# Analysis of Nonlinear Time Delay Systems Using the Sum of Squares Decomposition

Antonis Papachristodoulou, Matthew Peet and Sanjay Lall

**Abstract**

In this paper we present a methodology for verifying algorithmically the stability of equilibria of systems described by nonlinear functional differential equations, also known as time-delay systems. Delay-dependent and delay-independent stability analysis of such systems is performed in a unified way through the construction of appropriate Lyapunov certificates, Lyapunov-Krasovskii functionals. The construction is entirely algorithmic, and the computational methodology is based on the sum of squares decomposition of multivariate polynomials. The construction of parameterized Lyapunov-Krasovskii functionals allows the robust stability analysis of such systems. The methodology is applied to illustrative examples from population dynamics and the Internet.

**Index Terms**

Nonlinear time-delay systems, Lyapunov's direct method, stability analysis, robustness analysis, sum of squares decomposition, semidefinite programming.

## I. Introduction

Time delay systems have always been useful for representing systems that involve transport and propagation of data, such as communication systems, but also for modeling maturation and growth in population dynamics [1], [2]. It is known that the presence of delays may induce undesirable effects such as instabilities and performance degradation; ignoring them in the modeling process may result in inadequate designs and incorrect analysis conclusions. Inevitably the stability, robust stability and control of such systems have received a lot of interest in the

A. Papachristodoulou is with Control and Dynamical Systems, California Institute of Technology, Pasadena, CA 91125, USA. Email: antonis@cds.caltech.edu. Matthew Peet and Sanjay Lall are with the Department of Aeronautics and Astronautics, Stanford University, Stanford, CA 94305, USA. Email: {mmpeet,lall}@stanford.edu.

past few years. Recently this has intensified, as they provide the simplest adequate modeling framework for network congestion control for the Internet [3].

Time delay systems fall in the category of Functional Differential Equations (FDEs), which differ from Ordinary Differential Equations (ODEs) because of the infinite dimensional character of the state. The evolution of the system in an infinite dimensional space makes the problem of assessing the stability properties of steady states more difficult to answer, as more complicated analysis tools are required. The stability question can still be answered through the construction of Lyapunov-type certificates, i.e., functions of state that satisfy certain positivity conditions, as it is done in the case of ODE systems. However, while for the case of ODEs these certificates are functions, in the case of FDEs they are *functionals* owing to the fact that the state belongs in a function space itself.

Convex optimization has been used for the algorithmic construction of these functionals for linear time delay systems, through the solution of a set of Linear Matrix Inequalities (LMIs). However, in some cases the solution of parameterized (infinite-dimensional) LMIs is required, which is a difficult task. This has initially limited the functional structures that could be considered yielding conservative results on the delay bounds for stability [4]: non-conservative results could later be obtained by discretizing the Lyapunov functional and solving a set of LMIs whose size depends on the discretization level [5]. More recently a new approach was developed in [6] for constructing Lyapunov-Krasvoskii functionals. As far as nonlinear time delay systems are concerned, the only methodologies for stability analysis center on the manual construction of another type of Lyapunov certificates, Lyapunov-Razumikhin functions, for systems of low dimension [1].

In this paper we present a methodology for constructing Lyapunov-Krasovskii functionals for nonlinear time delay systems, using the Sum of Squares decomposition of multivariate polynomials as the computational tool. We construct appropriate Lyapunov-Krasovskii functionals both for delay-independent and delay-dependent stability. This stability classification is based on whether stability is retained independent of the size of the delay or whether it is lost as the delay size, seen as a parameter, is allowed to vary. The Lyapunov functionals we construct have polynomial kernels, and in the case in which the FDE we consider is linear, they reduce to the complete (i.e., necessary and sufficient) Lyapunov functional structures.

The paper is organized as follows. First, in section II we present some background information on systems described by Functional Differential Equations and the Lyapunov theorem that we will be using in the sequel. In section III we present a brief review on the sum of squares decomposition and SOSTOOLS, and discuss briefly how Lyapunov functions for nonlinear systems described by Ordinary Differential Equations can be constructed algorithmically. In section IV we present the methodology for robust stability analysis for nonlinear delayed systems, both in the delay-independent and delay-dependent sense, as well as for single and for multiple, incommensurate delays. In section V we illustrate our results with examples from population dynamics and network congestion control for the Internet. We conclude the paper in Section VI.

## II. Background and past results

In this section we will present some facts about autonomous Functional Differential Equations and introduce the notion of Lyapunov stability. A more detailed account can be found in [7].

Let $\mathbb{R}^n$ denote the $n$-dimensional real Euclidean space with norm $|\cdot|$. We denote $C([a,b],\mathbb{R}^n)$ the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b]=[-\tau,0]$ we let $C=C([-\tau,0],\mathbb{R}^n)$ and the norm on $C$ is defined as $\|\phi\|=\sup_{-\tau\leq\theta\leq 0}|\phi(\theta)|$. We denote by $C^\gamma$ the set defined by $C^\gamma=\{\phi\in C | \|\phi\|<\gamma\}$ with $\gamma>0$. For $\sigma\in\mathbb{R}$, $\rho\geq 0$ and $x\in C([\sigma-\tau,\sigma+\rho],\mathbb{R}^n)$ and for any $t\in[\sigma,\sigma+\rho]$, we define $x_t\in C$ as $x_t(\theta)=x(t+\theta),\theta\in[-\tau,0]$.

Assume $\Omega$ is a subset of $C$, $f:\Omega\to\mathbb{R}^n$ is a given function, and '$\dot{}$' represents the right-hand derivative. Then we call

$$\dot{x}(t)=f(x_t) \tag{1}$$

a retarded functional differential equation (RFDE) on $\Omega$. For a given $\phi\in C$, we say $x(t;\phi)$ is a solution to (1) at time $t$ with initial condition $\phi$, if there is a $\rho>0$ such that $x(t;\phi)$ is a solution of (1) on $[t-\tau,t+\rho)$ and $x(0;\phi)=\phi$. Such a solution exists and is unique under a certain Lipschitz condition on $f$ in $\Omega$ [7].

In this paper we concentrate on systems with $K$ incommensurate discrete delays, so that we have $f(x_t)=f(x(t),x(t-\tau_1),\ldots,x(t-\tau_K))$ where $\tau_i$ are the discrete delays in the system. Without loss of generality we assume that $0$ is a steady-state for the system. We now recall the following stability definitions.

*Definition 1:* The trivial solution of (1) is called

1) *stable* if for any $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $|x(t;\phi)| \leq \epsilon$ for any initial condition $\phi \in C^{\delta}$ and all $t \geq 0$; otherwise it is termed *unstable*.
2) *asymptotically stable* if it is stable and there is a $\gamma > 0$ such that $\lim_{t\to\infty} x(t;\phi) = 0$ for any initial condition $\phi \in C^{\gamma}$. The set of initial functions $\phi$ for which $\lim_{t\to\infty} x(t;\phi) = 0$ is called the *domain of attraction* of the trivial solution.

Establishing the stability of the steady-state is a rather difficult task if $f$ is a nonlinear function of $(x(t), x(t-\tau_1), \ldots, x(t-\tau_K))$. Even in the linear case, establishing the stability boundary, i.e., the exact values of $\tau_i$ for which stability is retained is NP-hard [8]. Just as in the case of nonlinear systems described by Ordinary Differential Equations (ODEs), a Lyapunov argument can be formulated, that proves useful when many, incommensurate delays appear in the system.

Consider a continuous functional $V : C \to \mathbb{R}$ and define:

$$\dot{V}(\phi) = \lim\sup_{h\to 0^+} \frac{1}{h}[V(x_h(\phi)) - V(\phi)]$$

This is the derivative of $V$ along a solution of (1). For autonomous systems we have the following theorem (Lyapunov-Krasovskii) [7]:

*Theorem 2:* Assume $f : \Omega \to \mathbb{R}^n$ is completely continuous and the solutions of (1) depend continuously on initial data. Suppose that $V : \Omega \to \mathbb{R}$ is continuous and there exist $a(s)$ and $b(s)$ nonnegative continuous, $a(0) = b(0) = 0$, $\lim_{s\to\infty} a(s) = +\infty$ such that:

$$V(\phi) \geq a(|\phi(0)|) \text{ on } \Omega$$

$$\dot{V}(\phi) \leq b(|\phi(0)|) \text{ on } \Omega$$

Then the solution $x = 0$ of (1) is stable, and every solution is bounded. If in addition, $b(s) > 0$ for $s > 0$, the $x = 0$ is asymptotically stable.

