Spacecraft and Aircraft Dynamics

Matthew M. Peet
Illinois Institute of Technology

Lecture 2: Invariants

Conclusion

In this Lecture, you learned:

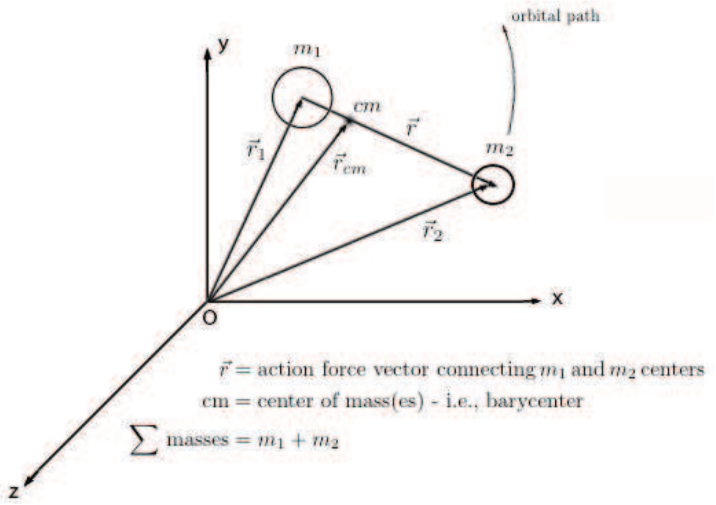
N-body Problem

- Introduction
- Invariants
  - Linear Momentum
  - Angular Momentum
  - Energy

Two-Body Problem

- How to calculate velocity given position
- How to calculate escape velocity

Universal Gravitation



$$\vec{F}_1 = G\frac{m_1 m_2}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

Relative motion

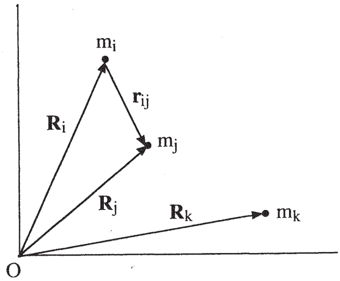
Recall the force on mass 1 due to mass 2 is

$$m_1\ddot{\vec{r}}_1 = \vec{F}_1 = G\frac{m_1 m_2}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

where we denote $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$. Clearly $\vec{r}_{12} = \vec{r}_{21}$. The motion of mass 2 due to mass 1 is

$$m_2\ddot{\vec{r}}_2 = \vec{F}_2 = G\frac{m_2 m_1}{\|\vec{r}_{21}\|^3}\vec{r}_{21}$$

The problem is nonlinear coupled ODE with 6 degrees of freedom.



Solution: Consider relative motion (only $\vec{r}_{12}$)

$$\ddot{\vec{r}}_{12} = -\frac{G(m_1 + m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

The N-body problem

In some situations, there are more than two bodies.

- The solar system
- The Earth-Moon system

In this case, the force on mass i due to all other masses is

$$m_i \ddot{\vec{R}}_i = \vec{F}_i = G \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij}$$

Definition 1.

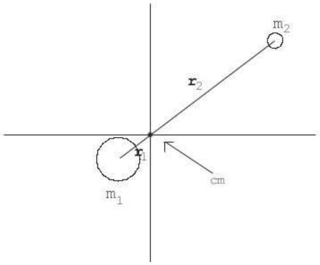
The center of mass of a collection of point masses m_i is

$$\vec{R}_{CM} = \frac{1}{\sum_i m_i} \sum_{i=1}^{N} m_i \vec{R}_i$$

The N-body problem

The key thing to note is that since $\vec{r}_{ij} = -\vec{r}_{ji}$,

$$\ddot{\vec{R}}_{CM} = \frac{1}{\sum_i m_i} \sum_{i=1}^{N} m_i \ddot{\vec{R}}_i$$
$$= \frac{G}{\sum_i m_i} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij} = 0$$



Therefore, The center of mass is *not accelerating*.

$\vec{R}_{CM} = 0$ makes an excellent choice for the origin of a **coordinate system**.

First Invariant Quantity: Linear Momentum

$$\sum_i m_i R_i = C_1 t + C_2$$

Thus the motion of the center of mass doesn't change with time.
We will return to this concept when we return to the 2-body problem.

Invariants in the N-body problem

We now define the two key invariant quantities which will define the motion.

- Energy
- Angular Momentum

These hold for both the 2 and N body problems. Begin with the Angular momentum.

Definition 2.

