Spacecraft Dynamics and Control

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Lecture 10: Rendezvous and Targeting - Lambert's Problem

Introduction

In this Lecture, you will learn:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Numerical Problem: Suppose we are in an equatorial parking orbit or radius r. Given a target with position $\vec{r}$ and velocity $\vec{v}$, calculate the Δv required to intercept the target before it reaches the surface of the earth.

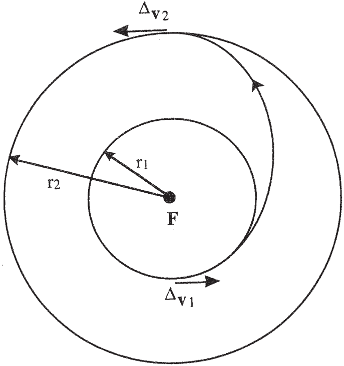
Problems we Have Solved

How to change our orbit to a desired one.

- Raise orbit
- Inclination change

Rendezvous?

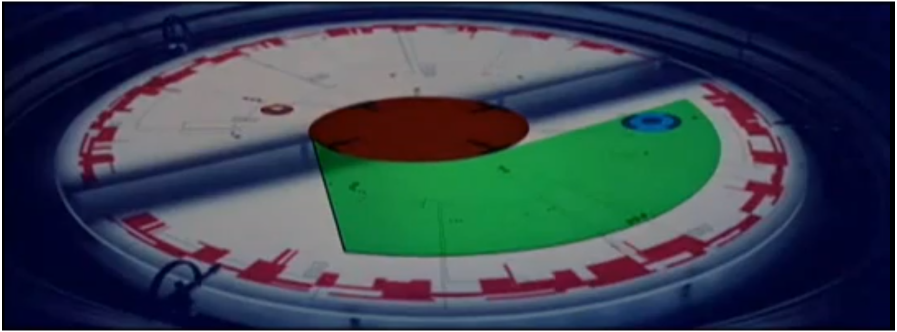
- OK, when mission is not time-sensitive.
- Must use phasing...



We need $\Delta\theta = 2\pi \frac{T_{hohman}}{T_{outer}}$, where $\dot{\theta} = n_{inner} - n_{outer}$.

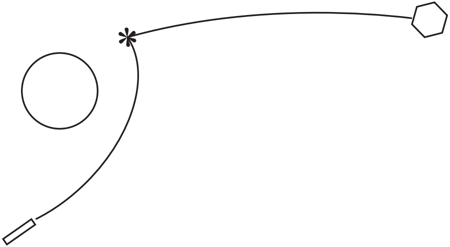
The Problem with phasing

Problem: We have to wait.



Remember what happened to the Death star?

Asteroid Interception



Suppose that:

- Our time to intercept is limited.
- The target trajectory is known.

Problem: Design an orbit starting from $\vec{r}_0$ which intersects the orbit of the asteroid at the same time as the asteroid.

- *Before* the asteroid intersects the earth (when $r(t) = 6378$)

Missile Defense

Problem: ICBM's have re-entry speeds in excess of 8km/s (Mach 26).

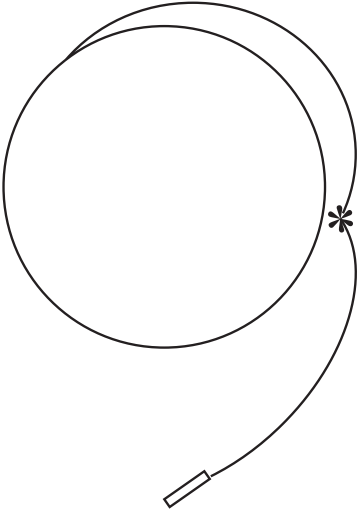
- Patriot missiles can achieve max of Mach 5.

