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Constructing Piecewise-Polynomial Lyapunov Functions for Nonlinear Systems Using Handelman's Theorem

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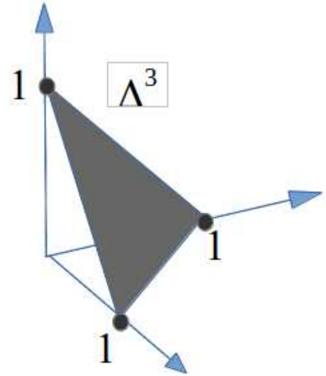
Solving Large-scale Problems in Control

- We designed a parallel version of Polya's algorithm to verify stability of **linear uncertain** systems with 100+ states. (TAC 2013)

$$\dot{x}(t) = A(\alpha)x(t), \quad \alpha^n \in \Delta^n$$

$$\Delta^n := \{\alpha \in \mathbb{R}^n : \|\alpha_i\|_1 = 1, \alpha_i \geq 0\}$$

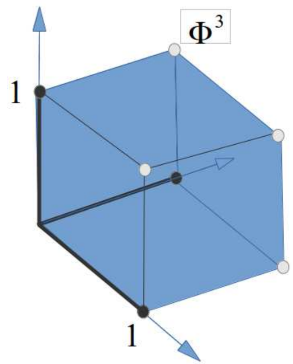


- We designed a parallel algorithm which uses a multi-simplex version of Polya's theorem to verify stability of

(CDC 2012) $\dot{x}(t) = A(\alpha)x(t), \quad \alpha \in \Phi^n$

(CDC 2013) $\dot{x}(t) = f(x(t)), \quad x(t) \in \Phi^n$

$$\Phi^n := \{x \in \mathbb{R}^n : |x_i| \leq r_i\}$$



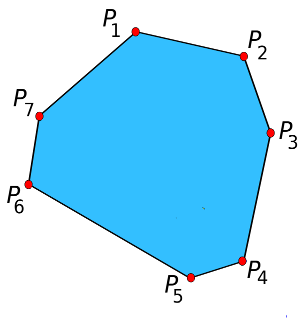
Analysis on More Complicated Geometries

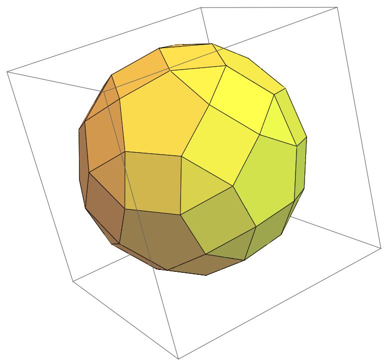
In this talk, we address local stability of **nonlinear** systems defined by polynomial vector fields

$$\dot{x}(t) = f(x) = \sum_{\|\alpha\|_1 \le d_f} b_\alpha x^\alpha = \sum_{\|\alpha\|_1 \le d_f} b_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

over **convex polytopes**:

$$\Gamma_p := \{x \in \mathbb{R}^n : x = \sum_{i=1}^{K} \mu_i p_i,\ \mu_i \in [0,1],\ \sum_{i=1}^{K} \mu_i = 1\}$$





Alternatively, convex polytopes can be represented as

$$\Gamma_K := \{x \in \mathbb{R}^n : w_i^T x + u_i \ge 0, i = 1, \cdots, K\}.$$

We Search for Lyapunov Polynomials on Polytopes

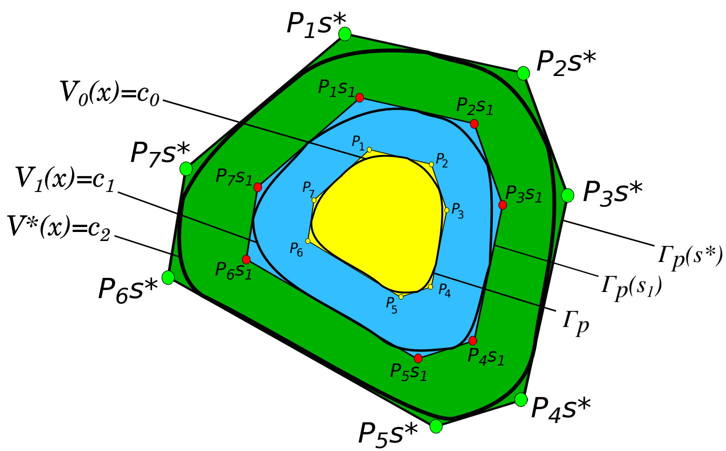
Given $p_i \in \mathbb{R}^n$, we would like to find a **polynomial** V which solves

$$s^* := \max_{V \in \mathbb{R}[x], s>0} \quad s$$

$$\text{subject to} \quad V(x) - \epsilon x^T x \geq 0 \quad \text{for all } x \in \Gamma_p(s)$$

$$\nabla V(x)^T f(x) + \epsilon x^T x \leq 0 \quad \text{for all } x \in \Gamma_p(s).$$

$$\Gamma_p(s) := \{x \in \mathbb{R}^n : x = \sum_{i=1}^K \mu_i p_i,\ \mu_i \in [0, s],\ \sum_{i=1}^K \mu_i = s\}$$



Then $\{x : \{y : V^*(y) \leq V^*(x)\} \subset \Gamma_p(s^*)\}$ is the ROA of the origin.

