A PIE Representation of coupled 2D PDEs and Stability Analysis using LPIs

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Lyapunov Stability Analysis of PDEs Requires an Algebra of Operators on L_2

Consider a 2D PDE

$$\dot{u}(t,x,y) = C[\partial_x + \partial_y]u(t,x,y), \qquad (x,y) \in [0,1] \times [0,1],$$
$$u(t,0,y) = u(t,x,0) = 0.$$

Letting $\mathbf{u}(t) = u(t) \in L_2^n\big[[0,1] \times [0,1]\big]$, stability can be certified with a quadratic Lyapunov Function (LF)

$$V(\mathbf{u}) = \langle \mathbf{u}, \mathcal{P}\mathbf{u} \rangle,$$

parameterized by some $\mathcal{P} : L_2^n \to L_2^n$.

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Representing the PDE as

$$\dot{\mathbf{u}}(t) = \mathcal{A}\mathbf{u}(t),$$
$$\mathbf{u}(t) \in X \subset L_2^n\big[[0,1] \times [0,1]\big],$$

the system is stable if and only if for some $\mathcal{P} > 0$,

$$\dot{V}(\mathbf{u}(t)) = \langle \mathbf{u}(t), [\mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A}]\mathbf{u}(t) \rangle \leq 0$$

along any solution $\mathbf{u}(t) \in X$.

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How do we parameterize a set of operators $\mathcal{P}$ that

1. act on infinite-dimensional states $\mathbf{u} \in L_2^n$,
2. is closed under addition and composition – i.e. is an **algebra**,
3. is suitably rich so as to avoid introducing significant conservatism?

An Algebra of Operators on **1D** States can be Parameterized by **3** Functions

For states $\mathbf{u} \in L_2^m[a,b]$ on a 1D domain $[a,b]$, we can parameterize an algebra of operators by 3 matrix-valued functions:

Definition 1 (1D-PI Operator)

For any $N = \{N_0, N_1, N_2\}$ *with* $N_i \in L_2^{n\times m}$*, we define the associated 1D-PI operator*

$$(\mathcal{P}[N]\mathbf{u})(x) = N_0(x)\mathbf{u}(x) + \int_a^x N_1(x,\theta)\mathbf{u}(\theta)d\theta + \int_x^b N_2(x,\theta)\mathbf{u}(\theta)d\theta \quad \in L_2^n[a,b],$$

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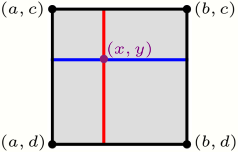
Proposition 2 (1D-PI Operators form an Algebra)

For any 1D-PI operators $\mathcal{B} : L_2^p \to L_2^n$ and $\mathcal{D} : L_2^m \to L_2^p$, there exists a 1D-PI operator $\mathcal{R} : L_2^{m\to n}$ such that for any $\mathbf{u} \in L_2^m$,

$$\Big(\mathcal{B}\big(\mathcal{D}\mathbf{u}\big)\Big)(x) = \big(\mathcal{R}\mathbf{u}\big)(x).$$

An Algebra of Operators on **2D** States can be Parameterized by **9** Functions

Composing PI operators on $[a, b]$ and $[c, d]$, we can define PI operators on the 2D hypercube $\Omega := [a, b] \times [c, d]$ as:



Definition 3 (2D-PI Operator)

For any $N = \begin{bmatrix} N_{00} & N_{01} & N_{02} \\ N_{10} & N_{11} & N_{12} \\ N_{20} & N_{21} & N_{22} \end{bmatrix}$ *with* $N_{ij} \in L_2^{n \times m}$*, we define the associated 2D-PI operator*

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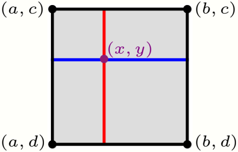

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PI-Based Stability Analysis of PDEs Requires an Alternative Representation

Consider a 2D PDE

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Then

- $\mathcal{A} = C[\partial_x + \partial_y]$,
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We want to represent the system in a way that

1. is parameterized by PI operators;
2. incorporates the BCs and continuity conditions into the dynamics.

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Linear, 2nd Order 2D PDEs are Parameterized by 9 Matrices

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This system can be represented as

$$\dot{\mathbf{u}}(t) = A_{10}\partial_x \mathbf{u}(t) + A_{01}\partial_y \mathbf{u}(t)$$
$$\mathbf{u}(t) \in X,$$

where $A_{10} = A_{01} = C$, and

$$X := \left\{ \mathbf{u} \in H_1^n\left[[0,1]\times[0,1]\right] \;\middle|\; \begin{bmatrix} \mathbf{u}(0,y) \\ \mathbf{u}(x,0) \end{bmatrix} = 0 \right\}$$

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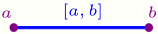
Any linear, 2nd order, 2D PDE can be represented as

$$\begin{aligned}
\dot{\mathbf{u}}(t,x,y) = {} & A_{00} \begin{bmatrix} \mathbf{u}_0(t,x,y) \\ \mathbf{u}_1(t,x,y) \\ \mathbf{u}_2(t,x,y) \end{bmatrix} + A_{01}\,\partial_y \begin{bmatrix} \mathbf{u}_1(t,x,y) \\ \mathbf{u}_2(t,x,y) \end{bmatrix} + A_{02}\,\partial_y^2\,\mathbf{u}_2(t,x,y) \\
& + A_{10}\,\partial_x \begin{bmatrix} \mathbf{u}_1(t,x,y) \\ \mathbf{u}_2(t,x,y) \end{bmatrix} + A_{11}\,\partial_x\partial_y \begin{bmatrix} \mathbf{u}_1(t,x,y) \\ \mathbf{u}_2(t,x,y) \end{bmatrix} + A_{12}\,\partial_x\partial_y^2\,\mathbf{u}_2(t,x,y) \\
& + A_{20}\,\partial_x^2\,\mathbf{u}_2(t,x,y) + A_{21}\,\partial_x^2\partial_y\,\mathbf{u}_2(t,x,y) + A_{22}\,\partial_x^2\partial_y^2\,\mathbf{u}_2(t,x,y) \qquad \forall (x,y)\in\Omega
\end{aligned}$$

