Systems Analysis and Control

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Lecture 5: Calculating the Laplace Transform of a Signal

Introduction

In this Lecture, you will learn:

Laplace Transform of Simple Signals.

- Step, Exponentials, Ramps.
- Impulse

Properties of the Laplace Transform.

- delay, scaling
- convolution
- etc.

Transfer Functions

- State Space to Transfer Function

Definition 1.

The **Laplace Transform** of the signal $u(t)$ is

$$\hat{u}(s) = \int_0^\infty e^{-st} u(t) dt$$

The Laplace Transform is not the same as the Fourier transform.

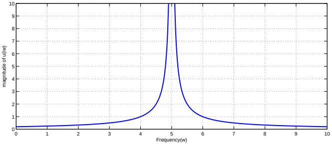
- Assumes that signals begin at time $t = 0$ and $u(t) = 0$ for $t < 0$.
- Tends to flatten out peaks.

Signal:

$$u(t) = \sin at$$

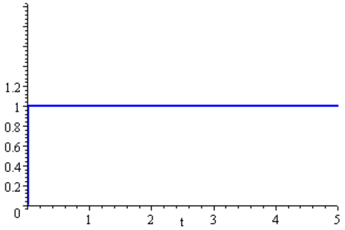
Laplace Transform:

$$\hat{u}(s) = \frac{1}{s-a}$$



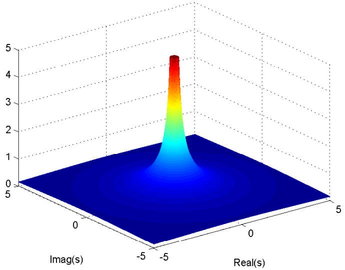
Consider the input signal:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



The Laplace transform is

$$\begin{aligned} \hat{u}(s) &= \int_0^\infty e^{-st} dt \\ &= \frac{1}{-s} \left[e^{-st} \right]_0^\infty = \frac{1}{-s} [0 - 1] \\ &= \frac{1}{s} \end{aligned}$$



Note that $\lim_{t \to \infty} e^{-st} = 0$.

- Because we assume the real part of s is negative.

Review: Integration by parts

A very useful formula for finding the Laplace Transform is

- integration by parts.

$$\int_a^b u(s)v'(s)ds = [u(s)v(s)]_{s=a}^{s=b} - \int_a^b u'(s)v(s)ds$$

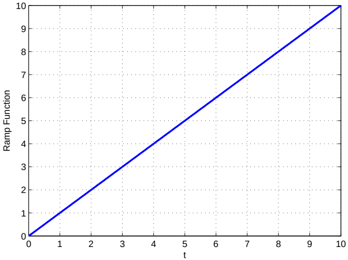
Consider the input signal:

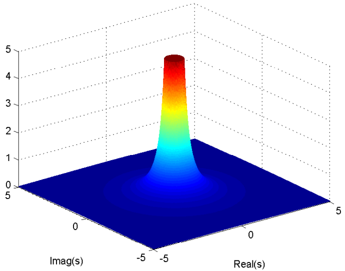
$$u(t) = \begin{cases} 0 & t < 0 \\ t & t \geq 0 \end{cases}$$
$$= t \cdot \mathbf{1}(t)$$

Recall $\mathbf{1}(t)$ is the step function.

The Laplace transform is

$$\begin{aligned} \hat{u}(s) &= \int_0^\infty te^{-st}dt \\ &= \left[\frac{1}{-s}te^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\ &= [0-0] - \frac{1}{s^2}[e^{-st}]_0^\infty dt \\ &= -\frac{1}{s^2}[0-1] = \frac{1}{s^2} \end{aligned}$$





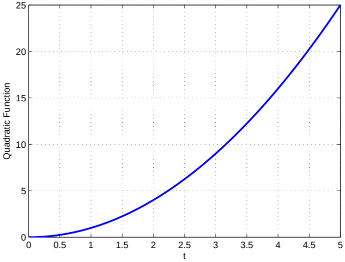
The Quadratic is similar to the Ramp.

