

Analysis of Polynomial Time Delay Systems Using the Sum of Squares Decomposition

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Abstract

We present a methodology for analyzing the stability of equilibria of systems described by nonlinear Delay Differential Equations by algorithmically constructing appropriate Lyapunov-Krasovskii functionals for independent-of-delay and delay-dependent stability. The technique uses the sum of squares decomposition of multivariate polynomials and semidefinite programming. Parameterized Lyapunov-Krasovskii functionals can be used for robust stability analysis under parametric uncertainty. We illustrate the methodology using an example from population dynamics.

I. INTRODUCTION

Delay differential equations (DDEs) are important for modelling systems that involve transport and propagation of data; examples include networked systems [1] and modeling maturation and growth in population dynamics [2], [3]. The stability, robust stability and control of such systems have received a lot of interest in the past few years [4], [5], as the presence of delays may induce undesirable effects such as instabilities and performance degradation.

DDEs fall in the category of Functional Differential Equations (FDEs), which differ from Ordinary Differential Equations (ODEs) because the system state belongs to an infinite dimensional space. Assuming local existence and uniqueness of solutions, appropriate Lyapunov functions can be used, just as in the case of ODEs. However, while for the case of ODEs these are functions, in the case of DDEs they are *functionals* owing to the fact that the state belongs in a function space itself.

In the case of linear DDEs, the form of these functionals that is necessary and sufficient for Delay-Dependent (DD) and strong Independent Of Delay (IOD) stability is known [6]–[8]. However, the DD conditions are difficult to test algorithmically. Under restrictions on their structure, convex optimization has been used to construct them but this yielded conservative results on the delay interval guaranteeing stability [9], [10]. This is because constructing the functional structure that is necessary and sufficient for stability amounts to parameterizing the set of positive operators on an infinite-dimensional space. One

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approach is to use Lyapunov functionals with piecewise-linear kernels – these functionals are constructed by solving a set of LMIs whose size depends on the discretization level [11], and as the discretization level is decreased, delay values closer to the boundary of stability can be tested. More recently a new approach was developed in [12] for constructing the complete Lyapunov-Krasovskii functional for linear DDEs, which is based on an explicit parameterization of positive operators and uses the Sum of Squares (SOS) decomposition and semidefinite programming as the computational tool.

As far as nonlinear time delay systems are concerned, the only methodologies centre on the construction of simple Lyapunov certificates for systems of low dimension through a judicious choice for a candidate Lyapunov function [2]. This is the case even for systems described by ODEs, where constructing Lyapunov functions is usually based on system structure and its properties (Volterra, gradient systems etc.). Recently, however, a computational methodology based on the SOS decomposition has been proposed [13]–[15].

In this paper we present a methodology for constructing Lyapunov-Krasovskii functionals with polynomial kernels to assess IOD and DD stability for polynomial time delay systems. Preliminary results have been presented in [16], [17] and the present paper offers significant improvements on the way these functionals can be constructed, as will be emphasized in the sequel. Applications of this approach to Internet congestion control problems have also appeared in [18] and preliminary results on state feedback stabilization have appeared in [19]. The methodology is the first to construct Lyapunov-Krasovskii functionals for polynomial time-delay systems in a completely algorithmic way, that unifies local and robust DD and IOD stability. However, only the single-delay case is presented in this paper in order to simplify the exposition: the case of multiple, incommensurate delays can be treated in a unified way.

In section II we present the proposed methodology and in section III we show how this can be used for the nominal, robust and local IOD and DD stability analysis of nonlinear delayed systems. In section IV we illustrate our results with an example from population dynamics. We conclude the paper in Section V.

A. Notation

\mathbb{R} denotes the reals and \mathbb{R}^n the n –dimensional Euclidean space. For $x \in \mathbb{R}^n$, $\mathbb{R}[x]$ denotes the ring of polynomials in x with real coefficients and $Z_d[x]$ the vector of monomials in x of degree d or less. $C([-\tau, 0], \mathbb{R}^n)$ denotes the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}^n with the topology of uniform convergence. The norm on C is $\|\phi\| = \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|$ where $|\cdot|$ is the infinity norm. Suppose $\sigma \in \mathbb{R}$, $\rho \geq 0$ and $x \in C([\sigma - \tau, \sigma + \rho], \mathbb{R}^n)$; then for any $t \in [\sigma, \sigma + \rho]$, define $x_t \in C$ by $x_t(\theta) = x(t + \theta)$, $\theta \in [-\tau, 0]$. We will be using symbolic independent variables to reference state and delayed state variables. $\hat{x}_l^{(i)}$ will reference $x_i(t - l\tau)$ where $l = 1, \dots, L$ and \hat{x}_l will denote the vector of $\hat{x}_l^{(i)}$'s, $i = 1, \dots, n$. In a similar fashion, $\hat{y}_l^{(i)}$ will be used for $x_i(t + \theta - l\tau)$ and $\hat{z}_l^{(i)}$ for $x_i(t + \xi - l\tau)$. Finally, we will use $\hat{X}_L = [\hat{x}_0, \hat{x}_1, \dots, \hat{x}_L]$ and $\hat{Y}_L = [\hat{y}_0, \hat{y}_1, \dots, \hat{y}_L]$.

II. THE PROPOSED METHODOLOGY

Here we describe an approach to construct Lyapunov functionals for polynomial time-delay systems. Background theory on stability and Lyapunov theory for DDEs can be found in [20].

Consider a time-delay system with a single delay of the form

$$\dot{x}(t) = f(x(t), x(t - \tau)) \quad (1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ with $f(0, 0) = 0$ is such that a unique solution to the above delay-differential equation exists from an appropriate initial condition close to 0. One type of Lyapunov functional that can be used to verify the DD stability of the zero steady-state takes the form

$$V(\phi) = \overbrace{\int_{-\tau}^0 v_1(\phi(0), \phi(\theta), \theta) d\theta}^{V_1} + \overbrace{\int_{-\tau}^0 \int_{-\tau}^0 v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi}^{V_2}. \quad (2)$$

This functional has a derivative of the form

$$\dot{V}(\phi) = \int_{-\tau}^0 v_3(\phi(0), \phi(-\tau), \phi(\theta), \theta) d\theta - \int_{-\tau}^0 \int_{-\tau}^0 \left(\frac{\partial v_2}{\partial \theta} + \frac{\partial v_2}{\partial \xi} \right) d\theta d\xi, \quad (3)$$

where the map v_1, v_2 to v_3 will be presented in later sections.

