# Systems Analysis and Control

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Lecture 12: Root Locus

# Overview

In this Lecture, you will learn:

Review of Feedback

- Closing the Loop
- Pole Locations

**Changing the Gain**

- Numerical Examples
  - Pole Locations
- Routh-Hurwitz vs. Root Locus

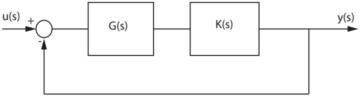
A Review of Complex Numbers

- Polar Form
- Multiplication-Division

# The Effect of Feedback

Feedback changes the open loop

$$\hat{G}(s) = \frac{n_G(s)}{d_G(s)} \qquad \hat{K}(s) = \frac{n_K(s)}{d_K(s)}$$




to

$$\frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)} = \frac{n_G(s)n_K(s)}{d_G(s)d_K(s)+n_G(s)n_K(s)}$$

The pole locations are the roots of

$$d_G(s)d_K(s)+n_G(s)n_K(s) = 0$$

**Objective:** A closed loop denominator.

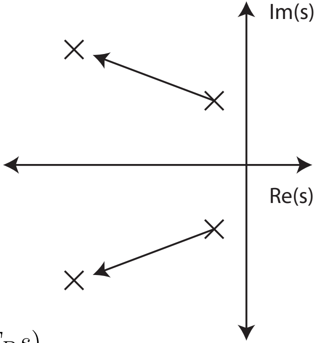
$$d(s) = (s-p_1)\cdots(s-p_n)$$

**Big Question:** How to choose $n_K(s)$ and $d_K(s)$ so that

$$d(s) = d_G(s)d_K(s)+n_G(s)n_K(s)$$

For a **Second-Order System**

$$\hat{G}(s) = \frac{1}{s^2 + as + b}$$




with PD feedback

$$\hat{K}(s) = K\,(1 + T_D s)$$

We can achieve any denominator

$$d(s) = s^2 + cs + d$$

**Question:** What happens for more complicated systems:

**Open Loop:**

$$\frac{s^2+s+1}{s^4+2s^3+3s^2+s+1}$$

**Closed Loop:**

$$\frac{K\left(1+T_D s\right)\left(s^2+s+1\right)}{s^4+2s^3+3s^2+s+1+K\left(1+T_D s\right)\left(s^2+s+1\right)}$$
$$=\frac{K\left(T_D s^3+(1+T_D)s^2+(1+T_D)s+1\right)}{s^4+(2+KT_D)s^3+(3+K+KT_D)s^2+(1+K+KT_D)s+1+K}$$

Given a desired denominator

$$d(s)=s^4+as^3+bs^2+cs+d$$

Which gives 4 equations and 2 unknowns

$$a=2+KT_D \qquad b=3+K+KT_D$$
$$c=1+K+KT_D \qquad d=1+K$$

There is No Solution!!!.

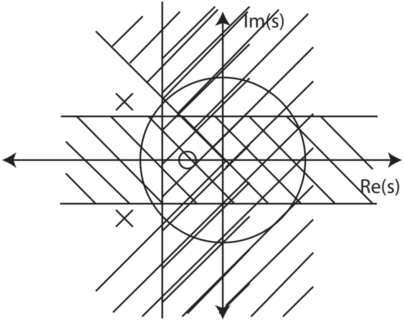
# The Effect of Feedback

## Solution

We rarely need to achieve a precise set of poles.

Performance Specifications Determine **Regions** of the Complex Plane.

- Stability
- Rise Time
- Settling time
- Overshoot




**New Question:** What controller will ensure all roots of

$$d_G(s)d_K(s) + n_G(s)n_K(s)$$

lie in the desired region of the complex plane.

