Spacecraft and Aircraft Dynamics

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Lecture 7: Converting to/from $\vec{r}$ and $\vec{v}$

Introduction

In this Lecture, you will learn:

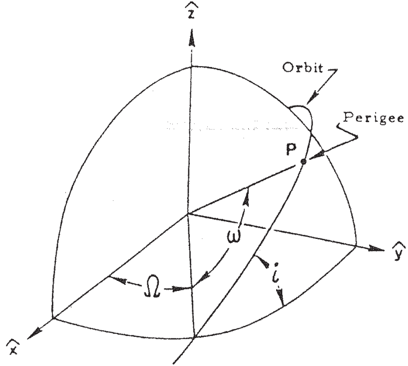
How to convert between

- a, e, i, Ω, ω, f
- $\vec{r}$ and $\vec{v}$

How to translate $\vec{r}$ and $\vec{v}$ into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Finding the Orbital Elements



In the previous lecture, we introduced three new orbital elements.

- Inclination, i
- RAAN, Ω
- Argument of Periapse, ω

We gave a numerical example to illustrate how to find these new elements

Step 1: Construct $\vec{h}$, $\vec{n}$ and $\vec{e}$.

$$\vec{h} = \vec{r} \times \vec{v}$$

Assume $\vec{r}$ and $\vec{v}$ are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h}$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r}$$

Alternatively

$$\vec{e} = \frac{1}{\mu} \left[\left(v^2 - \frac{\mu}{r} \right) \vec{r} - (\vec{r} \cdot \vec{v}) \, \vec{v} \right]$$

Calculate the scalars $r = \|\vec{r}\|$, $v = \|\vec{v}\|$, $e = \|\vec{e}\|$, $h = \|\vec{h}\|$.

Finding the Orbital Elements

Step 2: Calculate the 2D elements.

$$E = \frac{v^2}{2} - \frac{\mu}{r} \qquad \text{and} \qquad a = -\frac{\mu}{2E}$$

Step 3: Calculate the 3D elements. We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1}\left(\frac{\vec{h}}{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

No quadrant ambiguity.

$$\Omega = \cos^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|}\right)$$

Correct for quadrant.

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360 - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

Finding the Orbital Elements

Argument of perigee is the angle between $\vec{e}$ and $\vec{n}$.

$$\omega = \cos^{-1}\left(\frac{\vec{n}\cdot\vec{e}}{\|\vec{n}\|e}\right)$$

We resolve the quadrant ambiguity be checking

$$\omega = \begin{cases}\omega & \hat{z}\cdot\vec{e} \geq 0 \\ 360-\omega & \hat{z}\cdot\vec{e} < 0\end{cases}$$

True anomaly is the angle between $\vec{r}$ and $\vec{e}$.

$$f = \cos^{-1}\left(\frac{\vec{r}\cdot\vec{e}}{re}\right)$$

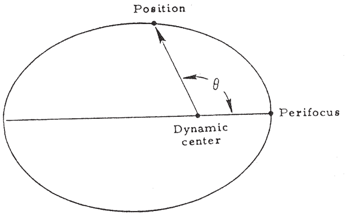
We resolve the quadrant ambiguity by checking

$$f = \begin{cases}f & \vec{r}\cdot\vec{v} \geq 0 \\ 360-f & \vec{r}\cdot\vec{v} < 0\end{cases}$$

Propagation in Time

All orbital elements can be determined from a single observation at t_0.

- Orbital motion is periodic
  - Orbital elements allow us to predict the motion for all time.



Given a future time, t_f, we can use Kepler's equation to predict $f(t_f)$
Step 1: Use true anomaly, $f(t_0)$ to find mean anomaly, $M(t_0)$.

$$E = 2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}\right)$$

$$M = E - e\sin E$$

Propagation in Time

Step 2: Determine mean anomaly at t_f

$$M(t_f) = M(t_0) + n(t_f - t_0)$$

Step 3: Use mean anomaly, $M(t_f)$ to find true anomaly, $f(t_f)$ using Kepler's equation.

$$M = E - e \sin E$$
$$f = 2 \tan^{-1} \left(\sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \right)$$

- The true anomaly, $f(t_f)$, tells us where the satellite is at time t_f.
- But how to translate that into $\vec{r}$ and $\vec{v}$?