$$\mathbf{u}(t) \in X_{\mathcal{B}}[\Omega] := \left\{ \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \in \begin{bmatrix} L_2^{n_0}[\Omega] \\ H_1^{n_1}[\Omega] \\ H_2^{n_2}[\Omega] \end{bmatrix} \;\middle|\; \mathcal{B}\,\Lambda_{\mathsf{bf}}\mathbf{u} = 0 \right\}$$

where we define $H_k^n[\Omega] := \{\mathbf{u} \mid \partial_x^{\alpha_1}\partial_y^{\alpha_2}\mathbf{u} \in L_2^n[\Omega],\; \forall \|\alpha\|_\infty \le k\}$.

BCs in 2D can be Expressed in terms of a Standardized Boundary State



We represent the BCs as $\mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = 0$,
where $\Lambda_{\mathsf{bf}}\mathbf{u}$ is the *full boundary state*.

In 1D, the boundary values are all finite-dimensional.

Then, we can define

$$\Lambda_{\mathsf{bf}}\mathbf{u} = \begin{bmatrix} \mathbf{u}_1(a) \\ \mathbf{u}_1(b) \\ \mathbf{u}_2(a) \\ \partial_x \mathbf{u}_2(a) \\ \mathbf{u}_2(b) \\ \partial_x \mathbf{u}_2(b) \end{bmatrix} \in \mathbb{R}^{2n_1+4n_2},$$

and impose boundary conditions as $B\Lambda_{\mathsf{bf}}\mathbf{u} = 0$ for a matrix B.

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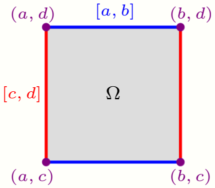
We represent the BCs as $\mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = 0$, where $\Lambda_{\mathsf{bf}}\mathbf{u}$ is the *full boundary state*.

In 2D, things are more complicated...

To define $\Lambda_{\mathsf{bf}}\mathbf{u}$, we seek a "minimal" representation of the boundary conditions.



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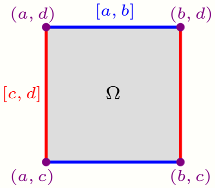

Example 1

For $\mathbf{u}_1 \in H_1[\Omega]$, we can decompose

$$\mathbf{u}_1(x, c) = \mathbf{u}_1(a, c) + \int_a^x \partial_x \mathbf{u}_1(\theta, c) d\theta$$

So, we may enforce $\mathbf{u}_1(x, c) = 0$ as $\mathbf{u}_1(a, c) = 0$ and $\partial_x \mathbf{u}_1(x, c) = 0$.

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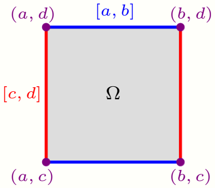

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Example 2

For $\mathbf{u}_2 \in H_2[\Omega]$, we can decompose

$$\mathbf{u}_2(b, y) = \mathbf{u}_2(b, d) + \partial_y \mathbf{u}_2(b, d)[y - d] + \int_d^y \partial_y^2 \mathbf{u}_2(b, \nu) d\nu$$

So, we may enforce $\mathbf{u}_2(b, y) = 0$ as $\mathbf{u}_2(b, d) = \partial_y \mathbf{u}_2(b, d) = 0$ and $\partial_y^2 \mathbf{u}_2(b, y) = 0$;

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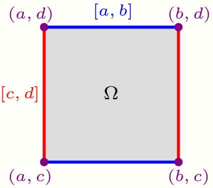
To define $\Lambda_{\text{bf}}\mathbf{u}$, we seek a "minimal" representation of the boundary conditions.



$$\Lambda_{\text{bf}}\mathbf{u} = \begin{bmatrix} \mathbf{u}_1(a,c) \\ \vdots \\ \mathbf{u}_1(b,d) \\ \mathbf{u}_2(a,c) \\ \vdots \\ \partial_x \mathbf{u}_2(a,c) \\ \vdots \\ \partial_y \mathbf{u}_2(a,c) \\ \vdots \\ \partial_x\partial_y \mathbf{u}_2(b,d) \\ \hline \partial_x \mathbf{u}_1(x,c) \\ \partial_x \mathbf{u}_1(x,d) \\ \partial_x^2 \mathbf{u}_2(x,c) \\ \vdots \\ \partial_x^2\partial_y \mathbf{u}_2(x,d) \\ \hline \partial_y \mathbf{u}_1(a,y) \\ \partial_y \mathbf{u}_1(b,y) \\ \partial_y^2 \mathbf{u}_2(a,y) \\ \vdots \\ \partial_x\partial_y^2 \mathbf{u}_2(b,y) \end{bmatrix}$$

Example 1

For $\mathbf{u}_1 \in H_1[\Omega]$, we can decompose

$$\mathbf{u}_1(x,c) = \mathbf{u}_1(a,c) + \int_a^x \partial_x \mathbf{u}_1(\theta,c)d\theta$$

So, we may enforce $\mathbf{u}_1(x,c) = 0$ as $\mathbf{u}_1(a,c) = 0$ and $\partial_x \mathbf{u}_1(x,c) = 0$.

