Systems Analysis and Control

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Lecture 15: Root Locus Continued

Overview

In this Lecture, you will learn:

What is the effect of small gain?

- Departure Angles

Which Poles go to Zeroes?

- Arrival Angles

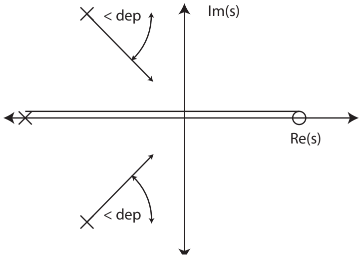
Picking Points?

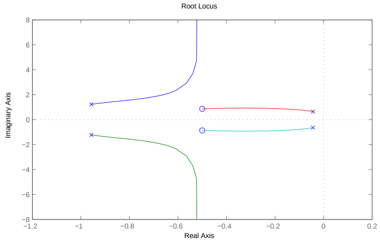
- Calculating the Gain
- Satisfying Performance Criteria

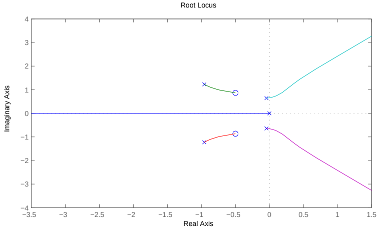
Departure Angle

The root locus starts at the poles.

- What it the effect of small gain?
- Do the poles become more or less stable?

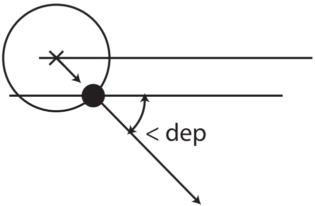


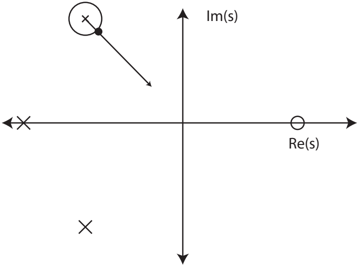




Departure Angle

To find the departure angle, we look at a very small region around the departure point.





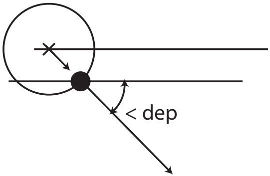
For a point to be on the root locus, we want phase of $180°$.

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = 180°$$

Departure Angle

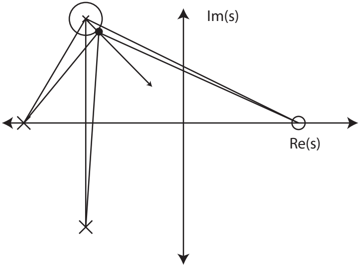
If we make the point s extremely close to the pole p.

- Most part of phase at s is the same as for p.
  - $\angle(s - z_i) \cong \angle(p - z_i)$ for all i
  - $\angle(s - z_i) \cong \angle(p - z_i)$ for all i
- The only difference is the phase from p itself.



The phase due to p is just the departure angle, $\angle_{dep}$

$$\angle(s - p) = \angle_{dep}$$



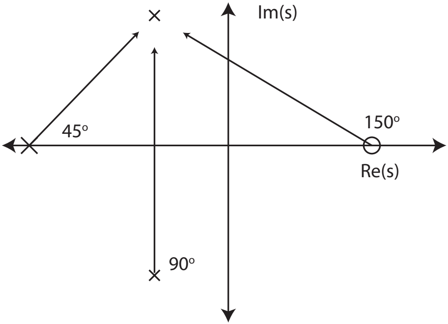
The total phase is

$$\angle G(s) = \angle G(p) - \angle_{dep} = 180^\circ$$

Thus the departure angle is

$$\angle_{dep} = \angle G(p) + 180^\circ$$

Therefore, to find the departure angle from pole p, just find the phase at p.



