A Convex Reformulation of the Controller Synthesis Problem for MIMO Single-Delay Systems

Matthew M. Peet
Arizona State University
Tempe, AZ USA

American Control Conference 2017
Seattle, WA

May 26, 2017

Control of Linear Systems with Delays

Consider a MIMO Linear Discrete-Delay system.

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t - \tau_i) + Bu(t) \qquad x(t) \in \mathbb{R}^n \qquad u(t) \in \mathbb{R}^m$$

Stability Analysis of linear discrete-delay systems is a CLOSED PROBLEM.

- Lets move on to optimal control.

We would like to use LMI and SOS methods to design controllers for this system.

- LMI methods optimize positive matrices
- SOS methods optimize positive polynomials

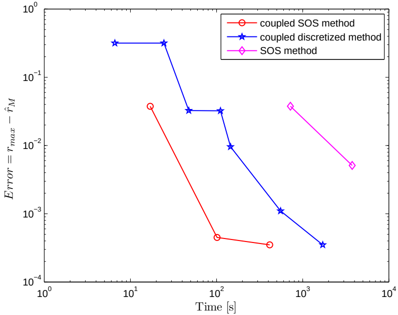


Figure: Comparison of asymptotic algorithms for maximum stable delay

Full-State Feedback Control of **ODE** Systems

Our Template is the LMI Framework

The goal is to find $K \in \mathbb{R}^{m\times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A+BK) + (A+BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A+BK)Q + Q(A+BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m\times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

In this Paper:

A Linear Operator Inequality (LOI) Framework for Synthesis

MAIN IDEA: Replace the Word MATRIX with **OPERATOR**.
An Operator Differential Equation:

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad u = \mathcal{K}x,$$

- $\mathcal{A} : \underbrace{W^2}_{X} \to \underbrace{L_2}_{Z}$ and $\mathcal{B} : \underbrace{\mathbb{R}^m}_{U} \to \underbrace{L_2}_{X}$, $\mathcal{K} : \underbrace{W^2}_{X} \to \underbrace{\mathbb{R}^m}_{U}$ are **OPERATORS**.
- We CAN Optimize Operators - to be discussed.

Primal Stability (No Feedback): $\mathcal{A}$ is exp. stable iff [Curtain, Zwart] there exists a $\mathcal{P} > 0$

$$\langle x, \mathcal{P}\mathcal{A}x \rangle_Z + \langle \mathcal{P}\mathcal{A}x, x \rangle_Z < 0 \qquad \forall x \in X$$

The First Main Result is Duality: $\mathcal{A}$ is exp. stable if there exists a $\mathcal{Q} : X \to X$ such that $\mathcal{Q} > 0$ and

$$\langle x, \mathcal{A}\mathcal{Q}x \rangle + \langle \mathcal{A}\mathcal{Q}x, x \rangle < 0$$

Controller Synthesis is then trivial (Mostly). Other Main Results:

- Solving LOIs with SDP
- Reconstructing $\mathcal{K}$ (Inverting the Controller).

The Duality Theorem

Formal Statement. Applies to any Strongly Continuous Semigroup

Theorem 1.

Suppose that $\mathcal{A}$ generates a strongly continuous semigroup on Hilbert space Z with domain X. Further suppose there exists an $\epsilon > 0$ and a bounded, coercive linear operator $\mathcal{P} : X \to X$ with $\mathcal{P}(X) = X$ and which is self-adjoint with respect to the Z inner product and satisfies

$$\langle \mathcal{A}\mathcal{P}z, z\rangle_Z + \langle z, \mathcal{A}\mathcal{P}z\rangle_Z \le -\epsilon\|z\|_Z^2$$

for all $z \in X$. Then $\dot{x}(t) = \mathcal{A}x(t)$ generates an exponentially stable semigroup.

Key Restriction: $\mathcal{P} : X \to X$. **Conservative?**

- Not when X is a Hilbert Subspace of Z.
- But this is not true for Delay systems.

