Systems Analysis and Control

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Lecture 21: Stability Margins and Closing the Loop

Overview

In this Lecture, you will learn:

Closing the Loop

- Effect on Bode Plot
- Effect on Stability

Stability Effects

- Gain Margin
- Phase Margin
- Bandwidth

Estimating Closed-Loop Performance using Open-Loop Data

- Damping Ratio
- Settling Time
- Rise Time

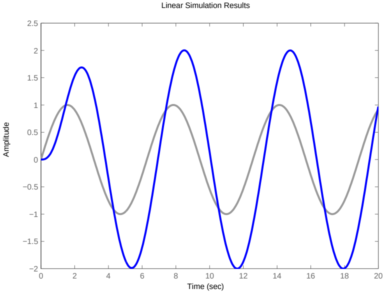
Recall: Frequency Response

Input:

$$u(t) = M\sin(\omega t + \phi)$$

Output: Magnitude and Phase Shift

$$y(t) = |G(\imath\omega)|M\sin(\omega t + \phi + \angle G(\imath\omega))$$



Frequency Response to $\sin\omega t$ is given by $G(\imath\omega)$

Review

Recall: Bode Plot

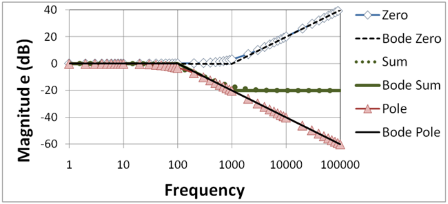
Definition 1.

The Bode Plot is a pair of $\log$-$\log$ and semi-$\log$ plots:

1. Magnitude Plot: $20\log_{10}|G(\imath\omega)|$ vs. $\log_{10}\omega$
2. Phase Plot: $\angle G(\imath\omega)$ vs. $\log_{10}\omega$

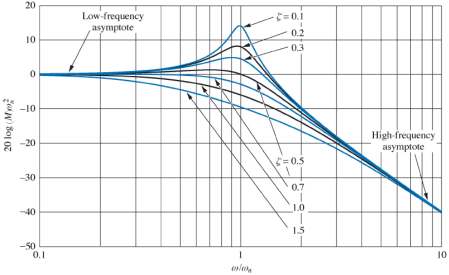
Bite-Size Chucks:

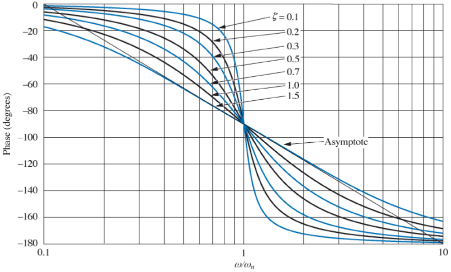
$$\angle G(\imath\omega) = \sum_i \angle G_i(\imath\omega)$$



We left off with **Complex Poles:**

$$G(s) = \frac{1}{\left(\left(\frac{s}{\omega_n}\right)^2 + 2\zeta\frac{s}{\omega_n} + 1\right)}$$





Complex Poles and Zeros

The amplification at the natural frequency, ω_n, is called resonance.

Figure: Frequency Sweeping with Resonance

Complex Poles and Zeros

Figure: Frequency Sweeping with Resonance

Complex Poles and Zeros

Figure: The Tacoma Narrows Bridge

Closing The Loop

Now we examine the effect of closing the loop on the Frequency Response.

Use simple Unity Feedback ($K = 1$).

Closed-Loop Transfer Function:

$$G_{cl}(\imath\omega) = \frac{G(\imath\omega)}{1 + G(\imath\omega)}$$

We are most concerned with magnitude:

$$|G_{cl}(\imath\omega)| = \frac{|G(\imath\omega)|}{|1 + G(\imath\omega)|}$$

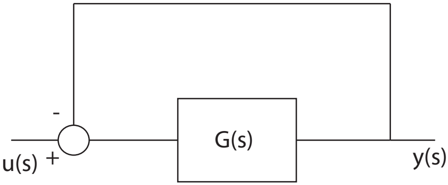


Figure: Unity Feedback

On the Bode Plot

$$20\log|G_{cl}(\imath\omega)| = 20\log|G(\imath\omega)| - 20\log|1+G(\imath\omega)|$$

Which is the combination of

- The original bode plot
- The new factor $\log|1+G(\imath\omega)|$

We are most concerned with the effect of the new term

$$-20\log|1+G(\imath\omega)|$$

Specifically, as $1+G(\imath\omega)\to 0$

$$\lim_{1+G(\imath\omega)\to 0} -20\log|1+G(\imath\omega)| = \infty$$

An unstable mode!