The positivity of V_1 in the Lyapunov functional (2) can be characterized using the following Theorem, the proof of which can be found in the appendix.

Theorem 1: Consider a continuous function $v(x, y(\theta), \theta)$ that satisfies the conditions of Lemma 14. Then

$$\int_{-\tau}^0 v(x, y(\theta), \theta) d\theta \geq 0 \text{ for all } x \in \mathbb{R}^m \text{ and } y \in C, \quad (4)$$

if and only if there exists a continuous function $r(x, \theta)$ such that for all $x \in \mathbb{R}^m, z \in \mathbb{R}^n$,

$$v(x, z, \theta) + r(x, \theta) \geq 0 \text{ for all } \theta \in [-\tau, 0], \quad \int_{-\tau}^0 r(x, \theta) d\theta = 0. \quad (5)$$

Non-negativity of V_2 can be guaranteed as follows:

Proposition 2: Given a continuous function $v(y, z, \theta, \xi)$ suppose there exists a continuous vector-valued function $s : (y, \theta) \mapsto \mathbb{R}^d$ for some d so that $v(y, z, \theta, \xi) = (s(y, \theta))^T s(z, \xi)$. Then we have:

$$\int_{-\tau}^0 \int_{-\tau}^0 v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi \geq 0.$$

Proof: Suppose that there exists such a decomposition for v . Then we have:

$$\int_{-\tau}^0 \int_{-\tau}^0 v(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi = \left(\int_{-\tau}^0 s(\phi(\theta), \theta) d\theta \right)^T \left(\int_{-\tau}^0 s(\phi(\xi), \xi) d\xi \right) \geq 0.$$

Therefore the result follows. ■

Theorem 1 and Proposition 2 allow us to reformulate the problem of testing positivity of an integral form to testing certain conditions on its kernel. However, even if we are given, for example, a function v_1 , it may be hard to ensure that condition (5) is satisfied, let alone *construct* it. In order to allow the algorithmic construction of these functions, we make the following assumption:

Assumption 3: We assume that v_1 and v_2 in (2) are *polynomials* in their arguments, and in particular that for some positive integer d ,

$$v_1(\hat{x}_0, \hat{y}_0, \theta) = Z_d^T[\hat{x}_0, \hat{y}_0] P(\theta) Z_d[\hat{x}_0, \hat{y}_0], \quad v_2(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d^T[\hat{y}_0, \theta] R Z_d[\hat{z}_0, \xi].$$

where $P(\theta)$ is a polynomial matrix and R is a matrix of appropriate dimensions. We also assume that the vector field $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1]$ as well as that $r(\hat{x}_0, \theta) \in \mathbb{R}[\hat{x}_0, \theta]$.

Eventhough the above assumption may seem restrictive, especially the restriction that the vector field be polynomial, it is true that in many cases polynomial vector fields arise in modeling of physical processes (e.g., using Volterra-type equations). In some other cases, one can employ a non-polynomial transformation to render a non-polynomial vector field polynomial. See [21], [22] for more details.

Under Assumption 3, all conditions in Theorem 1 and Proposition 2 are polynomial non-negativity conditions, which are in general difficult to test. In fact, it is known that testing non-negativity of polynomials of high degree (more than 4) is NP-hard. A sufficient condition for polynomial non-negativity which is worst-case polynomial-time verifiable by solving a Semidefinite Programme (SDP) is the existence of a sum of squares (SOS) decomposition. More details on positive polynomials and the SOS decomposition can be found in [13], [23]–[25]. Here, we denote by Σ the SOS cone and by Σ_d the subset of Σ of polynomials of degree d or less.

If $a \in \mathbb{R}[x]$ is an SOS, then it is globally nonnegative. In order to ensure that it is positive *definite* and radially unbounded we can use a polynomial ‘shaping’ function $\varphi(x)$:

Proposition 4: Given a polynomial $a(x)$ of degree $2d$, let

$$\varphi(x) = \sum_{i=1}^n \sum_{j=1}^d \epsilon_{ij} x_i^{2j}, \quad \sum_{j=1}^d \epsilon_{ij} \geq \gamma \quad \forall i = 1, \dots, n, \quad (6)$$

with γ a positive number, and $\epsilon_{ij} \geq 0$ for all i and j . Then the condition $a(x) - \varphi(x) \in \Sigma$ guarantees the positive definiteness of $a(x)$, i.e., $a(x) > 0, x \neq 0$. Moreover, $a(x)$ is radially unbounded.

Proof: The function $\varphi(x)$ is positive definite if ϵ_{ij} ’s satisfy the above. Then $a(x) - \varphi(x)$ being SOS implies that $a(x) \geq \varphi(x)$, and therefore $a(x)$ is positive definite. Moreover it is radially unbounded since φ is radially unbounded – it is the positive sum of monomials in only one variable squared. ■

A second issue is related to testing non-negativity of a polynomial over a bounded domain instead of globally. We have already come across Condition (5) in Theorem 1 where polynomial non-negativity is required only for $\theta \in [-\tau, 0]$. We will see similar conditions in the sequel: when studying local stability the non-negativity conditions will only be required to hold in some region of the state-space; when studying robust stability, these conditions will be required to hold for parameters inside a parameter set, etc. These conditional satisfiability conditions can be tested using a generalization to the S-procedure, which is based on Putinar’s representation [26] in Real Algebraic Geometry.

Given $p \in \mathbb{R}[x]$, suppose we want to ensure that $p(x) > 0$ on the set $D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \dots, N_1\}$. Then one can search for Lagrange-type multipliers $\lambda_i \in \Sigma_k$ so that $p(x) + \sum_{i=1}^{N_1} \lambda_i(x) g_i(x) \in \Sigma$. Searching for $\lambda_i(x)$ of a fixed degree k so that the above expression is SOS is a semidefinite programme. Note that if p and g_i are quadratic forms and λ_i are constants, the above test is indeed the S-procedure, which can fail if $i \geq 2$. However, it has been shown in [26] that if D is compact and another mild condition holds on the $g_i(x)$ (the highest degree homogeneous parts of the g_i ’s have no common zeros

in \mathbb{R}^n except at 0), then there is a k for which the above test will succeed – it is indeed a necessary and sufficient condition. Other tests can also be formulated, which may have different properties. See e.g. [27].

