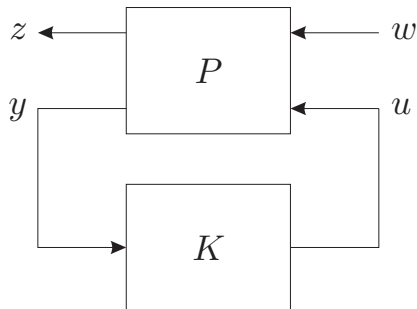


# LMI Methods in Optimal and Robust Control

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Lecture 09: An LMI for  $H_\infty$ -Optimal Full-State Feedback Control

# Recall: Linear Fractional Transformation



**Plant:**

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where} \quad P = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right]$$

**Controller:**

$$u = Ky \quad \text{where} \quad K = \left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$$

# Optimal Control

Choose  $K$  to minimize

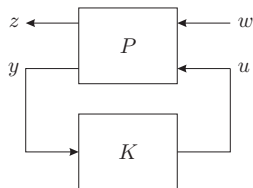
$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose  $\left[ \begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right]$  to minimize

$$\left\| \left[ \begin{array}{c|c} \left[ \begin{array}{cc} A & 0 \\ 0 & A_K \end{array} \right] + \left[ \begin{array}{cc} B_2 & 0 \\ 0 & B_K \end{array} \right] \left[ \begin{array}{cc} I & -D_K \\ -D_{22} & I \end{array} \right]^{-1} \left[ \begin{array}{cc} 0 & C_K \\ C_2 & 0 \end{array} \right] & \begin{array}{c} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{array} \\ \hline \left[ \begin{array}{cc} C_1 & 0 \end{array} \right] + \left[ \begin{array}{cc} D_{12} & 0 \end{array} \right] \left[ \begin{array}{cc} I & -D_K \\ -D_{22} & I \end{array} \right]^{-1} \left[ \begin{array}{cc} 0 & C_K \\ C_2 & 0 \end{array} \right] & D_{11} + D_{12} D_K Q D_{21} \end{array} \right\|_{H_\infty}$$

where  $Q = (I - D_{22}D_K)^{-1}$ .

# Optimal Full-State Feedback Control



For the full-state feedback case, we consider a controller of the form

$$u(t) = Fx(t)$$

**Controller:**

$$u = Ky \quad \text{where} \quad K = \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right]$$

**Plant:**

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \quad \text{where} \quad P = \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right]$$

# Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\underline{S}(\hat{P}, \hat{K}) = \left[ \begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array} \right]$$

By the KYP lemma,  $\|\underline{S}(\hat{P}, \hat{K})\|_{H_\infty} < \gamma$  if and only if there exists some  $X > 0$  such that

$$\begin{aligned} & \begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 \\ B_1^T X & -\gamma I \end{bmatrix} \\ & + \frac{1}{\gamma} \begin{bmatrix} (C_1 + D_{12} F)^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} (C_1 + D_{12} F) & D_{11} \end{bmatrix} < 0 \end{aligned}$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

# Optimal Full-State Feedback Control

Thus the closed-loop state-space representation is

$$\hat{G}(P, K) = \left[ \begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array} \right]$$

By the KYP lemma,  $\|\hat{G}(P, K)\|_{\mathcal{H}_\infty} < \gamma$  if and only if there exists some  $X > 0$  such that

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 \\ B_1^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} (C_1 + D_{12} F)^T \\ D_{11}^T \end{bmatrix} \begin{bmatrix} (C_1 + D_{12} F) & D_{11} \end{bmatrix} < 0$$

This is a matrix inequality, but is nonlinear

- Quadratic (Not Bilinear)
- May NOT apply variable substitution trick.

Recall the KYP Lemma

## Lemma 1.

Suppose

$$\hat{G}(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \leq \gamma$ .
- There exists a  $X > 0$  such that

$$\begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

# Schur Complement

The KYP condition is

$$\begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Recall the Schur Complement

## Theorem 2 (Schur Complement).

*For any  $S \in \mathbb{S}^n$ ,  $Q \in \mathbb{S}^m$  and  $R \in \mathbb{R}^{n \times m}$ , the following are equivalent.*

1.  $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} < 0$
2.  $Q < 0$  and  $M - RQ^{-1}R^T < 0$

In this case, let  $Q = -\frac{1}{\gamma}I < 0$ ,

$$M = \begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} \qquad R = \begin{bmatrix} C & D \end{bmatrix}^T$$

Note we are making the LMI **Larger**.

# Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the

## Full-State Feedback Condition

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 & (C_1 + D_{12} F)^T \\ B_1^T X & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in  $X$  and  $F$ .



# Schur Complement

The Schur Complement says that

$$\begin{bmatrix} A^T X + X A & X B \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

if and only if

$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

This leads to the

**Full-State Feedback Condition**

$$\begin{bmatrix} (A + B_2 F)^T X + X(A + B_2 F) & X B_1 & (C_1 + D_{12} F)^T \\ B_1^T X & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F) & D_{11} & -\gamma I \end{bmatrix} < 0$$

which is now bilinear in  $X$  and  $F$ .

## Statement of the Dilated KYP Lemma

### Lemma 3.

Suppose

$$\hat{G}(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

Then the following are equivalent.

- $\|\hat{G}\|_{H_\infty} \leq \gamma$ .
- There exists a  $X > 0$  such that

$$\begin{bmatrix} A^T X + X A & X B & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0$$

# Dual KYP Lemma

To apply the variable substitution trick, we must also construct the dual form of this LMI.

