Systems Analysis and Control

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Lecture 9: Dynamics of Response, Continued

Overview

In this Lecture, you will learn:

Characteristics of the Response

Complex Poles

- Rise Time
- Settling Time
- Percent Overshoot

Performance Specifications

- Geometric Pole Restrictions

Complex Poles

Recall: Damping and Frequency

3 Different Forms: Each with $y_{ss} = 1$

$$\hat{G}(s) = \frac{b}{s^2 + as + b}$$

$$\hat{G}(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\hat{G}(s) = \frac{\sigma^2 + \omega_d^2}{s^2 - 2\sigma s + (\sigma^2 + \omega_d^2)} = \frac{\sigma^2 + \omega_d^2}{(s - \sigma + \imath\omega_d)(s - \sigma - \imath\omega_d)}$$

Two Complex Poles at $s = \sigma \pm \imath\omega_d$.

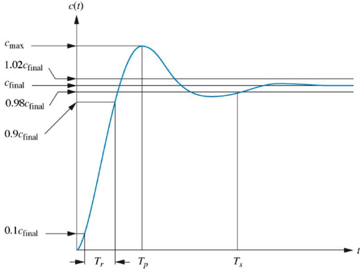
- **Damped Frequency:** ω_d
  - $\omega_d = \sqrt{b - \frac{a^2}{4}} = \omega_n\sqrt{1 - \zeta^2}$
- **Decay Rate:** σ
  - $\sigma = -\frac{a}{2} = \zeta\omega_n$
- **Natural Frequency:** ω_n
  - $\omega_n = \sqrt{b} = \sqrt{\sigma^2 + \omega_d^2}$
- **Damping Ratio:** ζ
  - $\zeta = \frac{a}{2\omega_n} = \frac{|\sigma|}{\omega_n}$

Recall:

- T_r is the time to go from $.1$ to $.9$ of the final value.

Suppose there were no damping ($\sigma = 0$). Then the normalized solution is

$$y(t) = 1 - \cos(\omega t)$$



The points t_1 and t_2 occur at

$$\omega t_1 = \cos^{-1}(.9) = .45, \qquad \omega t_2 = \cos^{-1}(.1) = 1.47$$

So that

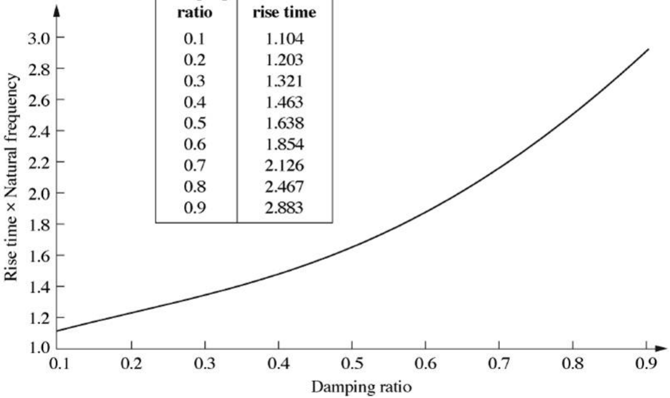
$$T_r = t_2 - t_1 = \frac{1.02}{\omega}$$

Complex Poles

Rise Time

However, with **Damping**, the situation changes.

- Damping *slows* the response.
- No good metric for rise time of a complex pole.
- When $\zeta = .5$, $T_r \cong \frac{1.8}{\omega_n}$



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

Complex Poles

Settling Time

Recall the step response

$$y(t) = 1 - e^{\sigma t}\left(\cos(\omega_d t) - \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

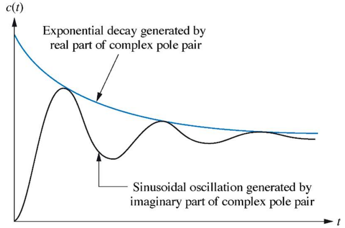
Oscillations are confined within an *Exponential Envelope*