Finally, recall that the number of monomials in n variables of degree less than or equal to $m/2$, where m is even is $\binom{n-1+m/2}{m/2}$. This is the size of the LMI that can be used to show that $p \in \mathbb{R}[x]$ in n variables of order m is SOS. The number of variables, however, can be large.

III. ANALYSIS OF POLYNOMIAL TIME DELAY SYSTEMS

Testing the stability of the steady-state of a time delay system is a difficult task even if the vector field is linear; for example, establishing whether a linear system with multiple delays is stable independent of the delay or stable for all $\tau_i \in [0, \bar{\tau}_i)$ with $\bar{\tau}_i \in \mathbb{R}_+$, $i = 1, \dots, K$ given, is known to be NP-Hard [28]. Although obtaining an exact answer to these questions would probably be computationally intractable, sufficient stability tests that are algorithmically verifiable can still be formulated which may answer the stability question for some problem instances. In this case it is convenient to provide a nested family of such tests, each of which is at least as powerful as the previous one, but comes with an increased computational cost. This is the approach we follow here.

A. Independent-of-Delay (IOD) stability

A steady-state of a time-delay system is IOD stable if it is stable for all fixed values of the delay. IOD stability conditions are used in controller synthesis when the size of the delay is unknown [29].

Given a linear time-delay system of the form $\dot{x} = A_0x(t) + A_1x(t - \tau)$, “*strong IOD stability*” is equivalent to $\det(sI_n - A_0 - zA_1) \neq 0$ for all $s = j\omega$, and z in the closed unit disk [7]. This property is stronger than the standard notion of IOD stability, for which only $\det(sI_n - A_0 - e^{-s\tau}A_1) \neq 0$ for all $s = j\omega$ and $\tau \geq 0$ is required. The property is, however, robust with respect to perturbations in A_0 and A_1 . In [7] the class of Lyapunov functionals necessary and sufficient for strong IOD stability of linear systems has been characterized. The system $\dot{x} = A_0x(t) + A_1x(t - \tau)$ is strongly IOD stable if and only if it possesses, for a certain $L \in \mathbb{N}$, a Lyapunov functional of the form

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^0 v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau)) d\theta \quad (7)$$

where v_0 and v_1 are *quadratic* polynomials.

Since for general nonlinear time-delay systems the appropriate structure of a Lyapunov functional is not known, we will generalize the above structure for polynomial time-delay systems. This is the improvement from the results that appeared in [16], [17], which used $L = 1$. We will assume that $f \in \mathbb{R}[\hat{x}_0, \hat{x}_1]$ and for the time being that 0 is the only steady-state of the system. We have the following conditions for IOD stability:

Proposition 5: Consider the system described by Equation (1). For a positive integer L , suppose there exist polynomials $v_0, v_1 : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$, a positive definite, radially unbounded polynomial $\varphi : \mathbb{R}^{n(L+1)} \rightarrow$

\mathbb{R} and a non-negative polynomial $\psi : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:

- 1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) \geq 0$,
- 2) $v_1(\hat{Y}_L) \geq 0$,
- 3) $\sum_{l=0}^L \nabla_{\hat{x}_l} v_0(\hat{X}_L)^T f(\hat{x}_l, \hat{x}_{l+1}) + v_1(\hat{x}_0, \dots, \hat{x}_L) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}) + \psi(\hat{X}_L) \leq 0$.

Then the 0 steady-state is globally IOD stable. If moreover ψ is positive definite, then the 0 steady-state is globally asymptotically IOD stable.

Proof: Consider the functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^0 v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau)) d\theta.$$

The first two constraints impose that $V(\phi)$ is non-negative and radially unbounded. The derivative of V along the trajectories of system (1) is:

$$\begin{aligned} \dot{V}(\phi) = & \sum_{l=0}^L \nabla_{\phi(-l\tau)} v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau))^T f(\phi(-l\tau), \phi(-(l+1)\tau)) \\ & + v_1(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) - v_1(\phi(-\tau), \phi(-2\tau), \dots, \phi(-(L+1)\tau)) \end{aligned}$$

Under the third condition the above derivative is non-positive. Therefore if all three conditions are satisfied, then by the Lyapunov-Krasovskii Theorem [20] the steady-state of the system given by (1) is globally stable; since the delay size does not appear explicitly in the above conditions, then the zero steady-state is globally stable independent of delay. Moreover if $\psi > 0$, then the zero steady-state is globally asymptotically stable independent of delay. ■

The functional V is the functional given by Equation (7), but we used it to analyze stability of polynomial systems. In particular, if the system (1) were linear, we recover the conditions given in [7]. Compared to the Lyapunov functional given by (2) that was investigated in the previous section, only the first term of that expression is used, and the kernel $v_1(\phi(0), \phi(\theta), \theta)$ is only a function of $\phi(0)$ and $\phi(\theta)$ with no cross-terms. These restrictions ensure that the delay, τ , does not appear explicitly either in the Lyapunov positivity or the derivative non-positivity conditions.

Making use of Assumption 3, the conditions in the above Proposition can be tested algorithmically using the SOS decomposition and SOSTOOLS [30]. The functions φ and ψ can be constructed as per Equation (6). For illustration, the following is a simple example.

Example 6: Consider the system:

$$\dot{x}_1(t) = -x_1(t) + x_2^2(t - \tau), \quad \dot{x}_2(t) = -x_2(t)$$

This system is IOD stable, a property which can be proven by constructing a Lyapunov functional $V(z_0)$ with v_0 and v_1 polynomials of bounded degree. When v_0 and v_1 are constrained to be second order polynomials, no certificate is found. However, when we search for fourth order polynomials, a certificate of stability is obtained. The functional and its derivative are

$$\begin{aligned}
V(x_t) &= x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 + \int_{-\tau}^0 x_2^4(t+\theta)d\theta. \\
-\dot{V}(x_t) &= (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) + x_2^2(t)x_2^2(t-\tau) + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2 + \frac{14}{16}x_1(t)^2.
\end{aligned}$$

To get this certificate, only a handful of SOSTOOLS commands are required.

