Modern Control Systems

Matthew M. Peet
Illinois Institute of Technology

Lecture 6: Back to Matrices

Eigenvalues and Spectrum are very different for matrices vs. operators.

Definition 1.

For a matrix $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$ is an **eigenvalue** of A, denoted $\lambda \in \text{eig}(A)$ if there exists some **eigenvector** $v_\lambda \in X$ such that

$$Av_\lambda = \lambda v_\lambda$$

Alternative Representations:

- $\lambda \in \text{eig}(A)$ if $\text{Ker}(\lambda I - A) \neq \{0\}$
- For matrices, $\lambda \in \text{eig}(A)$ if $\det(\lambda I - A) = 0$

Recall:

- $\det(A) = \prod \lambda_i(A)$
- $\text{trace}(A) = \sum \lambda_i(A)$

Eigenvalues and Eigenvectors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
[U,V]=eigs([1 2 3; 4 5 6; 7 8 9])
```

$$U = \begin{bmatrix} -.23 & -.78 & .41 \\ -.53 & -.09 & -.82 \\ -.82 & .61 & .41 \end{bmatrix}; \qquad V = \begin{bmatrix} 16.1 & 0 & 0 \\ 0 & -1.11 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which means:

$$\lambda_1(A) = 16.1, \quad \lambda_2(A) = -1.11, \quad \lambda_3(A) = 0,$$

$$v_1 = \begin{bmatrix} -.23 \\ -.53 \\ -.82 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -.78 \\ -.09 \\ .61 \end{bmatrix}, \quad v_3 = \begin{bmatrix} .41 \\ -.82 \\ .41 \end{bmatrix}$$

Characteristic Polynomials

Definition 2.

The **characteristic polynomial** of a matrix A is denoted

$$\text{char}A(\lambda) := \det(\lambda I - A)$$

Theorem 3.

$\lambda \in$ eig(A) if and only if char$A(\lambda) = 0$

Proof.

- $\lambda \in \text{eig}(A)$ if and only if $\text{Ker}(\lambda I - A) \neq \{0\}$
- $\text{Ker}(\lambda I - A) \neq \{0\}$ if and only if there exists some v such that

$$(\lambda I - A)v = 0$$

- This means λ is an eigenvalue of A

□

Characteristic Polynomials

Eigenvectors are **NOT** unique

- If v is an eigenvector, then so is αv
- If v_1 and v_2 are both eigenvectors for λ, so is $\alpha v_1 + \beta v_2$.

To avoid the problem of uniqueness, we use eigenspace

Definition 4.

For any $\lambda \in \text{eig}(A)$, there is a unique subspace, S_λ, called an **eigenspace**, such that $x \in S_\lambda$ if and only if

$$Ax = \lambda x.$$

Theorem 5.

If $\cup_i \{S_i\} = \mathbb{R}^n$, then there exist eigenvectors v_i such that

$$V = [v_1, \cdots, v_n] \qquad \textit{is invertible.}$$

Diagonalizability

Then

$$AV = [Av_1, \cdots, Av_n] = [\lambda_1 v_1, \cdots, \lambda_n v_n] = V\Lambda, \quad \text{where} \quad \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}.$$

If V is invertible, then

$$V^{-1}AV = \Lambda$$

Thus if the eigenspaces span the space, the matrix can be diagonalized via a *Similarity Transformation*.

- But when does this happen?
- Not all matrices can be diagonalized.

Example:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\lambda_1 = 0, \qquad v_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \lambda_2 = 0, \qquad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Jordan Form

Diagonalizibility is an important tool.

- A change of bases
- In new basis, the operator/matrix multiplies coordinate by a simple factor.
- Can all operators be so simple?
  - **NO**

An alternative which applies to ALL matrices is the Jordan Form

Definition 6.

A **Jordan Block** has the form

$$\lambda I + N = \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}$$

Almost Diagonal.

Jordan Form

$$N = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

is called **Nilpotent** as

- $N^d = 0$ for some $d > 0$.

Example

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; N^4 = 0$$

Jordan Form

λ is the sole eigenvalue of $\lambda I + N$ with multiplicity n and eigenvectors

$$v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad v_2 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}, \quad v_3 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad v_4 = \begin{bmatrix}-1\\0\\0\\0\end{bmatrix}$$

or a 1-Dimensional eigenspace $S_\lambda := \text{span}\ \{e_1\}$.

- Jordan blocks capture the part of a matrix which cannot be diagonalized.

Theorem 7.

For any $A \in \mathbb{C}^{n\times n}$, there exists an invertible T and Jordan Blocks J_i such that

$$TAT^{-1} = J = \begin{bmatrix}J_1 & & \\ & \ddots & \\ & & J_p\end{bmatrix}$$

ANY matrix is Jordan-Block Diagonalizable.
Different Jordan Blocks may have the same eigenvalue.

Symmetric Matrices

Definition 8.

A Matrix, $A \in \mathbb{R}^{n\times n}$ ($A \in \mathbb{C}^{n\times n}$) is self-adjoint, denoted $A \in \mathbb{S}^n$ ($A \in \mathbb{H}^n$) if $A = A^T$ ($A = A^*$).