The angular momentum of a collection of particles is

$$\vec{H}(t) = \sum_{i=1}^{N} m_i \vec{R}_i(t) \times \dot{\vec{R}}_i(t)$$

We will show next that

$$\frac{d}{dt}\vec{H}(t) = \sum_{i=1}^{N} m_i \left(\dot{\vec{R}}_i(t) \times \dot{\vec{R}}_i(t) + \vec{R}_i(t) \times \ddot{\vec{R}}_i(t) \right)$$
$$= \sum_{i=1}^{N} m_i \vec{R}_i(t) \times \ddot{\vec{R}}_i(t) = 0$$

Conservation of Angular Momentum under Gravity

Begin with the relation

$$m_i \ddot{\vec{R}}_i = \vec{F}_i = G \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{r}_{ij}.$$

Then

$$\sum_{i=1}^{N} m_i \left(\vec{R}_i \times \ddot{\vec{R}}_i \right) = G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{R}_i \times \vec{r}_{ij}.$$

However, we can use the identities

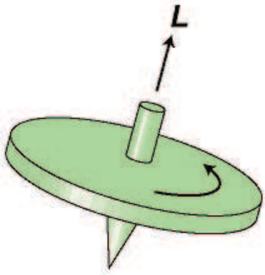
$$\vec{R}_i \times \vec{r}_{ij} = \vec{R}_i \times \left(\vec{R}_j - \vec{R}_i \right) = \vec{R}_i \times \vec{R}_j$$

$$\vec{R}_j \times \vec{r}_{ji} = \vec{R}_j \times \left(\vec{R}_i - \vec{R}_j \right) = -\vec{R}_i \times \vec{R}_j$$

Thus

$$\sum_{i=1}^{N} \vec{R}_i \times \ddot{\vec{R}}_i = G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \vec{R}_i \times \vec{R}_j = 0.$$

Conservation of Angular Momentum under Gravity



Thus we conclude that

$$\dot{\vec{H}} = 0$$

from whence we have that

$$\vec{H} = \sum_{i=1}^{N} m_i \vec{R}_i \times \dot{\vec{R}}_i = C_3$$

The concept of an invariant plane defined as normal to the vector $\vec{H}$.

Illustration of Conservation of Angular Momentum under Gravity

Milky Way + Andromeda

Conservation of Energy

Definition 3.

The Kinetic Energy of a particle is

$$T_i = \frac{1}{2} m_i \dot{\vec{R}}_i^T \dot{\vec{R}}_i$$

Thus the total kinetic energy for a system of particles is

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\vec{R}}_i^T \dot{\vec{R}}_i$$

Definition 4.

The Gravitational Potential Energy of a collection of particles is

$$V = -\frac{G}{2} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|}$$

Conservation of Energy

We show that the fourth invariant is

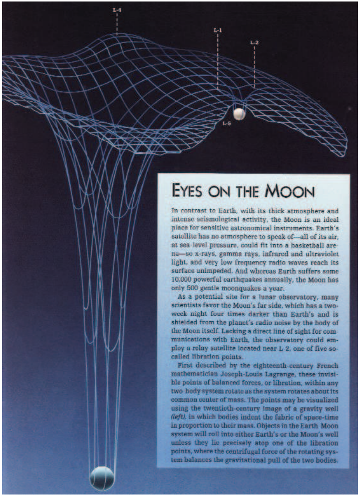
$$E = T + V = C_4$$

by showing $\dot{T} + \dot{V} = 0$.

$$\begin{aligned}\dot{T} &= \sum_{i=1}^{N} m_i \dot{\vec{R}}_i^T \ddot{\vec{R}}_i \\ &= G \sum_{i=1}^{N} \sum_{\substack{j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \dot{\vec{R}}_i^T \vec{r}_{ij}\end{aligned}$$

Which is complicated.

However, now look at $\dot{V}$



Conservation of Energy

Recall $\dot{T} = G \sum_{i=1}^{N} \sum_{\substack{j=1\\ i\neq j}}^{N} \frac{m_i m_j}{\|\vec{r}_{ij}\|^3} \dot{\vec{R}}_i^T \vec{r}_{ij}$ Now,

$$\dot{V} = -\frac{d}{dt}\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|} = -\frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j \frac{d}{dt}(\vec{r}_{ij}^T\vec{r}_{ij})^{-.5}$$

$$= \frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} m_i m_j (\vec{r}_{ij}^T\vec{r}_{ij})^{-1.5}\dot{\vec{r}}_{ij}^T\vec{r}_{ij}$$

$$= \frac{G}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|^3}(\dot{\vec{R}}_j - \dot{\vec{R}}_i)^T\vec{r}_{ij} = -G\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{m_i m_j}{\|\vec{r}_{ij}\|^3}\dot{\vec{R}}_i^T\vec{r}_{ij}$$

Which cancels $\dot{T}$. Therefore

$$\dot{E} = \dot{T} + \dot{V} = 0$$

and hence gravity is a conservative field with $T(t) + V(t) = C_4$.

