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SOS for Systems with Multiple Delays: Part 2. H_∞-Optimal Estimation

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***Abstract*— In this paper, we develop an SOS-based approach for design of observers for time-delay systems. The method is an extension of recently developed algorithms for control of infinite-dimensional systems. The observers we design are more general than the class of observers most commonly associated with time-delay systems in that they directly correct both the estimate of present state and the history of the state. As a result, the observer is itself a PDE. In this case the traditional notions of strong and weak observability do not apply and the resulting observer-based controllers can significantly outperform existing approaches.**

I. Introduction

In Part 1 of this paper [1], we presented a method for control synthesis for systems with multiple delays under the assumption that the full state is available for control feedback. However, state variables are often inaccessible in practice. Therefore, in this part, we consider the state estimation of such systems. Specifically, in Part 2 we consider the problem of state estimation for delayed systems of the form

$$\begin{aligned}\dot{x}(t) &= A_0 x(t) + \sum_{i=1}^K A_i x(t-\tau_i) + Bw(t),\\ z(t) &= C_{10}x(t) + \sum_{i=1}^K C_{1i} x(t-\tau_i),\\ y(t) &= C_2 x(t),\end{aligned} \tag{1}$$

where $y(t) \in \mathbb{R}^q$ is the measured output, $w(t) \in \mathbb{R}^r$ is the disturbance input, $z(t) \in \mathbb{R}^p$ is the regulated output, $x(t) \in \mathbb{R}^n$ are the state variables and $\tau_i > 0$ for $i \in [1, \cdots, K]$ are the delays ordered by increasing magnitude. We assume $x(s) = 0$ for $s \in [-\tau_K, 0]$. Our goal is to construct an observer of the form

$$\begin{aligned}\dot{\hat{x}}(t) &= A_0\hat{x}(t) + \sum_{i=1}^K A_i\hat{\phi}(t,-\tau_i) + L_1 e_0(t)\\ &\quad + \sum_{i=1}^K L_{2i} e_i(t-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 L_{3i}(\theta) e_i(t+s) d\theta,\\ \partial_t\hat{\phi}_i(t,s) &= \partial_s\hat{\phi}_i(t,s) + L_{4i}(s)e_0(t) + \sum_{j=1}^K L_{5ij}(s)e_j(t-\tau_j)\\ &\quad + L_{6i}(s)e_i(t+s) + \sum_{j=1}^K \int_{-\tau_i}^0 L_{7ij}(s,\theta)e_j(t+\theta)d\theta,\\ \hat{\phi}_i(t,0) &= \hat{x}(t),\end{aligned}$$

$$\begin{aligned}e_0(t) &= C_2\hat{x}(t) - y(t), \quad e_i(t+s) = C_2\phi_i(t,s) - y(t+s),\\ z_e(t) &= C_{10}e_0(t) + \sum_{i=1}^K C_{1i}e_i(t-\tau_i),\end{aligned} \tag{2}$$

where L_{3i}, L_{4i}, L_{5i} L_{6i} and L_{7i} are polynomials that are chosen to minimize $\gamma := \sup_{w\in L_2} \frac{\|z_e\|_{L_2}}{\|w\|_{L_2}}$.

The structure of the observer is a natural generalization of the Luenberger observer to infinite-dimensional systems. Specifically, when all gains are removed ($L_{ij} = 0$), System (2) reduces to the nominal System (1) without disturbance input w. The gains L_1 L_{2i}, and L_{3i} correct the estimate of the current state of the system. The gains L_{5i} L_{6i} and L_{7ij} correct the estimated history of the state.

The observer developed in Part 2 of this paper has the following characteristics: 1) The feasibility is expressed as LMIs; 2) It is not conservative in any significant sense; 3) The conditions are prima facie provable in that they are certified using Lyapunov-Krasovskii functionals; 4) The method is scalable to large numbers of states and delays; 5) The design process has an efficient real-time implementation; and 6) The algorithm is publicly available for verification via Code Ocean.

It should be pointed out, however, that our observer in the current form is not suitable when there is delay in the output or when the delays are unknown or time-varying. Furthermore, we do not yet have an effective method to accommodate the sensor noise in the measured output.

The result may be considered as a generalization of a well-known LMI for optimal estimation of ODEs. Specifically, when $A_i = 0$ and $C_{1i} = 0$ for $i > 0$, if there exist $P > 0$ and Z such that

$$\begin{bmatrix} PA + ZC_2 + (PA+ZC_2)^T & -PB - ZD & C_{10}^T \\ -(PB+ZD)^T & -\gamma I & 0 \\ C_{10} & 0 & -\gamma I\end{bmatrix} < 0. \tag{3}$$

then if we set $L_1 = P^{-1}Z$ and all other gains to zero we have $\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}$.

To generalize this LMI for multi-delay systems, we take the following approach, which parallels the developments for controller synthesis in Part 1 [1]. First, in Section III, Theorem 1 we show that an operator-valued version of LMI (3) holds for a general class of Distributed-Parameter Systems (DPS). In Theorem 1, however, the system matrices become operators. Similarly, the matrix variables P and Z are also replaced by linear operators. Next, in Section IV, we formulate the estimator design problem for multi-delay systems in the DPS framework, defining the system operators

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$\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{D}$ and using the PQRS framework to parameterize the operator variables $\mathcal{P}$ and $\mathcal{Z}$ as in Equations (8) and (10). The $PQRS$ framework (Eqn. (8)) uses matrix-valued polynomials to parameterize multiplier and integral operators and positivity of $PQRS$ operators can be enforced using LMI constraints on the coefficients of the polynomials, as defined in Section IV of Part 1 [1] and summarized here in Corollary 2. Having defined the system operators and parameterized the variables, in Section VI we provide the main result which shows that when these definitions and parameterizations are applied to Theorem 1, we obtain an optimal observer synthesis condition expressed as positivity and negativity of operators of the $PQRS$ format. These synthesis conditions can then be expressed as LMIs using the DelayTOOLs Matlab toolbox described in Section IX. To extract the optimal observer gains, however, one last step is required. Namely, we must find $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ — which requires inversion of $\mathcal{P}$. In Section VII, Theorem 4, we give an analytic expression for the inverse of a $PQRS$ operator. This inverse is itself a $PQRS$ operator and when composed with $\mathcal{Z}$, yields the optimal observer gains — an expression for which is found in Section VIII, Lemma 5. Finally, in Section X, these gains are implemented efficiently in Matlab and are shown to reject disturbances as per the predicted bounds on L_2-gain. Section X also performs an accuracy analysis when compared with observers designed using a high-order Padé approximation. The bounds obtained for our results are accurate in this metric to several decimal places in all problems considered. Furthermore, in Section X a computational complexity analysis is performed and it is shown that the algorithms can be implemented on desktop computers when the product of the number of states and the number of delays is less than 50.

