SOS Methods for Delay-Dependent Stability of Neutral Differential Systems

Matthew M. Peet, Catherine Bonnet and Hitay Özbay

***Abstract*— This paper gives a description of how "sum-of-squares" (SOS) techniques can be used to check conditions for the stability of neutral differential systems. We adapt an approach of Zhang et al. [10] and show how conditions associated with these results can be expressed as the infeasibility of certain semialgebraic sets. Then, using Positivstellensatz results from semi-algebraic geometry, we convert these infeasibility conditions to feasibility problems using sum-of-squares variables. By bounding the degree of the variables and using the Matlab toolbox SOSTOOLS [7], these conditions can be checked using semidefinite programming**

I. Introduction

In this paper, we consider the problem of computability of certain frequency-domain tests for delay-dependant stability of delay systems of the neutral type. We show that several frequency-domain conditions can be reduced to optimization problems on the cone of positive semidefinite matrices. Such an approach is an alternative to the classical graphical tests (e.g. the Nyquist criterion). The motivation for using optimization-based methods is as follows

a) The computational complexity of these methods is well-established. Computational complexity provides a standard benchmark for the difficulty of a given problem. A condition which is expressed as an optimization problem, therefore, will have well-known computational properties.
b) While the accuracy of a result based on graphical methods may be limited by the resolution and range of the plot, a feasible result from an optimization-based method serves as a readily verifiable certificate of stability.

The use of frequency domain criteria for analysis of linear systems has an extensive history and we will not make an attempt to catalogue the list of accomplishments in this field. We do note, however, that while for finite-dimensional systems, time-domain-based LMI methods currently compete successfully with frequency-domain-based graphical criteria, the same can not be said to be true for infinite-dimensional systems.

In [2], we have already considered some simple delay-independant stability conditions for systems of the neutral type using SOS techniques. Our aim here is to develop more sophisticated methods which will allow us to check the delay-dependant H_∞-stability of neutral systems. The paper is organized as follows. We begin in section II by recalling some background on polynomial optimization and "sum-of-squares". In section III, we show how a method based on dichotomy arguments which was proposed by Zhang *et al* [10] for the stability analysis of retarded-type delay systems can also be applied to the case of neutral-type systems. The main theorems which enable us to formulate delay-dependant stability in terms of feasibility of semi-algebraic sets and SOS polynomials conditions are given. It is shown that the problem of the appearance of asymptotic chains of roots, common for neutral systems, is solved by the Zhang method of rational over-approximation. In Section IV we give some numerical examples which show the efficacy of the proposed method. Finally a conclusion is given in section V.

H. Özbay is with the Department of Electrical and Electronics Engineering, Bilkent University, Bilkent, 06800 Ankara, Turkey `hitay@bilkent.edu.tr`.

M. M. Peet and C. Bonnet are with INRIA-Rocquencourt, Domaine de Voluceau, Rocquencourt BP105 78153, Le Chesnay Cedex, France. `matthew.peet@inria.fr`, `catherine.bonnet@inria.fr`.

II. The Positivstellensatz and Sum-of-Squares

A polynomial, p, is said to be *positive* on $G \subset \mathbb{R}^n$ if

$$p(x) \geq 0 \quad \text{for all } x \in G.$$

If G is not mentioned, then it is assumed $G = \mathbb{R}^n$. A *semialgebraic set* is a subset of $\mathbb{R}^n$ defined by polynomials p_i, as

$$G := \{x \in \mathbb{R}^n \,:\, p_i(x) \geq 0,\, i = 1, \ldots, k\}.$$

Given a polynomial, the question of whether it is positive has been shown to be NP-hard. A polynomial, p, is said to be sum-of-squares (SOS) in variables x, denoted $p \in \Sigma_s[x]$ if there exist a finite number of other polynomials, g_i such that

$$p(x) = \sum_{i=1}^{k} g_i(x)^2.$$

A sum-of-squares polynomial is positive, but a positive polynomial may not be sum-of-squares. A necessary and sufficient condition for the existence of a sum-of-squares representation for a polynomial, p, of degree $2d$ is the existence of a positive semidefinite matrix, Q, such that

$$p(x) = Z(x)^T Q Z(x),$$

where Z is any vector whose elements form a basis for the polynomials of degree d. Positivstellensatz results are "theorems of the alternative" which say that either a semialgebraic set is feasible or there exists a sum-of-squares refutation of feasibility. The Positivstellensatz that we use in this paper is that given by Stengle [9].

