int_0^1 \begin{bmatrix} c \\ x(s) \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) - \nu I \end{bmatrix} \begin{bmatrix} c \\ x(s) \end{bmatrix} ds \geq 0.$$

Now suppose that $M_{22}(s) - \nu I$ is not positive semidefinite for all $s \in [0, 1]$. Then there exists some $x_1 \in \mathbb{R}^n$ and $s_1 \in [0, 1]$ such that $x_1^T (M_{22}(s_1) - \nu I) x_1 < -2$. Let $\mathcal{B}(x, y)$ denote the ball centered at x with radius y . Due to piecewise-continuity of M_{22} , we can assume there exists some $\beta > 0$ such that M_{22} is continuous on $\mathcal{B}(s_1, 2\beta)$ and $x_1^T (M_{22}(s) - \nu I) x_1 < -1$ for $s \in \mathcal{B}(s_1, 2\beta)$. Now, let $c = 0$ and

$$x(s) = \begin{cases} ((s - (s_1 - 2\beta))/\beta) x_1 & s \in [s_1 - 2\beta, s_1 - \beta] \\ x_1 & s \in \mathcal{B}(s_1, \beta) \\ (1 - (s - (s_1 + \beta))/\beta) x_1 & s \in [s_1 + \beta, s_1 + 2\beta] \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in \mathcal{C}[0, 1]$ and we have the following.

$$\begin{aligned} & \int_0^1 \begin{bmatrix} c \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} ds - \nu \|x\|^2 \\ &= \int_{s_1 - \beta}^{s_1 + \beta} x_1^T (M_{22}(s) - \nu I) x_1 ds \\ &+ \int_{s_1 - 2\beta}^{s_1 - \beta} ((s - (s_1 - 2\beta))/\beta)^2 x_1^T (M_{22}(s) - \nu I) x_1 ds \\ &+ \int_{s_1 + \beta}^{s_1 + 2\beta} (1 - (s - (s_1 + \beta))/\beta)^2 x_1^T (M_{22}(s) - \nu I) x_1 ds \\ &\leq -2\beta \end{aligned}$$

Therefore, by contradiction, we have that $M_{22}(s) \geq \nu I$ for all $s \in [0, 1]$. Now define $\delta = \nu/2$ and $\tilde{M}_{22}(s) = M_{22}(s) - \delta I \geq \delta I$. Note that $\tilde{M}_{22}(s)^{-1}$ is piecewise continuous since \tilde{M}_{22} is upper and lower bounded. We can now prove statement 2 by construction. Define the following.

$$\begin{aligned} t_1(s, c) &:= \inf_{z \in \mathbb{R}^n} \begin{bmatrix} c \\ z \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & \tilde{M}_{22}(s) \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix} \\ t_0(c) &:= \int_0^1 t_1(s, c) ds \end{aligned}$$

Since we have shown that $\tilde{M}_{22}(s)$ is lower bounded, by the Schur complement formula we have

$$t_1(s, c) = c^T (M_{11}(s) - M_{12}(s) \tilde{M}_{22}^{-1}(s) M_{12}(s)^T) c.$$

Now define the following.

$$T_0 := \int_0^1 M_{11}(s) - M_{12}(s) \tilde{M}_{22}^{-1}(s) M_{12}(s)^T ds$$

$$T(s) := T_0 - (M_{11}(s) - M_{12}(s) \tilde{M}_{22}^{-1}(s) M_{12}(s)^T)$$

Since we have shown $\tilde{M}_{22}(s)^{-1}$ is piecewise-continuous, T and t_1 are likewise piecewise-continuous. Now note that $c^T T_0 c = t_0(c)$ and $c^T T(s) c = t_0(c) - t_1(s, c)$ and so we have that

$$\int_0^1 c^T T(s) c ds = t_0(c) - \int_0^1 t_1(s, c) ds = 0.$$

Furthermore, observe the following.

$$\begin{aligned} & \inf_{c, z} \begin{bmatrix} c \\ z \end{bmatrix}^T \begin{bmatrix} M_{11}(s) + T(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) - \delta I \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix} \\ &= \inf_c (t_1(s, c) + t_0(c) - t_1(s, c)) = \inf_c t_0(c) \\ &= \inf_c \int_0^1 \inf_z \begin{bmatrix} c \\ z \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & \tilde{M}_{22}(s) \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix} ds \\ &= \inf_c \int_0^1 \begin{bmatrix} c \\ \tilde{x}(s) \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & \tilde{M}_{22}(s) \end{bmatrix} \begin{bmatrix} c \\ \tilde{x}(s) \end{bmatrix} ds, \end{aligned}$$

where $\tilde{x}(s) := -\tilde{M}_{22}(s)^{-1} M_{12}(s)^T c$. Now, since \tilde{x} is piecewise-continuous, it can be shown that Statement 1 implies that

$$\begin{aligned} \inf_c t_0(c) &= \inf_c \int_0^1 \begin{bmatrix} c \\ \tilde{x}(s) \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & \tilde{M}_{22}(s) \end{bmatrix} \begin{bmatrix} c \\ \tilde{x}(s) \end{bmatrix} ds \\ &\geq \inf_{c, y \in \mathcal{C}} \int_0^1 \begin{bmatrix} c \\ y(s) \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & \tilde{M}_{22}(s) \end{bmatrix} \begin{bmatrix} c \\ y(s) \end{bmatrix} ds \\ &\geq \inf_{c, y \in \mathcal{C}} \int_0^1 \begin{bmatrix} c \\ y(s) \end{bmatrix}^T \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) - \nu I \end{bmatrix} \begin{bmatrix} c \\ y(s) \end{bmatrix} ds \\ &\geq 0 \end{aligned}$$