II. Notation

The symmetric matrices are denoted $\mathbb{S}^n \subset \mathbb{R}^{n\times n}$. An element of a symmetric matrix which can be deduced from symmetry is denoted with a $*$. We use $L_2^n[T]$ to denote the vector-valued Lesbesque square integrable functions which map $T \to \mathbb{R}^n$. In this paper, either $T = T_i := [-\tau_i, 0]$ or $T = [0, \infty]$. We occasionally denote the Sobolev space $W_2^n[T] := \{x \in L_2^n[T] \ : \ \dot{x} \in L_2^n[T]\}$. We also use the index shorthand $[K] := \{1, \cdots, K\}$.

III. Optimal Estimation of Distributed-Parameter Systems

In this section we consider the general class of distributed-parameter systems given by

$$\begin{aligned} &\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}w(t), && \mathbf{x}(0) = 0, \\ &z(t) = \mathcal{C}_1\mathbf{x}(t), && \mathbf{y}(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}w(t), \end{aligned} \tag{4}$$

where $\mathcal{A} : X \to Z, \mathcal{B} : \mathbb{R}^r \to Z, \mathcal{C}_1 : X \to \mathbb{R}^p, \mathcal{C}_2 : X \to Y$ and $\mathcal{D} : \mathbb{R}^r \to Y$. Now for a given operator $\mathcal{L} : Y \to Z$, we define the observer dynamics as follows

$$\dot{\hat{\mathbf{x}}}(t) = \mathcal{A}\hat{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\hat{\mathbf{x}}(t) - \mathbf{y}(t)\right), \tag{5}$$

where $\hat{\mathbf{x}}(0) = 0$. Now, defining the error state as $\mathbf{e}(t) = \hat{\mathbf{x}}(t) - \mathbf{x}(t)$, $z_e(t) = \mathcal{C}_1\hat{\mathbf{x}}(t) - z(t) = \mathcal{C}_1\mathbf{e}(t)$ we obtain the error dynamics as

$$\begin{aligned} &\dot{\mathbf{e}}(t) = (\mathcal{A} + \mathcal{L}\mathcal{C}_2)\mathbf{e}(t) - (\mathcal{B} + \mathcal{L}\mathcal{D})w(t), \\ &z_e(t) = \mathcal{C}_1\mathbf{e}(t), \qquad \mathbf{e}(0) = 0. \end{aligned} \tag{6}$$

The goal is to construct the operator $\mathcal{L}$ which minimizes $\gamma > 0$ such that $\|z_e\|_{L_2} \le \gamma \|w\|_{L_2}$ for any $\mathbf{e}$ and z_e which satisfy Equation (6). Note that although this paper is concerned with the H_∞ gain from disturbance w to the measured output z, it is also possible to extend this result to L_2 gain from the nonzero initial condition $\mathbf{e}(0)$ to z in a manner similar to [2].

Theorem 1: Suppose there exist bounded linear operators $\mathcal{P} : Z \to Z$ and $\mathcal{Z} : Y \to Z$, such that $\mathcal{P}$ is coercive and

$$\begin{aligned} &\langle(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_Z + \langle\mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_Z \\ &- \langle\mathbf{e}, (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w\rangle_Z - \langle(\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w, \mathbf{e}\rangle_Z \\ &- \gamma\|w\|^2 - \gamma\|v\|^2 + \langle v, \mathcal{C}_1\mathbf{e}\rangle + \langle\mathcal{C}_1\mathbf{e}, v\rangle \\ &\le -\epsilon\|\mathbf{e}\|^2 \qquad \forall \mathbf{e} \in X,\ w \in \mathbb{R}^r,\ v \in \mathbb{R}^q \end{aligned}$$

for some $\epsilon > 0$. Then $\mathcal{P}^{-1}$ exists and is a bounded linear operator and for $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ and any $w \in L_2^r$, the solution of Eqn. (6) satisfies

$$\|z_e\|_{L_2} \le \gamma\|w\|_{L_2}. \tag{7}$$

Proof: Since $\mathcal{P}$ is coercive and $P : Z \to Z$, $\mathcal{P}^{-1}$ is a bounded linear operator with $\mathcal{P}^{-1} : Z \to Z$. Let $\mathbf{e}$ satisfy Eqn. (6). Then $\mathbf{e}(t) \in X$. Define the storage function $V(e) = \langle\mathbf{e}, \mathcal{P}\mathbf{e}\rangle$, then $V(\mathbf{e}) \ge \delta\|\mathbf{e}\|^2$ for some $\delta > 0$ since $\mathcal{P}$ is coercive. Differentiating, we obtain

$$\begin{aligned} \dot{V}(\mathbf{e}(t)) &= \langle(\mathcal{P}\mathcal{A} + \mathcal{P}\mathcal{L}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_Z + \langle\mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{P}\mathcal{L}\mathcal{C}_2)\mathbf{e}\rangle_Z \\ &\quad - \langle\mathbf{e}, (\mathcal{P}\mathcal{B} + \mathcal{P}\mathcal{L}\mathcal{D})w\rangle_Z - \langle(\mathcal{P}\mathcal{B} + \mathcal{P}\mathcal{L}\mathcal{D})w, \mathbf{e}\rangle_Z. \end{aligned}$$