Example 2

For $\mathbf{u}_2 \in H_2[\Omega]$, we can decompose

$$\mathbf{u}_2(b,y) = \mathbf{u}_2(b,d) + \partial_y \mathbf{u}_2(b,d)[y-d] + \int_d^y \partial_y^2 \mathbf{u}_2(b,\nu)d\nu$$

So, we may enforce $\mathbf{u}_2(b,y) = 0$ as $\mathbf{u}_2(b,d) = \partial_y \mathbf{u}_2(b,d) = 0$ and $\partial_y^2 \mathbf{u}_2(b,y) = 0$;

For any PDE State, there Exists a Constraint-Free *Fundamental State*

We decompose the PDE state $\mathbf{u}$ using the fundamental theorem of calculus:

$$\begin{aligned}\mathbf{u}_1(x,y) = \mathbf{u}_1(a,c) + \int_a^x \partial_x \mathbf{u}_1(\theta,c)d\theta + \int_c^y \partial_y \mathbf{u}_1(a,\nu)d\nu \\ + \int_c^y \int_a^x \partial_x \partial_y \mathbf{u}_1(\theta,\nu)d\theta d\nu\end{aligned}$$

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For any PDE State, there Exists a Constraint-Free *Fundamental State*

We decompose the PDE state $\mathbf{u}$ using the fundamental theorem of calculus:

$$\begin{aligned}
\mathbf{u}_1(x,y) &= \mathbf{u}_1(a,c) + \int_a^x \partial_x \mathbf{u}_1(\theta,c)d\theta + \int_c^y \partial_y \mathbf{u}_1(a,\nu)d\nu \\
&\qquad + \int_c^y \int_a^x \partial_x \partial_y \mathbf{u}_1(\theta,\nu)d\theta d\nu \\
\mathbf{u}_2(x,y) &= \mathbf{u}_2(a,c) + (x-a)\partial_x \mathbf{u}_2(a,c) \\
&\quad + (y-c)\partial_y \mathbf{u}_2(a,c) + (y-c)(x-a)\partial_x \partial_y \mathbf{u}_2(a,c) \\
&\qquad + \int_a^x (x-\theta)\, \partial_x^2 \mathbf{u}_2(\theta,c)d\theta + \int_c^y (y-\nu)\, \partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\
&\qquad\quad + (y-c) \int_a^x (x-\theta)\, \partial_x^2 \partial_y \mathbf{u}_2(\theta,c)d\theta + (x-a) \int_c^y (y-\nu)\, \partial_x \partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\
&\qquad\qquad + \int_c^y \int_a^x (y-\nu)(x-\theta)\, \partial_x^2 \partial_y^2 \mathbf{u}_2(\theta,\nu)d\theta d\nu
\end{aligned}$$

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&\quad + \int_c^y \int_a^x \partial_x\partial_y \mathbf{u}_1(\theta,\nu)d\theta d\nu \\
\mathbf{u}_2(x,y) &= \mathbf{u}_2(a,c) + (x-a)\partial_x \mathbf{u}_2(a,c) \\
&\quad + (y-c)\partial_y \mathbf{u}_2(a,c) + (y-c)(x-a)\partial_x\partial_y \mathbf{u}_2(a,c) \\
&\quad + \int_a^x (x-\theta)\, \partial_x^2 \mathbf{u}_2(\theta,c)d\theta + \int_c^y (y-\nu)\, \partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\
&\quad + (y-c)\int_a^x (x-\theta)\, \partial_x^2\partial_y \mathbf{u}_2(\theta,c)d\theta + (x-a)\int_c^y (y-\nu)\, \partial_x\partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\
&\quad + \int_c^y \int_a^x (y-\nu)(x-\theta)\, \partial_x^2\partial_y^2 \mathbf{u}_2(\theta,\nu)d\theta d\nu
\end{aligned}
$$

Definition 5 (Fundamental State)

For an arbitrary PDE state $\mathbf{u} \in X_{\mathcal{B}}[\Omega]$, we define the corresponding fundamental state (PIE state) $\hat{\mathbf{u}} \in L_2^{n_0+n_1+n_2}[\Omega]$ as

$$
\hat{\mathbf{u}} = \begin{bmatrix} \hat{\mathbf{u}}_0 \\ \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \end{bmatrix} := \begin{bmatrix} \mathbf{u}_0 \\ \partial_x\partial_y \mathbf{u}_1 \\ \partial_x^2\partial_y^2 \hat{\mathbf{u}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I & & \\ & \partial_x\partial_y & \\ & & \partial_x^2\partial_y^2 \end{bmatrix}}_{\mathscr{D}} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \mathscr{D}\mathbf{u}
$$