Therefore we conclude that Statement 2 is true since

$$\begin{aligned} & \inf_{c, z} \begin{bmatrix} c \\ z \end{bmatrix}^T \begin{bmatrix} M_{11}(s) + T(s) & M_{12}(s) \\ M_{12}(s)^T & M_{22}(s) - \delta I \end{bmatrix} \begin{bmatrix} c \\ z \end{bmatrix} \\ &= \inf_c t_0(c) \geq 0. \end{aligned}$$

■

To use the theorem presented above, we need the following lemma which allows us to reduce positivity of the first component of the Lyapunov functional to positivity of the expression considered in Theorem 1. The lemma is stated in a general form which also applies to the derivative of the Lyapunov functional.

Lemma 2: Suppose $M : \mathbb{R} \rightarrow \mathbb{S}^{n(k+1)}$ is piecewise-continuous and $\tau_i \in [0, 1]$. Then the following are equivalent.

- 1) There exists $\nu > 0$ such that for all $x \in \mathcal{C}[0, 1]$ and $c \in \mathbb{R}^{nk}$,

$$\int_0^1 \begin{bmatrix} c \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} c \\ x(s) \end{bmatrix} ds \geq \nu \|x\|_2^2.$$

2) There exists $\epsilon > 0$ such that for all $x \in \mathcal{C}[0, 1]$,

$$\int_0^1 \begin{bmatrix} x(\tau_1) \\ \vdots \\ x(\tau_k) \\ x(s) \end{bmatrix}^T M(s) \begin{bmatrix} x(\tau_1) \\ \vdots \\ x(\tau_k) \\ x(s) \end{bmatrix} ds \geq \epsilon \|x\|_2^2.$$

Proof omitted.

To summarize the results so far, we have shown, through Theorem 1 and Lemma 2 that the convex cone of functions, M , which define certain positive quadratic forms can be represented using the cone of positive functions and a set of affine equality constraints. In Section IV, we will show that by approximating the cone of positive functions by the cone of positive polynomials, we are able to optimize over such functions using semidefinite programming. First, however, we consider the other cone of functions which define the set of positive Lyapunov functionals.

A. Positive Operators Define Positive Functionals

We now discuss how to enforce positivity of the second component of V , namely

$$V_2(\phi) := \int_0^1 \int_0^1 \phi(s)^T R(s, \omega) \phi(\omega) ds d\omega.$$

The set of R for which V_2 is positive is a convex cone. However, at present we have no way to efficiently test whether a specific function lies in this cone. We will now show briefly how one can construct such a test. We begin by noting that V_2 is a quadratic form on $L_2[0, 1]$ so that $V_2 = \langle \phi, B_R \phi \rangle$ where the function R defines the linear operator B_R as

$$(B_R \phi)(s) := \int_0^1 R(s, \omega) \phi(\omega) d\omega.$$

Then V_2 is positive if and only if B_R has positive eigenvalues. By the spectral theorem then, we have that R can be decomposed in the form $R(s, \omega) = G(s)^T G(\omega)$ for $G : \mathbb{R} \rightarrow \mathbb{R}^{m \times n}$ where m is the rank of B_R and thus may be infinite. To reduce the problem to finite dimensions, suppose that $\{\tilde{e}_i\}_{i=1}^d$ is an orthonormal basis for some subspace of $L_2[0, 1]$ and there exists a function Z which satisfies $\tilde{e}_i(s) = Z(s)^T e_i$ where $\{e_i\}_{i=1}^d$ is the standard basis for \mathbb{R}^d . Then the projection of B_R onto this subspace is given by $B_{R'}$, where $R'(s, \omega) = Z(s)^T Q Z(\omega)$ for some positive semidefinite matrix Q . Therefore, for a given set of basis functions, if we can construct the function Z , we can create a map from positive semidefinite matrices to a projection of the cone of functions, R , which define positive functionals V_2 . Construction of such functions, Z , is a subject which is considered in Section IV.

III. BACKGROUND ON LINEAR TIME-DELAY SYSTEMS

In the previous section, we have discussed how to characterize positivity of positive quadratic Lyapunov functionals of a particular type. To better understand the context of these results, we now return to the general theory of linear differential equations with delay and give an overview of this subject. Recall that the topic of this paper is stability of time-delay systems which can be expressed in the following

form, where we assume for convenience that τ_i are monotonically increasing and $\tau_0 = 0$. From now on, we relax the assumption that $\tau_k = 1$.

$$\dot{x}(t) = \sum_{i=0}^k A_i x(t - \tau_i) \quad (2)$$

We say that $x \in \mathcal{C}$ is a **solution** of Equation (2) with initial condition $x_0 \in \mathcal{C}[-\tau_k, 0]$ if $x(t) = x_0(t)$ for $t \in [-\tau_k, 0]$ and Equation (2) holds for all $t \geq 0$. It can be shown that elements of this class of system admit a unique solution for every initial condition $x_0 \in \mathcal{C}[-\tau_k, 0]$. Therefore, we can associate with systems of this form a solution map $G : \mathcal{C}[-\tau_k, 0] \rightarrow \mathcal{C}$, where $x = Gx_0$ if x is a solution of Equation (2) with initial condition x_0 . Stability is defined in terms of this solution map.

