A PIE Representation of Coupled Linear 2D PDEs and Stability Analysis using LPIs

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***Abstract*— This paper presents a Partial Integration Equation (PIE) representation of linear Partial Differential Equations (PDEs) in two spatial variables. PIEs are an algebraic state-space representation of infinite-dimensional systems and have been used to model 1D PDEs and time-delay systems without continuity constraints or boundary conditions – making these PIE representations amenable to stability analysis using convex optimization. To extend the PIE framework to 2D PDEs, we first construct an algebra of Partial Integral (PI) operators on the function space $L_2[x,y]$, providing formulae for composition, adjoint, and inversion. We then extend this algebra to $\mathbb{R}^n \times L_2[x] \times L_2[y] \times L_2[x,y]$ and demonstrate that, for any suitable coupled, linear PDE in 2 spatial variables, there exists an associated PIE whose solutions bijectively map to solutions of the original PDE – providing conversion formulae between these representations. Next, we use positive matrices to parameterize the convex cone of 2D PI operators – allowing us to optimize PI operators and solve Linear PI Inequality (LPI) feasibility problems. Finally, we use the 2D LPI framework to provide conditions for stability of 2D linear PDEs. We test these conditions on 2D heat and wave equations and demonstrate that the stability condition has little to no conservatism.**

I. INTRODUCTION

In this paper, we consider the problem of representation and stability analysis of linear Partial Differential Equations (PDEs) with multiple states evolving in 2 spatial dimensions.

First, consider how a PDE is defined. When we refer to a PDE, we are actually referring to 3 separate governing equations: The partial differential equation itself; a continuity constraint on the solution; and a set of boundary conditions (BCs). Any solution of the PDE is required to satisfy all three constraints at all times – leading to challenging questions of existence and uniqueness of solutions. Furthermore, suppose we seek to examine whether all solutions to a PDE exhibit a common *evolutionary* trait, such as stability or L_2-gain. How does each of the 3 governing equations affect this property? The fact that we have 3 governing equations significantly complicates the analysis and control of PDEs.

For comparison, consider the stability question for a system defined by a linear Ordinary Differential Equation (ODE) in state-space form, $\dot{x}(t) = Ax(t)$, where the ODE itself is the only constraint on the solutions of the system. In this case, a necessary and sufficient (N+S) condition for stability of the solutions of the system is the existence of a quadratic measure of energy (Lyapunov Function (LF)), $V(x) = x^T P x$ with $P > 0$, such that for any solution $x(t) \neq 0$ of the ODE, $V(x(t))$ is decreasing for all $t \geq 0$. A N+S condition for stability will then be existence of a matrix $P > 0$ such that $\dot{V}(x(t)) = x^T(t)(PA + A^T P)x(t) \leq 0$ for all $x(t) \in \mathbb{R}^n$. This constraint is in the form of a Linear Matrix Inequality (LMI), and can be solved using convex optimization algorithms [1].

Now, let us consider the problems with extending this state-space approach to stability analysis of 2D PDEs. In this case, the state at time t of the PDE is a function $\mathbf{u}(t,x,y)$ of 2 spatial variables – raising the question of how one can parameterize the convex cone of LFs which are positive on this 2D function space without introducing significant conservatism. Moreover, we recall that solutions to the 'PDE' are required to satisfy not only the PDE itself (e.g. $\mathbf{u}_t = \mathbf{u}_{xx} + \mathbf{u}_{yy}$), but are also required to satisfy both continuity constraints (e.g. $\mathbf{u}(t,\cdot,\cdot) \in H_2$) and boundary conditions (e.g. $\mathbf{u}(t,0,y) = \mathbf{u}(t,1,y) = \mathbf{u}(t,x,0) = \mathbf{u}(t,x,1) = 0$). The second problem is then, given a Lyapunov function of the form $V(\mathbf{u})$, how to determine whether $\dot{V}(\mathbf{u}(t)) < 0$ for all $\mathbf{u}$ satisfying all three constraints. In particular, how do the BCs and continuity constraints influence $\dot{V}(\mathbf{u}(t))$?

Regarding the first problem, it is known that for linear PDEs, as was the case for ODEs, existence of a decreasing quadratic LF is N+S for stability of solutions [2] - so that we may assume the LF has the form $V(\mathbf{u}) = \langle \mathbf{u}, \mathcal{P}\mathbf{u} \rangle_{L_2}$ for some positive operator $\mathcal{P} > 0$. However, it is unclear how to parameterize a set of linear operators which is suitably rich so as to avoid significant conservatism, whilst still allowing positivity of the operators to be efficiently enforced. As a result, most prior work has been restricted to employing variations of the identity operator for $\mathcal{P}$. For example, in [3], a Lypanunov function of the form $V = \alpha \|u\|_{L_2}^2$ was used. Meanwhile, in [4], [5], and [6], the authors assumed $\mathcal{P}$ to be a multiplier operator of the form $\mathcal{P}\mathbf{u} = M\mathbf{u}(s)$, with positivity implied by the matrix inequality $M > 0$. In [7], the authors extended these functionals somewhat, using polynomial multipliers $\mathcal{P}\mathbf{u} = M(s)\mathbf{u}(s)$ with the Sum-of-Squares (SOS) constraint $M(s) \geq 0$. However, in each of these cases, the use of multiplier operators (analagous to the use of diagonal matrices in an LMI) implies significant conservatism in any stability analysis using such results.

We now turn to the second problem with stability analysis of PDEs: enforcing negativity of the derivative. As stated, for linear ODEs $\dot{x}(t) = Ax(t)$, this condition is easy to enforce: $\dot{V}(x(t)) = x(t)^T(PA + A^T P)x(t) \leq 0$ for all solutions $x(t)$ and all $t \geq 0$ if and only if $PA + A^T P$ is a negative definite matrix - an LMI constraint. However, for a PDE $\dot{\mathbf{u}} = \mathcal{A}\mathbf{u}$, the state space has more structure. That is, while it is true that if $V = \langle \mathbf{u}, \mathcal{P}\mathbf{u} \rangle$ then $\dot{V}(\mathbf{u}) = \langle \mathbf{u}, [\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}]\mathbf{u} \rangle$, negativity of the operator $\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}$ is not a N+S condition for stability, as $\dot{V}(\mathbf{u}) \leq 0$ need only hold for solutions $\mathbf{u} \in X$ satisfying the BCs and continuity constraints. To account for this, in [4], [5], and [6], the authors consider particular types of BCs, allowing the use of known integral inequalities such as Poincare, Wirtinger, etc. to prove negativity. In [7], it was proposed to use a more general set of inqualities defined by Green's functions. In all these cases, however, the process of identification and application of useful inequalities to prove negativity requires significant expertise and insight.

To avoid having to manually integrate boundary conditions and continuity constraints into the expression for $\dot{V}$, [8]–[10] suggest representing the PDE as a Partial Integral Equation (PIE). A PIE is a unitary representation of the PDE whose solution is defined on L_2 and hence does not require boundary conditions or continuity constraints.

For autonomous systems, PIEs are parameterized by the Partial Integral (PI) operators $\mathcal{T}$ and $\mathcal{A}$, and take the form $\mathcal{T}\dot{\hat{\mathbf{u}}} = \mathcal{A}\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is the so-called "fundamental state". This *fundamental state* $\hat{\mathbf{u}} \in L_2$ is free of boundary and continuity constraints, and is associated to the PDE state through the transformation $\mathbf{u} = \mathcal{T}\hat{\mathbf{u}}$. PI operators form a Banach *-algebra, with analytic expressions for composition, adjoint, etc. Furthermore, positive PI operators can be parameterized by positive matrices – allowing us to solve Linear PI Inequality (LPI) optimization problems using semidefinite programming. Thus, the use of PIEs and PI operators also resolves the difficulty with parameterizing positive LFs, taking these to be of the form $V(\mathbf{u}) = \langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle$, where $\mathcal{P}$ is a PI operator. In 1D, the MATLAB toolbox PIETOOLS [8] can be used for parsing and solving LPI optimization problems.

Thus, through use of the LPI and PIE framework, stability analysis, as well as tasks like H_∞-optimal controller design [11], can be efficiently performed for almost any 1D PDE using convex optimization. However, as of yet, none of this architecture exists for 2D PDEs. Specifically, no concept of fundamental state has been defined for 2D PDEs, and there is no known algebra of 2D PI operators which could be used to parameterize a PIE representation, or be incorporated into some form of LPI optimization algorithm.

The goal of this paper, then, is to recreate the PIE framework for linear, 2nd order PDEs on a 2D domain $(x, y) \in [a, b] \times [c, d]$ with non-periodic boundary conditions. To this end, the paper makes the following contributions.

