LMI Methods in Optimal and Robust Control

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Lecture 15: Nonlinear Systems and Lyapunov Functions

Overview

Our next goal is to extend LMI's and optimization to nonlinear systems analysis.

Today we will discuss

1. Nonlinear Systems Theory
    1.1 Existence and Uniqueness
    1.2 Contractions and Iterations
    1.3 Gronwall-Bellman Inequality
2. Stability Theory
    2.1 Lyapunov Stability
    2.2 Lyapunov's Direct Method
    2.3 A Collection of Converse Lyapunov Results

The purpose of this lecture is to show that Lyapunov stability can be solved **Exactly** via optimization of polynomials.

Consider: A System of Nonlinear Ordinary Differential Equations

$$\dot{x}(t) = f(x(t))$$

Problem: Stability
Given a specific polynomial $f : \mathbb{R}^n \to \mathbb{R}^n$,
find the largest $X \subset \mathbb{R}^n$
such that for any $x(0) \in X$,
$\lim_{t\to\infty} x(t) = 0$.

Ordinary Nonlinear Differential Equations

Computing Stability and Domain of Attraction

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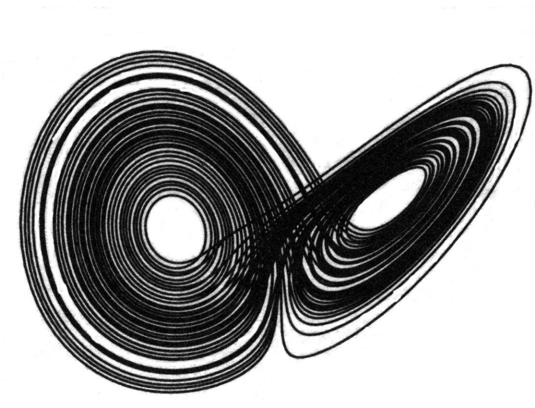
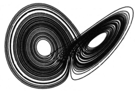
Linearity refers to the map from inputs to outputs vs. linearity in the RHS of the representation.



A model of atmospheric convection analyzed by E.N. Lorenz, Journal of Atmospheric Sciences, 1963.

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = xy - bz$$



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Nonlinear Systems may have

- Multiple Equilibria
- Regions of Attraction
- Limit Cycles
- Chaos
- Invariant Manifolds
- Non-exponential stability
- Finite-Escape Time
- Implicit (vs Excplicit) Algebraic Constraints

An oscillating circuit model:

$$\dot{y} = -x - (x^2 - 1)y$$
$$\dot{x} = y$$

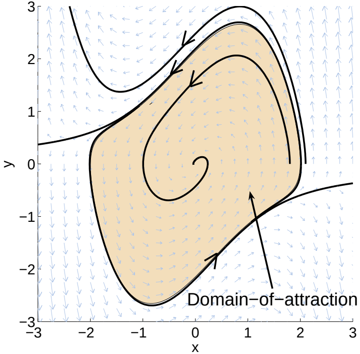


Figure: The van der Pol oscillator in reverse

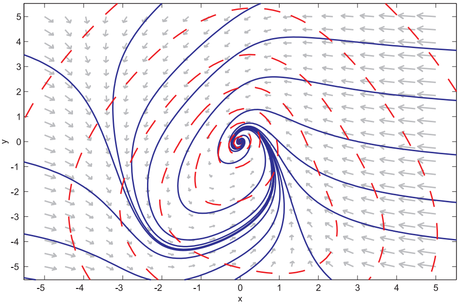
Theorem 1 (Poincaré-Bendixson).

Invariant sets in $\mathbb{R}^2$ always contain a limit cycle or fixed point.

Consider

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in \mathbb{R}^n$.



Theorem 2 (Lyapunov Stability).

Suppose there exists a continuous V and $\alpha, \beta, \gamma > 0$ where

$$\beta \|x\|^2 \leq V(x) \leq \alpha \|x\|^2$$
$$-\nabla V(x)^T f(x) \geq \gamma \|x\|^2$$

for all $x \in X$. Then any sub-level set of V in X is a **Domain of Attraction**.

A Sublevel Set: Has the form $V_\delta = \{x : V(x) \leq \delta\}$.

The first question people ask is the Cauchy problem:

For Autonomous (Uncontrolled) Systems:

Definition 3 (Cauchy Problem).

The Cauchy problem is to find a *unique*, *continuous* $x : [0, t_f] \to \mathbb{R}^n$ for some t_f such that $\dot{x}$ is defined and $\dot{x}(t) = f(t, x(t))$ for all $t \in [0, t_f]$.

If f is continuous, the solution must be continuously differentiable.

Controlled Systems:

- For a controlled system, we have $\dot{x}(t) = f(x(t), u(t))$ and assume $u(t)$ is given.
  - This precludes feedback
- In this lecture, we focus on the autonomous system.
  - Including t complicates the analysis.
  - However, results are almost all the same.

There exist many systems for which no solution exists or for which a solution only exists over a finite time interval.

