Analysis of Nonlinear Time Delay Systems Using the Sum of Squares Decomposition

Antonis Papachristodoulou, Matthew Peet and Sanjay Lall

Abstract

In this paper we present an algorithmic methodology for analyzing the stability of equilibria of systems described by nonlinear Delay Differential Equations (DDEs) with multiple incommensurate delays. Both delay-independent and delay-dependent stability tests are proposed, based on the construction of appropriate Lyapunov-Krasovskii functionals. The methodology is based on the sum of squares decomposition of multivariate polynomials and the algorithmic construction is achieved through the use of semidefinite programming. Moreover, robust stability analysis under parametric uncertainty is performed in a unified way, by considering parameterized Lyapunov-Krasovskii functionals. We illustrate the methodology using examples from population dynamics and the Internet.

Index Terms

Nonlinear time-delay systems, Lyapunov's direct method, stability analysis, robustness analysis, sum of squares decomposition, semidefinite programming.

I. Introduction

Delay differential equations are useful for modeling systems that involve transport and propagation of data, such as communication systems, but also for modeling maturation and growth in population dynamics [1], [2]. It is known that the presence of delays may induce undesirable effects such as instabilities and performance degradation; ignoring them in the modeling process may result in inadequate designs and incorrect analysis conclusions. Inevitably the stability, robust stability and control of such systems have received a lot of

A. Papachristodoulou is with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA. Email: antonis@cds.caltech.edu. Matthew Peet and Sanjay Lall are with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA. Email: {mmpeet, lall}@stanford.edu.

interest in the past few years. Recently this interest has intensified, as they provide the simplest adequate modeling framework for network congestion control for the Internet [3].

Delay Differential Equations fall in the category of Functional Differential Equations (FDEs), which differ from Ordinary Differential Equations (ODEs) because of the infinite dimensional character of the state. The evolution of the system in an infinite dimensional space makes the problem of assessing the stability properties of steady states more difficult to resolve, as more complicated analysis tools are required. The stability question can still be answered through the construction of Lyapunov-type certificates, i.e., functions of state that satisfy certain positivity conditions, as it is done in the case of ODE systems. However, while for the case of ODEs these certificates are functions, in the case of FDEs they are *functionals* owing to the fact that the state belongs in a function space itself.

In the case of linear DDEs, the form of these functionals that is necessary and sufficient for stability is known [4], [5], but they are inherently difficult to construct algorithmically. Under restrictions on their structure, convex optimization has been used to construct them algorithmically, through the solution of a set of Linear Matrix Inequalities (LMIs). Inevitably these functionals yielded conservative results on the delay interval guaranteeing stability, see for example the attempts in [6] and [7]. The reason for the difficulty in constructing the 'complete' (necessary and sufficient) structures is that it is equivalent to parameterizing the set of positive operators on an infinite-dimensional space. One approach is to use Lyapunov functionals with piecewise-linear kernels, which is known as the 'Discretized Lyapunov Functional Approach'; these functionals are constructed by solving a set of LMIs whose size depends on the discretization level [8], and as the discretization level is decreased, delay values closer to the boundary of stability can be tested. More recently a new approach was developed in [9] for constructing the complete Lyapunov-Krasovskii functional for linear DDEs, which is based on an explicit parameterization of positive operators and uses the Sum of Squares decomposition and semidefinite programming as the computational tool.

As far as nonlinear time delay systems are concerned, the only methodologies currently available for stability analysis center on the ad-hoc construction of simple Lyapunov certificates, for systems of low dimension [1]. Note that even for systems described by ODEs, constructing Lyapunov functions has been traditionally ad-hoc, but recently a computational methodology has been proposed in [10].

In this paper we present a methodology for constructing Lyapunov-Krasovskii functionals for nonlinear time delay systems with multiple incommensurate delays, using the Sum of

Squares decomposition of multivariate polynomials and semidefinite programming. We construct appropriate Lyapunov-Krasovskii functionals both for delay-independent and delay-dependent stability. This stability classification is based on whether stability is retained independent of the size of the delay or whether it is lost as the delay size, seen as a parameter, is allowed to vary. The Lyapunov functionals we construct have polynomial kernels, and in the case in which the DDE we consider is linear, they reduce to the complete (i.e., necessary and sufficient) Lyapunov functional structures.

The paper is organized as follows. First, in section II we present some background information on systems described by Functional Differential Equations and the Lyapunov theorem that we will be using in the sequel. In section III we present a brief review on the sum of squares decomposition and SOSTOOLS. In section IV we present the methodology for robust stability analysis for nonlinear delayed systems, both in the delay-independent and delay-dependent sense, as well as for single and for multiple, incommensurate delays. In section V we illustrate our results with examples from population dynamics and network congestion control for the Internet. We conclude the paper in Section VI.

II. Background and past results

In this section we will present some facts about autonomous Functional Differential Equations and introduce the notion of Lyapunov stability. A more detailed account can be found in [11].

The notation we use is standard. Let $\mathbb{R}^n$ denote the n-dimensional real vector space with norm the 2-norm denoted $|\cdot|$. Denote $C([a,b],\mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. Consider $[a,b]=[-\tau,0]$, and let $C=C([-\tau,0],\mathbb{R}^n)$; the norm on C is defined as $\|\phi\| = \sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. We denote by C^γ the set defined by $C^\gamma=\{\phi\in C | \|\phi\|<\gamma\}$ with $\gamma>0$. Suppose $\sigma\in\mathbb{R}$, $\rho\geq 0$ and $x\in C([\sigma-\tau,\sigma+\rho],\mathbb{R}^n)$; then for any $t\in[\sigma,\sigma+\rho]$, define $x_t\in C$ by $x_t(\theta)=x(t+\theta), \theta\in[-\tau,0]$.

Assume Ω is a subset of C, $f:\Omega\to\mathbb{R}^n$ is a given function, and '$\dot{}$' represents the right-hand derivative. Then we call

$$\dot{x}(t) = f(x_t) \tag{1}$$

a Retarded Functional Differential Equation (RFDE) on Ω. Given $\phi\in C$ and $\rho>0$, a function $x(\phi)\in\mathcal{C}([-\tau,\rho),\mathbb{R}^n)$ is said to be a solution to Equation (1) on $[-\tau,\rho)\times\Omega$ with initial condition ϕ, if $x_t \in \Omega$ for $t \in [0, \rho]$, $x(\phi)(t)$ satisfies (1) for $t \in [0, \rho)$ and $x_0(\phi) = \phi$. Such a solution exists and is unique if f continuous everywhere is Lipschitz continuous on every compact set in Ω; see [11] for more details. Such continuity conditions ensure that the solution $x(\phi)$ is continuous for t in some interval and $\phi \in \Omega$.

A state $x^* \in \mathbb{R}^n$ is called a steady-state of (1) if for some $t \geq 0$

$$x(\phi)(t) = x^* \Rightarrow x(\phi)(\sigma) = x^*, \quad \forall \sigma \geq t.$$

Without loss of generality we assume that 0 is a steady-state for the system; for if it is at x^*, one can employ a transformation $z(t) = x(t) - x_0$ so that the transformed system

$$\dot{z}(t) = f(z_t + x_0)$$

has the steady state z^* at 0.

We now recall the following stability definitions.

Definition 1: The trivial solution of (1) is called

1) ***stable*** if for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $|x(\phi)(t)| \leq \epsilon$ for any initial condition $\phi \in C^\delta$ and all $t \geq 0$; otherwise it is termed ***unstable***.
2) ***asymptotically stable*** if it is stable and there is a $\gamma > 0$ such that $\lim_{t \to \infty} x(\phi)(t) = 0$ for any initial condition $\phi \in C^\gamma$.

Just as in the case of nonlinear systems described by Ordinary Differential Equations (ODEs), a Lyapunov argument can be formulated for the stability analysis of RFDEs.

