LMI Methods in Optimal and Robust Control

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Lecture 11: Relationship between H_2, LQG and LGR and LMIs for state and output feedback H_2 synthesis

Conclusion

To solve the H_∞-optimal output-feedback problem, we solve

$$\min_{\gamma,X_1,Y_1,A_n,B_n,C_n,D_n} \gamma \qquad \text{such that}$$

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$$

$$\begin{bmatrix} AY_1+Y_1A^T+B_2C_n+C_n^TB_2^T & *^T & *^T & *^T \\ A^T+A_n+[B_2D_nC_2]^T & X_1A+A^TX_1+B_nC_2+C_2^TB_n^T & *^T & *^T \\ [B_1+B_2D_nD_{21}]^T & [XB_1+B_nD_{21}]^T & -\gamma I & \\ C_1Y_1+D_{12}C_n & C_1+D_{12}D_nC_2 & D_{11}+D_{12}D_nD_{21} & -\gamma I \end{bmatrix} < 0$$

Conclusion

Then, we construct our controller using

$$
\begin{aligned}
D_K &= (I + D_{K2}D_{22})^{-1}D_{K2}\\
B_K &= B_{K2}(I - D_{22}D_K)\\
C_K &= (I - D_K D_{22})C_{K2}\\
A_K &= A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\end{aligned}
$$

where

$$
\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2}\end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}.
$$

and where X_2 and Y_2 are any matrices which satisfy $X_2 Y_2^T = I - X_1 Y_1$.

- e.g. Let $Y_2 = I$ and $X_2 = I - X_1 Y_1$.
- The optimal controller is NOT uniquely defined.
- Don't forget to check invertibility of $I - D_{22}D_K$

Conclusion

The H_∞-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

Minimizes the effect of external input (w) on external output (z).

$$\|z\|_{L_2} \le \|\underline{\mathsf{S}}(P,K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain

H_2-optimal control
Motivation

H_2-optimal control minimizes the H_2-norm of the transfer function.

- The H_2-norm has no direct interpretation.

$$\|G\|_{H_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\imath\omega)^*\hat{G}(\imath\omega))d\omega$$

Motivation: Assume external input, w, is Gaussian noise with power spectral density $\hat{S}_w$. Then, the variance is given by

$$E[w(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{S}_w(\imath\omega))d\omega$$

Theorem 1.

For an LTI system P, if w is noise with spectral density $\hat{S}_w(\imath\omega)$ and $z = Pw$, then z is noise with density

$$\hat{S}_z(\imath\omega) = \hat{P}(\imath\omega)\hat{S}(\imath\omega)\hat{P}(\imath\omega)^*$$

Then the output $z = Gw$ has signal variance (Power)

$$E[z(t)^2] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\imath\omega)^* S(\imath\omega)\hat{G}(\imath\omega))d\omega$$
$$\le \|S\|_{H_\infty}\|G\|_{H_2}^2$$

If the input signal is white noise, then $\hat{S}(\imath\omega) = I$ and

$$E[z(t)^2] = \|\hat{G}\|_{H_2}^2$$

H_2-optimal control

Motivation

Then the output $z = Gw$ has signal variance (Power)

$$\begin{aligned} E[z(t)^2] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{Trace}(\hat{G}(\imath\omega)^* S(\imath\omega)\hat{G}(\imath\omega))d\omega \\ &\leq \|S\|_{H_\infty}\|G\|_{H_2}^2 \end{aligned}$$

If the input signal is white noise, then $\hat{S}(\imath\omega) = I$ and

$$E[z(t)^2] = \|\hat{G}\|_{H_2}^2$$

Hence the H_2 norm represents the power spectral density of the output of the system when the input is white noise.

- Thus H_2 optimal control is optimal in a certain sense when the input is expected to be white noise.
- However, this doesn't work when the noise is colored (concentrated at certain frequencies).
- For colored noise, however, we can use prefilters to obtain optimal controllers.