1) *Identify an algebra of 2D PI operators.*
In Section III, we parameterize a set of PI operators with domain $L_2[x, y]$, which we combine with the algebra of 4-PI operators on $\mathbb{R}^n \times L_2[x] \times L_2[y]$ (representing the boundary of the domain) to yield a Banach *-algebra of PI operators on $\mathbb{R}^n \times L_2[x] \times L_2[y] \times L_2[x, y]$. We demonstrate that this set of PI operators is closed under scalar multiplication, addition, adjoint, and composition – deriving analytic expressions for the result of each operation. In Section IV, we further derive analytic expressions for the inverse of a suitable PI operator on $\mathbb{R}^n \times L_2[x] \times L_2[y]$, and the composition of a differential operator with a suitable 2D PI operator, showing that the result of each is a PI operator.

2) *Identify the fundamental state for a 2D PDE.*
In Subsection VI-A, we differentiate ($\mathcal{D}$) the PDE state $\mathbf{u} \in X$ up to the maximal degree allowed per the continuity constraints. The resulting $\hat{\mathbf{u}} = \mathcal{D}\mathbf{u} \in L_2$ will then be free of boundary and continuity constraints.

3) *Reconstruct the PDE state from the fundamental state.*
Having defined a fundamental state $\hat{\mathbf{u}}$, we isolate a set of "core" and "full" boundary values of the PDE state as $\Lambda_{\text{bc}}\mathbf{u}$ and $\Lambda_{\text{bf}}\mathbf{u}$ respectively. Using the fundamental theorem of calculus, we can then express $\mathbf{u} = \mathcal{K}_1\Lambda_{\text{bc}}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}}$ and $\Lambda_{\text{bf}}\mathbf{u} = \mathcal{H}_1\Lambda_{\text{bc}}\mathbf{u} + \mathcal{H}_2\hat{\mathbf{u}}$, where $\mathcal{K}_1$, $\mathcal{K}_2$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are 2D PI operators. Next, we impose the boundary conditions as $\mathcal{B}\Lambda_{\text{bf}}\mathbf{u} = 0$, where $\mathcal{B}$ is a PI operator, allowing us to write $\mathcal{B}\mathcal{H}_1\Lambda_{\text{bc}}\mathbf{u} + \mathcal{B}\mathcal{H}_2\hat{\mathbf{u}} = 0$ and thus $\Lambda_{\text{bc}}\mathbf{u} = -(\mathcal{B}\mathcal{H}_1)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}}$. Finally, we retrieve the PDE state as $\mathbf{u} = \mathcal{T}\hat{\mathbf{u}} = [\mathcal{K}_2 - (\mathcal{B}\mathcal{H}_1)^{-1}\mathcal{B}\mathcal{H}_2\mathcal{K}_1]\hat{\mathbf{u}}$, where $\mathcal{T}$ is a 2D PI operator.

4) *Derive a PIE representation for a standardized PDE.*
In Section V, we present a standardized format for writing coupled PDEs. In Subsection VI-B, we then use the transformation $\mathbf{u} = \mathcal{T}\hat{\mathbf{u}}$ and the composition rules for differential operators with PI operators, to derive an equivalent PIE representation as $\mathcal{T}\dot{\hat{\mathbf{u}}} = \mathcal{A}\hat{\mathbf{u}}$.

5) *Derive an LPI stability test.*
In Subsection VII-A, we parameterize a LF $V(\hat{\mathbf{u}}) = \langle \mathcal{T}\hat{\mathbf{u}}, \mathcal{P}\mathcal{T}\hat{\mathbf{u}}\rangle$ as a PI operator $\mathcal{P}$. Using this LF, we prove that existence of a PI operator $\mathcal{P} > 0$ satisfying the LPI $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} < 0$ certifies stability of the PDE.

6) *Parameterize the convex cone of positive PI operators.*
In Subsection VII-B, we introduce a PI operator $\mathcal{Z}$, defined by monomial basis functions $Z(x, y)$. For any positive matrix $P > 0$, then, the product $\mathcal{P} = \mathcal{Z}^*P\mathcal{Z}$ will be a PI operator satisfying $\langle \hat{\mathbf{u}}, \mathcal{P}\hat{\mathbf{u}}\rangle > 0$ for any $\hat{\mathbf{u}} \in L_2[x, y]$.

7) *Implement this methodology in PIETOOLS.*
In Section VIII, we implement a class of 2D-PI operators in the MATLAB toolbox PIETOOLS 2021b, along with the formulae for constructing the PIE representation of a PDE. This allows an arbitrary PDE to be converted to an equivalent PIE, at which point stability may be tested by solving the LPI $\left[\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}\right] < 0$ for a positive operator $\mathcal{P} > 0$. Parameterizing the cone of positive operators as positive matrices, this problem may then be solved using semidefinite programming. We test this implementation on a heat equation and a wave equation in Section IX.

II. Notation

For a given domain $x \in [a, b]$ and $y \in [c, d]$, let $L_2^n[x, y]$ denote the set of $\mathbb{R}^n$-valued square-integrable functions on $[a, b] \times [c, d]$, with the standard inner product. $L_2^n[x]$ is defined similarly and we omit the domain when clear from context. For any $\alpha \in \mathbb{N}^2$, we denote $\|\alpha\|_\infty := \max\{\alpha_1, \alpha_2\}$. Then, we define $H_k^n[x, y]$ as a Sobolev subspace of $L_2^n[x, y]$, where

$$H_k^n[x, y] = \{\mathbf{u} \mid \partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathbf{u} \in L_2^n[x, y], \ \ \forall \|\alpha\|_\infty \le k\}.$$

As for L_2, we occasionally use $H_k^n := H_k^n[x, y]$ or $H_k^n := H_k^n[x]$ when the domain is clear from context. For $\mathbf{u} \in H_k^n[x, y]$, we use the norm

$$\|\mathbf{u}\|_{H_k} = \sum_{\|\alpha\|_\infty \le k} \left\|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathbf{u}\right\|_{L_2}.$$

For $\mathbf{u} \in H_k^n$ with $k > 0$, we denote the Dirac delta operators

$$[\Lambda_x^a \mathbf{u}](y) := \mathbf{u}(a,y) \quad \text{and} \quad [\Lambda_y^c \mathbf{u}](x) := \mathbf{u}(x,c).$$

III. Partial Integral Operators

In [10], we parameterized an algebra of PI operators whose domain was functions $L_2[x]$ of a single spatial variable, $x \in [a,b]$. These operators took the form

$$(\mathcal{P}[N]\mathbf{u})(x) = N_0(x)\mathbf{u}(x) + \int_a^x N_1(x,\theta)\mathbf{u}(\theta)d\theta \qquad (1)$$
$$+ \int_x^b N_2(x,\theta)\mathbf{u}(\theta)d\theta.$$

with polynomial parameters N_i. In [9], these PI operators were generalized, yielding operators with domain $\mathbb{R}^n \times L_2[x]$. These extended operators then had the form

$$\begin{bmatrix} Pv + \int_a^b \big(Q_1(\theta)\mathbf{u}(\theta)\big)d\theta \\ Q_2(x)v + (\mathcal{P}[N]\mathbf{u})(x) \end{bmatrix}$$

with parameters P, Q_1, Q_2, and $\mathcal{P}[N]$ – where we note that the third parameter is itself a parameterized PI operator on the domain L_2. In this paper, we are tasked with further generalizing the algebra of PI operators to functions of *two* spatial variables – i.e. $L_2[x,y]$. Operators in this rather more complicated algebra take the form of

$$\begin{aligned}(\mathcal{P}[N]\mathbf{u})(x,y) &:= N_{00}(x,y)\mathbf{u}(x,y) \qquad (2)\\
&+ \int_a^x N_{10}(x,y,\theta)\mathbf{u}(\theta,y)d\theta + \int_x^b N_{20}(x,y,\theta)\mathbf{u}(\theta,y)d\theta\\
&+ \int_c^y N_{01}(x,y,\nu)\mathbf{u}(x,\nu)d\nu + \int_y^d N_{02}(x,y,\nu)\mathbf{u}(x,\nu)d\nu\\
&+ \int_a^x \int_c^y N_{11}(x,y,\theta,\nu)\mathbf{u}(\theta,\nu)d\nu d\theta\\
&+ \int_x^b \int_c^y N_{21}(x,y,\theta,\nu)\mathbf{u}(\theta,\nu)d\nu d\theta\\
&+ \int_a^x \int_y^d N_{12}(x,y,\theta,\nu)\mathbf{u}(\theta,\nu)d\nu d\theta\\
&+ \int_x^b \int_y^d N_{22}(x,y,\theta,\nu)\mathbf{u}(\theta,\nu)d\nu d\theta.\end{aligned}$$

where we now have 9 polynomial parameters N_{ij}. Making matters worse, we will also need to include cross-terms from $\mathbb{R}^n$, $L_2[x]$ and $L_2[y]$ in our algebra. In the following subsections, to make presentation of this class of operators somewhat tractable, we will make heavy use of an "operator parameterization of operators" approach, whereby we parameterize simpler algebras of operators using polynomials and embed these simpler algebras in the more complex ones. To aid in keeping track of these algebras, we use, e.g. the term 0112-PI algebra to indicate its domain and range include $\mathbb{R}^n$ (indicated by '0'), $L_2[x]$ (indicated by 1), $L_2[y]$ (indicated by 1), and $L_2[x,y]$ (indicated by 2).

