A Convex Approach to Output Feedback Control of Parabolic PDEs Using Sum-of-Squares

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***Abstract*—In this paper we use optimization-based methods to design output-feedback controllers for a class of one-dimensional parabolic partial differential equations. The output may be distributed or point-measurements. The input may be distributed or boundary actuation. We use Lyapunov operators, duality, and the Luenberger observer framework to reformulate the synthesis problem as a convex optimization problem expressed as a set of Linear-Operator-Inequalities (LOIs). We then show how feasibility of these LOIs may be tested using Semidefinite Programming (SDP) and the Sum-of-Squares methodology.**

I. Introduction

Parabolic Partial Differential Equations (PDEs) are a simple class of system used to model processes such as diffusion, transport and reaction. Some examples of systems which have been modelled using Parabolic PDEs include plasma in a tokamak [50], heat propagation, and spatial dynamics of population in an ecosystem [28]. Despite the wide variety of physical phenomena modeled by partial-differential equations, our knowledge of how to control these systems is under-developed. While much attention has focused on the use of advanced computing strategies for simulation of partial-differential equations, relatively little work has focused on the development of numerical methods for control of PDEs. This is in particular contrast to the state of the art for linear Ordinary Differential Equations (ODEs), wherein Linear Matrix Inequalities and Convex optimization have been used to resolve a vast array of long-standing problems - e.g. H_∞-optimal output feedback. The goal of this paper, then, is to attempt to extend some of the computational methods for control of linear ODEs to control of linear PDEs.

Differential models incorporating multiple independent variables (e.g. time and space) have been around since the time of Newton. Indeed, many of the models we use today date from this time - e.g. D'Alembert and the wave equation; the Euler-Bernoulli beam; The Euler Equations. Although research into PDEs over the past century has mainly focused on constructing analytic or numerical solutions to these systems, an effort has also been made to define a framework for control. One facet of this research into defining a framework for control of PDEs has been to define a general class of forward-time PDE systems using the label of "strongly-continuous semigroup". For such systems, existence and continuity of solutions is guaranteed for bounded feedback operators. See [10], [2], [14], [25] for several excellent volumes on this subject. One of the advantages of a well-defined state-space is the ability to use Lyapunov analysis to prove properties of the state. Indeed, application of Lyapunov theory to infinite-dimensional systems has been studied for some time - See early results in [19], [11], [1].

PDE models of control can vary significantly based on the type of PDE, boundary conditions, measurements, etc. Unlike ODE systems, these differences may dramatically alter the definition of state and other mathematical properties of the solution. For instance, control of PDEs can be classified as either distributed input or boundary/point input. For distributed inputs, the control effort is spread over some measurable subset of the domain. For boundary/point inputs, the input precisely determines the state at a collection of points of zero measure. An example of a distributed input is RF heating of a plasma in a tokamak [4]. Examples of point actuation include a thermostat in HVAC regulation or the speaker in noise-cancelling headphones. In a similar manner, output may also be classified using either distributed or boundary/point measurements. A more subtle distinction is the classification as hyperbolic, parabolic or elliptic - a distinction determined by the number and type of partial derivatives. Additionally, we distinguish between isotropic and anisotropic systems. In isotropic systems, independent variables (spatial or temporal) do not appear in the coefficients, whereas the anisotropic form allows such dependence. Examples of anisotropic systems include heat conduction with non-homogeneous/time-varying conductive properties or a wave propagating through a medium of varying density. Finally, we classify the boundary conditions using terms such as Neumann/Dirichlet/Robin/etc. to denote which boundary points are specified or controlled. Classification of boundary conditions has a significant influence on the existence and mathematical properties of the solution [27].

In this paper we focus on the more difficult case of point actuation of a single-state anisotropic parabolic partial-differential equation in a single spatial variable using point observations and non-homogeneous boundary conditions.

There has been significant recent effort to understand and solve the problem of optimal control for PDE systems of this form. For instance, [48] solved certain distributed input/distributed output optimal control problems using infinite-dimensional Ricatti equations. Additionally, [25] and related work considered the problem of point actuation using Ricatti Equations and also discusses potential numerical methods for solving these equations. In [24], an extension of this approach to output feedback through the use of a Luenberger observer is developed. One relatively popular and practical method for controlling parabolic PDE systems has been backstepping [21] and its numerous extensions (e.g. [20], [43], [44], [42]). This method is attractive due to its straightforward explanation and implementation. However, it does have drawbacks including suboptimality due to the fixed structure of the controller and Lyapunov function. Additionally, we note some other recent use of Lyapunov functions for analysis and control of infinite dimensional systems including: a rotating beam [8]; quasilinear hyperbolic systems [7]; and control of systems governed by conservation laws [9].

Alternatively, Sturm-Liouville theory can also be used to devise stabilizing controllers for the class of PDEs we consider. In particular, the problem of searching for the eigenvalues of the differential operators defining the PDEs under consideration can be cast as a Sturm-Liouville eigenvalue problem. Thus, the eigenvalues of the differential operators can be found and consequently, stability properties can be inferred. Moreover, using the same approach, static output feedback controllers which stabilize the PDEs can also be found. Albeit rela-

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tively more complicated, the methodology presented in this work has various advantages over the Sturm-Liouville approach, chiefly among which is that we use Lyapunov functionals to achieve our results. Due to this, the presented work can be generalized to construct robust controllers for not only the systems under consideration, but also for nonlinear and uncertain PDEs. Moreover, the numerical examples provided in the paper show that the presented methodology is more effective in constructing stabilizing controllers.

To summarize, although there are a number of methods for control of PDEs, none of them are an ideal solution in the sense that if a controller exists, we have a practical and numerically efficient way to find it. Some previous work in this direction includes the use of Sum-of-Squares for stability analysis of nonlinear PDEs in [30] and was applied to fluid-flow in [46]. Additionally, the use of LMIs for stability analysis of semilinear parabolic and hyperbolic systems can be found in [13]. The results presented in this paper are a further step towards that ideal solution in the sense that the conditions are convex (meaning they are tractable) and asymptotically accurate (meaning that for any desired accuracy, we can find a convex set of conditions).

Specifically, in this paper, we consider a linear 1-D parabolic partial-differential equation with spatially- and temporally-varying coefficients. We focus on point actuation of Neumann-type boundary conditions, although the use of Dirichlet, Robin, or distributed inputs is also discussed. Our approach to controller synthesis is to use the semigroup framework to formulate the controller synthesis problem as a set of linear operator inequalities. These operator inequalities represent the conditions for existence of a decreasing quadratic Lyapunov function. For point observation, we use the Luenberger observer framework to construct additional inequalities which define the observer. Once we have defined our operator inequalities, we parameterize the set of solutions using operators with polynomial multipliers and kernels. This parametrization is convex and can be tested using recently developed methods for the optimization of positive polynomials such as Sum-of-Squares [37]. Some illustrative examples are also included. The results in this paper fall short of the ideal solution in that they rely on the Luenberger observer for state estimation - meaning the closed loop system may be suboptimal. In addition, the results in this paper cannot be directly applied to vector-valued PDE systems.

II. Notation

The set $\mathbb{R}^{m\times n}$ contains real matrices of dimensions m-by-n. The set $\mathbb{S}^n$ contains real symmetric matrices of dimension n-by-n. $C^1[X]$ is the space of continuously differentiable functions defined on X. The shorthand u_x denotes the partial derivative of u with respect to independent variable x. $(L_2[X])^n$ denotes the Hilbert space of Lebesgue measurable maps from X to $\mathbb{R}^n$. I_n is the identity matrix of dimension $n\times n$ and we denote $I=I_n$ when n is clear from context. We define $Z_d(x)$ to be the vector of monomials in variables x of degree d or less. We define $Z_{n,d}(x)=I_n\otimes Z_d(x)$ - the polynomial matrix whose rows form a basis for vector-valued polynomials of degree d or less.

Unless otherwise indicated, $\langle\cdot,\cdot\rangle$ denotes the inner product on L_2 and $\|\cdot\|=\|\cdot\|_{L_2}$ denotes the norm induced by the inner product. The Sobolev subspace of differentiable functions

$$H^n(0,1):=\{y\in L_2\,:\,y,\cdots,\frac{d^{n-1}y}{dt^{n-1}}\text{ are absolutely continuous with }\frac{d^ny}{dt^n}\in L_2(0,1)\}$$

is equipped with inner product $\langle x,y\rangle_{H^n}=\sum_{m=0}^n\left\langle\frac{d^mx}{dt^m},\frac{d^my}{dt^m}\right\rangle$. For Hilbert spaces X and Y, the set $\mathcal{L}(X,Y)$ includes bounded linear operators from X to Y endowed with the induced norm $\|\cdot\|_{\mathcal{L}}$.

III. Background

In this paper, we focus on the following class of parabolic PDE.

$$w_t(x,t)=a(x)w_{xx}(x,t)+b(x)w_x(x,t)+c(x)w(x,t),\quad x\in[0,1],\quad t\geq 0 \tag{1}$$

with mixed boundary conditions of the form

$$w(0,t)=0,\qquad w_x(1,t)=u(t). \tag{2}$$

For this paper, we assume w is scalar-valued ($w(x,t)\in\mathbb{R}$). Additionally, we assume that a, b and c are known polynomial functions with $a(x)\geq\alpha>0$, for $x\in[0,1]$. Note that the results of this paper can be readily modified to cover Dirichlet, Neumann or Robin-type boundary conditions or systems with time-varying uncertainty in the coefficients. In addition, note that conditions for well-posedness of this model under feedback have been established in, e.g. [47], [23], [22], [27].

In this paper, we will consider state-feedback of the form $u(t)=(\mathcal{F}w)(t)$ where $\mathcal{F}:H^1(0,1)\to\mathbb{R}$ is a bounded linear operator. It has been shown [13] that such feedback is well-posed with a unique local strong solution for any initial condition $w(x,0)=w_0(x)\in\mathcal{D}$, where we define the space

$$\mathcal{D}=\{z\in H^2(0,1):\quad z(0)=0,\quad z_x(1)=\mathcal{F}z\}. \tag{3}$$

For the purposes of stability analysis, we also define

$$\mathcal{D}_0=\{z\in H^2(0,1):\quad z(0)=0,\quad z_x(1)=0\}. \tag{4}$$

A. Sum-of-Squares Polynomials (SOSPs)

Sum-of-Squares (SOS) is an approach to the optimization of positive polynomial variables. A typical formalism for the polynomial optimization problem is given by

$$\max_x\ c^Tx,\quad\text{subject to}\quad\sum_{i=1}^m x_if_i(y)+f_0(y)\geq 0,$$

for all $y\in\mathbb{R}^n$, where the f_i are real polynomial functions. The key difficulty is that the feasibility problem of determining whether a polynomial is globally positive ($f(y)\geq 0$ for all $y\in\mathbb{R}^n$) is NP-hard [3]. To overcome this difficulty, there are a number of sufficient conditions for polynomial positivity. A particularly important such condition is that the polynomial, p, be a Sum-of-Squares,

$$p(x)=\sum_{i=1}^k g_i(x)^2,$$

where the g_i are polynomials and which is denoted $p\in\Sigma_s$. The importance of the SOS condition lies in the fact that it can be readily enforced using semidefinite programming. This is due to the easily proven fact that for a polynomial p of degree $2d$, $p\in\Sigma_s$ if and only if $p=Z(x)^TQZ(x)$ for some $Q\geq 0$, where $Z(x)$ is the vector of monomials of degree d or less. In this way, optimization of positive polynomials can be converted to semidefinite programming. The semidefinite-programming approach to polynomial positivity was described in the thesis work of [31] and also in [36]. See also [6] and [26] for contemporaneous work. MATLAB toolboxes for manipulation of SOS variables have been developed and can be found in [37] and [17].

SOS can also be used to optimize polynomials which are positive on a subset of $\mathbb{R}^n$ via Positivstellensatz (PS) results [45], [40], [38], [18]. To see this, consider a semialgebraic set

$$X:=\{x\in\mathbb{R}^n\,:\,g_i(x)\geq 0,\ \ i=1,\cdots,k\} \tag{5}$$

for polynomials g_i. A simplified form of PS result can be derived from [38] and summarized as follows.

Theorem 1. *For given polynomials g_i, suppose that X is defined as per Equation* (5)*. Further suppose that $\{x : g_i(x) \geq 0\}$ is compact for some i. If the polynomial f satisfies $f(x) > 0$ for $x \in X$, then there exist Sum-of-Squares polynomials $s_i \in \Sigma_s$ such that*

$$f(x) = s_0(s) + \sum_{i=1}^{m} s_i(s) g_i(s)$$

As an illustration of this result, suppose we can find Sums-of-Squares polynomials s_0 and s_1, such that $p\,(x) = s_0\,(x) + (1 - x^2)\; s_1\,(x)$. Then $p(x) \geq 0$ for $x^2 \leq 1$. The PS tells us that if p is strictly positive ($p(x) \geq \epsilon > 0$ for $x^2 \leq 1$), then such polynomials s_0 and s_1 will always exist. A summary of PS results can be found in [39].

IV. A Framework for Analysis and Synthesis of PDEs

The goal of this paper is to create a practical framework for controller synthesis akin to the LMI framework for ordinary differential equations. To motivate this approach, we recall some notation from the well-developed field of Semigroup theory discussed in the introduction. Within the semigroup framework are certain classes of systems which admit a continuously parameterized operator $S(t)$ which represents the solution map so that any solution $w(t)$ satisfies $S(s)w(t) = S(t+s)w(0)$. Associated with such systems is a possibly unbounded operator $\mathcal{A} : X \to Y$ known as the infinitesimal generator which satisfies $\dot{w}(t) = \mathcal{A}w(t)$ for any $w(t) = S(t)w(0)$ where X and Y are Hilbert spaces which depend on the system.