Theorem 1 (Stengle): The following are equivalent

1)
$$\left\{x : \begin{matrix} p_i(x) \geq 0 & i = 1, \ldots, k \\ q_j(x) = 0 & j = 1, \ldots, m \end{matrix}\right\} = \emptyset$$

2) There exist $t_i \in \mathbb{R}[x]$, $s_i, r_{ij}, \ldots \in \Sigma_s[x]$ such that
$$-1 = \sum_i q_i t_i + s_0 + \sum_i s_i p_i + \sum_{i \neq j} r_{ij} p_i p_j + \cdots$$

We use $\mathbb{R}[x]$ to denote the real-valued polynomials in variables x. For a given degree bound, the conditions associated with Stengle's Positivstellensatz can be represented as a semidefinite program. Note that, in general, no such upper bound on the degree bound will be known a-priori.

III. Stability of Neutral-Type Systems

In this section we consider aspects of the stability of general neutral-type systems of the form
$$\dot{x}(t) = \sum_{i=1}^{m} B_i \dot{x}(t - \tau_i) + \sum_{j=0}^{m} A_j x(t - \tau_j) \tag{1}$$
where $A_i, B_j \in \mathbb{R}^{n \times n}$ and $\tau_i \geq 0$.

A. Dichotomy Methods

The term *dichotomy* is used to describe methods based on the bisection of the complex plane into right and left half-planes. These methods are based on a continuity argument that states that if a system is stable for one value of a parameter and unstable for another value, then for some intermediate value of the parameter, the system must have a pole on the imaginary axis. For retarded and neutral-type systems which satisfy certain conditions, this argument is valid due to the following theorem from Datko [3].

Theorem 2 (Datko): Let
$$G_\alpha(s) := \left[s \left(I - \sum_{i=1}^{m} B_i e^{-\alpha s \gamma_i} \right) - \sum_{j=0}^{m} A_j e^{-\alpha s \gamma_j} \right].$$
If
$$\det \left[I - \sum_{i=1}^{m} B_i e^{-s \gamma_i} \right] = 0$$
has all roots lying in some left half-plane $\operatorname{Re} s \in (-\infty, -\beta_0]$, $\beta_0 > 0$, then
$$\sigma(G_\alpha) := \sup_{\det G_\alpha(s) = 0} \operatorname{Re} s$$
is continuous on $\alpha \in [0, \infty)$.

B. Generalization of a method of Zhang et al.

In this subsection, we consider the approach of [10], wherein the exponential term is "covered" by a set of rational transfer functions parameterized by a single parameter. The size of the set, or the range of values of the parameter, is determined by the degree of the rational functions. We begin with the definition of the Padé approximate of e^{-s}.
$$R_m(s) = \frac{P_m(s)}{P_m(-s)}$$
where
$$P_m(s) = \sum_{k=0}^{m} \frac{(2m-k)!m!(-s)^k}{(2m)!k!(m-k)!}.$$

Now define the sets of irrational and rational functions.
$$\Omega_d(\omega, h) := \{e^{-\imath \tau \omega} \,:\, \tau \in [0, h]\}$$
$$\Omega_m(\omega, h) := \{R_m(\imath \alpha_m \tau \omega) \,:\, \tau \in [0, h]\}$$

Where $\alpha_m := \frac{1}{2\pi} \min\{\omega > 0 \,|\, R_m(\imath \omega) = 1\}$. For example, $\alpha_3 = 1.2329$, $\alpha_4 = 1.0315$, and $\alpha_5 = 1.00363$.

The following is a key result of [10].

