Robust Analysis of Linear Systems with Uncertain Delays using PIEs

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Summary

PI operator is one basic element which forms the PIE representation and LPI constraints.

PIE (Partial Integral Equation) representation	LPI (Linear Partial Integral Inequality)
A set of differential equations that are parameterized by PI operators. An example: $\mathcal{T}\dot{\mathbf{z}}(t) = \mathcal{A}\mathbf{z}(t)$. where $\mathcal{T}, \mathcal{A}$ are PI operators.	An inequality constraint involved with PI variables to solve a convex feasibility/optimization problem. An example: $\mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} \prec 0$. where $\mathcal{T}, \mathcal{A}, \mathcal{P}$ are PI operators

(4-)PI (Partial Integral) operator

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}_{i=0}^2 \end{bmatrix}\begin{bmatrix} x \\ \Phi \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_{-1}^0 Q_1(s)\Phi(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}_{i=0}^2}\right)\Phi(s) \end{bmatrix}.$$

forms an algebra of bounded linear multiplier and integral operators defined jointly on $\mathbb{R}^n$ and L_2.

Many **ODE, PDE, DDE**, and delay differential (**DDF**) formulation can be converted into PIE format!

Nominal DDE:

$$\begin{aligned} \dot{x}(t) &= A_0 x(t) + B_0 w(t) + \sum_{i=1}^{k} A_i x(t-\tau_i) + B_i w(t-\tau_i), && x \in \mathbb{R}^n, w \in \mathbb{R}^r \\ z(t) &= C_0 x(t) + D_0 w(t) + \sum_{i=1}^{k} C_i x(t-\tau_i) + D_i w(t-\tau_i), && z \in \mathbb{R}^p. \end{aligned} \qquad (1)$$

DDE in PIE format:

$$\begin{aligned} \mathcal{T}\dot{\mathbf{z}}(t) + \mathcal{T}_w \dot{w}(t) &= \mathcal{A}\mathbf{z}(t) + \mathcal{B}w(t), \\ z(t) &= \mathcal{C}\mathbf{z}(t) + \mathcal{D}w(t). \end{aligned} \qquad (2)$$

$$\mathcal{T} = \mathcal{P}\left[\begin{matrix} I, & 0 \\ \begin{bmatrix} C_{r1} \\ \vdots \\ C_{rk} \end{bmatrix}, & \{0,0,-I\} \end{matrix}\right], \quad \mathcal{A} = \mathcal{P}\left[\begin{matrix} A_0 + \sum_{i=1}^{k} A_i, & \left[\left[-A_1, -B_1\right], \cdots, \left[-A_k, -B_k\right]\right] \\ 0, & \{H, 0, 0\} \end{matrix}\right],$$

$$\mathcal{B} = \mathcal{P}\left[\begin{matrix} B_0 + \sum_{i=1}^{k} B_i, & 0 \\ 0, & \{0,0,0\} \end{matrix}\right], \quad \mathcal{C} = \mathcal{P}\left[\begin{matrix} C_0 + \sum_{i=1}^{k} C_i, & \left[\left[-C_1, -D_1\right], \cdots, \left[-C_1, -D_1\right]\right] \\ 0, & \{0,0,0\} \end{matrix}\right],$$

$$\mathcal{D} = \mathcal{P}\left[\begin{matrix} D_0 + \sum_{i=1}^{k} D_i, & 0 \\ 0, & \{0,0,0\} \end{matrix}\right], \quad \mathcal{T}_w = \mathcal{P}\left[\begin{matrix} 0, & 0 \\ \begin{bmatrix} B_{r1} \\ \vdots \\ B_{rk} \end{bmatrix}, & \{0,0,-I\} \end{matrix}\right], \left[C_{ri}, B_{ri}\right] = I,$$

$$H = \mathrm{diag}\{-\frac{1}{\tau_1} I_{n_s}, \cdots, -\frac{1}{\tau_k} I_{n_s}\}, n = n + r.$$

PIE format is better:

- PIE representation are defined by bounded operators which form an algebra;
- no boundary constraints on the new state $\mathbf{z}$