Note that throughout this article we overload the $\mathcal{P}[N]$ notation for a PI operator with parameters $N \in \mathcal{N}$, where the structure and parameterization of the operator varies depending on the specific parameter space $\mathcal{N}$, where $\mathcal{N}$ may be $\mathcal{N}_{1D}$, $\mathcal{N}_{011}$, $\mathcal{N}_{2D}$, $\mathcal{N}_{1D\to 2D}$, $\mathcal{N}_{2D\to 1D}$, or $\mathcal{N}_{0112}$.

A. An Algebra of 2D to 2D PI Operators

We start by parameterizing operators on $L_2^m[x,y]$, for which we define the parameter space

$$\mathcal{N}_{2D}^{n\times m} := \begin{bmatrix} L_2^{n\times m}[x,y] & L_2^{n\times m}[x,y,\nu] & L_2^{n\times m}[x,y,\nu] \\ L_2^{n\times m}[x,y,\theta] & L_2^{n\times m}[x,y,\theta,\nu] & L_2^{n\times m}[x,y,\theta,\nu] \\ L_2^{n\times m}[x,y,\theta] & L_2^{n\times m}[x,y,\theta,\nu] & L_2^{n\times m}[x,y,\theta,\nu] \end{bmatrix}.$$

Then, for any $N := \left[\begin{smallmatrix} N_{00} & N_{01} & N_{02} \\ N_{10} & N_{11} & N_{12} \\ N_{20} & N_{21} & N_{22} \end{smallmatrix}\right] \in \mathcal{N}_{2D}^{n\times m}$, we let the associated 2D-PI operator $\mathcal{P}[N] : L_2^m[x,y] \to L_2^n[x,y]$ be as in Eqn. (2). Defining addition and scalar multiplication of the parameters $N, M \in \mathcal{N}_{2D}^{n\times m}$ in the obvious manner, it immediately follows that $\mathcal{P}[N] + \mathcal{P}[M] = \mathcal{P}[N+M]$, and $\lambda\mathcal{P}[N] = \mathcal{P}[\lambda N]$, for any $\lambda \in \mathbb{R}$. Further defining multiplication of 2D-PI operators as in the following lemma, we conclude that the set of 2D-PI operators is an algebra.

Lemma 1: For any $N \in \mathcal{N}_{2D}^{n\times p}$ and $M \in \mathcal{N}_{2D}^{p\times m}$, there exists a unique $Q \in \mathcal{N}_{2D}^{n\times m}$ such that $\mathcal{P}[N]\mathcal{P}[M] = \mathcal{P}[Q]$. Specifically, we may choose

$$Q = \mathcal{L}_{2D}(N,M) \in \mathcal{N}_{2D}^{n\times m}$$

where the linear parameter map $\mathcal{L}_{2D} : \mathcal{N}_{2D}^{n\times p} \times \mathcal{N}_{2D}^{p\times m} \to \mathcal{N}_{2D}^{n\times m}$ is as defined in Eqn. (26) in Appendix I-D of [12].

B. An Algebra of 011-PI Operators

Having defined an algebra of operators on $L_2[x,y]$, we now consider a parameterization of operators on $RL^{n_0,n_1} := \mathbb{R}^{n_0} \times L_2^{n_1}[x] \times L_2^{n_1}[y]$. To this end, we first let for any $N = \{N_0, N_1, N_2\} \in \mathcal{N}_{1D}^{n\times m} := L_2^{n\times m}[x] \times L_2^{n\times m}[x,\theta] \times L_2^{n\times m}[x,\theta]$, the associated 1D-PI operator be as in (1). Next, we let a parameter space for 011-PI operators be given by

$$\mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \end{smallmatrix}\right] := \begin{bmatrix} \mathbb{R}^{n_0\times m_0} & L_2^{n_0\times m_1}[x] & L_2^{n_0\times m_1}[y] \\ L_2^{n_1\times m_0}[x] & \mathcal{N}_{1D}^{n_1\times m_1} & L_2^{n_1\times m_1}[x,y] \\ L_2^{n_1\times m_0}[y] & L_2^{n_1\times m_1}[y,x] & \mathcal{N}_{1D}^{n_1\times m_1} \end{bmatrix}.$$

Then, for any $B := \left[\begin{smallmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{smallmatrix}\right] \in \mathcal{N}_{011}$, we define the associated PI operator $\mathcal{P}[B] : RL^{m_0,m_1} \to RL^{n_0,n_1}$ as

$$\mathcal{P}[B] := \begin{bmatrix} \mathbf{M}[B_{00}] & \int_{x=a}^b [B_{01}] & \int_{y=c}^d [B_{02}] \\ \mathbf{M}[B_{10}] & \mathcal{P}[B_{11}] & \int_{y=c}^d [B_{12}] \\ \mathbf{M}[B_{20}] & \int_{x=a}^b [B_{21}] & \mathcal{P}[B_{22}] \end{bmatrix},$$

where M is the multiplier operator and $\int$ is the integral operator, so that (through some abuse of notation)

$$(\mathbf{M}[N]\mathbf{u})(x,y) := N(x,y)\mathbf{u}(y),$$
$$\left(\int_{y=c}^{d} [N]\mathbf{u}\right)(x) := \int_c^d N(x,y)\mathbf{u}(y)dy.$$

Clearly then, $\mathcal{P}[B]+\mathcal{P}[D] = \mathcal{P}[B+D]$ and $\lambda\mathcal{P}[B] = \mathcal{P}[\lambda B]$ for any $B, D \in \mathcal{N}_{011}$ and $\lambda \in \mathbb{R}$. Moreover, we can also compose 011-PI operators, as per the following lemma.

Lemma 2: For any $B \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & p_0 \\ n_1 & p_1 \end{smallmatrix}\right]$, $D \in \mathcal{N}_{011}\left[\begin{smallmatrix} p_0 & m_0 \\ p_1 & m_1 \end{smallmatrix}\right]$, there exists a unique $R \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \end{smallmatrix}\right]$ such that $\mathcal{P}[B]\mathcal{P}[D] = \mathcal{P}[R]$. Specifically, we may choose

$$R = \mathcal{L}_{011}(B,D) \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \end{smallmatrix}\right],$$

where the linear parameter map $\mathcal{L}_{011} : \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & p_0 \\ n_1 & p_1 \end{smallmatrix}\right] \times \mathcal{N}_{011}\left[\begin{smallmatrix} p_0 & m_0 \\ p_1 & m_1 \end{smallmatrix}\right] \to \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \end{smallmatrix}\right]$ is as defined in Eqn. (24) in Appendix I-C of [12].

