Spacecraft Dynamics and Control

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Lecture 5: Hyperbolic Orbits

Introduction

In this Lecture, you will learn:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Given t, find r and v

For elliptic orbits:

1. Given time, t, solve for Mean Anomaly

$$M(t) = nt$$

2. Given Mean Anomaly, solve for Eccentric Anomaly

$$M(t) = E - e\sin E$$

3. Given eccentric anomaly, solve for true anomaly

$$\tan\frac{f}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

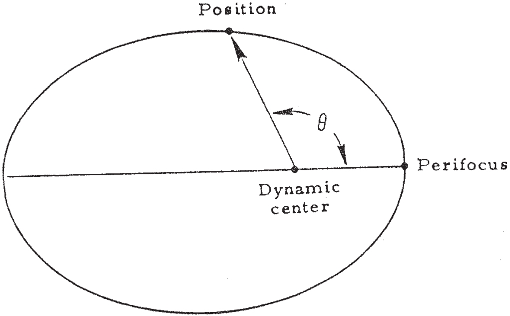
4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}$$

Does this work for Hyperbolic Orbits? Lets recall the angles.

What are these Angles?

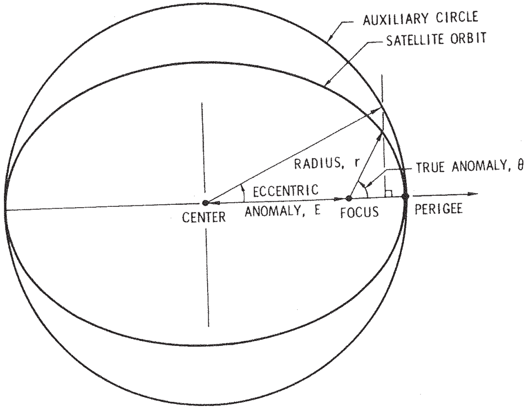
True Anomaly, f (θ)



- The angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$, measured COUNTERCLOCKWISE.
- The angle the position vector makes with periapse.

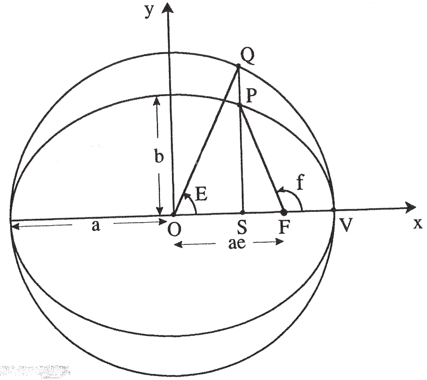
What are these Angles?

Eccentric Anomaly, E



- Measured from center of ellipse to a auxiliary reference circle.

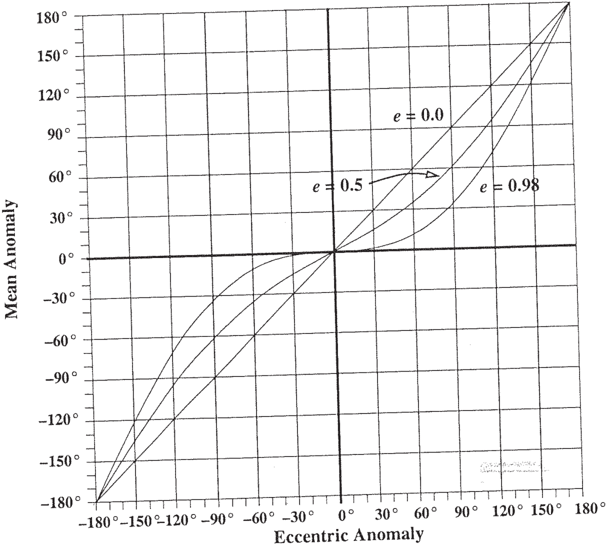
$$M(t) = 2\pi \frac{t}{T} = 2\pi \frac{A_{PFV}}{A_{Ellipse}}$$



- The fraction of area of the ellipse which has been swept out, in radians.

