Modern Control Systems

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Lecture 8: Controllability and Observability

Controllability

First add an input $u(t)$

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$$

The solution is

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

Use Leibniz rule for differentiation of integrals

$$\begin{aligned}\dot{x}(t) &= e^{A(t-t)} Bu(t) + \int_0^t Ae^{A(t-s)} Bu(s) ds \\ &= Bu(t) + Ax(t)\end{aligned}$$

Controllability asks whether we can "control" the system states through appropriate choice of $u(t)$.

- Note that we do not care how $u(t)$ is chosen.

We start with a weaker definition

Controllability

Definition 1.

For a given (A, B), the **state x_f is Reachable** if for any fixed T_f, there exists a $u(t)$ such that

$$x_f = \int_0^{T_f} e^{A(T_f - s)} B u(s) ds$$

Definition 2.

The **system (A, B) is reachable** if any point $x_f \in \mathbb{R}^n$ is reachable.

For a fixed t, the set of reachable states is defined as

$$R_t := \{x \,:\, x = \int_0^t e^{A(t-s)} B u(s) ds \text{ for some function } u.\}$$

Controllability

The mapping $\Gamma : u \mapsto x_f$ is linear. Let $u = \alpha u_1 + \beta u_2$

$$\begin{aligned}\Gamma u &= \int_0^{T_f} e^{A(T_f - s)} B \left(\alpha u_1(s) + \beta u_2(s)\right) ds \\ &= \alpha \int_0^{T_f} e^{A(T_f - s)} B u_1(s) ds + \beta \int_0^{T_f} e^{A(T_f - s)} B u_2(s) ds \\ &= \alpha \Gamma u_1 + \beta \Gamma u_2\end{aligned}$$

Thus $R_t = \text{Image}(\Gamma)$.

- R_t is a subspace.

Definition 3.

For a given system (A, B), the **Controllability Matrix** is

$$C(A, B) := \begin{bmatrix} B & AB & A^2 B & \cdots & A^{n-1} B \end{bmatrix}$$

where $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$.

In Williams-Lawrence, the controllability matrix is denoted P.

Definition 4.

For a given (A, B), the **Controllable Subspace** is

$$C_{AB} = \text{Image}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

Definition 5.

The system (A, B) is **controllable** if

$$C_{AB} = \operatorname{Im} C(A, B) = \mathbb{R}^n$$

Question: How does R_t relate to C_{AB}?

Controllability

Definition 6.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W_t := \int_0^t e^{As} B B^T e^{A^T s} ds$$

W_t is a positive semidefinite matrix.
The following relates these three concepts of controllability

Theorem 7.

For any $t \geq 0$,

$$R_t = C_{AB} = \textit{Image}\ (W_t)$$

or

$$\text{Image } \Gamma_t = \text{Image } C(A, B) = \text{Image } (W_t)$$

Controllability

The most important consequence is

- R_t does not depend on time!

If you can get there, you can get there arbitrarily fast.
This says nothing about how you get $u(t)$

- This $u(t)$ comes from the proof (and W_t)

We can test reachability of a point x by testing

$$x \in \text{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}$$

The system is controllable if $W_t > 0$. Summary

1. R_t is the set of reachable points
2. $C(A, B)$ is a fixed matrix, easily computable.
3. We need to find $u(t)$

Controllability

The following is a seminal result in state-space theory.

Theorem 8 (Cayley-Hamilton Theorem).

If

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

then

$$A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots a_0 I = 0$$

Proof Sketch.

The same principle as deriving the solution. Denote

$$\mathsf{char}_A(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = \det(sI - A)$$

Then if $A = T\Lambda T^{-1}$

$$\mathsf{char}_A(A) = T\mathsf{char}_A(\Lambda)T^{-1} = T \begin{bmatrix} \mathsf{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \mathsf{char}_A(\lambda_n) \end{bmatrix} T^{-1}$$

Sketch.

But the λ_i are eigenvalues of A, so

$$\text{char}_A(\lambda) = \det(\lambda I - A) = 0$$

hence

$$\text{char}_A(A) = T \begin{bmatrix} \text{char}_A(\lambda_1) & & \\ & \ddots & \\ & & \text{char}_A(\lambda_n) \end{bmatrix} T^{-1} = T \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} T = 0$$

The same approach works for Jordan Blocks. □

Cayley-Hamilton says

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

thus $A^n \in \text{span}(A^{n-1}, \cdots, I)$

- This is unsurprising since A has n^2 dimensions but is formed by n bases.

