Spacecraft Dynamics and Control

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Lecture 16: Euler's Equations

Attitude Dynamics

In this Lecture we will cover:

The Problem of Attitude Stabilization

- Actuators

Newton's Laws

- $\sum \vec{M}_i = \frac{d}{dt}\vec{H}$
- $\sum \vec{F}_i = m\frac{d}{dt}\vec{v}$

Rotating Frames of Reference

- Equations of Motion in Body-Fixed Frame
- Often Confusing

If in doubt, use the right-hand rules.

Figure: Positive Directions

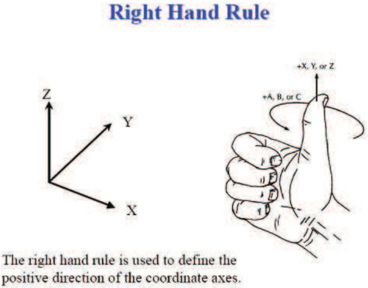
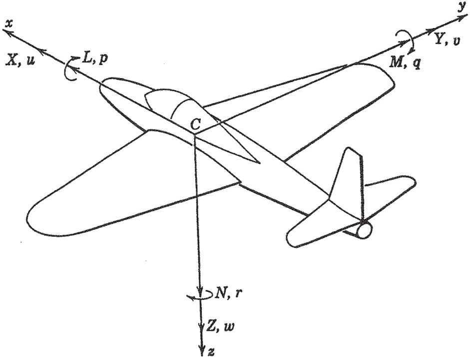


Figure: Positive Rotations

Review: Coordinate Rotations

Roll-Pitch-Yaw



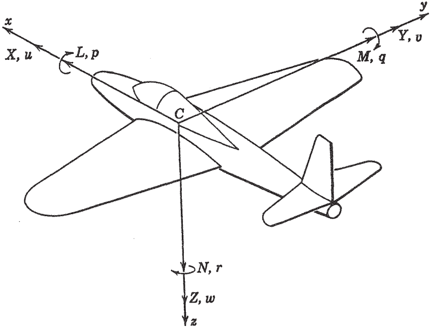
There are 3 basic rotations a vehicle can make:

- Roll = Rotation about x-axis
- Pitch = Rotation about y-axis
- Yaw = Rotation about z-axis
- Each rotation is a one-dimensional transformation.

Any two coordinate systems can be related by a sequence of 3 rotations.

Review: Forces and Moments

Forces

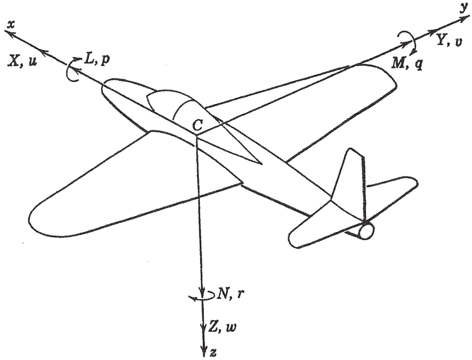


These forces and moments have standard labels. The Forces are:

X	Axial Force	Net Force in the positive x-direction
Y	Side Force	Net Force in the positive y-direction
Z	Normal Force	Net Force in the positive z-direction

Review: Forces and Moments

Moments



The Moments are called, intuitively:

L	Rolling Moment	Net Moment in the positive ω_x-direction
M	Pitching Moment	Net Moment in the positive ω_y-direction
N	Yawing Moment	Net Moment in the positive ω_z-direction

Newton's Second Law tells us that for a particle $F = ma$. In vector form:

$$\vec{F} = \sum_i \vec{F}_i = m\frac{d}{dt}\vec{V}$$

That is, if $\vec{F} = [F_x \; F_y \; F_z]$ and $\vec{V} = [u \; v \; w]$, then

$$F_x = m\frac{du}{dt} \qquad F_x = m\frac{dv}{dt} \qquad F_z = m\frac{dw}{dt}$$

Definition 1.

$m\vec{V}$ is referred to as **Linear Momentum**.

Newton's Second Law is only valid if $\vec{F}$ and $\vec{V}$ are defined in an *Inertial* coordinate system.

Definition 2.

A coordinate system is **Inertial** if it is not accelerating or rotating.