Even for something as simple as

$$\dot{x}(t) = x(t)^2 \qquad\qquad x(0) = x_0$$

has the solution

$$x(t) = \frac{x_0}{1 - x_0 t}$$

which clearly has escape time

$$t_e = \frac{1}{x_0}$$

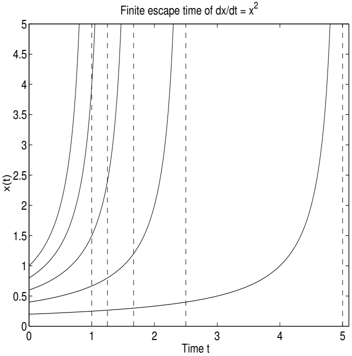


Figure: Simulation of $\dot{x} = x^2$ for several $x(0)$

A classical example of a system without a *unique* solution is

$$\dot{x}(t) = x(t)^{1/3} \qquad\qquad x(0) = 0$$

For the given initial condition, it is easy to verify that

$$x(t) = 0 \qquad \text{and} \qquad x(t) = \left(\frac{2t}{3}\right)^{3/2}$$

both satisfy the differential equation.

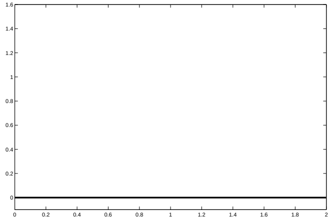


Figure: Matlab simulation of $\dot{x}(t) = x(t)^{1/3}$ with $x(0) = 0$

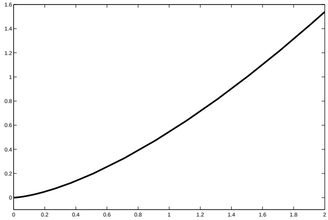


Figure: Matlab simulation of $\dot{x}(t) = x(t)^{1/3}$ with $x(0) = .000001$

Ordinary Differential Equations

Non-Uniqueness

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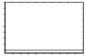

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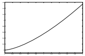


Figure: Matlab simulation of $\dot{x}(t) = x(t)^{1/3}$ with $x(0) = .000001$

- Systems without a unique solution are hard to simulate
- prone to numerical errors
- no smoothness with respect to initial conditions.

An Example of a system with *several solutions* is given by

$$\dot{x}(t) = \sqrt{x(t)} \qquad\qquad x(0) = 0$$

For the given initial condition, it is easy to verify that for any C,

$$x(t) = \begin{cases} \frac{(t-C)^2}{4} & t > C \\ 0 & t \le C \end{cases}$$

satisfies the differential equation.

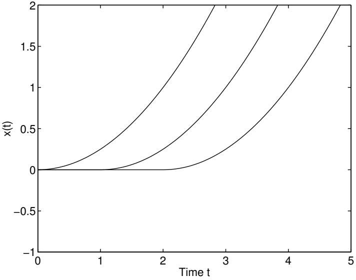
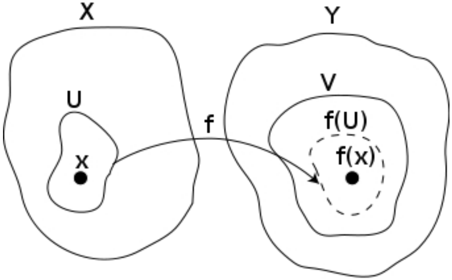


Figure: Several solutions of $\dot{x} = \sqrt{x}$

Nonlinear Stability requires some additional Math Definitions.

Definition 4 (Continuity at a Point).

For normed spaces X, Y, a function $f : X \to Y$ is **continuous at the point** $x_0 \in X$ if for any $\epsilon > 0$, there exists a $\delta > 0$ such that $\|x - x_0\| < \delta$ (U) implies $\|f(x) - f(x_0)\| < \epsilon$ (V).



Definition 5 (Continuity on a Set of Points (B)).

For normed spaces X, Y, a function $f : A \subset X \to Y$ is **continuous on** B if it is continuous at any point $x_0 \in B$. A function is simply **continuous** if $B = A$.

Dropping some of the notation,

Definition 6 (Uniform Continuity on a Set of Points (B)).

$f : A \subset X \to Y$ is **uniformly continuous on** B if for any $\epsilon > 0$, there exists a $\delta > 0$ such that for $x, y \in B$, $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \epsilon$.

Example: $f(x) = x^3$ is uniformly continuous on $B = [0, 1]$, but not $B = \mathbb{R}$

$$f'(x) = 3x^2 < 3 \text{ for } x \in [0, 1]$$

hence $|f(x) - f(y)| \leq 3|x - y|$. So given $\epsilon > 0$, choose $\delta < \frac{1}{3}\epsilon$.

Definition 7 (Lipschitz Continuity).

The function f is **Lipschitz continuous** on X if there exists some $L > 0$ such that

$$\|f(x) - f(y)\| \leq L\|x - y\| \qquad \text{for any } x, y \in X.$$

The constant L is referred to as the Lipschitz constant for f on X.