We now concentrate on local and robust IOD stability analysis.

1) *Local Stability*: Nonlinear systems may have more than one equilibria, or the stability properties of a steady-state may not be global. In order to obtain a local result, we have to restrict our attention to a region $\Omega \subset C$ around the equilibrium of interest. In the sequel, we will be using the following form for the region Ω , but other descriptions can also be accommodated:

$$\Omega = \left\{ x_t \in C : \|x_t\| = \sup_{-\tau \leq \theta \leq 0} |x(t+\theta)| \leq \gamma \right\}$$

Related to this is the semi-algebraic set $\{y \in \mathbb{R} : h(y) := (y - \gamma)(y + \gamma) \leq 0\}$, which we will use to describe (conservatively) the set Ω . We now formulate conditions for stability that impose validity of the corresponding Lyapunov conditions when $x_t \in \Omega$.

Proposition 7: Let 0 be a steady-state of system (1) and given γ , let $h(y) = (y - \gamma)(y + \gamma)$. For an integer $L \geq 0$, let there exist functions $v_0, v_1 : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$, a positive definite function $\varphi : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$, non-negative functions $\psi, \check{p}_{ij} : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$, $i = 1, \dots, n$, $j = 0, \dots, L$ and non-negative functions $\check{q}_{ij} : \mathbb{R}^{n(L+2)} \rightarrow \mathbb{R}$ for $i = 1, \dots, n$, $j = 0, \dots, L+1$ such that the following hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, and $\hat{x}_{L+1} \in \mathbb{R}^n$:

- 1) $v_0(\hat{X}_L) - \varphi(\hat{X}_L) + \sum_{j=0}^L \sum_{i=1}^n \check{p}_{ij}(\hat{X}_L) h(\hat{x}_j^{(i)}) \geq 0$,
- 2) $v_1(\hat{Y}_L) \geq 0$,
- 3) $\sum_{l=0}^L \nabla_{\hat{x}_l} v_0(\hat{X}_L)^T f(\hat{x}_l, \hat{x}_{l+1}) + v_1(\hat{x}_0, \dots, \hat{x}_l) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}) + \psi(\hat{X}_L) - \sum_{j=0}^{L+1} \sum_{i=1}^n \check{q}_{ij}(\hat{X}_{L+1}) h(\hat{x}_j^{(i)}) \leq 0$,

where $\hat{X}_{L+1} = [\hat{x}_0, \hat{x}_1, \dots, \hat{x}_{L+1}]$. Then 0 is (locally) IOD stable. If moreover $\psi(\hat{X}_L) > 0$, then 0 is (locally) IOD asymptotically stable.

Proof: Consider the functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^0 v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau)) d\theta.$$

When $h(\phi_i(-j\tau)) \leq 0$ and $\check{p}_{ij}(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) \geq 0$ for $i = 1, \dots, n$ and $j = 0, \dots, L$, we have from the first two conditions:

$$\begin{aligned}
V(\phi) &= v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) + \int_{-\tau}^0 v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau)) d\theta \\
&\geq - \sum_{j=0}^L \sum_{i=1}^n \check{p}_{ij}(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) h(\phi_i(-j\tau)) + \varphi(\phi(0), \phi(-\tau), \dots, \phi(-L\tau)) > 0
\end{aligned}$$

and so the first Lyapunov condition is satisfied, i.e., $V > 0$ on Ω . The same is true for the derivative condition, given constraint (3) above, and so the zero steady-state of system (1) is locally IOD stable. If $\psi > 0$, then $-\frac{dV}{dt} > 0$ on Ω and so asymptotic stability independent of delay is concluded. ■

The conditions in the above Proposition can be tested algorithmically if we assume a polynomial structure of $v_0(\hat{X}_L)$, $v_1(\hat{Y}_L)$, $\tilde{p}_{ij}(\hat{X}_L)$ for $i = 1, \dots, n$, $j = 0, \dots, L$ and $\tilde{q}_{ij}(\hat{X}_{L+1})$ for $i = 1, \dots, n$, $j = 0, \dots, L+1$; construct $\varphi(\hat{X}_L)$ and possibly $\psi(\hat{X}_L)$ using Equation (6); and replace non-negativity conditions by the existence of an SOS decomposition for them. SOSTOOLS can then be used to construct these polynomial functions algorithmically.

At this point we should note that having obtained a Lyapunov function that is valid locally and proves asymptotic stability, the domain of attraction of the steady-state can be estimated as the maximal level set of V that is contained in Ω^L . This can also be formulated as an SOS programme, but it is beyond the scope of this paper.

2) *Robust Stability*: Another important issue is robust stability under parametric uncertainty, which can be treated in a unified way as we will see in the sequel. Consider a time-delay system of the form (1) with an uncertain parameter p :

$$\dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), \tilde{x}(t - \tau), p), \quad (8)$$

where $p \in P$, where P is a semi-algebraic set defined by

$$P = \{p \in \mathbb{R}^l \mid q_i(p) \leq 0, \ i = 1, \dots, N\} \quad (9)$$

where $q_i \in \mathbb{R}[p]$. Define a new variable $x(t) := \tilde{x}(t) - \tilde{x}^*$, where \tilde{x}^* is a steady-state of (8), which satisfies $\tilde{f}(\tilde{x}^*, \tilde{x}^*, p) = 0$, and may change as the parameters $p \in P$ vary. Then we have:

$$\dot{x}(t) = \tilde{f}(x(t) + \tilde{x}^*, x(t - \tau) + \tilde{x}^*, p) = f(x(t), x(t - \tau), p, \tilde{x}^*) \quad (10)$$

$$0 = \tilde{f}(\tilde{x}^*, \tilde{x}^*, p) \quad (11)$$

The transformed system has a steady-state at the origin. We assume for simplicity that there is a single steady-state for any given values of the parameters. Then the stability of this steady-state can be tested by constructing a *Parameter-Dependent* Lyapunov functional. See the remark at the end of this section for systems with multiple equilibria. The following Proposition constructs a Lyapunov functional which is only parameterized by p , even though the Lyapunov functional could also be parameterized by \tilde{x}^* .