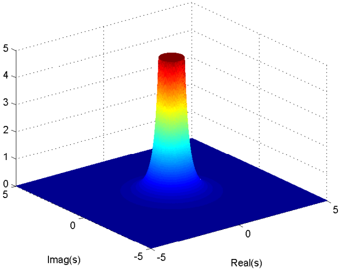
$$u(t) = \begin{cases} 0 & t < 0 \\ t^2 & t \ge 0 \end{cases}$$
$$= t^2 \mathbf{1}(t)$$

Recall $\mathbf{1}(t)$ is the step function.

The Laplace transform is

$$\hat{u}(s) = \int_0^\infty t^2 e^{-st} dt$$
$$= \left[\frac{1}{-s} t^2 e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty t e^{-st} dt$$
$$= \frac{1}{s^3}$$



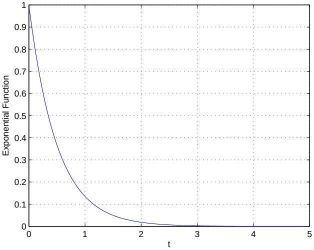


The Sinusoid is created from the complex exponentials.

- Lets do exponentials first.

Consider the input signal:

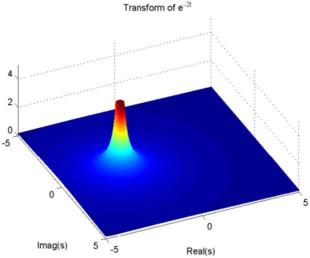
$$u(t) = \begin{cases} 0 & t < 0 \\ e^{at} & t \geq 0 \end{cases}$$
$$= e^{at}\mathbf{1}(t)$$



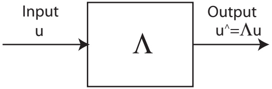
Recall $\mathbf{1}(t)$ is the step function.

The Laplace transform is

$$\hat{u}(s) = \int_0^\infty e^{(a-s)t}dt$$
$$= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^\infty$$
$$= \frac{1}{s-a}$$



The Laplace Transform
Linearity



The Laplace Transform is itself a LINEAR SYSTEM.

- The *Input* is u
- The *Output* is $\hat{u}$
- Lets call the system $\mathcal{L}$

$$\hat{u} = \mathcal{L}u$$

- As for any linear system,

$$\begin{aligned}\mathcal{L}(\alpha u_1 + \beta u_2) &= \alpha \mathcal{L}u_1 + \beta \mathcal{L}u_2 \\ &= \alpha \hat{u}_1 + \beta \hat{u}_2\end{aligned}$$

We can compute the Laplace transform by using simple parts.
Example: Sinusoids

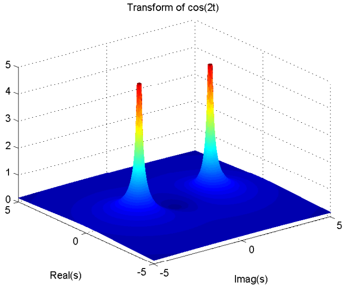
$$\cos \omega t = \frac{e^{\imath \omega t} + e^{-\imath \omega t}}{2}$$

Recall that

$$\cos \omega t = \frac{e^{\imath\omega t} + e^{-\imath\omega t}}{2}$$

But the Laplace transform of e^{at} is

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$



So the Laplace transforms of $e^{\imath\omega t}$ and $e^{-\imath\omega t}$ are

$$\hat{u}_1(s) = \mathcal{L}(e^{\imath\omega t}) = \frac{1}{s-\imath\omega} \qquad \text{and} \qquad \hat{u}_2(s) = \mathcal{L}(e^{-\imath\omega t}) = \frac{1}{s+\imath\omega}$$

Which shows us how to find $\mathcal{L}(\cos(\omega t))$

$$\mathcal{L}(\cos(\omega t)) = \mathcal{L}\left(\frac{e^{\imath\omega t} + e^{-\imath\omega t}}{2}\right) = \mathcal{L}\left(\frac{e^{\imath\omega t}}{2}\right) + \mathcal{L}\left(\frac{e^{-\imath\omega t}}{2}\right) = \frac{\hat{u}_1}{2} + \frac{\hat{u}_2}{2}$$

