Spacecraft Dynamics and Control

Matthew M. Peet
Arizona State University

Lecture 3: Elliptic Orbits

Introduction

In this Lecture, you will learn:

Geometry of the orbit

- How to use conservation of angular momentum and energy to derive geometric properties of the orbit
- The types of orbit
  - Elliptic
  - Parabolic
  - Hyperbolic

Different Representations

- How to convert between
  - Energy and Momentum
  - Position and velocity
  - geometric properties

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.

Recall Invariant Quantities

Energy and Angular Momentum

If we don't care about motion of the planet, the 2-body problem is actually a 1-body problem.

$$\ddot{\vec{r}} = -\frac{\mu}{\|\vec{r}\|^3}\vec{r}$$

Last lecture, we showed that the angular momentum vector,

$$\vec{h} = \vec{r} \times \dot{\vec{r}}$$

is invariant. i.e. $\dot{\vec{h}} = 0$.

Furthermore, the angular momentum vector is orthogonal to the orbital plane. That is

$$\vec{r}(t) \times \vec{h} = 0 \qquad \text{for all } t.$$

Definition 1.

The **Eccentricity Vector** can be defined as

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

Now we show that the eccentricity vector is invariant. Since we have already shown $\dot{h} = 0$, we get

$$\dot{\vec{e}} = \left(\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} - \frac{\dot{\vec{r}}}{\|\vec{r}\|} + \frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right)$$

Expanding the first term, we get

$$\frac{1}{\mu}\ddot{\vec{r}} \times \vec{h} = -\frac{\mu}{\|\vec{r}\|^3}\vec{r} \times \vec{r} \times \dot{\vec{r}}$$

We use the triple cross identity

$$a \times b \times c = (a \cdot c)b - (a \cdot b)c$$

Eccentricity Vector

Applying the triple cross identity,

$$\begin{aligned}\frac{1}{\mu}\ddot{\vec{r}}\times\vec{h} &= -\frac{\mu}{\|\vec{r}\|^3}\vec{r}\times\vec{r}\times\dot{\vec{r}}\\ &= -\frac{1}{\|\vec{r}\|^3}\left((\vec{r}\cdot\dot{\vec{r}})\vec{r}-(\vec{r}\cdot\vec{r})\dot{\vec{r}}\right)\\ &= -\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}\vec{r}+\frac{1}{\|\vec{r}\|}\dot{\vec{r}}\end{aligned}$$

Substituting,

$$\begin{aligned}\dot{\vec{e}} &= \left(\frac{1}{\mu}\ddot{\vec{r}}\times\vec{h}-\frac{\dot{\vec{r}}}{\|\vec{r}\|}+\frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right)\\ &= \left(-\frac{\vec{r}\cdot\dot{\vec{r}}}{\|\vec{r}\|^3}\vec{r}+\frac{1}{\|\vec{r}\|}\dot{\vec{r}}-\frac{\dot{\vec{r}}}{\|\vec{r}\|}+\frac{\vec{r}}{\|\vec{r}\|^3}\vec{r}^T\dot{\vec{r}}\right)=0\end{aligned}$$

Definition 2 (First Orbital Element).

The eccentricity of an orbit is

$$e=\|\vec{e}\|$$

Eccentricity

The eccentricity vector has several important properties.
Expand the dot product $\mu\vec{r}\cdot(\vec{e}-\vec{e})=0$

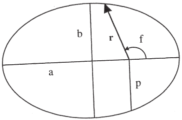
$$\vec{r}\cdot\left(\dot{\vec{r}}\times\vec{h}-\mu\frac{\vec{r}}{\|\vec{r}\|}-\vec{e}\mu\right)=\vec{r}\cdot(\dot{\vec{r}}\times h)-\mu\|\vec{r}\|-\mu\vec{r}\cdot\vec{e}$$
$$=\|\vec{h}\|^2-\mu\|\vec{r}\|-\mu\vec{r}\cdot\vec{e}=0$$

Since $\vec{r}\cdot(\dot{\vec{r}}\times h)=(\vec{r}\times\dot{\vec{r}})\cdot h$ and $\vec{r}\times\dot{\vec{r}}=\vec{h}$.