Objective: Intercept ballistic trajectory before missile re-entry

- *Before* the missile intersects the atmosphere
- When $r(t) = 6378km + \cong 200km$

Complications:

- Plane changes may be required.
- The required time-to-intercept may be small.
  - Hohman transfer is not possible



The Targeting Problem

Step 1: Determine the orbit of the target

Step 1 can be accomplished one of two ways:

Method 1:

1. Given $\vec{r}(t_0)$ and $\vec{v}(t_0)$, find $a, e, i, \omega_p, \Omega$ and $f(t_0)$
   - we have covered this approach in Lecture 6.
2. Unfortunately, it is difficult to measure $\vec{v}$

Method 2:

1. Given two observations $\vec{r}(t_1)$ and $\vec{r}(t_2)$, find $a, e, i, \omega_p, \Omega$ and $f(t_0)$.
   - Alternatively, find $\vec{v}(t_1)$ and $\vec{v}(t_2)$
2. This is referred to as Lambert's problem (the topic of this lecture)

Note: This is a *boundary-value* problem:

- We know some states at two points.
- In contrast to the *initial value* problem, where we know all states at the initial time.
- Unlike initial-value problems, boundary-value problems cannot always be solved.

Carl Friedrich Gauss (1777-1855)

The Problem of orbit determination was originally solved by C. F. Gauss

Mathematician First

- Astronomer Second

Boring/Conservative.

Considered by many as one of the greatest mathematicians

- Professor of Astronomy in Göttingen
- He was OK.

Discovered

- Gaussian Distributions
- Gauss' Law

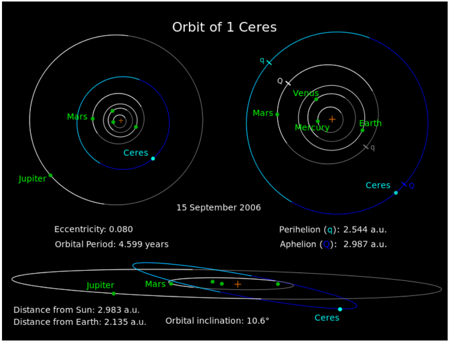
Discovery and Rediscovery of Ceres

The pseudo-planet Ceres was discovered by G. Piazzi

- Observed 12 times between Jan. 1 and Feb. 11, 1801
- Planet was then lost.

Complication:

- Observation was only declination and right-ascension.
- Observations were only spread over 1% of the orbit.
    - No ranging info.
- For this case, three observations are needed.



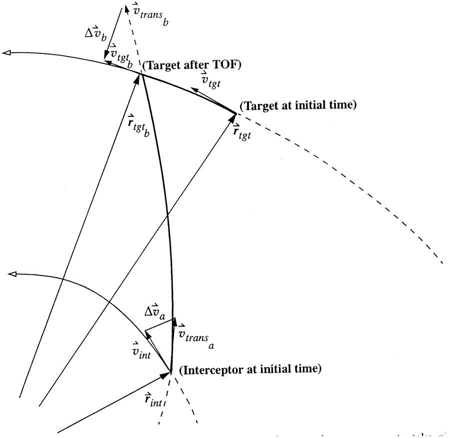
C. F. Gauss solved the orbit determination problem and correctly predicted the location.

- Planet was re-found on Dec 31, 1801 in the correct location.

Once we have found the orbit of the target, we can determine where the target will be at the desired time of impact, t_f.

Procedure:

- The difference $t_f - t_0$ is the Time of Flight (TOF)
- Calculate $M(t_f) = M(t_0) + n(t_f - t_0)$
- Use $M(t_f)$ to find $E(t_f)$.
- Use $E(t_f)$ to find $f(t_f)$.
- Use $f(t_f)$ to find $\vec{r}(t_f)$.



For a given

- Initial Position, $\vec{r}_1$
- Final Position, $\vec{r}_2$
- Time of Flight, TOF

the transfer orbit is uniquely determined.

Challenge: Find that orbit!!!

Difficulties:

- Where is the second focus?
- May require initial plane-change.
- May use LOTS of fuel.

