Spacecraft Dynamics and Control

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Lecture 12: Orbit Perturbations

Introduction

In this Lecture, you will learn:

Perturbation Basics

- The Satellite-Normal Coordinate System
- Equations for
  - $\dot{a}$, $\dot{i}$, $\dot{\Omega}$, $\dot{\omega}$, $\dot{e}$

Drag Perturbations

- Models of the atmosphere.
- Orbit Decay
- Δv budgeting.
- Effect on eccentricity.

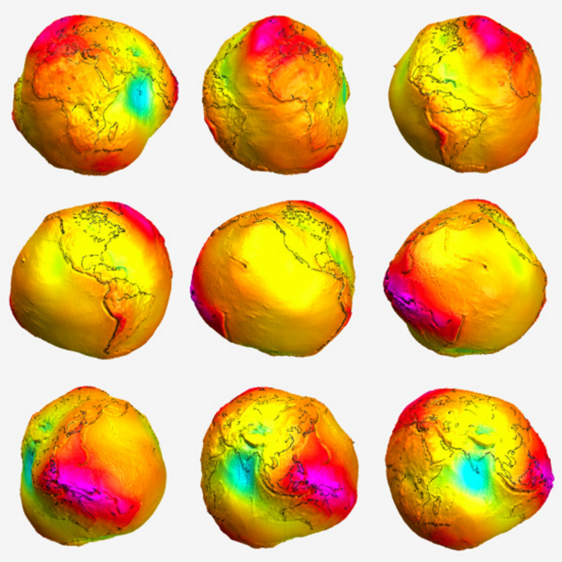
Introduction to Perturbations

So far, we have only discussed idealized orbits.

- Solutions to the 2-body problem.
- All orbital elements are fixed (except f).

In reality, there are many other forces at work:

- Drag
- Non-spherical Earth
- Lunar Gravity
- Solar Radiation
- Tidal Effects



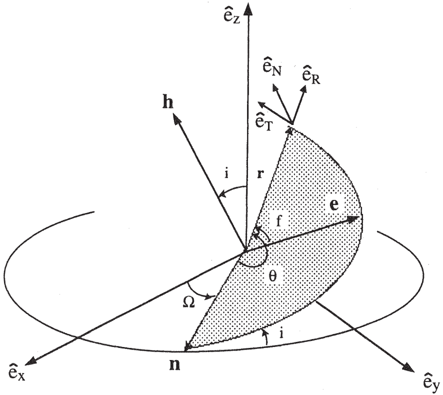
By definition, perturbations don't point to the center of mass

- Where do they point?
- Need a new coordinate system.

$$\vec{F} = N\hat{e}_N + R\hat{e}_R + T\hat{e}_T$$

Satellite-Normal CS (R-T-N):

- $\hat{e}_R$ points along the earth $\rightarrow$ satellite vector.
- $\hat{e}_N$ points in the direction of $\vec{h}$
- $\hat{e}_T$ is defined by the RHR
  - $\hat{e}_T \cdot v > 0$.



Generalized Perturbation Analysis

Now suppose we have an expression for the disturbing force:

$$\vec{F} = R\hat{e}_R + T\hat{e}_T + N\hat{e}_N$$

How does this affect $\dot{a}$, $\dot{i}$, $\dot{\Omega}$, $\dot{\omega}$, $\dot{e}$?

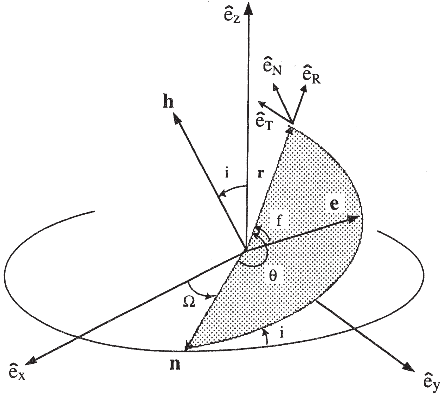
Most elements depend on $\vec{h}$ **and** E**:**

$$a = -\frac{\mu}{2E}$$

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$$

$$\cos i = \frac{h_z}{h}$$

$$\tan \Omega = \frac{h_x}{-h_y}$$



Energy and Momentum Perturbation

We have the orbital elements in terms of $\vec{h}$ and E.