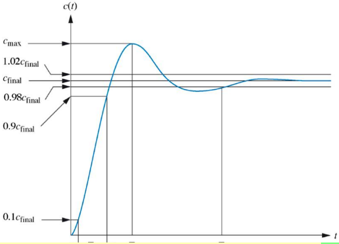
- The exponential envelop decays at rate σ.
- **Settling Time** for a complex pole is given by contraction of the envelope

$$\|1 - y(t)\| \le e^{\sigma t}$$

- The same as for a real pole at $s = \sigma$, the settling time is

$$T_s = \frac{4.6}{-\sigma}$$

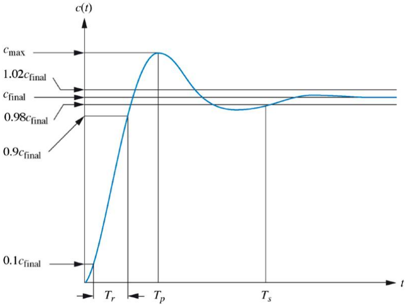




Complex Poles

Time to Peak

For complex poles, we can be interested in when the maximum motion happens.



Definition 1.

The **Peak Time**, T_p is time at which the signal obtains its maximum value.

To calculate T_p, we must find when

$$y(t) = 1 - e^{\sigma t}\left(\cos(\omega_d t) - \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)$$

Achieves its maximum.

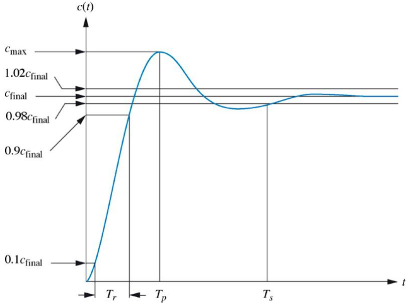
Peak time is an indication of rise time, but less reliable

- Better to use rise time if available.
- May be useful for timing.

To find the extrema, we set $\dot{y}(t) = 0$, where

$$\dot{y}(t) = \frac{\omega_n^2}{\omega_d} e^{\sigma t} \sin \omega_d t$$

So $\dot{y}(t) = 0$ when $t = \frac{n\pi}{\omega_d}$.



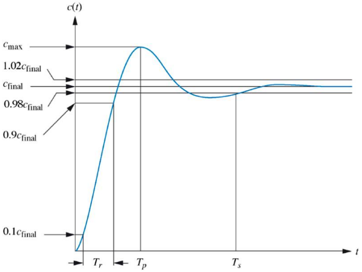
Because of the exponential envelope, the first peak will always be largest. Thus

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Unique to complex poles is the concept of overshoot:

Definition 2.

The **Percent Overshoot**, M_p is the peak value of the signal, as a percentage of steady-state.



To calculate M_p, we must find the maximum of $y(t)$.

- The value at time $T_p = \frac{\pi}{\omega_d}$.

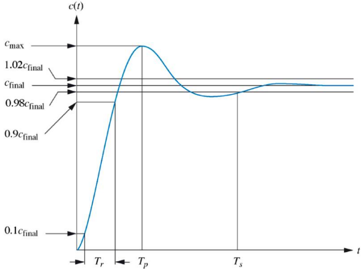
Systems with high overshoot may move violently before settling.

- May diverge from acceptable path
- Can cause crashes, un-modeled dynamics, etc.

To calculate M_p, we must find the maximum of $y(t)$.

- The value at time $T_p = \frac{\pi}{\omega_d}$.

Since we already have T_p,



$$\begin{aligned} M_p &= y(T_p) - 1 \\ &= -e^{\sigma T_p}\left(\cos(\omega_d T_p) - \frac{\sigma}{\omega_d}\sin(\omega_d T_p)\right) \\ &= -e^{\sigma T_p}\left(\cos(\pi) - \frac{\sigma}{\omega_d}\sin(\pi)\right) \\ &= e^{\sigma T_p} = e^{\frac{\pi\sigma}{\omega_d}} = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \end{aligned}$$

$$M_p = e^{\frac{\pi\sigma}{\omega_d}} = e^{\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

M_p depends only on ζ.

