On Positive Quadratic Forms and the Stability of Linear Time-Delay Systems

Matthew Peet [*,1], Antonis Papachristodoulou [**],and Sanjay Lall [1]

Abstract

We consider the problem of constructing Lyapunov functions for linear differential equations with delays. For such systems it is known that stability implies that there exists a quadratic Lyapunov function on the state space, although this is in general infinite dimensional. We give an explicit parametrization of a finite-dimensional subset of the cone of Lyapunov functions using positive semidefinite matrices. This allows the computation to be formulated as a semidefinite program.

Key words: Time-delay, Infinite dimensional systems, Stability, Semidefinite programming, Parametric uncertainty

1 Introduction

In this paper we present an approach to the construction of Lyapunov functions for systems with time-delays. Specifically, we are interested in systems of the form

$$\dot{x}(t) = \sum_{i=1}^{k} A_i x(t - h_i) + \int_0^h B(s)x(t-s)ds, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$. In the simplest case we are given information about the the delays $h_0, \ldots, h_k$, the matrices $A_0, \ldots, A_k$, and a matrix of polynomials, B, and we would like to determine whether the system is stable.

For such systems it is known that if the system is stable, then this property can be proven using a Lyapunov functional of the form

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t)\phi(t) \, ds \, dt,$$

where M and N are piecewise-continuous matrix-valued functions. Here $\phi : [-h, 0] \to \mathbb{R}^n$ is an element of the state space, which for delayed systems is the infinite-dimensional space of continuous functions mapping $[-h, 0]$ to $\mathbb{R}^n$. The derivative of V has a similar structure to V and is also defined by matrix functions which are affine transformations of M and N.

The goal of this paper is to show how to use semidefinite programming to construct piecewise-continuous functions M and N such that the functional V is positive. Roughly speaking, in Section 3, for a piecewise-continuous function M, we show that

$$V_1(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$

is positive for all ϕ if and only if there exists a piecewise

* Department of Aeronautics and Astronautics 4035, Stanford University, Stanford CA 94305-4035, U.S.A.
**Department of Engineering Science, University of Oxford, Oxford, OX1 3PJ, UK

Email addresses: `mmpeet@stanford.edu` (Matthew Peet), `antonis@eng.ox.ac.uk` (Antonis Papachristodoulou), `lall@stanford.edu` (Sanjay Lall).

continuous matrix-valued function T such that

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for all } t$$

$$\int_{-h}^{0} T(t)\, dt = 0$$

That is, we convert positivity of the integral to *pointwise* positivity. This result is stated precisely in Theorem 6. Pointwise positivity may then be easily enforced, and in particular in this case is equivalent to a sum-of-squares constraint. The constraint that T integrates to zero is a simple linear constraint on the coefficients. The condition that the derivative of the Lyapunov function be negative is similarly enforced. Notice that the sufficient condition that $M(s)$ be pointwise nonnegative is conservative, and as the equivalence above shows it is easy to generate examples where V_1 is nonnegative even though $M(s)$ is not pointwise nonnegative.

In Section 4, we show that if N is defined by polynomial N_{ij} on the interval $I_i \times I_j$, then

$$V_2(\phi) = \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s,t) \phi(t)\, ds\, dt$$

is positive if and only is there exists a $Q \geq 0$ such that

$$N_{ij}(s,t) = Z(s)^T Q_{ij} Z(t).$$

Here Z is a monomial basis. This result is stated precisely in Theorem 9. This result allows us to express our positivity conditions directly in terms of positive matrices. Note that simple positivity of N in this case is not even sufficient for positivity of V_2 as illustrated by a simple counterexample such as $N(s,t) = (s-t)^2$, $h = 2$ and

$$\phi(s) = s + 1.$$

1.1 Prior Work

The use of Lyapunov functionals on an infinite dimensional space to analyze differential equations with delay originates with the work of Krasovskii [9]. For linear systems, quadratic Lyapunov functions were first considered by Repin [15]. The book of Gu, Kharitonov, and Chen [4] presents many useful results in this area, and further references may be found there as well as in Hale and Lunel [5], Kolmanovskii and Myshkis [8] and Niculescu [12]. The idea of using sum-of-squares polynomials together with semidefinite programming to construct Lyapunov functions originates in Parrilo [13].

1.2 Notation

Let $\mathbb{N}$ denote the set of nonnegative integers. Let $\mathbb{S}^n$ be the set of $n \times n$ real symmetric matrices, and for $S \in \mathbb{S}^n$ we write $S \succeq 0$ to mean that S is positive semidefinite. For X any Banach space and $I \subset \mathbb{R}$ any interval let $\Omega(I, X)$ be the space of all functions

$$\Omega(I, X) = \{\, f \colon I \to X \,\}$$

and let $C(I, X)$ be the Banach space of bounded continuous functions

$$C(I, X) = \{\, f : I \to X \mid f \text{ is continuous and bounded}\,\}$$

equipped with the norm

$$\|f\| = \sup_{t \in I} \|f(t)\|_X$$

We will omit the range space when it is clear from the context; for example we will simply write $C[a, b]$ to mean $C([a, b], X)$. A function $f \in \Omega[a, b]$ is called piecewise C^1 if there exists a finite number of points $a \leq h_1 < \cdots < h_k \leq b$ such that f is continuous at all $x \in [a, b] \backslash \{h_1, \ldots, h_k\}$. We will write, as usual

$$f(a+) = \lim_{t \to a+} f(t).$$

Define also the projection $H_t : \Omega[-h, \infty) \to \Omega[-h, 0]$ for $t \geq 0$ and $h > 0$ by

$$(H_t x)(s) = x(t+s) \quad \text{for all } s \in [-h, 0].$$

We follow the usual convention and denote $H_t x$ by x_t.

We consider polynomials in n variables. For $\alpha \in \mathbb{N}^n$ define the monomial in n variables x^α by $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. We say M is a real matrix polynomial in n variables if for some finite set $W \subset \mathbb{N}^n$ we have

$$M(x) = \sum_{\alpha \in W} A_\alpha x^\alpha$$

where A_α is a real matrix for each $\alpha \in W$.

