Systems Analysis and Control

Matthew M. Peet
Arizona State University

Lecture 8: Response Characteristics

Overview

In this Lecture, you will learn:

The Effects of Feedback on Dynamic Response

- Changes in Transfer Function
- Stability
- Impact of Poles on Dynamic Response

Real Poles

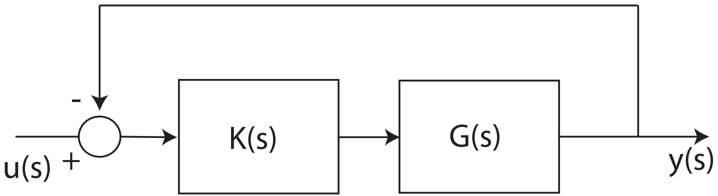
- Steady-State Error
- Rise Time
- Settling Time

Complex Poles

- Complex Pole Locations
- Damped/Natural Frequency
- Damping and Damping Ratio

The Effect of Feedback Control

Recall the Upper Feedback Interconnection



Feedback:

- **Controller:** $\hat{u}(s) = K(s)(\hat{u}(s) - \hat{y}(s))$
- **Plant:** $\hat{y}(s) = \hat{G}(s)\hat{u}(s)$

The output signal is $\hat{y}(s)$,

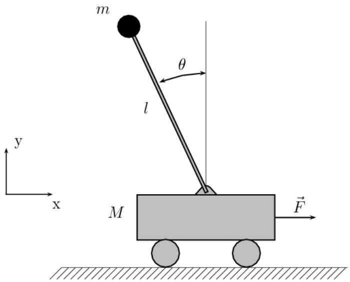
$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$

Controlling the Inverted Pendulum Model

Open Loop Transfer Function

$$\hat{G}(s) = \frac{1}{Js^2 - \frac{Mgl}{2}}$$

Controller: Static Gain: $\hat{K}(s) = K$
Input: Impulse: $\hat{u}(s) = 1$.



Closed Loop: Lower Feedback Interconnection

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1+\hat{G}(s)\hat{K}(s)}\hat{u}(s) = \frac{\frac{K}{Js^2-\frac{Mgl}{2}}}{1+\frac{K}{Js^2-\frac{Mgl}{2}}} = \frac{K}{Js^2 - \frac{Mgl}{2}+K}$$

Effect of Feedback on the Inverted Pendulum Model

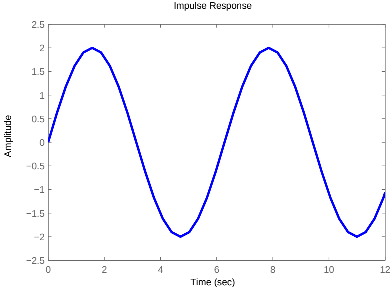
Closed Loop Impulse Response:

Lower Feedback Interconnection

$$\hat{y}(s) = \frac{\frac{K}{J}}{s^2 - \frac{Mgl}{2J} + \frac{K}{J}}$$

Traits:

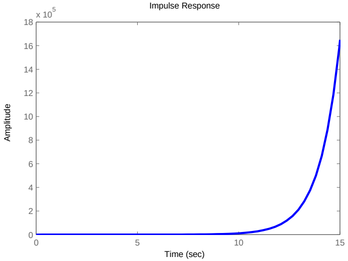
- Infinite Oscillations
- Oscillates about 0.



Open Loop Impulse Response:

$$\hat{y}(s) = \frac{1}{J}\sqrt{\frac{2J}{Mgl}}\left(\frac{1}{s - \sqrt{\frac{Mgl}{2J}}} - \frac{1}{s + \sqrt{\frac{Mgl}{2J}}}\right)$$

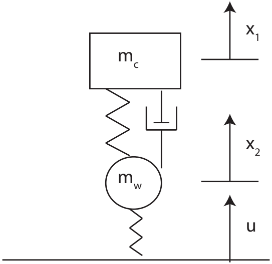
Unstable!



Controlling the Suspension System

Open Loop Transfer Function:
Set $m_c = m_w = g = c = K_1 = K_2 = 1$.

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}$$



Controller: Static Gain: $\hat{K}(s) = k$

Closed Loop: Lower Feedback Interconnection:

$$\hat{y}(s) = \frac{\hat{G}(s)\hat{K}(s)}{1 + \hat{G}(s)\hat{K}(s)}\hat{u}(s)$$
$$= \frac{k(s^2 + s + 1)}{s^4 + 2s^3 + (3+k)s^2 + (1+k)s + (1+k)}$$

Closed Loop Step Response:

$$\hat{y}(s) = \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}\frac{1}{s}$$

High k:

- Overshot the target
- Quick Response
- Closer to desired value of f

Low k:

- Slow Response
- No overshoot
- Final value is farther from 1.