Now since $\mathcal{Z} = \mathcal{P}\mathcal{L}$, we have

$$\begin{aligned} \dot{V}(\mathbf{e}(t)) &= \langle(\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_Z + \langle\mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_Z \\ &\quad - \langle\mathbf{e}, (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w\rangle_Z - \langle(\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w, \mathbf{e}\rangle_Z \\ &< \gamma\|w(t)\|^2 + \gamma\|v(t)\|^2 - \langle v(t), \mathcal{C}_1\mathbf{e}(t)\rangle - \langle\mathcal{C}_1\mathbf{e}(t), v(t)\rangle \\ &= \gamma\|w(t)\|^2 + \gamma\|v(t)\|^2 - \langle v(t), z_e(t)\rangle - \langle z_e(t), v(t)\rangle \end{aligned}$$

for all $[\mathbf{e}(t)\ w(t)\ v(t)] \ne 0$. Let $v(t) = \frac{1}{\gamma}z_e(t)$. Then we get

$$\begin{aligned} \dot{V}(\mathbf{e}(t)) &< \gamma\|w(t)\|^2 + \frac{1}{\gamma}\|z_e(t)\|^2 - \frac{2}{\gamma}\|z_e(t)\|^2 \\ &= \gamma\|w(t)\|^2 - \frac{1}{\gamma}\|z_e(t)\|^2. \end{aligned}$$

Integration of this inequality yields

$$V(\mathbf{e}(t)) - V(\mathbf{e}(0)) + \frac{1}{\gamma}\int_0^t \|z_e(s)\|^2 ds < \gamma\int_0^t \|w(s)\|^2 ds.$$

As $V(e(0)) = 0$ and $V(\mathbf{e}(t)) \ge 0$, if we let $t \to \infty$, we see that the above implies (7). ∎

Note that the conditions of the theorem also establish $\lim_{t\to\infty}\mathbf{e}(t) = 0$ when $\lim_{t\to\infty}w(t) = 0$.

IV. Application to Systems with Multiple State Delays

Theorem 1 gives a convex formulation of the observer synthesis problem for a general class of distributed-parameter systems. In this section and the next, we apply Theorem 1 to the case of systems with multiple delays. Specifically, we consider solutions to the system of equations given by Equation (1). First, we express System (1) in the abstract form of (4). Following the mathematical formalism developed in [3], we define the inner-product space $Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$ and for $\{x, \phi_1, \cdots, \phi_K\} \in Z_{m,n,K}$, we define the following shorthand notation

$$\begin{bmatrix} x \\ \phi_i \end{bmatrix} := \{x, \phi_1, \cdots, \phi_K\},$$

which allows us to simplify expression of the inner product on $Z_{m,n,K}$, which we define to be

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s) ds.$$

When $m = n$, we simplify the notation using $Z_{n,K} := Z_{n,n,K}$. The state-space for System (1) is defined as

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \, : \, \substack{\phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K]} \right\}.$$

We now represent the infinitesimal generator, $\mathcal{A} : X \to Z_{n,K}$, of Eqn. (1) as

$$\mathcal{A} \begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i \phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}.$$

and the operators $\mathcal{B} : \mathbb{R}^r \to Z_{n,n,K}$, $\mathcal{C}_1 : X \to \mathbb{R}^p$, $\mathcal{C}_2 : X \to Z_{q,q,K}$ are defined as

$$(\mathcal{B}w)(s) := \begin{bmatrix} Bw \\ 0 \end{bmatrix}, \quad \left(\mathcal{C}_2 \begin{bmatrix} x_0 \\ x_i \end{bmatrix} \right)(s) := \begin{bmatrix} C_2 x_0 \\ C_2 x_i(s) \end{bmatrix}$$

$$\left(\mathcal{C}_1 \begin{bmatrix} x_0 \\ x_i \end{bmatrix} \right) := \begin{bmatrix} C_{10} x_0 + \sum_{i=1}^{K} C_{1i} x_i(-\tau_i) \end{bmatrix}.$$

In this paper, we set $\mathcal{D} = 0$ since a realistic model of sensor noise requires an augmented state space. We leave inclusion of sensor noise for future work.

A. The PQRS Framework

Now that we have specified the operators $\mathcal{A}, \mathcal{B}, \mathcal{C}_1, \mathcal{C}_2$, and $\mathcal{D}$ which define the nominal system, we next use matrices and matrix-valued polynomials to define the decision variables in Theorem 1. Specifically, we now introduce a class of operators $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} : Z_{m,n,K} \to Z_{m,n,K}$, parameterized by matrix P and matrix-valued functions $Q_i \in W_2^{m\times n}[T_i]$, $S_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ as

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right)(s) := \tag{8}$$

$$\begin{bmatrix} Px + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Q_i(s)\phi_i(s) ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} R_{ij}(s,\theta)\phi_j(\theta)\, d\theta \end{bmatrix}.$$

We now turn to the operator $\mathcal{Z}$. To obtain the estimator form defined in (2), the error injection operator $\mathcal{L} : Z_{q,q,K} \to Z_{n,n,K}$ must have the form

$$\mathcal{L} \begin{bmatrix} y_0 \\ y_i \end{bmatrix}(s) = \begin{bmatrix} L_0 y_0 + \sum_{i=1}^{K} L_{2i} y_i(-\tau_i) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} L_{3,i}(\theta) y_i(\theta) d\theta \\ l_i(s) \end{bmatrix}$$

$$l_i(s) = L_{4i}(s) y_0 + \sum_{j=1}^{K} L_{5ij}(s) y_j(-\tau_j) + L_{6i}(s) y_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} L_{7ij}(s,\theta) y_j(\theta) d\theta. \tag{9}$$