For Banach spaces X and Y, define an integral operator A with kernel function $k : I \times I \to \{f : X \to Y\}$ to be the map $A : \Omega(I, X) \to \Omega(I, Y)$ such that

$$Ax(s) = \int_I k(s, t) x(t)\, dt$$

is in $\Omega(I, Y)$. Similarly, a multiplier operator B with multiplier $m : I \to \{f : X \to Y\}$ is defined to be the map $B : \Omega(I, X) \to \Omega(I, Y)$ such that

$$Bx(s) = m(s) x(s)$$

If $X = Y$ and $\Omega(I, X)$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, then an operator, C, is positive with respect to $\Omega(I, X)$ if

$$\langle x, Cx \rangle \geq 0 \quad \text{for all } x \in \Omega(I, X)$$

2 System Formulation

Suppose $0 = h_0 < h_1 < \cdots < h_k = h$, $H = \{-h_0, \ldots, -h_k\}$, and let $H^c = [-h, 0] \backslash H$. Suppose $A_0, \ldots, A_k \in \mathbb{R}^{n \times n}$ and B is continuous. We consider linear differential equations with delay, of the form

$$\dot{x}(t) = \sum_{i=0}^{k} A_i x(t - h_i) + \int_0^h B(s) x(t-s) ds \quad \text{for all } t \geq 0 \tag{2}$$

where the trajectory $x : [-h, \infty) \to \mathbb{R}^n$. The boundary conditions are specified by a given function $\phi : [-h, 0] \to \mathbb{R}^n$ and the constraint

$$x(t) = \phi(t) \quad \text{for all } t \in [-h, 0]. \tag{3}$$

If $\phi \in C[-h, 0]$, then there exists a unique differentiable function x satisfying (2) and (3). The system is called **exponentially stable** if there exists $\sigma > 0$ and $a \in \mathbb{R}$ such that for every initial condition $\phi \in C[-h, 0]$ the corresponding solution x satisfies

$$\|x(t)\| \leq a e^{-\sigma t} \|\phi\| \quad \text{for all } t \geq 0.$$

We write the solution as an explicit function of the initial conditions using the map $G : C[-h, 0] \to \Omega[-h, \infty)$, defined by

$$(G\phi)(t) = x(t) \quad \text{for all } t \geq -h$$

where x is the unique solution of (2) and (3) corresponding to initial condition ϕ. Also for $s \geq 0$ define the *flow map* $\Gamma_s : C[-h, 0] \to C[-h, 0]$ by

$$\Gamma_s \phi = H_s G \phi$$

which maps the state of the system x_t to the state at a later time $x_{t+s} = \Gamma_s x_t$.

2.1 Lyapunov Functions

Suppose $V : C[-h, 0] \to \mathbb{R}$. We use the notion of derivative as follows. Define the **Lie derivative** of V with respect to Γ by

$$\dot{V}(\phi) = \limsup_{r \to 0+} \frac{1}{r} \big(V(\Gamma_r \phi) - V(\phi)\big)$$

We will use the notation $\dot{V}$ for both the Lie derivative and the usual derivative, and state explicitly which we

mean if it is not clear from context. We will consider the set X of quadratic functions, where $V \in X$ if there exists bounded piecewise C^1 functions $M : [-h, 0) \to \mathbb{S}^{2n}$ and $N : [-h, 0) \times [-h, 0) \to \mathbb{R}^{n \times n}$ such that

$$V(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T M(s) \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T N(s, t) \phi(t) \, ds \, dt \tag{4}$$

The following important result shows that for linear systems with delay, the system is exponentially stable if and only if there exists a quadratic Lyapunov function.

Theorem 1 *The linear system defined by equations (2) and (3) is exponentially stable if and only if there exists a Lie-differentiable function $V \in X$ and $\varepsilon > 0$ such that for all $\phi \in C[-h, 0]$*

$$V(\phi) \geq \varepsilon \|\phi(0)\|^2$$
$$\dot{V}(\phi) \leq -\varepsilon \|\phi(0)\|^2 \tag{5}$$

Further $V \in X$ may be chosen such that the corresponding functions M and N of equation (4) have the following smoothness property: $M(s)$ and $N(s, t)$ are continuous at all s, t such that $s \in H^c$ and $t \in H^c$.

PROOF. See Kharitonov and Hinrichsen [6] for a recent proof.

3 Positivity of Integrals

We would like to be able to computationally find functions $V \in X$ which satisfy the positivity conditions (5). In this section, we address the first part of the functional,

$$V_1(y) = \int_{-h}^{0} \begin{bmatrix} y(0) \\ y(t) \end{bmatrix}^T M \begin{bmatrix} y(0) \\ y(t) \end{bmatrix} dt \geq 0. \tag{6}$$

The following three lemmas are necessary to prove the main result, Theorem 6.

Lemma 2 *Suppose $f : [-h, 0] \to \mathbb{R}$ is piecewise continuous. Then the following are equivalent.*

(i) $\displaystyle \int_{-h}^{0} f(t) \, dt \geq 0$

(ii) there exists a function $g : [-h, 0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t) + g(t) \geq 0 \quad \text{for all } t$$
$$\int_{-h}^{0} g(t) \, dt = 0$$

PROOF. The direction (ii) $\implies$ (i) is immediate. To show the other direction, suppose (i) holds, and let g be

$$g(t) = -f(t) + \frac{1}{h}\int_{-h}^{0} f(s)\,ds \quad \text{for all } t$$

Then g satisfies (ii).

Lemma 3 *Suppose $H = \{-h_0, \ldots, -h_k\}$ and let $H^c = [-h, 0]\backslash H$. Let $f\colon [-h, 0]\times \mathbb{R}^n \to \mathbb{R}$ be continuous on $H^c \times \mathbb{R}^n$, and suppose there exists a bounded function $z\colon [-h, 0] \to \mathbb{R}$, continuous on H^c, such that for all $t \in [-h, 0]$*

$$f\big(t, z(t)\big) = \inf_x f(t, x)$$

Further suppose for each bounded set $X \subset \mathbb{R}^n$ the set

$$\{\, f(t, x) \mid x \in X, t \in [-h, 0]\,\}$$

is bounded. Then

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt = \int_{-h}^{0} \inf_x f(t, x)\,dt \qquad (7)$$

PROOF. Let

$$K = \int_{-h}^{0} \inf_x f(t, x)\,dt$$

It is easy to see that

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq K$$

since if not there would exist some continuous function y and some interval on which

$$f\big(t, y(t)\big) < \inf_x f(t, x)$$

which is clearly impossible.

