An LMI formulation for analysis of coupled Linear PDE systems

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For a linear ODE with inputs and outputs,

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

we find the H_∞-norm by solving the LMIs $\rightarrow$

min γ, s.t.

$$P \succ 0$$

$$\begin{bmatrix} -\gamma I & D^* & B^T P \\ D & -\gamma I & C \\ PB & C^* & A^T P + PA \end{bmatrix} \preccurlyeq 0.$$

Goal: We want to formulate solvable, LMI-type inequalities/tests for PDEs.

Example: Finding a bound on L_2 gain of the PDE (in abstract state-space form)

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t),$$
$$y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}u(t),$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are operators via

Operator-valued inequality:

min γ, s.t.

$$\mathcal{P} \succ 0$$

$$\begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^* \mathcal{P} \\ \mathcal{D} & -\gamma I & \mathcal{C} \\ \mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^* \mathcal{P} + \mathcal{P}\mathcal{A} \end{bmatrix} \preccurlyeq 0$$

Dynamics of a linear PDE can be represented as

$$\dot{\mathbf{x}}_p(t,s) = A_0(s)\underbrace{\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}}_{\mathbf{x}_p}(t,s) + A_1(s)\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}_s(t,s) + A_2(s)[\mathbf{x}_3]_{ss}(t,s).$$

This equation does not have a **unique** solution without

Boundary Conditions: and **Continuity Constraint:**

$$B\begin{bmatrix}x_2(0)\\ x_2(L)\\ x_3(0)\\ x_3(L)\\ x_{3,s}(0)\\ x_{3,s}(L)\end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

$$\mathbf{x}_p \in L^2_{n_1} \times H^2_{n_2} \times H^2_{n_3} := X.$$

$$u_{tt}(t,s) = u_{ss}(t,s) \qquad\qquad s \in [0,L]$$

$$\text{BCs:} \qquad u(t,0) = 0 \quad u_s(t,L) = -k u_t(t,L)$$

Let

$$u_1(t,s) = u_s(t,s), \quad u_2(t,s) = u_t(t,s).$$

Then

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A_1} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s}_{x_2}, \quad A_0 = 0, \quad A_2 = []$$

and boundary conditions

$$u_2(t,0) = 0, \quad u_1(t,L) = -k u_2(t,L) \implies \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix}}_{B} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0.$$

$$u_{tt}(t,s) = u_{ss}(t,s) - 2au_t(t,s) - a^2u(t,s) \qquad s \in [0,1]$$

BCs: $\quad u(t,0) = 0, \qquad u_s(t,1) = -ku_t(t,1)$

Change the variables to $u_1 = u_t$ and $u_2 = u$. This yields a diffusive form:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \underbrace{\begin{bmatrix} -2a & -a^2 \\ 1 & 0 \end{bmatrix}}_{A_0} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_2} \underbrace{u_{2ss}}_{x_3}, \quad A_1 = 0.$$

BCs: $\quad u_2(t,0) = 0, \quad u_{2s}(t,1) = -ku_1(t,1), \quad u_1(t,0) = 0$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_1(L) \\ u_2(0) \\ u_2(L) \\ u_{2s}(0) \\ u_{2s}(L) \end{bmatrix} = 0.$$

BCs force us to make u_1 a hyperbolic state.

(Boundary conditions strongly influence dynamics)

Original Form:

$$\dot{\mathbf{x}}_p(t,s) = A_0(s)\underbrace{\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}}_{\mathbf{x}_p}(t,s) + A_1(s)\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}_s(t,s) + A_2(s)[\mathbf{x}_3]_{ss}(t,s)$$

$$\dot{\mathbf{x}}_p(t) = \mathcal{A}_d\mathbf{x}_p(t), \qquad \mathcal{A}_d \text{ is a differential operator.}$$

$$B\left[x_2(0)\ x_2(L)\ x_3(0)\ x_3(L)\ x_{3,s}(0)\ x_{3,s}(L)\right]^T = 0$$

PIE Format: Write the PDE as a Partial Integral Equation!

$$\mathcal{H}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) \qquad \mathbf{x} := \left[x_1\ x_{2s}\ x_{3ss}\right]^T$$

where $\mathcal{H}, \mathcal{A}$ are 3-PI Operators (bounded).