1. Find expressions for $\dot{\vec{h}}$ and $\dot{E}$.
2. Translate into expressions for $\dot{a}$, $\dot{e}$, etc.

Example 1: Semimajor axis.

$$a = -\frac{\mu}{2E}$$

Chain Rule:

$$\begin{aligned}\dot{a} &= \frac{da}{dE}\frac{dE}{dt} \\ &= \frac{\mu}{2E^2}\dot{E}\end{aligned}$$

Example 2: Eccentricity.

$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$$

Chain Rule:

$$\begin{aligned}\dot{e} &= \frac{de}{dh}\frac{dh}{dt} + \frac{de}{dE}\frac{dE}{dt} \\ &= \frac{1}{2e}(e^2 - 1)\left[2\frac{\dot{h}}{h} - \frac{\dot{E}}{E}\right]\end{aligned}$$

Energy and Momentum Perturbation

So now the key is to find expressions for $\dot{h}$ and $\dot{E}$.

$$\vec{F} = \begin{bmatrix} R \\ T \\ N \end{bmatrix}$$

Energy: Energy is Force times distance.

$$dE = \vec{F} \cdot d\vec{r}$$

So in RTN coordinates,

$$\begin{aligned}\dot{E} &= \vec{F} \cdot \vec{v} \\ &= \vec{F} \cdot \left(\dot{r}\hat{e}_R + r\dot{\theta}\hat{e}_T \right) \\ &= \dot{r}R + r\dot{\theta}T\end{aligned}$$

Momentum: Newton's Second Law:

$$\begin{aligned}\dot{\vec{h}} &= \vec{r} \times \vec{F} \\ &= rT\hat{e}_N - rN\hat{e}_T\end{aligned}$$

With magnitude

$$\begin{aligned}\dot{h} &= \frac{\vec{h} \cdot \dot{\vec{h}}}{h^2} \\ &= rT\end{aligned}$$

Energy and Momentum Perturbation

Using $r = \dfrac{h^2/\mu}{1+e\cos f}$ and the approximation $\dot{\theta} = d/dt(\omega + f) \cong \dot{f} = h/r^2$, we get

Semi-major Axis

$$\dot{a} = 2\frac{a^2}{\mu}\left[R\frac{\mu e \sin f}{h} + T\frac{h}{r}\right]$$

or, in terms of a, e, and f,

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[eR\sin f + T(1+e\cos f)\right]$$

Eccentricity:

$$\dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[R\sin f + T(\cos f + \cos E_{ecc})\right]$$

where E_{ecc} is eccentric anomaly,

$$\tan\frac{E_{ecc}}{2} = \sqrt{\frac{1-e}{1+e}}\tan\frac{f}{2}$$

Inclination: From

$$\cos i = \frac{h_z}{h}$$

we have from the chain rule

$$\frac{d}{dt} i = \frac{1}{\sin i} \frac{h\dot{h}_z - \dot{h} h_z}{h^2}$$

from which we can get

$$\frac{d}{dt} i = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \cos(\omega + f)}{1 + e \cos f}$$

RAAN: From

$$\tan \Omega = \frac{h_x}{-h_y}$$

we have from the chain rule

$$\dot{\Omega} = \cos^2 \Omega \frac{h_x \dot{h}_y - \dot{h}_x h_y}{h_y^2}$$

from which we can get

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i (1 + e \cos f)}$$

Although complicated, we can also find $\dot{\omega}$.

$$\dot{\omega} = -\dot{\Omega} \cos i + \sqrt{\frac{a(1-e^2)}{e^2 \mu}} \left(-R \cos f + T \frac{(2 + e \cos f) \sin f}{1 + e \cos f} \right)$$

Levitated Orbit Example

Problem: Suppose a satellite of 100kg in circular polar orbit of 42,164km experiences a continuous solar pressure of $.1$ Newton in $\hat{e}_N$ direction. How do the orbital elements vary with time?

