Bootstrap current density maximization in Tokamaks

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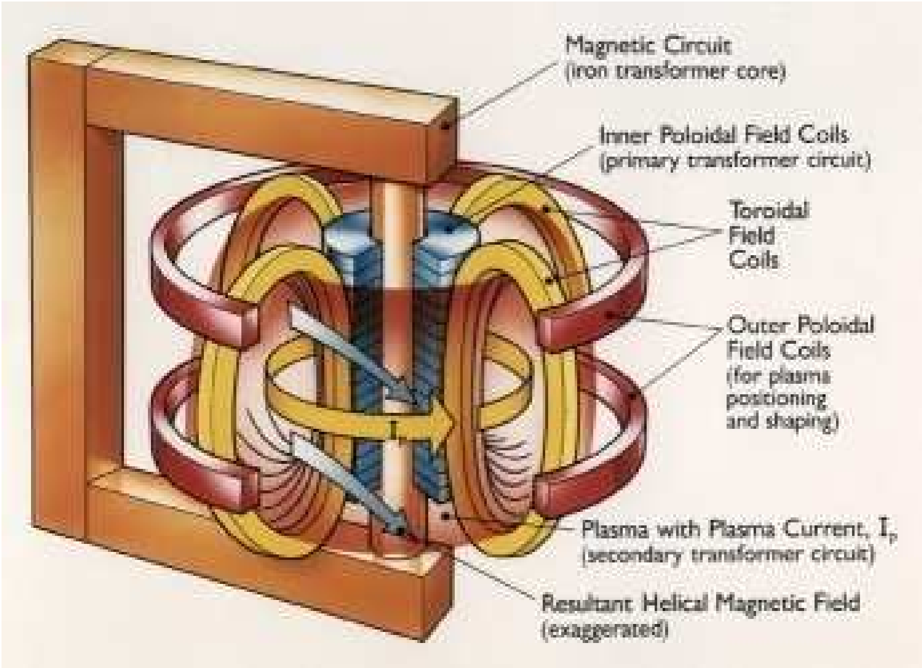
Tokamak

- Toroidal chambered device with poloidal and toroidal magnetic fields.



Tokamak

- To heat the plasma (resistive heating) and generate the poloidal field, plasma current is required.

- Plasma current is primarily generated by the transformer action of the ohmic coil (central coil).

- This renders the Tokamak a pulsed device in addition to the external energy inputs provided to the central coil.

Bootstrap current density

- Internally generated current due to particles (ions and electrons) trapped in the low magnetic field side of the vessel.
- This causes a reduced dependence on the externally induced current from the transformer action.
- Also allows for the Tokamak to run for pulses of longer duration.
- A higher value of bootstrap current has been identified as a crucial factor for the steady state operation of Tokamaks.

Poloidal magnetic flux ψ

- Bootstrap current density is inversely related to the gradient of the poloidal magnetic flux ψ_x.

- The goal of the controller, thus, is to minimize $\|\psi_x\|_{L_2}$.

- We will use a linearized model for the evolution of ψ_x and use Lyapunov functional based methods.

- Lyapunov functionals will be constructed using Sum-of-Squares (SOS) polynomials.

Model for ψ_x

The linearized evolution model is given by

$$\frac{d\psi_x}{dt}(x,t) = \frac{1}{\mu_0 a^2}\frac{\partial}{\partial x}\left(\frac{\eta_{||}(x,t)}{x}\frac{\partial}{\partial x}(x\psi_x)\right)$$
$$+ R_0\frac{\partial}{\partial x}\Big(\eta_{||}(x,t)j_{eni}(x,t)\Big) + +R_0\frac{\partial}{\partial x}\Big(\eta_{||}(x,t)u(x,t)\Big),$$

where

$x \in [0,1]$, $\quad t \geq 0$, $\quad \mu_0 =$ Permeability of free space
$\eta_{||} : [0,1] \times \mathbb{R}_+ \to \mathbb{R} =$ Plasma resistivity,
$a =$ Plasma minor radius, $R_0 =$ Plasma major radius,
$j_{eni} =$ external non-induced current deposit (control input) and
$u =$ distributed disturbance (other actuators' effects).

Control Input and Actuators

- External non-inductive current density $j_{eni}(x,t)$ is the control input.

- $j_{eni} = j_{lh} + j_{ec}$ where j_{lh} is the current deposited by the Lower hybrid current density (LHCD) antennae and j_{ec} is provided by electron cyclotron current drive (ECCD) antennae.

- The *Tore Supra* has 2 LHCD and 6 ECCD antennae.

