Systems Analysis and Control

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Lecture 10: Routh-Hurwitz Stability Criterion

Overview

In this Lecture, you will learn:

The Routh-Hurwitz Stability Criterion:

- Determine whether a system is stable.
- An easy way to make sure feedback isn't destabilizing
- Construct the Routh Table

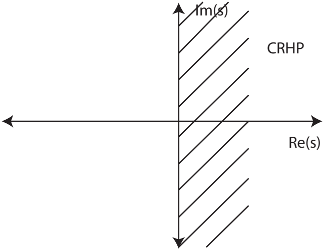
A Stability Test

We know that for a system with Transfer function

$$\hat{G}(s) = \frac{n(s)}{d(s)}$$

Input-Output Stability implies that

- all roots of $d(s)$ are in the Left Half-Plane
  - All have negative real part.



Question: How do we determine if all roots of $d(s)$ have negative real part?

Example:

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}$$

A Stability Test

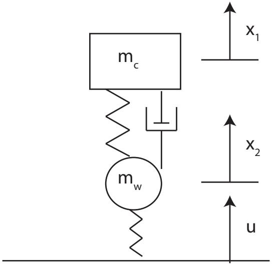
Another Variation

Determining stability is not that hard (Matlab).
Now suppose we add feedback:

Controller: Static Gain: $\hat{K}(s) = k$

Closed Loop Transfer Function:

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$



Closed Loop Transfer Function:

$$\frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3 + k)s^2 + (1 + k)s + (1 + k)}$$

We know that increasing the gain reduces steady-state error.

- But how high can we go?

What is the maximum value of k for which we have stability?

A Stability Test

Suppose we are given a polynomial denominator

$$d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

Fact: $\frac{n(s)}{d(s)}$ is unstable if any roots of $d(s)$ have negative real part.

Question: How to determine if any roots of $a(s)$ have negative real part

Simple Case All Real Roots.

- Suppose all the roots of $d(s)$ had negative real parts.

$$d(s) = (s - p_1)(s - p_2)\cdots(s - p_n)$$

Observe what happens as we expand out the roots:

$$\begin{aligned}
d(s) &= (s - p_1)(s - p_2)(s - p_3)(s - p_4)\cdots(s - p_n) \\
&= (s^2 - (p_1 + p_2)s + p_1p_2)(s - p_3)(s - p_4)\cdots(s - p_n) \\
&= (s^3 - (p_1 + p_2 + p_3)s^2 + (p_1p_2 + p_2p_3 + p_1p_3)s - p_1p_2p_3)(s - p_4)\cdots(s - p \\
&= \cdots \\
&= s^n - (p_1 + p_2 + \cdots + p_n)s^{n-1} + (p_1p_2 + p_1p_3 + \cdots)s^{n-2} \\
&\qquad - (p_1p_2p_3 + p_1p_2p_4 + \cdots)s^{n-3} + \cdots + (-1)^n p_1p_2\cdots p_n
\end{aligned}$$

A Stability Test

So if we write

$$d(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$$

we get

$$\begin{aligned} a_{n-1} &= -(p_1 + p_2 + \cdots + p_n) \\ a_{n-2} &= (p_1 p_2 + \cdots) \\ a_{n-3} &= -(p_1 p_2 p_3 + \cdots) \end{aligned}$$

Critical Point: If $d(s)$ is stable, all the p_i are negative.

- $a_{n-1} = -(p_1 + p_2 + \cdots + p_n) > 0$
- $a_{n-2} = (p_1 p_2 + \cdots) > 0$
- $a_{n-3} = -(p_1 p_2 p_3 + \cdots) > 0$

Conclusion: All the coefficients of $d(s)$ are positive!!!

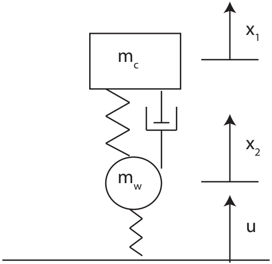
- Also true if the p_i are complex
  - Harder to show.
- If any coefficient is negative, $d(s)$ is unstable.
- **Note!** If all a_i are positive, that proves nothing.