Although we do not explicitly use semigroup theory in this paper, it provides a convenient shorthand for presenting and interpreting our results. Specifically, for PDEs in the form of Equation (1), we define the first-order differential form

$$\dot{w}(t) = \mathcal{A}w(t) + \mathcal{B}u(t) \tag{6}$$

where the operator $\mathcal{A} : \mathcal{D}_0 \subset L_2(0,1) \to L_2(0,1)$ is defined as

$$(\mathcal{A}w)(x) := a(x)\frac{d^2}{dx^2}w(x) + b(x)\frac{d}{dx}w(x) + c(x)w(x), \tag{7}$$

and the space $\mathcal{D}_0$ has been defined in Equation (4). Moreover, analogous to the examples in [48] and [5], it can be established that

$$\begin{aligned}&(\mathcal{B}u(t))(x) = \delta_1(x)u(t) \quad \text{and}\\ &y(t) = \mathcal{C}w(t) = \langle \delta_1(\cdot), w(t)\rangle = w(1,t),\end{aligned}$$

where δ_1 is the Dirac delta functional centered at $x = 1$. It can be established that the operator $\mathcal{A}$, with domain $\mathcal{D}_0$, generates a strongly continuous semigroup $S(t)$ on $L_2(0,1)$ [10]. Let $\mathcal{D}_1 = \mathcal{D}_0$ with the norm $\|x\|_1 = \|(\alpha I - \mathcal{A})x\|$, $\alpha \in \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the resolvent set of $\mathcal{A}$. Additionally, let $\mathcal{D}_{-1}$ be the completion of $L_2(0,1)$ with respect to the norm $\|x\|_{-1} = \|(\alpha I - \mathcal{A})^{-1}x\|$. Then, it has been shown in [16], using the results presented in [29] and [49], that $\mathcal{B} \in \mathcal{L}(\mathbb{R}, \mathcal{D}_{-1})$ and $\mathcal{C} \in \mathcal{D}_1^\star$, the dual space of $\mathcal{D}_1$. Additionally, it has been shown that Equation (6) has a continuous state strong solution $w(\cdot) \in C([0,\infty]; L_2(0,1))$ for any $u \in L_2([0,T]; L_2(0,1))$, for all $0 < T < \infty$.

One of the advantages of the operator framework associated with the Semigroup approach is a simplified treatment of Lyapunov functions. Specifically, it is known [10] that the strongly continuous semigroup $S(t)$ generated by $\dot{w} = \mathcal{A}w$ is exponentially stable if and only if there exists a positive operator $\mathcal{P} : X \to X$ such that

$$\langle \mathcal{A}w, \mathcal{P}w\rangle_X + \langle \mathcal{P}w, \mathcal{A}w\rangle_X \leq -\|w\|^2. \tag{8}$$

We refer to the feasibility of Condition (8) as a *Linear Operator Inequality* (LOI). This condition in particular is equivalent to the existence of a decreasing Lyapunov function of the form $V(w) = \langle w, \mathcal{P}w\rangle_X$. Of course, there have been many Lyapunov stability tests proposed in the literature for analysis of infinite-dimensional systems. The goal of this paper, however, is to extend these results to controller and observer synthesis.

Roughly speaking, the approach we take in this paper is to formulate linear operator inequalities similar to Condition (8) and interpret these inequalities using Lyapunov functions of the form $V(w) = \langle w, \mathcal{P}w\rangle$ where the operator $\mathcal{P}$ is parameterized using polynomials. Positivity is enforced using Sum-of-Squares and the results in [32]. The sections in this paper are defined by the particular form of LOI problem which we hope to solve. Specifically, we have the following problems.

1) Stability
$$\langle \mathcal{A}w, \mathcal{P}w\rangle + \langle \mathcal{P}w, \mathcal{A}w\rangle \leq -\epsilon\|w\|^2,$$
2) Controller Synthesis
$$\langle (\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})w, w\rangle + \langle w, (\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})w\rangle \leq -\epsilon\|w\|^2, \tag{9}$$
3) Observer Synthesis
$$\langle (\mathcal{P}\mathcal{A} + \mathcal{V}\mathcal{C})w, \mathcal{P}w\rangle + \langle w, (\mathcal{P}\mathcal{A} + \mathcal{V}\mathcal{C})w\rangle \leq -\epsilon\|w\|^2, \tag{10}$$

for $w \in \mathcal{D}_0$. In the inequalities above, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are as defined previously. Furthermore, we parameterize the operators $\mathcal{P}$, $\mathcal{Z}$ and $\mathcal{V}$ as follows.

$$\begin{aligned}(\mathcal{P}w)(x) = M(x)w(x) + &\int_0^x K_1(x,y)w(y)dy\\ + &\int_y^1 K_2(x,y)w(y)dy,\end{aligned} \tag{11}$$

where $M(x) : [0,1] \to \mathbb{S}^n$ and $K_1(x,y), K_2(x,y) : [0,1]\times[0,1] \to \mathbb{R}^{n\times n}$ are polynomial matrices and $w \in L_2(0,1)^n$.

The operator $\mathcal{Z} : H^1(0,1) \to \mathbb{R}$ is parameterized using $R_1 \in \mathbb{R}$ and polynomial R_2 as

$$\mathcal{Z}w := R_1 w(1) + \int_0^1 R_2(y)w(y)dy. \tag{12}$$

The operator $\mathcal{V} : \mathbb{R} \to L_2(0,1)$ is parameterized using polynomial G_0 as

$$(\mathcal{V}r)\,(y) := G_0(y)r. \tag{13}$$

V. Positive operators and semi-separable polynomial kernels

In this paper, our results are expressed as optimization over a set of positive operators. To solve these optimization problems, we use positive matrices to parameterize a subset of positive operators on $(L_2(0,1))^n$ as described in [32]. We consider operators of the form

$$\begin{aligned}(\mathcal{P}w)(x) = M(x)w(x) + &\int_0^x K_1(x,y)w(y)dy\\ + &\int_x^1 K_2(x,y)w(y)dy,\end{aligned} \tag{14}$$

where $M(x) : [0,1] \to \mathbb{S}^n$ and $K_1(x,y), K_2(x,y) : [0,1]\times[0,1] \to \mathbb{R}^{n\times n}$ are polynomial matrices and $w \in L_2(0,1)^n$. In [34], we gave necessary and sufficient conditions for positivity of multiplier and integral operators of similar form using pointwise constraints on the functions M, K_1 and K_2. Recently, in [32], these conditions was sharpened - See Theorem 2.

Theorem 2. *Given $d_1, d_2, n \in \mathbb{N}$ and $\epsilon \in \mathbb{R}$, $\epsilon > 0$, let $Z_1(x) = Z_{n,d_1}(x)$ and $Z_2(x,y) = Z_{n,d_2}(x,y)$. Suppose there exists a matrix*

U such that

$$U = \begin{bmatrix} U_{11} - \epsilon I & U_{12} & U_{13} \\ \star & U_{22} & U_{23} \\ \star & \star & U_{33} \end{bmatrix} \geq 0,$$

where the U_{ij} are a partition of U. Let

$$\begin{aligned}
&M(s) = Z_1(x)^T Q_{11} Z_1(x),\\
&K_1(x,y) = Z_1(x)^T U_{12} Z_2(x,y) + Z_2(y,x) U_{31} Z_1(y)\\
&+ \int_0^y Z_2(\theta,x)^T U_{33} Z_2(\theta,y) d\theta + \int_y^x Z_2(\theta,x)^T U_{32} Z_2(\theta,y) d\theta\\
&+ \int_x^1 Z_2(\theta,x)^T U_{22} Z_2(\theta,y) d\theta,
\end{aligned}$$

and

$$\begin{aligned}
&K_2(x,y) = Z_1(x)^T U_{13} Z_2(x,y) + Z_2(y,x) U_{21} Z_1(y)\\
&+ \int_0^x Z_2(\theta,x)^T U_{33} Z_2(\theta,y) d\theta + \int_x^y Z_2(\theta,x)^T U_{23} Z_2(\theta,y) d\theta\\
&+ \int_y^1 Z_2(\theta,x)^T U_{22} Z_2(\theta,y) d\theta.
\end{aligned}$$

Then the operator $\mathcal{P}$, defined by Equation (14) *is self-adjoint and satisfies*

$$\langle \mathcal{P} w, w \rangle \geq \epsilon \|w\|^2, \text{ for all } w \in L_2(0,1)^n.$$

Proof: See [32] for a proof. ■

For convenience, we define the set of multipliers and kernels which satisfy Theorem 2.

$$\Xi_{\{d_1,d_2,\epsilon\}} = \{M, K_1, K_2 : M, K_1, K_2 \text{ satisfy the conditions of Theorem 2 for } d_1, d_2, \epsilon.\}$$

VI. Inverses of Positive Operators

As is the case for the finite-dimensional equivalents of Operator Inequalities (9) and (10), reconstruction of the controller ($u = \mathcal{F}w$) and observer ($\dot{\hat{w}} = \mathcal{A}\hat{w} + \mathcal{O}(\hat{y} - y)$) from a feasible solution of the LOI requires inversion of the operator $\mathcal{P}$ as $\mathcal{F} = \mathcal{Z}\mathcal{P}^{-1}$ and $\mathcal{O} = \mathcal{P}^{-1}\mathcal{V}$. Thus, if we are to use the parametrization of positive operators described in Section V, then given such a positive operator, we must have a reliable way of finding its inverse. For operators without joint positivity, this procedure has been presented in [35] and expanded in [33]. In this subsection, we further expand these results by proposing a numerical method for constructing inverses for the class of operators considered in Subsection V. Specifically, for scalar valued polynomials $M(x)$, $K_1(x,\xi)$ and $K_2(x,\xi)$ which satisfy the conditions of Theorem 2, we will provide a method to construct $\mathcal{P}^{-1}$.

Naturally, all positive operators in the sense of Theorem 2 are invertible. Our approach is to use a power series expansion with terms which are readily constructed from the matrices described in Theorem 2. A closely related result for operators which consist of the identity plus a Volterra operator can be found in [41, Sec 1.99]. Our case is slightly different in that we have a positive multiplier and the Volterra operator is combined with its transpose. Note that the conditions of this theorem are very conservative. In our experience, the series converges whenever $\mathcal{P}$ is positive.

Theorem 3. *Suppose $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ for some $d_1, d_2 \in \mathbb{N}$ and $\epsilon > 0$. Additionally assume that*

$$|K_1(x,y)| < \epsilon \quad \text{and} \quad |K_2(x,y)| < \epsilon \quad \text{for all } (x,y) \in [0,1]\times[0,1].$$

Then for the operator $\mathcal{P}$ defined as $\mathcal{P} = \mathcal{T} + \mathcal{S}$, where

$$(\mathcal{T}w)(x) = M(x)w(x) \text{ and}$$

$$(\mathcal{S}w)(x) = \int_0^x K_1(x,y)w(y)dy + \int_x^1 K_2(x,y)w(y)dy,$$

the inverse is given by

$$\mathcal{P}^{-1} = \left(\sum_{k=0}^{\infty} (-\mathcal{T}^{-1}\mathcal{S})^k\right)\mathcal{T}^{-1},$$

where

$$(\mathcal{T}^{-1}w)(x) = M(x)^{-1}w(x).$$

Proof: We begin by noting that since $M, K_1, K_2 \in \Xi_{d_1,d_2,\epsilon}$, $M(x) \geq \epsilon > 0$ for all $x \in [0,1]$. Thus

$$(\mathcal{T}^{-1}w)(x) = M(x)^{-1}w(x), \text{ for all } w \in L_2(0,1).$$

Consequently, $\mathcal{P} = \mathcal{T} + \mathcal{S} = \mathcal{T}(I + \mathcal{T}^{-1}\mathcal{S})$ is well defined. The small-gain theorem states that if $\|\mathcal{T}^{-1}\mathcal{S}\| < 1$ then $(I + \mathcal{T}^{-1}\mathcal{S})^{-1}$ exists, is bounded and is given by the convergent series

$$(I + \mathcal{T}^{-1}\mathcal{S})^{-1} = \sum_{k=0}^{\infty} (-\mathcal{T}^{-1}\mathcal{S})^k.$$

First we examine $\mathcal{T}^{-1}$.

$$\begin{aligned}
\|\mathcal{T}^{-1}\| &= \sup_{\|w\|=1} |\langle \mathcal{T}^{-1}w, w\rangle| = \sup_{\|w\|=1} \left| \int_0^1 \frac{w(x)^2}{M(x)} dx \right|\\
&\leq \frac{1}{\epsilon} \sup_{\|w\|=1} \int_0^1 w(x)^2 dx = \frac{1}{\epsilon}.
\end{aligned} \tag{15}$$

Now, looking at $\mathcal{S}$,

$$\begin{aligned}
\|\mathcal{S}\| = &\sup_{\|w\|=1} \left| \int_0^1 \int_0^x w(x)K_1(x,y)w(y)dydx \right.\\
&\left. + \int_0^1 \int_x^1 w(x)K_2(x,y)w(y)dydx \right|\\
\leq &\sup_{\|w\|=1} \left(\int_0^1 \int_0^x |w(x)||K_1(x,y)||w(y)|dydx \right.\\
&\left. + \int_0^1 \int_x^1 |w(x)||K_2(x,y)||w(y)|dydx \right).
\end{aligned}$$