Optimization of Polynomials is NP-hard

The problem

$$s^* := \max_{V \in \mathbb{R}[x], s>0} \quad s$$

$$\text{subject to} \quad V(x) - \epsilon x^T x \geq 0 \qquad \text{for all } x \in \Gamma_p(s)$$

$$\nabla V(x)^T f(x) + \epsilon x^T x \leq 0 \qquad \text{for all } x \in \Gamma_p(s)$$

is an instance of the more general problem of **Optimization Of Polynomials (OOP)**:

$$\gamma^* := \max_{x, F_0} \; c^T x$$

$$\text{subject to} \quad F(x,y) := F_0(y) + \sum_{i=1}^{n} x_i F_i(y) \geq 0 \text{ for all } \; y \in \mathbb{R}^n$$

- The OOP is a convex optimization problem
- The OOP is an NP-hard problem. Difficult part is to verify $F \geq 0$
  ↳ Indeed the question: "Is $p(x) \geq 0$ for all $x \in \mathbb{R}^n$?" is NP-hard

Tests for Non-negativity of Polynomials

- ***Quantifier Elimination*** (QE) algorithms yield **exact** solutions to OOPs

  ↳ Tarski-Seidenberg (1954): Computational complexity grows double-exponentially with the number of variables

  ↳ Basu-Pollack (1996): Computational complexity $\sim d^{O(n)}$

- ***Sum-of-Squares programming*** and Positivstellensatz results

  ↳ If $f(x) > 0$ for all $x \in \{x \in \mathbb{R}^n : g_i(x) \geq 0\}$, then

$$f = s_0 + \sum_i s_i g_i \quad \text{for some} \quad s_i = \sum_j h_j(x)^2$$

  ↳ The search for s_i is a Semi-Definite Program with complexity $\sim n^{O(d)}$

Schweighofer's Parameterization of Positive Polynomials

Suppose the semi-algebraic set

$$S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x], i = 1, \cdots, K\}$$

is bounded and define the Cone

$$\Theta_d := \left\{ \sum_{\lambda \in \mathbb{N}^K : \lambda_1 + \cdots + \lambda_K \leq d} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K} \; : \; s_\lambda \text{ are SOS} \right\}.$$

Theorem(*Schweighofer's parameterization*): If polynomial f satisfies

1. $f \geq 0$ on S
2. $f = q_1 p_1 + q_1 p_2 + \cdots$ for some $q_i \in \Theta_d$ and $p_i > 0$ on $S \cap \{x \in \mathbb{R}^n : f(x) = 0\}$

then, $f \in \Theta_d$.

We are interested in a **special case** of this parameterization:

$$g_i \text{ are affine}, \;\; p_i = 1 \text{ and } s_\lambda = Constant \geq 0$$

Handelman's Theorem: A Parameterization Using Polytopes

Handelman's Theorem:
Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, suppose

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$$

is bounded. If polynomial $f(x) > 0$ on Γ , then there exist $b_\alpha \geq 0$ with $\alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,

$$f(x) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \alpha_1 + \cdots + \alpha_K \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

- Theorem uses **products of affine functions** as a basis to parameterize polynomials that are positive on convex polytopes
- Converse of theorem yields a test for nonnegativity on polytopes

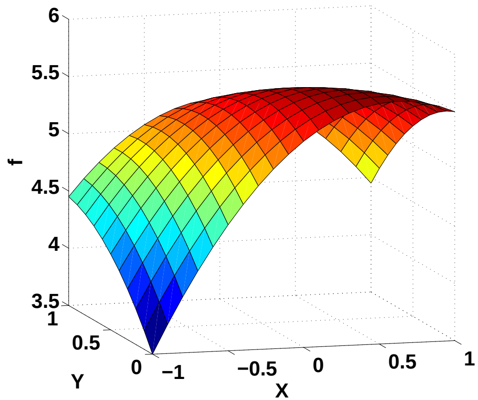
Handelman's Theorem: A Test for Positivity

Example 1:

Is $f(x,y) = -\frac{4}{5}x^2 - xy + x - \frac{3}{4}y^2 + 0.7y + 5.3 \geq 0$
over $\Gamma := [-1,1] \times [0,1]$?