Proposition 8: Consider the system given by (10), where $p \in P$ as defined by (9). For a positive integer L , suppose that there exist functions $v_0, v_1 : \mathbb{R}^{n(L+1)} \times \mathbb{R}^m \rightarrow \mathbb{R}$, a positive definite radially unbounded function $\varphi : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$ and a non-negative function $\psi : \mathbb{R}^{n(L+1)} \rightarrow \mathbb{R}$ such that the following conditions hold for all $\hat{X}_L, \hat{Y}_L \in \mathbb{R}^{n(L+1)}$, $\hat{x}_{L+1} \in \mathbb{R}^n$ and $p \in P$:

- 1) $v_0(\hat{X}_L, p) - \varphi(\hat{X}_L) \geq 0$,
- 2) $v_1(\hat{Y}_L, p) \geq 0$,
- 3) $\sum_{l=0}^L \nabla_{\hat{x}_l} v_0(\hat{X}_L, p)^T f(\hat{x}_l, \hat{x}_{l+1}, p, \tilde{x}^*) + v_1(\hat{x}_0, \dots, \hat{x}_L, p) - v_1(\hat{x}_1, \dots, \hat{x}_{L+1}, p) + \psi(\hat{X}_L) \leq 0$, when (11) is satisfied.

Then the steady-state 0 of the system given by (10–11) is robustly globally IOD stable for all $p \in P$. Moreover, if $\psi(\hat{x}_L) > 0$, 0 is IOD robustly globally asymptotically stable for all $p \in P$.

The proof is based on the fact that the following functional

$$V(\phi) = v_0(\phi(0), \phi(-\tau), \dots, \phi(-L\tau), p) + \int_{-\tau}^0 v_1(\phi(\theta), \phi(\theta - \tau), \dots, \phi(\theta - L\tau), p) d\theta \quad (12)$$

is a Lyapunov-Krasovskii functional, and is omitted for brevity. The constraints defining the set P can be adjoined in the three conditions above using non-negative multipliers. Polynomial multipliers can also be used to adjoin condition (11) to the third constraint.

Remark 9: Many times there are multiple steady-states in (8) that move as p is allowed to vary in P ; in this case we seek a local result, and the parameter set P should be extended to include the ‘motion’ of the steady-state \tilde{x}^* and the region Ω has to be sufficiently small so that no other equilibria cross into Ω as the parameters change within P . An example of this case can be found in Section IV.

B. Delay-Dependent (DD) stability

When the stability properties of the steady-state change as the delay size, seen as a static parameter, changes, the stability is termed delay-dependent. In this case, a different type of Lyapunov functional has to be used to allow for the delay size to appear explicitly in the Lyapunov conditions. The structure of the Lyapunov-Krasovskii functional we will be considering will take the form (2) which we repeat here for convenience:

$$V(\phi) = \int_{-\tau}^0 v_1(\phi(0), \phi(\theta), \theta) d\theta + \int_{-\tau}^0 \int_{-\tau}^0 v_2(\phi(\xi), \phi(\theta), \theta, \xi) d\theta d\xi$$

Note that this functional reduces to the complete Lyapunov functional used for linear time-delay systems when v_1 and v_2 are quadratic in ϕ [8]. We have the following result:

Proposition 10: Let 0 be a steady-state for the system given by (1) with a polynomial vector field. Given $Z_d[\hat{y}_0, \theta]$, let $Q \succeq 0$ be of appropriate dimensions and define

$$v_2(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d[\hat{y}_0, \theta]^T Q Z_d[\hat{z}_0, \xi]. \quad (13)$$

Suppose also that there exists a function $v_1 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, a radially unbounded positive definite function $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$, as well as a non-negative function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$, and functions $r_1 : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ and $r_2 : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$, all polynomials, that satisfy the following conditions for all $\hat{x}_0, \hat{x}_1, \hat{y}_0, \hat{z}_0 \in \mathbb{R}^n$:

- 1) $v_1(\hat{x}_0, \hat{y}_0, \theta) + r_1(\hat{x}_0, \theta) - \varphi(\hat{x}_0) \geq 0$, for $\theta \in [-\tau, 0]$,
- 2) $\nabla_{\hat{x}_0} v_1(\hat{x}_0, \hat{y}_0, \theta)^T f(\hat{x}_0, \hat{x}_1) - \frac{\partial v_1(\hat{x}_0, \hat{y}_0, \theta)}{\partial \theta} + \frac{1}{\tau} v_1(\hat{x}_0, \hat{x}_0, 0) - \frac{1}{\tau} v_1(\hat{x}_0, \hat{x}_1, -\tau) + 2v_2(\hat{y}_0, \hat{x}_0, \theta, 0) - 2v_2(\hat{y}_0, \hat{x}_1, \theta, -\tau) + r_2(\hat{x}_0, \hat{x}_1, \theta) + \psi(\hat{x}_0) \leq 0$ for $\theta \in [-\tau, 0]$,
- 3) $\int_{-\tau}^0 r_1(\hat{x}_0, \theta) d\theta = 0$,
- 4) $\int_{-\tau}^0 r_2(\hat{x}_0, \hat{x}_1, \theta) d\theta = 0$.
- 5) There exists a $R \succeq 0$ so that $\frac{\partial v_2}{\partial \theta}(\hat{y}_0, \hat{z}_0, \theta, \xi) + \frac{\partial v_2}{\partial \xi}(\hat{y}_0, \hat{z}_0, \theta, \xi) = Z_d[\hat{y}_0, \theta]^T R Z_d[\hat{z}_0, \xi]$.

Then the steady-state 0 of the system given by (1) is *globally stable* for delay size τ . Moreover, if $\psi(\hat{x}_0) > 0$, then 0 is *globally asymptotically stable* for delay size τ .

Proof: Consider (2) where v_2 is given by (13). The first term is positive definite owing to conditions 1) and 3) in the above Proposition, and the fact that $\varphi(\phi(0)) > 0$. The second term is positive semidefinite by construction, as $Q \succeq 0$. Therefore the first Lyapunov condition is satisfied.