We note that apart from the Lyapunov-Krasovskii theorem for stability analysis, there is the Lyapunov-Razumikhin theorem which uses functions as the certificates for stability instead of functionals. In [9] connections between appropriate Lyapunov-Razumikhin conditions and Input-to-State Stability small-gain are made, and relaxed Razumikhin-type conditions guaranteeing global asymptotic stability are derived.

In this paper we are interested in *algorithmic* methodologies for constructing Lyapunov-type certificates. The *convexity* in the Lyapunov conditions is important – the conditions in Theorem 2 are convex, whereas those in the corresponding Lyapunov-Razumikhhin theorem are not, because of the derivative condition and its connection to the Lyapunov function itself. For simple linear systems and properly structured Lyapunov functionals, LMI criteria can still be written, but they lead to conservative conclusions as the structures considered are not rich enough — see [4]. Under convexification assumptions, the Lyapunov-Razuminkin criteria can also be tested by solving LMIs for linear FDEs, but are in general more conservative than the Lyapunov-Krasovskii ones [8], and therefore we will not investigate the construction of Lyapunov-Razumikhin functions in this paper.

The stability of time-delay systems is classified as *delay-dependent* or *delay-independent*, based on the persistence of stability as the delay size, seen as a parameter, is increased. We say that a system is *delay-independent* stable, if the stability property is retained for all positive (finite) values of the delays in the system, i.e., the system is robust with respect to the delay size. On the other hand, we say that a system is *delay-dependent* stable, if the stability is preserved for some values of delays and is lost for some others. In general the former condition is more conservative, and indeed a condition that depends on the value of the delay is usually required for stability analysis. The existence of complete quadratic Lyapunov-Krasovskii functionals necessary and sufficient for strong delay-independent [10] and delay-dependent stability [8] of linear time delay systems is known, and so is their structure, but there is an inherent difficulty in constructing them.

## III. Sum of Squares: Constructing Lyapunov functions for nonlinear systems algorithmically

In this section we present the methodology that we propose to use for the algorithmic construction of Lyapunov-Krasovksii functionals for nonlinear systems. This is based on the Sum of Squares (SOS) decomposition of multivariate polynomials. This methodology was introduced by Parrilo in his thesis [11].

*Definition 3:* A multivariate polynomial $p(x)$, $x \in \mathbb{R}^n$ is a *Sum of Squares* (SOS) if there exist polynomials $f_i(x)$, $i = 1, \ldots, M$ such that $p(x) = \sum_{i=1}^{M} f_i^2(x)$.

An equivalent characterization of SOS polynomials is given in the following proposition, the proof of which can be found in [11].

*Proposition 4:* [11] A polynomial $p(x)$ of degree $2d$ is SOS if and only if there exists a positive semidefinite matrix $Q$ and a vector $Z(x)$ containing monomials in $x$ of degree $\leq d$ so that

$$p = Z(x)^T Q Z(x)$$

In general, the monomials in $Z(x)$ are not algebraically independent. Expanding $Z(x)^T Q Z(x)$ and equating the coefficients of the resulting monomials to the ones in $p(x)$, we obtain a set of affine relations in the elements of $Q$. Since $p(x)$ being SOS is equivalent to $Q \geq 0$, the problem of finding a $Q$ which proves that $p(x)$ is an SOS can be cast as a semidefinite program (SDP) [11]. Therefore the problem of seeking a $Q$ such that $p$ is a SOS can be formulated as an LMI. Note that if a polynomial $p(x)$ is a sum of squares, then it is globally nonnegative. The converse is not always true: not all positive semi-definite polynomials can be written as SOS — in fact, testing global non-negativity of a polynomial $p(x)$ is known to be NP-hard when the degree of $p(x)$ is greater than 4 [12], whereas checking whether $p$ can be written as a SOS is computationally tractable - it can be formulated as an SDP which has a worst-case polynomial-time complexity. The construction of the SDP related to the SOS conditions can be performed efficiently using SOSTOOLS [13], a software that formulates general sum of squares programmes as SDPs and calls semidefinite programming solvers to solve them.

If the monomials in the polynomial $p(x)$ have *unknown* coefficients then the search for feasible values of those coefficients such that $p(x)$ is nonnegative is also an SDP, a fact that is important for the construction of Lyapunov functions and other S-procedure type multipliers.

The SOS technique has been used to construct Lyapunov functions for nonlinear systems, by relaxing the non-negativity conditions to SOS conditions [11], [14]. The only complication is that the Lyapunov function $V(x)$ be positive *definite*, which can be achieved as follows.

*Proposition 5:* Given a polynomial $V(x)$ of degree $2d$, let $\varphi(x) = \sum_{i=1}^{n} \sum_{j=1}^{d} \epsilon_{ij} x_i^{2j}$ such that:

$$\sum_{j=1}^{m} \epsilon_{ij} \geq \gamma \quad \forall\ i = 1, \ldots, n,$$

with $\gamma$ a positive number, and $\epsilon_{ij} \geq 0$ for all $i$ and $j$. Then the condition

$$V(x) - \varphi(x) \text{ is a sum of squares} \tag{2}$$

guarantees the positive definiteness of $V(x)$. Moreover, $V(x)$ is radially unbounded.

*Proof:* The function $\varphi(x)$ as defined above is positive definite if $\epsilon_{ij}$'s satisfy the conditions mentioned in the proposition. Moreover it is radially unbounded by construction, as it is the positive sum of simple monomials (i.e. in only one variable) squared. Then $V(x)-\varphi(x)$ being SOS implies that $V(x) \geq \varphi(x)$, and therefore $V(x)$ is positive definite. Since $\varphi$ is radially unbounded, so is $V$. ■

For an autonomous system described by an ODE of the form:

$$\dot{x} = f(x), \tag{3}$$

where $x \in \mathbb{R}^n$ and $f : D \rightarrow \mathbb{R}^n$ is a locally Lipschitz map with $f(0) = 0$, and where $D \subset \mathbb{R}^n$ is a region of the state space containing the equilibrium described by

$$D = \{x \in \mathbb{R}^n : g_i(x) \leq 0, \quad i = 1, \ldots, m\} \tag{4}$$

with $f$ and $g_i$ polynomials, a polynomial Lyapunov function that ensures asymptotic stability of the zero equilibrium can be constructed using the following SOS program:

Find a polynomial $V(x)$ and $\varphi(x) > 0, \psi(x) > 0$ constructed using Proposition 5

and sum of squares $p_{1i}(x)$ and $p_{2i}(x)$

Such that:

$$V(x) - \varphi(x) + \sum_{i=1}^{m} p_{1i}(x) g_i(x) \text{ is a SOS,} \tag{5}$$

$$-\frac{\partial V}{\partial x} f(x) - \psi(x) + \sum_{i=1}^{m} p_{2i}(x) g_i(x) \text{ is a SOS.} \tag{6}$$

Indeed conditions (5–6) are sufficient for verifying local stability of the equilibrium of system (3). To see this, while $x \in D$, (5) guarantees that

$$V(x) \geq \varphi(x) - \sum_{i=1}^{m} p_{1i}(x) g_i(x) > 0 \tag{7}$$

as $g_i(x) \leq 0$ for $x \in D$ and $p_{1i} \geq 0$ (since they are SOS). Similarly (6) guarantees that $-\frac{\partial V}{\partial x} f(x) > 0$ for $x \in D$.

Note the construction of appropriate SOS multipliers – rather than non-negative constants – that are used to adjoin the conditions $g_i \geq 0$, $i = 1, \ldots, m$ that describe the region $D$ in (5–6). In fact this reduces to the well-known S-procedure for testing conditional satisfiability, when the multipliers are non-negative constants. The above formulation is a particular instance of a Positivstellensatz construction for testing emptiness of a semi-algebraic set, which yields a nested family of conditions of non-decreasing strength. More details can be found in [11].

Robust stability of uncertain systems can be treated in the same spirit – see [14] for more details. All the above sum of squares programs can be formulated and solved using SOSTOOLS [13]. Also, a discussion of how to treat the non-polynomial case can be found in [15].

We now turn to the main purpose of this paper, stability analysis of time-delay systems. The conditions for stability are given by Theorem 2, which differs from the Lyapunov theorem for ODE systems.