Using this parametrization of $\mathcal{P}$ and $\mathcal{L}$, we may compose these operators to show that $\mathcal{Z} = \mathcal{P}\mathcal{L}$ likewise has the form

$$\mathcal{Z} \begin{bmatrix} y_0 \\ y_i \end{bmatrix}(s) = \begin{bmatrix} Z_0 y_0 + \sum_{i=1}^{K} Z_{2i} y_i(-\tau_i) + \sum_{i=1}^{K} \int_{-\tau_i}^{0} Z_{3,i}(\theta) y_i(\theta) d\theta \\ \tau_K z_i(s) \end{bmatrix}$$

$$z_i(s) = Z_{4i}(s) y_0 + \sum_{j=1}^{K} Z_{5ij}(s) y_j(-\tau_j) + Z_{6i}(s) y_i(s) + \sum_{j=1}^{K} \int_{-\tau_j}^{0} Z_{7ij}(s,\theta) y_j(\theta) d\theta. \tag{10}$$

The expressions for the gains Z_i in terms of $PQRS$ and L_i are omitted for brevity. Now that we have defined the problem and parameterized our decision variables, we may apply these results to Theorem 1 to obtain a synthesis condition expressed entirely in the PQRS framework. However, before we do this, we need to show how LMIs can be used to enforce positivity of operators in the PQRS format. Furthermore, we need to show how to construct $\mathcal{P}^{-1}$ when $\mathcal{P}$ is in PQRS format.

V. Enforcing Operator Inequalities in the $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ Framework

The problem of enforcing operator positivity on $Z_{m,n,K}$ in the $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ framework was solved in [3]. To avoid repetition, we refer to Section IV in Part 1 of this paper [1] for a detailed discussion. However, for convenience, we repeat the main result of that section.

Corollary 2: Let $\Xi_{d,m,K}$ and $\mathcal{L}_1$ be as defined in Section IV of Part 1 [1]. Suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{m\times m}$, polynomials Q_i, S_i, R_{ij} for $i, j \in [K]$ such that

$$\mathcal{L}_1(P, Q_i, S_i, R_{ij}) \in \Xi_{d,m,nK}.$$

Then $\langle \mathbf{x}, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \mathbf{x} \rangle_{Z_{m,n,K}} \geq 0$ for all $\mathbf{x} \in Z_{m,n,K}$.

VI. Reformulation of the Synthesis Condition using $Z_{2n+r,n}$

In this section, we reformulate the conditions of Theorem 1 as a linear operator inequality where all operators are of the form of Equation (8). Specifically, we show that for $\mathbf{e} \in X$,

$$\langle (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e} \rangle_Z + \langle \mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e} \rangle_Z$$
$$- \langle \mathbf{e}, (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w \rangle_Z - \langle (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w, \mathbf{e} \rangle_Z$$
$$- \gamma \|w\|^2 + \frac{1}{\gamma} \|\mathcal{C}_1 \mathbf{e}\|^2 = \langle \mathbf{h}, \mathcal{P}_{\{D,E_i,F_i,G_{ij}\}} \mathbf{h} \rangle_{Z_{r+q+n(K+1),n,K}}$$

where

$$\mathbf{h} = \begin{bmatrix} w^T & v^T & e_0^T & e_1(-\tau_1)^T & \cdots & e_K(-\tau_K)^T & e_i^T \end{bmatrix}^T \in Z_{r+q+n(K+1),n,K}.$$

Theorem 3: For any $\gamma > 0$, suppose there exist $d \in \mathbb{N}$, constant $\epsilon > 0$, matrix $P \in \mathbb{R}^{n\times n}$, polynomials $S_i, Q_i \in W_2^{n\times n}[T_i]$, $R_{ij} \in W_2^{n\times n}[T_i \times T_j]$ for $i,j \in [K]$, matrices $Z_0, Z_{2i} \in \mathbb{R}^{n\times q}$ and polynomials $Z_{3i}, Z_{4i}, Z_{5ji}, Z_{6i} \in W_2^{n\times q}[T_i]$ and $Z_{7ij} \in W_2^{n\times q}[T_i \times T_j]$ for $i,j \in [K]$ such that

$$\mathcal{L}_1(P-\epsilon I_n, Q_i, S_i - \epsilon I_n, R_{ij}) \in \Xi_{d,n,nK}$$
$$-\mathcal{L}_1(D+\epsilon \hat{I}, E_i, F_i + \epsilon I_n, G_{ij}) \in \Xi_{d,r+q+n(K+1),nK},$$

where

$$D = \begin{bmatrix} -\frac{\gamma}{\tau_K} I & 0 & -T_{w0}^T & 0 & \dots & 0 \\ *^T & -\frac{\gamma}{\tau_K} I & \frac{C_{10}}{\tau_K} & \frac{C_{11}}{\tau_K} & \dots & \frac{C_{1K}}{\tau_K} \\ *^T & *^T & T_{00} & T_{01} & \dots & T_{0K} \\ *^T & *^T & *^T & S_1(-\tau_1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ *^T & *^T & *^T & *^T & \dots & S_K(-\tau_K) \end{bmatrix}$$

$$E_i(s) = \begin{bmatrix} -\Phi_{wi}(s) & 0 & \Phi_{0i}(s) & \Phi_{1i}(s) & \dots & \Phi_{Ki}(s) \end{bmatrix}^T,$$
$$F_i = \dot{S}_i(s) + Z_{6i}(s)C_2 + C_2^T Z_{6i}(s)^T,$$
$$G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta) + \tau_K \left(Z_{7ij}(s,\theta)C_2 + C_2^T Z_{7ji}(\theta,s)^T \right),$$

and

$$T_{w0} = PB,$$
$$T_{00} = PA_0 + A_0^T P + \sum_{k=1}^K Q_k(0) + Q_k(0)^T + S_k(0) + Z_1 C_2 + C_2^T Z_1^T,$$
$$T_{0i} = PA_i - Q_i(-\tau_i) + Z_{2,i}C_2,$$
$$\Phi_{wi}(s) = Q_i(s)^T B,$$
$$\Phi_{0i}(s) = A_0^T Q_i(s) + \frac{1}{\tau_K}\sum_{k=1}^K R_{ik}^T(s,0) - \dot{Q}_i(s) + Z_{4i}(s)C_2 + C_2^T Z_{3,i}(s)^T,$$
$$\Phi_{ji}(s) = A_j^T Q_i(s) - \frac{1}{\tau_K} R_{ij}^T(s,-\tau_j) + Z_{5i,j}(s)C_2,$$

$\hat{I} = \mathrm{diag}(0_{r+q}, I_n, 0_{nK})$, and $\mathcal{L}_1$ is as defined in Section IV of [1]. Then if $\mathcal{L} = \mathcal{P}^{-1}_{\{P,Q_i,S_i,R_{ij}\}}\mathcal{Z}$, where $\mathcal{Z}$ is as defined in Equation (10), then $\mathcal{L}$ has the form of Eqn. (9) and any solution of Eqns (1) coupled with Eqns. (5) satisfies $\|z_e\|_{L_2} < \gamma \|w\|_{L_2}$.

