Modern Control Systems

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Lecture 14: Linear Operators

An operator is simply any map between normed spaces. $P : X \to Y$

- This includes the implicit assumption that $\|Px\|$ is bounded.
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Definition 1.

An operator P is uniformly bounded if there exists some K such that

$$\|Px\| \le K\|x\|$$

A **Linear Operator** is uniformly bounded if and only if it is bounded.

Definition 2.

Let X, Y be Banach spaces. The operator $P : X \to Y$ is a bounded linear operator if

1. It is linear. i.e.
$$F(\alpha x + \beta z) = \alpha F(x) + \beta F(z)$$
for all $x, z \in X$ and $\alpha, \beta \in \mathbb{R}$.
2. It is uniformly bounded. i.e. there exists a K such that
$$\|Px\|_Y \le K\|x\|_X$$
for all $x \in X$

Definition 3.

The normed space of bounded linear operators from X to Y is **denoted** $\mathcal{L}(X,Y)$ with norm

$$\|P\|_{\mathcal{L}(X,Y)} := \sup_{\substack{x\in X\\ x\neq 0}} \frac{\|Px\|_Y}{\|x\|_X} = K$$

- The norm is the bound (an induced norm)
- Notation: $\mathcal{L}(X) := \mathcal{L}(X,X)$
- If X is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space

Properties: Suppose $G_1 \in \mathcal{L}(X,Y)$ and $G_1 \in \mathcal{L}(Y,Z)$

- Define

$$G_{12}(x) = G_2(G_1(x))$$

- Then $G_{12} \in \mathcal{L}(X,Z)$.
- $\|G_{12}\|_{\mathcal{L}(X,Z)} \leq \|G_2\|_{\mathcal{L}(y,Z)}\|G_1\|_{\mathcal{L}(X,Y)}$.
- Not Cauchy Schwartz
- Composition forms an *algebra*.

Any linear system defines an operator

$$y = Gu$$

Feedback: $u = Ky$.

- $y = G(u - Ky)$
- $Y = (I + GK)^{-1}Gu$

Question:

- Is $(I + GK)^{-1}G$ a bounded linear operator?
- If so, the feedback is stable

Linear Operators

Example

The space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ using the $\|\cdot\|_2$ on $\mathbb{R}^n$ and $\mathbb{R}^m$.

- $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{m\times n}$
- For $A \in \mathbb{R}^{m\times n}$,

$$\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \bar{\sigma}(A)$$

Like matrices, the set of convolution operators is equivalent to the causal time-invariant subspace of $\mathcal{L}(L_2)$.

Definition 4.

For a given f, define $y = Fu$ by

$$y(t) = (Fu)(t) := \int_0^t f(t-s)u(s)ds$$

- Clearly, F is linear
- If $f \in L_1$, then $F \in \mathcal{L}(L_p)$ for any $p > 0$.
- Young's Inequality: $\|y\|_{L_r} \le \|f\|_{L_p} \|u\|_{L_q}$ for any p, q, r with

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

The Convolution Operator

$$(Fu)(t) := \int_0^t f(t-s)u(s)ds$$

Lets consider the case where $f \in L_1$.

Theorem 5.

Suppose $f \in L_1$, then $F : L_\infty[0,\infty) \to L_\infty[0,\infty)$ with

$$\|F\|_{\mathcal{L}(L_\infty)} = \|f\|_{L_1}$$

Proof.

To show that $\|F\|_{\mathcal{L}(L_\infty)} = \|f\|_{L_1}$,

- we will show $\|F\|_{\mathcal{L}(L_\infty)} \leq \|f\|_{L_1}$
- We will show $\|F\|_{\mathcal{L}(L_\infty)} \geq \|f\|_{L_1}$

$\square$

Proof.

To show that $\|F\|_{\mathcal{L}(L_\infty)} \le \|f\|_{L_1}$, let $y = Fu$. then

$$\begin{aligned}
|y(t)| = |(Fu)(t)| &= \left|\int_0^t f(t-s)u(s)ds\right| \\
&\le \int_0^t |f(t-s)u(s)|\, ds \\
&\le \int_0^t |f(t-s)|\,|u(s)|\, ds \\
&\le \int_0^t |f(t-s)|\,\|u\|_{L_\infty} ds \\
&= \|u\|_{L_\infty} \int_0^t |f(t-s)|\, ds \\
&\le \|u\|_{L_\infty} \|f\|_{L_\infty} ds
\end{aligned}$$

Thus $L \in \mathcal{L}(L_\infty)$ with $\|F\|_{\mathcal{L}(L_\infty)} \le \|f\|_{L_1}$ □

The Convolution Operator

Proof.

To show $\|F\|_{\mathcal{L}(L_\infty)} = \|f\|_{L_1}$, we need only show that $\|F\|_{\mathcal{L}(L_\infty)} \geq \|f\|_{L_1}$.

- To show that $\|F\|_{\mathcal{L}(L_\infty)} \geq \|f\|_{L_1}$, we will show that for any $\epsilon > 0$, $\|F\|_{\mathcal{L}(L_\infty)} \geq \|f\|_{L_1} - \epsilon$. We proceed by construction.
- Since $f \in L_1$,
$$\|f\|_{L_1} = \lim_{T \to \infty} \int_0^T |f(s)|\, ds$$
- Therefore, for any $\epsilon > 0$, there exists a $T_\epsilon > 0$ such that
$$\|f\|_{L_1} - \int_0^{T_\epsilon} |f(s)|\, ds < \epsilon$$

□

The Convolution Operator

Proof.

- Let $u(t) = \frac{f(T_\epsilon - t)}{|f(T_\epsilon - t)|}$. Then $\|u\|_{L_\infty} = 1$ and

$$\begin{aligned}
Fu(T_\epsilon) &= \int_0^{T_\epsilon} f(T_\epsilon - s)u(s)ds \\
&= \int_0^{T_\epsilon} \frac{f(T_\epsilon - s)^2}{|f(T_\epsilon - s)|}ds \\
&= \int_0^{T_\epsilon} |f(T_\epsilon - s)|ds \\
&> \|f\|_{L_1} - \epsilon
\end{aligned}$$

- Thus $\|Fu\|_\infty \geq \|f\|_{L_1} - \epsilon$
- Thus $\|F\|_\infty = \sup_{\|u\|_{L_\infty}=1} \|Fu\| \geq \|f\|_{L_1} - \epsilon$
- Thus $\|F\|_\infty \geq \|f\|_{L_1}$ (Implicit Contradiction)

$\square$

Convolution Operators

To conclude, if $f \in L_1$, then the convolution operator maps $L_\infty \to L_\infty$. Recall the input-output map for a linear system is

$$y(t) = \int_0^t Ce^{A(t-s)}Bu(s)ds + Du(t)$$

Conclusion:

- If $Ce^{At}B \in L_1$, then G is stable on L_∞

In fact, we usually don't work with systems $G : L_\infty \to L_\infty$
Question: When is $G \in \mathcal{L}(L_p)$

- Young's inequality:
  - When
$$\frac{1}{t} + \frac{1}{p} = \frac{1}{p} + 1$$
- Thus we need $f \in L_1$ for any p.
- A Sufficient Condition