Definition 3: The solution map G defined by equation (2) is **stable** if G is continuous at 0 with respect to the supremum norms on \mathcal{C} and $\mathcal{C}[-\tau_k, 0]$.

Definition 4: The solution map G defined by equation (2) is **asymptotically stable** if $x = Gx_0$ implies $\lim_{t \rightarrow \infty} x(t) = 0$ for any $x_0 \in \mathcal{C}[-\tau_k, 0]$.

With any solution x and time $t \geq 0$, one can denote the element $x_t \in \mathcal{C}[-\tau_k, 0]$, where $x_t(\theta) = x(t + \theta)$ for all $\theta \in [-\tau_k, 0]$. This element is considered to be the **state** of the system at time t . For a given solution map, we define the **flow map** $\Gamma : \mathcal{C}[-\tau_k, 0] \times \mathbb{R}_+ \rightarrow \mathcal{C}[-\tau_k, 0]$ by $y = \Gamma(x_t, \Delta t) = (Gx_t)_{\Delta t} = x_{t+\Delta t}$ which maps the state at time t to the state at time $t + \Delta t$.

A. Stability of Systems with Delay

Having defined the properties of systems of differential equations with delay, we now discuss methods of proving stability of these systems. Stability proofs for linear systems with delay are generally grouped into those based on analysis either in the frequency-domain or in the time-domain. Frequency-domain techniques typically attempt to determine whether all roots of the characteristic equation of the system lie in the left half-plane. Time-domain techniques typically use Lyapunov-based analysis, an approach which was extended to infinite dimensional systems by N. N. Krasovskii in [5]. Stability proofs are defined to be either delay-dependent or delay-independent. If a delay-dependent condition holds, then stability is guaranteed for a specific value or range of values of the delay. If a delay-independent condition holds, then the system is stable for all possible values of the delay.

The algorithms presented in this paper are used to prove delay-dependent stability and are based on analysis in the time-domain. Our approach is based on the Lyapunov framework, which is exemplified by the following general theorem [2] which can be applied to the nonlinear as well as the linear case.

Theorem 5: Consider a solution map G defined by Equation (2). Suppose $u, v, w : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are continuous nondecreasing functions, and that $u(s) > 0, v(s) > 0$ for $s > 0$ and $u(0) = v(0) = 0$. Suppose there exists a continuous $V : \mathcal{C}[-\tau_k, 0] \rightarrow \mathbb{R}$ such that for all $\phi \in$

$\mathcal{C}[-\tau_k, 0]$, $u(|\phi(0)|) \leq V(\phi) \leq v(\|\phi\|)$ and the following holds.

$$\dot{V}(\phi) := \limsup_{\Delta t \rightarrow 0^+} \frac{1}{\Delta t} (V(\Gamma(\phi, \Delta t)) - V(\phi)) \leq -w(|\phi(0)|)$$

Here Γ is the flow map defined by G . Then the solution map G is stable. If $w(s) > 0$ for $s > 0$, then G is asymptotically stable.

B. Complete Quadratic Functionals

There have been a number of results on necessary and sufficient conditions for stability of linear systems with delay in terms of the existence of Lyapunov-Krasovskii functionals. These results are significant in that they allow us to restrict the search for a Lyapunov-Krasovskii functional to a specific class of quadratic forms. Recall that we consider differential equations which can be expressed in the form of Equation 2. The following comes from Gu et al. [1].

Definition 6: We say that a functional $V : \mathcal{C}[-\tau_k, 0] \rightarrow \mathbb{R}$ is of the **complete quadratic type** if there exist a matrix P and functions Q , S and R such that the following holds.

$$V(\phi) = \phi(0)^T P \phi(0) + 2\phi(0)^T \int_{-\tau_k}^0 Q(s)\phi(s)ds + \int_{-\tau_k}^0 \phi(s)^T S(s)\phi(s)ds + \int_{-\tau_k}^0 \int_{-\tau_k}^0 \phi(s)^T R(s, \omega)\phi(\omega)dsd\omega$$

Theorem 7: Suppose the system described by Equation (2) is asymptotically stable. Then there exists a complete quadratic functional V and $\eta > 0$ such that $V(\phi) \geq \eta\|\phi(0)\|^2$ and $\dot{V}(\phi) \leq -\eta\|\phi(0)\|^2$ for all $\phi \in \mathcal{C}[-\tau_k, 0]$. Furthermore, the functions which define V can be taken to be continuous everywhere except possibly at points $s, \omega = -\tau_i$, for $i = 1, \dots, k-1$.

Note: The complete quadratic functionals discussed here have the same structure as those discussed in Sections I and II, subject only to a scaling of time.

IV. CONVERTING TO FINITE DIMENSIONS

In this section, we expand on the results of Section II to provide a methodology for construction of the complete quadratic Lyapunov functionals necessary for stability of linear delay-differential equations using positive semidefinite matrices.