Potential Energy

We can see velocity increases as we descend into the well

Potential Energy

We can see velocity increases as we descend into the well

Back to the Two-Body problem

Going back to the two-body problem.... Recall our equation of motion

$$\ddot{\vec{r}}_{12} = -\frac{G(m_1 + m_2)}{\|\vec{r}_{12}\|^3}\vec{r}_{12}$$

but this considers only relative motion. How to recover absolute position? Use a coordinate system centered at the center of mass so that

$$\vec{R}_{CM} = \frac{\vec{R}_1 m_1 + \vec{R}_2 m_2}{m_1 + m_2} = 0$$

Then we can recover from $\vec{r}_{12}$

$$\vec{R}_1 = \frac{m_2}{m_1 + m_2}\vec{r}_{12}$$

and

$$\vec{R}_2 = \frac{m_1}{m_1 + m_2}\vec{r}_{12}$$

If m_1 is a satellite and m_2 is a planet, then $\frac{m_1}{m_1+m_2} \cong 0$ and $\frac{m_1}{m_1+m_2} \cong 1$ and so

$$\vec{R}_2 \cong 0 \qquad \text{and} \qquad \vec{R}_1 \cong \vec{r}_{12}.$$

The Orbital Parameter, μ

While we are considering **orbits**, we can make some simplifications. Our first simplification is to write

$$\mu = G(m_1 + m_2).$$

If $m_2 >> m_1$, then

$$\mu \cong Gm_2$$

- Each central body has its own μ
- The size of the orbit varies with μ
- Needed to convert orbital elements to $\vec{r}$ and $\vec{v}$.
- Values are tabulated

The Sun

Mass $= 1.989 \cdot 10^{30}$ kg

Radius $= 6.9599 \cdot 10^{5}$ km

$\mu_{sun} = Gm_{sun} = 1.327 \cdot 10^{11}\ km^3/s^2$

The Earth

Mass $= 5.974 \cdot 10^{24}$ kg

Radius $= 6.37812 \cdot 10^{3}$ km

$\mu_{earth} = Gm_{earth} = 3.986 \cdot 10^{5}\ km^3/s^2$

Mean distance from sun = 1 au = $1.495978 \cdot 10^{8}$ km

The Moon

Mass $= 7.3483 \cdot 10^{22}$ kg

Radius $= 1.738 \cdot 10^{3}$ km

$\mu_{moon} = Gm_{moon} = 4.903 \cdot 10^{3}\ km^3/s^2$

Mean distance from earth $= 3.844 \cdot 10^{5}$ km

Orbit eccentricity = 0.0549

Orbit inclination (to ecliptic) = $5° 09'$

A note on the Gravitational Constant, G

The calculation of G is non-trivial

- Given G, it is easy to calculate the mass of any planet.
- The search for G was another major scientific quest.

In principle, it is easy to calculate:

- take two objects of known mass (m_1, m_2) and measure the attraction, F. Then

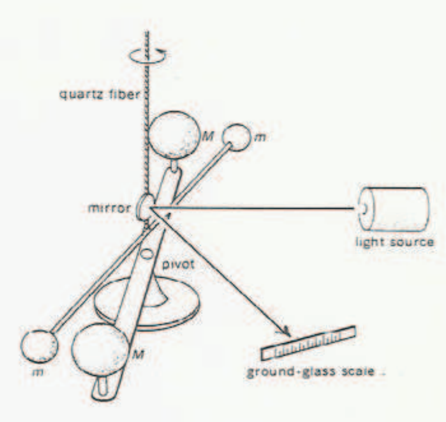
$$G = \frac{Fr^2}{m_1 m_2}$$

Unfortunately, the force is infinitesimal for all but planet-size objects. So how to calculate G?

The Cavendish Balance

The first accurate measurement of G was made by Cavendish in 1798.

1. Suspend two small spheres, separated by length, l, by a quartz filament.
2. Move two large spherical masses within known (small) distance of test masses.
3. Gravity will produce a moment on the quartz fiber, causing a deflection.
4. Deflection is measured by movement of a mirror on glass filament.
5. Rotation of mirror causes movement of reflected light.