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We decompose the PDE state $\mathbf{u}$ using the fundamental theorem of calculus:

$$\mathbf{u}_1(x,y) = \mathbf{u}_1(a,c) + \int_a^x \partial_x \mathbf{u}_1(\theta,c)d\theta + \int_c^y \partial_y \mathbf{u}_1(a,\nu)d\nu + \int_c^y \int_a^x \partial_x\partial_y \mathbf{u}_1(\theta,\nu)d\theta d\nu$$

$$\begin{aligned}\mathbf{u}_2(x,y) &= \mathbf{u}_2(a,c) + (x-a)\partial_x \mathbf{u}_2(a,c) \\ &\quad + (y-c)\partial_y \mathbf{u}_2(a,c) + (y-c)(x-a)\partial_x\partial_y \mathbf{u}_2(a,c) \\ &\quad + \int_a^x (x-\theta)\, \partial_x^2 \mathbf{u}_2(\theta,c)d\theta + \int_c^y (y-\nu)\, \partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\ &\quad + (y-c)\int_a^x (x-\theta)\, \partial_x^2\partial_y \mathbf{u}_2(\theta,c)d\theta + (x-a)\int_c^y (y-\nu)\, \partial_x\partial_y^2 \mathbf{u}_2(a,\nu)d\nu \\ &\quad + \int_c^y \int_a^x (y-\nu)(x-\theta)\, \partial_x^2\partial_y^2 \mathbf{u}_2(\theta,\nu)d\theta d\nu\end{aligned}$$

$$\Lambda_{\mathsf{bc}}\mathbf{u} = \begin{bmatrix} \mathbf{u}_1(a,c) \\ \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_2(a,c) \\ \partial_y \mathbf{u}_2(a,c) \\ \partial_x\partial_y \mathbf{u}_2(a,c) \\ \hline \partial_x \mathbf{u}_1(x,c) \\ \partial_x^2 \mathbf{u}_2(x,c) \\ \partial_x^2\partial_y \mathbf{u}_2(x,c) \\ \hline \partial_y \mathbf{u}_1(a,y) \\ \partial_y^2 \mathbf{u}_2(a,y) \\ \partial_x\partial_y^2 \mathbf{u}_2(a,y) \end{bmatrix}$$

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For an arbitrary PDE state $\mathbf{u} \in X_{\mathcal{B}}[\Omega]$, we define the corresponding fundamental state (PIE state) $\hat{\mathbf{u}} \in L_2^{n_0+n_1+n_2}[\Omega]$ as

$$\hat{\mathbf{u}} = \begin{bmatrix} \hat{\mathbf{u}}_0 \\ \hat{\mathbf{u}}_1 \\ \hat{\mathbf{u}}_2 \end{bmatrix} := \begin{bmatrix} \mathbf{u}_0 \\ \partial_x\partial_y \mathbf{u}_1 \\ \partial_x^2\partial_y^2 \hat{\mathbf{u}}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I & & \\ & \partial_x\partial_y & \\ & & \partial_x^2\partial_y^2 \end{bmatrix}}_{\mathscr{D}} \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} = \mathscr{D}\mathbf{u}$$

The PDE State can be Expressed in terms of the Fundamental State through PI Operators

$$
\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}
= \overbrace{\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & \int_a^x [I] & 0 & 0 & \int_c^y [I] & 0 & 0 \\
0 & I & [x-a] & [y-c] & [y-c][x-a] & 0 & \int_a^x [x-\theta] & [y-c]\int_a^x [x-\theta] & 0 & \int_c^y [y-\nu] & [x-a]\int_c^y [y-\nu]
\end{bmatrix}}^{\mathcal{K}_1}
\overbrace{\begin{bmatrix}
\mathbf{u}_1(a,c) \\ \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_2(a,c) \\ \partial_y \mathbf{u}_2(a,c) \\ \partial_x \partial_y \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_1(\theta,c) \\ \partial_x^2 \mathbf{u}_2(\theta,c) \\ \partial_x^2 \partial_y \mathbf{u}_2(\theta,c) \\ \partial_y \mathbf{u}_1(a,\nu) \\ \partial_y^2 \mathbf{u}_2(a,\nu) \\ \partial_x \partial_y^2 \mathbf{u}_2(a,\nu)
\end{bmatrix}}^{\Lambda_{\mathsf{bc}} \mathbf{u}}
$$

$$
+ \overbrace{\begin{bmatrix}
I & & \\
& \int_a^x [I] \int_c^y [I] & \\
& & \int_a^x [x-\theta] \int_c^y [y-\nu]
\end{bmatrix}}^{\mathcal{K}_2}
\overbrace{\begin{bmatrix}
\mathbf{u}_0 \\ \partial_x \partial_y \mathbf{u}_1 \\ \partial_x^2 \partial_y^2 \hat{\mathbf{u}}_2
\end{bmatrix}}^{\hat{\mathbf{u}}}
= \mathcal{K}_1 \Lambda_{\mathsf{bc}} \mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}
$$

The PDE State can be Expressed in terms of the Fundamental State through PI Operators

$$
\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}
= \overbrace{\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & \int_a^x [I] & 0 & 0 & \int_c^y [I] & 0 & 0 \\
0 & I & [x-a] & [y-c] & [y-c][x-a] & 0 & \int_a^x [x-\theta] & [y-c]\int_a^x [x-\theta] & 0 & \int_c^y [y-\nu] & [x-a]\int_c^y [y-\nu]
\end{bmatrix}}^{\mathcal{K}_1}
\overbrace{\begin{bmatrix}
\mathbf{u}_1(a,c) \\ \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_2(a,c) \\ \partial_y \mathbf{u}_2(a,c) \\ \partial_x\partial_y \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_1(\theta,c) \\ \partial_x^2 \mathbf{u}_2(\theta,c) \\ \partial_x^2\partial_y \mathbf{u}_2(\theta,c) \\ \partial_y \mathbf{u}_1(a,\nu) \\ \partial_y^2 \mathbf{u}_2(a,\nu) \\ \partial_x\partial_y^2 \mathbf{u}_2(a,\nu)
\end{bmatrix}}^{\Lambda_{\mathsf{bc}}\mathbf{u}}
$$

$$
+ \overbrace{\begin{bmatrix}
I & & \\
& \int_a^x [I] \int_c^y [I] & \\
& & \int_a^x [x-\theta] \int_c^y [y-\nu]
\end{bmatrix}}^{\mathcal{K}_2}
\overbrace{\begin{bmatrix} \mathbf{u}_0 \\ \partial_x\partial_y \mathbf{u}_1 \\ \partial_x^2\partial_y^2 \hat{\mathbf{u}}_2 \end{bmatrix}}^{\hat{\mathbf{u}}}
= \mathcal{K}_1 \Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}
$$