Introduction

Solving stability/H$_\infty$ performance problems of DDE in PIE framework

$$\text{DDE in PIE format: } \mathcal{T}\dot{\mathbf{z}}(t) + \mathcal{T}_w \dot{w}(t) = \mathcal{A}\mathbf{z}(t) + \mathcal{B}w(t),$$
$$z(t) = \mathcal{C}\mathbf{z}(t) + Dw(t),$$

Stability condition in LPIs	H$_\infty$ performance condition in LPIs
$\mathcal{P} \succ 0,\quad \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} \prec 0$	$\min\limits_{\gamma,\mathcal{P}} \gamma, \mathcal{P} \succ 0, \begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P}\mathcal{T} & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} & -\gamma I & D^T \\ \mathcal{C} & D & -\gamma I \end{bmatrix} \prec 0$

Using MATLAB PIETOOLS to convert and solving LPIs! Follow the steps:
Input the system parameters: `A_0 = …, B_0 = …,` $\cdots$;
Converting the system to PIE format: `convert_PIETOOLS_DDE`;
Setting Optimization parameters: `settings_`
Executive the condition in LPI:
Link:

Question: what if the parametric uncertainty enters the DDE?

Examples:

- Uncertain delays.
- Valued parametric uncertainties.

Linear DDE system with uncertain delays and parametric uncertainties

$$\dot{x}(t) = (A_0 + \Delta A_0)x(t) + (B_0 + \Delta B_0)w(t) + \sum_{i=1}^{k} ((A_i + \Delta A_i)x(t-\tau_i) + (B_i + \Delta B_i)w(t-\tau_i))$$
$$z(t) = (C_0 + \Delta C_0)x(t) + (D_0 + \Delta D_0)w(t) + \sum_{i=1}^{k} ((C_i + \Delta C_i)x(t-\tau_i) + (D_i + \Delta D_i)w(t-\tau_i))$$
$$x(s) = x_0, s \in [-\tau, 0], \tau = \max\{\tau_1, \cdots, \tau_k\}$$

where $\tau_i \in [\tau_i^0, \tau_i^1]$, $\quad \|\Delta X_0\| \leq m_X, \|\Delta X_i\| \leq m_{xi}$ for $X = A, B, C, D$.

Questions:

- The PIE format contains uncertain parameters ΔX and ΔX_i, $X = A, B, C, D$.
- Is the PIE framework still applicable to deal with the stability and H_∞ norm of uncertain DDE?

The PIE system with parametric uncertainties is given

$$\begin{aligned}\mathcal{T}\dot{\mathbf{z}}(t) &= \mathcal{A}(v_1,\cdots,v_l)\mathbf{z}(t) + \mathcal{B}(v_1,\cdots,v_l)w(t)\\ z(t) &= \mathcal{C}(v_1,\cdots,v_l)\mathbf{z}(t) + D(v_1,\cdots,v_l)w(t) \quad v_i \in \Delta_i\end{aligned} \tag{3}$$

where $w \in \mathbb{R}^p$, $z \in \mathbb{R}^q$, $v_i \in \mathbb{R}$, $\mathcal{T}, \mathcal{A}(v_1,\cdots,v_l) : Z_{m,n} \to Z_{m,n}$, $\mathcal{B}(v_1,\cdots,v_l) : \mathbb{R}^p \to Z_{m,n}$, $\mathcal{C}(v_1,\cdots,v_l) : Z_{m,n} \to \mathbb{R}^q$, and $D(v_1,\cdots,v_l) : \mathbb{R}^p \to \mathbb{R}^q$ are PI operators.

Definition (Robust stability of the uncertain PIE)

The PIE system (5) defined by $\{\mathcal{T}, \mathcal{A}(v_1,\cdots,v_l)\}$ $(w(t) \equiv 0)$ is said to be robustly stable over Δ if the PIE system (5) defined by $\{\mathcal{T}, \mathcal{A}(v_1,\cdots,v_l)\}$ is stable for any given $v_i \in \Delta_i$.

Can we solve the robust stability and H_∞ norm of the uncertain PIE system and enrich the current PIE framework?

Main Results

Uncertain PIE: Definition, Robust Stability, H_∞ performance

To make it clearer, two independent uncertainties are considered (can be extended for single/multiple uncertainties case).