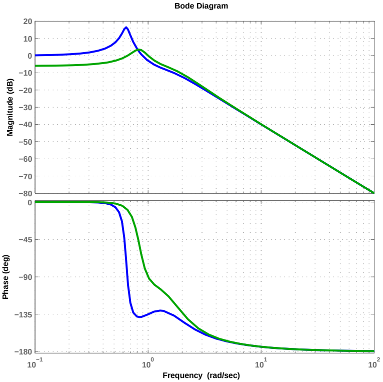


Figure: Open Loop: Blue, CL: Green

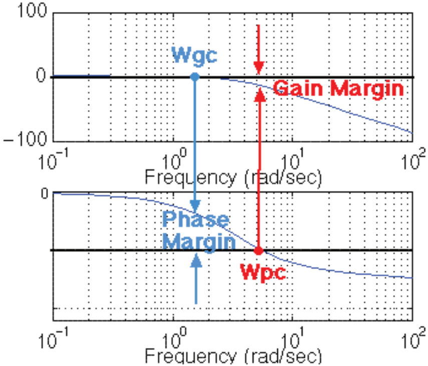
Instability occurs when

$$1 + G(\imath\omega) = 0$$

For this to happen, we need:

- $|G(\imath\omega)| = 1$
- $\angle G(\imath\omega) = -180°$

Stability Margins measure how far we are from the point $(|G| = 1, \angle G = -180°)$.



Definition 2.

The **Gain Crossover Frequency**, ω_{gc} is the frequency at which $|G(\imath\omega_c)| = 1$.

This is the danger point:

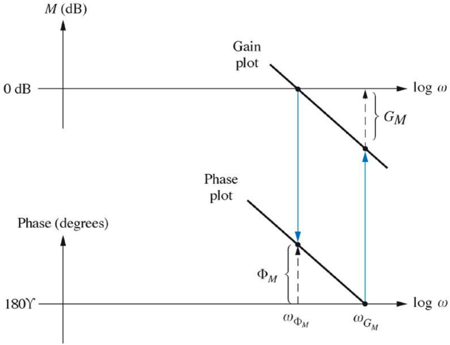
- If $\angle G(\imath\omega_c) = 180°$, we are unstable

Definition 3.

The **Phase Margin**, Φ_M is the phase relative to $180°$ when $|G| = 1$.

- $\Phi_M = |180° - \angle G(i\omega_{gc})|$

- ω_{gc} is also known as the phase-margin frequency, ω_{Φ_M}



Definition 4.

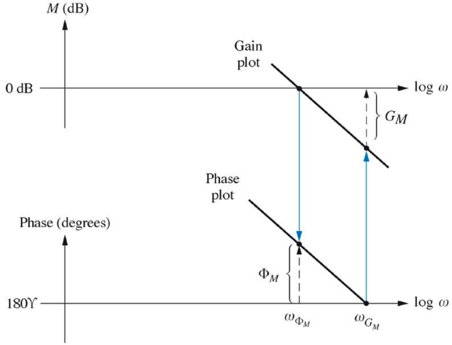
The **Phase Crossover Frequency**, ω_{pc} is the frequency (frequencies) at which $\angle G(\imath\omega_{pc}) = 180°$.

Definition 5.

The **Gain Margin**, G_M is the gain relative to $0dB$ when $\angle G = 180°$.

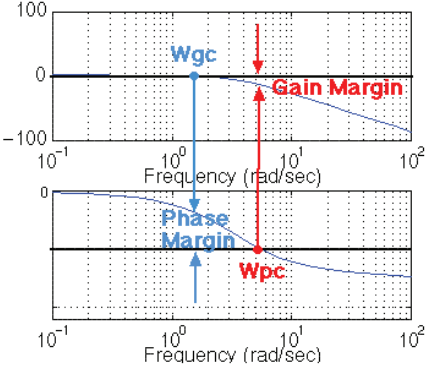
- $G_M = -20\log|G(\imath\omega_{pc})|$

G_M is the gain (in dB) which will destabilize the system in closed loop.