Consider a continuous functional $V : C \to \mathbb{R}$ and define:

$$\dot{V}(\phi) = \lim \sup_{h \to 0^+} \frac{1}{h}[V(x_h(\phi)) - V(\phi)]$$

This is the upper Lie derivative of V along a solution of (1). For autonomous systems we have the following theorem (Lyapunov-Krasovskii) [11]:

Theorem 2: Assume $f : \Omega \to \mathbb{R}^n$ is completely continuous[1] and the solutions of (1) depend continuously on initial data. Suppose that $V : \Omega \to \mathbb{R}$ is continuous and there exist nonnegative continuous functions $a(s)$ and $b(s)$ satisfying $a(0) = b(0) = 0$, $a(s)$ strictly increasing, such that:

$$V(\phi) \geq a(|\phi(0)|) \text{ on } \Omega$$

$$\dot{V}(\phi) \leq b(|\phi(0)|) \text{ on } \Omega$$

[1]Recall that if Ω is a subset of a Banach space C and $A : \Omega \to C$, then A is *completely continuous* if A is continuous and for any bounded set $B \subseteq \Omega$, the closure of AB is compact.

Then the solution $x = 0$ of (1) is stable, and every solution is bounded. If in addition, $b(s) > 0$ for $s > 0$, the $x = 0$ is asymptotically stable.

We note that apart from the Lyapunov-Krasovskii theorem for stability analysis, there is the Lyapunov-Razumikhin theorem which uses functions as the certificates for stability instead of functionals — see [11] for more details. In this paper, however, we are interested in *algorithmic* methodologies for constructing Lyapunov-type certificates. The *convexity* in the Lyapunov conditions is important in that respect – the conditions in Theorem 2 are convex in $V(x)$, whereas those in the corresponding Lyapunov-Razumikhhin theorem are not, as the derivative condition involves the Lyapunov function itself. Under convexification assumptions, one can construct Lyapunov-Razumikhin functions using convex optimization, but the criteria are at least as conservative as the relevant Lyapunov-Krasovskii ones [12]. Therefore, we will not investigate the construction of Lyapunov-Razumikhin functions.

The stability of steady-states of time-delay systems is classified as *delay-dependent* or *delay-independent*, based on the persistence of stability as the delay size, seen as a parameter, is increased. We say that a system is *delay-independent* stable, if the stability property is retained for all positive values of the delays in the system, i.e., the system is robust with respect to the delay size. On the other hand, we say that a system is *delay-dependent* stable if the stability is preserved for some values of delays and is lost for some others. The property of delay-independent stability is therefore stronger than that of delay-dependent stability. As discussed in the Introduction, the existence of complete quadratic Lyapunov-Krasovskii functionals necessary and sufficient for strong delay-independent [5] and delay-dependent stability [12] of linear time delay systems is known, and so is their structure, but there is an inherent difficulty in constructing them, as they cannot be fully parameterized using a finite dimensional space.

In this paper we employ a recently developed methodology that aids in the construction of Lyapunov-Krasovksii functionals for nonlinear DDEs with multiple, noncommensurate delays.

III. The Proposed Methodology: Sum of Squares

In this section we introduce the proposed algorithmic methodology that we will be using. It uses notion from real algebraic geometry and employs semidefinite programming as the computational tool.

Let $\mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_m]$ denote the ring of polynomials in x with real coefficients. An element $p \in \mathbb{R}[x]$ is said to be a *Sum of Squares*, if it admits the following decomposition:

Definition 3: $p \in \mathbb{R}[x]$ is a *Sum of Squares* (SOS) if there exist polynomials $f_i(x)$, $i = 1, \ldots, M$ such that

$$p(x) = \sum_{i=1}^{M} f_i^2(x).$$

The set of polynomials that admit a SOS decomposition is denoted by Σ; similarly, the subset of Σ for which the degree of the polynomials is $2d$ is denoted by Σ_d. An equivalent characterization of SOS polynomials is given in the following proposition, the proof of which can be found in [13].

Proposition 4: A polynomial $p \in \Sigma_d$ is SOS if and only if there exists a positive semidefinite matrix Q and a vector $Z(x)$ containing monomials in x of degree $\leq d$ so that

$$p(x) = Z(x)^T Q Z(x).$$

Searching for such a Q can be performed using semidefinite programming. To see this, note first that monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T Q Z(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of Q. This implies that $p(x)$ being SOS is equivalent to $Q \geq 0$ under some affine relations in the elements of Q; so it can be cast as a semidefinite program (SDP), i.e., a set of Linear Matrix Inequalities (LMIs) [13]. If the monomials in the polynomial $p(x)$ have *unknown* coefficients then the search for feasible values of those coefficients such that $p(x)$ is nonnegative is also an SDP, a fact that is important for the construction of Lyapunov functions and other S-procedure type multipliers.

Note that if a polynomial $p(x)$ is a sum of squares, then it is globally nonnegative. The converse is not always true: not all positive semi-definite polynomials can be written as SOS — in fact, testing global non-negativity of a polynomial $p(x)$ is known to be NP-hard when the degree of $p(x)$ is greater than 4 [14], whereas checking whether p can be written as a SOS is computationally tractable, as it can be formulated as an SDP which has a worst-case polynomial-time complexity. The construction of the SDP related to the SOS conditions can be performed efficiently using SOSTOOLS [15], a software that formulates general sum of squares programs as SDPs and calls semidefinite programming solvers to solve them.

The SOS technique has been used to construct Lyapunov functions for nonlinear systems described by ODEs, by relaxing the non-negativity conditions to SOS conditions [13], [10].

The only complication is that the Lyapunov function $V(x)$ be positive *definite*, which can be achieved as follows.

Proposition 5: Given a polynomial $V(x)$ of degree $2d$, let $\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}$ such that:

$$\sum_{j=1}^{m} \epsilon_{ij} \geq \gamma \quad \forall\ i = 1, \dots, n,$$

with γ a positive number, and $\epsilon_{ij} \geq 0$ for all i and j. Then the condition

$$V(x) - \varphi(x) \text{ is a sum of squares} \tag{2}$$

guarantees the positive definiteness of $V(x)$. Moreover, $V(x)$ is radially unbounded.

Proof: The function $\varphi(x)$ as defined above is positive definite if ϵ_{ij}'s satisfy the conditions mentioned in the proposition. Moreover it is radially unbounded by construction, as it is the positive sum of simple monomials (i.e. in only one variable) squared. Then $V(x) - \varphi(x)$ being SOS implies that $V(x) \geq \varphi(x)$, and therefore $V(x)$ is positive definite. Since φ is radially unbounded, so is V. ■

In this paper, we concentrate on systems with K incommensurate delays, and we assume for convenience that $f \in \mathbb{R}[x(t), x(t-\tau_1), \dots, x(t-\tau_K)]$ where $x \in \mathbb{R}^n$ and $\tau_i, i = 1, \dots, K$ are the discrete, incommensurate delays in the system. Note that through a recasting procedure, one can render a non-polynomial vector field polynomial [16], and so analysis of nonlinear time-delay systems with non-polynomial vector fields is also possible.

With this structure for f, it is worth mentioning that establishing the stability of the steady-state of a system of the form (1) is a difficult task even if f is linear; for example, establishing whether the system is stable independent of the delay or stable for all $\tau_i \in [0, \overline{\tau}_i)$ with $\overline{\tau}_i$, $i = 1, \dots, K$, given, is shown to be NP-Hard [17]. However, sufficient stability tests that are algorithmically verifiable can still be formulated, which may be able to give an answer to the stability question. It is customary for such problems to provide a nested family of such tests, each of which is at least as powerful as the previous one, but comes with an increased computational requirement.

The proposed methodology uses, along with the sum of squares decomposition, another tool: a central theorem in Real Algebraic Geometry, *Positivstellensatz*. This theorem can be used, e.g., to strengthen the conditions for conditional satisfiability and provide a more general S-procedure to test for it. In particular, suppose one is interested in ensuring that

$p \in \mathbb{R}[x]$ is positive on the set

$$D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, N\} \tag{3}$$

Then one can search for Lagrange-type multipliers $\lambda_i \in \Sigma_k$ so that:

$$p(x) + \sum_{i=1}^{N} \lambda_i(x) g_i(x) \in \Sigma$$

For fixed k, searching for $\lambda_i(x)$ so that the above expression is SOS is, as argued earlier, a semidefinite programme. Note that if p and q_i are quadratic forms and λ_i are constants, the above test is indeed the S-procedure, which can fail to give a positive answer even if the condition is true. However, it has been shown in [18] that if D is compact and another mild condition holds on the $g_i(x)$, then there is a k for which the above test will succeed – it is indeed a necessary and sufficient condition. Note that the test may fail for some k but give a positive answer for $k+1$; of course the computational requirements increase as k increases. Other tests can also be formulated, which may have different properties. See e.g. [19].