Control Input and Actuators

- We will use only 1 LHCD antenna as an actuator, this is typical since the other actuators are required for shape control, position control etc.

- Even though the control input is distributed, it is shape constrained by the actuators' engineering parameters, power and refractive index.

Control Input and Actuators

- Using measurements from *Tore Supra*, an empirical model was developed.

- This model utilizes Gaussians to model the shape of the current deposit with control authority on the mean, amplitude and variance.

$$j_{eni}(x,t) = v_{lh}(t)e^{-\frac{(\mu_{lh}(t)-x)^2}{2\sigma_{lh}(t)}},$$

where $v_{lh}(t) \in [0, 1.22MA]$, $\mu_{lh}(t) \in [0.14, 0.33]$ and $\sigma_{lh}(t) \in [0.016, 0.073]$ for all $t \geq 0$.

Controller Synthesis Outline

- We will construct full state feedback based controllers since on-line state reconstruction is available.

- The small latency (order of few milliseconds) in the state reconstruction will be neglected.

- We will employ the following empirical model of the plasma resistivity for the *Tore Supra*
$$\eta_{\|}(x,t) = \mathbf{a}(t)e^{\lambda(t)x},$$
where $0 < \underline{\mathrm{a}} \leq \mathbf{a}(t) \leq \bar{a} < \infty$ and $0 < \underline{\lambda} \leq \lambda(t) \leq \bar{\lambda} < \infty$ for all $t \geq 0$.

Controller Synthesis Outline

- We assume the solution $\psi_x \in C_2^1(0,T;C_2^2[0,1])$ for all $0 < T < \infty$ and disturbance $u \in L_2(0,\infty;C_2^2[0,1])$.

- The goal is to construct a controller j_{eni} such that there exists a strictly positive $M(x)$ for which

$$V(t) = \int_0^1 \psi_x^2 M(x)^{-1} f(x) dx, \quad f(x) = x^2$$

satisfies

$$\frac{dV(t)}{dt} \leq \frac{1}{\gamma} \|u(x,t)\|_{C_2^2[0,1]} - \gamma \|\psi_x(x,t)\|_{C_2^{2,M^{-2}}[0,1]}$$

for some $\gamma > 0$.

Controller Synthesis Outline

- If our controller allows us to construct such a V, then

$$\|\psi_x\|^2_{C_2^1(0,\infty;C_2^{2,M-2}[0,1])} \leq \frac{1}{\gamma^2}\|u\|^2_{L_2(0,\infty;C_2^2[0,1])} + \frac{V(0)}{\gamma}.$$

- Thus γ defines an upper bound on the norm of ψ_x and has to be maximized.

Controller Synthesis Outline

- The first step is to formulate an optimization problem whose solution will maximize γ.

- The second step is to add shape constraints to obtain the complete problem.

Controller Synthesis with Unconstrained Shape

Theorem 1. *Suppose that for a given* γ *there exist polynomials* $M, R : [0,1] \to \mathbb{R}$ *such that*

$$M(x) > 0 \text{ for all } x \in [0,1],$$
$$\Omega(x,\lambda) + \Theta \le 0 \text{ for all } (x,\lambda) \in [0,1] \times [\underline{\lambda}, \bar{\lambda}] \text{ and}$$
$$2A_4 + 2B_2 + A_2(1) \le 0,$$

where

$$\Omega(x,\lambda) = \begin{bmatrix} 2A_1(x) & 0 & -R_0\mu_0 a^2 f(x) \\ 0 & A_0(x,\lambda) & -R_0\mu_0 a^2 f_x(x) \\ -R_0\mu_0 a^2 f(x) & -R_0\mu_0 a^2 f_x(x) & 0 \end{bmatrix},$$

$$\Theta = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\mu_0 a^2 \gamma}{\underline{a}} & 0 \\ 0 & 0 & -\frac{\mu_0 a^2}{\bar{a} e^{\bar{\lambda}} \gamma} \end{bmatrix},$$

$$\begin{aligned}
A_0(x,\lambda) &= 2A_3(x) - \lambda A_2(x) - A_{2,x}(x) + 2B_1(x,\lambda), \\
A_1(x) &= -f(x)M(x), \\
A_2(x) &= -\bar{f}(x)M(x) - f(x)M_x(x) - f_x(x)M(x), \\
A_3(x) &= -2M(x) - f_x(x)M_x(x), \\
A_4 &= M(1), \\
B_1(x) &= -\frac{f_x(x)R(x)}{2} + \frac{f(x)R_x(x)}{2} + \lambda\frac{f(x)R(x)}{2}, \\
B_2 &= \frac{R(1)}{2}, \quad f(x) = x^2 \textit{ and } \bar{f}(x) = x.
\end{aligned}$$