By hypothesis we have that $|K_1(x,y)| < \epsilon$ and $|K_2(x,y)| < \epsilon$ and from the triangle, submultiplicative and Holder inequalities we have

$$\begin{aligned}
\|\mathcal{S}\| &< \epsilon \sup_{\|w\|=1} \left(\int_0^1 \int_0^x |w(x)||w(y)|dydx \right.\\
&\qquad \left. + \int_0^1 \int_x^1 |w(x)||w(y)|dydx \right)\\
&= \epsilon \sup_{\|w\|=1} \left(\int_0^1 \int_0^1 |w(x)||w(y)|dydx \right)\\
&= \epsilon \sup_{\|w\|=1} \left(\int_0^1 |w(x)|dx \right)^2\\
&\leq \epsilon \sup_{\|w\|=1} \int_0^1 (w(x))^2 dx = \epsilon.
\end{aligned} \tag{16}$$

Thus from (15) and (16),

$$\|\mathcal{T}^{-1}\mathcal{S}\| \leq \|\mathcal{T}^{-1}\|\|\mathcal{S}\| < 1.$$

Hence $(I + \mathcal{T}^{-1}\mathcal{S})^{-1} = \sum_{k=0}^{\infty} (-\mathcal{T}^{-1}\mathcal{S})^k$, which implies

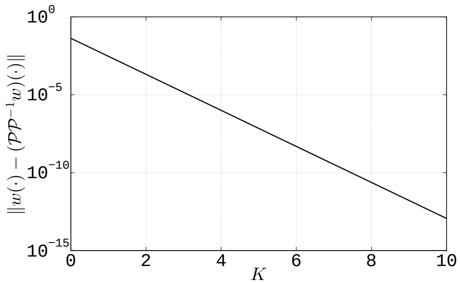
$$\mathcal{P}^{-1} = (\mathcal{T} + \mathcal{S})^{-1} = (I + \mathcal{T}^{-1}\mathcal{S})^{-1}\mathcal{T}^{-1} = \left(\sum_{k=0}^{\infty} (-\mathcal{T}^{-1}\mathcal{S})^k\right)\mathcal{T}^{-1}.$$

■

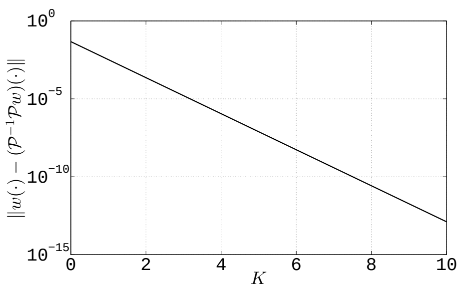
For convenience, we define the set of multipliers and kernels which satisfy the conditions of both Theorem 2 and Theorem 3.

$$\Omega_{d_1,d_2,\epsilon} = \{M, K_1, K_2 : M, K_1, K_2 \in \Xi_{d_1,d_2,\epsilon} \text{ and satisfy the conditions of Theorem 3 for } d_1, d_2, \epsilon\}.$$

To construct the inverse, then, we use the MuPAD symbolic engine of MATLAB to evaluate the series $\left(\sum_{k=0}^{K}\left(-\mathcal{T}^{-1}\mathcal{S}\right)^k\right)\mathcal{T}^{-1}$ for some finite K where K is chosen sufficiently large so that the series adequately approximates the inverse. In practice, we have found that only a few terms are required for convergence. To illustrate, in Figures 1(a) and 1(b) we find some $(M, K_1, M_2) \in \Omega_{2,2,2}$ and find $\mathcal{P}_K^{-1} = \left(\sum_{k=0}^{K}\left(-\mathcal{T}^{-1}\mathcal{S}\right)^k\right)\mathcal{T}^{-1}$ for several values of K. Then we plot $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|_{L_2}$ and $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|_{L_2}$ as a function of K for the arbitrarily chose function $w(x) = \sin(5\pi x)/(x+1)$. In this case, $K = 10$ yields norm error of order $\approx 10^{-12}$.



(a) $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|_{L_2}$



(b) $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|_{L_2}$

Fig. 1: $\|w - \mathcal{P}\mathcal{P}_K^{-1}w\|_{L_2}$ and $\|w - \mathcal{P}_K^{-1}\mathcal{P}w\|_{L_2}$ as a function of K.

VII. Stability Analysis

In this section, we address the simpler problem of stability of PDE systems of the Form (1). Roughly speaking, we are looking for a positive operator in the form of Equation (14) which satisfies the inequality

$$\langle \mathcal{A}x, \mathcal{P}x\rangle + \langle x, \mathcal{P}\mathcal{A}x\rangle \le -\epsilon\|x\|^2$$

for all $x \in \mathcal{D}_0$ where the operator $\mathcal{A}$ is defined in Equation (7). The main result relies primarily on the following upper-bound - the proof of which is included in the appendix.

$$\langle \mathcal{A}x, \mathcal{P}x\rangle + \langle x, \mathcal{P}\mathcal{A}x\rangle \le \left\langle \begin{bmatrix} x(1) \\ x \end{bmatrix}, \mathcal{Q}\begin{bmatrix} x(1) \\ x \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + \int_0^1 x_s(0)Q_3(s)x(s)ds, \tag{17}$$

where we define the operator $\mathcal{Q}$ as

$$(\mathcal{Q}y)(s) := Q_0(s)\begin{bmatrix} y(1) \\ y(s) \end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0 \\ 0 & Q_1(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt + \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & Q_2(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt,$$

where $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$ and where the linear operator $\mathcal{M}_\epsilon$ is defined as follows.

Definition 1. *We say* $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$ *if the following hold*

$$Q_0(s)_{1,1} = \left[(b(1) - a_s(1))\, M(1) - a(1)M_s(1)\right], \tag{18}$$

$$Q_0(s)_{1,2} = Q_0(s)_{2,1} = \left[(b(1) - a_s(1))\, K_1(1,s) - a(1)K_{1,s}(1,s)\right], \tag{19}$$

$$Q_0(s)_{2,2} = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)M(s)\right] - b(s)M(s)\right] + 2M(s)c(s) + \left[\frac{\partial}{\partial s}\left[2a(s)\left(K_1(s,t) - K_2(s,t)\right)\right]\right]_{t=s} - \frac{\pi^2}{2}\alpha\epsilon, \tag{20}$$

$$Q_1(s,t) = \left(\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)K_1(s,t)\right] - b(s)K_1(s,t)\right] + c(s)K_1(s,t)\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left[a(t)K_1(s,t)\right] - b(t)K_1(s,t)\right] + c(t)K_1(s,t)\right), \tag{21}$$

$$Q_2(s,t) = \left(\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)K_2(s,t)\right] - b(s)K_2(s,t)\right] + c(s)K_2(s,t)\right) + \left(\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left[a(t)K_2(s,t)\right] - b(t)K_2(s,t)\right] + c(t)K_2(s,t)\right) \text{ and} \tag{22}$$

$$Q_3(s) = -2a(0)K_2(0,s), \tag{23}$$

where $K_{1,s}(1,s) = \left[K_{1,s}(s,t)|_{s=1}\right]_{t=s}$.

Theorem 4. *Suppose that there exist* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ *and* $\epsilon, \delta > 0$ *such that*

$$\left\{-Q_{0_{2,2}} - 2\delta M, -Q_1 - 2\delta K_1, -Q_2 - 2\delta K_2\right\} \in \Xi_{d_1,d_2,0},$$
$$Q_{0_{1,1}} = 0, \quad Q_{0_{1,2}} = 0 \quad \text{and} \quad K_2(0,x) = 0,$$

where $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$*. Then, for any initial condition* $w(0) \in \mathcal{D}_0$*, the solution* $w(x,t)$ *of Equations* (1)-(2) *with* $u(t) = 0$ *satisfies*

$$\|w(\cdot,t)\|_{L_2} \le e^{-\delta t}\sqrt{\frac{\langle w_0, \mathcal{P}w_0\rangle}{\epsilon}}, \quad t > 0,$$

where

$$(\mathcal{P}z)(x) = M(x)z(x) + \int_0^x K_1(x,\xi)z(\xi)d\xi + \int_x^1 K_2(x,\xi)z(\xi)d\xi.$$

Proof: Consider the following Lyapunov function $V(w) = \langle w, \mathcal{P}w\rangle_{L_2}$. Taking the derivative along trajectories of the system, we have

$$\frac{d}{dt}V(w(t)) = \langle w_t(t), (\mathcal{P}w(t))\rangle + \langle w(t), (\mathcal{P}w_t(t))\rangle = \langle \mathcal{A}w(t), \mathcal{P}w(t)\rangle + \langle w(t), \mathcal{P}\mathcal{A}w(t)\rangle.$$

Since the initial condition $w(0) \in \mathcal{D}_0$, $w(t) \in \mathcal{D}_0$ exists for all $t \ge 0$. For $\mathcal{P}$ as defined in (14) and $\mathcal{M}_\epsilon$ as defined in Definition 1, it is shown in the Appendix that if $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$, then

$$\frac{d}{dt}V(w(t)) = \langle \mathcal{A}w(t), \mathcal{P}w(t)\rangle + \langle w(t), \mathcal{P}\mathcal{A}w(t)\rangle \le \left\langle \begin{bmatrix} w(1,t) \\ w(\cdot,t) \end{bmatrix}, \mathcal{Q}\begin{bmatrix} w(1,t) \\ w(\cdot,t) \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2(0,1)} + \int_0^1 w_x(0,t)Q_3(x)w(x,t)dx.$$

	$d=3$	4	5	6	7
$\delta=0.1$	0.55	2.19	2.35	2.36	2.36
$\delta=0.01$	0.59	2.19	2.448	2.451	2.452
$\delta=0.001$	0.59	2.19	2.457	2.46	2.461

TABLE I: Maximum λ as a function of polynomial degree, $d_1 = d_2 = d$ for $w_t = w_{xx} + \lambda w$ and different exponential decay rates δ.

Now, by definition, $Q_3(x) = -2a(0)K_2(0,x)$ and since by assumption $K_2(0,x) = 0$, we have $Q_3 = 0$. Moreover, since $Q_{0_{1,1}} = 0$ and $Q_{0_{1,2}} = Q_{0_{2,1}} = 0$, we have

$$\frac{d}{dt}V(w(t)) \leq \int_0^1 w(x,t)\Bigg(Q_0(x)_{2,2}(x)w(x,t) + \int_0^x Q_1(x,s)w(s,t)ds + \int_x^1 Q_2(x,s)w(s,t)ds\Bigg)dx.$$

Since

$$\left\{-Q_{0_{2,2}} - 2\delta M, -Q_1 - 2\delta K_1, -Q_2 - 2\delta K_2\right\} \in \Xi_{d_1,d_2,0},$$

we have that

$$\int_0^1 w(x,t)\left(Q_0(x)_{2,2}(x)w(x,t) + \int_0^x Q_1(x,s)w(s,t)ds + \int_x^1 Q_2(x,s)w(s,t)ds\right)dx \leq -2\delta \left\langle w(\cdot,t), \mathcal{P}w(\cdot,t)\right\rangle.$$

Hence we conclude that

$$\frac{d}{dt}V(w(t)) \leq -2\delta V(w(t)), \quad t > 0.$$

Integrating in time yields $\langle w(\cdot,t), (\mathcal{P}w)(\cdot,t)\rangle \leq e^{-2\delta t}\langle w_0, \mathcal{P}w_0\rangle$ and since, $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$, we have

$$\epsilon\|w(\cdot,t)\|^2 \leq \langle w(\cdot,t), (\mathcal{P}w)(\cdot,t)\rangle \leq e^{-2\delta t}\langle w_0, \mathcal{P}w_0\rangle, \quad t > 0$$

which implies

$$\|w(\cdot,t)\|_{L_2} \leq e^{-\delta t}\sqrt{\frac{\langle w_0, \mathcal{P}w_0\rangle}{\epsilon}}, \quad t > 0.$$

■

A. Stability Analysis Numerical Results

Example 1: To illustrate the accuracy of the stability test, we perform several numerical experiments. For the first test, we check the conditions of Theorem 4 on a system whose stability properties are known a priori - $w_t = w_{xx} + \lambda w$. The system is defined by Equations (1) - (2) with $u(t) = 0$, $a(x) = 1$, $b(x) = 0$ and $c(x) = \lambda$, where $\lambda > 0$. The analytic solution to this PDE is given by

$$w(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t}\langle w_0, \phi_n\rangle \phi_n(x),$$

where $\lambda_n = \lambda - \frac{(2n-1)^2\pi^2}{4}$, $\phi_n(x) = \sqrt{2}\sin\left(\frac{2n-1}{2}\pi x\right)$ and w_0 is the initial condition. Thus, one can see that the boundary-value problem is stable for $\lambda \in [0, \frac{\pi^2}{4})$. Table I presents the accuracy of Theorem 4 when applied to the problem of determination of the maximum stable λ. Note that an increase in the degree of polynomials $d = d_1 = d_2$ increases the accuracy of the test in terms of the maximum detectable stable value of λ. For degree 7, we can construct a Lyapunov function which proves stability for $\lambda = 2.461$, with $\delta = 0.001$, which is 99.74% of the stability margin $\frac{\pi^2}{4} = 2.4674$.