Estimate:

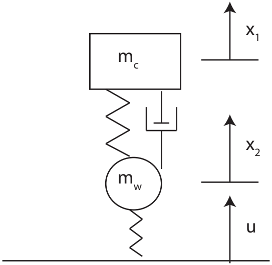
- Rise/Peak Time
- Percent Overshoot

Complex Poles

Numerical Example

Lets look at the suspension problem
Open Loop:

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 3s^2 + s + 1}$$



The poles are:

- $p_{1,2} = -.9567 \pm 1.2272\imath$
- $p_{3,4} = -.0433 \pm .6412\imath$

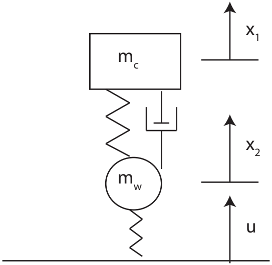
Because there are two sets of poles, we should consider both.

$$\sigma_1 = -.9567 \qquad \omega_{d,1} = 1.2272$$
$$\sigma_2 = -.0433 \qquad \omega_{d,2} = .6412$$

$$\omega_{n,1} = \sqrt{\sigma_1^2 + \omega_{d,1}^2} = 1.5561 \qquad \zeta_1 = \frac{|\sigma|}{\omega_{n,1}} = .6148$$
$$\omega_{n,2} = .6427 \qquad \zeta_2 = .0674$$

Closed Loop: Let $k = 1$

$$\hat{G}(s) = \frac{s^2 + s + 1}{s^4 + 2s^3 + 4s^2 + 2s + 2}$$



The poles are:

- $p_{1,2} = -.8624 \pm 1.4391\imath$
- $p_{3,4} = -.1376 \pm .8316\imath$

Consider both sets of poles.

$$\sigma_1 = -.8624 \qquad \omega_{d,1} = 1.4391$$
$$\sigma_2 = -.1376 \qquad \omega_{d,2} = .8316$$

The natural frequency and damping ratios are

$$\omega_{n,1} = 1.6777 \qquad \zeta_1 = .5140$$

$$\omega_{n,2} = .8429 \qquad \zeta_2 = .1632$$

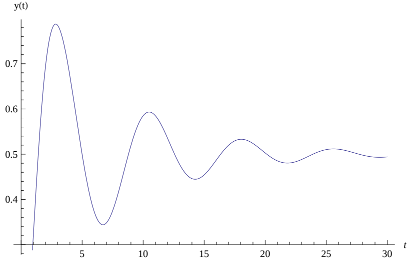


Figure: Closed Loop

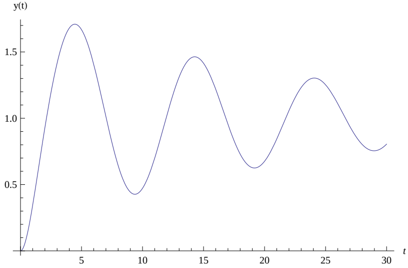


Figure: Open Loop

Overshoot: Open Loop (easiest to use σ and ω directly)

$$M_{p,1} = e^{\frac{\pi \sigma_1}{\omega_1}} = .0864 \qquad M_{p,2} = .8088$$

Overshoot: Closed Loop

$$M_{p,1} = .152 \qquad M_{p,2} = .5946$$

A substantial improvement in performance.