The polytope Γ is defined by the inequalities:

$$x+1 \geq 0, \quad 1-x \geq 0, \quad y \geq 0, \quad 1-y \geq 0$$



Choosing $d=2$, polytope Γ yields the following Handelman basis:

$$\{1, x+1, 1-x, y, 1-y, (x+1)^2, (x+1)(1-x), (x+1)y, y^2,$$
$$(x+1)(1-y), (1-x)^2, (1-x)y, (1-x)(1-y), y(1-y), (1-y)^2\}$$

Example 1: Continued ...

Write polynomial f as

$$f(x,y) = -\frac{4}{5}x^2 - xy + x - \frac{3}{4}y^2 + 0.7y + 5.3 = \begin{bmatrix} 5.3 & 1 & 0.7 & \frac{-4}{5} & -1 & \frac{-3}{4} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

Convert the Handelman basis to the monomial basis:

$$\begin{bmatrix} 1 \\ x+1 \\ 1-x \\ y \\ 1-y \\ (x+1)^2 \\ (x+1)(1-x) \\ (x+1)y \\ y^2 \\ (x+1)(1-y) \\ (1-x)^2 \\ (1-x)y \\ (1-x)(1-y) \\ y(1-y) \\ (1-y)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

Example 1: Continued ...

From Handelman's Theorem, we search for $b_\alpha \geq 0$ such that

$$f(x,y) = \sum_{\alpha \in \mathbb{N}^K : \alpha_1 + \cdots + \alpha_K \leq 2} b_\alpha (x+1)^{\alpha_1} (1-x)^{\alpha_2} y^{\alpha_3} (1-y)^{\alpha_4}.$$

Substitute for the RHS:

$$f(x,y) = \begin{bmatrix} 5.3 \\ 1 \\ 0.7 \\ \frac{-4}{5} \\ -1 \\ \frac{-3}{4} \end{bmatrix}^T \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix} = \begin{bmatrix} b_{[0,0,0,0]} \\ b_{[1,0,0,0]} \\ b_{[0,1,0,0]} \\ b_{[0,0,1,0]} \\ b_{[0,0,0,1]} \\ b_{[2,0,0,0]} \\ b_{[1,1,0,0]} \\ b_{[1,0,1,0]} \\ b_{[1,0,0,1]} \\ b_{[0,2,0,0]} \\ b_{[0,1,1,0]} \\ b_{[0,1,0,1]} \\ b_{[0,0,2,0]} \\ b_{[0,0,1,1]} \\ b_{[0,0,0,2]} \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ x^2 \\ xy \\ y^2 \end{bmatrix}$$

Example 1: Continued ...

Finally, positivity of f on $\Gamma := [-1, 1] \times [0, 1]$ can be expressed as:

Find $b_\alpha \geq 0$ subject to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & -1 & 0 \\ 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} b_{[0,0,0,0]} \\ b_{[1,0,0,0]} \\ b_{[0,1,0,0]} \\ b_{[0,0,1,0]} \\ b_{[0,0,0,1]} \\ b_{[2,0,0,0]} \\ b_{[1,1,0,0]} \\ b_{[1,0,1,0]} \\ b_{[1,0,0,1]} \\ b_{[0,2,0,0]} \\ b_{[0,1,1,0]} \\ b_{[0,1,0,1]} \\ b_{[0,0,2,0]} \\ b_{[0,0,1,1]} \\ b_{[0,0,0,2]} \end{bmatrix} = \begin{bmatrix} 5.3 \\ 1 \\ 0.7 \\ \frac{-4}{5} \\ -1 \\ \frac{-3}{4} \end{bmatrix}$$

Example 1: Finished!

The problem is in the dual form of **Linear Program**:

$$\min_{b_\alpha \geq 0} \quad c^T b_\alpha$$

$$\text{subject to} \quad A\, b_\alpha = g$$

By solving the LP we get

$$b_\alpha = [1.5 \;\; 0 \;\; 0 \;\; 2.2 \;\; 0 \;\; 0 \;\; 0.8 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0 \;\; 0.75 \;\; 3.5 \;\; 2]$$

By plugging b_α in

$$f(x,y) = \sum_{\alpha \in \mathbb{N}^K : \alpha_1 + \cdots + \alpha_K \leq 2} b_\alpha (x+1)^{\alpha_1} (1-x)^{\alpha_2} y^{\alpha_3} (1-y)^{\alpha_4}.$$