Using Calculus, momentum can be extended to rigid bodies by integration over all particles.

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

Definition 3.

Where $\vec{H} = \int (\vec{r}_c \times \vec{v}_c) dm$ is the **angular momentum**.

Angular momentum of a rigid body can be found as

$$\vec{H} = I\vec{\omega}_I$$

where $\vec{\omega}_I = [p,\ q,\ r]^T$ is the angular rotation vector of the body about the center of mass.

- $p = \omega_x$ is rotation about the x-axis.
- $q = \omega_y$ is rotation about the y-axis.
- $r = \omega_z$ is rotation about the z-axis.
- ω_I is defined in an *Inertial* Frame.

The matrix I is the *Moment of Inertia Matrix*.

The moment of inertia matrix is defined as

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

$$I_{xy} = I_{yx} = \int\int\int xydm \qquad I_{xx} = \int\int\int \left(y^2 + z^2\right) dm$$

$$I_{xz} = I_{zx} = \int\int\int xzdm \qquad I_{yy} = \int\int\int \left(x^2 + z^2\right) dm$$

$$I_{yz} = I_{zy} = \int\int\int yzdm \qquad I_{zz} = \int\int\int \left(x^2 + y^2\right) dm$$

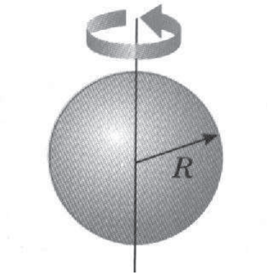
So

$$\begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p_I \\ q_I \\ r_I \end{bmatrix}$$

where p_I, q_I and r_I are the rotation vectors as expressed in the inertial frame corresponding to x-y-z.

Moment of Inertia

Examples:

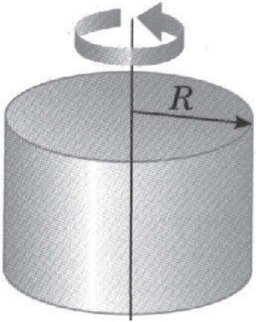


Homogeneous Sphere

$$I_{sphere} = \frac{2}{5} m r^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ring

$$I_{ring} = m r^2 \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Homogeneous Disk

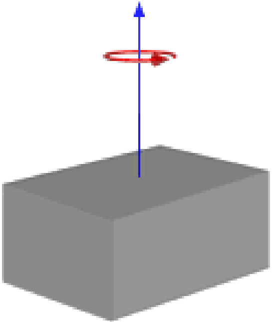
$$I_{disk} = \frac{1}{4}mr^2 \begin{bmatrix} 1+\frac{1}{3}\frac{h}{r^2} & 0 & 0 \\ 0 & 1+\frac{1}{3}\frac{h}{r^2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

F/A-18

$$I = \begin{bmatrix} 23 & 0 & 2.97 \\ 0 & 15.13 & 0 \\ 2.97 & 0 & 16.99 \end{bmatrix} kslug-ft^2$$

Moment of Inertia

Examples:

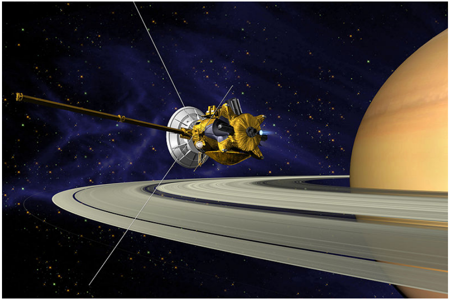


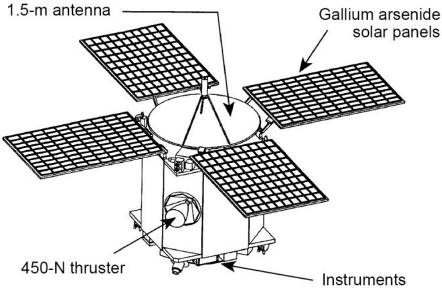
Cube

$$I_{cube} = \frac{2}{3} l^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Box

$$I_{box} = \begin{bmatrix} \frac{b^2+c^2}{3} & 0 & 0 \\ 0 & \frac{a^2+c^2}{3} & 0 \\ 0 & 0 & \frac{a^2+b^2}{3} \end{bmatrix}$$





