

Spacecraft and Aircraft Dynamics

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Lecture 7: Converting to/from \vec{r} and \vec{v}

Introduction

In this Lecture, you will learn:

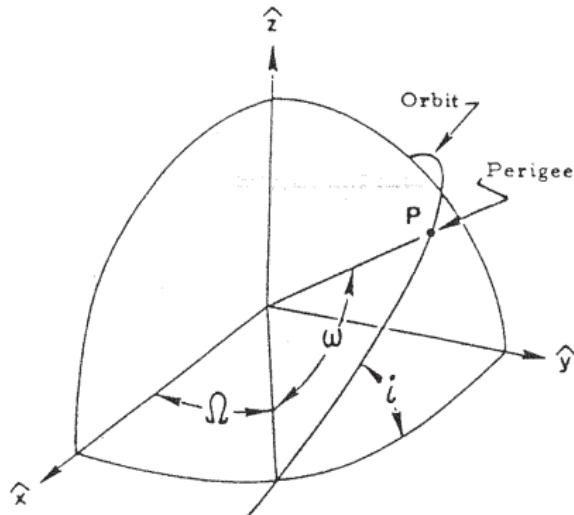
How to convert between

- $a, e, i, \Omega, \omega, f$
- \vec{r} and \vec{v}

How to translate \vec{r} and \vec{v} into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Finding the Orbital Elements



In the previous lecture, we introduced three new orbital elements.

- Inclination, i
- RAAN, Ω
- Argument of Periapse, ω

We gave a numerical example to illustrate how to find these new elements

Finding the Orbital Elements

Summary

Step 1: Construct \vec{h} , \vec{n} and \vec{e} .

$$\vec{h} = \vec{r} \times \vec{v}$$

Assume \vec{r} and \vec{v} are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h}$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r}$$

Alternatively

$$\vec{e} = \frac{1}{\mu} \left[\left(v^2 - \frac{\mu}{r} \right) \vec{r} - (\vec{r} \cdot \vec{v}) \vec{v} \right]$$

Calculate the scalars $r = \|\vec{r}\|$, $v = \|\vec{v}\|$, $e = \|\vec{e}\|$, $h = \|\vec{h}\|$.

Finding the Orbital Elements

Step 2: Calculate the 2D elements.

$$E = \frac{v^2}{2} - \frac{\mu}{r} \quad \text{and} \quad a = -\frac{\mu}{2E}$$

Step 3: Calculate the 3D elements. We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1} \left(\frac{\vec{h}}{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

No quadrant ambiguity.

$$\Omega = \cos^{-1} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right)$$

Correct for quadrant.

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360 - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

Finding the Orbital Elements

Argument of perigee is the angle between \vec{e} and \vec{n} .

$$\omega = \cos^{-1} \left(\frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\| e} \right)$$

We resolve the quadrant ambiguity by checking

$$\omega = \begin{cases} \omega & \hat{z} \cdot \vec{e} \geq 0 \\ 360 - \omega & \hat{z} \cdot \vec{e} < 0 \end{cases}$$

True anomaly is the angle between \vec{r} and \vec{e} .

$$f = \cos^{-1} \left(\frac{\vec{r} \cdot \vec{e}}{re} \right)$$

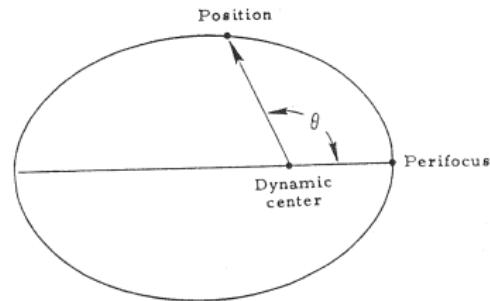
We resolve the quadrant ambiguity by checking

$$f = \begin{cases} f & \vec{r} \cdot \vec{v} \geq 0 \\ 360 - f & \vec{r} \cdot \vec{v} < 0 \end{cases}$$

Propagation in Time

All orbital elements can be determined from a single observation at t_0 .

- Orbital motion is periodic
 - ▶ Orbital elements allow us to predict the motion for all time.