We now show that the left-hand side of (7) is also less than or equal to K, and hence equals K. We need to show that for any $\varepsilon > 0$ there exists $y \in C[-h, 0]$ such that

$$\int_{-h}^{0} f\big(t, y(t)\big)\,dt < K + \varepsilon$$

To do this, for each $n \in \mathbb{N}$ define the set $H_n \subset \mathbb{R}$ by

$$H_n = \bigcup_{i=1}^{k-1} (h_i - \alpha/n, h_i + \alpha/n)$$

and choose $\alpha > 0$ sufficiently small so that $H_1 \subset (-h, 0)$. Let z be as in the hypothesis of the lemma, and pick M and R so that

$$M > \sup_t \|z(t)\|$$
$$R = \sup\big\{\, |f(t, x)| \mid t \in [-h, 0], \|x\| \leq M\,\big\}$$

For each n choose a continuous function $x_n : [-h, 0] \to \mathbb{R}^n$ such that $x_n(t) = z(t)$ for all $t \notin H_n$ and

$$\sup_{t \in [-h, 0]} \|x_n(t)\| < M$$

This is possible, for example, by linear interpolation. Now we have, for the continuous function x_n

$$\int_{-h}^{0} f\big(t, x_n(t)\big)\,dt = K + \int_{-h}^{0} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big)\Big)\,dt$$
$$= K + \int_{H_n} \Big(f\big(t, x_n(t)\big) - f\big(t, z(t)\big)\Big)\,dt$$
$$\leq K + 4R\alpha(k-1)/n$$

This proves the desired result.

Lemma 4 *Suppose $f\colon [-h, 0] \times \mathbb{R}^n \to \mathbb{R}$ and the hypotheses of Lemma 3 hold. Then the following are equivalent.*

(i) For all $y \in C[-h, 0]$

$$\int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq 0$$

(ii) There exists $g : [-h, 0] \to \mathbb{R}$ which is piecewise continuous and satisfies

$$f(t, z) + g(t) \geq 0 \quad \text{for all } t, z$$
$$\int_{-h}^{0} g(t)\,dt = 0$$

PROOF. Again we only need to show that (i) implies (ii). Suppose (i) holds, then

$$\inf_{y \in C[-h,0]} \int_{-h}^{0} f\big(t, y(t)\big)\,dt \geq 0$$

and hence by Lemma 3 we have

$$\int_{-h}^{0} r(t)\,dt \geq 0$$

where $r : [-h, 0] \to \mathbb{R}^n$ is given by

$$r(t) = \inf_x f(t, x) \quad \text{for all } t$$

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The function r is continuous on H^c since f is continuous on $H^c \times \mathbb{R}^n$. Hence by Lemma 2, there exists g such that condition (ii) holds, as desired.

Remark 5 *Lemma 4 is stated in a way which applies to general functions given specific conditions are satisfied. The following main result shows that these conditions are satisfied in the quadratic case.*

Theorem 6 *Suppose $M : [-h, 0] \to \mathbb{S}^{m+n}$ is piecewise continuous, and there exists $\varepsilon > 0$ such that for all $t \in [-h, 0]$ we have*

$$M_{22}(t) \geq \varepsilon I$$
$$M(t) \leq \varepsilon^{-1} I$$

Then the following are equivalent.

(i) For all $x \in \mathbb{R}^m$ and continuous $y : [-h, 0] \to \mathbb{R}^n$

$$\int_{-h}^{0} \begin{bmatrix} x \\ y(t) \end{bmatrix}^T M \begin{bmatrix} x \\ y(t) \end{bmatrix} dt \geq 0 \qquad (8)$$

(ii) There exists a function $T : [-h, 0] \to \mathbb{S}^m$ which is piecewise continuous and satisfies

$$M(t) + \begin{bmatrix} T(t) & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for all } t \in [-h, 0]$$
$$\int_{-h}^{0} T(t)\, dt = 0$$

PROOF. Again we only need to show (i) implies (ii). Suppose $x \in \mathbb{R}^n$, and define

$$f(t, z) = \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \quad \text{for all } t, z$$

Since by the hypothesis M_{22} has a lower bound, it is invertible for all t and its inverse is piecewise continuous. Therefore $z(t) = -M_{22}(t)^{-1} M_{21}(t) x$ is the unique minimizer of $f(t, z)$ with respect to z. By the hypothesis (i), we have that for all $y \in C[-h, 0]$

$$\int_{-h}^{0} f(t, y(t))\, dt \geq 0$$

Hence by Lemma 4 there exists a function g such that

$$g(t) + f(t, z) \geq 0 \quad \text{for all } t, z$$
$$\int_{-h}^{0} g(t)\, dt = 0 \qquad (9)$$

The proof of Lemma 2 gives one such function as

$$g(t) = -f(t, z(t)) + \int_{-h}^{0} f(s, z(s))\, dt$$

We have

$$f(t, z(t)) = x^T \big(M_{11}(t) - M_{12}(t) M_{22}^{-1}(t) M_{21}(t) \big) x$$

and therefore $g(t)$ is a quadratic function of x, say $g(t) = x^T T(t) x$, and $T : [-h, 0] \to \mathbb{S}^m$ is continuous on H^c. Then equation (9) implies

$$x^T T(t) x + \begin{bmatrix} x \\ z \end{bmatrix}^T M(t) \begin{bmatrix} x \\ z \end{bmatrix} \geq 0 \quad \text{for all } t, z, x$$

as required.

We have now shown that the convex cone of functions M such that the first term of (4) is nonnegative is exactly equal to the sum of the cone of pointwise nonnegative functions and the linear space of functions whose integral is zero. Note that in (8) the vectors x and y are allowed to vary independently, whereas (4) requires that $x = y(0)$. It is however straightforward to show that this additional constraint does not change the result, using the technique in the proof of Lemma 3.

The key benefit of this is that it is easy to parametrize the latter class of functions, and in particular when M is a polynomial these constraints are semidefinite representable constraints on the coefficients of M.

