LMI Methods in Optimal and Robust Control

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Lecture 03: Relaxations, Duality, Cones, Positive Matrices, LMIs

What is Optimization?

An Optimization Problem has 3 parts.

$$\begin{aligned} \min_{x \in \mathbb{F}} \quad f(x) : & \qquad \text{subject to} \\ g_i(x) \le 0 & \qquad i = 1, \cdots K_1 \\ h_i(x) = 0 & \qquad i = 1, \cdots K_2 \end{aligned}$$

Variables: $x \in \mathbb{F}$

- The things you must choose.
- Typically vectors or matrices.

Objective: $f(x)$

- A function which assigns a *scalar* value to any choice of variables.

Constraints: $g(x) \le 0;\ h(x) = 0$

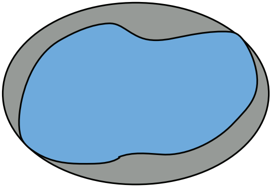
- Defines what is a minimally acceptable choice of variables.
- Equality and Inequality constraints are common.

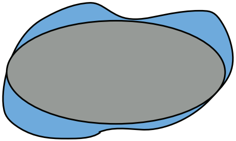
New Concept: Relaxations and Tightenings

How to Approximate a Non-Convex Problem (Using a Convex Approximation)

Original Problem:

$$\gamma^* := \min_{x \in \mathbb{R}} \quad f(x): \qquad g(x) \geq 0 \;\; (FS)$$





Definition 1.

In a **Relaxation**, we remove or loosen one of the constraints.

$$\gamma_R^* := \min_{x \in \mathbb{R}} \quad f(x): \qquad g(x) \geq -1$$

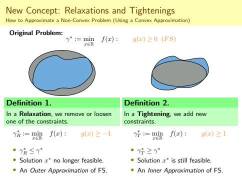
- $\gamma_R^* \leq \gamma^*$
- Solution x^* no longer feasible.
- An *Outer Approximation* of FS.

Definition 2.

In a **Tightening**, we add new constraints.

$$\gamma_T^* := \min_{x \in \mathbb{R}} \quad f(x): \qquad g(x) \geq 1$$

- $\gamma_T^* \geq \gamma^*$
- Solution x^* is still feasible.
- An *Inner Approximation* of FS.



- FS stands for Feasible Set
  - The set of values of x which satisfy the constraints
- Relaxations *Increase* the size of the feasible set
  - The solution may not be feasible for the original problem
- Tightenings *Decrease* the size of the feasible set
  - The solution may not be optimal for the original problem

MAX-CUT: Original Problem

$$\max_{x_i^2=1} \frac{1}{2}\sum_{i,j} w_{i,j}(1-x_i x_j)$$

Solution: $\gamma^* = 4$

- $x_1 = x_3 = x_4 = 1$
- $x_2 = x_5 = -1$

MAX-CUT: Relaxed Problem

$$\max_{x_i^2\le 1} \frac{1}{2}\sum_{i,j} w_{i,j}(1-x_i x_j)$$

Solution: $\gamma_R^* = 4$

- $x_2 = x_5 = 1$
- $x_1 = x_4 = -1$
- $x_3 = 0$

YALMIP Code:

```
> x = sdpvar(5,1);
> F=[-1 <= x <= 1];
> obj=2.5-.5*x(1)*x(2)-.5*x(2)*x(3)
>       -.5*x(3)*x(4)-.5*x(4)*x(5)-.5*x(1)*x(5);
> optimize(F,-obj);
> value(x);
```

Link: Download the Free version of IBM solver CPLEX and add path to MATLAB

Relaxations and Tightenings

Examples

MAX-CUT: Original Problem

$$\max_{x_i^2=1} \frac{1}{2} \sum_{i,j} w_{i,j}(1 - x_i x_j)$$

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MAX-CUT: Relaxed Problem

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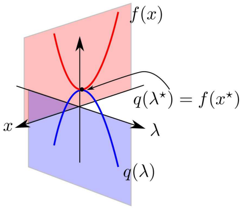
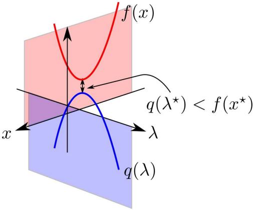
Note the solution to the relaxed Max Cut problem is not feasible for the original problem.