Assumption

$\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}$ is linear in the uncertain parameters α,β and the parameters lie in a polytope so that

$$\mathcal{A}(\alpha,\beta) := \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\alpha_i\beta_j\mathcal{A}_{ij}, \Delta_x = \{x\in\mathbb{R}^{N_x} : x_i\in[0,1], \sum_{i=1}^{N_x} x_i = 1\}, \textit{for} \quad x=\alpha,\beta. \tag{4}$$

The same to uncertain PI operators $\mathcal{B}(\alpha,\beta),\mathcal{C}(\alpha,\beta),\mathcal{D}(\alpha,\beta)$.

Then the uncertain PIE is parameterized by the vertex values $\mathcal{A}_{ij},\mathcal{B}_{ij},\mathcal{C}_{ij},\mathcal{D}_{ij}$ as follows

$$\mathcal{T}\dot{\mathbf{z}}(t) = \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\alpha_i\beta_j\left(\mathcal{A}_{ij}\mathbf{z}(t)+\mathcal{B}_{ij}w(t)\right), \mathbf{z}(0)=0 \tag{5}$$

$$z(t) = \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta}\alpha_i\beta_j\left(\mathcal{C}_{ij}\mathbf{z}(t)+D_{ij}w(t)\right) \quad \alpha\in\Delta_\alpha,\beta\in\Delta_\beta \tag{6}$$

Theorem (Robust stability criterion in LPIs of the uncertain PIE)

Suppose there exist a PI operator $\mathcal{P}$ satisfying $\mathcal{P}^* = \mathcal{P} \succ 0$ and

$$\mathcal{A}_{ij}^* \mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}_{ij} \prec 0, i = 1, 2, \cdots, N_\alpha, j = 1, 2, \cdots, N_\beta. \tag{7}$$

Then PIE system

$$\mathcal{T}\dot{\mathbf{z}} = \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \alpha_i \beta_j \mathcal{A}_{ij}\mathbf{z}(t)$$

is robustly stable over $\Delta_\alpha \times \Delta_\beta$ where $\Delta_x = \{x \in \mathbb{R}^{N_x} : x_i \in [0,1], \sum_{i=1}^{N_x} x_i = 1\}$, for $x = \alpha, \beta$..

Consider the Lyapunov candidate function: $V(\mathbf{z}) = \langle \mathcal{T}\mathbf{z}, \mathcal{P}\mathcal{T}\mathbf{z} \rangle$.

Theorem (H_∞ performance in LPIs of the uncertain PIE)

Suppose there exist a positive scalar γ *and a bounded PI operator* $\mathcal{P}$ *satisfying* $\mathcal{P}^* = \mathcal{P} \succ 0$ *and*

$$\begin{bmatrix} \mathcal{T}^*\mathcal{P}\mathcal{A}_{ij} + (\mathcal{A}_{ij})^*\mathcal{P}\mathcal{T}^* & \mathcal{T}^*\mathcal{P}\mathcal{B}_{ij} & \mathcal{C}_{ij}^* \\ \mathcal{B}_{ij}^*\mathcal{P}\mathcal{T} & -\gamma I & D_{ij}^T \\ \mathcal{C}_{ij} & D_{ij} & -\gamma I \end{bmatrix} \prec 0, \quad i = 1, 2, \cdots, N_\alpha, j = 1, \cdots, N_\beta. \tag{8}$$

Then if $\mathbf{z}_0 \equiv 0$*, for any* $w \in L_2$*, any solution of the PIE system*

$$\mathcal{T}\dot{\mathbf{z}}(t) = \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \alpha_i \beta_j \left(\mathcal{A}_{ij}\mathbf{z}(t) + \mathcal{B}_{ij} w(t)\right), \mathbf{z}(0) = 0$$

$$z(t) = \sum_{i=1}^{N_\alpha} \sum_{j=1}^{N_\beta} \alpha_i \beta_j \left(\mathcal{C}_{ij}\mathbf{z}(t) + D_{ij} w(t)\right) \quad \alpha \in \Delta_\alpha, \beta \in \Delta_\beta$$

satisfies $\|z\|_{L_2[0,\infty]} \leq \gamma \|w\|_{L_2[0,\infty]}$ *over* $\Delta_\alpha \times \Delta_\beta$ *where* $\Delta_x = \{x \in \mathbb{R}^{N_x} : x_i \in [0,1], \sum_{i=1}^{N_x} x_i = 1\}$*, for* $x = \alpha, \beta$.