The phase at p is based on geometry.

$$\angle G(p) = 150^\circ - 90^\circ - 45^\circ = 15^\circ$$

So the departure angle is easy to calculate.

$$\angle_{dep} = \angle G(p) + 180^\circ = 195^\circ$$

Root Locus Demo 1

Wiley+ Root Locus Demo 1

$$G(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + 1s + 1}$$

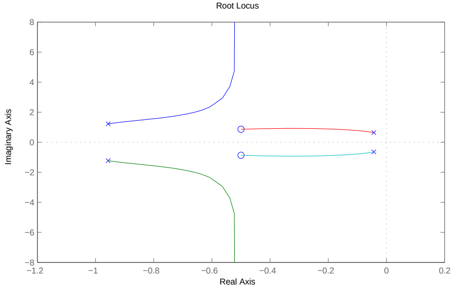
Poles at

- $p_{1,2} = -.957 \pm 1.23$
- $p_{3,4} = -.0433 \pm .641$

Zeroes at

- $z_{1,2} = -.5 \pm .866\imath$

Problem:
Find departure angle at $p_1 = -.957 + 1.23$.



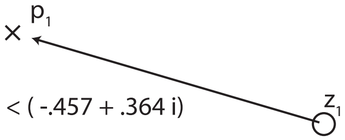
$$\angle_{dep} = 180^\circ + \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4)$$

The difficulty is calculating the phase.

Departure Angle

Numerical Examples

$$\begin{aligned}\angle(p_1 - z_1) &= \angle(-.957 + 1.23\imath + .5 - .866\imath)\\ &= \angle(-.457 + .364\imath)\\ &= \tan^{-1}\left(\frac{.364}{-.457}\right)\\ &= 141.46^\circ\end{aligned}$$



$$\angle(p_1 - z_2) = \angle(-.457 + 2.096\imath) = 102.3^\circ$$

Obviously,

$$\angle(p_1 - p_2) = 90^\circ$$

$$\angle(p_1 - p_3) = 147.2^\circ, \qquad \angle(p_1 - p_4) = 116.03^\circ$$

Departure Angle

Numerical Examples

Now that we have all the angles:

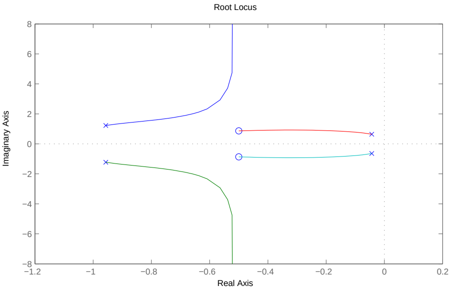
$$\begin{aligned}\angle G(p_1) &= \angle(p_1 - z_1) + \angle(p_1 - z_2) - \angle(p_1 - p_2) - \angle(p_1 - p_3) - \angle(p_1 - p_4)\\ &= 141.46^\circ + 102.3^\circ - 90^\circ - 147.2^\circ - 116.03^\circ\\ &= -109.47^\circ\end{aligned}$$

We conclude

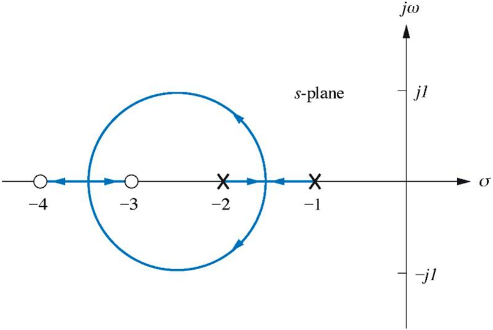
$$\angle_{dep,p_1} = \angle G(p_1) + 180^\circ = 70.53^\circ$$

By symmetry we could find

$$\angle_{dep,p_2} = -70.53^\circ$$



What about a pole on the real axis?

