# Spacecraft Dynamics and Control

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Lecture 8: Impulsive Orbital Maneuvers

# Introduction

In this Lecture, you will learn:

Coplanar Orbital Maneuvers

- Impulsive Maneuvers
  - $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

**Numerical Problem:** Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and $\Delta v$).

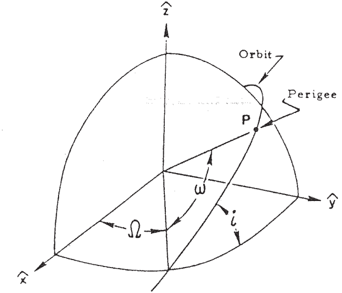
# Changing Orbits

Suppose we have designed our ideal orbit.

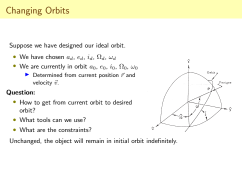
- We have chosen $a_d$, $e_d$, $i_d$, $\Omega_d$, $\omega_d$
- We are currently in orbit $a_0$, $e_0$, $i_0$, $\Omega_0$, $\omega_0$
  - Determined from current position $\vec{r}$ and velocity $\vec{v}$.

**Question:**

- How to get from current orbit to desired orbit?
- What tools can we use?
- What are the constraints?



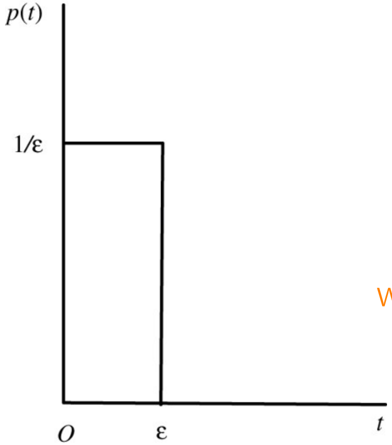

Unchanged, the object will remain in initial orbit indefinitely.




- For now, we don't care about $f$ (time)
  - Lambert's Problem
  - Can correct using phasing
- Don't care about efficiency
- true anomaly $(f)$ determines phasing within the orbit and is easily altered post-insertion.

# How to create a $\Delta v$

$\Delta v$ is our tool for changing orbits




Velocity change is caused by thrust.

- For constant thrust, $F$,

$$v(t) = v(0) + \frac{F}{m}\Delta t$$

- for a desired $\Delta v$, the time needed is

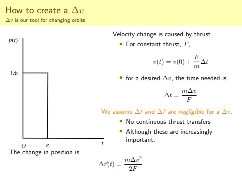
$$\Delta t = \frac{m\Delta v}{F}$$

We assume $\Delta t$ and $\Delta \vec{r}$ are *negligible* for a $\Delta v$.

- No continuous thrust transfers
- Although these are increasingly important.

The change in position is

$$\Delta \vec{r}(t) = \frac{m\Delta v^2}{2F}$$




- For fixed $\Delta v$, if $\frac{m}{F}$ is small, the $\Delta\vec{r}$ is small
- We will assume $\Delta\vec{r} = 0$

$$v(t) = v(0) + \frac{F}{m}t$$

so

$$t = \Delta v \frac{m}{F}$$

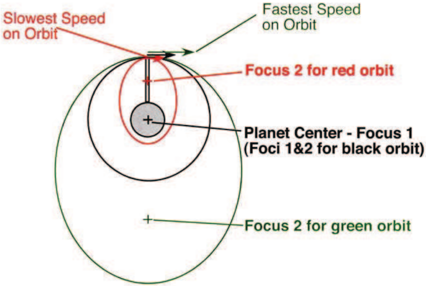
Now,

$$\begin{aligned} r(t) &= r(0) + v(0)t + \frac{F}{2m}t^2 \\ \Delta r &= v(0)t + \frac{F}{2m}\Delta v^2 \frac{m^2}{F^2} \\ &= v(0)\Delta v \frac{m}{F} + \frac{\Delta v^2}{2}\frac{m}{F} = \left(v(0)\Delta v + \frac{\Delta v^2}{2}\right)\frac{m}{F} \end{aligned}$$

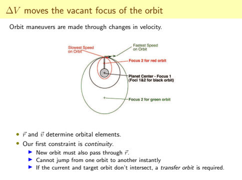
However, we can ignore the $v(0)$ if we are considering deviation from a nominal path.

