Modern Control Systems

Matthew M. Peet
Illinois Institute of Technology

Lecture 13: Linear Analysis

Linear Analysis

We will now temporarily skip realization theory in favor of linear analysis.

$$y_e = Gu_e$$

We now view systems only in terms of inputs and outputs.

- We also have control inputs and outputs

$$\begin{bmatrix} y_e \\ y_c \end{bmatrix} = G \begin{bmatrix} u_e \\ u_c \end{bmatrix}$$

- More on this later

Normed Spaces

Recall about normed Spaces

Definition 1.

A **Norm** on a vector space, V, is a function $\|\cdot\| : V \to \mathbb{R}^+$ such that

1. $\|x\| = 0$ if and only if $\|x\| = 0$
2. $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$
3. $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$

Norms only satisfy Pythagorean Theorem

Definition 2.

A vector space with an associated norm is called a **Normed Space**.

Normed Spaces

Recall examples of normed spaces
On $\mathbb{R}^n$:

- $\|x\|_1 = \sum_{i=1}^n |x_i|$ (Taxicab norm)
- $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$ (Euclidean norm)
- $\|x\|_p = \sqrt[p]{\sum_{i=1}^n x_i^p}$
- $\|x\|_\infty = \max |x_i|$

On infinite sequences $g : \mathbb{N} \to \mathbb{R}$

- $\|f\|_{\ddagger_1} = \sum_{i=1}^\infty |g_i|$
- $\|f\|_{\ddagger_2} = \sqrt{\sum_{i=1}^\infty g_i^2}$
- $\|f\|_{\ddagger_p} = \sqrt[p]{\sum_{i=1}^\infty g_i^p}$
- $\|f\|_{\ddagger_\infty} = \max_{i=1,\ldots,\infty} |g_i|$

On functions $f : [0,1] \to \mathbb{R}$

- $\|f\|_{L_1} = \int_0^1 |f(s)| ds$
- $\|f\|_{L_2} = \sqrt{\int_0^1 f(s)^2 ds}$
- $\|f\|_{L_p} = \sqrt[p]{\int_0^1 f(s)^p ds}$
- $\|f\|_{L_\infty} = \sup_{s \in [0,1]} |f(s)|$

Norms define what is meant by convergence of a sequence.

Definition 3.

We say that

$$\lim_{i\to\infty} x_i = y$$

if for every $\epsilon > 0$, there exists a N such that

$$\|y - x_i\| \le \epsilon \qquad \text{for all } i > N.$$

Or the limit of a function $f : X \to V$.

Definition 4.

For normed spaces X and Y, we say that

$$\lim_{x\to y} f(x) = z$$

if for every $\epsilon > 0$, there exists a β such that

$$\|x - y\|_X \le \beta$$

implies

$$\|f(x) - z\|_Y \le \epsilon.$$

For function $f : X \to V$, suppose that

$$\lim_{x \to y} f(x) = z$$

Question: does this imply that $z \in V$?
Question: Does every function have a limit?
Answer: It depends on the norm of V

Definition 5.

A sequence x_i is a **Cauchy Sequence** if for any $\epsilon > 0$, there exists an N such that

$$\|x_i - x_j\| \leq \epsilon$$

for all $i, j > N$.

This is a definition of a convergent sequence without the inconvenience of requiring the existence of a limit

- Otherwise, we need to find the limit to prove convergence.
- Now we just show the elements get closer together.

Question: Are all convergent sequences Cauchy?

Lemma 6.

Any convergent sequence is Cauchy.

Whether all Cauchy sequences converge depends on the norm.

Definition 7.

A normed space, V, is **Complete** if every Cauchy sequence converges to a point in V.

- A complete normed space is called a **Banach Space**

In a Banach Space, if a sequence converges, it converges to a point in the space

Banach Space

Example

For any p, the space of of functions $L_p(-\infty,\infty)$ is a Banach Space.

On infinite sequences $g:\mathbb{N}\to\mathbb{R}$

- $\|f\|_{\updownarrow_1}=\sum_{i=1}^{\infty}|g_i|$
- $\|f\|_{\updownarrow_2}=\sqrt{\sum_{i=1}^{\infty}g_i^2}$
- $\|f\|_{\updownarrow_p}=\sqrt[p]{\sum_{i=1}^{\infty}g_i^p}$
- $\|f\|_{\updownarrow_\infty}=\max_{i=1,\ldots,\infty}|g_i|$

On functions $f:[-\infty,\infty]\to\mathbb{R}$

- $\|f\|_{L_1}=\int_{-\infty}^{\infty}|f(s)|ds$
- $\|f\|_{L_2}=\sqrt{\int_{-\infty}^{\infty}f(s)^2ds}$
- $\|f\|_{L_p}=\sqrt[p]{\int_{-\infty}^{\infty}f(s)^pds}$
- $\|f\|_{L_\infty}=\sup_{s\in[-\infty,\infty]}|f(s)|$

Also the $L_p[0,1]$ spaces are complete

Lemma 8.