The time derivative of $V(\phi)$ is:

$$\begin{aligned} \dot{V}(\phi) = \int_{-\tau}^0 & \left(\nabla_{\phi(0)} v_1(\phi(0), \phi(\theta), \theta)^T f(\phi(0), \phi(-\tau)) - \frac{\partial v_1}{\partial \theta}(\phi(0), \phi(\theta), \theta) \right. \\ & + \frac{1}{\tau} v_1(\phi(0), \phi(0), 0) - \frac{1}{\tau} v_1(\phi(0), \phi(-\tau), -\tau) \\ & \left. + 2v_2(\phi(\theta), \phi(0), \theta, 0) - 2v_2(\phi(\theta), \phi(-\tau), \theta, -\tau) \right) d\theta \\ & - \int_{-\tau}^0 \int_{-\tau}^0 \left(\frac{\partial v_2}{\partial \theta}(\phi(\xi), \phi(\theta), \theta, \xi) + \frac{\partial v_2}{\partial \xi}(\phi(\xi), \phi(\theta), \theta, \xi) \right) d\theta d\xi. \end{aligned}$$

This derivative condition has a form similar to (3). Conditions 2) and 4) in the Proposition guarantee that the first term of \dot{V} is non-positive for $\theta \in [-\tau, 0]$. Condition 5) guarantees that the second term in \dot{V} is non-positive. So (2) is a Lyapunov-Krasovskii functional, and the zero steady-state is *stable*. Since there is no constraint on the state-space, and φ is radially unbounded, the result holds globally. Moreover if $\psi > 0$, then the steady-state is globally asymptotically stable for delay τ . ■

The Lyapunov-Krasovskii functional can be constructed algorithmically in a way similar to the other cases in this paper. Different Lyapunov-Krasovskii structures can also be used and similar conditions can be imposed, that may have better convergence properties. In particular if v_1 is not a function of θ , a different functional can be used, see the example in Section IV. For the case in which v_1 is a function of θ , a significant improvement from the result in [16] is the introduction of the functions r_i .

Example 11: The delayed logistic (Hutchinson's) equation $\dot{x}(t) = \alpha x(t-1)[1+x(t)]$ models single species growth with delay. A stability condition was derived, $\alpha < \frac{37}{24}$ for $x_t > -1$ without using a Lyapunov argument [2]. The Proposition above can be used to construct algorithmically a Lyapunov-Krasovskii functional to verify the (local) asymptotic stability of the zero equilibrium. For example, for $\|x_t\| < 0.2$ and v_1, v_2 of order 2 we get a stability condition for $\alpha = 1.23$, while if we set $r_i = 0$ then we can only ensure stability for $\alpha = 1.11$. This shows the improvement of using the functions r_i .

An important issue which is unique to the case of DD stability, is to ensure that the stability properties hold for a *delay interval* rather than for a specific value of the delay. The conditions in Proposition 10 are not affine in τ . However, one can consider τ as a static parameter, itself being allowed to vary within the interval. One can then view this as a robustness problem and construct a (possibly parameterized) Lyapunov-Krasovskii functional that guarantees DD stability for the whole interval. Similar arguments allow the construction of Lyapunov-Krasovskii functionals for *local* DD stability and robust DD stability under parameter variations.

C. Computational Considerations

The largest LMI constraint in the semidefinite programme that is related to the above Propositions is the one related to the derivative condition, which has lower order 2. Suppose that the vector field is of order k and that the candidate v_i 's are of order m .

In the derivative condition of Proposition 5 there are $n(L + 1)$ variables and the highest degree is $(k - 1) + m$. Therefore the size of the LMI is $\binom{n(L + 1) - 1 + ((k - 1) + m)/2}{((k - 1) + m)/2} - n - 1$. For example, if $L = 1$, $k = 3$ and $n = 4$ and $m = 4$ this gives an LMI of size 115. If n is increased to 6, this gives 359, while if m is increased to 6 this gives an LMI of size 325. All these are sizes that current semidefinite programming solvers can handle well if the number of variables is not large.

For the case of delay-dependent stability (Proposition 10), in the derivative condition there are $3n + 1$ variables and the largest degree is $(k - 1) + m$. Therefore the size of the LMI is $\binom{3n + ((k - 1) + m)/2}{((k - 1) + m)/2} - 3n$. If $k = 3$ and $n = 4$, then a Lyapunov function of order 4 already gives an LMI of size 443. Hence one can see that the delay-dependent conditions are more computationally expensive to test.

Of course it is expected that one should start with the simplest Lyapunov functional structures first, before increasing the order of the functional sought or considering more complicated structures. Also, the above considerations do not take into account sparsity, which could reduce the LMI size considerably, or even symmetry reduction which would result in better conditioned semidefinite programmes.

IV. EXAMPLE: A POPULATION DYNAMICS MODEL

A realistic predator-prey model which takes into account *maturation* of the predator population takes the form [31]:

$$\dot{x}(t) = x(t)[b - ax(t) - k_1y(t)], \quad \dot{y}(t) = -\sigma y(t) + k_2x(t - \tau)y(t - \tau), \quad (14)$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators. Here x and y are the prey and predator populations, b is the rate of increase of prey, k_1 and k_2 are the coefficients of the effect of predation on x and y and σ is the death rate of y . We assume that a, b, k_1, k_2 and σ are positive. The equilibria (x^*, y^*) of the above system are $(x^*, y^*) = (0, 0)$, $(x^*, y^*) = (b/a, 0)$ and

$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1k_2} \right), \quad (15)$$

the last of which is the equilibrium of interest. We further assume that $(bk_2 - a\sigma) > 0$ which ensures that (15) is in the first quadrant. Linearization of the system about this equilibrium and subsequent analysis gives the following result, which is proven in [16]:

Proposition 12: Consider the linearization of system (14) about the equilibrium point (15). Then if $(bk_2 - 3a\sigma) < 0$ the zero equilibrium is stable independent of the delay. If $(bk_2 - 3a\sigma) > 0$ the zero equilibrium is stable if the delay satisfies $\tau < \tau^*$ and is unstable otherwise, where τ^* is given by:

$$\tau^* = \frac{1}{\omega} \operatorname{atan} \left[\omega \frac{(a\sigma^2 - \omega^2k_2)k_2 - \sigma(2a\sigma - bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega^2k_2)} \right].$$

where ω solves $\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0$.

Consider now (14) with parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$. Denote $x_1 = x - x^*$ and $x_2 = y - y^*$.