So now we have An LOI for Controller Synthesis!!!
Find Q, Z such that $\mathcal{Q} : X \to X$,

$$\langle (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x, x\rangle + \langle x, (\mathcal{A}\mathcal{Q} + \mathcal{B}\mathcal{Z})x\rangle_Z < 0 \qquad \mathcal{Q} > 0, \quad \mathcal{Z} \in \mathcal{L}(X, U)$$

Question: Is this a tractable problem????

What is an LOI

And How do I solve one?

First Rule of LOIs: NO DISCRETIZATION

Formal Definition:
An LOI is a TUPLE (Z, X, $\mathbb{P}$, $\mathcal{H}$, $\mathcal{G}$) which defines the feasibility problem: Find $\mathcal{P} \in \mathbb{P}$ such that

$$\mathcal{H}\mathcal{P}\mathcal{G} + (\mathcal{H}\mathcal{P}\mathcal{G})^* > 0, \qquad \mathcal{P} \in \mathbb{P}$$

where the inequality is shorthand for

$$\langle \mathcal{H}\mathcal{P}\mathcal{G}x, x\rangle_Z + \langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z \geq 0 \qquad \text{for all } x \in X \subset Z$$

The key features of an LOI are

1. Inner Product Space Z is an inner-product space (the meaning of ≥ 0).

2. State Space $X \subset Z$ quantifies "for all $x \in X$".

3. Variables The operator $\mathcal{P}$ is constrained to lie in set $\mathbb{P}$.

4. Data $\mathcal{H}$ and $\mathcal{G}$ are operators and may be unbounded.

5. Well-posed Given $\mathcal{H}$ and $\mathcal{G}$, the inner product $\langle x, \mathcal{H}\mathcal{P}\mathcal{G}x\rangle_Z$ must be well-defined for all $\mathcal{P} \in \mathbb{P}$ and $x \in X$.

Applying the LOI Framework to Delay Systems

Represent

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{K} A_i x(t-\tau_i) + Bu(t)$$

as **An Operator Differential Equation:**

$$\dot{x} = \mathcal{A}x + \mathcal{B}u, \qquad u = \mathcal{K}x,$$

In this case

$$\mathcal{A}\begin{bmatrix} x \\ \phi_i \end{bmatrix}(s) := \begin{bmatrix} A_0 x + \sum_{i=1}^{K} A_i \phi_i(-\tau_i) \\ \dot{\phi}_i(s) \end{bmatrix}, \qquad (\mathcal{B}u)(s) := \begin{bmatrix} Bu \\ 0 \end{bmatrix}.$$

Furthermore, we define the inner product and state spaces as

$$Z = Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$$

where

$$\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s) ds.$$

$$X := \left\{ \begin{bmatrix} x \\ \phi_i \end{bmatrix} \in Z_{n,K} \; : \; \substack{\phi_i \in W_2^n[-\tau_i, 0] \text{ and} \\ \phi_i(0) = x \text{ for all } i \in [K]} \right\}.$$

Solving LOIs

Steps To Solving an LOI

1. Parameterize your Operator $\mathbb{P} \subset \mathbb{R}^m$.
2. Reduce your LOI to one which has already been solved.
3. Done

Chapter 1: The Variables. In this talk, our operators look like this:

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \begin{bmatrix} x \\ \phi_i \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\,d\theta. \end{bmatrix} \tag{1}$$

Conclusions:

- Polynomials P, Q_i, S_i, R_{ij} parameterize the operator.
- Real numbers parameterize the polynomials. Restrict the degree to $\leq d$!

Chapter 2: Reduction to a Solveable LOI

- But which LOIs are Solveable?

