Spacecraft Dynamics and Control

Matthew M. Peet
Arizona State University

Lecture 9: Bi-elliptics and Out-of-Plane Maneuvers

Introduction

In this Lecture, you will learn:

Bi-elliptic Maneuvers

- 3-burn Maneuvers
- Comparison with Hohmann
- Numerical Example
  - Elliptic
  - Circular

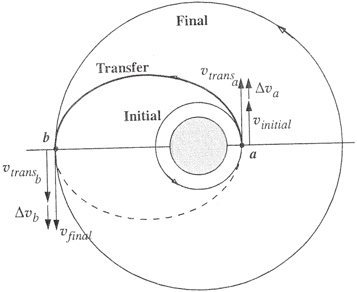
Out-of-Plane Maneuvers

- Inclination Change
- Right Ascension Change

Numerical Problem: Suppose we are in a circular parking orbit at altitude $191km$. We desire a final altitude of $376,310km$. Design the energy optimal orbital maneuvers necessary to reach our desired orbit.

The Oberth Effect

Generally it is better to make the initial burn at perigee.



For a burn at velocity v, the change in kinetic energy is

$$\Delta T = \frac{1}{2}\left(v + \Delta v\right)^2 - \frac{1}{2}v^2 = \frac{1}{2}\Delta v^2 + v \cdot \Delta v$$

For a fixed Δv, the second term $v \cdot \Delta v$ is much greater when v is large.

- For an elliptic orbit, maximum velocity is at perigee
- Lower orbits move faster
- It is much easier to achieve escape velocity when in low earth orbit

The Oberth Effect: Energy Explanation

Propulsive force results from expulsion of particles at high velocity.

Kinetic Energy of Propellant

- Suppose craft moving at velocity v_s.
- Particles are ejected with relative velocity $\Delta v_p > v_s$
- Absolute velocity of particles is $v_s - \Delta v_p$.
- Kinetic Energy of particles is

$$T_p \cong (v_s - \Delta v_p)^2$$

- The closer v_s is to Δv, the lower the kinetic energy.

Potential Energy of Propellant

- The potential energy of the propellant is

$$V = -\sqrt{\frac{\mu}{r}}$$

- the lower the propellant is ejected, the lower the potential energy

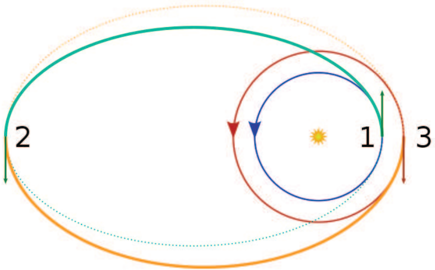
Conclusion: Propellant used at perigee has much less energy.

The energy not spent on propellant is retained by the spacecraft.

The Bi-Elliptic Transfer

The Hohmann transfer is the energy-optimal *2-impulse* transfer.

- Addition Energy savings can be bought at the expense of additional time.
- a 3-impulse trajectory

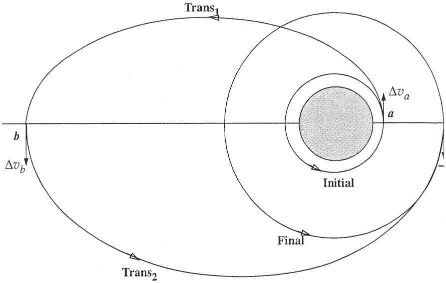


The bi-elliptic transfer uses the Oberth effect

1. initial impulse close to escape velocity.
2. perigee-raising maneuver at apogee.
3. apogee-*lowering* maneuver at perigee.

The Bi-Elliptic Transfer

Suppose we want to raise a circular orbit of radius r_1 to radius r_2.