We now show how to use these ideas to provide an algorithmic methodology for assessing the stability of nonlinear time-delay systems.

IV. Nonlinear Time Delay Systems

In this section, we will propose tests for delay-independent and delay-dependent stability of steady-states of nonlinear time-delay systems with or without parametric uncertainty. In the first part, we consider systems with a single delay, and then we consider the case of systems with multiple incommensurate time delays. These system descriptions can be treated using the sum of squares technique in a unified way. Indeed this is the only proposed methodology thus far to handle the analysis of nonlinear time-delay systems in an algorithmic, unified fashion.

A. Delay-independent stability

Recall that a steady-state of a time-delay system is delay-independent stable if it is stable for all finite values of the delay. Delay-independent stability conditions may be conservative, in the sense that the system may still be stable in a delay-dependent fashion even if these conditions are violated. However this property is used in controller synthesis when the size of the delay is uncertain.

In general, finding the proper structure for a Lyapunov functional in the case of nonlinear systems involves some guessing. For this reason, the intuition we gain from the structures used in the linear case is invaluable. For delay-dependent stability and the case of what is called strong delay-independent stability [5] for linear systems, the class of such Lyapunov functionals has been completely characterized. The following example illustrates the case of delay-independent stability.

Example 6: (*Delay-independent stability of Linear Time Delay Systems*) For system

$$\dot{x} = A_0 x(t) + A_1 x(t-\tau) \tag{4}$$

a Lyapunov-Krasovskii candidate that would yield a delay-independent condition is

$$V(x_t) = x(t)^T P x(t) + \int_{-\tau}^{0} x(t+\theta)^T S x(t+\theta) d\theta \tag{5}$$

Sufficient conditions on $V(x_t)$ to be positive definite are $P > 0$, $S \geq 0$. Evaluating $\dot{V}(x_t)$ we get:

$$\begin{aligned} \dot{V}(x_t) &= x^T(t)\left(A_0^T P + P A_0 + S\right)x(t) + x^T(t-\tau)A_1^T P x(t) \\ &\quad + x^T(t) P A_1 x(t-\tau) - x^T(t-\tau) S x(t-\tau) \end{aligned}$$

Therefore if we impose

$$\begin{bmatrix} A_0^T P + P A_0 + S & P A_1 \\ A_1^T P & -S \end{bmatrix} + \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0,$$

for $\epsilon > 0$, then $\dot{V}(x_t) < 0$ and the conditions for stability (see Theorem 2) can be written as Linear Matrix Inequalities (LMIs) with P and S as the unknowns; in other words, if we can find P and S such that

$$P > 0, \quad S \geq 0, \quad \begin{bmatrix} A_0^T P + P A_0 + S & P A_1 \\ A_1^T P & -S \end{bmatrix} + \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \leq 0, \quad \epsilon > 0,$$

then (4) is stable independent of the delay (as the delay size does not appear in the conditions).

The (strong) delay-independent stability of (4) is equivalent to

$$\det(sI_n - A_0 - zA_1) \neq 0$$

for all $s = j\omega$, $\tau \geq 0$ and z in the closed unit disk. This is a stronger notion of delay-independent stability, for which only $\det(sI_n - A_0 - e^{-s\tau}A_1) \neq 0$ for all $s = j\omega$ and $\tau \geq 0$ is required; it is however robust with respect to perturbations in A_0 and A_1.

As illustrated in [5], the Lyapunov functional used above may not suffice to prove stability for a general linear delay-independent stable system, as the proposed structure is not sufficiently 'rich'. However, the following Lyapunov structure has been proven to be 'complete' for *strong* delay-independent stability in the case of linear time delay systems. For a given $x \in \mathcal{C}([-\tau, \rho], \mathbb{R}^n)$ and integer $L \geq 0$, define $z_L \in \mathcal{C}([(L-1)\tau, \rho], \mathbb{R}^{n\times(L+1)})$ as follows.

$$z_L(t) := [x(t), x(t-\tau), \ldots, x(t-L\tau)]. \tag{6}$$

Then the complete structure for strong delay-independent stability in the single delay case is

$$W_L(t) = V_0(z_L(t)) + \int_{-\tau}^{0} V_1(z_L(t+\theta))d\theta \tag{7}$$

where V_0 and V_1 are *quadratic* polynomials in their arguments. This is essentially a Lyapunov-Krasovskii functional for an extension of the system, replicated backwards in time $L-1$ times.

Let us now turn to nonlinear time-delay systems, and consider for simplicity a system with one discrete delay given by:

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{8}$$

where we assume that $f \in \mathbb{R}[x(t), x(t-\tau)]$ and that without any loss of generality 0 is a steady-state of the system. Later on we will allow for multiple, incommensurate delays. We have the following conditions for delay-independent stability:

Proposition 7: Consider the system described by Equation (8). For a positive integer L, if there exist functions $V_0(z_L(t))$ and $V_1(z_L(t+\theta))$, a positive definite, radially unbounded function $\varphi(z_L(t))$ and a non-negative function $\psi(z_L(t))$ such that:

1) $V_0(z_L(t)) - \varphi(z_L(t)) \geq 0$,
2) $V_1(z_L(t+\theta)) \geq 0$,
3) $\sum_{m=0}^{L} \frac{\partial V_0(z_L(t))}{\partial x(t-m\tau)} f(x(t-m\tau), x(t-(m+1)\tau)) + V_1(z_L(t)) - V_1(z_L(t-\tau)) - \psi(z_L(t)) \leq 0$,

then the 0 steady-state is globally delay-independent stable. If moreover ψ is positive definite, then the 0 steady-state is globally asymptotically delay-independent stable.

Proof: Consider the functional

$$W_L(t) \;=\; V_0(z_L(t)) + \int_{-\tau}^{0} V_1(z_L(t+\theta))d\theta \tag{9}$$

The first two constraints impose that

$$W_L(t) = V_0(z_L(t)) + \int_{-\tau}^{0} V_1(z_L(t+\theta))d\theta \geq \varphi(z_L(t)) > 0$$

so the first Lyapunov-Krasovskii condition is satisfied and moreover W_L is radially unbounded. The derivative of W_L along the trajectories of system (8) is:

$$\dot{W}_L(t) = \sum_{m=0}^{L} \frac{\partial V_0(z_L(t))}{\partial x(t-m\tau)} f(x(t-m\tau), x(t-(m+1)\tau)) + V_1(z_L(t)) - V_1(z_L(t-\tau))$$

Under the third condition the above derivative is non-positive. Therefore if all three conditions are satisfied the steady-state of the system given by (8) is globally stable; since the delay size does not appear explicitly in the above conditions, then the zero steady-state is globally stable independent of delay. Moreover if $\psi(z_L(t)) > 0$, then the zero steady-state is globally asymptotically stable independent of delay. ■

The functional W_L is the functional given by Equation (7), but here we used it to analyze stability of nonlinear systems. In particular, if the system (8) is linear, we recover the conditions given in [5].

The conditions in the above proposition can be algorithmically tested by considering the construction of bounded-degree *polynomials* (i.e., of any order instead of just *quadratic forms*) V_i etc, and formulating appropriate sum of squares conditions. In particular, in this case the three conditions become polynomial non-negativity conditions that can be reduced to Sum of Squares conditions which can then be tested using SOSTOOLS. The function φ and possibly ψ can be constructed using Proposition 5. Here is a simple example of how this is done.