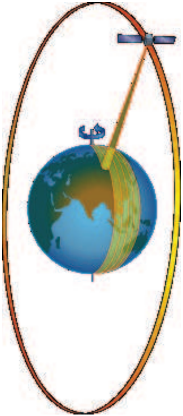
Solution: The Force per unit mass is
$N = F/m = .001m/s^2 = 1E-6km/s^2$.
Since $T = R = e = 0$, and $f \cong E_{ecc} \cong M = nt$

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[eR\sin f + T(1+e\cos f)\right] = 0$$

$$\dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[R\sin f + T(\cos f + \cos E_{ecc})\right] = 0$$

For inclination, we have

$$\frac{d}{dt}i = N\sqrt{\frac{a(1-e^2)}{\mu}}\frac{\cos(\omega+f)}{1+e\cos f} = N\sqrt{\frac{a}{\mu}}\cos nt$$



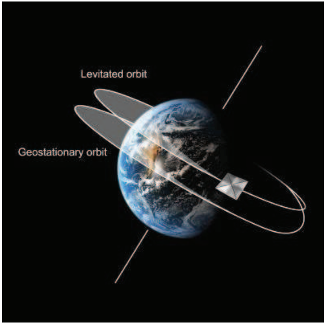
Levitated Orbit Example

The formula for inclination integrates out to

$$\Delta i(t) = N\sqrt{\frac{a}{\mu}}\frac{1}{n}\sin nt = .00446\sin nt \; radians$$

Similarly, since $i \cong 90°$

$$\dot{\Omega} = N\sqrt{\frac{a(1-e^2)}{\mu}}\frac{\sin(\omega+f)}{\sin i(1+e\cos f)} = N\sqrt{\frac{a}{\mu}}\sin nt$$



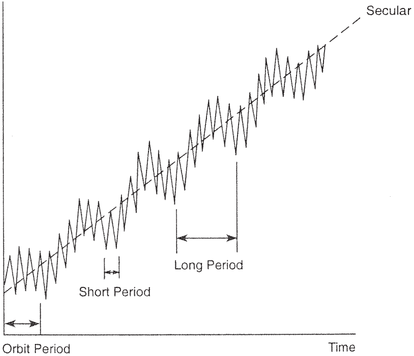
We have

$$\Delta\Omega(t) = -N\sqrt{\frac{a}{\mu}}\frac{1}{n}\cos nt = -.00446\cos nt \; radians$$

The effect is a "Displaced" orbit. The size of the displacement is .0045rad * 42164 km = 188km. See "Light Levitated Geostationary Cylindrical Orbits are Feasible" by S. Baig and C. R. McInnes.

Periodic and Secular Variation

The preceding example illustrated the effect of periodic variation.



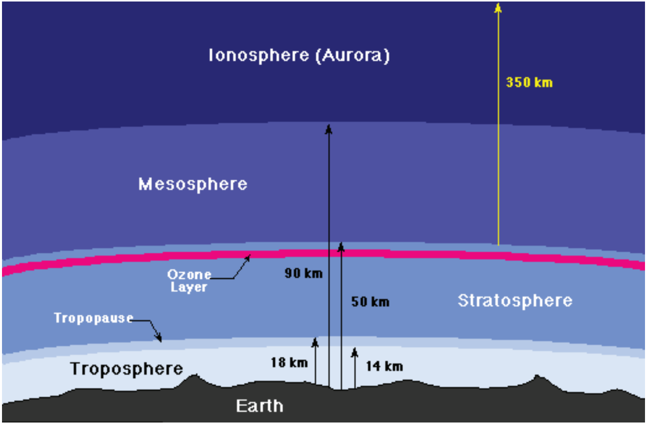
There are three types of disturbances

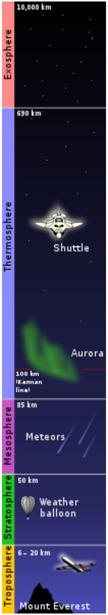
- **Short Periodic** - Cycles every orbital period.
- **Long Periodic** - Cycles last longer than one orbital period.
- **Secular** - Does not cycle. Disturbances mount over time.