3-PI Operators ($\{N_i\}$):

$$\left(\mathcal{P}_{\{N_0,N_1,N_2\}}\mathbf{x}\right)(s) := N_0(s)\mathbf{x}(s)ds + \int_a^s N_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b N_2(s,\theta)\mathbf{x}(\theta)d\theta$$

Tip damped wave Equation:

$$u_{tt}(t,s) = u_{ss}(t,s), \qquad s \in [0,L]$$

$$\textbf{BCs:} \qquad u(t,0) = 0, \quad u_s(t,L) = -0.5u_t(t,L).$$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_s, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & k & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0$$

$$\int_0^s \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{bmatrix}_s (t,\eta) d\eta + \int_s^L \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{u}}_1 \\ \dot{\mathbf{u}}_2 \end{bmatrix}_s (t,\eta) d\eta = \begin{bmatrix} \mathbf{u}_2 \\ \mathbf{u}_1 \end{bmatrix}_s (t,s)$$

$$\mathcal{P}_{\{0,H_1,H_2\}} \dot{\mathbf{u}}(t) = \mathcal{P}_{\{A,0,0\}} \mathbf{u}(t), \qquad \mathbf{u} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix}_s \qquad \text{(no BCs)}$$

where $H_1 = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$ and $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

A UNIVERSAL Transformation from PDE to PIE

$$\dot{\mathbf{x}}_p(t) = A_0(s) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} (t,s) + A_1(s) \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}_s (t,s) + A_2(s) \begin{bmatrix} x_3 \end{bmatrix}_{ss} (t,s)$$

Boundary Conditions:

$$B \begin{bmatrix} x_2(0) \\ x_2(L) \\ x_3(0) \\ x_3(L) \\ x_{3s}(0) \\ x_{3s}(L) \end{bmatrix} = 0, \quad \text{rank}(B) = n_2 + 2n_3$$

Becomes:

$$\mathcal{E}\dot{\mathbf{x}} = \mathcal{A}\mathbf{x}(t), \qquad \mathcal{E} = \mathcal{P}_{\{G_i\}}, \qquad \mathcal{A} = \mathcal{P}_{\{J_i\}}, \quad \mathbf{x} := \begin{bmatrix} x_1 \\ x_{2s} \\ x_{3ss} \end{bmatrix}$$

for appropriate choice of G_i and J_i (There is a formula)[1].

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[1] Peet, Matthew M. "Discussion Paper: A New Mathematical Framework for Representation and Analysis of Coupled PDEs." *3rd IFAC/IEEE CSS Workshop on Control of Distributed Parameter Systems* (CPDS 2019).

The Need for 4-PI operators: System with finite-dimensional I/O

$$\begin{bmatrix} y(t) \\ \mathcal{H}\dot{\mathbf{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{D} & \mathcal{C} \\ \mathcal{B} & \mathcal{A} \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} u(t) \\ \mathbf{x}(t) \end{bmatrix}$$

$\mathbf{x}(t) \in L_2^n$, $y(t) \in \mathbb{R}^m$ and $u(t) \in \mathbb{R}^q$. Then $\mathcal{T} : \mathbb{R}^q \times L_2^n \to \mathbb{R}^m \times L_2^n$.

4-PI Operators $\mathcal{P} : \mathbb{R}^p \times L_2^q \to \mathbb{R}^m \times L_2^n$

$$\left(\mathcal{P} \left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} \right)(s) := \begin{bmatrix} Px + \int_0^L Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s) \end{bmatrix}$$

4-PI operators, $\mathcal{P} \left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\}$**, include a 3-PI operator,** $\mathcal{P}_{\{R_i\}}$

$$\mathcal{H}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x} + \mathcal{B}w(t), \quad y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}w(t)$$

Theorem 1 (KYP and H_∞-Gain).

Suppose there exists operator $\mathcal{P} = \mathcal{P}\left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\} \succ 0$, *such that*

$$\underbrace{\begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ \mathcal{D} & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix}}_{\mathcal{T}} \preccurlyeq 0$$

where $\mathcal{H}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *are PI operators. Then* $\|y\|_{L_2} \le \gamma \|\omega\|_{L_2}$.