A. Sum-of-Squares and Convex Optimization

In this first part of the section, we provide a few notes on a relatively new approach to optimization of polynomial functions known as sum-of-squares [6]. These results allow us to embed the cone of positive semidefinite matrices within the cone of positive functions. To understand the relevance of these results, recall that the conditions expressed in Theorem 1 for positivity of the first part of the Lyapunov functional were expressed using the cone of positive functions. Unfortunately, however, no algorithm exists for the direct solution of optimization problems expressed over this cone. Therefore, we approximate the cone of positive functions by the cone of positive polynomials of bounded degree and use the sum-of-squares approach to compute solutions using semidefinite programming. We begin by noting that many difficult problems in analysis and control can be reformulated as convex optimization problems of the following form.

$$\max \gamma : \quad f_0(x) - \gamma \in \mathcal{P}^Y$$

Here \mathcal{P}^Y is the cone of real polynomial functions which are positive on some subset Y . Although the formulation of the problem is convex for arbitrary f_0 , the problem is not tractable since there exists no efficient test for membership in the set \mathcal{P}^Y . Indeed, the question of whether $f \in \mathcal{P}^+$, that is, $f(x) \geq 0$ for all $x \in \mathbb{R}^n$, is, in general, NP hard [7]. Thus there is unlikely to exist a computationally tractable set membership test for \mathcal{P}^+ unless $P = NP$. However, there may exist some convex cone embedded in \mathcal{P}^+ for which the set membership test is computationally tractable. One such cone is defined as follows.

Definition 8: A polynomial is a **sum-of-squares polynomial**, denoted $s \in \Sigma$, if it can be represented as $s(x) = \sum_{i=1}^m g_i(x)^2$ for some finite set of polynomials g_i , $i = 1, \dots, m$. The set Σ^d is defined as the elements of Σ of degree d or less.

In [6], it was shown that the set membership $f \in \Sigma$ can be represented as a semidefinite programming constraint.

Definition 9: Define $Z_d(x)$ to be the vector function of monomials of x of degree d or less. For example, $Z_3(s)^T := [1 \ s \ s^2 \ s^3]$.

Lemma 10: $f \in \Sigma_{2d}$ if and only if there exists some matrix $Q \geq 0$ such that

$$f(x) = Z_d(x)^T Q Z_d(x).$$

Note that the equality constraint in Lemma 10 is affine in the monomial coefficients of f . Therefore, set membership in Σ can be tested using semidefinite programming. The question of how well Σ represents \mathcal{P}^+ , however, has been a topic on ongoing research for some time. We refer the reader to the survey paper by Reznick [7] for an overview of results on this subject. However, two important cases when $s \in \mathcal{P}^+$ implies $s \in \Sigma$ occur when s is quadratic or is a function of 2 variables.

Representation of \mathcal{P}^Y has also received significant recent attention. Consider a semialgebraic set Y defined by $Y := \{x \in \mathbb{R}^n : f_i(x) \geq 0, i = 1, \dots, n_K\}$. A condition for membership in \mathcal{P}^Y can be found in versions of a Positivstellensatz result from Putinar [8] for semialgebraic sets which satisfy a condition slightly stronger than compactness.

Theorem 11: Suppose Y , as defined above, satisfies the necessary conditions and p lies in the interior of \mathcal{P}^Y . Then there exist $s_i \in \Sigma$ for $i = 1, \dots, n_K$, such that

$$p(x) - \sum_{i=1}^{n_K} s_i(x)f_i(x) \in \Sigma.$$

B. SOS Constraints for Matrix-Valued Functions

In order to use the “sum-of-squares” technique for systems evolving in \mathbb{R}^n , we extend the above results to matrices of polynomials. Of course, one can always test whether a matrix function M satisfies $M(x) \geq 0$ for all x by testing whether $y^T M(x)y \in \Sigma$. However, the resulting introduction of auxiliary variables y dramatically increases the complexity of the problem. An alternative condition is given as follows.

Definition 12: We say that a matrix of polynomials $M : \mathbb{R}^m \rightarrow \mathbb{S}^n$ is a **sum-of-squares matrix**, denoted $M \in \Sigma^n$ if there exist matrices of polynomials G_i such that $M(x) =$

$\sum_{i=1}^n G_i(x)^T G_i(x)$. Σ_d^n are the elements of Σ^n of degree d or less.

The following lemma states that membership in Σ^n can be tested using semidefinite programming.

Definition 13: Define the matrix function $Z_d^n(x) := I_n \otimes Z_d(x)$ as the matrix with n copies of Z_d as diagonal entries. For example,

$$Z_3^3(x) := \begin{bmatrix} Z_3(x) & & \\ & Z_3(x) & \\ & & Z_3(x) \end{bmatrix}$$

Lemma 14: $M \in \Sigma_{2d}^n$ if and only if there exists some matrix $Q \geq 0$ such that $M(x) = Z_d^n(x)^T Q Z_d^n(x)$.

The complexity of the membership test associated with Lemma 14 is considerably lower than that associated with the introduction of auxiliary variables. Specifically, the variables associated with this test are of order $n \binom{m+d}{d}$ as opposed to order $\binom{n+m+d+1}{d+1}$ when $x \in \mathbb{R}^m$. Furthermore, the use of the test associated with Lemma 14 does not increase conservativity since it can be shown that $M(x) \in \Sigma^n$ if and only if $y^T M(x) y \in \Sigma$. As was the case for Σ , the general question of how well Σ^n approximates the cone of positive polynomial matrices is open. However, it is critical to note that for the univariate functions associated with Theorem 1 in Section II, the approximation is exact, as was shown in the paper by Choi et al. [9].

Lemma 15: For a matrix of polynomials, $M : \mathbb{R} \rightarrow \mathbb{S}^n$, $M(s) \geq 0$ for all $s \in \mathbb{R}$ if and only if $M \in \Sigma^n$.