# $\Delta V$ moves the vacant focus of the orbit

Orbit maneuvers are made through changes in velocity.




- $\vec{r}$ and $\vec{v}$ determine orbital elements.
- Our first constraint is *continuity*.
  - New orbit must also pass through $\vec{r}$.
  - Cannot jump from one orbit to another instantly
  - If the current and target orbit don't intersect, a *transfer orbit* is required.




Equations involving velocity

$$v_c = \sqrt{\frac{\mu}{r_c}}$$ circular orbit

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$ vis-viva

$$v_p = \sqrt{\frac{\mu}{a}\left(\frac{1+e}{1-e}\right)}$$ periapse velocity

$$v_a = \sqrt{\frac{\mu}{a}\left(\frac{1-e}{1+e}\right)}$$ apoapse velocity

$$v_{esc} = \sqrt{\frac{2\mu}{r}}$$ escape velocity

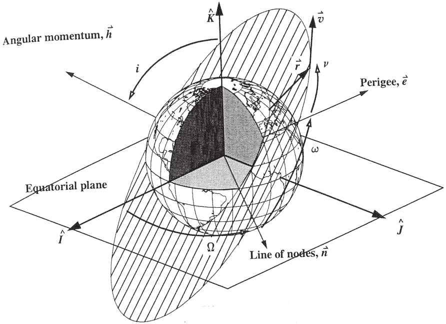
$$\vec{v} = \begin{bmatrix} -\frac{\mu}{h}\left(\cos\Omega\left(\sin(\omega+f) + e\sin\omega\right) + \sin\Omega\left(\cos(\omega+f) + e\cos\omega\right)\cos i\right) \\ -\frac{\mu}{h}\left(\sin\Omega\left(\sin(\omega+f) + e\sin\omega\right) - \cos\Omega\left(\cos(\omega+f) + e\cos\omega\right)\cos i\right) \\ \frac{\mu}{h}\left(\cos(\omega+f) + e\cos\omega\right)\sin i \end{bmatrix}$$

# What can we do with a $\Delta v$ Maneuver?

$\Delta v$ refers to the difference between the initial and final velocity vectors.

A $\Delta v$ maneuver can:

- Raise/lower the apogee/perigee
- A change in inclination
- Escape
- Reduction/Increase in period
- Change in RAAN
- Begin a 2+ maneuver sequence of burns.
  - Creates a **Transfer Orbit**.




We'll start by talking about coplanar maneuvers.

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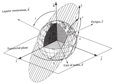



We'll start by talking about coplanar maneuvers.

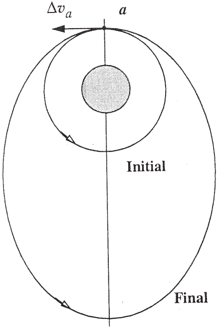
- Raise/lower the apogee/perigee is performed at perigee/apogee
- A change in inclination is *usually* performed at the equatorial plane (any inclination achievable from this point).
- Small changes in period help with phase changes $f(t)$.
- Change in RAAN should be done as far from equatorial plane as possible.

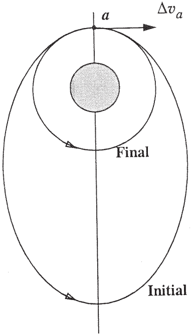
# Single Burn Coplanar Maneuvers

Apogee or Perigee raising or lowering.

## Definition 1.

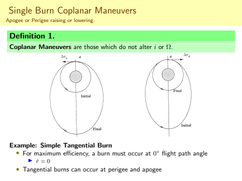
**Coplanar Maneuvers** are those which do not alter $i$ or $\Omega$.




**Example: Simple Tangential Burn**

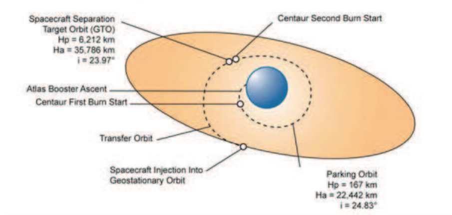
- For maximum efficiency, a burn must occur at $0^\circ$ flight path angle
  - $\dot{r} = 0$
- Tangential burns can occur at perigee and apogee

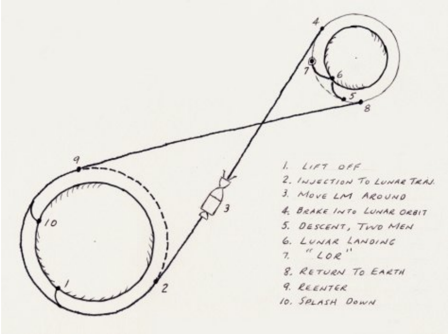



We will explain why we want $\angle FPA = 0^\circ$ in Lecture 9, when we discuss the Oberth effect.