Example 8: Consider the system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t-\tau), \quad \dot{x}_2(t) = -x_2(t)$$

This system is delay-independent stable, and we prove this by constructing a Lyapunov functional $V = W_0$ with V_0 and V_1 polynomials of bounded degree. When V_0 and V_1 are second order polynomials, no certificate is found. However, when their order is increased, a certificate of stability is obtained. In fact the two conditions become

$$\begin{aligned} V(x_t) &= x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 + \int_{-\tau}^{0} x_2^4(t+\theta)d\theta. \\ -\dot{V}(x_t) &= (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) + \\ &\quad + x_2^2(t)x_2^2(t-\tau) + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2 + \frac{14}{16}x_1(t)^2. \end{aligned}$$

To get this certificate, only a handful of SOSTOOLS commands are required.

In the next two subsections, we will concentrate on local stability and robust stability analysis for delay-independent stability. In section IV-C we will also investigate delay-independent stability under multiple delays.

1) Local Stability: Nonlinear systems may have more than one equilibria, or the stability properties of a steady-state may not be global. In order to obtain a local result, we have to use the region Ω in Theorem 2. Here we use the following form of local stability region.

$$\Omega = \left\{ x_t \in C : \|x_t\| = \sup_{-\tau \le \theta \le 0} |x(t+\theta)|_\infty \le \gamma \right\}.$$

Here $|\cdot|_\infty := \max_i |x_i|$ is the ∞-norm on $\mathbb{R}^n$. This region can be represented using the following inequalities.

$$\begin{aligned} h_{i0} &:= (x_i(t)-\gamma)(x_i(t)+\gamma) \le 0, \quad i = 1,\ldots,n \\ h_{i1} &:= (x_i(t-\tau)-\gamma)(x_i(t-\tau)+\gamma) \le 0, \quad i = 1,\ldots,n \end{aligned}$$

In some cases, we may need to introduce replicas of the system backwards in time so as to prove stability, as argued in the previous section; in this case we require that $|x(t+\theta)| \le \gamma, \ \forall\ \theta \in [-L\tau, 0]$ where $L > 1$ is an integer; this in turn produces the following constraints:

$$h_{ij} \ := \ (x_i(t-j\tau)-\gamma)(x_i(t-j\tau)+\gamma) \le 0, \quad i = 1,\ldots,n, \quad j = 0,\ldots,L+1$$

Having captured the set Ω using the above inequality constraints, we can formulate conditions for stability that impose validity of the corresponding Lyapunov conditions when $x_t \in \Omega$, i.e., when $|x(t+\theta)| \le \gamma, \forall \theta$. In particular, we have the following result:

Proposition 9: Let 0 be a steady-state of system (8) and given γ, consider the state-space constraints of the form h_{ij} shown above for $i = 1\ldots,n$, $j = 0,\ldots,L+1$. For an integer $L > 1$, let there exist functions $V_0(z_L(t))$ and $V_1(z_L(t+\theta)$, a positive definite function $\varphi(z_L(t))$, a non-negative function $\psi(z_L(t))$ and non-negative functions $p_{ij}(z_L(t))$ for $i = 1,\ldots,n$, $j = 0,\ldots,L$ and $q_{ij}(z_{L+1}(t))$ for $i = 1,\ldots,n$, $j = 0,\ldots,L+1$ such that:

1) $V_0(z_L(t)) - \varphi(z_L(t)) + \sum_{j=0}^{L}\sum_{i=1}^{n} p_{ij}(z_L(t))h_{ij}(z_{L+1}(t)) \ge 0,$
2) $V_1(z_L(t+\theta)) \ge 0,$
3) $\sum_{m=0}^{L} \frac{\partial V_0(z_L(t))}{\partial x(t-m\tau)} f(x(t-m\tau), x(t-(m+1)\tau)) + V_1(z_L(t)) - V_1(z_L(t-\tau)) - \psi(z_L(t)) + \sum_{j=0}^{L+1}\sum_{i=1}^{n} q_{ij}(Z_{L+1}(t))h_{ij}(Z_{L+1}(t)) \ge 0.$

Then 0 is (locally) delay-independent stable. If moreover $\psi(z_L(t)) > 0$, then 0 is (locally) delay-independent asymptotically stable.

Proof: Consider the functional

$$W_L(t) = V_0(z_L(t)) + \int_{-\tau}^{0} V_1(z_L(t+\theta))d\theta. \tag{10}$$

While the elements of $z_L(t)$ satisfy $h_{ij}(z_L(t)) \leq 0$ and $p_{ij}(z_L(t)) \geq 0$ for $i = 1, \ldots, n$ and $j = 0, \ldots, L$ we have:

$$W_L(t) = V_0(z_L(t)) + \int_{-\tau}^{0} V_1(z_L(t+\theta))d\theta \geq \varphi(z_L(t)) - \sum_{j=0}^{L} \sum_{i=1}^{n} p_{ij}(z_L(t))h_{ij}(z_L(t)) > 0,$$

and so the first Lyapunov condition is satisfied, i.e. $V > 0$ on Ω. The same is true for the derivative condition, given constraint (3) above, and so the zero steady-state of system (8) is locally delay-independent stable. If $\psi > 0$, then

$$-\frac{dV}{dt} > 0 \text{ on } \Omega$$

and so asymptotic stability is concluded. ■

The conditions in the above proposition can be tested algorithmically if we assume a polynomial structure of $V_0(z_L(x(t)))$, $V_1(z_L(x(t+\theta)))$, $p_{ij}(z_L(x(t)))$ for $i = 1, \ldots, n$, $j = 0, \ldots, L$ and $q_{ij}(z_{L+1}(x(t)))$ for $i = 1, \ldots, n$, $j = 0, \ldots, L+1$; construct $\varphi(z_L(x(t)))$ and possibly $\psi(z_L(x(t)))$ using Proposition (5); and replace non-negative non-negativity conditions by the existence of a sum of squares decomposition for them. SOSTOOLS can then be used to construct these polynomial functions algorithmically.

At this point we should note that having obtained a Lyapunov function that is valid locally and proves asymptotic stability, the domain of attraction of the steady-state can be estimated as the maximal level set of V that is contained in Ω. This can also be formulated as a SOS program, but it is beyond the scope of this paper. We suffice to say that the system is locally stable for some sufficiently small region about the origin.

2) Robust Stability: Another important issue is robust stability under parametric uncertainty, which can be treated in a unified way as we will see in the sequel. Consider a time-delay system of the form (1) with an uncertain parameter p:

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{x}(t-\tau), p), \tag{11}$$

where $p \in P$, where P is a semi-algebraic set defined by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \leq 0,\ i = 1, \ldots, N\} \qquad \text{for } q_i \in \mathbb{R}[p], \tag{12}$$

Define a new variable $x(t) := \tilde{x}(t) - \tilde{x}^*$, where $\tilde{x}^*$ is a steady-state of (11), which satisfies $f(\tilde{x}^*, \tilde{x}^*, p) = 0$, and may change as the parameters $p \in P$ vary. Then we have:

$$\dot{x}(t) = f(x(t) + \tilde{x}^*, x(t-\tau) + \tilde{x}^*, p) = \tilde{f}(x(t), x(t-\tau), p) \tag{13}$$

$$0 = f(\tilde{x}^*, \tilde{x}^*, p) \tag{14}$$

This system has a steady-state at the origin. We assume for simplicity that there is only one steady-state whose stability can be tested by constructing a *Parameter Dependent* Lyapunov functional – see the remark at the end of this section for systems with multiple equilibria. The robust stability properties of the above system can be tested using the following proposition:

Proposition 10: Consider the system given by (13), where $p \in P$ as defined by (12). For an integer $L \geq 0$, suppose that there exist functions $V_0(z_L(t), p)$ and $V_1(z_L(t+\theta), p)$, a positive definite radially unbounded function $\varphi(z_L(t))$ and a non-negative function $\psi(z_L(t))$ such that the following conditions hold for $p \in P$:

1) $V_0(z_L(t), p) - \varphi(z_L(t)) \geq 0$,
2) $V_1(z_L(t+\theta), p) \geq 0$,
3) $\sum_{m=0}^{L} \frac{\partial V_0(z_L(t))}{\partial x(t-m\tau)} \tilde{f}(x(t-m\tau), x(t-(m+1)\tau), p) + V_1(z_L(t), p) - V_1(z_L(t-\tau), p) + \psi(z_L(t)) \leq 0$, when (14) is satisfied.