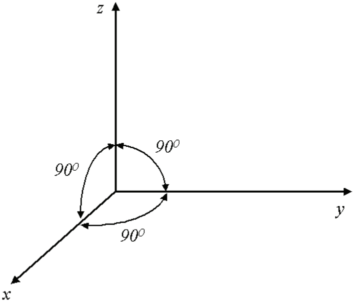
Coordinate Systems

A coordinate system

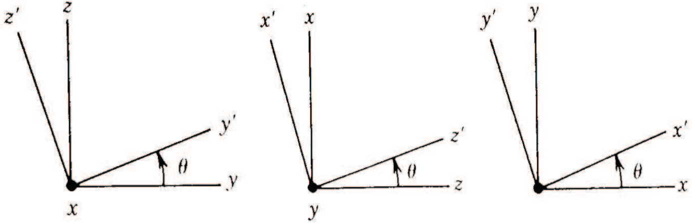
- defines position variables
- defines positivity

A coordinate system may be

- inertial
  - $F = ma$
- translating
- rotating



A cartesian coordinate system has right angles and is right-handed.



Example: Given a vector $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and a rotation θ, about the y-axis,

$$\vec{v}' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \end{bmatrix}$$

The matrix is called a *rotation matrix*.

There is also a right-hand rule for Rotation.

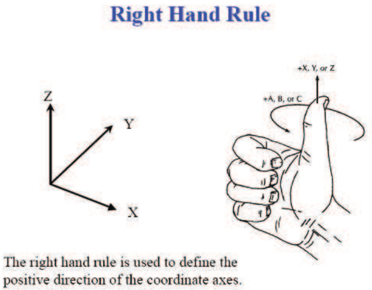
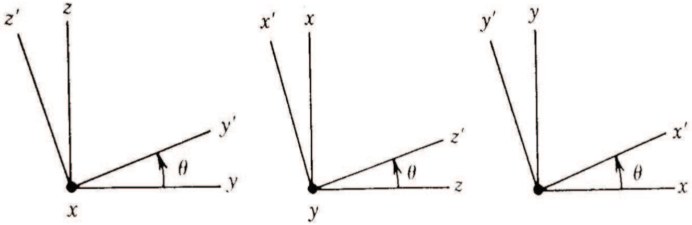


Figure: Positive Rotations

Rotation is counterclockwise when axis is pointing toward your eye.



Rotation matrices can be used to calculate the effect of **ANY** rotation.

X-Axis, ϕ :

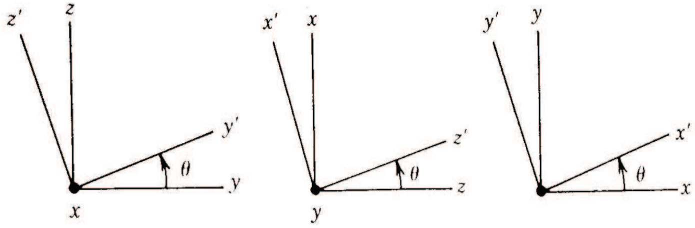
$$\vec{v}' = R_1(\phi)\vec{v}$$

Y-Axis θ :

$$\vec{v}' = R_2(\theta)\vec{v}$$

Z-Axis ψ :

$$\vec{v}' = R_3(\psi)\vec{v}$$



The rotation matrices are (for reference):

Roll (X-Axis) (ϕ) :

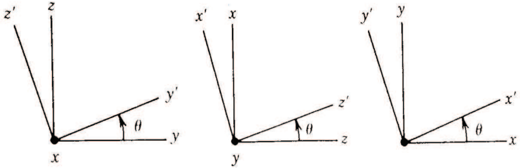
$$R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

Pitch (Y-Axis) (θ):