## Lemma 4 (KYP Dual).

*Suppose*

$$\hat{G}(s) = \left[ \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right].$$

*Then the following are equivalent.*

- $\|G\|_{H_\infty} \leq \gamma.$
- *There exists a  $Y > 0$  such that*

$$\begin{bmatrix} Y A^T + A Y & B & Y C^T \\ B^T & -\gamma I & D^T \\ C Y & D & -\gamma I \end{bmatrix} < 0$$

# Dual KYP Lemma

## Proof.

Let  $X = Y^{-1}$ . Then

$$\begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} < 0 \quad \text{and} \quad Y > 0$$

if and only if  $X > 0$  and

$$\begin{aligned} & \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} YA^T + AY & B & YC^T \\ B^T & -\gamma I & D^T \\ CY & D & -\gamma I \end{bmatrix} \begin{bmatrix} Y^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \\ &= \begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

By the Schur complement this is equivalent to

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

By the KYP lemma, this is equivalent to  $\|G\|_{H_\infty} \leq \gamma$ .



# Full-State Feedback Optimal Control

We can now apply this result to the state-feedback problem.

## Theorem 5.

*The following are equivalent:*

- *There exists an  $F$  such that*

$$\left\| \underline{S} \left( \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right], \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right] \right) \right\|_{H_\infty} \leq \gamma.$$

- *There exist  $Y > 0$  and  $Z$  such that*

$$\begin{bmatrix} Y A^T + A Y + Z^T B_2^T + B_2 Z & B_1 & Y C_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

*Then  $F = ZY^{-1}$ .*

# Full-State Feedback Optimal Control

## Proof.

Suppose there exists an  $F$  such that  $\|\underline{S}(P, K(0, 0, 0, F))\|_{H_\infty} \leq \gamma$ . By the Dual KYP lemma, this implies there exists a  $Y > 0$  such that

$$\begin{bmatrix} Y(A + B_2F)^T + (A + B_2F)Y & B_1 & Y(C_1 + D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let  $Z = FY$ . Then

$$\begin{aligned} & \begin{bmatrix} YA^T + Z^T B_2^T + AY + B_2Z & B_1 & YC_1^T + Z^T D_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}Z & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} YA^T + YF^T B_2^T + AY + B_2FY & B_1 & YC_1^T + YF^T D_{12}^T)^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1Y + D_{12}FY & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} Y(A + B_2F)^T + (A + B_2F)Y & B_1 & Y(C_1 + D_{12}F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12}F)Y & D_{11} & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

# Full-State Feedback Optimal Control

Proof.

Suppose there exists an  $F$  such that  $\|\underline{z}(P, K(0, 0, 0, F))\|_{\infty} \leq \gamma$ . By the Dual KVP lemma, this implies there exists a  $Y > 0$  such that

$$\begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)^T Y & B_1^T & Y(C_1 + D_{12} F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F)^T & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let  $Z = FY$ . Then

$$\begin{aligned} & \begin{bmatrix} Y A^T + Z^T B_2^T + A Y + B_2 Z & B_1^T & Y C_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} Y A^T + Y F^T B_2^T + A Y + B_2 F Y & B_1^T & Y C_1^T + Y F^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} F Y & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)^T Y & B_1^T & Y(C_1 + D_{12} F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F)^T & D_{11} & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

For convenience, we use

$$\underline{S}(P, K(0, 0, 0, F)) = \underline{S} \left( \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right], \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right] \right)$$

# Full-State Feedback Optimal Control

## Proof.

Now suppose there exists a  $Y > 0$  and  $Z$  such that

$$\begin{bmatrix} Y A^T + Z^T B_2^T + AY + B_2 Z & B_1 & Y C_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0.$$

Let  $F = ZY^{-1}$ . Then

$$\begin{aligned} & \begin{bmatrix} Y(A + B_2 F)^T + (A + B_2 F)Y & B_1 & Y(C_1 + D_{12} F)^T \\ B_1^T & -\gamma I & D_{11}^T \\ (C_1 + D_{12} F)Y & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} Y A^T + Y F^T B_2^T + AY + B_2 F Y & B_1 & Y C_1^T + Y F^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} F Y & D_{11} & -\gamma I \end{bmatrix} \\ &= \begin{bmatrix} Y A^T + Z^T B_2^T + AY + B_2 Z & B_1 & Y C_1^T + Z^T D_{12}^T \\ B_1^T & -\gamma I & D_{11}^T \\ C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0. \end{aligned}$$

# Full-State Feedback Optimal Control

Therefore the following optimization problems are equivalent

**Form A**

$$\min_F \left\| \underline{S} \left( \left[ \begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ I & 0 & 0 \end{array} \right], \left[ \begin{array}{c|c} 0 & 0 \\ \hline 0 & F \end{array} \right] \right) \right\|_{H_\infty}$$

**Form B**

$$\min_{\gamma, Y, Z} \gamma : \begin{bmatrix} -Y & 0 & 0 & 0 \\ 0 & Y A^T + A Y + Z^T B_2^T + B_2 Z & B_1 & Y C_1^T + Z^T D_{12}^T \\ 0 & B_1^T & -\gamma I & D_{11}^T \\ 0 & C_1 Y + D_{12} Z & D_{11} & -\gamma I \end{bmatrix} < 0$$

The optimal controller is given by  $F = ZY^{-1}$ .

**Next Lecture: Optimal Output Feedback**