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POLYNOMIAL OPTIMIZATION WITH APPLICATIONS TO STABILITY ANALYSIS AND CONTROL - ALTERNATIVES TO SUM OF SQUARES

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Abstract. In this paper, we explore the merits of various algorithms for solving polynomial optimization and optimization of polynomials, focusing on alternatives to sum of squares programming. While we refer to advantages and disadvantages of Quantifier Elimination, Reformulation Linear Techniques, Blossoming and Groebner basis methods, our main focus is on algorithms defined by Polya's theorem, Bernstein's theorem and Handelman's theorem. We first formulate polynomial optimization problems as verifying the feasibility of semi-algebraic sets. Then, we discuss how Polya's algorithm, Bernstein's algorithm and Handelman's algorithm reduce the intractable problem of feasibility of semi-algebraic sets to linear and/or semi-definite programming. We apply these algorithms to different problems in robust stability analysis and stability of nonlinear dynamical systems. As one contribution of this paper, we apply Polya's algorithm to the problem of H_∞ control of systems with parametric uncertainty. Numerical examples are provided to compare the accuracy of these algorithms with other polynomial optimization algorithms in the literature.

1. **Introduction.** Consider problems such as portfolio optimization, structural design, local stability of nonlinear ordinary differential equations, stability of time-delay systems and control of systems with uncertainties. These problems can all be formulated using *polynomial optimization* and/or *optimization of polynomials*. In this paper, we survey a variety of ways that computation can be used to solve these classes of problems. One example of polynomial optimization is $\beta^* = \min_{x \in Q} p(x)$, where $p : \mathbb{R}^n \to \mathbb{R}$ is a multi-variate polynomial and $Q \subset \mathbb{R}^n$. In general, since $p(x)$ and Q are not convex, this is not a convex optimization problem. In fact, it has been proved that polynomial optimization is NP-hard [11]. Fortunately, algorithms such as branch-and-bound can find arbitrarily precise solutions to polynomial optimization problems by repeatedly partitioning Q into subsets Q_i and computing

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lower and upper bounds on $p(x)$ over each Q_i. To find an upper bound for $p(x)$ over each Q_i, one could use a local optimization algorithm such as sequential quadratic programming. To find a lower bound on $p(x)$ over each Q_i, one can solve the following optimization problem.

$$\begin{aligned} \beta^* = & \max_{y\in\mathbb{R}} \; y \\ \text{subject to} \quad & p(x) - y \geq 0 \text{ for all } x \in Q_i. \end{aligned} \tag{1}$$

Problem (1) is in fact an instance of the problem of optimization of polynomials. Optimization of polynomials is convex, yet again NP-hard.

One approach to find lower bounds on the optimal objective β^* is to apply *Sum of Squares (SOS) programming* [50, 48]. A polynomial p is SOS if there exist polynomials q_i such that $p(x) = \sum_{i=1}^r q_i(x)^2$. The set $\{q_i \in \mathbb{R}[x], i = 1, \cdots, r\}$ is called an *SOS decomposition* of $p(x)$, where $\mathbb{R}[x]$ is the ring of real polynomials. An SOS program is an optimization problem of the form

$$\begin{aligned} & \min_{x\in\mathbb{R}^m} \qquad c^T x \\ & \text{subject to} \quad A_{i,0}(y) + \sum_{j=1}^m x_j A_{i,j}(y) \text{ is SOS, } i = 1, \cdots, k, \end{aligned} \tag{2}$$

where $c \in \mathbb{R}^m$ and $A_{i,j} \in \mathbb{R}[y]$ are given. If $p(x)$ is SOS, then clearly $p(x) \geq 0$ on $\mathbb{R}^n$. While verifying $p(x) \geq 0$ on $\mathbb{R}^n$ is NP-hard, verifying whether $p(x)$ is SOS - hence non-negative - can be done in polynomial time [50]. It was first shown in [50] that verifying the existence of a SOS decomposition is a Semi-Definite Program (SDP). Fortunately, there exist several algorithms [43, 31, 2] and solvers [76, 73, 75] that solve SDPs to arbitrary precision in polynomial time. To find lower bounds on $\beta^* = \min_{x\in\mathbb{R}^n} p(x)$, consider the SOS program

$$y^* = \max_{y\in\mathbb{R}} y \;\text{ subject to }\; p(x) - y \text{ is SOS.}$$

Clearly $y^* \leq \beta^*$. One can compute y^* by performing a bisection search on y and using semi-definite programming to verify $p(x) - y$ is SOS. SOS programming can also be used to find lower bounds on the global minimum of polynomials over a semi-algebraic set $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, h_j(x) = 0\}$ generated by $g_i, h_j \in \mathbb{R}[x]$. Given problem (1) with $x \in S$, *Positivstellensatz* results [72, 58, 66] define a sequence of SOS programs whose optimal objective values form a sequence of lower bounds on the global minimum β^*. It is shown that under certain conditions on S, the sequence of lower bounds converges to the global minimum [58]. See [41] for a comprehensive discussion on the Positivstellensatz results.

In this paper, we explore the merits of some of the alternatives to SOS programming. There exist several results in the literature that can be applied to polynomial optimization; e.g., Quantifier Elimination (QE) algorithms [19] for testing the feasibility of semi-algebraic sets, Reformulation Linear Techniques (RLTs) [70, 71] for linearizing polynomial optimizations, Polya's theorem [30] for positivity over the positive orthant, Bernstein's [12, 42] and Handelman's [29] theorems for positivity over simplices and convex polytopes, and other results based on Groebner bases [1] and Blossoming [60]. We will discuss Polya's, Bernstein's and Handelman's results in more depth. The discussion of the other results are beyond the scope of this paper, however the ideas behind these results can be summarized as follows.

QE algorithms apply to First-Order Logic formulae, e.g.,

$$\forall x\, \exists y\, (f(x,y) \geq 0 \Rightarrow ((g(a) < xy) \wedge (a > 2)),$$

to eliminate the *quantified* variables x and y (preceded by quantifiers $\forall, \exists$) and construct an equivalent formula in terms of the *unquantified* variable a. The key result underlying QE algorithms is the Tarski-Seidenberg theorem [74]. The theorem implies that for every formula of the form $\forall x \in \mathbb{R}^n\, \exists y \in \mathbb{R}^m (f_i(x,y,a) \geq 0)$, where $f_i \in \mathbb{R}[x,y,a]$, there exists an equivalent quantifier-free formula of the form $\wedge_i (g_i(a) \geq 0) \vee_j (h_j(a) \geq 0)$ with $g_i, h_j \in \mathbb{R}[a]$. QE implementations [13, 23] with a bisection search yield the exact solution to optimization of polynomials, however the complexity scales double exponentially in the dimension of variables x, y.

RLT was initially developed to find the convex hull of feasible solutions of zero-one linear programs [68]. It was later generalized to address polynomial optimizations of the form $\min_x p(x)$ subject to $x \in [0,1]^n \cap S$ [70]. RLT constructs a $\delta-$hierarchy of linear programs by performing two steps. In the first step (reformulation), RLT introduces the new constraints $\prod_i x_i \prod_j (1-x_j) \geq 0$ for all $i,j : i+j = \delta$. In the second step (linearization), RTL defines a linear program by replacing every product of variables x_i by a new variable. By increasing δ and repeating the two steps, one can construct a $\delta-$hierarchy of lower bounding linear programs. A combination of RLT and branch-and-bound partitioning of $[0,1]^n$ was developed in [71] to achieve tighter lower bounds on the global minimum. For a survey of different extensions of RLT see [69].

Groebner bases can be used to reduce polynomial optimization over a semi-algebraic set $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0,\, h_j(x) = 0\}$ to the problem of finding the roots of univariate polynomials [15]. First, one needs to construct the system of polynomial equations

$$[\nabla_x L(x,\lambda,\mu), \nabla_\lambda L(x,\lambda,\mu), \nabla_\mu L(x,\lambda,\mu)] = 0, \tag{3}$$

where $L := p(x) + \sum_i \lambda_i g_i(x) + \sum_j \mu_j h_j(x)$ is the Lagrangian. It is well-known that the set of solutions to (3) is the set of extrema of the polynomial optimization $\min_{x \in S} p(x)$. Let

$$[f_1(x,\lambda,\mu), \cdots, f_N(x,\lambda,\mu)] := [\nabla_x L(x,\lambda,\mu), \nabla_\lambda L(x,\lambda,\mu), \nabla_\mu L(x,\lambda,\mu)].$$

Using the elimination property [1] of the Groebner bases, the minimal Groebner basis of the ideal of $f_1, \cdots, f_N$ defines a triangular-form system of polynomial equations. This system can be solved by calculating one variable at a time and back-substituting into other polynomials. The most computationally expensive part is the calculation of the Groebner basis, which in the worst case scales double-exponentially in the number of decision variables.

The blossoming technique involves a bijective map between the space of polynomials $p : \mathbb{R}^n \to \mathbb{R}$ and the space of multi-affine functions $q : \mathbb{R}^{d_1+d_2+\cdots+d_n} \to \mathbb{R}$ (polynomials that are affine in each variable), where d_i is the degree of p in variable x_i. For instance, the blossom of a cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$ is the multi-affine function

$$q(z_1,z_2,z_3) = az_1z_2z_3 + \frac{b}{3}(z_1z_2 + z_1z_3 + z_2z_3) + \frac{c}{3}(z_1+z_2+z_3) + d.$$

It can be shown that any polynomial $p \in \mathbb{R}[x]$ with degree d_i in variable x_i and its corresponding blossom q satisfy the so-called *diagonal* property [60], i.e.,

$$p(z_1, z_2, \cdots, z_n) = q(\underbrace{z_1, \cdots, z_1}_{d_1 \text{ times}}, \cdots, \underbrace{z_n, \cdots, z_n}_{d_n \text{ times}}) \quad \text{for all } z \in \mathbb{R}.$$

By using this property, one can reformulate any polynomial optimization problem $\min_{x \in S} p(x)$ as

$$\begin{aligned} &\min_{z \in Q} \quad q(z) \\ &\text{subject to} \quad z_{\phi(i)} = z_{\phi(i)-j} \quad \text{for } i = 1, \cdots, n \text{ and for } j = 1, \cdots, d_i - 1, \end{aligned} \tag{4}$$

where $\phi(i) := \sum_{k=1}^{i} d_i$ and Q is the set defined by the blossoms of the generating polynomials of S. In the special case, where S is a hypercube, it is shown in [4] that the Lagrangian dual to Problem (4) is a linear program. Hence, the optimal objective value of this linear program is a lower bound on the minimum of $p(x)$ over the hypercube. Application of blossoming in estimation of reachability sets of discrete-time dynamical systems can be found in [5].

While QE, RLT, Groebner bases and blossoming are all useful techniques with advantages and disadvantages (such as exponential complexity), the focus of this paper is on Polya's, Bernstein's and Handelman's theorems - results which yield polynomial-time tests for polynomial positivity. Polya's theorem yields a basis which represents the cone of polynomials that are positive on the positive orthant. Bernstein's and Handelman's theorems yield bases which represent the cones of polynomials that are positive on simplices and convex polytopes, respectively. Similar to SOS programming, one can find certificates of positivity using Polya's, Bernstein's and Handelman's representations by solving a sequence of Linear Programs (LPs) and/or SDPs. However, unlike the SDPs associated with SOS programming, the SDPs associated with these theorems have a block-diagonal structure. This structure has been exploited in [34] to design parallel algorithms for optimization of polynomials of high degree with several independent variables. Unfortunately, unlike the SOS methodology, the bases given by Polya's theorem and Handelman's theorem cannot be used to represent non-negative polynomials which have zeros in the interior of the simplices and polytopes. There do, however, exist variants of Polya's theorem which consider zeros at the corners [54] and edges [14] of simplices. In addition, other variants of Polya's theorem provide certificates of positivity over hypercubes [45, 35], the intersection of semi-algebraic sets and the positive orthant [22], and the entire space $\mathbb{R}^n$ [20], or apply to polynomials with rational exponents [21].

We organize this paper as follows. In Section 2, we place Polya's, Bernstein's, Handelman's and the Positivstellensatz results in the broader topic of research on polynomial positivity. In Section 3, we first define polynomial optimization and optimization of polynomials. Then, we formulate optimization of polynomials as the problem of verifying the feasibility of semi-algebraic sets. To verify the feasibility of different semi-algebraic sets, we present algorithms based on the different variants of Polya's, Bernstein's, Handelman's and Positivstellensatz results. In Section 4, we discuss how these algorithms apply to robust stability analysis [59, 34, 46] and nonlinear stability [26, 36, 64, 37]. Finally, one contribution of this paper is to apply Polya's algorithm to the problem of H_∞ control synthesis for systems with parametric uncertainties.

2. **Background on positivity of polynomials.** In 1900, Hilbert published a list of mathematical problems, one of which was: For every non-negative $f \in \mathbb{R}[x]$, does there exist some non-zero $q \in \mathbb{R}[x]$ such that $q^2 f$ is a sum of squares? In other words, is every non-negative polynomial a sum of squares of rational functions? This question was motivated by his earlier works [32, 33], in which he proved: 1- Every non-negative bi-variate degree 4 *homogeneous* polynomial (A polynomial whose monomials all have the same degree) is a SOS of three polynomials. 2- Every bi-variate non-negative polynomial is a SOS of four rational functions. 3- Not every homogeneous polynomial with more than two variables and degree greater than 5 is a SOS of polynomials. While there exist systematic ways (e.g., semi-definite programming) to prove that a non-negative polynomial is SOS, proving that a non-negative polynomial is not a SOS of polynomials is not straightforward. Indeed, the first example of a non-negative non-SOS polynomial was published eighty years after Hilbert posed his 17^{th} problem. Motzkin [44] constructed a non-negative degree 6 polynomial with three variables which is not SOS:

$$M(x_1, x_2, x_3) = x_1^4 x_2^2 + x_1^2 x_2^4 - 3x_1^2 x_2^2 x_3^2 + x_3^6.$$

Robinson [61] generalized Motzkin's example as follows. Polynomials of the form $(\prod_{i=1}^n x_i^2) f(x_1, \cdots, x_n) + 1$ are not SOS if polynomial f of degree $< 2n$ is not SOS. Hence, although the non-homogeneous Motzkin polynomial $M(x_1, x_2, 1) = x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) + 1$ is non-negative it is not SOS.

In 1927, Artin answered Hilbert's problem in the following theorem [3].