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

Yaw (Z-Axis) (ψ):

$$R_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



Rotation matrices, can be used to calculate a sequence of rotations:

Roll-Pitch-Yaw:

$$\vec{v}_{RPY} = R_3(\psi)R_2(\theta)R_1(\phi)\vec{v}$$

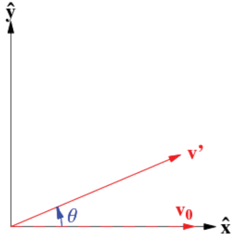
Note the *order* of multiplication is critical.

$$\vec{v}_{RPY} = \left(R_3(\psi)\left(R_2(\theta)\left(R_1(\phi)\vec{v}\right)_1\right)_2\right)_3$$

Case 1: Rotation of a vector in a fixed coordinate system.

Consider rotation of $\vec{r}$ around the $\hat{x}$ axis by θ and around the $\hat{z}$ axis by ω

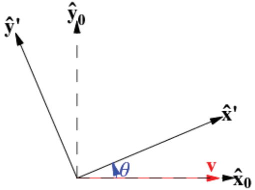
$$\vec{r}' = R_3(\omega)R_1(\theta)\vec{r}$$



Case 2: Expression of a fixed vector in a new coordinate system.

Consider what happens if we rotate the coordinates (F1) about the $\hat{x}$ axis by θ (F2) and then rotate the coordinates about the $\hat{z}$ axis by ω (F3)

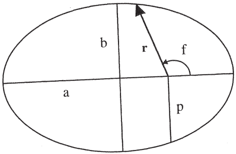
$$\vec{r}_{F3} = R_3(-\omega)\vec{r}_{F2} = R_3(-\omega)R_1(-\theta)\vec{r}_{F1}$$



Note: Our method is slightly different than the book. You are free to take either approach.

Perifocal Coordinates:

- $\hat{x} = \vec{e}/e$.
- $\hat{z} = \vec{h}/h$
- $\hat{y}$ by RHR



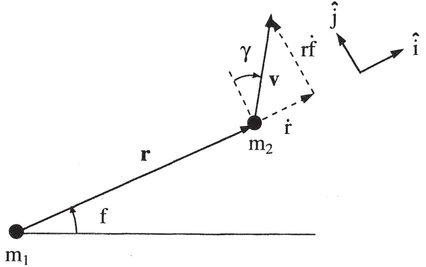
Position in perifocal frame is simple.

$$\vec{r} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \text{where} \qquad r = \frac{p}{1 + e\cos f}$$

Velocity in the Perifocal Frame

Recall our original expression for 2D velocity.

$$\vec{v}_o = \dot{r}\hat{i} + r\dot{f} = \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix}$$



- To get to the perifocal frame we rotate backwards by angle f.
- Can use rotation matrix $R_3(-f)$.

$$\vec{v}_{PQW} = R_3(-f) \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{r}\cos f - r\dot{f}\sin f \\ \dot{r}\sin f + r\dot{f}\cos f \\ 0 \end{bmatrix}$$

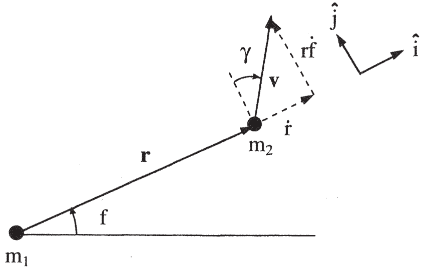
Velocity in the Perifocal Frame

Now recall $h = r^2 \dot{f}$. Hence we can simplify

$$r\dot{f} = \sqrt{\frac{\mu}{p}}\,(1 + e\cos f)\,.$$

by differentiating the orbit equation and using the above expression, we get

$$\dot{r} = \sqrt{\frac{\mu}{p}}\,(e\sin f)$$



Plugging these expressions in, we get the following

$$\vec{v}_{PQW} = R_3(-f)\begin{bmatrix}\dot{r}\\ r\dot{f}\\ 0\end{bmatrix} = \begin{bmatrix}\dot{r}\cos f - r\dot{f}\sin f\\ \dot{r}\sin f + r\dot{f}\cos f\\ 0\end{bmatrix} = \begin{bmatrix}-\sqrt{\frac{\mu}{p}}\sin f\\ \sqrt{\frac{\mu}{p}}\,(e+\cos f)\\ 0\end{bmatrix}$$