Definition 8 (Local Lipschitz Continuity).

The function f is **Locally Lipschitz continuous** on X if for every $x \in X$, there exists a neighborhood, B of x such that f is Lipschitz continuous on B.

Definition 9.

The function f is **Globally Lipschitz** if it is Lipschitz on its entire domain.

Example: $f(x) = x^3$ is Locally Lipschitz on $[-1, 1]$ with $L = 3$.

- But $f(x) = x^3$ is NOT Globally Lipschitz on $\mathbb{R}$
- L is typically just a bound on the derivative.

Let $B(x_0, r)$ be the unit ball, centered at x_0 of radius r.

Theorem 10 (A Typical Existence Theorem).

Suppose $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and there exist L, r such that for any $x, y \in B(x_0, r)$,

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

and $\|f(x)\| \leq c$ for $x \in B(x_0, r)$. Let $t_f < \min\{\frac{1}{L}, \frac{r}{c}\}$. Then there exists a unique differentiable $x : [0, t_f] \mapsto \mathbb{R}^n$, such that $x(0) = x_0$, $x(t) \in B(x_0, r)$ and $\dot{x}(t) = f(x(t))$.

Solution Map: If solutions are well-defined, we may define the solution map $g : [0, t_f] \times \mathbb{R}^n$ as the unique functions such that

$$g(0, x) = x, \qquad \dot{g}(t, x) = f(g(t, x))$$

A Theorem on Existence of Solutions
Existence and Uniqueness

Let $B(x_0, r)$ be the unit ball, centered at x_0 of radius r.

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The solution map is a rather important conceptual tools

- An explicit representation of the solutions of the system (as opposed to solutions implicit in the ODE)
- Encodes every possible solution of the system
- It is almost impossible to find an analytic expression for the solution map (except for linear systems)

Counterexamples on Existence of Solutions

Theorem 11 (A Typical Existence Theorem).

Suppose $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and there exist L, r such that for any $x, y \in B(x_0, r)$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

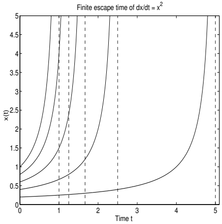
and $\|f(x)\| \le c$ for $x \in B(x_0, r)$. Let $t_f < \min\{\frac{1}{L}, \frac{r}{c}\}$. Then there exists a unique differentiable solution on interval $[0, t_f]$.

Recall:

$$\dot{x}(t) = x(t)^2 \qquad\qquad x(0) = x_0$$

has the solution

$$x(t) = \frac{x_0}{1 - x_0 t}$$



Lets take $r = 1$, $x_0 = 1$. Then $L = \sup_{x\in[0,2]} |f'(x)| = 4$.
$c = \sup_{x\in[0,2]} |f(x)| = 4$. Then we have a solution for $t_f < \min\{\frac{1}{L}, \frac{r}{c}\} = \min\{\frac{1}{4}, \frac{1}{4}\} = \frac{1}{4}$ and where $|x(t)| < 2$ for $t \in [0, t_f]$.
We can verify that the solution $x(t) = \frac{1}{1-t} < \frac{4}{3}$ for $t < t_f$.

Theorem 12 (A Typical Existence Theorem).

Suppose $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ *and there exist* L, r *such that for any* $x, y \in B(x_0, r)$,

$$\|f(x) - f(y)\| \le L\|x - y\|$$

and $\|f(x)\| \le c$ *for* $x \in B(x_0, r)$*. Let* $t_f < \min\{\frac{1}{L}, \frac{r}{c}\}$*. Then there exists a unique differentiable solution on interval* $[0, t_f]$*.*

Recall the system without a *unique* solution is

$$\dot{x}(t) = x(t)^{1/3} \qquad\qquad x(0) = 0$$

The problem here is that $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}$.

$$L = \sup_{x\in[0,2]} |f'(x)| = \sup_{x\in[0,2]} \left|\frac{1}{3x^{\frac{2}{3}}}\right| = \infty$$

Since $\frac{1}{0} = \infty$. So there is no Lipschitz Bound.

Concepts of State and Solution Maps

Definition 13.

The **State** of the system ($x \in X$) is the knowledge needed to propagate the solution forward in time.

- For every state, one and only one solution should exist, and small changes in state should cause small changes in solution.

Examples:

NDEs: $x(t) \in \mathbb{R}^n$, PDEs: $x_{ss}(t, \cdot) \in L_2$, TDS: $x(t)$ and $x(t+s)$ for $s \in [-\tau, 0]$.

Definition 14.

The **Solution Map** $g : \mathbb{R}^+ \times X \to X$ is a function of both time and state.

- $g(x, t)$ is the state at time t if $x(0) = x$.