Lemma 3 (Zhang et al.): For every integer $m \geq 3$, the following statements hold.

1) All poles of $R_m(s)$ are in the open left half complex plane.
2) $\Omega_d(\omega, h) \subset \Omega_m(\omega, h)$ for any $h \geq 0$ and $\omega \geq 0$.
3) $\lim_{m \to \infty} \alpha_m = 0$

Typically, Lemma 3 is used to prove that a delay-differential system has no poles in the closed right half-plane for an interval of delay of the form $[0, h]$. This is illustrated by the following theorem.

Theorem 4: Suppose G satisfies the conditions of Theorem 2 and $\det G(s) := \sum_{i=1}^{n} q_i(s) e^{-\tau_i s}$. Let $m \geq 3$, and suppose that
$$\{s \in \mathcal{C} \,:\, \sum_{i=1}^{n} q_i(s) = 0,\, \operatorname{Re} s \geq 0\} = \emptyset$$
and
$$\{\omega \geq 0,\, \tau_i \in [\,0,\, h_i\,] \,:\, \sum_{i=1}^{n} q_i(\omega \imath) R_m(\alpha_m \tau_i \omega \imath) = 0\} = \emptyset.$$
Then
$$\{s \in \mathcal{C},\, \tau_i \in [\,0,\, h_i\,] \,:\, \sum_{i=1}^{n} q_i(s) e^{-\tau_i s} = 0,\, \operatorname{Re} s \geq 0\} = \emptyset.$$

Theorem 4 is a trivial generalization of the work of Zhang et al. [10] to neutral-type systems. For retarded-type systems, the work of [10] proposed the construction of a parameter-dependent state-space system. In a later work, [1] proposed the use of a generalized version of the KYP lemma to check the conditions associated with the retarded case through the construction of a perturbed singular system. In this paper, we use a "sum-of-squares" approach based on the application of the Positivstellensatz results described in Section II.

For convenience, define the following functions.
$$g_r(\omega, \tau) := \operatorname{Re}\Big(\sum_{i=1}^{n} q_i(\omega \imath) P_m(\alpha_m \tau_i \omega \imath) \prod_{\substack{j=1 \\ j \neq i}}^{n} P_m(-\alpha_m \tau_j \omega \imath) \Big)$$
$$g_i(\omega, \tau) := \operatorname{Im}\Big(\sum_{i=1}^{n} q_i(\omega \imath) P_m(\alpha_m \tau_i \omega \imath) \prod_{\substack{j=1 \\ j \neq i}}^{n} P_m(-\alpha_m \tau_j \omega \imath) \Big)$$

Lemma 5: The following are equivalent

1) The following set is infeasible.

$$\{\omega \geq 0,\, \tau_i \in [\,0,\, h_i\,] \,:\, \sum_{i=1}^{n} q_i(\omega\imath) R_m(\alpha_m \tau_i \omega \imath) = 0\}$$

2) The following real semi-algebraic set is infeasible.

$$\Big\{\omega, \tau_i \in \mathbb{R} \,:\, \omega \geq 0,\;\; \tau(h_i - \tau_i) \geq 0,\\ g_r(\omega, \tau_i) = 0,\, g_i(\omega, \tau_i) = 0\Big\}$$

3) There exist polynomials $t_1, t_2 \in \mathbb{R}[\omega, \tau]$ and SOS polynomials $s_i \in \Sigma_s[\omega, \tau]$ such that

$$\begin{aligned}-1 =& s_0 + t_1 g_r + t_2 g_i + \omega s_1 + \tau_1(h_1 - \tau_1) s_2 \\ &+ \omega\tau_1(h_1 - \tau_1) s_3 + \omega\tau_2(h_2 - \tau_2) s_4 + \cdots\end{aligned}$$

Proof: By Lemma 3, all roots of $P_m(-s)$ are in the open left half complex plane, and so Statement 1 is equivalent to infeasibility of the following set.