A Third Option: Duality

A Cool Word, but Meaning is Vague



strong duality



weak duality

Definition 3.

Two **Optimization Problems are Dual** if any feasible solution to one has objective value which bounds the solution to the other problem.

Primal Problem:

$$\min_{x \in \mathbb{R}} \quad f(x) : \qquad x \in S$$

Dual Problem:

$$\max_{y \in \mathbb{R}} \quad f_D(y) : \qquad y \in S_D$$

Relationship:

- if $y \in S_D$, then $f_D(y) \le f(x)$ for any $x \in S$.
- if $x \in S$, then $f(x) \ge f_D(y)$ for any $y \in S_D$.

Lagrangian Duality

$$\min_{x \in \mathbb{F}} f_0(x) : \qquad \text{subject to } f_i(x) \geq 0 \qquad i = 1, \cdots k$$

Note that

$$\max_{\alpha > 0} -\alpha f_i(x) = \begin{cases} \infty & f_i(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

Equivalent Form:

$$\gamma^* = \min_{x \in \mathbb{F}} \max_{\alpha_i > 0} \; f_0(x) - \sum_i \alpha_i f_i(x) \; = \; \min_{x \in \mathbb{F}} \max_{\alpha_i > 0} \; L(x, \alpha)$$

The function $L(x, \alpha) = f_0(x) - \sum_i \alpha_i f_i(x)$ is called the **Lagrangian**.

The **Dual Problem** switches the min-max:

$$\lambda^* = \max_{\alpha_i > 0} \min_{x \in \mathbb{F}} \; f_0(x) - \sum_i \alpha_i f_i(x)$$

Or if we define $g(\alpha) = \min_{x \in \mathbb{F}} f_0(x) - \sum_i \alpha_i f_i(x)$,

$$\lambda^* = \max_{\alpha_i > 0} \; g(\alpha)$$

For convex optimization, $\lambda^* = \gamma^*$. However, $x^* \neq \alpha^*$.

Note: We always have $\max_x \min_y g(x, y) \leq \min_y \max_x g(x, y)$ (2-player game).

Lagrangian Duality

$$\min_{x\in\mathbb{F}} f_0(x): \qquad \text{subject to } f_i(x) \geq 0 \qquad i = 1, \cdots k$$

Note that

$$\max_{\alpha>0} -\alpha f_i(x) = \begin{cases} \infty & f_i(x) < 0 \\ 0 & \text{otherwise} \end{cases}$$

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The function $L(x,\alpha) = f_0(x) - \sum_i \alpha_i f_i(x)$ is called the **Lagrangian**.

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Note: We always have $\max_x \min_y g(x,y) \leq \min_y \max_x g(x,y)$ (2-player game).

The property (Weak Duality)

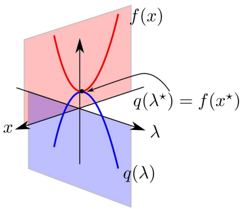
$$\max_x \min_y g(x,y) \leq \min_y \max_x g(x,y)$$

always holds for smooth functions and when equality holds can be interpreted as a Nash equilibrium (See window drawings in "A Beautiful Mind").

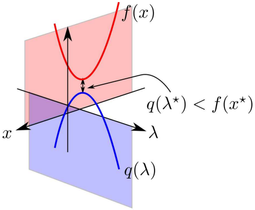
The player which moves second always wins. Note the second player here is the inner optimization problem ($\min$ on LHS and $\max$ on RHS). To verify, just use the function

$$g(x,y) = xy$$

Strong vs. Weak Duality (In the Lagrangian Sense)



strong duality



weak duality

Primal Problem:

$$\gamma^* = \min_{x \in \mathbb{R}} \quad f(x): \qquad x \in S$$

Dual Problem:

$$\lambda^* = \max_{y \in \mathbb{R}} \quad f_D(y): \qquad y \in S_D$$

Definition 4.