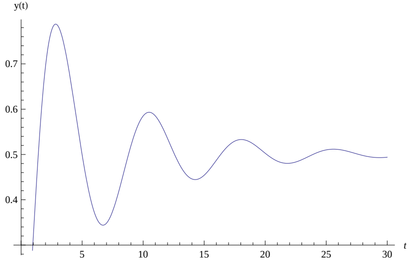


Figure: Closed Loop

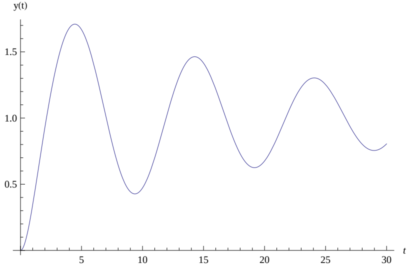


Figure: Open Loop

Settling Time: Open Loop

$$T_{s,1} = \frac{4.6}{-\sigma} = 4.81 \qquad T_{s,2} = 106.23$$

Settling Time: Closed Loop

$$T_{s,1} = 5.33 \qquad T_{s,2} = 33.43$$

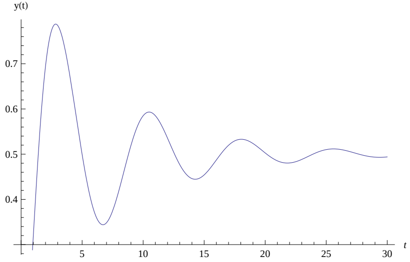


Figure: Closed Loop

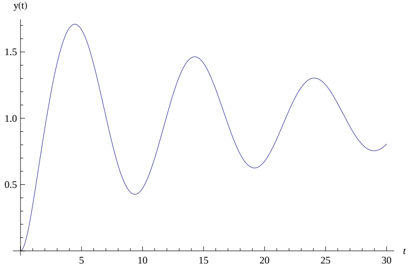


Figure: Open Loop

Peak Time: Open Loop

$$T_{p,1} = \frac{\pi}{\omega} = 2.56 \qquad T_{p,2} = 4.90$$

Peak Time: Closed Loop

$$T_{p,1} = 2.18 \qquad T_{p,2} = 3.78$$

Complex Poles

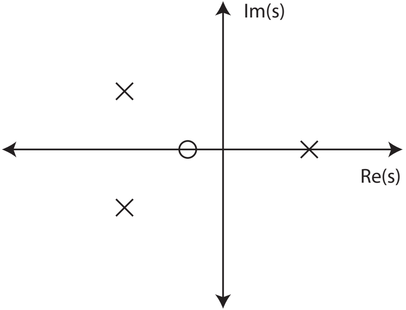
Pole Locations

As we see, *pole locations* predict almost all control objectives.

- Except Steady-State Error

In the numerical example, we showed that dynamic response **improved** with feedback.

- We found the poles before and after feedback.
- However, our choice of $k = 1$ was just a guess.



Question: How do we design controllers?
Answer: The goal of a controller is to change the location of the poles.

Performance Specifications create Geometric Constraints in the Complex Plane.

Complex Poles

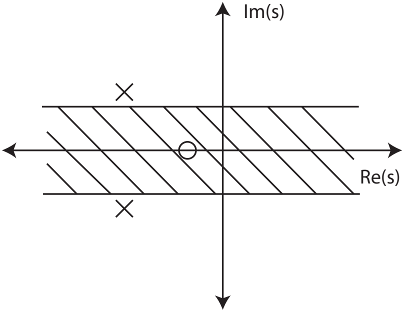
Peak Time

Suppose we are given performance specifications

- T_p, T_r, M_p etc.

we can solve instead for limits on pole locations σ, ω.

Desired Peak Time: $T_{p,desired}$.



We generally want a fast response, so we require $T_p < T_{p,desired}$.

$$\frac{\pi}{\omega_d} = T_p < T_{p,desired}$$

Thus we require

$$\omega_d > \frac{\pi}{T_{p,desired}}$$

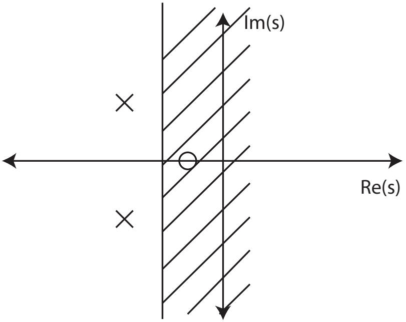
The geometric interpretation is that the imaginary part be sufficiently large.

Desired Settling Time: $T_{s,desired}$.