Example: Suspension Problem

Controller: Static Gain: $\hat{K}(s) = k$

Closed Loop Transfer Function:

$$\frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}$$



Examine the denominator:

$$d(s) = s^4 + 2s^3 + (3+k)s^2 + (1+k)s + (1+k)$$

All coefficients are positive for all positive $k > 0$

Conclusion: We don't know anything new.

Example: Another Example

Consider the very simple transfer function

$$\hat{G}(s) = \frac{1}{s^3 + s^2 + s + 2}$$

The coefficients of

$$d(s) = s^3 + s^2 + s + 2$$

are all positive.

However, the roots of $d(s)$ are at

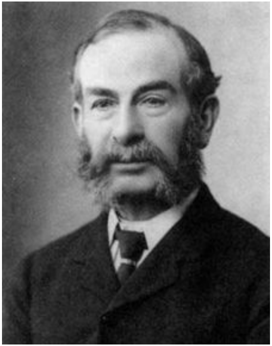
- $p_1 = -1.35$
  - Stable
- $p_{2,3} = .177 \pm 1.2\imath$
  - Positive Real Part - Unstable

Routh's Method

Introduced in 1874

- Generalizes the previous method
- Introduces additional combinations of coefficients
- Based on Sturm's theorem.

Central is the idea of the "Routh Table"



Step 1: Write the polynomial as

$$d(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh's Method

Step 2

Write the coefficients in 2 rows

- First row starts with a_n
- Second row starts with a_{n-1}
- Other coefficients alternate between rows
- Both rows should be same length
  - Continue until no coefficients are left
  - Add zero as last coefficient if necessary

TABLE 6.1 Initial layout for Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Routh's Method

Step 3

Complete the third row.

- Call the new entries $b_1, \cdots, b_k$
  - The third row will be the same length as the first two

$$b_1 = -\frac{\det \begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} \qquad b_2 = -\frac{\det \begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} \qquad b_3 = -\frac{\det \begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3}$$

- The denominator is the first entry from the previous row.
- The numerator is the determinant of the entries from the previous two rows:
  - The first column
  - The next column following the coefficient

$$b_k = -\frac{\det \begin{vmatrix} a_n & a_{n-2k} \\ a_{n-1} & a_{n-2k-1} \end{vmatrix}}{a_{n-1}}$$

  - If a coefficient doesn't exist, substitute 0.

Routh's Method

Step 4

TABLE 6.2 Completed Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\dfrac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\dfrac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\dfrac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\dfrac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\dfrac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\dfrac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\dfrac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Treat each following row in the same way as the third row

- There should be $n + 1$ rows total, including the first row.

Routh's Method

Step 4

Now examine the first column

TABLE 6.3 **Completed Routh table for Example 6.1**

s^3	1	31	0
s^2	$\cancel{10}$ 1	$\cancel{1030}$ 103	0
s^1	$\dfrac{-\begin{vmatrix}1 & 31\\1 & 103\end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix}1 & 0\\0 & 0\end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\1 & 0\end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix}1 & 103\\-72 & 0\end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix}1 & 0\\-72 & 0\end{vmatrix}}{-72} = 0$

Theorem 1.

The number of sign changes in the first column of the Routh table equals the number of roots of the polynomial in the Closed Right Half-Plane (CRHP).

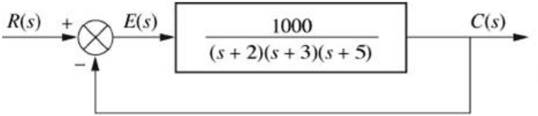
Note: Any row can be multiplied by any positive constant without changing the result.

Suppose we have a stable transfer function

$$\hat{G}(s) = \frac{1}{(s+2)(s+3)(s+5)}$$

To improve performance, we close the loop with a gain of 1000

Controller: $\hat{K}(s) = 1000$



(a)

The Closed-Loop Transfer Function is

$$\frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

Question: Have we destabilized the system?

TABLE 6.3 **Completed Routh table for Example 6.1**

s^3	1	31	0
s^2	$\cancel{10}\ \ 1$	$\cancel{1030}\ \ 103$	0
s^1	$\dfrac{-\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}{1} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$\dfrac{-\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$\dfrac{-\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$

- We divide the second row by 10
- There are **two** sign changes: $1 \to -72$ and $-72 \to 103$
  - Two poles in the CRHP.