Illustration: Primal Stability of Time-Delay Systems

Theorem: Stability of

$$\dot{x}(t) = Ax(t) + \sum_i A_i x(t - \tau_i)$$

is equivalent [Datko] to existence of P, Q_i, S_i, R_{ij} such that

$$\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq 0 \qquad \text{and} \qquad \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} < 0$$

where

$$D_1 := \begin{bmatrix} \Delta_0 & \Delta_1 & \cdots & \Delta_K \\ \Delta_1^T & S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ \Delta_K^T & 0 & 0 & S_K(-\tau_K) \end{bmatrix},$$

$$\Delta_0 = PA_0 + A_0^T P + \sum_{k=1}^{K} Q_k(0) + Q_k(0)^T + S_k(0), \quad \Delta_j = PA_j - Q_j(-\tau_j),$$

$$V_i(s) = \begin{bmatrix} \Pi_{0,i}(s)^T & \ldots & \Pi_{K,i}(s)^T \end{bmatrix}^T, \quad \Pi_{0j}(s) = A_0^T Q_j(s) + \sum_{k=1}^{K} R_{jk}^T(s,0) - \dot{Q}_j(s)$$

$$\Pi_{ij}(s) = A_i^T Q_j(s) - R_{ji}^T(s, -\tau_i), \quad G_{ij}(s,\theta) = -\frac{\partial}{\partial s} R_{ij}(s,\theta) - \frac{\partial}{\partial \theta} R_{ij}(s,\theta).$$

In Lyapunov Form: $V = \langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x \rangle_Z \geq 0$ for all $x \in X$ and $\dot{V}(x) = \langle z, \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} z \rangle_{Z'} \leq 0$ for all $z \in Z' = Z_{n(K+1),n,K}$.

We CAN solve LOIs on $X = L_2[-\tau_K, 0]$ using SDP

We can solve tuples of the following form (Z, X, $\mathbb{P}$, $\mathcal{H}$, $\mathcal{G}$)

1. $Z = L_2$
2. $X = L_2$
3. $\mathcal{P} \in \mathbb{P} := \{\mathcal{P} \,:\, (\mathcal{P}_{M,N} x)(s) := M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta.\}$ where M, N are piecewise Polynomial
4. $H, G \in \mathbb{P}$

Then $\mathcal{H}\mathcal{P}\mathcal{G} \in \mathbb{P}$, and we can test whether $\mathcal{P}_{M,N} > 0$

Theorem 2.

For any functions Y_1, Y_2, let

$$M(s) = Y_1(s)^T Q_{11} Y_1(s)$$

$$N(s,\theta) = Y_1(s) Q_{12} Y_2(s,\theta) + Y_2(\theta,s)^T Q_{12}^T Y_1(\theta) + \int_{-\tau_K}^{0} Y_2(\omega,s)^T Q_{22} Y_2(\omega,\theta)\, d\omega$$

where $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \geq 0$. *Then* $\langle \mathcal{P}_{M,N} x, x \rangle_{L_2} > 0$ *for all* $x \in L_2^n[-\tau_K, 0]$.

We CAN solve LOIs on $X = \mathbb{R}^m \times L_2^n[-\tau_K, 0]$

By reduction to an LOI on $X = L_2^{m+n}[-\tau_K, 0]$

Again with the tuples: $(L_2,\ \mathbb{R}^m \times L_2^n,\ \mathcal{P} \in \mathbb{P},\ \mathcal{P} > 0)$ is feasible iff

$$(L_2, L_2^{m+n}, (\mathcal{P} \in \mathbb{P},\ \mathcal{T} \in \mathbb{T}), \mathcal{P} + \mathcal{T} > 0)$$

is feasible, where

$$\mathbb{T} := \{\mathcal{P}_{F,H}\ \ :\ \ \text{such that for some functions } K, L_{11}, L_{12}, L_{21},$$

$$F(s) = \begin{bmatrix} K(s) + \int\limits_{-\tau_K}^{0} \int\limits_{-\tau_K}^{0} \frac{L_{11}(\omega,t)}{\tau_K} d\omega dt & \int\limits_{-\tau_K}^{0} L_{12}(\omega, s) d\omega \\ \int\limits_{-\tau_K}^{0} L_{21}(s,\omega) d\omega & 0 \end{bmatrix}$$

$$H(s,\theta) = -\begin{bmatrix} L_{11}(s,\theta) & L_{12}(s,\theta) \\ L_{21}(s,\theta) & 0 \end{bmatrix}, \qquad \int_{-\tau_K}^{0} K(s) ds = 0\}$$

$\mathbb{T}$ imposes structure on $\mathcal{T}$

- We call these spacing operators.