# The Effect of Feedback

## Proportional Feedback

More fundamentally, how does changing $n_K(s)$ and $d_K(s)$ change the roots of

$$d_G(s)d_K(s) + n_G(s)n_K(s)?$$

The answer is complicated

- Must account for the effect of each term in $n_K$ and $d_K$

So simplify, lets consider a controller with only a single free parameter.

$$\hat{K}(s) = k$$

Other options include:

- **PD Control**: $\hat{K}(s) = 1 + T_D s$
- **PI Control**: $\hat{K}(s) = 1 + \frac{1}{T_I s}$

**Question:** How do the roots of

$$d_G(s) + kn_G(s)?$$

change with $k$?

$$\hat{G}(s) = \frac{n_G(s)}{d_G(s)}$$

**Definition 1.**

The **Root Locus** of $\hat{G}(s)$ is the set of all poles of

$$\frac{k\hat{G}(s)}{1 + k\hat{G}(s)}$$

as $k$ ranges from $0$ to $\infty$

**Alternatively:**

- The roots of $1 + k\hat{G}(s)$ for $k \geq 0$
- The roots of $d_G(s) + kn_G(s)$ for $k > 0$
- The solutions of $\hat{G}(s) = \frac{-1}{k}$ for $k \geq 0$

We can estimate the root locus by finding the roots for several different values of $k$

**Example:** Video Surveillance System.

Pole at $s = 0$ to eliminate steady-state error.

**Open Loop:**

$$\hat{G}(s) = \frac{1}{s(s+10)}$$

**Closed Loop:**

$$\frac{k}{s^2 + 10s + k}$$

**Pole Locations:**

$$p_{1,2} = -5 \pm \frac{1}{2}\sqrt{100 - 4k}$$

**TABLE 8.1** **Pole location as function of gain for the system of Figure 8.4**

| $K$ | Pole 1 | Pole 2 |
|---|---|---|
| 0 | $-10$ | 0 |
| 5 | $-9.47$ | $-0.53$ |
| 10 | $-8.87$ | $-1.13$ |
| 15 | $-8.16$ | $-1.84$ |
| 20 | $-7.24$ | $-2.76$ |
| 25 | $-5$ | $-5$ |
| 30 | $-5 + j2.24$ | $-5 - j2.24$ |
| 35 | $-5 + j3.16$ | $-5 - j3.16$ |
| 40 | $-5 + j3.87$ | $-5 - j3.87$ |
| 45 | $-5 + j4.47$ | $-5 - j4.47$ |
| 50 | $-5 + j5$ | $-5 - j5$ |

$$p_{1,2} = -5 \pm \frac{1}{2}\sqrt{100 - 4k}$$

We can visualize the effect of changing $k$ by plotting the poles on the complex plane.

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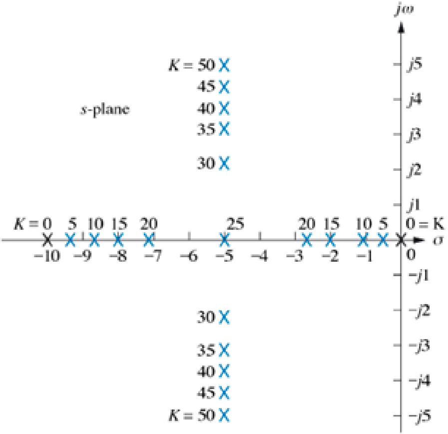
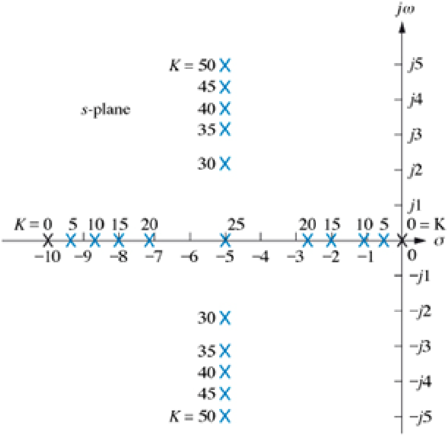
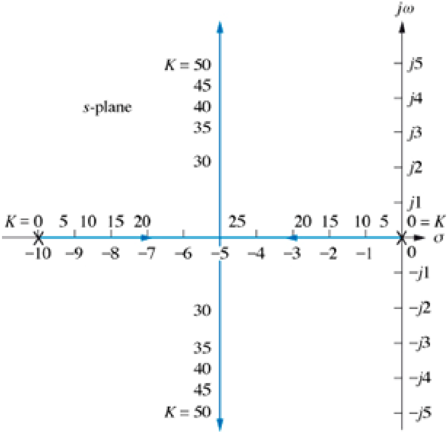


Plotting every possible value of $k$ yields the *root locus*.