- Real self-adjoint matrices are called symmetric. ($\mathbb{S}$)
- Complex self-adjoint matrices are called Hermetian. ($\mathbb{H}$)

Lemma 9.

Both Hermetian and Symmetric Matrices have **real** *Eigenvalues.*

Proof.

We show that is $A \in \mathbb{H}$ and $Av = \lambda v$, then $\lambda = \lambda^*$.

- If $Av = \lambda v$, then

$$\lambda v^* v = v^* A v = (A^* v)^* v = (Av)^* v = (\lambda v)^* v = \lambda^* v^* v$$

- Hence $\lambda = \lambda^*$, which means λ is real.

$\square$

Unitary Matrices

Unitary matrices are a special case of coordinate transformation.

Definition 10.

Two vectors, $x, y \in R^n$ are

- **orthogonal** if $x^T y = 0$
- **orthonormal** if they are orthogonal and $\|x\| = \|y\| = 1$

A basis, $\{v_i\}$ is an **orthonormal basis** if all v_i are orthonormal.

- A coordinate transformation to an orthonormal basis is a *unitary* operator.

Given a basis $\{v_1, \cdots, v_n\}$, we can construct an orthonormal basis

$$x_1 = \frac{v_1}{\|v_1\|}$$

$$x_2 = \frac{v_2 - (v_2^T x_1)x_1}{\|v_2^T x_1)x_1\|}$$

$$x_3 = \frac{v_3 - (v_3^T x_2)x_2 - (v_3^T x_1)x_1}{v_3 - (v_3^T x_2)x_2 - (v_3^T x_1)x_1}$$

- Clearly, all vectors are of unit length
- To see orthogonality, note

$$x_2^T x_1 = \frac{v_2^T x_1 - (v_2^T x_1)x_1^T x_1}{\|v_1\|\|v_2^T x_1)x_1\|} = \frac{v_2^T x_1 - v_2^T x_1}{\|v_1\|\|(v_2^T x_1)x_1\|} = \frac{v_2^T x_1 - v_2^T x_1}{\|v_1\|\|v_2^T x_1)x_1\|}$$

Unitary Matrices

Definition 11.

A matrix U is **Unitary** if $U^*U = I$.

- The columns of U form an orthonormal basis

$$\begin{bmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_n^* \end{bmatrix} \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1^*u_1 & u_1^*u_2 & \cdots & u_1^*u_n \\ u_2^*u_1 & u_2^*u_1 & & u_2^*u_n \\ \vdots & & \ddots & u_1^*u_n \\ u_n^*u_1 & & \cdots & u_n^*u_n \end{bmatrix} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

- $U^* = U^{-1}$
  - The columns of U^* also form an orthonormal basis

Length Preservation: A basis change which preserves norms.

- If $y = Ux$, then

$$\|y\|^2 = x^*U^*Ux = x^*x = \|x\|^2$$

- Why the $\|\cdot\|_2$-norm?

Symmetric matrices can be diagonalized via unitary matrices.

Theorem 12.

If $A \in \mathbb{H}$, then there exists a unitary matrix U and a real diagonal Λ such that

$$A = U \Lambda U^*$$

There is also a spectral theorem for operators.

- What is a diagonal operator?

Positive Matrices

Definition 13.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Definition 14.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \qquad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

Positive Matrices

Lemma 15.

$P \in \mathbb{S}^n$ *is positive definite if and only if all its eigenvalues are positive.*

Proof.

($\Rightarrow$) To show sufficiency, use the spectral decomposition. For $x \neq 0$,

$$x^T P x = x^T U^T \Lambda U x = v^T \Lambda v = \sum_{\lambda_i} v_i^2 > 00$$

($\Leftarrow$) To show necessity, Suppose λ is an eigenvalue and $Px = \lambda x$ for $x \neq 0$, then

$$\lambda x^T x = x^T P x > 0$$

Hence

$$\lambda = \frac{x^T P x}{x^T x} > 0$$

$\square$

Easy Proofs:

- A Positive Definite matrix is invertible.
- The inverse of a positive definite matrix is positive definite.

Positive Matrices

Easy Proofs:

- If $P > 0$, then $TPT^T \geq 0$ for any T. If T is invertible, then $TPT^T > 0$.

Lemma 16.

For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ such that $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

Proof.

By the spectral decomposition $A = U \Lambda U^T$, where $\Lambda = \text{diag}(\lambda_1, \cdots, \lambda_n)$.

- Let $\Lambda^{\frac{1}{2}} = \text{diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$.
- Then $\Lambda^{\frac{1}{2}} > 0$, $\Lambda = \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}}$, and

$$
\begin{aligned}
A &= U \Lambda U^T \\
&= U \Lambda^{\frac{1}{2}} \Lambda^{\frac{1}{2}} U^T \\
&= U \Lambda^{\frac{1}{2}} U^T U \Lambda^{\frac{1}{2}} U^T
\end{aligned}
$$

- Thus $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$, where $P^{\frac{1}{2}} = U \Lambda^{\frac{1}{2}} U^T > 0$

$\square$

Positive Matrices

Other Notes

The properties of positive matrices extend to positive operators.

What is an inequality? What does ≥ 0 mean?

- An inequality implies a **partial ordering**:
  - $x \geq y$ if $x - y \geq 0$
- Any convex cone, C defines a partial ordering:
  - $x - y \geq 0$ if $x - y \in C$
- The ordering is only partial because $x \not\leq 0$ does not imply $x \geq 0$
  - $-x \notin C$ does not imply $x \in C$.
  - x may be indefinite.