For Multiple Delay, the inner product is $Z_{m,n,K}$

We Can reduce this to an LOI on $\mathbb{R}^m \times L_2^n[-\tau_K, 0]$

$$Z = Z_{m,n,K} := \{\mathbb{R}^m \times L_2^n[-\tau_1, 0] \times \cdots \times L_2^n[-\tau_K, 0]\}$$

where

$$\left\langle \begin{bmatrix} y \\ \psi_i \end{bmatrix}, \begin{bmatrix} x \\ \phi_i \end{bmatrix} \right\rangle_{Z_{m,n,K}} = \tau_K y^T x + \sum_{i=1}^{K} \int_{-\tau_i}^{0} \psi_i(s)^T \phi_i(s) ds.$$

This inner product is a bit unusual, but we can go back to L_2

Lemma 3.

Let $M(s) = M_i(s)$ and $N(s,\theta) = N_{ij}(s,\theta)$ for $s \in [-\tau_i, -\tau_{i-1}]$, $\theta \in [-\tau_j, -\tau_{j-1}]$ and

$$\left(\mathcal{P}_{M,N} x\right)(s) := M(s)x(s) + \int_{-\tau_K}^{0} N(s,\theta)x(\theta)d\theta.$$

If $\langle x, \mathcal{P}_{M,N} x\rangle_{L_2^{m+n}} \geq \alpha \|x\|^2_{L_2^{m+n}}$ for some $\alpha > 0$ and all $x \in \mathbb{R}^m \times L_2^n[-\tau_K, 0]$, then $\langle x, \mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} x\rangle_{Z_{m,n,K}} \geq \alpha \|x\|^2_{Z_{m,n,K}}$ for all $x \in Z_{m,n,K}$.

$$M_i(s) = \begin{bmatrix} P & \frac{\tau_K}{a_i} Q_i(\frac{s+\tau_{i-1}}{a_i}) \\ \frac{\tau_K}{a_i} Q_i(\frac{s+\tau_{i-1}}{a_i})^T & \frac{\tau_K}{a_i} S_i(\frac{s+\tau_{i-1}}{a_i}) \end{bmatrix}, \quad N_{ij}(s,\theta) = R_{ij}\left(\frac{s+\tau_{i-1}}{a_i}, \frac{\theta+\tau_{j-1}}{a_j}\right)$$

How to ensure $\mathcal{P}(X) = X$

Recall we have operators of the form

$$\left(\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}\begin{bmatrix}x\\ \phi_i\end{bmatrix}\right)(s) := \tag{2}$$

$$\begin{bmatrix} Px + \sum_{i=1}^K \int_{-\tau_i}^0 Q_i(s)\phi_i(s)ds \\ \tau_K Q_i(s)^T x + \tau_K S_i(s)\phi_i(s) + \sum_{j=1}^K \int_{-\tau_j}^0 R_{ij}(s,\theta)\phi_j(\theta)\, d\theta. \end{bmatrix}$$

with

$$X := \left\{ \begin{bmatrix}x\\ \phi_i\end{bmatrix} \in Z_{n,K} \; : \; \begin{smallmatrix}\phi_i \in W_2^n[-\tau_i,0] \text{ and} \\ \phi_i(0)=x \text{ for all } i\in[K]\end{smallmatrix} \right\}.$$

Lemma 4.

Suppose that S_i, R_{ij} are polynomial , $S_i = S_i^T$, $R_{ij}(s,\theta) = R_{ji}(\theta,s)^T$, $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$ and $Q_j(s) = R_{ij}(0,s)$. Then $\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}}(X) = X$.