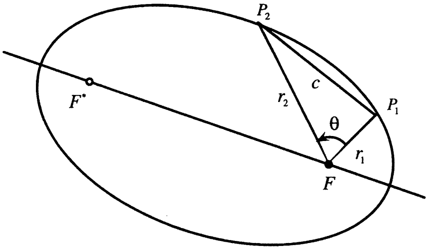


Figure: For given P_1 and P_2 and TOF, the transfer ellipse is uniquely determined.

On the Plus Side:

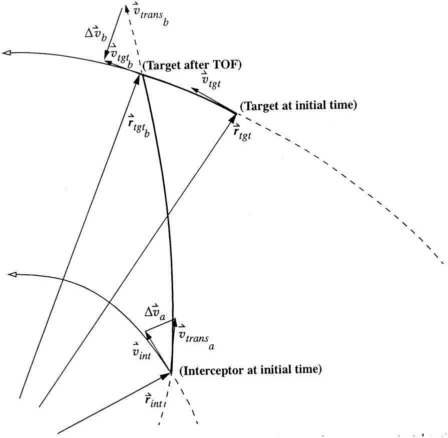
- We know the change in true anomaly, Δf...
- For this geometry, TOF only depends on a.

The Targeting Problem

Step 4: Calculate the Δv

Once we have found the transfer orbit,

- Calculate $\vec{v}_{tr}(t_0)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_0)$
  - If a ground-launch, use rotation of the earth.
  - If in orbit, use orbital elements.
- Calculate $\Delta v = \vec{v}_{tr}(t_0) - \vec{v}(t_0)$



Finding the transfer orbit (The Hard Part)

Semi-major axis, a and Focus

The difficult part is to determine a.

- TOF only depends on a

The value of a dictates the location of the focus.

Focal Circle 1: For given $\vec{r}_i$ and a,

- The set of potential locations for the second focus is a circle.
  - Let $r_1 = \|\vec{r}\|$ be the distance to the known focus.
  - Then $r_{f1} = 2a - r_1$ is the distance to the unknown focus.
- The unknown focus lies on a circle of radius r_{f1} around our current position.

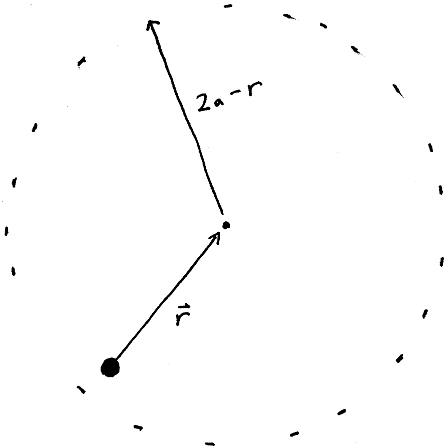


Figure: Potential Locations of Second Focus

Finding Semi-major axis, a and Focus

Focal Circle 2: The same geometry holds for the final position vector, $\vec{r}_f$.

- The set of potential locations for the second focus is a circle.
  - Let $r_2 = \|\vec{r}_f\|$ be the distance to the known focus.
  - Then $r_{f2} = 2a - r_2$ is the distance to the unknown focus.
- The unknown focus lies on a circle of radius r_{f2} around the target position.

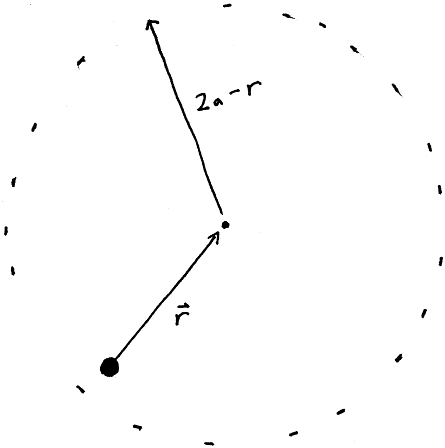


Figure: Potential Locations of Second Focus using Final Destination

Finding Semi-major axis, a and Focus

Geometry: By intersecting the two circles about $\vec{r}_i$ and $\vec{r}_f$,

- We find two potential locations for the focus

We can discard the more distant focus.