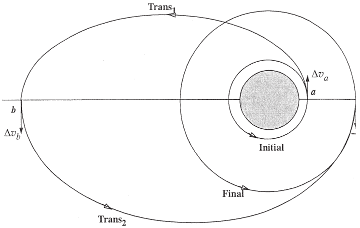


3 burns are required. Given r_1 and r_2, choose transfer radius $r_* >> r_f$.

1. Convert circular orbit at radius r_1 to elliptic orbit with perigee $r_p = r_1$ and apogee $r_a = r_*$.
2. At apogee, raise perigee of elliptic orbit to r_2.
3. At Perigee, circularize the orbit by lowering perigee to r_2.

The Bi-Elliptic Transfer

Suppose we want to raise a circular orbit of radius r_1 to radius r_2.



This time there are 2 transfer ellipses

Ellipse 1:

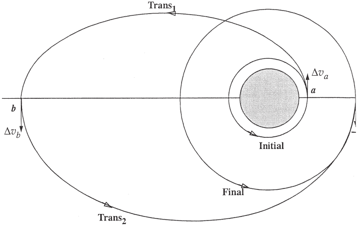
$$a_1 = \frac{r_1 + r_*}{2}$$
$$e = \frac{r_* - r_1}{r_* + r_1}$$

Ellipse 2:

$$a_2 = \frac{r_2 + r_*}{2}$$
$$e = \frac{r_* - r_2}{r_* + r_2}$$

The Bi-Elliptic Transfer

Suppose we want to raise a circular orbit of radius r_1 to radius r_2.



We can calculate the 3 burns as:

Burn 1:

$$\Delta v_1 = v_{1,p} - v_i = \sqrt{2\mu \frac{r_*}{r_1(r_1 + r_*)}} - \sqrt{\frac{\mu}{r_1}}$$

Burn 2:

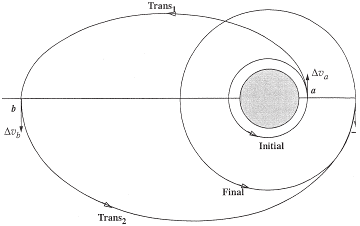
$$\Delta v_2 = v_{2,a} - v_{1,a} = \sqrt{2\mu \frac{r_2}{r_*(r_2 + r_*)}} - \sqrt{2\mu \frac{r_1}{r_*(r_1 + r_*)}}$$

Burn 3:

$$\Delta v_3 = v_f - v_{2,p} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu \frac{r_*}{r_2(r_2 + r_*)}}$$

Notes on the Bi-Elliptic Transfer

Suppose we want to raise a circular orbit of radius r_1 to radius r_2.



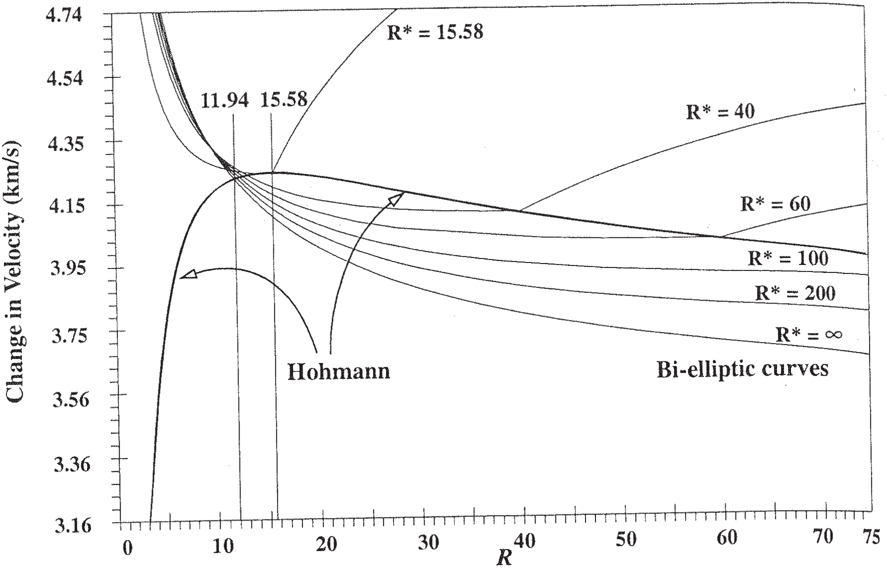
Note that the third burn is retrograde.