In the single delay case, this constraint eliminates P and Q entirely:

$$\left(\mathcal{P}\begin{bmatrix}x\\ \phi\end{bmatrix}\right)(s) = \begin{bmatrix} \tau(R(0,0)+S(0))x + \int_{-\tau}^0 R(0,s)\phi(s)ds \\ \tau R(s,0)\phi(0) + \tau S(s)\phi(s) + \int_{-\tau}^{0} R(s,\theta)\phi(\theta)d\theta \end{bmatrix}$$

Dual Stability Theorem for Time-Delay Systems

Theorem:
$$\dot{x}(t) = Ax(t) + \sum_i A_i x(t-\tau_i)$$
is stable if there exist P, Q_i, S_i, R_{ij} such that $P = \tau_K Q_i(0)^T + \tau_K S_i(0)$, $Q_j(s) = R_{ij}(0,s)$ and

$$\mathcal{P}_{\{P,Q_i,S_i,R_{ij}\}} \geq 0 \qquad \text{and} \qquad \mathcal{P}_{\{D_1,V_i,\dot{S}_i,G_{ij}\}} < 0$$

where

$$D_1 := \begin{bmatrix} C_0 + C_0^T & C_1 & \cdots & C_k \\ C_1^T & -S_1(-\tau_1) & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ C_{kT} & 0 & 0 & -S_k(-\tau_K) \end{bmatrix},$$

$$C_0 := A_0 P + \tau_K \sum_{i=1}^{K} (A_i Q_i(-\tau_i)^T + \frac{1}{2} S_i(0)), \qquad C_i := \tau_K A_i S_i(-\tau_i),$$

$$V_i(s) := \begin{bmatrix} B_i(s)^T & 0 & \cdots & 0 \end{bmatrix}^T, \qquad B_i(s) := A_0 Q_i(s) + \dot{Q}_i(s) + \sum_{j=1}^{K} R_{ji}(-\tau_j, s),$$

$$G_{ij}(s,\theta) := \frac{\partial}{\partial s} R_{ij}(s,\theta) + \frac{\partial}{\partial \theta} R_{ji}(s,\theta)^T.$$

$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$ is stable if there exist R, S such that

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} \tau(R(0,0)+S(0)) & \tau R(0,s) \\ \tau R(s,0) & \tau S(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T R(s,\theta)\phi(\theta) d\theta ds \geq \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|^2$$

and

$$V_D(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11}+D_{11}^T & D_{12} & \tau D_{13}(s) \\ D_{12}^T & -S(-\tau) & 0_n \\ \tau D_{13}(s)^T & 0_n & \tau \dot{S}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds}R(s,\theta) + \frac{d}{d\theta}R(s,\theta) \right) \phi(\theta) d\theta ds \leq -\epsilon \left\| \begin{bmatrix} \phi(0) \\ \phi \end{bmatrix} \right\|.$$

where

$$D_{11} := \tau A_0(R(0,0)+S(0)) + \tau A_1 R(-\tau,0) + \frac{1}{2}S(0),$$
$$D_{12} := \tau A_1 S(-\tau),$$
$$D_{13}(s) := A_0 R(0,s) + A_1 R(-\tau,s) + \dot{R}(s,0)^T.$$

IMPORTANT: V_D is NOT the derivative of V!!!

Compare with the Primal L-K Formulation

Note Reduced Sparsity

$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau)$ is stable if there exist M, N such that

$$V(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} M_{11} & \tau M_{12}(s) \\ \tau M_{21}(s) & \tau M_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(s) \end{bmatrix} ds$$
$$+ \tau \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s)^T N(s,\theta) \phi(\theta) d\theta ds \geq \left\| [\phi] \right\|^2$$

and

$$V_D(\phi) = \int_{-\tau}^{0} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix}^T \begin{bmatrix} D_{11} + D_{11}^T & D_{12} & \tau D_{13}(s) \\ D_{12}^T & -M_{22}(-\tau) & \tau D_{23}(s) \\ \tau D_{13}(s)^T & \tau D_{23}(s)^T & -\tau \dot{M}_{22}(s) \end{bmatrix} \begin{bmatrix} \phi(0) \\ \phi(-\tau) \\ \phi(s) \end{bmatrix} ds$$
$$- \tau \int_{-\tau}^{0} \int_{-\tau}^{0} \phi(s)^T \left(\frac{d}{ds} N(s,\theta) + \frac{d}{d\theta} N(s,\theta) \right) \phi(\theta) d\theta ds \leq -\epsilon \left\| [\phi(0)] \right\|^2.$$

where

$$D_{11} = M_{11} A_0 + M_{12}(0) + \frac{1}{2} M_{22}(0),$$
$$D_{12} = M_{11} A_1 - M_{12}(-\tau), \quad D_{23} = A_1^T M_{12}(s) - N(-\tau, s)$$
$$D_{13} = A_0^T M_{12}(s) - \dot{M}_{12}(s) + N(0,s).$$