Proof: Define the Lyapunov candidate function (storage function) as $V(\mathbf{z}) = \langle \mathcal{T}\mathbf{z}, \mathcal{P}\mathcal{T}\mathbf{z}\rangle$. Then we find

$$\begin{aligned}
&\dot V(\mathbf{z}(t)) - \gamma\|w(t)\|^2 - \gamma\|v(t)\|^2 + 2\langle v(t), z(t)\rangle \\
&= \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta} \alpha_i\beta_j \Big(\langle \mathcal{T}\mathbf{z}(t), \mathcal{P}\mathcal{B}_{ij}w(t)\rangle + \langle \mathcal{B}_{ij}w(t), \mathcal{P}\mathcal{T}\mathbf{z}(t)\rangle - \gamma\|w(t)\|^2 - \gamma\|v(t)\|^2 \\
&\quad + \langle v(t),\mathcal{C}_{ij}\mathbf{z}(t)\rangle + \langle \mathcal{C}_{ij}\mathbf{z}(t),v(t)\rangle + \langle v(t),\mathcal{D}_{ij}w(t)\rangle + \langle \mathcal{D}_{ij}w(t),v(t)\rangle + \langle \mathcal{T}\mathbf{z}(t), \mathcal{P}\mathcal{A}_{ij}\mathbf{z}(t)\rangle \\
&\quad + \langle \mathcal{A}_{ij}\mathbf{z}(t), \mathcal{P}\mathcal{T}\mathbf{z}(t)\rangle\Big) \\
&= \sum_{i=1}^{N_\alpha}\sum_{j=1}^{N_\beta} \alpha_i\beta_j \begin{bmatrix}\mathbf{z}(t)\\ w(t)\\ v(t)\end{bmatrix}^T \begin{bmatrix}\mathcal{T}^*\mathcal{P}\mathcal{A}_{ij} + +(\mathcal{A}_{ij})^*\mathcal{P}\mathcal{T}^* & \mathcal{T}^*\mathcal{P}\mathcal{B}_{ij} & \mathcal{C}_{ij}^* \\ \mathcal{B}_{ij}^*\mathcal{P}\mathcal{T} & -\gamma I & D_{ij}^T \\ \mathcal{C}_{ij} & D_{ij} & -\gamma I\end{bmatrix}\begin{bmatrix}\mathbf{z}(t)\\ w(t)\\ v(t)\end{bmatrix}.
\end{aligned}$$

Therefore, if Eqn (8) is satisfied, we have $\dot V(\mathbf{z}(t)) - \gamma\|w(t)\|^2 + \frac{1}{\gamma}\|z(t)\|^2 < 0$ Integration of this inequality with respect to t yields

$$V(\mathbf{z}(t)) - V(\mathbf{z}(0)) - \gamma\int_0^t \|w(s)\|^2\,ds + \frac{1}{\gamma}\int_0^t \|z(s)\|^2\,ds < 0.$$

Since $V(\mathbf{z}(0)) = 0$ and $V(\mathbf{z}(t)) \geq 0$ for any $t \geq 0$, then as $t \to \infty$, any solution of the PIE system satisfies $\|z\|_{L_2[0,\infty]} \leq \gamma\|w\|_{L_2[0,\infty]}$ over $\Delta_\alpha \times \Delta_\beta$.

Main Results

Application to DDEs with uncertain delays

Consider the uncertain delay case

$$\dot{x}(t) = A_0 x(t) + B_0 w(t) + \sum_{i=1}^{k} A_i x(t-\tau_i)$$

$$z(t) = C_{10} x(t) + D_{10} w(t) + \sum_{i=1}^{k} C_{1i} x(t-\tau_i)$$

$$x(s) = x_0, s \in [-\tau, 0], \tau = \max\{\tau_1, \cdots, \tau_k\}. \tag{9}$$

where $x(t) \in \mathbb{R}^m$ is the system state with the initial function $x_0 \in L_2[-\tau, 0]$. $w(t) \in \mathbb{R}^p$ is the disturbance input. $z(t) \in \mathbb{R}^q$ is the regulated output. The delay parameters $\tau_i, i = 1, 2, \cdots, k$ are time-invariant but uncertain and

$$\tau \in \Delta_\tau := \{\tau \in \mathbb{R}^k_+ : \tau_i \in \left[\tau_i^{[0]}, \tau_i^{[1]}\right], i = 1, 2, \cdots, k\}. \tag{10}$$

where τ_i^0 and τ_i^1 are known positive constants defining the lower and upper bound of the τ_i respectively.