What are we trying to do?

- Find $\mathcal{T}$
- Solve operator-valued inequalities $\mathcal{P} \succ 0$ and $\mathcal{T} \preccurlyeq 0$

A general operator $\mathcal{P}\left\{\begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix}\right\} : \mathbb{R}^p \times L_2^q[a,b] \to \mathbb{R}^m \times L_2^n[a,b]$

$$\left(\mathcal{P}\left\{\begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}\right)(s) = \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{x}(s)ds \\ Q_2(s)x + \left(\mathcal{P}_{\{R_i\}}\mathbf{x}\right)(s) \end{bmatrix}.$$

MATLAB structure has following elements.

1. `opvar P`: declares P to be a 4-PI operator object.
2. `P.P`: a $m \times p$ matrix
3. `P.Q1, P.Q2`: $m \times q$ and $n \times p$ matrix valued polynomials in s, respectively
4. `P.R`: a structure with entities R_0, R_1, and R_2
5. `P.R.R0` : $n \times q$ matrix valued polynomial in s
6. `P.R.R1, P.R.R2` : $n \times q$ matrix valued polynomials in s and θ

Composition of 3-PI

$$\mathcal{P}_{\{R_i\}} = \mathcal{P}_{\{B_i\}}\mathcal{P}_{\{N_i\}}$$

where

$$\begin{aligned}
R_0(s) &= B_0(s)N_0(s)\\
R_1(s,\theta) &= B_0(s)N_1(s,\theta) + B_1(s,\theta)N_0(\theta) + \int_a^\theta B_1(s,\xi)N_2(\xi,\theta)d\xi\\
&\qquad + \int_\theta^s B_1(s,\xi)N_1(\xi,\theta)d\xi + \int_s^b B_2(s,\xi)N_1(\xi,\theta)d\xi\\
R_2(s,\theta) &= B_0(s)N_2(s,\theta) + B_2(s,\theta)N_0(\theta) + \int_a^s B_1(s,\xi)N_2(\xi,\theta)d\xi\\
&\qquad + \int_s^\theta B_2(s,\xi)N_2(\xi,\theta)d\xi + \int_\theta^b B_2(s,\xi)N_1(\xi,\theta)d\xi
\end{aligned}$$

Notation: $\{R_i\} = \{B_i\} \times \{N_i\}$

MATLAB code:

```
opvar P1 P2; P1=..; P2=..;
Pcomp = P1*P2;
```

Composition of 4-PI

$$\mathcal{P}\left\{\begin{smallmatrix} A, B_1 \\ B_2, \{C_i\}\end{smallmatrix}\right\}\mathcal{P}\left\{\begin{smallmatrix} L, M_1 \\ M_2, \{N_i\}\end{smallmatrix}\right\}\begin{bmatrix} x \\ \mathbf{x}\end{bmatrix}$$
$$= \begin{bmatrix} \left(AL + \int_a^b B_1(s)M_2(s)\right)x + \int_a^b AM_1(s)\mathbf{x}(s)ds + \int_a^b B_1(s)\left(\mathcal{P}_{\{N_i\}}\mathbf{x}\right)(s)ds \\ \left(B_2(s)L + \mathcal{P}_{\{C_i\}}M_2(s)\right)x + B_2(s)\int_a^b M_1(\theta)\mathbf{x}(\theta)d\theta + \left(P_{\{C_i\}}\mathcal{P}_{\{N_i\}}\mathbf{x}\right)(s)\end{bmatrix}$$

Notation: $\begin{Bmatrix} P, Q_1 \\ Q_2, \{R_i\}\end{Bmatrix} = \begin{Bmatrix} A, B_1 \\ B_2, \{C_i\}\end{Bmatrix} \times \begin{Bmatrix} L, M_1 \\ M_2, \{N_i\}\end{Bmatrix}$

MATLAB code:

```
opvar P1 P2;P1=..; P2=..;
Pcomp = P1*P2;
```

For PI operators, composition is **closed** and **associative**.