Strong Duality holds if $\lambda^* = \gamma^*$. **Weak Duality** holds if $\lambda^* < \gamma^*$.

- Strong Duality holds if f, S are convex and S has non-empty interior.
- Weak Duality always holds.
- The Lagrangian Dual Problem is **ALWAYS** Convex.

Lagrangian Duality Examples

Two Ways to Solve the Same Problem

Primal LP:

$$\max_{x\in\mathbb{R}} \quad c^T x :$$
$$Ax \le b$$
$$x \ge 0$$

Dual LP:

$$\min_{y\in\mathbb{R}} \quad b^T y :$$
$$A^T y \ge c$$
$$y \ge 0$$

$$g(\alpha) = \max_{x\ge 0} c^T x - \alpha^T (Ax - b) = \max_{x \ge 0} (c - A^T\alpha)^T x + \alpha^T b = \begin{cases} b^T\alpha & A^T\alpha \ge c \\ \infty & \text{otherwise} \end{cases}$$

Primal SDP:

$$\min_{x\in\mathbb{R}} \quad \sum_{i=1}^m c_i x_i$$
$$X = \sum_{i=1}^m F_i x_i - F_0 \ge 0$$

Dual SDP:

$$\max_{y\in\mathbb{R}} \quad \text{trace}(F_0 Y) :$$
$$\text{trace}(F_i Y) = c_i \quad (i = 1, \cdots, m)$$
$$Y \ge 0$$

The trace notation simply means $\text{trace}(FY) = \sum_{i,j} F_{ij} Y_{ij}$.

Lagrangian Duality Examples

Two Ways to Solve the Same Problem

Primal LP:

$$\max_{x\in\mathbb{R}} \; c^T x : \quad Ax \le b, \quad x \ge 0$$

Dual LP:

$$\min_{y\in\mathbb{R}} \; b^T y : \quad A^T y \ge c, \quad y \ge 0$$

$$g(\alpha) = \max_{x\ge 0} c^T x - \alpha^T(Ax-b) = \max_{x\ge 0}(c - A^T\alpha)^T x + \alpha^T b = \begin{cases} b^T\alpha & A^T\alpha \ge c \\ \infty & \text{otherwise} \end{cases}$$

Primal SDP:

$$\min_{x\in\mathbb{R}} \; \sum_{i=1}^{m} c_i x_i \quad X = \sum_{i=1}^{m} F_i x_i - F_0 \ge 0$$

Dual SDP:

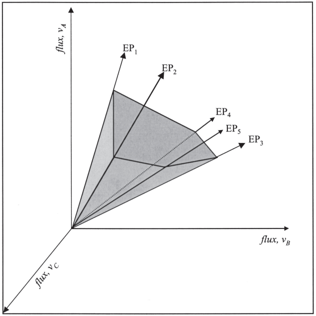
$$\max_{y\in\mathbb{R}} \; \text{trace}(F_0 Y) : \quad \text{trace}(F_i Y) = c_i \quad (i = 1, \cdots, m), \quad Y \ge 0$$

The trace notation simply means trace$(FY) = \sum_{i,j} F_{ij} Y_{ij}$.

$$
\begin{aligned}
&\max_{x\ge 0,\, Ax-b\le 0} c^T x \\
&= \max_{x\ge 0} \; \min_{\alpha\ge 0} \; c^T x - \alpha^T(Ax-b) \\
&\le \min_{\alpha\ge 0} \; \max_{x\ge 0} \; c^T x - \alpha^T(Ax-b) \\
&= \min_{\alpha\ge 0} \; \max_{x\ge 0} \; (c^T - \alpha^T A)x + \alpha^T b \\
&= \min_{\alpha\ge 0,\, c^T-\alpha^T A\le 0} \alpha^T b \\
&= \min_{\alpha\ge 0,\, c\le A^T\alpha} b^T \alpha
\end{aligned}
$$

Question: What does $f(x) \geq 0$ mean.