Proof: First let $\mathcal{P} = \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. Then

$$\langle \mathbf{e}, \mathcal{P}\mathbf{e}\rangle = \left\langle \mathbf{e}, \mathcal{P}_{\{P-\epsilon I, Q_i, S_i-\epsilon I, R_{ij}\}}\mathbf{e}\right\rangle + \epsilon \|\mathbf{e}\|^2_{Z_{n,K}} \geq \epsilon \|\mathbf{e}\|^2_{Z_{n,K}},$$

hence $\mathcal{P}$ is coercive. Next, we show that for $\mathbf{e} \in X$,

$$\langle (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}, \mathbf{e}\rangle_Z + \langle \mathbf{e}, (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_Z - \langle \mathbf{e}, (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w\rangle_Z - \langle (\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})w, \mathbf{e}\rangle_Z - \gamma \|w\|^2 + \frac{1}{\gamma}\|\mathcal{C}_1\mathbf{e}\|^2 = \left\langle \mathbf{h}, \mathcal{P}_{\{D,E_i,F_i,G_{ij}\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1),n,K}}$$

where

$$\mathbf{h} = \begin{bmatrix} w^T & v^T & e_0^T & e_1(-\tau_1)^T & \cdots & e_K(-\tau_K)^T & e_i^T \end{bmatrix}^T = \begin{bmatrix} h_0 \\ h_i \end{bmatrix} \in Z_{r+q+n(K+1),n,K}.$$

and apply Theorem 1. We do this in parts by reformulating each element separately and then summing up. The first term is complicated, but has been well-studied [3].

$$\langle \mathbf{e}, \mathcal{P}\mathcal{A}\mathbf{e}\rangle_{Z_{n,K}} + \langle \mathcal{A}\mathbf{e}, \mathcal{P}\mathbf{e}\rangle_{Z_{n,K}} = \left\langle \mathbf{h}, \mathcal{P}_{\{D_1, E_{1i}, F_{1i}, G_{1ij}\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1),n,K}},$$

where

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ 0 & 0 & \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ 0 & 0 & \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^K Q_k(0) + Q_k(0)^T + S_k(0),$$
$$\Delta_j = PA_j - Q_j(-\tau_j),$$
$$E_{1i}(s) = \begin{bmatrix} 0 & 0 & \Pi_{0,i}(s)^T & \dots & \Pi_{K,i}(s)^T \end{bmatrix}^T,$$
$$\Pi_{0j}(s) = A_0^T Q_j(s) + \frac{1}{\tau_K}\sum_{k=1}^K R_{jk}^T(s,0) - \dot{Q}_j(s),$$
$$\Pi_{ij}(s) = A_i^T Q_j(s) - \frac{1}{\tau_K} R_{ji}^T(s,-\tau_i),$$
$$F_{1i} = \dot{S}_i(s)$$
$$G_{1ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

For the second term, we have

$$\mathcal{Z}\mathcal{C}_2 \begin{bmatrix} e_0 \\ e_i \end{bmatrix}(s) := \begin{bmatrix} g_0 \\ \tau_K g_i(s) \end{bmatrix},$$

where

$$g_0 = Z_0 C_2 e_0 + \sum_{i=1}^K Z_{2i} C_2 e_i(-\tau_i) + \sum_{i=1}^K \int_{-\tau_i}^0 Z_{3,i}(\theta) C_2 e_i(\theta) d\theta,$$
$$g_i(s) = Z_{4i}(s) C_2 e_0 + \sum_j Z_{5ij}(s) C_2 e_j(-\tau_j) + Z_{6i}(s) C_2 e_i(s) + \sum_j \int_{-\tau_j}^0 Z_{7ij}(s,\theta) C_2 e_j(\theta) d\theta.$$

Hence

$$\langle \mathbf{e}, \mathcal{Z}\mathcal{C}_2\mathbf{e}\rangle_{Z_{n,K}} + \langle \mathcal{Z}\mathcal{C}_2\mathbf{e}, \mathbf{e}\rangle_{Z_{n,K}} = \left\langle \mathbf{h}, \mathcal{P}_{\{D_2, E_{2i}, F_{2i}, G_{2ij}\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1),n,K}},$$

where

$$D_2=\begin{bmatrix}0&0&0&0&\dots&0\\0&0&0&0&\dots&0\\0&0&Z_0C_2+C_2^TZ_0^T&Z_{2,1}C_2&\dots&Z_{2,K}C_2\\0&0&C_2^TZ_{2,1}^T&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&C_2^TZ_{2,K}^T&0&\dots&0\end{bmatrix},$$

$E_{2i}(s)=$

$$\begin{bmatrix}0 & 0 & Z_{4i}(s)C_2+C_2^TZ_{3i}(s)^T & Z_{5i1}(s)C_2 & \cdots & Z_{5iK}(s)C_2\end{bmatrix}^T,$$

$F_{2i}=Z_{6i}(s)C_2+C_2^TZ_{6i}(s)^T,$

$G_{2ij}(s,\theta)=Z_{7ij}(s,\theta)C_2+C_2^TZ_{7ji}(\theta,s)^T.$

For the third term, we have

$$\mathcal{P}\mathcal{B}\begin{bmatrix}e_0\\e_i\end{bmatrix}(s):=\begin{bmatrix}PBw\\\tau_KQ_i(s)^TBw\end{bmatrix}.$$