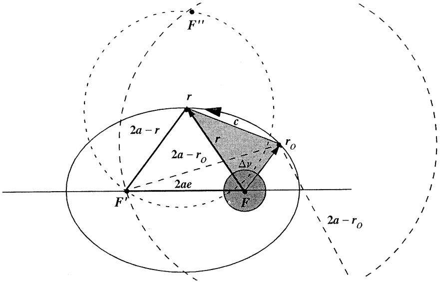


Figure: Potential Locations of Second Focus for a given a

Finding Semi-major axis, a and Focus

Note: As we vary a, the set of foci form a *hyperbola*.

Conclusions

- There is a minimum-energy transfer corresponding to minimum a.
  - $a_{\min} = \frac{r_0+r+c}{4}$
  - Focus lies on the line between $\vec{r}_1$ and $\vec{r}_2$.
  - But we don't want a *minimum energy* transfer!
  - Still, its nice to have a lower bound.

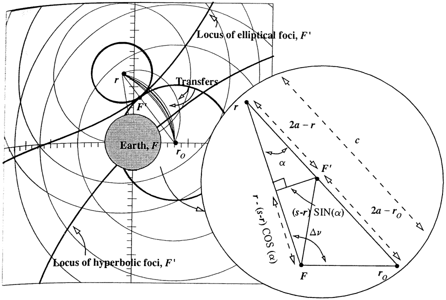


Figure: Potential Locations of Second Focus for a given a

Question: How to use TOF to find a?

We know

- $\theta = f(t_f) - f(t_0)$ - change in true anomaly.
- $\Delta t = TOF$

How to find a?

Kepler's Equation:

$$\sqrt{\frac{\mu}{a^3}}\Delta t = E(t_f) - E(t_0) - e(\sin E(t_f) - \sin E(t_0))$$

Problem: We know Δf, but not ΔE!

- E depends on e as well as f and a
- We don't know e

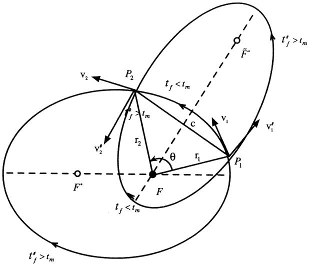


Figure: Geometry of the Problem

First: Calculate some lengths

- $c = \|\vec{r}_1 - \vec{r}_2\|$ is the *chord*.
- $s = \frac{c+r_1+r_2}{2}$ is the *semi-perimeter*.
  - NOT semiparameter.

Then we get **Lambert's Equation:**

$$\Delta t = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

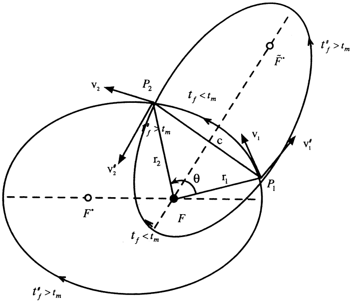


Figure: Geometry of the Problem

Conclusion: We can express TOF, solely as a function of a.

- albeit through a complicated function.

There are several ways to solve Lambert's Equation

- Newton Iteration
  - More Complicated than Kepler's Equation
- Series Expansion
  - Probably the easiest...
- Bisection
  - Relatively Slow, but easy to understand

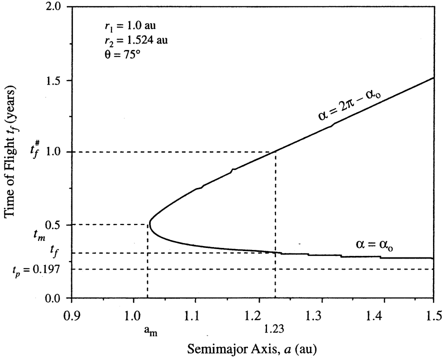


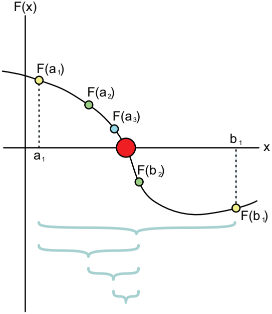
Figure: Geometry of the Problem

Define $g(a) = \sqrt{\frac{a^3}{\mu}}\left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$.