Cassini

$$I = \begin{bmatrix} 8655.2 & -144 & 132.1 \\ -144 & 7922.7 & 192.1 \\ 132.1 & 192.1 & 4586.2 \end{bmatrix} kg \cdot m^2$$

NEAR Shoemaker

$$I = \begin{bmatrix} 473.924 & 0 & 0 \\ 0 & 494.973 & 0 \\ 0 & 0 & 269.83 \end{bmatrix} kg{\cdot}m^2$$

Problem:

The Body-Fixed Frame

The moment of inertia matrix, I, is fixed in the body-fixed frame. However, Newton's law only applies for an inertial frame:

$$\vec{M} = \sum_i \vec{M}_i = \frac{d}{dt}\vec{H}$$

If the body-fixed frame is rotating with rotation vector $\vec{\omega}$, then for any vector, $\vec{a}$, $\frac{d}{dt}\vec{a}$ in the inertial frame is

$$\frac{d\vec{a}}{dt}\Big|_I = \frac{d\vec{a}}{dt}\Big|_B + \vec{\omega} \times \vec{a}$$

Specifically, for Newton's Second Law

$$\vec{F} = m\frac{d\vec{V}}{dt}\Big|_B + m\vec{\omega} \times \vec{V}$$

and

$$\vec{M} = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

Equations of Motion

Displacement

The equation for acceleration (which we will ignore) is:

$$\begin{aligned}\begin{bmatrix}F_x\\F_y\\F_z\end{bmatrix} &= m\frac{d\vec{V}}{dt}\bigg|_B + m\vec{\omega}\times\vec{V}\\ &= m\begin{bmatrix}\dot{u}\\\dot{v}\\\dot{w}\end{bmatrix} + m\det\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \omega_x & \omega_y & \omega_z\\ u & v & w\end{bmatrix}\\ &= m\begin{bmatrix}\dot{u}+\omega_y w-\omega_z v\\ \dot{v}+\omega_z u-\omega_x w\\ \dot{w}+\omega_x v-\omega_y u\end{bmatrix}\end{aligned}$$

As we will see, displacement and rotation in space are **decoupled**.

- no aerodynamic forces.

Equations of Motion

The equations for rotation are:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \frac{d\vec{H}}{dt}\Big|_B + \vec{\omega} \times \vec{H}$$

$$= \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} + \vec{\omega} \times \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z \\ -I_{xy}\dot{\omega}_x + I_{yy}\dot{\omega}_y - I_{yz}\dot{\omega}_z \\ -I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + I_{zz}\dot{\omega}_z \end{bmatrix} + \vec{\omega} \times \begin{bmatrix} \omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz} \\ -\omega_x I_{xy} + \omega_y I_{yy} - \omega_z I_{yz} \\ -\omega_x I_{xz} - \omega_y I_{yz} + \omega_z I_{zz} \end{bmatrix}$$

$$= \begin{bmatrix} I_{xx}\dot{\omega}_x - I_{xy}\dot{\omega}_y - I_{xz}\dot{\omega}_z + \omega_y(\omega_z I_{zz} - \omega_x I_{xz} - \omega_y I_{yz}) - \omega_z(\omega_y I_{yy} - \omega_x I_{xy} - \omega_z I_{yz}) \\ I_{yy}\dot{\omega}_y - I_{xy}\dot{\omega}_x - I_{yz}\dot{\omega}_z - \omega_x(\omega_z I_{zz} - \omega_y I_{yz} - \omega_x I_{xz}) + \omega_z(\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}) \\ I_{zz}\dot{\omega}_z - I_{xz}\dot{\omega}_x - I_{yz}\dot{\omega}_y + \omega_x(\omega_y I_{yy} - \omega_x I_{xy} - \omega_z I_{yz}) - \omega_y(\omega_x I_{xx} - \omega_y I_{xy} - \omega_z I_{xz}) \end{bmatrix}$$

Which is too much for any mortal. We simplify as:

- For spacecraft, we have $I_{yz} = I_{xy} = I_{xz} = 0$ (**two planes of symmetry**).
- For aircraft, we have $I_{yz} = I_{xy} = 0$ (**one plane of symmetry**).