4   Positivity of Double Integrals

The previous section dealt only with the first part of the functional. We now give results which allow us to treat the second part of the functional. We first define the following functions. Let Z_d be the vector of monomials in variable y of degree d or less, for example

$$Z_4(y)^T = \begin{bmatrix} 1 & y & y^2 & y^3 & y^4 \end{bmatrix}$$

Define the function Z_d^n by

$$Z_d^n(y) = I \otimes Z_d(y)$$

This means the entries of Z_d^n are monomials in y. The rows of Z_d^n define a basis for the degree d polynomials in n dimensions.

Theorem 7 *Suppose $N : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ is a polynomial of degree $2d$. Then the following are equivalent:*

(1) The following hold for all $x \in \mathcal{C}$

$$\int_I \int_I x(s) N(s,t) x(t) ds dt \geq 0$$

(2) There exists a $Q \geq 0$ such that

$$N(s,t) + N(t,s)^T = Z_d^n(s)^T Q Z_d^n(t)$$

PROOF. That 2 implies 1 is clear. Now suppose N is a polynomial of degree d, then N can be represented as

$$N(s,t) = \sum_{i,j=0}^{2d} A_{ij} s^i t^j = Z_{2d}^n(s)^T A Z_{2d}^n(t).$$

Now we have

$$N(s,t) + N(t,s)^T = Z_{2d}^n(s)^T (A + A^T) Z_{2d}^n(t).$$

Let $Q = A + A^T$. Now suppose there exists some vector u such that $u^T Q u < 0$. Let

$$B = \int_I Z_{2d}^n(s) Z_{2d}^n(s)^T ds.$$

Since the rows of Z_{2d}^n are linearly independent functions, we have that B is invertible. Now let $x(s) = Z_{2d}(s)^T B^{-1} u$. Then we have the following.

$$2 \int_I \int_I x(s)^T N(s,t) x(t) ds dt$$
$$= \int_I \int_I x(s)^T (N(s,t) + N(t,s)^T) x(t) ds dt$$
$$= \int_I \int_I x(s)^T Z_{2d}^n(s)^T (A + A^T) Z_{2d}^n(t) x(t) ds dt$$
$$= \int_I (Z_{2d}^n(s) Z_{2d}^n(s)^T B^{-1} u)^T ds Q \int_I Z_{2d}^n(t) Z_{2d}^n(t)^T B^{-1} u dt$$
$$= (B B^{-1} u)^T Q B B^{-1} u$$
$$= u^T Q u < 0$$

Thus we have by contradiction that $Q \geq 0$. Now, divide Q as

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix},$$

where $Q_{22} \in \mathbb{R}^{d \times d}$. Then $Q_{22} = 0$ since N is of degree $2d$. Therefore $Q_{12} = Q_{12}^T = 0$ since $Q \geq 0$. Thus $Q_{11} \geq 0$ and

$$N(s,t) + N(t,s)^T = Z_d^n(s)^T Q_{11} Z_d^n(t)$$

4.1 Piecewise-Continuous Kernels

We now deal with piecewise-continuous functions. First, some notation. Define the intervals

$$H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1 \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \ldots, k \end{cases}$$

Let $I = [-h, 0]$ and define the vector of indicator functions $g : I \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \ldots, k$ and all $t \in [-h, 0]$.

Define a_i and b_i to be the scalars such that $\{a_i s + b_i : s \in [-h, 0]\} = [h_i, h_{i-1}]$ for $i = 1, \ldots, k$.

Lemma 8 *Suppose the function $N : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ is continuous on $H^c \times H^c$. Define $N_{ij}(s,t) = N(s,t) g_i(s) g_j(t)$. Suppose $R : \mathbb{R}^2 \to \mathbb{R}^{nK \times nK}$ is given by*

$$R(s,t) = \sum_{i,j=1}^{k} N_{ij}(a_i s + b_i, a_j t + b_j) \otimes e_i e_j^T$$
$$= \begin{bmatrix} N_{11} & N_{12} & \cdots & N_{1k} \\ N_{21} & & & \\ \vdots & & \ddots & \\ N_{k1} & & & N_{kk} \end{bmatrix}$$

where e_i are the unit vectors in $\mathbb{R}^k$. Then N defines a positive integral operator if and only if R defines a positive integral operator.

PROOF. Suppose R defines a positive integral operator. Let $x_i(s) = g_i(s) x(s)$ and $\bar{x}_i(s) = a_i x_i(a_i s + b_i)$. Let $s_i = a_i s + b_i$ and $t_i = a_i t + b_i$. Then since R is positive, we have

$$\int_I \int_I x(s)^T N(s,t) x(t) ds dt$$
$$= \sum_{i,j=1}^{k} \int_{H_i} \int_{H_j} x_i(s_i)^T N_{ij}(s_i, t_j) x_j(t_j) ds_i dt_j$$
$$= \sum_{i,j=1}^{k} \int_I \int_I \bar{x}_i(s)^T N_{ij}(a_i s + b_i, a_j t + b_j) \bar{x}_j(t) ds dt$$
$$= \int_I \int_I \bar{x}(s)^T R(s,t) \bar{x}(t) ds dt \geq 0$$

since $\bar{x}$ is continuous on I.

For the converse, suppose N defines a positive operator, then for any $x \in \mathcal{C}$, partition x as

$$x = \left[x_1^T, \cdots, x_k^T \right]^T.$$

and define $\bar{x}_i(s) = x_i(s/a_i - b_i/a_i)/a_i$ and

$$\bar{x}(s) = \sum_{i=1}^{k} \bar{x}_i(s) g_i(s).$$

Then $\bar{x}(s)$ is continuous on H^c and

$$= \int_I \int_I x(s)^T R(s,t) x(t) ds dt$$

$$= \sum_{i,j=1}^{k} \int_I \int_I x_i(s)^T N_{ij}(a_i s + b_i, a_j t + b_j) x_j(s) ds dt$$

$$= \sum_{i,j=1}^{k} \int_{H_i} \int_{H_j} \bar{x}_i(s_i)^T N_{ij}(s_i, t_j) \bar{x}_j(t_j) ds_i dt_j$$

$$= \sum_{i,j=1}^{k} \int_{H_i} \int_{H_j} \bar{x}_i(s)^T g_i(s) N(s,t) g_j(t) \bar{x}_j(t) ds dt$$

$$= \sum_{i,j=1}^{k} \int_{H_i} \int_{H_j} \bar{x}(s)^T N(s,t) \bar{x}(t) ds dt$$

$$= \int_I \int_I \bar{x}(s)^T N(s,t) \bar{x}(t) ds dt \geq 0.$$

The last follows since it can be shown that positivity for continuous functions on $[-h, 0]$ implies positivity for continuous functions on H^c.