$$\Big\{\omega \geq 0,\, \tau_i \in [\,0,\, h_i\,] \,: \\ \sum_{i=1}^{n} q_i(\omega\imath) P_m(\alpha_m \tau_i \omega\imath) \prod_{\substack{j=1\\ j\neq i}}^{n} P_m(-\alpha_m \tau_j \omega\imath) = 0\Big\}$$

Now, $\tau \in [0, h]$ is equivalent to $\tau(h - \tau) \geq 0$ and so Statement 1 is equivalent to infeasibility of the set in Statement 2. Furthermore, the real or imaginary part of a complex polynomial is a polynomial in the real and complex parts of the complex argument. Therefore, the set in Statement 2 is real semi-algebraic.

That Statement 2 is equivalent to Statement 3 is an immediate consequence of Theorem 1. ■

We now give delay-dependent stability conditions for neutral-type systems.

Lemma 6: Define

$$r_m := \\ \inf\left\{\left\|\sum_{i=1}^{n} q_i(\omega\imath) R_m(\alpha_m \tau \omega\imath)\right\| \,:\, \omega \geq 0,\, \tau_i \in [\,0,\, h_i\,]\right\}$$

and

$$r_e := \inf\left\{\left\|\sum_{i=1}^{n} q_i(\omega\imath) e^{-\tau_i \omega\imath}\right\| \,:\, \omega \geq 0,\, \tau_i \in [\,0,\, h_i\,]\right\}.$$

Then for any $m \geq 3$, $r_e \geq r_m$.

Proof: Suppose there exists an $\omega_w \geq 0$ and $\tau_{i,w} \in [0, h_i]$ such that

$$w = \left\|\sum_{i=1}^{n} q_i(\omega_w\imath) e^{-\tau_{i,w}\omega_w\imath}\right\|$$

Now, by Lemma 3, there exists $\tau'_{i,w} \in [0, h_i]$ such that

$$\left\|\sum_{i=1}^{n} q_i(\omega_w\imath) R_m(\alpha_m \tau'_{i,w} \omega_w\imath)\right\| \\ = \left\|\sum_{i=1}^{n} q_i(\omega_w\imath) e^{-\tau_{i,w}\omega_w\imath}\right\| = w$$

Therefore, $r_m \leq w$ and hence $r_m \leq r_e$. ■

Theorem 7: Suppose G_τ satisfies the conditions of Theorem 2 and $\det G_\tau(s) := \sum_{i=1}^{n} q_i(s) e^{-\tau_i s}$. Let $m \geq 3$ and suppose that

$$\{s \in \mathcal{C} \,:\, \sum_{i=1}^{n} q_i(s) = 0,\, \operatorname{Re} s \geq 0\} = \emptyset$$

and that there exist polynomials $t_1, t_2 \in \mathbb{R}[\omega, \tau]$ and SOS polynomials $s_i \in \Sigma_s[\omega, \tau]$ such that

$$\begin{aligned}-1 =& s_0 + t_1 g_r + t_2 g_i + \omega s_1 + \tau_1(h_1 - \tau_1) s_2 \\ &+ \omega\tau_1(h_1 - \tau_1) s_3 + \omega\tau_2(h_2 - \tau_2) s_4 + \cdots\end{aligned}$$

Then the system defined by Equation 1 is H_∞ stable for all $\tau_i \in [0, h_i]$

Proof: If the conditions of the theorem are satisfied, then by Lemma 5

$$H_m(s) := \sum_{i=1}^{n} q_i(s) R_m(\alpha_m \tau_i s)$$

has no roots on the imaginary axis for any $\tau_i \in [0, h_i]$. Therefore, by Theorem 4 $\det G_\tau(s)$ has no roots on the closed right half plane. Therefore $\det G_\tau(s)^{-1}$ is analytic on the closed right half plane. It remains to show that $\det G_\tau(s)^{-1}$ is bounded on the imaginary axis.