Suppose we launch from the surface of the earth.

- This creates an initial elliptic orbit which will re-enter.
- To circularize the orbit, we plan on using a burn at apogee.




**Problem:** We are given $a$ and $e$ of the initial elliptic orbit. Calculate the $\Delta v$ required at apogee to circularize the orbit.

# Example: Insertion into a Parking Orbit

A perigee raising maneuver

**Calculating the $\Delta v$:** To raise the perigee, we burn at apogee. At apogee, we have that

$$r_{a_0} = a_0(1+e_0)$$

From the vis-viva equation, we can calculate the velocity at apogee.

$$v_{a_0} = \sqrt{\mu\left(\frac{2}{r_{a_0}} - \frac{1}{a_0}\right)} = \sqrt{\frac{\mu}{a_0}\left(\frac{1-e_0}{1+e_0}\right)}$$

Our target orbit is circular with radius $r_d = a_d = r_{a_0}$. The velocity of the target orbit is constant at

$$v_c = \sqrt{\frac{\mu}{r_a}} = \sqrt{\frac{\mu}{a(1+e)}}$$

Therefore, the $\Delta v$ required to circularize the orbit is

$$\Delta v = v_c - v_{a_0} = \sqrt{\frac{\mu}{a_0(1+e_0)}} - \sqrt{\frac{\mu}{a_0}\left(\frac{1-e_0}{1+e_0}\right)}$$

- It is unusual to launch directly into the desired orbit. Instead we use the parking orbit while waiting for more complicated orbital maneuvers.

Lets generalize the parking orbit example to the case of a transfer orbit.

**Definition 2.**

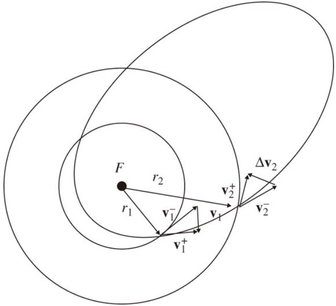
- The **Initial Orbit** is the orbit we want to leave.
- The **Target Orbit** is the orbit we want to achieve.
- The **Transfer Orbit** is an orbit which intersects both the initial orbit and target orbit.

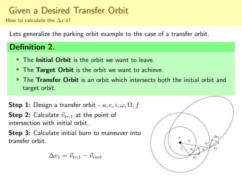
**Step 1:** Design a transfer orbit - $a, e, i, \omega, \Omega, f$

**Step 2:** Calculate $\vec{v}_{tr,1}$ at the point of intersection with initial orbit.

**Step 3:** Calculate initial burn to maneuver into transfer orbit.

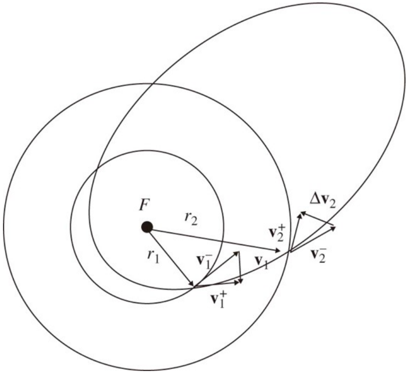
$$\Delta v_1 = \vec{v}_{tr,1} - \vec{v}_{init}$$






- Note that in the illustration, the transfer orbit is not a Hohman transfer, which is the most common type of transfer orbit.
- Step 1 may be **VERY HARD** because you may not know what $f_{tr}$ will be in your transfer orbit!