Transpose/addition of 4-PI operators is a 4-PI operator

Addition

$$\mathcal{P}\left\{\begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix}\right\} = \mathcal{P}\left\{\begin{smallmatrix} A, B_1 \\ B_2, \{C_i\} \end{smallmatrix}\right\} + \mathcal{P}\left\{\begin{smallmatrix} L, M_1 \\ M_2, \{N_i\} \end{smallmatrix}\right\}$$

where

$$P = A + L, \hat{Q}_i(s) = B_i + M_i, R_i = C_i + N_i.$$

Transpose/Adjoint

$$\left\langle \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}, \mathcal{P}\left\{\begin{smallmatrix} \hat{P}, \hat{Q}_1 \\ \hat{Q}_2, \{\hat{R}_i\} \end{smallmatrix}\right\} \begin{bmatrix} y \\ \mathbf{y} \end{bmatrix} \right\rangle_{\mathbb{R}\times L_2} = \left\langle \mathcal{P}\left\{\begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix}\right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} y \\ \mathbf{y} \end{bmatrix} \right\rangle_{\mathbb{R}\times L_2}$$

where

$$\hat{P} = P^T, \hat{Q}_1(s) = Q_2(s)^T, \hat{Q}_2(s) = Q_1(s)^T,$$

$$\hat{R}_0(s) = R_0(s)^T, \hat{R}_1(s,\eta) = R_2(\eta, s)^T, \hat{R}_2(s,\eta) = R_1(\eta, s)^T.$$

Notation: $\left\{\begin{matrix} \hat{P}, \hat{Q}_1 \\ \hat{Q}_2, \{\hat{R}_i\} \end{matrix}\right\} = \left\{\begin{matrix} P, Q_1 \\ Q_2, \{R_i\} \end{matrix}\right\}^*$

MATLAB code:

```
Padd = P1+P2; %addition
Padj = P'; %adjoint
```

Definition: $\left\langle \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} y \\ \mathbf{y} \end{bmatrix} \right\rangle_{\mathbb{R}\times L_2} := x^T y + \int_0^L \mathbf{x}(s)^T \mathbf{y}(s) ds$

Concatenation of 4 PI operators is a 4 PI operator

Stacking-vertical

$$\mathcal{P}\left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\} \begin{bmatrix} x \\ y \\ \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\left\{ \begin{smallmatrix} A, B_1 \\ B_2, \{C_i\} \end{smallmatrix} \right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} \\ \mathcal{P}\left\{ \begin{smallmatrix} L, M_1 \\ M_2, \{N_i\} \end{smallmatrix} \right\} \begin{bmatrix} y \\ \mathbf{y} \end{bmatrix} \end{bmatrix},$$

where

$$P = \begin{bmatrix} A \\ L \end{bmatrix}, \quad Q_i(s) = \begin{bmatrix} B_i(s) \\ M_i(s) \end{bmatrix}, \quad R_i = \begin{bmatrix} C_i \\ N_i \end{bmatrix}.$$

Stacking-horizontal

$$\mathcal{P}\left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathcal{P}\left\{ \begin{smallmatrix} A, B_1 \\ B_2, \{C_i\} \end{smallmatrix} \right\} & \mathcal{P}\left\{ \begin{smallmatrix} L, M_1 \\ M_2, \{N_i\} \end{smallmatrix} \right\} \end{bmatrix} \begin{bmatrix} x \\ \mathbf{x} \end{bmatrix},$$

where

$$P = \begin{bmatrix} A & L \end{bmatrix}, \quad Q_i(s) = \begin{bmatrix} B_i(s) & M_i(s) \end{bmatrix}, \quad R_i = \begin{bmatrix} C_i & N_i \end{bmatrix}.$$

MATLAB code:

```
Pv = [P1; P2]; %vertical stacking
```

MATLAB code:

```
Ph = [P1 P2]; %horizontal stacking
```

$$\mathcal{H}\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x} + \mathcal{B}w(t), \quad y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}w(t)$$

Theorem 2 (KYP and H_∞-Gain).