Feedback is **Destabilizing**!

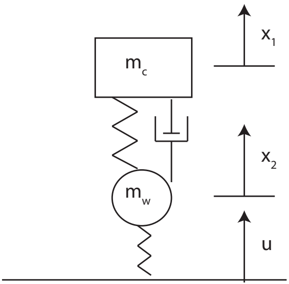
Another Numerical Example

Recall the suspension Problem with feedback:
Closed Loop Transfer Function:

$$\frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}$$

Question: Can feedback destabilize the suspension system?

- Is it stable for any $k>0$???



Lets start the Routh Table:

$$\begin{array}{c|lll}
s^4 & 1 & 3+k & 1+k \\
s^3 & 2 & 1+k & 0 \\
s^2 & b_1 & b_2 & b_3
\end{array}$$

We need to find the b_i.

Another Numerical Example

Start by calculating the coefficients in the first row:

$$b_1 = -\frac{\det\begin{vmatrix}1 & 3+k\\ 2 & 1+k\end{vmatrix}}{2} = \frac{1}{2}(5+k)$$

and

$$b_2 = -\frac{\det\begin{vmatrix}1 & 1+k\\ 2 & 0\end{vmatrix}}{2} = 1+k$$

which gives

s^4	1	$3+k$	$1+k$
s^3	2	$1+k$	0
s^2	$\frac{1}{2}(5+k)$	1+k	0
s	c_1	0	0

So far, so good.

- Now calculate the next row.

Another Numerical Example

The coefficients for the next row are

$$c_1 = -\frac{\det\begin{vmatrix} 2 & 1+k \\ \frac{1}{2}(5+k) & 1+k \end{vmatrix}}{\frac{1}{2}(5+k)} = \frac{k^2+2k+1}{5+k}$$

and $c_2 = 0$.

s^4	1	$3+k$	$1+k$
s^3	2	$1+k$	0
s^2	$\frac{1}{2}(5+k)$	$1+k$	0
s	$\frac{k^2+2k+1}{5+k}$	0	0
1	d_1	0	0

Again, the first column is all positive for any $k > 0$

- Now calculate the final row.

Another Numerical Example

There is only one non-zero coefficient in the last row.

$$d_1 = -\frac{\det \begin{vmatrix} \frac{1}{2}(5+k) & 1+k \\ \frac{k^2+2k+1}{5+k} & 0 \end{vmatrix}}{\frac{k^2+2k+1}{5+k}} = k+1$$

s^4	1	$3+k$	$1+k$
s^3	2	$1+k$	0
s^2	$\frac{1}{2}(5+k)$	1+k	0
s	$\frac{k^2+2k+1}{5+k}$	0	0
1	$1+k$	0	0

Conclusion: No matter what $k > 0$ is, the first column is always positive.

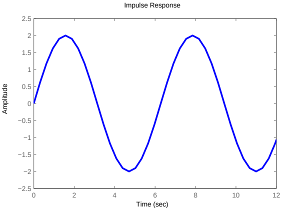
- No sign changes for any k.
- Stable for any k.
- We'll find out why later on.

Feedback **CANNOT** destabilize the suspension system.

Stability of Quadratics

What about a simple second-order system?

$$\frac{1}{s^2 + bs + c}$$



We know the poles are at

$$p_{1,2} = -b \pm \sqrt{b^2 - 4c}$$

Calculate

$$-\frac{\det \begin{vmatrix} 1 & c \\ b & 0 \end{vmatrix}}{b} = -\frac{-bc}{b} = c$$

The Routh table is

$$\begin{array}{c|ccc} \hline s^2 & 1 & c & 0 \\ s & b & 0 & 0 \\ 1 & c & 0 & 0 \end{array}$$

Thus a quadratic is stable if and only if both coefficients are positive.