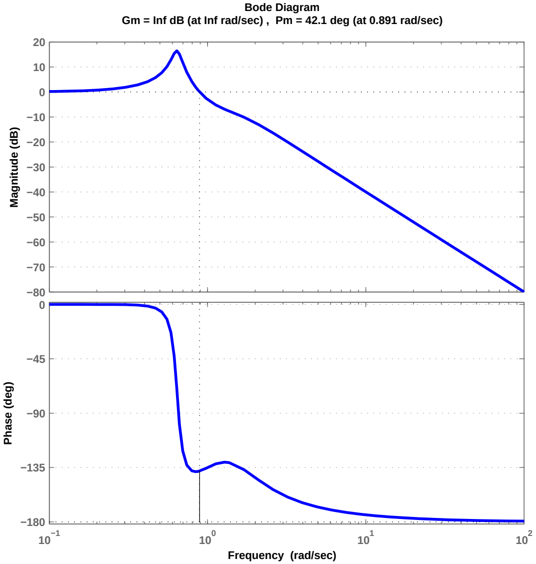


ω_{pc} is also known as the gain-margin frequency, ω_{G_M}

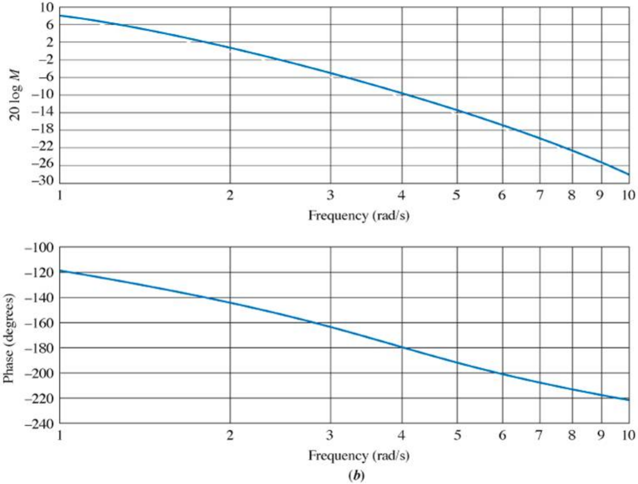
Gain and Phase Margin tell how stable the system would be in Closed Loop.



These quantities can be read from the **Open-Loop Data**.



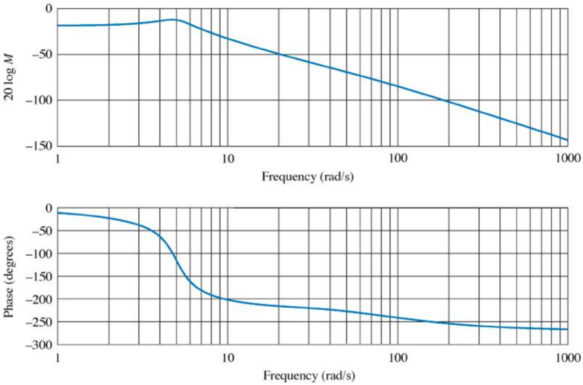
Bode Diagram
Gm = Inf dB (at Inf rad/sec) , Pm = 42.1 deg (at 0.891 rad/sec)
Magnitude (dB)
20
10
0
−10
−20
−30
−40
−50
−60
−70
−80
Phase (deg)
0
−45
−90
−135
−180
10^−1
10^0
10^1
10^2
Frequency (rad/sec)



- $\Phi_M = 35^\circ$
- $G_M = 10\text{dB}$

Note that sometimes the margins are undefined

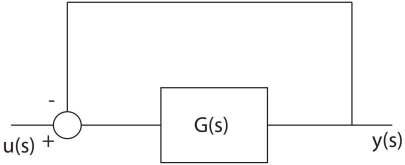
- When there is no crossover at 0dB
- When there is no crossover at $180°$

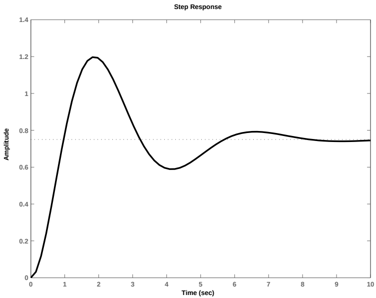


Question: What happens when we Close the Loop?