Example 2: For the second numerical test, we consider a completely arbitrary system defined by Equations (1) - (2) with $u(t) = 0$, $a(x) = x^3 - x^2 + 2$, $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$. Again, we seek to determine the maximum value of λ for which the system is exponentially stable. The maximum stable λ predicted by Theorem 4 is shown in Table II for $\epsilon = 0.001$. For this system, there is no analytic solution and hence if we wish to determine the accuracy of our results, we must use finite difference methods to simulate the system and hence estimate the true maximum stable value of λ. This work is presented in Figure 2, which suggests that the system is unstable for $\lambda > 4.66$. The maximum λ for which we can prove the exponential stability for is $\lambda = 4.62$, which is 99.14% of the predicted stability margin of 4.66. Finally, although the Lyapunov function generated for this system is too complicated for print, Figure 3 illustrates the evolution of this Lyapunov functional time derivative.

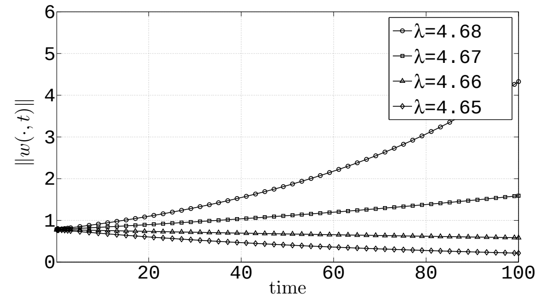


Fig. 2: State norm evolution for different λ for Example 2.

	$d=3$	4	5	6	7
$\delta=0.1$	4.27	4.51	4.51	4.52	4.52
$\delta=0.01$	4.36	4.60	4.60	4.61	4.61
$\delta=0.001$	4.37	4.61	4.61	4.62	4.62

TABLE II: Maximum stable λ as a function of polynomial degree for Example 2.

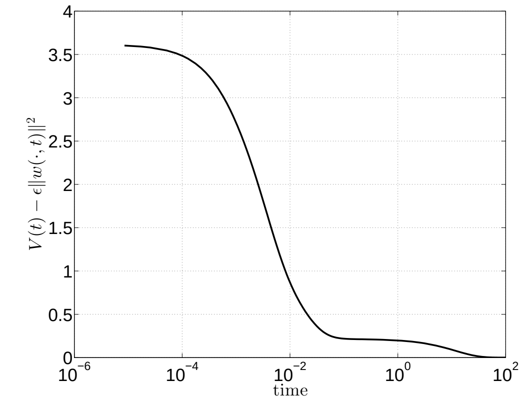
Example 3: In this numerical example we wish to examine if we achieve any performance improvement in the stability analysis by including the integral kernels K_1 and K_2 in the Lyapunov functional operator $\mathcal{P}$. Thus, we apply Theorem 4, with $K_1 = K_2 = 0$, on the systems considered in Examples 1 and 2. Table III presents the results. Comparing Table III to Tables I and II shows that for the system considered in Example 1, the integral kernels K_1 and K_2 do not have an effect. However, the inclusion of K_1 and K_2 increases the precision in predicting the stability margin for the system considered in Example 2. Thus, this numerical experiment indicates that for systems with distributed coefficients, including K_1 and K_2 produces sharper results for stability analysis.

Example 4: For the final numerical test, we wish to examine the effectiveness of the presented method on a system with different boundary conditions. In particular, we consider $w_t = w_{xx} + \lambda w$ with Dirichlet boundary conditions $w(0,t) = w(1,t) = 0$. The analytic solution of this PDE can be calculated as

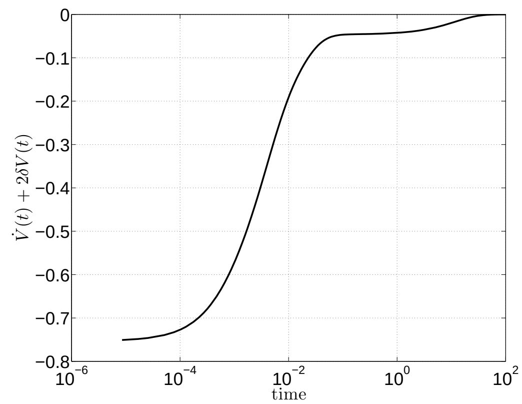
$$w(x,t) = \sum_{n=1}^{\infty} e^{\lambda_n t}\left\langle w_0, \phi_n\right\rangle \phi_n(x),$$

	Example 1	Example 2
λ	2.461	4.38

TABLE III: Maximum stable λ, for $K_1 = K_2 = 0$, for Examples 1 and 2 for $\delta = 0.001$.



(a) Illustration of $V(t) \geq \epsilon \|w(\cdot,t)\|^2$.



(b) Illustration of $\dot{V}(t) \leq -2\delta V(t)$.

Fig. 3: Evolution of the Lyapunov functional and its time derivative for $a(x) = x^3 - x^2 + 2$, $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$ with $\lambda = 4.62$ and $\delta = \epsilon = 0.001$.

	$d = 4$	5	6	7	8
$\delta = 0.1$	1.4	4.9	7.59	9.61	9.7
$\delta = 0.01$	1.5	5.1	7.69	9.63	9.79
$\delta = 0.001$	1.8	5.3	7.99	9.66	9.82

TABLE IV: Maximum λ as a function of polynomial degree, $d_1 = d_2 = d$ for $w_t = w_{xx} + \lambda w$ with Dirichlet boundary conditions and different exponential decay rates δ.

where $\lambda_n = \lambda - n^2\pi^2$, $\phi_n(x) = \sqrt{2}\sin(n\pi x)$ and $w_0(x)$ is the initial condition. Thus, the system is stable for $\lambda < \pi^2$. The conditions of Theorem 4 can be easily modified to analyze this system.

Table IV presents the accuracy of the modified Theorem 4 when applied to the problem of determination of the maximum stable λ for $w_t = w_{xx} + \lambda w$ with $w(0,t) = w(1,t) = 0$. For degree 8, we can construct a Lyapunov function which proves stability for $\lambda = 9.82$, with $\delta = 0.001$, which is 99.49% of the stability margin π^2.

VIII. State-Feedback Controller Synthesis

In this section, we use a dual version of the stability condition in Theorem 4 to synthesize full-state feedback controllers. Roughly speaking, the dual stability condition is expressed as the search for a positive operator, $\mathcal{P}$, of the form of Equation (14) which satisfies the inequality

$$\langle \mathcal{A}\mathcal{P}x, x\rangle + \langle x, \mathcal{A}\mathcal{P}x\rangle \leq -\epsilon\|x\|^2.$$

When we include an input of the form $w_x(1,t) = u(t) = \mathcal{F}w(t)$, this becomes

$$\langle (\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})x, x\rangle + \langle x, (\mathcal{A}\mathcal{P} + \mathcal{B}\mathcal{Z})x\rangle \leq -\epsilon\|x\|^2$$

where $\mathcal{F} = \mathcal{Z}\mathcal{P}^{-1}$. Recall the dynamics in Equation (1):

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) \tag{24}$$

with

$$w(0,t) = 0, \qquad w_x(1,t) = u(t) \tag{25}$$

with initial condition $w(\cdot,0) = w_0 \in \mathcal{D}$. As before our main result uses an upper-bound of the form

$$\begin{aligned}\langle \mathcal{A}\mathcal{P}x, x\rangle + \langle x, \mathcal{A}\mathcal{P}x\rangle \leq &\left\langle \begin{bmatrix} x(1) \\ x \end{bmatrix}, \mathcal{T}\begin{bmatrix} x(1) \\ x \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} \\ &+ x(0)(T_3x(0) + T_4x_s(0)),\end{aligned} \tag{26}$$

where the operator $\mathcal{T}$ is defined as

$$\begin{aligned}(\mathcal{T}y)(s) := T_0(s)\begin{bmatrix} y(1) \\ y(s) \end{bmatrix} + &\int_0^s \begin{bmatrix} 0 & 0 \\ 0 & T_1(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt \\ + &\int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & T_2(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt,\end{aligned}$$

where $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$ and where the linear operator $\mathcal{N}_\epsilon$ is defined as follows.

Definition 2. *We say* $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$ *if*

$$T_0(s)_{1,1} = \left[-a(1)M_s(1) + (b(1) - a_s(1))M(1)\right], \tag{27}$$

$$T_0(s)_{1,2} = T_0(s)_{2,1} = -a(1)K_{1,s}(1,s), \tag{28}$$

$$\begin{aligned}T_0(s)_{2,2} = &\left[(a_{ss}(s) - b_s(s))M(s) + b(s)M_s(s)\right] + 2M(s)c(s) \\ &+ a(s)\left[M_{ss}(s) + 2\frac{\partial}{\partial s}\left[K_1(s,t) - K_2(s,t)\right]\right]_{t=s} \\ &- \frac{\pi^2}{2}\alpha\epsilon,\end{aligned} \tag{29}$$

$$\begin{aligned}T_1(s,t) = &a(s)K_{1,ss}(s,t) + b(s)K_{1,s}(s,t) + c(s)K_1(s,t) \\ &+ a(t)K_{1,tt}(s,t) + b(t)K_{1,t}(s,t) + c(t)K_1(s,t),\end{aligned} \tag{30}$$

$$\begin{aligned}T_2(s,t) = &a(s)K_{2,ss}(s,t) + b(s)K_{2,s}(s,t) + c(s)K_2(s,t) \\ &+ a(t)K_{2,tt}(s,t) + b(t)K_{2,t}(s,t) + c(t)K_2(s,t),\end{aligned} \tag{31}$$

$$T_3 = a_x(0)M(0) - a(0)M_x(0) - b(0)M(0) + \frac{\pi^2}{2}\alpha\epsilon \text{ and} \tag{32}$$

$$T_4 = -2a(0)M(0). \tag{33}$$

Theorem 5 (Dual Stability)**.** *Suppose there exist* $\{M, K_1, K_2\} \in \Omega_{d_1,d_2,\epsilon}$ *and* $\epsilon, \delta > 0$ *such that*

$$\begin{aligned}&\left\{-T_{0_{2,2}} - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\right\} \in \Xi_{d_1,d_2,0}, \\ &T_{0_{1,1}} = 0, \quad T_{0_{1,2}} = 0 \quad \text{and} \quad K_2(0,x) = 0,\end{aligned}$$

where $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$.

Then any solution w *of* (24) - (25) *with* $u(t) = 0$ *and* $w_0 \in \mathcal{D}_0$ *satisfies*

$$\|w(\cdot,t)\| \leq \|P\|_{\mathcal{L}} e^{-\delta t}\sqrt{\frac{\langle w_0, P^{-1}w_0\rangle}{\epsilon}},$$

where

$$(\mathcal{P}v)(x) = M(x)v(x) + \int_0^x K_1(x,\xi)v(\xi)d\xi + \int_x^1 K_2(x,\xi)v(\xi)d\xi.$$

The proof of Theorem 5 will be implied by the proof of Theorem 6.

Theorem 6 (Controller Synthesis). *For $\epsilon, \delta > 0$, $d_1, d_2 \in \mathbb{N}$, suppose there exist $\{M, K_1, K_2\} \in \Omega_{d_1,d_2,\epsilon}$ such that*

$$\left\{-T_{0_{2,2}} - 2\delta M, -T_1 - 2\delta K_1, -W_2 - 2\delta K_2\right\} \in \Xi_{d_1,d_2,0} \text{ and } K_2(0,x) = 0,$$

where $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$.

Define the operator $\mathcal{F} := \mathcal{Z}\mathcal{P}^{-1}$ where

$$(\mathcal{Z}y) = R_1 y(1) + \int_0^1 R_2(x) y(x) dx,$$
$$R_1 = -\frac{T_{0_{1,1}}}{2a(1)}, \quad R_2 = -\frac{T_{0_{1,2}}}{a(1)}.$$

Then any solution w of (24) - (25) with $u(t) = (\mathcal{F}w)(t)$ and $w_0 \in \mathcal{D}$ satisfies

$$\|w(\cdot,t)\| \le \|\mathcal{P}\|_{\mathcal{L}} e^{-\delta t} \sqrt{\frac{\langle w_0, \mathcal{P}^{-1} w_0\rangle}{\epsilon}}, \quad t > 0.$$

Proof: Consider the following Lyapunov function $V(w) = \langle w, \mathcal{P}^{-1} w\rangle$. Taking the time derivative along trajectories of the system, we have

$$\frac{d}{dt} V(w(t)) = \left\langle \mathcal{A}w(t), \mathcal{P}^{-1} w(t)\right\rangle + \left\langle \mathcal{P}^{-1} w(t), \mathcal{A}w(t)\right\rangle,$$

where we have used the fact that $\mathcal{P} = \mathcal{P}^\star$ implies $\mathcal{P}^{-1} = (\mathcal{P}^\star)^{-1}$. Now let $y = \mathcal{P}^{-1} w$. Then $y \in P^{-1}\mathcal{D}$ and

$$\frac{d}{dt} V(w(t)) = \langle \mathcal{A}\mathcal{P}y(t), y(t)\rangle + \langle y(t), \mathcal{A}\mathcal{P}y(t)\rangle.$$

From Corollary 2, we have

$$\begin{aligned}
&\frac{d}{dt} V(w(t)) = \langle \mathcal{A}\mathcal{P}y(t), y(t)\rangle + \langle y(t), \mathcal{A}\mathcal{P}y(t)\rangle \\
&\le \left\langle \begin{bmatrix} y(1,t) \\ y(\cdot,t)\end{bmatrix}, \mathcal{T}\begin{bmatrix} y(1,t) \\ y(\cdot,t)\end{bmatrix}\right\rangle_{\mathbb{R}\times L_2(0,1)} \\
&\quad + y(0,t)(T_3 y(0,t) + T_4 y_x(0,t)) + 2y(1,t)a(1)M_x(1)y(1,t) \\
&\quad + 2y(1,t)a(1)\left(\int_0^1 K_{1,x}(1,x) y(x,t) dx + M(1) y_x(1,t)\right).
\end{aligned} \tag{34}$$