- Δv_3 is clearly wasted energy.
- For this reason, bielliptics only work when $r_2 >> r_1$ ($R \cong 11.94$).
  - $v_f << v_i$

Note that r_* is a free parameter.

- As $r_* \to \infty$, the bielliptic gets *more* efficient.
  - Escape and reinsertion.
- As $r_* \to \infty$, $\Delta t \to \infty$.
  - A tradeoff between time and efficiency.

Notes on the Bi-Elliptic Transfer ($R = \frac{r_2}{r_1}$)



Numerical Example

Problem: Suppose we are in a circular parking orbit at altitude $191km$. We desire a final altitude of $376,310km$. Design the energy optimal orbital maneuvers necessary to reach our desired orbit.

Solution: First we choose between Hohmann and bi-elliptic. Note

$r_1 = 191km + 1ER = 1.03ER$ and $r_2 = 376,310km + 1ER = 60ER$

Thus our ratio $R \cong 60$. In this case, it is clear that the bi-elliptic is better.

We choose a transfer radius of $r_* = 80ER$.

Ellipse 1: Our first transfer ellipse will have $a_1 = \frac{r_1 + R_*}{2} = 40.5ER$. We have the following data

$$v_i = .985ER/TU$$
$$v_{1,p} = 1.385ER/TU$$
$$v_{1,a} = .0178ER/TU$$

Thus our initial velocity change is

$$\Delta v_1 = v_{1,p} - v_i = 1.385 - .985 = .4ER/TU$$

Numerical Example

Ellipse 2: Our second transfer ellipse will have $a_2 = \frac{r_2+R_*}{2} = 70ER$. We have the following data

$$\begin{aligned} v_f &= .129ER/TU \\ v_{2,p} &= .138ER/TU \\ v_{2,a} &= .103ER/TU \end{aligned}$$

Our change from ellipse 1 to ellipse 2 requires

$$\Delta v_2 = v_{2,a} - v_{1,a} = .103 - .0178 = .0857ER/TU$$

Our final circularization requires

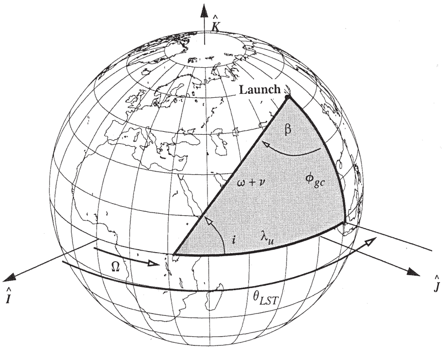
$$\Delta v_3 = v_f - v_{2,p} = .129 - .138 = -.009ER/TU$$

Conclusion:

- Our total Δv budget is $.4938ER/TU = 3.9km/s$.
- Budget for Hohmann is $4.0km/s$.
- The total duration of transit is 2650 TU = 593.9hr = 24.75 days.

Most satellites are launched from the surface of the earth.

- Launch Geometry dictates the orbital plane.

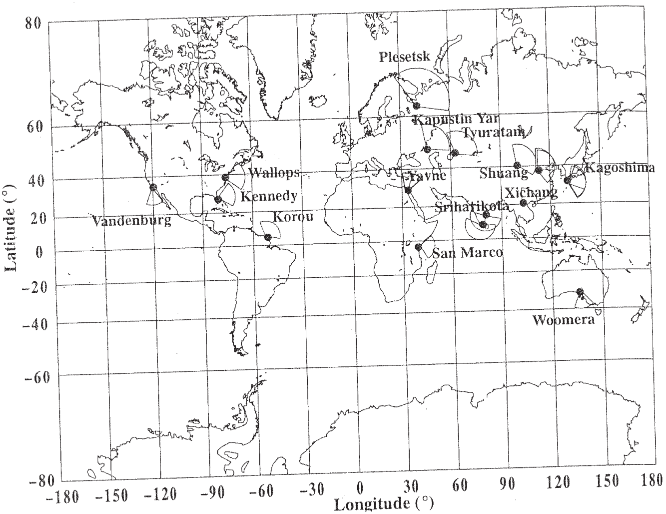


The two geometric features of launch are:

- latitude of the launch site, ϕ_{gc}
- launch azimuth (direction), β.

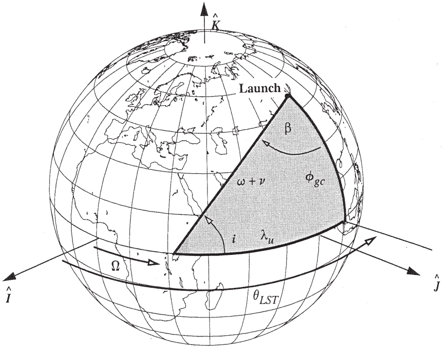
Launch Geometry

Site Restrictions



- The set of launch sites is restricted
- The range of launch azimuth is restricted

Launch geometry determines the **inclination** of the orbital plane of the parking orbit.



$$\cos i = \cos\phi_{gc} \sin\beta$$

Launch Geometry

Site Restrictions

Site	Latitude (°)	Longitude (°)	Azimuth Min (°)	Azimuth Max (°)
Vandenberg	34.600 000	−120.600 000	147	201
Cape Kennedy	28.500 000	−80.550 000	37	112
Wallops	37.850 000	−75.466 67	30	125
Kourou	5.200 000	−52.800 000	340	100
San Marco	−2.933 333	40.200 000	50	150
Plesetsk	62.800 000	40.600 000	330	90
Kapustin Yar	48.400 000	45.800 000	350	90
Tyuratam	45.600 000	63.400 000	340	90
Sriharikota	13.700 000	80.250 000	100	290
Shuang-Ch'Eng-Tzu	40.416 667	99.833 333	350	120
Xichang	28.250 000	102.200 000	94	105
Tai-yuan	37.766 667	112.500 000	90	190
Kagoshima	31.233 333	131.083 333	20	150
Woomera	−30.950 000	136.500 000	350	15
Yavne	31.516 667	34.450 000	350	120

Typically, different sites are used for different purposes.

Site	Latitude (°)	Longitude (°)	Azimuth Min (°)	Azimuth Max (°)
Vandenberg	34.600 000	−120.600 000	147	201
Cape Kennedy	28.500 000	−80.550 000	37	112
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Example: Vandenburg has

$$\phi_{gc} = 34.6\,\text{deg} \qquad \text{and} \qquad \beta \in [147\,\text{deg}, 201\,\text{deg}]$$

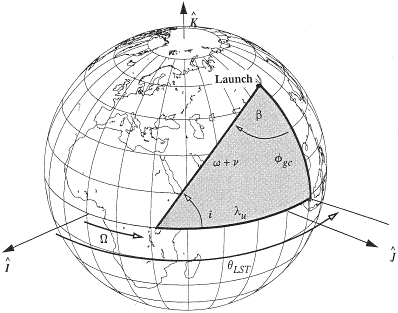
Therefore

$$-.295 < \cos i < .4483$$

So the inclination is restricted as

$$63.36\,\text{deg} < i < 107.16\,\text{deg}$$

Unlike inclination, the Right Ascension of the orbital plane can be chosen by **Launch Window**.