Root-Finding Problem:

Find a :

such that $g(a) = \Delta t$

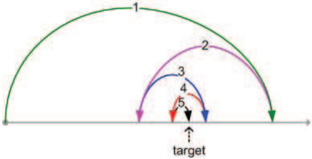


Bisection Algorithm:

1. Choose $a_{\min} = \frac{s}{2} = \frac{r_0+r+c}{4}$
2. Choose $a_{\max} >> a_{\min}$
3. Set $a = \frac{a_{\max}+a_{\min}}{2}$
4. If $g(a) > \Delta t$, set $a_{\min} = a$
5. If $g(a) < \Delta t$, set $a_{\max} = a$
6. Goto 3

This is guaranteed to converge to the unique solution (if it exists).

- We assume *Elliptic* solutions.



Bisection

Some Implementation Notes

Make Sure a Solution Exists!!

- First calculate the minimum TOF.
- This corresponds to a parabolic trajectory

$$\Delta t_p = \frac{2}{3}\sqrt{\frac{s^3}{\mu}}\left(1-\left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right)$$

  - Can get there even faster by using a hyperbolic approach (Not Covered).
- One might also calculate the Maximum TOF

$$\Delta t_{\max} = \sqrt{\frac{a_{\min}^3}{\mu}}\left(\alpha_{\max}-\beta_{\max}-\left(\sin\alpha_{\max}-\sin\beta_{\max}\right)\right)$$

where

$$\sin\left[\frac{\alpha_{\max}}{2}\right] = \sqrt{\frac{s}{2a_{\min}}}, \qquad \sin\left[\frac{\beta_{\max}}{2}\right] = \sqrt{\frac{s-c}{2a_{\min}}}$$

- One can exceed this by going the long way around (Not Covered Here)

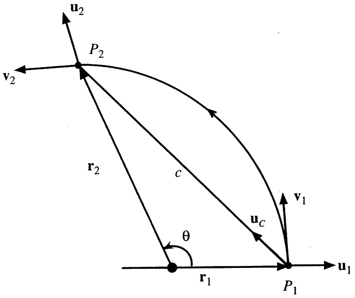
Calculating $\vec{v}(t_0)$ and $\vec{v}(t_f)$

Once we have a, calculating $\vec{v}$ is not difficult.

$$\vec{v}(t_0) = (B + A)\vec{u}_c + (B - A)\vec{u}_1, \qquad \vec{v}(t_f) = (B + A)\vec{u}_c - (B - A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\alpha}{2}\right), \qquad B = \sqrt{\frac{\mu}{4a}} \cot\left(\frac{\beta}{2}\right)$$



and the unit vectors

- $\vec{u}_1$ and $\vec{u}_2$ point to positions 1 and 2.

$$\vec{u}_1 = \frac{\vec{r}(t_0)}{r_1}, \qquad \vec{u}_2 = \frac{\vec{r}(t_f)}{r_2}$$

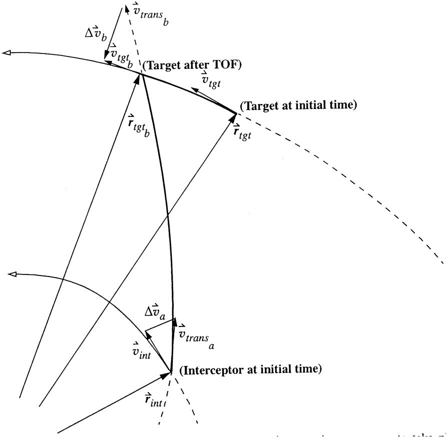
- $\vec{u}_c$ points from position 1 to 2.