# Coplanar Two-Impulse Orbit Transfers



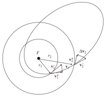

**Step 4:** Calculate $\vec{v}_{tr,2}$ at the point of intersection with target orbit.

**Step 5:** Calculate velocity of the target orbit, $\vec{v}_{fin}$, at the point of intersection with transfer orbit.

**Step 6:** Calculate the final burn to maneuver into target orbit.

$$\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}$$

Coplanar Two-Impulse Orbit Transfers



**Step 4:** Calculate $\vec{v}_{tr,2}$ at the point of intersection with target orbit.
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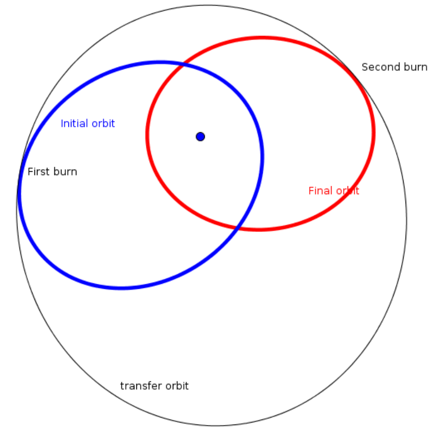
**Step 6:** Calculate the final burn to maneuver into target orbit.

$$\Delta v_2 = \vec{v}_{fin} - \vec{v}_{tr,2}$$

Step 4 may be easy if you can find the point of intersection, $\vec{r}_{tr,2}$, since you can then use the polar equation to find $f_{tr,2}$. However, finding the point of intersection may be hard.

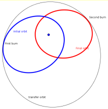
# The Choice of Transfer Orbit




Constraints:

- The transfer orbit must intersect both current and target orbit
- The $\Delta v$'s for entering transfer and orbital insertion are limited by $\Delta v$ budget
  - Typically limits us to elliptic transfers.
- There may be constraints on elapsed time.

The Choice of Transfer Orbit

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The image is of an unproven conjecture that the most efficient 2-burn transfer between 2 coplanar orbits always uses a tangential burn.

- Feel free to find a counterexample!
- Use a brute force search approach using Lambert's problem to calculate $\Delta v's$

# The Choice of Transfer Orbit

## Continuity Constraints affect range of $a$ and $e$

There are many orbits which intersect both the initial and target orbits.

However, there are some constraints.

Consider

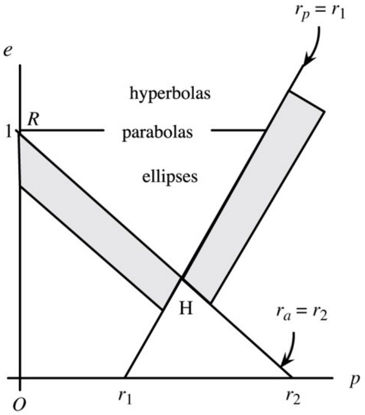
- Circular initial orbit of radius $r_1$
- Circular target orbit of radius $r_2 > r_1$

Obviously, the transfer orbit must satisfy

$$r_p = \frac{p}{1+e} \leq r_1$$

and

$$r_a = \frac{p}{1-e} \geq r_2$$



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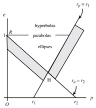



- The $r_p$ constraint says the transfer orbit must intersect the initial orbit.
- The $r_a$ constraint says the transfer orbit must intersect the target orbit.
- The plot illustrates the range of realizable $p$ and $e$ for given initial and target radii
- The lines represent

$$p \ge r_2(1-e) \qquad \leftrightarrow \qquad e > 1 - \frac{p}{r_2}$$

and

$$p < r_1(1+e) \qquad \leftrightarrow \qquad e > \frac{p}{r_1} - 1$$

- Again, we assume no constraint on timing or phasing.

Occasionally, we want to arrive at

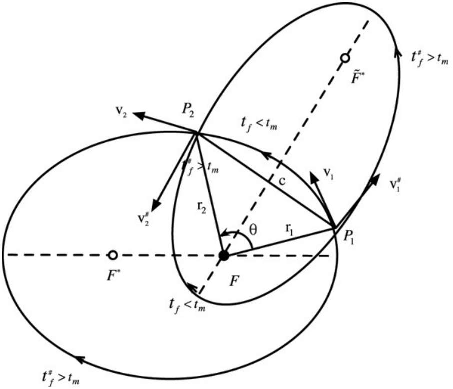
- A certain point in the target orbit, $\vec{r}_2$
- at a certain time, $t_f$

Finding the necessary transfer orbit is **Lambert's Problem**.

Primary Applications are:

- Targeting
- Rendez-vous

We will come back to the section on Lambert's problem.