Examples:

NDEs: $\partial_t g(t, x) = f(g(t, x)), \quad g(0, x) = x$

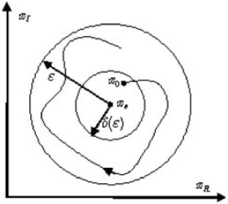
PDEs: $y_t(s, t) = A_0(s) y(s, t) + A_1(s) y_s(s, t) + A_2(s) y_{ss}(s, t), \qquad y(s, t) = \int_a^s (s - \eta) g(x_{ss}, t)(\eta) d\eta$

TDS: $\partial_t \begin{bmatrix} g_1(\phi, t) \\ g_2(\phi, t) \end{bmatrix} = \begin{bmatrix} A_0 g_1(\phi, t) + A_1 g_2(\phi, t)(-\tau) \\ \partial_s g_2(\phi, t)(s) \end{bmatrix}$ and $x_t(s) = \begin{bmatrix} x(t) \\ x(t+s) \end{bmatrix}$ for $s \in [-\tau, 0]$.

Stability Definitions

Whenever you are trying to prove stability, *Please* define your notion of stability!

Denote the set of bounded continuous functions by $\bar{\mathcal{C}} := \{x \in \mathcal{C} : \|x(t)\| \le r, r \ge 0\}$ with norm $\|x\| = \sup_t \|x(t)\|$.



We define $g : D \to \bar{\mathcal{C}}$ to be the *solution map*: $g(x_0, t)$ if

$$\frac{\partial}{\partial t} g(x_0,t) = f(g(x_0,t)) \quad \text{and} \quad g(x_0,0) = x_0 \qquad x_0 \in D$$

Definition 15.

The system is **locally Lyapunov stable** on D where D contains an open neighborhood of the origin if it defines a unique map $g : D \to \bar{\mathcal{C}}$ $(x \mapsto g(x,\cdot))$ which is continuous at the origin $(x_0 = 0)$.

The system is locally Lyapunov stable on D if for any $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that for $\|x(0)\| \le \delta(\epsilon)$, $x(0) \subset D$ we have $\|x(t)\| \le \epsilon$ for all $t \ge 0$

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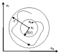

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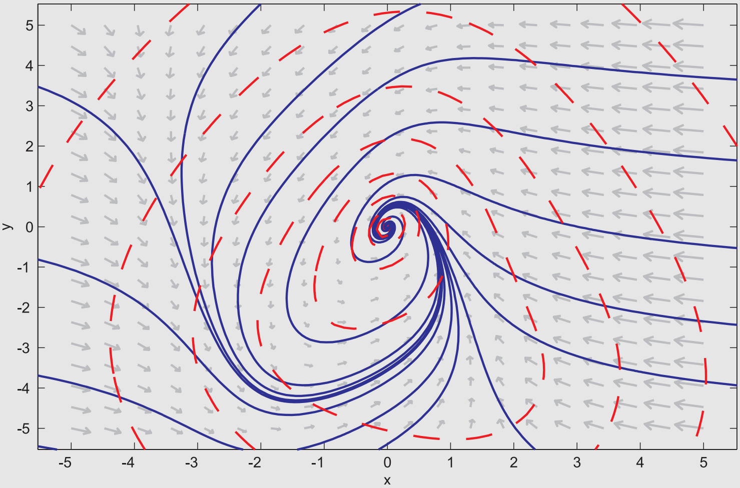


Stability Definitions

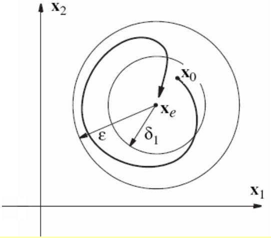
Definition 16.

The system is **globally Lyapunov stable** if it defines a unique map $g : \mathbb{R}^n \to \bar{\mathcal{C}}$ which is continuous at the origin.

We define the subspace of bounded continuous functions which tend to the origin by $G := \{x \in \bar{\mathcal{C}} : \lim_{t\to\infty} x(t) = 0\}$ with norm $\|x\| = \sup_t \|x(t)\|$.

Definition 17.

The system is **locally asymptotically stable** on D where D contains an open neighborhood of the origin if it defines a map $g : D \to G$ which is continuous at the origin.



Stability Definitions

Definition 18.

The system is **globally asymptotically stable** if it defines a map $g : \mathbb{R}^n \to G$ which is continuous at the origin.

Definition 19.

The system is **locally exponentially stable** on D if it defines a map $g : D \to G$ where

$$\|g(x,t)\| \le K e^{-\gamma t}\|x\|$$

for some positive constants $K, \gamma > 0$ and any $x \in D$.

Definition 20.

The system is **globally exponentially stable** if it defines a map $g : \mathbb{R}^n \to G$ where

$$\|g(x,t)\| \le K e^{-\gamma t}\|x\|$$

for some positive constants $K, \gamma > 0$ and any $x \in \mathbb{R}^n$.