1) *Delay-independent stability analysis:* The system (14) has many equilibria, and so we need to define a region around the steady-state of interest to obtain a stability condition. We let:

$$\|x_{1t}\| \leq \gamma_1 x^*, \quad \|x_{2t}\| \leq \gamma_2 y^*, \quad (16)$$

where the steady-state (x^*, y^*) is given by (15). We consider b to be a parameter in the problem. From Proposition 12, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of a, σ and k_2 , the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating (x^*, y^*) we use a value of $b = 20/3$. The steady-state (x^*, y^*) of system (14) does not move as b changes, however the other two equilibria cross through the region defined by (16). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady-state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = v_0(x_1(t), x_2(t), b) + \int_{-\tau}^0 v_1(x_1(t+\theta), x_2(t+\theta), b) d\theta.$$

We use Proposition 8 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When v_0 is second order and v_1 is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval.

2) *Delay-dependent stability analysis:* Now we will try to test values for τ for which stability is retained using the same parameters as before and fixed $b = 15$. Given these parameters $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on x_1 and x_2 on the state-space given by (16) with $\gamma_1 = \gamma_2 = 0.1$. We can construct the Lyapunov functional $V(x_t)$ given by (2) with v_1 0th order with respect to θ and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When v_1 is quartic with respect to all variables but θ (which is kept at 0 order) then we can construct this $V(x_t)$ for $\tau = 0.053$. Here instead of the term V_2 in (2) we use

$$V_2 = \int_{-\tau}^0 \int_{t+\theta}^t v_2(\phi(\zeta)) d\zeta d\theta.$$

Terms like these were used in [9] and are useful when v_1 is zeroth order in θ . The corresponding SDP is bigger as the functional is more complicated, but we can see that values of the delay closer to the stability boundary can be tested. We have also tested the case $b = 10.5$, for which $\tau^* = 0.307$ and found that the above functional can show stability for $\tau = 0.292$, which is again very close to the stability boundary.

Functionals of the form (2) that are functions of θ can also be used at an increased computational cost; however the simple functional shown above results in an adequate estimate for the largest delay bound in this example without the need for higher complexity, emphasizing the fact that an appropriate choice of Lyapunov functional structure can lead to a successful stability test at a lower computational cost.

V. CONCLUDING REMARKS

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is achieved through the solution of a set of Linear Matrix Inequalities (LMIs). Linear and nonlinear IOD and DD stability can now be treated using the same tools, and the results can be easily extended to the cases of multiple, distributed and time-varying delays. A judicious choice for the structure of the Lyapunov functional would be required in these cases.

APPENDIX I

PROOF OF THEOREM 1

We first develop a lemma that will be used in the proof of Theorem 1. We start by giving a quantitative definition of global minimum.

Definition 13: We say that a function $f(y)$ has a unique global minimum, if there exists a y^* such that for any $\epsilon > 0$, there exists a $\beta > 0$ such that $f(x) \geq f(y^*) + \beta$ for all x such that $\|x - y^*\| \geq \epsilon$.

Lemma 14: Suppose there exists $\alpha > 0$ such that for all $t \in [-\tau, 0]$, $\alpha\|y\|^2 \leq f(t, y)$ where $f(t, y)$ is continuous in y and piecewise continuous in t . Suppose f has a unique global minimum for every $t \in [-\tau, 0]$. Then there exists a piecewise continuous function z , such that $\min_y f(t, y) = f(t, z(t))$ for all $t \in [-\tau, 0]$.

Proof: Let $z(t) = \arg \min_{y \in \mathbb{R}^n} f(t, y)$. We now demonstrate that z is piecewise continuous. Suppose $f(t, y)$ is continuous on a closed interval $t \in I$. We first show that $z(t)$ is bounded on I . Choose an arbitrary $z_0 \in \mathbb{R}^n$ and let $B = \sup_{t \in I} f(t, z_0)$. B is finite because f is continuous in t on the compact interval I . Now choose $r > \sqrt{B/\alpha}$. Then if $\|z(t_1)\| \geq r$ for some $t_1 \in I$, we have that $f(t_1, z(t_1)) \geq \alpha\|z(t_1)\|^2 > B$, while $f(t_1, z_0) \leq B$, which contradicts the definition of $z(t_1)$. Thus we have that $z(t) \leq r$ for all $t \in I$.

Now suppose we are given some arbitrary $\epsilon > 0$ and a point t in the interior of I . To show that z is continuous at t , we must find a $\gamma > 0$ such that $|t - s| \leq \gamma$ implies $\|z(t) - z(s)\| \leq \epsilon$. Since $z(t)$ is the unique global minimizer of f at t , there exists a $\beta > 0$ such that

$$f(t, z(t)) \leq f(t, y) - \beta \quad \text{for any } y \text{ such that } \|y - z(t)\| \geq \epsilon$$

By continuity of f at t , there exists a $\delta > 0$ such that for any $s \in I$ with $|t - s| \leq \delta$, we have $\|f(t, y) - f(s, y)\| \leq \frac{\beta}{4}$ for all $\|y\| \leq r$.

Now choose $\gamma < \delta$ such that $s \in I$ for all $|s - t| \leq \gamma$. Suppose there exists an s , with $|t - s| \leq \gamma$ and $|z(t) - z(s)| > \epsilon$. Then

$$f(s, z(s)) \geq f(t, z(s)) - \beta/4 \geq f(t, z(t)) + \beta - \beta/4 \geq f(s, z(t)) + \beta - \beta/2 = f(s, z(t)) + \beta/2.$$

This contradicts the definition of $z(s)$ as being the minimizer of f at s . Therefore, $\|z(t) - z(s)\| \leq \epsilon$ for all $|s - t| \leq \gamma$, as desired and z is continuous at t . Concluding, since $f(t, y)$ is piecewise continuous in t ,

if f is continuous at a point t , then it is continuous in an open neighborhood of t . Since we have shown that this implies z is also continuous at point t , we have that z is continuous at every point for which $f(t, z)$ is continuous. Since f is piecewise continuous, this proves that z is piecewise continuous. ■

Another ingredient that we will need to prove Theorem 1 is following Theorem from [32].