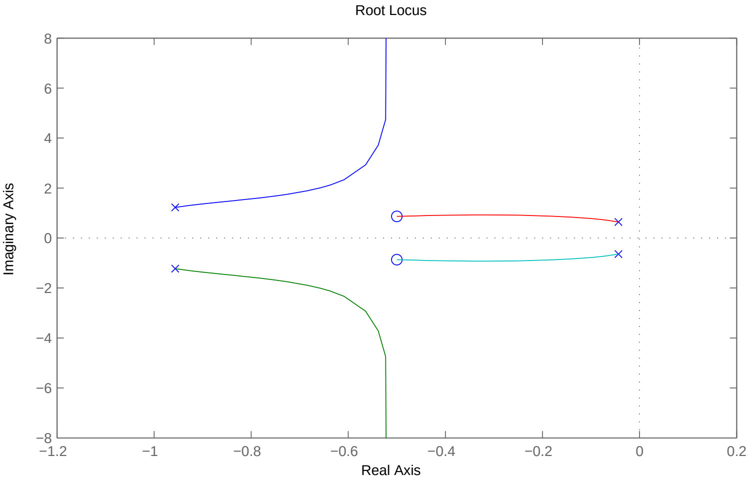



Connect the dots.

# Root Locus

## Example: Suspension System

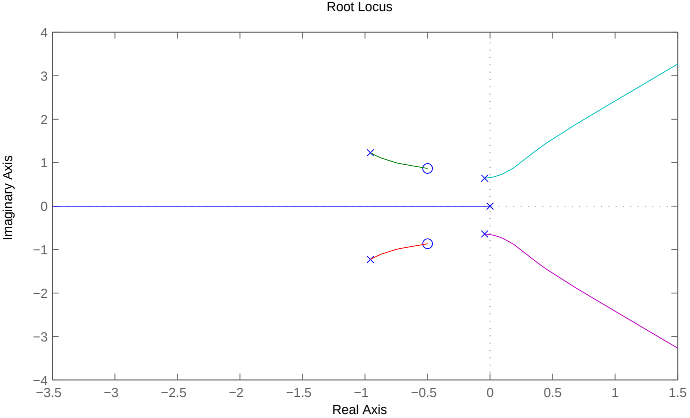



**From Routh Test:** Stable for all $k > 0$.

# Root Locus

## Example: Suspension System with Integral Feedback

Now, if we add integral feedback: $\hat{K}(s) = k\frac{1}{s}$

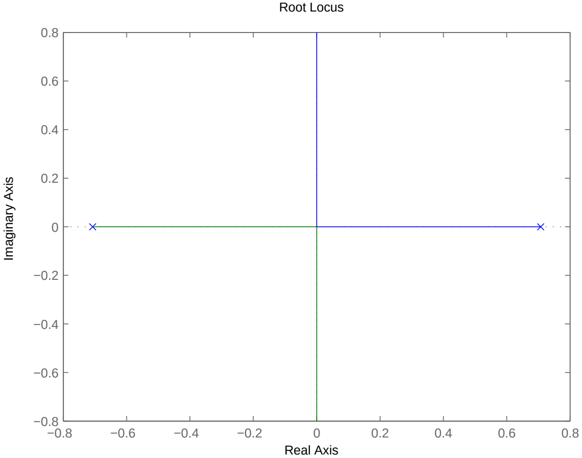



**From Routh Test:** Stable for all $k < .1$.

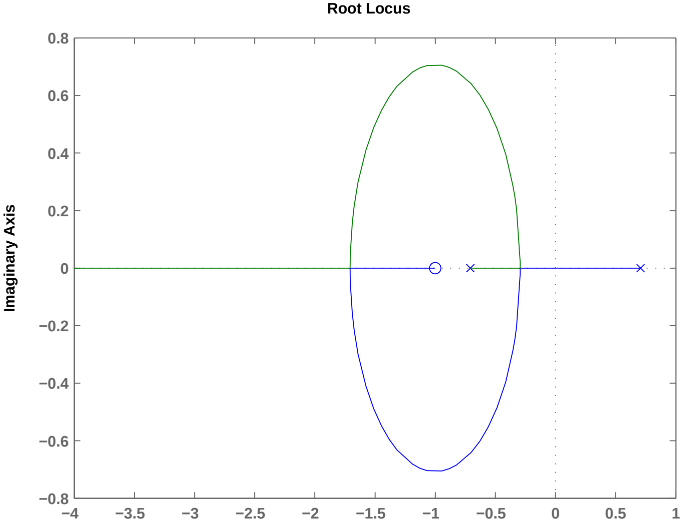
# Root Locus

## Example: Inverted Pendulum Model

$$\hat{G}(s) = \frac{1}{s^2 - \frac{1}{2}}$$

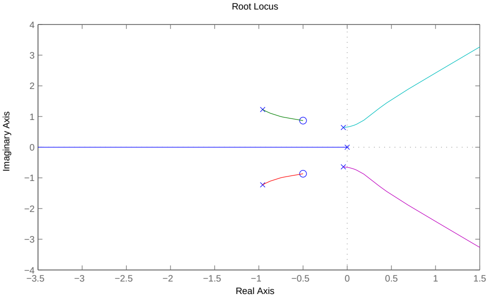


Now an inverted pendulum with some derivative feedback: $\hat{K}(s) = k(1+s)$



# Root Locus

## Complex Numbers




When Matlab calculates the root locus, it plots every point.

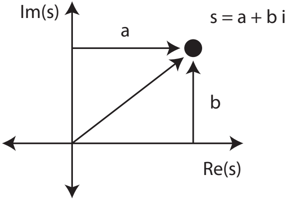
- Impractical for students
- Yields no intuition.
  - Root Locus is only one parameter.
  - We must know how to manipulate the root locus by changes in controller type.