$$\vec{u}_c = \frac{\vec{r}(t_f) - \vec{r}(t_0)}{c}$$

Calculating Δv

Once we have found the transfer orbit,

- Calculate $\vec{v}_{tr}(t_0)$ of the transfer orbit.
- Calculate our current velocity, $\vec{v}(t_0)$
  - If a ground-launch, use rotation of the earth.
  - If in orbit, use orbital elements.
- Calculate $\Delta v = \vec{v}_{tr}(t_0) - \vec{v}(t_0)$



Numerical Example of Missile Targeting

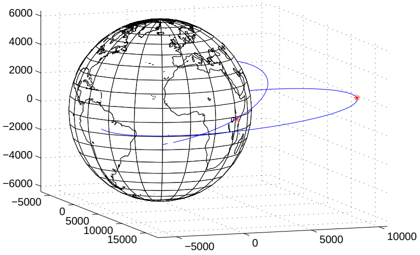
Problem: Suppose that Brasil launches an ICBM at Bangkok, Thailand.

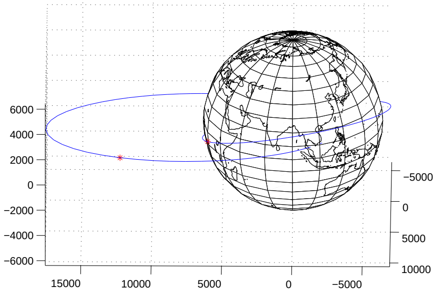
- We have an interceptor in the air with position and velocity

$$\vec{r} = \begin{bmatrix} 6045 & 3490 & 0 \end{bmatrix} km \qquad \vec{v} = \begin{bmatrix} -2.457 & 6.618 & 2.533 \end{bmatrix} km/s.$$

- We have tracked the missile at $r_t = \begin{bmatrix} 12214.839 & 10249.467 & 2000 \end{bmatrix} km$ heading $\vec{v} = \begin{bmatrix} -3.448 & .924 & 0 \end{bmatrix} km/s$.

Question: Determine the Δv required to intercept the missile before re-entry, which occurs in 30 minutes.



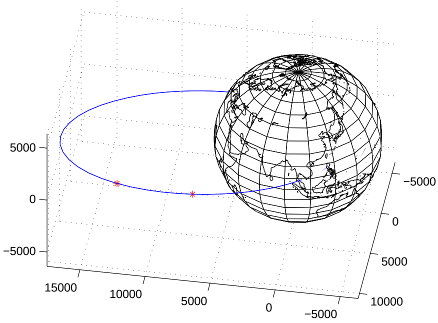


Numerical Example of Missile Targeting

The first step is to determine the position of the ICBM in $t + 30$min.

Recall: To propagate an orbit in time:

1. Use $\vec{r}_t$ and $\vec{v}_t$ to find the orbital elements, including $M(t_0)$.
2. Propagate Mean anomaly $M(t_f) = M(t_0) + n\Delta t$ where $\Delta t = 1800s$.
3. Use $M(t_f)$ to find true anomaly, $f(t_f)$.
   - Requires iteration to solve Kepler's Equation.
4. Use the orbital elements, including $f(t_f)$ to find $\vec{r}(t_f)$



Numerical Example of Missile Targeting

The next step is to determine whether an intercept orbit is feasible using TOF$=30$min.

Geometry of the Problem:

$$r_1 = \|\vec{r}\| = 6,980km, \quad r_2 = \|\vec{r}_t(t_f)\| = 12,282km,$$

$$c = \|\vec{r} - \vec{r}_t(t_f)\| = 7,080km, \quad s = \frac{c + r_1 + r_2}{2} = 13,171km$$

Minimum Flight Time: Using the formula, the minimum (parabolic) flight time is

$$t_{\min} = t_p = \frac{2}{3}\sqrt{\frac{s^3}{\mu}}\left(1 - \left(\frac{s-c}{s}\right)^{\frac{3}{2}}\right) = 18.2min$$

Thus we have more than enough time.