Definition 3 (Second Orbital Element).

The **True Anomaly**, $f(t)$ is the time-varying angle between $\vec{e}$ and $\vec{r}(t)$.

$$\cos f(t)=\frac{\vec{r}(t)\cdot\vec{e}}{\|\vec{r}(t)\|\|\vec{e}\|}$$



Eccentricity

Continuing, we get

$$\begin{aligned}
&\vec{r}\cdot(\dot{\vec{r}}\times h)-\mu\|\vec{r}\|-\mu\vec{r}\cdot\vec{e}\\
&=\|\vec{h}\|^2-\mu\|\vec{r}\|-\mu\|\vec{r}\|\|\vec{e}\|\cos f\\
&=\|\vec{h}\|^2-\mu\|\vec{r}\|\left(1+\|\vec{e}\|\cos f\right)\\
&=0
\end{aligned}$$

From which we get the important equation

$$\|\vec{r}(t)\|=\frac{\|\vec{h}\|^2}{\mu\left(1+\|\vec{e}\|\cos f(t)\right)}$$

or

$$r(t)=\frac{h^2}{\mu\left(1+e\cos f(t)\right)}$$

Is this a solution to the two-body problem?

Solution to the Two-Body Problem

Examine the equation

$$\|\vec{r}(t)\| = \frac{\|\vec{h}\|^2}{\mu\,(1+\|\vec{e}\|\cos f(t))} = \frac{h^2}{\mu\,(1+e\cos f(t))}$$

The **true anomaly**, f, is defined as the angle between $\vec{e}$ and $\vec{r}$. Since $\vec{e}$ is fixed, this equation gives the path on the orbital plane as f moves from $0\,\mathrm{deg}$ to $360\,\mathrm{deg}$ There are three cases

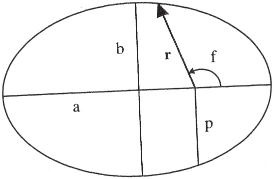


Figure: $\|\vec{e}\| < 1$

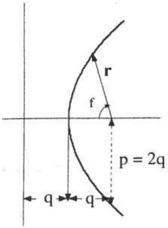


Figure: $\|\vec{e}\| = 1$

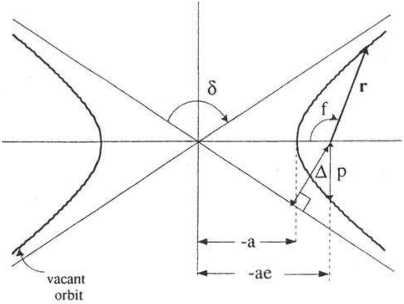


Figure: $\|\vec{e}\| > 1$

The case of Elliptic Orbits ($e < 1$)

Consider several points of

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

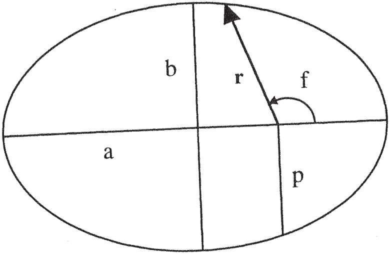
Case 1 ($f = \frac{\pi}{2}$):

- $\vec{r}$ is orthogonal to $\vec{e}$. $\vec{r}$ points up from the major axis, parallel to the minor axis.
- This length is called the *semi-latus rectum*, denoted by p.

$$p = \frac{h^2}{\mu}$$

Although difficult to measure, p is a useful quantity

- Directly represents h^2/μ.
- Also called "`the parameter`" of the ellipse.