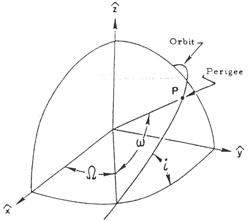
Coordinate Rotations

Perifocal to ECI Transformation

$$\vec{r}_{PQW} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \vec{v}_{PQW} = \begin{bmatrix} -\sqrt{\frac{\mu}{p}}\sin f \\ \sqrt{\frac{\mu}{p}}\left(e+\cos f\right) \\ 0 \end{bmatrix}$$

The Perifocal coordinates can be reached from ECI via 3 rotations.

1. Rotate Ω about $\hat{z}$
2. Rotate i about $\hat{x}$
3. Rotate ω about $\hat{z}$



As mentioned, rotating coordinates has the *opposite* effect of rotating the vector. Thus a vector $\vec{r}_{ECI}$ in ECI coordinates can be expressed as

$$\vec{r}_{PQW} = R_3(-\omega)R_1(-i)R_3(-\Omega)\vec{r}_{ECI}$$

Perifocal to ECI Transformation

Thus to convert a PQW vector to ECI, we can

$$\vec{r}_{ECI} = R_3(\Omega) R_1(i) R_3(\omega) \vec{r}_{PQW} = R_{PQW \to ECI} \vec{r}_{PQW}$$

$$R_{PQW \to ECI} = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i \\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i \\ \sin\omega\sin i & \cos\omega\sin i & \cos i \end{bmatrix}$$

$\vec{r}$ and $\vec{v}$ in ECI

Finally, we can express our $\vec{r}$ and $\vec{v}$ in ECI.

$$\vec{r}_{ECI} = R_{PQW\rightarrow ECI}\begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} \qquad \vec{v}_{ECI} = R_{PQW\rightarrow ECI}\begin{bmatrix} -\sqrt{\frac{\mu}{p}}\sin f \\ \sqrt{\frac{\mu}{p}}\left(e+\cos f\right) \\ 0 \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} r(\cos\Omega\cos(\omega+f)-\sin\Omega\sin(\omega+f)\cos i) \\ r(\sin\Omega\cos(\omega+f)+\cos\Omega\sin(\omega+f)\cos i) \\ r\sin(\omega+f)\sin i \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} -\frac{\mu}{h}\left(\cos\Omega\left(\sin(\omega+f)+e\sin\omega\right)+\sin\Omega\left(\cos(\omega+f)+e\cos\omega\right)\cos i\right) \\ -\frac{\mu}{h}\left(\sin\Omega\left(\sin(\omega+f)+e\sin\omega\right)-\cos\Omega\left(\cos(\omega+f)+e\cos\omega\right)\cos i\right) \\ \frac{\mu}{h}\left(\cos(\omega+f)+e\cos\omega\right)\sin i \end{bmatrix}$$

Numerical Example: $\vec{r}$ and $\vec{v}$ in ECI

Problem: Given the following orbital elements, find $\vec{r}$ and $\vec{v}$.

$$a = 35,960km = 5.64ER \qquad e = .832 \qquad f = 92.335\deg$$
$$i = 87.87\deg \qquad \Omega = 227.9\deg \qquad \omega = 53.39\deg$$

Solution: First solve for r and h.

$$p = a(1-e^2) = 1.735ER$$

$$r = \frac{p}{1+e\cos f} = 1.7947$$

$p = h^2/\mu$, so

$$h = \sqrt{p} = 1.3172.$$

Now in perifocal coordinates

$$\vec{r}_{PQW} = \begin{bmatrix} -.07319 \\ 1.7947 \\ 0 \end{bmatrix} \qquad \vec{v}_{PQW} = \begin{bmatrix} -.7585 \\ .6013 \\ 0 \end{bmatrix}$$