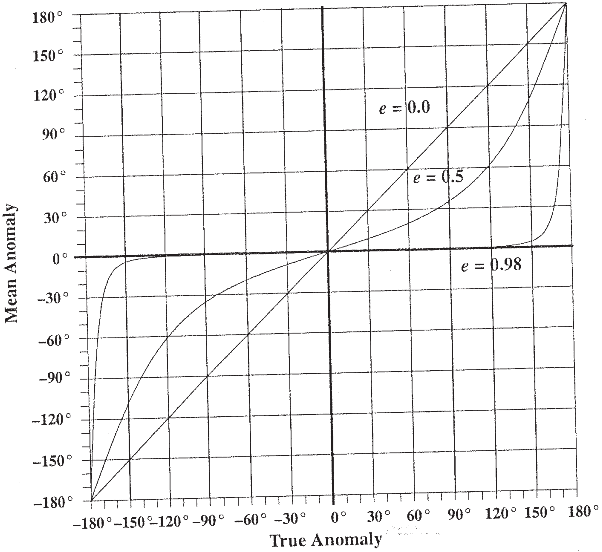
Relationships between M, E, and f

M vs. E

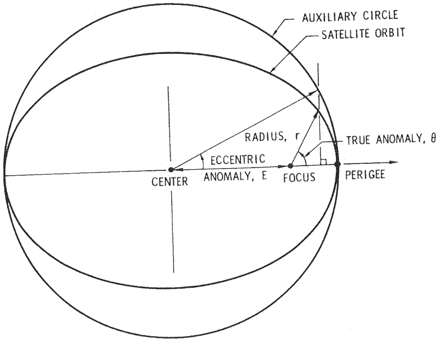


Relationships between M, E, and f

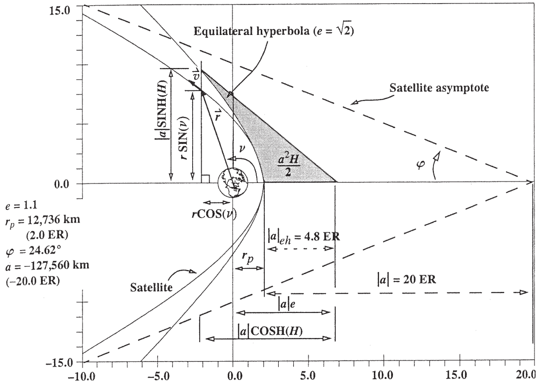
M vs. f



- The orbit does not repeat (no period, T)
  - We can't use
$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$
  - What is mean motion, n?
- No reference circle
  - Eccentric Anomaly is Undefined



Note: In our treatment of hyperbolae, we do **NOT** use the *Universal Variable* approach of Prussing/Conway and others.



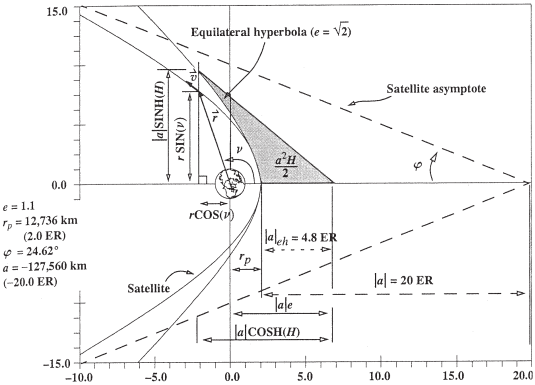
Eccentricity is still

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

and $e = \|\vec{e}\|$.