Since $w = \mathcal{P}y$, we have

$$w(x,t) = M(x)y(x,t) + \int_0^x K_1(x,\xi) y(\xi,t) d\xi + \int_x^1 K_2(x,\xi) y(\xi,t) d\xi.$$

Thus boundary condition $w(0,t) = 0$ and the hypothesis $K_2(0,x) = 0$ imply

$$y(0,t) = 0. \tag{35}$$

Similarly, $u(t) = w_x(1,t)$ implies

$$u(t) = M(1) y_x(1,t) + M_x(1) y(1,t) + \int_0^1 K_{1,x}(1,x) y(x,t) dx.$$

Combining this with $u(t) = (\mathcal{F}w)(t) = (\mathcal{Z}\mathcal{P}^{-1}w)(t) = (\mathcal{Z}y)(t)$, we obtain

$$\begin{aligned}
&(R_1 - M_x(1)) y(1,t) + \int_0^1 R_2(x) y(x,t) dx \\
&= \int_0^1 K_{1,x}(1,x) y(x,t) dx + M(1) y_x(1,t).
\end{aligned} \tag{36}$$

Substituting (35) and (36) into (34) and using the definitions of R_1 and $R_2(x)$ produces

$$\begin{aligned}
&\frac{d}{dt} V(w(t)) = \langle \mathcal{A}\mathcal{P}y(t), y(t)\rangle + \langle y(t), \mathcal{A}\mathcal{P}y(t)\rangle \\
&\le \int_0^1 y(x,t)\Bigg(T_0(x)_{2,2} y(x,t) + \int_0^x T_1(x,s) y(s,t) ds \\
&\qquad + \int_x^1 T_2(x,s) y(s,t) ds\Bigg) dx,
\end{aligned}$$

where we have used the fact that R_1 and $R_2(x)$ cancel the boundary terms $T_{0_{1,1}}$ and $T_{0_{1,2}}$. From the Theorem hypotheses,

$$\left\{-T_{0_{2,2}} - 2\delta M, -T_1 - 2\delta K_1, -T_2 - 2\delta K_2\right\} \in \Xi_{d_1,d_2,0}.$$

Thus we conclude that

$$\frac{d}{dt} V(w(t)) \le -2\delta V(w(t)), \quad t > 0.$$

Integrating in time yields

$$\begin{aligned}
V(w(t)) \le e^{-2\delta t} V(w(0)) \Rightarrow &\langle \mathcal{P}y(\cdot,t), y(\cdot,t)\rangle \\
&\le e^{-2\delta t} \langle w_0, \mathcal{P}^{-1} w_0\rangle.
\end{aligned}$$

Since $\{M, K_1, K_2\} \in \Xi_{d_1, d_2 \epsilon}$, $\epsilon\|y(\cdot,t)\|^2 \le \langle \mathcal{P}y(\cdot,t), y(\cdot,t)\rangle$ and thus

$$\|y(\cdot,t)\| \le e^{-\delta t} \sqrt{\frac{\langle w_0, \mathcal{P}^{-1} w_0\rangle}{\epsilon}}.$$

Hence,

$$\begin{aligned}
\|w(\cdot,t)\| = \|(\mathcal{P}y)(\cdot,t)\| &\le \|\mathcal{P}\|_{\mathcal{L}} \|y(\cdot,t)\| \\
&\le \|\mathcal{P}\|_{\mathcal{L}} e^{-\delta t} \sqrt{\frac{\langle w_0, \mathcal{P}^{-1} w_0\rangle}{\epsilon}}.
\end{aligned}$$

Which concludes the proof. ■

A. Numerical Results for Full-State Feedback Synthesis

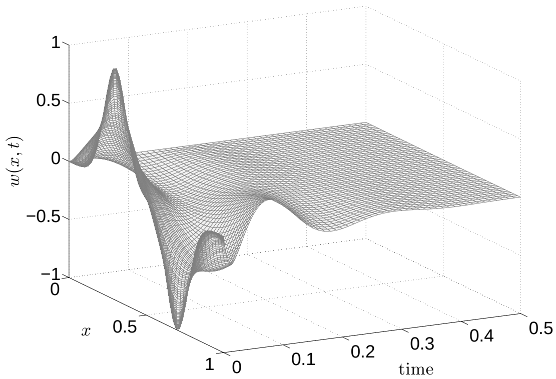
Example 5: In this example, we apply Theorem 6 to Example 2 from the section on stability analysis. Specifically, System (24) - (25) with $a(x) = x^3 - x^2 + 2$ and $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$. Table V presents the maximum λ, for which a controller can be constructed, as a function of degree $d = d_1 = d_2$. The maximum λ for which we can construct an exponentially stabilizing controller for is $\lambda = 35$, which is 651.1% increase over the stability margin of 4.66 which was predicted using finite-difference methods in the previous section. A static controller of the form $u(t) = -kw(1,t)$, $k > 0$, can also be devised using Sturm-Liouville theory [12, Chapter 5]. Such a static controller can stabilize the system for $\lambda < 17.58$. The presented methodology can stabilize the system for $\lambda = 35$, which is an increase of 99.09% over $\lambda = 17.58$.

	$d=4$	5	6	7
λ	15	18	25.9	35

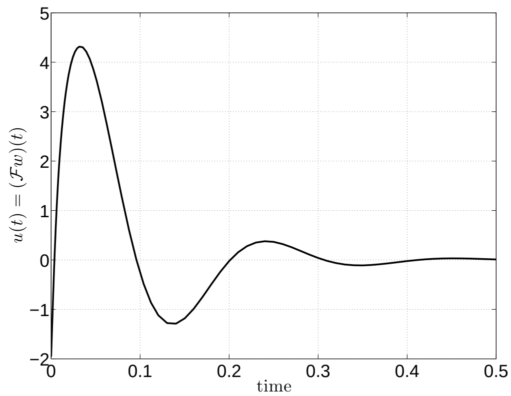
TABLE V: Maximum λ under feedback as a function of polynomial degree, $d = d_1 = d_2$ for Example 5 with $\delta = 0.1$ and $\epsilon = 0.001$.

Figure 4 illustrates the state evolution of the controlled system for $\lambda = 35$, $\delta = 0.1$ and $\epsilon = 0.001$ and the required control effort. Finally, Figure 5 illustrates the Lyapunov functional and its time derivative for the controlled system. The initial condition is chosen arbitrarily as

$$w_0(x) = e^{-\frac{(x-0.3)^2}{2(0.07)^2}} - e^{-\frac{(x-0.7)^2}{2(0.07)^2}}. \tag{37}$$

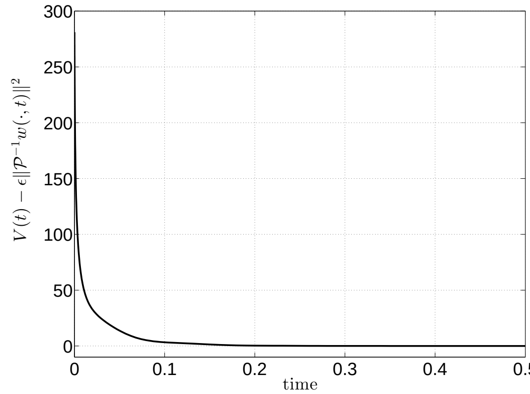


(a) State evolution

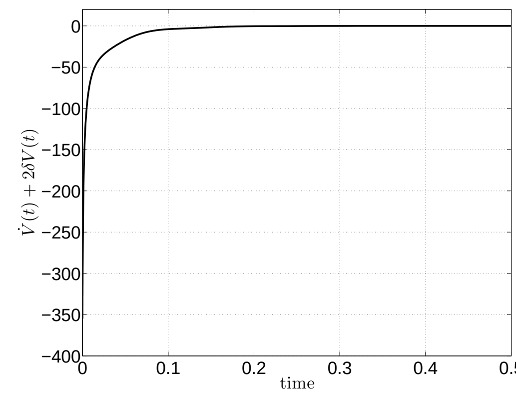


(b) Control effort $u(t) = (\mathcal{F}w)(t) = (\mathcal{Z}\mathcal{P}^{-1}w)(t)$

Fig. 4: Evolution of state and input for $a(x) = x^3 - x^2 + 1$ and $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$ with $\lambda = 35$ and $\delta = 0.1$ in Example 5.



(a) Illustration of $V(t) \geq \epsilon\|w(\cdot,t)\|^2$.



(b) Illustration of $\dot{V}(t) \leq -2\delta V(t)$.

Fig. 5: Lyapunov functional and its derivative for the controlled system with $\delta = 0.1$ and $\epsilon = 0.001$.

Example 6: In this example, we apply Theorem 6 to System (24) - (25) with $a(x) = x^3 - x^2 + 2$ and $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 6.7$. These values render the system unstable as verified by numerical simulation in Figure 6. We wish to find the maximum exponential decay rate δ for which we can construct a controller. Table VI presents the results. As we see, the maximum δ

	$d = 4$	5	6	7
δ	1.7	2.9	20.9	22

TABLE VI: Maximum decay rate δ under feedback as a function of polynomial degree, $d = d_1 = d_2$ for Example 5 with $\epsilon = 0.001$.

for which we can construct an exponentially stabilizing controller is $\delta = 22$. This is an increase of 89.98% over $\delta = 11.58$ for which an exponentially stabilizing controller can be constructed using Sturm-Liouville theory.

Example 7: The presence of the integral kernels K_1 and K_2 in the Lyapunov functional operator $\mathcal{P}$ necessitates the inclusion of R_2 in the control operator $\mathcal{Z}$. As a result, if we wish to use this controller with only an output, instead of the complete state, available for design, an observer is required to be constructed. Thus, it is important to establish the performance improvement gained by the inclusion of K_1, K_2 and R_2. For this purpose, we compare the results obtained in Example 5 to the results obtained for a simple static output feedback based controller which is achieved by setting $K_1 = K_2 = 0$ and $R_2 = 0$. We apply Theorem 6, for $K_1 = K_2 = 0$ and $R_2 = 0$, on the System considered in Example 5, that is, with $a(x) = x^3 - x^2 + 2$ and $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$. Table VII presents the maximum λ, for which a static controller can be constructed, as a function of degree $d = d_1 = d_2$. Upon comparing

	$d = 4$	5	6	7
λ	9.1	9.24	9.24	9.24

TABLE VII: Maximum λ, for $K_1 = K_2 = 0$ and $R_2 = 0$, as a function of polynomial degree, $d = d_1 = d_2$ for Example 7 with $\delta = 0.1$ and $\epsilon = 0.001$.

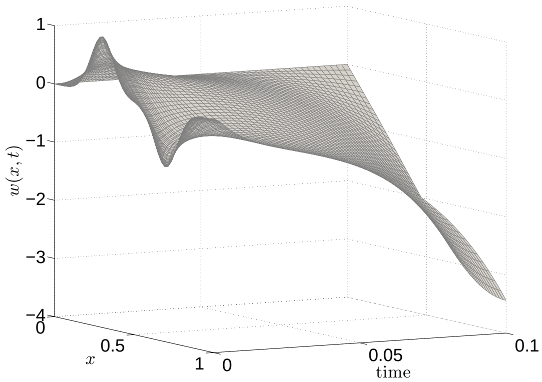
these results with the ones presented in Table V, it is evident that the inclusion of K_1, K_2 and R_2 produces much sharper results.

IX. Observer Synthesis

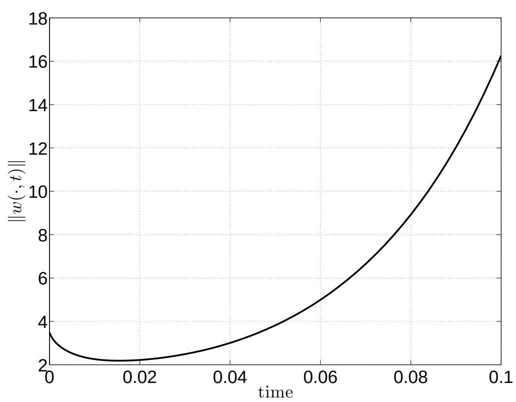
Recall the dynamics of System (1):

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) \qquad (38)$$

with output $z(t) = \mathcal{C}w(t) = w(1,t)$. Because of the infinite-dimensional nature of PDEs of the Form (1), real-time measurement of the state is not possible. For this reason, any realistic approach to control must include an observer and must account for the error dynamics in the closed-loop response. The simplest form of observer for which it is possible to verify closed-loop stability is the Luenberger



(a) State evolution



(b) State norm evolution

Fig. 6: Evolution of autonomous state for $a(x) = x^3 - x^2 + 2$ and $b(x) = 3x^2 - 2x$ and $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 6.7$.

observer. In our version of the Luenberger observer, the dynamics of the state estimate, $\hat{w}$ are defined by operator $\mathcal{O} : L_2(0,1) \to L_2(0,1)$ and $O_1 \in \mathbb{R}$ as

$$\begin{aligned}\hat{w}_t(x,t) =& a(x)\hat{w}_{xx}(x,t) + b(x)\hat{w}_x(x,t) + c(x)\hat{w}(x,t)\\ &+ (\mathcal{O}(\hat{z}(t) - z(t)))(x),\end{aligned} \tag{39}$$

where $\hat{z}(t) = \mathcal{C}\hat{w}(t) = \hat{w}(1,t)$ with boundary conditions

$$\hat{w}(0,t) = 0, \qquad \hat{w}_x(1,t) = O_1(\hat{z}(t) - z(t)) + u(t), \tag{40}$$

where recall that in feedback $u(t) = \mathcal{F}\hat{w}(t)$ and hence the state itself satisfies

$$w_t(x,t) = a(x)w_{xx}(x,t) + b(x)w_x(x,t) + c(x)w(x,t) \tag{41}$$

with output $z(t) = w(1,t)$ and boundary conditions

$$w(0,t) = 0, \qquad w_x(1,t) = u(t) = \mathcal{F}\hat{w}(t). \tag{42}$$

A block-diagram of the coupled dynamics can be found in Figure 7.