Complexity and Accuracy of Dual Stability Conditions

$$\dot{x}(t) = -x(t-\tau)$$

d	1	2	3	4	analytic
$\tau_{\max}$	1.408	1.5707	1.5707	1.5707	1.5707
CPU sec	.18	.21	.25	.47	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.6581	1.716	1.7178	1.7178	1.7178
$\tau_{\min}$	.10019	.10018	.10017	.10017	.10017
CPU sec	.25	.344	.678	1.725	

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & .1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t-\tau/2) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(t-\tau)$$

d	1	2	3	4	limit
$\tau_{\max}$	1.33	1.371	1.3717	1.3718	1.372
CPU sec	2.13	6.29	24.45	79.0	

$$\dot{x}(t) = (A-BKC)x(t) + (A+BKC)x(t-\tau),$$

where $K = 1$, $\tau = 3$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

d	1	2	3	4
CPU sec	1.45	5.99	24.78	63.21

Complexity Scaling Results: Single Delay Case

- **10 State Example (d=2):** 22s
- **20 State Example (d=2):** 951s

Further reduction possible using Differential-Difference Formulation.

Now Recall Our ODE Roadmap

The goal is to find $K \in \mathbb{R}^{m\times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A+BK) + (A+BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A+BK)Q + Q(A+BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m\times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

Recall the Controller Synthesis LOI

Find $\mathcal{P}$, $\mathcal{Z}$ such that $\mathcal{P}(X) = X$, $\mathcal{P} > 0$

$$\begin{aligned}\langle \mathcal{A}\mathcal{P}x, x\rangle + \langle x, \mathcal{A}\mathcal{P}x\rangle_Z + \langle \mathcal{B}\mathcal{Z}x, x\rangle + \langle x, \mathcal{B}\mathcal{Z}x\rangle_Z \\ = \langle x, \mathcal{D}x\rangle + \langle \mathcal{B}\mathcal{Z}x, x\rangle + \langle x, \mathcal{B}\mathcal{Z}x\rangle_Z < 0\end{aligned}$$

We already discussed $\langle x, \mathcal{D}x\rangle$. Now examine the new variable $\mathcal{Z} = \mathcal{K}\mathcal{P}$.

- Since $\mathcal{B}$ is not differential, it helps to let $\mathcal{K}$ have the form

$$\left(\mathcal{K}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = K_0 x + K_1\phi(-\tau) + \int_{-\tau}^{0} K_2(s)\phi(s)ds,$$

- Then if $\mathcal{Z} = \mathcal{K}\mathcal{P}$, $\mathcal{Z}$ has the form

$$\left(\mathcal{Z}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = Z_0 x + Z_1\phi(-\tau) + \int_{-\tau}^{0} Z_2(s)\phi(s)ds,$$

$\mathcal{B}$ is simply $(\mathcal{B}u)(s)$

$$\left(\mathcal{B}\mathcal{Z}\begin{bmatrix} x \\ \phi \end{bmatrix}\right)(s) = \begin{bmatrix} BZ_0 x + BZ_1\phi(-\tau) + \int_{-\tau}^{0} BZ_2(s)\phi(s)ds \\ 0 \end{bmatrix}$$