Theorem 2.1. *(Artin's theorem) A polynomial $f \in \mathbb{R}[x]$ satisfies $f(x) \geq 0$ on $\mathbb{R}^n$ if and only if there exist SOS polynomials N and $D \neq 0$ such that $f(x) = \frac{N(x)}{D(x)}$.*

Although Artin settled Hilbert's problem, his proof was neither constructive nor gave a characterization of the numerator N and denominator D. In 1939, Habicht [28] showed that if f is positive definite and can be expressed as $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ for some polynomial g, then one can choose the denominator $D = \sum_{i=1}^n x_i^2$. Moreover, he showed that by using $D = \sum_{i=1}^n x_i^2$, the numerator N can be expressed as a sum of squares of monomials. Habicht used Polya's theorem ([30], Theorem 56) to obtain the above characterizations for N and D.

Theorem 2.2. *(Polya's theorem) Suppose a homogeneous polynomial p satisfies $p(x) > 0$ for all $x \in \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i \neq 0\}$. Then $p(x)$ can be expressed as*

$$p(x) = \frac{N(x)}{D(x)},$$

where $N(x)$ and $D(x)$ are homogeneous polynomials with all positive coefficients. For every homogeneous $p(x)$ and some $e \geq 0$, the denominator $D(x)$ can be chosen as $(x_1 + \cdots + x_n)^e$.

To see Habicht's result, suppose f is homogeneous and positive on the positive orthant and can be expressed as $f(x_1, \cdots, x_n) = g(x_1^2, \cdots, x_n^2)$ for some homogeneous polynomial g. Let $y_i = x_i^2$. By using Polya's theorem $g(y) = \frac{N(y)}{D(y)}$, where $y := (y_1, \cdots, y_n)$ and polynomials N and D have all positive coefficients. By Theorem 2.2 we may choose $D(y) = \left(\sum_{i=1}^n y_i\right)^e$. Then, $\left(\sum_{i=1}^n y_i\right)^e g(y) = N(y)$. Therefore, $\left(\sum_{i=1}^n x_i^2\right)^e f(x_1, \cdots, x_n) = N(x_1^2, \cdots, x_n^2)$. Since N has all positive coefficients, $N(x_1^2, \cdots, x_n^2)$ is a sum of squares of monomials. Unlike the case of positive definite polynomials, it is shown that there exists no single SOS polynomial $D \neq 0$ which satisfies $f = \frac{N}{D}$ for every positive semi-definite f and some SOS polynomial N [63].

As in the case of positivity on $\mathbb{R}^n$, there has been an extensive research regarding positivity of polynomials on bounded sets. A pioneering result on local positivity is Bernstein's theorem (1915) [6]. Bernstein's theorem uses the polynomials $h_{i,j} = (1+x)^i(1-x)^j$ as a basis to parameterize univariate polynomials which are positive on $[-1, 1]$.

Theorem 2.3. *(Bernstein's theorem) If a polynomial $f(x) > 0$ on $[-1, 1]$, then there exist $c_{i,j} > 0$ such that*

$$f(x) = \sum_{\substack{i,j \in \mathbb{N} \\ i+j=d}} c_{i,j}(1+x)^i(1-x)^j$$

for some $d > 0$.

Reference [55] used Goursat's transform of f to find an upper bound on d. Unfortunately, the bound itself is a function of the minimum of f on $[-1, 1]$. In order to reduce the computational complexity of testing positivity, [12] proposed a decomposition scheme for breaking $[-1, 1]$ into a collection of sub-intervals. Subsequently, Bernstein theorem was applied to f over each sub-interval to find a certificate of positivity over each sub-interval. An extension of this technique was proposed in [42] to verify positivity over simplices (a simplex is the convex hull of $n + 1$ vertices in $\mathbb{R}^n$). Moreover, [42] provided a degree bound as a function of the minimum of f over the simplex, the number of variables in f, the degree of f and the maximum of certain affine combinations of the coefficients $c_{i,j}$.

In 1988, Handelman [29] also used products of affine functions as a basis (the Handelman basis) to extend Bernstein's theorem to multi-variate polynomials which are positive over arbitrary convex polytopes.

Theorem 2.4. *(Handelman's Theorem) Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, define the polytope $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. If a polynomial $f(x) > 0$ on Γ^K, then there exist $b_\alpha \geq 0$, $\alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,*

$$f(x) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \alpha_1 + \cdots + \alpha_K \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}. \tag{5}$$

A generalization of Handelman's theorem was made by Schweighofer [67] to verify non-negativity of polynomials over compact semi-algebraic sets. Schweighofer used the cone of polynomials defined in (7) to parameterize any polynomial f which has the following properties:

1. f is non-negative over the compact semi-algebraic set S in (6)
2. $f = q_1 p_1 + q_2 p_2 + \cdots$ for some q_i in the cone (7) and for some $p_i > 0$ over $S \cap \{x \in \mathbb{R}^n : f(x) = 0\}$

Theorem 2.5. *(Schweighofer's theorem) Suppose*

$$S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x] \text{ for } i = 1, \cdots, K\} \tag{6}$$

is compact. Define the following set of polynomials which are positive on S.

$$\Theta_d := \left\{ \sum_{\lambda \in \mathbb{N}^K : \lambda_1 + \cdots + \lambda_K \leq d} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K} : s_\lambda \text{ are SOS} \right\} \tag{7}$$

If $f \geq 0$ on S and there exist $q_i \in \Theta_d$ and polynomials $p_i > 0$ on $S \cap \{x \in \mathbb{R}^n : f(x) = 0\}$ such that $f = \sum_i q_i p_i$ for some d, then $f \in \Theta_d$.

On the assumption that g_i are affine functions, $p_i = 1$ and s_λ are constant, Schweighofer's theorem gives the same parameterization of f as in Handelman's theorem. Another special case of Schweighofer's theorem is when $\lambda \in \{0,1\}^K$. In this case, Schweighofer's theorem reduces to Schmudgen's Positivstellensatz [66]. Schmudgen's Positivstellensatz states that the cone

$$\Lambda_g := \left\{ \sum_{\lambda \in \{0,1\}^K} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K} : s_\lambda \text{ are } SOS \right\} \subset \Theta_d \tag{8}$$

is sufficient to parameterize every $f > 0$ over the semi-algebraic set S generated by $\{g_1, \cdots, g_K\}$. Unfortunately, the cone Λ_g contains 2^K products of g_i, thus finding a representation of Form (8) for f requires a search for 2^K SOS polynomials. Putinar's Positivstellensatz [58] reduces the complexity of Schmudgen's parameterization in the case where the quadratic module M_g (as defined in (9)) of g_i is *Archimedean*, i.e., there exist $N \in \mathbb{N}$ such that

$$N - \sum_{i=1}^n x_i^2 \in M_g.$$

Theorem 2.6. *(Putinars's Positivstellensatz) Let $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x]$ for $i = 1, \cdots, K\}$ and define*

$$M_g := \left\{ s_0 + \sum_{i=1}^K s_i g_i : s_i \text{ are } SOS \right\}. \tag{9}$$

If there exist some $N > 0$ such that $N - \sum_{i=1}^n x_i^2 \in M_g$, then M_g is Archimedean. If M_g is Archimedean and $f > 0$ over S, then $f \in M_g$.

Finding a representation of the Form (9) for f, only requires a search for $K+1$ SOS polynomials using SOS programming. Verifying the Archimedean condition $N - \sum_{i=1}^n x_i^2 \in M_g$ in Theorem 2.6 is also a SOS program. Observe that the Archimedean condition implies the compactness of S. The following theorem lifts the compactness requirement for the semi-algebraic set S.

Theorem 2.7. *(Stengle's Positivstellensatz) Let $S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, g_i \in \mathbb{R}[x]$ for $i = 1, \cdots, K\}$ and define*

$$\Lambda_g := \left\{ \sum_{\lambda \in \{0,1\}^K} s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K} : s_\lambda \text{ are } SOS \right\}.$$

If $f > 0$ on S, then there exist $p, g \in \Lambda_g$ such that $qf = p + 1$.

Notice that the Parameterziation (5) in Handelman's theorem is affine in f and the coefficients b_α. Likewise, the parameterizations in Theorems 2.5 and 2.6, i.e., $f = \sum_\lambda s_\lambda g_1^{\lambda_1} \cdots g_K^{\lambda_K}$ and $f = s_0 + \sum_i s_i g_i$ are affine in f, s_λ and s_i. Thus, one can use convex optimization to find b_α, s_λ, s_i and f. Unfortunately, since the parameterization $qf = p + 1$ in Stengle's Positivstellensatz is non-convex (bilinear in q and f), it is more difficult to verify $qf = p + 1$ compared to Handelman's and Putinar's parameterizations.

For a comprehensive discussion on the Positivstellensatz and other results on polynomial positivity in algebraic geometry see [41, 65, 57].

3. **Algorithms for polynomial optimization.** In this Section, we first define polynomial optimization, optimization of polynomials and an equivalent feasibility problem using semi-algebraic sets. Then, we introduce some algorithms to verify the feasibility of different semi-algebraic sets. We observe that combining these algorithms with bisection yields lower bounds on optimal objective values of polynomial optimization problems.

3.1. **Polynomial optimization and optimization of polynomials.** Given f, g_i, $h_j \in \mathbb{R}[x]$ for $i = 1, \cdots, m$ and $j = 1, \cdots, r$, define a semi-algebraic set S as

$$S := \{y \in \mathbb{R}^n : g_i(y) \geq 0,\, h_j(y) = 0 \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, r\}. \tag{10}$$

We then define *polynomial optimization* problems as

$$\beta^* = \min_{x \in S} \; f(x). \tag{11}$$

For example, the integer program

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} && p(x) \\ &\text{subject to} && a_i^T x \geq b_i \text{ for } i = 1, \cdots, m, \\ &&& x \in \{-1, 1\}^n, \end{aligned} \tag{12}$$

with given $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}$ and $p \in \mathbb{R}[x]$, can be formulated as a polynomial optimization problem by setting $f = p$ in (11) and setting

$$\begin{aligned} g_i(x) &= a_i^T x - b_i && \text{for } i = 1, \cdots, m \\ h_j(x) &= x_j^2 - 1 && \text{for } j = 1, \cdots, n. \end{aligned}$$

in the definition of S in (10).

Given $c \in \mathbb{R}^n$ and g_i, $h_j \in \mathbb{R}[x]$ for $i = 1, \cdots, m$ and $j = 1, \cdots, r$, we define *Optimization of polynomials* problems as

$$\begin{aligned} &\gamma^* = \max_{x \in \mathbb{R}^q} && c^T x \\ &\text{subject to} && F(x, y) := F_0(y) + \sum_{i=1}^{q} x_i F_i(y) \geq 0 \text{ for all } y \in S, \end{aligned} \tag{13}$$

where S is defined in (10) and

$$F_i(y) := \sum_{\alpha \in E_{d_i}} F_{i,\alpha} y_1^{\alpha_1} \cdots y_n^{\alpha_n}$$

with $E_{d_i} := \{\alpha \in \mathbb{N}^n : \sum_{i=1}^n \alpha_i \leq d_i\}$, where coefficients $F_{i,\alpha} \in \mathbb{R}^{t \times t}$, $i = 0, \cdots, q$ are given. If the goal is to optimize over a polynomial variable, $p(y)$, this may be achieved using a basis of monomials for $F_i(y)$ so that the polynomial variable becomes $p(y) = \sum_i x_i F_i(y)$. Optimization of polynomials can be used to find β^* in (11). For example, we can compute the optimal objective value η^* of the polynomial optimization problem

$$\begin{aligned} &\eta^* = \min_{x \in \mathbb{R}^n} && p(x) \\ &\text{subject to} && a_i^T x - b_i \geq 0 && \text{for } i = 1, \cdots, m, \\ &&& x_j^2 - 1 = 0 && \text{for } j = 1, \cdots, n, \end{aligned}$$

by solving the problem

$$\begin{aligned}\eta^* = &\max_{\eta \in \mathbb{R}} \quad \eta \\ &\text{subject to} \quad p(y) \geq \eta \qquad \text{for } y \in \{y \in \mathbb{R}^n : a_i^T y \geq b_i,\, y_j^2 - 1 = 0 \text{ for } i = 1, \cdots, m \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{and } j = 1, \cdots, n\}, \end{aligned} \tag{14}$$

where Problem (14) can be expressed in the Form (13) by setting

$$c = 1, \qquad q = 1, \qquad t = 1, \qquad F_0 = p \qquad F_1 = -1,$$
$$S := \{y \in \mathbb{R}^n : a_i^T y \geq b_i,\, y_j^2 - 1 = 0 \text{ for } i = 1, \cdots, m, \text{ and } j = 1, \cdots, n\}.$$

Optimization of polynomials (13) can be reformulated as the feasibility problem

$$\begin{aligned}\gamma^* = &\min_{\gamma} \ \gamma \\ &\text{subject to } S_\gamma := \left\{x \in \mathbb{R}^q : c^T x > \gamma,\, F(x, y) \geq 0 \text{ for all } y \in S\right\} = \emptyset,\end{aligned} \tag{15}$$

where c and F are given and

$$S := \{y \in \mathbb{R}^n : g_i(y) \geq 0,\, h_j(y) = 0 \text{ for } i = 1, \cdots, m \text{ and } j = 1, \cdots, r\},$$

where polynomials g_i and h_j are given. The question of feasibility of a semi-algebraic set is NP-hard [11]. However, if we have a test to verify $S_\gamma = \emptyset$, we can find γ^* by performing a bisection on γ. In Section 3.2, we use the results of Section 2 to provide sufficient conditions, in the form of Linear Matrix Inequalities (LMIs), for $S_\gamma = \emptyset$.

3.2. **Algorithms.** In this section, we discuss how to find lower bounds on β^* for different classes of polynomial optimization problems. The results in this section are primarily expressed as methods for verifying $S_\gamma = \emptyset$ and can be used with bisection to solve optimization of polynomials problems.

Case 1. Optimization over the standard simplex Δ^n:

Define the standard unit simplex as

$$\Delta^n := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}. \tag{16}$$

Consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Delta^n} \quad f(x),$$

where f is a homogeneous polynomial of degree d. If f is not homogeneous, we can homogenize it by multiplying each monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in f by $(\sum_{i=1}^n x_i)^{d - \|\alpha\|_1}$. Notice that since $\sum_{i=1}^n x_i = 1$ for all $x \in \Delta^n$, the homogenized f is equal to f for all $x \in \Delta^n$. To find γ^*, one can solve the following optimization of polynomials problem.

$$\begin{aligned}\gamma^* = &\max_{\gamma \in \mathbb{R}} \quad \gamma \\ &\text{s.t. } f(x) \geq \gamma \text{ for all } x \in \Delta^n\end{aligned} \tag{17}$$

It can be shown that Problem (17) is equivalent to the feasibility problem

$$\begin{aligned}\gamma^* = &\min_{\gamma \in \mathbb{R}} \quad \gamma \\ &\text{s.t. } S_\gamma := \{x \in \mathbb{R}^n : f(x) - \gamma < 0, \sum_{i=1}^n x_i = 1,\, x_i \geq 0\} = \emptyset.\end{aligned}$$

For a given γ, we use the following version of Polya's theorem to verify $S_\gamma = \emptyset$.