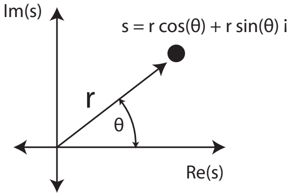
Before we analyze the root locus, we begin with a review of Complex Numbers.

Consider a Complex Number:

$$s = a + b\imath$$

The *Complex Plane* is the a-b plane.




A complex number can also be represented in polar form

$$s = r\left(\cos\theta + \imath\sin\theta\right)$$

Recall the Euler equation

$$e^{\imath\theta} = \cos\theta + \imath\sin\theta$$

Euler yields the more practical form:

$$s = re^{\theta\imath}$$

**Rectilinear**

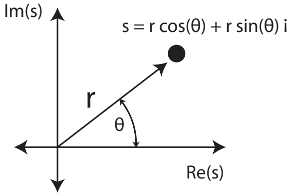
$$s = a + b\imath$$

Notation:

- $r$ is called the **Magnitude**
  - Denoted $r = |s|$
- $\theta =$ is called the **Phase**
  - Denoted $\theta = \angle s$

**Polar**

$$s = re^{\theta\imath}$$




The relationship between Polar and Rectilinear coordinates is obvious

- $\theta = \tan^{-1}\left(\dfrac{b}{a}\right)$
- $r = \sqrt{a^2 + b^2}$

- $a = r\cos\theta$
- $b = r\sin\theta$

In polar form, Multiplying and Dividing complex numbers is cleaner.