Hence

$$\begin{aligned}&\langle \mathbf{e},\mathcal{P}\mathcal{B}w\rangle_{Z_{n,K}}+\langle \mathcal{P}\mathcal{B}w,\mathbf{e}\rangle_{Z_{n,K}}\\&\quad=\left\langle \mathbf{h},\mathcal{P}_{\{D_3,E_{3i},0,0\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1)},n,K},\end{aligned}$$

where

$$D_3=\begin{bmatrix}0&0&B^TP&0&\dots&0\\0&0&0&0&\dots&0\\PB&0&0&0&\dots&0\\0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&\dots&0\end{bmatrix},$$

$$E_{3i}(s)=\begin{bmatrix}Q_i(s)^TB & 0 & 0 & 0 & \dots & 0\end{bmatrix}^T.$$

Finally, we have $-\gamma\left\|w\right\|^2-\gamma\left\|v\right\|^2+\langle v,\mathcal{C}_1\mathbf{e}\rangle+\langle\mathcal{C}_1\mathbf{e},v\rangle$. First,

$$\langle v,\mathcal{C}_1\mathbf{e}\rangle=v^T(C_0e_0+\sum_{i=1}^K C_ie_i(-\tau_i))$$

hence

$$\begin{aligned}&-\gamma\left\|w\right\|^2-\gamma\left\|v\right\|^2+\langle v,\mathcal{C}_1\mathbf{e}\rangle+\langle\mathcal{C}_1\mathbf{e},v\rangle\\&=\tau_Kh_0^T\underbrace{\frac{1}{\tau_K}\begin{bmatrix}-\gamma I&0&0&0&\dots&0\\0&-\gamma I&C_0&C_1&\dots&C_K\\0&C_0^T&0&0&\dots&0\\0&C_1^T&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&C_K^T&0&0&\dots&0\end{bmatrix}}_{D_4}h_0\\&\quad=\left\langle \mathbf{h},\mathcal{P}_{\{D_4,0,0,0\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1)},n,K}.\end{aligned}$$

Summing all the terms and noting the $\mathcal{D}=0$, we have

$$\begin{aligned}&\langle(\mathcal{P}\mathcal{A}+\mathcal{Z}\mathcal{C}_2)\mathbf{e},\mathbf{e}\rangle_Z+\langle\mathbf{e},(\mathcal{P}\mathcal{A}+\mathcal{Z}\mathcal{C}_2)\mathbf{e}\rangle_Z\\&-\langle\mathbf{e},(\mathcal{P}\mathcal{B}+\mathcal{Z}\mathcal{D})w\rangle_Z-\langle(\mathcal{P}\mathcal{B}+\mathcal{Z}\mathcal{D})w,\mathbf{e}\rangle_Z\\&-\gamma\|w\|^2+\frac{1}{\gamma}\left\|\mathcal{C}_1\mathbf{e}\right\|^2=\left\langle\mathbf{h},\mathcal{P}_{\{D,E_i,F_i,G_{ij}\}}\mathbf{h}\right\rangle_{Z_{r+q+n(K+1)},n,K}\end{aligned}$$

where

$$\begin{aligned}D&=D_1+D_2-D_3+D_4,\\E_i&=E_{1i}+E_{2i}-E_{3i},\\F_i&=F_{1i}+F_{2i},\\G_{ij}&=G_{1ij}+G_{2ij}.\end{aligned}$$

■

Numerical implementation of the conditions of Theorem 3 using the DELAYTOOLS mod pack for SOSTOOLS is relatively straightforward. An implementation of this test, the estimator reconstruction, and simulations keyed to this paper can be found at [4].

VII. Inverting the Operator

Now that we have an observer synthesis condition, we address the question of reconstructing the observer which attains the desired H_∞ gain condition. Recall the observer gain is of the form $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$. Clearly, we need an expression for the inverse of an operator of the form $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$. Such an inverse was presented in Part 1 [1] for the case where Q_i,R_{ij},S_i are polynomials as a generalization of the result in [5] to the case of multiple delays. This inverse is also of the form $\mathcal{P}_{\{\hat{P},\hat{Q}_i,\hat{S}_i,\hat{R}_{ij}\}}$ where expressions for the matrix $\hat{P}$ and functions $\hat{Q}_i,\hat{R}_{ij},\hat{S}_i$ are given in the following theorem, which is repeated here for convenience as it is used in the following section.

Theorem 4: Suppose that $Q_i(s) = H_iZ(s)$ and $R_{ij}(s,\theta)=Z(s)^T\Gamma_{ij}Z(\theta)$ and $\mathcal{P}:=\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}$ is a coercive operator where $\mathcal{P}:X\to X$ and $\mathcal{P}=\mathcal{P}^*$. Define

$$H=\begin{bmatrix}H_1&\dots&H_K\end{bmatrix}\quad\text{and}\quad\Gamma=\begin{bmatrix}\Gamma_{11}&\dots&\Gamma_{1K}\\\vdots&&\vdots\\\Gamma_{K,1}&\dots&\Gamma_{K,K}\end{bmatrix}.$$

Now let $K_i=\int_{-\tau_i}^0 Z(s)S_i(s)^{-1}Z(s)^Tds$,

$$\begin{aligned}K&=\mathrm{diag}(K_1,\cdots,K_K),\\\hat{H}&=P^{-1}H\left(KH^TP^{-1}H-I-K\Gamma\right)^{-1},\\\hat{\Gamma}&=-(\hat{H}^TH+\Gamma)(I+K\Gamma)^{-1},\end{aligned}$$

$$\begin{bmatrix}\hat{H}_1&\dots&\hat{H}_K\end{bmatrix}=\hat{H},\qquad\begin{bmatrix}\hat{\Gamma}_{11}&\dots&\hat{\Gamma}_{1K}\\\vdots&&\vdots\\\hat{\Gamma}_{K,1}&\dots&\hat{\Gamma}_{K,K}\end{bmatrix}=\hat{\Gamma}.$$