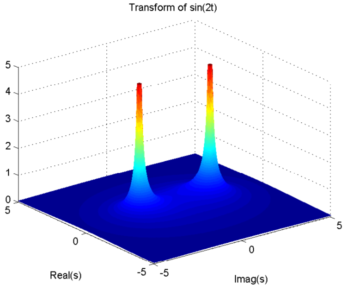
$$= \frac{1}{2}\left(\frac{1}{s-\imath\omega} + \frac{1}{s+\imath\omega}\right) = \frac{1}{2}\left(\frac{2s}{(s-\imath\omega)(s+\imath\omega)}\right) = \frac{s}{s^2+\omega^2}$$

The Laplace Transform

Example: Sinusoids

Likewise, for the $\sin$ function,

$$\sin \omega t = \frac{e^{\imath \omega t} - e^{-\imath \omega t}}{2i}$$



Which shows us how to find $\mathcal{L}(\cos(\omega t))$

$$\mathcal{L}(\sin(\omega t)) = \frac{\hat{u}_1}{2} - \frac{\hat{u}_2}{2i}$$

$$= \frac{1}{2i}\left(\frac{1}{s - \imath\omega} - \frac{1}{s + \imath\omega}\right)$$

$$= \frac{1}{2i}\left(\frac{2\imath\omega}{(s - \imath\omega)(s + \imath\omega)}\right) = \frac{\omega}{s^2 + \omega^2}$$

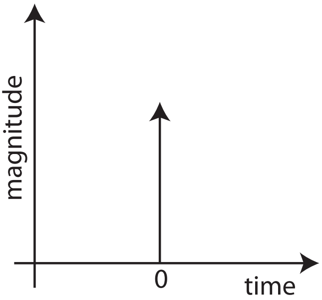
Definition 2.

An **Impulse** is a significant strike which occurs in an infinitesimally short time. Mathematically, we model the impulse as a Dirac delta function, $\delta(t)$.

Question: What is the Laplace transform of an *Impulse*?

Answer: By definition of the Dirac Delta:

$$\begin{aligned}\hat{\delta}(s) &= \int_0^\infty e^{-st}\delta(t)dt \\ &= e^{-st}|_{t=0} \\ &= 1\end{aligned}$$



- The Impulse has a Uniform Frequency Response!!!!
- The opposite of the step function ($\mathbf{1}(s) = \mathcal{L}\delta(t)$).

Laplace Transform Table

Simple Signals

Summary: We can find the Laplace Transform for many simple signals

	$u(t)$	$\hat{u}(s)$
Step	$\mathbf{1}(t)$	$\frac{1}{s}$
Power	t^m	$\frac{m!}{s^{m+1}}$
Exponential	e^{at}	$\frac{1}{s+a}$
Power Exponential	$\frac{t^{m-1}e^{-at}}{(m-1)!}$	$\frac{1}{(s+a)^m}$
Sine	$\sin(at)$	$\frac{a}{s^2+a^2}$
Cosine	$\cos(at)$	$\frac{s}{s^2+a^2}$
Impulse	$\delta(t)$	1

Remember that all functions are only for $t \geq 0$.

- We always have $u(t) = 0$ for $t < 0$

Question: How to use these functions to solve for more complex functions?

Properties: What properties of the Laplace transform can we exploit?

- We have already discussed *Linearity*.

Other Examples:

- **Delay:** The signal is delayed

$$u(t) \to u(t-\tau)$$

- **Time-Scale:** The signal is slowed down or speeded up

$$u(t) \to u(at)$$

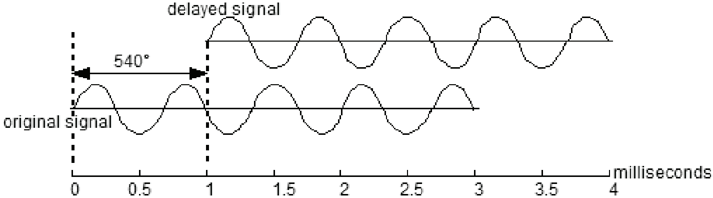
- **Differentiation:** The signal is a differential in time:

$$u(t) \to u'(t)$$

How do these changes affect the the Laplace transform?

$$\hat{u}(s) \to ?????$$

What is the effect of a delay on the Laplace Transform?