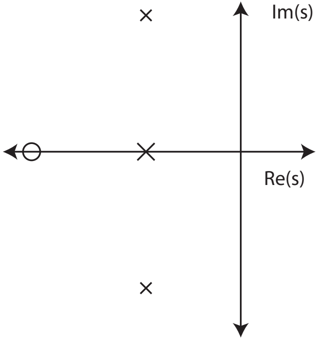


$$\angle G(p) = 0^\circ \quad \text{or} \quad 180^\circ$$

Calculating the Departure Angle

DIY Example

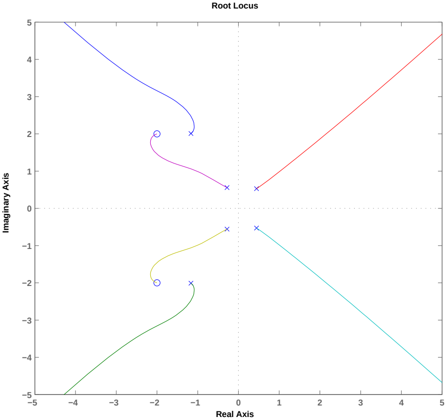
$$G(s) = \frac{s+2}{(s+1)(s^2+2s+2)}$$



Arrival Angle

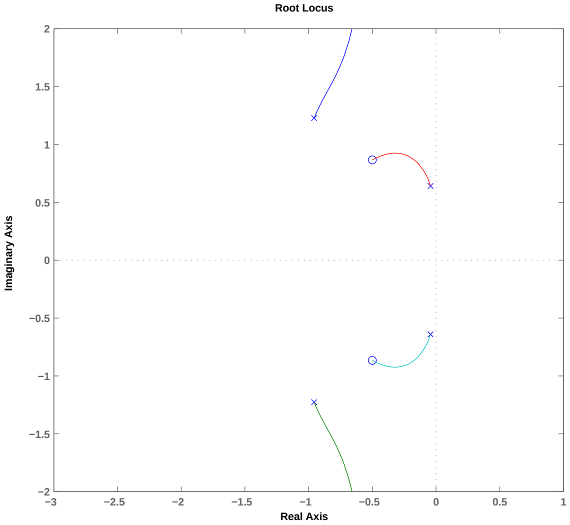
Similar to Departure Angles, we can find arrival angles

$$G(s) = \frac{s^2 + 4s + 8}{s^6 + 2s^5 + 5s^4 - s^3 + 2s^2 + 1}$$



Arrival Angle

Suspension Problem

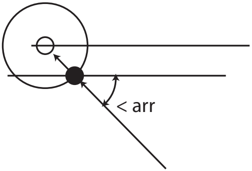


Arrival Angle

- We treat arrival angles like departure angles.
- To find the arrival angle, we look at a very small region around the arrival point.

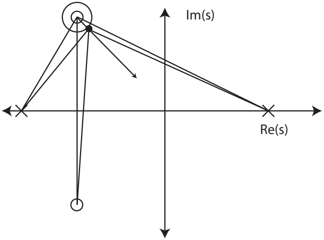
For a point to be on the root locus, we want phase of 180°.

$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = 180^\circ$$



If we make the point s extremely close to z.

- Most of phase at s is same as phase at z.
  - Most of phase is $\angle G(z)$
- The only difference is the phase from z itself, $\angle(s - z)$.



Arrival Angle

The phase due to z is just the arrival angle, $\angle_{arr}$

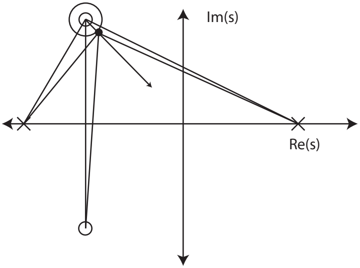
$$\angle(s-z) = \angle_{arr}$$

The total phase is

$$\angle G(s) = \angle G(z) + \angle_{arr} = 180^\circ$$

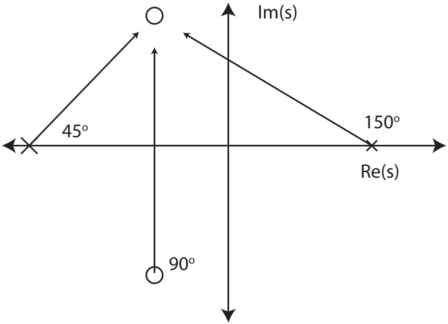
Thus the departure angle is

$$\angle_{arr} = 180^\circ - \angle G(z)$$



Arrival Angle

Numerical Examples



The phase at z is based on geometry.