Numerical Example: $\vec{r}$ and $\vec{v}$ in ECI

We can find the rotation matrices in Matlab using the following commands:

```
R3w = [cosd(w) -sind(w) 0; sind(w) cosd(w) 0; 0 0 1 ];
R1 = [1 0 0; 0 cosd(i) -sind(i) ; 0 sind(i) cosd(i)];
R3Om = [cosd(Om) -sind(Om) 0; sind(Om) cosd(Om) 0; 0 0 1 ];
```

Then compute the position and velocity vectors:

```
rECI = R3O*R1*R3w*rPQW
vECI = R3O*R1*R3w*vPQW
```

which yields

$$\vec{r}_{ECI} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER \qquad \vec{v}_{ECI} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix} ER/TU$$

Of course we could have simply used the formulae.

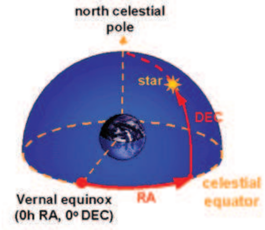
Pointing Coordinates

Right Ascension and Declination

Question: Now that we have $\vec{r}$ and $\vec{v}$, what do we do with them?

Answer:

- Tracking
- Communication
- Interception
- Astronomy



For all of these applications, we need to know where to look.

1. The sky is big.
2. Satellites are small.

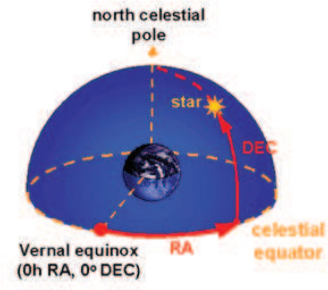
To track a satellite or star, the position vector must be translated into a direction.

These directions are declination and right ascension.

Right Ascension

Definition 1.

Right Ascension, α is the angle the position vector makes with the FPOA when projected onto the reference plane.



Initially suppose we are at the center of the earth. If $[\vec{r}] = \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix}$, the projection is simply $\begin{bmatrix} r_1 & r_2 \end{bmatrix}$. Thus

$$\tan(\alpha) = \frac{r_2}{r_1}$$

Definition 2.

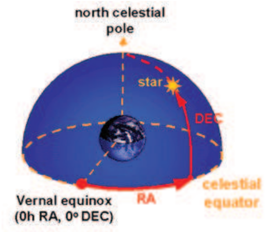
Declination, δ is the angle the position vector makes with the reference plane.

Again, simple geometry yields

$$\sin\delta = \frac{r_3}{r}$$

or

$$\tan\delta = \frac{r_3}{\sqrt{r_1^2 + r_2^2}}$$

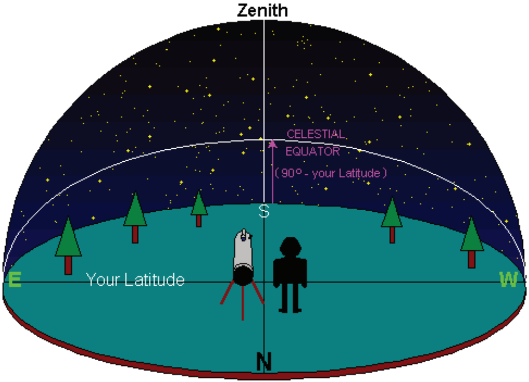


For a point on the surface of the earth, the observer must use

$$\vec{r}_{rel} = \vec{r}_{sat} - \vec{r}_{site}$$

to calculate the right ascension and declination.
Question: How to find Jupiter?

Observation using α and δ

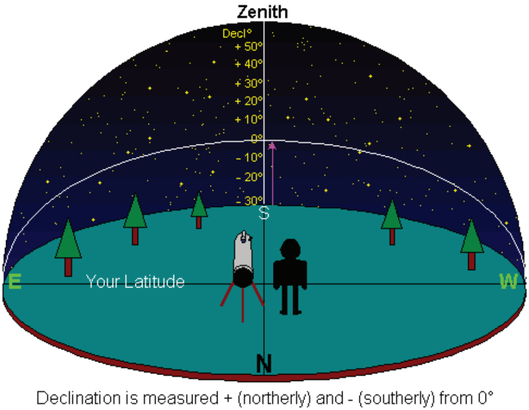


The Celestial Equator is up (90°- your Latitude) from the S horizon

Step 1: Locate the Equatorial plane.