Lemma 6

Let $\mathbf{u} \in \begin{bmatrix} L_2[\Omega] \\ H_1[\Omega] \\ H_2[\Omega] \end{bmatrix}$*, and define* $\hat{\mathbf{u}} = \mathscr{D}\mathbf{u}$*. Then, there exist PI operators* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *such that*

$$\mathbf{u} = \mathcal{K}_1 \Lambda_{bc}\mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}.$$

The PDE State can be Expressed in terms of the Fundamental State through PI Operators

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}$$

$$= \overbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & \int_a^x [I] & 0 & 0 & \int_c^y [I] & 0 & 0 \\ 0 & I & [x-a] & [y-c] & [y-c][x-a] & 0 & \int_a^x [x-\theta] & [y-c] \int_a^x [x-\theta] & 0 & \int_c^y [y-\nu] & [x-a] \int_c^y [y-\nu] \end{bmatrix}}^{\mathcal{K}_1} \overbrace{\begin{bmatrix} \mathbf{u}_1(a,c) \\ \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_2(a,c) \\ \partial_y \mathbf{u}_2(a,c) \\ \partial_x \partial_y \mathbf{u}_2(a,c) \\ \partial_x \mathbf{u}_1(\theta,c) \\ \partial_x^2 \mathbf{u}_2(\theta,c) \\ \partial_x^2 \partial_y \mathbf{u}_2(\theta,c) \\ \partial_y \mathbf{u}_1(a,\nu) \\ \partial_y^2 \mathbf{u}_2(a,\nu) \\ \partial_x \partial_y^2 \mathbf{u}_2(a,\nu) \end{bmatrix}}^{\Lambda_{\mathsf{bc}} \mathbf{u}}$$

$$+ \overbrace{\begin{bmatrix} I & & \\ & \int_a^x [I] \int_c^y [I] & \\ & & \int_a^x [x-\theta] \int_c^y [y-\nu] \end{bmatrix}}^{\mathcal{K}_2} \overbrace{\begin{bmatrix} \mathbf{u}_0 \\ \partial_x \partial_y \mathbf{u}_1 \\ \partial_x^2 \partial_y^2 \hat{\mathbf{u}}_2 \end{bmatrix}}^{\hat{\mathbf{u}}} = \mathcal{K}_1 \Lambda_{\mathsf{bc}} \mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}$$

Lemma 6

Let $\mathbf{u} \in \begin{bmatrix} L_2[\Omega] \\ H_1[\Omega] \\ H_2[\Omega] \end{bmatrix}$*, and define* $\hat{\mathbf{u}} = \mathscr{D}\mathbf{u}$*. Then, there exist PI operators* $\mathcal{K}_1$ *and* $\mathcal{K}_2$ *such that*

$$\mathbf{u} = \mathcal{K}_1 \Lambda_{bc} \mathbf{u} + \mathcal{K}_2 \hat{\mathbf{u}}.$$

Corollary 7

Let $\mathbf{u} \in \begin{bmatrix} L_2[\Omega] \\ H_1[\Omega] \\ H_2[\Omega] \end{bmatrix}$*, and define* $\hat{\mathbf{u}} = \mathscr{D}\mathbf{u}$*. Then there exist PI operators* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *such that*

$$\Lambda_{bf} \mathbf{u} = \mathcal{H}_1 \Lambda_{bc} \mathbf{u} + \mathcal{H}_2 \hat{\mathbf{u}}.$$

There Exists a Direct Map from Fundamental State to PDE State

For appropriate PI operators $\mathcal{H}_1$, $\mathcal{H}_2$, we can represent

$$\Lambda_{\mathsf{bf}}\mathbf{u} = \mathcal{H}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{H}_2\hat{\mathbf{u}},$$

where $\hat{\mathbf{u}} = \mathscr{D}\mathbf{u}$.
Given BCs $\mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = 0$, it follows that

$$0 = \mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = \mathcal{B}\mathcal{H}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{B}\mathcal{H}_2\hat{\mathbf{u}},$$

and therefore

$$\Lambda_{\mathsf{bc}}\mathbf{u} = -\big(\mathcal{B}\mathcal{H}_1\big)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}}.$$

There Exists a Direct Map from Fundamental State to PDE State

For appropriate PI operators $\mathcal{H}_1$, $\mathcal{H}_2$, we can represent

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and therefore

$$\Lambda_{\mathsf{bc}}\mathbf{u} = -\big(\mathcal{B}\mathcal{H}_1\big)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}}.$$

For appropriate PI operators $\mathcal{K}_1$, $\mathcal{K}_2$, we can also represent

$$\mathbf{u} = \mathcal{K}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}},$$

and thus

$$\begin{aligned}
\mathbf{u} &= \mathcal{K}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}} \\
&= -\mathcal{K}_1\big(\mathcal{B}\mathcal{H}_1\big)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}} + \mathcal{K}_2\hat{\mathbf{u}} \\
&= \underbrace{\Big[\mathcal{K}_2 - \mathcal{K}_1\big(\mathcal{B}\mathcal{H}_1\big)^{-1}\mathcal{B}\mathcal{H}_2\Big]}_{\mathcal{T}}\hat{\mathbf{u}} = \mathcal{T}\hat{\mathbf{u}}
\end{aligned}$$