Secular Disturbances must be corrected.

Atmospheric Drag

Earth's atmosphere extends into space.





The ionosphere extends well past 350km.

- ISS orbit lies between 330 and 400km.

The Ionosphere

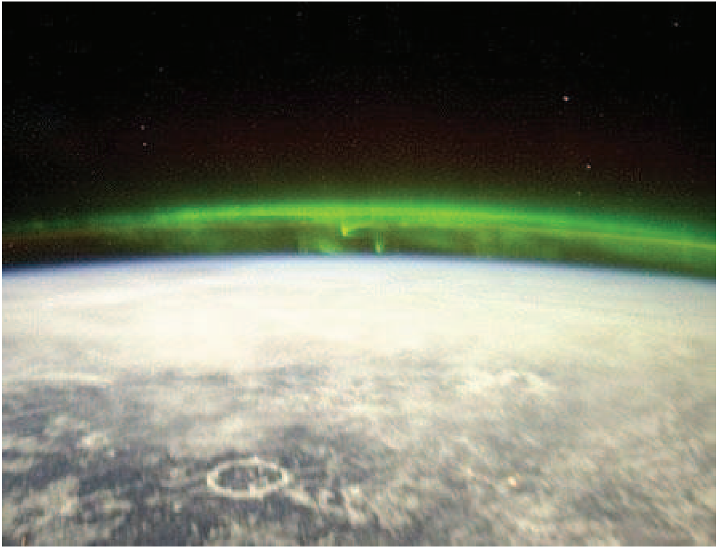


Figure: The Aurora Borealis Shows the Ionosphere Extending Well into Orbital Range

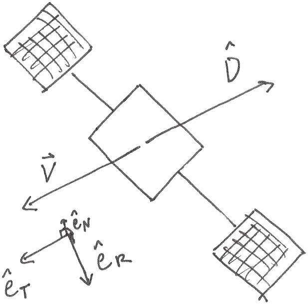
The Drag Perturbation

Drag force for satellites is the same as for aircraft

$$F_D = C_D Q A = \frac{1}{2}\rho v^2 C_D A$$

By definition, drag is opposite to the velocity vector.

- Since by definition, $\vec{v} \perp \vec{h}$, $N = 0$
- For now, ignore the rotation of the earth (adds $\Delta v = \omega_e r \cong .5km/s$).
- For now, assume circular orbit, so $\vec{v} = v\hat{e}_T$.



Ballistic Coefficient:

$$B = \frac{C_D A}{m}$$

Then as first approximation,

$$N = R = 0, \qquad T = -\frac{1}{2}\frac{\rho}{m}C_D A v^2 = -\frac{1}{2}B\rho v^2$$

First note that since $N = 0$, the orbital plane does not change

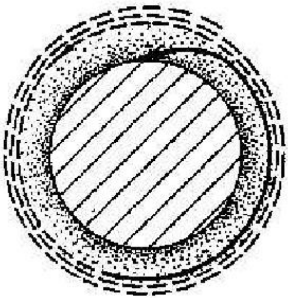
- $\dot{\Omega} = 0$.
- $\frac{d}{dt} i = 0$.

Semi-Major Axis: Since $e = 0$, the dominant effect is on a.