Given a future time, t_f , we can use Kepler's equation to predict $f(t_f)$
Step 1: Use true anomaly, $f(t_0)$ to find mean anomaly, $M(t_0)$.

$$E = 2 \tan^{-1} \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2} \right)$$

$$M = E - e \sin E$$

Propagation in Time

Step 2: Determine mean anomaly at t_f

$$M(t_f) = M(t_0) + n(t_f - t_0)$$

Step 3: Use mean anomaly, $M(t_f)$ to find true anomaly, $f(t_f)$ using Kepler's equation.

$$M = E - e \sin E$$

$$f = 2 \tan^{-1} \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$

- The true anomaly, $f(t_f)$, tells us where the satellite is at time t_f .
- But how to translate that into \vec{r} and \vec{v} ?

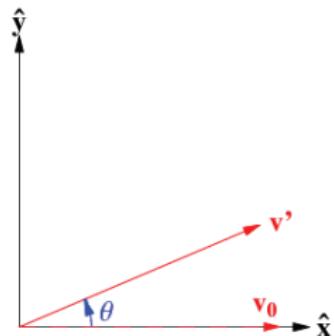
Review: Coordinate Rotations

Coordinate rotations are different than vector rotations

Case 1: Rotation of a vector in a fixed coordinate system.

Consider rotation of \vec{r} around the \hat{x} axis by θ and around the \hat{z} axis by ω

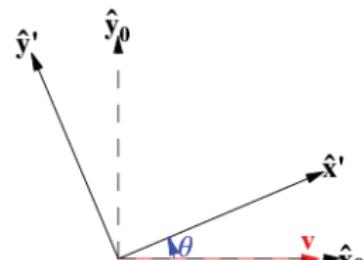
$$\vec{r}' = R_3(\omega)R_1(\theta)\vec{r}$$



Case 2: Expression of a fixed vector in a new coordinate system.

Consider what happens if we rotate the coordinates (F1) about the \hat{x} axis by θ (F2) and then rotate the coordinates about the \hat{z} axis by ω (F3)

$$\vec{r}_{F3} = R_3(-\omega)\vec{r}_{F2} = R_3(-\omega)R_1(-\theta)\vec{r}_{F1}$$



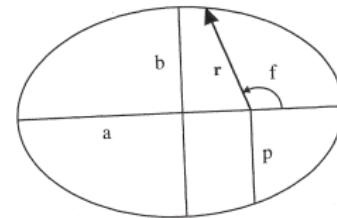
Finding \vec{r} and \vec{v}

The Perifocal Frame

Note: Our method is slightly different than the book. You are free to take either approach.

Perifocal Coordinates:

- $\hat{x} = \vec{e}/e$.
- $\hat{z} = \vec{h}/h$
- \hat{y} by RHR



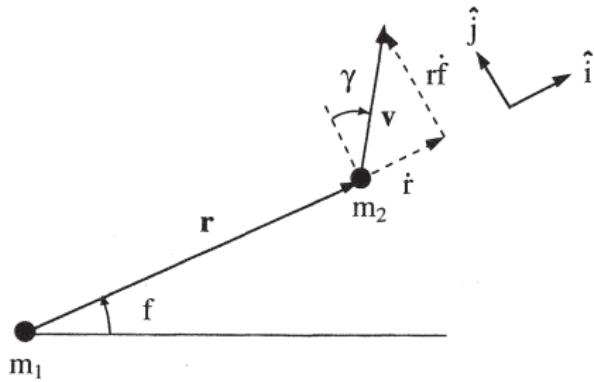
Position in perifocal frame is simple.

$$\vec{r} = \begin{bmatrix} r \cos f \\ r \sin f \\ 0 \end{bmatrix} \quad \text{where} \quad r = \frac{p}{1 + e \cos f}$$

Velocity in the Perifocal Frame

Recall our original expression for 2D velocity.

$$\vec{v}_o = \dot{r}\hat{i} + r\dot{\theta}\hat{j} = \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix}$$



- To get to the perifocal frame we rotate backwards by angle f .
- Can use rotation matrix $R_3(-f)$.

$$\vec{v}_{PQW} = R_3(-f) \begin{bmatrix} \dot{r} \\ r\dot{\theta} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{r} \cos f - r\dot{\theta} \sin f \\ \dot{r} \sin f + r\dot{\theta} \cos f \\ 0 \end{bmatrix}$$