Solutions for Hyperbolic Orbits

Semimajor axis

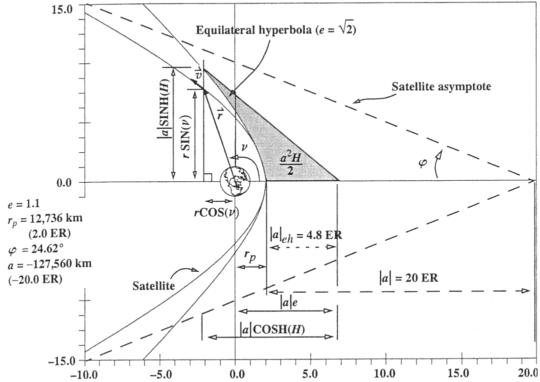


Semimajor axis can still be defined by energy as

$$E = -\frac{\mu}{2a} = \frac{1}{2}v_{excess}^2$$

The periapse is still

$$r_p = a(1-e)$$



Hyperbolic Orbits still satisfy the polar equation

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

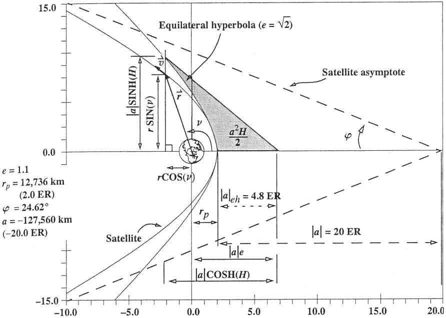
Solutions for Hyperbolic Orbits

Velocity

Velocity can still be calculated from the vis - viva equation

$$v = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a} \right)}$$

Hyperbolic Anomaly is defined by the projection onto a reference hyperbola.

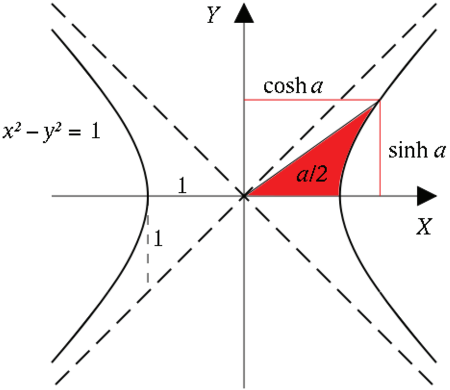


- defined using the reference hyperbola, tangent at perigee

$$x^2 - y^2 = a^2$$

Hyperbolic anomaly is the hyperbolic angle using the area enclosed by the center of the hyperbola, the point of perifocus and the point on the reference hyperbola directly above the position vector.

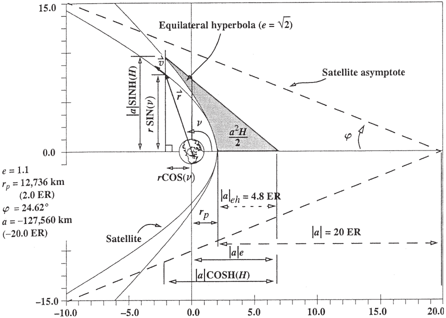
Consider $\qquad x^2 - y^2 = 1$



$cosh$ and $sinh$ relate area swept out by the reference hyperbola to lengths.

- Yet another branch of mathematics developed for solving orbits (Lambert).

Hyperbolic Anomaly



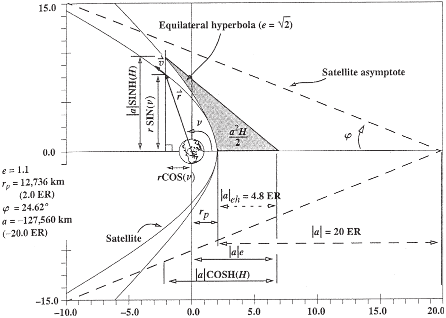
- Hyperbolic Trig (which I won't get into) gives a relationship to true anomaly, which is

$$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f}{2}\right)$$

- Alternatively,

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

Hyperbolic Anomaly



Using hyperbolic anomaly, we can give a simpler form of the polar equation.

$$r = a(1 - e\cosh H)$$

Hyperbolic Kepler's Equation

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

$$n = \sqrt{\frac{\mu}{-a^3}}$$

Definition 1 (Hyperbolic Kepler's Equation).

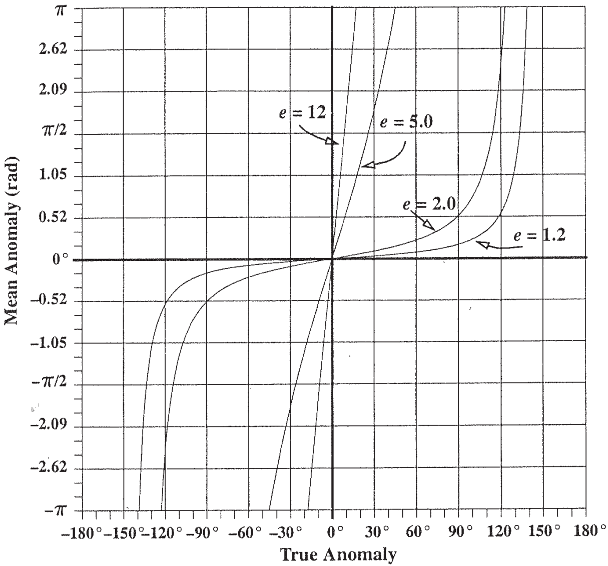
$$nt = \sqrt{\frac{\mu}{-a^3}}t = M = e\sinh(H) - H$$