Since H_m is a polynomial, then for any $\tau_i \in [0, h_i]$, it has a finite number of roots, none of which are on the imaginary axis. We conclude that for any fixed $\tau_i \in [0, h_i]$,

$$\inf_{\omega \geq 0}\left\|\sum_{i=1}^{n} q_i(\omega\imath) R_m(\alpha_m \tau_i \omega\imath)\right\| > 0$$

Therefore, since the $[0, h_i]$ are compact sets,

$$r_m := \\ \inf\left\{\left\|\sum_{i=1}^{n} q_i(\omega\imath) R_m(\alpha_m \tau_i \omega\imath)\right\| \,:\, \omega \geq 0,\, \tau_i \in [\,0,\, h_i\,]\right\} \\ > 0.$$

Therefore, by Lemma 6,

$$r_e := \\ \inf\left\{\left\|\sum_{i=1}^{n} q_i(\omega\imath) e^{-\tau_i \omega\imath}\right\| \,:\, \omega \geq 0,\, \tau_i \in [\,0,\, h_i\,]\right\} \\ \geq r_m > 0.$$

This proves that

$$\|\det G_\tau^{-1}\|_\infty = \sup_{\omega \geq 0} \|\det G_\tau(\imath\omega)\|^{-1} \leq \frac{1}{r_m}$$

and so $\det G_\tau(s)^{-1} \in H_\infty$ for any $\tau_i \in [0, h_i]$. ■

C. Verifying the Datko Conditions

In this section, we briefly consider the problem of verifying the conditions associated with Theorem 2. In particular, we would like to show that

$$\det\left[I - \sum_{i=1}^{m} B_i e^{-s\gamma_i}\right] = 0$$

has all roots lying in some left half-plane $\operatorname{Re} s \in (-\infty, -\beta_0]$, $\beta_0 > 0$.

A simple sufficient condition, also proposed in [3], is given by the following.

Proposition 8: Suppose $B_i = 0$ for $i = 1, \ldots m-1$ and $\det(\lambda I - B_m)$ has all roots in the disc $|\lambda| < 1$. Then the conditions of Theorem 2 are satisfied.

More generally, if the delays are commensurate, then the condition is equivalent to exponential stability of an expanded discrete-time linear system of the general form

$$\begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix} = \begin{bmatrix} B_1 & \cdots & B_n \\ I & & \\ & I & \end{bmatrix} \begin{bmatrix} x_1(k) \\ \vdots \\ x_n(k) \end{bmatrix}.$$

If the delays are non-commensurate, then an SOS condition can also be given.

Lemma 9: Let $\epsilon > 0$. If

$$\left\{ z_i \in \mathbb{C} : \det\left(I + \sum_{i=1}^{m} B_i z_i\right) = 0, |z_i| \le e^{\epsilon T_i} \right\} = \emptyset,$$

then

$$\det\left[I + \sum_{i=1}^{m} B_i e^{-s\gamma_i}\right] = 0$$

has all roots in the left half-plane $\{\operatorname{Re} s \le -\epsilon\}$ for all $\gamma_i \le T_i$.

Proof: Proof by contradiction. Suppose that there exist $(\gamma_1, \cdots, \gamma_n) \in \mathbb{R}^n$ with $\gamma_i \le T_i$ for $i = 1, \cdots, m$ and $s_0 \in \{\operatorname{Re} s \ge -\epsilon\}$ such that

$$\det\left(I + \sum_{i=1}^{m} B_i e^{-\gamma_i s_0}\right) = 0.$$

Let $s = s_0$ and $z_i = e^{-\gamma_i s_0}$. Then $s_0 \in \{\operatorname{Re} s \ge -\epsilon\}$ and $|z_i| \le e^{\epsilon T_i}$ with

$$\det\left(I + \sum_{i=1}^{n} B_i z_i\right) = 0,$$

which contradicts the statement of the lemma. ■

The conditions of Lemma 9 can be verified using SOS by application of Theorem 1.