Questions:

- Which Traits are important?
- How to predict the behaviour?

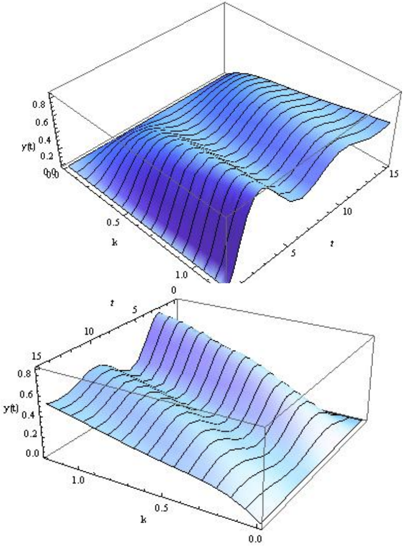


Figure: Step Response for different k

Stability

The most basic property is **Stability**:

Definition 1.

A system, G is **Stable** if there exists a $K > 0$ such that

$$\|Gu\|_{L_2} \le K\|u\|_{L_2}$$

Note: Although this is the true definition for systems defined by transfer functions, it is rarely used.

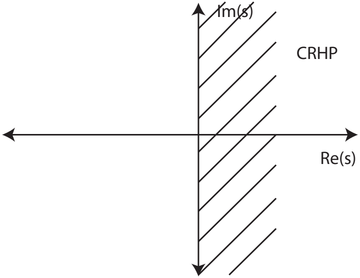
- Bounded input means bounded output.
- Stable means $y(t) \to 0$ when $u(t) \to 0$.

Stability Depends on Pole Locations

Definition 2.

The **Closed Right Half-Plane**, $CRHP$ is the set of complex numbers with non-negative real part.

$$\{s \in \mathbb{C} \,:\, \text{Real}(s) \geq 0\}$$



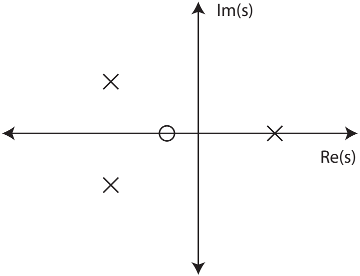


Figure: Unstable

Theorem 3.

A system G is stable if and only if it's transfer function $\hat{G}$ has no poles in the Closed Right Half Plane.

- Check stability by checking poles.
- x is a pole
- o is a zero

Predicting Steady-State Error

Definition 4.

Steady-State Error for a stable system is the final difference between input and output.

$$e_{ss} = \lim_{t\to\infty} u(t) - y(t)$$

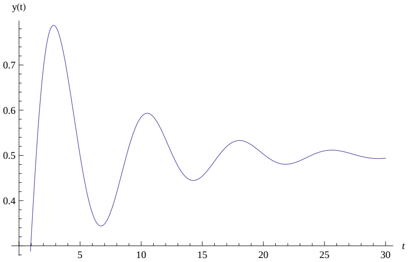
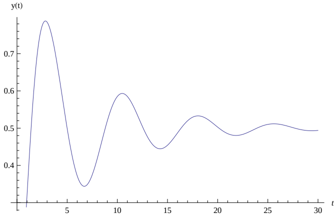


Figure: Suspension Response for $k = 1$

- Usually measured using the step response.
  - Since $u(t) = 1$, $e_{ss} = 1 - \lim_{t\to\infty} y(t)$

Predicting Steady-State Value Using the Residue



Recall: For any system G, by partial fraction expansion:

$$\hat{y}(s) = \hat{G}(s)\frac{1}{s} = \frac{r_0}{s} + \frac{r_1}{s-p_1} + \ldots + \frac{r_n}{s-p_n}$$

So

$$y(t) = r_0 \mathbf{1}(t) + r_1 e^{p_1 t} + \ldots + r_n e^{p_n t}$$

which means

$$\lim_{t\to\infty} y(t) = r_0$$

and hence

$$e_{ss} = 1 - r_0$$

The Final Value Theorem

The steady-state error is given by r_0.

$$e_{ss} = 1 - r_0$$

Recall: The residue at $s = 0$ is r_0 and is found as

$$r_0 = \hat{G}(s)|_{s=0} = \lim_{s\to 0} \hat{G}(s)$$

Thus the steady-state error is

$$e_{ss} = 1 - \lim_{s\to 0} \hat{G}(s)$$

This can be generalized to find the limit of any signal:

Theorem 5 (Final Value Theorem).