If we want to solve this, we get a different Newton iteration.
Newton Iteration for Hyperbolic Anomaly:

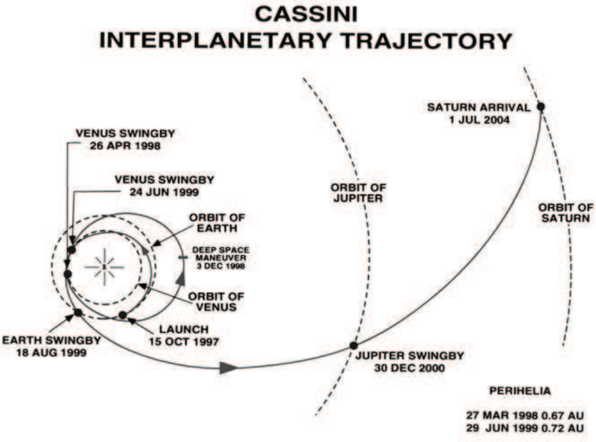
$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

with $H_1 = M$.

Relationship between M and f for Hyperbolic Orbits



Example: Jupiter Flyby



Problem: Suppose we want to make a flyby of Jupiter. The relative velocity at approach is $v_\infty = 10km/s$. To achieve the proper turning angle, we need an eccentricity of $e = 1.07$. Radiation limits our time within radius $r = 100,000km$ to 1 hour (radius of Jupiter is $71,000km$). Will the spacecraft survive the flyby?

Example: Jupiter Flyby

Example Continued

Solution: First solve for a and p. $\mu = 1.267E8$.

- The total energy of the orbit is given by

$$E_{tot} = \frac{1}{2} v_\infty^2$$

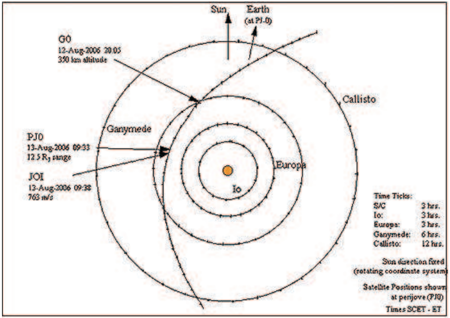
- The total energy is expressed as

$$E = -\frac{\mu}{2a} = \frac{1}{2} v_\infty^2$$

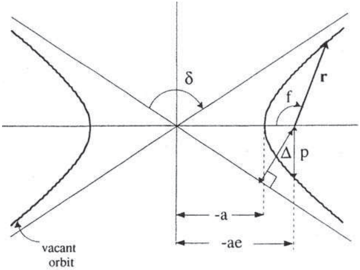
which yields

$$a = -\frac{\mu}{v_\infty^2} = -1.267E6$$

- The parameter is $p = a(1 - e^2) = 1.8359E5$



Example Continued



We need to find the time between $r_1 = 100,000km$ and $r_2 = 100,000km$. Find f at each of these points.

- Start with the conic equation:

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

- Solving for f,

$$f_{1,2} = \cos^{-1}\left(\frac{1}{e} - \frac{r}{ep}\right) = \pm 64.8\,\mathrm{deg}$$

Example Continued

Given the true anomalies, $f_{1,2}$, we want to find the associated times, $t_{1,2}$.

- Only solve for t_2, get t_1 by symmetry.
- First find Hyperbolic Anomaly,

$$H_2 = \tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f_2}{2}\right)\right) = .1173$$

- Now use Hyperbolic anomaly to find mean anomaly

$$M_2 = e\sinh(H_2) - H_2 = .0085$$

  - This is the “easy” direction.
  - No Newton iteration required.
- t_2 is now easy to find

$$t_2 = M_2\sqrt{\frac{-a^3}{\mu}} = 1076.6$$

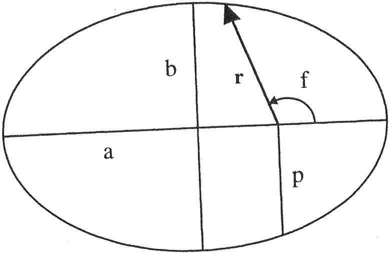
Finally, we conclude $\Delta t = 2 * t_2 = 2153s = 35min$.