Theorem 3.1. *(Polya's theorem, simplex version) If a homogeneous matrix-valued polynomial F satisfies $F(x) > 0$ for all $x \in \Delta^n := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, x_i \geq 0\}$, then there exists $e \geq 0$ such that all the coefficients of*

$$\left(\sum_{i=1}^n x_i\right)^e F(x)$$

are positive definite.

The converse of the theorem only implies $F \geq 0$ over the unit simplex. Given $\gamma \in \mathbb{R}$, it follows from the converse of Theorem 3.1 that $S_\gamma = \emptyset$ if there exist $e \geq 0$ such that

$$\left(\sum_{i=1}^n x_i\right)^e \left(f(x) - \gamma \left(\sum_{i=1}^n x_i\right)^d\right) \tag{18}$$

has all positive coefficients, where recall that d is the degree of f. We can compute lower bounds on γ^* by performing a bisection on γ. For each γ of the bisection, if there exists some $e \geq 0$ such that all of the coefficients of (18) are positive, then $\gamma \leq \gamma^*$.

Case 2. Optimization over the hypercube Φ^n:

Given $r_i \in \mathbb{R}$, define the hypercube

$$\Phi^n := \{x \in \mathbb{R}^n : |x_i| \leq r_i, i = 1, \cdots, n\}. \tag{19}$$

Define the set of n-variate *multi-homogeneous* polynomials of degree vector $d \in \mathbb{N}^n$ as

$$\left\{ p \in \mathbb{R}[x,y] : p(x,y) = \sum_{\substack{h,g \in \mathbb{N}^n \\ h+g=d}} p_{h,g} x_1^{h_1} y_1^{g_1} \cdots x_n^{h_n} y_n^{g_n},\, p_{h,g} \in \mathbb{R} \right\}. \tag{20}$$

In a more general case, if the coefficients $p_{h,g}$ are matrices, we call p a *matrix-valued multi-homogeneous* polynomial. It is shown in [35] that for every polynomial $f(z)$ with $z \in \Phi^n$, there exists a multi-homogeneous polynomial p such that

$$\{f(z) \in \mathbb{R} : z \in \Phi^n\} = \{p(x,y) \in \mathbb{R} : \ x, y \in \mathbb{R}^n \text{ and } (x_i, y_i) \in \Delta^2 \text{ for } i = 1, \cdots, n\}. \tag{21}$$

For example, consider $f(z_1, z_2) = z_1^2 + z_2$, with $z_1 \in [-2, 2]$ and $z_2 \in [-1, 1]$. Let $x_1 = \frac{z_1+2}{4} \in [0, 1]$ and $x_2 = \frac{z_2+1}{2} \in [0, 1]$. Then, define

$$q(x_1, x_2) := f(4x_1 - 2, 2x_2 - 1) = 16x_1^2 - 16x_1 + 2x_2 + 3.$$

By homogenizing q we obtain the multi-homogeneous polynomial

$$\begin{aligned} p(x,y) =& 16x_1^2(x_2 + y_2) - 16x_1(x_1 + y_1)(x_2 + y_2) + 2x_2(x_1 + y_1)^2 \\ &+ 3(x_1 + y_1)^2(x_2 + y_2), \quad (x_1, y_1), (x_2, y_2) \in \Delta^2 \end{aligned}$$

with degree vector $d = [2, 1]$, where $d_1 = 2$ is the sum of exponents of x_1 and y_1 in every monomial of p, and $d_2 = 1$ is the sum of exponents of x_2 and y_2 in every monomial of p. See Section 3 of [35] for an algorithm which computes the multi-homogeneous polynomial p for an arbitrary f defined on a hypercube.

Now consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Phi^n} f(x).$$

To find γ^*, one can solve the following feasibility problem.

$$\begin{aligned} \gamma^* = & \min_{\gamma \in \mathbb{R}} \quad \gamma \\ & \text{subject to} \quad S_{\gamma,r} := \{x \in \mathbb{R}^n : f(x) - \gamma < 0,\, |x_i| \leq r_i,\, i = 1, \cdots, n\} = \emptyset \end{aligned} \tag{22}$$

For a given γ, one can use the following version of Polya's theorem to verify $S_{\gamma,r} = \emptyset$.

Theorem 3.2. *(Polya's theorem, multi-homogeneous version) A matrix-valued multi-homogeneous polynomial F satisfies $F(x, y) > 0$ for all $(x_i, y_i) \in \Delta^2, i = 1, \cdots, n$, if there exist $e \geq 0$ such that all the coefficients of*

$$\left(\prod_{i=1}^{n} (x_i + y_i)^e\right) F(x, y)$$

are positive definite.

The converse of the theorem only implies non-negativity of F over the multi-simplex. To find lower bounds on γ, we first obtain the multi-homogeneous form p of the polynomial f in (22) by using the algorithm in Section 3 of [35]. Given $\gamma \in \mathbb{R}$ and $r \in \mathbb{R}^n$, it follows from the converse of Theorem 3.2 that $S_{\gamma,r}$ defined in (22) is empty if there exists some $e \geq 0$ such that

$$\left(\prod_{i=1}^{n} (x_i + y_i)^e\right) \left(p(x, y) - \gamma \left(\prod_{i=1}^{n} (x_i + y_i)^{d_i}\right)\right) \tag{23}$$

has all positive coefficients, where d_i is the degree of x_i in $p(x, y)$. We can compute lower bounds on γ^* by performing a bisection on γ. For each γ of the bisection, if there exists some $e \geq 0$ such that all of the coefficients of (23) are positive, then $\gamma \leq \gamma^*$.

Case 3. Optimization over the convex polytope Γ^K:

Given $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$, define the convex polytope

$$\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}.$$

Suppose Γ^K is bounded. Consider the polynomial optimization problem

$$\gamma^* = \min_{x \in \Gamma^K} f(x),$$

where f is a polynomial of degree d_f. To find γ^*, one can solve the feasibility problem

$$\begin{aligned} \gamma^* = & \min_{\gamma \in \mathbb{R}} \quad \gamma \\ & \text{subject to} \quad S_{\gamma,K} := \{x \in \mathbb{R}^n : f(x) - \gamma < 0,\, w_i^T x + u_i \geq 0,\, i = 1, \cdots, K\} = \emptyset. \end{aligned}$$

Given γ, one can use Handelman's theorem (Theorem 2.4) to verify $S_{\gamma,K} = \emptyset$ as follows. Consider the Handelman basis associated with polytope Γ^K defined as

$$B_s := \left\{\lambda_\alpha \in \mathbb{R}[x] : \lambda_\alpha(x) = \prod_{i=1}^{K} \left(w_i^T x + u_i\right)^{\alpha_i}, \alpha \in \mathbb{N}^K, \sum_{i=1}^{K} \alpha_i \leq s\right\}.$$

Basis B_s spans the space of polynomials of degree s or less, however it is not minimal. Given polynomial $f(x)$ of degree $d_f, \gamma \in \mathbb{R}$ and $d_{\max} \in \mathbb{N}$, if there exist

$$c_\alpha \geq 0 \text{ for all } \alpha \in \{\alpha \in \mathbb{N}^K : \|\alpha\|_1 \leq d\} \tag{24}$$

such that

$$f(x) - \gamma = \sum_{\|\alpha\|_1 \leq d} c_\alpha \prod_{i=1}^{K} (w_i^T x + u_i)^{\alpha_i} \tag{25}$$

for some $d \geq d_f$, then $f(x) - \gamma \geq 0$ for all $x \in \Gamma^K$. Thus, $S_{\gamma,K} = \emptyset$. Feasibility of Conditions (24) and (25) can be determined using linear programming. If (24) and (25) are infeasible for some d, then one can increase d up to some $d_{\max}$. From Handelman's theorem, if $f(x) - \gamma > 0$ for all $x \in \Gamma^K$, then for some $d \geq d_f$, Conditions (24) and (25) hold. However, computing upper bounds for d is difficult [56, 42].

Similar to Cases 1 and 2, we can compute lower bounds on γ^* by performing a bisection on γ. For each γ of the bisection, if there exists some $d \geq d_f$ such that Conditions (24) and (25), then $\gamma \leq \gamma^*$.

Case 4. Optimization over compact semi-algebraic sets:

Recall that we defined a semi-algebraic set as

$$S := \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, \cdots, m,\ h_j(x) = 0, j = 1, \cdots, r\}. \tag{26}$$

Suppose S is compact. Consider the polynomial optimization problem

$$\begin{aligned} \gamma^* = \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \geq 0 \text{ for } i = 1, \cdots, m \\ & h_j(x) = 0 \text{ for } j = 1, \cdots, r. \end{aligned}$$

Define the following cone of polynomials which are positive over S.

$$M_{g,h} := \left\{ m \in \mathbb{R}[x] : m(x) - \sum_{i=1}^{m} s_i(x) g_i(x) - \sum_{i=1}^{r} t_i(x) h_i(x) \text{ is SOS}, s_i \in \Sigma_{2d}, t_i \in \mathbb{R}[x] \right\}, \tag{27}$$

where Σ_{2d} denotes the set of SOS polynomials of degree $2d$. From Putinar's Positivstellensatz (Theorem 2.6) it follows that if the Cone (27) is Archimedean, then the solution to the following SOS program is a lower bound on γ^*. Given $d \in \mathbb{N}$, define

$$\begin{aligned} \gamma^d = & \max_{\gamma \in \mathbb{R}, s_i, t_i} \ \gamma \\ & \text{subject to } \ f(x) - \gamma - \sum_{i=1}^{m} s_i(x) g_i(x) - \sum_{i=1}^{r} t_i(x) h_i(x) \text{ is SOS }, \ t_i \in \mathbb{R}[x], s_i \in \Sigma_{2d}. \end{aligned} \tag{28}$$

For given $\gamma \in \mathbb{R}$ and $d \in \mathbb{N}$, Problem (28) is the following linear matrix inequality.

$$\begin{aligned} &\text{Find} \quad Q_i \geq 0, P_j \quad \text{for } i = 0, \cdots, m \text{ and } j = 1, \cdots, r \\ &\text{such that } \ f(x) - \gamma = z_d^T(x) \left(Q_0 + \sum_{i=1}^{m} Q_i g_i(x) + \sum_{j=1}^{r} P_j h_j(x) \right) z_d(x), \end{aligned} \tag{29}$$

where $Q_i, P_j \in \mathbb{S}^N$, where $\mathbb{S}^N$ is the subspace of symmetric matrices in $\mathbb{R}^{N \times N}$ and $N := \binom{n+d}{d}$, and where $z_d(x)$ is the vector of monomial basis of degree d or less. See [48] and [7] for methods of solving SOS programs. It is shown in [40] that if the Cone (27) is Archimedean, then $\lim_{d \to \infty} \gamma^d = \gamma^*$.

If the Cone (27) is not Archimedean, then we can use Schmudgen's Positivstellensatz to obtain the following SOS program with solution $\gamma^d \leq \gamma^*$.

$$\begin{aligned}\gamma^d = &\max_{\gamma\in\mathbb{R}, s_i, t_i} \quad \gamma \\ &\text{subject to} \;\; f(x) - \gamma = 1 + \sum_{\lambda\in\{0,1\}^m} s_\lambda(x) g_1(x)^{\lambda_1}\cdots g_m(x)^{\lambda_m} + \sum_{i=1}^{r} t_i(x) h_i(x), \\ &\qquad\qquad t_i \in \mathbb{R}[x], s_\lambda \in \Sigma_{2d}. \end{aligned} \tag{30}$$

The Positivstellensatz and SOS programming can also be applied to polynomial optimization over a more general form of semi-algebraic sets defined as

$$T := \{x \in \mathbb{R}^n : g_i(x) \geq 0,\, i = 1, \cdots, m,\, h_j(x) = 0,\, j = 1, \cdots, r,\, q_k(x) \neq 0,\, k = 1, \cdots, l\}.$$

It can be shown that $T = \emptyset$ if and only if

$$\hat{T} := \{(x, y) \in \mathbb{R}^{n+l} : g_i(x) \geq 0,\, i = 1, \cdots, m,\, h_j(x) = 0,\, j = 1, \cdots, r,\, y_k q_k(x) = 1,\, k = 1, \cdots, l\} = \emptyset.$$

Thus, for any $f \in \mathbb{R}[x]$, we have

$$\min_{x\in T} f(x) = \min_{(x,y)\in\hat{T}} f(x).$$

Therefore, to find lower bounds on $\min_{x\in T} f(x)$, one can apply SOS programming and Putinar's Positivstellensatzs to $\min_{(x,y)\in\hat{T}} f(x)$.

Case 5. Tests for non-negativity on $\mathbb{R}^n$:

The following theorem [62], gives a test for non-negativity of homogeneous polynomials over $\mathbb{R}^n$.

Theorem 3.3. *For every homogeneous polynomial f that satisfies $f(x_1, \cdots, x_n) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$, there exists some $e \geq 0$ such that all of the coefficients of*

$$\left(\sum_{i=1}^{n} x_i^2\right)^e f(x_1, \cdots, x_n) \tag{31}$$

are positive. In particular, the product is a sum of squares of monomials.

Using this theorem, one can verify non-negativity of any homogeneous polynomial f over $\mathbb{R}^n$ by multiplying f repeatedly by $\sum_{i=1}^n x_i^2$. If for some $e \in \mathbb{N}$, the Product (31) has all positive coefficients, then $f \geq 0$. An alternative test for non-negativity on $\mathbb{R}^n$ is given in the following theorem [20].