## Transfer Orbits in Fixed Time

### Constraints on Transfer Time

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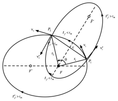


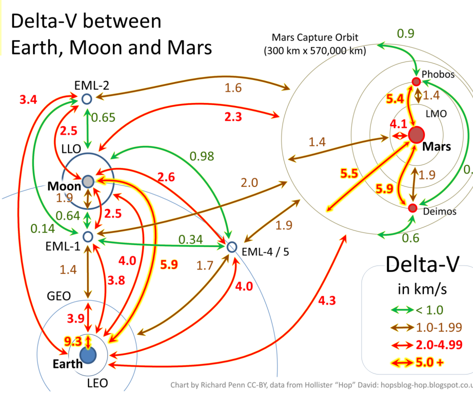

The plot shows 2 possible transfer orbits between point $P_1$ and $P_2$.

# Constraints on $\Delta v$ budget

## What is a minimum energy transfer orbit?

The critical resource in space travel is $\Delta v$.

- The $\Delta v$ budget is fixed at takeoff.
- Refueling is not usually possible.
- If you run out of $\Delta v$, bad things happen.

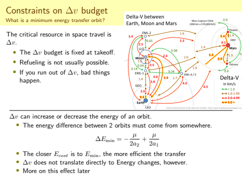



$\Delta v$ can increase or decrease the energy of an orbit.

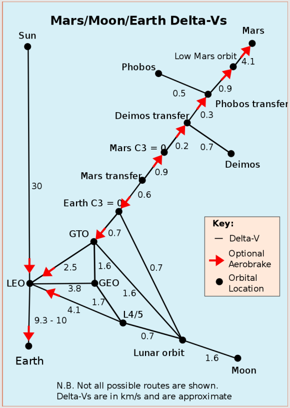
- The energy difference between 2 orbits must come from somewhere.

$$\Delta E_{\min} = -\frac{\mu}{2a_2} + \frac{\mu}{2a_1}$$

- The closer $E_{cost}$ is to $E_{\min}$, the more efficient the transfer
- $\Delta v$ does not translate directly to Energy changes, however.
- More on this effect later



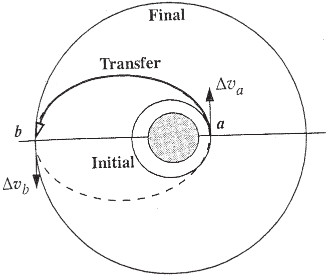

- The energy GAIN for each $\Delta v$ is actually larger - depending on the initial velocity. We will discuss this more carefully next lecture.
- Note energy is NOT conserved here, so $\frac{\Delta v^2}{2} \neq \Delta E_{\min}$.

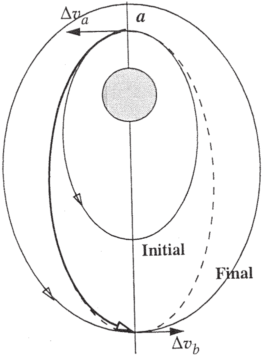


# The Hohmann Transfer

## A Minimum Energy Orbit?

The Hohmann transfer is the energy-optimal two burn maneuver between any two coaxial elliptic orbits.




- Proposed by Hohmann (1925)
  - Why?
- Proven for circular target orbits by Lawden (1952)
- Proven for coaxial elliptical initial and target orbits by Thompson (1986)

### The Hohmann Transfer
A Minimum Energy Orbit?

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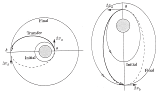


- Proposed by Hohmann (1925)
  - Why?
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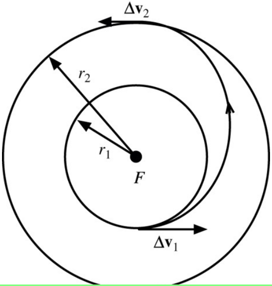
- Proven for coaxial elliptical initial and target orbits by Thompson (1986)

- Also first proposed use of separable lunar landers
- Did not participate in Nazi rocket program.
- Died of hunger/stress after allied bombardment of Essen
- Optimality was originally a conjecture.

- Published in "Die Erreichbarkeit der Himmelskörper (The Attainability of Celestial Bodies)" (1925) [PDF Available Here]

# The Hohmann Transfer

We will first consider the circular case.