Equations of Motion

Euler Moment Equations

With $I_{xy} = I_{yz} = I_{xz} = 0$, we get: **Euler's Equations**

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy}) \\ I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) \\ I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx}) \end{bmatrix}$$

Thus:

- Rotational variables $(\omega_x, \omega_y, \omega_z)$ do not depend on translational variables (u,v,w).
  - For spacecraft, Moment forces (L,M,N) do not depend on rotational and translational variables.
  - Can be decoupled
- However, translational variables (u,v,w) depend on rotation $(\omega_x, \omega_y, \omega_z)$.
  - But we don't care.

Euler Equations

Torque-Free Motion

Notice that even in the absence of external moments, the dynamics are still active:

$$\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}I_x\dot{\omega}_x + \omega_y\omega_z(I_z - I_y)\\ I_y\dot{\omega}_y + \omega_x\omega_z(I_x - I_z)\\ I_z\dot{\omega}_z + \omega_x\omega_y(I_y - I_x)\end{bmatrix}$$

which yield the 3-state nonlinear ODE:

$$\dot{\omega}_x = -\frac{I_z - I_y}{I_x}\omega_y(t)\omega_z(t)$$

$$\dot{\omega}_y = -\frac{I_x - I_z}{I_y}\omega_x(t)\omega_z(t)$$

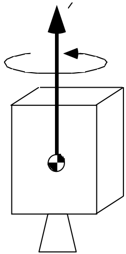
$$\dot{\omega}_z = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t)$$

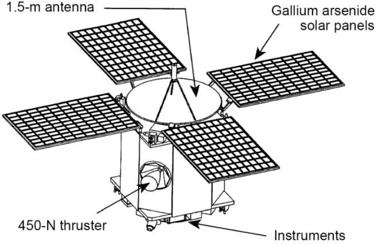
Thus even in the absence of external moments

- The axis of rotation $\vec{\omega}$ will evolve
- Although the angular momentum vector $\vec{h}$ will NOT.
  - occurs because tensor I changes in inertial frame.
- This can be problematic for spin-stabilization!

We can use Euler's equation for study Spin Stabilization.

There are two important cases:



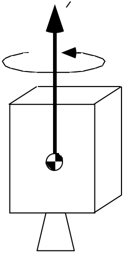


Axisymmetric: $I_x = I_y$

$$I = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_x & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

Non-Axisymmetric: $I_x \neq I_y$

$$I = \begin{bmatrix} 473.924 & 0 & 0 \\ 0 & 494.973 & 0 \\ 0 & 0 & 269.83 \end{bmatrix} kg{\cdot}m^2$$



An important case is spin-stabilization of an axisymmetric spacecraft.

- Assume symmetry about z-axis ($I_x = I_y$)

Then recall

$$\dot{\omega}_z = -\frac{I_y - I_x}{I_z}\omega_x(t)\omega_y(t) = 0$$

Thus $\omega_z = constant$.

The equations for ω_x and ω_y are now

$$\begin{bmatrix}\dot{\omega}_x \\ \dot{\omega}_y\end{bmatrix} = \begin{bmatrix} 0 & -\frac{I_z - I_y}{I_x}\omega_z \\ -\frac{I_x - I_z}{I_y}\omega_z & 0 \end{bmatrix}\begin{bmatrix}\omega_x(t) \\ \omega_y(t)\end{bmatrix}$$

Which is a linear ODE.

Fortunately, linear systems have closed-form solutions.
let $\lambda = \frac{I_z - I_x}{I_x}\omega_z$. Then

$$\dot{\omega}_x(t) = -\lambda\omega_y(t)$$
$$\dot{\omega}_y(t) = \lambda\omega_x(t)$$

Combining, we get

$$\ddot{\omega}_x(t) = -\lambda^2\omega_x(t)$$

which has solution

$$\omega_x(t) = \omega_x(0)\cos(\lambda t) + \frac{\dot{\omega}_x(0)}{\lambda}\sin(\lambda t)$$

Differentiating, we get

$$\begin{aligned}\omega_y(t) &= -\frac{\dot{\omega}_x(t)}{\lambda} = \omega_x(0)\sin(\lambda t) - \frac{\dot{\omega}_x(0)}{\lambda}\cos(\lambda t)\\ &= \omega_x(0)\sin(\lambda t) + \omega_y(0)\cos(\lambda t)\\ \omega_x(t) &= \omega_x(0)\cos(\lambda t) - \omega_y(0)\sin(\lambda t)\end{aligned}$$

Spin Stabilization

Axisymmetric Case

Define $\omega_{xy} = \sqrt{\omega_x^2 + \omega_y^2}$.