Theorem 9 *Suppose the function $N : \mathbb{R}^2 \to \mathbb{R}^{n \times n}$ is piecewise-continuous with N_{ij} polynomials of degree $2d$. Then the following are equivalent:*

(1) For all $x \in \mathcal{C}$,

$$\int_I \int_I x(s) N(s,t) x(t) ds dt \geq 0$$

(2) There exists a $Q \geq 0$ such that if Q_{ij} is the i,jth block of Q, then

$$N_{ij}(s,t) = Z_d^n(s)^T Q_{ij} Z_d^n(t)$$

PROOF. Suppose N satisfies condition 1. Let R be given by

$$R(s,t) = \sum_{i,j=1}^{k} N_{ij}(a_i s + b_i, a_j t + b_j) \otimes e_i e_j^T$$

By Lemma 8 and Theorem 7,

$$R(s,t) = Z_d^{nk}(s) \bar{Q} Z_d^{nk}(t)$$

for some $\bar{Q} \geq 0$ and so

$$N_{ij}(s,t) = Z_d^n \left(\frac{s}{a_i} - \frac{b_i}{a_i} \right)^T \bar{Q}_{ij} Z_d^n \left(\frac{t}{a_j} - \frac{b_j}{a_j} \right),$$

where $\bar{Q}_{ij}$ is the i,jth block of $\bar{Q}$. Now Define T_i to be the matrix such that

$$Z_d^n \left(\frac{s}{a_i} - \frac{b_i}{a_i} \right) = T_i Z_d^n(s).$$

Existence and invertibility of T_i can be shown by simple construction. Define

$$T = \sum_{i=1}^{k} T_i \otimes e_i e_i^T$$

where e_i are the unit vectors in $\mathbb{R}^k$. Then T_i are the block-diagonal elements of T. Let $Q = T^T \bar{Q} T$. Then $Q \geq 0$, $Q_{ij} = T_i^T \bar{Q}_{ij} T_j$ and

$$N_{ij}(s,t) = Z_d^n \left(\frac{s}{a_i} - \frac{b_i}{a_i} \right)^T \bar{Q}_{ij} Z_d^n \left(\frac{t}{a_j} - \frac{b_j}{a_j} \right)$$
$$= Z_d^n(s)^T T_i^T \bar{Q}_{ij} T_j Z_d^n(t)$$
$$= Z_d^n(s)^T Q_{ij} Z_d^n(t).$$

Therefore, we have condition 2.

Now suppose that condition 2 holds. Let $\bar{Q} = T^{-T} Q T^{-1} \geq 0$ and define

$$R(s,t) = Z_d^{nk}(s)^T \bar{Q} Z_d^{nk}(t).$$

Then

$$R(s,t) = Z_d^{nk}(s)^T T^{-T} Q T^{-1} Z_d^{nk}(t)$$

$$= \sum_{i,j=1}^{k} Z_d^n(s)^T T_i^{-T} Q_{ij} T_j^{-1} Z_d^n(t) \otimes e_i e_j^T$$

$$= \sum_{i,j=1}^{k} Z_d^n(a_i s + b_i)^T Q_{ij} Z_d^n(a_j t + b_j) \otimes e_i e_j^T$$

$$= \sum_{i,j=1}^{k} N_{ij}(a_i s + b_i, a_j t + b_j) \otimes e_i e_j^T$$

Therefore, by Theorem 7 and Lemma 8, N defines a positive operator and thus condition 2 holds.

5 Lie Derivatives

5.1 Single Delay Case

We first present the single delay case, as it will illustrate the formulation in the more complicated case. Suppose that $V \in X$ is given by (4), where $M : [-h, 0] \to \mathbb{S}^{2n}$ and $N : [-h, 0] \times [-h, 0] \to \mathbb{R}^{n \times n}$. Since there is only one delay, if the system is exponentially stable then there always exists a Lyapunov function of this form with C^1 functions M and N. Then the Lie derivative of V is

$$
\dot{V}(\phi) = \int_{-h}^{0} \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(0) \\ \phi(-h) \\ \phi(s) \end{bmatrix} ds
$$
$$
+ \int_{-h}^{0} \int_{-h}^{0} \phi(s)^T E(s,t) \phi(t) \, ds \, dt \quad (10)
$$

Partition D and M as

$$
M(t) = \begin{bmatrix} M_{11} & M_{12}(t) \\ M_{21}(t) & M_{22}(t) \end{bmatrix} \quad D(t) = \begin{bmatrix} D_{11} & D_{12}(t) \\ D_{21}(t) & D_{22}(t) \end{bmatrix}
$$

so that $M_{11} \in \mathbb{S}^n$ and $D_{11} \in \mathbb{S}^{2n}$. Without loss of generality we can assume M_{11} and D_{11} are constant. The functions D and E are linearly related to M and N by

$$
D_{11} = \begin{bmatrix} A_0^T M_{11} + M_{11} A_0 & M_{11} A_1 \\ A_1^T M_{11} & 0 \end{bmatrix}
$$
$$
+ \frac{1}{h} \begin{bmatrix} M_{12}(0) + M_{21}(0) & -M_{12}(-h) \\ -M_{21}(-h) & 0 \end{bmatrix}
$$
$$
+ \frac{1}{h} \begin{bmatrix} M_{22}(0) & 0 \\ 0 & -M_{22}(-h) \end{bmatrix}
$$
$$
D_{12}(t) = \begin{bmatrix} A_0^T M_{12}(t) - \dot{M}_{12}(t) + N(0,t) \\ A_1^T M_{12}(t) - N(-h,t) \end{bmatrix}
$$
$$
D_{22}(t) = -\dot{M}_{22}(t)
$$
$$
E(s,t) = \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}
$$