There Exists a Direct Map from Fundamental State to PDE State

For appropriate PI operators $\mathcal{H}_1$, $\mathcal{H}_2$, we can represent

$$\Lambda_{\mathsf{bf}}\mathbf{u} = \mathcal{H}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{H}_2\hat{\mathbf{u}},$$

where $\hat{\mathbf{u}} = \mathscr{D}\mathbf{u}$.
Given BCs $\mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = 0$, it follows that

$$0 = \mathcal{B}\Lambda_{\mathsf{bf}}\mathbf{u} = \mathcal{B}\mathcal{H}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{B}\mathcal{H}_2\hat{\mathbf{u}},$$

and therefore

$$\Lambda_{\mathsf{bc}}\mathbf{u} = -\left(\mathcal{B}\mathcal{H}_1\right)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}}.$$

For appropriate PI operators $\mathcal{K}_1$, $\mathcal{K}_2$, we can also represent

$$\mathbf{u} = \mathcal{K}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}},$$

and thus

$$\begin{aligned}\mathbf{u} &= \mathcal{K}_1\Lambda_{\mathsf{bc}}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}} \\ &= -\mathcal{K}_1\left(\mathcal{B}\mathcal{H}_1\right)^{-1}\mathcal{B}\mathcal{H}_2\hat{\mathbf{u}} + \mathcal{K}_2\hat{\mathbf{u}} \\ &= \underbrace{\left[\mathcal{K}_2 - \mathcal{K}_1\left(\mathcal{B}\mathcal{H}_1\right)^{-1}\mathcal{B}\mathcal{H}_2\right]}_{\mathcal{T}}\hat{\mathbf{u}} = \mathcal{T}\hat{\mathbf{u}}\end{aligned}$$

Theorem 8

Let $\mathcal{B}$ be a given PI operator, and such that the operator $\mathcal{B}\mathcal{H}_1$ is invertible. Then, there exists a 2D-PI operator $\mathcal{T}$ such that for any $\mathbf{u} \in X_{\mathcal{B}}[\Omega]$ and $\hat{\mathbf{u}} \in L_2[\Omega]$, we have

$$\mathbf{u} = \mathcal{T}\mathscr{D}\mathbf{u} \qquad \textit{and} \qquad \hat{\mathbf{u}} = \mathscr{D}\mathcal{T}\hat{\mathbf{u}},$$

where $\mathscr{D} := \begin{bmatrix} I & & \\ & \partial_x\partial_y & \\ & & \partial_x^2\partial_y^2 \end{bmatrix}$.

Any Well-Posed, Linear, 2nd Order, 2D PDE can be Equivalently Represented as a PIE

Using the relation $\mathbf{u} = \mathcal{T}\hat{\mathbf{u}}$, the PDE defined by $\{A_{ij}, \mathcal{B}\}$,

$$\dot{\mathbf{u}}(t) = \mathscr{A}\mathbf{u}(t) = A_{00}\mathbf{u}(t) + \ldots + A_{22}\ \partial_x^2\partial_y^2\mathbf{u}_2(t),$$

$$\mathbf{u}(t) \in X_{\mathcal{B}}[\Omega] := \left\{ \begin{bmatrix}\mathbf{u}_0\\ \mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} \in \begin{bmatrix} L_2^{n_0}[\Omega]\\ H_1^{n_1}[\Omega]\\ H_2^{n_2}[\Omega]\end{bmatrix} \middle| \mathcal{B}\,\Lambda_{\mathsf{bf}}\mathbf{u} = 0 \right\},$$

can be equivalently represented as a Partial Integral Equation (PIE)

$$\mathcal{T}\dot{\hat{\mathbf{u}}}(t) = \mathcal{A}\hat{\mathbf{u}}(t),$$
$$\hat{\mathbf{u}}(t) \in L_2[\Omega]$$

defined by $\{\mathcal{T}, \mathcal{A}\}$:

Lemma 9

For given $\{A_{ij}, \mathcal{B}\}$, let

$$\mathcal{T} := \left[\mathcal{K}_2 - \mathcal{K}_1\left(\mathcal{B}\mathcal{H}_1\right)^{-1}\mathcal{B}\mathcal{H}_2\right], \qquad \textit{and} \qquad \mathcal{A} := \mathscr{A}\mathcal{T},$$

where $\mathscr{A}$ is defined by $\{A_{ij}\}$, and where $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{K}_1$ and $\mathcal{K}_2$ are such that

$$\Lambda_{bf}\mathbf{u} = \mathcal{H}_1\Lambda_{bc}\mathbf{u} + \mathcal{H}_2\hat{\mathbf{u}},$$
$$\mathbf{u} = \mathcal{K}_1\Lambda_{bc}\mathbf{u} + \mathcal{K}_2\hat{\mathbf{u}}.$$

Then, for any $\hat{\mathbf{u}}_I \in L_2^n[\Omega]$, $\hat{\mathbf{u}}(t)$ solves the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ with the initial condition $\hat{\mathbf{u}}_I$ if and only if $\mathbf{u}(t) = \mathcal{T}\hat{\mathbf{u}}(t)$ solves the PDE defined by $\{A_{ij}, \mathcal{B}\}$ with the initial condition $\mathbf{u}_I = \mathcal{T}\hat{\mathbf{u}}_I$.