C. An Algebra of 0112-PI Operators

We now combine the 011-PI algebra and the 2D-PI algebra to obtain an algebra of operators on $RL^{n_0,n_1} \times L_2^{n_2}[x,y] = \mathbb{R}^{n_0} \times L_2^{n_1}[x] \times L_2^{n_1}[y] \times L_2^{n_2}[x,y]$. Specifically, for any

$$C = \begin{bmatrix} & & & C_{03} \\ & B & & C_{13} \\ & & & C_{23} \\ C_{30} & C_{31} & C_{32} & N \end{bmatrix} \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right] :=$$

$$\begin{bmatrix} & & & L_2^{n_0\times m_2}[x,y] \\ & \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \end{smallmatrix}\right] & & \mathcal{N}_{2D\to 1D}^{n_1\times m_2} \\ & & & \mathcal{N}_{2D\to 1D}^{n_1\times m_2} \\ L_2^{n_2\times m_0}[x,y] & \mathcal{N}_{1D\to 2D}^{n_2\times m_1} & \mathcal{N}_{1D\to 2D}^{n_2\times m_1} & \mathcal{N}_{2D}^{n_2\times n_2} \end{bmatrix},$$

where

$$\mathcal{N}_{2D\to 1D}^{n\times m} := L_2^{n\times m}[x,y] \times L_2^{n\times m}[x,y,\theta] \times L_2^{n\times m}[x,y,\theta],$$
$$\mathcal{N}_{1D\to 2D}^{n\times m} := L_2^{n\times m}[x,y] \times L_2^{n\times m}[x,y,\nu] \times L_2^{n\times m}[x,y,\nu],$$

we define the 0112-PI operator $\mathcal{P}[C] : RL^{m_0,m_1} \times L_2^{m_2}[x,y] \to RL^{n_0,n_1} \times L_2^{n_2}[x,y]$ as

$$\mathcal{P}[C] = \begin{bmatrix} & & & \int_{x=a}^{b}[I] \circ \int_{y=c}^{d}[C_{03}] \\ & \mathcal{P}[B] & & \mathcal{P}[C_{13}] \\ & & & \mathcal{P}[C_{23}] \\ \mathbf{M}[C_{30}] & \mathcal{P}[C_{31}] & \mathcal{P}[C_{32}] & \mathcal{P}[N] \end{bmatrix}$$

where for $D = \{D_0, D_1, D_2\} \in \mathcal{N}_{2D\to 1D}^{n\times m}$, we have

$$(\mathcal{P}[D]\mathbf{u})(x) := \int_c^d \Bigg[D_0(x,y)\mathbf{u}(x,y)$$
$$+ \int_a^x D_1(x,y,\theta)\mathbf{u}(\theta,y)d\theta + \int_x^b D_2(x,y,\theta)\mathbf{u}(\theta,y)d\theta \Bigg] dy,$$

and for $E = \{E_0, E_1, E_2\} \in \mathcal{N}_{1D\to 2D}^{n\times m}$, we have

$$(\mathcal{P}[E]\mathbf{u})(x,y) := E_0(x,y)\mathbf{u}(y)$$
$$+ \int_c^y E_1(x,y,\nu)\mathbf{u}(\nu)d\nu + \int_y^d E_2(x,y,\nu)\mathbf{u}(\nu)d\nu.$$

Since the set of operators parameterized in this manner is clearly closed under addition and scalar multiplication, by the following lemma, it is also an algebra.

Lemma 3: For any $B \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & p_0 \\ n_1 & p_1 \\ n_2 & p_2 \end{smallmatrix}\right]$ and $D \in \mathcal{N}_{0112}\left[\begin{smallmatrix} p_0 & m_0 \\ p_1 & m_1 \\ p_2 & m_2 \end{smallmatrix}\right]$, there exists a unique $R \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right]$ such that $\mathcal{P}[B]\mathcal{P}[D] = \mathcal{P}[R]$. Specifically, we may choose

$$R = \mathcal{L}_{0112}(B,D) \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right]$$

where the linear parameter map $\mathcal{L}_{0112} : \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & p_0 \\ n_1 & p_1 \\ n_2 & p_2 \end{smallmatrix}\right] \times \mathcal{N}_{0112}\left[\begin{smallmatrix} p_0 & m_0 \\ p_1 & m_1 \\ p_2 & m_2 \end{smallmatrix}\right] \to \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right]$ is as defined in Eqn. (36) in Appendix I-G of [12].

For the purpose of implementation in Section VIII, we will be considering only PI operators parameterized by polynomial functions, exploiting the following result.

Corollary 4: For any polynomial parameters $B \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & p_0 \\ n_1 & p_1 \\ n_2 & p_2 \end{smallmatrix}\right]$ and $D \in \mathcal{N}_{0112}\left[\begin{smallmatrix} p_0 & m_0 \\ p_1 & m_1 \\ p_2 & m_2 \end{smallmatrix}\right]$, the composite parameters $R = \mathcal{L}_{0112}(B,D) \in \mathcal{N}_{0112}\left[\begin{smallmatrix} n_0 & m_0 \\ n_1 & m_1 \\ n_2 & m_2 \end{smallmatrix}\right]$ are also polynomial.

IV. Useful Properties of PI Operators

Having defined the different algebras of PI operators, we now derive several critical properties of such operators. Specifically, we focus on obtaining an analytic expression for the inverse of a 011-PI operator, using a generalization of the formula in [13]. This result will be used to enforce the boundary conditions when deriving the mapping from fundamental state to PDE state in Section VI. In addition, we prove that the composition of a differential and a PI operator is a PI operator, which will be used to relate differential operators in the PDE to PI operators in the PIE in Section VI. Finally, we consider the adjoint of a PI operator, necessary for deriving and enforcing LPI stability conditions in Section VII. For a more complete description and proofs of these results, we refer to the Arxiv version of this paper [12].

A. Inverse of 011-PI Operators

First, given a 011-PI operator $\mathcal{P}[Q]$ defined by parameters $Q \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right]$, we prove that $\mathcal{P}[Q]^{-1} = \mathcal{P}[\hat{Q}]$ is a 011-PI operator and obtain an analytic expression for the parameters $\hat{Q} \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right]$. As is typical, we restrict ourselves to the case where the operator has a separable structure and the parameters are polynomial.

Lemma 5: Suppose

$$Q = \begin{bmatrix} Q_{00} & Q_{0x} & Q_{0y} \\ Q_{x0} & Q_{xx} & Q_{xy} \\ Q_{y0} & Q_{yx} & Q_{yy} \end{bmatrix} \in N_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right]$$

where $Q_{xx} = \{Q_{xx}^0, Q_{xx}^1, Q_{xx}^1\} \in \mathcal{N}_{1D}^{n_1\times n_1}$ (separable) and $Q_{yy} = \{Q_{yy}^0, Q_{yy}^1, Q_{yy}^1\} \in \mathcal{N}_{1D}^{n_1\times n_1}$ (separable). Suppose further that the parameters can be decomposed as

$$\begin{bmatrix} & Q_{0x}(x) & Q_{0y}(y) \\ Q_{x0}(x) & Q_{xx}^1(x,\theta) & Q_{xy}(x,y) \\ Q_{y0}(y) & Q_{yx}(x,y) & Q_{yy}^1(y,\nu) \end{bmatrix} =$$
$$\begin{bmatrix} & H_{0x}Z(x) & H_{0y}Z(y) \\ Z^T(x)H_{x0} & Z^T(x)\Gamma_{xx}Z(\theta) & Z^T(x)\Gamma_{xy}Z(y) \\ Z^T(y)H_{y0} & Z^T(y)\Gamma_{yx}Z(x) & Z^T(y)\Gamma_{yy}Z(\nu) \end{bmatrix}$$

where $Z \in L_2^{q\times n_1}$ for some $q \in \mathbb{N}$, and

$$\begin{bmatrix} & H_{0x} & H_{0y} \\ H_{x0} & \Gamma_{xx} & \Gamma_{xy} \\ H_{y0} & \Gamma_{yx} & \Gamma_{yy} \end{bmatrix} \in \begin{bmatrix} & \mathbb{R}^{n_0\times q} & \mathbb{R}^{n_0\times q} \\ \mathbb{R}^{q\times n_0} & \mathbb{R}^{q\times q} & \mathbb{R}^{q\times q} \\ \mathbb{R}^{q\times n_0} & \mathbb{R}^{q\times q} & \mathbb{R}^{q\times q} \end{bmatrix},$$

Now suppose that $\hat{Q} = \mathcal{L}_{\text{inv}}(Q) \in \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right]$, where $\mathcal{L}_{\text{inv}} : \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right] \to \mathcal{N}_{011}\left[\begin{smallmatrix} n_0 & n_0 \\ n_1 & n_1 \end{smallmatrix}\right]$ is as defined in Eqn. (37) in [12]. Then for any $\mathbf{u} \in \mathbb{R}^{n_0} \times L_2^{n_1}[x] \times L_2^{n_1}[y]$,

$$(\mathcal{P}[\hat{Q}] \circ \mathcal{P}[Q])\mathbf{u} = (\mathcal{P}[Q] \circ \mathcal{P}[\hat{Q}])\mathbf{u} = \mathbf{u}.$$

B. Differentiation and 2D-PI Operators

While differentiation is an unbounded operator and PI operators are bounded, we now show that for a PI operator with no multipliers, composition of a differential operator with this PI operator is a PI operator and hence bounded. This will be used in Section VI to show that the dynamics of a PDE can be equivalently represented using only bounded operators (the PIE representation).

