Spacecraft and Aircraft Dynamics

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Lecture 4: Position and Velocity

Introduction

In this Lecture, you will learn:

Motion of a satellite in time

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - True Anomaly
- How to convert between them
  - Kepler's Equation

Problem: Let $a = 25,512km$ and $e = .625$. Find r, v at $t = 4hr$.

Recall the Conic Equation

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

Which we have shown describes elliptic, parabolic of hyperbolic motion.

Question: What is $f(t)$?

Response: There is no closed-form expression for $f(t)$!

What to do?

Start with Kepler's Second Law: Equal Areas in Equal Time.

$$\frac{dA}{dt} = \frac{h}{2} = constant$$

But how does $A(t)$ relate to $f(t)$?

A useful geometric tool is to inscribe the ellipse in a circle.

The equation of a conic section is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

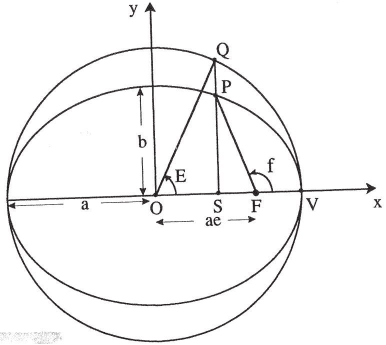
Solving for y,

$$y_e = \frac{b}{a}\sqrt{a^2 - x^2}$$

but for a circle of radius a, $y_c(x) = \sqrt{a^2 - x^2}$. Thus

$$y_e = \frac{b}{a} y_c$$

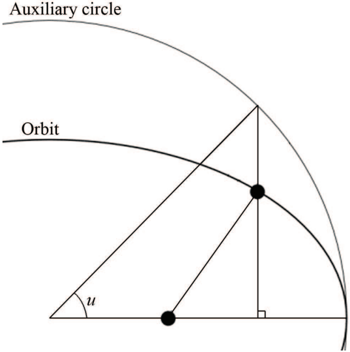
This is the ellipse scaling law.



The Eccentric Anomaly

The **Eccentric Anomaly** is an artificial angle

- From the *Center* of the ellipse
- To the projection of r on a fictional circular orbit of radius a



- Measured from center of ellipse (not focus).
- No physical interpretation.
- A mathematical convenience

The Ellipse Revisited

For convenience, suppose $t = 0$ at periapse. The area swept out is FVP

Kepler's Second Law say that

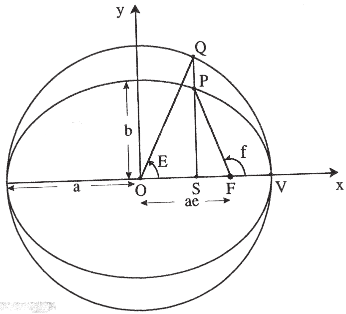
$$\frac{t}{T} = \frac{\text{Area of FVP}}{\text{Area of ellipse}} = \frac{A_{FVP}}{\pi ab}$$

But what is A_{FVP}?

$$A_{FVP} = A_{PSV} - A_{PSF}$$

PSF is a triangle, so

$$A_{PSF} = \frac{1}{2}(ae - a\cos E) \cdot \frac{b}{a}(a \sin E)$$



E is the **Eccentric Anomaly**.

The conversion from E to f (or vice-versa) is not difficult.

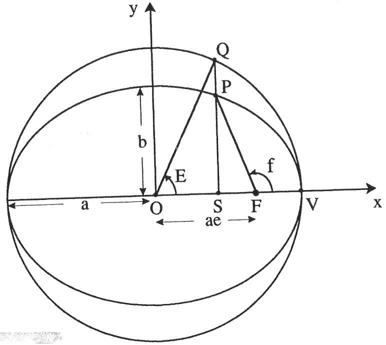
The Ellipse Revisited

It is easy to see by the scaling law that $A_{PSV} = \frac{b}{a} A_{QSV}$. A_{QSV} is easily calculated as

$$\begin{aligned} A_{QSV} &= A_{QOV} - A_{QOS} \\ &= \frac{1}{2}a^2 E - \frac{1}{2} a\cos E \cdot a \sin E \end{aligned}$$

where E **is in radians**. Thus we conclude

$$\begin{aligned} A_{FVP} &= A_{PSV} - A_{PSF} \\ &= \frac{1}{2}ab(E - \cos E \sin E) \\ &\qquad - \frac{1}{2}ab(e - \cos E)\sin E \\ &= \frac{1}{2}ba(E - e\sin E) \end{aligned}$$



Mean Anomaly

The conclusion is that

$$\frac{t}{T} = \frac{A_{FVP}(t)}{\pi ab} = \frac{E(t) - e\sin E(t)}{2\pi}$$

Since by Kepler's third law,

$$T = \sqrt{\frac{4\pi^2 a^3}{\mu}}$$

we have

$$\frac{E(t) - e\sin E(t)}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{\mu}{a^3}}t$$

- Thus we have an expression for t in terms of $E(t)$.
- What we really want is an expression for E in terms of t.
- Unfortunately no such solution exists.
  - Equation must be solved numerically for each value of t.
  - Prompted invention of first known numerical algorithm, Newton's Method.

Mean Anomaly

We define some terms

Definition 1.

The mean motion, n is defined as

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$

Definition 2.

The mean anomaly, $M(t)$ is defined as

$$M(t) = nt = \sqrt{\frac{\mu}{a^3}}t$$

Neither of these have good physical interpretations.

$$M(t) = E(t) - e\sin E(t)$$

Converting Between E and f

Once we get E from solving Kepler's equation, we still need to find the angle f in order to recover position. Going back to the ellipse,

We express the line OS using both E and f.

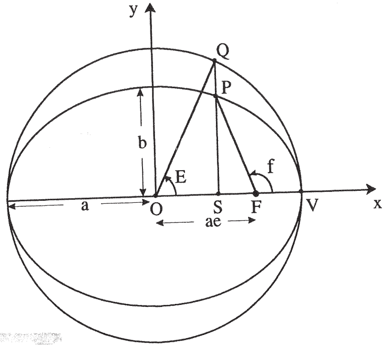
$$OS = a\cos E$$
$$= ae + r\cos f$$

But $r = \frac{a(1-e^2)}{1+e\cos f}$, so

$$\cos E = (1-e^2)\frac{\cos f}{1+e\cos f}$$

Using the half-angle formula, we can get the expression

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}$$



Given f, we can find E.

Converting Between E and f

Alternatively, given E, we can find f.

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

We can also now directly express the orbit equation using E,

$$r(t) = a(1 - e\cos E(t))$$

Example

Problem: Given an orbit with $a = 10,000km$ and $e = .5$, determine the times at which $r = 14,147km$.

Solution: First solve for the true anomaly, f. we have

$$r(t) = \frac{a(1-e^2)}{1 + e\cos f(t)}$$

which yields

$$\cos f(t) = \frac{a(1-e^2) - r(t)}{er(t)} = -.9397$$

Solving for f yields two solutions $f = 160\,\mathrm{deg}, 200\,\mathrm{deg}$.

Now we want to find $E(t)$.

$$\tan\frac{E}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2} = \pm 3.27$$

This yields

$$E = \pm 146.0337\,\mathrm{deg},$$

Example

Solving for mean anomaly (*in radians!!*),

$$M(t) = E(t) - e \sin E(t) = 2.2694rad, 4.0138rad$$

Now the mean motion is

$$n = \sqrt{\frac{\mu}{a^3}} = 6.3135E-4$$

So finally, the times of arrival are

$$t = \frac{M(t)}{n} = 3594s, 6357s$$

Note: In this way, it is easy to find the time between any 2 points in the orbit. e.g. from $f = 160 \deg$ to $f = 200 \deg$ takes time $\Delta t = 6357 - 3594 = 2763s$.

Problem 2

Given t, find r and v

Generally speaking we can follow the previous steps in reverse.

1. Given time, t, solve for Mean Anomaly

$$M(t) = nt$$

2. Given Mean Anomaly, solve for Eccentric Anomaly
   - How???
3. Given eccentric anomaly, solve for true anomaly

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}$$

The Missing Piece is how to solve for Eccentric Anomaly, E given Mean Anomaly, M.

Solving the Kepler Equation

Given M, find E

$$M = E - e \sin E$$

- A Transcendental Equation
- No Closed-Form Solution
- However, for any M, there is a unique E.

To Solve Kepler's Equation, Newton had to redefine the meaning of a solution.

Iterative Methods (Algorithms):

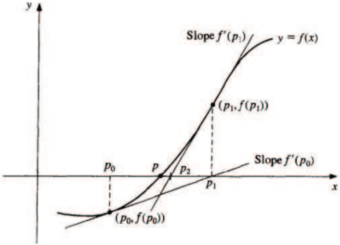
Instead of solving a single equation, we solve a sequence of equations until a stopping criterion (usually error tolerance) is met.

- The solution is never exact.
- Perfect for implementation on computers
- Dramatically increased to set of solvable problems.
- Today, most problems are solved via Algorithms.

Newton-Raphson Iteration

An Algorithm for solving equations

$$f(x) = 0$$



Start by guessing the solution x_k.

- Approximate $f(x) = f(x_k) + f'(x_k)(x - x_k)$.
- Solve $f(x_k) + f'(x_k)(x - x_k) = 0$

$$x = x_k - \frac{f(x_k)}{f'(x_k)}$$

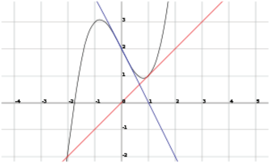
- Update your guess, $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$
- Repeat until $\|f(x_k)\|$ is sufficiently small.

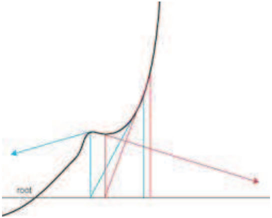
Newton's Method

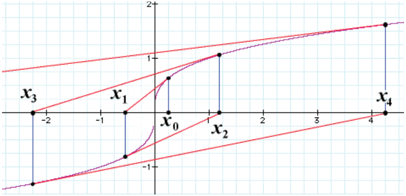
Illustration

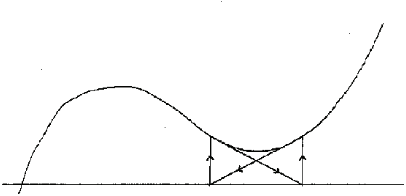
Failure of Newton-Raphson Iteration

When Newton's Method Works, it works well



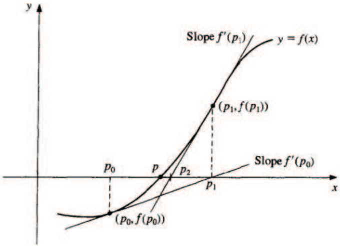






Newton always converges for *convex functions*, ($f''(x) > 0$)

Applied to Kepler's Equation



Given M, we want to solve

$$f(E) = M - E + e\sin E = 0 \qquad \text{then,} \qquad f'(E) = -1 + e\cos E$$

Algorithm: Choose $E_1 = M$.

- Update
$$E_{k+1} = E_k - \frac{M - E_k + e\sin E_k}{e\cos E_k - 1}$$
- If $\|M - E_k + e\sin E_k\| < .001$ or whatever, quit.
- Otherwise repeat.

Example

Problem: Let $a = 25,512km$ and $e = .625$. Find r, v at $t = 4hr$.

Solution: First, solve for Mean Anomaly.

$$n = \sqrt{\frac{\mu}{a^3}} = 1.549E - 4s^{-1}$$

Thus

$$M(t) = nt = 1.549 \cdot 10^{-4} * 4 * 3600 = 2.231rad$$

Newton Iteration: Now to solve for E, we set $E_1 = M$, an iterate

$$E_2 = E_1 - \frac{2.231 - E_1 + .625 \sin E_1}{.625 \cos E_1 - 1} \qquad = 2.588$$

$$f(E_2) = 2.231 - E_2 + .625 \sin E_2 \qquad = -.0284$$

We verify that $\|f(E_2)\| = .0284 > .001$, so continue:

$$E_3 = E_2 - \frac{2.231 - E_2 + .625 \sin E_2}{.625 \cos E_2 - 1} \qquad = 2.570$$

$$f(E_3) = 2.231 - E_3 + .625 \sin E_3 \qquad = -.000892$$

Example

Now $\|f(E_3)\| < .001$, so quit. $E = E_3 = 2.570$. Now Solve for true anomaly

$$f = 2\tan^{-1}\left(\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}\right) = 2.861rad$$

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)} = 38920km$$

Now via vis-viva,

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = 2.2043km/s$$

Conclusion

In this Lecture, you learned:

- How to predict position given time.
- New Angles
  - Mean Anomaly
  - Eccentric Anomaly
  - True Anomaly
- How to convert between them
  - How to Solve Kepler's Equation

Key Equations:

$$n = \sqrt{\frac{\mu}{a^3}}$$

$$M(t) = nt$$

$$M(t) = E(t) - e \sin E(t)$$

$$\tan \frac{E}{2} = \sqrt{\frac{1-e}{1+e}} \tan \frac{f}{2}$$

Newton Iteration:

$$E_0 = M$$

$$E_{k+1} = E_k - \frac{M - E_k + e \sin E_k}{e \cos E_k - 1}$$