Modern Control Systems

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Lecture 15: Small Gain Theorem

A algebra is a vector space with a distributive multiplication operator.

Definition 1.

A **Banach Algebra**, X, is a Banach space with an associated mapping $\cdot : X \times X \to X$ such that

1. **Identity:** There exists some $I \in X$ such that
$$F \cdot I = I \cdot F = F$$
for all $F \in X$.
2. **Distributivity:** $F \cdot (G \cdot H) = (F \cdot G) \cdot H$ for all $F, G, H \in X/$
3. **Associativity:** $F \cdot (G + H) = F \cdot G + F \cdot H$ for all $F, G, H \in X$.
4. $F \cdot (\alpha G) = (\alpha F) \cdot G$ for all $F, G \in X$ and $\alpha \in \mathbb{R}$.
5. **Submultiplicative Inequality:**
$$\|F \cdot G\| \leq \|F\| \|G\|$$

Some algebras have extra properties

- **Inverse Property (Group):** For any $F \in X$, there exists a $F^{-1} \in X$ such that
$$F \cdot F^{-1} = F^{-1} \cdot F = I$$
- **Commutative Algebra, Abelian Group:** $F \cdot G = G \cdot F$

Square Matrices:

- Matrix multiplication using the $\bar{\sigma}$ norm.

Vectors:

- Pointwise addition/multiplication **only**.

Linear Operators: $\mathcal{L}(X)$

- Using the composition operation and induced norm.

Some elements of a Banach Algebra may have an inverse.

Definition 2.

For $J \in X$, where X is a Banach Algebra, we say K is the **inverse** of J if $K \in X$ and

$$J \cdot K = K \cdot J = I$$

If such a K exists, we say J is **invertible**.

Note that if the inverse exists, it is unique **Observation:** The feedback

interconnection yields

$$y = (I + GK)^{-1} Gu$$

Question: Given G and K, how to tell whether $(I + GK)$ is invertible?

- How to tell whether anything is invertible?

Answer: Spectral Theory

- When is $\lambda I - A$ invertible?

Small Gain Theorem

The simplest form of spectral theory.

Theorem 3 (Small Gain Theorem).

Suppose B is a Banach Algebra and $Q \in B$. If $\|G\| < 1$, then $(I-Q)^{-1}$ exists and furthermore

$$(I-Q)^{-1} = \sum_{k=0}^{\infty} Q^k$$

Clearly holds for $B = \mathbb{R}$ since

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} = (1-r)^{-1}$$

Small Gain Theorem

Proof.

Relatively Simple: Show $\sum_{k=0}^\infty Q^k$ converges and that it is the inverse.

- To show that the sequence $T_i = \sum_{k=0}^i Q^k$ converges, we show it is Cauchy.
- Suppose $m > n$. Then

$$
\begin{aligned}
\|T_m - T_n\| &= \| \sum_{k=n+1}^{m} Q^k \| \leq \| \sum_{k=n+1}^{m} \|Q^k\| \| \quad \text{Triangle Inequality} \\
&\leq \| \sum_{k=n+1}^{m} \|Q\|^k \| \quad \text{Submultiplicative Inequality} \\
&= \|Q\|^{n+1} \| \sum_{k=0}^{m-n-1} \|Q\|^k \| \\
&\leq \|Q\|^{n+1} \| \sum_{k=0}^{\infty} \|Q\|^k \| \\
&= \|Q\|^{n+1} \frac{1}{1 - \|Q\|} \quad \text{Scalar Power Series}
\end{aligned}
$$

Small Gain Theorem

Proof.

- Thus for $m > n$
$$\lim_{n\to\infty}\|T_m - T_n\| = \lim_{n\to\infty}\|Q\|^{n+1}\frac{1}{1-\|Q\|} = 0$$
since $\|Q\| < 1$.
- Since the sequence is Cauchy and the space Banach, $\sum_{k=0}^{\infty} Q^k \in B$

We have shown that $\sum_{k=0}^{\infty} Q^k \in B$. Now we show that is is the inverse.

- Start by showing it is a right inverse
$$\begin{aligned}(I-Q)\sum_{k=0}^{\infty} Q^k &= \sum_{k=0}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k \\ &= Q^0 + \sum_{k=1}^{\infty} Q^k - \sum_{k=1}^{\infty} Q^k = I\end{aligned}$$
- Showing $\sum_{k=0}^{\infty} Q^k(I-Q) = I$ is identical.
- Thus $(I-Q)^{-1}$ exists, so $(I-Q)$ is invertible.