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}} \left[eR\sin f + T(1 + e\cos f) \right]$$

$$= -\sqrt{\frac{a^3}{\mu}} \frac{\rho}{m} C_D A v^2 = -\sqrt{\frac{a^3}{\mu} \frac{\mu^2}{a^2}} \frac{\rho}{m} C_D A$$

$$= -\sqrt{a\mu}\rho B$$



Integrating with respect to time (assuming constant ρ) yields

$$a(t) = \left(\sqrt{a(0)} - \sqrt{\mu}\rho B t \right)^2$$

Example: International Space Station

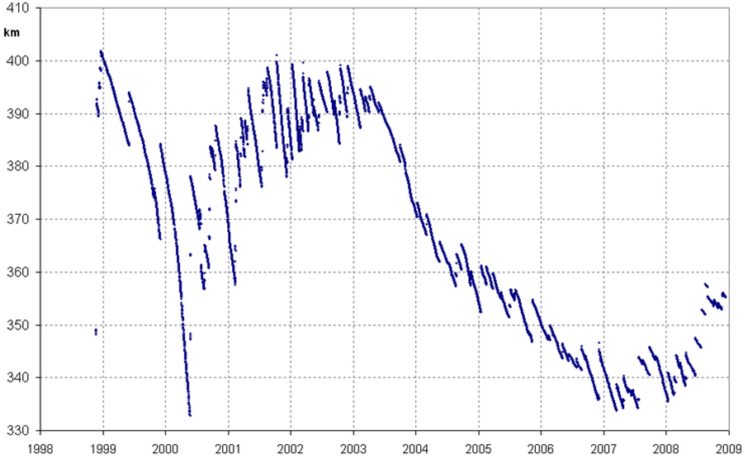


Figure: Orbit Decay of the International Space Station

Density Variation

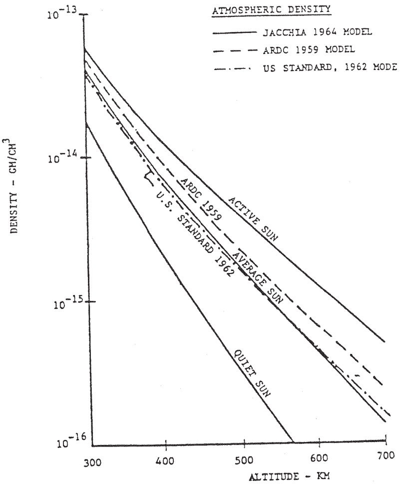
The atmospheric density is not even remotely constant

Exponential Growth:

- Extends to $1.225 * 10^{-3} g/cm^3$ at sea level.
- Orbits below Kármán Line (100km) will not survive a single orbit.
    - Suborbital flight.

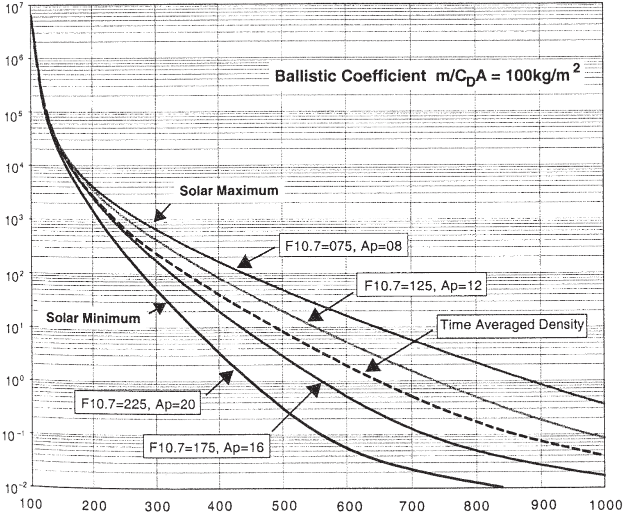
Solar Activity: We have different models of the atmosphere depending on solar activity level.

- Unlike aircraft applications
- Variation mainly occurs in ionosphere
- Solar wind changes earth's EM field



Stationkeeping

All Satellites must budget Δv (m/s/yr) to compensate for atmospheric drag.