Velocity in the Perifocal Frame

Now recall $h = r^2 \dot{f}$. Hence we can simplify

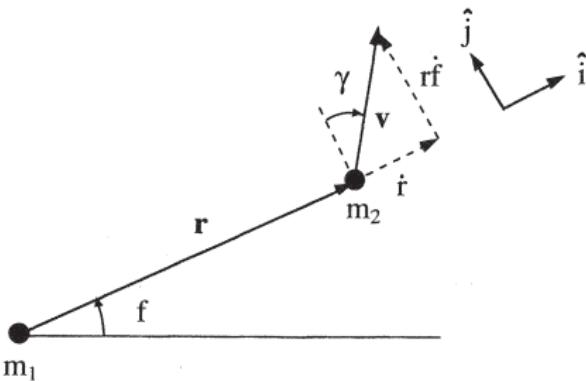
$$r\dot{f} = \sqrt{\frac{\mu}{p}} (1 + e \cos f).$$

by differentiating the orbit equation and using the above expression, we get

$$\dot{r} = \sqrt{\frac{\mu}{p}} (e \sin f)$$

Plugging these expressions in, we get the following

$$\vec{v}_{PQW} = R_3(f) \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{r} \cos f - r\dot{f} \sin f \\ \dot{r} \sin f + r\dot{f} \cos f \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}} \sin f \\ \sqrt{\frac{\mu}{p}} (e + \cos f) \\ 0 \end{bmatrix}$$



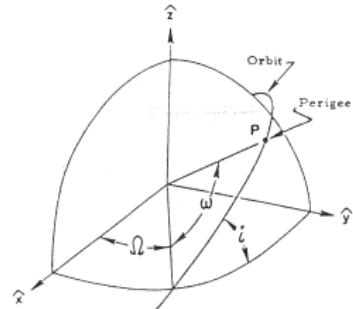
Coordinate Rotations

Perifocal to ECI Transformation

$$\vec{r}_{PQW} = \begin{bmatrix} r \cos f \\ r \sin f \\ 0 \end{bmatrix} \quad \vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}} \sin f \\ \sqrt{\frac{\mu}{p}} (e + \cos f) \\ 0 \end{bmatrix}$$

The Perifocal coordinates can be reached from ECI via 3 rotations.

1. Rotate Ω about \hat{z}
2. Rotate i about \hat{x}
3. Rotate ω about \hat{z}



As mentioned, rotating coordinates has the *opposite* effect of rotating the vector. Thus a vector \vec{r}_{ECI} in ECI coordinates can be expressed as

$$\vec{r}_{PQW} = R_3(-\omega)R_1(-i)R_3(-\Omega)\vec{r}_{ECI}$$

Perifocal to ECI Transformation

Thus to convert a PQW vector to ECI, we can

$$\vec{r}_{ECI} = R_3(\Omega)R_1(i)R_3(\omega)\vec{r}_{PQW} = R_{PQW \rightarrow ECI}\vec{r}_{PQW}$$

$$R_{PQW \rightarrow ECI} = \begin{bmatrix} \cos \Omega & -\sin \Omega & 0 \\ \sin \Omega & \cos \Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos \omega & -\sin \omega & 0 \\ \sin \omega & \cos \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \cos \Omega \cos \omega - \sin \Omega \sin \omega \cos i & -\cos \Omega \sin \omega - \sin \Omega \cos \omega \cos i & \sin \Omega \sin i \\ \sin \Omega \cos \omega + \cos \Omega \sin \omega \cos i & -\sin \Omega \sin \omega + \cos \Omega \cos \omega \cos i & -\cos \Omega \sin i \\ \sin \omega \sin i & \cos \omega \sin i & \cos i \end{bmatrix}$$

\vec{r} and \vec{v} in ECI

Finally, we can express our \vec{r} and \vec{v} in ECI.

$$\vec{r}_{ECI} = R_{PQW \rightarrow ECI} \begin{bmatrix} r \cos f \\ r \sin f \\ 0 \end{bmatrix} \quad \vec{v}_{ECI} = R_{PQW \rightarrow ECI} \begin{bmatrix} -\sqrt{\frac{\mu}{p}} \sin f \\ \sqrt{\frac{\mu}{p}} (e + \cos f) \\ 0 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} r(\cos \Omega \cos(\omega + f) - \sin \Omega \sin(\omega + f) \cos i) \\ r(\sin \Omega \cos(\omega + f) + \cos \Omega \sin(\omega + f) \cos i) \\ r \sin(\omega + f) \sin i \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -\frac{\mu}{h} (\cos \Omega (\sin(\omega + f) + e \sin \omega) + \sin \Omega (\cos(\omega + f) + e \cos \omega) \cos i) \\ -\frac{\mu}{h} (\sin \Omega (\sin(\omega + f) + e \sin \omega) - \cos \Omega (\cos(\omega + f) + e \cos \omega) \cos i) \\ \frac{\mu}{h} (\cos(\omega + f) + e \cos \omega) \sin i \end{bmatrix}$$