## IV. Nonlinear Time Delay Systems

In this section, we concentrate on delay-independent and delay-dependent stability analysis of nonlinear time delay systems with or without parametric uncertainty. Later in this section, we consider the case of systems with multiple incommensurate time delays. These system descriptions can be treated using the sum of squares technique in a unified way.

As it was already mentioned, previous attempts to analyze stability of nonlinear time delay systems centered on manual constructions of Lyapunov-Razumikhin functions [1] and sometimes Lyapunov-Krasovskii functionals. The methodology proposed here is the first to handle nonlinear time-delay systems in an algorithmic fashion.

### *A. Delay-independent stability*

An equilibrium of a time-delay system is delay-independent stable, if it is stable for all finite values of the delay. Delay-independent stability conditions may be conservative, in the sense that the system may still be stable in a delay-dependent fashion even if these condition are violated. It is however used in controller synthesis for delay systems where the size of the delay is uncertain. Delay-independent stability and stabilization for nonlinear systems has been investigated under Lyapunov-Razumikhin conditions [16]. In [17] a relationship between a criterion obtained using a Lyapunov-Krasovskii functional and the delay-independent small gain theorem was established for a special class of nonlinear time-delay systems.

In general, finding the proper structure for a Lyapunov functional in the case of nonlinear systems involves some guessing. The intuition we gain from the structures used in the linear case is invaluable. For the case of what is called strong delay-independent stability (for definitions and details see [17]) for linear systems, the class of such Lyapunov functionals has been completely characterized.

*Example 6:* For system

$$\dot{x} = A_0 x(t) + A_1 x(t - \tau) \tag{8}$$

a Lyapunov-Krasovskii candidate that would yield a delay-independent condition is

$$V(x_t) = x(t)^T P x(t) + \int_{-\tau}^{0} x(t+\theta)^T S x(t+\theta) d\theta \tag{9}$$

Sufficient conditions on $V(x_t)$ to be positive definite are $P > 0,\ S \geq 0$. Evaluating $\dot{V}(x_t)$ we get:

$$V(x_t) = x^T(t)\left(A_0^T P + PA_0 + S\right)x(t) + x^T(t-\tau)A_1^T P x(t) + x^T(t) P A_1 x(t-\tau) - x^T(t-\tau) S x(t-\tau)$$

Therefore if we impose

$$\begin{bmatrix} A_0^T P + PA_0 + S & PA_1 \\ A_1^T P & -S \end{bmatrix} + \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} < 0,$$

for $\epsilon > 0$, then $\dot{V}(x_t) < 0$ and the conditions for stability (see Theorem 2) can be written as Linear Matrix Inequalities (LMIs) with $P$ and $S$ as the unknowns; in other words, if we can find $P$ and $S$ such that

$$P > 0, \quad S \geq 0, \quad \begin{bmatrix} A_0^T P + PA_0 + S & PA_1 \\ A_1^T P & -S \end{bmatrix} + \epsilon \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \leq 0, \quad \epsilon > 0,$$

then (8) is stable independent of the delay (as the delay size does not appear in the conditions).

The Lyapunov functional used above may not suffice to prove stability for a general delay-independent stable system, as the proposed structure is not adequate for the stability proof. However, the Lyapunov structure in [10] has been proven to be 'complete' for *strong* delay-independent stability in the case of linear time delay systems. Denoting

$$z_k(t) = [x(t), x(t-\tau), \ldots, x(t-(k-1)\tau)] \tag{10}$$

then the complete structure in the single delay case is

$$V_k(x_t) = V_0(z_k(t)) + \int_{-\tau}^{0} V_1(z_k(t+\theta)) d\theta \tag{11}$$

where $V_0$ and $V_1$ are *quadratic* polynomials in their arguments.

Let us now consider a nonlinear system with one discrete delay given by:

$$\dot{x}(t) = f(x(t), x(t-\tau)), \tag{12}$$

where we assume that $f$ is a nonlinear, polynomial function of its arguments, although this assumption may be lifted through a change of variables [15]. It is assumed without any loss of generality that 0 is a steady-state of the system. We have the following conditions for delay-independent stability:

*Proposition 7:* Consider the system described by Equation (12). If either

- Case 1: There exist functions $V_0(z_k(t))$ and $V_1(z_k(t+\theta))$ (where $z_k$ is defined by (10)), a positive definite, radially unbounded function $\varphi_1(z_k(t))$ and a non-negative function $\psi_1(z_k(t))$ such that:
  1) $V_0(z_k(t)) - \varphi_1(z_k(t)) \geq 0$,
  2) $V_1(z_k(t+\theta)) \geq 0$,
  3) $\sum_{m=0}^{k-1} \frac{\partial V_0}{\partial x(t-m\tau)} f(x(t-m\tau), x(t-(m+1)\tau)) + V_1(z_k(t)) - V_1(z_k(t-\tau)) - \psi_1(z_k(t)) \leq 0$.
- Case 2: There exist polynomial functions $V_0(x(t))$ and $V_1(x(t+\theta))$ and a positive definite, radially unbounded function $\varphi_2(x(t))$ and a non-negative function $\psi_2(x(t))$ such that
  1) $V_0(x(t)) - \varphi_2(x(t)) \geq 0$,
  2) $V_1(x(t+\theta)) \geq 0$,
  3) $\frac{\partial V_0}{\partial x(t)} f + V_1(x(t)) - V_1(x(t-\tau)) - \psi_2(x(t)) \leq 0$.

then the 0 steady-state is globally delay-independent stable. If moreover $\psi_1$ and $\psi_2$ are positive definite, then the 0 steady-state is globally asymptotically delay-independent stable.

*Proof:* Consider first the functional

$$W_k(x_t) \quad = \quad V_0(z_k(t)) + \int_{-\tau}^{0} V_1(z_k(t+\theta))d\theta \tag{13}$$

and the conditions in the first case. The first two constraints impose that $W_k \geq \varphi_1(z_k(t)) > 0$, so the first Lyapunov-Krasovskii condition is satisfied and moreover $W_k$ is radially unbounded. The derivative of $W_k$ along the trajectories of system (12) is:

$$\dot{W}_k = \sum_{m=0}^{k-1} \frac{\partial V_0}{\partial x(t-m\tau)} f(x(t-m\tau), x(t-(m+1)\tau)) + V_1(z_k(t)) - V_1(z_k(t-\tau))$$

Under the third condition in Case 1, the above derivative is non-positive. Therefore if all three conditions are satisfied the equilibrium of the system given by (12) is globally stable; since the delay size does not appear explicitly in the above conditions, then the zero steady-state is globally stable independent of delay. Moreover if $\psi_1(z_k(t)) > 0$, then the zero steady-state is globally asymptotically stable independent of delay.

For case 2, consider the functional:

$$V(x_t) \quad = \quad V_0(x(t)) + \int_{-\tau}^{0} V_1(x(t+\theta))d\theta \tag{14}$$

Again the first two conditions in Case 2 impose that $V > 0$ and radial unboundedness, and the third condition imposes that $V$ is a Lyapunov-Krasovksii functional. Therefore the equilibrium of the system given by (12) is globally stable independent of delay. If $\psi_2(x(t)) > 0$ then the zero steady-state is globally asymptotically stable independent of delay. ■

The functional $W_k$ is the functional given by Equation (11), but here we use it to analyze stability of nonlinear systems. The functional $V$ is similar to (9).

In order to algorithmically construct the above certificates of stability, we consider the construction of bounded-degree *polynomial* (i.e., of any order instead of just *quadratic*) kernel Lyapunov-Krasovskii functionals, by formulating relevant sum of squares conditions. The three conditions then become polynomial non-negativity conditions that can be reduced to Sum of Squares conditions which can be tested using SOSTOOLS. The functions $\varphi_1$ and $\varphi_2$, and possibly $\psi_1$ and $\psi_2$ can be constructed using Proposition 5. Here is a simple example of how this is done.

*Example 8:* Consider the system:

$$\dot{x}_1 = -x_1(t) + x_2^2(t-\tau), \quad \dot{x}_2 = -x_2(t)$$

This system is delay-independent stable, and we prove this by constructing a Lyapunov functional of the form $V_2$ (given by 14) with $a_2$ and $b_2$ polynomials of bounded degree - note that now $x(t) = [x_1(t), x_2(t)]^T$. When $a_2$ and $b_2$ are second order polynomials, no certificate is found. However, when their order is increased, a certificate of stability is obtained. In fact the two conditions become

$$\begin{aligned} V(x_t) &= x_2^2(t) + \frac{3}{4}x_1^2(t) + (0.5x_1(t) + x_2^2(t))^2 + \int_{-\tau}^{0} x_2^4(t+\theta)d\theta. \\ -\dot{V}(x_t) &= (x_1(t) + x_2^2(t) - x_2^2(t-\tau))^2 + 2x_2^2(t) + \\ &\quad + x_2^2(t)x_2^2(t-\tau) + 2(x_2^2(t) + \frac{1}{4}x_1(t))^2 + \frac{14}{16}x_1(t)^2. \end{aligned}$$

To get this certificate, only a handful of SOSTOOLS commands are required.