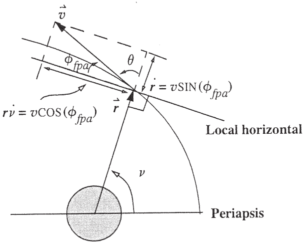
Referring to the triangle our desired launch time (in Local Sidereal Time) is given by

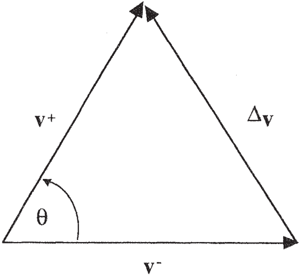
$$\theta_{LST} = \Omega + \lambda_u$$

where λ_u can be found from β and i as

$$\cos \lambda_u = \frac{\cos \beta}{\sin i}$$

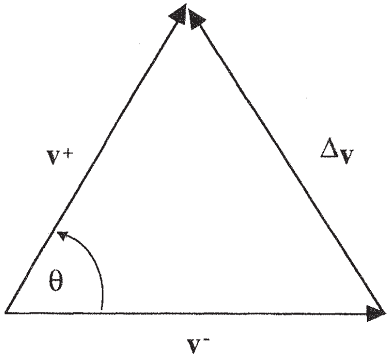
Suppose we want to change inclination without changing any other orbital element.





- Cannot change magnitude of v (a)
- Cannot change flight path angle (e, f, ω)
- Must occur at ascending node (Ω)

Inclination Only Plane Changes



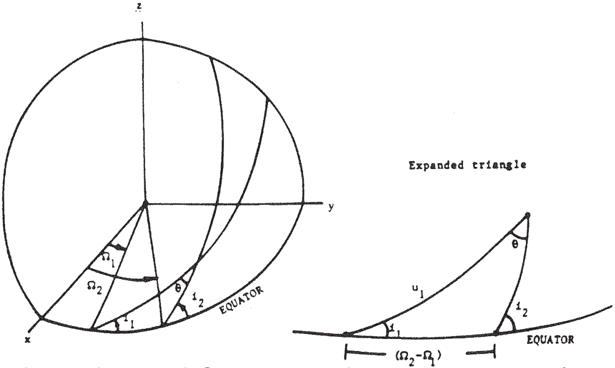
The Δv required can be calculated as

$$\Delta v = 2v \sin \frac{\theta}{2}$$

The direction of thrust is

$$90^\circ + \frac{\theta}{2}$$

Plane changes can be made anywhere in the orbit. However, this affects both i and Ω.

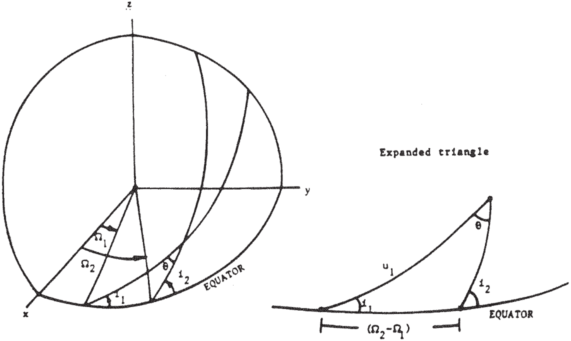


Given an initial orbit with i_1 and Ω_1, a plane change at $u = \omega + f$ yields the spherical geometry.

$$\cos i_2 = \cos i_1 \cos\theta - \sin i_1 \sin\theta \cos u$$

$$\cos(\Omega_2 - \Omega_1) = \frac{\cos\theta - \cos i_1 \cos i_2}{\sin i_1 \sin i_2}$$

Changing both Ω and i simultaneously is always more efficient than changing them separately.