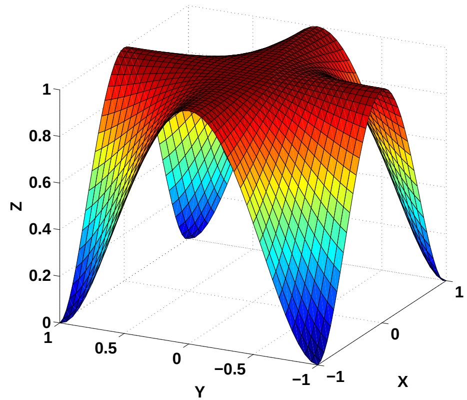
we get

$$\begin{aligned} f(x,y) &= -\frac{4}{5}x^2 - xy + x - \frac{3}{4}y^2 + 0.7y + 5.3 \\ &= 1.5 + 2.2y + 0.8(x+1)(1-x) + y^2 \\ &+ 0.75(1-x)(1-y) + 3.5y(1-y) + 2(1-y)^2 \end{aligned}$$

Handelman's Theorem: A Test for Positivity

Example 2: (Motzkin Polynomial)
Is $f(x,y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 \geq 0$ over
$\Gamma := [-1,1] \times [-1,1]$?

- It is well-known that Motzkin polynomial is not SOS of polynomials.
- However $f(x,y)$ it can be represented in Handelman basis with all positive coefficients:



$$
\begin{aligned}
f(x,y) = 0.125(&\lambda_1^3\lambda_3^2\lambda_4^2 + \lambda_1^2\lambda_2^2\lambda_3^3 + \lambda_1^2\lambda_2^2\lambda_4^3 + \lambda_2^3\lambda_3^2\lambda_4^2 + \lambda_1^3\lambda_2\lambda_3\lambda_4) \\
&+0.0625(\lambda_1^2\lambda_2\lambda_3^3\lambda_4 + \lambda_1\lambda_2^3\lambda_3^2\lambda_4 + \lambda_1\lambda_2^3\lambda_3\lambda_4^2 + \lambda_1\lambda_2^2\lambda_3^3\lambda_4 \\
&+ \lambda_1\lambda_2\lambda_3^2\lambda_4^3 + \lambda_1\lambda_2\lambda_3\lambda_4^4)
\end{aligned}
$$

$$
\lambda_1(x) := 1-x, \quad \lambda_2(x) := 1+x, \quad \lambda_3(y) := 1-y, \quad \lambda_4(y) := 1+y
$$

Handelman's theorem precludes interior zeros

Recall the problem of stability analysis:

$$\begin{aligned} s^* := \max_{V \in \mathbb{R}[x], s>0} \quad & s \\ \text{subject to} \quad & V(x) - \epsilon x^T x \geq 0 && \text{for all } x \in \Gamma_p(s) \\ & \nabla V(x)^T f(x) + \epsilon x^T x \leq 0 && \text{for all } x \in \Gamma_p(s). \end{aligned}$$

- We need to search over polynomials $V(x) \geq 0$ with $V(0) = 0$
- Handelman's theorem allows for zeros on the vertices of Γ_p

  ↳ Motzkin example

- Handelman's theorem does **NOT** parameterize positive polynomials with zeros in the interior of Γ_p (why?)

Handelman's theorem precludes interior zeros

Proof:

Suppose $f(a)=0$ for some $a \in \text{int}(\Gamma)$, where

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}.$$

Suppose there exist $b_\alpha \geq 0, \alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,

$$f(x) = \sum_{\alpha \in \mathbb{N}^K : \|\alpha_i\|_1 \leq d} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

Then,

$$f(a) = \sum_{\alpha \in \mathbb{N}^K : \|\alpha_i\|_1 \leq d} b_\alpha (w_1^T a + u_1)^{\alpha_1} \cdots (w_K^T a + u_K)^{\alpha_K} = 0.$$

Since $a \in \text{int}(\Gamma)$, $w_i^T a + u_i > 0$ for $i = 1, \cdots, K$.
Hence there exist some $\alpha \in \{\alpha \in \mathbb{N}^K : \|\alpha\|_1 \leq d\}$ such that

$$b_\alpha < 0.$$

This contradicts with the assumption that all $b_\alpha \geq 0$.

Applying Handelman's Theorem to Stability Analysis

Step 1: Decomposition

Decompose the polytope

$$\Gamma := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$$

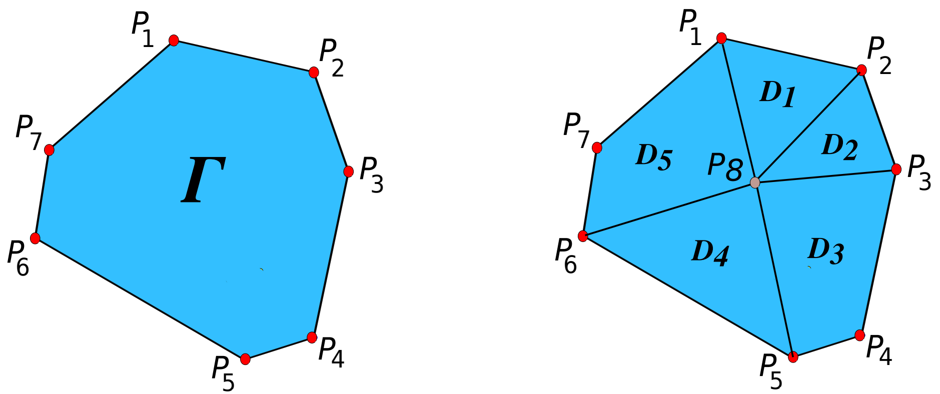
into L subpolytopes

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$$

such that

$$\cup_{i=1}^L D_i = \Gamma, \;\; \cap_{i=1}^L D_i = \{0\}, \;\; \text{int}(D_i) \cap \text{int}(D_j) = \emptyset$$