- What does $y \geq 0$ mean?



Definition 5.

A set is a **cone** if for any $x \in Q$,

$$\{\mu x \,:\, \mu \geq 0\} \subset Q.$$

Examples:

- **Positive Orthant:** $y \geq 0$ if $y_i \geq 0$ for $i = 1, \cdots, n$.
- **Half-space:** $y \geq 0$ if $\sum y_i \geq 0$ $(\mathbf{1}^T y \geq 0)$.
  - More generally, $y \geq 0$ if $\mathbf{a}^T y + b \geq 0$.
- **Intersection of Half-spaces:** $y \geq 0$ if $a_i^T y + b_i \geq 0$ for $i = 1, \cdots, n$.
- **Positive Matrices:** $P \geq 0$ if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$.
- **Positive Functions:** $f \geq 0$ if $f(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Generalized Inequalities and Convex Cones

What is an inequality? What does ≥ 0 mean?

- An inequality implies a **partial ordering**:
  - $x \geq y$ if $x - y \geq 0$
- Any convex cone, C defines a partial ordering:
  - $x - y \geq 0$ if $x - y \in C$
- The ordering is only partial because $x \not\leq 0$ does not imply $x \geq 0$
  - $-x \notin C$ does not imply $x \in C$.
  - x may be indefinite.
- The Cone of Positive Matrices is a partial ordering.
  - A matrix may have both positive and negative eigenvalues.

Dual Cones (on an inner-product space)

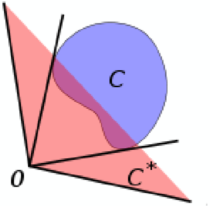
Definition 6.

Two **Sets are Dual (X and Y)** if $x \in X$ implies $\langle x, y\rangle \geq 0$ for all $y \in Y$.

For every point $y \in Y$, the angle between y and every point in X is less than $90°$ (and vice-versa holds be definition).

We can now consider **Self-Dual Cones**:

- **Positive Orthant:** $y \geq 0$ if $y_i \geq 0$ for $i = 1, \cdots, n$.
  - $\langle x, y\rangle = x^T y = \|x\|\|y\| \cos\theta$.
- **Positive Matrices:** $P \geq 0$ if $x^T P x \geq 0$ for all $x \in \mathbb{R}^n$.
  - $\langle X, Y\rangle = \text{trace}(XY)$



This is why we refer to both primal and dual versions of SDP and LP.

- Their dual problems are of the same form.
- Allows Primal-Dual Algorithms
- Faster Convergence

Linear Algebra Review: Symmetric Matrices

Definition 7.

A square matrix $P \in \mathbb{R}^{n \times n}$ is **Symmetric**, denoted $P \in \mathbb{S}^n$ if $P = P^T$.

$$P = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 2 & 3 \\ *^T & 4 & 5 \\ *^T & *^T & 6 \end{bmatrix}$$

- Symmetric Matrices have **Real** Eigenvalues: $\{\lambda \,:\, Px = \lambda x,\ x \in \mathbb{R}^n\} \subset \mathbb{R}$

Definition 8.

A matrix U is **Unitary** (orthogonal) if $U^{-1} = U^T$.

Unitary Matrices have the pleasant property that $\|Ux\| = \|x\|$ for any $x \in \mathbb{R}^n$.

- e.g. Rotation Matrices $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

Symmetric Matrices can be diagonalized by a Unitary matrix.

$$P = U\Lambda U^T \quad \text{or} \quad U^T P U = \Lambda \qquad \Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \quad (\lambda_i \text{ are eigenvalues})$$

Linear Algebra Review: Singular Value Decomposition

For ANY non-symmetric matrices, $P = U\Sigma V^T$, with U, V unitary and $\Sigma = \text{diag}(\sigma_1, \cdots, \sigma_n)$ diagonal, positive.