$$s_1 = r_1 e^{\theta_1 \imath}$$
$$s_2 = r_2 e^{\theta_2 \imath}$$

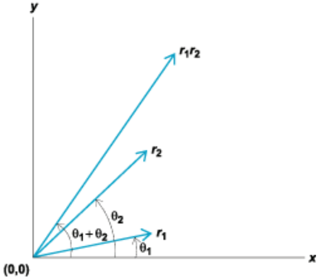
$$s_1 = a_1 + b_1 \imath$$
$$s_2 = a_2 + b_2 \imath$$

$$\begin{aligned} s_1 s_2 &= r_1 e^{\theta_1 \imath} r_2 e^{\theta_2 \imath} = r_1 r_2 e^{\theta_1 \imath} e^{\theta_2 \imath} \\ &= r_1 r_2 e^{(\theta_1 + \theta_2)\imath} \end{aligned}$$

$$\begin{aligned} s_1 \cdot s_2 &= (a_1 + b_1 \imath)(a_2 + b_2 \imath) \\ &= (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)\imath \end{aligned}$$

For multiplication

- magnitudes and phases *decouple*.
- magnitudes multiply
- phases add



For *Division*, the benefit is even greater.

$$s_1 = r_1 e^{\theta_1 \imath}$$
$$s_2 = r_2 e^{\theta_2 \imath}$$

$$s_1 = a_1 + b_1 \imath$$
$$s_2 = a_2 + b_2 \imath$$

$$s_1/s_2 = r_1 e^{\theta_1 \imath} r_2^{-1} e^{-\theta_2 \imath} = \\ = \frac{r_1}{r_2} e^{(\theta_1 - \theta_2)\imath}$$

$$s_1 \cdot s_2 = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2} \imath$$

For division in polar form,

- Again, magnitudes and phases *decouple*.
- magnitudes divide
- phases subtract

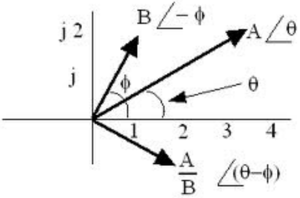



**Fig. 17: Division using Polar Form**

What does this mean for the root locus?
Recall the root locus is the set of $s$ such that

$$1 + k\hat{G}(s) = 0$$

In other words,

$$\hat{G}(s) = -\frac{1}{k}$$

In polar coordinates, this means

$$\hat{G}(s) = \frac{1}{k}e^{\pi\imath}$$

- Magnitude is $1/k$
- Phase is $\pi$rad $= 180^\circ$

Since $k$ can be anything greater than 0:

- Root locus is all point such that

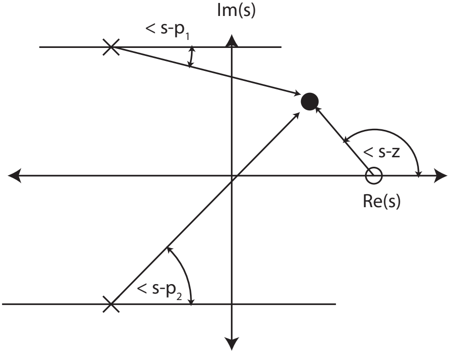
$$\angle\hat{G}(s) = 180^\circ$$

Since

$$\hat{G}(s) = \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

Then

$$\begin{aligned}\angle \hat{G}(s) &= \angle (s - z_1) + \cdots + \angle (s - z_m) \\ &\qquad - \angle (s - p_1) - \cdots - \angle (s - p_n) \\ &= \sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i)\end{aligned}$$




For a point on the root locus:

$$\sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i) = -180^\circ$$

# Summary

What have we learned today?

Review of Feedback

- Closing the Loop
- Pole Locations

**The Effect of Changes in Gain**

- Numerical Examples
  - Pole Locations
- Routh-Hurwitz

A Review of Complex Numbers

- Polar Form
- Multiplication-Division

**Next Lecture: Constructing the Root Locus**