A subspace of a Banach Space is complete if and only if it is closed.

Example: The subspace

$$L_p[0,\infty):=\{f\in L_p(-\infty,\infty):f(t)=0\quad \text{for } t<0\}$$

Question: is $L_p(0,\infty)$ closed?

Banach Space

Example

Let $C[0,1]$ be the set of **continuous** functions with norm

$$\|f\| = \|f\|_{L_1} = \int_0^1 |f(s)| ds$$

To show that this is **NOT** a Banach space, define the sequence of functions $x_i \in C[0,1]$

$$x_i(t) = \begin{cases} 0 & t \le \frac{1}{2} - \frac{1}{n} \\ 1 - \frac{n}{2} + nt & t \in [\frac{1}{2} - \frac{1}{n}, \frac{1}{2}] \\ 1 & t \ge \frac{1}{2} \end{cases}$$

The sequence is Cauchy since

$$\|x_i - x_j\| = \frac{1}{2} \left|1/i - 1/j\right| \to 0$$

However, there is obviously no **continuous** limit.

Inner Product Spaces

Now we get to a really important concept

Definition 9.

An **Inner Product** on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, such that

1. $\langle x, x \rangle \geq 0$ for all $x \in V$.
2. $\langle x, x \rangle = 0$ if and only if $x = 0$.
3. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$. [**Linearity**]
4. $\langle x, y \rangle = \langle y, x \rangle$.

Definition 10.

A vector space with an inner product is called a **Inner Product Space**

- Any inner product space is a normed space using

$$\|x\|_V^2 = \langle x, x \rangle_V$$

An inner product space has the concept of an angle between vectors.

Theorem 11 (Cauchy Schwartz).

If $\|x\|^2 = \langle x, x \rangle$, *then*

$$|\langle x, y \rangle| \le \|x\| \|y\|$$

IMPORTANT:

- Only norms derived from inner products satisfy Cauchy-Schwartz.

Pythagorean Theorem

Inner Product Spaces allow for “right angles”.

Definition 12.

x and y are orthogonal in inner product space V, denoted $x \perp y$, if

$$\langle x, y \rangle_V = 0$$

Pythagorean Theorem

Theorem 13.

For x and y in inner product space V,

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2$$

if and only if $x \perp y$.

Inner Product Spaces

Some inner product spaces:
Euclidean Space On R^n

$$\langle x, y\rangle_2 = x^T y = \sum_{i=1}^{n} x_i y_i$$

The Frobenius Norm on Matrices $\mathbb{R}^{n\times m}$

$$\langle A, B\rangle = \operatorname{trace}(A^T B) = \sum_{i=1}^{n}\sum_{i=1}^{m} A_{ij}B_{ij}$$

which induces the Frobenius norm

$$\|X\|^2 = \langle X, X\rangle = \sum_{i=1}^{n}\sum_{i=1}^{m} X_{ij}^2$$

Definition 14.

An inner product space which is complete in the norm $\|x\|^2 = \langle x, x \rangle$ is called a **Hilbert Space**.

Hilbert spaces are actually quite unusual.

Example: Define the following inner product on $L_2[0, \infty)$:

$$\langle x, y \rangle_{L_2} := \int_0^\infty x^T(s) y(s) ds$$

Then

$$\|x\|_{L_2}^2 = \int_0^\infty \|x(s)\|^2 ds$$

And since L_2 is complete in this norm, $L_2[0, \infty)$ is a Hilbert Space.

$\updownarrow_p$-Spaces

- $\|f\|_{\updownarrow_p} = \sqrt[p]{\sum_{i=1}^{\infty} g_i^p}$
- $\|f\|_{\updownarrow_\infty} = \max_{i=1,\ldots,\infty} |g_i|$

L_p-Spaces

- $\|f\|_{L_p} = \sqrt[p]{\int_{-\infty}^{\infty} f(s)^p ds}$
- $\|f\|_{L_\infty} = \sup_{s\in[-\infty,\infty]} |f(s)|$

Neither $\updownarrow_p$ nor L_p are Hilbert spaces for $p \neq 2$.

Definition 15.

$C[0,\infty)$ is the space of continuous functions with norm

$$\|f\|_\infty = \sup_t \|f(t)\|$$

- $C[0,\infty)$ is a Banach Space.
- $C[0,\infty)$ is not a Hilbert space.