Therefore, we are given an initial orbit with i_1 and Ω_1, along with desired elements i_2 and Ω_1, then required plane change and position are given by

$$\cos\theta = \cos i_1 \cos i_2 + \sin i_1 \sin i_2 \cos(\Omega_2 - \Omega_1)$$

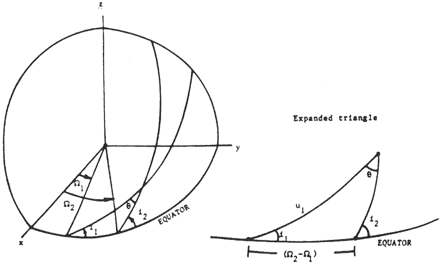
$$\cos(\omega + f) = \frac{\cos i_1 \cos\theta - \cos i_2}{\sin i_1 \sin\theta}$$

Combined Maneuvers

Inclination changes are by definition inefficient

$$\Delta v = 2v \sin\frac{\theta}{2}$$

- Up to 200% of total energy.
- Changes become more efficient as $\lim v \to 0$.
  - $v \to 0$ as $r \to \infty$.



It is often worth boosting the orbit to improve the efficiency of a plane change (See Homework.)

A typical strategy is to combine a plane change with a bi-elliptic transfer

Numerical Example: Combined Change

Problem: Suppose we are in an orbit with inclination $i = 55°$, $\Omega = 0°$ and $a = 1.8ER$. Determine the timing and Δv required to change the inclination to $i = 40°$ and RAAN to $\Omega = 45°$.

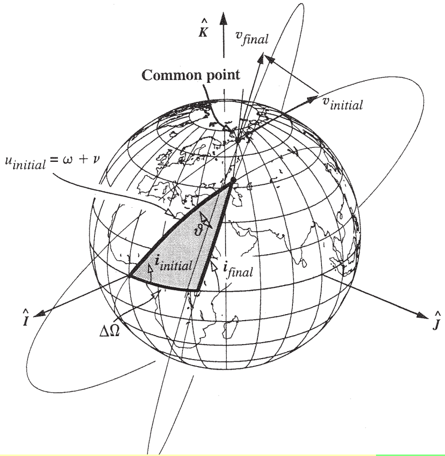
Solution: First find the plane change required

Using our formula,

$$\begin{aligned}\cos\theta &= \cos 55° \cos 40° \\ &\quad + \sin 55° \sin 40° \cos 45° \\ &= .8117\end{aligned}$$

Thus $\theta = 35.74°$. The timing for the Δv can be calculated from $u = \omega + f$ as

$$\begin{aligned}\cos u &= \frac{\cos 55° \cos 35.74° - \cos 40°}{\sin 35.74° \sin 55°} \\ &= -.628\end{aligned}$$



Numerical Example: Combined Change

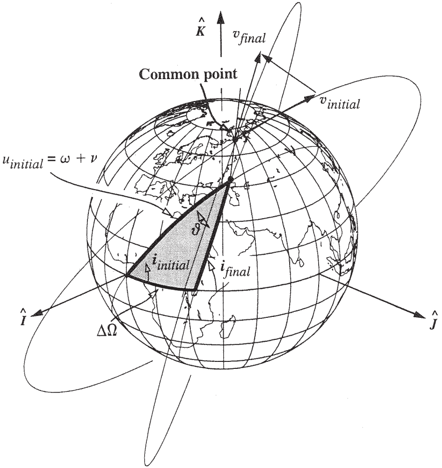
Since $\cos u = -.628$, we have that $u = 128.9°$.

Since the orbit is circular, $\omega = 0$ (or neglected). Thus the burn occurs at

$$f = 128.9°.$$

To calculate the Δv, we must first find the v at the desired point in the orbit. Since the orbit is circular, this is not difficult.

$$v = \sqrt{\frac{\mu}{r}} = .745ER/TU$$



Then the required Δv can be calculated as

$$\Delta v = 2 * v * \sin\frac{\theta}{2} = .457ER/TU$$

Summary

This Lecture you have learned:

Bi-elliptic Maneuvers

- 3-burn Maneuvers
- Comparison with Hohmann
- Numerical Example
  - Elliptic
  - Circular

Out-of-Plane Maneuvers

- Inclination Change
- Right Ascension Change

Next Lecture: Interplanetary Mission Planning.