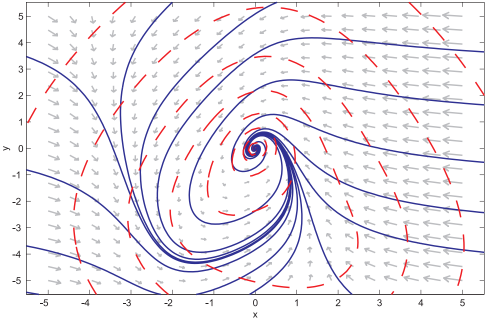
What are Lyapunov Functions?

Necessary and Sufficient Condition for Stability

Consider

$$\dot{x}(t) = f(x(t))$$

with $x(0) \in X$.



Theorem 21 (Lyapunov Stability).

Suppose there exists a V where

$$V(x) > 0 \quad \text{for } x \neq 0, \qquad \text{and} \qquad V(0) = 0$$
$$\dot{V}(x) = \nabla V(x)^T f(x) \leq 0$$

for all $x \in X$. Then any sub-level set of V in X is a **Domain of Attraction**.

Lyapunov Theorem for Lyapunov Stability

Consider the system:

$$\dot{x} = f(x), \qquad f(0) = 0$$

Theorem 22.

Let $V : D \to \mathbb{R}$ be a continuously differentiable function and D compact such that

$$\begin{aligned} V(0) &= 0 \\ V(x) &> 0 \quad \textit{for } x \in D,\, x \neq 0 \\ \nabla V(x)^T f(x) &\leq 0 \quad \textit{for } x \in D. \end{aligned}$$

- *Then $\dot{x} = f(x)$ is well-posed and locally Lyapunov stable on the largest sublevel set $V_\gamma = \{x : V(x) \leq \gamma\}$ of V contained in D.*
- *Furthermore, if $\nabla V(x)^T f(x) < 0$ for $x \in D$, $x \neq 0$, then $\dot{x} = f(x)$ is locally asymptotically stable on the largest sublevel set $V_\gamma = \{x : V(x) \leq \gamma\}$ contained in D.*

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Proof Notes for Lyapunov Theorem

Sublevel Set: For a given Lyapunov function V and positive constant γ, we denote the set $V_\gamma = \{x : V(x) \le \gamma\}$.

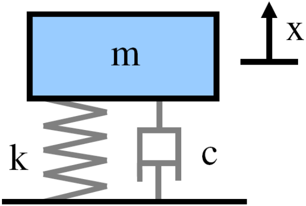
Existence: Denote the largest bounded sublevel set of V contained in the interior of D by V_{γ^*}. Because $\dot{V}(x(t)) = \nabla V(x(t))^T f(x(t)) \le 0$, if $x(0) \in V_{\gamma^*}$, then $x(t) \in V_{\gamma^*}$ for all $t \ge 0$. Therefore since f is locally Lipschitz continuous on the compact V_{γ^*}, by the extension theorem, there is a unique solution for any initial condition $x(0) \in V_{\gamma^*}$.

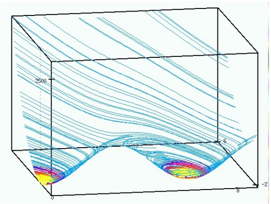
Lyapunov Stability: Given any $\epsilon' > 0$, choose $\epsilon < \epsilon'$ with $B(\epsilon) \subset V_{\gamma^*}$, choose γ_i such that $V_{\gamma_i} \subset B(\epsilon)$. Now, choose $\delta > 0$ such that $B(\delta) \subset V_{\gamma_i}$. Then $B(\delta) \subset V_{\gamma_i} \subset B(\epsilon)$ and hence if $x(0) \in B(\delta)$, we have $x(0) \in V_{\gamma_i} \subset B(\epsilon) \subset B(\epsilon')$.

Asymptotic Stability:

- V monotone decreasing implies $\lim_{t \to} V(x(t)) = 0$.
- $V(x) = 0$ implies $x = 0$.

Examples of Lyapunov Functions





Mass-Spring:

$$\ddot{x} = -\frac{c}{m}\dot{x} - \frac{k}{m}x$$

$$V(x) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

$$\begin{aligned}\dot{V}(x) &= \dot{x}(-c\dot{x} - kx) + kx\dot{x} \\ &= -c\dot{x}^2 - k\dot{x}x + kx\dot{x} \\ &= -c\dot{x}^2 \le 0\end{aligned}$$

Pendulum:

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 \quad \dot{x}_1 = x_2$$

$$V(x) = (1 - \cos x_1)gl + \frac{1}{2}l^2x_2^2$$

$$\begin{aligned}\dot{V}(x) &= glx_2 \sin x_1 - glx_2 \sin x_1 \\ &= 0\end{aligned}$$

A Lyapunov Function for Every Purpose ...

Mathematical Optimization and Curly's Law:
Curly: Do you know what the secret of life is?
Curly: One thing (metric). Just one thing. You stick to that (metric) and the rest don't mean ****.

Given a performance metric

- In a well-posed system, your current state tells you everything you need to know about the future (no inputs, disturbances).
- The Lyapunov function says how well that future performs in your metric.

Definition 23.