Theorem 3.4. *Define $E_n := \{-1, 1\}^n$. Suppose a polynomial $f(x_1, \cdots, x_n)$ of degree d satisfies $f(x_1, \cdots, x_n) > 0$ for all $x \in \mathbb{R}^n$ and its homogenization is positive definite. Then,*

1. *For every $e \in E^n$, there exist $\lambda_e \geq 0$ and coefficients $c_\alpha \in \mathbb{R}$ such that*
$$\left(1 + e^T x\right)^{\lambda_e} f(x_1, \cdots, x_n) = \sum_{\alpha\in I_e} c_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n}, \tag{32}$$
where $I_e := \{\alpha \in \mathbb{N}^n : \|\alpha\|_1 \leq d + \lambda_e\}$ and $sgn(c_\alpha) = e_1^{\alpha_1}\cdots e_n^{\alpha_n}$.
2. *There exist positive $N, D \in \mathbb{R}[x_1^2, \cdots, x_n^2, f^2]$ such that $f = \frac{N}{D}$.*

Based on the converse of Theorem 3.4, we can propose the following test for non-negativity of polynomials over the cone $\Lambda_e := \{x \in \mathbb{R}^n : sgn(x_i) = e_i, i = 1, \cdots, n\}$ for some $e \in E_n$. Multiply a given polynomial f repeatedly by $1 + e^T x$ for some $e \in E_n$. If there exists some $\lambda_e \geq 0$ such that $sgn(c_\alpha) = e_1^{\alpha_1} \cdots e_n^{\alpha_n}$, then (32) clearly implies $f(x) \geq 0$ for all $x \in \Lambda_e$. Since $\mathbb{R}^n = \cup_{e \in E_n} \Lambda_e$, we can repeat the test 2^n times to obtain a test for non-negativity of f over $\mathbb{R}^n$.

The second part of Theorem 3.4 gives a solution to the Hilbert's problem in Section 2. See [20] for an algorithm which computes polynomials N and D.

4. **Applications of polynomial optimization.** In this section, we discuss how the algorithms in Section 3.2 apply to stability analysis and control of dynamical systems. We consider robust stability analysis of linear systems with parametric uncertainty, stability of nonlinear systems and robust controller synthesis for systems with parametric uncertainty.

4.1. **Robust stability analysis.** Consider the linear system

$$\dot{x}(t) = A(\alpha)x(t), \tag{33}$$

where $A(\alpha) \in \mathbb{R}^{n \times n}$ is a polynomial and $\alpha \in Q \subset \mathbb{R}^l$ is a vector of uncertain parameters, where Q is compact. From converse Lyapunov theory [39] and existence of polynomial solutions for feasible parameter-dependent LMIs [8] it follows that System (33) is asymptotically stable if and only if there exist matrix-valued polynomial $P(\alpha) \in \mathbb{S}^n$ such that

$$P(\alpha) > 0 \text{ and } A^T(\alpha)P(\alpha) + P(\alpha)A(\alpha) < 0 \text{ for all } \alpha \in Q. \tag{34}$$

If Q is a semi-algebraic set, then asymptotic stability of System (33) is equivalent to positivity of γ^* in the following optimization of polynomials problem for some $d \in \mathbb{N}$.

$$\gamma^* = \max_{\gamma \in \mathbb{R}, C_\beta \in \mathbb{S}^n} \gamma$$
$$\text{subject to} \begin{bmatrix} \sum_{\beta \in E_d} C_\beta \alpha^\beta & 0 \\ 0 & -A^T(\alpha)\Big(\sum_{\beta \in E_d} C_\beta \alpha^\beta\Big) - \Big(\sum_{\beta \in E_d} C_\beta \alpha^\beta\Big)A(\alpha) \end{bmatrix} - \gamma I \alpha^T \alpha \geq 0, \alpha \in Q, \tag{35}$$

where we have denoted $\alpha_1^{\beta_1} \cdots \alpha_l^{\beta_l}$ by α^β and

$$E_d := \{\beta \in \mathbb{N}^l : \sum_{i=1}^n \beta_i \leq d\}. \tag{36}$$

Given stable systems of the form defined in (33) with different classes of polynomials $A(\alpha)$, we discuss different algorithms for solving (35). Solutions to (35) yield Lyapunov functions of the form $V = x^T(\sum_{\beta \in E_d} C_\beta \alpha^\beta)x$ proving stability of System (33).

Case 1. $A(\alpha)$ **is affine with** $\alpha \in \Delta^l$**:** Consider the case where $A(\alpha)$ belongs to the polytope

$$\Lambda_l := \left\{ A(\alpha) \in \mathbb{R}^{n \times n} : A(\alpha) = \sum_{i=1}^l A_i \alpha_i, A_i \in \mathbb{R}^{n \times n}, \alpha_i \in \Delta^l \right\},$$

where A_i are the vertices of the polytope and Δ^l is the standard unit simplex defined as in (16). Given $A(\alpha) \in \Lambda_l$, we address the problem of stability analysis of System (33) for all $\alpha \in \Delta^l$.

A sufficient condition for asymptotic stability of System (33) is to find a matrix $P > 0$ such that the Lyapunov inequality $A^T(\alpha)P + PA(\alpha) < 0$ holds for all $\alpha \in \Delta^l$. If $A(\alpha) = \sum_{i=1}^l A_i\alpha_i$, then from convexity of A it follows that the condition

$$A^T(\alpha)P + PA(\alpha) < 0 \text{ for all } \alpha \in \Delta^l$$

is equivalent to positivity of γ^* in the following semi-definite program.

$$\begin{aligned}
\gamma^* = &\max_{\gamma\in\mathbb{R}, P\in\mathbb{S}^n} \ \gamma \\
&\text{subject to} \begin{bmatrix} P & 0 & \cdots & 0 \\ 0 & -A_1^TP - PA_1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & -A_l^TP - PA_l \end{bmatrix} - \gamma I \geq 0
\end{aligned} \tag{37}$$

Any $P \in \mathbb{S}^n$ that satisfies the LMI in (37) for some $\gamma > 0$, yields a Lyapunov function of the form $V = x^TPx$. However for many systems, this class of Lyapunov functions can be conservative (see Numerical Example 1).

More general classes of Lyapunov functions such as parameter-dependent functions of the forms $V = x^T(\sum_{i=1}^l P_i\alpha_i)x$ [59, 24, 47] and $V = x^T(\sum_\beta P_{\beta\in E_d}\alpha^\beta)x$ [46, 34] have been utilized in the literature. As shown in [59], given $A_i \in \mathbb{R}^{n\times n}$, $x^TP(\alpha)x$ with $P(\alpha) = \sum_{i=1}^l P_i\alpha_i$ is a Lyapunov function for (33) with $\alpha \in \Delta^l$ if the following LMI conditions hold.

$$\begin{aligned}
&P_i > 0 && \text{for } i = 1, \cdots, l \\
&A_i^TP_i + P_iA_i < 0 && \text{for } i = 1, \cdots, l \\
&A_i^TP_j + A_j^TP_i + P_jA_i + P_iA_j < 0 && \text{for } i = 1, \cdots, l-1,\ j = i+1, \cdots, l
\end{aligned}$$

In [9], it is shown that given continuous functions $A_i, B_i : \Delta^l \to \mathbb{R}^{n\times n}$ and continuous function $R : \Delta^l \to \mathbb{S}^n$, if there exists a continuous function $X : \Delta^l \to \mathbb{S}^n$ which satisfies

$$\sum_{i=1}^N \left(A_i(\alpha)X(\alpha)B_i(\alpha) + B_i(\alpha)^TX(\alpha)A_i(\alpha)^T\right) + R(\alpha) > 0 \ \text{ for all } \alpha \in \Delta^l, \tag{38}$$

then there exists a homogeneous polynomial $Y : \Delta^l \to \mathbb{S}^n$ which also satisfies (38). Motivated by this result, [46] uses the class of homogeneous polynomials of the form

$$P(\alpha) = \sum_{\beta\in I_d} P_\beta\alpha_1^{\beta_1}\cdots\alpha_l^{\beta_l}, \tag{39}$$

with

$$I_d := \left\{\beta \in \mathbb{N}^l : \sum_{i=1}^l \beta_i = d\right\} \tag{40}$$

to provide the following necessary and sufficient LMI condition for stability of System (33). Given $A(\alpha) = \sum_{i=1}^l A_i\alpha_i$, System (33) is asymptotically stable for all $\alpha \in \Delta^l$ if and only if there exist some $d \geq 0$ and positive definite $P_\beta \in \mathbb{S}^n, \beta \in I_d$ such that

$$\sum_{i=1,\cdots,l;\, \beta_i>0} \left(A_i^TP_{\beta-e_i} + P_{\beta-e_i}A_i\right) < 0 \text{ for all } \beta \in I_{d+1}, \tag{41}$$

where $e_i = [0 \cdots 0 \underbrace{1}_{i^{th}} 0 \cdots 0] \in \mathbb{N}^l, i = 1, \cdots, l$ form the canonical basis for $\mathbb{R}^l$.

Numerical Example 1. Consider the system $\dot{x}(t) = A(\alpha, \eta)x(t)$ from [17], where $A(\alpha, \eta) = (A_0 + A_1\eta)\alpha_1 + (A_0 + A_2\eta)\alpha_2 + (A_0 + A_3\eta)\alpha_3$, where

$$A_0 = \begin{bmatrix} -2.4 & -0.6 & -1.7 & 3.1 \\ 0.7 & -2.1 & -2.6 & -3.6 \\ 0.5 & 2.4 & -5.0 & -1.6 \\ -0.6 & 2.9 & -2.0 & -0.6 \end{bmatrix}, A_1 = \begin{bmatrix} 1.1 & -0.6 & -0.3 & -0.1 \\ -0.8 & 0.2 & -1.1 & 2.8 \\ -1.9 & 0.8 & -1.1 & 2.0 \\ -2.4 & -3.1 & -3.7 & -0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.9 & 3.4 & 1.7 & 1.5 \\ -3.4 & -1.4 & 1.3 & 1.4 \\ 1.1 & 2.0 & -1.5 & -3.4 \\ -0.4 & 0.5 & 2.3 & 1.5 \end{bmatrix}, A_3 = \begin{bmatrix} -1.0 & -1.4 & -0.7 & -0.7 \\ 2.1 & 0.6 & -0.1 & -2.1 \\ 0.4 & -1.4 & 1.3 & 0.7 \\ 1.5 & 0.9 & 0.4 & -0.5 \end{bmatrix}$$

and $(\alpha_1, \alpha_2, \alpha_3) \in \Delta^3$, $\eta \geq 0$. We would like to find $\eta^* = \max \eta$ such that $\dot{x}(t) = A(\alpha, \eta)x(t)$ is asymptotically stable for all $\eta \in [0, \eta^*]$.

By performing a bisection on η and verifying the inequalities in (41) for each η of the bisection algorithm, we obtained lower bounds on η^* (see Figure 1) using degrees $d = 0, 1, 2$ and 3. For comparison, we have also plotted the lower bounds computed in [17] using the Complete Square Matricial Representation (CSMR) of the Lyapunov inequalities in (34). Both methods found $\max \eta = 2.224$, however the method in [17] used a lower d to find this bound.

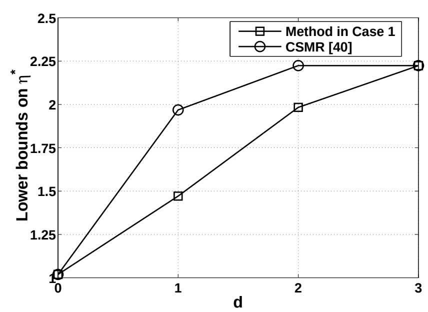

FIGURE 1. lower-bounds for η^* computed using the LMIs in (41) and the method in [17]

Case 2. $A(\alpha)$ **is a polynomial with** $\alpha \in \Delta^l$**:**

Given $A_h \in \mathbb{R}^{n\times n}$ for $h \in I_d$ as defined in (40), we address the problem of stability analysis of System (33) with $A(\alpha) = \sum_{h \in I_d} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}$ for all $\alpha \in \Delta^l$. Using Lyapunov theory, this problem can be formulated as the following optimization of polynomials problem.

$$\gamma^* = \max_{\gamma \in \mathbb{R}, P \in \mathbb{R}[\alpha]} \gamma$$
$$\text{subject to } \begin{bmatrix} P(\alpha) & 0 \\ 0 & -A(\alpha)^T P(\alpha) - P(\alpha)A(\alpha) \end{bmatrix} - \gamma I \geq 0 \text{ for all } \alpha \in \Delta^l \quad (42)$$

System (33) is asymptotically stable for all $\alpha \in \Delta^l$ if and only if $\gamma^* > 0$. As in Case 1 of Section 3.2, one can apply a bisection algorithm on γ while using Polya's theorem (Theorem 3.1) as a test for feasibility of Constraint (42), thereby finding a lower bound on γ^*. Suppose P and A are homogeneous matrix-valued polynomials. Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.1 that the inequality condition in (42) holds for all $\alpha \in \Delta^l$ if there exist some $e \geq 0$ such that

$$\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(P(\alpha) - \gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p}\right) \tag{43}$$

and

$$-\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(A(\alpha)^T P(\alpha) + P(\alpha)A(\alpha) + \gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p+d_a}\right) \tag{44}$$

have all positive coefficients, where d_p is the degree of P and d_a is the degree of A. Let P and A be of the form

$$P(\alpha) = \sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, P_h \in \mathbb{S}^n \quad \text{and} \quad A(\alpha) = \sum_{h \in I_{d_a}} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, A_h \in \mathbb{R}^{n \times n}. \tag{45}$$

By combining (45) with (43) and (44) it follows that for a given $\gamma \in \mathbb{R}$, the inequality condition in (42) holds for all $\alpha \in \Delta^l$ if there exist some $e \geq 0$ such that

$$\left(\sum_{i=1}^{l} \alpha_i\right)^e \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l} - \gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p}\right) = \sum_{g \in I_{d_p+e}} \left(\sum_{h \in I_{d_p}} f_{g,h} P_h\right) \alpha_1^{g_1} \cdots \alpha_l^{g_l} \tag{46}$$

and

$$\begin{aligned}
-\left(\sum_{i=1}^{l} \alpha_i\right)^e \Bigg(\left(\sum_{h \in I_{d_a}} A_h^T \alpha^h\right)\left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}\right) + \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}\right)\left(\sum_{h \in I_{d_a}} A_h \alpha^h\right) \\
+\gamma I \left(\sum_{i=1}^{l} \alpha_i\right)^{d_p+d_a}\Bigg) = \sum_{q \in I_{d_a+d_p+e}} \left(\sum_{h \in I_{d_p}} M_{h,q}^T P_h + P_h M_{h,q}\right) \alpha_1^{q_1} \cdots \alpha_l^{q_l}
\end{aligned} \tag{47}$$

have all positive coefficients, i.e.,

$$\begin{aligned}
&\sum_{h \in I_{d_p}} f_{h,g} P_h > 0 \quad \text{for all } g \in I_{d_p+e} \\
&\sum_{h \in I_{d_p}} \left(M_{h,q}^T P_h + P_h M_{h,q}\right) < 0 \quad \text{for all } q \in I_{d_p+d_a+e},
\end{aligned} \tag{48}$$

where we define $f_{h,g} \in \mathbb{R}$ as the coefficient of $P_h \alpha_1^{g_1} \cdots \alpha_l^{g_l}$ after expanding (46). Likewise, we define $M_{h,q} \in \mathbb{R}^{n \times n}$ as the coefficient of $P_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after expanding (47). See [38] for recursive formulae for $f_{h,g}$ and $M_{h,q}$. Feasibility of Conditions (48) can be verified by the following semi-definite program.