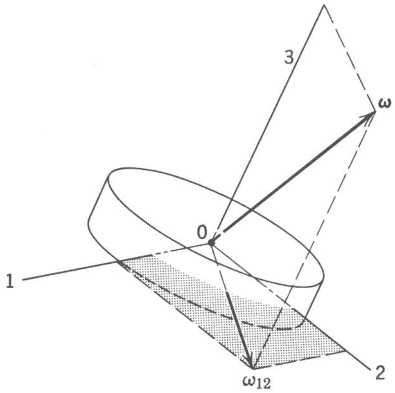
$$\begin{aligned}
\omega_{xy}^2 &= (\omega_x(0)\sin(\lambda t) + \omega_y(0)\cos(\lambda t))^2 + (\omega_x(0)\cos(\lambda t) - \omega_y(0)\sin(\lambda t))^2 \\
&= \omega_x(0)^2 \sin^2(\lambda t) + \omega_y(0)^2 \cos^2(\lambda t) + 2\omega_x(0)\omega_y(0)\cos(\lambda t)\sin(\lambda t) \\
&\qquad\qquad + \omega_x(0)^2 \cos^2(\lambda t) + \omega_y(0)^2 \sin^2(\lambda t) - 2\omega_x(0)\omega_y(0)\cos(\lambda t)\sin(\lambda t) \\
&= \omega_x(0)^2(\sin^2(\lambda t) + \cos^2(\lambda t)) + \omega_y(0)^2(\cos^2(\lambda t) + \sin^2(\lambda t)) \\
&= \omega_x(0)^2 + \omega_y(0)^2
\end{aligned}$$

Thus

- ω_z is constant
  - rotation about axis of symmetry
- $\sqrt{\omega_x^2 + \omega_y^2}$ is constant
  - rotation perpendicular to axis of symmetry

This type of motion is often called **Precession**!

Circular Motion in the Body-Fixed Frame



Thus

$$\omega(t) = \begin{bmatrix}\omega_x(t)\\ \omega_y(t)\\ \omega_z(t)\end{bmatrix} = \begin{bmatrix}\cos(\lambda t) & -\sin(\lambda t) & 0\\ \sin(\lambda t) & \cos(\lambda t) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\omega_x(0)\\ \omega_y(0)\\ \omega_z(0)\end{bmatrix} = R_3(\lambda t)\begin{bmatrix}\omega_x(0)\\ \omega_y(0)\\ \omega_z(0)\end{bmatrix}$$

Prolate vs. Oblate

The speed of the precession is given by the natural frequency:

$$\lambda = \frac{I_z - I_x}{I_x}\omega_z$$

with period $T = \frac{2\pi}{\lambda} = \frac{2\pi I_x}{I_z - I_x}\omega_z^{-1}$.

Direction of Precession: There are two cases

Definition 4 (Direct).

An axisymmetric (about z-axis) rigid body is **Prolate** if $I_z < I_x = I_y$.

Definition 5 (Retrograde).

An axisymmetric (about z-axis) rigid body is **Oblate** if $I_z > I_x = I_y$.

Thus we have two cases:

- $\lambda > 0$ if object is *Oblate*
- $\lambda < 0$ if object is *Prolate*

Note that these rotations are in the **Body-Fixed** Frame.

Pay Attention to the Body-Fixed Axes

Figure: Prolate Precession

Figure: Oblate Precession

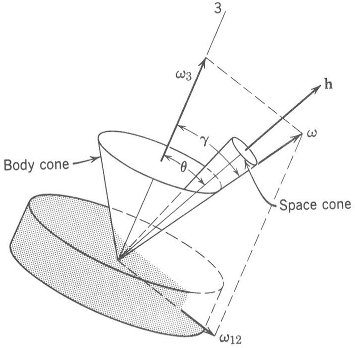
Motion in the Inertial Frame

As these videos illustrate, we are typically interested in motion in the **Inertial Frame**.