Then the steady-state 0 of the system given by (13–14) is robustly globally delay-independent stable for all $p \in P$. Moreover, if $\psi(x(t)) > 0$, 0 is delay-independent robustly globally asymptotically stable for all $p \in P$.
The proof is based on functional

$$W_L(t) = W_0(z_L(t), p) + \int_{-\tau}^{0} W_1(z_L(t+\theta), p) d\theta \tag{15}$$

which is modified from (10); it is omitted for brevity. The constraints defining the set P can be adjoined in the three conditions above using non-negative multipliers. Multipliers can also be used to adjoin condition (14) to the third constraint. If we assume that all functions involved above are polynomial, then these multipliers can be chosen to be sum of squares and polynomial respectively, allowing their construction to be done using sum of squares programming and SOSTOOLS.

Here we illustrate the construction of the sum of squares program for $L = 0$. This program ensures robust asymptotic stability independent of delays.

Find polynomials $V_0(x(t),p)$ and $V_1(x(t+\theta),p)$

and $\varphi(x(t))$ and $\psi(x(t))$ constructed using Proposition 5

and sum of squares $\sigma_{1i}(x(t),p), \sigma_{2i}(x(t+\theta),p)$ and $\sigma_{3i}(x(t),x(t-\tau),p)$

and a polynomial $r(x(t),x(t-\tau),p)$

Such that:

$$V_0(x(t),p) - \varphi(x(t)) + \sum_i \sigma_{1i}(x(t),p)q_i(p) \text{ is a SOS} \tag{16}$$

$$V_1(x(t+\theta),p) + \sum_i \sigma_{2i}(x(t+\theta),p)q_i(p) \text{ is a SOS} \tag{17}$$

$$\begin{aligned} -\frac{dV_0(x(t),p)}{dx(t)}\tilde{f}(x(t),x(t-\tau),p) - V_1(x(t),p) + V_1(x(t-\tau),p) + \psi(x(t)) \\ + \sum_i \sigma_{3i}(x(t),x(t-\tau),p)q_i(p) + r(x(t),x(t-\tau),p)f(\tilde{x}^*,p) \text{ is a SOS} \end{aligned} \tag{18}$$

Remark 11: Sometimes there are more than one steady-states in (11) that move as p is allowed to vary in P and the result we seek in that case is a local result. In this case, the parameter set P should be extended to include the 'motion' of the steady-state $\tilde{x}^*$ and the region Ω has to be sufficiently small so that no other equilibria cross into Ω as the parameters change within P. An example of this case can be found in Section V.

B. *Delay-dependent stability*

When the stability properties of the steady-state change as the delay size, seen as a static parameter, changes, the stability is termed *delay-dependent*. In this case, a different type of Lyapunov functional has to be used to allow for the delay size to appear explicitly in the SOS conditions.

Recall that for a linear time-delay system of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i)$$

where A_i are fixed matrices, the complete Lyapunov structure for delay-dependent stability is:

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + 2x^T(t)\int_{-\tau}^{0} Q(\theta)x(t+\theta)d\theta \\ &\quad + \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+\theta)R(\theta,\xi)x(t+\xi)d\xi d\theta + \int_{-\tau}^{0} x^T(t+\theta)S(\theta)x(t+\theta)d\theta \end{aligned} \tag{19}$$

where $P \in \mathbb{S}^n$ (the set of real symmetric $n \times n$ matrices), $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$, $S : \mathbb{R} \to \mathbb{S}^n$ and $R : \mathbb{R}^2 \to \mathbb{R}^{n\times n}$ satisfies $R(\theta,\eta) = R^T(\eta,\theta)$. The matrices Q, S and R are continuous except possibly at points $\eta, \theta = \tau_i$ for $i = 1, \ldots, K-1$. Here $\tau = \max_i \tau_i$. The conditions for positivity of V and non-negativity of the derivative condition are infinite dimensional and difficult to test. One approach is to approximate them using piecewise linear functions [12]. Recently, other techniques have been proposed based on sum of squares methods [9] that essentially parameterize these operators by a particular choice of basis functions.

In this paper we will construct Lyapunov-Krasovskii functionals for delay-dependent stability of nonlinear time delay systems. To do that, we consider kernels in the Lyapunov functional that are polynomials, and we use the Sum of Squares decomposition to construct them - the structures will resemble the structure of (19). We have the following result, again referring to the system given by (1). The multiple delay case will be considered in the next section.

Proposition 12: Let 0 be a steady-state for the system given by (8). Let there exist functions $V_0(x(t))$, $V_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))$ and a positive definite, radially unbounded function $\varphi(x(t))$, a non-negative function $\psi(x(t))$, and functions $t_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))$ and $t_2(\theta,\xi,x(t),x(t-\tau),x(t+\theta),x(t+\xi))$ such that:

1) $V_0(x(t)) - \varphi(x(t)) \geq 0$,
2) $V_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi)) + t_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi)) \geq 0$ for $\theta \in [-\tau,0]$, $\xi \in [-\tau,0]$,
3) $\tau V_1(0,\xi,x(t),x(t),x(t+\xi)) - \tau V_1(-\tau,\xi,x(t),x(t-\tau),x(t+\xi)) + \tau V_1(\theta,0,x(t),x(t+\theta),x(t)) - \tau V_1(\theta,-\tau,x(t),x(t+\theta),x(t-\tau)) + \frac{dV_0(x(t))}{dx(t)} f(x(t),x(t-\tau))$ $+ \tau^2 \left(\frac{\partial V_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))}{\partial x(t)} f(x(t),x(t-\tau)) - \frac{\partial V_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi))}{\partial \theta} - \frac{\partial V_1}{\partial \xi} \right)$ $+ t_2(\theta,\xi,x(t),x(t-\tau),x(t+\theta),x(t+\xi)) \leq 0$ for $\theta \in [-\tau,0]$ and $\xi \in [-\tau,0]$,
4) $\int_{-\tau}^0 \int_{-\tau}^0 t_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi)) d\theta d\xi = 0$,
5) $\int_{-\tau}^0 \int_{-\tau}^0 t_2(\theta,\xi,x(t),x(t-\tau),x(t+\theta),x(t+\xi)) d\theta d\xi = 0$.

Then the steady-state 0 of the system given by (1) is *globally stable* for delay size τ. Moreover, if $\psi(x(t)) > 0$, then 0 is *globally asymptotically stable* for delay size τ.

Proof: Consider the following functional:

$$V(x_t) \quad = \quad V_0(x(t)) + \int_{-\tau}^0 \int_{-\tau}^0 V_1(\theta,\xi,x(t),x(t+\theta),x(t+\xi)) d\theta d\xi \tag{20}$$

Integrating the second condition and adding the first condition, taking into account condition 4), we get that $V(x_t) \geq \varphi(x(t))$: therefore the first Lyapunov condition is satisfied. The time derivative of $V(x_t)$ is:

$$\begin{aligned}
\dot{V}(x_t) &= \frac{dV_0}{dx(t)}f + \int_{-\tau}^{0} \left(V_1(0,\xi,x(t),x(t),x(t+\xi)) - V_1(-\tau,\xi,x(t),x(t-\tau),x(t+\xi))\right) d\xi \\
&+ \int_{-\tau}^{0} \left(V_1(\theta,0,x(t),x(t+\theta),x(t)) - V_1(\theta,-\tau,x(t),x(t+\theta),x(t-\tau))\right) d\theta \\
&+ \int_{-\tau}^{0}\int_{-\tau}^{0} \left(\frac{\partial V_1}{\partial x(t)}f - \frac{\partial V_1}{\partial \theta} - \frac{\partial V_1}{\partial \xi}\right) d\theta d\xi \\
&= \frac{1}{\tau^2}\int_{-\tau}^{0}\int_{-\tau}^{0} \left\{ \begin{array}{c} \tau\left(V_1(0,\xi,x(t),x(t),x(t+\xi)) - V_1(-\tau,\xi,x(t),x(t-\tau),x(t+\xi))\right) \\ +\tau\left(V_1(\theta,0,x(t),x(t+\theta),x(t)) - V_1(\theta,-\tau,x(t),x(t+\theta),x(t-\tau))\right) \\ +\frac{dV_0}{dx(t)}f + \tau^2\left(\frac{\partial V_1}{\partial x(t)}f - \frac{\partial V_1}{\partial \theta} - \frac{\partial V_1}{\partial \xi}\right) + t_2 \end{array} \right\} d\theta.
\end{aligned}$$