**Theorem 3 (The Hohmann Conjecture).**

*The energy-optimal transfer orbit between two circular orbits of radii $r_1$ and $r_2$ is an elliptic orbit with*

$$r_p = r_1 \qquad \textit{and} \qquad r_a = r_2$$

This yields the orbital elements of the transfer orbit ($a$, $e$) as

$$a = \frac{r_a + r_p}{2} = \frac{r_1 + r_2}{2} \qquad \text{and} \qquad e = 1 - \frac{r_p}{a} = \frac{r_2 - r_1}{r_2 + r_1}$$

# The Hohmann Transfer

To calculate the required $\Delta v_1$ and $\Delta v_2$, the initial velocity is the velocity of a circular orbit of radius $r_1$

$$v_{init} = \sqrt{\frac{\mu}{r_1}}$$

The required initial velocity is that of the transfer orbit at perigee. From the vis-viva equation,

$$v_{trans,p} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = \sqrt{2\mu}\sqrt{\frac{1}{r_1} - \frac{1}{r_1 + r_2}} = \sqrt{2\mu\frac{r_2}{r_1(r_1 + r_2)}}$$

So the initial $\Delta v_1$ is

$$\Delta v_1 = v_{trans,p} - v_{init} = \sqrt{2\mu\frac{r_2}{r_1(r_1 + r_2)}} - \sqrt{\frac{\mu}{r_1}} = \sqrt{\frac{\mu}{r_1}}\left(\sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1\right)$$

The velocity of the transfer orbit at apogee is

$$v_{trans,a} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = \sqrt{2\mu\frac{r_1}{r_2(r_1 + r_2)}}$$

# The Hohmann Transfer

The required velocity for a circular orbit at apogee is

$$v_{fin} = \sqrt{\frac{\mu}{r_2}}$$

So the final $\Delta v_2$ is

$$\Delta v_2 = v_{fin} - v_{trans,a} = \sqrt{\frac{\mu}{r_2}} - \sqrt{2\mu \frac{r_1}{r_2(r_1 + r_2)}} = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}} \right)$$

Thus we conclude to raise a circular orbit from radius $r_1$ to radius $r_2$, we use

$$\Delta v_1 = \sqrt{\frac{\mu}{r_1}} \left( \sqrt{\frac{2r_2}{(r_1 + r_2)}} - 1 \right)$$

$$\Delta v_2 = \sqrt{\frac{\mu}{r_2}} \left( 1 - \sqrt{\frac{2r_1}{(r_1 + r_2)}} \right)$$

# Hohmann Transfer Illustration

# The Hohmann Transfer

## Transfer Time

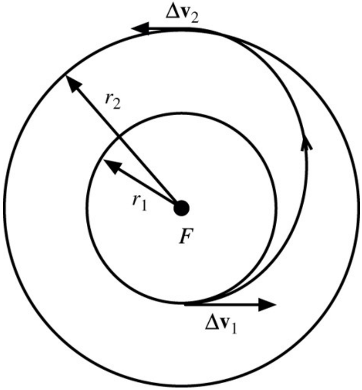
The Hohmann transfer is optimal

- Only for impulsive transfers
  - Continuous Thrust is not considered
- Only for **two** impulse transfers
  - A three impulse transfer can be better
  - Bi-elliptics are better

The transfer time is simply half the period of the orbit. Hence

$$\Delta t = \frac{\tau}{2} = \pi\sqrt{\frac{a^3}{\mu}}$$

$$= \pi\sqrt{\frac{(r_1 + r_2)^3}{8\mu}}$$




The Hohmann transfer is also the *Maximum Time 2-impulse Transfer*.

- Always a tradeoff between time and efficiency
- Bielliptic Transfers extend this tradeoff.

## The Hohmann Transfer

Transfer Time

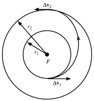
The Hohmann transfer is optimal

- Only for impulsive transfers
  - Continuous Thrust is not considered
- Only for **two** impulse transfers
  - A three impulse transfer can be better
  - Bi-elliptics are better

The transfer time is simply half the period of the orbit. Hence

$$\Delta t = \frac{\tau}{2} = \pi\sqrt{\frac{a^3}{\mu}}$$

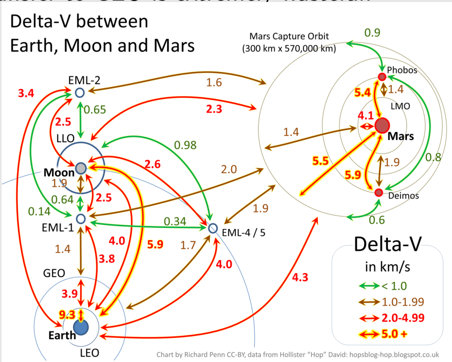
$$= \pi\sqrt{\frac{(r_1+r_2)^3}{8\mu}}$$




The Hohmann transfer is also the *Maximum Time 2-impulse Transfer*.