Lemma 6: Suppose

$$N=\begin{bmatrix}0&0&0\\N_{10}&N_{11}&N_{12}\\N_{20}&N_{21}&N_{22}\end{bmatrix}\in\begin{bmatrix}0&0&0\\H_1&H_1&H_1\\H_1&H_1&H_1\end{bmatrix}\subset\mathcal{N}_{2D}.$$

Then letting $M\in\mathcal{N}_{2D}$ be as defined in Eqn. (3) in [12], for any $\mathbf{u}\in L_2[x,y]$,

$$\partial_x(\mathcal{P}[N]\mathbf{u})(x,y)=(\mathcal{P}[M]\mathbf{u})(x,y).$$

Lemma 7: Suppose

$$N=\begin{bmatrix}0&N_{01}&N_{02}\\0&N_{11}&N_{12}\\0&N_{21}&N_{22}\end{bmatrix}\in\begin{bmatrix}0&H_1&H_1\\0&H_1&H_1\\0&H_1&H_1\end{bmatrix}\subset\mathcal{N}_{2D}.$$

Then letting $M\in\mathcal{N}_{2D}$ be as defined in Eqn. (4) in [12], for any $\mathbf{u}\in L_2[x,y]$,

$$\partial_y(\mathcal{P}[N]\mathbf{u})(x,y)=(\mathcal{P}[M]\mathbf{u})(x,y).$$

For the remainder of this paper, and specifically in Lemma 14, whenever a 2D-PI operator $\mathcal{P}[N]$ has no multipliers along the x-direction (as in Lemma 6), or along the y-direction (as in Lemma 7), we will denote the composition of the differential operator ∂_x or ∂_y with this PI operator in the obvious manner as $\partial_x\mathcal{P}[N]$ or $\partial_y\mathcal{P}[N]$, such that, e.g.

$$\left[\left(\partial_x\mathcal{P}[N]\right)\mathbf{u}\right](x,y)=\partial_x\left(\mathcal{P}[N]\mathbf{u}\right)(x,y).$$

C. Adjoint of 2D-PI Operators

Finally, we consider the adjoint of a 2D-PI operator.

Lemma 8: Suppose $N\in\mathcal{N}_{2D}^{n\times m}$, and let $\hat{N}\in\mathcal{N}_{2D}^{m\times n}$ be as defined in Eqn. (5) in [12]. Then for any $\mathbf{u}\in L_2^m$ and $\mathbf{v}\in L_2^n$,

$$\langle\mathbf{v},\mathcal{P}[N]\mathbf{u}\rangle_{L_2}=\left\langle\mathcal{P}[\hat{N}]\mathbf{v},\mathbf{u}\right\rangle_{L_2}.$$

V. A Standardized PDE Format in 2D

Having formulated the required hierarchy of algebras of PI operators, we now introduce a class of linear 2D PDEs, the solutions of which may be represented using 2D PIEs. These 2D PDEs are represented in a standardized format, allowing for efficient construction of a general mapping between the PDE and PIE state spaces – this construction being found in Section VI.

Consider a coupled linear PDE of the form

$$\dot{\mathbf{u}}(t,x,y)=\sum_{i,j=0}^{2}A_{ij}\partial_x^i\partial_y^j\Big(N_{\max\{i,j\}}\mathbf{u}(t,x,y)\Big),\tag{3}$$

where we use the matrices

$$\begin{aligned}N_0&=I_{n_0+n_1+n_2},\\N_1&=\begin{bmatrix}0_{(n_1+n_2)\times n_0}&I_{n_1+n_2}\end{bmatrix},\\N_2&=\begin{bmatrix}0_{n_2\times n_0}&0_{n_2\times n_1}&I_{n_2}\end{bmatrix},\end{aligned}\tag{4}$$

to partition the states according to differentiability, so that $\mathbf{u}(t)\in X(\mathcal{B})$ for all $t\geq0$, where $X(\mathcal{B})$ is the domain of the PDE and includes both continuity constraints and boundary conditions, where the boundary conditions are parameterized by the PI operator $\mathcal{B}:=\mathcal{P}[B]$ as

$$X(\mathcal{B}):=\left\{\begin{bmatrix}\mathbf{u}_0\\\mathbf{u}_1\\\mathbf{u}_2\end{bmatrix}\in\begin{bmatrix}L_2^{n_0}\\H_1^{n_1}\\H_2^{n_2}\end{bmatrix}\;\middle|\;\mathcal{B}\Lambda_{\text{bf}}\mathbf{u}=0\right\},\tag{5}$$

where $B\in\mathcal{N}_{011}\left[\begin{smallmatrix}n_1+4n_2&4n_1+16n_2\\n_1+2n_2&2n_1+4n_2\end{smallmatrix}\right]$ and where Λ_{bf} allows us to list all the possible boundary values for the state components $\mathbf{u}_1$ and $\mathbf{u}_2$, and as limited by differentiability. In particular,

$$\Lambda_{\text{bf}}=\begin{bmatrix}\mathcal{C}_1\\\mathcal{C}_2\\\mathcal{C}_3\end{bmatrix}:L_2^{n_0}\times H_1^{n_1}\times H_2^{n_2}\to\begin{bmatrix}\mathbb{R}^{4n_1+16n_2}\\L_2^{2n_1+4n_2}[x]\\L_2^{2n_1+4n_2}[y]\end{bmatrix},\tag{6}$$

where

$$\mathcal{C}_1:=\begin{bmatrix}0&\Lambda_1&0\\0&0&\Lambda_1\\0&0&\Lambda_1\partial_x\\0&0&\Lambda_1\partial_y\\0&0&\Lambda_1\partial_{xy}\end{bmatrix},\quad\begin{bmatrix}\mathcal{C}_2\\\mathcal{C}_3\end{bmatrix}:=\begin{bmatrix}0&\Lambda_2\partial_x&0\\0&0&\Lambda_2\partial_x^2\\0&0&\Lambda_2\partial_x^2\partial_y\\0&\Lambda_3\partial_y&0\\0&0&\Lambda_3\partial_y^2\\0&0&\Lambda_3\partial_x\partial_y^2\end{bmatrix}$$

and where we use the Dirac operators Λ_k defined as

$$\Lambda_1=\begin{bmatrix}\Lambda_x^a\Lambda_y^c\\\Lambda_x^b\Lambda_y^c\\\Lambda_x^a\Lambda_y^d\\\Lambda_x^b\Lambda_y^d\end{bmatrix},\quad\Lambda_2=\begin{bmatrix}\Lambda_y^c\\\Lambda_y^d\end{bmatrix},\quad\Lambda_3=\begin{bmatrix}\Lambda_x^a\\\Lambda_x^b\end{bmatrix}.$$

This formulation is very general and allows us to express almost any linear 2D PDE. For illustrations of this representation, see the examples in Section IX, along with additional examples in the Arxiv version of this paper [12].

a) Definition of Solution: For a given initial condition $\mathbf{u}_\text{I}\in X(\mathcal{B})$, we say that a function $\mathbf{u}(t)$ satisfies the PDE defined by $\{A_{ij},\mathcal{B}\}$ if $\mathbf{u}$ is Frechét differentiable, $\mathbf{u}(0)=\mathbf{u}_\text{I}$, and for all $t\geq0$, $\mathbf{u}(t)\in X(\mathcal{B})$ and $\mathbf{u}(t)$ satisfies Eqn. (3).

Definition 9: We say that a solution $\mathbf{u}$ with initial condition $\mathbf{u}_\text{I}$ of the PDE defined by $\{A_{ij},\mathcal{B}\}$ is exponentially stable in L_2 if there exist constants $K,\gamma>0$ such that

$$\|\mathbf{u}(t)\|_{L_2}\leq Ke^{-\gamma t}\|\mathbf{u}_\text{I}\|_{L_2}$$

We say the PDE defined by $\{A_{ij},\mathcal{B}\}$ is exponentially stable in L_2 if any solution $\mathbf{u}$ of the PDE is exponentially stable in L_2.