Controllability

Proof: Show $R_t \subset C_{AB}$ for any $t \geq 0$. Expand

$$e^{At} = \left[I + At + \cdots + \frac{A^m t^m}{m!} + \cdots \right]$$

By Cayley-Hamilton

$$A^n = -a_{n-1}A^{n-1} + \cdots + -a_0 I$$

Grouping by A^i, we get

$$e^{At} = \left[I\phi_0(t) + A^1\phi_1(t) + \cdots + A^{n-1}\phi_{n-1}(t) \right]$$

for some scalar functions $\phi_i(t)$. Because the ϕ_i are scalars,

$$\begin{aligned}\Gamma_t u &= \int_0^t e^{A(t-s)}Bu(s)ds \\ &= B\int_0^t \phi_0(t-s)u(s)ds + \cdots + A^{n-1}B\int_0^t \phi_{n-1}(t-s)u(s)ds\end{aligned}$$

Controllability

Let

$$y_i = \int_0^t \phi_i(t-s)u(s)ds,$$

then

$$\begin{aligned}\Gamma_t u &= By_0 + \cdots + A^{n-1}By_{n-1} \\ &= \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix} = C(A,B) \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}\end{aligned}$$

Thus $\Gamma_t u \in \text{Im} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix}$. Therefore, $R_t \subset C_{AB}$.

2 new concepts: perp space

Definition 9.

The **Orthogonal Complement** of a subspace, $S \subset X$, is denoted

$$S^\perp := \left\{ x \in \mathbb{R}^n \ : \ \langle x, y \rangle = x^T y = 0 \qquad \text{for all } y \in S \right\}$$

Properties

- $\dim(S^\perp) = n - \dim(S)$
- For any $x \in \mathbb{R}^n$,

$$x = x_S + x_{S^\perp} \quad \text{for } x_S \in S \text{ and } x_{S^\perp} \in S^\perp$$

  - x_S and $x_{S^\perp}$ are unique.

Controllability

Definition 10.

The Projection operator P_S is defined by $x_S = P_S x$ if $x_S \in S$ and $x - x_S \in S^\perp$.

Generalizes to any Hilbert space

Theorem 11.

For any $M \in \mathbb{R}^{n\times m}$, $[Im(M)]^\perp = Ker\left[M^T\right]$.

Proof.

We need to show $[\text{Im}(M)]^\perp \subset \text{Ker}\left[M^T\right]$ and $\text{Ker}\left[M^T\right] \subset [\text{Im}(M)]^\perp$.

- Suppose $x \in [\text{Im}(M)]^\perp$. If $x^T y = 0$ for any $y \in \text{Im}[M]$, then $x^T M z = 0$ for all z.
- Thus $z^T M^T x$ for all z. Let $z = M^T z$.
- Then $x^T M M^T x = \|M^T x\|^2 = 0$.
- Thus $x \in \text{Ker}\left[M^\perp\right]$, which implies $[\text{Im}(M)]^\perp \subset \text{Ker}\left[M^T\right]$.

Next we show $\text{Ker}\left[M^T\right] \subset [\text{Im}(M)]^\perp$. □

Controllability

Proof.

We need to show $\text{Ker}\left[M^T\right] \subset [\text{Im}(M)]^\perp$.

- Suppose $x \in \text{Ker}\left[M^T\right]$. Then

$$y^T M^T x = x^T M y = x^T z = 0$$

  for any $z \in \text{Im}(M)$. Thus $x \in [\text{Im}(M)]^\perp$.
- This proves that $[\text{Im}(M)]^\perp = \text{Ker}\left[M^T\right]$.

□

We would like to prove that

$$R_t = \text{Im}(W_t) = \text{Im}(C(A,B))$$

To do this, we will prove that

- $\text{Im}(W_t) \subset R_t$
- $R_t \subset \text{Im}(C(A,B))$
- $\text{Im}(C(A,B)) \subset \text{Im}(W_t)$