Maximum Flight Time: Geometry yields a minimum semi-major axis of

$$a_{\min} = \frac{s}{2} = 6,586km$$

Plugging this into Lambert's equation yields a maximum flight time of $t_{\max} = 37.3$min.

Numerical Example of Missile Targeting

What remains is to solve Lambert's equation:

$$\Delta t = \sqrt{\frac{a^3}{\mu}} \left(\alpha - \beta - (\sin\alpha - \sin\beta)\right)$$

where

$$\sin\left[\frac{\alpha}{2}\right] = \sqrt{\frac{s}{2a}}, \qquad \sin\left[\frac{\beta}{2}\right] = \sqrt{\frac{s-c}{2a}}$$

Initialize our search parameters using $a \in [a_l, a_h] = [a_{\min}, 2s]$.

1. $a_1 = \frac{a_l + a_h}{2} = 8,232$ - $TOF = 21.14$min - too low, decrease a
   1.1 Set $a_h = a_1$
2. $a_2 = \frac{a_l + a_h}{2} = 7,409$ - $TOF = 24$min - too low, decrease a
   2.1 Set $a_h = a_2$
3. $a_3 = \frac{a_l + a_h}{2} = 6,997$ - $TOF = 26.76$min - too low, decrease a
   3.1 Set $a_h = a_3$
4. $\cdots$

K. $a_k = \frac{a_l + a_h}{2} = 6,744$ - $TOF = 29.99$
   K.1 Close Enough!

Numerical Example of Missile Targeting

Now we need to calculate Δv.

$$\vec{v}(t_0) = (B+A)\vec{u}_c + (B-A)\vec{u}_1, \qquad \vec{v}(t_f) = (B+A)\vec{u}_c - (B-A)\vec{u}_2$$

where

$$A = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\alpha}{2}\right) = .597, \qquad B = \sqrt{\frac{\mu}{4a}}\cot\left(\frac{\beta}{2}\right) = 4.2363$$

and the unit vectors

$$\vec{u}_1 = \begin{bmatrix} .866 \\ .5 \\ 0 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} .52 \\ .8414 \\ .1451 \end{bmatrix}, \quad \vec{u}_c = \begin{bmatrix} .0493 \\ .9666 \\ .2516 \end{bmatrix}$$

which yields

$$\vec{v}_t(t_0) = \begin{bmatrix} 3.3901 & 6.4913 & 1.2163 \end{bmatrix} km/s$$

Calculating Δv

$$\Delta v = \vec{v}_t(t_0) - \vec{v} = \begin{bmatrix} 5.847 & -.1267 & -1.3167 \end{bmatrix} km/s$$

For a total impulse of 6km/s.

Numerical Example of Missile Targeting

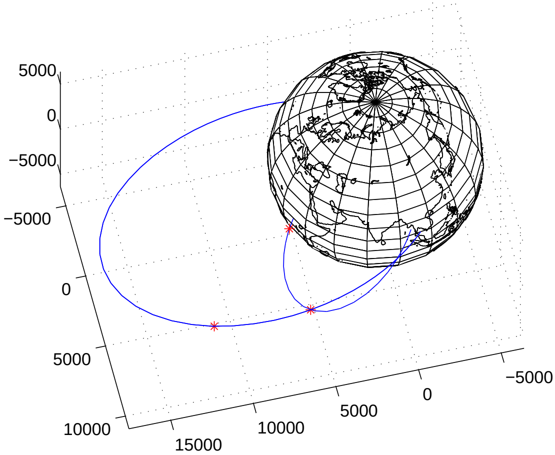


Figure: Intercept Trajectory

Summary

This Lecture you have learned:

Introduction to Lambert's Problem

- The Rendezvous Problem
- The Targeting Problem
  - Fixed-Time interception

Solution to Lambert's Problem

- Focus as a function of semi-major axis, a
- Time-of-Flight as a function of semi-major axis, a
  - Fixed-Time interception
- Calculating Δv.

Next Lecture: Rocketry.