$$\max_{\eta\in\mathbb{R},P_h\in\mathbb{S}^n_+} \eta$$

$$\text{subject to} \begin{bmatrix} \sum_{h\in I_{d_p}} f_{h,g^{(1)}}P_h & 0 & \cdots & & 0 \\ 0 & \ddots & & & \\ & & \sum_{h\in I_{d_p}} f_{h,g^{(L)}}P_h & & \vdots \\ \vdots & & & -\sum_{h\in I_{d_p}}\left(M^T_{h,q^{(1)}}P_h+P_hM_{h,q^{(1)}}\right) & \\ & & & \ddots & 0 \\ 0 & \cdots & 0 & & -\sum_{h\in I_{d_p}}\left(M^T_{h,q^{(M)}}P_h+P_hM_{h,q^{(M)}}\right) \end{bmatrix} -\eta I \geq 0, \tag{49}$$

where we have denoted the elements of I_{d_p+e} by $g^{(i)}\in\mathbb{N}^l, i=1,\cdots,L$ and have denoted the elements of $I_{d_p+d_a+e}$ by $q^{(i)}\in\mathbb{N}^l, i=1,\cdots,M$. For any $\gamma\in\mathbb{R}$, if there exist $e\geq 0$ such that SDP (49) is feasible, then $\gamma\leq\gamma^*$ as defined in (42). If for a positive γ, there exist $e\geq 0$ such that SDP (49) has a solution $P_h, h\in I_{d_p}$, then $V=x^T\left(\sum_{h\in I_{d_p}} P_h\alpha_1^{h_1}\cdots\alpha_l^{h_l}\right)x$ is a Lyapunov function proving stability of $\dot{x}(t)=A(\alpha)x(t), \alpha\in\Delta^l$. See [34] for a complexity analysis of the resulting SDP (49).

SDPs such as (49) can be solved in polynomial time using interior-point algorithms such as the central path primal-dual algorithms in [43, 31, 2]. Fortunately, Problem (49) has block-diagonal structure. Block-diagonal structure in SDP constraints can be used to design massively parallel algorithms, an approach which was applied to Problem (49) in [34].

Numerical Example 2. Consider the system $\dot{x}(t)=A(\alpha)x(t)$, where

$$A(\alpha)=A_1\alpha_1^3+A_2\alpha_1^2\alpha_2+A_3\alpha_1\alpha_3^2+A_4\alpha_1\alpha_2\alpha_3+A_5\alpha_2^3+A_6\alpha_3^3,$$

where

$$\alpha\in T_L:=\{\alpha\in\mathbb{R}^3:\sum_{i=1}^3\alpha_i=2L+1, L\leq\alpha_i\leq 1\}$$

and

$$A_1=\begin{bmatrix}-0.57 & -0.44 & 0.33 & -0.07\\ -0.48 & -0.60 & 0.30 & 0\\ -0.22 & -1.12 & 0.08 & -0.24\\ 1.51 & -0.42 & 0.67 & -1.00\end{bmatrix} \quad A_2=\begin{bmatrix}-0.09 & -0.16 & 0.3 & -1.13\\ -0.15 & -0.17 & -0.02 & 0.82\\ 0.14 & 0.06 & 0.02 & -1\\ 0.488 & 0.32 & 0.97 & -0.71\end{bmatrix}$$

$$A_3=\begin{bmatrix}-0.70 & -0.29 & -0.18 & 0.31\\ 0.41 & -0.76 & -0.30 & -0.12\\ -0.05 & 0.35 & -0.59 & 0.91\\ 1.64 & 0.82 & 0.01 & -1\end{bmatrix} \quad A_4=\begin{bmatrix}0.72 & 0.34 & -0.64 & 0.31\\ -0.21 & -0.51 & 0.59 & 0.07\\ 0.27 & 0.49 & -0.84 & -0.94\\ -1.89 & -0.66 & 0.27 & 0.41\end{bmatrix}$$

$$A_5=\begin{bmatrix}-0.51 & -0.47 & -1.38 & 0.17\\ 1.18 & -0.62 & -0.29 & 0.35\\ -0.65 & 0.01 & -1.44 & -0.04\\ -0.74 & -1.22 & 0.60 & -1.47\end{bmatrix} \quad A_6=\begin{bmatrix}-0.201 & -0.19 & -0.55 & 0.07\\ 0.803 & -0.42 & -0.20 & 0.24\\ -0.440 & 0.01 & -0.98 & -0.03\\ 0 & -0.83 & 0.41 & -1\end{bmatrix}.$$

We would like to solve the following optimization problem.

$$\begin{aligned} L^* = \min \quad & L \\ \text{subject to} \quad & \dot{x}(t)=A(\alpha)x(t) \text{ is stable for all } \alpha\in T_L. \end{aligned} \tag{50}$$

We first represent T_L using the unit simplex Δ^3 as follows. Define the map $f : \Delta^3 \to T_L$ as

$$f(\alpha) = [f_1(\alpha), f_2(\alpha), f_3(\alpha)],$$

where $f_i(\alpha) = 2|L|(\alpha_i - 0.5)$. Then, we have $\{A(\alpha) : \alpha \in T_L\} = \{A(f(\beta)), \beta \in \Delta^3\}$. Thus, the following optimization problem is equivalent to Problem (50).

$$\begin{aligned} L^* = \min \quad & L \\ \text{subject to} \quad & \dot{x}(t) = A(f(\beta))x(t) \text{ is stable for all } \beta \in \Delta^3. \end{aligned} \tag{51}$$

We solved Problem (51) using bisection on L. For each L, we used Theorem 3.1 to verify the inequality in (42) using Polya's exponents $e = 1$ to 7 and $d_p = 1$ to 4 as degrees of P. Figure 2 shows the computed upper-bounds on L^* for different e and d_p. The best upper-bound found by the algorithm is -0.0504.

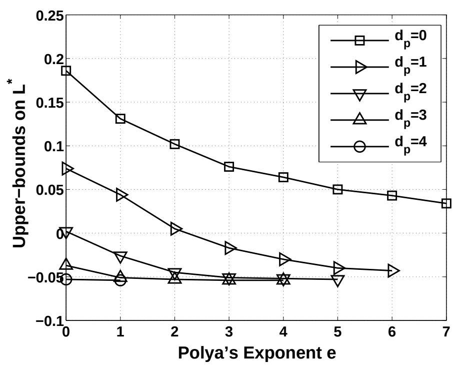

FIGURE 2. Upper-bounds for L^* in Problem (50) for different Polya's exponents e and different degrees of P

For comparison, we solved the same problem using SOSTOOLS [48] and Putinar's Positivstellensatz (see Case 4 of Section 3.2). By computing a Lyapunov function of degree two in x and degree one in β, SOSTOOLS certified $L = -0.0504$ as an upper-bound for L^*. This is the same as the upper-bound computed by Polya's algorithm. The CPU time required for SOSTOOLS to compute the upper-bound on a Core i7 machine with 64 GB of RAM was 22.3 minutes, whereas the Polya's algorithm only required 7.1 seconds to compute the same upper-bound.

Case 3. $A(\alpha)$ **is a polynomial with** $\alpha \in \Phi^l$**:**

Given $A_h \in \mathbb{R}^{n\times n}$ for $h \in E_d$ as defined in (36), we address the problem of stability analysis of System (33) with $A(\alpha) = \sum_{h\in E_d} A_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}$ for all $\alpha \in \Phi^l := \{x \in \mathbb{R}^n : |x_i| \le r_i\}$. From Lyapunov theory, System (33) is asymptotically stable for all $\alpha \in \Phi^l$ if and only if $\gamma^* > 0$ in the following optimization of polynomials problem.

$$\begin{aligned} \gamma^* = & \max_{\gamma\in\mathbb{R}, P\in\mathbb{R}[\alpha]} \quad \gamma \\ & \text{subject to } \begin{bmatrix} P(\alpha) & 0 \\ 0 & -A(\alpha)^T P(\alpha) - P(\alpha)A(\alpha) \end{bmatrix} - \gamma I \ge 0 \text{ for all } \alpha \in \Phi^l \end{aligned} \tag{52}$$

As in Case 2 of Section 3.2, by applying a bisection algorithm on γ and using a multi-simplex version of Polya's algorithm (such as Theorem 3.2) as a test for feasibility of Constraint (52) we can compute lower bounds on γ^*. Suppose there exists a matrix-valued multi-homogeneous polynomial (defined in (20)) Q of degree vector $d_q \in \mathbb{N}^l$ (d_{q_i} is the sum of the exponents of β_i and η_i in monomials of Q) such that

$$\{P(\alpha) \in \mathbb{S}^n : \alpha \in \Phi^l\} = \{Q(\beta,\eta) \in \mathbb{S}^n : \beta,\eta \in \mathbb{R}^l \text{ and } (\beta_i,\eta_i) \in \Delta^2 \text{ for } i = 1,\cdots,l\}. \tag{53}$$

Likewise, suppose there exists a matrix-valued multi-homogeneous polynomial B of degree vector $d_b \in \mathbb{N}^l$ such that

$$\{A(\alpha) \in \mathbb{S}^n : \alpha \in \Phi^l\} = \{B(\beta,\eta) \in \mathbb{S}^n : \beta,\eta \in \mathbb{R}^l \text{ and } (\beta_i,\eta_i) \in \Delta^2 \text{ for } i = 1,\cdots,l\}.$$

Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.2 that the inequality condition in (52) holds for all $\alpha \in \Phi^l$ if there exist $e \geq 0$ such that

$$\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^e\right)\left(Q(\beta,\eta) - \gamma I\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^{d_{p_i}}\right)\right) \tag{54}$$

and

$$-\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^e\right)\left(B^T(\alpha,\beta)Q(\beta,\eta) + Q(\beta,\eta)B(\beta,\eta) + \gamma I\left(\prod_{i=1}^{l}(\beta_i+\eta_i)^{d_{pa_i}}\right)\right), \tag{55}$$

have all positive coefficients where d_{p_i} is the degree of α_i in $P(\alpha)$ and d_{pa_i} is the degree of α_i in $P(\alpha)A(\alpha)$. Suppose Q and B are of the forms

$$Q(\beta,\eta) = \sum_{\substack{h,g\in\mathbb{N}^l\\ h+g=d_q}} Q_{h,g}\beta_1^{h_1}\eta_1^{g_1}\cdots\beta_l^{h_l}\eta_l^{g_l} \tag{56}$$

and

$$B(\beta,\eta) = \sum_{\substack{h,g\in\mathbb{N}^l\\ h+g=d_b}} B_{h,g}\beta_1^{h_1}\eta_1^{g_1}\cdots\beta_l^{h_l}\eta_l^{g_l}. \tag{57}$$

By combining (56) and (57) with (54) and (55) we find that for a given $\gamma \in \mathbb{R}$, the inequality condition in (52) holds for all $\alpha \in \Phi^l$ if there exist some $e \geq 0$ such that

$$\begin{aligned} &\sum_{\substack{h,g\in\mathbb{N}^l\\ h+g=d_q}} f_{\{q,r\},\{h,g\}}Q_{h,g} > 0 \quad \text{for all} \quad q,r \in \mathbb{N}^l : q+r = d_q + e\cdot\mathbf{1} \quad \text{and} \\ &\sum_{\substack{h,g\in\mathbb{N}^l\\ h+g=d_q}} M^T_{\{s,t\},\{h,g\}}Q_{h,g} + Q_{h,g}M_{\{s,t\},\{h,g\}} < 0 \text{ for all } s,t \in \mathbb{N}^l : s+t = d_q + d_b + e\cdot\mathbf{1}, \end{aligned} \tag{58}$$

where $\mathbf{1} \in \mathbb{N}^l$ is the vector of ones and where we define $f_{\{q,r\},\{h,g\}} \in \mathbb{R}$ to be the coefficient of $Q_{h,g}\beta^q\eta^r$ after expansion of (54). Likewise, we define $M_{\{s,t\},\{h,g\}} \in \mathbb{R}^{n\times n}$ to be the coefficient of $Q_{h,g}\beta^s\eta^t$ after expansion of (55). See [35] for recursive formulae for calculating $f_{\{q,r\},\{h,g\}}$ and $M_{\{s,t\},\{h,g\}}$. Similar to Case 2, Conditions (58) are an SDP (See [35] for a complexity analysis of this SDP). For any $\gamma \in \mathbb{R}$, if there exist $e \geq 0$ and $\{Q_{h,g}\}$ that satisfy (58), then $\gamma \leq \gamma^*$ as defined in (52).

Furthermore, if γ is positive, then $\dot{x}(t) = A(\alpha)x(t)$ is asymptotically stable for all $\alpha \in \Phi^l$.