Small Gain Theorem

Recall the feedback interconnection:

$$y = (I + GK)^{-1} G u$$

Question: When is $(I + GK)^{-1} G \in \mathcal{L}(L_\infty)$?

Answer: When $\|GK\| < 1$.

- Using Banach Algebra of Composition

$$\|GK\| \le \|G\| \|K\|$$

- We can require $\|K\| < \frac{1}{\|G\|}$ or $\|K\| < 1$ and $\|G\| \le 1$.
- Can also express as $\int_0^\infty \left| \|Ce^{As}B\| \right| ds \le 1$

Note: Small Gain Theorem works for **ANY** Banach space (unusual).

- For most results we need a Hilbert space.

Small Gain Theorem

Example

Take the Banach Algebra of square matrices. $(\mathbb{R}^{n\times n}, \|\cdot\| = \bar{\sigma}(\cdot))$.

- Let $Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$, with norm $\bar{\sigma}(Q) = \frac{1}{2}$.
- By small gain $(I-Q)^{-1}$ exists. Further

$$(I-Q)^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}^{-1} = \sum_{k=0}^{\infty} Q^k$$

- Now

$$Q = \begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}, \quad Q^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad Q^3 = \begin{bmatrix} 0 & \frac{1}{8} \\ \frac{1}{8} & 0 \end{bmatrix}, \quad Q^4 = \begin{bmatrix} \frac{1}{16} & 0 \\ 0 & \frac{1}{16} \end{bmatrix}$$

- We conclude that

$$Q = \begin{bmatrix} \frac{1}{4^k} & \frac{1}{2}\frac{1}{4^k} \\ \frac{1}{2}\frac{1}{4^k} & \frac{1}{4^k} \end{bmatrix} = \frac{1}{4^k}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$$

- Thus

$$\sum_{k=0}^{\infty} Q^k = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \sum_{k=0}^{\infty} \frac{1}{4^k} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \frac{1}{1-\frac{1}{4}} = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \frac{4}{3} = \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

Unfortunately, the small gain theorem is conservative.

- Let
$$Q = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}$$
- Then $\bar{\sigma}(Q) = 10$, yet
$$(I - Q)^{-1} = \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}^{-1}$$
- Furthermore, $(I - Q)^{-1} = \sum_{k=0}^{\infty} Q^k$.

Spectral Theorem

An extension of the concept of eigenvalues.

- with important differences.

Definition 4.

Let B be a Banach Space and $M \in \mathcal{L}(B)$. The **Spectrum** of M is:

$$\sigma(M) := \{\lambda \in \mathbb{C} \,:\, (\lambda I - M) \text{ is not invertible in } \mathcal{L}(B)\}$$

The **Spectral Radius** is

$$\rho(M) := \max\{|\lambda| \,:\, \lambda \in \sigma(M)\}$$

Fact: $\sigma(M)$ is non-empty and closed.

- all you can say.

Consider a multiplication operator.

$$M : u(t) \mapsto e^{-t}u(t)$$

Theorem 5.

$M \in \mathcal{L}(L_\infty[0,\infty))$ and $\sigma(M) = [0,1]$.

Proof.

First we show that $[0,1] \subset \sigma(M)$.

- Since $((\lambda I - M)u)(t) = (\lambda - e^{-t})\, u(t)$, we can construct the inverse

$$\left((\lambda I - M)^{-1} u\right)(t) = \frac{1}{\lambda - e^{-t}} u(t)$$

- If $\lambda \in [0,1]$, then $\lim_{t \to -\log(\lambda)} \frac{1}{\lambda - e^{-t}} = \infty$.
- Therefore, $(\lambda I - M)^{-1}$ is unbounded.
- Thus $[0,1] \subset \sigma(M)$.

$\square$

Proof.

Now we show that if $\lambda \notin [0,1]$, then $\lambda \notin \sigma(M)$.

- If $\lambda < 0$, then

$$\left|\left((\lambda I - M)^{-1} u\right)(t)\right| = \frac{1}{\lambda - e^{-t}} u(t) \leq \frac{1}{|\lambda|} |u(t)|$$

- If $\lambda > 1$, then

$$\left|\left((\lambda I - M)^{-1} u\right)(t)\right| = \frac{1}{\lambda - e^{-t}} u(t) \leq \frac{1}{|\lambda + 1|} |u(t)|$$

$\square$