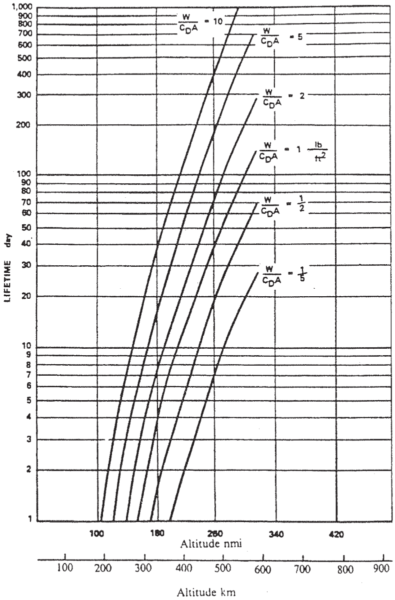
The problem with budgeting is predicting solar activity.

Spacecraft Lifetime

Without stationkeeping, orbits will decay quickly.

Definition 1.

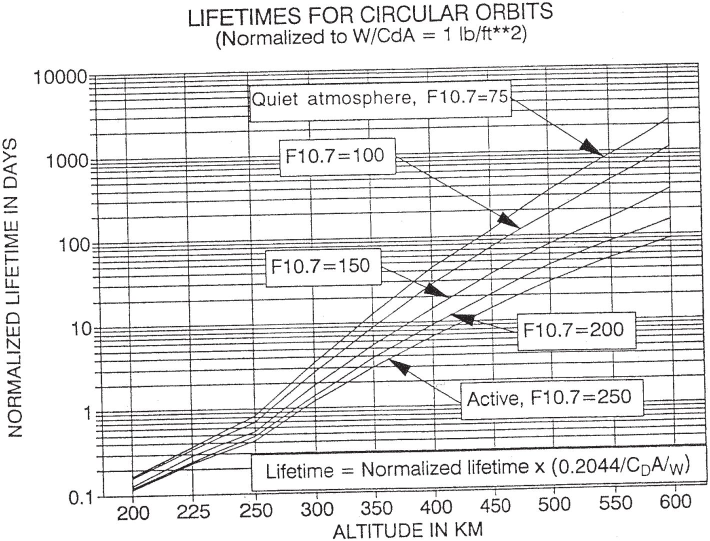
The **Lifetime** of a spacecraft is the time it takes to reach the 100km Kármán Line.



- The Figure shows mean value of lifetime.
- Actual values will depend on solar activity.