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\hat{y}(s)$$

- Assumes the limit exists (Stability)
- Can be used to find response to other inputs
  - Ramp, impulse, etc.

The Final Value Theorem for Systems in Lower Feedback

Lower Feedback Interconnection:

$$\hat{y}(s) = \frac{G(s)K}{1+G(s)K}\hat{u}(s)$$

Error Response for Lower Feedback Interconnection:

$$\begin{aligned}\hat{e}(s) &= \hat{u}(s) - \hat{y}(s)\\ &= \frac{1+G(s)K}{1+G(s)K}\hat{u}(s) - \frac{G(s)K}{1+G(s)K}\hat{u}(s) = \frac{1+G(s)K-G(s)K}{1+G(s)K}\hat{u}(s)\\ &= \frac{1}{1+G(s)K}\hat{u}(s)\end{aligned}$$

So the steady-state error is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\hat{e}(s) = \lim_{s\to 0}\frac{s}{1+G(s)K}\hat{u}(s)$$

Error in Step Response: If $\hat{u}(s) = \frac{1}{s}$, then

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0}\frac{1}{1+G(s)K}$$

Conclusion: Increasing K always reduces steady-state error to step!

The Final Value Theorem for Systems in Upper Feedback

Upper Feedback Interconnection:

$$\hat{y}(s) = \frac{G(s)}{1 + G(s)K}\hat{u}(s)$$

Error Response for Upper Feedback Interconnection:

$$\begin{aligned}\hat{e}(s) &= \hat{u}(s) - \hat{y}(s) \\ &= \frac{1 + G(s)K}{1 + G(s)K}\hat{u}(s) - \frac{G(s)}{1 + G(s)K}\hat{u}(s) = \frac{1 + G(s)K - G(s)}{1 + G(s)K}\hat{u}(s) \\ &= \frac{1 + G(s)(K-1)}{1 + G(s)K}\hat{u}(s)\end{aligned}$$

Error in Step Response: If $\hat{u}(s) = \frac{1}{s}$, then

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1 + G(s)(K-1)}{1 + G(s)K}$$

Conclusion: Increasing K doesn't help with step response!

Steady-State Error

Numerical Example

$$\hat{G}(s) = \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)}$$

The steady-state response is

$$\begin{aligned} y_{ss} &= \lim_{s\to 0} s\hat{y}(s) = \lim_{s\to 0} \hat{G}(s) \\ &= \lim_{s\to 0} \frac{k(s^2+s+1)}{s^4+2s^3+(3+k)s^2+(1+k)s+(1+k)} \\ &= \frac{k}{1+k} \end{aligned}$$

The steady-state error is

$$\begin{aligned} e_{ss} &= 1 - y_{ss} = 1 - \frac{k}{1+k} \\ &= \frac{1}{1+k} \end{aligned}$$

- When $k = 0$, $e_{ss} = 1$
- As $k \to \infty$, $e_{ss} = 0$

Ramp Response for Systems in Lower Feedback

Lower Feedback Interconnection: Recall the steady-state error is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\hat{e}(s) = \lim_{s\to 0} \frac{s}{1+G(s)K}\hat{u}(s)$$

Error in Ramp Response: If $\hat{u}(s) = \frac{1}{s^2}$, then

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{s(1+G(s)K)}$$

Conclusion: $\lim_{t\to\infty} e(t) = \infty$!

Preview of Integral Feedback: However, if we choose $K \to \frac{K}{s}$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{s(1+G(s)\frac{K}{s})} = \lim_{s\to 0} \frac{1}{s+G(s)K}$$

Conclusion: Increasing K improves the ramp response!

- More on this Later....

From Partial Fractions Expansion, you know that motion is determined by the poles of the Transfer Function!

$$\frac{n(s)}{(s^2+as+b)(s-p_1)\cdots(s-p_n)}\hat{u}(s) = \frac{k_1 s + k_2}{s^2+as+b}\hat{u}(s) + \frac{r_1}{s-p_1}\hat{u}(s) + \cdots + \frac{r_n}{s-p_n}\hat{u}(s)$$

- Simplify the response by considering response of each pole.