Example:



Applying Handelman's Theorem to Stability Analysis

Step 2: Enforcing $V(0)=0$

- For each subpolytope D_i with m edges, let

$$V_i(x) = \sum_{\alpha \in \mathbb{N}^m : \|\alpha\|_1 \le d} b_{i,\alpha} \lambda_1(x)^{\alpha_1} \cdots \lambda_m(x)^{\alpha_m}$$

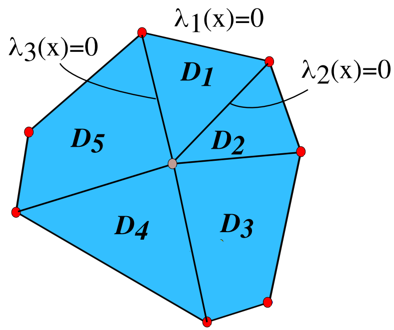
where $\lambda_j(x) := h_{i,j}^T x + g_{i,j}$ define the edges of D_i.

- To enforce $V_i(0)=0$, set

$$b_{i,\alpha} = 0 \ \text{ for all } \ \alpha \in \{\alpha : \alpha_j = 0 \text{ for all } j : \lambda_j(0) = 0\}$$

Example:

$$V_1(x) = \cancel{b_{1,[0,0,0]}} + \cancel{b_{1,[1,0,0]}}\lambda_1 + b_{1,[0,1,0]}\lambda_2 + b_{1,[0,0,1]}\lambda_3$$
$$+ \cancel{b_{1,[2,0,0]}}\lambda_1^2 + b_{1,[1,1,0]}\lambda_1\lambda_2 + b_{1,[1,0,1]}\lambda_1\lambda_3 + b_{1,[0,1,1]}\lambda_2\lambda_3$$
$$+ b_{1,[0,2,0]}\lambda_2^2 + b_{1,[0,1,1]}\lambda_2\lambda_3 + b_{1,[0,0,2]}\lambda_3^2$$

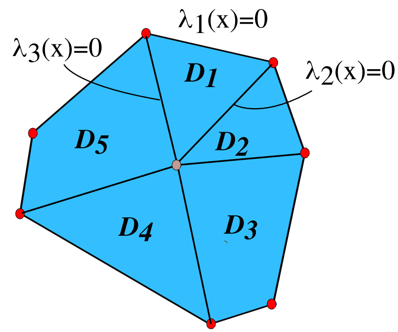


Applying Handelman's Theorem to Stability Analysis

Step 3: Ensuring continuity of $V(x)$

- For any two adjacent subpolytopes D_i and D_j with $D_i \cap D_j = \{x : \lambda(x) = 0\}$, set

$$V_i(x) \mid_\lambda = V_j(x) \mid_\lambda$$



Example:

- Define V_1 on D_1 as

$$V_1(x) = b_{1,[0,1,0]}\lambda_2 + b_{1,[0,0,1]}\lambda_3 + b_{1,[1,1,0]}\lambda_1\lambda_2 + b_{1,[1,0,1]}\lambda_1\lambda_3 + b_{1,[0,1,1]}\lambda_2\lambda_3 + b_{1,[0,2,0]}\lambda_2^2 + b_{1,[0,1,1]}\lambda_2\lambda_3 + b_{1,[0,0,2]}\lambda_3^2$$

- Restriction of V_1 to $\lambda_2(x) = 0$:

$$V_1(x) \mid_{\lambda_2} = b_{1,[0,0,1]}\lambda_3 + b_{1,[1,0,1]}\lambda_1\lambda_3 + b_{1,[0,0,2]}\lambda_3^2 = [l_1(b_1) \;\; l_2(b_1) \;\; l_3(b_1) \;\; l_4(b_1) \;\; l_5(b_1)]\,[1 \;\; x \;\; y \;\; xy \;\; x^2 \;\; y^2]^T$$

- Similarly, On D_2 we have

$$V_2(x) \mid_{\lambda_2} = [g_1(b_2) \;\; g_2(b_2) \;\; g_3(b_2) \;\; g_4(b_2) \;\; g_5(b_2)]\,[1 \;\; x \;\; y \;\; xy \;\; x^2 \;\; y^2]^T$$