Then if

$$j_{eni}(x,t) = \frac{K(x)}{R_0\mu_0 a^2} Z(x,t)$$

where $K(x) = R(x)M^{-1}(x)$*, then* ψ_x *is upper bounded bounded as follows.*

$$\|Z(x,t)\|^2_{C^1_2((0,\infty);C^{2,M-2}_2[0,1])} \leq \frac{1}{\gamma^2}\|u(x,t)\|^2_{L_2((0,\infty);C^2_2[0,1])} + \frac{V(0)}{\gamma}$$

- The search for the maximum γ can be performed using a bisection method.

Controller Shape Constraints

- We want that the controller $j_{eni}(x,t) = \frac{K(x)}{R_0 \mu_0 a^2} \psi_x(x,t)$ should resemble feasible Gaussian shapes.

- We construct an additional constraint of the form
$$g_1(x) \leq \frac{K(x)}{R_0 \mu_0 a^2} \psi_x(x,t) \leq g_2(x),$$
where $g_1 < g_2$ are polynomial approximations of two heuristically selected Gaussians.

- Since both $K(x)$ and $\psi_x(x,t)$ are continuous, the shape of j_{eni} will be bounded by the constraint envelope defined by g_1 and g_2.

Controller Shape Constraints

- Additionally we assume that

$$\psi_x(x,t) = \alpha(t)\psi_{x,1}(x) + (1-\alpha(t))\psi_{x,2}(x,t),$$

  where $\alpha(t) \in [0,1]$, $\psi_{x,1}(x)$ is the polynomial approximation of the open loop steady state and $\psi_{x,2}(x)$ is the polynomial approximation of the steady state under maximum actuation.

- Using $K(x) = R(x)/M(x)$, the shape constraint becomes

  $R_0\mu_0 a^2 M(x) g_1(x) \leq R(x)(\alpha\psi_{x,1}(x) + (1-\alpha)\psi_{x,2}(x,t))$
  $R_0\mu_0 a^2 M(x) g_2(x)$ for all $(x,\alpha) \in [0,1] \times [0,1]$.