The orbit equation is simplified to

$$r(t) = \frac{p}{1 + e\cos f}$$

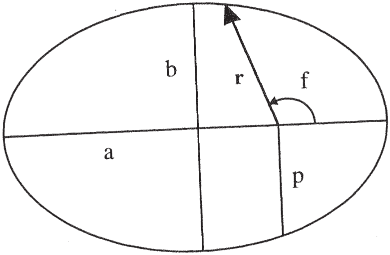
The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

Case 2 ($f = 0$): $\vec{r}$ is aligned with $\vec{e}$. The radius is *minimum*. Thus the radius at periapse is

$$r_p = \frac{h^2}{\mu(1+e)} = \frac{p}{1+e}$$

The point of minimum radius is also referred to as

- *Perigee* for orbits around Earth.
- *Perihelion* for orbits around the Sun.
- *Perilune* for lunar orbits.



The vector $\vec{e}$ always points toward the periapse. This orients the orbit in space. Note that $\|\vec{e}\| \neq r_p$.

The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

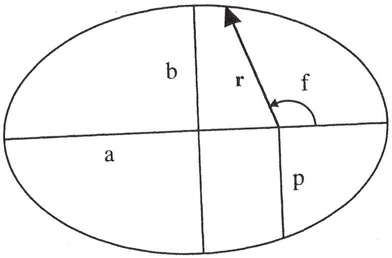
Case 3 ($f = \pi$):

- $\vec{r}$ is opposite to $\vec{e}$. Thus $\vec{r}$ points away from periapse.
- The radius now *maximum*. Thus the radius at *apoapse* is

$$r_a = \frac{h^2}{\mu(1-e)} = \frac{p}{1-e}$$

The point of maximum radius is also referred to as

- *Apogee* for orbits around Earth.
- *Aphelion* for orbits around the Sun.
- *Apolune* for lunar orbits.



Combine periapse and apoapse.

Definition 4 (3rd Orbital Element).

The **semimajor axis** of the orbit is

$$a = \frac{r_p + r_a}{2} = \frac{p}{1-e^2}$$

The case of Elliptic Orbits ($\|\vec{e}\| < 1$)

This relationship between r_p, r_a, a, and e can be simplified by eliminating $p = h^2/\mu$.For example,

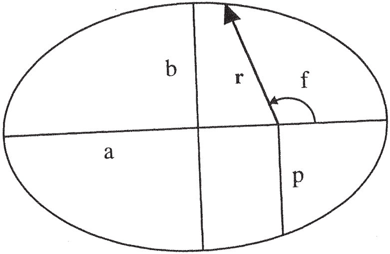
$$p = \frac{h^2}{\mu} = r_a(1-e) = r_p(1+e) = a(1-e^2)$$

Thus we can solve for h^2/μ given

- r_a and e
- r_p and e
- a and e

Alternatively, we can solve directly

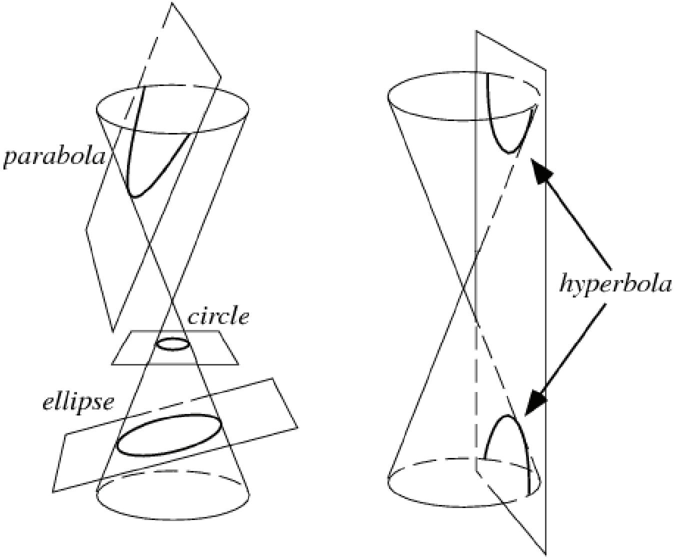
$$r_p = a(1-e)$$
$$r_a = a(1+e)$$



Elliptic Orbits

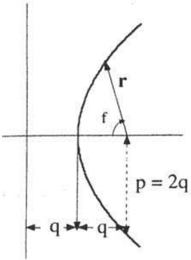
We have established Kepler's First law: Elliptic Motion.