$$\angle G(z) = 90^\circ - 150^\circ - 45^\circ = -105^\circ$$

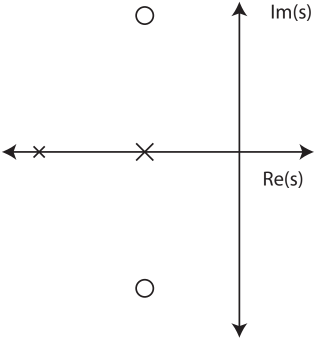
So the departure angle is easy to calculate.

$$\angle_{arr} = 180 - \angle G(z) = 285^\circ$$

Calculating the Arrival Angle

DIY Example

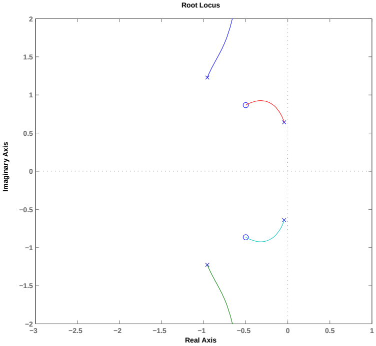
$$G(s) = \frac{s^2 + 2s + 2}{(s+1)(s+2)}$$



Suppose we have drawn our root locus.

Now we want:

- A point with 20% overshoot
- A point with 4s settling time
- A point with 2s rise time.



We can see that acceptable points are on the root locus.
Question: How to achieve these points?

Calculating Gain

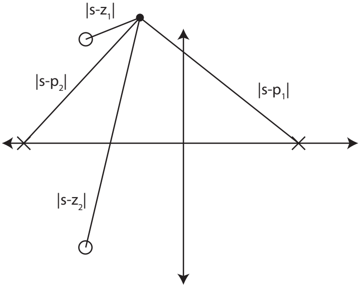
Problem: Given a point on the root locus, $s_{desired}$, find the gain which achieves that point.

Answer: We know that for a point on the root locus,

$$1 + kG(s) = 0$$

Therefore, the gain at the point $s_{desired}$ is

$$k = \left| -\frac{1}{G(s)} \right| = \frac{1}{|G(s)|}$$



- The gain is determined by the *magnitude* of $G(s)$.

Note: Even is a point is not *EXACTLY* on the root locus, the formula still gives an approximate gain

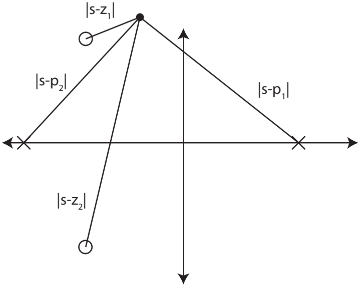
Calculating Gain

Calculating the magnitude of $G(s)$ is similar to calculating the phase

$$G(s) = \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

Multiplication and division properties of complex numbers:

$$|r_1 \cdot r_2| = |r_1| \cdot |r_2| \qquad \left|\frac{r_1}{r_2}\right| = \frac{|r_1|}{|r_2|}$$



To calculate $|G(s)|$ we can use

$$|G(s)| = \frac{|s - z_1| \cdots |s - z_m|}{|s - p_1| \cdots |s - p_n|}$$

$$k(s) = \frac{|s - p_1| \cdots |s - p_n|}{|s - z_1| \cdots |s - z_m|}$$

Calculating Gain

Numerical Example

It is somewhat hard to find k. Easiest in factored form.

$$G(s) = \frac{1}{s(s+4)(s+6)}$$

Lets find the gain at $s = -1.8$ Pole 1:

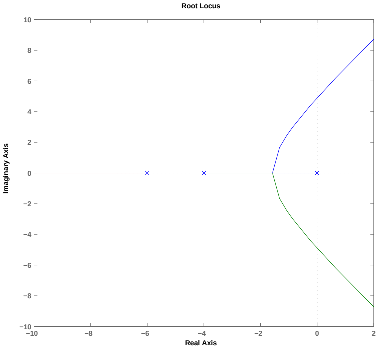
$$|s| = 1.8$$

Pole 2:

$$|s+4| = 2.2$$

Pole 3:

$$|s+6| = 4.2$$



$$k = \frac{|s-p_1|\cdots|s-p_n|}{|s-z_1|\cdots|s-z_m|} = \frac{1.8 \cdot 2.2 \cdot 4.2}{1} = 16.63$$

Calculating Gain

Numerical Example

Points on the real axis are easiest. Lets try the point at $s \cong -1 + 2\imath$

$$G(s) = \frac{1}{s(s+4)(s+6)}$$

Pole 1:

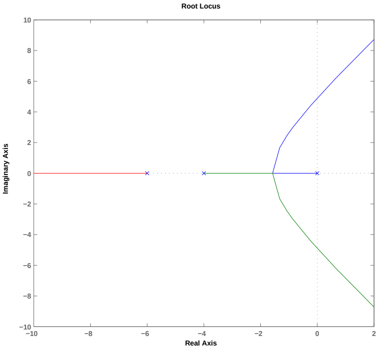
$$|s| = \sqrt{1+2^2} = \sqrt{5} = 2.236$$

Pole 2:

$$|s+4| = \sqrt{3^2+2^2} = \sqrt{13} = 3.6$$

Pole 3:

$$|s+6| = \sqrt{5^2+2^2} = \sqrt{29} = 5.385$$



$$k = \frac{|s-p_1|\cdots|s-p_n|}{|s-z_1|\cdots|s-z_m|} = \sqrt{5 \cdot 13 \cdot 29} = 43.4$$

Calculating Gain

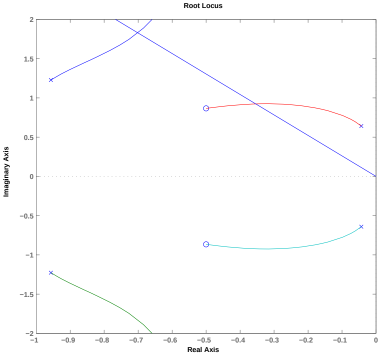
Can the Suspension system achieve 30% overshoot using proportional feedback?

$$G(s) = \frac{s^2+s+1}{s^4+2s^3+3s^2+s+1}$$

- $M_P = .3$ defines the line at

$$\omega = \frac{\pi}{\ln(M_{p,desired})}\sigma$$

- Examine the gain at
  - $s_1 = -.3536 + .922\imath$
  - $s_2 = -.7 + 1.83\imath$

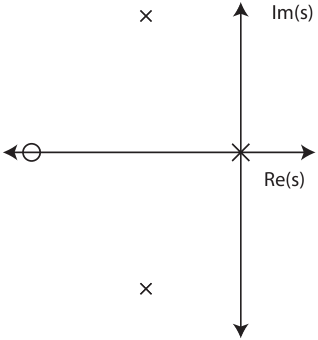


$$k(s_1) = 3.58$$
$$k(s_2) = 2.6153$$

Calculating Gain

DIY Example

$$G(s) = \frac{s+2}{s(s^2+2s+2)}$$



Find $T_s \leq 8s$

The Matlab syntax for root locus is

```
> rlocus(n,d)
```

where

- `n` is a vector of the coefficients of the numerator of G
- `d` is a vector of the coefficients of the denominator of G

Example:

$$G(s) = \frac{s^2 + 4s + 8}{s^6 + 2s^5 - s^3 + 2s^2 + 1}$$

- `> n = [1 4 8]`
- `> d = [1 2 0 -1 2 0 1]`

To find the gain at a point on the root locus:

- Plot the root locus.
- `> rlocfind(n,d)`
- Use the cursor to select the point.

Summary

What have we learned today?

What is the effect of small gain?

- Departure Angles

Which Poles go to Zeroes?

- Arrival Angles

Picking Points?

- Calculating the Gain
- Satisfying Performance Criteria

Next Lecture: Generalized Root Locus and Design Problems