Modern Control Systems

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Lecture 9: Controllability and Observability

Controllability

We would like to prove that

$$R_t = \mathsf{Im}(W_t) = \mathsf{Im}(C(A, B))$$

To do this, we will prove that

- $\mathsf{Im}(W_t) \subset R_t$
- $R_t \subset \mathsf{Im}(C(A, B))$
- $\mathsf{Im}(C(A, B)) \subset \mathsf{Im}(W_t)$

So far, we have show that

$$R_t \subset C_{AB}$$

Next, we will show that

$$\mathsf{Im}(W_t) \subset R_t$$

Controllability: $\text{Im}(W_t) \subset R_t$

Theorem 1.

$$\mathit{Im}(W_t) \subset R_t$$

Proof.

First, suppose that $x \in \text{Im}(W_t)$ for some $t > 0$. Then $x = W_t z$ for some z.

- Now let $u(s) = B^T e^{A^T(t-s)} z$. Then

$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)} B u(s) ds \\ &= \int_0^t e^{A(t-s)} B B^T e^{A^T(t-s)} z ds \\ &= W_t z = x\end{aligned}$$

- Thus $x \in \text{Im}(\Gamma_t) = R_t$.

We conclude that $\text{Im}(W_t) \subset R_t$ $\square$

Proof by Contradiction: $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$

The last proof is a proof by contradiction.

> *Once you eliminate the impossible, whatever remains, no matter how improbable, must be the truth.*

There are two mutually exclusive possibilities

1. $x \in \mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$
2. There exists some $x \in \mathrm{Im}(C(A,B))$ such that $x \notin \mathrm{Im}(W_t)$.

We eliminate the second possibility by showing that:

- If $x \notin \mathrm{Im}(W_t)$ then $x \notin \mathrm{Im}(C(A,B))$.

In shorthand:

$$(\neg 2 \Rightarrow \neg 1) \;\Leftrightarrow\; (1 \to 2)$$

An alternative would be to find an x which disproves the second possibility.

Proof by Contradiction: $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$

Theorem 2.

$$\mathit{Im}(C(A,B)) \subset \mathit{Im}(W_t).$$

Proof.

Suppose $x \notin \mathrm{Im}(W_t)$. Then $x \in \mathrm{Im}(W_t)^\perp$.

- As we have shown, this means $x \in \ker(W_t)$, so $W_t x = 0$.
- Thus

$$\begin{aligned} x^T W_t x &= \int_0^t x^T e^{A(t-s)} B B^T e^{A^T(t-s)} x ds \\ &= \int_0^t u(s)^T u(s) ds = 0 \end{aligned}$$

  where $u(s) = B^T e^{A^T(t-s)} x$.
- This implies $u(s) = B^T e^{A^T(t-s)} x = 0$ for all $s \in [0,t]$.

Proof by Contradiction: $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$

Proof.

- This means that for all $s \in [0,t]$,

$$\frac{d^k}{ds^k} B^T e^{A^T s} x = B^T \left(A^T\right)^k e^{A^T s} x = 0$$

- At $s = 0$, this implies $B^T \left(A^T\right)^k x = 0$ for all k.
- We conclude that

$$x^T \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0$$

- Thus $C(A,B)^T x = 0$, so $x \in \ker C(A,B)^T$. As before, this means $x \in \mathrm{Im}(C(A,B))^\perp$.
- We conclude that $x \notin \mathrm{Im}(C(A,B))$. This proves by contradiction that $\mathrm{Im}(C(A,B)) \subset \mathrm{Im}(W_t)$.

□

Summary: $R_t = \mathsf{Im}(W_t) = \mathsf{Im}(C(A,B))$

We have shown that

$$R_t = \mathsf{Im}(W_t) = \mathsf{Im}(C(A,B))$$

Moreover, we have shown that for any $x_d \in R_{T_f}$, we can find a controller

- Choose any z such that $x_d = W_{T_f} z$. ($z = W_{T_f}^{-1} x$ if W_{T_f} is invertible)
- Let $u(t) = B^T e^{A^T(T_f - t)} z$.
- Then the system $\dot{x}(t) = Ax(t) + B(t)$ with $x(0) = 0$ has solution with $x(T_f) = x_d$.
- $x_d = \Gamma_{T_f} u$.

Representation and Controllability

Question: Is the representation (A, B, C, D) of the system $y = Gu$,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

unique?