The positivstellensatz result of Putinar has an extension to matrix functions and this result can be found in [10]. Technically, use of the positivstellensatz is not necessary to enforce the conditions associated with Theorem 1, since a function positive on an interval always has a globally positive extension. However, we have found in practice that its use significantly increases the convergence rate of the algorithms presented in the following chapter.

C. Positive Matrices and Positive Operators

In Section II, it was argued that one can represent a projection of the cone of functions, R , which define positive functionals of the following form using the space of positive semidefinite matrices.

$$V_2(\phi) := \int_0^1 \int_0^1 \phi(s)^T R(s, \omega) \phi(\omega) ds d\omega.$$

Such an approach requires the construction of a mapping function Z from bases in \mathbb{R}^d to a set of bases in $L_2[0, 1]$. We now show how to construct such a function for three sets of basis functions which are relevant to the results of this paper. For practical purposes, we will relax the constraint that the basis functions be orthonormal.

Matrix Functions Consider the case when $R : \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n}$. Now the basis $\{\tilde{e}_i\}_{i=1}^{n(d+1)}$ is defined to be the vector functions with monomial entries of degree d or less. For example, $\tilde{e}_i = s^{i-1} e_1$ for $i = 1, \dots, d$ and $\tilde{e}_i = s^{i-1-d} e_2$ for $i = d+1, \dots, 2d+1$, etc. Again, let Z_d^n be as in Definition 13 presented previously in this section. Then $\tilde{e}_i = Z_d^n(s) e_i$ for $i = 1, \dots, n(d+1)$. Thus the linear map $Q \mapsto R$ is defined as $R(s, \omega) = Z_d^n(s)^T Q Z_d^n(\omega)$.

Piecewise-Continuous Matrix Functions Now we consider the somewhat more difficult case of piecewise-continuous functions R . This time we use a basis of vector monomials similar to the previous case, but which only have support on the distinct intervals $[\tau_{i-1}, \tau_i]$ for $i = 1, \dots, k$ where the τ_i are monotonically increasing. In this case we define $Z : \mathbb{R} \rightarrow \mathbb{R}^{n(d+1)(k-1) \times n}$ separately on each interval. We use the function Z_d^n to construct Z . Specifically, we equipartition Z into $k-1$ blocks and for $s \in [\tau_{i-1}, \tau_i]$, define Z to be zero except that the i th block of Z is Z_d^n . For example, $Z(s) = [Z_d^n(s) \quad \mathcal{O}]^T$ for $s \in [0, \tau_1]$, where \mathcal{O} is the $n \times n(d+1)(k-1)$ matrix of zeros. The linear map $Q \mapsto R$ is now defined as the following for $i, j = 1, \dots, k-1$.

$$R(s, \omega) = Z_d^n(s)^T Q_{ij} Z_d^n(\omega) \quad \text{for } \begin{cases} s \in [\tau_i, \tau_{i+1}], \\ \omega \in [\tau_j, \tau_{j+1}] \end{cases}$$

Here we have equipartitioned Q so that Q_{ij} is the ij th block of Q . For example, when $k = 4$,

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}.$$

V. AN ALGORITHM FOR STABILITY

In this section we provide the details on how to determine the stability of a system of linear differential equations with delay using semidefinite programming. Let us first state the problem of stability abstractly in terms of a convex optimization problem. Recall we consider systems of the form defined by Equation 2 where $\tau_i \in [0, \tau_k]$ are monotonically increasing and $\tau_1 = 0$.

Theorem 16: The system defined by Equation 2 is asymptotically stable if and only if there exists $\epsilon > 0$ and piecewise-continuous functions M and R such that for all $\phi \in \mathcal{C}[-\tau_k, 0]$,

$$\begin{aligned} & \int_{-\tau_k}^0 \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds \\ & + \int_{-\tau_k}^0 \int_{-\tau_k}^0 \phi(s) R(s, \omega) \phi(\omega) ds d\omega \geq \epsilon \|\phi(0)\|_2^2 \\ & \int_{-\tau_k}^0 \begin{bmatrix} \phi(-\tau_1) \\ \vdots \\ \phi(-\tau_k) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(-\tau_1) \\ \vdots \\ \phi(-\tau_k) \\ \phi(s) \end{bmatrix} ds \\ & + \int_{-\tau_k}^0 \int_{-\tau_k}^0 \phi(s) L(s, \omega) \phi(\omega) ds d\omega \leq -\epsilon \|\phi(0)\|_2^2, \end{aligned}$$

subject to the constraint that M , R , D , and L are related as follows where we define $I_i := [-\tau_i, -\tau_{i-1}]$.

$$M(s) = M_i(s) = \begin{bmatrix} P & \tau_k Q_i(s) \\ \tau_k Q_i(s)^T & \tau_k S_i(s) \end{bmatrix} \quad \text{for all } s \in I_i$$

$$R(s, \omega) = R_{ij}(s, \omega) \quad \text{for all } s \in I_i, \omega \in I_j$$

$$L(s, \omega) = \frac{\delta}{\delta s} R(s, \omega) + \frac{\delta}{\delta \omega} R(s, \omega) \quad \text{for all } s \in I_i, \omega \in I_j$$

$$D(s) = D_i(s) = \begin{bmatrix} D11 & \tau_k D12_i(s) \\ \tau_k D12_i(s)^T & \tau_k D22_i(s) \end{bmatrix} \quad \text{for all } s \in I_i$$