5.2 Multiple-delay case

We now define the class of functions under consideration for the Lyapunov functions. Define the intervals

$$
H_i = \begin{cases} [-h_1, 0] & \text{if } i = 1 \\ [-h_i, -h_{i-1}) & \text{if } i = 2, \dots, k \end{cases}
$$

For the Lyapunov function V, define the sets of functions

$$
Y_1 = \Big\{ M : [-h, 0] \to \mathbb{S}^{2n} \mid
$$
$$
M_{11}(t) = M_{11}(s) \quad \text{for all } s, t \in [-h, 0]
$$
$$
M \text{ is } C^1 \text{ on } H_i \quad \text{for all } i = 1, \dots, k \Big\}
$$
$$
Y_2 = \Big\{ N : [-h, 0] \to \mathbb{S}^n \mid
$$
$$
N(s,t) = N(t,s)^T \quad \text{for all } s, t \in [-h, 0]
$$
$$
N \text{ is } C^1 \text{ on } H_i \times H_j \quad \text{for all } i, j = 1, \dots, k \Big\}
$$

and for its derivative, define

$$
Z_1 = \Big\{ D : [-h, 0] \to \mathbb{S}^{(k+2)n} \mid
$$
$$
D_{ij}(t) = D_{ij}(s) \quad \text{for all } s, t \in [-h, 0]
$$
$$
\text{for } i, j = 1, \dots, 3
$$
$$
D \text{ is } C^0 \text{ on } H_i \quad \text{for all } i = 1, \dots, k \Big\}
$$
$$
Z_2 = \Big\{ E : [-h, 0] \to \mathbb{S}^n \mid
$$
$$
E(s,t) = E(t,s)^T \quad \text{for all } s, t \in [-h, 0]
$$
$$
E \text{ is } C^0 \text{ on } H_i \times H_j \quad \text{for all } i, j = 1, \dots, k \Big\}
$$

Here $D \in Z_1$ is partitioned according to

$$
D(t) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14}(t) \\ D_{21} & D_{22} & D_{23} & D_{24}(t) \\ D_{31} & D_{32} & D_{33} & D_{34}(t) \\ D_{41}(t) & D_{42}(t) & D_{43}(t) & D_{44}(t) \end{bmatrix} \quad (11)
$$

where $D_{11}, D_{33}, D_{44} \in \mathbb{S}^n$ and $D_{22} \in S^{(k-1)n}$. Let $Y = Y_1 \times Y_2$ and $Z = Z_1 \times Z_2$. Notice that if $M \in Y_1$, then M need not be continuous at h_i for $1 \leq i \leq k-1$, however, it must be right continuous at these points. We also define the derivative $\dot{M}(t)$ at these points to be the right-hand derivative of M. We define the continuity and derivatives of functions in Y_2, Z_1 and Z_2 similarly.

We define the jump values of M and N at the discontinuities as follows.

$$
\Delta M(h_i) = M(-h_i+) - M(-h_i-)
$$

for each $i = 1, \dots, k-1$, and similarly define

$$
\Delta N(h_i, t) = N(-h_i+, t) - N(-h_i-, t)
$$

Definition 10 *Define the map $L : Y \to Z$ by $(D, E) =$*

$L(M,N)$ *if for all* $t, s \in [-h, 0]$ *we have*

$$D_{11} = A_0^T M_{11} + M_{11} A_0$$
$$+ \frac{1}{h}\big(M_{12}(0) + M_{21}(0) + M_{22}(0)\big)$$

$$D_{12} = \begin{bmatrix} M_{11}A_1 & \dots & M_{11}A_{k-1} \end{bmatrix}$$
$$- \begin{bmatrix} \Delta M_{12}(h_1) & \dots & \Delta M_{12}(h_{k-1}) \end{bmatrix}$$

$$D_{13} = \frac{1}{h}\big(M_{11}A_k - M_{12}(-h)\big)$$

$$D_{22} = \frac{1}{h}\operatorname{diag}\big(-\Delta M_{22}(h_1), \dots, -\Delta M_{22}(h_{k-1})\big)$$

$$D_{23} = 0$$

$$D_{33} = -\frac{1}{h}M_{22}(-h)$$

$$D_{14}(t) = N(0,t) + A_0^T M_{12}(t) - \dot{M}_{12}(t)$$

$$D_{24}(t) = \begin{bmatrix} \Delta N(-h_1, t) + A_1^T M_{12}(t) \\ \vdots \\ \Delta N(-h_{k-1}, t) + A_{k-1}^T M_{12}(t) \end{bmatrix}$$

$$D_{34}(t) = A_k^T M_{12}(t) - N(-h, t)$$

$$D_{44}(t) = -\dot{M}_{22}(t)$$

and

$$E(s,t) = \frac{\partial N(s,t)}{\partial s} + \frac{\partial N(s,t)}{\partial t}$$

Here D is partitioned as in (11) and the remaining entries are defined by symmetry.

The map L is the Lie derivative operator applied to the set of functions specified by (4); this is stated precisely below. Notice that this implies that L is a linear map.

Lemma 11 *Suppose $M \in Y_1$ and $N \in Y_2$ and V is given by (4). Let $(D, E) = L(M, N)$. Then the Lie derivative of V is given by*

$$\dot{V}(\phi) = \int_{-h}^0 \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix}^T D(s) \begin{bmatrix} \phi(-h_0) \\ \vdots \\ \phi(-h_k) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-h}^0 \int_{-h}^0 \phi(s)^T E(s,t)\phi(t)\, ds\, dt \quad (12)$$

PROOF. The proof is straightforward by differentiation and integration by parts of (4).