In the next two subsections, we will concentrate on local stability and robust stability analysis for delay-independent stability.

*1) Local Stability:* Nonlinear systems may have more than one equilibria, or the stability properties of an equilibrium may not be global. In order to obtain a local result, we have to use the region $\Omega$ in Theorem 2. For this, we define the set:

$$\Omega = \left\{ x_t \in C : \|x_t\| = \sup_{-\tau \leq \theta \leq 0} |x(t+\theta)| \leq \gamma \right\}.$$

In particular this means that $|x(t+\theta)| \leq \gamma, \ \forall \ \theta \in [-\tau, 0]$, where $|\cdot|$ is the $\infty$-norm. In effect, this gives rise to the following conditions:

$$\begin{aligned} h_{1i} &:= (x_i(t) - \gamma)(x_i(t) + \gamma) \leq 0, \quad i = 1, \ldots, n \\ h_{2i} &:= (x_i(t-\tau) - \gamma)(x_i(t-\tau) + \gamma) \leq 0, \quad i = 1, \ldots, n \end{aligned}$$

Having captured the set $\Omega$ using the above inequality constraints, we can formulate conditions for stability that impose validity of the corresponding Lyapunov conditions when $x_t \in \Omega$, i.e., when $|x(t+\theta)| \leq \gamma, \quad \forall \ \theta$.

We have the following result:

*Proposition 9:* Let 0 be a steady-state of system (12), and let there exist functions $V_0(x(t))$ and $V_1(x(t))$, a positive definite function $\varphi(x(t))$, a non-negative function $\psi(x(t))$ and non-negative functions $p_i(x(t))$, $q_{1i}(x(t), x(t-\tau))$ and $q_{2i}(x(t), x(t-\tau))$, $i = 1, \ldots, n$ such that:

1) $V_0(x(t)) - \varphi(x(t)) + \sum_{i=1}^n p_i h_{1i} \geq 0$,
2) $V_1(x(t+\theta)) \geq 0$,
3) $-\frac{\partial V_0}{\partial x(t)} f - V_1(x(t)) + V_1(x(t-\tau)) - \psi_2(x(t)) + \sum_{i=1}^n (q_{1i} h_{1i} + q_{2i} h_{2i}) + \psi(x(t)) \geq 0$.

Then 0 is delay-independent stable. If moreover $\psi(x(t)) > 0$, then 0 is delay-independent asymptotically stable.

*Proof:* Consider the functional given by (14). While $x(t)$ satisfies $h_{1i} \leq 0$ and $p_i(x(t))$ is a SOS we have:

$$V(x_t) = V_0(x(t)) + \int_{-\tau}^{0} V_1(x(t+\theta)) d\theta \geq \varphi(x(t)) - \sum_i p_i h_{1i} > 0,$$

and so the first Lyapunov condition is satisfied, i.e. $V > 0$ on $\Omega$. The same is true for the derivative condition, given constraint (3) above, and so the zero steady-state of system (12) is locally delay-independent stable. If $\psi > 0$, then

$$-\frac{dV}{dt} > 0 \text{ on } \Omega$$

and so asymptotic stability is concluded. ■

A similar proposition can be written using the structure (13). Having obtained a Lyapunov function that is valid locally and proves asymptotic stability, the domain of attraction of the equilibrium can also be estimated as the maximal level set of $V$ that is contained in $\Omega$. This can also be formulated as a SOS programme, but it is beyond the scope of this paper.

*2) Robust Stability:* Another important issue is robust stability under parametric uncertainty, which can be treated in a unified way as we will see in the sequel. Consider a time-delay system of the form (1) with an uncertain parameter $p$:

$$\dot{z}(t) = f(z_t, p), \tag{15}$$

where $p \in P$, where $P$ is given by

$$P = \{p \in \mathbb{R}^m \,|\, q_i(p) \leq 0,\ i = 1, \ldots, N\}, \tag{16}$$

i.e., the uncertainty set is captured by certain inequalities. Let $x(t) = z(t) - z^*$, where $z^*$ is the steady-state of (15), which satisfies $f(z^*, p) = 0$, i.e., it is allowed to change as the parameters $p \in P$ vary. Then we have:

$$\dot{x}(t) = f(x_t + z^*, p) \tag{17}$$

$$0 = f(z^*, p) \tag{18}$$

This system has a steady-state $x^*$ at the origin. We assume for simplicity that there is only one equilibrium whose stability can be tested by constructing a *Parameter Dependent* Lyapunov functional – see the remark at the end of this section for systems with multiple equilibria. The robust stability properties of the above system can be tested using the following proposition:

*Proposition 10:* Consider the system given by (17), where $p \in P$ as defined by (16). Suppose that there exist functions $V_0(x(t), p)$ and $V_1(x(t+\theta), p)$, a positive definite radially unbounded function $\varphi(x(t))$ and a non-negative function $\psi(x(t))$ such that the following conditions hold for $p \in P$:

1) $V_0(x(t), p) - \varphi(x(t)) \geq 0$,
2) $V_1(x(t+\theta), p) \geq 0$,
3) $\frac{dV_0}{dx(t)} f + V_1(x(t), p) - V_1(x(t-\tau), p) + \psi(x(t)) \leq 0$, when (18) is satisfied.

Then the steady-state $0$ of the system given by (17–18) is robustly globally delay-independent stable for all $p \in P$. Moreover, if $\psi(x(t)) > 0$, 0 is delay-independent robustly globally asymptotically stable for all $p \in P$.
The proof is based on functional

$$V(x_t, p) = V_0(x(t), p) + \int_{-\tau}^{0} V_1(x(t+\theta), p) d\theta \tag{19}$$

which is modified from (14).

Again, if we assume that all functions involved are polynomial functions, the condition $p \in P$ can be adjoined to conditions (1)–(3) above using sum of squares multipliers and tested using SOSTOOLS. Similarly, the condition (18) can be adjoined using polynomial multipliers. In particular the following Sum of Squares program can be written, which ensures robust asymptotic stability independent of delays.

Find polynomials $V_0(x(t), p)$ and $V_1(x(t+\theta), p)$

and $\varphi(x(t))$ and $\psi(x(t))$ constructed using Proposition 5

and sum of squares $\sigma_{1i}(x(t), p), \sigma_{2i}(x(t+\theta), p)$ and $\sigma_{3i}(x(t), x(t-\tau), p)$

and a polynomial $r(x(t), x(t-\tau), p)$

Such that:

$$V_0(x(t), p) - \varphi(x(t)) + \sum_i \sigma_{1i}(x(t), p) q_i(p) \text{ is a SOS} \tag{20}$$

$$V_1(x(t+\theta), p) + \sum_i \sigma_{2i}(x(t+\theta), p) q_i(p) \text{ is a SOS} \tag{21}$$

$$\begin{aligned} &- \frac{dV_0}{dx(t)} f - V_1(x(t), p) + V_1(x(t-\tau), p) + \psi(x(t)) + \sum_i \sigma_{3i}(x(t), x(t-\tau), p) q_i(p) \\ &\qquad + r(x(t), x(t-\tau), p) f(z^*, p) \text{ is a SOS} \end{aligned} \tag{22}$$

*Remark 11:* Sometimes there are more than one equilibria in 15, that move as $p$ is allowed to vary in $P$ and the result we seek in that case is a local result. In this case, the parameter set $p$ should be extended to include the uncertainty in the equilibrium $z^*$ of interest and a region $\Omega$ has to be defined appropriately. Examples of how this can be used can be found in Section V.

*B. Delay-dependent stability*

When the stability properties of the equilibrium change as the delay size, seen as a static parameter, changes, the stability is termed *delay-dependent*. In this case, a different type of Lyapunov functionals has to be used to allow for the delay size to appear explicitly in the SOS conditions.