Numerical Example 3a. Consider the system $\dot{x}(t) = A(\alpha)x(t)$, where

$$A(\alpha) = A_0 + A_1\alpha_1^2 + A_2\alpha_1\alpha_2\alpha_3 + A_3\alpha_1^2\alpha_2\alpha_3^2,$$

$$\alpha_1 \in [-1, 1],\ \alpha_2 \in [-0.5, 0.5],\ \alpha_3 \in [-0.1, 0.1],$$

where

$$A_0 = \begin{bmatrix} -3.0 & 0 & -1.7 & 3.0 \\ -0.2 & -2.9 & -1.7 & -2.6 \\ 0.6 & 2.6 & -5.8 & -2.6 \\ -0.7 & 2.9 & -3.3 & -2.1 \end{bmatrix} A_1 = \begin{bmatrix} 2.2 & -5.4 & -0.8 & -2.2 \\ 4.4 & 1.4 & -3.0 & 0.8 \\ -2.4 & -2.2 & 1.4 & 6.0 \\ -2.4 & -4.4 & -6.4 & 0.18 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -8.0 & -13.5 & -0.5 & -3.0 \\ 18.0 & -2.0 & 0.5 & -11.5 \\ 5.5 & -10.0 & 3.5 & 9.0 \\ 13.0 & 7.5 & 5.0 & -4.0 \end{bmatrix} A_3 = \begin{bmatrix} 3.0 & 7.5 & 2.5 & -8.0 \\ 1.0 & 0.5 & 1.0 & 1.5 \\ -0.5 & -1.0 & 1.0 & 6.0 \\ -2.5 & -6.0 & 8.5 & 14.25 \end{bmatrix}.$$

The problem is to investigate asymptotic stability of this system for all α in the given intervals using the method in Case 3 of Section 4.1. We first represented $A(\alpha)$ over the hypercube $[-1, 1] \times [-0.5, 0.5] \times [-0.1, 0.1]$ by a multi-homogeneous polynomial $B(\beta, \eta)$ with $(\beta_i, \eta_i) \in \Delta^2$ and with the degree vector $d_b = [2, 1, 2]$ (see [35] for an algorithm which finds B and see Case 2 of Section (3.2) for an example). Then, by applying Theorem 3.2 (as in (54) and (55)) with $\gamma = 0.1, e = 1$ and $d_p = [1, 1, 1]$, we set-up the inequalities in (58) with $d_q = [1, 1, 1]$. We then applied an SDP solver to the inequalities. The result is the following Lyapunov function as a certificate for asymptotic stability of $\dot{x}(t) = A(\alpha)x(t)$ for all $\alpha_1 \in [-1, 1]$, $\alpha_2 \in [-0.5, 0.5]$, $\alpha_3 \in [-0.1, 0.1]$.

$$\begin{aligned} V(x, \beta, \eta) = x^T Q(\beta, \eta)x = x^T (&\beta_1(Q_1\beta_2\beta_3 + Q_2\beta_2\eta_3 + Q_3\eta_2\beta_3 + Q_4\eta_2\eta_3) \\ &+\eta_1(Q_5\beta_2\beta_3 + Q_6\beta_2\eta_3 + Q_7\eta_2\beta_3 + Q_8\eta_2\eta_3))x, \end{aligned}$$

where $\beta_1 = 0.5\alpha_1 + 0.5, \beta_2 = \alpha_2 + 0.5, \beta_3 = 5\alpha_3 + 0.5, \eta_1 = 1 - \beta_1, \eta_2 = 1 - \beta_2, \eta_3 = 1 - \beta_3$ and

$$Q_1 = \begin{bmatrix} 5.807 & 0.010 & -0.187 & -1.186 \\ 0.010 & 5.042 & -0.369 & 0.227 \\ -0.187 & -0.369 & 8.227 & -1.824 \\ -1.186 & 0.227 & -1.824 & 8.127 \end{bmatrix} Q_2 = \begin{bmatrix} 7.409 & -0.803 & 1.804 & -1.594 \\ -0.803 & 6.016 & 0.042 & -0.538 \\ 1.804 & 0.042 & 7.894 & -1.118 \\ -1.594 & -0.538 & -1.118 & 8.590 \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} 6.095 & -0.873 & 0.512 & -1.125 \\ -0.873 & 5.934 & -0.161 & 0.503 \\ 0.512 & -0.161 & 7.417 & -0.538 \\ -1.125 & 0.503 & -0.538 & 6.896 \end{bmatrix} Q_4 = \begin{bmatrix} 5.388 & 0.130 & -0.363 & -0.333 \\ 0.130 & 5.044 & -0.113 & -0.117 \\ -0.363 & -0.113 & 6.156 & -0.236 \\ -0.333 & -0.117 & -0.236 & 5.653 \end{bmatrix}$$

$$Q_5 = \begin{bmatrix} 7.410 & -0.803 & 1.804 & -1.594 \\ -0.803 & 6.016 & 0.042 & -0.538 \\ 1.804 & 0.042 & 7.894 & -1.118 \\ -1.594 & -0.538 & -1.118 & 8.590 \end{bmatrix} Q_6 = \begin{bmatrix} 5.807 & 0.010 & -0.187 & -1.186 \\ 0.010 & 5.042 & -0.369 & 0.227 \\ -0.187 & -0.369 & 8.227 & -1.824 \\ -1.186 & 0.227 & -1.824 & 8.127 \end{bmatrix}$$

$$Q_7 = \begin{bmatrix} 5.388 & 0.130 & -0.363 & -0.333 \\ 0.130 & 5.044 & -0.113 & -0.117 \\ -0.363 & -0.113 & 6.156 & -0.236 \\ -0.333 & -0.117 & -0.236 & 5.653 \end{bmatrix} Q_8 = \begin{bmatrix} 6.095 & -0.873 & 0.512 & -1.125 \\ -0.873 & 5.934 & -0.161 & 0.503 \\ 0.512 & -0.161 & 7.417 & -0.538 \\ -1.125 & 0.503 & -0.538 & 6.896 \end{bmatrix}.$$

Numerical Example 3b. In this example, we used the same method as in Example 3a to find lower bounds on $r^* = \max r$ such that $\dot{x}(t) = A(\alpha)x(t)$ with

$$A(\alpha) = A_0 + \sum_{i=1}^{4} A_i \alpha_i,$$

$$A_0 = \begin{bmatrix} -3.0 & 0 & -1.7 & 3.0 \\ -0.2 & -2.9 & -1.7 & -2.6 \\ 0.6 & 2.6 & -5.8 & -2.6 \\ -0.7 & 2.9 & -3.3 & -2.4 \end{bmatrix} A_1 = \begin{bmatrix} 1.1 & -2.7 & -0.4 & -1.1 \\ 2.2 & 0.7 & -1.5 & 0.4 \\ -1.2 & -1.1 & 0.7 & 3.0 \\ -1.2 & -2.2 & -3.2 & -1.4 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1.6 & 2.7 & 0.1 & 0.6 \\ -3.6 & 0.4 & -0.1 & 2.3 \\ -1.1 & 2 & -0.7 & -1.8 \\ -2.6 & -1.5 & -1.0 & 0.8 \end{bmatrix} A_3 = \begin{bmatrix} -0.6 & 1.5 & 0.5 & -1.6 \\ 0.2 & -0.1 & 0.2 & 0.3 \\ -0.1 & -0.2 & -0.2 & 1.2 \\ -0.5 & -1.2 & 1.7 & -0.1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.4 & -0.1 & -0.3 & 0.1 \\ 0.1 & 0.3 & 0.2 & 0.0 \\ 0.0 & 0.2 & -0.3 & 0.1 \\ 0.1 & -0.2 & -0.2 & 0.0 \end{bmatrix}.$$

is asymptotically stable for all $\alpha \in \{\alpha \in \mathbb{R}^4 : |\alpha_i| \le r\}$. Table 1 shows the computed lower bounds on r^* for different degree vectors d_q (degree vector of polynomial Q in (53)). In all of the cases, we set the Polya's exponent $e = 0$. For comparison, we have also included the lower-bounds computed by the methods of [10] and [16] in Table 1.

Table 1. The lower-bounds on r^* computed by the method in Case 3 of Section 4.1 and the methods in [10] and [16] - i^{th} entry of d_q is the sum of the exponents of β_i and η_i in (56)

	d_q =[0,0,0,0]	d_q =[0,1,0,1]	d_q =[1,0,1,0]	d_q =[1,1,1,1]	d_q =[2,2,2,2]	Ref.[10]	Ref.[16]
bound on r^*	0.494	0.508	0.615	0.731	0.840	0.4494	0.8739

4.2. **Nonlinear stability analysis.** Consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t)), \tag{59}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a degree d_f polynomial. Suppose the origin is an isolated equilibrium of (59). In this section, we address local stability of the origin in the following sense.

Lemma 4.1. *Consider the System (59) and let $Q \subset \mathbb{R}^{n\times n}$ be an open set containing the origin. Suppose there exists a continuously differentiable function V which satisfies*

$$V(x) > 0 \text{ for all } x \in Q \setminus \{0\}, V(0) = 0 \tag{60}$$

and

$$\langle \nabla V, f(x) \rangle < 0 \text{ for all } x \in Q \setminus \{0\}. \tag{61}$$

Then, the origin is an asymptotically stable equilibrium of System (59), meaning that for every $x(0) \in \{x \in \mathbb{R}^n : \{y : V(y) \le V(x)\} \subset Q\}$, $\lim_{t\to\infty} x(t) = 0$.

Since existence of polynomial Lyapunov functions is necessary and sufficient for stability of (59) on any compact set [51], we can formulate the problem of stability analysis of (59) as follows.

$$\gamma^* = \max_{\gamma, c_\beta \in \mathbb{R}} \; \gamma$$
$$\text{subject to } \begin{bmatrix} \sum_{\beta \in E_d} c_\beta x^\beta - \gamma x^T x & 0 \\ 0 & -\langle \nabla \sum_{\beta \in E_d} c_\beta x^\beta, f(x) \rangle - \gamma x^T x \end{bmatrix} \geq 0 \text{ for all } x \in Q. \tag{62}$$

Conditions (60) and (61) hold if and only if there exist $d \in \mathbb{N}$ such that $\gamma^* > 0$. In Sections 4.2.1 and 4.2.2, we discuss two alternatives to SOS programming for solving (62). These methods apply Polya's theorem and Handelman's theorem to Problem (62) (as described in Cases 2 and 3 in Section 3.2) to find lower bounds on γ^*. See [64] for a different application of Handelman's theorem to the nonlinear stability problem. Also, see [26] for a method of computing continuous piecewise affine Lyapunov functions for nonlinear stability analysis on polytopes using linear programming and a *triangulation* scheme.

4.2.1. *Application of Handelman's theorem in nonlinear stability analysis.* Recall that every convex polytope can be represented as

$$\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\} \tag{63}$$

for some $w_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$. Suppose Γ^K is bounded and the origin is in the interior of Γ^K. In this section, we would like to investigate asymptotic stability of the equilibrium of System (59) by verifying positivity of γ^* in Problem (62) with $Q = \Gamma^K$.

Unfortunately, Handelman's theorem (Theorem 2.4) cannot be readily used to represent polynomials which have zeros in the interior of a given polytope. To see this, suppose a polynomial g (g is not identically zero) is zero at $x = a$, where a is in the interior of a polytope $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$. Suppose there exist $b_\alpha \geq 0, \alpha \in \mathbb{N}^K$ such that for some $d \in \mathbb{N}$,

$$g(x) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \|\alpha\|_1 \leq d}} b_\alpha (w_1^T x + u_1)^{\alpha_1} \cdots (w_K^T x + u_K)^{\alpha_K}.$$

Then,

$$g(a) = \sum_{\substack{\alpha \in \mathbb{N}^K \\ \|\alpha\|_1 \leq d}} b_\alpha (w_1^T a + u_1)^{\alpha_1} \cdots (w_K^T a + u_K)^{\alpha_K} = 0.$$

From the assumption $a \in \text{int}(\Gamma^K)$ it follows that $w_i^T a + u_i > 0$ for $i = 1, \cdots, K$. Since g is not identically zero, $b_\alpha < 0$ for at least one $\alpha \in \{\alpha \in \mathbb{N}^K : \|\alpha\|_1 \leq d\}$. This contradicts with the assumption that all $b_\alpha \geq 0$.

Based on the above reasoning, one cannot readily use Handelman's theorem to verify the Lyapunov inequalities in (60). In [37], a combination of Handelman's theorem and a decomposition scheme was applied to Problem (62) with $Q = \Gamma^K$. Here we outline this result. First, consider the following definitions.

Definition 4.2. Given a bounded polytope of the form $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$, we call

$$\zeta^i(\Gamma^K) := \{x \in \mathbb{R}^n : w_i^T x + u_i = 0 \text{ and } w_j^T x + u_j \geq 0 \text{ for } j \in \{1, \cdots, K\}\}$$

the $i-$th facet of Γ^K if $\zeta^i(\Gamma^K) \neq \emptyset$.

Definition 4.3. Given a bounded polytope of the form $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$, we call $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$ a $D-$*decomposition* of Γ^K if

$$D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\} \quad \text{for some } h_{i,j} \in \mathbb{R}^n,\ g_{i,j} \in \mathbb{R}$$

such that $\cup_{i=1}^L D_i = \Gamma^K$, $\cap_{i=1}^L D_i = \{0\}$ and $\text{int}(D_i) \cap \text{int}(D_j) = \emptyset$.

Consider System (59) with polynomial f of degree d_f. Given $w_i, h_{i,j} \in \mathbb{R}^n$ and $u_i, g_{i,j} \in \mathbb{R}$, let $\Gamma^K := \{x \in \mathbb{R}^n : w_i^T x + u_i \geq 0, i = 1, \cdots, K\}$ with $D-$decomposition $D_\Gamma := \{D_i\}_{i=1,\cdots,L}$, where $D_i := \{x \in \mathbb{R}^n : h_{i,j}^T x + g_{i,j} \geq 0, j = 1, \cdots, m_i\}$. Let us denote the elements of the set

$$E_{d,n} := \{\alpha \in \mathbb{N}^n : \sum_{i=1}^n \alpha_i \leq d\}$$

by $\lambda^{(k)}, k = 1, \cdots, B$, where B is the cardinality of $E_{d,e}$. For any $\lambda^{(k)}$, let $p_{\{\lambda^{(k)},\alpha,i\}}$ be the coefficient of $b_{i,\alpha}\, x^{\lambda^{(k)}}$ in

$$P_i(x) := \sum_{\alpha \in E_{d,m_i}} b_{i,\alpha} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j},\ x \in \mathbb{R}^n, b_{i,\alpha} \in \mathbb{R}. \tag{64}$$

Let us denote the cardinality of E_{d,m_i} by N_i and denote the vector of all coefficients $b_{i,\alpha}$ by $b_i \in \mathbb{R}^{N_i}$. Given coefficients $b_i \in \mathbb{R}^{N_i}$ and degree d, let $F_i(b_i, d)$ be the vector of coefficients of $P_i(x)$ after expansion. Mathematically, we show the map $F_i : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^B$ as

$$F_i(b_i, d) := \left[\sum_{\alpha \in E_{d,m_i}} p_{\{\lambda^{(1)},\alpha,i\}} b_{i,\alpha}\ ,\ \cdots\ ,\ \sum_{\alpha \in E_{d,m_i}} p_{\{\lambda^{(B)},\alpha,i\}} b_{i,\alpha} \right]^T$$

for $i = 1, \cdots, L$. Likewise, let $H_i(b_i, d)$ be the vector of coefficients of square terms of $P_i(x)$ after expansion. The map $H_i : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^Q$ can be shown as

$$H_i(b_i, d) := \left[\sum_{\alpha \in E_{d,m_i}} p_{\{\delta^{(1)},\alpha,i\}} b_{i,\alpha}\ ,\ \cdots\ ,\ \sum_{\alpha \in E_{d,m_i}} p_{\{\delta^{(Q)},\alpha,i\}} b_{i,\alpha} \right]^T$$

for $i = 1, \cdots, L$, where we have denoted the elements of $\{\delta \in \mathbb{N}^n : \delta = 2e_j \text{ for } j = 1, \cdots, n\}$ by $\delta^{(k)}, k = 1, \cdots, Q$, where e_j are the canonical basis for $\mathbb{N}^n$.