Full-State Feedback Controllers

Theorem:
$$\dot{x}(t) = A_0 x(t) + A_1 x(t-\tau) + Bu(t)$$
is full-state-feedback stabilizable if there exist S, R, Z_0, Z_1 and Z_2 such that

$$\mathcal{P}_{\{R(0,0)+S(0),R(0,s),S,R\}} \geq 0 \qquad \text{and} \qquad \mathcal{P}_{\{C,V,\dot{S},G\}} < 0$$

where

$$C := \begin{bmatrix} D_{11} + D_{11}^T & S_{12} \\ D_{12}^T & D_{22} \end{bmatrix} + \begin{bmatrix} L_{11} + L_{11}^T & L_{12} \\ L_{12}^T & 0 \end{bmatrix}, \quad V(s) = \begin{bmatrix} D_{13}(s) + L_{13}(s) \\ 0 \end{bmatrix},$$

$$D_{11} := \tau A_0 (R(0,0) + S(0)) + \tau A_1 R(-\tau, 0) + \frac{1}{2} S(0),$$

$$D_{12} := \tau A_1 S(-\tau), \qquad D_{22} := -S(-\tau),$$

$$D_{13}(s) := A_0 R(0,s) + A_1 R(-\tau, s) + \dot{R}(s,0)^T,$$

$$G(s,\theta) := \frac{d}{ds} R(s,\theta) + \frac{d}{d\theta} R(s,\theta),$$

$$L_{11} = B_0 Z_0, \quad L_{12} = B_0 Z_1, \quad L_{13}(s) = \tau B_0 Z_2(s).$$

Again Recall Our ODE Roadmap

The goal is to find $K \in \mathbb{R}^{m \times n}$ such that

$$\dot{x} = Ax + Bu, \qquad u = Kx \qquad\qquad \text{is Stable}$$

Step 1: DUALITY says the following are equivalent for fixed A, B, K:

1. $\exists P > 0$ such that $P(A+BK) + (A+BK)^T P < 0$.
2. $\exists Q > 0$ such that $(A+BK)Q + Q(A+BK)^T < 0$.

Step 2: Variable Substitution - Define variable $Z = KQ$. The Synthesis condition becomes

$$AQ + BZ + QA^T + Z^T B^T < 0 \qquad Q > 0, \quad Z \in \mathbb{R}^{m \times n}$$

Step 3: Controller Reconstruction. Given solution Q, Z, the controller is

$$K = ZQ^{-1}$$

Analytic Formula for Operator Inversion [Keqin's Result]

Suppose $\mathcal{P} > 0$ where

$$\begin{aligned}
\mathcal{P}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) &= \begin{bmatrix} P\psi + \int_{-\tau}^{0} Q(\theta)\phi(\theta)d\theta \\ \tau Q^T(s)\psi + \int_{-\tau}^{0} R(s,\theta)\phi(\theta)d\theta + S(s)\phi(s)\end{bmatrix}\\
R(s,\theta) &= Y^T(s)\Gamma Y(\theta), \qquad Q(s) = HY(s),
\end{aligned}$$

Then the inverse $\mathcal{P}^{-1}$ is given by

$$\mathcal{P}^{-1}\begin{bmatrix}\psi\\ \phi\end{bmatrix}(s) = \begin{bmatrix} \hat{P}\psi + \int_{-\tau}^{0} \hat{Q}(\theta)\phi(\theta)d\theta \\ \tau\hat{Q}^T(s)\psi + \hat{S}(s)\phi(s) + \int_{-\tau}^{0} \hat{R}(s,\theta)\phi(\theta)d\theta\end{bmatrix},$$

where $\hat{R}(s,\theta)$, $\hat{Q}(\theta)$ and $\hat{S}(s)$ are given as follows

$$\begin{aligned}
\hat{R}(s,\theta) &= \hat{Y}^T(s)\hat{\Gamma}\hat{Y}(\theta),\\
\hat{Q}(\theta) &= \hat{H}\hat{Y}(\theta), \qquad \hat{S}(s) = S^{-1}(s), \qquad \hat{Y}(s) = Y(s)S^{-1}(s)\\
\hat{H} &= -P^{-1}HT, \qquad \hat{P} = [I + rP^{-1}HTKH^T]P^{-1},\\
\hat{\Gamma} &= [rT^TH^TP^{-1}H - \Gamma](I+K\Gamma)^{-1}, \quad T = (I + K\Gamma - \tau KH^TP^{-1}H)^{-1},
\end{aligned}$$