VI. The Fundamental State on 2D

In this section, we provide the main technical result of the paper, wherein we show that for a suitably well-posed set of boundary conditions, $\mathcal{B}$, there exists a unitary 2D-PI operator $\mathcal{T}:L_2\to X(\mathcal{B})$ (where $X(\mathcal{B})$ is defined in Eqn. (5)) such that, if we define the differentiation operator

$$\mathcal{D}:=\begin{bmatrix}I&&\\&\partial_x\partial_y&\\&&\partial_x^2\partial_y^2\end{bmatrix}\tag{7}$$

then for any $\mathbf{u}\in X(\mathcal{B})$ and $\hat{\mathbf{u}}\in L_2^{n_0+n_1+n_2}$, we have

$$\mathbf{u}=\mathcal{T}\mathcal{D}\mathbf{u}\qquad\text{and}\qquad\hat{\mathbf{u}}=\mathcal{D}\mathcal{T}\hat{\mathbf{u}}.$$

This implies that for any $\mathbf{u} \in X(\mathcal{B})$, there exists a *unique* $\hat{\mathbf{u}} \in L_2$ where the map from $\hat{\mathbf{u}}$ to $\mathbf{u}$ is defined by a 2D-PI operator. Because differentiation of a PI operator is a PI operator (Section IV-B), this implies that derivatives of $\mathbf{u}$ can be expressed in terms of a PI operator acting on $\hat{\mathbf{u}}$. Using these results, in Thm. 12, we show that for any suitable PDE defined by $\{A_{ij}, \mathcal{B}\}$, there exist 2D-PI operators $\mathcal{T}, \mathcal{A}$ such that $\hat{\mathbf{u}} \in L_2$ satisfies

$$\mathcal{T}\dot{\hat{\mathbf{u}}}(t) = \mathcal{A}\hat{\mathbf{u}}(t)$$

if and only if $\mathcal{T}\hat{\mathbf{u}} \in X(\mathcal{B})$ satisfies the PDE.

A. Map From Fundamental State to PDE State

As mentioned above, given the boundary-constrained "PDE state" $\mathbf{u} \in X(\mathcal{B})$, we associate a corresponding "fundamental state" $\hat{\mathbf{u}} \in L_2^{n_0+n_1+n_2}$, defined as

$$\hat{\mathbf{u}}(t) := \begin{bmatrix} \hat{\mathbf{u}}_0(t) \\ \hat{\mathbf{u}}_1(t) \\ \hat{\mathbf{u}}_2(t) \end{bmatrix} = \begin{bmatrix} I & & \\ & \partial_x \partial_y & \\ & & \partial_x^2 \partial_y^2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_0(t) \\ \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \end{bmatrix} \tag{8}$$

In the following lemma, we temporarily ignore boundary conditions and use the fundamental theorem of calculus to express any $\mathbf{u} \in L_2 \times H_1 \times H_2$ in terms of $\hat{\mathbf{u}}$, and a set of "core" boundary values.

Lemma 10: Let $\mathbf{u} \in L_2[x,y] \times H_1[x,y] \times H_2[x,y]$. If $\mathcal{K}_1 = \mathcal{P}[K_1]$ and $\mathcal{K}_2 = \mathcal{P}[K_2]$ where $K_1 \in L_2 \times \mathcal{N}_{1D\to 2D} \times \mathcal{N}_{1D\to 2D}$ and $K_2 \in \mathcal{N}_{2D}$ are as defined in Eqn. (10) in [12], then

$$\mathbf{u} = \mathcal{K}_1 \Lambda_{\text{bc}} \mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}$$

where $\hat{\mathbf{u}} = \mathcal{D}\mathbf{u}$ and

$$\Lambda_{\text{bc}} := \begin{bmatrix} 0 & \Lambda_x^a \Lambda_y^c & 0 \\ 0 & 0 & \Lambda_x^a \Lambda_y^c \\ 0 & 0 & \Lambda_x^a \Lambda_y^c \partial_x \\ 0 & 0 & \Lambda_x^a \Lambda_y^c \partial_y \\ 0 & 0 & \Lambda_x^a \Lambda_y^c \partial_x \partial_y \\ 0 & \Lambda_y^c \partial_x & 0 \\ 0 & 0 & \Lambda_y^c \partial_x^2 \\ 0 & 0 & \Lambda_y^c \partial_x^2 \partial_y \\ 0 & \Lambda_x^a \partial_y & 0 \\ 0 & 0 & \Lambda_x^a \partial_y^2 \\ 0 & 0 & \Lambda_x^a \partial_x \partial_y^2 \end{bmatrix}$$

Proof: A proof can be found in [12] ■

Corollary 11: Let $\mathbf{u} \in L_2[x,y] \times H_1[x,y] \times H_2[x,y]$. Then, if $\mathcal{H}_1 = \mathcal{P}[H_1]$ and $\mathcal{H}_2 = \mathcal{P}[H_2]$ where $H_1 \in \mathcal{N}_{011}$ and $H_2 \in L_2 \times \mathcal{N}_{2D\to 1D} \times \mathcal{N}_{2D\to 1D}$ are as defined in Eqn. (11) in [12], then

$$\Lambda_{\text{bf}} \mathbf{u} = \mathcal{H}_1 \Lambda_{\text{bc}} \mathbf{u} + \mathcal{H}_2 \hat{\mathbf{u}}$$

where $\hat{\mathbf{u}} = \mathcal{D}\mathbf{u}$, and Λ_{bf} is as defined in Eqn. (6).

With these definitions, we can express an arbitrary PDE state $\mathbf{u} \in X(\mathcal{B})$ in terms of a corresponding state $\hat{\mathbf{u}} \in L_2^{n_0 \times n_1 \times n_2}$ and $\Lambda_{\text{bc}}\mathbf{u}$. In the following theorem, we describe this relation as a PI operator, incorporating the boundary conditions to describe a map from $L_2^{n_0 \times n_1 \times n_2}$ to $X(\mathcal{B})$.

Theorem 12: Let $B = \left[\begin{smallmatrix} B_{00} & B_{01} & B_{02} \\ B_{10} & B_{11} & B_{12} \\ B_{20} & B_{21} & B_{22} \end{smallmatrix}\right] \in \mathcal{N}_{011}$ with $B_{11} = \{B_{11}^0, B_{11}^1, B_{11}^1\} \in \mathcal{N}_{1D}$ and $B_{22} = \{B_{22}^0, B_{22}^1, B_{22}^1\} \in \mathcal{N}_{1D}$ be given, and such that the operator $\mathcal{B}\mathcal{H}_1$ is invertible, where $\mathcal{B} = \mathcal{P}[B]$ and where $\mathcal{H}_1$ is as in Cor. 11. Let associated parameters $T \in \mathcal{N}_{2D}$ be as defined in Eqn. (13) in [12]. Then if $\mathcal{T} = \mathcal{P}[T]$, for any $\mathbf{u} \in X(\mathcal{B})$ and $\hat{\mathbf{u}} \in L_2[x,y]$, we have

$$\mathbf{u} = \mathcal{T}\mathcal{D}\mathbf{u} \qquad \text{and} \qquad \hat{\mathbf{u}} = \mathcal{D}\mathcal{T}\hat{\mathbf{u}}.$$

a) Outline of Proof: Theorem 12 combines several results. First, we use Lemma 10 to express $\mathbf{u}$ in terms of $\hat{\mathbf{u}} = \mathcal{D}\mathbf{u}$ and the "core" boundary values, $\mathbf{u}_{\text{bc}} := \Lambda_{\text{bc}}\mathbf{u}$ as $\mathbf{u} = \mathcal{K}_1\mathbf{u}_{\text{bc}} + \mathcal{K}_2\hat{\mathbf{u}}$ where $\mathcal{K}_1$ and $\mathcal{K}_2$ are PI operators. Then, using Corollary 11 we express the "full" boundary values $\mathbf{u}_{\text{bf}} := \Lambda_{\text{bf}}\mathbf{u}$ in terms of $\mathbf{u}_{\text{bc}}$ and $\hat{\mathbf{u}}$ as $\mathbf{u}_{\text{bf}} = \mathcal{H}_1\mathbf{u}_{\text{bc}} + \mathcal{H}_2\hat{\mathbf{u}}$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are PI operators. Next, we show that $\mathcal{B}\Lambda_{\text{bf}}\mathbf{u} = 0$ implies $\mathcal{B}\mathbf{u}_{\text{bf}} = \mathcal{B}\mathcal{H}_1\mathbf{u}_{\text{bc}} + \mathcal{B}\mathcal{H}_2\hat{\mathbf{u}} = 0$ and use the inversion formula in Lemma 5 to obtain the PI operator $\hat{\mathcal{E}} = (\mathcal{B}\mathcal{H}_1)^{-1}$ which gives us $\mathbf{u}_{\text{bc}} = -\hat{\mathcal{E}}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}}$. Finally, we substitute this expression for the "core" boundary values into the expression from Lemma 10 to obtain

$$\mathbf{u} = \mathcal{K}_1\mathbf{u}_{\text{bc}} + \mathcal{K}_2\hat{\mathbf{u}} = -\mathcal{K}_1\mathcal{E}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}} + \mathcal{K}_2\hat{\mathbf{u}} = (\mathcal{K}_2 - \mathcal{K}_1\hat{\mathcal{E}}\mathcal{B}\mathcal{H}_2)\hat{\mathbf{u}}.$$

Thus, we obtain $\mathcal{T} = (\mathcal{K}_2 - \mathcal{K}_1(\mathcal{B}\mathcal{H}_1)^{-1}\mathcal{B}\mathcal{H}_2)$ which, by the algebraic property of PI operators, is a PI operator.

