Modern Control Systems

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Lecture 7.5: Positive Matrices and SVD

Manipulations of Positive Matrices

Positivity will not change with coordinate transformations.

- $P > 0$ if and only if $T^* P T > 0$
- What about $T^{-1} P T$?

Theorem 1 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n\times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} > 0$
2. $Q > 0$ *and* $M - RQ^{-1}R^T > 0$

Proof.

First, we show that 2) implies 1). Suppose that $Q > 0$ and $M - RQ^{-1}R^T > 0$. Then

$$\begin{bmatrix} M - RQ^{-1}R^T & 0 \\ 0 & Q \end{bmatrix} > 0$$

Thus

$$\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} = \begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M - RQ^{-1}R^T & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} I & RQ^{-1} \\ 0 & I \end{bmatrix}^T > 0$$

Schur Complement

Proof.

We first show that 1) implies 2). Suppose that

$$\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} > 0$$

Then for any $x \in \mathbb{R}^m$, $x \neq 0$,

$$\begin{bmatrix} 0 \\ x \end{bmatrix}^T \begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} 0 \\ x \end{bmatrix} = x^T Q x > 0.$$

Thus $Q > 0$ and hence Q^{-1} exists. Now, since $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} > 0$,

$$\begin{bmatrix} I & -RQ^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} \begin{bmatrix} I & -RQ^{-1} \\ 0 & I \end{bmatrix}^T = \begin{bmatrix} M - RQ^{-1}R^T & 0 \\ 0 & Q \end{bmatrix} > 0$$

Therefore $M - RQ^{-1}R^T > 0$ □

Singular Value Decomposition

Theorem 2.

Let $A \in \mathbb{R}^{m,n}$ and $p = \min(m,n)$. There exist unitary matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that $A = U \Sigma V^T$, where

$$\Sigma_{i,j} = \begin{cases} 0 & i \neq j \\ \sigma_i > 0 & i = j \end{cases} \qquad \Sigma = \begin{bmatrix} \sigma_1 & & & 0 & \cdots & 0 \\ & \ddots & & & & \vdots \\ & & \sigma_p & 0 & \cdots & 0 \end{bmatrix}$$

Assume that $\sigma_1 > \sigma_2 > \cdots > \sigma_p$.

Singular Value Decomposition

Proof.

By Spectral Theorem,

$$A^T A = V \begin{bmatrix} \Lambda & \\ & 0 \end{bmatrix} V^T$$

where $\Lambda > 0$ is diagonal and V is unitary. Let $\Sigma = \Lambda^{\frac{1}{2}}$. Then

$$v^T A^T A V = \begin{bmatrix} \Sigma^2 & \\ & 0 \end{bmatrix}$$

Let $X = AV \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix}$. Then

$$\begin{aligned} x^T X &= \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix}^T V^T A A V \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix} \\ &= \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix}^T \begin{bmatrix} \Sigma^2 & \\ & 0 \end{bmatrix} \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \end{aligned}$$

□

Singular Value Decomposition

Proof.

Thus if $X = \begin{bmatrix} x_1 & \cdots x_n \end{bmatrix}$, then

$$x_i^T x_j = \begin{cases} 1 & i = j, \;\; i = 1 \dots k < n \\ 0 & \text{otherwise} \end{cases}$$

Thus the first k columns are orthonormal and the rest are zero. Now define $U_1 = \begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix}$ so that $X = \begin{bmatrix} U_1 & 0 \end{bmatrix}$. Now complete the basis as $U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}$, where $U_2 = \begin{bmatrix} v_{k+1} & \cdots & v_n \end{bmatrix}$ is a arbitrarily chosen set of orthonormal vectors. Then

$$X = AV \begin{bmatrix} \Sigma^{-1} & \\ & I \end{bmatrix} = \begin{bmatrix} U_1 & 0 \end{bmatrix}$$

$$\begin{aligned} A &= \begin{bmatrix} U_1 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & \\ & I \end{bmatrix} V^T = \begin{bmatrix} U_1 & 0 \end{bmatrix} \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix} V^T \\ &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix} V^T = U \begin{bmatrix} \Sigma & \\ & 0 \end{bmatrix} V^T \end{aligned}$$

The Maximum Singular Value

The σ_i^2 are the eigenvalues of $A^T A$ or AA^T.

Definition 3.

We denote the Maximum Singular Value of a Matrix, M, as

$$\bar{\sigma}(M) = \max_i \sigma_i(M)$$

The maximum singular value of a matrix is a matrix norm with many pleasing properties.

- An induced norm

$$\bar{\sigma}(A) = \sup_v \frac{\|Av\|}{\|v\|} = \sup_{\|v\|=1} \|Av\|$$