Given coefficients $b_i \in \mathbb{R}^{N_i}$, degree d and $k \in \{1, \cdots, m_i\}$, let $J_i(b_i, d, k)$ be the vector of coefficients of

$$\sum_{\substack{\alpha \in E_{d,m_i} \\ \alpha_k = 0}} b_{i,\alpha} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j}$$

after expansion. The map $J_i : \mathbb{R}^{N_i} \times \mathbb{N} \times \{1, \cdots, m_i\} \to \mathbb{R}^B$ can be shown as

$$J_i(b_i, d, k) := \left[\sum_{\substack{\alpha \in E_{d,m_i} \\ \alpha_k = 0}} p_{\{\lambda^{(1)},\alpha,i\}} b_{i,\alpha}\ ,\ \cdots\ ,\ \sum_{\substack{\alpha \in E_{d,m_i} \\ \alpha_k = 0}} p_{\{\lambda^{(B)},\alpha,i\}} b_{i,\alpha} \right]^T$$

for $i, k = 1, \cdots, L$. Define $R_i(b_i, d) : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^C$ as

$$R_i(b_i, d) := \left[b_{i,\beta^{(1)}}\ ,\ \cdots\ ,\ b_{i,\beta^{(C)}} \right]^T,$$

for $i = 1, \cdots, L$, where we have denoted the elements of the set

$$\{\beta \in E_{d,m_i} : \beta_j = 0 \text{ for } j \in \{j \in \mathbb{N} : g_{i,j} = 0\}\}$$

by $\beta^{(k)}$ for $k = 1, \cdots, C$. Given $d, d_f, n \in \mathbb{N}$, let us denote the elements of the index set

$$E_{d+d_f-1,n} := \{\alpha \in \mathbb{N}^n : \sum_{i=1}^{n} \alpha_i \le d + d_f - 1\}$$

by $\eta^{(k)}$ for $k = 1, \cdots, Z$, where it can be shown that

$$Z = \binom{n + d + d_f - 1}{n}.$$

Then, let $G_i(b_i, d)$ be the vector of coefficients of $\langle \nabla P_i(x), f(x)\rangle$. The map $G_i : \mathbb{R}^{N_i} \times \mathbb{N} \to \mathbb{R}^Z$ can be shown as

$$G_i(b_i, d) := \left[\sum_{\alpha \in E_{d,m_i}} q_{\{\eta^{(1)},\alpha,i\}} b_{i,\alpha} \ , \ \cdots \ , \ \sum_{\alpha \in E_{d,m_i}} q_{\{\eta^{(Z)},\alpha,i\}} b_{i,\alpha} \right]^T$$

for $i = 1, \cdots, L$, where for any $\eta^{(k)} \in E_{d+d_f-1,n}$, we have denoted the coefficient of $b_{i,\alpha}\, x^{\eta^{(k)}}$ in $\langle \nabla P_i(x), f(x)\rangle$ by $q_{\{\eta^{(k)},\alpha,i\}}$.

Finally, given $i, j \in \{1, \cdots, L\}, i \ne j$, let

$$\Lambda_{i,j} := \left\{ k, l \in \mathbb{N} : k \in \{1, \cdots, m_i\}, l \in \{1, \cdots, m_j\} : \zeta^k(D_i) \ne \emptyset \text{ and } \zeta^k(D_i) = \zeta^l(D_j) \right\}.$$

If there exist $d \in \mathbb{N}$ such that $\max \gamma$ in the linear program

$$\begin{aligned}
&\max_{\gamma \in \mathbb{R}, b_i \in \mathbb{R}^{N_i}, c_i \in \mathbb{R}^{M_i}} && \gamma \\
&\text{subject to} && b_i \ge \mathbf{0} && \text{for } i = 1, \cdots, L \\
& && c_i \le \mathbf{0} && \text{for } i = 1, \cdots, L \\
& && R_i(b_i, d) = \mathbf{0} && \text{for } i = 1, \cdots, L \\
& && H_i(b_i, d) \ge \gamma \cdot \mathbf{1} && \text{for } i = 1, \cdots, L \\
& && H_i(c_i, d + d_f - 1) \le -\gamma \cdot \mathbf{1} && \text{for } i = 1, \cdots, L \\
& && G_i(b_i, d) = F_i(c_i, d + d_f - 1) && \text{for } i = 1, \cdots, L \\
& && J_i(b_i, d, k) = J_j(b_j, d, l) && \text{for } i, j = 1, \cdots, L \text{ and } k, l \in \Lambda_{i,j}
\end{aligned} \tag{65}$$

is positive, then the origin is an asymptotically stable equilibrium for System (59) and

$$V(x) = V_i(x) = \sum_{\alpha \in E_{d,m_i}} b_{i,\alpha} \prod_{j=1}^{m_i} (h_{i,j}^T x + g_{i,j})^{\alpha_j} \text{ for } x \in D_i, i = 1, \cdots, L$$

is a piecewise polynomial Lyapunov function proving stability of System (59). See [37] for a comprehensive discussion on the computational complexity of the LP defined in (65) in terms of the state-space dimension and the degree of $V(x)$.

Numerical Example 4. Consider the following nonlinear system [18].

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -2x_1 - x_2 + x_1 x_2^2 - x_1^5 + x_1 x_2^4 + x_2^5.
\end{aligned}$$

Using the polytope

$$\Gamma^4 = \{x_1, x_2 \in \mathbb{R}^2 : 1.428x_1 + x_2 - 0.625 \geq 0, -1.428x_1 + x_2 + 0.625 \geq 0, \\ 1.428x_1 + x_2 + 0.625 \geq 0, -1.428x_1 + x_2 - 0.625 \geq 0\}, \quad (66)$$

and $D-$decomposition

$$D_1 := \{x_1, x_2 \in \mathbb{R}^2 : -x_1 \geq 0, x_2 \geq 0, -1.428x_1 + x_2 - 0.625 \geq 0\}$$
$$D_2 := \{x_1, x_2 \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, 1.428x_1 + x_2 + 0.625 \geq 0\}$$
$$D_3 := \{x_1, x_2 \in \mathbb{R}^2 : x_1 \geq 0, -x_2 \geq 0, -1.428x_1 + x_2 + 0.625 \geq 0\}$$
$$D_4 := \{x_1, x_2 \in \mathbb{R}^2 : -x_1 \geq 0, -x_2 \geq 0, 1.428x_1 + x_2 + 0.625 \geq 0\},$$

we set-up the LP in (65) with $d = 4$. The solution to the LP certified asymptotic stability of the origin and yielded the following piecewise polynomial Lyapunov function. Figure 3 shows the largest level set of $V(x)$ inscribed in the polytope Γ^4.

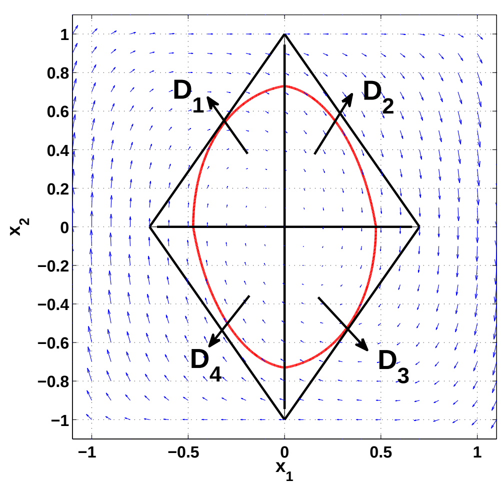

FIGURE 3. The largest level-set of Lyapunov function (67) inscribed in Polytope (66)

$$V(x) = \begin{cases} 0.543x_1^2 + 0.233x_2^2 + 0.018x_2^3 - 0.074x_1x_2^2 - 0.31x_1^3 \\ +0.004x_2^4 - 0.013x_1x_2^3 + 0.015x_1^2x_2^2 + 0.315x_1^4 & \text{if } x \in D_1 \\ 0.543x_1^2 + 0.329x_1x_2 + 0.233x_2^2 + 0.018x_2^3 + 0.031x_1x_2^2 \\ +0.086x_1^2x_2 + 0.3x_1^3 + 0.004x_2^4 + 0.009x_1x_2^3 + 0.015x_1^2x_2^2 \\ +0.008x_1^3x_2 + 0.315x_1^4 & \text{if } x \in D_2 \\ 0.0543x_1^2 + 0.0233x_2^2 - 0.0018x_2^3 + 0.0074x_1x_2^2 + 0.03x_1^3 \\ +0.004x_2^4 - 0.013x_1x_2^3 + 0.015x_1^2x_2^2 + 0.315x_1^4 & \text{if } x \in D_3 \\ 0.543x_1^2 + 0.329x_1x_2 + 0.233x_2^2 - 0.018x_2^3 - 0.031x_1x_2^2 \\ -0.086x_1^2x_2 - 0.3x_1^3 + 0.004x_2^4 + 0.009x_1x_2^3 + 0.015x_1^2x_2^2 \\ +0.008x_1^3x_2 + 0.315x_1^4 & \text{if } x \in D_4 \end{cases} \quad (67)$$

See Numerical Example 5 for a comparison of this method with the method in Section 4.2.2.

4.2.2. *Application of Polya's theorem in nonlinear stability analysis.* In this section, we discuss an algorithm based on a multi-simplex version of Polya's theorem (Theorem 3.2) to verify local stability of nonlinear systems of the form

$$\dot{x} = A(x)x(t), \tag{68}$$

where $A(x) \in \mathbb{R}^{n\times n}$ is a matrix-valued polynomial and $A(0) \neq 0$.

Unfortunately, Polya's theorem cannot certify positivity of polynomials which have zeros in the interior of the unit simplex (see [54] for an elementary proof of this). From the same reasoning as in [54] it follows that the multi-simplex version of Polya's theorem (Theorem 3.2) cannot parameterize polynomials which have zeros in the interior of a multi-simplex. Moreover, if $f(z)$ in (21) has a zero in the interior of Φ^n, then any multi-homogeneous polynomial $p(x, y)$ that satisfies (21) has a zero in the interior of the multi-simplex $\Delta^2 \times \cdots \times \Delta^2$ - hence cannot be parameterized by Polya's theorem. One way to enforce the condition $V(0) = 0$ in (60) is to search for coefficients of a matrix-valued polynomial P which defines a Lyapunov function of the form $V(x) = x^T P(x)x$. It can be shown that $V(x) = x^T P(x)x$ is a Lyapunov function for System (68) if and only if γ^* in the following optimization of polynomials problem is positive.

$$\begin{aligned}\gamma^* = &\max_{\gamma\in\mathbb{R}, P\in\mathbb{R}[x]} \ \gamma\\ &\text{subject to} \begin{bmatrix} P(x) & 0\\ 0 & -Q(x)\end{bmatrix} - \gamma I \geq 0 \quad \text{for all } x \in \Phi^n,\end{aligned} \tag{69}$$

where

$$Q(x) = A^T(x)P(x) + P(x)A(x) + \frac{1}{2}\left(A^T(x) \begin{bmatrix} x^T \frac{\partial P(x)}{\partial x_1}\\ \vdots \\ x^T \frac{\partial P(x)}{\partial x_n}\end{bmatrix} + \begin{bmatrix} x^T \frac{\partial P(x)}{\partial x_1}\\ \vdots \\ x^T \frac{\partial P(x)}{\partial x_n}\end{bmatrix}^T A(x)\right).$$

As in Case 2 of Section 3.2, by applying a bisection algorithm on γ and using Theorem 3.2 as a test for feasibility of Constraint (69), we can compute lower bounds on γ^*. Suppose there exists a matrix-valued multi-homogeneous polynomial S of degree vector $d_s \in \mathbb{N}^n$ (d_{s_i} is the sum of the exponents of y_i and z_i in monomials of S) such that

$$\{P(x) \in \mathbb{S}^n : x \in \Phi^n\} = \{S(y, z) \in \mathbb{S}^n : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n\}.$$

Likewise, suppose there exist matrix-valued multi-homogeneous polynomials B and C of degree vectors $d_b \in \mathbb{N}^n$ and $d_c = d_s \in \mathbb{N}^n$ such that

$$\{A(x) \in \mathbb{R}^{n\times n} : x \in \Phi^n\} = \{B(y, z) \in \mathbb{R}^{n\times n} : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n\}$$

and

$$\left\{\left[\tfrac{\partial P(x)}{\partial x_1}x, \cdots, \tfrac{\partial P(x)}{\partial x_n}x\right] \in \mathbb{R}^{n\times n} : x \in \Phi^n\right\} = \left\{C(y, z) \in \mathbb{R}^{n\times n} : (y_i, z_i) \in \Delta^2, i = 1, \cdots, n\right\}.$$

Given $\gamma \in \mathbb{R}$, it follows from Theorem 3.2 that the inequality condition in (69) holds for all $\alpha \in \Phi^l$ if there exist $e \geq 0$ such that

$$\left(\prod_{i=1}^n (y_i + z_i)^e\right)\left(S(y, z) - \gamma I \left(\prod_{i=1}^n (y_i + z_i)^{d_{p_i}}\right)\right) \tag{70}$$

and

$$\left(\prod_{i=1}^{n}(y_i+z_i)^e\right)\left(B^T(y,z)S(y,z)+S(y,z)B(y,z)\right.$$
$$\left.+\frac{1}{2}\left(B^T(y,z)C^T(y,z)+C(y,z)B(y,z)\right)-\gamma I\left(\prod_{i=1}^{n}(y_i+z_i)^{d_{q_i}}\right)\right) \tag{71}$$

have all positive coefficients, where d_{p_i} is the degree of x_i in $P(x)$ and d_{q_i} is the degree of x_i in $Q(x)$. Let S, B and C have the following forms.

$$S(y,z)=\sum_{\substack{h,g\in\mathbb{N}^n\\ h+g=d_s}} S_{h,g}y_1^{h_1}z_1^{g_1}\cdots y_n^{h_n}z_n^{g_n} \tag{72}$$

$$B(y,z)=\sum_{\substack{h,g\in\mathbb{N}^n\\ h+g=d_b}} B_{h,g}y_1^{h_1}z_1^{g_1}\cdots y_n^{h_n}z_n^{g_n} \tag{73}$$

$$C(y,z)=\sum_{\substack{h,g\in\mathbb{N}^n\\ h+g=d_c}} C_{h,g}y_1^{h_1}z_1^{g_1}\cdots y_n^{h_n}z_n^{g_n} \tag{74}$$

By combining (72), (73) and (74) with (70) and (71) it follows that for a given $\gamma\in\mathbb{R}$, the inequality condition in (69) holds for all $\alpha\in\Phi^n$ if there exist some $e\geq 0$ such that

$$\sum_{\substack{h,g\in\mathbb{N}^n\\ h+g=d_s}} f_{\{q,r\},\{h,g\}}S_{h,g}>0 \quad \text{for all } \ q,r\in\mathbb{N}^n: q+r=d_s+e\cdot\mathbf{1} \quad \text{and}$$
$$\sum_{\substack{h,g\in\mathbb{N}^n\\ h+g=d_s}} M^T_{\{u,v\},\{h,g\}}S_{h,g}+S_{h,g}M_{\{u,v\},\{h,g\}}+N^T_{\{u,v\},\{h,g\}}C^T_{h,g}+C_{h,g}N_{\{u,v\},\{h,g\}}<0$$
$$\text{for all } u,v\in\mathbb{N}^n: u+v=d_s+d_b+e\cdot\mathbf{1}, \tag{75}$$

where similar to Case 3 of Section 4.1, we define $f_{\{q,r\},\{h,g\}}$ to be the coefficient of $S_{h,g}y^qz^r$ after combining (72) with (70). Likewise, we define $M_{\{u,v\},\{h,g\}}$ to be the coefficient of $S_{h,g}y^uz^v$ and $N_{\{u,v\},\{h,g\}}$ to be the coefficient of $C_{h,g}y^uz^v$ after combining (73) and (74) with (71). Conditions (75) are an SDP (See [37] for a complexity analysis of this SDP). For any $\gamma\in\mathbb{R}$, if there exist $e\geq 0$ and $\{S_{h,g}\}$ such that Conditions (75) hold, then γ is a lower bound for γ^* as defined in (69). Furthermore, if γ is positive, then the origin is an asymptotically stable equilibrium for System (68).