Theorem 15: Suppose $f : [-\tau, 0] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous on $[-\tau, 0] \times \mathbb{R}^n$ and suppose there exists a bounded function $z : [-\tau, 0] \rightarrow \mathbb{R}$, continuous on $[-\tau, 0]$, such that for all $t \in [-\tau, 0]$, $f(t, z(t)) = \inf_x f(t, x)$. Further suppose that for each bounded set $X \subset \mathbb{R}^n$ the set $\{f(t, x) | x \in X, t \in [-\tau, 0]\}$ is bounded. Then the following are equivalent:

- (i) For all $y \in C[-\tau, 0]$, $\int_{-\tau}^0 f(t, y(t))dt \geq 0$.
- (ii) There exists $g : [-\tau, 0] \rightarrow \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t, z) + g(t) \geq 0 \text{ for all } t, z, \quad \int_{-\tau}^0 g(t)dt = 0.$$

Proof: (Of Theorem 1) That (5) implies (4) follows from a simple integration argument. The converse follows by combining the results in Theorem 15 and Lemma 14. ■

REFERENCES

- [1] R. Srikant, *The Mathematics of Internet Congestion Control*. Birkhäuser, 2003.
- [2] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, 1993. Mathematics in Science and Engineering (191).
- [3] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Mathematics and its Applications (74), Kluwer Academic Publishers, 1992.
- [4] S.-I. Niculescu, *Delay Effects on Stability: A Robust Control Approach*. Lecture Notes in Control and Information Sciences (269), Springer-Verlag, 2001.
- [5] V. Kolmanovskii and A. Myshkis, *Introduction to the Theory and Applications of Functional Differential Equations*. Kluwer Academic Publishers, 1999.
- [6] M. Y. Repin, “Quadratic Lyapunov functionals for systems with delay,” *Prikl. Mat. Meh.*, vol. 29, pp. 564–566, 1965.
- [7] P.-A. Bliman, “Lyapunov equation for the stability of linear delay systems of retarded and neutral type,” *IEEE Trans. Automat. Contr.*, vol. 47, no. 2, pp. 327–335, 2002.
- [8] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay systems*. Birkhäuser, 2003.
- [9] V. B. Kolmanovskii, S.-I. Niculescu, and J.-P. Richard, “On the Lyapunov-Krasovskii functionals for stability analysis of linear delay systems,” *International Journal of Control*, vol. 72, no. 4, pp. 374–384, 1999.
- [10] P. Park, “A delay-dependent stability criterion for systems with uncertain time-invariant delays,” *IEEE Trans. Automat. Contr.*, vol. 44, pp. 876–877, 1999.
- [11] K. Gu, “Discretised LMI set in the stability problem of linear uncertain time-delay systems,” *International Journal of Control*, vol. 68, pp. 155–163, 1997.
- [12] A. Papachristodoulou, M. Peet, and S. Lall, “Constructing Lyapunov-Krasovskii functionals for linear time delay systems,” in *Proceedings of the American Control Conference*, 2005.
- [13] P. A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, CA, 2000.
- [14] A. Papachristodoulou and S. Prajna, “On the construction of Lyapunov functions using the sum of squares decomposition,” in *Proceedings IEEE Conference on Decision and Control*, 2002.
- [15] D. Henrion and J.-B. Lasserre, “Solving nonconvex optimization problems,” *IEEE Control Systems Magazine*, vol. 24, no. 3, pp. 72–83, 2004.

- [16] A. Papachristodoulou, "Analysis of nonlinear time delay systems using the sum of squares decomposition," in *Proceedings of the ACC*, 2004.
- [17] M. Peet and S. Lall, "Constructing Lyapunov functions for nonlinear delay-differential equations using semidefinite programming," in *Proceedings of NOLCOS*, 2004.
- [18] A. Papachristodoulou, J. C. Doyle, and S. H. Low, "Analysis of nonlinear delay differential equation models of TCP/AQM protocols using sums of squares," in *Proceedings of the IEEE CDC*, 2004.
- [19] A. Papachristodoulou, "Robust stabilization of nonlinear time delay systems using convex optimization," in *Proceedings of the IEEE CDC*, 2005.
- [20] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. Applied Mathematical Sciences (99), Springer-Verlag, 1993.
- [21] A. Papachristodoulou and S. Prajna, "Analysis of non-polynomial systems using the sum of squares decomposition," in *Positive Polynomials in Control* (H. Didier and A. Garulli, eds.), vol. 312 of *Lecture Notes in Control and Information Sciences*, pp. 23–43, Springer Berlin/Heidelberg, 2005.
- [22] G. Chesi, "Domain of attraction: estimates for non-polynomial systems via LMIs," in *16th IFAC World Congress on Automatic Control*, 2005.
- [23] J.-B. Lasserre, "Global optimization with polynomials and the problem of moments," *SIAM Journal of Optimization*, vol. 11, pp. 796–817, 2001.
- [24] Y. Nesterov, "Squared functional systems and optimization problems," in *High Performance Optimization Methods.*, pp. 405–439, Kluwer Academic Publishers, 2000.
- [25] D. Henrion and A. Garulli, *Positive Polynomials in Control*. Springer, 2005.
- [26] M. Putinar, "Positive polynomials on compact semialgebraic sets," *Indiana University Mathematics Journal*, vol. 42, no. 3, pp. 969–984, 1993.
- [27] J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*. Springer-Verlag, 1998.
- [28] O. Toker and H. Ozbay, "Complexity issues in robust stability of linear delay-differential systems," *Math., Control, Signals, Syst.*, vol. 9, pp. 386–400, 1996.
- [29] F. Mazenc and S.-I. Niculescu, "Lyapunov stability analysis for nonlinear delay systems," *Systems and Control Letters*, vol. 42, pp. 245–251, 2001.
- [30] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "SOSTOOLS – Sum of Squares Optimization Toolbox, User's Guide." Available at <http://www.cds.caltech.edu/sostools>, 2002.
- [31] P. J. Wangersky and W. J. Cunningham, "Time lag in prey-predator population models," *Ecology*, vol. 38, no. 1, pp. 136–139, 1957.
- [32] M. Peet, A. Papachristodoulou, and S. Lall, "Stability analysis of linear time-delay systems," in *Proceedings of the IEEE Conference on Decision and Control*, 2006.