Stability of a PIE can be Tested as an LMI

For a PIE defined by $\{\mathcal{T}, \mathcal{A}\}$,

$$\begin{aligned}\mathcal{T}\dot{\mathbf{u}}(t) &= \mathcal{A}\mathbf{u}(t),\\ \mathbf{u}(t) &\in L_2^n[\Omega],\end{aligned}$$

a LF can be parameterized by a 2D-PI operator $\mathcal{P}$ as

$$V\big(\hat{\mathbf{u}}(t)\big) = \langle \mathcal{T}\hat{\mathbf{u}}(t), \mathcal{P}\mathcal{T}\hat{\mathbf{u}}(t)\rangle_{L_2},$$

for which

$$\begin{aligned}\dot{V}\big(\mathbf{u}(t)\big) &= \langle \mathcal{T}\dot{\mathbf{u}}, \mathcal{P}\mathcal{T}\mathbf{u}\rangle_{L_2} + \langle \mathcal{T}\mathbf{u}, \mathcal{P}\mathcal{T}\dot{\mathbf{u}}\rangle_{L_2}\\ &= \langle \mathbf{u}, [\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}]\mathbf{u}\rangle_{L_2}.\end{aligned}$$

Then, stability of the PIE can be verified by solving the Linear PI Inequality (LPI)

$$\begin{aligned}\mathcal{P} &> 0,\\ \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} &\leq 0.\end{aligned}$$

Stability of a PIE can be Tested as an LMI

For a PIE defined by $\{\mathcal{T}, \mathcal{A}\}$,

$$\mathcal{T}\dot{\mathbf{u}}(t) = \mathcal{A}\mathbf{u}(t),$$
$$\mathbf{u}(t) \in L_2^n[\Omega],$$

a LF can be parameterized by a 2D-PI operator $\mathcal{P}$ as

$$V\big(\hat{\mathbf{u}}(t)\big) = \langle \mathcal{T}\hat{\mathbf{u}}(t), \mathcal{P}\mathcal{T}\hat{\mathbf{u}}(t)\rangle_{L_2},$$

for which

$$\begin{aligned}\dot{V}\big(\mathbf{u}(t)\big) &= \langle \mathcal{T}\dot{\mathbf{u}}, \mathcal{P}\mathcal{T}\mathbf{u}\rangle_{L_2} + \langle \mathcal{T}\mathbf{u}, \mathcal{P}\mathcal{T}\dot{\mathbf{u}}\rangle_{L_2} \\ &= \langle \mathbf{u}, [\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}]\mathbf{u}\rangle_{L_2}.\end{aligned}$$

Then, stability of the PIE can be verified by solving the Linear PI Inequality (LPI)

$$\begin{aligned}\mathcal{P} &> 0, \\ \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} &\leq 0.\end{aligned}$$

To enforce positivity of PI operators, let

$$\mathcal{P} := \mathcal{Z}^* P \mathcal{Z},$$

for some fixed 2D-PI operator $\mathcal{Z}$. Then if $P > 0$,

$$\begin{aligned}\langle \mathbf{u}, \mathcal{P}\mathbf{u}\rangle_{L_2} &= \langle \mathcal{Z}\mathbf{u}, P\mathcal{Z}\mathbf{u}\rangle_{L_2} \\ &= \left\langle P^{\frac{1}{2}}\mathcal{Z}\mathbf{u}, P^{\frac{1}{2}}\mathcal{Z}\mathbf{u}\right\rangle_{L_2} > 0,\end{aligned}$$

for any $\mathbf{u} \in L_2^n[\Omega]$.

Stability of a PIE can be Tested as an LMI

For a PIE defined by $\{\mathcal{T}, \mathcal{A}\}$,

$$\mathcal{T}\dot{\mathbf{u}}(t) = \mathcal{A}\mathbf{u}(t),$$
$$\mathbf{u}(t) \in L_2^n[\Omega],$$

a LF can be parameterized by a 2D-PI operator $\mathcal{P}$ as

$$V\big(\hat{\mathbf{u}}(t)\big) = \langle \mathcal{T}\hat{\mathbf{u}}(t), \mathcal{P}\mathcal{T}\hat{\mathbf{u}}(t)\rangle_{L_2},$$

for which

$$\begin{aligned}\dot{V}\big(\mathbf{u}(t)\big) &= \langle \mathcal{T}\dot{\mathbf{u}}, \mathcal{P}\mathcal{T}\mathbf{u}\rangle_{L_2} + \langle \mathcal{T}\mathbf{u}, \mathcal{P}\mathcal{T}\dot{\mathbf{u}}\rangle_{L_2} \\ &= \langle \mathbf{u}, [\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A}]\mathbf{u}\rangle_{L_2}.\end{aligned}$$

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for any $\mathbf{u} \in L_2^n[\Omega]$.

As such, if there exist matrices $P > 0$ and $Q \leq 0$ such that, for given PI operators $\mathcal{Z}_1$ and $\mathcal{Z}_2$,

$$\mathcal{P} = \mathcal{Z}_1^* P \mathcal{Z}_1,$$
$$\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} = \mathcal{Z}_2^* Q \mathcal{Z}_2,$$

then the PIE defined by $\{\mathcal{T}, \mathcal{A}\}$ is stable.