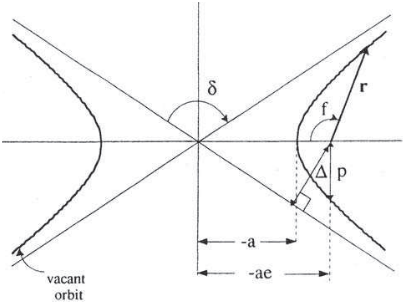
Conic Sections



The case of Parabolic/Hyperbolic Orbits ($\|\vec{e}\| \geq 1$)

$$r(t) = \frac{h^2}{\mu\,(1 + e\cos f(t))}$$





Definition 5.

If $e = 1$, the orbit is **parabolic**.
$\delta = 180\,\text{deg}$.

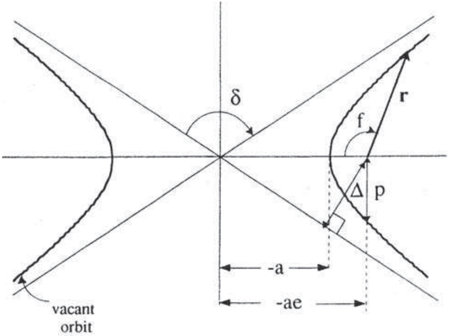
Definition 6.

If $e > 1$, the orbit is **hyperbolic**.
$\delta < 180\,\text{deg}$

$$r(t) = \frac{h^2}{\mu\left(1 + e\cos f(t)\right)}$$

If $e > 1$, then $\lim_t r(t) \to \infty$ when $1 + e\cos f = 0$.

- The orbit is asymptotic at $f = \pm\cos^{-1}\frac{1}{e}$.
  - $f = -\cos^{-1}\frac{1}{e}$ is the incoming asymptote.
  - $f = \cos^{-1}\frac{1}{e}$ is the outgoing asymptote.
- The angle between incoming and outgoing asymptotes is the *Turning Angle*, δ.
  - $\delta = 2\cos^{-1}\frac{1}{e}$



When $f \geq \delta/2$ of $f \leq -\delta/2$, the orbit is fictional.

- Hyperbolic orbits do not repeat.

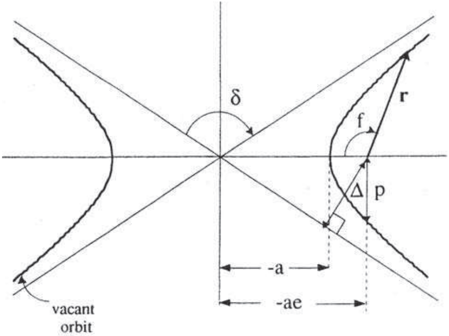
- When $f = 0$, we have closest approach.

$$r_p = \frac{h^2}{\mu}\frac{1}{1+e} = \frac{p}{1+e}$$

- For hyperbolic orbits, $r_a = \infty$, we explicitly define semimajor axis as

$$a = \frac{p}{1-e^2}$$

so that $r_p = a(1-e)$. For hyperbolic orbits, this means $a < 0$.



For $f = \pi$, r has a minimum on the fictional orbit

$$r_a = \frac{p}{1+e} = a(1-e)$$

Parabolic Orbits ($\|\vec{e}\| = 1$)

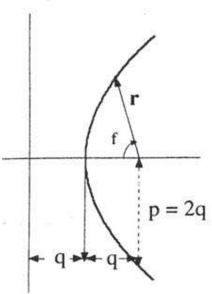
$$r(t) = \frac{h^2}{\mu\,(1 + \cos f(t))}$$

Turning Angles: ($f \to \pm\pi$): If $e = 1$, then $\lim_t r(t) \to \infty$ as $\lim_t f(t) \to \pm\pi$.