H_2-optimal control

Colored Noise

Now suppose the noise is colored with density $\hat{S}_w(\imath\omega)$. Now define $\hat{H}$ as $\hat{H}(\imath\omega)\hat{H}(\imath\omega)^* = \hat{S}_w(\imath\omega)$ and the filtered system

$$\hat{P}_s(s) = \begin{bmatrix} \hat{P}_{11}(s)\hat{H}(s) & \hat{P}_{12}(s) \\ \hat{P}_{21}(s)\hat{H}(s) & \hat{P}_{22}(s) \end{bmatrix}.$$

Now, applying feedback to the filtered plant, we get

$$\underline{\mathsf{S}}(P_s, K)(s) = P_{11}H + P_{12}(I - KP_{22})^{-1}KP_{21}H = \underline{\mathsf{S}}(P, K)H$$

Now the spectral density, $\hat{S}_z$ of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

$$\begin{aligned} \hat{S}_z(s) &= \underline{\mathsf{S}}(P, K)(s)\hat{S}_w(s)\underline{\mathsf{S}}(P, K)(s)^* \\ &= \underline{\mathsf{S}}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\underline{\mathsf{S}}(P, K)(s)^* = \hat{S}(P_s, K)(s)\hat{S}(P_s, K)(s)^* \end{aligned}$$

Thus if K minimizes the H_2-norm of the filtered plant ($\|\hat{S}(P_s, K)\|_{H_2}^2$), it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$.

H_2-optimal control

Colored Noise

Now suppose the noise is colored with density $\hat{S}_w(\imath\omega)$. Now define $\hat{H}$ as $\hat{H}(\imath\omega)\hat{H}(\imath\omega)^* = \hat{S}_w(\imath\omega)$ and the filtered system

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Now the spectral density, $\hat{S}_z$ of the output of the true plant using colored noise equals the output of the artificial plant under white noise. i.e.

$$\begin{aligned} \hat{S}_z(s) &= \underline{S}(P, K)(s)\hat{S}_w(s)\underline{S}(P, K)(s)^* \\ &= \underline{S}(P, K)(s)\hat{H}(s)\hat{H}(s)^*\underline{S}(P, K)(s)^* = \hat{S}(P_s, K)(s)\hat{S}(P_s, K)(s)^* \end{aligned}$$

Thus if K minimizes the H_2-norm of the filtered plant ($\|\hat{S}(P_s, K)\|_{H_2}^2$), it will minimize the variance of the true plant under the influence of colored noise with density $\hat{S}_w$.

In this case, the response of the prefiltered system to white noise is the same as the unfiltered system response to colored noise.

Alternatively, we can write

$$\min_K \|S(P, K)w\|_{\substack{var \\ w=colored}} = \min_K \|S(P, K)Hu\|_{\substack{var \\ u=white}} = \min_K \|S(P, K)H\|_{H_2}$$

H_2-optimal control

Theorem 2.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. *A is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$.*
2. *There exists some $X > 0$ such that*

$$\begin{aligned} \operatorname{trace} B^T X B &< \gamma^2 \\ A^T X + XA + C^T C &< 0 \end{aligned}$$

Recall that the Controllability Grammian is a solution!

- Recall how the proof works.
- But this time use the observability grammian.

H_2-optimal control

Proof.

Suppose A is Hurwitz and $\|\hat{P}\|_{H_2} < \gamma$. Then the Observability Grammian is defined as

$$X_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

Now recall the Laplace transform

$$\begin{aligned}
\left(\Lambda e^{At}\right)(s) &= \int_0^\infty e^{At} e^{-ts} dt \\
&= \int_0^\infty e^{-(sI-A)t} dt \\
&= -(sI-A)^{-1} e^{-(sI-A)t} dt \Big|_{t=0}^{t=-\infty} \\
&= (sI-A)^{-1}
\end{aligned}$$

Hence $\left(\Lambda C e^{At} B\right)(s) = C(sI-A)^{-1}B$. □

H_2-optimal control

Proof.