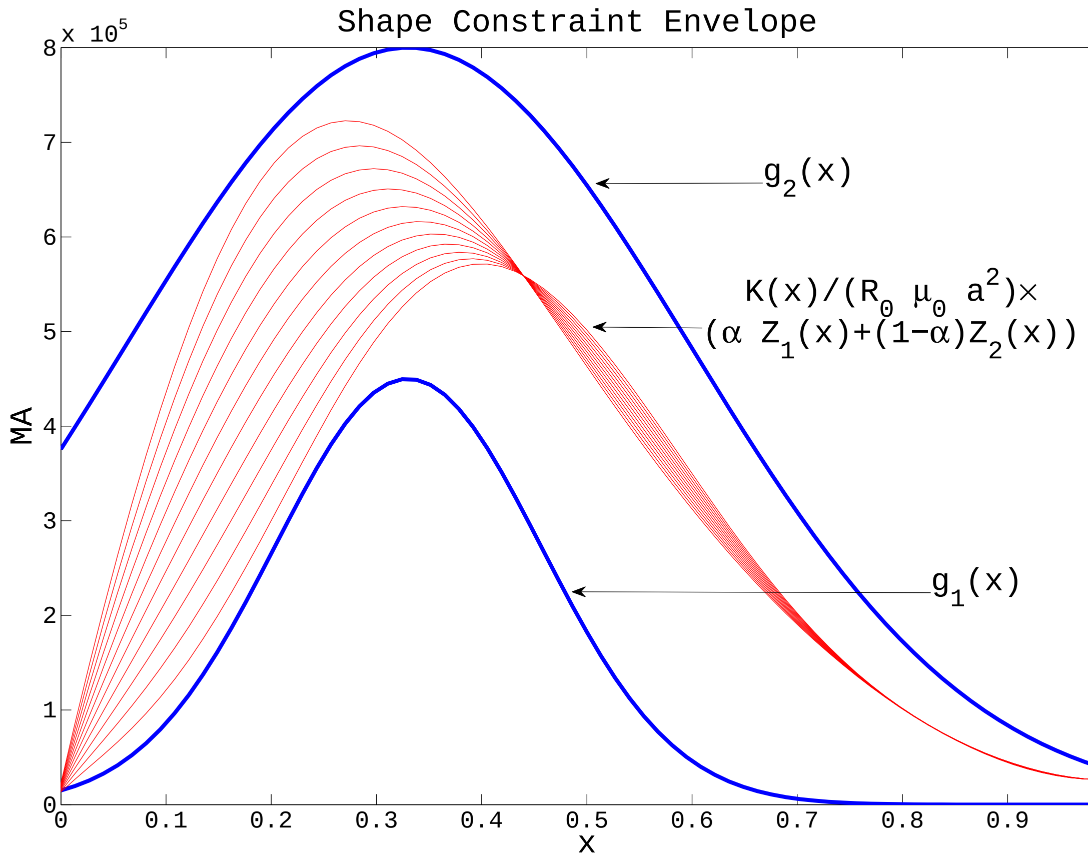
Simulation

- Simulations are done for *Tore Supra*.

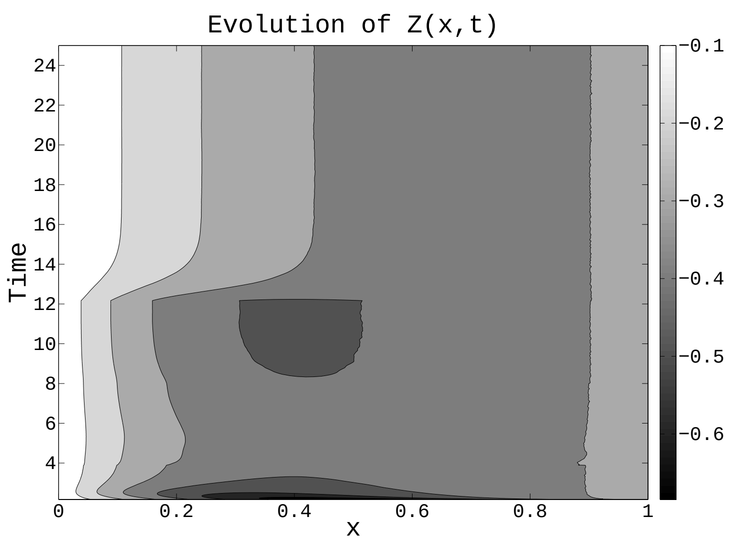
- The optimization problem is solved using SOSTOOLS.

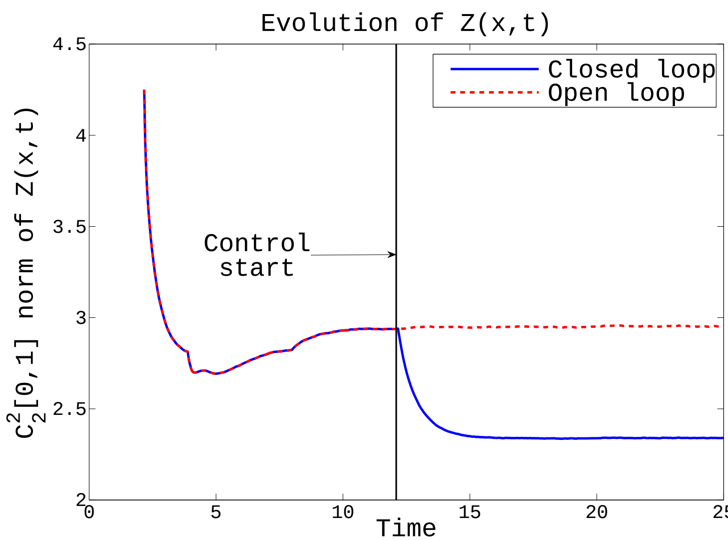
- A maximum of $\gamma \approx 10^4$ is obtained for polynomials of degree 12 in x.