If $h : L_2 \to \mathbb{R}^+$ is your metric and $g : X \to L_2$ is your solution map, the **Lyapunov Function** is $V(x) = h(g(x, \cdot))$.

Note: Lyapunov Functions are simpler than solution maps because they contain less information.

- $V : X \to \mathbb{R}^+$ vs. $g : X \times t \to X$
  - "the rest don't mean ****"
- It is impossible to find solution maps except for Linear ODEs.

Example: Some Solutions are Better than Others

Consider: Linear Ordinary Differential Equations with a regulated output:

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad\qquad y(t) = Cx(t)$$

Question: Which solutions, $x(\cdot)$, are better?
Answer: Our metric is $\int_0^\infty \|y(t)\|^2 dt$

Question: How to compute $V(x) = \int_0^\infty \|Cg(x,t)\| dt$?
Answer: The solution map is

$$x(t) = g(x_0, t) = e^{At} x_0,$$

Hence the performance is

$$V(x_0) = \int_0^\infty x_0^T e^{A^T t} C^T C e^{At} x_0 dt = x_0^T \left(\int_0^\infty e^{A^T t} C^T C e^{At} dt \right) x_0 = x_0^T P_o x_0$$

$V(x)$ is our first Lyapunov function. P_o is called the observability Grammian.
But to find it, we solve $\dot{V}(x) = -\|y(t)\|^2$ or

$$A^T P_o + P_o A = -C^T C$$

Lyapunov Theorem for Exponential Stability

Theorem 24.

Suppose there exists a continuously differentiable function V and constants $c_1, c_2, c_3 >$ and radius $r > 0$ such that the following holds for all $x \in B(r)$.

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2$$
$$\nabla V(x)^T f(x) \le -c_3\|x\|^2$$

Then $\dot{x} = f(x)$ is exponentially stable on any ball contained in the largest sublevel set contained in $B(r)$.

Exponential Stability allows a quantitative prediction of system behavior.

Lyapunov Theorem for Exponential Stability

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Exponential Stability allows a quantitative prediction of system behavior.

The proof of exponential stability is so short and so widely used, we give an overview

- Easily extended to PDEs, switched systems, delay systems, etc.

Lemma 25 (Gronwall-Bellman).

Let λ be continuous and μ be continuous and nonnegative. Let y be continuous and satisfy for $t \le b$,

$$y(t) \le \lambda(t) + \int_a^t \mu(s) y(s) ds.$$

Then

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\mu(s) \exp\left[\int_s^t \mu(\tau) d\tau\right] ds$$

If λ and μ are constants, then

$$y(t) \le \lambda e^{\mu t}.$$

For $\lambda(t) = y_0$, the condition is equivalent to

$$\dot{y}(t) \le \mu(t) y(t), \qquad y(0) = y(t).$$

The Gronwall-Bellman Inequality

Proof of Exponential Stability

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The application of Gronwall Bellman to Lyapunov functions is rather simple.

- It is important that the function $y(t)$ be a scalar
- We don't use vector-valued Lyapunov functions

$$\dot{V}(t) \leq \mu(t) V(t)$$

becomes

$$V(t) \leq V(0) + \int_0^t \mu(s) V(s) ds$$

Proof.

We begin by noting that we already satisfy the conditions for existence, uniqueness and asymptotic stability and that $x(t) \in B(r)$.
Now, observe that

$$\dot{V}(x(t)) \le -c_3\|x(t)\|^2 \le -\frac{c_3}{c_2}V(x(t))$$

Which implies by the **Gronwall-Bellman** inequality $\left(\mu = \frac{-c_3}{c_2},\ \lambda = V(x(0))\right)$ that

$$V(x(t)) \le V(x(0))e^{-\frac{c_3}{c_2}t}.$$

Hence

$$\|x(t)\|^2 \le \frac{1}{c_1}V(x(t)) \le \frac{1}{c_1}e^{-\frac{c_3}{c_2}t}V(x(0)) \le \frac{c_2}{c_1}e^{-\frac{c_3}{c_2}t}\|x(0)\|^2.$$

□

Problem Statement 1: Global Lypunov Stability

Given:

- Vector field, $f(x)$

Find: function V, non-negative scalars α_i, β_i such that $\sum_i \alpha_i = .01$, $\sum_i \beta_i = .01$ and

$$V(x) \geq \sum_{i=1}^{p} \alpha_i (x^T x)^i \qquad \text{for all } x$$

$$V(x) \leq \sum_{i=1}^{p} \beta_i (x^T x)^i \qquad \text{for all } x$$

$$\nabla V(x)^T f(x) \leq 0 \qquad \text{for all } x$$

Conclusion:

- Lyapunov stability for any $x(0) \in \mathbb{R}^n$.
- Can replace $V(x) \leq \sum_{i=1}^{p} \beta_i (x^T x)^i$ with $V(0) = 0$ if it is well-behaved.