- Use of Rotation Matrices is complicated.
  - Which coordinate system to use???
- Lets consider motion relative to $\vec{h}$.
  - Which is fixed in inertial space.

We know that in Body-Fixed coordinates,

$$\vec{h} = I\vec{\omega} = \begin{bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{bmatrix}$$



Now lets find the orientation of ω and $\hat{z}$ with respect to this fixed vector.

Motion in the Inertial Frame

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ define the body-fixed unit vectors.

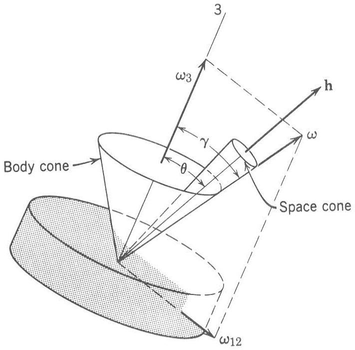
We first note that since $I_x = I_y$ and

$$\begin{aligned}\vec{h} &= I_x\omega_x\hat{x} + I_y\omega_y\hat{y} + I_z\omega_z\hat{z} \\ &= I_x(\omega_x\hat{x} + \omega_y\hat{y} + \omega_z\hat{z}) + (I_z - I_x)\omega_z\hat{z} \\ &= I_x\vec{\omega} + (I_z - I_x)\omega_z\hat{z}\end{aligned}$$

we have that

$$\vec{\omega} = \frac{1}{I_x}\vec{h} + \frac{I_x - I_z}{I_x\omega_z}\hat{z}$$

which implies that $\vec{\omega}$ lies in the $\hat{z} - \vec{h}$ plane.



Motion in the Inertial Frame

We now focus on two constants of motion

- θ - The angle $\vec{h}$ makes with the body-fixed $\hat{z}$ axis.
- γ - The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

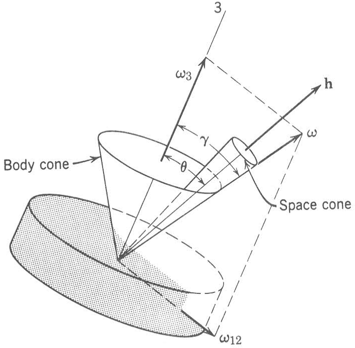
Since

$$\vec{h} = \begin{bmatrix} I_x\omega_x \\ I_y\omega_y \\ I_z\omega_z \end{bmatrix}$$

The angle θ is defined by

$$\tan\theta = \frac{\sqrt{h_x^2 + h_y^2}}{h_z} = \frac{I_x}{I_z}\frac{\omega_{xy}}{\omega_z}$$

Since ω_{xy} and ω_z are fixed, θ is a constant of motion.



Motion in the Inertial Frame

The second angle to consider is

- γ - The angle $\vec{\omega}$ makes with the body-fixed $\hat{z}$ axis.

As before, the angle γ is defined by

$$\tan\gamma = \frac{\sqrt{\omega_x^2 + \omega_y^2}}{\omega_z} = \frac{\omega_{xy}}{\omega_z}$$

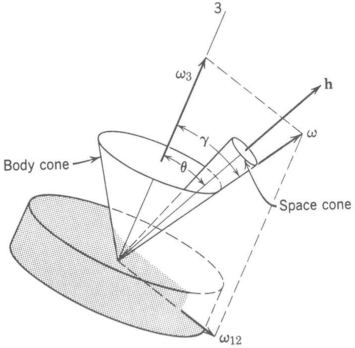
Since ω_{xy} and ω_z are fixed, γ is a constant of motion.