Numerical Example: \vec{r} and \vec{v} in ECI

Problem: Given the following orbital elements, find \vec{r} and \vec{v} .

$$a = 35,960 \text{ km} = 5.64ER$$

$$e = .832$$

$$f = 92.335 \text{ deg}$$

$$i = 87.87 \text{ deg}$$

$$\Omega = 227.9 \text{ deg}$$

$$\omega = 53.39 \text{ deg}$$

Solution: First solve for r and h .

$$p = a(1 - e^2) = 1.735ER$$

$$r = \frac{p}{1 + e \cos f} = 1.7947$$

$$p = h^2/\mu, \text{ so}$$

$$h = \sqrt{p} = 1.3172.$$

Now in perifocal coordinates

$$\vec{r}_{PQW} = \begin{bmatrix} -.07319 \\ 1.7947 \\ 0 \end{bmatrix} \quad \vec{v}_{PQW} = \begin{bmatrix} -.7585 \\ .6013 \\ 0 \end{bmatrix}$$

Numerical Example: \vec{r} and \vec{v} in ECI

We can find the rotation matrices in Matlab using the following commands:

```
R3w = [cosd(w) -sind(w) 0; sind(w) cosd(w) 0; 0 0 1];  
R1 = [1 0 0; 0 cosd(i) -sind(i); 0 sind(i) cosd(i)];  
R30m = [cosd(0m) -sind(0m) 0; sind(0m) cosd(0m) 0; 0 0 1];
```

Then compute the position and velocity vectors:

```
rECI = R30*R1*R3w*rPQW
```

```
vECI = R30*R1*R3w*vPQW
```

which yields

$$\vec{r}_{ECI} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER \quad \vec{v}_{ECI} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix} ER/TU$$

Of course we could have simply used the formulae.

Pointing Coordinates

Right Ascension and Declination

Question: Now that we have \vec{r} and \vec{v} , what do we do with them?

Answer:

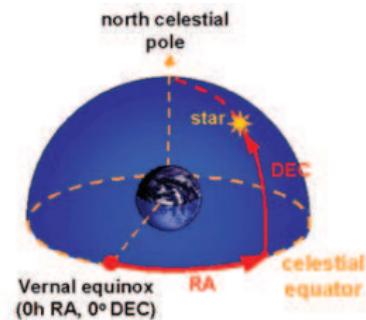
- Tracking
- Communication
- Interception
- Astronomy

For all of these applications, we need to know where to look.

1. The sky is big.
2. Satellites are small.

To track a satellite or star, the position vector must be translated into a direction.

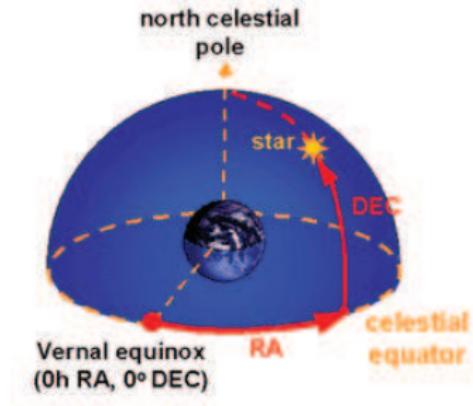
These directions are declination and right ascension.



Right Ascension

Definition 1.

Right Ascension, α is the angle the position vector makes with the FPOA when projected onto the reference plane.



Initially suppose we are at the center of the earth. If $\vec{r} = [r_1 \quad r_2 \quad r_3]$, the projection is simply $[r_1 \quad r_2]$. Thus

$$\tan(\alpha) = \frac{r_2}{r_1}$$

Declination

Definition 2.