$\left(\Lambda C e^{At} B\right)(s) = C(sI - A)^{-1}B$ implies

$$\begin{aligned}\|\hat{P}\|_{H_2}^2 &= \|C(sI-A)^{-1}B\|_{H_2}^2 \\ &= \frac{1}{2\pi}\int_0^\infty \text{Trace}((C(\imath\omega I - A)^{-1}B)^*(C(\imath\omega I - A)^{-1}B))d\omega \\ &= \text{Trace}\int_{-\infty}^\infty B^T e^{A^T t} C^T C e^{At} B dt \\ &= \text{Trace} B^T X_o B\end{aligned}$$

Thus $X_o \geq 0$ and $\text{Trace} B^T X_o B = \|\hat{P}\|_{H_2}^2 < \gamma^2$. □

The rest of the proof we can skip.

Lets consider the full-state feedback problem

$$\hat{G}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I & 0 & 0 \end{array}\right]$$

- D_{12} is the weight on control effort.
- $D_{11} = 0$ is a feed-through term and must be 0.
- $C_2 = I$ as this is state-feedback.

$$\hat{K}(s) = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right]$$

Theorem 3.

The following are equivalent.

1. $\|S(K,P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ *for some* Z *and* $X > 0$ *where*

$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0$$

$$\textit{Trace} \left[C_1 X + D_{12} Z \right] X^{-1} \left[C_1 X + D_{12} Z \right] < \gamma^2$$

However, this is nonlinear, so we need to reformulate using the Schur Complement.

H_2-optimal control

Applying the Schur Complement gives the alternative formulation convenient for control.

Theorem 4.

Suppose $\hat{P}(s) = C(sI - A)^{-1}B$. Then the following are equivalent.

1. A *is Hurwitz and* $\|\hat{P}\|_{H_2} < \gamma$.
2. *There exists some* $X, W > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} < 0, \qquad \begin{bmatrix} X & C^T \\ C & W \end{bmatrix} > 0, \qquad \mathit{Trace} W < \gamma^2$$

Theorem 5.

The following are equivalent.

1. $\|S(K,P)\|_{H_2} < \gamma$.
2. $K = ZX^{-1}$ *for some* Z *and* $X > 0$ *where*

$$\begin{bmatrix} A & B_2 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} X & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B_2^T \end{bmatrix} + B_1 B_1^T < 0$$

$$\begin{bmatrix} X & (C_1 X + D_{12} Z)^T \\ C_1 X + D_{12} Z & W \end{bmatrix} > 0$$

$$\mathit{Trace} W < \gamma^2$$

Thus we can solve the H_2-optimal static full-state feedback problem.

The LQR Problem:

- Full-State Feedback
- Choose K to minimize the cost function

$$\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt$$

subject to dynamic constraints

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

Trying to minimize the effect of x_0 on a weighted-L_2-norm of the regulated output.

H_2-optimal control

Relationship to LQR

To solve the LQR problem using H_2 optimal state-feedback control, let

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$,
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$ and $D_{11} = 0$,
- $B_2 = B$ and $B_1 = I$.

So that

$$\underline{\mathsf{S}}(P,K) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C_1 + D_{12}K & D_{11} \end{array}\right] = \left[\begin{array}{c|c} A + BK & I \\ \hline \begin{matrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K \end{matrix} & 0 \end{array}\right]$$

And solve the H_2 full-state feedback problem. Then if

$$\begin{aligned} \dot{x}(t) &= A_{CL}x(t) = (A+BK)x(t) = Ax(t) + Bu(t) \\ u(t) &= Kx(t), \qquad x(0) = x_0 \end{aligned}$$

Then $x(t) = e^{A_{CL}t}x_0$.

H_2-optimal control

Relationship to LQR

To solve the LQR problem using H_2 optimal state-feedback control, let

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$,
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$ and $D_{11} = 0$,
- $B_2 = B$ and $B_1 = I$.

So that

$$\underline{S}(P,K) = \left[\begin{array}{c|c} A + B_2 K & B_1 \\ \hline C_1 + D_{12}K & D_{11} \end{array}\right] = \left[\begin{array}{c|c} A + BK & I \\ \hline \begin{matrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}} K \end{matrix} & 0 \end{array}\right]$$

And solve the H_2 full-state feedback problem. Then if

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

Then $x(t) = e^{A_{CL}t}x_0$.