- To enforce continuity on $\{x : \lambda_2(x) = 0\}$, set

$$l(b_1) = g(b_2)$$

Applying Handelman's Theorem to Stability Analysis

Step 4: Enforcing $\dot{V} < 0$ on Γ

- For each subpolytope D_i, put

$$\dot{V}_i(x) = \langle \nabla V_i, f(x) \rangle = \nabla \left(\sum_{\|\alpha\|_1 \le d} b_{i,\alpha} \lambda_1(x)^{\alpha_1} \cdots \lambda_m(x)^{\alpha_m} \right)^T f(x)$$

$$= [h_1(b_i) \;\; h_2(b_i) \cdots] \, [1 \;\; x_1 \;\; x_2 \;\; \cdots]^T$$

$$= \sum_{\|\beta\|_1 \le d + d_f - 1} c_{i,\beta} \lambda_1(x)^{\beta_1} \cdots \lambda_m(x)^{\beta_m}$$

$$= [q_1(c_i) \;\; q_2(c_i) \cdots] \, [1 \;\; x_1 \;\; x_2 \;\; \cdots]^T$$

- To enforce $\dot{V} < 0$ on D_i we need to solve:

Find	$b_{i,\alpha} \in \mathbb{R}$ and $c_{i,\beta} \le 0$
such that	$q(c_i) = h(b_i)$

Applying Handelman's Theorem to Stability Analysis

Step 1: Decomposition of original polytope Γ into D_i

$$\cup_{i=1}^{L} D_i = \Gamma, \quad \cap_{i=1}^{L} D_i = \{0\}, \quad \text{int}(D_i) \cap \text{int}(D_j) = \emptyset$$

Step 2: Enforcing $V(0) = 0$

$$V_i(x) = \sum_{\alpha \in \mathbb{N}^m : \|\alpha\|_1 \leq d} b_{i,\alpha} \lambda_1(x)^{\alpha_1} \cdots \lambda_m(x)^{\alpha_m}$$

$$b_{i,\alpha} = 0 \ \text{ for all } \ \alpha \in \{\alpha : \alpha_j = 0 \text{ for all } j : \lambda_j(0) = 0\}$$

Step 3: Continuity of V on λ

$$V_i(x) \mid_{\lambda} = V_j(x) \mid_{\lambda} \ \Leftrightarrow \ \ l(b_i) = g(b_j)$$

where l and g are affine in b_i and b_j

Step 4: Enforcing $\dot{V} < 0$ on Γ

$$q(c_i) = h(b_i), \quad c_i \leq \mathbf{0}$$

where q and h are affine in c_i and b_i

The Four Steps Define a Linear Program

Given a polytope, the stability analysis problem can be expressed as the following **Linear Program**:

$$\min_{\substack{b_{i,\alpha}\geq 0 \\ c_{i,\alpha}\leq 0}} \quad c^T[b_{1,\alpha},\cdots,b_{L,\alpha}]$$

$$\begin{array}{lll} \text{subject to} & b_{i,\alpha}=0 & \text{for all } \alpha\in\{\alpha:\alpha_j=0,\forall j:\lambda_j(0)=0\} \\ & l(b_i)=g(b_j) & \text{for all } i,j\in\{1,\cdots,L\}:D_i\cap D_j\neq\{0\} \\ & q(c_i)=h(b_i) & \text{for all } i\in\{1,\cdots,L\} \end{array}$$

A solution to the Linear Program yields the Lyapunov function

$$V(x)=V_i(x)=\sum_{\|\alpha\|_1\leq d} b_{i,\alpha}\lambda_1(x)^{\alpha_1}\cdots\lambda_m(x)^{\alpha_m}$$

for $x\in D_i,\ i=1,\cdots,L$

Numerical Example: Van-Der-Pol Oscillator

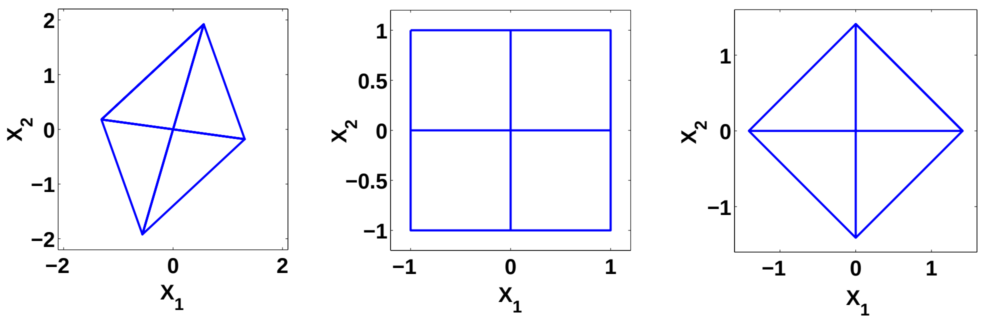
For the Van-Der-Pol oscillator in reverse-time

$$\dot{x}_1(t) = -x_2(t), \quad \dot{x}_2(t) = x_1(t) + x_2(t)\left(x_1^2(t) - 1\right)$$

