Spacecraft Dynamics and Control

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Lecture 18: Feedback Control of Attitude Dynamics

Attitude Dynamics

In this Lecture we will cover:
An Review of Feedback control for Attitude Dynamics

- Transfer Functions
- PID Control
- Root Locus

Problem: 3-axis Stabilization

- Detumble (if $\vec{\omega} \ncong 0$)
- Attitude Tracking (assuming $\vec{\omega} \cong 0$)

Linearizing Euler's Equations

Recall:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy}) \\ I_{yy}\dot{\omega}_y + \omega_x\omega_z(I_{xx} - I_{zz}) \\ I_{zz}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx}) \end{bmatrix}$$

Non-axisymmetric Case $I_x \neq I_y \neq I_z$.

We first assume the spacecraft has been detumbled, so we have
Small Spin Assumption: $\omega_x = \omega_y = \omega_z \cong 0$.

- Nominal motion is

$$\omega_0(t) = \begin{bmatrix} \omega_{x,0}(t) \\ \omega_{y,0}(t) \\ \omega_{z,0}(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

In this case, the Linearized dynamics become:

$$\begin{bmatrix} L \\ M \\ N \end{bmatrix} = \begin{bmatrix} I_x\dot{\omega}_x \\ I_y\dot{\omega}_y \\ I_z\dot{\omega}_z \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \frac{M}{I_z} \\ \frac{M}{I_y} \\ \frac{N}{I_z} \end{bmatrix}$$

The Dynamics are all uncoupled.

Kinematics and Euler Angles

Problem: We don't measure rotation *rates*. We measure rotation *angles*.

- Now we need to choose an inertial coordinate system.

The Euler Angles define the transformation from the body-fixed to inertial coordinates

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \vec{\omega}_{/B} = R_1(\phi)R_2(\theta)R_3(\psi)\vec{\omega}_{/I} = R(\phi)R(\theta)R(\psi) \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\phi} - \dot{\psi}\sin\theta \\ \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \dot{\psi}\cos\theta\cos\psi - \dot{\theta}\sin\phi \end{bmatrix}$$

Of course, we often have $\vec{\omega}_{/B}$ and are trying to find $\vec{\omega}_{/I}$. In this case, the rotation matrices can be inverted to obtain

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} p + (q\sin\phi r\cos\phi)\tan\theta \\ q\cos\phi - r\sin\phi \\ (q\sin\phi + r\cos\phi)\sec\theta \end{bmatrix}$$

Notice the Singularity at $\theta = \pm 90^\circ$(can be avoided with quaternions). Equations are also different for 2-1-3 and 1-2-3 rotation sequences.

Kinematics and Euler Angles

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} p + (q \sin \phi r \cos \phi) \tan \theta \\ q \cos \phi - r \sin \phi \\ (q \sin \phi + r \cos \phi) \sec \theta \end{bmatrix}$$

where

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \dot{\omega}_x \\ \dot{\omega}_y \\ \dot{\omega}_z \end{bmatrix} = \begin{bmatrix} \frac{L}{I_x} \\ \frac{M}{I_y} \\ \frac{N}{I_z} \end{bmatrix}$$

Which is a set of 6 nonlinear coupled differential equations.

- We have already linearized the second set.
- We should also linearize the first set.
    - Assume $\theta \cong 0$, $\phi \cong 0$, and $\psi \cong 0$.

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Decoupling the Equations

Combining these two sets of equations, we get

$$\begin{bmatrix} \ddot{\phi} = \frac{L}{I_x} \\ \ddot{\theta} = \frac{M}{I_y} \\ \ddot{\psi} = \frac{N}{I_z} \end{bmatrix}$$

If L, M, and N are decoupled, we can decouple the equations

- Orthogonal reaction wheel for each body-fixed axis.

Lets design a controller for roll

$$\ddot{\phi} = \frac{L}{I_x}$$

Consider **Proportional Feedback**

$$\frac{L(t)}{I_x} = -K(\phi(t) - \phi_0)$$

Then the closed-loop poles are at $s = \pm\imath\sqrt{K}$

- Neutrally Stable

Lets look at the effect of PD control on a 2nd-order system:

$$\hat{G}(s) = \frac{1}{s^2 + bs + c}$$

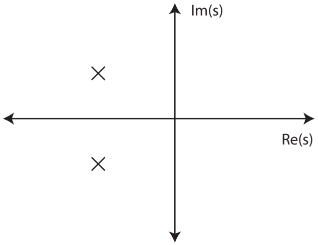
Controller: $\hat{K}(s) = -K\left[1 + T_D s\right]$
Closed Loop Transfer Function:

$$\frac{\hat{K}(s)\hat{G}(s)}{1 + \hat{K}(s)\hat{G}(s)} = \frac{K\left[1 + T_D s\right]}{s^2 + bs + c + K\left[1 + T_D s\right]}$$
$$= \frac{K\left[1 + T_D s\right]}{s^2 + (b + KT_D)s + (c + K)}$$

The poles of the system are freely assignable for a 2nd order system.

- T_D and K allow us to construct any denominator we desire.

Suppose we want poles at $s = p_1, p_2$.



- We want the closed loop of the form:

$$\frac{1}{(s-p_1)(s-p_2)} = \frac{1}{(s^2-(p_1+p_2)s+p_1p_2)}$$

Thus we want

- $c + K = p_1 p_2$ which means $K = p_1 p_2 - c$.
- $b + KT_D = -(p_1+p_2)$ which means $T_D = -\frac{p_1+p_2+b}{K} = -\frac{p_1+p_2+b}{p_1p_2-c}$

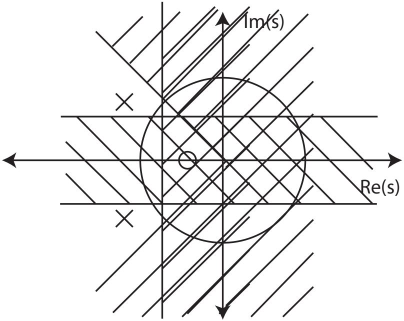
PD feedback gives Total Control over a 2nd-order system.