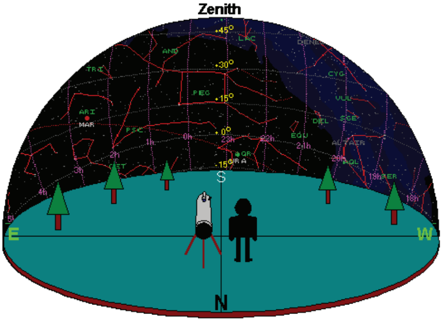
- When facing due south, the equatorial plane will be at $90\deg - \lambda$, where λ is your latitude.

Observation using α and δ



Step 2: From the Equatorial plane, measure up/down to declination line.

Observation using α and δ

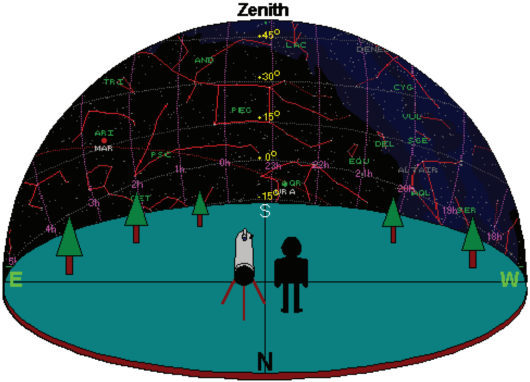


Lines of Right Ascension

Step 3: Determine right ascension, α_S of due south.

- This is given by Local Sidereal Time.
- Consult a table or do the conversion (not covered here).
- There *is* an app for that.
  - Sidereal Time, Skyfari, Star Map Pro, Emerald Geneva for iphone

Observation using α and δ



Lines of Right Ascension

Step 4: Find desired α relative to α_S.

Observation using α and δ

RA/Dec Coordinates

RA/Dec Coordinates

Numerical Example

Targeting

Problem: Suppose we are in a spacecraft in the following orbit

$$a = 60,000km = 9.41ER \qquad e = .9 \qquad f = 130\deg$$
$$i = 80\deg \qquad \Omega = 220\deg \qquad \omega = 70\deg$$

We would like to use our laser cannon to destroy a defense satellite in the following orbit.

$$a = 35,960km = 5.64ER \qquad e = .832 \qquad f = 92.335\deg$$
$$i = 87.87\deg \qquad \Omega = 227.9\deg \qquad \omega = 53.39\deg$$

What range, Right Ascension and declination should we give to the targeting computer?

Step 1: Find our position vector. We use the same Matlab script as before.

$$\vec{r}_1 = \begin{bmatrix} 4.71 \\ 5.97 \\ -8.74 \end{bmatrix} ER$$

Numerical Example

Step 2: The position vector

$$\vec{r}_2 = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix} ER$$

Step 3: The relative position vector

$$\vec{r}_2 - \vec{r}_1 = \begin{bmatrix} -3.7 \\ -4.89 \\ 9.75 \end{bmatrix}$$

Step 4: Translate into RA and declination. Use Matlab commands

```
dec=atan2(rrel(3),sqrt(rrel(1)^2 + rrel(2)^2))
RA=atan2(rrel(2),rrel(1))
```

Yields $\delta = 1.0097 rad$, $\alpha = -2.2173 rad$.

Summary

This Lecture you have learned:

How to convert between

- a, e, i, Ω, ω, f
- $\vec{r}$ and $\vec{v}$

How to translate $\vec{r}$ and $\vec{v}$ into pointing data for telescope/radio

- Right Ascension
- Declination
- Tracking

Next Lecture: Transfer Orbits