Condition 3) in the proposition above states that the kernel of the above integral is non-positive for $\theta \in [-\tau, 0]$ and $\xi \in [-\tau, 0]$. So (20) is a Lyapunov-Krasovskii functional, and the zero steady-state is *stable*. Since there is no constraint on the state-space, and φ is radially unbounded, the result holds globally. Moreover if $\psi > 0$, then the steady-state is asymptotically stable. ■

The first term in the functional is added to impose positive definiteness of V and the second term is a generalization of the terms that appear in the complete Lyapunov-Krasovskii functional (19). The above proposition imposes sufficient conditions to test stability properties of the zero steady-state. Here the functions t_1 and t_2 are so as to avoid imposing the non-negativity positions for the kernel of V point-wisely, but rather allow it to become negative in $[-\tau, 0]$, but still asking that it integrates to a non-negative (or non-positive) value.

In order to algorithmically construct the Lyapunov-Krasovskii functional for the nonlinear system, we can use the above proposition in a similar way as described in the delay-independent case. All functions are assumed to be bounded degree polynomials, and the $\varphi > 0$ function (and ψ, in case of asymptotic stability) is constructed using Proposition 5. To impose the conditions $\theta \in [-\tau, 0]$ and $\xi \in [-\tau, 0]$, we use a process similar to the S-procedure, as explained in the previous section, using Sum of Squares multipliers to adjoin them. Then we get three SOS conditions in a relevant Sum of Squares program which can be solved using SOSTOOLS [15]. Note that the integral constraints 4) and 5) above are still affine in the unknown coefficients of the polynomials t_1 and t_2 and so they resulting program is still a Semidefinite Program. Different Lyapunov-Krasovskii structures can also be used that can reduce the computational burden significantly at the expense of being more conservative.

An important issue, that is unique in the case of delay-dependent stability, is to ensure that the stability properties hold for a delay interval rather than for a specific value of the delay. The conditions in Proposition 12 are not affine in τ (that would depend on the structure of V_1). One can consider, however the τ as a static parameter, itself being allowed to vary within the interval $\tau \in [0, \overline{\tau}]$. In that case, one can consider the problem in a robustness setting and construct a Lyapunov function that guarantees delay-dependent stability for a delay interval.

Similar arguments allow the construction of Lyapunov-Krasovskii functionals for *local* delay-dependent stability. A modified version of Proposition 9 can be developed. We will still need to specify $\Omega = \{x_t \in C : \|x_t\| \leq \gamma\}$, and adjoin the relevant conditions on $x(t)$, $x(t-\tau)$ and $x(t+\theta)\ \forall\ \theta \in [-\tau, 0]$ to the relevant kernels of the Lyapunov functionals using the extended S-procedure. The examples that will follow will illustrate how this is done in practice. Robust stability can also be dealt with in a unified manner, in a similar way as it was done for the delay-independent case.

In the next section we turn to the problem of investigating stability for nonlinear time-delay systems with multiple delays.

C. The Multiple-Delay Case

The presence of multiple, many times incommensurate delays in a functional differential equation causes significant complications in analysis. Necessary and sufficient conditions for stability in this case are very difficult to test, and the reason lies in the heart of computational complexity theory — the problem of deciding stability is NP-hard, i.e., there is no known polynomial-time algorithm that can decide whether the system is stable or not, especially as the number of delays increases and the 'boundary' of the stability/instability as a function of the delay sizes is approached. This of course does not mean that for certain delay values, the stability property cannot be verified easily, by considering certificates of a particular structure: this is the approach we follow in this section, in which we propose tests for delay-independent and delay-dependent stability.

1) Delay-Independent Stability: Here we propose a test for delay-independent stability for nonlinear systems described by DDEs with multiple, incommensurate delays.

Consider the system:

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K)) \triangleq f(x_t) \tag{21}$$

We assume that $0 \leq \tau_1 \leq \ldots \leq \tau_K = \tau$, i.e., the delays are put in non-decreasing order, with $\tau_0 = 0$, and f is polynomial in its arguments. Again we assume that 0 is the steady-state for this system. Simple sufficient delay-independent stability conditions for global stability can be found in the following proposition:

Proposition 13: Consider the system given by (21) with 0 a steady state. Suppose there exist functions $V_0(x(t))$ and $V_i(x(t+\theta))$ for $i = 1, \ldots, K$, a positive definite radially unbounded function $\varphi(x(t)) > 0$, a non-negative function $\psi(x(t))$ such that:

1) $V_0(x(t)) - \varphi(x(t)) \geq 0$,
2) $V_i(x(t+\theta)) \geq 0$ for all $i = 1, \ldots, K$,
3) $\frac{\partial V_0}{\partial x(t)} f + \sum_{i=1}^{K} \left(V_i(x(t)) - V_i(x(t-\tau_i))\right) + \psi(x(t)) \leq 0$.

Then the steady-state is globally stable. If $\psi(x(t)) > 0$, then the steady-state is globally asymptotically stable.

The proof uses the following as a Lyapunov-Krasovskii functional, and is omitted for brevity.

$$V = V_0(x(t)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} V_i(x(t+\theta))d\theta.$$

The more interesting delay-dependent stability analysis with multiple, possibly incommensurate delays is dealt with next.

2) Multiple-Delay Case: Delay-Dependent Stability: In this section we concentrate on delay-dependent stability of systems with multiple, incommensurate delays of the form (21). Sufficient conditions for delay-dependent stability can be written as follows:

Proposition 14: Consider the system given by (21) with 0 a steady state. Suppose there exist functions $V_0(x(t))$, $V_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi))$, $t_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi))$ and $s_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi))$ for $i, j = 1, \ldots, K$, a positive definite radially unbounded function $\varphi(x(t)) > 0$, a non-negative function $\psi(x(t))$ such that:

1) $V_0(x(t)) + \tau_K^2 V_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi)) - \varphi(x(t)) + s_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi)) \geq 0$, for all $\theta \in [-\tau_i, -\tau_{i-1}]$ and $\xi \in [-\tau_j, -\tau_{j-1}]$ for each $i, j = 1, \ldots, K$,
2) $\tau_K V_{ij}(\theta, -\tau_{j-1}, x(t), x(t+\theta), x(t-\tau_{j-1})) - \tau_K V_{ij}(\theta, -\tau_j, x(t), x(t+\theta), x(t-\tau_j)) + \tau_K V_{ij}(-\tau_{i-1}, \xi, x(t), x(t-\tau_{i-1}), x(t+\xi)) - \tau_K V_{ij}(-\tau_i, \xi, x(t), x(t-\tau_i), x(t+\xi)) + \frac{\partial V_0}{\partial x(t)} f + \tau_K^2 \left(\frac{\partial V_{ij}}{\partial x(t)} f - \frac{\partial V_{ij}}{\partial \theta} - \frac{\partial V_{ij}}{\partial \xi}\right) + \psi(x(t)) + t_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi)) \leq 0$ for all $\theta \in [-\tau_i, -\tau_{i-1}]$ and $\xi \in [-\tau_j, -\tau_{j-1}]$ for each $i, j = 1, \ldots, K$,
3) $\sum_{i,j}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} s_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi)) d\theta d\xi = 0$,
4) $\sum_{i,j}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{-\tau_j}^{-\tau_{j-1}} t_{ij}(\theta, \xi, x(t), x(t+\theta), x(t+\xi)) d\theta d\xi = 0$.

Then the steady-state is globally stable. If $\psi(x(t)) > 0$, then the steady-state is globally asymptotically stable.