For the coupled dynamics, we consider the following coupled initial conditions

$$w(x,0) = w_0(x) \in H^2(0,1) \quad \text{and} \quad \hat{w}(x,0) = \hat{w}_0(x) \in H^2(0,1), \tag{43}$$

where we assume the initial conditions are consistent with the equations as .

$$w_0(0) = 0, \quad \hat{w}_0(0) = 0, \quad w_{0,x}(1) = \mathcal{F}\hat{w}_0, \quad \text{and}$$

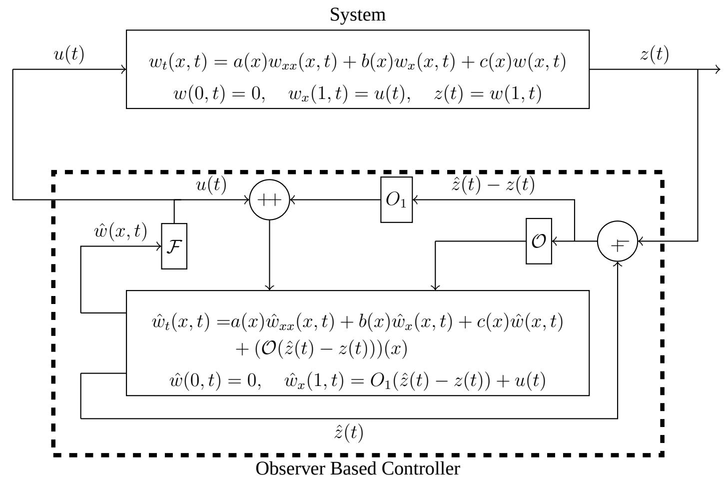


Fig. 7: Schema representing the coupled dynamics (39)-(42)

$$\hat{w}_{0,x}(1) = O_1(\hat{w}_0(1) - w_0(1)) + \mathcal{F}\hat{w}_0. \tag{44}$$

In finite-dimensional systems, the Luenberger observer has the property that the eigenvalues of the closed-loop system is the union of the eigenvalues of $\mathcal{A} + \mathcal{LC}$ and the eigenvalues of $\mathcal{A} + \mathcal{BF}$. This implies that stability in closed-loop is equivalent to stability of these two subsystems.

In the following theorem, we prove the analogue of this result for System (1) in feedback using the Luenberger observer. Our conditions have the form of the following Linear Operator Inequality.

$$\langle(\mathcal{AP} + \mathcal{BZ})x, x\rangle + \langle x, (\mathcal{AP} + \mathcal{BZ})x\rangle \le -\epsilon\|x\|^2 \tag{45}$$

$$\langle(\mathcal{PA} + \mathcal{VC})x, \mathcal{P}x\rangle\, \langle x, (\mathcal{PA} + \mathcal{VC})x\rangle \le -\epsilon\|x\|^2 \tag{46}$$

Theorem 7. *Suppose there exist*

$$\{M_c, K_{1,c}, K_{2,c}\} \in \Omega_{d_1,d_2,\epsilon}, \quad \{M_o, K_{1,o}, K_{2,o}\} \in \Omega_{d_1,d_2,\epsilon}$$

and $\epsilon, \delta > 0$, *such that*

$$\begin{aligned}&\left\{-T_{0_{2,2}} - 2\delta M_c, -T_1 - 2\delta K_{1,c}, -T_2 - 2\delta K_{2,c}\right\} \in \Xi_{d_1,d_2,0},\\ &\left\{-Q_{0_{2,2}} - 2\delta M_o, -Q_1 - 2\delta K_{1,o}, -Q_2 - 2\delta K_{2,o}\right\} \in \Xi_{d_1,d_2,0},\\ &K_{2,o}(0,x) = 0 \quad \textit{and} \quad K_{2,c}(0,x) = 0.\end{aligned}$$

where

$$\begin{aligned}(P_c v)(x) = M_c(x)v(x) &+ \int_0^x K_{1,c}(x,\xi)v(\xi)d\xi\\ &+ \int_x^1 K_{2,c}(x,\xi)v(\xi)d\xi,\\ (P_o v)(x) = M_o(x)v(x) &+ \int_0^x K_{1,o}(x,\xi)v(\xi)d\xi\\ &+ \int_x^1 K_{2,o}(x,\xi)v(\xi)d\xi,\end{aligned}$$

$$\begin{aligned}\{T_0, T_1, T_2, T_3, T_4\} =& \ \mathcal{N}_\epsilon(M_c, K_{1,c}, K_{2,c}), \ \textit{and}\\ \{Q_0, Q_1, Q_2, Q_3\} =& \ \mathcal{M}_\epsilon(M_o, K_{1,o}, K_{2,o}).\end{aligned}$$

Let

$$\mathcal{F}w := \mathcal{Z}\mathcal{P}_c^{-1}w \qquad \textit{and} \qquad \mathcal{O}w := \mathcal{P}_o^{-1}\mathcal{V}w$$

where

$$(\mathcal{Z}y) = R_1 y(1) + \int_0^1 R_2(x)y(x)dx, \quad R_1 = -\frac{T_{0_{1,1}}}{2a(1)},$$

$$R_2 = -\frac{T_{0_{1,2}}}{a(1)}, \quad O_1 = \frac{1}{2a(1)M_o(1)}\left(a_x(1)M(1) + a(1)M_{o,x}(1)\right),$$

and

$$\mathcal{V}r = [(a_x(1) - O_1 a(1) - b(1))\, K_{1,o}(1,x) + a(1) K_{1,o,x}(1,x)]\, r.$$

Then, for initial conditions w_0 and $\hat{w}_0$ given in (43)-(44)*, there exists a constant $\omega > 0$ such that any solution $\{w, \hat{w}\}$ of* (39)-(42) *satisfies*

$$\left\| \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} \right\|_{L_2(0,1)} \leq \omega e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ \hat{w}_0 \end{bmatrix} \right\|_{L_2(0,1)}.$$

Proof: We begin by defining the state estimation error $e(x,t) = \hat{w}(x,t) - w(x,t)$, the dynamics of which are given by

$$e_t(x,t) = a(x) e_{xx}(x,t) + b(x) e_x(x,t) + c(x) e(x,t) + (\mathcal{O}e(1,t))(x) \tag{47}$$

with boundary conditions

$$e(0,t) = 0, \qquad e_x(1,t) = O_1 e(1,t). \tag{48}$$

For the error system, we define the following Lyapunov functional

$$V(e(t)) = \langle e(t), \mathcal{P}_o e(t) \rangle.$$

Taking the time derivative yields

$$\begin{aligned} \frac{d}{dt} V(t) =& \langle e_t(t), \mathcal{P}_o e(t) \rangle + \langle e(t), \mathcal{P}_o e_t(t) \rangle \\ =& \langle \mathcal{A} e(t), \mathcal{P}_o e(t) \rangle + \langle e(t), \mathcal{P}_o \mathcal{A} e(t) \rangle + 2 \langle \mathcal{O} e(1,t), \mathcal{P}_o e(t) \rangle. \end{aligned}$$

Let $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M_o, K_{1,o}, K_{2,o})$ then $K_{2,o}(0,x) = 0$ implies $Q_3(x) = 0$, and hence Corollary 1 and $e_x(1,t) = O_1(\hat{z}(t) - z(t)) = O_1 e(1,t)$ imply

$$\begin{aligned} &\frac{d}{dt} V(t) \\ &\leq \left\langle \begin{bmatrix} e(1,t) \\ e(\cdot,t) \end{bmatrix}, \mathcal{Q} \begin{bmatrix} e(1,t) \\ e(\cdot,t) \end{bmatrix} \right\rangle_{\mathbb{R} \times L_2(0,1)} \\ &\quad + 2 \int_0^1 (\mathcal{O} e(1,t))(x) (\mathcal{P}_o e)(x,t) dx + 2 O_1 a(1) M_o(1) e^2(1,t) \\ &\quad + 2 e(1,t) O_1 \int_0^1 a(1) K_{1,o}(1,x) e(x,t) dx. \end{aligned} \tag{49}$$

where

$$\begin{aligned} (\mathcal{Q} y)(s) :=& Q_0(s) \begin{bmatrix} y(1) \\ y(s) \end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0 \\ 0 & Q_1(s,t) \end{bmatrix} \begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt \\ &+ \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & Q_2(s,t) \end{bmatrix} \begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt. \end{aligned}$$

Now,

$$\int_0^1 (\mathcal{O} e(1,t))(x) (\mathcal{P}_o e)(x,t) dx = \int_0^1 (\mathcal{P}_o \mathcal{O} e(1,t))(x) e(x,t) dx.$$

and $\mathcal{V} = \mathcal{P}_o \mathcal{O}$ implies

$$\begin{aligned} &\int_0^1 (\mathcal{O} e(1,t))(x) (\mathcal{P}_0 e)(x,t) dx = \int_0^1 (\mathcal{V} e(1,t))(x) e(x,t) dx \\ &= e(1,t) \int_0^1 \Big((a_x(1) - O_1 a(1) - b(1)) K_{1,o}(1,x) \\ &\qquad\qquad + a(1) K_{1,o,x}(1,x) \Big) e(x,t) dx. \end{aligned} \tag{50}$$

Substituting Equation (50) into (49), yields

$$\begin{aligned} &\frac{d}{dt} V(t) \\ &\leq \int_0^1 e(x,t) \Big(Q_0(x)_{2,2} e(x,t) + \int_0^x Q_1(x,s) e(s,t) ds \\ &\qquad\qquad + \int_x^1 Q_2(x,s) e(s,t) ds \Big) dx, \end{aligned}$$

where, the boundary terms have been canceled due to $\mathcal{V}$ and O_1.

Since we have

$$\left\{ -Q_{0_{2,2}} - 2\delta M_o, -Q_1 - 2\delta K_{1,o}, -Q_2 - 2\delta K_{2,o} \right\} \in \Xi_{d_1, d_2, 0},$$

we conclude that

$$\frac{d}{dt} V(t) \leq -2\delta V(t), \quad t > 0.$$

Since $\{M_o, K_{1,o}, K_{2,o}\} \in \Xi_{d_1, d_2, \epsilon}$, we have

$$\|e(\cdot,t)\| \leq e^{-\delta t} \sqrt{\frac{\langle e_0, \mathcal{P}_o e_0 \rangle}{\epsilon}}, \quad t > 0.$$

Now, since the state satisfies

$$w_t(x,t) = a(x) w_{xx}(x,t) + b(x) w_x(x,t) + c(x) w(x,t) \tag{51}$$

with $w(0,t) = 0$ and $w_x(1,t) = \mathcal{F}\hat{w}(t) = \mathcal{F}w(t) + \mathcal{F}e(t)$ then by applying $\mathcal{P}_c$ to Theorem 6, we conclude exponential stability of the coupled system. which implies the existence of an $\omega > 0$ such that

$$\left\| \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} \right\|_{L_2(0,1)^2} \leq \omega e^{-\delta t} \left\| \begin{bmatrix} w_0 \\ \hat{w}_0 \end{bmatrix} \right\|_{L_2(0,1)^2}.$$

■

Note that in this theorem we have chosen a common positivity margin $\epsilon > 0$ and exponential decay rate $\delta > 0$ for the controller and observer synthesis conditions. In practice, it is customary to choose a faster decay rate for the observer than the controller. In this case, the conditions should be modified accordingly.

A. Observer Synthesis Numerical Results

a) Example 8: In this final section, we perform numerical experiments on the same example presented in Section VIII-A. Specifically, we apply Theorem 7 to System (39)-(42) with $a(x) = x^3 - x^2 + 2$, $b(x) = 3x^2 - 2x$, $c(x) = -0.5x^3 + 1.3x^2 - 1.5x + 0.7 + \lambda$. The results presented here are simulations obtained using the observer based controller $u(t) = (\mathcal{F}\hat{w})(t)$ given by the conditions of Theorem 7 and obtained using the operator inversion technique described in Theorem 3.

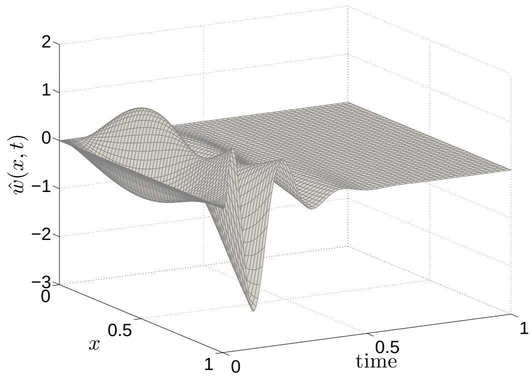
Table VIII presents the maximum λ for which an observer can be constructed using $\epsilon = 0.001$ and $\delta = 0.1$ as a function of degree $d = d_1 = d_2$. Figure 8 illustrates the evolution of the trajectory of the state estimate $\hat{w}(x,t)$, system state $w(x,t)$ and the error state $e(x,t) = \hat{w}(x,t) - w(x,t)$ for $\delta = 0.1$. Finally, Figure 9 illustrates the Lyapunov functional defined in the proof of Theorem 7 for the error dynamics. The initial condition $w_0(x)$ is given in Equation (37) and for the observer we choose $\hat{w}_0(x) = 0$.