Stability of 2D PDEs can be Numerically Tested with PIETOOLS

Combining all steps, we can perform stability analysis of 2D PDEs using PIETOOLS (`https://control.asu.edu/pietools/`):

1. Represent the PDE in the standardized format by $\{A_{ij}, \mathcal{B}\}$:

```
PDE.n.n_pde = [n0,n1,n2];   PDE.dom = [a,b;c,d];   PDE.vars = [x,tt;y,nu];
PDE.PDE.A = ...;            PDE.BC.Ebb = ...;
```

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```

2. Convert the PDE to a PIE defined by $\{\mathcal{T}, \mathcal{A}\}$:

```
PIE = convert_PIETOOLS_PDE(PDE);
T = PIE.T;         A = PIE.A;
```

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```

3. Test for existence of a PI operator $\mathcal{P} > 0$ such that $\mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} \leq 0$:

```
prog = sosprogram([x y tt nu]);
[prog, P] = poslpivar_2d(prog,n,dom,deg);
P = P + eps;     Q = - A'*P*T - T'*P*A;
prog = lpi_ineq_2d(prog,Q);
prog = sossolve(prog);
```

Ex. 1: Advection Equation

For the simple advection equation

$$\dot{u}(t,x,y) = C\partial_x u(t,x,y) + C\partial_y u(t,x,y)$$
$$u(t,0,y) = u(t,x,0) = 0, \qquad (x,y) \in [0,1]^2$$

we have $\mathbf{u}(t) = \mathbf{u}_1(t) = u(t) \in H_1\left[[0,1]^2\right]$.
The associated fundamental state is
$\hat{\mathbf{u}} = \partial_x \partial_y \mathbf{u}$, with corresponding PIE

$$\int_0^x \int_0^y \dot{\hat{\mathbf{u}}}(t,\theta,\nu) d\nu d\theta$$
$$= C \int_0^y \hat{\mathbf{u}}(t,x,\nu) d\nu + C \int_0^x \hat{\mathbf{u}}(t,\theta,y) d\theta.$$

Examples

Ex. 1: Advection Equation

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Ex. 2: Wave Equation

The wave equation on $(x,y) \in [0,1]^2$

$$\ddot{u}(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y)$$
$$u(0,y) = \partial_x u(0,y) = u(x,0) = \partial_y u(x,0) = 0$$

can be represented in the standardized format by defining $\mathbf{u}_1 = u$ and $\mathbf{u}_2 = \dot{u}$, as

$$\begin{bmatrix}\dot{\mathbf{u}}_1\\ \dot{\mathbf{u}}_2\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\partial_x^2\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\partial_y^2\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix} + \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}.$$

The associated fundamental state is $\hat{\mathbf{u}} = \partial_x^2\partial_y^2\mathbf{u}$, with corresponding PIE

$$\int_0^x\int_0^y \begin{bmatrix}(x-\theta)(y-\nu) & 0\\ 0 & (x-\theta)(y-\nu)\end{bmatrix}\begin{bmatrix}\dot{\hat{\mathbf{u}}}_1\\ \dot{\hat{\mathbf{u}}}_2\end{bmatrix}d\nu d\theta$$
$$= \int_0^y \begin{bmatrix}0 & 0\\ (y-\nu) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\nu + \int_0^x \begin{bmatrix}0 & 0\\ (x-\theta) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\theta$$
$$+ \int_0^x\int_0^y \begin{bmatrix}0 & (x-\theta)(y-\nu)\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\nu d\theta$$

For BCs $u \equiv 0$ on $\partial\Omega$, stability can be verified with PIETOOLS.

Examples

Ex. 1: Advection Equation

For the simple advection equation

$$\dot{u}(t,x,y) = C\partial_x u(t,x,y) + C\partial_y u(t,x,y)$$
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we have $\mathbf{u}(t) = \mathbf{u}_1(t) = u(t) \in H_1\left[[0,1]^2\right]$. The associated fundamental state is $\hat{\mathbf{u}} = \partial_x\partial_y \mathbf{u}$, with corresponding PIE

$$\int_0^x \int_0^y \dot{\hat{\mathbf{u}}}(t,\theta,\nu)d\nu d\theta = C\int_0^y \hat{\mathbf{u}}(t,x,\nu)d\nu + C\int_0^x \hat{\mathbf{u}}(t,\theta,y)d\theta.$$

Ex. 3: Reaction-Diffusion Equation

For the 2D reaction-diffusion equation on $(x,y) \in [0,1]^2$,

$$\dot{u}(x,y) = u_{xx}(x,y) + u_{yy}(x,y) + \lambda u(x,y)$$
$$u(0,y) = u(1,y) = u(x,0) = u(x,1) = 0,$$

stability can be proven analytically for any $\lambda \leq 2\pi^2 = 19.739....$ Using PIETOOLS, stability was verified for any $\lambda \leq 19.736$.

Ex. 2: Wave Equation

The wave equation on $(x,y) \in [0,1]^2$

$$\ddot{u}(x,y) = \partial_x^2 u(x,y) + \partial_y^2 u(x,y)$$
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The associated fundamental state is $\hat{\mathbf{u}} = \partial_x^2\partial_y^2\mathbf{u}$, with corresponding PIE

$$\int_0^x\int_0^y \begin{bmatrix}(x-\theta)(y-\nu) & 0\\ 0 & (x-\theta)(y-\nu)\end{bmatrix}\begin{bmatrix}\dot{\hat{\mathbf{u}}}_1\\ \dot{\hat{\mathbf{u}}}_2\end{bmatrix}d\nu d\theta$$
$$= \int_0^y \begin{bmatrix}0 & 0\\ (y-\nu) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\nu + \int_0^x \begin{bmatrix}0 & 0\\ (x-\theta) & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\theta$$
$$+ \int_0^x\int_0^y \begin{bmatrix}0 & (x-\theta)(y-\nu)\\ 0 & 0\end{bmatrix}\begin{bmatrix}\hat{\mathbf{u}}_1\\ \hat{\mathbf{u}}_2\end{bmatrix}d\nu d\theta$$

For BCs $u \equiv 0$ on $\partial\Omega$, stability can be verified with PIETOOLS.