Recall that for a linear time-delay system of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i)$$

where $A_i$ are fixed matrices, the complete Lyapunov structure for delay-dependent stability is:

$$\begin{aligned} V(x_t) &= x^T(t)Px(t) + 2x^T(t)\int_{-\tau}^{0} P_1(\theta)x(t+\theta)d\theta \\ &+ \int_{-\tau}^{0}\int_{-\tau}^{0} x^T(t+\theta)P_2(\theta,\xi)x(t+\xi)d\xi d\theta + \int_{-\tau}^{0} x^T(t+\theta)Qx(t+\theta)d\theta \end{aligned} \quad (23)$$

where appropriate conditions on $P, P_1, P_2$ and $Q$ have to be imposed, and $\tau = \max_i \tau_i$. The derivative condition gives rise to Paramererized LMIs (PLMIs) that are NP-hard to solve; see [8] for the general setup and how these PLMIs can be solved algorithmically using SOSTOOLS [6].

In this paper we will construct Lyapunov-Krasovskii functionals for delay-dependent stability of nonlinear time delay systems. We will consider kernels that are polynomials, and we will use the Sum of Squares decomposition to construct them - the structures will resemble the structure of (23). We have the following result, again referring to the system given by (1).

*Proposition 12:* Let 0 be a steady-state for the system given by (1). Let there exist functions $V_0(x(t))$, $V_1(\theta, x(t), x(t+\theta))$ and $V_2(x(\zeta))$, a positive definite, radially unbounded function $\varphi(x(t))$, a non-negative function $\psi(x(t))$, and functions $t_1(x(t),\theta)$ and $t_2(x(t),\theta)$ such that:

1) $V_0(x(t)) - \varphi(x(t)) \geq 0$,
2) $V_1(\theta, x(t), x(t+\theta)) + t_1(x(t),\theta) \geq 0$ for $\theta \in [-\tau, 0]$,
3) $V_2(x(\zeta)) \geq 0$,
4) $\tau \frac{\partial V_1}{\partial x(t)} f + \frac{dV_0}{dx(t)} f - \tau \frac{\partial V_1}{\partial \theta} + \tau V_2(x(t)) - \tau V_2(x(t+\theta)) + V_1(0, x(t), x(t)) - V_1(-\tau, x(t), x(t-\tau)) + \psi(x(t)) + t_2(x(t),\theta) \leq 0$ for $\theta \in [-\tau, 0]$,
5) $\int_{-\tau}^{0} t_1(x(t),\theta) = 0, \quad \int_{-\tau}^{0} t_2(x(t),\theta) = 0.$

Then the steady-state $0$ of the system given by (1) is *globally stable* for delay size $\tau$. Moreover, if $\psi(x(t)) > 0$, then 0 is *globally asymptotically stable* for delay size $\tau$.

*Proof:* Consider the following functional:

$$V(x_t) = V_0(x(t)) + \int_{-\tau}^{0} V_1(\theta, x(t), x(t+\theta))d\theta + \int_{-\tau}^{0}\int_{t+\theta}^{t} V_2(x(\zeta))d\zeta d\theta \qquad (24)$$

Integrating the second and third conditions and adding the first condition to get that $V(x_t) \geq \varphi(x(t))$: therefore the first Lyapunov condition is satisfied. The time derivative of $V(x_t)$ is:

$$\begin{aligned}\dot{V}(x_t) &= \frac{dV_0}{dx(t)}f + V_1(0, x(t), x(t)) - V_1(-\tau, x(t), x(t-\tau)) \\ &+ \int_{-\tau}^{0}\left(\frac{\partial V_1}{\partial x(t)}f - \frac{\partial V_1}{\partial \theta} + V_2(x(t)) - V_2(x(t+\theta))\right)d\theta \\ &= \frac{1}{\tau}\int_{-\tau}^{0}\left\{\begin{array}{c}\frac{dV_0}{dx(t)}f + \tau\frac{\partial V_1}{\partial x(t)}f - \tau\frac{\partial V_1}{\partial \theta} + V_1(0, x(t), x(t)) - V_1(-\tau, x(t), x(t-\tau)) \\ +\tau V_2(x(t)) - \tau V_2(x(t+\theta))\end{array}\right\}d\theta.\end{aligned}$$

Condition 4) in the proposition above states that the kernel of the above integral is nonpositive for $\theta \in [-\tau, 0]$. So (24) is a Lyapunov functional, and the zero steady-state is *stable*. Since there is no constraint on the state-space, and $\varphi$ is radially unbounded, the result holds globally. Moreover if $\psi > 0$, then the steady-state is asymptotically stable. ■

The first term in the functional is added to impose positive definiteness of $V$ and the last term is added for convenience, as it will be used in the derivative condition to 'complete the squares'. The above proposition imposes sufficient conditions to test stability properties of the zero steady-state. Here the functions $t_1$ and $t_2$ are so as to avoid imposing the non-negativity positions for the kernel of $V$ pointwisely, but rather allow it to become negative over time, but integrate to a non-negative value.

In order to algorithmically construct the Lyapunov-Krasovskii functional for the nonlinear system, we can use the above proposition in a similar way as described in the delay-independent case. All functions are assumed to be bounded degree polynomials, and the $\varphi > 0$ function (and $\psi$, in case of asymptotic stability) is constructed using Proposition 5. To impose the conditions $\theta \in [-\tau, 0]$, we use a process similar to the S-procedure, as explained in the previous section, using Sum of Squares multipliers to adjoin them. Then we get four SOS conditions in a relevant Sum of Squares programme which can be solved using SOSTOOLS [13]. Different Lyapunov-Krasovskii structures can also be used, that may have better properties. For example the complete Lyapunov-Krasovksii functional for linear time-delay systems (23) can be used instead.

An important issue, that is unique in the case of delay-dependent stability, is to ensure that the stability properties hold for a delay interval rather than for a specific value of the delay. The conditions in Proposition 12 are not affine in $\tau$ (that would depend on the structure of $V_1$). One can consider, however the $\tau$ as a static parameter, itself being allowed to vary within the interval $\tau \in [0, \overline{\tau}]$. In that case, one can consider the problem in a robustness setting and construct a Lyapunov function that guarantees delay-dependent stability for a delay interval.

Similar arguments allow the construction of Lyapunov-Krasovskii functionals for *local* delay-dependent stability. A modified version of Proposition 9 can be developed. We will still need to specify $\Omega = \{x_t \in C : \|x_t\| \leq \gamma\}$, and adjoin the relevant conditions on $x(t)$, $x(t-\tau)$ and $x(t+\theta)\ \forall\ \theta \in [-\tau, 0]$ to the relevant kernels of the Lyapunov functionals using the extended S-procedure. The examples that will follow will illustrate how this is done in practice. Robust stability can also be dealt with in a unified manner, in a similar way as it was done for the delay-independent case.

In the next section we turn to the problem of investigating stability for nonlinear time-delay systems with multiple delays.

### C. Multiple-Delay Case

The presence of multiple, many times incommensurate delays in a functional differential equation causes significant complications in analysis. Necessary and sufficient conditions for stability in this case are very difficult to test, and the reason lies in the heart of computational complexity theory — the problem of deciding stability is NP-hard, i.e., there is no known polynomial-time algorithm that can decide whether the system is stable or not, especially as the number of delays increases and the 'boundary' of the stability/instability as a function of the delay sizes is approached. This of course does not mean that for certain delay values, the stability property cannot be verified easily, by considering simple certificates: this is the approach we follow in this section, in which we propose a test for incommensurate delays.

Consider the following form of functional differential equation which contains multiple discrete delays.

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_K)) \triangleq f(x_t) \tag{25}$$

We assume that $0 \leq \tau_1 \leq \ldots \leq \tau_K$, i.e., the delays are put in non-decreasing order, with $\tau_0 = 0$, and $f$ is polynomial in its arguments. Again we assume that 0 is the equilibrium for this system. Sufficient conditions for stability can then be found in the following proposition:

*Proposition 13:* Suppose there exist functions $V(x(t))$, $V_{1i}(\theta, x(t), x(t+\theta))$, $V_{2i}(x(\zeta))$, $i = 1,\ldots,K$, a positive definite radially unbounded function $\varphi_1(x(t)) > 0$ and a non-negative function $\varphi_2(x(t))$ such that:

1) $\frac{1}{\tau_K} V_0(x(t)) + \sum_{j=i}^{K} V_{1j}(\theta, x(t), x(t+\theta)) - \varphi_1(x(t)) \geq 0$, for all $\theta \in [-\tau_i, -\tau_{i-1}]$, for each $i = 1,\ldots,K$,
2) $V_{2i}(x(\zeta)) \geq 0$ for all $i = 1,\ldots,K$,
3) $\frac{1}{\tau_K} \frac{\partial V_0}{\partial x(t)} f + \sum_{j=i}^{K} \left( \frac{\partial V_{1j}}{\partial x(t)} f - \frac{\partial V_{1j}}{\partial \theta} \right) + V_{2i}(x(t)) - V_{2i}(x(t+\theta)) + \frac{1}{\tau_i - \tau_{i-1}} \sum_{j=i}^{K} V_{1j}(-\tau_{i-1}, x(t), x(t-\tau_{i-1})) - \frac{1}{\tau_i - \tau_{i-1}} \sum_{j=i}^{K} V_{1j}(-\tau_i, x(t), x(t-\tau_i)) + \varphi_2(x(t)) \leq 0$ for all $i = 1,\ldots,K$.

