Modern Control Systems

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Lecture 5: Normed Spaces, Matrix Properties

Coordinate Systems

Recall:

- A basis, $\{x_i\}$ is a set of independent elements which span a vector space.
- A minimal basis defines a **Coordinate System**.

Consider a vector space, X.

Definition 1.

For any $x \in X$, the **Coordinates** of x in basis $B = \{b_i\}$ is the unique set of scalars $\{\alpha_i\}$ such that

$$x = \sum_i \alpha_i b_i.$$

We denote the coordinates of x in basis $B = \{b_i\}$ as

$$x_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Note: Some bases are better for certain applications

Consider the vector

$$x = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

Canonical Basis:

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Then

$$x_{B_1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T$$

Alternative Basis:

$$B_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Then

$$x_{B_2} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

Coordinate Systems

Examples

The polynomial $p(x) = x^3 + 2$

Monomial Basis:

$$B_1 = \left\{1, x, x^2, x^3\right\}$$

Then

$$x_{B_1} = \begin{bmatrix} 2 & 0 & 0 & 1 \end{bmatrix}^T$$

Chebyshev Basis:

$$B_2 = \left\{1, x, 2x^2 - 1, 4x^3 - 3x\right\}$$

Then

$$x_{B_2} = \begin{bmatrix} 2 & 3/4 & 0 & 1/4 \end{bmatrix}^T$$

It is often necessary to convert from one coordinate system to another.

- Given bases $\{v_i\}$ and $\{w_i\}$ and coordinates $x = \sum_i \alpha_i v_i$, find $\{\beta_i\}$ such that $\sum_i \alpha_i v_i = \sum_i \beta_i w_i$

Coordinate Transformations

Convert ($x_{B_1} \mapsto x_{B_2}$) Coordinates in basis $B_1 = \{v_j\}$ to coordinates in basis $B_2 = \{w_j\}$.

Let t_i be the coordinates of v_i in basis $\{w_j\}$ so

$$v_i = \sum_j t_{i,j} w_j.$$

If

$$x_{B_1} = \begin{bmatrix} \alpha_1 & \dots & \alpha_n \end{bmatrix}^T$$

Then

$$x = \sum_i \alpha_i v_i = \sum_i \alpha_i \sum_j t_{i,j} w_j = \sum_j (\sum_i \alpha_i t_{i,j}) w_j$$

So

$$x_{B_2} = \begin{bmatrix} t_{1,1} & \dots & t_{1,n} \\ \vdots & \ddots & \vdots \\ t_{n,1} & \dots & t_{n,n} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = T x_{B_1}$$

Coordinate Transformations

The coordinate transformation $T : x_{B_1} \mapsto x_{B_2}$ is a linear mapping. **Notable Examples of Coordinate Transforms**

- Laplace Transform
- Fourier Transform
- Z-transform

Any coordinate transformation is invertible since

- Surjective (e.g. $\mathbb{R}^n \rightarrow \mathbb{R}^n$)
- Injective ($Im(T) = \text{span}(\{w_i\}) = \mathbb{R}^n$)

Question: What about Operators?

- A linear operator on a vector space X defines a linear operator on the coefficient space.
- The operator A for the map $y = Ax$ defines a linear operator A_B on the space of coefficients for basis B.

$$y_B = A_B x_B$$

where x_B and y_B are the coefficients of x and y in basis B

Coordinate Transformations

The representation of the map A in another basis B_2 can be found easily

- Suppose $y_{B_1} = A_{B_1} x_{B_1}$ defines A_{B_1}.
- Suppose the coordinate transform from B_1 to B_2 is $x_{B_2} = T_{B_1 \to B_2} B_1$.
- Then

$$y_{B_2} = T_{B_1 \to B_2} y_{B_1} = T_{B_1 \to B_2} A_{B_1} x_{B_1} = T_{B_1 \to B_2} A_{B_1} T^{-1}_{B_1 \to B_2} x_{B_2}$$

To simplify

$$A_{B_2} = T A_{B_1} T^{-1}$$

This is called a similarity transformation

- e.g. Frequency Domain $\leftrightarrow$ Time Domain.
- We will return to this in the next chapter.

Normed Spaces

A norm is used to express a concept of distance.

- A concept of energy for signal spaces.

Definition 2.

A **norm** on a vector space, X is a mapping, $\|\cdot\| : X \to \mathbb{R}$ which satisfies

1. $\|x\| \geq 0$ for all $x \in X$. (Positivity)
2. $\|x\| = 0$ if and only if $x = 0$. (Non-Degeneracy)
3. $\|\lambda x\| = |\lambda| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$.
4. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$. (Triangle Inequality)

By Definition:

- Triangle Inequality means space satisfies the *Pythagorean Theorem*.

Note: The submultiplicative inequality is often NOT SATISFIED:

$$\|AB\| \not\leq \|A\| \|B\|$$

- For this we need multiplication
- A normed space with the submultiplicative inequality is called a **Normed Algebra**.

Normed Spaces

Definition 3.

A **Normed Space** is a vector space with an associated norm.