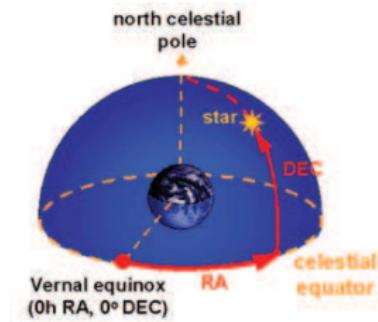
Declination, δ is the angle the position vector makes with the reference plane.

Again, simple geometry yields

$$\sin \delta = \frac{r_3}{r}$$

or

$$\tan \delta = \frac{r_3}{\sqrt{r_1^2 + r_2^2}}$$



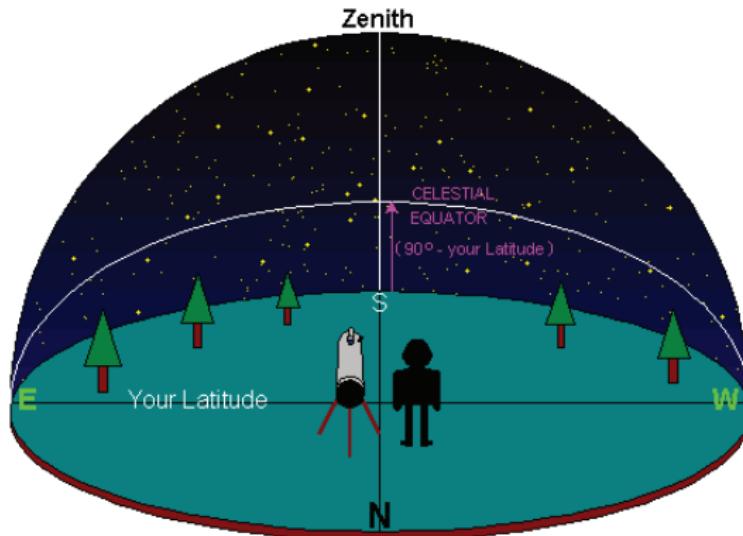
For a point on the surface of the earth, the observer must use

$$\vec{r}_{rel} = \vec{r}_{sat} - \vec{r}_{site}$$

to calculate the right ascension and declination.

Question: How to find Jupiter?

Observation using α and δ

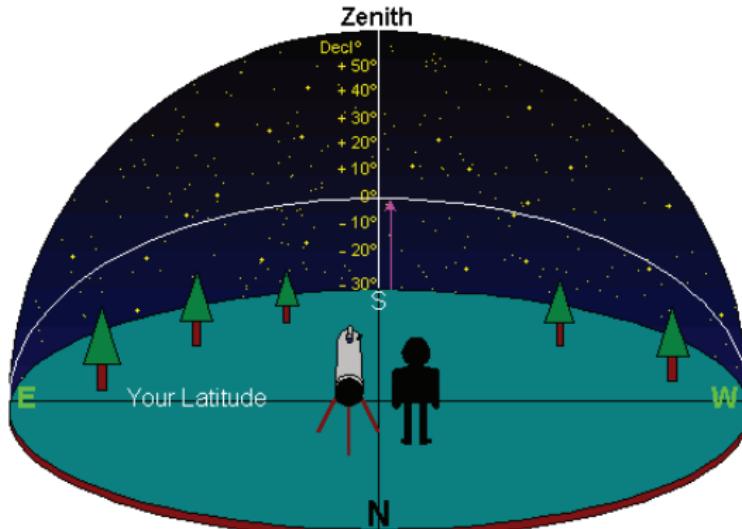


The Celestial Equator is up (90° - your Latitude) from the S horizon

Step 1: Locate the Equatorial plane.

- When facing due south, the equatorial plane will be at $90^\circ - \lambda$, where λ is your latitude.

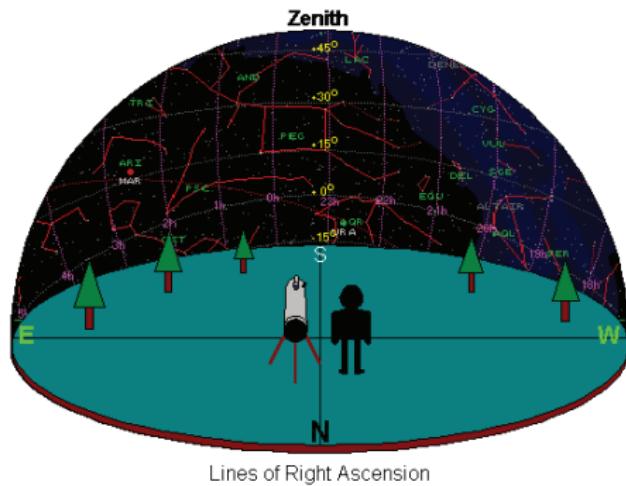
Observation using α and δ



Declination is measured + (northerly) and - (southerly) from 0°

Step 2: From the Equatorial plane, measure up/down to declination line.