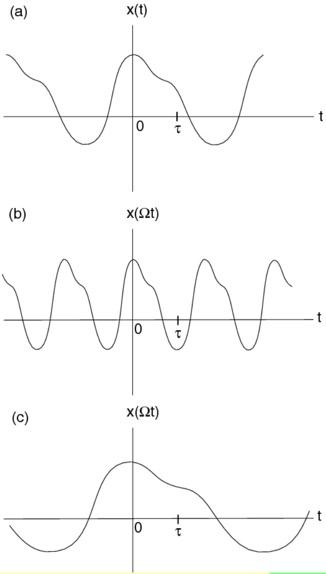
Use a change of variables: $t' = t - \tau$, $dt' = dt$

$$\begin{aligned}
\hat{u}_{delayed}(s) = \mathcal{L}(u(t-\tau)) &= \int_0^\infty e^{-st} u(t-\tau) dt \\
&= \int_0^\infty e^{-s(t'+\tau)} u(t') dt' \\
&= e^{-s\tau} \int_0^\infty e^{-st'} u(t') dt' \\
&= e^{-s\tau} \hat{u}(s)
\end{aligned}$$

What is the effect of scaling time?

Use a change of variables: $t' = at$, $dt' = adt$

$$\begin{aligned}\hat{u}_{scaled}(s) &= \mathcal{L}(u(at)) \\ &= \int_0^\infty e^{-st} u(at) dt \\ &= \frac{1}{a}\int_0^\infty e^{-\frac{st'}{a}} u(t') dt' \\ &= \frac{1}{a}\hat{u}\left(\frac{s}{a}\right)\end{aligned}$$



Example: Watch The Order of Operations

What is the Laplace Transform of

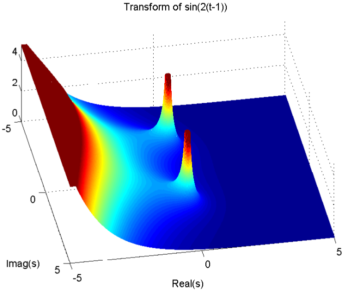
$$u(t) = \sin(a(t-\tau))$$

We know that $\hat{u}_{\sin}(s) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$.

- The change is $\sin t \to \sin(at) \to \sin(a(t-\tau))$
- **Time-Scaling**: $\hat{u}_s(s) = \frac{1}{a}\hat{u}\left(\frac{s}{a}\right)$
- **Time-Delay**: $\hat{u}_d(s) = e^{-\tau s}\hat{u}\left(s\right)$

Thus we have

$$\begin{aligned}\hat{u}_{ds}(s) &= e^{-\tau s}\hat{u}_s(s)\\ &= e^{-\tau s}\left(\frac{\frac{1}{a}}{\left(\frac{s}{a}\right)^2+1}\right)\\ &= e^{-\tau s}\frac{a}{s^2+a^2}\end{aligned}$$



A very common case is when a signal has been differentiated. e.g.:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

In this case we use integration by parts:

$$\begin{aligned}
\mathcal{L}\left(\dot{x}(t)\right) &= \int_0^\infty e^{-st}\dot{x}(t)dt \\
&= \left[e^{-st}\dot{x}(t)\right]_0^\infty + s\int_0^\infty e^{-st}x(t)dt \\
&= -\dot{x}(0) + s\int_0^\infty e^{-st}x(t)dt \\
&= -\dot{x}(0) + s\hat{x}(s)
\end{aligned}$$

The differentiation property can be applied recursively:

$$\begin{aligned}\mathcal{L}\left(\dddot{x}(t)\right) &= -\ddot{x}(0) + s\mathcal{L}(\ddot{x}(t)) \\ &= -\ddot{x}(0) - s\dot{x}(0) + s^2\mathcal{L}(\dot{x}(t)) \\ &= -\ddot{x}(0) - s\dot{x}(0) - s^2 x(0) + s^3\hat{x}(s)\end{aligned}$$

General Formula:

$$\mathcal{L}(x^{(n)}(t)) = s^n\hat{x}(s) - s^{n-1}x(0) - \cdots - x^{n-1}(0)$$

The Laplace Transform Properties
Integration

Importantly, the Laplace transform can also be applied to integration:

- An immediate consequence of differentiation

$$\mathcal{L}\left(\dot{f}(t)\right) = -f(0) + s\mathcal{L}(f(t))$$

Let $f(t) = \int_0^t x(\tau)d\tau$.