- This is called the **Singular Value Decomposition (SVD)**.
- $\sigma_i \geq 0$ (**Singular Values**) are the square roots of the eigenvalues of $P^T P$.
- The maximum σ_i is denoted $\bar{\sigma}(P)$. Note that $\bar{\sigma}(P) = \max_x \frac{\|Px\|}{\|x\|}$.

Matlab Code: `> [U,S,V]=svd([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}}_{U} \underbrace{\begin{bmatrix}11.34 & 0 & 0\\ 0 & .52 & 0\\ 0 & 0 & .17\end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix}-.33 & .74 & -.59\\ -.59 & .33 & .74\\ -.74 & -.59 & -.33\end{bmatrix}^T}_{V^T}$$

NOTE: This is not quite the same as Diagonalization! Unless....

Matlab Code: `> [U,S]=schur([1 2 3;2 4 5;3 5 6]);`

$$\begin{bmatrix}1 & 2 & 3\\ 2 & 4 & 5\\ 3 & 5 & 6\end{bmatrix} = \underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}}_{U} \underbrace{\begin{bmatrix}11.34 & 0 & 0\\ 0 & -.52 & 0\\ 0 & 0 & .17\end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix}-.33 & -.74 & -.59\\ -.59 & -.33 & .74\\ -.74 & .59 & -.33\end{bmatrix}^T}_{U^T}$$

Linear Algebra Review: Matrix Positivity - Definition

Try not to define positivity using eigenvalues. (Eigenvalues don't add)

Definition 9.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Semidefinite**, denoted $P \geq 0$ if

$$x^T P x \geq 0 \qquad \text{for all } x \in \mathbb{R}^n$$

Definition 10.

A symmetric matrix $P \in \mathbb{S}^n$ is **Positive Definite**, denoted $P > 0$ if

$$x^T P x > 0 \qquad \text{for all } x \neq 0$$

- P is **Negative Semidefinite** if $-P \geq 0$
- P is **Negative Definite** if $-P > 0$
- A matrix which is neither Positive nor Negative Semidefinite is **Indefinite**

The set of positive or negative matrices is a *convex cone*.

Pleasant Properties of Positive Matrices

Lemma 11.

$P \in \mathbb{S}^n$ is positive definite if and only if all its eigenvalues are positive.

In this case, the SVD and Unitary (Schur) Diagonalization are the same.

$$\begin{bmatrix} 4 & 1 & 2 \\ 1 & 5 & 3 \\ 2 & 3 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -.37 & .82 & -.44 \\ -.58 & -.58 & -.58 \\ -.73 & .04 & .69 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 9.4 & 0 & 0 \\ 0 & 3.4 & 0 \\ 0 & 0 & 2.2 \end{bmatrix}}_{\Lambda=\Sigma} \underbrace{\begin{bmatrix} -.37 & .82 & -.44 \\ -.58 & -.58 & -.58 \\ -.73 & .04 & .69 \end{bmatrix}^T}_{U^T=V^T}$$

Fact: If T is invertible, then $P > 0$ is equivalent to $T^T P T > 0$.

- $P > 0 \to (Tx)^T P (Tx) = x^T T^T P T x > 0$
- $T^T P T > 0 \to (T^{-1}x)^T T^T P T (T^{-1}x) = x^T P x > 0$

Fact: A Positive Definite matrix is invertible: $P^{-1} = U \Sigma^{-1} U^T$.

Fact: The inverse of a positive definite matrix is positive definite: $\Sigma^{-1} > 0$

Fact: For any $P > 0$, there exists a positive square root, $P^{\frac{1}{2}} > 0$ where $P = P^{\frac{1}{2}} P^{\frac{1}{2}}$.

$$P^{\frac{1}{2}} = U \Sigma^{\frac{1}{2}} U^T > 0 \qquad P^{\frac{1}{2}} P^{\frac{1}{2}} = U \Sigma^{\frac{1}{2}} U^T U \Sigma^{\frac{1}{2}} U^T = U \Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} U^T = U \Sigma U^T = P$$

Building Linear Matrix Inequalities

Fact: $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, implies both $X > 0$ and $Z > 0$.