- We have the relationship

$$\tan\theta = \frac{I_x}{I_z}\frac{\omega_{xy}}{\omega_z} = \frac{I_x}{I_z}\tan\gamma$$

Thus we have two cases:

1. $I_x > I_z$ - Then $\theta > \gamma$
2. $I_x < I_z$ - Then $\theta < \gamma$ (As Illustrated)



Motion in the Inertial Frame

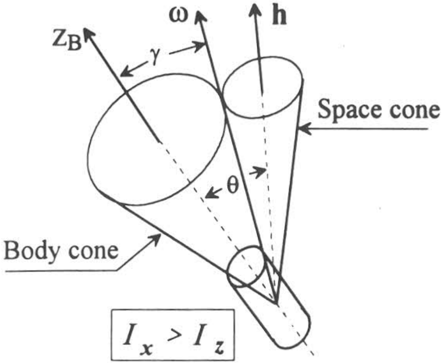


Figure: The case of $I_x > I_z$ ($\theta > \gamma$)

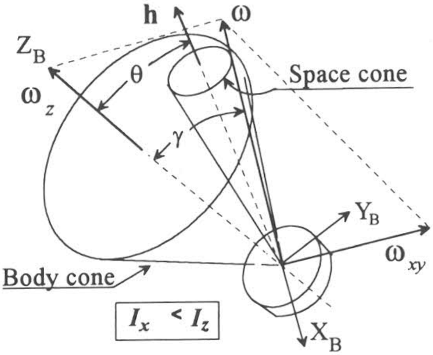
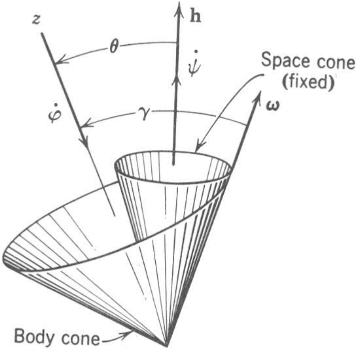


Figure: The case of $I_z > I_x$ ($\gamma > \theta$)

Motion in the Inertial Frame

The orientation of the body in the inertial frame is defined by the sequence of Euler rotations

- ψ - R_3 rotation about $\vec{h}$.
- θ - R_1 rotation to body-fixed $\hat{z}$ vector.
  - We have shown that this angle is fixed!
  - $\dot{\theta} = 0$.
- ϕ - R_3 rotation about body-fixed $\hat{z}$ vector.



The Euler angles are related to the angular velocity vector as

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} \dot{\psi} \sin\theta \sin\phi \\ \dot{\psi} \sin\theta \cos\phi \\ \dot{\phi} + \dot{\psi}\cos\theta = constant \end{bmatrix}$$

Motion in the Inertial Frame

To find the motion of ω, we differentiate

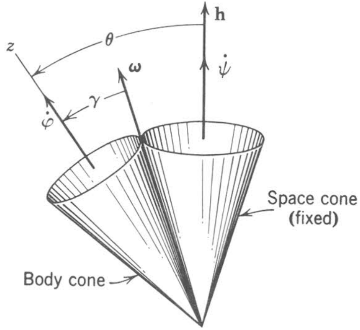
$$\begin{bmatrix}\dot{\omega}_x\\ \dot{\omega}_y\\ \dot{\omega}_z\end{bmatrix} = \begin{bmatrix}\dot{\psi}\dot{\phi}\sin\theta\cos\phi\\ -\dot{\psi}\dot{\phi}\sin\theta\sin\phi\\ 0\end{bmatrix}$$

Now, substituting into the Euler equations yields

$$\dot{\psi} = \frac{I_z}{(I_x - I_z)\cos\theta}\dot{\phi}$$

There are two cases here:

- $I_x > I_z$ - **Direct** precession
  - $\dot{\psi}$ and $\dot{\phi}$ aligned.
- $I_y > I_x$ - **Retrograde** precession
  - $\dot{\psi}$ and $\dot{\phi}$ are opposite.