- Always a tradeoff between time and efficiency
- Bielliptic Transfers extend this tradeoff.

- The slowest part of the orbit is at apogee.
- Due to Oberth effect, you want to use as much $\Delta v$ budget as possible at low altitude. Bi-elliptics use this to further reduce $\Delta v$ at apogee (Next Lecture)
- Hohmann transfer to GEO is extremely wasteful!



# Numerical Example (Parking Orbit to GEO)

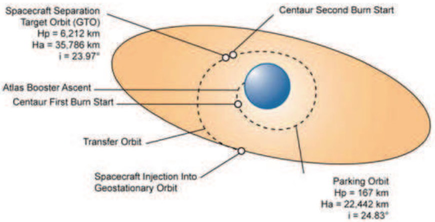
**Problem:** Suppose we are in a circular parking orbit at an altitude of 191.34km and we want to raise our altitude to 35,781km. Describe the required orbital maneuvers (time and $\Delta v$).

**Solution:** We will use a Hohmann transfer between circular orbits of

$r_1 = 191.35km + 1ER = 1.03ER$ and $r_2 = 35781km + 1ER = 6.61ER$

The initial velocity is

$$v_i = \sqrt{\frac{\mu}{r_1}} = .985\frac{ER}{TU}$$




The transfer ellipse has $a = \frac{r_1 + r_2}{2} = 3.82ER$. The velocity at perigee is

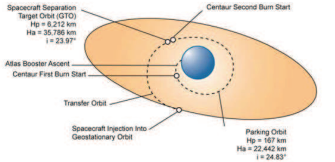
$$v_{trans,1} = \sqrt{\frac{2\mu}{r_1} - \frac{\mu}{a}} = 1.296\frac{ER}{TU}$$

Thus the initial $\Delta v$ is $\Delta v_1 = 1.296 - .985 = .315\frac{ER}{TU}$.

# Numerical Example

The velocity at apogee is

$$v_{trans,1} = \sqrt{\frac{2\mu}{r_2} - \frac{\mu}{a}} = .202\frac{ER}{TU}$$




However, the required velocity for a circular orbit at radius $r_2$ is

$$v_f = \sqrt{\frac{\mu}{r_2}} = .389\frac{ER}{TU}$$

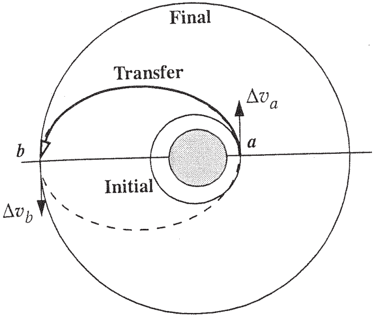
Thus the final $\Delta v$ is $\Delta v_2 = .389 - .202 = .182\frac{ER}{TU}$. The second $\Delta v$ maneuver should be made at time

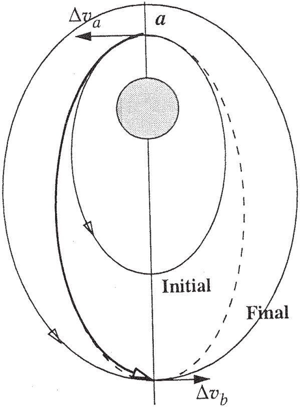
$$t_{fin} = \pi\sqrt{\frac{a^3}{\mu}} = 23.45TU = 5.256hr$$

The total $\Delta v$ budget is $.497ER/TU$.

# The Elliptic Hohmann Transfer

The Hohmann transfer is also energy optimal for coaxial elliptic orbits.




The only ambiguity is whether to make the initial burn at perigee or apogee.

- Need to check both cases
- Often better to make initial burn at perigee
  - Due to Oberth Effect

# Summary

This Lecture you have learned:

Coplanar Orbital Maneuvers

- Impulsive Maneuvers
  - $\Delta v$
- Single Burn Maneuvers
- Hohmann transfers
  - Elliptic
  - Circular

**Next Lecture:** Oberth Effect, Bi-elliptics, Out-of-plane maneuvers.