Proof: True since $\begin{bmatrix} 0 \\ z \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} > 0$ and $\begin{bmatrix} x \\ 0 \end{bmatrix}^T \begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix} > 0$

Fact: $X > 0$ and $Z > 0$ is equivalent to $\begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} > 0$.

Proof: True since $x^T X x > 0$ and $z^T Z z > 0$ implies

$$\begin{bmatrix} x \\ z \end{bmatrix}^T \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = x^T X x + z^T Z z > 0.$$

Theorem 12 (Schur Complement).

$$\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X & 0 \\ 0 & Z - Y^T X^{-1} Y \end{bmatrix} > 0 \Leftrightarrow \begin{bmatrix} X - Y Z^{-1} Y^T & 0 \\ 0 & Z \end{bmatrix} > 0$$

Diagonal Dominance: If X and Z are big enough, Y doesn't matter.

Leftover Factoids on Positive Matrices

Things which are true:

- $P > 0$ and $Q > 0$ implies $P + Q > 0$.
- $P > 0$ implies $\mu P > 0$ for any positive scalar $\mu > 0$.
- $M^T M \geq 0$ for any matrix, M.
- $P > 0$ implies $M^T P M > 0$ if nullspace of M is empty.

Things which are NOT TRUE (Fallacies):

- $P > 0$ implies $TPT^{-1} > 0$.
- $P > 0$ and $Q > 0$ implies $PQ > 0$.
- $P > 0$ implies $T^T P + PT > 0$
- $P \geq 0$ implies P invertible.
- A has positive eigenvalues implies $A + A^T > 0.([1\ -3; 0\ 1])$

Semidefinite Programming - Dual Form

$$\begin{array}{lll} \text{minimize} & \operatorname{trace} CX & \\ \text{subject to} & \operatorname{trace} A_i X = b_i & \text{for all } i \\ & X \succeq 0 & \end{array}$$

- The variable X is a symmetric matrix
- $X \succeq 0$ is another way to say X is positive semidefinite
- The feasible set is the intersection of an affine set with the *positive semidefinite cone*

$$\left\{ X \in \mathbb{S}^n \mid X \succeq 0 \right\}$$

Recall $\operatorname{trace} CX = \sum_{i,j} C_{i,j} X_{j,i}$.

SDPs with Explicit Variables - Primal Form

We can also explicitly parametrize the affine set to give

$$\begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & F_0 + x_1 F_1 + x_2 F_2 + \cdots + x_n F_n \preceq 0 \end{array}$$

where $F_0, F_1, \ldots, F_n$ are symmetric matrices.

The inequality constraint is called a *Linear Matrix Inequality (LMI)*; e.g.,

$$\begin{bmatrix} x_1 - 3 & x_1 + x_2 & -1 \\ x_1 + x_2 & x_2 - 4 & 0 \\ -1 & 0 & x_1 \end{bmatrix} \preceq 0$$

which is equivalent to

$$\begin{bmatrix} -3 & 0 & -1 \\ 0 & -4 & 0 \\ -1 & 0 & 0 \end{bmatrix} + x_1 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \preceq 0$$

Linear Matrix Inequalities

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.
Common Form:

$$\text{Find } X:$$
$$\sum_i A_i X B_i + Q > 0$$

There are several very efficient **LMI/SDP Solvers** which interface with YALMIP:

- **SeDuMi**
  - Fast, but somewhat unreliable.
  - Link: http://sedumi.ie.lehigh.edu/
- **LMI Lab** (Part of Matlab's Robust Control Toolbox)
  - Universally disliked, but you already have it.
  - Link: http://www.mathworks.com/help/robust/lmis.html
- **MOSEK** (commercial, but free academic licenses available)
  - Probably the most reliable
  - Link: https://www.mosek.com/resources/academic-license