$$D11_{11} = P A_0 + A_0^T P + Q_1(0) + Q_1(0)^T + S_1(0)$$

$$D11_{ij} = \begin{cases} PA_{i-1} - Q_{i-1}(-\tau_{i-1}) + Q_i(-\tau_{i-1}) & i, j = 1, 2 \dots k \\ S_i(-\tau_{i-1}) - S_{i-1}(-\tau_{i-1}) & i = j = 2 \dots k \\ PA_k - Q_k(-\tau_k) & i, j = 1, k+1 \\ -S_k(-\tau_k) & i = j = k+1 \\ 0 & \text{otherwise} \end{cases}$$

$$D12_j(s) = \begin{bmatrix} R_{1j}(0, s) + A_0^T Q_j(s) - \dot{Q}_j(s) \\ R_{2j}(-\tau_1, s) - R_{1j}(-\tau_1, s) + A_1^T Q_j(s) \\ \vdots \\ R_{(k)j}(-\tau_{k-1}, s) - R_{(k-1)j}(-\tau_{k-1}, s) + A_{k-1}^T Q_j(s) \\ A_k^T Q_j(s) - R_{(k)j}(-\tau_k, s) \end{bmatrix}$$

$$D22_i(s) = -\dot{S}_i(s)$$

The proof is a standard result and can be found in sources such as [1].

The map from piecewise-continuous functions M and R to D and L , as described in Theorem 16 will be used throughout this section. We can now apply the results of Sections II and IV to provide a condition for stability. This condition is expressed as a proto-algorithm, meaning that, for clarity, that it is not explicitly stated as a semidefinite program. The explicit construction of the semidefinite program will follow and is stated afterwards in Lemma 18. First define the function $g(s) := g_i(s) = -(s + \tau_i)(s + \tau_{i-1})$ for $s \in I_i$. Then $g(s) \geq 0$ for $s \in [-\tau_k, 0]$ and we have the following.

Lemma 17: Suppose there exist constant $\epsilon > 0$ and piecewise-continuous functions $M : \mathbb{R} \rightarrow \mathbb{S}^{2n}$, $T_1 : \mathbb{R} \rightarrow \mathbb{S}^n$, $T_2 : \mathbb{R} \rightarrow \mathbb{S}^{n(k+1)}$, $R : \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n}$, $L : \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n}$, G_1 , G_2 , S_1 , S'_1 , S'_2 and S_2 such that

$$\begin{aligned} M(s) + \begin{bmatrix} T_1(s) & 0 \\ 0 & -\epsilon I \end{bmatrix} &= S_1(s)^T S_1(s) + g(s) S'_1(s)^T S'_1(s), \\ D(s) + \begin{bmatrix} T_2(s) & 0 \\ 0 & \epsilon I \end{bmatrix} &= -S_2(s)^T S_2(s) - g(s) S'_2(s)^T S'_2(s), \\ R(s, \omega) &= G_1(s)^T G_1(\omega), \quad L(s, \omega) = G_2(s)^T G_2(\omega), \\ \int_{-\tau_k}^0 T_1(s) ds &= 0, \quad \int_{-\tau_k}^0 T_2(s) ds = 0, \end{aligned}$$

where the map from M and R to D and L is given in Theorem 16. Then the system given by Equation 2 is asymptotically stable.

We now show how this optimization problem can be solved via a semidefinite programming algorithm. Although the existence of such an algorithm is implicit in the conditions of Lemma 17, in this case we give the conditions explicitly so as to facilitate implementation. Recall that Z_d and Z'_d are defined as in the previous section. We have the following.

Lemma 18: Suppose there exist positive semidefinite matrices $Q1_i$, $Q1'_i$, $Q2$, $Q3_i$, $Q3'_i$, $Q4$ and matrices $B1_i \in \mathbb{R}^{n \times n(d+1)}$, $B2_i \in \mathbb{R}^{n(k+1) \times n(k+1)(d+1)}$ for $i = 1, \dots, k$. Let $Q2_{ij}$, $Q4_{ij} \in \mathbb{R}^{n(d+1) \times n(d+1)}$ be the ij th blocks of $Q2$ and $Q4$ respectively. Suppose that that piecewise-continuous functions M, R, D , and L satisfy the following for $s \in I_i$,

$\omega \in I_j$, and $i, j = 1, \dots, k$.

$$\begin{aligned} M(s) &= \begin{bmatrix} B1_i^T Z_d^n(s) & 0 \\ 0 & 0 \end{bmatrix} + Z_d^{2n}(s)^T Q1_i Z_d^{2n}(s), \\ &\quad + g_i(s) Z_d^{2n}(s)^T Q1'_i Z_d^{2n}(s) \\ R(s, \omega) &= Z_d^n(s)^T Q2_{ij} Z_d^n(\omega), \\ D(s) &= \begin{bmatrix} B2_i^T Z_d^{n(k+1)}(s) & 0 \\ 0 & 0 \end{bmatrix} + Z_d^{n(k+2)}(s)^T Q3_i Z_d^{n(k+2)}(s), \\ &\quad + g_i(s) Z_d^{n(k+2)}(s)^T Q3'_i Z_d^{n(k+2)}(s) \\ L(s, \omega) &= Z_d^n(s)^T Q4_{ij} Z_d^n(\omega), \end{aligned}$$

Further suppose the map from M and R to D and L is defined as in Theorem 16 and

$$\sum_{i=1}^k B1_i^T C_i = \sum_{i=1}^k B2_i^T F_i = 0,$$

$$C_i := \int_{-\tau_i}^{-\tau_{i-1}} Z_d^n(s) ds, \quad F_i := \int_{-\tau_i}^{-\tau_{i-1}} Z_d^{n(k+1)}(s) ds.$$

Then the system defined by Equation 2 is asymptotically stable.