Numerical Example 5. Consider the reverse-time Van Der Pol oscillator defined as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-x_2\\ x_1+x_2(x_1^2-1)\end{bmatrix}=A(x)x,$$

where $A(x)=\begin{bmatrix}0 & -1\\ 1 & x_1^2-1\end{bmatrix}$. By using the method in Section 4.2.2, we solved Problem (69) using the hypercubes

$$
\begin{aligned}
\Phi_1^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1, |x_2| \leq 1\}\\
\Phi_2^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.5, |x_2| \leq 1.5\}\\
\Phi_3^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.7, |x_2| \leq 1.8\}\\
\Phi_4^2 &= \{x \in \mathbb{R}^2 : |x_1| \leq 1.9, |x_2| \leq 2.4\} \qquad (76)
\end{aligned}
$$

and $d_p = 0, 2, 4, 6$ as the degrees of $P(x)$. For each hypercube Φ_i^2 in (76), we computed a Lyapunov function of the form $V_i(x) = x^T P_i(x)x$. In Figure 4, we have plotted the largest level-set of V_i, inscribed in Φ_i^2 for $i = 1, \cdots, 4$. For all the cases, we used the Polya's exponent $e = 1$.

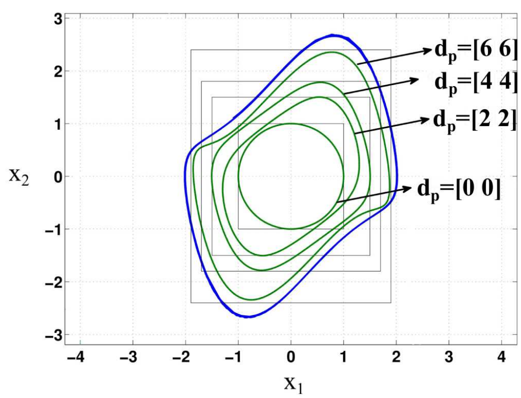

FIGURE 4. Level-sets of the Lyapunov functions $V(x) = x^T P(x)x$ computed by the method in Section 4.2.2 - d_p is the degree of $P(x)$

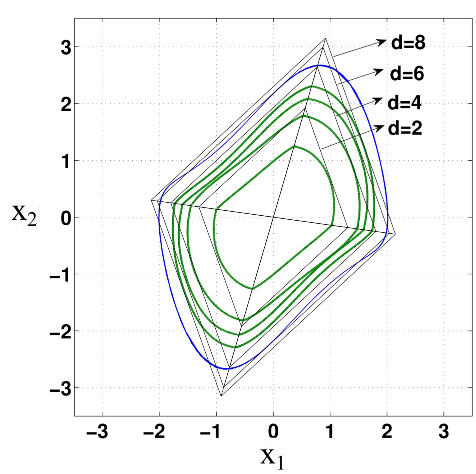

FIGURE 5. Level-sets of the Lyapunov functions computed for Van Der Pol oscillator using the method in Section 4.2.1 - d is the degree of the Lyapunov functions

We also used the method in Section 4.2.1 to solve the same problem (see Figure 5) using the polytopes

$$\Gamma_\nu := \left\{ x \in \mathbb{R}^2 : x = \sum_{i=1}^{4} \rho_i v_i : \rho_i \in [0,\nu], \sum_{i=1}^{4} \rho_i = \nu \right\}$$

with $\nu = 0.83, 1.41, 1.52, 1.64$, where

$$v_1 = \begin{bmatrix} -1.31 \\ 0.18 \end{bmatrix}, v_2 = \begin{bmatrix} 0.56 \\ 1.92 \end{bmatrix}, v_3 = \begin{bmatrix} -0.56 \\ -1.92 \end{bmatrix} \text{ and } v_4 = \begin{bmatrix} 1.31 \\ -0.18 \end{bmatrix}.$$

From Figures 4 and 5 we observe that in both methods, computing larger invariant sets requires an increase in the degree of Lyapunov functions. The CPU time required for computing the Lyapunov functions associated with the largest invariant subsets in Figures 4 and 5 were 88.9 minutes (using the method of Section 4.2.2) and 3.78 minutes (using the method of Section 4.2.1), respectively using a core i7 machine with 64 GB of RAM.

4.3. **Robust H_∞ control synthesis.** Consider plant G with the state-space formulation

$$\begin{aligned} \dot{x}(t) &= A(\alpha)x(t) + \begin{bmatrix} B_1(\alpha) & B_2(\alpha) \end{bmatrix} \begin{bmatrix} \omega(t) \\ u(t) \end{bmatrix} \\ \begin{bmatrix} z(t) \\ y(t) \end{bmatrix} &= \begin{bmatrix} C_1(\alpha) \\ C_2(\alpha) \end{bmatrix} x(t) + \begin{bmatrix} D_{11}(\alpha) & D_{12}(\alpha) \\ D_{21}(\alpha) & 0 \end{bmatrix} \begin{bmatrix} \omega(t) \\ u(t) \end{bmatrix}, \end{aligned} \tag{77}$$

where $\alpha \in Q \subset \mathbb{R}^l$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control input, $\omega(t) \in \mathbb{R}^p$ is the external input and $z(t) \in \mathbb{R}^q$ is the external output. According to [25], if $(A(\alpha), B_2(\alpha))$ is stabilizable and $(C_2(\alpha), A(\alpha))$ is detectable for all $\alpha \in Q$, then there exists a state feedback gain $K(\alpha) \in \mathbb{R}^{m\times n}$ such that

$$\|S(G, K(\alpha))\|_{H_\infty} \le \gamma, \text{ for all } \alpha \in Q,$$

if and only if there exist $P(\alpha) > 0$ and $R(\alpha) \in \mathbb{R}^{m\times n}$ such that $K(\alpha) = R(\alpha)P^{-1}(\alpha)$ and

$$\begin{bmatrix} \begin{bmatrix} A(\alpha) & B_2(\alpha) \end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha) \end{bmatrix}\begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha) \end{bmatrix} & \star & \star \\ B_1^T(\alpha) & -\gamma I & \star \\ \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha) \end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} & D_{11}(\alpha) & -\gamma I \end{bmatrix} < 0, \tag{78}$$

for all $\alpha \in Q$, where $\gamma > 0$ and $S(G, K(\alpha))$ is the map from the external input ω to the external output z of the closed loop system with static full state feedback controller $u(t) = K(\alpha)x(t)$. The symbol $\star$ denotes the symmetric blocks in the matrix inequality. To find a solution to the robust H_∞-optimal static state-feedback controller problem with optimal feedback gain $K(\alpha) = R(\alpha)P^{-1}(\alpha)$, one can solve the following optimization of polynomials problem.

$$\gamma^* = \min_{P, R \in \mathbb{R}[\alpha], \gamma \in \mathbb{R}} \gamma$$

subject to

$$\begin{bmatrix} -P(\alpha) & \star & \star & \star \\ 0 & \begin{bmatrix} A(\alpha) & B_2(\alpha) \end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha) \end{bmatrix}\begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha) \end{bmatrix} & \star & \star \\ 0 & B_1^T(\alpha) & 0 & \star \\ 0 & \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha) \end{bmatrix}\begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} & D_{11}(\alpha) & 0 \end{bmatrix} - \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} < 0$$

$$\text{for all } \alpha \in Q. \tag{79}$$

In Problem (79), if $Q = \Delta^l$ as defined in (16), then we can apply Polya's theorem (Theorem 3.1) as in the algorithm in Case 1 of Section 3.2 to find a $\gamma \leq \gamma^*$ and P and R which satisfy the inequality in (79). Suppose $P, A, B_1, B_2, C_1, D_{11}$ and D_{12} are homogeneous polynomials (otherwise use the procedure in Case 1 of Section 3.2 to homogenize them). Let

$$F(P(\alpha), R(\alpha)) := \begin{bmatrix} -P(\alpha) & \star & \star & \star \\ 0 & \begin{bmatrix} A(\alpha) & B_2(\alpha) \end{bmatrix} \begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} + \begin{bmatrix} P(\alpha) & R^T(\alpha) \end{bmatrix} \begin{bmatrix} A^T(\alpha) \\ B_2^T(\alpha) \end{bmatrix} & \star & \star \\ 0 & B_1^T(\alpha) & 0 & \star \\ 0 & \begin{bmatrix} C_1(\alpha) & D_{12}(\alpha) \end{bmatrix} \begin{bmatrix} P(\alpha) \\ R(\alpha) \end{bmatrix} & D_{11}(\alpha) & 0 \end{bmatrix},$$

and denote the degree of F by d_f. Given $\gamma \in \mathbb{R}$, the inequality in (79) holds if there exist $e \geq 0$ such that all of the coefficients of the polynomial

$$\left(\sum_{i=1}^{l} \alpha_i \right)^e \left(F(P(\alpha), R(\alpha)) - \gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \left(\sum_{i=1}^{l} \alpha_i \right)^{d_f} \right) \tag{80}$$

are negative-definite. Let P and R be of the forms

$$P(\alpha) = \sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, P_h \in \mathbb{S}^n \text{ and } R(\alpha) = \sum_{h \in I_{d_r}} R_h \alpha_1^{h_1} \cdots \alpha_l^{h_l}, R_h \in \mathbb{R}^{n \times n}, \tag{81}$$

where I_{d_p} and I_{d_r} are the exponent sets defined in (40). By combining (81) and (80) it follows from Theorem (3.1) that for a given γ, the inequality in (79) holds, if there exist $e \geq 0$ such that

$$\sum_{h \in I_{d_p}} \left(M_{h,q}^T P_h + P_h M_{h,q} \right) + \sum_{h \in I_{d_r}} \left(N_{h,q}^T R_h^T + R_h N_{h,q} \right) < 0 \quad \text{for all } q \in I_{d_f + e}, \tag{82}$$

where we define $M_{h,q} \in \mathbb{R}^{n \times n}$ as the coefficient of $P_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after substituting (81) into (80), and where I_{d_f+e} denotes the set of all exponents of an l-variate homogeneous polynomial of degree $d_f + e$. Likewise, $N_{h,q} \in \mathbb{R}^{n \times n}$ is the coefficient of $R_h \alpha_1^{q_1} \cdots \alpha_l^{q_l}$ after substituting (81) into (80). For given $\gamma > 0$, if there exists some $e \geq 0$ such that LMI (82) has a solution, say $P_h, h \in I_{d_p}$ and $R_g, g \in I_{d_r}$, then

$$K(\alpha) = \left(\sum_{g \in I_{d_r}} R_g \alpha_1^{g_1} \cdots \alpha_l^{g_l} \right) \left(\sum_{h \in I_{d_p}} P_h \alpha_1^{h_1} \cdots \alpha_l^{h_l} \right)^{-1}$$

defines an H_∞-suboptimal static state-feedback controller $u(t) = K(\alpha)x(t)$ for System (77). By performing bisection search on γ and solving (82) for each γ in the bisection, the controller becomes H_∞-optimal.

In Problem (79), if $Q = \Phi^l$ as defined in (19), then by applying the algorithm in Case 2 of section 3.2 to Problem (79), we can find a solution P, Q, γ to (79), where $\gamma \leq \gamma^*$. See Case 3 of Section 4.1 and Section 4.2.2 for similar applications of this theorem.

If $Q = \Gamma^l$ as defined in (63), then we can use Handelman's theorem (Theorem 2.4) as in the algorithm in Case 3 of section 3.2 to find a solution to Problem (79). We have provided a similar application of Handelman's theorem in Section 4.2.1.

If Q is a compact semi-algebraic set, then for given $d \in \mathbb{N}$, one can apply the Positivstellensatz results in Case 4 of Section 3.2 to the inequality in (79) to obtain a SOS program of the Form (28). A solution to the SOS program yields a solution to Problem (79).

5. Conclusion.

SOS programming, the moment approach and their application to optimization of polynomials has been thoroughly reviewed in existing literature. To promote diversity in the tools available to the control and optimization community, we have focused this paper on some of the alternatives to the standard paradigm of SOS programming. In particular, we focused on the algorithms defined by Polya's theorem, Bernstein's theorem and Handelman's theorem. We discussed how these algorithms can be applied to optimization of polynomials problems defined on simplices, hypercubes and arbitrary convex polytopes. Moreover, we demonstrated some of the applications of Polya's and Handelman's algorithms in stability analysis of nonlinear systems and stability analysis and H_∞ control of systems with parametric uncertainty. For most of these applications, we have provided numerical examples to compare the conservativeness of Polya's and Handelman's algorithms with other algorithms including SOS programming.

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