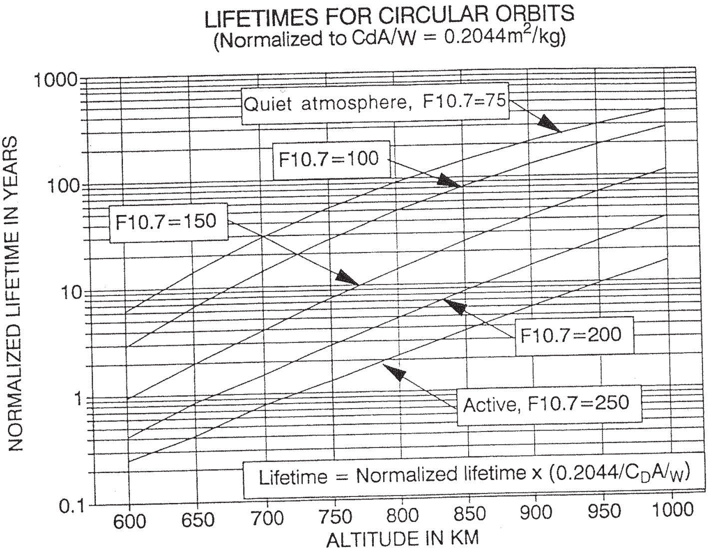
Spacecraft Lifetime

Solar Activity Effect



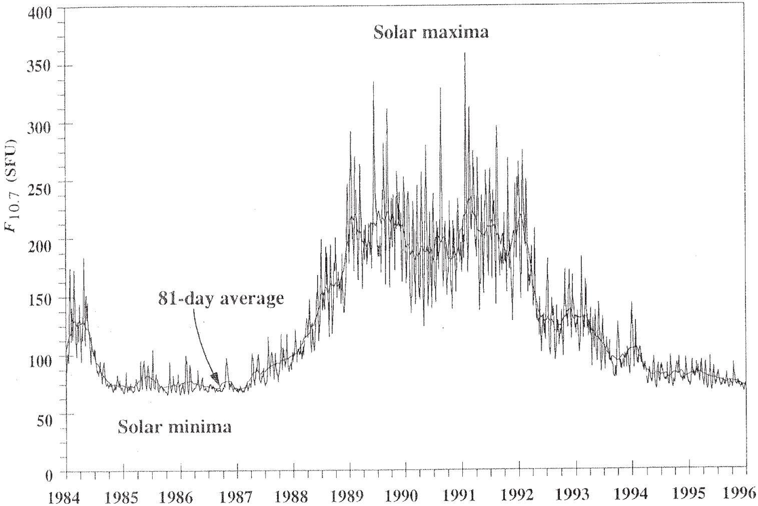
Spacecraft Lifetime

Solar Activity Effect



Solar Activity

Solar Activity varies substantially with time. $F_{10.7}$ measures normalized solar power flux at EM wavelength 10.7cm.



Solar Activity is Hard to Predict

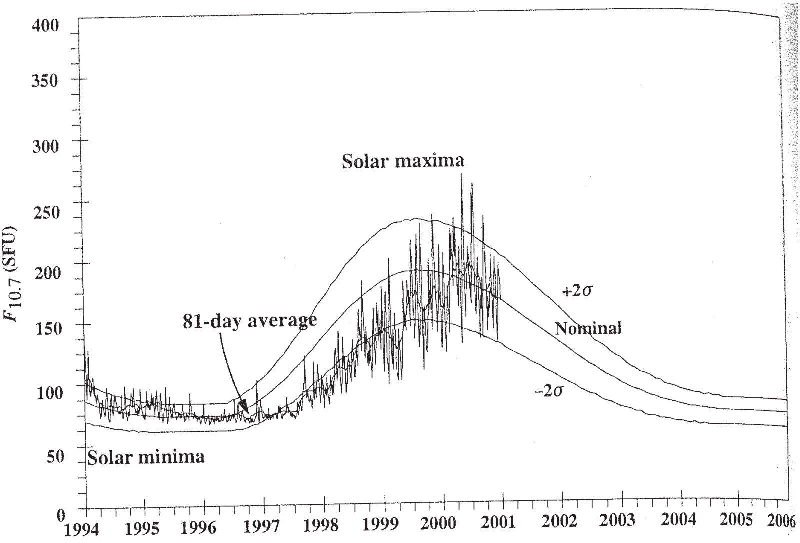
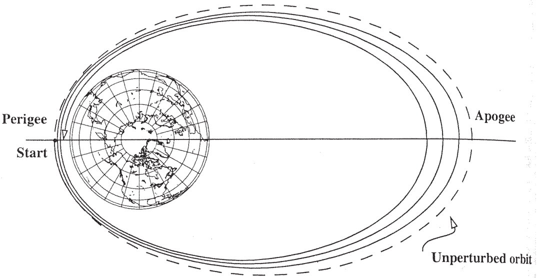


Figure: Shatten Prediction Model with Actual Data

Drag Effects on Eccentric Orbits

Eccentric orbits are particularly prone to drag.