A Matlab implementation of this algorithm can be found on the author's home page [11].

A. Computational Complexity

Before continuing, we address the important question of computational complexity of the proposed algorithm. Specifically, let n denote the number of variables contained in x , let k denote the number of delays and let d denote the order of the desired monomials. Then we have that the variable of highest dimension is $Q3_i \in \mathbb{S}^{n(k+2)(d+1)}$. Since there are k such matrix variables, the number of scalar variables is of order $k(n(k+2)(d+1))^2$. The number of constraints scale as the number of scalar variables.

B. Parameter-Dependent Spacing Functions

In order to expand on the results of this section, we consider the case which arises when the dynamics of the system depend on some uncertain time-invariant vector of parameters. Let the system be described as follows where $A_i : \mathbb{R}^p \rightarrow \mathbb{R}^{n \times n}$ are polynomial, $\tau_i \in [0, \tau_k]$ are monotonically increasing and $y \in \mathbb{R}^p$ is an uncertain vector of parameters.

$$\dot{x}(t) = \sum_{i=0}^k A_i(y) x(t - \tau_i) \quad (3)$$

We suggest an approach based on the extension of Putinar's Positivstellensatz to matrix functions [10] to give sufficient conditions for stability in the presence of static uncertainty. We assume that the uncertainty vector, y , lies in some semialgebraic set $\mathcal{P} \subset \mathbb{R}^p$ where \mathcal{P} is defined by scalar polynomials p_i as follows where we assume for convenience that p_1 is unity.

$$\mathcal{P} := \{y \in \mathbb{R}^p : p_i(y) \geq 0 \text{ for } i = 1, \dots, b\}$$

Lemma 17 can be extended as follows.

Lemma 19: Suppose there exist constant $\epsilon > 0$ and piecewise-continuous functions $M : \mathbb{R}^{p+1} \rightarrow \mathbb{S}^{2n}$, $T_1 : \mathbb{R}^{p+1} \rightarrow \mathbb{S}^n$, $T_2 : \mathbb{R}^{p+1} \rightarrow \mathbb{S}^{n(k+1)}$, $R : \mathbb{R}^{p+2} \rightarrow \mathbb{R}^{n \times n}$,

$L : \mathbb{R}^{p+2} \rightarrow \mathbb{R}^{n \times n}$, $S1'$, $S2'$, $G1_i$, $G2_i$, $S1_i$ and $S2_i$ for $i = 1, \dots, b$ such that

$$M(s, y) = \begin{bmatrix} T_1(s, y) & 0 \\ 0 & -\epsilon I \end{bmatrix} + \sum_{i=1}^b p_i(y) S1_i(s, y)^T S1_i(s, y) + g(s) S1'(s, y)^T S1'(s, y),$$

$$D(s, y) + \begin{bmatrix} T_2(s, y) & 0 \\ 0 & \epsilon I \end{bmatrix} = - \sum_{i=1}^b p_i(y) S2_i(s, y)^T S2_i(s, y) + g(s) S2'(s, y)^T S2'(s, y),$$

and where

$$R(s, \omega, y) = \sum_{i=1}^b p_i(y) G1_i(s, y)^T G1_i(\omega, y),$$

$$L(s, \omega, y) = \sum_{i=1}^b p_i(y) G2_i(s, y)^T G2_i(\omega, y),$$

$$\int_{-\tau_k}^0 T_1(s, y) ds = 0, \quad \int_{-\tau_k}^0 T_2(s, y) ds = 0,$$

and where the map from M and R to D and L is given in Theorem 16. Then the system given by Equation 3 is asymptotically stable for all $y \in \mathcal{P}$.

Note: An important application for Lemma 19 is the case when the delays τ_i appear as uncertain parameters.

VI. NUMERICAL EXAMPLES

To demonstrate the effectiveness of the proposed algorithm, we present a series of numerical examples.

A. Example with a Single Delay

In this example, we compare our results with the discretized Lyapunov functional approach used by Gu et al. in [1] in the case of a system with a single delay. Although numerous other papers have also given sufficient conditions for stability of time-delay systems, e.g. [12], [13], [14], we use the approach introduced by Gu since it has demonstrated a particularly high level of precision. When we are comparing with the piecewise linear approach here and throughout this paper, we will only consider examples which have been presented in the work [1] and we will compare our results with the numbers that are cited therein. We use SOSTOOLS [15] and SeDuMi [16] for solution of all semidefinite programming problems. Now consider the following system of delay differential equations.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

The problem is to estimate the upper and lower bounds on τ for which the differential equation remains stable. Using the method presented in this paper and by sweeping τ in increments of .1, we estimate the region of stability to be an interval. We then use a bisection method to find the minimum and maximum stable delay. Our results are summarized in Table I and are compared to the analytical limit and to the results obtained in [1].