Translating to the input-output formulation, recall we apply the problem setup to

$$P = \left[\begin{array}{c|c} A & \begin{bmatrix} B_1 & B_2 \end{bmatrix} \\ \hline \begin{bmatrix} C_1 \\ I \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} \\ 0 & 0 \end{bmatrix} \end{array}\right]$$

$$K = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K \end{array}\right]$$

H_2-optimal control
Relationship to LQR

Ignoring the regulated outputs for now, we have

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

then $x(t) = e^{A_{CL}t}x_0$, $u(t) = Ke^{A_{CL}t}x_0$ and

$$\int_0^\infty x(t)^T Qx(t) + u(t)^T Ru(t)dt = \int_0^\infty x_0^T e^{A_{CL}^T t}(Q + K^T RK)e^{A_{CL}t}x_0 dt$$
$$= \text{Trace} \int_0^\infty x_0^T e^{A_{CL}^T t} \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix}^T \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix} e^{A_{CL}t}x_0 dt$$
$$= \|x_0\|^2 \text{Trace} \int_0^\infty B_1^T e^{A_{CL}^T t}(C_1 + D_{12}K)^T(C_1 + D_{12}K)e^{A_{CL}t}B_1 dt$$
$$= \|x_0\|^2 B_1^T X_0 B_1 = \|x_0\|^2 \|S(P, K)\|_{H_2}^2$$

Thus LQR reduces to a special case of H_2 static state-feedback.

H_2-optimal control

Relationship to LQR

Ignoring the regulated outputs for now, we have

$$\dot{x}(t) = A_{CL}x(t) = (A + BK)x(t) = Ax(t) + Bu(t)$$
$$u(t) = Kx(t), \qquad x(0) = x_0$$

then $x(t) = e^{A_{CL}t}x_0$, $u(t) = Ke^{A_{CL}t}x_0$ and

$$\begin{aligned}
&\int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) dt = \int_0^\infty x_0^T e^{A_{CL}^T t}(Q + K^T R K)e^{A_{CL}t}x_0 dt \\
&= \text{Trace} \int_0^\infty x_0^T e^{A_{CL}^T t} \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix}^T \begin{bmatrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{bmatrix} e^{A_{CL}t}x_0 dt \\
&= \|x_0\|^2 \text{Trace} \int_0^\infty B_1^T e^{A_{CL}^T t}(C_1 + D_{12}K)^T(C_1 + D_{12}K)e^{A_{CL}t}B_1 dt \\
&= \|x_0\|^2 B_1^T X_0 B_1 = \|x_0\|^2 \|S(P,K)\|_{H_2}^2
\end{aligned}$$

Thus LQR reduces to a special case of H_2 static state-feedback.

Recall that

- $C_1 = \begin{bmatrix} Q^{\frac{1}{2}} \\ 0 \end{bmatrix}$,
- $D_{12} = \begin{bmatrix} 0 \\ R^{\frac{1}{2}} \end{bmatrix}$ and $D_{11} = 0$,
- $B_2 = B$ and $B_1 = I$.

So that

$$\underline{S}(P,K) = \left[\begin{array}{c|c} A + B_2K & B_1 \\ \hline C_1 + D_{12}K & D_{11} \end{array}\right] = \left[\begin{array}{c|c} A + BK & I \\ \hline \begin{matrix} Q^{\frac{1}{2}} \\ R^{\frac{1}{2}}K \end{matrix} & 0 \end{array}\right]$$

H_2-optimal output feedback control

Theorem 6 (Scherer, Gahinet).