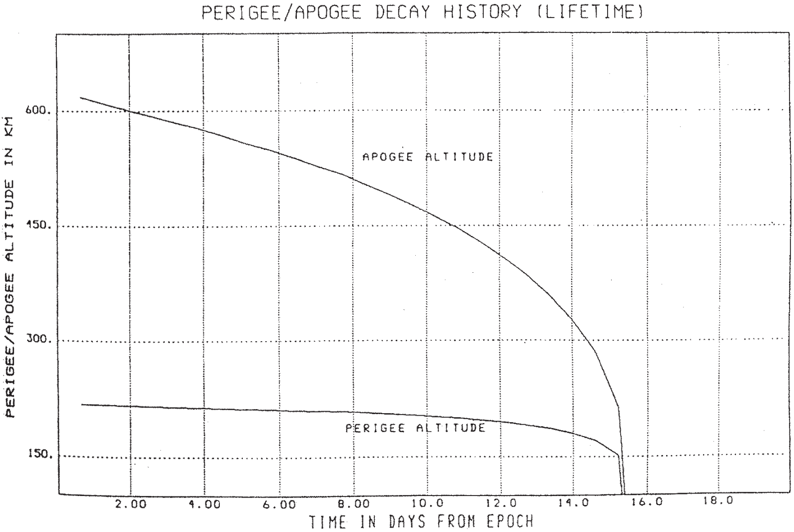
- Even if a is large, drag at perigee is high.
- Very difficult to integrate, due to changing density
- Using Exponential Density model,

$$\Delta e_{rev} = -2\pi \frac{C_D A}{m} a \rho_{perigee} e^{-ae/H} [I_1 + e(I_0 + I_2)/2]$$

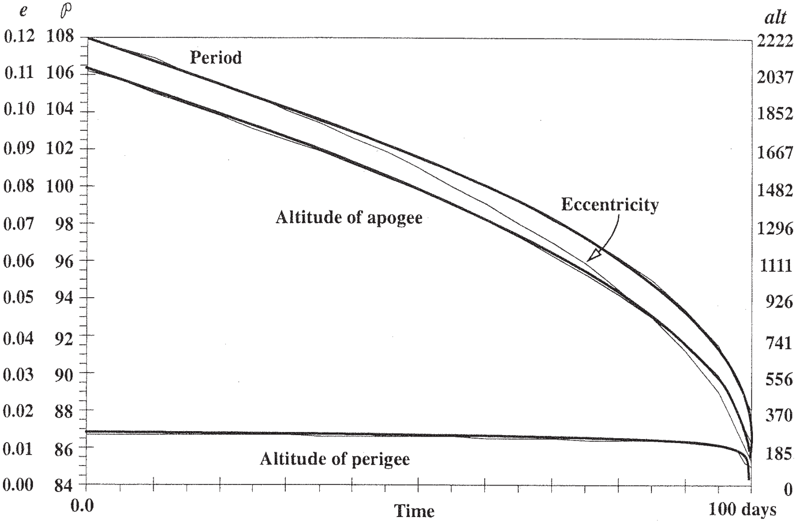
  - ρ_p is density at perigee. H is a height constant. I_i are Bessel functions
- Δa is also complicated.

Decay of Eccentricity

Although drag occurs at perigee, apogee is lowered.



Drag Effects on Eccentric Orbits



Hayabusa Re-entry

Summary

This Lecture you have learned:

Perturbation Basics

- The Satellite-Normal Coordinate System
- Equations for
    - $\dot{a}$, $\dot{i}$, $\dot{\Omega}$, $\dot{\omega}$, $\dot{e}$

Drag Perturbations

- Models of the atmosphere.
- Orbit Decay
- Δv budgeting.
- Effect on eccentricity.

Next Lecture: Earth's Shape and Sun-synchronous Orbits.

Equations

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[eR\sin f + T(1+e\cos f)\right]$$

$$\dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[R\sin f + T(\cos f + \cos E_{ecc})\right]$$

$$\frac{d}{dt}i = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\cos(\omega+f)}{1+e\cos f}$$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\sin(\omega+f)}{\sin i(1+e\cos f)}$$

$$\dot{\omega} = -\dot{\Omega}\cos i + \sqrt{\frac{a(1-e^2)}{e^2\mu}}\left(-R\cos f + T\frac{(2+e\cos f)\sin f}{1+e\cos f}\right)$$

Drag (circular orbit):

$$N = R = 0, \qquad T = -\frac{1}{2}B\rho v^2 = -\frac{1}{2}B\rho\frac{\mu}{a}.$$