	$d = 4$	5	6	7
λ	15	18	25.9	35

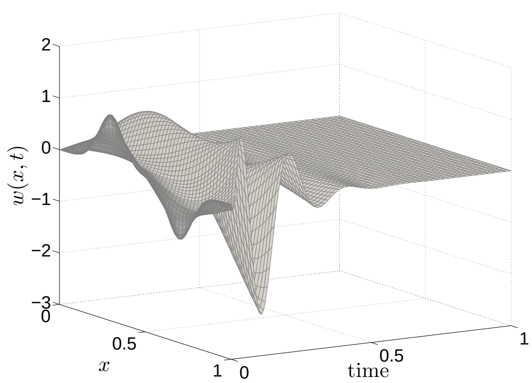
TABLE VIII: Maximum λ of the error system as a function of $d = d_1 = d_2$ for Numerical Example 8.

X. Conclusion

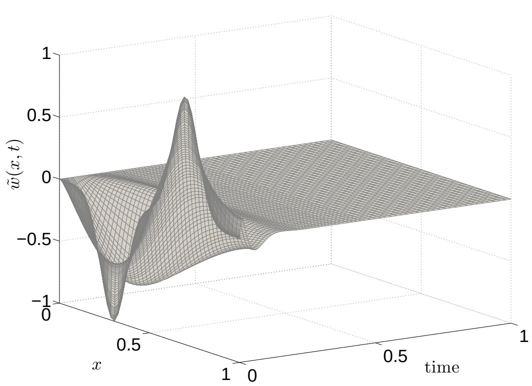
In this paper, we have developed a algorithmic approach to the design of observer-based controllers for a general class of scalar parabolic partial differential equations using point measurements and feedback at the boundary. The results use the sum-of-squares methodology to parameterize a convex set of positive operators. In this way we cast the problem of controller synthesis in the framework of convex optimization - a class of optimization problems for which we have efficient numerical algorithms. Furthermore, we have applied our results to a difficult numerical example in order to demonstrate



(a) Observer state evolution



(b) System state evolution



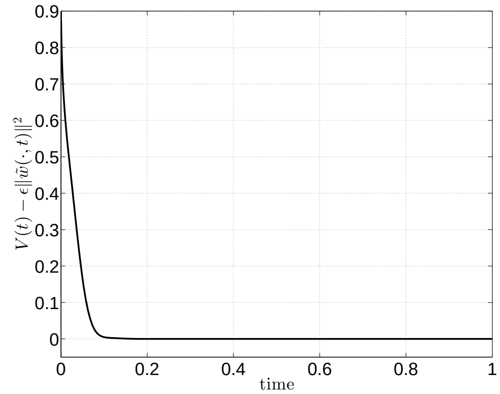
(c) Error in the estimate of the state

Fig. 8: Evolution of the observer state $\hat{w}(x,t)$, the system state $w(x,t)$ and the error state $e(x,t)$.

that our results are practical and effective. The reader is invited to contemplate natural extensions of this work including the development of methods for control of coupled partial-differential equations. We also speculate that the conditions as stated are conservative and may be improved through a generalization of the Wirtinger inequality, or some other method for relating state parameters $w, w_s, w_{ss}, w(1)$, etc. Additional possibilities include application to other classes of PDE system.

Appendix

First, recall the variation of Wirtinger's Inequality.



(a) Illustration of $V(t) \geq \epsilon\|\tilde{w}(\cdot,t)\|^2$.

Fig. 9: Lyapunov functional for the error system with $\delta = 0.1$ and $\epsilon = 0.001$.

Lemma 1 ([15],[21])**.** *let $z \in H^2(0,1)$ be a scalar function. Then*

$$\int_0^1 (z(s))^2 ds \leq (z(0))^2 + \frac{4}{\pi^2}\int_0^1 (z_s(s))^2 ds.$$

Now recall the definition of $\mathcal{M}_\epsilon$.

Definition 3. *We say $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$ if the following hold*

$$Q_0(s)_{1,1} = \left[(b(1) - a_s(1))\, M(1) - a(1)M_s(1)\right], \tag{52}$$

$$\begin{aligned} Q_0(s)_{1,2} &= Q_0(s)_{2,1} \\ &= \left[(b(1) - a_s(1))\, K_1(1,s) - a(1)K_{1,s}(1,s)\right], \end{aligned} \tag{53}$$

$$\begin{aligned} Q_0(s)_{2,2} &= \frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)M(s)\right] - b(s)M(s)\right] + 2M(s)c(s) \\ &+ \left[\frac{\partial}{\partial s}\left[2a(s)\left(K_1(s,t) - K_2(s,t)\right)\right]\right]_{t=s} - \frac{\pi^2}{2}\alpha\epsilon, \end{aligned} \tag{54}$$

$$\begin{aligned} &Q_1(s,t) \\ &= \left(\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)K_1(s,t)\right] - b(s)K_1(s,t)\right] + c(s)K_1(s,t)\right) \\ &+ \left(\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left[a(t)K_1(s,t)\right] - b(t)K_1(s,t)\right] + c(t)K_1(s,t)\right), \end{aligned} \tag{55}$$

$$\begin{aligned} &Q_2(s,t) = Q_1(t,s) \\ &= \left(\frac{\partial}{\partial s}\left[\frac{\partial}{\partial s}\left[a(s)K_2(s,t)\right] - b(s)K_2(s,t)\right] + c(s)K_2(s,t)\right) \\ &+ \left(\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}\left[a(t)K_2(s,t)\right] - b(t)K_2(s,t)\right] + c(t)K_2(s,t)\right) \text{ and} \end{aligned} \tag{56}$$

$$Q_3(s) = -2a(0)K_2(0,s), \tag{57}$$

where $K_{1,s}(1,s) = [K_{1,s}(s,t)|_{s=1}]_{t=s}$.

Lemma 2. *Suppose we are given $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ and $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$. Then, for $\mathcal{A}$ as defined in Equation* (7) *and $\mathcal{P}$ as defined in Equation* (14)*, we have that*

$$\begin{aligned} \langle \mathcal{A}w, \mathcal{P}w\rangle + \langle w, \mathcal{P}\mathcal{A}w\rangle \leq \left\langle \begin{bmatrix} w(1) \\ w \end{bmatrix}, \mathcal{Q}\begin{bmatrix} w(1) \\ w \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} \\ + \int_0^1 w_s(0)Q_3(s)w(s)ds \end{aligned}$$

for any $w \in \mathcal{D}_0$ where $\mathcal{D}_0$ is defined in Equation (4) *and where $\mathcal{Q}$*

is defined as

$$(\mathcal{Q}y)(s) := Q_0(s)\begin{bmatrix} y(1) \\ y(s)\end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0 \\ 0 & Q_1(s,t)\end{bmatrix}\begin{bmatrix} y(1) \\ y(t)\end{bmatrix} dt + \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & Q_2(s,t)\end{bmatrix}\begin{bmatrix} y(1) \\ y(t)\end{bmatrix} dt.$$

Proof: We begin by considering the following decomposition

$$\begin{aligned}&\langle \mathcal{A}w, \mathcal{P}w\rangle + \langle w, \mathcal{P}\mathcal{A}w\rangle \\ &= 2\int_0^1 \left(a(s)w_{ss}(s) + b(s)w_s(s) + c(s)w(s)\right)(\mathcal{P}w)(s)ds \\ &= 2\left(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5\right),\end{aligned} \tag{58}$$

where

$$\Gamma_1 = \int_0^1 w_{ss}(s)a(s)M(s)w(s)ds,$$

$$\Gamma_2 = \int_0^1 w_s(s)b(s)M(s)w(s)ds,$$

$$\Gamma_3 = \int_0^1 w_{ss}(s)a(s)\left(\int_0^s K_1(s,t)w(t)dt + \int_s^1 K_2(s,t)w(t)dt\right)ds,$$

$$\Gamma_4 = \int_0^1 w_s(s)b(s)\left(\int_0^s K_1(s,t)w(t)dt + \int_s^1 K_2(s,t)w(t)dt\right)ds$$

and

$$\Gamma_5 = \int_0^1 w(s)^2 M(s)c(s)ds + \int_0^1\int_0^s w(s)c(s)K_1(s,t)w(t)dtds + \int_0^1\int_s^1 w(s)c(s)K_2(s,t)w(t)dtds.$$

Applying integration by parts and using the boundary condition $w(0) = 0$ yields

$$\begin{aligned}\Gamma_1 = &-\int_0^1 w_s(s)^2 a(s)M(s)ds + \int_0^1 w(s)^2\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)M(s)\right]\right)ds \\ &- w(1)^2\left(\frac{1}{2}\left(a_s(1)M(1) + a(1)M_s(1)\right)\right) \\ &+ w_s(1)a(1)M(1)w(1).\end{aligned}$$

Since $a(s) \geq \alpha$ and $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$, we have $a(s)M(s) \geq \alpha\epsilon$. Thus, by application of the Wirtinger Inequality and boundary condition $w(0) = 0$, we have

$$-\int_0^1 w_s(s)^2 a(s)M(s)ds \leq -\frac{\pi^2}{4}\alpha\epsilon\int_0^1 w(s)^2 ds.$$

We conclude that

$$\begin{aligned}\Gamma_1 \leq &\int_0^1 w(s)^2\left[\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)M(s)\right]\right) - \frac{\pi^2}{4}\alpha\epsilon\right]dx \\ &- w(1)^2\left(\frac{1}{2}\left(a_s(1)M(1) + a(1)M_s(1)\right)\right) \\ &+ w_s(1)w(1)a(1)M(1).\end{aligned} \tag{59}$$

Through integration by parts and application of boundary conditions, we also obtain

$$\Gamma_2 = -\int_0^1 w(s)^2\left(\frac{1}{2}\frac{\partial}{\partial s}\left[b(s)M(s)\right]\right)ds + (w(1))^2\left(\frac{1}{2}b(1)M(1)\right). \tag{60}$$

Now, note that for $(M, K_1, K_2) \in \Xi_{d_1,d_2,\epsilon}$, we have $K_1(x,y) = K_2(y,x)$. Exploiting this property, we find

$$\begin{aligned}\Gamma_3 = &\int_0^1 w(s)^2\left(\left[\frac{\partial}{\partial s}\left[a(s)(K_1(s,t) - K_2(s,t))\right]\right]_{t=s}\right)ds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{\partial^2}{\partial s^2}\left[a(s)K_1(s,t)\right]\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{\partial^2}{\partial s^2}\left[a(s)K_2(s,t)\right]\right)w(t)dtds \\ &- w(1)\int_0^1\left(a_s(1)K_1(1,s) + a(1)K_{1,s}(1,s)\right)w(s)ds \\ &+ w_s(1)\int_0^1 a(1)K_1(1,s)w(s)ds.\end{aligned}$$

We can re-write the previous expression as

$$\begin{aligned}\Gamma_3 = &\int_0^1 w(s)^2\left(\left[\frac{\partial}{\partial s}\left[a(s)(K_1(s,t) - K_2(s,t))\right]\right]_{t=s}\right)ds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_1(s,t)\right]\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_2(s,t)\right]\right)w(t)dtds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_1(s,t)\right]\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_2(s,t)\right]\right)w(t)dtds \\ &- w(1)\int_0^1\left(a_s(1)K_1(1,s) + a(1)K_{1,s}(1,s)\right)w(s)ds \\ &+ w_s(1)\int_0^1 a(1)K_1(1,s)w(s)ds.\end{aligned}$$

Changing the order of integration in the last two double integrals and switching the variables s and t,

$$\begin{aligned}\Gamma_3 = &\int_0^1 w(s)^2\left(\left[\frac{\partial}{\partial s}\left[a(s)(K_1(s,t) - K_2(s,t))\right]\right]_{t=s}\right)ds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_1(s,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial t^2}\left[a(t)K_1(s,t)\right]\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}\frac{\partial^2}{\partial s^2}\left[a(s)K_2(s,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial t^2}\left[a(t)K_2(s,t)\right]\right)w(t)dtds \\ &- w(1)\int_0^1\left(a_s(1)K_1(1,s) + a(1)K_{1,s}(1,s)\right)w(s)ds \\ &+ w_s(1)\int_0^1 a(1)K_1(1,s)w(s)ds.\end{aligned} \tag{61}$$

Similarly,

$$\begin{aligned}\Gamma_4 = &-\int_0^1\int_0^s w(s)\left(\frac{1}{2}\frac{\partial}{\partial s}\left[b(s)K_1(s,t)\right] + \frac{1}{2}\frac{\partial}{\partial t}\left[b(t)K_1(s,t)\right]\right)w(t)dtds \\ &-\int_0^1\int_s^1 w(s)\left(\frac{1}{2}\frac{\partial}{\partial s}\left[b(s)K_2(s,t)\right] + \frac{1}{2}\frac{\partial}{\partial t}\left[b(s)K_2(s,t)\right]\right)w(t)dtds \\ &+ w(1)\int_0^1 b(1)K_1(1,s)w(s)ds.\end{aligned} \tag{62}$$

Finally, employing a change of order of integration produces

$$\Gamma_5 = \int_0^1 w(s)^2 M(s)c(s)ds + \int_0^1\int_0^s w(s)\left(\frac{1}{2}\left[c(s)+c(t)\right]K_1(s,t)\right)w(t)dtds + \int_0^1\int_s^1 w(s)\left(\frac{1}{2}\left[c(s)+c(t)\right]K_2(s,t)\right)w(t)dtds. \quad (63)$$