Motion in the Inertial Frame

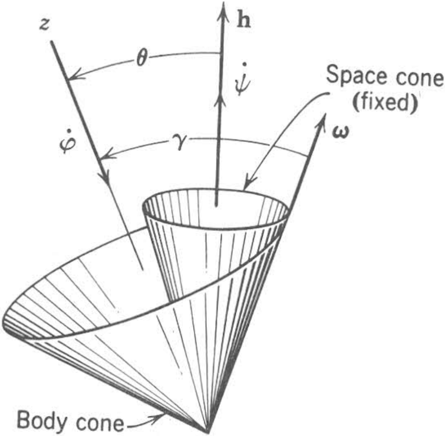


Figure: Retrograde Precession ($I_z > I_x$)

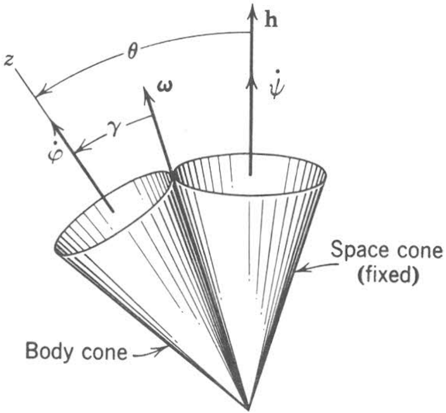


Figure: Direct Precession ($I_z < I_x$)

Mathematica Demonstrations

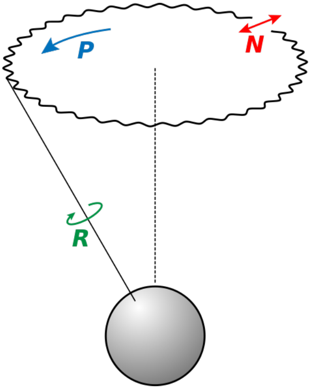
Mathematica Precession Demonstration

Prolate and Oblate Spinning Objects

Figure: Prolate Object: $I_x = I_y = 4$ and $I_z = 1$

Figure: Oblate Object: Vesta

Next Lecture



Note Bene: Precession of a spacecraft is often called nutation (θ is called the nutation angle).

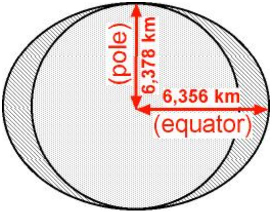
- By most common definitions, for torque-free motions, $N = 0$
  - Free rotation has NO nutation.
  - This is confusing

Problem: The earth is 42.72 km wider than it is tall. How quickly will the rotational axis of the earth precess due to this effect?

Solution: for an axisymmetric ellipsoid with height a and width b, we have $I_x = I_y = \frac{1}{5}m(a^2 + b^2)$ and $I_z = \frac{2}{5}mb^2$.

Thus $b = 6378km$, $a = 6352km$ and we have $(m_e = 5.974 \cdot 10^{24}kg)$

$$I_x = 9.68 \cdot 10^{37kg-m^2}, \qquad I_z = I_y = 9.72 \cdot 10^{37kg-m^2}$$



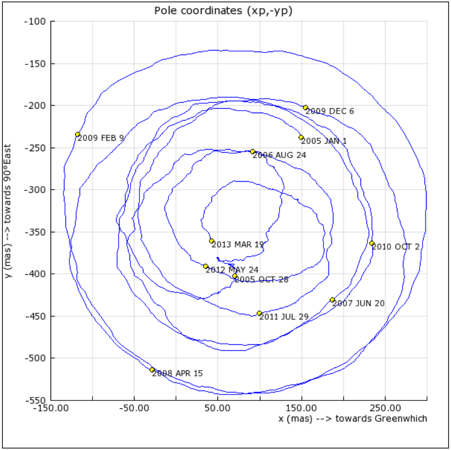
If we take $\omega_z = \frac{2\pi}{T} \cong 2\pi day^{-1}$, then we have

$$\lambda = \frac{I_z - I_x}{I_x}\omega_z = .0041day^{-1}$$

That gives a period of $T = \frac{2\pi}{\lambda} = 243.5days$. This motion of the earth is known as the **Chandler Wobble**.

Note: This is only the Torque-free precession.

Precession



- Actual period is 434 days
  - Actual $I_x = I_y = 8.008 \cdot 10^{37} kg - m^2$.
  - Actual $I_z = 8.034 \cdot 10^{37} kg - m^2$.
  - Which would predict $T = 306$days

Next Lecture

In the next lecture we will cover

Non-Axisymmetric rotation

- Linearized Equations of Motion
- Stability

Energy Dissipation

- The effect on stability of rotation