- By definition, $\dot{f}(t) = x(t)$, and $f(0) = 0$, so

$$\begin{aligned}\hat{x}(s) = \mathcal{L}(x(t)) &= \mathcal{L}\left(\dot{f}(t)\right)\\ &= -f(0) + s\mathcal{L}(f(t))\\ &= s\mathcal{L}\left(\int_0^t x(\tau)d\tau\right)\end{aligned}$$

- Hence

$$\mathcal{L}\left(\int_0^t x(\tau)d\tau\right) = \frac{\hat{x}(s)}{s}$$

The integration property is important in finding the Laplace transform of the output.

Recall the solution of a state-space system:

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

This is a special kind of integral called the *Convolution Integral*.

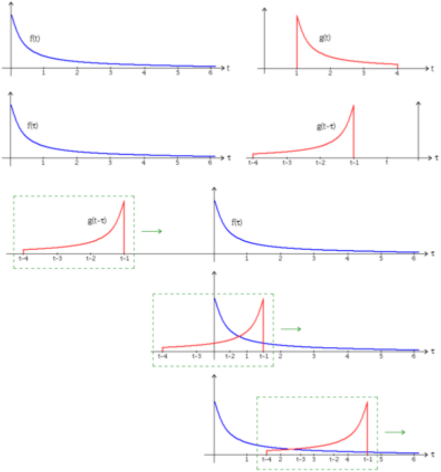
Definition 3.

The **Convolution Integral** of $u(t)$ with $v(t)$ is defined as

$$(u * v)(t) := \int_0^t u(\tau)v(t-\tau)d\tau$$

Thus roughly speaking

$$y(t) = u(t) * \left(Ce^{At}B\right)$$



$$(f * g)(t) := \int_0^t f(\tau)g(t-\tau)d\tau$$

The Laplace Transform Properties

Convolution

The convolution of two signals has a remarkable property:

$$\mathcal{L}(u * v) = \hat{u}(s)\hat{v}(s)$$

Proof.

Since $u(t) = v(t) = 0$ for $t < 0$, we have that $u(\tau)v(t-\tau) = 0$ for $\tau < 0$ or $\tau > t$. Thus:

$$\int_0^t u(\tau)v(t-\tau)d\tau = \int_0^\infty u(\tau)v(t-\tau)d\tau$$

Now using the delay property:

$$\begin{aligned}
\mathcal{L}(u * v) &= \int_0^\infty e^{-st}\int_0^t u(\tau)v(t-\tau)d\tau dt = \int_0^\infty e^{-st}\int_0^\infty u(\tau)v(t-\tau)d\tau dt \\
&= \int_0^\infty u(\tau)\int_0^\infty e^{-st}v(t-\tau)dtd\tau \\
&= \int_0^\infty u(\tau)e^{-s\tau}\hat{v}(s)d\tau = \hat{u}(s)\hat{v}(s)
\end{aligned}$$

$\square$

Summary: Simple functions and properties can be combined to calculate the Laplace Transform.

	$u(t)$	$\hat{u}(s)$
Delay	$u(t-\tau)$	$e^{-\tau s}\hat{u}(s)$
Time-Scaling	$u(at)$	$\frac{1}{a}\hat{u}\left(\frac{s}{a}\right)$
Differentiation	$u'(t)$	$s\hat{u}(s)-u(0)$
Integration	$\int_0^t u(\tau)d\tau$	$\frac{1}{s}\hat{u}(s)$
Frequency Differentiation	$tu(t)$	$\frac{d}{ds}\hat{u}(s)$
Frequency Shift	$e^{at}u(t)$	$\hat{u}(s-a)$
Convolution	$\int_0^t u(\tau)v(t-\tau)d\tau$	$\hat{u}(s)\hat{v}(s)$

Consider the simple case of a cosine with an exponential:

$$u(t) = e^{\sigma t} \sin \omega t$$

Approach:

- $\mathcal{L}(\cos \omega t) = \frac{\omega}{s^2+\omega^2}$
- Exponential means frequency shift: $e^{\sigma t}u(t) \mapsto \hat{u}(s-\sigma)$.

Hence the Laplace transform is

$$\hat{u}(s) = \frac{\omega}{(s-\sigma)^2 + \omega^2} = \frac{\omega}{s^2 - 2\sigma s + (\sigma^2 + \omega^2)}$$

Problem: Can we go the other direction? Calculate $u(t)$ if

$$\hat{u}(s) = \frac{1}{s^2 + bs + c}$$

Solution: Use a combination sinusoid and frequency shift First recall that

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2}$$

An Important Case, continued

If $\hat{u}(s) = \frac{1}{s^2+bs+c}$, the roots of the denominator are

$$s_{1,2} = -b/2 \pm \frac{1}{2}\sqrt{b^2 - 4c}$$

Hence

$$\begin{aligned}\hat{u}(s) &= \frac{1}{(s + b/2 + \frac{1}{2}\sqrt{b^2-4c})(s + b/2 - \frac{1}{2}\sqrt{b^2-4c})} \\ &= \frac{1}{((s + b/2) + \frac{1}{2}\sqrt{b^2-4c})((s + b/2) - \frac{1}{2}\sqrt{b^2-4c})} \\ &= \frac{1}{((s+b/2)^2 - \frac{1}{4}(b^2-4c))}\end{aligned}$$

Now assume $4c > b^2$. Then $\hat{u}(s) = \hat{h}(s + b/2)$, where

$$\hat{h}(s) = \frac{1}{s^2 + \frac{1}{4}(4c - b^2)}.$$

Numerical Example

$$\hat{h}(s) = \frac{1}{s^2 + \frac{1}{4}(4c - b^2)}.$$

This is a simple cosine function:

$$h(t) = \frac{1}{\omega}\sin(\omega t)$$

where $\omega = \sqrt{c - \frac{b^2}{4}}$

- Recall the frequency-shift property: $\mathcal{L}^{-1}\hat{u}(s) = \mathcal{L}^{-1}\hat{h}(s + \frac{b}{2}) = e^{-\frac{b}{2}t}h(t)$.
- We conclude that

$$u(t) = \frac{1}{\sqrt{4c - \frac{b^2}{4}}} e^{-\frac{b}{2}t} \sin\left(\sqrt{4c - \frac{b^2}{4}} t\right)$$

Numerical Example: Let $\hat{u}(s) = \frac{1}{s^2+2s+2}$. Taking the roots, we find

$$\hat{u}(s) = \frac{1}{(s+1)^2 + 1}.$$

This is a sine function with a frequency shift of 1. Therefore

$$u(t) = e^{-t}\sin t$$

The Laplace Transform of a State-Space System

Transfer Functions

Recall the State-Space System

$$\dot{x} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \qquad x(0) = 0$$

Applying the Laplace Transform to the first signal:

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$

where we used

- Linearity
- Differentiation Property

Let $\dot{x}(0) = 0$ and we have

$$(sI - A)\hat{x}(s) = B\hat{u}(s) \qquad \text{or} \qquad \hat{x}(s) = (sI - A)^{-1}B\hat{u}(s)$$

Applying the Laplace Transform to the output equation,

$$y(t) = Cx(t) + Du(t)$$

we get

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

Solving for $\hat{y}(s)$, we get

$$\hat{y}(s) = \left(C(sI - A)^{-1}B + D\right)\hat{u}(s)$$

Thus we have a Transfer Function for the system which can be used to construct a solution for any input using the frequency-domain representation.

Theorem 4.

The **Transfer Function** *for a state-space system is*

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

Summary

What have we learned today?

Laplace Transform of Simple Signals.

- Step, Exponentials, Ramps.
- Impulse

Properties of the Laplace Transform.

- delay, scaling
- convolution
- etc.

Transfer Functions

- State Space to Transfer Function

Next Lecture: More on Transfer Functions