Then the steady-state is globally stable. If $\phi_2(x(t)) > 0$, then the steady-state is globally asymptotically stable.

*Proof:* The proof of this proposition is similar to the one given in the earlier section. It is based on ensuring that the following functional is a Lyapunov-Krasovksii functional:

$$\begin{aligned} V(x_t) &= V_0(x(t)) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} V_{1i}(\theta, x(t), x(t+\theta)) d\theta + \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \int_{t+\theta}^{t} V_{2i}(x(\zeta)) d\zeta d\theta \\ &= \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} \left( \frac{1}{\tau_K} V_0(x(t)) + \sum_{j=i}^{K} V_{1j}(\theta, x(t), x(t+\theta)) + \int_{t+\theta}^{t} V_{2i}(x(\zeta)) d\zeta \right) d\theta \end{aligned} \tag{26}$$

Indeed the first three conditions ensure that $V(x_t) > 0$, so the first Lyapunov condition is satisfied and moreover that $V$ is radially unbounded.

The derivative of this functional along the trajectories of (25) is:

$$\begin{aligned} \dot{V}(x_t) = \sum_{i=1}^{K} \int_{-\tau_i}^{-\tau_{i-1}} &\left( \frac{1}{\tau_K} \frac{\partial V_0}{\partial x(t)} f + \sum_{j=i}^{K} \left( \frac{\partial V_{1j}}{\partial x(t)} f - \frac{\partial V_{1j}}{\partial \theta} \right) + V_{2i}(x(t)) - V_{2i}(x(t+\theta)) \right) d\theta \\ &+ \sum_{i=1}^{K} \sum_{j=i}^{K} \left( V_{1j}(-\tau_{i-1}, x(t), x(t-\tau_{i-1})) - V_{1j}(-\tau_i, x(t), x(t-\tau_i)) \right) \end{aligned}$$

The non-positivity of this time derivative is ensured by the 4th condition, and so $V(x_t)$ is a Lyapunov-Krasovskii functional that proves global stability of the equilibrium. If $\phi_2(x(t)) > 0$, then $\dot{V}(x_t) < 0$ and the steady-state is globally asymptotically stable. ■

It is sometimes beneficial to consider other criteria for the stability analysis, such as combining all the above derivative conditions into one in more variables. For example, we can consider another Lyapunov function of the form:

$$V(x_t) \;=\; V_0(x(t)) + \sum_{i=1}^{K}\int_{-\tau_i}^{0} V_{1i}(\theta_i, x(t), x(t+\theta_i))d\theta_i + \sum_{i=1}^{K}\int_{-\tau_i}^{-\tau_{i-1}}\int_{t+\theta_i}^{t} V_{2i}(x(\zeta))d\zeta d\theta_i \tag{27}$$

This can serve as a Lyapunov function under the following proposition:

*Proposition 14:* Suppose there exist functions $V(x(t))$, $V_{1i}(\theta_i, x(t), x(t+\theta_i))$, $V_{2i}(x(\zeta))$, $i = 1, \ldots, K$, a positive definite radially unbounded function $\varphi_1(x(t))$ and a non-negative function $\varphi_2(x(t))$ such that:

1) $V_0(x(t)) - \phi_1(x(t)) \geq 0$
2) $\sum_{i=1}^{K} \tau_i V_{1j}(\theta_i, x(t), x(t+\theta_i)) \geq 0$, for all $\theta_i \in [-\tau_i, 0]$,
3) $V_{2i}(x(\zeta)) \geq 0$ for all $i = 1, \ldots, K$,
4) $\frac{\partial V_0}{\partial x(t)} f + \sum_{i=1}^{K} \tau_i \left(\frac{\partial V_{1i}}{\partial x(t)} f - \frac{\partial V_{1i}}{\partial \theta_i} + V_{2i}(x(t)) - V_{2i}(x(t+\theta_i))\right) + \sum_{i=1}^{K} V_{1j}(0, x(t), x(t)) - V_{1j}(-\tau_i, x(t), x(t-\tau_i)) + \phi_2(x(t)) \leq 0$ for all $i = 1, \ldots, K$ and $\theta_i \in [-\tau_i, 0]$

Then the steady-state is globally stable. If $\phi_2(x(t)) > 0$, then the steady-state is globally asymptotically stable.

The proof of this proposition is omitted, as it is similar to the one given earlier.

In Section V, we will see an example of a system from network congestion control for the Internet, where the inhomogeneous communication delays are taken into account in the stability test.

## V. Examples

In this section we present two examples, one from population dynamics, and one from network congestion control for the Internet.

### *A. Stability analysis of a predator-prey model*

A simple model of predator-prey interactions is

$$\dot{x} = bx - k_1 xy, \quad \dot{y} = k_2 xy - \sigma y,$$

where $x$ and $y$ are the prey and predator populations, $b$ is the rate of increase of prey, $k_1$ and $k_2$ are the coefficients of the effect of predation on $x$ and $y$ and $\sigma$ is the death rate of $y$. So the cause of death of the prey is due to predation alone, and the growth of the predator population has as the only limitation the number of prey. These equations give rise to Lotka-Volterra predator-prey cycles, but the model is not biologically meaningful because it is *conservative* giving rise to a family of closed trajectories rather than a single limit cycle [18].

The above equations describe ideal populations that can react instantaneously to any change in the environment; in real populations this change comes with a delay that represents *maturation* of the predator population. A more realistic set of equations is [19]:

$$\dot{x}(t) = x(t)[b - ax(t) - k_1 y(t)],$$
$$\dot{y}(t) = -\sigma y(t) + k_2 x(t-\tau) y(t-\tau),$$

where $-ax(t)^2$ limits the growth of the prey, and $\tau \geq 0$ is a constant capturing the average period between death of prey and birth of a subsequent number of predators.

*Assumption 15:* $a, b, k_1, k_2$ and $\sigma$ are positive.

The equilibria $(x^*, y^*)$ of the above system are:

$$(x^*, y^*) = (0,0), \quad (x^*, y^*) = (b/a, 0),$$
$$(x^*, y^*) = \left(\frac{\sigma}{k_2}, \frac{bk_2 - a\sigma}{k_1 k_2}\right). \tag{28}$$

We are only interested in the steady-state given by (28).

*Assumption 16:* $(bk_2 - a\sigma) > 0$.

Assumption 16 ensures that the steady-state (28) is in the first quadrant. We now shift the coordinates to $(x_1, x_2) = (x - \frac{\sigma}{k_2}, y - \frac{bk_2 - a\sigma}{k_1 k_2})$ to get:

$$\dot{x}_1(t) = \left[x_1(t) + \frac{\sigma}{k_2}\right][-ax_1(t) - k_1 x_2(t)] \tag{29}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1} x_1(t-\tau) + k_2 x_1(t-\tau) x_2(t-\tau) \tag{30}$$

We can linearize the above system about $(0,0)$ to get:

$$\dot{x}_1(t) = \frac{\sigma}{k_2}[-ax_1(t) - k_1 x_2(t)] \tag{31}$$

$$\dot{x}_2(t) = -\sigma x_2(t) + \sigma x_2(t-\tau) + \frac{bk_2 - a\sigma}{k_1} x_1(t-\tau) \tag{32}$$

For the linearised system, we have the following result:

*Proposition 17:* Consider the system (31–32) under the assumptions (15,16). Then if $(bk_2 - 3a\sigma) < 0$ the zero steady-state is stable independent of the delay. If $(bk_2 - 3a\sigma) > 0$ the zero steady-state is stable if the delay satisfies $\tau < \tau^*$ and is unstable otherwise, where $\tau^*$ is given by:

$$\tau^* = \frac{1}{\omega} \text{ atan } \left[\omega \frac{(a\sigma^2 - \omega k_2)k_2 - \sigma(2a\sigma + bk_2)(k_2 + a)}{k_2\sigma\omega^2(k_2 + a) + (2a\sigma - bk_2)(a\sigma^2 - \omega k_2)}\right]$$

where $\omega$ solves

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{33}$$

*Proof:* In the absence of delay, and under the two assumptions, the system is *asymptotically stable*. Substituting $s = j\omega$ in the characteristic equation and separating real and imaginary parts we get:

$$\begin{aligned} -\omega^2 + \frac{a\sigma^2}{k_2} &= \sigma\omega \sin(\omega\tau) + \sigma\left(\frac{2a\sigma}{k_2} - b\right)\cos(\omega\tau) \\ \sigma\left[1 + \frac{a}{k_2}\right]\omega &= \sigma\omega \cos(\omega\tau) - \sigma\left(\frac{2a\sigma}{k_2} - b\right)\sin(\omega\tau) \end{aligned}$$

Squaring the two equations and adding we get:

$$\omega^4 + \frac{a^2\sigma^2}{k_2^2}\omega^2 + \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2) = 0. \tag{34}$$

Denoting $p_1 = \frac{a^2\sigma^2}{k_2^2}$ and $p_2 = \frac{\sigma^2}{k_2^2}(bk_2 - a\sigma)(3a\sigma - bk_2)$ the roots of this equation are:

$$\omega^2 = -\frac{p_1}{2} \pm \frac{\sqrt{p_1^2 - 4p_2}}{2}. \tag{35}$$

Under assumption 16, if $(bk_2 - 3a\sigma) < 0$ (i.e $p_2 > 0$), then there are no real solutions to (34). Since the steady-state is stable when the delay is zero, and there is no $\omega$ for which poles cross to the RHP, we conclude that (31–32) is delay-independent stable.

Under assumption 16 and $(bk_2 - 3a\sigma) > 0$ then $p_2 < 0$ and one of the two roots of (35) is positive and the other one is negative. Therefore the poles cross the imaginary axis at only one $\omega$ — there is no possibility for *stability reversal*. If $\omega$ is the solution to the above equation, then at $\tau = \tau^*$ given in the statement of the Proposition a Hopf bifurcation occurs; the system is stable for $\tau < \tau^*$ and unstable for $\tau > \tau^*$. ■

We now analyze the nonlinear description of the system (29–30) using the methodology that was developed in the previous sections. We choose as nominal values for the parameters $\sigma = 10$, $a = 1$, $k_1 = 1$, and $k_2 = 3$.

*1) Delay-independent stability analysis:* The system (29–30) has many equilibria, and so we need to define a region around the zero steady-state to obtain a stability condition (this is the region $\Omega$ in Theorem 2). We let:

$$|x_{1_t}| \leq \gamma_1 x^*, \quad |x_{2_t}| \leq \gamma_2 y^*, \tag{36}$$

where the steady-state $(x^*, y^*)$ is given by (28). We consider $b$ to be a parameter in the problem. From Proposition 17, the linear version of this system is delay-independent stable when $\frac{a\sigma}{k_2} < b < \frac{3a\sigma}{k_2}$. For the given values of $a, \sigma$ and $k_2$, the system is delay-independent stable for $10/3 < b < 10$. For the purpose of calculating $(x^*, y^*)$ we use a value of $b = 20/3$. The steady-state $(0, 0)$ of system (29–30) does not move as $b$ changes, however the other two equilibria cross through the region defined by (36). If we choose $\gamma_1 = \gamma_2 = 0.1$, then no other steady-state enters this region for $11/3 < b < 10$.

We consider the following Lyapunov structure:

$$V(x_t) = V_0(x_1(t), x_2(t), b) + \int_{-\tau}^{0} V_1(x_1(t+\theta), x_2(t+\theta), b)d\theta.$$

We use a variant of Proposition 10 to obtain parameter regions for which robust delay-independent stability of the origin can be proven. When the order of $V_0$ is second order and $V_1$ is 4th order, we can construct $V(x_t)$ for $4.56 \leq b \leq 7.11$. When they are 4th order and 6th order respectively, then this region becomes $3.67 \leq b \leq 9.95$, which is essentially the full interval.

*2) Delay-dependent stability analysis:* Now we will try to test values for $\tau$ for which stability is retained using the same parameters as before and fixed $b = 15$. Given these parameters $\tau^* = 0.0541$. The system has several equilibria and so we use the same constraints on $x_1$ and $x_2$ on the state-space given by (36) with $\gamma_1 = \gamma_2 = 0.1$.

We can construct the Lyapunov functional $V(x_t)$ given by (24) with $V_1$ 0th order with respect to $\theta$ and 2nd order with respect to the rest of the variables for $\tau = 0.04$. When $V_1$ is quartic with respect to all variables but $\theta$ (which is kept at 0 order) then we can construct this $V(x_t)$ for $\tau = 0.053$. The corresponding SDP is bigger as the functional is more complicated, but we can see that larger values of the delay closer to the stability boundary can be tested.

### *B. Network Congestion Control for the Internet*

Internet congestion control [3] is a distributed algorithm to allocate available bandwidth to competing sources so as to avoid congestion collapse by ensuring that link capacities are not

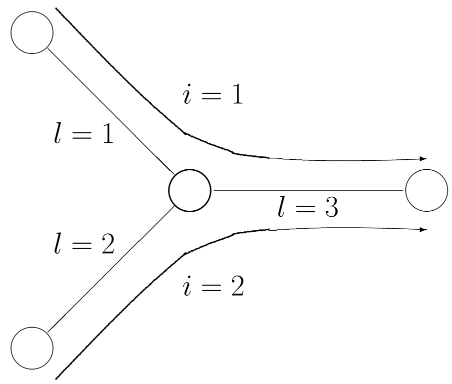


Fig. 1. A network topology under consideration.

exceeded. The need for congestion control for the Internet emerged in the mid-1980s, when congestion collapse resulted in unreliable file transfer. In 1988 Jacobson [20] proposed an admittedly ingenious scheme for congestion control. The shortcomings of this scheme and its successors such as TCP Reno and Vegas have only recently become apparent: they are not scalable to arbitrary networks with very large capacities and multiple, non-commensurate time-delays. New designs of Active Queue Management (AQM) and/or Transmission Control Protocol (TCP) dynamics have been proposed that provide scalable stability in the presence of heterogeneous delays, which can be verified at least for the linearization about a steady-state.

The simplest adequate modeling framework for network congestion control is in the form of nonlinear deterministic delay-differential equations [21], [22], [23]. Some work has been done on the analysis of such systems usually for the single-bottleneck link case, using either Lyapunov-Razumikhin functions or Lyapunov-Krasovskii functionals, IQC methods, passivity etc.

In this section we analyze a simple network instantiation of what is known a ‘primal’ congestion control scheme, shown in Figure 1. It consists of $L = 3$ links, labeled $l = 1, 2, 3$ and $S = 2$ sources, $i = 1, 2$. For this network, we define an $L \times S$ routing matrix $R$ which is 1 if source $i$ uses link $l$ and 0 otherwise:

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \tag{37}$$

The architecture of network congestion control is shown in Figure 2. To each source $i$ we associate a transmission rate $x_i$. All sources whose flow passes through resource $l$ contribute to the *aggregate rate* $y_l$ for resource $l$, the rates being added with some forward time delay $\tau^f_{i,l}$. Hence we have:

$$y_l(t) = \sum_{i=1}^{S} R_{li} x_i(t - \tau^f_{i,l}) \triangleq r_f(x_i, \tau^f_{i,l}) \tag{38}$$

The resources $l$ react to the aggregate rate $y_l$ by setting congestion information $p_l$, the price at resource $l$. This is the Active Queue Management part of the picture. The prices of all the links that source $i$ uses are aggregated to form $q_i$, the *aggregate price* for source $i$, again through a delay $\tau^b_{i,l}$:

$$q_i(t) = \sum_{l=1}^{L} R_{li} p_l(t - \tau^b_{i,l}) \triangleq r_b(p_l, \tau^b_{i,l}) \tag{39}$$

The prices $q_i$ can then be used to set the rate of source $i$, $x_i$, which completes the loop. The forward and backward delays can be combined to yield the Round Trip Time (RTT):

$$\tau_i = \tau^f_{i,l} + \tau^b_{i,l}, \quad \forall\ l \tag{40}$$

The capacity of link $l$ is given by $c_l$. For a general network, the interconnection is shown in Figure 2. In this section we choose the control laws for TCP and AQM as follows:

$$\begin{aligned} p_l(t) &= \left(\frac{y_l(t)}{c_l}\right)^B \\ \dot{x}_i(t) &= 1 - x_i(t-\tau_i) q_i(t) \end{aligned}$$

Here $p_l(t)$ corresponds to the probability that the queue length exceeds B in a $M/M/1$ queue with arrival rate $y_l(t)$ and capacity $c_l$, and the source law corresponds to a queue length with proportionally fair source dynamics.