The following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma$.
- *There exist* $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ *such that*

$$\begin{bmatrix} AY_1 + Y_1A^T + B_2C_n + C_n^TB_2^T & *^T & *^T \\ A^T + A_n + [B_2D_nC_2]^T & X_1A + A^TX_1 + B_nC_2 + C_2^TB_n^T & *^T \\ [B_1 + B_2D_nD_{21}]^T & [X_1B_1 + B_nD_{21}]^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y_1 & I & *^T \\ I & X_1 & *^T \\ C_1Y_1 + D_{12}C_n & C_1 + D_{12}D_nC_2 & Z \end{bmatrix} > 0,$$

$$D_{11} + D_{12}D_nD_{21} = 0, \qquad \text{trace}(Z) < \gamma^2$$

H_2-optimal output feedback control

As before, the controller can be recovered as

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}$$

for any full-rank X_2 and Y_2 such that

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

To find the actual controller, we use the identities:

$$\begin{aligned} D_K &= (I + D_{K2} D_{22})^{-1} D_{K2} \\ B_K &= B_{K2}(I - D_{22} D_K) \\ C_K &= (I - D_K D_{22}) C_{K2} \\ A_K &= A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K \end{aligned}$$

An LMI for Mixed H_2-H_∞ optimal output feedback control

Theorem 7.

The following are equivalent.

- *There exists a* $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_2} < \gamma_1$ *and* $\|S(K,P)\|_{H_\infty} < \gamma_2$.
- *There exist* $X_1, Y_1, Z, A_n, B_n, C_n, D_n$ *such that*

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -I \end{bmatrix} < 0,$$

$$\begin{bmatrix} Y_1 & I & *^T \\ I & X_1 & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & Z \end{bmatrix} > 0,$$

$$D_{11} + D_{12} D_n D_{21} = 0, \qquad \mathrm{trace}(Z) < \gamma_1^2$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma_2 I & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma_2 I \end{bmatrix} < 0$$

An LMI for the Kalman Filter! - Continuous Time

System:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t)$$
$$y(t) = Cx(t) + v(t)$$

Filter:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = C\hat{x}(t)$$

Error:
$$\dot{e}(t) = (A + LC)e(t) + w(t) + Lv(t)$$

The Kalman Filter chooses L to minimize the cost $J = \mathbf{E}[e^T e]$.

$$L = \Sigma C^T V_2^{-1}$$

where $V_1 = \mathbf{E}[\mathbf{w(t)w(t)^T}]$ and $V_2 = \mathbf{E}[\mathbf{v(t)v(t)^T}]$ and Σ satisfies

$$A\Sigma + \Sigma A^T + V_1 = \Sigma C^T V_2^{-1} C\Sigma$$

If we choose $u(t) = K\hat{x}(t)$ where $A + BK$ is stable,

- $A + LC$ is stable if system is observable (not detectable).
- Closed-Loop is stable by the separation principle (has Luenberger form).
- A Dual to the LQR problem. Replace (A, B, Q, R, K) with $(A^T, C^T, V_1, V_2, L^T)$

Kalman Filter - Discrete Time

Assume the system is driven by noise w_k (no feedback)

$$x_{k+1} = Ax_k + w_k, \qquad y_k = Cx_k + v_k$$

The steady-state Kalman filter is an estimator of the form:

$$\hat{x}_{k+1} = A\hat{x}_k + L(C\hat{x}_k - y_k),$$

where v_k is sensor noise. This gives error ($e_k = x_k - \hat{x}_k$) dynamics

$$e_{k+1} = (A + LC)e_k$$

For the Kalman Filter, we choose $L = A\Sigma C^T(C\Sigma C^T + V)^{-1}$ where $V = \mathbf{E}[\mathbf{v_k v_k^T}]$ and Σ is the steady-state covariance of the error in the estimated state and satisfies

$$\Sigma = A\Sigma A^T + W - A\Sigma C^T(C\Sigma C^T + V)^{-1}C\Sigma A^T$$

where $W = \mathbf{E}[\mathbf{w_k w_k^T}]$. For the unsteady Kalman filter, Σ_k is updated at each time-step.

- If (A, W) controllable and (C, A) observable, then $A + LC$ is stable.
- Again, dual to discrete-time LQR (which we haven't solved here!)