Then if we define

$$\hat{P}=\left(I-\hat{H}KH^T\right)P^{-1},\qquad\hat{Q}_i(s)=\hat{H}_iZ(s)S_i(s)^{-1},$$
$$\hat{S}_i(s)=S_i(s)^{-1},\quad\hat{R}_{ij}(s,\theta)=\hat{S}_i(s)Z(s)^T\hat{\Gamma}_{ij}Z(\theta)\hat{S}_j(\theta),$$

then for $\hat{\mathcal{P}}:=\mathcal{P}_{\left\{\hat{P},\frac{1}{\tau_K}\hat{Q}_i,\frac{1}{\tau_K^2}\hat{S}_i,\frac{1}{\tau_K}\hat{R}_{ij}\right\}}$, we have $\hat{\mathcal{P}}=\hat{\mathcal{P}}^*$, $\hat{\mathcal{P}}:X\to X$, and $\hat{\mathcal{P}}\mathcal{P}\mathbf{x}=\mathcal{P}\hat{\mathcal{P}}\mathbf{x}=\mathbf{x}$ for any $\mathbf{x}\in Z_{m,n,K}$.

Proof: See [6] for the proof. ■

VIII. Constructing the Observer Gains

Armed with this inverse, we may define the observer gains.

Lemma 5: Let $\mathcal{L}=\hat{\mathcal{P}}\mathcal{Z}$ where $\hat{\mathcal{P}}$ is as in Theorem 4 and $\mathcal{Z}$ is as in Eqn. (10) with polynomial representation of the Z_i as $Z_{4i}(s)=Z(s)^TW_{4i}$, $Z_{5ij}(s)=Z(s)^TW_{5ij}$, $Z_{6i}(s)=$

$Z(s)^T W_{6i}$, and $Z_{7ij}(s,\theta) = Z(s)^T W_{7ij} Z(\theta)$. Then $\mathcal{L}$ is as in Eqn. (9) where

$$L_0 = \hat{P} Z_0 + \sum_{i=1}^K \hat{H}_i T_i W_{4i},$$

$$L_{2i} = \hat{P} Z_{2i} + \sum_{j=1}^K \hat{H}_j T_j W_{5ji},$$

$$L_{3,i}(\theta) = \hat{P} Z_{3,i}(\theta) + \hat{H}_i V_i(\theta) W_{6i} + \sum_j \hat{H}_j T_j W_{7ji} Z(\theta),$$

$$L_{4i}(s) = X_i(s) \left(\hat{H}_i^T Z_0 + W_{4i} + \sum_{j=1}^K \hat{\Gamma}_{ij} T_j W_{4j} \right),$$

$$L_{5ij}(s) = X_i(s) \left(\hat{H}_i^T Z_{2j} + W_{5ij} + \sum_{k=1}^K \hat{\Gamma}_{ik} T_k W_{5kj} \right),$$

$$L_{6i}(s) = X(s) W_{6i},$$

$$L_{7ij}(s,\theta) = X_i(s) \Big(\hat{H}_i^T Z_{3,j}(\theta) + W_{7ij} Z(\theta) + \hat{\Gamma}_{ij} V_j(\theta) W_{6j} + \sum_{k=1}^K \hat{\Gamma}_{ik} T_k W_{7kj} Z(\theta) \Big),$$

$$X_i(s) = \hat{S}_i(s) Z(s)^T, \qquad V_i(s) = Z(s) \hat{S}_i(s) Z(s)^T,$$

$$T_i = \int_{-\tau_i}^0 Z(s) \hat{S}_i(s) Z(s)^T ds.$$

Proof: The proof is a straightforward composition of operators and is omitted for brevity. ∎

Note that if we constrain $Q_i = 0$ and $Z_{3i} = Z_{4i} = Z_{5ij} = Z_{6i} = Z_{7ij} = 0$, we recover an observer with corrections only to the present state.

The advantage of this representation is that there is only a single integration to find the matrices T_i and $\hat{S}$ only appears in the auxiliary functions X_i and V_i. This significantly improves numerical reliability and decreases computational complexity of implementation.

IX. Numerical Implementation

In this section, we address two issues significant for the efficient construction and implementation of the observers. The first is numerical computation of the inverse. The second is real-time simulation of the observer dynamics.

A. Computing the Inverse

There are two steps to computing the inverse which may be difficult for the reader. Both steps are implemented in the m-file `P_PQRS_Inverse_joint_sep` in the Matlab package associated with this paper. The first step is to calculate the uniquely defined H_i and Γ_{ij} matrices. Each element of these matrices is defined by a single coefficient in the matrix-valued polynomials $Q_i(s)$ and $R_{ij}(s,\theta)$. This can be cumbersome in the `pvar` framework, so we have constructed Matlab functions `decomposition_multiplier.m` and `decomposition_kernel.m` which automate this process (See Code available from [4]).

The next step is to Calculate $S^{-1}(s)$. However, because of the representation in Lemma 5, we do not need a polynomial formulation of this inverse. Rather, we evaluate $S(s)$ at discrete points where the value is needed, and invert the matrix at each point. This results in a reliable computation of the integral T_i.

B. Implementation of the Observer

The PDE governing the observer dynamics is a generalization of the transport equation. Therefore, we use a forward difference approximation based on a number of lumped states, N. Typically 20 states is more than sufficient to obtain accurate results. In the code associated with this paper, we verified the observers and H_∞-gain bounds using several different methods. A complete description is not given here due to space limitation, however, and hence we refer to that code for additional details.