- We want **Performance Specs!**
- We only have open-loop data.
- Φ_M and G_M can help us.

Unity Feedback:





We want:

- Damping Ratio
- Settling Time
- Peak Time

Assume the closed loop system is the quadratic

$$G_{cl} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Then the open-loop system must be

$$G_{ol} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s}$$

Assume that our open-loop system is G_{ol}

- Use Φ_M to solve for ζ
- Set $20\log|G_{cl}| = -3dB$ to solve for ω_n

The Quadratic Approximation gives the **Closed-Loop Damping Ratio** as

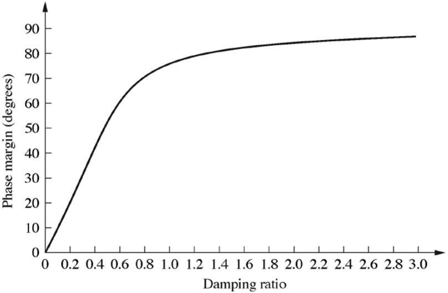
$$\Phi_M = \tan^{-1} \frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}}$$

Φ_M is from the *Open-Loop Data!*

A Handy approximation is

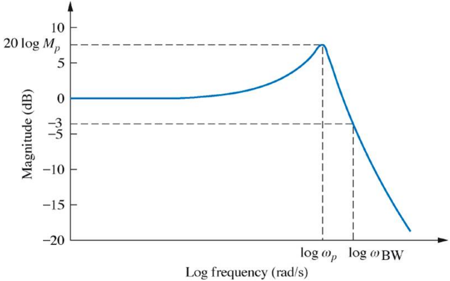
$$\zeta = \frac{\Phi_M}{100}$$

- Only valid out to $\zeta \cong .7$.
- Given Φ_M, we find closed-loop ζ.



We find closed-loop ζ from Phase margin.

- We can find closed-loop Natural Frequency ω_n from the closed-loop Bandwidth.



Definition 6.

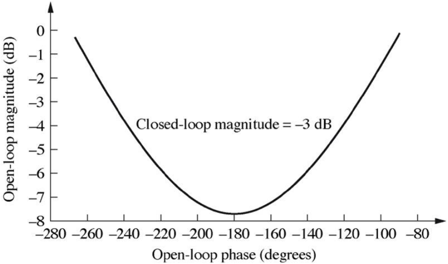
The **Bandwidth**, ω_{BW} is the frequency at gain
$20 \log |G(\imath\omega_{BW}) = |20 \log |G(0)| - 3\text{dB}$.

- Closely related to crossover frequency.
- The Bandwidth measures the range of frequencies in the output.
- For 2nd order, Bandwidth is related to natural frequency by

$$\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Question: How to find closed-loop bandwidth?

Finding the closed-loop bandwidth from open-loop data is tricky.

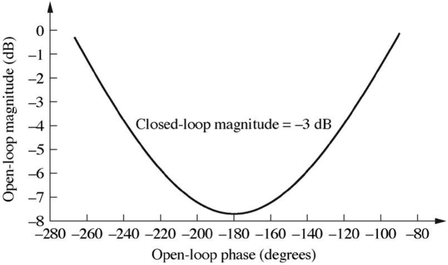


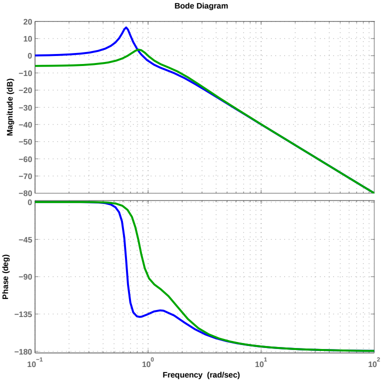
Have to find the frequency when the Bode plot intersects this curve.

- **Heuristic:** Check the frequency at -6dB and see if phase is $\cong 180°$.

