Constructive Representation of Functions in N-Dimensional Sobolev Space

Declan S. Jagt, Matthew M. Peet

Arizona State University
Tempe, AZ USA

January 19, 2024

We Consider the Challenge of Representation of Differentiable Functions

We Expand a Function in 1D using the Fundamental Theorem of Calculus

Consider $\mathbf{u} \in W^{d}[a,b] := \{\mathbf{u} \mid \partial_x^k \mathbf{u} \in L_2[a,b],\ \forall 0 \le k \le d\}$.

$d = 1$: Then, by the Fundamental Theorem of Calculus (FTC)

$$\boxed{\mathbf{u}(x) = \mathbf{u}(a) + \int_a^x \partial_x \mathbf{u}(\theta) d\theta}$$

$d = 2$: Then $\partial_x \mathbf{u} \in W^1[a,b]$, so by FTC

$$\begin{aligned}
\mathbf{u}(x) &= \mathbf{u}(a) + \int_a^x \overbrace{\left[\partial_x \mathbf{u}(a) + \int_a^\eta \partial_x (\partial_x \mathbf{u})(\theta) d\theta\right]}^{\partial_x \mathbf{u}(\eta)} d\eta \\
&= \mathbf{u}(a) + \partial_x \mathbf{u}(a) \int_a^x d\eta + \int_a^x \int_a^\eta \partial_x^2 \mathbf{u}(\theta) d\theta d\eta \\
&= \mathbf{u}(a) + [x-a]\partial_x \mathbf{u}(a) + \int_a^x [x-\theta]\partial_x^2 \mathbf{u}(\theta) d\theta
\end{aligned}$$

Define boundary operator

$$(\mathfrak{b}^k \mathbf{u})(x) = \begin{cases} \mathbf{u}(a), & k < 0, \\ \mathbf{u}(x), & k = 0 \end{cases}$$

Then
$d = 1:$

$$\mathbf{u}(x) = (\mathfrak{b}^{-1}\mathbf{u}) + \int_a^x (\mathfrak{b}^0 \partial_x \mathbf{u})(\theta) d\theta$$

$d = 2:$

$$\mathbf{u}(x) = (\mathfrak{b}^{-2}\mathbf{u}) + [x-a](\mathfrak{b}^{-1}\partial_x \mathbf{u}) + \int_a^x [x-\theta$$

We can Express a 1D Function in terms of its Highest-order Derivative

Sobolev expansion of Functions in 1D

Suppose $\mathbf{u} \in W^d[a,b]$ for $d \in \mathbb{N}$ and $[a,b] \subseteq \mathbb{R}$. Then,

$$\mathbf{u}(x) = \sum_{k=0}^{d} (g_k^d \mathfrak{b}^{k-d} \partial_x^k \mathbf{u})(x), \qquad x \in [a,b],$$

Moreover, for any $\{\mathbf{v}^k \in L_2[\Omega^{(k-\delta_i)\mathsf{e}_i}] \mid 0 \le k \le \delta_i\}$, if

$$\mathbf{u}(s) = \sum_{k=0}^{\delta_i} (g_{i,k}^{\delta_i} \mathbf{v}^k)(s), \qquad s \in \Omega,$$

then, $\mathbf{v}^k = \mathfrak{b}_i^{k-\delta_i} \partial_{s_i}^k \mathbf{u}$ for all $0 \le k \le \delta_i$.

Example: 3rd-Order Differentiable Function

For $\mathbf{u} \in W^3[a,b]$,

$$(\mathfrak{b}^{-3}\partial_x^0 \mathbf{u}) = \mathbf{u}(a), \qquad (\mathfrak{b}^{-2}\partial_x^1 \mathbf{u}) = \partial_x \mathbf{u}(a),$$

Define boundary operator

$$(\mathfrak{b}^k \mathbf{u})(x) = \begin{cases} \mathbf{u}(a), & k < 0, \\ \mathbf{u}(x), & k = 0. \end{cases}$$

Define integral operator

$$(g_k^d \mathbf{u})(x) = \begin{cases} \mathbf{u}(x), & 0 = k = d, \\ \mathbf{p}_k(x-a)\mathbf{u}(x), & 0 \le k < d, \\ \int_a^x \mathbf{p}_{k-1}(x-\theta)\mathbf{u}(\theta)d\theta & 0 < k = d. \end{cases}$$

where $\mathbf{p}_k(z) = \frac{z^k}{k!}$.