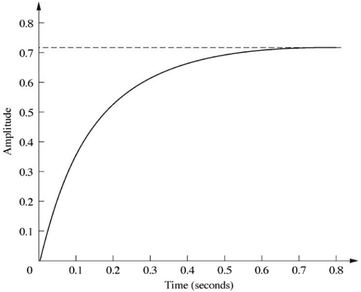


Figure: Real Pole

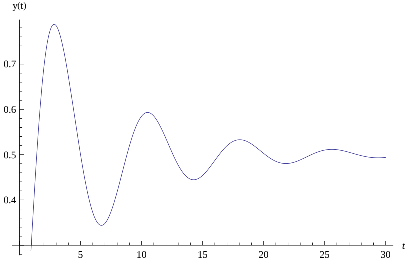


Figure: Complex Pair of Poles

We start with **Real Poles**

Step Response Characteristics

Real Poles

The Solution: Step response of a real pole.

$$\hat{y}(s) = \frac{r}{s-p}\hat{u}(s) = \frac{r}{s-p}\frac{1}{s} = \frac{\frac{r}{p}}{s-p} - \frac{\frac{r}{p}}{s} \qquad y(t) = \frac{r}{p}\left(e^{pt} - 1\right)$$

- Is it stable? $(p < 0?)$

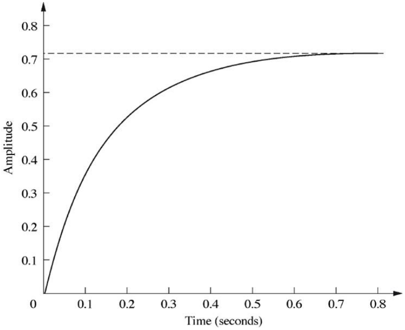
Cases: Stability or Instability?

- If $p > 0$, then $\lim_{t\to\infty} y(t) \to \infty$
- If $p < 0$, then $\lim_{t\to\infty} y(t) \to -\frac{r}{p}$

Final Value:

$$y_{ss} = -\frac{r}{p}$$

Question: How quickly do we reach the final value?



Step Response Characteristics

Rise Time

$$y(t) = \frac{r}{p}\left(e^{pt} - 1\right), \qquad y_{ss} = -\frac{r}{p}$$

Definition 6.

The rise time, T_r, is the time it takes to go from .1 to .9 of the final value.

If $y(t_1) = .1 y_{ss} = -.1\frac{r}{p}$, then t_1 is:

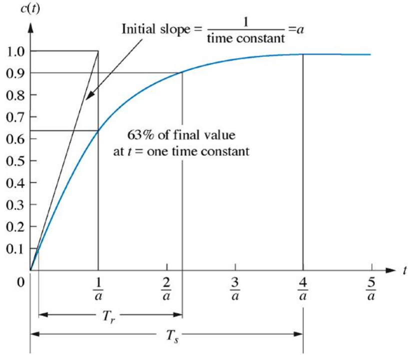
$$-.1 = e^{pt_1} - 1$$

$$\ln(1 - .1) = pt_1$$

$$t_1 = \frac{\ln .9}{p} = \frac{.11}{-p}$$

Likewise if $y(t_2) = -.9\frac{r}{p}$, then

$$t_2 = \frac{\ln .1}{p} = \frac{2.31}{-p}$$



Rise time (T_r) for a **Single Pole** is:

$$T_r = t_2 - t_1 = \frac{2.31}{-p} - \frac{.11}{-p} = \frac{2.2}{-p}$$

Definition 7.

The **Settling Time**, T_s, is the time it takes to reach and stay within .99 of the final value.

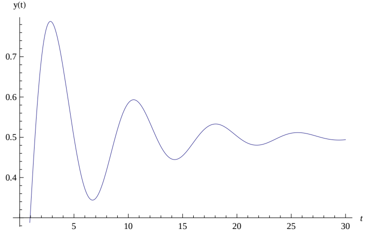


Figure: Complex Pair of Poles

LINK: Bouncing Balls
LINK: More Bouncing Balls!
LINK: Newton's Cradle

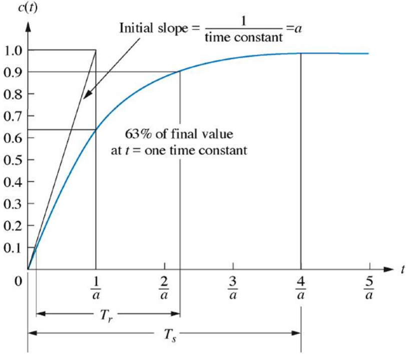
$$y(t) = \frac{r}{p}\left(e^{pt} - 1\right), \qquad y_{ss} = -\frac{r}{p}$$

Definition 8.

The **Settling Time**, T_s, is the time it takes to reach and stay within .99 of the final value.

Solve $y(T_s) = \frac{r}{p}\left(e^{pT_s} - 1\right) = -.99\frac{r}{p}$ for T_s:

$$-.99 = e^{pT_s} - 1$$
$$\ln(.01) = pT_s$$
$$T_s = \frac{\ln .01}{p} = -\frac{4.6}{p}$$



The settling time for a **Single Pole** is:

$$T_s = \frac{4.6}{-p}$$

Solution for Complex Poles

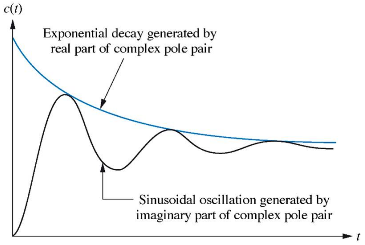
$$\hat{y}(s) = \frac{\omega_d^2 + \sigma^2}{s^2 + 2\sigma s + \omega_d^2 + \sigma^2}\frac{1}{s} = \frac{\sigma^2 + \omega_d^2}{(s+\sigma)^2 + \omega_d^2}\frac{1}{s} = -\frac{s + 2\sigma}{(s+\sigma)^2 + \omega_d^2} + \frac{1}{s}$$

The poles are at $s = -\sigma \pm \omega_d \imath$. The solution is:

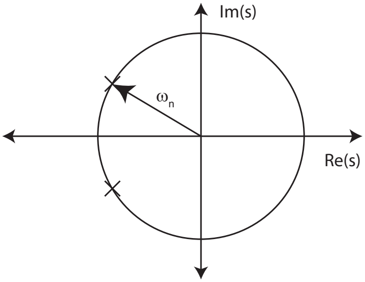
$$y(t) = 1 - e^{\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

The result is oscillation with an Exponential Envelope.

- Envelope decays at rate σ
- Speed of oscillation is ω_d, the **Damped Frequency**



Besides ω_d, there is another way to measure oscillation



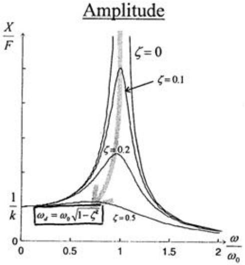
Definition 9.

The **Natural Frequency** of a pole at $p = \sigma + \imath\omega_d$ is $\omega_n = \sqrt{\sigma^2 + \omega_d^2}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\omega_n = \sqrt{b}$.
- Radius of the pole in complex plane.

Resonant Frequency.

- Also known as resonant frequency

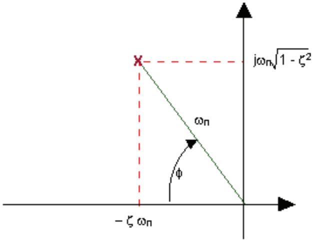


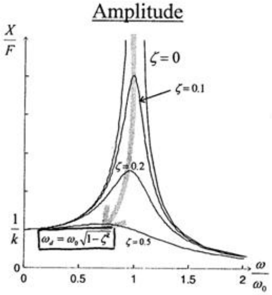
Besides σ, there are other ways to measure damping

Definition 10.

The **Damping Ratio** of a pole at $p = \sigma + \imath\omega$ is $\zeta = \frac{|\sigma|}{\omega_n}$.

- for $\hat{y}(s) = \frac{1}{s^2+as+b}\frac{1}{s}$, $\zeta = \frac{a}{2\sqrt{b}}$.
- Gives the ratio by which the amplitude decreases per oscillation (almost...).

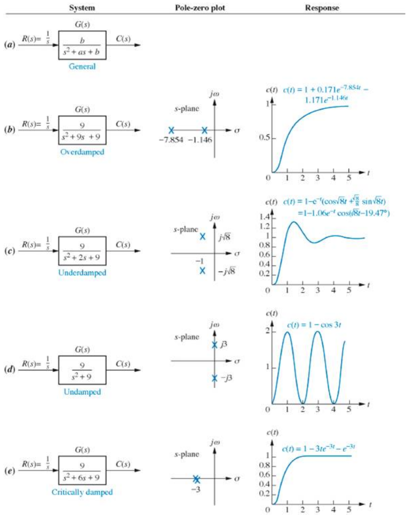




Damping

We use several adjectives to describe exponential decay:

- Undamped
  - Oscillation continues forever, $\sigma = \zeta = 0$
- Underdamped
  - Oscillation continues for many cycles. $\zeta < 1$
- Critically Damped
  - No oscillation or overshoot. $\zeta = 1$, $\omega_d = 0$
- Overdamped
  - When $\zeta > 1$, poles are real (not complex)



Summary

What have we learned today?

Characteristics of the Response

Real Poles

- Steady-State Error
- Rise Time
- Settling Time

Complex Poles

- Complex Pole Locations
- Damped/Natural Frequency
- Damping and Damping Ratio

Continued in Next Lecture