where $K = \int_{-\tau}^{0} Y(s)S^{-1}(s)Y^T(s)ds$,

A Full-State Feedback Controller

Finally, we recover the controller as

$$u(t) = K_0 x(t) + K_1 x(t-\tau) + \int_{-\tau}^{0} K_2(s) x(t+s) ds$$

where

$$\begin{aligned}
K_0 &= Z_0 \hat{P} + \tau Z_1 \hat{Q}^T(-\tau) + \tau \int_{-\tau}^{0} Z_2(s) \hat{Q}^T(s) ds, \\
K_1 &= Z_1 \hat{S}(-\tau), \\
K_2(s) &= Z_0 \hat{Q}(s) + Z_1 \hat{R}(-\tau, s) + Z_2(s) \hat{S}(s) + \int_{-\tau}^{0} Z_2(\theta) \hat{R}(\theta, s) d\theta.
\end{aligned}$$

Note: This is *Full-State* Feedback.

- Contrast with output feedback: $u(t) = Kx(t)$ or $u(t) = Kx(t-\tau)$.

Response: Design an Observer.

- Ongoing Research.

Full-state Feedback Controller: Numerical Example

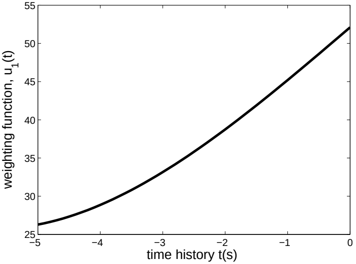
Consider a numerical example.

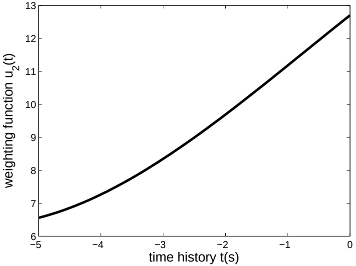
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -.5 \\ 0 & -1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Using a value of $\tau = 5s$, we compute the following controller:

$$u(t) = \begin{bmatrix} -3601 \\ -944 \end{bmatrix}^T x(t) + \begin{bmatrix} -.00891 \\ .872 \end{bmatrix}^T x(t-\tau)$$

$$+ \int_{-5}^{0} \begin{bmatrix} 52.1 + 6.98s + .00839s^2 - .0710s^3 \\ 12.7 + 1.50s - .0407s^2 - .0190s^3 \end{bmatrix}^T x(t+s)ds$$





Numerical Example

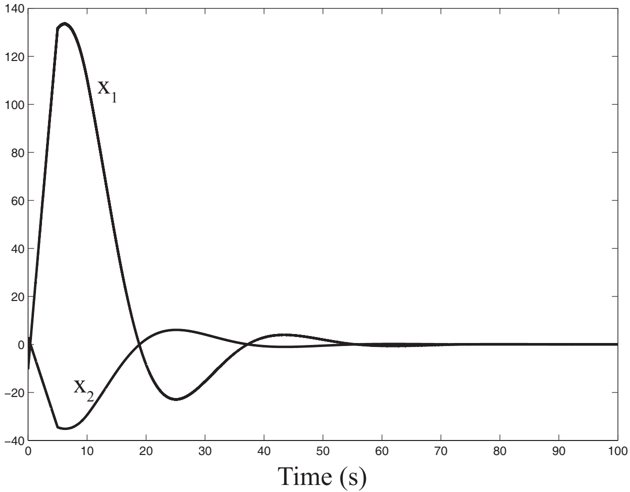


Figure: Trajectory of a delayed system ($\tau = 5s$) with full-state feedback

Conclusions:

- A Dual approach to controller synthesis
  - Convexifies the problem
  - Can be applied to any Lyapunov-Krasovskii-based approach.
  - NOT limited to SOS.
- Practical Implications
  - First numerical solution to **Full-State Feedback** of multi-state delayed systems.
  - No Analytic Solution to operator inversion in multi-delay case.

Numerical Code Produced:

- LOI Toolbox
  - Packaged as DelayTools
  - But limited Functionality
  - Can declare L_2-positive operator variables.

Available for download at
`http://control.asu.edu`

- Next Talk:
  - Observer-Based Controller Synthesis
  - Preliminary Work by Guoying