Question: Do there exist $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ such that y and u also satisfy,

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}u(t)$$
$$y(t) = \hat{C}x(t) + \hat{D}u(t)$$

Answer: Of Course! Recall the similarity transform: $z(t) = Tx(t)$ for any invertible T. Then y and u also satisfy,

$$\begin{aligned}
\dot{z}(t) &= T\dot{x}(t) = TAx(t) + TBu(t) \\
&= TAT^{-1}z(t) + TBu(t) \\
y(t) &= Cx(t) + Du(t) \\
&= CT^{-1}x(t) + Du(t)
\end{aligned}$$

Representation and Controllability

Thus the pair $(TAT^{-1}, TB, CT^{-1}, D)$ is also a representation of the map $y = Gu$.

- Furthermore $x(t) \to 0$ if and only if $z(t) \to 0$.
- So internal stability is unaffected.

Controllability is Unaffected:

$$\begin{aligned} C(TAT^{-1}, TB) &= \begin{bmatrix} TB & TAT^{-1}TB & TAT^{-1}TAT^{-1}TB & \cdots & TA^{n-1}B \end{bmatrix} \\ &= TC(A, B) \end{aligned}$$

Invariant Subspaces

Definition 3.

A subspace, $W \subset X$, is **Invariant** under the operator $A : X \to X$ if $x \in W$ implies $Ax \in W$.

For a linear operator, only subspaces can be invariant.

Proposition 1.

If W if A-invariant, then there exists an invertible T, such that

$$TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \qquad \textit{and} \qquad TW = \textit{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}$$

That is, for any $x \in W$, $Tx = \begin{bmatrix} \bar{x}_1 \\ 0 \end{bmatrix}$, which is clearly $\bar{A}$-invariant.

Invariant Subspaces

Proposition 2.

C_{AB} is A-invariant.

Proof.

The proof is direct

$$A \cdot C(A,B) = A \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = \begin{bmatrix} AB & A^2B & \cdots & A^nB \end{bmatrix}$$

But, by Cayley-Hamilton,

$$A^n = \sum_{i=0}^{n-1} a_i A^i$$

so if $x \in C_{AB}$, there exists a z such that

$$\begin{aligned} Ax &= A \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} z \\ &= \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} z_n a_0 \\ z_1 + z_n a_1 \\ \vdots \\ z_{n+1} + z_n a_{n-1} \end{bmatrix} \in C_{AB} \end{aligned}$$

Controllability Form

Since C_{AB} is an invariant subspace of A, there exists an invertible T such that

$$TAT^{-1} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$$

and $Tx = \begin{bmatrix} \bar{x} \\ 0 \end{bmatrix}$ for any $x \in C_{AB}$.

- Clearly $B \in C_{AB}$.
- Thus $TB = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$.

Definition 4.

The pair (A, B) is in **Controllability Form** when

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

and the pair (A_{11}, B_1) is controllable.

Controllability Form

When a system is in controllability form, the dynamics have special structure

$$\begin{aligned}\dot{x}_1(t) &= A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)\\ \dot{x}_2(t) &= A_{22}x_2(t)\end{aligned}$$

The uncontrolled dynamics are autonomous.

- Cannot be stabilized or controlled.

We can formulate a procedure for putting a system in Controllability Form

1. Find an orthonormal basis, $\begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}$ for C_{AB}.
2. Complete the basis in $\mathbb{R}^n$: $\begin{bmatrix} v_{r+1} & \cdots & v_n \end{bmatrix}$.
3. Define $T = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$.
4. Construct $\bar{A} = TAT^{-1}$ and $\bar{B} = TB$
    - Works for ANY invariant subspace.

Controllability Form

Example

Let

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Construct $C(A,B) = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$.

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},$$

$$A^2B = A(AB) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$C(A,B) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\operatorname{rank} C(A,B) = 2 < n = 3$ which means not controllable.

Controllability Form

Example Continued

Using Gramm-Schmidt, we can construct an orthonormal basis for C_{AB}

$$C_{AB} = \text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} = \text{span}\left\{v_1, v_2\right\}$$

Let $v_3 = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^T$. Then

$$T^{-1} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = I$$

So $T = I$, which is because the system is already in controllability form. We could also have used

$$T^{-1} = \begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 1\end{bmatrix} \quad \text{to get} \quad TAT^{-1} = \begin{bmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & -1\end{bmatrix} = A$$