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Strict Positivity and negativity is a bit more challenging in the nonlinear case

$$\ge \epsilon I$$

means

$$\ge \epsilon x^T x$$

which we relax to the weaker condition:

$$\ge \sum_{i=1}^p \alpha_i (x^T x)^i$$

Problem Statement 2: Global Exponential Stability

Given:

- Vector field, $f(x)$, exponent, p

Find: function V, positive scalars $\alpha, \beta, \delta > 0$, such that

$$\begin{aligned} V(x) &\geq \alpha(x^T x)^p && \text{for all } x \\ V(x) &\leq \beta(x^T x)^p && \text{for all } x \\ \nabla V(x)^T f(x) &\leq -\delta V(x) && \text{for all } x \end{aligned}$$

Conclusion:

- Exponential stability for any $x(0) \in \mathbb{R}^n$.

Convergence Rate:

$$\|x(t)\| \leq \sqrt[2p]{\frac{\beta_{\max}}{\alpha_{\min}}} \|x(0)\| e^{-\frac{\delta}{2p} t}$$

Problem Statement 2: Global Exponential Stability

Example

Consider: Attitude Dynamics of a rotating Spacecraft:

$$J_1\dot{\omega}_1 = (J_2 - J_3)\omega_2\omega_3$$
$$J_2\dot{\omega}_2 = (J_3 - J_1)\omega_3\omega_1$$
$$J_3\dot{\omega}_3 = (J_1 - J_2)\omega_1\omega_2$$

What about:

$$V(x) = \omega_1^2 + \omega_2^2 + \omega_3^2?$$

$$\begin{aligned}
\nabla V(x)^T f(x) &= \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}^T \begin{bmatrix}\frac{J_2-J_3}{J_1}\omega_2\omega_3\\ \frac{J_3-J_1}{J_2}\omega_3\omega_1\\ \frac{J_1-J_2}{J_3}\omega_1\omega_2\end{bmatrix}\\
&= \left(\frac{J_2-J_3}{J_1} + \frac{J_3-J_1}{J_2} + \frac{J_1-J_2}{J_3}\right)\omega_1\omega_2\omega_3\\
&= \left(\frac{J_2^2J_3 - J_3^2J_2 + J_3^2J_1 - J_1^2J_3 + J_2J_1^2 - J_2^2J_1}{J_1J_2J_3}\right)\omega_1\omega_2\omega_3
\end{aligned}$$

OK, maybe not. Try $u_i = -k_i\omega_i$.

Problem Statement 3: Local Exponential Stability

Given:

- Vector field, $f(x)$, exponent, p
- Ball of radius r, $B_r := \{x \in \mathbb{R}^n \,:\, x^T x \le r^2\}$

Find: function V, positive scalars $\alpha, \beta, \delta > 0$, such that

$$\begin{aligned} V(x) &\ge \alpha (x^T x)^p && \text{for all } x \,:\, x^T x \le r^2 \\ V(x) &\le \beta (x^T x)^p && \text{for all } x \,:\, x^T x \le r^2 \\ \nabla V(x)^T f(x) &\le -\delta V(x) && \text{for all } x \,:\, x^T x \le r^2 \end{aligned}$$

Conclusion: A Domain of Attraction! of the origin

- Exponential stability for $x(0) \in V_\gamma := \{x \,:\, V(x) \le \gamma\}$ if $V_\gamma \subset B_r$.

Sub-Problem: Given, V, r,

$$\max_{\gamma} \ \gamma \qquad \text{such that}$$

$$V(x) \le \gamma \qquad \text{for all} \qquad x \in \{x^T x \le r\}$$

Domain of Attraction

The van der Pol Oscillator

An oscillating circuit: (in reverse time)

$$\dot{x} = -y$$
$$\dot{y} = x + (x^2 - 1)y$$

Choose:

$$V(x) = x^2 + y^2, r = 1$$

Derivative

$$\nabla V(x)^T f(x) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}^T \begin{bmatrix} y \\ -x - (x^2 - 1)y \end{bmatrix}$$
$$= -xy + xy + (x^2 - 1)y^2$$
$$\leq 0 \quad \text{for} \quad x^2 \leq 1$$

Level Set:

$$V_{\gamma=1} = \{(x, y) : x^2 + y^2 \leq 1\} = B_1$$

So $B_1 = V_{\gamma=1}$ is a Domain of Attraction!