Clearly, the results for this test case illustrate a high rate of convergence to the analytical limit. However, although the results presented here give reasonable estimates for the interval of stability, they do not prove stability over any

Our results			Piecewise Functional		
d	τ_{\min}	τ_{\max}	N_2	τ_{\min}	τ_{\max}
1	.10017	1.6249	1	.1006	1.4272
2	.10017	1.7172	2	.1003	1.6921
3	.10017	1.71785	3	.1003	1.7161
Analytic	.10017	1.71785			

TABLE I

τ_{\max} AND τ_{\min} FOR DISCRETIZATION LEVEL N_2 USING THE PIECEWISE-LINEAR LYAPUNOV FUNCTIONAL AND FOR DEGREE d USING OUR APPROACH AND COMPARED TO THE ANALYTICAL LIMIT

interval, but rather only at the specific values of τ for which the algorithm was tested. To provide a more rigorous analysis, we now include τ as an uncertain parameter and search for parameter dependent Lyapunov functionals which prove stability over an interval. The results from this test are given in Table III.

d in τ	d in θ	τ_{\min}	τ_{\max}
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

TABLE II

STABILITY ON THE INTERVAL $[\tau_{\min}, \tau_{\max}]$ VS. DEGREE USING A PARAMETER-DEPENDENT FUNCTIONAL

B. Example with Parametric Uncertainty

In this example, we illustrate the flexibility of our algorithm through a simplistic control design and analysis problem. Suppose we wish to control a simple inertial mass remotely using a PD controller. Now suppose that the derivative control is half of the proportional control. Then we have the following dynamical system.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

It is easy to show that this system is stable for all positive values of a . However, because we are controlling the mass remotely, some delay may be introduced due to, for example, the fixed speed of light. We assume that this delay is known and changes sufficiently slowly so that for the purposes of analysis, it may be taken to be fixed. Now we have the following delay-differential equation with uncertain, time-invariant parameters a and τ .

$$\ddot{x}(t) = -ax(t - \tau) - \frac{a}{2}\dot{x}(t - \tau)$$

Whereas before the system was stable for all positive values of a , now, for any fixed value of a , there exists a τ for which the system will be unstable. In order to determine which values of a are stable for any fixed value of τ , we divide the parameter space into regions of the form $a \in [a_{\min}, a_{\max}]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. This type of region is compact and can be represented as a semi-algebraic set using the polynomials $p_1(a) = (a - a_{\min})(a - a_{\max})$ and $p_2(\tau) = (\tau - \tau_{\min})(\tau - \tau_{\max})$. By using these polynomials, we are able to construct parameter dependent Lyapunov functionals which prove stability over a number of parameter regions. These regions are illustrated in Figure 1.

C. Example with Multiple Delays

Consider the following system of delay-differential equations.

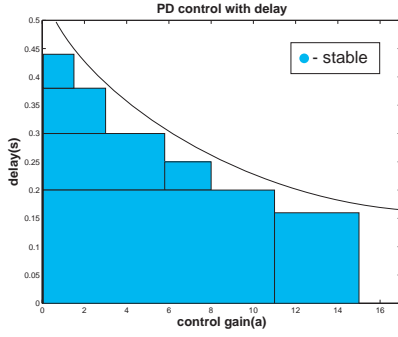


Fig. 1. Regions of Stability for Example 2

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{10} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - \frac{\tau}{2}) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

Again, the system is stable when τ lies on some interval. The problem is to search for the minimum and maximum value of τ for which the system remains stable. In applying the methods of this paper, we again use a bisection method to find the minimum and maximum value of τ for which the system remains stable. Our results are summarized in Table III and are compared to the analytical limit as well as piecewise-linear functional method. For the piecewise functional method, N_2 is the level of both discretization and subdiscretization.

Our Approach			Piecewise Functional		
d	τ_{\min}	τ_{\max}	N_2	τ_{\min}	τ_{\max}
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

TABLE III

τ_{\max} AND τ_{\min} USING THE PIECEWISE-LINEAR LYAPUNOV FUNCTIONAL OF GU ET AL. AND OUR APPROACH AND COMPARED TO THE ANALYTICAL LIMIT

VII. CONCLUSION

The general question of stability of linear differential equations with delay is NP-hard [17].

In this paper, we have shown that the question of stability can be expressed as a convex optimization problem within the function space $\mathcal{C}[0, 1]$. Furthermore, through the use of projections from $\mathcal{C}[0, 1]$ onto the space of polynomials of degree d , we have shown how to semidefinite programming to compute solutions to this optimization problem. Thus we have created a sequence of sufficient conditions for stability of linear time-delay systems, indexed by d , and whose complexity is of order $k(n(d+1)(k+2))^2$. As $d \rightarrow \infty$, the validity of the polynomial approximation increases and the conditions become increasingly accurate, as adequately illustrated by the numerical examples.

Aside from the accuracy of the results, the approach taken in this paper has an important advantage. Because of the flexibility of the polynomial approach, we are able to make various extensions of our results to other problems which can be expressed using convex optimization in $\mathcal{C}[0, 1]$. Specifically, in this paper we have addressed the problem of systems with parametric uncertainty using a variant of the

S-procedure. Additionally, extension to nonlinear systems with time-delay is easily addressed, although in this case conservatism arises from a number of sources. A more extensive treatment of nonlinear time-delay systems is the topic of another paper currently in preparation.

We conclude by mentioning that the methods of this paper would seem to allow us to synthesize stabilizing controllers for linear time-delay systems. This observation is motivated by viewing the functions M and R as defining a full rank operator on $L_2[0, 1]$. We have observed that for given functions, the inverse of such an operator can be computed numerically. By computing Lyapunov functionals for the adjoint system constructed by Delfour and Mitter [18], this invertibility result seems to imply that one can construct stabilizing controllers. This work is ongoing.

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