Suppose there exists operator $\mathcal{P} = \mathcal{P}\left\{\begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix}\right\} \succ 0$, *such that*

$$\underbrace{\begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ \mathcal{D} & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix}}_{\mathcal{T}} \preccurlyeq 0$$

where $\mathcal{H}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *are PI operators. Then* $\|y\|_{L_2} \leq \gamma\|\omega\|_{L_2}$.

What are we trying to do?

- Find $\mathcal{T}$ ✓
    - Composition, addition, transpose and stacking of 4-PI operators gives another 4-PI operator, i.e. $\mathcal{T}$ is 4-PI.
- Solve operator-valued inequalities $\mathcal{P} \succ 0$ and $\mathcal{T} \preccurlyeq 0$

Theorem 3.

Suppose
$$\begin{Bmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{Bmatrix} = \begin{Bmatrix} I, & 0 \\ 0, & \{Z_i\} \end{Bmatrix}^* \times \begin{Bmatrix} T_1, & T_2 \\ T_2^T, & \{T_3, 0, 0\} \end{Bmatrix} \times \begin{Bmatrix} I, & 0 \\ 0, & \{Z_i\} \end{Bmatrix}$$

where $T = \begin{bmatrix} T_1 & T_2 \\ T_2^T & T_3 \end{bmatrix} \geq 0$ *and*

$$\{Z_i\} := \left\{ \begin{bmatrix} \sqrt{g(s)} Z_{d1}(s) & & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & \sqrt{g(s)} Z_{d2}(s,\theta) & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & \\ & & \sqrt{g(s)} Z_{d2}(s,\theta) \end{bmatrix} \right\}.$$

where $g(s) = s(L - s)$ *or* $g = 1$ *and* Z_d *is the vector of monomials. Then* $\mathcal{P}\begin{Bmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{Bmatrix} \geq 0$.

MATLAB Code:

```
P = sospos_RL2RL_psatz(prog,n,s,th,d);
```

Almost complete MATLAB code,

Define $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{H}$ $\rightarrow$

Constrain $\mathcal{P}_{\{N_i\}} > 0$ $\rightarrow$

```
pvar s, th, gam; opvar A,B,C,H;
prog = sosprogram([s; th],gam);
A=..; C=..; H =..; B=..; D=..;
[prog, P] = sospos_L2L(prog,np,X,s,th,d);
T = [-gam*I   D'       B'*P*H;
       D       -gam*I   C;
       H'*P*B   C'       H'*P*A+A'*P*H];
```

$$\mathcal{T} = \begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^*\mathcal{P}\mathcal{H} \\ \mathcal{D} & -\gamma I & \mathcal{C} \\ \mathcal{H}^*\mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P}\mathcal{H} + \mathcal{H}^*\mathcal{P}\mathcal{A} \end{bmatrix}$$

Constrain $\mathcal{T} \leq 0$ $\rightarrow$

```
[prog, Pa] = sospos_RL2RL(prog,n,X,s,th,d);
[prog, Pb] = sospos_RL2RL_psatz(prog,n,X,s,th,d);
prog = sosopeq(prog,Pa+Pb+Pkyp);
prog = sossolve(prog);
```

$$u_{tt}(t,s) = u_{ss}(t,s) + w(t), \qquad u(t,0) = 0, \qquad u_s(t,L) = -ku_t(t,L).$$

Recall,

$$\mathcal{P}_{\{0,H_1,H_2\}}\dot{\mathbf{u}}(t) = \mathcal{P}_{\{I,0,0\}}\mathbf{u}(t)$$

where $\mathbf{u}_f = \begin{bmatrix} u_s \\ u_t \end{bmatrix}$, $H_1 = \begin{bmatrix} 0 & -2 \\ 0 & -1 \end{bmatrix}$ and $H_2 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

How does the disturbance affect tip displacement? (i.e. $u(t,L)$)

$$y(t) = u(t,L) = \int_0^L u_1(t,s)ds$$

$\boxed{\text{We get } \|u(L)\|_{L_2} \le 0.5\|w\|_{L_2} \text{ for } k = 2.}$

Consider,

$$
\begin{aligned}
u_t(s,t) &= A_0(s)u(s,t) + A_1(s)u_s(s,t) \\
&\quad + A_2(s)u_{ss}(s,t) + w(t), \\
u(0,t) &= 0, \quad u_s(L,t) = 0 \\
y(t) &= \int_0^L u(s,t)ds
\end{aligned}
$$

where

$$
\begin{aligned}
&A_0(s) = (-0.5s^3 + 1.3s^2 - 1.5s + 0.7 + \lambda) \\
&A_1(s) = (3s^2 - 2s), \quad A_2(s) = (s^3 - s^2 + 2)
\end{aligned}
$$

and $\lambda = 4.6$.