Proof: The proof of this proposition is based on ensuring that the following functional is a Lyapunov-Krasovksii functional:

$$\begin{aligned} V(x_t) &= V_0(x(t)) + \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\int_{-\tau_j}^{-\tau_{j-1}} V_{ij}(\theta,\xi,x(t),x(t+\theta),x(t+\xi))d\theta d\xi \\ &= \frac{1}{\tau_K^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\int_{-\tau_j}^{-\tau_{j-1}} \left(V_0(x(t)) + \tau_K^2 V_{ij}(\theta,\xi,x(t),x(t+\theta),x(t+\xi))\right) d\theta d\xi \end{aligned} \tag{22}$$

Indeed the first condition ensures that $V(x_t) > 0$, so the first Lyapunov condition is satisfied and moreover that V is radially unbounded.

The derivative of this functional along the trajectories of (21) is:

$$\begin{aligned} \dot{V}(x_t) = & \frac{\partial V_0}{\partial x(t)} f + \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} V_{ij}(\theta,-\tau_{j-1},x(t),x(t+\theta),x(t-\tau_{j-1}))d\theta \\ & - \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}} V_{ij}(\theta,-\tau_j,x(t),x(t+\theta),x(t-\tau_j))d\theta \\ & + \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_j}^{-\tau_{j-1}} V_{ij}(-\tau_{i-1},\xi,x(t),x(t-\tau_{i-1}),x(t+\xi))d\xi \\ & - \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_j}^{-\tau_{j-1}} V_{ij}(-\tau_i,\xi,x(t),x(t-\tau_i),x(t+\xi))d\xi \\ & + \sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\int_{-\tau_j}^{-\tau_{j-1}} \left(\frac{\partial V_{ij}}{\partial x(t)} f - \frac{\partial V_{ij}}{\partial \theta} - \frac{\partial V_{ij}}{\partial \xi}\right) d\theta d\xi \\ = & \frac{1}{\tau_K^2}\sum_{i=1}^{K}\sum_{j=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\int_{-\tau_j}^{-\tau_{j-1}} \begin{pmatrix} \tau_K V_{ij}(\theta,-\tau_{j-1},x(t),x(t+\theta),x(t-\tau_{j-1})) \\ -\tau_K V_{ij}(\theta,-\tau_j,x(t),x(t+\theta),x(t-\tau_j)) \\ +\tau_K V_{ij}(-\tau_{i-1},\xi,x(t),x(t-\tau_{i-1}),x(t+\xi)) \\ -\tau_K V_{ij}(-\tau_i,\xi,x(t),x(t-\tau_i),x(t+\xi)) \\ +\frac{\partial V_0}{\partial x(t)} f + \tau_K^2\left(\frac{\partial V_{ij}}{\partial x(t)} f - \frac{\partial V_{ij}}{\partial \theta} - \frac{\partial V_{ij}}{\partial \xi}\right) \end{pmatrix} d\theta d\xi \end{aligned}$$

The non-positivity of this time derivative is ensured by the 4th condition, and so $V(x_t)$ is a Lyapunov-Krasovskii functional that proves global stability of the steady-state. If $\phi_2(x(t)) > 0$, then $\dot{V}(x_t) < 0$ and the steady-state is globally asymptotically stable. ■

In Section V, we will see an example of a system from network congestion control for the Internet, where the inhomogeneous communication delays are taken into account in the stability test.

V. Examples

In this section we present two examples, one from population dynamics, and one from network congestion control for the Internet.

A. Stability analysis of a predator-prey model

A simple model of predator-prey interactions is

$$\dot{x} = bx - k_1 xy, \quad \dot{y} = k_2 xy - \sigma y,$$

where x and y are the prey and predator populations, b is the rate of increase of prey, k_1 and k_2 are the coefficients of the effect of predation on x and y and σ is the death rate of y. So the cause of death of the prey is due to predation alone, and the growth of the predator population has as the only limitation the number of prey. These equations give rise to Lotka-Volterra predator-prey cycles, but the model is not biologically meaningful because it is *conservative* giving rise to a family of closed trajectories rather than a single limit cycle [20].

The above equations describe ideal populations that can react instantaneously to any change in the environment; in real populations this change comes with a delay that represents *maturation* of the predator population. A more realistic set of equations is [21]:

$$\dot{x}(t) = x(t)[b - ax(t) - k_1 y(t)],$$

$$\dot{y}(t) = -\sigma y(t) + k_2 x(t-\tau) y(t-\tau),$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators.

Assumption 15: a, b, k_1, k_2 and σ are positive.

The equilibria (x^*, y^*) of the above system are:

$$(x^*, y^*) = (0, 0), \quad (x^*, y^*) = (b/a, 0),$$

$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1 k_2}\right). \tag{23}$$

We are only interested in the steady-state given by (23).

Assumption 16: $(bk_2 - a\sigma) > 0$.

Assumption 16 ensures that the steady-state (23) is in the first quadrant. We now shift the coordinates to $(x_1, x_2) = (x - \frac{\sigma}{k_2}, y - \frac{bk_2 - a\sigma}{k_1 k_2})$ to get:

$$\dot{x}_1(t) = \left[x_1(t) + \frac{\sigma}{k_2}\right][-ax_1(t) - k_1 x_2(t)] \tag{24}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1} x_1(t-\tau) + k_2 x_1(t-\tau) x_2(t-\tau) \tag{25}$$

We can linearize the above system about $(0,0)$ to get:

$$\dot{x}_1(t) = \frac{\sigma}{k_2}\left[-ax_1(t) - k_1x_2(t)\right] \tag{26}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1}x_1(t-\tau) \tag{27}$$

For the linearised system, we have the following result:

Proposition 17: Consider the system (26–27) under the assumptions (15,16). Then if $(bk_2 - 3a\sigma) < 0$ the zero steady-state is stable independent of the delay. If $(bk_2 - 3a\sigma) > 0$ the zero steady-state is stable if the delay satisfies $\tau < \tau^*$ and is unstable otherwise, where τ^* is given by:

$$\tau^* = \frac{1}{\omega}\ \text{atan}\ \left[\omega\frac{(a\sigma^2 - \omega k_2)k_2 - \sigma(2a\sigma + bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega k_2)}\right]$$

where ω solves

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{28}$$

Proof: In the absence of delay, and under the two assumptions, the system is *asymptotically stable*. Substituting $s = j\omega$ in the characteristic equation and separating real and imaginary parts we get:

$$\begin{aligned} -\omega^2 + \frac{a\sigma^2}{k_2} &= \sigma\omega\sin(\omega\tau) + \sigma\left(\frac{2a\sigma}{k_2} - b\right)\cos(\omega\tau) \\ \sigma\left[1 + \frac{a}{k_2}\right]\omega &= \sigma\omega\cos(\omega\tau) - \sigma\left(\frac{2a\sigma}{k_2} - b\right)\sin(\omega\tau) \end{aligned}$$

Squaring the two equations and adding we get:

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{29}$$

Denoting $p_1 = \frac{a^2\sigma^2}{k_2^2}$ and $p_2 = \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2)$ the roots of this equation are:

$$\omega^2 = -\frac{p_1}{2} \pm \frac{\sqrt{p_1^2 - 4p_2}}{2}. \tag{30}$$

Under assumption 16, if $(bk_2 - 3a\sigma) < 0$ (i.e $p_2 > 0$), then there are no real solutions to (29). Since the steady-state is stable when the delay is zero, and there is no ω for which poles cross to the RHP, we conclude that (26–27) is delay-independent stable.

Under assumption 16 and $(bk_2 - 3a\sigma) > 0$ then $p_2 < 0$ and one of the two roots of (30) is positive and the other one is negative. Therefore the poles cross the imaginary axis at only one ω — there is no possibility for *stability reversal*. If ω is the solution to the above equation, then at $\tau = \tau^*$ given in the statement of the Proposition a Hopf bifurcation occurs; the system is stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. ■

We now analyze the nonlinear description of the system (24–25) using the methodology that was developed in the previous sections. We choose as nominal values for the parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$.