So the spacecraft survives.

The Orbital Elements

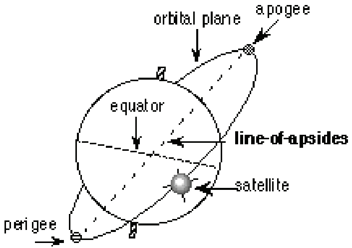
So far, all orbits are parameterized by 3 parameters

- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

Note: We have shown how to use a, e and f to find the scalars r and v.

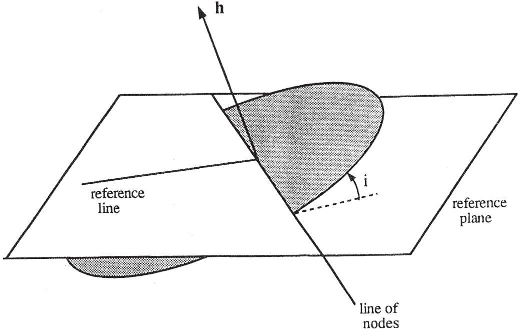


Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

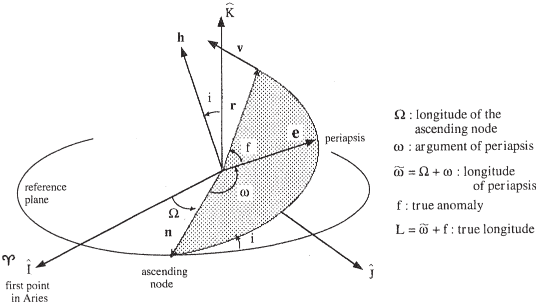
- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.

Inclination, i



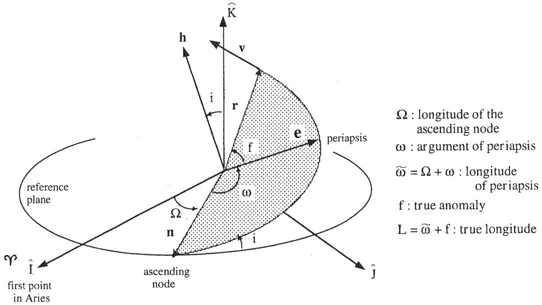
Angle the orbital plane makes with the reference plane.

Right Ascension of Ascending Node, Ω



Angle measured from reference direction in the reference plane to intersection with orbital plane.

Argument of Periapse, ω



Angle measured from reference plane to point of periapse.

Summary

This Lecture you have learned:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Summary

Properties of Keplerian Orbits

Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis $a < 0$ b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 + b^2/a^2} \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a\,(1 - e)$	$q = p/2$	$q = a\,(1 - e)$
Velocity, V, at distance, r, from Focus	$V^2 = \mu/r$	$V^2 = \mu\,(2/r - 1/a)$	$V^2 = 2\mu/r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, $\mathcal{E}$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = 0$	$\mathcal{E} = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu/a^3}$	$n = \sqrt{\mu}$	$n = \sqrt{\mu/(-a)^3}$
Period, P	$P = 2\pi/n$	$P = 2\pi/n$	$P = \infty$	$P = \infty$
Anomaly	$v = M = E$	Eccentric anomaly, E $\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2}\tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{v}{2} = D/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{v}{2} = \left(\frac{e+1}{e-1}\right)^{1/2}\tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e\sin E$	$M = qD + (D^3/6)$	$M = (e\sinh F) - F$
Distance from Focus, $r = q\,(1+e)/(1+e\cos v)$	$r = a$	$r = a\,(1 - e\cos E)$	$r = q + (D^2/2)$	$r = a\,(1 - e\cosh F)$
$r\,dr/dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu}\sin E$	$r\dot{r} = \sqrt{\mu}\,D$	$r\dot{r} = e\sqrt{(-a)\mu}\sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2}r^2\frac{dv}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$	$\frac{dA}{dt} = \sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$

$\mu = GM$ is the gravitational constant of the central body; v is the true anomaly, and $M \equiv n\,(t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion.