Polynomial Lyapunov Functions for Exponential Stability of Nonlinear Systems on Bounded Regions

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Abstract: This paper presents a proof that the use of polynomial Lyapunov functions is not conservative for studying exponential stability properties of nonlinear ordinary differential equations on bounded regions. The main result implies that if there exists an n-times continuously differentiable Lyapunov function which proves exponential decay on a bounded subset of $\mathbb{R}^n$, then there exists a polynomial Lyapunov function which proves that same rate of decay on the same region. Our investigation is motivated by the use of semidefinite programming to construct polynomial Lyapunov functions for delayed and nonlinear systems of differential equations.

Keywords: Nonlinear Systems, Stability, Lyapunov functions, Semidefinite Programming, Polynomial Optimization, Polynomials, Polynomial Approximation, Sobolev Spaces.

1. INTRODUCTION

In Weierstrass (1885), it was demonstrated that real-valued polynomial functions can approximate real-valued continuous functions arbitrarily well with respect to the supremum norm on a compact interval. Various structural generalizations of the Weierstrass approximation theorem have focused on generalized mappings, as in Stone (1948), and on alternate topologies, as in Krein (1945). Polynomial approximation of differentiable functions has been studied in numerical analysis of differential and partial differential equations. Results of relevance include the lemma by Bramble and Hilbert (1971) and its generalization in Dupont and Scott (1980), and the theorem of Jackson (1911), generalizations of which can be found in the work of Timan (1960).

The approximation of Sobolev spaces by smooth functions has been studied in a number of contexts. In Everitt and Littlejohn (1993), the density of polynomials of a single variable in certain Sobolev spaces was discussed. The density of infinitely continuously differentiable test functions in Sobolev spaces has been studied in the context of partial differential equations. Important results include the theorem of Meyers and Serrin (1964), extensions of which can be found in Adams (1975) or Evans (1998).

Recently, the density of polynomials in the space of continuous functions has been used to motivate research on optimization over parameterized sets of positive polynomial functions. Relevant results include an extension of the Weierstrass approximation theorem to polynomials subject to affine constraints in Peet and Bliman (2007). Examples of algorithms for optimization over cones of positive polynomials include SOSTOOLS, which optimizes over the cone of sums of squares of polynomials, and Gloptipoly, which optimizes over the dual space to obtain bounds on the primal objective.

One application of polynomial optimization which has received particular attention has been construction of polynomial Lyapunov functions for ordinary nonlinear differential equations of the form

$$\dot{x}(t) = f(x(t)),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$. The use of polynomial optimization for the construction of polynomial Lyapunov functions has been studied extensively in Tan (2006) and Wang (2007).

Several converse theorems have shown that local exponential stability of ordinary differential equations implies the existence of an exponentially decreasing Lyapunov function. The reader is referred to Hahn (1967) and Krasovskii (1963) for an extensive treatment of converse theorems of Lyapunov. It is not generally known, however, under what conditions exponential stability implies the existence of a polynomial Lyapunov function.

A significant result of this paper is summarized in Theorem 8. This theorem shows that for a vector field which is bounded on compact set X, the existence of a sufficiently smooth Lyapunov function which decreases exponentially on X with rate γ implies the existence of a polynomial Lyapunov function which decreases exponentially on X at rate γ' for any $\gamma' < \gamma$. The smoothness condition is automatically satisfied if the function is n-times continuously differentiable.

To establish this result, we prove an extension to the Weierstrass approximation theorem. In Theorem 6, we combine a second order Taylor series expansion with the Weierstrass approximation theorem to find polynomials which approximate functions with a pointwise weight on the error given by

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$$w(x) = \frac{1}{x^T x}.$$

A corollary of this result states that polynomials can be used to approximate continuously differentiable multivariate functions arbitrarily well in a variety of norms, including the Sobolev norm $W^{1,\infty}$. This means that for any continuously differentiable function and any $\gamma > 0$, we can find a polynomial which approximates the function with error γ and whose partial derivatives approximate those of the function with error γ. The proof is based on a construction using approximations to the partial derivatives.

This extension is applied to Lyapunov functions in Theorem **??** which states that if there exists a sufficiently smooth continuous Lyapunov function which proves exponential stability on a bounded set, then there exists a polynomial Lyapunov function which proves exponential stability on the same set. In Section 4, we briefly discuss implications for the use of polynomial optimization to prove stability of nonlinear ordinary differential equations.

2. NOTATION AND BACKGROUND

Let $\mathbb{N}^n$ denote the set of length n vectors of non-negative natural numbers. Denote the unit cube in $\mathbb{N}^n$ by $Z^n := \{\alpha \in \mathbb{N}^n : \alpha_i \in \{0, 1\}\}$. For $x \in \mathbb{R}^n$, $\|x\|_\infty = \max_i |x_i|$ and $\|x\|_2 = \sqrt{x^T x}$. Define the unit cube in $\mathbb{R}^n$ by $B := \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. Let $\mathcal{C}(B)$ be the Banach space of scalar continuous functions defined on $B \subset \mathbb{R}^n$ with norm

$$\|f\|_\infty := \sup_{x \in B} \|f(x)\|_\infty.$$

For operators $h_i : X \to X$, let $\prod_i h_i : X \to X$ denote the sequential composition of the h_i. i.e.

$$\prod_i h_i := h_1 \circ h_2 \circ \cdots \circ h_{n-1} \circ h_n.$$

For a sufficiently regular function $f : \mathbb{R}^n \to \mathbb{R}$ and $\alpha \in \mathbb{N}^n$, we will often use the following shorthand to denote the partial derivative

$$D^\alpha f(x) := \frac{\partial^\alpha}{\partial x^\alpha} f(x) = \prod_{i=1}^n \frac{\partial^{\alpha_i}}{\partial x^{\alpha_i}} f(x),$$

where naturally, $\partial^0 f / \partial x_i^0 = f$. For $\Omega \subset \mathbb{R}^n$, we define the following sets of differentiable functions.

$$\mathcal{C}_1^i(\Omega) := \Big\{ f \,:\, D^\alpha f \in \mathcal{C}(\Omega) \quad \text{for any } \alpha \in \mathbb{N}^n \text{ such that } \|\alpha\|_1 = \sum_{j=1}^n \alpha_j \leq i. \Big\}$$

$$\mathcal{C}_\infty^i(\Omega) := \Big\{ f \,:\, D^\alpha f \in \mathcal{C}(\Omega) \quad \text{for any } \alpha \in \mathbb{N}^n \text{ such that } \|\alpha\|_\infty = \max_j \alpha_j \leq i. \Big\}$$

$\mathcal{C}^\infty(B) = \mathcal{C}_1^\infty(B) = \mathcal{C}_\infty^\infty(B)$ is the logical extension to infinitely continuously differentiable functions. Note that in n-dimensions, $\mathcal{C}_1^i(B) \subset \mathcal{C}_\infty^i(B) \subset \mathcal{C}_1^{in}(B)$. We will occasionally refer to the Banach spaces $W^{k,p}(\Omega)$, which denote the standard Sobolev spaces of locally summable functions $u : \Omega \to \mathbb{R}$ with ***weak*** derivatives $D^\alpha u \in L_p(\Omega)$ for $|\alpha|_1 \leq k$ and norm

$$\|u\|_{W^{k,p}} := \sum_{|\alpha|_1 \leq k} \|D^\alpha u\|_{L_p}.$$

The following version of the Weierstrass approximation theorem in multiple variables comes from Timan (1960).

Theorem 1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous and $G \subset \mathbb{R}^n$ is compact. Then there exists a sequence of polynomials which converges to f uniformly in G.

3. APPROXIMATION OF DIFFERENTIABLE FUNCTIONS

The technical results of this paper concern a constructive method of approximating a differentiable function of several variables using approximations to the partial derivatives of that function. Specifically, the result states that if one can find polynomials which approximate the partial derivatives of a given function to accuracy $\gamma/2^n$, then one can construct a polynomial which approximates the given function to accuracy γ, and all of whose partial derivatives approximate those of the given function to accuracy γ.

The difficulty of this problem is a consequence of the fact that, for a given function, the partial derivatives of that function are not independent. Therefore, given approximations to the partial derivatives of a function, these approximations will, in general, not be the partial derivatives of any function. The problem is, for each partial derivative approximation, how to extract the information which is unique to that partial derivative in order to form an approximation to the original function. The following construction shows how this can be done.

Definition 2. Let X be the space of 2^n-tuples of continuous functions indexed using the 2^n elements $\alpha \in Z^n$. Thus if functions $f_\alpha \in \mathcal{C}(B)$ for all $\alpha \in Z^n$, then these functions define an element of X, denoted $\{f_\alpha\}_{\alpha \in Z^n} \in X$. Define the linear map $K : X \to \mathcal{C}_\infty^1(B)$ as

$$K\left(\{f_\alpha\}_{\alpha \in Z^n}\right) = \sum_{\alpha \in Z^n} G_\alpha f_\alpha$$

where $G_\alpha : \mathcal{C}(B) \to \mathcal{C}_\infty^1(B)$ is given by

$$G_\alpha h = \left(\prod_{i=1}^n g_{i,\alpha_i} \right) h$$

and where

$$(g_{i,j}h)(x_1, \ldots, x_n) = \begin{cases} h(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) & j = 0 \\ \int_0^{x_i} h(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_n)\, ds & j = 1. \end{cases}$$

In practice, the functions f_α represent either the partial derivatives of a function or approximations to those partial derivatives. The following examples illustrate the construction.

Example: If $p = K(\{q_\alpha\}_{\alpha \in Z^2})$, then

$$\begin{aligned} p(x_1, x_2) = &\int_0^{x_1} \int_0^{x_2} q_{(1,1)}(s_1, s_2)\, ds_1\, ds_2 \\ &+ \int_0^{x_1} q_{(1,0)}(s_1, 0)\, ds_1 + \int_0^{x_2} q_{(0,1)}(0, s_2)\, ds_2 \\ &+ q_{(0,0)}(0, 0). \end{aligned}$$

The following lemma shows that when the f_α are the partial derivatives of a function, we recover the original function. The proof is simple and works by using a single integral identity to repeatedly expand the function.

Lemma 3. For $v \in \mathcal{C}^1_\infty(B)$, if $f_\alpha = D^\alpha v$ for all $\alpha \in Z^n$, then $K(\{f_\alpha\}_{\alpha \in Z^n}) = v$.

Proof. By assumption, $v \in \mathcal{C}^1_\infty(B)$ and so $D^\alpha v$ exist for all $\alpha \in Z^n$. The statement then follows from application of the identity $v = g_{i,0}v + g_{i,1}\frac{\partial}{\partial x_i}v$ recursively for $i = 1, \ldots, n$. That is, we make the sequential substitutions $v \mapsto g_{1,0}v + g_{1,1}\frac{\partial}{\partial x_1}v$, $v \mapsto g_{2,0}v + g_{2,1}\frac{\partial}{\partial x_1}v$, ... , $v \mapsto g_{n,0}v + g_{n,1}\frac{\partial}{\partial x_n}v$. Then we have the following expansion.

$$
\begin{aligned}
v=&g_{1,0}v + g_{1,1}D^{(1,0,\ldots,0)}v \\
=&g_{1,0}g_{2,0}v + g_{1,0}g_{2,1}D^{(0,1,\ldots,0)}v + g_{2,0}g_{1,1}D^{(1,0,\ldots,0)}v \\
&+ g_{2,1}g_{1,1}D^{(1,1,\ldots,0)}v \\
=&g_{1,0}g_{2,0}g_{3,0}v + g_{1,0}g_{2,0}g_{3,1}D^{(0,0,1,0,\ldots,0)}v \\
&+ g_{1,0}g_{2,1}g_{3,0}D^{(0,1,\ldots,0)}v + g_{1,0}g_{2,1}g_{3,1}D^{(0,1,1,0,\ldots,0)}v \\
&+ g_{2,0}g_{1,1}g_{3,0}D^{(1,0,\ldots,0)}v + g_{2,0}g_{1,1}g_{3,1}D^{(1,0,1,0,\ldots,0)}v \\
&+ g_{2,1}g_{1,1}g_{3,0}D^{(1,1,\ldots,0)}v + g_{2,1}g_{1,1}g_{3,1}D^{(1,1,1,0,\ldots,0)}v \\
=&\cdots \\
=&K(\{D^\alpha v\}_{\alpha \in Z^n}),
\end{aligned}
$$

as desired.

Recall we would like to demonstrate that polynomial Lyapunov functions can be used to approximate continuous Lyapunov functions. To be a Lyapunov function, a polynomial approximation must satisfy certain constraints. In particular, if v is a Lyapunov function and p is a polynomial approximation to v, then p is also a Lyapunov function if it satisfies an error bound of the form

$$\left\| \frac{v(x) - p(x)}{x^T x} \right\|_\infty \leq \epsilon.$$

For p to prove exponential stability, the derivatives of p and v must satisfy a similar bound. The justification for this form of the error bound is that the error should be everywhere bounded on a compact set, but in addition must decay to zero near the origin.

The idea behind our proof of the existence of such a polynomial, p, is to combine a Taylor series approximation with the Weierstrass approximation theorem. Specifically, a second order Taylor series expansion about a point, x_0, has the property that the error, or residue, R, satisfies

$$\frac{R(x, x_0)}{x^T x} \to 0$$

as $x \to x_0$. However, the error in the Taylor series is not guaranteed to converge uniformly over an arbitrary compact set as the order of the expansion increases. The Weierstrass approximation theorem, on the other hand, gives approximations which converge uniformly on a compact set, but in general no Weierstrass approximation will have the residual convergence property mentioned above for any point, x_0. Our approach, then, is to use a second order Taylor series expansion to guarantee accuracy near the origin. We then use a Weierstrass polynomial approximation to the error between the Taylor series and the function away from the origin to cancel out this error and guarantee a uniform bound.

We begin by combining the second order Taylor series expansion and the Weierstrass approximation.

Lemma 4. Suppose $v \in \mathcal{C}^2_1(B)$. Then for any $\epsilon > 0$, there exists a polynomial p such that

$$\left\| \frac{p(x) - v(x)}{x^T x} \right\|_\infty \leq \epsilon.$$

Proof. Let the polynomial m be defined using the second order Taylor series expansion for v about $x = 0$ as

$$m(x) = v(0) + \sum_{i=1}^n x_i \frac{\partial v}{\partial x_i}(0) + \frac{1}{2}\sum_{i,j=1}^n x_i x_j \frac{\partial^2 v}{\partial x_i \partial x_j}(0).$$

Then m approximates v near the origin and specifically

$$(v - m)(0) = \frac{\partial(v - m)}{\partial x_i}(0) = \frac{\partial^2(v - m)}{\partial x_i \partial x_j}(0) = 0$$

for $i, j = 1, \ldots, n$.

Now define

$$h(x) = \begin{cases} 0 & x = 0 \\ \dfrac{v(x) - m(x)}{x^T x} & \text{otherwise.} \end{cases}$$

Then from Taylor's theorem(See, e.g. Marsden and Tromba (1988)), we have that

$$
\begin{aligned}
v(x) =&v(0) + \sum_{i=1}^n x_i \frac{\partial v}{\partial x_i}(0) \\
&+ \frac{1}{2}\sum_{i,j=1}^n x_i x_j \frac{\partial^2 v}{\partial x_i \partial x_j}(0) + R_2(x),
\end{aligned}
$$

where $\frac{R_2(x)}{x^T x} \to 0$ as $x \to 0$. Therefore

$$h(x) = \frac{v(x) - m(x)}{x^T x} = \frac{R_2(x)}{x^T x} \to 0$$

as $x \to 0$ and so $h(x)$ is continuous at 0. Since $v(x) - m(x)$ and $x^T x$ are continuous and $x^T x \neq 0$ on every domain not containing $x = 0$ and every point $x \neq 0$ has a neighborhood not containing $x = 0$, we conclude that $h(x)$ is continuous at every point $x \in \mathbb{R}^n$.

We can now use the Weierstrass approximation theorem, which states that there exists some polynomial q such that

$$\|q - h\|_\infty \leq \epsilon.$$

The Taylor and Weierstrass approximations are now combined as $p(x) = m(x) + q(x)x^T x$. Then p is polynomial and

$$
\begin{aligned}
&\left\| \frac{p(x) - v(x)}{x^T x} \right\|_\infty \\
&= \left\| \frac{m(x) + q(x)x^T x - v(x)}{x^T x} \right\|_\infty \\
&= \left\| \frac{m(x) - v(x)}{x^T x} + h(x) + (q(x) - h(x)) \right\|_\infty \\
&= \|q(x) - h(x)\|_\infty \leq \epsilon
\end{aligned}
$$

The following lemma states that the linear map K is Lipschitz continuous in an appropriate sense. This means that a small error in the partial derivatives, f_α, results in a small error of the construction $K(\{f_\alpha\}_{\alpha \in Z^n})$ and all of its partial derivatives.

Lemma 5. Let $p = \{p_\alpha\}_{\alpha \in Z^n} \in X$. Then

$$\max_{\beta \in Z^n} \left\| \frac{D^\beta K p(x)}{x^T x} \right\|_\infty \leq 2^n \max_{\alpha \in Z} \left\| \frac{p_\alpha(x)}{x^T x} \right\|_\infty .$$

Proof. From the definition of $g_{j,k}$, we have that

$$\frac{\partial}{\partial x_i} g_{j,k} f = \begin{cases} g_{j,k} \frac{\partial}{\partial x_i} f & i \neq j \\ f & i = j, k = 1 \\ 0 & i = j, k = 0 \end{cases},$$

which implies

$$D^\beta G_\alpha f = \begin{cases} 0 & \alpha_i < \beta_i \text{ for some } i \\ \left(\prod_{\substack{i=1 \\ \beta_i \neq 1}}^{n} g_{i,\alpha_i} \right) f & \text{otherwise.} \end{cases}$$

Now consider the term

$$\frac{1}{x^T x}(g_{i,j} f)(x).$$

We would like to obtain bounds on this function. For $j = 1$, and for any $x \in B$,

$$\begin{aligned} &\left| \frac{1}{x^T x}(g_{i,1} f)(x) \right| \\ &= \left| \int_0^{x_i} \frac{f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)}{\sum_{k=1}^n x_k^2} \, dt \right| \\ &\leq \sup_{\nu \in [-|x_i|, |x_i|]} \frac{|f(x_1, \ldots, x_{i-1}, \nu, x_{i+1}, \ldots, x_n)|}{\sum_{k=1}^n x_k^2} \\ &\leq \sup_{\nu \in [-|x_i|, |x_i|]} \frac{|f(\ldots, x_{i-1}, \nu, x_{i+1}, \ldots)|}{\nu^2 + \sum_{k \neq i}^n x_k^2} \leq \left\| \frac{f(s)}{s^T s} \right\|_\infty . \end{aligned}$$

Here the first inequality is due to the mean value theorem and that $|x_i| \leq 1$ and the second inequality follows since $x_i^2 \geq \nu^2$ for $\nu \in [-|x_i|, |x_i|]$. Therefore, we have

$$\left\| \frac{1}{x^T x}(g_{i,1} f)(x) \right\|_\infty \leq \left\| \frac{1}{x^T x} f(x) \right\|_\infty .$$

Similarly, if $j = 0$, then for any $x \in B$,

$$\begin{aligned} &\left| \frac{1}{x^T x}(g_{i,0} f)(x) \right| \\ &= \left| \frac{f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}{x^T x} \right| \\ &\leq \left| \frac{f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)}{\sum_{k \neq i}^n x_k^2} \right| \leq \left\| \frac{f(s)}{s^T s} \right\|_\infty , \end{aligned}$$

where the first inequality follows since $x_i^2 \geq 0$. Therefore, we have that for $j \in \{0, 1\}$ and $i = 1, \ldots n$,

$$\left\| \frac{1}{x^T x}(g_{i,j} f)(x) \right\|_\infty \leq \left\| \frac{1}{x^T x} f(x) \right\|_\infty .$$

Since the terms G_α are compositions of the $g_{i,j}$, we can apply the above bounds inductively. Specifically, we see that for any $\beta \in Z^n$,

$$\begin{aligned} &\left\| \frac{1}{x^T x} \left(D^\beta G_\alpha f \right)(x) \right\|_\infty \\ &= \left\| \frac{1}{x^T x} \left(\left(\prod_{\substack{i=1 \\ \beta_i \neq 1}}^{n} g_{i,\alpha_i} \right) f \right)(x) \right\|_\infty \\ &\leq \left\| \frac{1}{x^T x} \left(\left(\prod_{\substack{i=2 \\ \beta_i \neq 1}}^{n} g_{i,\alpha_i} \right) f \right)(x) \right\|_\infty \leq \cdots \leq \left\| \frac{f(x)}{x^T x} \right\|_\infty . \end{aligned}$$

Now that we have bounds on the G_α, we can use the triangle inequality to deduce that for any $\beta \in Z^n$,

$$\begin{aligned} \left\| \frac{1}{x^T x} D^\beta K p(x) \right\|_\infty &\leq \sum_{\alpha \in Z^n} \left\| \frac{1}{x^T x} \left(D^\beta G_\alpha p_\alpha \right)(x) \right\|_\infty \\ &\leq \sum_{\alpha \in Z^n} \left\| \frac{p_\alpha(x)}{x^T x} \right\|_\infty \leq 2^n \max_{\alpha \in Z^n} \left\| \frac{p_\alpha(x)}{x^T x} \right\|_\infty . \end{aligned}$$

The following theorem gives the main approximation result of the paper. It combines Lemmas 4 and 5 to show that polynomials are dense in the space $\mathcal{C}_1^{n+2}$ with respect to the weighted $W^{1,\infty}$ norm with weight $1/x^T x$, among others.

Theorem 6. Suppose v is a function with partial derivatives

$$D^\alpha v \in \mathcal{C}_1^2(B)$$

for all $\alpha \in Z^n$. Then for any $\epsilon > 0$, there exists a polynomial p, such that

$$\max_{\alpha \in Z^n} \left\| \frac{D^\alpha p(x) - D^\alpha v(x)}{x^T x} \right\|_\infty \leq \epsilon.$$

Proof. By assumption, $D^\alpha v \in \mathcal{C}_1^2(B)$ for all $\alpha \in Z^n$. By Lemma 4, there exist polynomial functions r_α such that

$$\max_{\alpha \in Z^n} \left\| \frac{r_\alpha(x) - D^\alpha v(x)}{x^T x} \right\|_\infty \leq \frac{\epsilon}{2^n}.$$

Let $r = \{r_\alpha\}_{\alpha \in Z^n}$ and $p = Kr$. Then p is polynomial since the r_α are polynomial. Let $h = \{D^\alpha v\}_{\alpha \in Z^n}$. Then by Lemma 3, $v = Kh$. Therefore by Lemma 5, we have

$$\begin{aligned} &\max_{\alpha \in Z^n} \left\| \frac{D^\alpha p(x) - D^\alpha v(x)}{x^T x} \right\|_\infty \\ &= \max_{\alpha \in Z^n} \left\| \frac{D^\alpha K r(x) - D^\alpha K h(x)}{x^T x} \right\|_\infty \\ &\leq 2^n \max_{\alpha \in Z} \left\| \frac{r_\alpha(x) - D^\alpha v(x)}{x^T x} \right\|_\infty \leq \epsilon, \end{aligned}$$

as desired.

The following result is of interest. However, the proof is omitted for brevity.

Corollary 7. Suppose $v \in \mathcal{C}_\infty^1(B)$. Then for any $\epsilon > 0$, there exists a polynomial p, such that

$$\max_{\alpha \in Z^n} \|D^\alpha p - D^\alpha v\|_\infty \leq \epsilon.$$

4. LYAPUNOV STABILITY

We begin the section by showing that the existence of a sufficiently smooth Lyapunov function implies the existence of a polynomial Lyapunov function.

Theorem 8. Let $\Omega \subset \mathbb{R}^n$ be bounded with radius r in norm $\|\cdot\|_\infty$ and $f(x)$ be uniformly bounded on $B_r := \{x \in \mathbb{R}^n : \|x\|_\infty \leq r\}$. Suppose there exists a $v : B_r \to \mathbb{R}$ with $D^\alpha v \in \mathcal{C}_1^2(B_r)$ for all $\alpha \in Z^n$ and such that

$$\beta_0 \|x\|^2 \leq v(x) \leq \gamma_0 \|x\|^2$$
$$\nabla v(x)^T f(x) \leq -\delta_0 \|x\|^2,$$

for some $\beta_0 > 0$, $\gamma_0 > 0$ and $\delta_0 > 0$ and all $x \in \Omega$. Then for any $\beta < \beta_0$, $\gamma > \gamma_0$ and $\delta < \delta_0$ there exists a polynomial p such that

$$\beta\|x\|^2 \le p(x) \le \gamma\|x\|^2$$
$$\nabla p(x)^T f(x) \le -\delta\|x\|^2$$

for all $x \in \Omega$.

Proof. Let $\hat{v}(x) = v(rx)$ and

$$b = \|f\|_\infty = \sup_{\|x\|_\infty \le r} \|f(x)\|_\infty.$$

Choose $0 < d < \min\{\beta_0 - \beta, \gamma - \gamma_0, \frac{\delta_0 - \delta}{nb}\}$. By Theorem 6, there exists a polynomial, $\hat{p}$, such that for $\|x\|_\infty \le 1$,

$$\left|\frac{\hat{p}(x) - \hat{v}(x)}{x^T x}\right| \le \frac{d}{r^2}$$

and

$$\left|\frac{\frac{\partial \hat{p}}{\partial x^i}(x) - \frac{\partial \hat{v}}{\partial x^i}(x)}{x^T x}\right| \le \frac{d}{r^2}$$

for $i = 1, \ldots, n$. Now let $p(x) = \hat{p}(x/r)$. Then for $x \in \Omega$, $\|x\|_\infty \le r$ and so $\|x/r\|_\infty \le 1$. Therefore we have the following for all $x \in \Omega$,

$$\begin{aligned} p(x) &= v(x) + \hat{p}(x/r) - \hat{v}(x/r) \\ &= v(x) + \frac{\hat{p}(x/r) - \hat{v}(x/r)}{(x/r)^T (x/r)} r^2 x^T x \\ &\ge (\beta_0 - d) x^T x \\ &\ge \beta x^T x. \end{aligned}$$

Likewise,

$$\begin{aligned} p(x) &= v(x) + \frac{\hat{p}(x/r) - \hat{v}(x/r)}{(x/r)^T (x/r)} r^2 x^T x \\ &\le (\gamma_0 + d) x^T x \\ &\le \gamma x^T x. \end{aligned}$$

Finally,

$$\begin{aligned} &\nabla p(x)^T f(x) \\ &= \frac{\nabla(\hat{p}(x/r) - \hat{v}(x/r))^T f}{x^T x} x^T x + \nabla v(x)^T f(x) \\ &\le \sum_{i=1}^n \left(r^2 \frac{\frac{\partial \hat{p}}{\partial x_i}(x/r) - \frac{\partial \hat{v}}{\partial x_i}(x/r)}{(x/r)^T (x/r)} f_i(x) \right) x^T x - \delta_0 x^T x \\ &\le n\, d\, b\, x^T x - \delta_0 x^T x \\ &\le -\delta x^T x. \end{aligned}$$

Thus the proposition holds for $x \in \Omega$.

A consequence of Theorem 8 is that when estimating exponential rates of decay, using polynomial Lyapunov functions does not result in a reduction of accuracy. i.e. if there exists a continuous Lyapunov function proving an exponential rate of decay with bound α_0, then for any $0 < \alpha < \alpha_0$, there exists a polynomial Lyapunov function which proves an exponential rate of decay with bound α.

Consider the system

$$\dot{x}(t) = f(x(t)) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $f(0) = 0$ and $x(0) = x_0$. We assume that there exists an $r \ge 0$ such that for any $\|x_0\|_\infty \le r$, Equation (1) has a unique solution for all $t \ge 0$. We define the solution map $A : \mathbb{R}^n \to \mathcal{C}([0, \infty))$ by

$$(Ay)(t) = x(t)$$

for $t \ge 0$, where x is the unique solution of Equation (1) with initial condition y.

Theorem 9. Let $\Omega \subset \mathbb{R}^n$ and suppose there exists a $v \in \mathcal{C}_1^1(\Omega)$ such that

$$\beta\|x\|^2 \le v(x) \le \gamma\|x\|^2$$
$$\nabla v(x)^T f(x) \le -\delta\|x\|^2,$$

for some $\beta > 0$, $\gamma > 0$ and $\delta > 0$ and all $x \in \Omega$. Then there exists an $\alpha > 0$ such that

$$\|(Ax_0)(t)\|_2 \le \|x_0\|_2 e^{-\alpha t}$$

for all $t \ge 0$ and all $x_0 \in Y_\gamma$ for any γ such that $Y_\gamma := \{x : v(x) \le \gamma\} \subset \Omega$

In polynomial optimization, it is common to use Positivstellensatz results to find locally positive polynomial Lyapunov functions in a manner similar to the S-procedure. When the polynomial v can be assumed to be positive, i.e. $v(x) > 0$ for all x, these conditions are necessary and sufficient. See Stengle (1973), Schmüdgen (1991), and Putinar (1993) for strong theoretical contributions. Unfortunately, the polynomial Lyapunov functions are not positive since $v(0) = 0$, and so these conditions are no longer necessary and sufficient. However, Positivstellensatz results still allow us to search over polynomial Lyapunov functions in a manner which has proven very effective in practice.

Definition 10. A polynomial, p, is ***sum-of-squares***, if there exists a $K > 0$ and polynomials g_i for $i = 1, \ldots, K$ such that

$$p(x) = \sum_{i=1}^K g_i(x)^2.$$

Proposition 11. Consider the system defined by Equation (1) where f is polynomial. Suppose there exists a polynomial $v : \mathbb{R}^n \to \mathbb{R}$, a constant $\epsilon > 0$, and sum-of-squares polynomials $s_1, s_2, t_1, t_2 : \mathbb{R}^n \to \mathbb{R}$ such that

$$v(x) - s_1(x)(r - x^T x) - s_2(s) - \epsilon\, x^T x = 0$$

and

$$-\nabla v(x)^T f(x) - t_1(x)(r - x^T x) - t_2(x) - \epsilon\, x^T x = 0$$

Then there exist constants $\mu, \delta, r > 0$ such that

$$\|(Ax_0)(t)\|_2 \le \mu \|x_0\|_2 e^{-\delta t}$$

for all $t \ge 0$ and $x_0 \in Y(v, r)$ where $Y(v, r)$ is the largest sublevel set of v contained in the ball $\|x\|^2 \le r$.

See Papachristodoulou and Prajna (2002), Wang (2007), or Tan (2006) for a proof and more details on using semidefinite programming to construct solutions to this polynomial optimization problem. For an extension of the sum-of-squares approach to stability of delay-differential equations, we refer to the recent work of Peet (2006).

5. CONCLUSION

The main conclusion of this paper is that polynomial Lyapunov functions are as good as continuous Lyapunov functions for the analysis of local stability of ordinary differential equations. An important application of polynomial programming is the search for a polynomial Lyapunov function which proves local exponential stability. Our results, therefore, tend to support continued research into improving polynomial optimization algorithms.

In addition, as a byproduct of our proof, we were able to give a method for constructing polynomial approximations to differentiable functions. The interesting feature of this construction is the guaranteed convergence of the derivatives of the approximation. Another consequence of the results of this paper is that the polynomials are dense in $\mathcal{C}^1_\infty(B)$ with respect to the Sobolev norm $W^{1,\infty}(B)$.

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