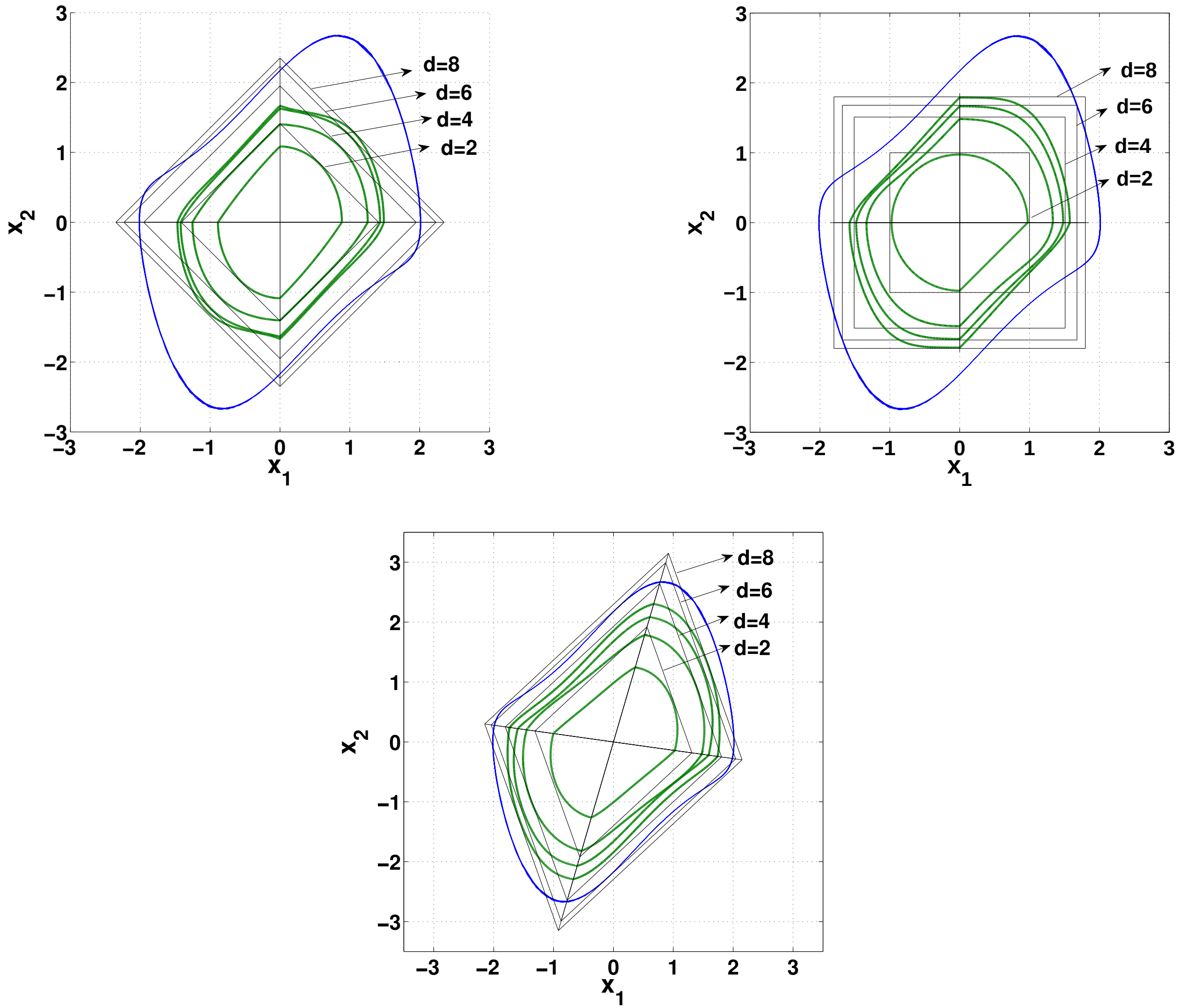
We solved

$$\begin{aligned} s^* := \max_{V \in \mathbb{R}[x], s>0} \quad & s \\ \text{subject to} \quad & V(x) - \epsilon x^T x \geq 0 && \text{for all } x \in \Gamma_p(s) \\ & \nabla V(x)^T f(x) + \epsilon x^T x \leq 0 && \text{for all } x \in \Gamma_p(s) \end{aligned}$$