$$
\begin{aligned}
&\mathcal{T} := \mathcal{P}\begin{bmatrix} I, & 0 \\ I, & \{0, 0, -I\} \end{bmatrix}, \mathcal{B} := \mathcal{P}\begin{bmatrix} B_0, & 0 \\ 0, & \{0\} \end{bmatrix}, \\
&\mathcal{C} := \mathcal{P}\begin{bmatrix} C_{10} + \sum_{j=1}^{k} C_{1j}, & -\begin{bmatrix} C_{11} & \cdots & C_{1k} \end{bmatrix} \\ 0, & \{0, 0, 0\} \end{bmatrix}, D := D_{10}.
\end{aligned} \tag{11}
$$

Note that none of these PI operators depend on the τ_i. The effect of delay parameter is felt only in the generator $\hat{\mathcal{A}}(\hat{\tau})$ where $\hat{\tau} \in \mathbb{R}^k_+$ represents the vector of uncertain delay parameters in the uncertain TDS. Specifically, define $\hat{\mathcal{A}}(\hat{\tau})$ as

$$
\begin{aligned}
&\hat{\mathcal{A}}(\hat{\tau}) := \mathcal{P}\begin{bmatrix} A_0 + \sum_{j=1}^{k} A_j, & -\begin{bmatrix} A_1 & \cdots & A_k \end{bmatrix} \\ 0, & \left\{ \mathrm{diag}(\hat{\tau})^{-1}, 0, 0 \right\} \end{bmatrix} \\
&\mathrm{diag}(\hat{\tau}) = \mathrm{diag}\{\hat{\tau}_1 I_m, \cdots, \hat{\tau}_k I_m\}.
\end{aligned} \tag{12}
$$

Main Results

Application to DDEs with uncertain delays

Lemma

Given positive constants $\tau_i^{[0]}, \tau_i^{[1]}, i = 1, \cdots, k$, *suppose that* $\mathcal{T}$ *satisfies Eqn* (11), $\hat{\mathcal{A}}(\tau)$ *satisfies Eqn* (12), *the* $\hat{\mathcal{A}}_i$ *are as defined in Eqn* (**??**), *and* Δ_τ *is as defined in Eqn* (10). *Then the PIE system* (5) *defined by* $\{\mathcal{T}, \hat{\mathcal{A}}(\tau)\}$ *is robustly stable over* Δ_τ *if and only if the PIE system* (5) *defined by* $\{\mathcal{T}, \sum_{i=1}^{2^k} \beta_i \hat{\mathcal{A}}_i\}$ *is robustly stable over* $\Delta_\beta = \{\beta \in \mathbb{R}^{2^k} : \beta_i \in [0,1], \sum_{i=1}^{2^k} \beta_i = 1\}$.

We now give the

Theorem

Given positive constants $\tau_i^{[0]}, \tau_i^{[1]}, i = 1, \cdots, k$, *Suppose there exist a constant real-valued matrix* $P \in \mathbb{R}^{m\times m}$ *and matrix-valued polynomials* $Q : [a, b] \to \mathbb{R}^{m\times n}$, $R_0 : [a, b] \to, \mathbb{R}^{n\times n}$, *and* $R_1, R_2 : [a, b] \times [a, b] \to \mathbb{R}^{n\times n}$, *such that* $\mathcal{P} := \mathcal{P}\left[\begin{smallmatrix} P, & Q \\ Q^T, & \{R_0, R_1, R_2\} \end{smallmatrix}\right]$ *satisfies* $\mathcal{P}^* = \mathcal{P} \succ 0$ *and*

$$\hat{\mathcal{A}}_i^* \mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\hat{\mathcal{A}}_i \prec 0, i = 1, 2, \cdots, 2^k \tag{13}$$

where $n = m \cdot k$, $\hat{\mathcal{A}}_i$ *is as defined in Eqn* (**??**) *and* $\mathcal{T}$ *is as defined in Eqn* (11). *Then the linear TDS* (9) *with* $w \equiv 0$ *is robustly stable over* Δ_τ *as defined in Eqn* (10).