- When $f = 0$, we have closest approach.

$$q = \frac{h^2}{2\mu} = \frac{p}{2}$$

- When $f \geq \pi$ of $f \leq -\pi$, the orbit is fictional.
  - Parabolic orbits do not repeat.
- The turning angle is $180\,\mathrm{deg}$.
- No "excess velocity"



Kepler's Second Law

Equal Areas in Equal Time

Now its *Time* for Kepler's Second Law.

Kepler's Second Law

Conservation of Angular Momentum

We need to find an expression for dA/dt.

Consider an infinitesimal section of area:

$$dA = \frac{1}{2}r^2 df$$

where we denote $r = \|\vec{r}\|$. To first order,

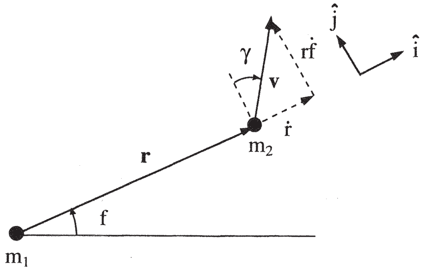
$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{df}{dt}$$

- Note that $\dot{r} \neq \dot{\vec{r}}$. In the orbital plane,

$$\dot{\vec{r}} = \dot{r}\hat{i} + r\dot{f}\hat{j}$$

Now examine angular momentum

$$\vec{h} = \vec{r} \times \dot{\vec{r}} = \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} \dot{r} \\ r\dot{f} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r^2\dot{f} \end{bmatrix}$$



Thus from conservation of angular momentum, we have

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{f} = \frac{\|\vec{h}\|}{2} = constant$$

Which proves Kepler's Second Law

The period of an orbit is the time taken to sweep out the entire ellipse.

Fortunately, the area of an ellipse is known

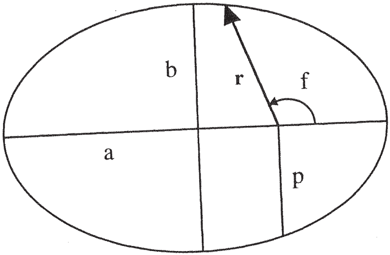
$$A_{ellipse} = \pi a b$$

where b is the **semi-minor axis** of the ellipse.

$$b = a\sqrt{1-e^2}$$

Therefore $dA = \frac{h}{2}dt$ implies

$$T_{orbit} = \frac{2A_{ellipse}}{h} = \frac{2\pi a^2\sqrt{1-e^2}}{\sqrt{\mu a(1-e^2)}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$



Which proves Kepler's second Law

$$\frac{T^2}{a^3} = \frac{4\pi^2}{\mu} = constant$$

For Kepler, μ was μ_{sun}.

Energy Methods

Now, lets talk about energy in terms of geometry. Recall

$$E = \frac{1}{2}v^2 - \frac{\mu}{r}$$

- But what is the energy associated with a geometry?
- We already know angular momentum in terms of geometry

$$p = \frac{h^2}{\mu}, \qquad \text{implies} \qquad h = \sqrt{p\mu}$$

Consider r and v at periapse. Then $r_p = a(1-e)$ and $r_p v_p = h$, so

$$E = \frac{1}{2}v_p^2 - \frac{\mu}{r_p} \qquad\qquad = \frac{1}{2}\frac{p\mu}{a^2(1-e)^2} - \frac{\mu}{a(1-e)}$$

$$= \frac{1}{2}\frac{\mu a(1-e^2)}{a^2(1-e)^2} - \frac{\mu}{a(1-e)} \qquad\qquad = \mu\frac{\frac{1}{2}(1+e)-1}{a(1-e)} = -\frac{\mu}{2a}$$

Therefore, the total energy of an orbit is

$$E = -\frac{\mu}{2a}$$

$$E = -\frac{\mu}{2a}$$

Note for an elliptic orbit, the total energy is negative, as expected.