Stability of 3rd order systems

Now consider a third order system:

$$\frac{1}{s^3 + as^2 + bs + c}$$

$$-\frac{\det \begin{vmatrix} 1 & b \\ a & c \end{vmatrix}}{a} = -\frac{c - ab}{a} = b - \frac{c}{a}$$

$$-\frac{\det \begin{vmatrix} a & c \\ b - \frac{c}{a} & 0 \end{vmatrix}}{b - \frac{c}{a}} = c$$

The Routh table is

$$\begin{array}{c|ccc} \hline s^3 & 1 & b & 0 \\ s^2 & a & c & 0 \\ s & b - \frac{c}{a} & 0 & 0 \\ 1 & c & 0 & 0 \end{array}$$

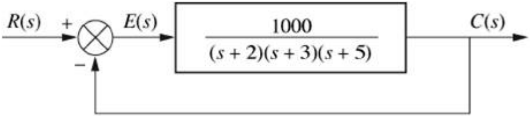
So for 3rd order, stability is equivalent to:

- $a > 0$
- $c > 0$
- $b > \frac{c}{a}$

Now lets look at the previous example to determine the maximum gain:
We have the stable transfer function

$$\hat{G}(s) = \frac{1}{(s+2)(s+3)(s+5)}$$

We close the loop with a gain of size k
Controller: $\hat{K}(s) = k$



(a)

The Closed-Loop Transfer Function is

$$\frac{k}{s^3 + 10s^2 + 31s + 30 + k}$$

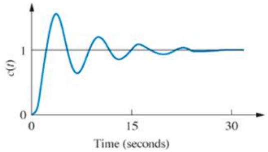
But this is a third order system!

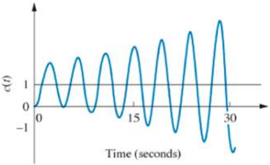
For the third-order system,

$$\frac{k}{s^3 + 10s^2 + 31s + 30 + k}$$

we require

- $a > 0$, which means $10 > 0$
- $c > 0$, which means $30 + k > 0$
- $b > \frac{c}{a}$





The last requirement implies $31 > \frac{k+30}{10}$ or

$$k < 310 - 30 = 280$$

So our gain is limited to $k < 280$

Limited Special Cases

Consider the transfer function

$$\hat{G}(s) = \frac{1}{s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10}$$

The Routh Table begins:

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0

The next entry in the table will be

$$-\frac{\det\begin{vmatrix} 2 & 4 \\ 0 & 6 \end{vmatrix}}{0} = \frac{-12}{0}$$

Which is problematic.

Note: If there is a zero in the first column, the system is only marginally stable

- Small changes in the coefficients lead to instability.

Limited Special Cases

The solution is to use ϵ instead of 0 in the first column.

$$\begin{array}{c|ccc} \hline s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & \epsilon & 6 & 0 \end{array}$$

Now the next entry in the table will be

$$-\frac{\det \begin{vmatrix} 2 & 4 \\ \epsilon & 6 \end{vmatrix}}{\epsilon} = \frac{-(12-4\epsilon)}{\epsilon}$$

Because ϵ is infinitely small, we let $12 - 4\epsilon = 12$.
Assume $\epsilon > 0$

- We have at least one sign change
- At least one unstable pole.

Limited Special Cases

We can keep calculating if necessary.

s^5	1	2	11
s^4	2	4	10
s^3	0	6	0
s^3	ϵ	6	0
s^2	$\dfrac{4\epsilon-12}{\epsilon}$	$\frac{10\epsilon}{\epsilon}$	0
s^2	$\dfrac{-12}{\epsilon}$	10	0
s	$\frac{10\epsilon^2+72}{12}$	0	0
s	6	0	0
1	10	0	0

So there are two sign changes

- Two unstable poles

Limited Special Cases

Consider the transfer function

$$\hat{G}(s) = \frac{1}{s^5 + 7s^4 + 6s^3 + 42s^2 + 8s + 56}$$

The Routh Table begins:

s^5	1	6	8
s^4	7	42	56
s^3	0	0	0

The next entry in the table will be

$$-\frac{\det \begin{vmatrix} 7 & 42 \\ 0 & 0 \end{vmatrix}}{0} = \frac{0}{0}$$

Which is even more problematic - the whole row is zero.
We won't cover this case.

- However, it can be done - see book.

Summary

What have we learned today?

The Routh-Hurwitz Stability Criterion:

- Determine whether a system is stable.
- An easy way to make sure feedback isn't destabilizing
- Construct the Routh Table

Next Lecture: PID Control