Corollary 13: Let $\mathcal{T}$ be as defined in Theorem 12. Then $\mathcal{T} : L_2 \to X(\mathcal{B})$ is unitary with respect to the inner product $\langle \mathbf{u}, \mathbf{v} \rangle_X := \langle \mathcal{D}\mathbf{u}, \mathcal{D}\mathbf{v} \rangle_{L_2}$.

Proof: A proof can be found in [12] ■

B. PDE to PIE conversion

We now demonstrate that, given a PDE defined by $\{A_{ij}, \mathcal{B}\}$, for appropriate choice of $\mathcal{A}, \mathcal{T}$, we may define a Partial Integral Equation (PIE) whose solutions are equivalent to those of the PDE. Specifically, for given PI operators $\mathcal{T}, \mathcal{A}$, and an initial condition $\hat{\mathbf{u}}_{\text{I}}$, we say $\hat{\mathbf{u}}(t)$ solves the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ for initial condition $\hat{\mathbf{u}}_{\text{I}}$ if $\hat{\mathbf{u}}(0) = \hat{\mathbf{u}}_{\text{I}}$, $\hat{\mathbf{u}}(t) \in L_2^n[x,y]$ for all $t \geq 0$ and for all $t \geq 0$

$$\mathcal{T}\dot{\hat{\mathbf{u}}}(t) = \mathcal{A}\hat{\mathbf{u}}(t). \tag{9}$$

The following result shows that if $\mathcal{T}$ is as defined in Theorem 12, then $\mathbf{u}(t)$ satisfies the PDE defined by $\{A_{ij}, \mathcal{B}\}$ if and only if $\hat{\mathbf{u}}(t) = \mathcal{T}\mathbf{u}(t)$ satisfies the PIE defined by $\{\mathcal{A}, \mathcal{T}\}$.

Lemma 14: Suppose $\mathcal{T}$ is as defined in Theorem 12, and let

$$\mathcal{A} = \sum_{i,j=0}^{2} A_{ij}\left(\partial_x^i \partial_y^j \left[N_{\max\{i,j\}}\mathcal{T}\right]\right), \tag{10}$$

where the matrices N_0, N_1, N_2 are as defined in Eqn. (4). Then, given $\hat{\mathbf{u}}_{\text{I}} \in L_2^{n_0+n_1+n_2}[x,y]$, $\hat{\mathbf{u}}(t)$ solves the PIE (9) defined by $\{\mathcal{T}, \mathcal{A}\}$ with the initial condition $\hat{\mathbf{u}}_{\text{I}}$ if and only if $\mathbf{u}(t) = \mathcal{T}\hat{\mathbf{u}}(t)$ satisfies the PDE defined by $\{A_{ij}, \mathcal{B}\}$ with the initial condition $\mathbf{u}_{\text{I}} = \mathcal{T}\hat{\mathbf{u}}_{\text{I}}$.

A proof of this result may be found in the Arxiv version [12]. Specific examples of PDEs and their PIE equivalents are given in Section IX, with additional examples in [12]. In the following section, we propose stability conditions for the PIE which can be enforced using LMIs.

VII. Stability as an LPI

Having derived an equivalent PIE representation of PDEs, we now show how this representation can be used for stability analysis. First, we show that existence of a quadratic Lyapunov function for a PIE can be posed as a convex Linear PI Inequality (LPI) optimization problem, with variables of the form $\mathcal{P} = \mathcal{P}[P]$ for $P \in \mathcal{N}_{2D}$, and inequality constraints of the form $\mathcal{P} \geq 0$, where positivity is with respect to the L_2 inner product. Next, we show how to use LMIs to parameterize the cone of positive 2D-PI operators - allowing us to test the Lyapunov stability criterion. A proof of the resulting Thm. 15 and Prop. 16 can be found in [12].

A. Lyapunov Stability Criterion

We first express the problem of existence of a quadratic Lyapunov function as an LPI, whose feasibility implies stability of the associated PIE and PDE. Specifically, the following theorem tests for existence of a quadratic Lyapunov function $V(\mathbf{u}) = \langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle_{L_2} = \langle \hat{\mathbf{u}}, \mathcal{T}^*\mathcal{P}\mathcal{T}\hat{\mathbf{u}}\rangle_{L_2} \geq \epsilon \|\mathbf{u}\|_{L_2}^2$, such that $\dot{V}(\mathbf{u}(t)) \leq -\delta \|\mathbf{u}(t)\|_{L_2}^2$ for any solution $\mathbf{u}$ of the PDE defined by $\{A_{ij}, \mathcal{B}\}$.

Theorem 15: Suppose $\mathcal{T}$ and $\mathcal{A}$ are as defined in Thm. 12 and Lem. 14 respectively, and that there exist $\epsilon, \delta > 0$ and $P \in \mathcal{N}_{2D}$ such that the PI operator $\mathcal{P} := \mathcal{P}[P]$ satisfies $\mathcal{P} = \mathcal{P}^*$, $\mathcal{P} \geq \epsilon I$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \leq -\delta\mathcal{T}^*\mathcal{T}.$$

Finally, let $\zeta = \|\mathcal{P}\|_{\mathcal{L}_{L_2}}$. Then, any solution $\mathbf{u}(t) \in X(\mathcal{B})$ of the PDE defined by $\{A_{ij}, \mathcal{B}\}$ satisfies

$$\|\mathbf{u}(t)\|_{L_2}^2 \leq \frac{\zeta}{\epsilon}\|\mathbf{u}(0)\|_{L_2}^2 e^{-\frac{\delta}{\zeta}t}.$$

In this stability condition, the decision variable is $P \in \mathcal{N}_{2D}^{n\times n}$ and the constraints are operator inequalities on the inner product space L_2. While the decision variables may be readily parameterized using polynomials, to numerically enforce the inequality constraints, we need to parameterize the cone of operators on $\mathcal{N}_{2D}$ which are positive semidefinite. This problem will be addressed in the following subsection.

B. A Parameterization of Positive PI Operators

Having posed the PDE stability problem as an LPI, we now show how to parameterize the cone of positive 2D-PI operators using positive matrices.

Proposition 16: For any $Z \in L_2^{q\times n}$ and $g \in L_2$ satisfying $g(x,y) \geq 0$ for all $x, y \in [a,b]\times[c,d]$, let $\mathcal{L}_{\text{PI}} : \mathbb{R}^{9q\times 9q} \to \mathcal{N}_{2D}^{n\times n}$ be as defined in Eqn. (50) in Appendix II-D of [12]. Then for any $P \geq 0$, if $N = \mathcal{L}_{\text{PI}}(P)$, we have that $\mathcal{P}^*[N] = \mathcal{P}[N]$, and $\langle \mathbf{u}, \mathcal{P}[N]\mathbf{u}\rangle_{L_2} \geq 0$ for any $\mathbf{u} \in L_2^n[x,y]$.

Prop. 16 tests whether there exists a 2D-PI operator $\mathcal{Z}$ defined in terms of Z and g such that $\mathcal{P}[N] = \mathcal{Z}^*P\mathcal{Z}$, allowing us to enforce an LPI with variable $\mathcal{P}[N]$ using an LMI constraint on P. For this test, we use a monomial basis, Z_d to define $\mathcal{Z}$, yielding polynomial paramaters $N = \mathcal{L}_{\text{PI}}(P)$. For the scalar function $g(x,y) \geq 0$, we may use $g(x,y) = 1$, implying the inequality is valid over any domain, or $g(x,y) = (x-a)(b-x)(y-c)(d-y)$, implying the operator is positive only on the domain $(x,y) \in [a,b]\times[c,d]$.

In the following section, we will apply Prop. 16 to obtain an LMI for stability of a PDE. For this section we denote

$$\begin{aligned}\Omega_d := \{\mathcal{P}[N] + \mathcal{P}[M] \mid N, M \in \mathcal{N}_{2D} &\text{ are as in Prop. 16}\\ \text{with } Z = Z_d \text{ and } g(x,y) = 1 &\text{ and respectively}\\ g(x,y) = (x-a)(b-x)&(y-c)(d-y)\}\end{aligned}$$

where now $\mathcal{P} \in \Omega_d$ is an LMI constraint implying $\mathcal{P} > 0$.