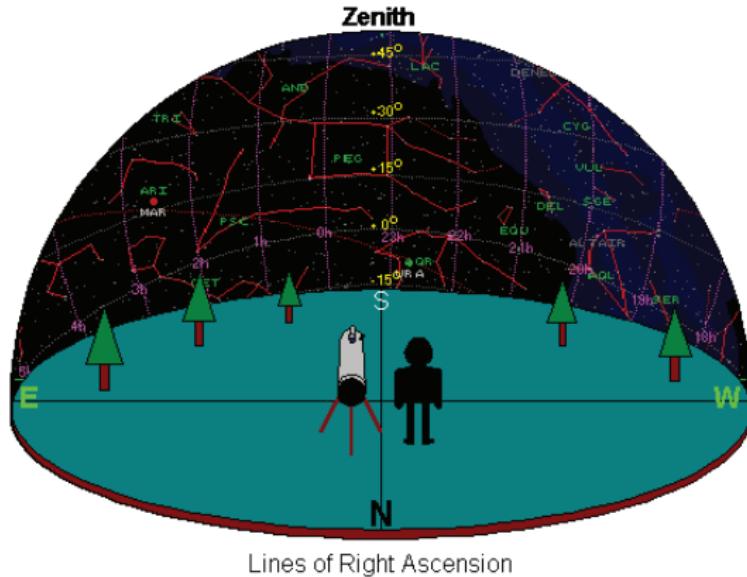
Observation using α and δ



Step 3: Determine right ascension, α_S of due south.

- This is given by Local Sidereal Time.
- Consult a table or do the conversion (not covered here).
- There *is* an app for that.
 - ▶ Sidereal Time, Skyfari, Star Map Pro, Emerald Geneva for iphone

Observation using α and δ



Step 4: Find desired α relative to α_S .

Observation using α and δ

Celestial Sphere

RA/Dec Coordinates

RA/Dec Coordinates

Numerical Example

Targeting

Problem: Suppose we are in a spacecraft in the following orbit

$$a = 60,000 \text{ km} = 9.41ER$$

$$e = .9$$

$$f = 130 \text{ deg}$$

$$i = 80 \text{ deg}$$

$$\Omega = 220 \text{ deg}$$

$$\omega = 70 \text{ deg}$$

We would like to use our laser cannon to destroy a defense satellite in the following orbit.

$$a = 35,960 \text{ km} = 5.64ER$$

$$e = .832$$

$$f = 92.335 \text{ deg}$$

$$i = 87.87 \text{ deg}$$

$$\Omega = 227.9 \text{ deg}$$

$$\omega = 53.39 \text{ deg}$$

What range, Right Ascension and declination should we give to the targeting computer?

Step 1: Find our position vector. We use the same Matlab script as before.

$$\vec{r}_1 = \begin{bmatrix} 4.71 \\ 5.97 \\ -8.74 \end{bmatrix} ER$$

Numerical Example

Step 2: The position vector

$$\vec{r}_2 = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER$$

Step 3: The relative position vector

$$\vec{r}_2 - \vec{r}_1 = \begin{bmatrix} -3.7 \\ -4.89 \\ 9.75 \end{bmatrix}$$

Step 4: Translate into RA and declination. Use Matlab commands

```
dec=atan2(rrel(3),sqrt(rrel(1)^2 + rrel(2)^2))  
RA=atan2(rrel(2),rrel(1))
```

Yields $\delta = 1.0097\text{rad}$, $\alpha = -2.2173\text{rad}$.

Summary

This Lecture you have learned:

How to convert between

- $a, e, i, \Omega, \omega, f$
- \vec{r} and \vec{v}

How to translate \vec{r} and \vec{v} into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Next Lecture: Transfer Orbits