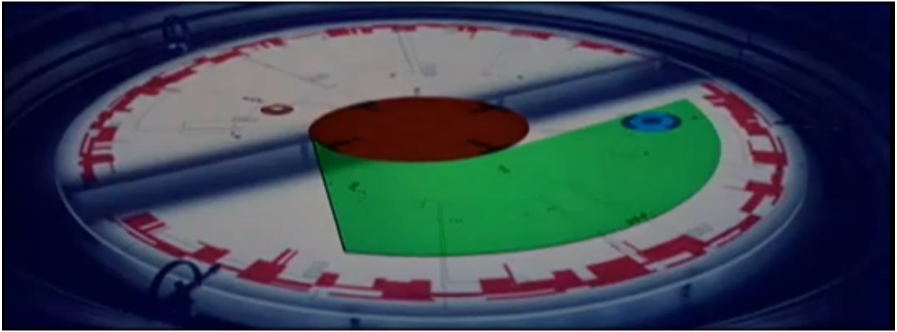
This yields the famous Vis-Viva relationship

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Also yields the "Excess Velocity":

$$v_{excess} = \lim_{r\to\infty} v(r) = \sqrt{-\frac{\mu}{a}}$$

Only real when $a \leq 0$.



For circular orbits,

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{r}\right) = \frac{\mu}{r} \qquad \text{or} \qquad v = \sqrt{\frac{\mu}{r}}$$

Example

Problem: Suppose we observe at perigee, $r_p = 15000km$ and at apogee, $r_a = 25000km$. At time t_0, we observe $r(t_0) = 20000km$. Determine $v(t_0)$.
Solution: We first find the total energy, E from

$$E = -\frac{\mu}{2a}$$

where a can be found from

$$a = \frac{r_p + r_a}{2} = 20000km$$

Therefore

$$E = -\frac{\mu}{2a} = -9.96$$

Now we use

$$E = v^2/2 - \mu/r = -9.96$$

to find

$$v = \sqrt{2\left((E + \frac{\mu}{r}\right)} = \sqrt{2\left(-9.96 + 19.93\right)} = 4.464km/s$$

Example

Problem: Suppose we want to position a satellite in equatorial orbit which always maintains the same position above the earth. Find a and e for such an orbit.

Solution: The earth rotates 363.25 times a year (one day comes from motion about the sun). Thus the earth rotates once every 23 hours, 56 minutes and 4 seconds. Thus the period of the satellite must be $\tau = 86164s$.

From Kepler's Third Law, we have

$$\begin{aligned} a &= \left(\frac{\mu\tau^2}{4\pi^2}\right)^{1/3} \\ &= 42,164km \end{aligned}$$

Since the rotation rate of the earth is constant, we want $e = 0$.

Summary

Some Important Scalar Relations

$$
\begin{aligned}
r(t) &= \frac{h^2}{\mu\left(1 + e\cos f(t)\right)} \\
r_p &= a(1-e) \\
r_a &= a(1+e) \\
p &= a(1-e^2) = \frac{h^2}{\mu} \\
b &= a\sqrt{1-e^2} \\
E &= \frac{1}{2}v^2 - \frac{\mu}{r} = -\frac{\mu}{2a} \\
h &= r_p v_p = r_a v_a \cong r^2 \dot{f}
\end{aligned}
$$

Summary

Some Important Vector Relations

$$
\begin{aligned}
\dot{\vec{e}} &= 0 \\
\dot{\vec{h}} &= 0 \\
\vec{h} &= \vec{r} \times \dot{\vec{r}} \\
\vec{e} &= \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu\frac{\vec{r}}{\|\vec{r}\|}\right) \\
\vec{r}(t) \times \vec{h} &= 0 \\
\vec{e} \times \vec{h} &= 0 \\
\dot{\vec{r}} &= \dot{r}\hat{i} + r\dot{f}\hat{j}
\end{aligned}
$$