- ▷ Discretization needs large number of points
- ▷ Error is larger near stability limits

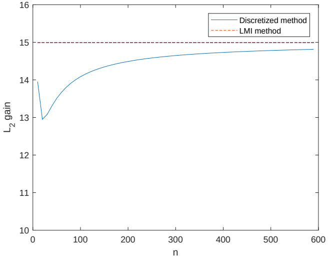

Figure 1: Mesh size vs L_2 gain

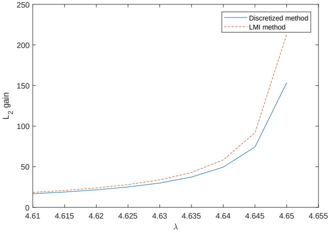

Figure 2: λ vs L_2 gain

Bounds from discretization are not provable

Example 1:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) + w(t) \qquad x(0) = x(1) = 0 \qquad y(t) = \int_0^1 x(t,s)ds$$

γ value: 8.214(LMI method), 8.253(Discretization method, $dx = 1/100$)

Example 2:

$$\dot{x}(t,s) = \lambda x(t,s) + x_{ss}(t,s) + w(t) \qquad x(0) = x_s(1) = 0 \qquad y(t) = \int_0^1 x(t,s)ds$$

γ value: 12.03(LMI method), 12.3(Discretization method, $dx = 1/100$)

Example 3: From Valmorbida,2014,

$$\dot{x}(t,s) = \begin{bmatrix} 1 & 1.5 \\ 5 & .2 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(t), \quad x(0) = x_s(1) = 0 \quad y(t) = \int_0^1 x_1(t,s)ds$$

γ value: 1.67(LMI method), 1.66(Discretization method, $dx = 1/100$).

Example 4: From Valmorbida,2016,

$$\dot{x}(t,s) = \begin{bmatrix} 0 & 0 & 0 \\ s & 0 & 0 \\ s^2 & -s^3 & 0 \end{bmatrix} x(t,s) + R^{-1} x_{ss}(t,s) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} w(t), \; x(0) = x_s(1) = 0, \; y(t) = \int_0^1 x_2(t,s)ds$$

γ value: 3.58(LMI method), 3.97(Discretization method, $dx = 1/200$).

Consider,

$$u_{t,i}(s,t) = \lambda u_i(s,t) + \sum_{k=1}^{i} u_{ss,k}(s,t) + w(t)$$
$$u(0,t) = 0 \qquad u(L,t) = 0.$$

We use, $\lambda = 0.5\pi^2$ for all i.

i	1	2	3	4	5	10	20
CPU time(s)	0.60	1.45	5.22	13.7	36.5	2317	27560

Table 1: Runtime to solve LMIs for increasing number of coupled PDEs, i.

- Can solve systems involving 20 coupled PDEs within 8 hours on a standard desktop computer

$\mathcal{P}\left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\}$ Operators extend LMI techniques to PDEs.

- Boundary conditions and dynamics are combined into a single equation
- No Conservatism
- Computationally Efficient
- Easily Extended to New Problems
  - e.g. Observer synthesis, coupled ODE-PDEs, systems with input delay et c.

Future work:

- IQCs for Nonlinearity
  - H_2 Gain
- Optimal Dynamic state/output Feedback
- Inversion of the $\mathcal{P}\left\{ \begin{smallmatrix} P, Q_1 \\ Q_2, \{R_i\} \end{smallmatrix} \right\}$ Operator
  - When $R_1 \neq R_2$

Its a toolbox that anyone can use.

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Thank you