Finding Closed-Loop Bandwidth from Open-Loop Data

Example





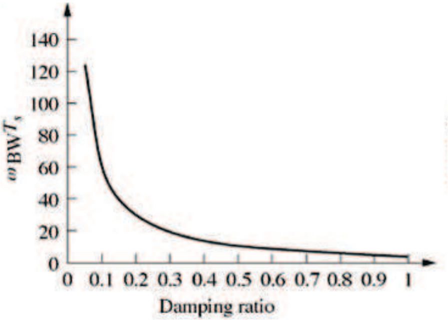
At phase 135°, -5dB, we get closed loop $\omega_{BW} \cong 1$.

Transient Response

Bandwidth and Settling Time

We can use the expression $T_s = \frac{4}{\zeta\omega_n}$ to get

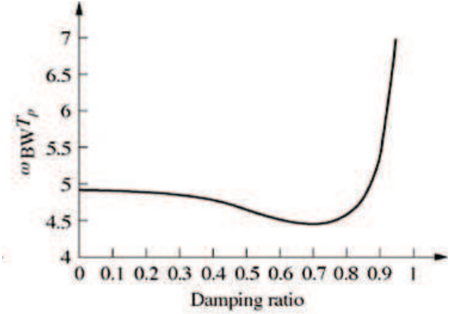
$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$



Given closed-loop ζ and ω_{BW}, we can find T_s.

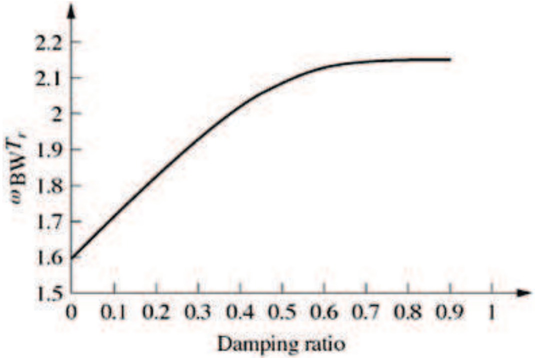
We can use the expression $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ to get

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$



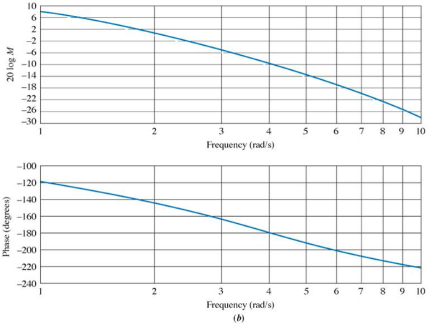
Given closed-loop ζ and ω_{BW}, we can find T_p.

Using an expression for T_r, we get a relationship between $\omega_{BW} \cdot T_r$ and ζ.



Given closed-loop ζ and ω_{BW}, we can find T_r.

Question: Using Frequency Response Data, find T_r, T_s, T_p after unity feedback.



First Step: Find the phase Margin.

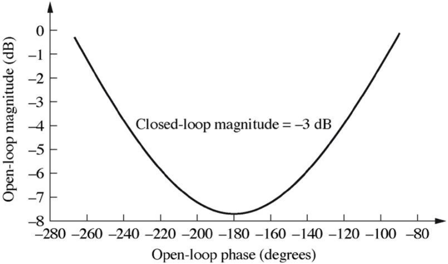
- Frequency at 0dB is $\omega_{gc} \cong 2$
- $\angle G(2) \cong -145°$
- $\Phi_M = 180° - 145° = 35°$