6 Matrix Sums of Squares

In this paper we use polynomial matrices as a conveniently parametrized class of functions to represent the functions M and N defining the Lyapunov function (4) and its derivative. Theorem 6 has reduced nonnegativity of the first term of (4) to pointwise nonnegativity of a matrix polynomial in one variable. A matrix polynomial in one variable is pointwise nonnegative semidefinite if and only if it is a sum of squares; see Choi, Lam, and Reznick [1]. We now briefly describe the matrix sum-of-squares construction. Suppose z is a vector of N monomials in variables y, for example

$$z(y)^T = \begin{bmatrix} 1 & y_1 & y_1 y_2^2 & y_3^4 \end{bmatrix}$$

For each y, we can define $K(y) : \mathbb{R}^n \to \mathbb{R}^{Nn}$ by

$$\big(K(y)\big)x = x \otimes z(y)$$

This means $K(y) = I \otimes z(y)$, and the entries of K are monomials in y. Now suppose $U \in \mathbb{S}^{Nn}$ is a symmetric matrix, and let

$$M(y) = K^T(y)UK(y)$$

Then each entry of the matrix M is a polynomial in y. The matrix M is called a *matrix sum-of-squares* if U is positive semidefinite, in which case $M(y) \succeq 0$ for all y. Given a matrix polynomial $G(y)$, we can test whether it is a sum-of-squares by finding a matrix U such that

$$G(y) = K^T(y)UK(y) \quad (13)$$
$$U \succeq 0 \quad (14)$$

Equation (13) is interpreted as equating the two polynomials G and $K^T U K$. Equating their coefficients gives a family of linear constraints on the matrix U. Therefore to find such a U we need to find a positive semidefinite matrix subject to linear constraints, and this is therefore a semidefinite program. See Scherer and Hol [16] and Kojima [7] for more details on matrix sums-of-squares and their properties, and Vandenberghe and Boyd [19] for background on semidefinite programming.

6.1 Piecewise matrix sums-of-squares.

Recall we define the vector of indicator functions $g : [-h, 0] \to \mathbb{R}^k$ by

$$g_i(t) = \begin{cases} 1 & \text{if } t \in H_i \\ 0 & \text{otherwise} \end{cases}$$

for all $i = 1, \dots, k$ and all $t \in [-h, 0]$. Now let $z(t)$ be the vector of monomials

$$z(t)^T = \begin{bmatrix} 1 & t & t^2 & \dots & t^d \end{bmatrix}$$

Define the function $Z_{n,d} : [-h, 0] \to \mathbb{R}^{(d+1)kn \times n}$ by

$$Z_{n,d}(t) = I \otimes z(t) \otimes g(t)$$

Then, define the sets $P_{n,d}$ and $\Sigma_{n,d}$ by

$$P_{n,d} = \left\{ Z_{n,d}^T(t) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk} \right\}$$
$$\Sigma_{n,d} = \left\{ Z_{n,d}^T(t) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk}, U \succeq 0 \right\}$$

These sets are defined so that if $M \in P_{n,d}$ then each entry of the $n \times n$ matrix M is a piecewise polynomial in t. If $M \in \Sigma_{n,d}$, then it is called a *piecewise matrix sum-of-squares*. We will omit the subscripts n, d where possible in order to lighten the notation. If we are given a function $G : [-h, 0] \to \mathbb{R}^n$ which is piecewise polynomial, and polynomial on each interval H_i, then it is a piecewise sum of squares of degree $2d$ if and only if there exists a matrix U such that

$$G(t) = Z_{n,d}^T(t) U Z_{n,d}(t) \tag{15}$$
$$U \succeq 0 \tag{16}$$

which, as for the non-piecewise case discussed above is a semidefinite program.

6.2 Kernel functions.

We can also consider functionals of the form of the second term of the Lyapunov function (4) using the notation presented here. We define the set of piecewise polynomial positive kernels in two variables s, t by

$$\Gamma_{n,d} = \left\{ Z_{n,d}^T(s) U Z_{n,d}(t) \mid U \in \mathbb{S}^{(d+1)nk}, U \succeq 0 \right\}$$

For any function $N \in \Gamma_{n,d}$, by Theorem 9, we have

$$\int_{-h}^0 \int_{-h}^0 \phi(s)^T N(s,t) \phi(t) \, ds \, dt \geq 0$$

for all $\phi \in C([-h, 0], \mathbb{R}^n)$.

6.3 Stability Conditions

Theorem 12 *Suppose there exist $d \in \mathbb{N}$ and piecewise matrix polynomials M, T, N, D, U, E such that*

$$M + \begin{bmatrix} T & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{2n,d}$$

$$-D + \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \in \Sigma_{(k+2)n,d}$$

$$N \in \Gamma_{n,d}$$
$$-E \in \Gamma_{n,d}$$
$$(D, E) = L(M, N)$$
$$\int_{-h}^0 T(s) \, ds = 0$$
$$\int_{-h}^0 U(s) \, ds = 0$$
$$M_{11} \succ 0$$
$$D_{11} \prec 0$$

Then the system defined by equations (2) and (3) is exponentially stable.

PROOF. Assume M, T, N, D, U, E satisfy the above conditions, and define the function V by (4). Then Lemma 11 implies that $\dot{V}$ is given by (12). The function V is the sum of two terms, each of which is nonnegative by Theorem 6 and the second is nonnegative since $N \in \Gamma_{n,d}$. The same is true for $\dot{V}$. The strict positivity conditions of equations (5) hold since $M_{11} > 0$ and $D_{11} < 0$, and Theorem 1 then implies stability.

Remark 13 *The feasibility conditions of Theorem 12 are semidefinite-representable. In particular the condition that a piecewise polynomial matrix lie in Σ is a set of linear and positive semidefiniteness constraints on its coefficients. Similarly, the condition that T and U integrate to zero is simply a linear equality constraint on its coefficients. Standard semidefinite programming codes may therefore be used to efficiently find such piecewise polynomial matrices. Most such codes will also return a dual certificate of infeasibility if no such polynomials exist.*

As in the Lyapunov analysis of nonlinear systems using sum-of-squares polynomials, the set of candidate Lyapunov functions is parametrized by the degree d. This allows one to search first over polynomials of low degree, and increase the degree if that search fails.

There are various natural extensions of this result. The first is to the case of uncertain systems, where we would like to prove stability for all matrices A_i in some given

semialgebraic set. This is possible by extending Theorem 12 to allow Lyapunov functions which depend polynomially on unknown parameters. A similar approach may be used to check stability for systems with uncertain delays. It is also straightforward to extend the class of Lyapunov functions, since it is not necessary that each piece of the piecewise sums-of-squares functions be nonnegative on the whole real line. To do this, one can use techniques for parameterizing polynomials nonnegative on an interval; for example, every polynomial $p(x) = f(x) - (x-1)(x-2)g(x)$ where f and g are sums of squares is nonnegative on the interval $[1, 2]$.

7 Numerical Examples

7.1 Example with a Single Delay

In this example, we compare our results with the discretized Lyapunov functional approach used by Gu et al. in [4] in the case of a system with a single delay. Although numerous other papers have also given sufficient conditions for stability of time-delay systems, e.g. [3, 10, 11], the approach introduced by Gu has demonstrated a particularly high level of precision. When we are comparing results we will only consider examples which have been presented in the literature. We use SOSTOOLS [14] and SeDuMi [17] for solution of all semidefinite programming problems.

Consider the following system of delay differential equations.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

The problem is to estimate the upper and lower bounds on τ for which the differential equation remains stable. Using the algorithm presented in this paper and by gridding the parameter space, we can determine the approximate region of stability. We then use a bisection method to find the minimum and maximum values of this region. Our results are summarized in Table 1 and are compared to the analytical limit and to the results obtained in [4].

Our results			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.10017	1.6249	1	.1006	1.4272
2	.10017	1.7172	2	.1003	1.6921
3	.10017	1.71785	3	.1003	1.7161
Analytic	.10017	1.71785			

Table 1
τ_{max} and τ_{min} for discretization level N_2 using the piecewise-linear Lyapunov functional and for degree d using our approach and compared to the analytical limit

The results for this test case illustrate a high rate of convergence of the accuracy as the parameter d increases.

Note that the algorithm only proves stability at the input points. It makes no claim for points not tested. To provide a stability proof over certain parameter regions, the parameter-dependent algorithm should be used. In the next examples, we now include τ as an uncertain parameter and search for parameter-dependent Lyapunov functionals which prove stability over the region. The proposed regions are based on data from the deterministic algorithm. Results from this test are given in Table 2.

d in τ	d in θ	τ_{min}	τ_{max}
1	1	.1002	1.6246
1	2	.1002	1.717
Analytic		.10017	1.71785

Table 2
Stability on the interval $[\tau_{min}, \tau_{max}]$ vs. degree using a parameter-dependent functional

7.2 Example with Multiple Delays

Consider the following system of delay-differential equations.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{10} \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t - \frac{\tau}{2}) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t - \tau)$$

Again, the problem is to estimate the stable region. These results are summarized in Table 3. For the piecewise functional method, N_2 is the level of both discretization and subdiscretization.

Our Approach			Piecewise Functional		
d	$\tau_{\min}$	$\tau_{\max}$	N_2	$\tau_{\min}$	$\tau_{\max}$
1	.20247	1.354	1	.204	1.35
2	.20247	1.3722	2	.203	1.372
Analytic	.20246	1.3723			

Table 3
τ_{max} and τ_{min} using the piecewise-linear Lyapunov functional of Gu et al. and our approach and compared to the analytical limit

7.3 Example with Parametric Uncertainty

In this example, we illustrate the flexibility of our algorithm with a simplistic control design and analysis problem. Suppose we are controlling a simple inertial mass remotely using a PD controller. Now suppose that the derivative control is half of the proportional control. Then we have the following dynamical system.

$$\ddot{x}(t) = -ax(t) - \frac{a}{2}\dot{x}(t)$$

It is easy to show that this system is stable for all positive values of a. However, because we are controlling the mass remotely, some delay will be introduced due to, for example, the fixed speed of light. We assume that this delay is known slowly varying. Now we have the following delay-differential equation with uncertain, time-invariant parameters a and τ.

$$\ddot{x}(t) = -ax(t-\tau) - \frac{a}{2}\dot{x}(t-\tau)$$

Whereas before the system was stable for all positive values of a, now, for any fixed value of a, there exists a τ for which the system is unstable. In order to determine which values of a are stable for any fixed value of τ, we apply the algorithm from this paper to a gridded parameter space. Based on these results, we propose stability regions of the form $a \in [a_{\min}, a_{\max}]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. This type of region is compact and can be represented as a semi-algebraic set using the polynomials $p_1(a) = (a - a_{\min})(a - a_{\max})$ and $p_2(\tau) = (\tau - \tau_{\min})(\tau - \tau_{\max})$. By now applying the parameter-dependent version of the algorithm, we are able to construct Lyapunov functionals which prove stability over a number of parameter regions. These regions are illustrated in Figure 1.
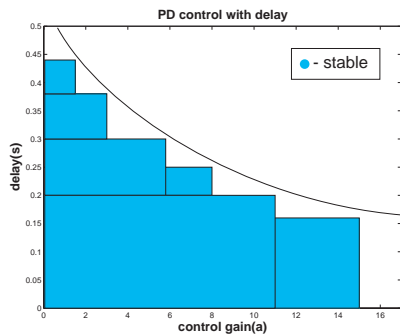

Fig. 1. Regions of Stability for Example 2

8 Conclusion

The general question of stability of linear differential equations with delay is NP-hard [18]. In this paper, we have shown that the question of stability can be expressed as a convex optimization problem within the function space $C[0,1]$. Furthermore, through the use of polynomial programming techniques and projections from $C[0,1]$ onto the space of polynomials, we have shown how to use semidefinite programming to compute solutions to this optimization problem. Thus we have created a sequence of sufficient conditions for stability of linear time-delay systems, indexed by polynomial degree, d, and whose complexity is of order $k(n(d+1)(k+2))^2$. As $d \to \infty$, the validity of the polynomial approximation increases and the conditions become increasingly accurate, as adequately illustrated by the numerical examples.

Aside from the accuracy of the results, the approach taken in this paper has an important advantage. Because of the flexibility of the polynomial approach, we are able to make various extensions of our results to other problems which can be expressed using convex optimization in $C[0,1]$. Specifically, in this paper we have addressed the problem of systems with parametric uncertainty using a variant of the S-procedure. Additionally, extension to nonlinear systems with time-delay is easily addressed, although in this case conservatism arises from a number of sources. A more extensive treatment of nonlinear time-delay systems is the topic of another paper currently in preparation.

We conclude by mentioning that the methods of this paper would seem to allow us to synthesize stabilizing controllers for linear time-delay systems. This observation is motivated by viewing the functions M and R as defining a full rank operator on $L_2[0,1]$. We have observed that for given functions, the inverse of such an operator can be computed numerically. By computing Lyapunov functionals for the adjoint system constructed by Delfour and Mitter [2], this invertibility result seems to imply that one can construct stabilizing controllers. This work is ongoing.

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