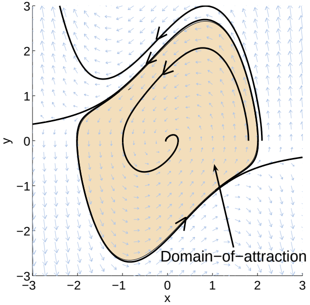
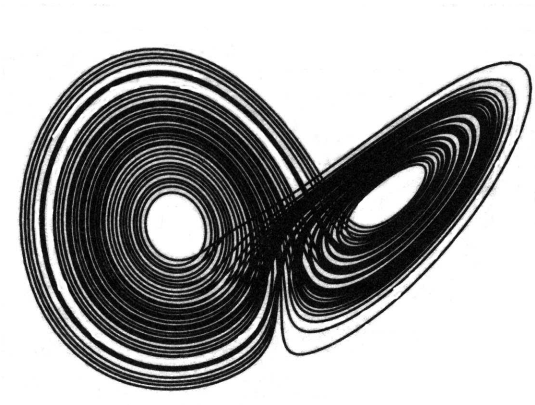


Figure: The van der Pol oscillator in reverse

Recall the Problem of Invariant Manifolds

Finding the Lorentz Attractor



A model of atmospheric convection analyzed by E.N. Lorenz, Journal of Atmospheric Sciences, 1963.

$$\dot{x} = \sigma(y - x) \qquad \dot{y} = rx - y - xz \qquad \dot{z} = xy - bz$$

Sometimes, we want to prove convergence to a set. Recall

$$V_\gamma = \{x\,,\, V(x) \le \gamma\}$$

Definition 26.

A set, X, is **Positively Invariant** if $x(0) \in X$ implies $x(t) \in X$ for all $t \ge 0$.

Theorem 27.

Suppose that there exists some continuously differentiable function V such that

$$\begin{aligned} V(x) &> 0 \quad \textit{for } x \in D,\, x \neq 0 \\ \nabla V(x)^T f(x) &\le 0 \quad \textit{for } x \in D. \end{aligned}$$

for all $x \in D$. Then for any γ such that the level set $X = \{x : V(x) = \gamma\} \subset D$, we have that V_γ is positively invariant.

Problem Statement 4: Invariant Regions/Manifolds

Given:

- Vector field, $f(x)$, exponent, p
- Ball of radius r, B_r

Find: function V, positive scalars $\alpha, \beta, \delta > 0$, such that

$$\begin{aligned} V(x) &\ge \alpha(x^T x)^p && \text{for all } x \;:\; x^T x \ge r^2 \\ V(x) &\le \beta(x^T x)^p && \text{for all } x \;:\; x^T x \ge r^2 \\ \nabla V(x)^T f(x) &\le -\delta V(x) && \text{for all } x \;:\; x^T x \ge r^2 \end{aligned}$$

Conclusion: Choose γ such that $B_r \subset V_\gamma$. Then

- There exist a T such that $x(t) \in \{x \;:\; V(x) \le \gamma\}$ for all $t \ge T$.

Sub-Problem: Given, V, r,

$$\min_\gamma \; \gamma \qquad \text{such that}$$
$$x^T x \ge r^2 \qquad \text{for all} \qquad x \in \{V(x) \ge \gamma\}$$

Problem Statement 5: Controller Synthesis (Local)

Suppose

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \qquad u(t) = k(x(t))$$

Given:

- Vector fields, $f(x)$, $g(x)$, exponent, p

Find: functions, k, V, positive scalars $\alpha, \beta, \delta > 0$, such that

$$V(x) \geq \alpha(x^T x)^p \qquad \text{for all } x \;:\; x^T x \leq r^2$$

$$V(x) \leq \beta(x^T x)^p \qquad \text{for all } x \;:\; x^T x \leq r^2$$

$$\nabla V(x)^T f(x) + \nabla V(x)^T g(x)k(x) \leq 0 \qquad \text{for all } x \;:\; x^T x \leq r^2 \qquad \text{(BILINEAR)}$$

Conclusion:

- Controller $u(t) = k(x(t))$ stabilizes the system for $x(0) \in \{x \;:\; V(x) \leq \gamma\}$ if $V_\gamma \subset B_r$.

Problem Statement 6: **Output** Feedback Controller Synthesis (Global Exponential)

Suppose

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) \qquad u(t) = k(y(t))$$
$$y(t) = h(x(t))$$

Given:

- Vector fields, $f(x)$, $g(x)$, $h(x)$ exponent, p

Find: function functions, k, V, positive scalars $\alpha, \beta, \delta > 0$, such that

$$V(x) \geq \alpha(x^T x)^p \qquad \text{for all } x$$
$$V(x) \leq \beta(x^T x)^p \qquad \text{for all } x$$
$$\nabla V(x)^T f(x) + \nabla V(x)^T g(x)k(h(x)) \leq -\delta V(x) \qquad \text{for all } x$$

Conclusion:

- Controller $u(t) = k(y(t))$ exponentially stabilizes the system for any $x(0) \in \mathbb{R}^n$.

How to Solve these Problems?

General Framework for solving these problems

Convex Optimization of Functions: Variables $V \in \mathcal{C}[\mathbb{R}^n]$ and $\gamma \in \mathbb{R}$

$$\max_{V,\gamma} \; \gamma$$

subject to

$$V(x) - x^T x \geq 0 \qquad \forall x \in X$$
$$\nabla V(x)^T f(x) + \gamma x^T x \leq 0 \qquad \forall x \in X$$

Going Forward

- Assume all functions are polynomials or rationals.
- Assume $X := \{x \; : \; g_i(x) \geq 0\}$ (Semialgebraic)