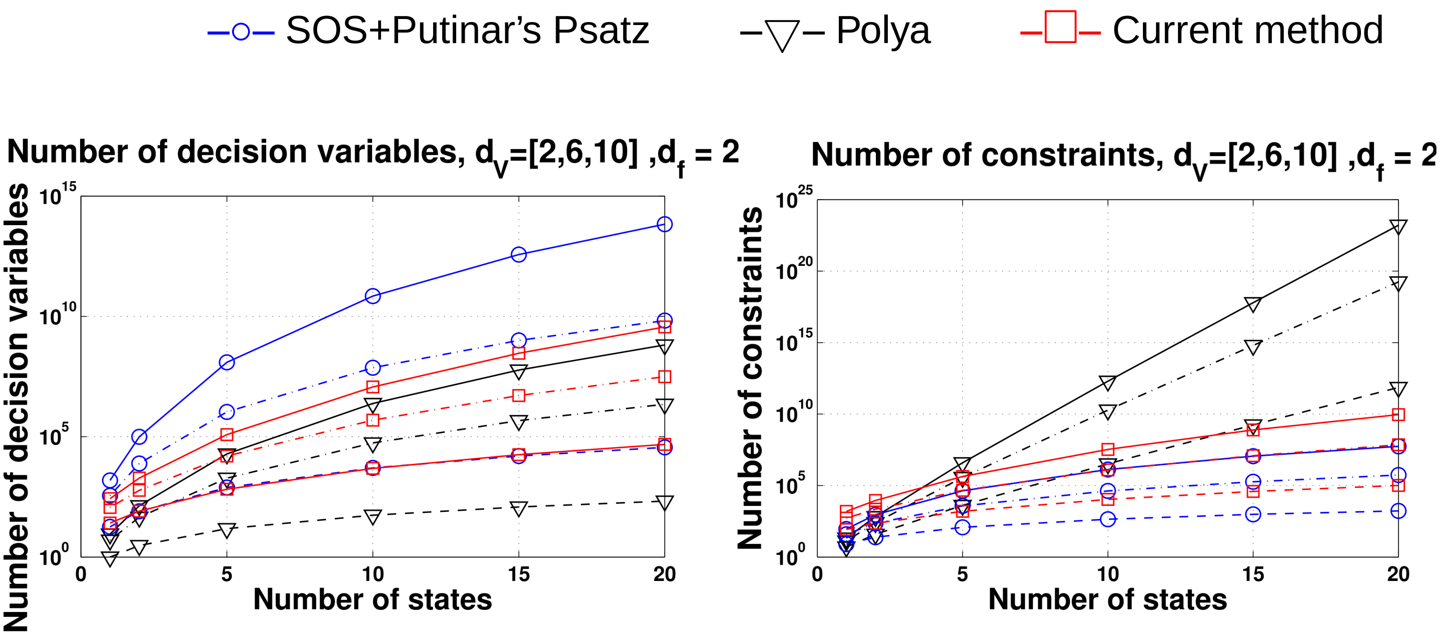
using the following polytopes as Γ_p



For Fixed d, Quadrilateral Results in Better Estimation



Complexity Scales Polynomially in State Space Dimension



	SOS	Polya	Current method
Complexity of LP/SDP	$\sim n^{3.5(d_V+d_f)-3}$	$\sim (d_V+d_f+e-2)^{3n}$	$\sim n^{3(d_V+d_f)}$

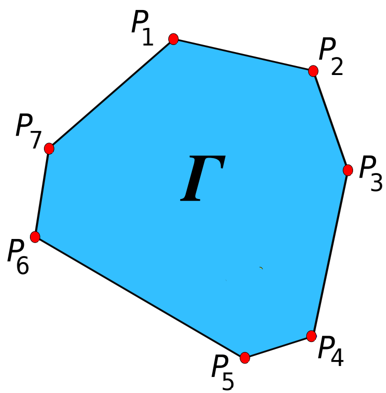
d_V: degree of $V(x)$ d_f: degree of $f(x)$ n: No. of states

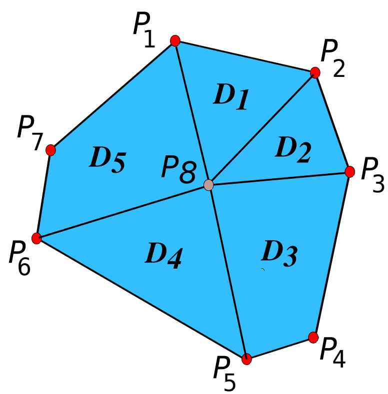
Conclusion

- We proposed a methodology based on Handelman's theorem to perform stability analysis on nonlinear ODEs

$$\dot{x}(t) = f(x(t)) \qquad x(t) \in \Gamma \text{ (Convex polytopes)}$$

using a decomposition of the polytope.





- The method can be readily applied to stability analysis of

$$\dot{x}(t) = f(x(t), \alpha) \qquad x(t), \alpha \in \Gamma \text{ (Convex polytopes)}$$