Factor in Constraints on the Step Response:

$$\sigma < -\frac{4.6}{T_{s,desired}}$$

$$\omega_d < \frac{\pi}{\ln(M_{p,desired})}\sigma$$

$$\omega_n > \frac{1.8}{T_{r,desired}}$$



Any pole locations not prohibited are allowed.

- σ is real part of s
- ω_d is imaginary part of s
- ω_n is magnitude of s
- T_s is setlling time
- M_p is percent overshoot
- T_r is rise time

Steady-State Error

Definition 1.

Steady-State Error (e_{ss}) for a stable system G is the final difference between input and output.

$$e_{ss} = \lim_{t \to \infty} u(t) - y(t)$$

Theorem 2 (Final Value Theorem).

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} s\hat{y}(s)$$

For Step response: $\hat{u} = \frac{1}{s}$. So if $G(s)$ is the transfer function of G,

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)K(s)}$$

Now, for roll control, $G(s) = \frac{1}{s^2}$,

$$e_{ss} = \lim_{s \to 0} \frac{1}{1 + G(s)K(s)} = \lim_{s \to 0} \frac{s^2}{s^2 + KT_D s + K} = \frac{0}{K} = 0$$

So $e_{ss} = 0$!

Steady-State Error

Ramp Response - Important for tracking (Since spacecraft is moving). In this case the input is

$$\hat{u}(s) = \frac{1}{s^2}$$

The stead-state error can be found as:

$$e_{ss} = \lim_{s \to 0} \left(\frac{1}{s^2} - \frac{G(s)K(s)}{1 + G(s)K(s)} \frac{1}{s^2} \right) s = \lim_{s \to 0} \frac{1}{(1 + G(s)K(s))\, s}$$

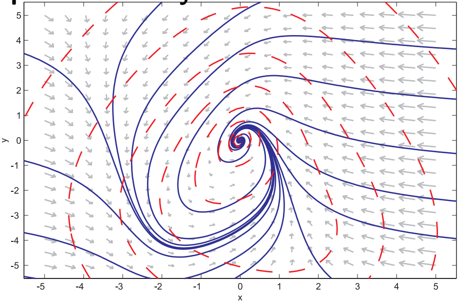
Now, for roll control, $G(s) = \frac{1}{s^2}$,

$$e_{ss} = \lim_{s \to 0} \frac{1}{s(1 + G(s)K(s))} = \lim_{s \to 0} \frac{s}{s + KT_D s + K} = \frac{0}{K} = 0$$

So still, $e_{ss} = 0$!

Conclusion: Integral Control is not necessary in space!!

A VERY Brief Introduction to Lyapunov Stability



Consider a Nonlinear ODE

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in \mathbb{R}^n$.

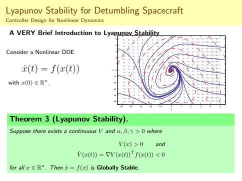
Theorem 3 (Lyapunov Stability).

Suppose there exists a continuous V and $\alpha, \beta, \gamma > 0$ where

$$V(x) > 0 \qquad \textit{and}$$

$$\dot{V}(x(t)) = \nabla V(x(t))^T f(x(t)) < 0$$

for all $x \in \mathbb{R}^n$. Then $\dot{x} = f(x)$ is **Globally Stable**.



The literature on Lyapunov functions is vast

- BY FAR the most commonly used tool for control of nonlinear systems.
- Represents the potential energy of a particular state.
  - Potential Energy as measured by the size (square integral) of the resulting trajectory.

Lyapunov Stability for Detumbling Spacecraft

The Dynamics of a tumbling spacecraft with input torque $u(t)$:

- Uses the matrix form of cross-product ($\omega_\times$) and inertia tensor (I)

$$I\dot{\omega}(t) = -\omega_\times(t) I\omega(t) + u(t)$$

Now we Propose a Lyapunov Function:

$$V(\omega) = \frac{1}{2}\omega^T I \omega = \frac{1}{2}\|I^{\frac{1}{2}}\omega\|^2 > 0$$

Take the time-derivative of this Lyapuonv Function:

- Uses the matrix form of cross-product ($\omega_\times$)

$$\begin{aligned}\dot{V}(t) &= \omega(t)^T \, I \, \dot{\omega}(t) = \omega(t)^T(-\omega_\times(t)\omega(t) + u(t)) \\ &= -\omega(t)^T(\omega(t) \times \omega(t)) + \omega(t)^T u(t)) \\ &= \omega(t)^T u(t)\end{aligned}$$

Choose Controller

$$u(t) = -P\omega(t) \qquad \text{where} \qquad P > 0$$

Since all the eigenvalues of P are positive, $\omega^T P \omega > 0$ and hence

$$\dot{V}(t) = -\omega(t) P \omega(t) < 0$$

Which proves global stability!

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Which proves global stability!

- I is a positive matrix, so we can take its square root.
- Note this does not control to any particular orientation
- Assumes spacecraft capable of large torques
- Typically we use nonlinear control for detumble and piecewise-linearized control for attitude tracking.

Attitude Dynamics

In this Lecture we have covered:
Kinematics Coupled with Dynamics

- Linearized to uncoupled version

Feedback Control

- Proportion-Differential Feedback
- Steady-State Error.