1) Delay-independent stability analysis: The system (24–25) has many equilibria, and so we need to define a region around the zero steady-state to obtain a stability condition (this is the region Ω in Theorem 2). We let:

$$|x_{1_t}| \leq \gamma_1 x^*, \quad |x_{2_t}| \leq \gamma_2 y^*, \tag{31}$$

where the steady-state (x^*, y^*) is given by (23). We consider b to be a parameter in the problem. From Proposition 17, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of a, σ and k_2, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating (x^*, y^*) we use a value of $b = 20/3$. The steady-state $(0, 0)$ of system (24–25) does not move as b changes, however the other two equilibria cross through the region defined by (31). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady-state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = V_0(x_1(t), x_2(t), b) + \int_{-\tau}^{0} V_1(x_1(t+\theta), x_2(t+\theta), b) d\theta.$$

We use a variant of Proposition 10 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When the order of V_0 is second order and V_1 is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th order and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval.

2) Delay-dependent stability analysis: Now we will try to test values for τ for which stability is retained using the same parameters as before and fixed $b = 15$. Given these parameters $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on x_1 and x_2 on the state-space given by (31) with $\gamma_1 = \gamma_2 = 0.1$.

We can construct the Lyapunov functional $V(x_t)$ given by (20) with V_1 0th order with respect to θ and ξ and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When V_1 is quartic with respect to all variables but θ and ξ (which are kept at 0 order) then we can construct this $V(x_t)$ for $\tau = 0.053$. The corresponding SDP is bigger as the functional is more complicated, but we can see that larger values of the delay closer to the stability boundary can be tested.

B. Network Congestion Control for the Internet

Internet congestion control [3] is a distributed algorithm to allocate available bandwidth to competing sources so as to avoid congestion collapse by ensuring that link capacities are not exceeded. The need for congestion control for the Internet emerged in the mid-1980s, when congestion collapse resulted in unreliable file transfer. In 1988 Jacobson [22] proposed an admittedly ingenious scheme for congestion control. The shortcomings of this scheme and its successors such as TCP Reno and Vegas have only recently become apparent: they are not scalable to arbitrary networks with very large capacities and multiple, non-commensurate time-delays. New designs of Active Queue Management (AQM) and/or Transmission Control Protocol (TCP) dynamics have been proposed that provide scalable stability in the presence of heterogeneous delays, which can be verified at least for the linearization about a steady-state.

The simplest adequate modeling framework for network congestion control is in the form of nonlinear deterministic delay-differential equations [23], [24], [25]. Some work has been done on the analysis of such systems usually for the single-bottleneck link case, using either Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals, IQC methods, passivity etc.

In this section we analyze a simple network instantiation of what is known a ‘primal’ congestion control scheme, shown in Figure 1. It consists of $L = 3$ links, labeled $l = 1, 2, 3$ and $S = 2$ sources, $i = 1, 2$. For this network, we define an $L \times S$ routing matrix R which

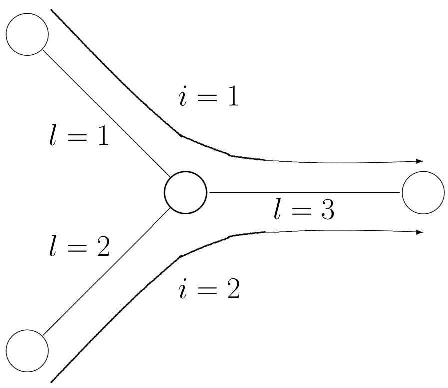

Fig. 1. A network topology under consideration.

is 1 if source i uses link l and 0 otherwise:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{32}$$

The architecture of network congestion control is shown in Figure 2. To each source i we associate a transmission rate x_i. All sources whose flow passes through resource l contribute to the *aggregate rate* y_l for resource l, the rates being added with some forward time delay $\tau^f_{i,l}$. Hence we have:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}) \tag{33}$$

The resources l react to the aggregate rate y_l by setting congestion information p_l, the price at resource l. This is the Active Queue Management part of the picture. The prices of all the links that source i uses are aggregated to form q_i, the *aggregate price* for source i, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}) \tag{34}$$

The prices q_i can then be used to set the rate of source i, x_i, which completes the loop. The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l}, \quad \forall\ l \tag{35}$$

The capacity of link l is given by c_l. For a general network, the interconnection is shown in Figure 2. In this section we choose the control laws for TCP and AQM as follows:

$$\begin{aligned} p_l(t) &= \left(\frac{y_l(t)}{c_l}\right)^B \\ \dot{x}_i(t) &= 1 - x_i(t - \tau_i) q_i(t) \end{aligned}$$

Here $p_l(t)$ corresponds to the probability that the queue length exceeds B in a $M/M/1$ queue with arrival rate $y_l(t)$ and capacity c_l, and the source law corresponds to a queue length with proportionally fair source dynamics.

For the network shown in Figure 1, the closed loop dynamics become:

$$\begin{aligned} \dot{x}_1 &= 1 - x_1(t-\tau_1)\left[\left(\frac{x_1(t-\tau_1)}{c_1}\right)^B + \left(\frac{x_1(t-\tau_1) + x_2(t - \tau^b_{1,3} - \tau^f_{2,3})}{c_3}\right)^B\right] \\ \dot{x}_2 &= 1 - x_2(t-\tau_2)\left[\left(\frac{x_2(t-\tau_2)}{c_2}\right)^B + \left(\frac{x_2(t-\tau_2) + x_1(t - \tau^f_{1,3} - \tau^b_{2,3})}{c_3}\right)^B\right] \end{aligned}$$

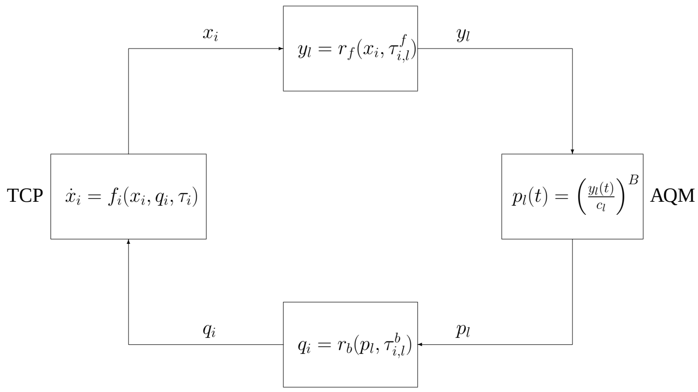

Fig. 2. The internet as an interconnection of sources and links through delays.

Let $B = 2$, $c_1 = c_2 = 3$, and $c_3 = 1$. Then the steady-state of this system is $(x_1^*, x_2^*) = (0.6242, 0.6242)$. We consider delay sizes such that $\tau_{1,3}^b = 63ms$, $\tau_{1,3}^f = 93ms$, $\tau_{2,3}^f = 49ms$, $\tau_{2,3}^b = 77ms$. The system equations about the new steady-state become:

$$\begin{aligned}
\dot{x}_1' &= 1 - [x_1'(t-0.154) + x_1^*] \\
&\quad \left[\left(\frac{x_1'(t-0.154)+x_1^*}{3}\right)^2 + (x_1'(t-0.154) + x_2'(t-0.14) + x_1^* + x_2^*)^2\right] \\
\dot{x}_2' &= 1 - [x_2'(t-0.126) + x_2^*] \\
&\quad \left[\left(\frac{x_2'(t-0.126)+x_2^*}{3}\right)^2 + (x_2'(t-0.126) + x_1'(t-0.14) + x_1^* + x_2^*)^2\right]
\end{aligned}$$

where $x_i(t) = x_i'(t) + x^*$. The linearization of this system about the steady-state is stable, as a Lyapunov function of the form (22) can be constructed. The same Lyapunov function can be constructed in a region of the state-space satisfying $\|x_{1_t}\| \leq 0.8x_1^*$ and $\|x_{2_t}\| \leq 0.8x_2^*$, thus showing that the steady-state is nonlinearly stable.

VI. Concluding Remarks

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is achieved through the solution of a set of Linear Matrix Inequalities (LMIs). Linear and nonlinear delay-independent and delay-dependent stability can now be treated using the same tools.

The above methods can be easily extended to systems with many delays, either commensurate or not. Still a judicious choice for the structure of the Lyapunov functional would be required. Functional differential equations of neutral type can also be treated in a unified way.

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