VIII. PIETOOLS Implementation

In this section, we show how the PIETOOLS 2021b toolbox may be used to perform stability analysis of PDEs. This toolbox offers a framework for implementation and manipulation of PI operators in MATLAB, allowing e.g. Lyapunov stability analysis [9], robust stability analysis [14], and H_∞-optimal control [11] of PDEs. For a detailed manual of the PIETOOLS toolbox we refer to [8].

To implement PI operators in MATLAB, the `dpvar` class of polynomial objects is used to define the polynomial functions N parameterizing PI operators $\mathcal{P}[N]$. A class of 0112-PI operators is then defined as `opvar2d` objects, and overloaded with standard operations such as multiplication (*), addition (+) and adjoint (') presented in earlier sections. Defining decision operators `dopvar2d` in terms of positive matrices, we may also enforce positivity conditions $\mathcal{P} \in \Omega_d$, allowing stability to be tested with any LMI solver. Such a test may be run in PIETOOLS 2021b as follows:

1) Define the spatial variables in the PDE, and initialize an optimization program structure `X`.
```
pvar x y tt nu;
X = sosprogram([x y tt nu]);
```
2) Construct the PDE, defining the sizes n_0, n_1, n_2 of the state variables, the matrices A_{ij} defining the PDE, and an `opvar2d` object `Ebb` defining $\mathcal{B}$. Compute the associated PIE, extracting operators $\mathcal{T}$ and $\mathcal{A}$.
```
PDE.n.n_pde = [n0,n1,n2];
PDE.dom = [a,b;c,d];
PDE.PDE.A = ...; PDE.BC.Ebb = ...;
PIE = convert_PIETOOLS_PDE(PDE);
T = PIE.T;          A = PIE.A;
```
3) Declare a positive operator $\mathcal{P} \in \Omega_d$ for monomial degree $d \in \mathbb{N}$, and add a small constant $\epsilon << 1$ to ensure strict positivity. Impose the additional requirement $-\delta\mathcal{T}^*\mathcal{T} - \mathcal{A}^*\mathcal{P}\mathcal{T} - \mathcal{T}^*\mathcal{P}\mathcal{A} \in \Omega_d$ for some $\delta > 0$.
```
[X, P] = poslpivar_2d(X,n,dom,d);
P = P + eps;
D = -del*(T'*T) - A'*P*T - T'*P*A;
X = lpi_ineq_2d(X,D);
```
4) Call the SDP solver, and extract the solution $\mathcal{P}$.
```
X = sossolve(X);
Psol = getsol_lpivar_2d(X,P);
```

IX. Illustrative Examples

To illustrate the techniques described in the previous sections, we will apply them to two simple examples, showing how the PDE may be expressed in the standardized format, defining the associated PIE, and numerically testing stability.

A. Heat Equation

As a first example, we consider a 2D heat equation

$$u_t(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y),\\ u(x,0) = u_y(x,0) = u(0,y) = u_x(0,y) = 0.$$

To describe this system in the standardized Format (3), we use PDE state $\mathbf{u} = \mathbf{u}_2 = u \in H_2^{n_2}[x,y]$ with $n_2 = 1$, and define $A_{20} = A_{02} = 1$, setting $A_{ij} = 0$ for all other $i,j \in \{0,1,2\}$. For the boundary conditions, we require $\mathbf{u}(1,0) = \mathbf{u}_x(0,0) = \mathbf{u}_y(0,1) = \mathbf{u}_{xy}(0,0) = 0$, and $\mathbf{u}_{xx}(x,0) = \mathbf{u}_{xxy}(x,0) = \mathbf{u}_{yy}(0,y) = \mathbf{u}_{xyy}(0,y) = 0$, which can be expressed as $B\Lambda_{\text{bf}}\mathbf{u} = 0$ for an appropriate choice of $B \in \mathbb{R}^{8\times 24}$. Then, we may describe the system as a PDE defined by $\{A_{ij}, B\}$, for which we obtain a corresponding PIE representation

$$\mathcal{T}\dot{\hat{\mathbf{u}}} = \int_0^x \int_0^y (x-\theta)(y-\nu)\dot{\hat{\mathbf{u}}}(\theta,\nu)d\nu d\theta\\ = \int_0^x (x-\theta)\hat{\mathbf{u}}(\theta,y)d\theta + \int_0^y (y-\nu)\hat{\mathbf{u}}(x,\nu)d\nu = \mathcal{A}\hat{\mathbf{u}},$$

where $\hat{\mathbf{u}} = u_{xxyy}$ is the fundamental state.

For the purpose of testing accuracy of the stability analysis, we consider the following system, as presented in [15],

$$u_t(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y) + ru(t,x,y),\\ u(x,0) = u(0,y) = u(x,1) = u(1,y) = 0.$$

Stability of this system can be proven analytically whenever $r \le 2\pi^2 = 19.739....$ Using PIETOOLS 2021b, performing bisection on r and using monomial degree $d = 3$, exponential stability can be verified for any $r \le 19.736$.

B. Wave Equation

As a second example, we consider a 2D wave equation

$$\ddot{u}(t,x,y) = u_{xx}(t,x,y) + u_{yy}(t,x,y),\\ u(x,0) = u_y(x,0) = u_x(0,y) = u(0,y) = 0.$$

To write this system in the standardized form, we define $\mathbf{u}_1 = u$ and $\mathbf{u}_2 = \dot{u}$, so that the PDE may be denoted as

$$\begin{bmatrix}\dot{\mathbf{u}}_1\\ \dot{\mathbf{u}}_2\end{bmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}0&0\\1&0\end{bmatrix}\partial_x^2\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}0&0\\1&0\end{bmatrix}\partial_y^2\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}\\ = A_{00}\mathbf{u} + A_{20}\partial_x^2\mathbf{u} + A_{02}\partial_y^2\mathbf{u}$$

Requiring $\mathbf{u}(1,0) = \mathbf{u}_x(0,0) = \mathbf{u}_y(0,1) = \mathbf{u}_{xy}(0,0) = 0$, and $\mathbf{u}_{xx}(x,0) = \mathbf{u}_{xxy}(x,0) = \mathbf{u}_{yy}(0,y) = \mathbf{u}_{xyy}(0,y) = 0$, we may write the boundary conditions as $B\Lambda_{\text{bf}}\mathbf{u} = 0$ for appropriate $B \in \mathbb{R}^{16\times 48}$ (see PIETOOLS 2021b examples), defining the PDE through $\{A_{ij}, B\}$. Letting $\hat{\mathbf{u}} = \mathbf{u}_{xxyy}$, the associated PIE representation is

$$\mathcal{T}\dot{\hat{\mathbf{u}}} = \int_0^x\int_0^y \begin{bmatrix}(x-\theta)(y-\nu) & 0\\ 0 & (x-\theta)(y-\nu)\end{bmatrix}\dot{\hat{\mathbf{u}}}(\theta,\nu)d\nu d\theta\\ = \int_0^x \begin{bmatrix}0 & (x-\theta)\\ 0 & 0\end{bmatrix}\hat{\mathbf{u}}(\theta,y)d\theta + \int_0^y \begin{bmatrix}0 & (y-\nu)\\ 0 & 0\end{bmatrix}\hat{\mathbf{u}}(x,\nu)d\nu\\ + \int_0^x\int_0^y \begin{bmatrix}0 & 0\\ (x-\theta)(y-\nu) & 0\end{bmatrix}\hat{\mathbf{u}}(\theta,\nu)d\nu d\theta = \mathcal{A}\hat{\mathbf{u}}.$$

Simulation suggests the system is neutrally stable with boundary conditions $u(x,0) = u(x,1) = u(0,y) = u(1,y) = 0$. Setting $\delta = 0$, this can be verified with PIETOOLS.

X. Conclusion

In this paper, we have derived formulae to convert any well-posed, linear, second order 2D PDE to an equivalent PIE. To this end, we have introduced different PI operators in 2D, showing that their product, inverse, adjoint, and compositions with differential operators each return PI operators. Exploiting these relations, we derived a mapping $\mathbf{u} = \mathcal{T}\hat{\mathbf{u}}$ between the PDE state $\mathbf{u} \in X(\mathcal{B})$, constrained by boundary conditions as described by $\mathcal{B}$, and a fundamental state $\hat{\mathbf{u}} \in L_2$, free of any such constraints. This allowed us to describe a stability condition as an LPI, which we demonstrated could be numerically solved through semidefinite programming, as implemented in the MATLAB toolbox PIETOOLS 2021b.

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