X. Numerical Examples

Significant care must be taken in the choice of numerical examples to correctly demonstrate the advantages and limitations of the proposed observer design. Specifically, most examples in the literature are 2-state and have disturbance inputs of the form $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. That is, a single disturbance affects both states equally. In such cases our observers can achieve very small H_∞ norms — typically less than .001 (we do not test smaller gains due to potential numerical difficulties in verification). We can achieve these gains because the observers we design can indirectly observe the disturbance through the measured output and use this information to correct the state estimate. However, we feel that this approach is not fair or realistic and hence use independent channels to disturb all states. For this reason, several of the examples given below have been modified from their original form. Because most codes are not available online, the result is that we only include numerical comparisons for the results in [7], for which we were able to reproduce the tests given in that paper. However, the readers should bear in mind that using the original systems and results from, e.g. [8], [9], the observers in we provide improve the achieved H_∞ gains by several orders of magnitude (Specifically, the H_∞ gains using our algorithm can be made *arbitrarily* ($< .001$) small) and their omission is not due to poor performance with respect to these earlier works.

In all cases, in order to show that the observers we design are not significantly conservative, we have used a 10th order Padé approximation to construct an ODE approximation of the original multi-delay system. We then applied the LMI in Equation (3) to obtain an estimate of the minimal achievable closed-loop H_∞ norm bound. These results are indicated in a table associated with each numerical example and compared to results obtained through implementation of Theorem 3 using the value of d indicated in the table.

a) Example 1: In this example, we consider the unstable system with $\tau = .3$

$$\dot{x}(t) = \begin{bmatrix} -3 & 4 \\ 2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$

$$y(t) = \begin{bmatrix} 0 & 7 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t).$$

Applying the Ricatti approach in [10] with $\epsilon = .001$ we obtain a L_2-gain of $\gamma = .580$. Applying the conditions of Theorem 3, we obtain an L_2-gain of .2357. Of all the systems we tested, this one showed the least improvement

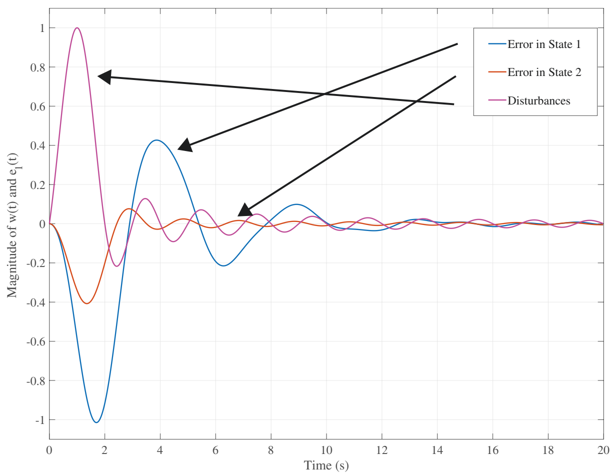

Fig. 1. A Matlab simulation of the error dynamics of System 12 coupled with the observer from Theorem 3 with gain 2.33 and delay $\tau = 1s$. The image displays $w(t)$ and $e_1(t) = \hat{x}(t) - x(t)$.

in performance. Note, however, that the lack of improvement is due to the fact that a lower H_∞ bound is not achievable (See comparison with Padé).

	$d=1$	$d=2$	$d=3$	Padé	[10]
$\gamma_{\min}$	.2357	.2357	.2357	.2357	.580
CPU sec	.433	.918	2.29	2.92	N/A

b) Example 2: In this example, we consider the following unstable system which is modified from the result in [8].

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{11}$$

Using the original system with $\tau = 1$, a closed-loop gain of 22.8 was obtained in [8]. For this problem, the Ricatti approach in [10] was infeasible for any value of gain. Applying the conditions of Theorem 3, we obtained a closed-loop gain of 2.327 using polynomials of degree 4. A Simulation of the error and disturbance dynamics is shown in Figure 1. Note that only the values of $w(t)$ and $e_1(t) = \hat{x}(t) - x(t)$ are shown in this figure. The input is a sinc function and the numerically calculated L_2 gain for this observer using the sinc function is 1.186.

	$d=1$	$d=2$	$d=3$	Padé	[8]
$\gamma_{\min}$	2.3323	2.3270	2.3270	2.3270	22.8
CPU sec	.968	.668	1.99	2.98	N/A

c) Example 3: In this example, we further modify the problem in [8] to obtain a 2-delay system.

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 \\ 0 & -.9 \end{bmatrix} \frac{x(t-.5) + x(t-1)}{2} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w(t),$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t), \quad z(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{12}$$

	$d=1$	$d=2$	$d=3$	Padé
$\gamma_{\min}$	1.3511	1.3501	1.3501	1.3501
CPU sec	4.77	7.57	33.59	5.45

$K \downarrow n \rightarrow$	1	2	3	5	10
1	.516	.218	.375	2.203	24.094
2	.219	.547	2.141	19.282	875.137
3	.3910	1.782	9.484	113.236	4742.7
5	1.375	12.454	109.939	1859.9	62069
10	18.406	582.945	4717.2	66033	N/A

TABLE I

CPU SEC INDEXED BY # OF STATES (n) AND # OF DELAYS (K)

d) Example 4: In this example, we examine the computational complexity of the proposed algorithm using an unstable n-D system with K delays, a single disturbance $w(t)$, a single regulated output and a single sensed output.

$$\dot{x}(t) = -J \sum_{i=1}^{K} \frac{x(t - i/K)}{K} + \mathbf{1} w(t)$$
$$y(t) = z(t) = \mathbf{1}^T x(t)$$

where J is the n-dimensional Jordan block and $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones. The resulting computation time is listed in Table I as CPU sec on a Intel i7-5960X processor and omits preprocessing and postprocessing times. Note that computational complexity is approximately a function of the product of the number of delays and number of states.

XI. Conclusion

We have proposed an LMI approach to H_∞-optimal observer design for systems with multiple time delays. These observers correct both the estimates of present state and history. Given a solution to the LMI, the observer gains can be reconstructed using algebraic techniques and implemented using discretization.

The Matlab code associated with this paper performs all these tasks and is freely available online. The numerical testing and validation indicates little if any conservatism in the H_∞ bound. The observers in this paper outperform existing observers, often by several orders of magnitude – to the extent that new test cases had to be created to fully understand the limitations of the approach.

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