For the network shown in Figure 1, the closed loop dynamics become:

$$\begin{aligned} \dot{x}_1 &= 1 - x_1(t-\tau_1)\left[\left(\frac{x_1(t-\tau_1)}{c_1}\right)^B + \left(\frac{x_1(t-\tau_1) + x_2(t-\tau^b_{1,3}-\tau^f_{2,3})}{c_3}\right)^B\right] \\ \dot{x}_2 &= 1 - x_2(t-\tau_2)\left[\left(\frac{x_2(t-\tau_2)}{c_2}\right)^B + \left(\frac{x_2(t-\tau_2) + x_1(t-\tau^f_{1,3}-\tau^b_{2,3})}{c_3}\right)^B\right] \end{aligned}$$

Let $B = 2$, $c_1 = c_2 = 3$, and $c_3 = 1$. Then the steady-state of this system is $(x_1^*, x_2^*) = (0.6242, 0.6242)$. We consider delay sizes such that $\tau^b_{1,3} = 63ms$, $\tau^f_{1,3} = 93ms$, $\tau^f_{2,3} = 49ms$,

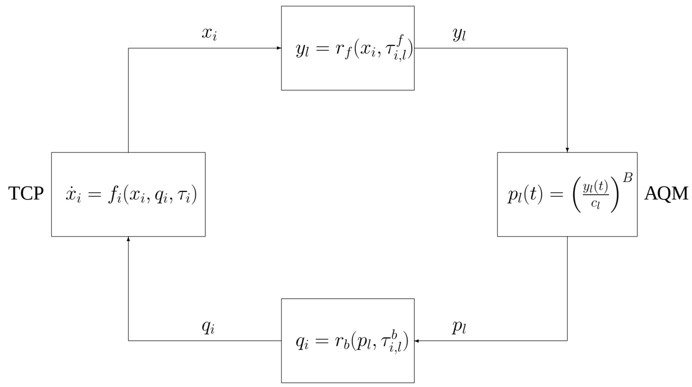


Fig. 2. The internet as an interconnection of sources and links through delays.

$\tau_{2,3}^b = 77ms$. The system equations about the new steady-state become:

$$
\begin{aligned}
\dot{x}_1' &= 1 - [x_1'(t-0.154) + x_1^*] \\
&\quad \left[ \left( \frac{x_1'(t-0.154) + x_1^*}{3} \right)^2 + (x_1'(t-0.154) + x_2'(t-0.14) + x_1^* + x_2^*)^2 \right] \\
\dot{x}_2' &= 1 - [x_2'(t-0.126) + x_2^*] \\
&\quad \left[ \left( \frac{x_2'(t-0.126) + x_2^*}{3} \right)^2 + (x_2'(t-0.126) + x_1'(t-0.14) + x_1^* + x_2^*)^2 \right]
\end{aligned}
$$

where $x_i(t) = x_i'(t) + x^*$. The linearization of this system about the steady-state is stable, as a Lyapunov function of the form (27) can be constructed. The same Lyapunov function can be constructed in a region of the state-space satisfying $\|x_{1_t}\| \leq 0.8x_1^*$ and $\|x_{2_t}\| \leq 0.8x_2^*$, thus showing that the equilibrium is nonlinearly stable.

## VI. Concluding Remarks

In this paper we presented a methodology to construct Lyapunov-Krasovskii functionals for time delay systems based on the Sum of Squares decomposition. The construction is entirely algorithmic and is achieved through the solution of a set of Linear Matrix Inequalities (LMIs). Linear and nonlinear delay-independent and delay-dependent stability can now be treated using the same tools.

The above methods can be easily extended to systems with many delays, either commensurate or not. Still a judicious choice for the structure of the Lyapunov functional would be required. Functional differential equations of neutral type can also be treated in a unified way.

# REFERENCES


[1] Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*. Academic Press, 1993. Mathematics in Science and Engineering (191).

[2] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*. Mathematics and its Applications (74), Kluwer Academic Publishers, 1992.

[3] R. Srikant, *The Mathematics of Internet Congestion Control*. Birkhäuser, 2003.

[4] V. B. Kolmanovskii, S.-I. Niculescu, and J.-P. Richard, "On the Lyapunov-Krasovskii functionals for stability analysis of linear delay systems," *International Journal of Control*, vol. 72, no. 4, pp. 374–384, 1999.

[5] K. Gu, "Discretised LMI set in the stability problem of linear uncertain time-delay systems," *International Journal of Control*, vol. 68, pp. 155–163, 1997.

[6] A. Papachristodoulou, M. Peet, and S. Lall, "Constructing Lyapunov-Krasovksii functionals for linear time delay systems," in *Proceedings of the American Control Conference*, 2005.

[7] J. K. Hale and S. M. V. Lunel, *Introduction to Functional Differential Equations*. Applied Mathematical Sciences (99), Springer-Verlag, 1993.

[8] K. Gu, V. L. Kharitonov, and J. Chen, *Stability of Time-Delay systems*. Birkhäuser, 2003.

[9] A. R. Teel, "Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 7, pp. 960–964, 1998.

[10] P.-A. Bliman, "Lyapunov equation for the stability of linear delay systems of retarded and neutral type," *IEEE Trans. Automat. Contr.*, vol. 47, no. 2, pp. 327–335, 2002.

[11] P. A. Parrilo, *Structured Semidefinite Programs and Semialgebraic Geometry Methods in Robustness and Optimization*. PhD thesis, California Institute of Technology, Pasadena, CA, 2000. Available at `http://www.control.ethz.ch/~parrilo/pubs/index.html`.

[12] K. G. Murty and S. N. Kabadi, "Some NP-complete problems in quadratic and nonlinear programming," *Mathematical Programming*, vol. 39, pp. 117–129, 1987.

[13] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "SOSTOOLS – Sum of Squares Optimization Toolbox, User's Guide." Available at `http://www.cds.caltech.edu/sostools` and `http://www.aut.ee.ethz.ch/~parrilo/sostools`, 2002.

[14] A. Papachristodoulou and S. Prajna, "On the construction of Lyapunov functions using the sum of squares decomposition," in *Proceedings IEEE Conference on Decision and Control*, 2002.

[15] A. Papachristodoulou and S. Prajna, "Analysis of non-polynomial systems using the sum of squares decomposition." In *Positive Polynomials in Control*, Springer-Verlag, 2005.

[16] F. Mazenc and S.-I. Niculescu, "Lyapunov stability analysis for nonlinear delay systems," *Systems and Control Letters*, vol. 42, pp. 245–251, 2001.

[17] P.-A. Bliman, "Stability of non-linear delay systems: delay-independent small gain theorem and frequency domain interpretation of the Lyapunov-Krasovskii method," *Int. J. Control*, vol. 75, no. 4, pp. 265–274, 2002.

[18] J. D. Murray, *Mathematical Biology*. Springer-Verlag, second ed., 1993.

[19] P. J. Wangersky and W. J. Cunningham, "Time lag in prey-predator population models," *Ecology*, vol. 38, no. 1, pp. 136–139, 1957.

[20] V. Jacobson, "Congestion avoidance and control," *Proceedings of SIGCOMM'88, ACM*, August 1988. An updated version is available via `ftp://ftp.ee.lbl.gov/papers/congavoid.ps.Z`.

[21] S. Shakkottai and R. Srikant, "How good are deterministic fluid models of internet congestion control," in *Proc. of IEEE Infocom,*, 2002.

[22] M. Mathis, J. Semske, J. Mahdavi, and T. Ott, "The macroscopic behavior of the TCP congestion avoidance algorithm," *Computer Communication Review*, 1997.

[23] J. Padhye, V. Firoiu, D. Towsley, and J. Kurose, "Modeling TCP throughput: A simple model and its empirical validation," in *ACM Sigcomm*, 2001.