The same vector space may define several different normed spaces:
On $\mathbb{R}^n$:

- $\|x\|_1 = \sum_{i=1}^n |x_i|$ (Taxicab norm)
- $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ (Euclidean norm)
- $\|x\|_p = \sqrt[p]{\sum_{i=1}^n x_i^p}$
- $\|x\|_\infty = \max |x_i|$

On infinite sequences $g : \mathbb{N} \to \mathbb{R}$

- $\|f\|_{\updownarrow_1} = \sum_{i=1}^\infty |g_i|$
- $\|f\|_{\updownarrow_2} = \sqrt{\sum_{i=1}^\infty g_i^2}$
- $\|f\|_{\updownarrow_p} = \sqrt[p]{\sum_{i=1}^\infty g_i^p}$
- $\|f\|_{\updownarrow_\infty} = \max_{i=1,\ldots,\infty} |g_i|$

On functions $f : [0,1] \to \mathbb{R}$

- $\|f\|_{L_1} = \int_0^1 |f(s)| ds$
- $\|f\|_{L_2} = \sqrt{\int_0^1 f(s)^2 ds}$
- $\|f\|_{L_p} = \sqrt[p]{\int_0^1 f(s)^p ds}$
- $\|f\|_{L_\infty} = \sup_{s\in[0,1]} |f(s)|$

Normed Spaces

On a normed space, we define the following common subsets

- Closed unit ball/disk

$$\{x : \|x\| \le 1\}$$

- Open unit ball/disk

$$\{x : \|x\| < 1\}$$

- Unit sphere/circle

$$\{x : \|x\| = 1\}$$

For each norm, the unit ball is different.

- In $\|\cdot\|_\infty$, the unit ball is a cube!
- $\|\cdot\|_1$?

Note: Norms are often associated to a coordinate system.

- All our norms on $\mathbb{R}^n$ use Euclidean coordinates
- Define a polar norm?

Open and Closed Sets

Define the closed ball of radius r centered at x_0.

$$B(r, x_0) := \{x : \|x - x_0\| \leq r\}$$

Definition 4.

A subset $Q \subset X$ is **Open** if for any $x \in \mathbb{Q}$, there exists a closed ball, centered at x, which is contained in Q.
i.e. For any x_0, there exists some $r > 0$ such that

$$B(r, x_0) \subset Q$$

Open and Closed Sets

Example

Lemma 5.

The open unit ball, $B_o := \{x : \|x\| < 1\}$, is open.

Proof.

- For any $x_0 \in B_o$, $\|x_0\| < 1$.
- Let $\epsilon = 1 - \|x_0\| > 0$ and $r = \epsilon/2$. Then for any $y \in B(r, x_0)$, we have $\|y - x_0\| \le \epsilon/2$.
- Thus for any $y \in B(r, x_0)$

$$\begin{aligned}\|y\| &= \|x_0 + y - x_0\| \\ &\le \|x_0\| + \|y - x_0\| \\ &\le 1 - \epsilon + \epsilon/2 \\ &= 1 - \epsilon/2 < 1\end{aligned}$$

- Thus $B(r, x_0) \subset B_o$. Since x_0 is arbitrary, this proves that B_o is open.

$\square$

Open and Closed Sets

Closed Sets

Definition 6.

The **Complement** of a subset $Q \subset X$ in X is

$$Q^c = X/Q := \{x \in X \,:\, x \notin Q\}$$

Definition 7.

$Q \subset X$ is **closed** in X if X/Q is open.

Definition 8.

The **closure** of Q in X is the set of points in X which are infinitely close to Q.

$$\bar{Q} := \{x \in X \,:\, B(r,x) \cap Q \neq \emptyset \text{ for every } r > 0\}$$

The closure of a set is the smallest closed set containing the set.

Definition 9.

The **interior** of Q in X is

$$\text{int}Q := \{x : B(r,x) \subset Q \text{ for some } r > 0\}$$

Definition 10.

The **boundary** of Q in X is $\bar{Q}/\text{int}Q$

Definition 11.

A set is **bounded** if it is contained in some ball. There exists an $r > 0$ such that

$$Q \subset B(r,0)$$

Definition 12.

In $\mathbb{R}^n$, a set is **compact** if it is closed and bounded.

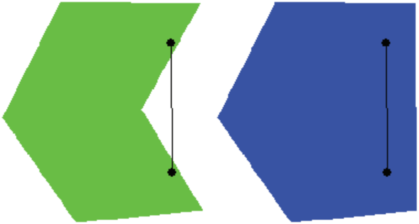
Convexity

Definition 13.

A set is **convex** if for any $x, y \in Q$,

$$\{\mu x + (1-\mu)y \,:\, \mu \in [0,1]\} \subset Q.$$

The line connecting any two points lies in the set.



Convex Cones

Definition 14.

A set is a **cone** if for any $x \in Q$,

$$\{\mu x \,:\, \mu \geq 0\} \subset Q.$$

A subspace is a cone but not all cones are subspaces.

- If the cone is also convex, it is a convex cone.
- Cones are convex if they are closed under addition.