Shape Constraint Envelope

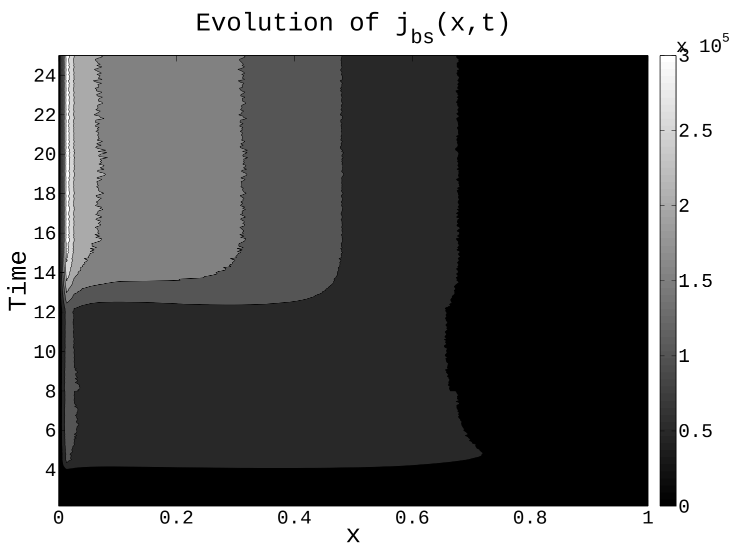


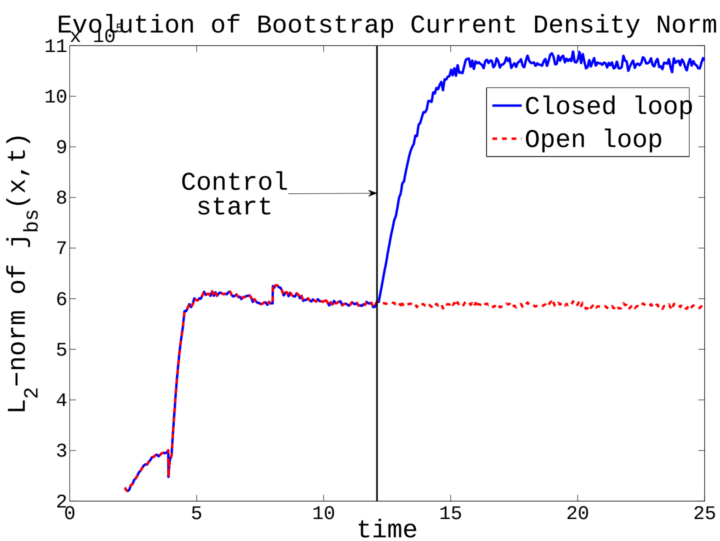
Evolution of ψ_x



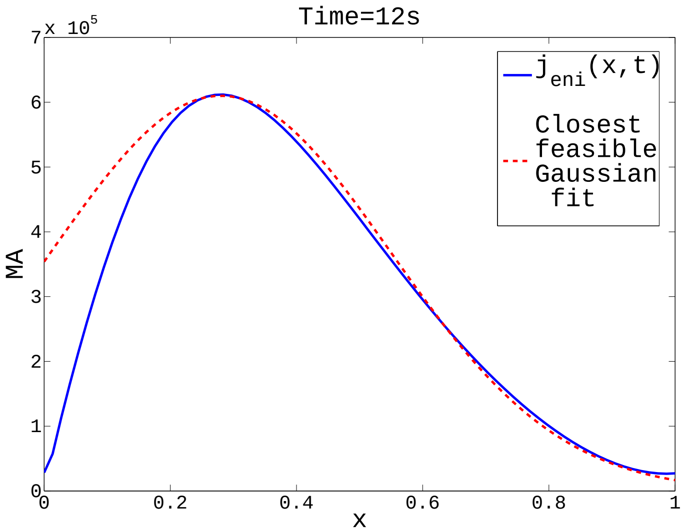


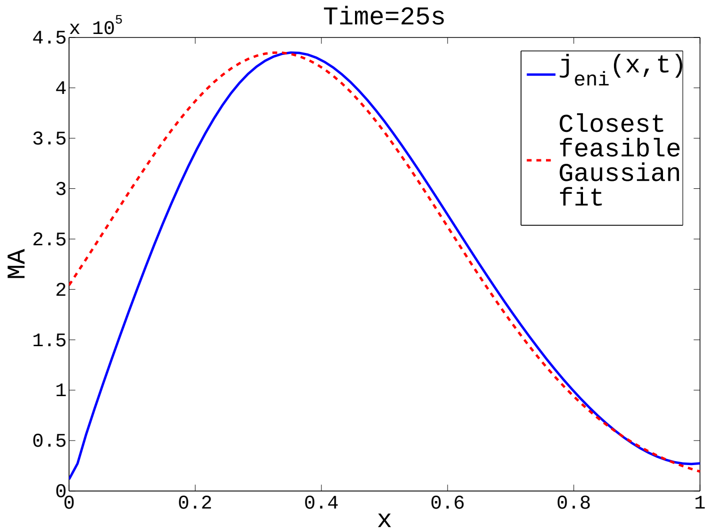
Evolution of j_{bs}





Control Demand and Gaussian Fits





Future Work

- Implementation of the controllers of the form
$$j_{eni} = K(x, \alpha, \beta, \omega) \int_0^1 |\psi_x| dx.$$

- Incorporate delay (state reconstruction) in the control actuation.

- Formulate control cost function to include parameters like confinement efficiency.

- Implement controllers on METIS.