Measurement revealed

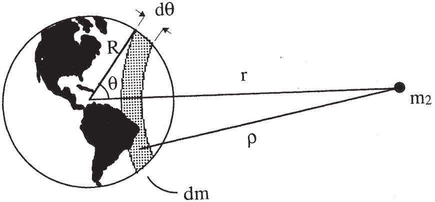
$$G = 6.67428 \pm .00067 * 10^{-11} m^3 kg^{-1} s^{-2}$$

A Problem with the Model?

Non-point masses

- Our equation of motion assumes that the masses are concentrated at a point.
- Most masses are actually quite large (planet-sized)

Question: Is this a problem?
Answer: Not if there is symmetry about the line $\vec{r}_{12}$.



The *sphere* is symmetric about any line passing through the center.

- Most planets are spheres.
- Exception is the orbit perturbation effect due earth oblateness.
- See book for proof that the effect reduces to a point mass.

Energy in Orbits

Now lets return to the question of potential fields and energy For an orbit, we will hence only consider motion of the satellite, which we denote

$$\vec{r}(t) = \vec{R}_1(t) \cong \vec{r}_{12}(t)$$

and $m = m_1$. $\dot{r} = v$. We will ignore motion of the larger body.

$$T = \frac{1}{2}\|\vec{v}\|^2$$

and

$$V = -\frac{\mu}{\|\vec{r}\|}$$

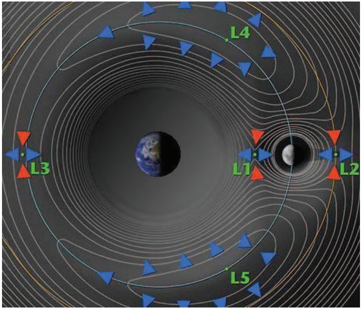
and T+V is conserved. This can already be used to solve problems

Question: Suppose a satellite of earth is initially tracked at radius of $r_1 = 20,000km$ at a velocity of $1000m/s$. The satellite is later spotted at a radius of $10,000km$. Determine the velocity of the satellite.

Solution: Find the Energy at the initial time and use it to find the kinetic energy at the final time.

$$E = T_1 + V_1 = \frac{1}{2}\|\vec{v_1}\|^2 - \frac{\mu}{\|\vec{r_1}\|}$$

$$= .5 - \frac{398601}{20,000} = -19.43$$



$$V_2 = -\frac{\mu}{\|\vec{r_2}\|} = -\frac{398601}{10,000} = -39.86.$$

So $T_1 + V_1 = T_2 + V_2$ implies

$$T_2 = E - V_2 = -19.43 + 39.86 = 20.43$$

$$\|v_2\| = \sqrt{2T_2} = \sqrt{40.86} = 6.392km/s$$

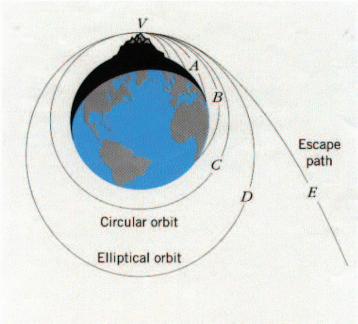
Example: Escape Velocity

Escape velocity is the kinetic energy needed to leave the sphere of influence of a planet. To achieve escape, net energy, E must be positive, so that as $r \to \infty$,

we still have forward motion.

At $r \to \infty$, $V_\infty = \lim_{r\to\infty} \frac{\mu}{\|\vec{r}\|} = 0$, so

$$E = T_\infty + V_\infty = T_\infty$$



Question: Find the escape velocity at $r = 20,000km$.

Solution: As we know, at $r = 20,000$, $V = -19.93$. So in order for $E > 0$, we need

$$E = T + V = T - 19.93 > 0$$

So we need $T > 19.93$. This yields a velocity of

$$\|v\| = \sqrt{2T} > 6.313$$

Conclusion

In this Lecture, you learned:

N-body Problem

- Introduction
- Invariants
  - Linear Momentum
  - Angular Momentum
  - Energy

Two-Body Problem

- How to calculate velocity given position
- How to calculate escape velocity

Next Lecture: The two-body problem continued

Derivation of Kepler's First Law

- eccentricity vector
  - How to calculate
  - circular orbits
  - elliptic orbits
  - parabolic orbits
  - hyperbolic orbits
- Solution to the two-body problem