Substituting (59)-(63) into (58) gives us

$$\langle \mathcal{A}w, \mathcal{P}w\rangle + \langle w, \mathcal{P}\mathcal{A}w\rangle \le \left\langle \begin{bmatrix} w(1)\\ w\end{bmatrix}, \mathcal{Q}\begin{bmatrix} w(1)\\ w\end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + w_s(0)\int_0^1 Q_3(s)w(s)ds + 2w_s(1)\left(a(1)M(1)w(1) + \int_0^1 a(1)K_1(1,s)w(s)ds\right). \quad (64)$$

Since $w \in \mathcal{D}_0$, $w_s(1) = 0$. This gives us the desired result. ∎

Corollary 1. *Suppose we are given* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ *and* $\{Q_0, Q_1, Q_2, Q_3\} = \mathcal{M}_\epsilon(M, K_1, K_2)$. *Then, for* $\mathcal{A}$ *as defined in Equation* (7) *and* $\mathcal{P}$ *as defined in Equation* (14), *we have that*

$$\langle \mathcal{A}w, \mathcal{P}w\rangle + \langle w, \mathcal{P}\mathcal{A}w\rangle \le \left\langle \begin{bmatrix} w(1)\\ w\end{bmatrix}, \mathcal{Q}\begin{bmatrix} w(1)\\ w\end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + w_s(0)\int_0^1 Q_3(s)w(s)ds + 2w_s(1)\left(a(1)M(1)w(1) + \int_0^1 a(1)K_1(1,s)w(s)ds\right). \quad (65)$$

for any $w \in H^2(0,1)$ *with* $w(0) = 0$ *where* $\mathcal{Q}$ *is defined as*

$$(\mathcal{Q}y)(s) := Q_0(s)\begin{bmatrix} y(1)\\ y(s)\end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0\\ 0 & Q_1(s,t)\end{bmatrix}\begin{bmatrix} y(1)\\ y(t)\end{bmatrix}dt + \int_s^1 \begin{bmatrix} 0 & 0\\ 0 & Q_2(s,t)\end{bmatrix}\begin{bmatrix} y(1)\\ y(t)\end{bmatrix}dt.$$

Proof: Omit the last line in the proof of Lemma 1. ∎

The following lemma gives a result which is dual to Lemma 3.

Definition 4. *We say* $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$ *if the following hold*

$$T_0(s)_{1,1} = \left[-a(1)M_s(1) + (b(1) - a_s(1))M(1)\right], \quad (66)$$

$$T_0(s)_{1,2} = T_0(s)_{2,1} = -a(1)K_{1,s}(1,s), \quad (67)$$

$$T_0(s)_{2,2} = \left[(a_{ss}(s) - b_s(s))M(s) + b(s)M_s(s)\right] + 2M(s)c(s) + a(s)\left[M_{ss}(s) + 2\frac{\partial}{\partial s}\left[K_1(s,t) - K_2(s,t)\right]\right]_{t=s} - \frac{\pi^2}{2}\alpha\epsilon, \quad (68)$$

$$T_1(s,t) = a(s)K_{1,ss}(s,t) + b(s)K_{1,s}(s,t) + c(s)K_1(s,t) + a(t)K_{1,tt}(s,t) + b(t)K_{1,t}(s,t) + c(t)K_1(s,t), \quad (69)$$

$$T_2(s,t) = a(s)K_{2,ss}(s,t) + b(s)K_{2,s}(s,t) + c(s)K_2(s,t) + a(t)K_{2,tt}(s,t) + b(t)K_{2,t}(s,t) + c(t)K_2(s,t), \quad (70)$$

$$T_3 = a_x(0)M(0) - a(0)M_x(0) - b(0)M(0) + \frac{\pi^2}{2}\alpha\epsilon \text{ and} \quad (71)$$

$$T_4 = -2a(0)M(0). \quad (72)$$

Lemma 3. *Suppose* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ *and* $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$. *Then, for* $\mathcal{A}$ *as defined in Equation* (7) *and* $\mathcal{P}$ *as defined in Equation* (14), *we have that*

$$\langle \mathcal{A}\mathcal{P}w, w\rangle + \langle w, \mathcal{A}\mathcal{P}w\rangle \le \left\langle \begin{bmatrix} w(1)\\ w\end{bmatrix}, \mathcal{T}\begin{bmatrix} w(1)\\ w\end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + w(0)\left(T_3 w(0) + T_4 w_s(0)\right).$$

for any $w \in \mathcal{P}^{-1}\mathcal{D}_0$ *where* $\mathcal{D}_0$ *is defined in Equation* (4) *and*

$$(\mathcal{T}y)(s) := T_0(s)\begin{bmatrix} y(1)\\ y(s)\end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0\\ 0 & T_1(s,t)\end{bmatrix}\begin{bmatrix} y(1)\\ y(t)\end{bmatrix}dt + \int_s^1 \begin{bmatrix} 0 & 0\\ 0 & T_2(s,t)\end{bmatrix}\begin{bmatrix} y(1)\\ y(t)\end{bmatrix}dt.$$

Proof: We begin by considering the following decomposition

$$\langle \mathcal{A}\mathcal{P}w, w\rangle + \langle w, \mathcal{A}\mathcal{P}w\rangle = 2\int_0^1 \left(a(s)\frac{\partial^2}{\partial s^2}\left[(\mathcal{P}w)(s)\right] + b(s)\frac{\partial}{\partial s}\left[(\mathcal{P}w)(s)\right] + c(s)(\mathcal{P}w)(s)\right)w(s)ds = 2\left(\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 + \Gamma_5\right), \quad (73)$$

where

$$\Gamma_1 = \int_0^1 w(s)a(s)\frac{\partial^2}{\partial s^2}\left[M(s)w(s)\right]ds,$$

$$\Gamma_2 = \int_0^1 w(s)b(s)\frac{\partial}{\partial s}\left[M(s)w(s)\right]ds,$$

$$\Gamma_3 = \int_0^1 w(s)a(s)\frac{\partial^2}{\partial s^2}\left[\int_0^s K_1(s,t)w(t)dt + \int_s^1 K_2(s,t)w(t)dt\right]ds,$$

$$\Gamma_4 = \int_0^1 w(s)b(s)\frac{\partial}{\partial s}\left[\int_0^s K_1(s,t)w(t)dt + \int_s^1 K_2(s,t)w(t)dt\right]ds$$

and

$$\Gamma_5 = \int_0^1 w(s)^2 M(s)c(s)ds + \int_0^1\int_0^s w(s)c(s)K_1(s,t)w(t)dtds + \int_0^1\int_s^1 w(s)c(s)K_2(s,t)w(t)dtds.$$

Applying integration by parts,

$$\Gamma_1 = -\int_0^1 w_s(s)^2 a(s)M(s)ds + \frac{1}{2}\int_0^1 w(s)^2\left[a_{ss}(s)M(s) + a(s)M_{ss}(s)\right]ds + \frac{1}{2}w(1)^2\left[a(1)M_s(1) - a_s(1)M(1)\right] + w(1)a(1)M(1)w_s(1) + \frac{1}{2}w(0)^2\left[a_s(0)M(0) - a(0)M_s(0)\right] - w(0)a(0)M(0)w_s(0).$$

Since $a(s)M(s) \ge \alpha\epsilon$, applying Lemma 1 yields

$$-\int_0^1 w_s(s)^2 a(s)M(s)ds \le -\frac{\pi^2}{4}\alpha\epsilon\int_0^1 w(s)^2 ds + \frac{\pi^2}{4}\alpha\epsilon w(0)^2.$$

Thus

$$\Gamma_1 \le \int_0^1 w(s)^2\left(\frac{1}{2}\left[a_{ss}(s)M(s) + a(s)M_{ss}(s)\right] - \frac{\pi^2}{4}\alpha\epsilon\right)ds + \frac{1}{2}w(1)^2\left[a(1)M_s(1) - a_s(1)M(1)\right] + w(0)^2\left(\frac{1}{2}\left[a_s(0)M(0) - a(0)M_s(0)\right] + \frac{\pi^2}{4}\alpha\epsilon\right) + w(1)a(1)M(1)w_s(1) - w(0)a(0)M(0)w_s(0). \quad (74)$$

Similarly

$$\Gamma_2 = \frac{1}{2}\int_0^1 w(s)^2 \left[b(s)M_s(s) - b_s(s)M(s)\right] ds + \frac{1}{2}w(1)^2 b(1)M(1) - \frac{1}{2}w(0)^2 b(0)M(0). \tag{75}$$

Applying integration by parts and using the fact that $K_1(s,s) = K_2(s,s)$, we get

$$\Gamma_3 = \int_0^1 w(s)^2 \left(a(s)\left[\frac{\partial}{\partial s}[K_1(s,t) - K_2(s,t)]\right]_{t=s}\right) ds + \int_0^1\int_0^s w(s)a(s)K_{1,ss}(s,t)w(t)dtds + \int_0^1\int_s^1 w(s)a(s)K_{2,ss}(s,t)w(t)dtds.$$

Using a change of order of integration as applied in Equation (61) in Lemma 2, we obtain

$$\begin{aligned}\Gamma_3 = &\int_0^1 w(s)^2 \left(a(s)\left[\frac{\partial}{\partial s}[K_1(s,t) - K_2(s,t)]\right]_{t=s}\right) ds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}a(s)K_{1,ss}(s,t)\right)w(t)dtds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}a(t)K_{1,tt}(s,t)\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}a(s)K_{2,ss}(s,t)\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}a(t)K_{2,tt}(s,t)\right)w(t)dtds \end{aligned} \tag{76}$$

Similarly,

$$\begin{aligned}\Gamma_4 = &\int_0^1\int_0^s w(s)\left(\frac{1}{2}b(s)K_{1,s}(s,t) + \frac{1}{2}b(t)K_{1,t}(s,t)\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}b(s)K_{2,s}(s,t) + \frac{1}{2}b(t)K_{2,t}(s,t)\right)w(t)dtds \end{aligned} \tag{77}$$

and

$$\begin{aligned}\Gamma_5 = &\int_0^1 w(s)^2 M(s)c(s)ds \\ &+ \int_0^1\int_0^s w(s)\left(\frac{1}{2}c(s)K_1(s,t) + \frac{1}{2}c(t)K_1(s,t)\right)w(t)dtds \\ &+ \int_0^1\int_s^1 w(s)\left(\frac{1}{2}c(s)K_2(s,t) + \frac{1}{2}c(t)K_2(s,t)\right)w(t)dtds. \end{aligned} \tag{78}$$

Substituting (74)-(78) in (73),

$$\begin{aligned}&\langle \mathcal{A}\mathcal{P}w, w\rangle + \langle w, \mathcal{A}\mathcal{P}w\rangle \\ &\le \left\langle \begin{bmatrix} w(1) \\ w \end{bmatrix}, \mathcal{T}\begin{bmatrix} w(1) \\ w \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + w(0)\left(T_3 w(0) + T_4 w_s(0)\right) \\ &\quad + 2\int_0^1 w(1)a(1)K_{1,s}(1,s)w(s)ds + 2w(1)a(1)M(1)w_s(1) \\ &\quad + 2w(1)a(1)M_s(1)w(1). \end{aligned} \tag{79}$$

Since $w \in \mathcal{P}^{-1}\mathcal{D}_0$, there exists a $y \in \mathcal{D}_0$ such that $w = \mathcal{P}^{-1}y$ which implies $y = \mathcal{P}w$. Hence, we obtain the boundary condition

$$y_s(1) = M_s(1)w(1) + M(1)w_s(1) + \int_0^1 K_{1,s}(1,s)w(s)ds.$$

Since $y \in D_0$, $y_s(1) = 0$ and hence

$$M_s(1)w_1(1) = -M(1)w_s(1) - \int_0^1 K_{1,s}(1,s)w(s)ds.$$

Substituting this boundary condition into the last term of (81) gives us the desired result. ∎

Corollary 2. *Suppose we are given* $\{M, K_1, K_2\} \in \Xi_{d_1,d_2,\epsilon}$ *and* $\{T_0, T_1, T_2, T_3, T_4\} = \mathcal{N}_\epsilon(M, K_1, K_2)$*. Then, for* $\mathcal{A}$ *as defined in Equation* (7) *and* $\mathcal{P}$ *as defined in Equation* (14)*, we have that*

$$\langle \mathcal{A}\mathcal{P}w, w\rangle + \langle w, \mathcal{A}\mathcal{P}w\rangle \tag{80}$$

$$\begin{aligned}&\le \left\langle \begin{bmatrix} w(1) \\ w \end{bmatrix}, \mathcal{T}\begin{bmatrix} w(1) \\ w \end{bmatrix}\right\rangle_{\mathbb{R}\times L_2} + w(0)\left(T_3 w(0) + T_4 w_s(0)\right) \\ &\quad + 2\int_0^1 w(1)a(1)K_{1,s}(1,s)w(s)ds + 2w(1)a(1)M(1)w_s(1) \\ &\quad + 2w(1)a(1)M_s(1)w(1). \end{aligned} \tag{81}$$

for any $w \in \mathcal{P}^{-1}\mathcal{D}$ *where* $\mathcal{D}$ *is defined in Equation* (4) *and*

$$(\mathcal{T}y)(s) := T_0(s)\begin{bmatrix} y(1) \\ y(s) \end{bmatrix} + \int_0^s \begin{bmatrix} 0 & 0 \\ 0 & T_1(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt + \int_s^1 \begin{bmatrix} 0 & 0 \\ 0 & T_2(s,t) \end{bmatrix}\begin{bmatrix} y(1) \\ y(t) \end{bmatrix} dt.$$

The proof of Corollary 2 is implied by the proof of Lemma 3 in Inequality (79).

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