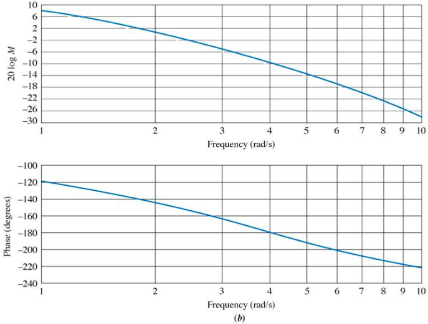
Transient Response

Example

Step 2: Closed-Loop Damping Ratio

$$\zeta \cong \frac{\Phi_M}{100} = .35$$



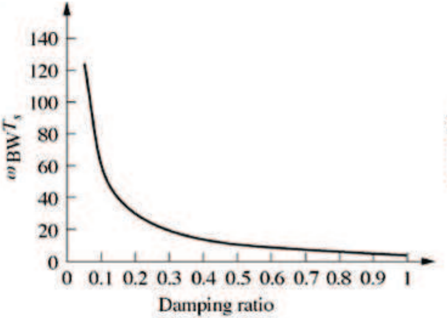


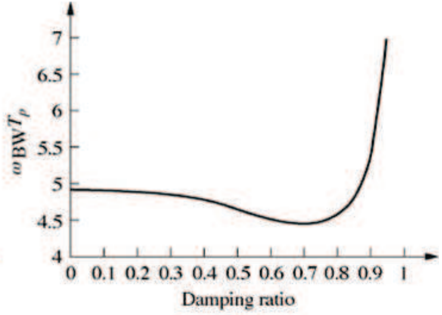
Step 3: Closed-Loop Bandwidth

- Intersect at $\cong (|G| = -6dB, \angle G = 170°)$
- Frequency at intersection is $\omega_{BW} \cong 3.7$

Step 4: Settling Time and Peak Time

- $\omega_{BW} = 3.7$, $\zeta = .35$

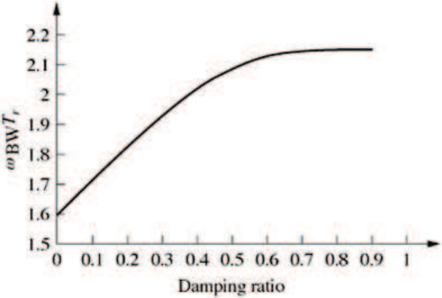




- $\omega_{BW}T_s = 20$ implies $T_s = 5.4s$
- $\omega_{BW}T_p = 4.9$ implies $T_p = 1.32$

Step 5: Rise Time

- $\omega_{BW} = 3.7$, $\zeta = .35$



- $\omega_{BW} T_r = 1.98$ implies $T_r = .535$

Step 6: Experimental Validation.

Use the plant

$$G(s) = \frac{50}{s(s+3)(s+6)}$$

We find

- $T_p = 1.6s$
  - predicted 1.32
- $T_r = .7s$
  - predicted $.535$
- $T_s = 4s$
  - Predicted 5.4

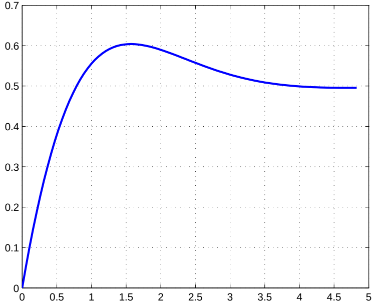


Figure: Step Response

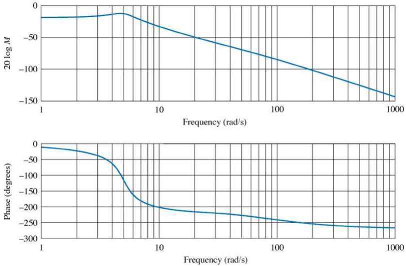
Steady-State Error

Finally, we want steady-state error.

- Steady-State Step response is

$$\lim_{s\to 0} G(s) = \lim_{\omega\to 0} G(\imath\omega)$$

Steady-state response is the Low-Frequency Gain, $|G(0)|$.



Close The Loop to get steady-state error

$$e_{ss} = \frac{1}{1 + |G(0)|}$$

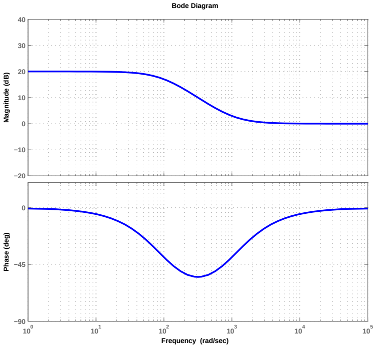
A Lag Compensator

$$\lim_{\omega\to 0} 20\log G(\imath\omega) = 20dB$$

So $\lim_{\omega\to 0} |G(\imath\omega)| = 10$.

Steady-State Error:

$$e_{ss} = \frac{1}{1+G(0)} = \frac{1}{11} = .091$$



Summary

What have we learned today? **Closing the Loop**

- Effect on Bode Plot
- Effect on Stability

Stability Effects

- Gain Margin
- Phase Margin
- Bandwidth

Estimating Closed-Loop Performance using Open-Loop Data

- Damping Ratio
- Settling Time
- Rise Time

Next Lecture: Compensation in the Frequency Domain