Main Results

Application to DDEs with uncertain delays

Theorem

Given positive constants $\tau_i^{[0]}, \tau_i^{[1]}, i = 1, \cdots, k$, suppose there exist a positive scalar γ, a constant real-valued matrix $P \in \mathbb{R}^{m\times m}$, matrix-valued polynomials $Q : [a,b] \to \mathbb{R}^{m\times n}, R_0 : [a,b] \to, \mathbb{R}^{n\times n}$, and $R_1, R_2 : [a,b] \times [a,b] \to \mathbb{R}^{n\times n}$, such that $\mathcal{P} := \mathcal{P}\left[\begin{smallmatrix} P, & Q \\ Q^T, & \{R_0, R_1, R_2\}\end{smallmatrix}\right]$ satisfying $\mathcal{P}^* = \mathcal{P} \succ 0$ and

$$\begin{bmatrix} \mathcal{T}^*\mathcal{P}\hat{\mathcal{A}}_i + \hat{\mathcal{A}}_i^*\mathcal{P}\mathcal{T} & \mathcal{T}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* \\ \mathcal{B}^*\mathcal{P}\mathcal{T} & -\gamma I & D^T \\ \mathcal{C} & D & -\gamma I \end{bmatrix} \prec 0, \quad i = 1, 2, \cdots, 2^k \tag{14}$$

*where $n = m \cdot k$, $\hat{\mathcal{A}}_i$ is defined by Eqn (**??**) and $\mathcal{T}$, $\mathcal{B}, \mathcal{C}, D$ are as defined in Eqn (11). Then for $x_0 \equiv 0$, for any $w \in L_2$, the solution of the linear TDS (9) satisfies $\|z\|_{L_2[0,\infty]} \le \gamma \|w\|_{L_2[0,\infty]}$ for any $\tau \in \Delta_\tau$ where Δ_τ is as defined in Eqn (10).*

Numerical Examples

Consider the following linear TDS.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

Here τ is a constant delay satisfying $\tau \in [\tau^{[0]}, \tau^{[1]}]$. The robust stability region of this system has been well-studied and the analytical delay interval is known to be $[0.100169, 1.7178]$, as listed in Table 1. It is worth noting that using Theorem 3 we are able to prove robust stability for $\tau \in [0.100169, 1.7178]$ - precisely matching the analytical results.

Table 1 – The maximum admissible range of τ

Methods	Delay interval
(**Seuret:2013**)	[0.1003, 1.5406]
(**Park:2015**)(Theorem 1)	[0.1002, 1.5954]
(**Zeng:2015**)	[0.100169, 1.7122]
(**Li:2017**)	[0.100169, 1.7146]
Theorem 3	[0.100169, 1.7178]
the analytical range of τ	[0.100169, 1.7178]

Numerical Examples

Consider the linear system with commensurate delays as follows

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} x(t-\tau)$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 0 \end{bmatrix} x(t-2\tau) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & -2 \end{bmatrix} x(t-3\tau).$$

From (**Chen:1995**), this system is stable for $\tau \leq 0.3783$. We get by Theorem 3 the maximum delay interval which can assure the robust stability is $\tau \in \left[1.0 \times 10^{-11}, 0.3786\right]$. Fig 1. plots the state response when $\tau = 0.3786$, which shows the system is stable.

Numerical Examples

Consider the following linear TDS system

$$\dot{x}(t) = \begin{bmatrix} -3.09 & 2.67 \\ -9.80 & 2.83 \end{bmatrix} x(t) + \begin{bmatrix} 0.57 & 0.02 \\ 1.26 & 0.80 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t)$$
$$z(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} x(t) + 0.5w(t).$$

When $w(t) = 0$, the exact delay bound is found in (**Roozbehani:2005**) to be $\tau \in [0.2319, 0.8609]$. Using Theorem 3, a maximum delay interval is derived as $[0.2319, 0.8609]$ exactly matching the analytical result. When $w(t) \neq 0$, we compute the robust H_∞ performance via Theorem 4 to obtain an L_2 gain bound. When $\tau \in [0.28, 0.6]$, the analytical γ^*_{min} is 4.962. While the result in (**Roozbehani:2005**) obtains a bound of $\gamma_{min} = 5.200$, our results based on Theorem 4 yield $\gamma_{min} = 4.9692$, which is much closer to the analytical γ^*_{min}.