IV. Numerical Example

Example 1: In this example, we consider a somewhat arbitrarily chosen example to illustrate the delay-dependent stability condition associated with Theorem7. We use the example chosen by [4], [6], and [5] among many others.

$$\dot{x}(t) - B\dot{x}(t-\tau) = A_0 x(t) + A_1 x(t-\tau)$$

where

$$A_0 = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix},$$
$$B = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}.$$

To test stability, we first verify the Datko conditions, which hold by Proposition 8. We now replace the characteristic equation

$$g(s) = \det\left(sI - A_0 - A_1 e^{-\tau s} - Bse^{-\tau s}\right)$$

with the family of rational approximations R_m to get a new characteristic polynomial family

$$H_m(s,\tau) = \det\Big(sP_m(-\alpha_m \tau s)I - A_0 P_m(-\alpha_m \tau s) - A_1 P_m(\alpha_m \tau s) - BsP_m(\alpha_m \tau s)\Big)$$

We then create the real polynomial functions

$$g_i(\omega,\tau) := \operatorname{Re} H_m(\imath\omega,\tau)$$

and

$$g_r(\omega,\tau) := \operatorname{Re} H_m(\imath\omega,\tau)$$

This can be done automatically in Matlab using the function `cpoly2rpoly` contained in the software package available online at [8].

We now use SOSTools [7] to find polynomials $t_1, t_2 \in \mathbb{R}[\omega,\tau]$ and SOS polynomials $s_i \in \Sigma_s[\omega,\tau]$ for $i = 0, \ldots, 3$ such that

$$-1 = s_0 + t_1 g_r + t_2 g_i + \omega s_1 + \tau(h-\tau)s_2 + \omega\tau(h-\tau)s_3$$

By using this method, we are able to prove stability of the system for the values of delay listed in Table I. This table also lists the degree of the refutation necessary. The accuracy is roughly comparable to what is currently available using existing time-domain methods. Note that the accuracy is restricted only by the value of α_m.

TABLE I
Stability Regions

	$m=3$	$m=4$	$m=5$	actual value
maximum τ	1.805	2.157	2.217	2.2255
required degree	14	18	22	

V. Conclusion

We have a proposed a method to check delay-dependant stability of neutral-type delay systems which involves the Padé approximate of e^{-s} and uses sum-of-squares methods to prove the infeasibility of certain semi-algebraic sets. The method is applied to a standard example from the literature.

REFERENCES

[1] P.-A. Bliman and T. Iwasaki, "Lmi characterisation of robust stability for time-delay systems: singular perturbation approach," in *IEEE Conference on Decision and Control*, 2006.

[2] C. Bonnet and M. M. Peet, "Using the positivstellensatz for stability analysis of neutral delay systems in the frequency domain," in *Seventh IFAC Workshop on Time-Delay Systems*, Sept. 2007.

[3] R. Datko, "A procedure for determination of the exponential stability of certain differential-difference equations," *Quarterly of Applied Mathematics*, vol. 36, pp. 279–292, 1978.

[4] E. Fridman, "New lyapunov-krasovskii functionals for tability of linear retarded and neutral-type systems," *Automatica*, vol. 43, pp. 309–319, 2001.

[5] Q.-L. Han, "On stability of linear neutral systems with mixed delays: A discretized lyapunov functional approach," *Automatica*, vol. 41, no. 7, pp. 1209–1218, 2005.

[6] Y. He, M. Wu, J.-H. She, and G.-P. Liu, "Delay-dependent robust stability criteria for uncertain neutral systems with mixed delays," *Systems & Controls Letters*, vol. 51, no. 1, pp. 57–65, 2004.

[7] P. A. Parrilo, "Web site for SOSTOOLS," 2004, `http://www.cds.caltech.edu/sostools/`.

[8] M. Peet, "Web site for Matthew M. Peet," 2006, `http://www-rocq.inria.fr/~peet`.

[9] G. Stengle, "A nullstellensatz and a positivstellensatz in semialgebraic geometry," *Mathematische Annalen*, vol. 207, pp. 87–97, 1973.

[10] J. Zhang, C. Knospe, and P. Tsiotras, "Stability of linear time-delay systems: A delay-dependant criterion with a tight conservatism bound," in *American Control Conference*, 2002.