Numerical Examples

Consider the linear TDS

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -100 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0.2 \end{bmatrix} x(t-\tau)$$

where τ is a constant delay. The upper bound of delay parameter which keep the system stability are derived by using Theorem 3 with $\tau^{[0]} = \tau^{[1]}$. Table 2 lists the computed upper bounds by different methods showing a larger delay bound using our method than previous results.

Table 2 – The maximum admissible range of τ

Methods	Upper bound τ_M
(**Park:2015**)	0.126
(**Hien:2015**)	0.577
(**Zhao:2017**)	0.675
(**Tian:2019**)	0.728
(**Tao:2018**)	0.7495
Theorem 3	0.7519

Numerical Examples

Consider the linear TDS with uncertain delays and uncertain valued parameters ()

$$\dot{x}(t) = (-5+\delta)x(t) - 2x(t-\tau_1) + 4w(t) - (2+\delta)w(t-\tau)$$
$$z(t) = (2+\delta)x(t) - 2x(t-\tau_1) + 4w(t) - (2+\delta_1)x(t) - 5x(t-\tau)$$

where $\tau \in \begin{bmatrix} 0.95 & 1.05 \end{bmatrix}$ is a uncertain delay and $\delta \in \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}$. The H_∞ norm derived derived using Theorem with $\tau^0 = \tau^1 = \tau$ and $\delta = 0$ is 10.9263 – exactly match the analytical bound proposed in ().
For the case with uncertain delays and $\delta \in \begin{bmatrix} -0.1 & 0.1 \end{bmatrix}$, we get H_∞ norm $\gamma_{\mathsf{min}} = 11.6603$—close to the analytical bound 11.3622 proposed in ().
paper: Computing the robust H-infinity norm of time-delay LTI systems with real-valued and structured uncertainties

Numerical Examples

Consider the linear TDS with two uncertain valued, bounded parameters in the system matrices() and direct feed-through.

$$\begin{aligned} \dot{x}(t) &= (-2+\delta_1)x(t) + (1+\delta_2)x(t-1) - w(t) + (-0.5+\delta_1)w(t-2) \\ z(t) &= (-2+2\delta_2)x(t) + x(t-2) + (5+4\delta_1)w(t) + 1.5w(t-1) + (-3+\delta_1)w(t-2) \end{aligned}$$

where $\delta_1 \in \begin{bmatrix} -0.2 & 0.2 \end{bmatrix}$ and $\delta_2 \in \begin{bmatrix} -0.3 & 0.3 \end{bmatrix}$. Using Theorem , we get H_∞ norm $\gamma_{\mathsf{min}} = 10.5286$—close to the analytical robust strong asymptotic H_∞ norm 10.1 proposed in ().
paper: Computing the robust H-infinity norm of time-delay LTI systems with real-valued and structured uncertainties

Numerical Examples

A Tab. 3 é um exemplo de tabela inserida usando o ambiente LaTeX "table" e numerada automaticamente.

Table 3 – Exemplo de legenda de tabela.

L [m]	L^2 [m^2]	L^3 [m^3]	L^4 [m^4]
1	1	1	1
2	4	8	16
3	9	27	81
4	16	64	256
5	25	125	625

Source: autoria própria.

Para gerar ou editar tabelas em LaTeX, pode-se utilizar a ferramenta "Tables Generator", entre outras.

Informações e dicas sobre TeX/LaTeX

- LaTeX Project.
- Comprehensive TeX Archive Network (CTAN).
- TeX Users Group (TUG).
- LaTeX — Wikibooks.
- TeX-LaTeX Stack Exchange.

Conclusion

Descrição das Conclusões Obtidas

Lista de conclusões

- Conclusão 1.
- Conclusão 2.
- Conclusão 3.
- Conclusão 4.
- Conclusão 5.

References