We generally want quick convergence.

- So we require $T_s < T_{s,desired}$.

$$\frac{4.6}{-\sigma} = T_s < T_{s,desired}$$



Thus we require

$$\sigma < -\frac{4.6}{T_{s,desired}}$$

The geometric interpretation is that the real part be sufficiently negative.

Desired Overshot: $M_{p,desired}$.

We don't like hitting things,

- So we don't want overshoot $M_p < M_{p,desired}$.

$$e^{\frac{\pi\sigma}{\omega_d}} = M_p < M_{p,desired}$$

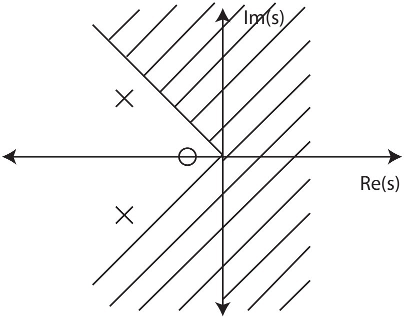
Thus we require

$$\frac{\pi\sigma}{\omega_d} < \ln(M_{p,desired})$$

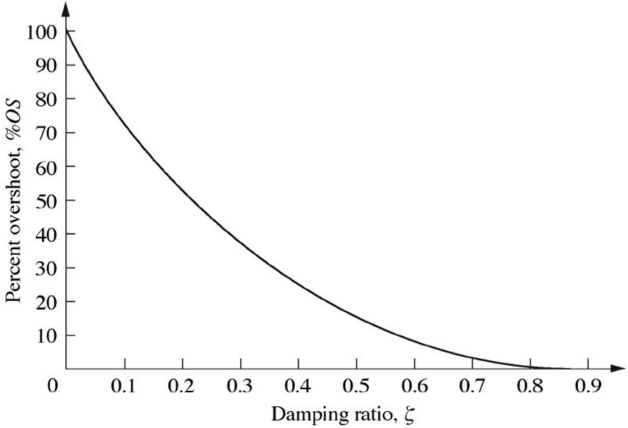
or

$$\sigma < \frac{\ln(M_{p,desired})}{\pi}\omega_d$$

A linear relation between σ and ω.



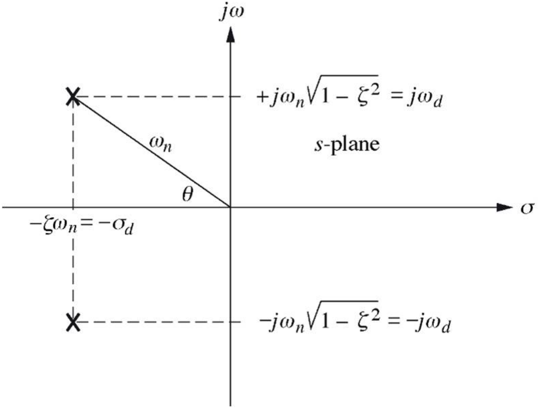
Alternatively, $M_{p,desired}$ is determined by damping ratio alone:



Invert:

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta_2}}}$$

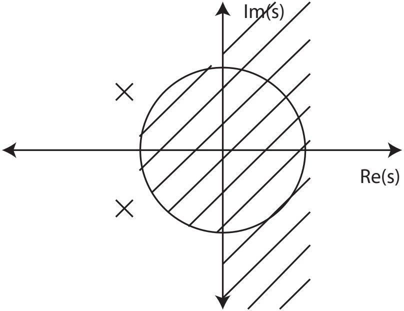
A fixed $\zeta_{desired}$ defines an angle in the complex plane.



$$\theta = \frac{\pi}{2} - \sin^{-1}(\zeta_{desired})$$

Although find rise time is complicated, for $\zeta = .5$, we can approximate

$$T_r \cong \frac{1.8}{\omega_n}$$



Desired Rise Time: $T_{r,desired}$.

- We generally want quick convergence.
- We require $T_r < T_{r,desired}$.

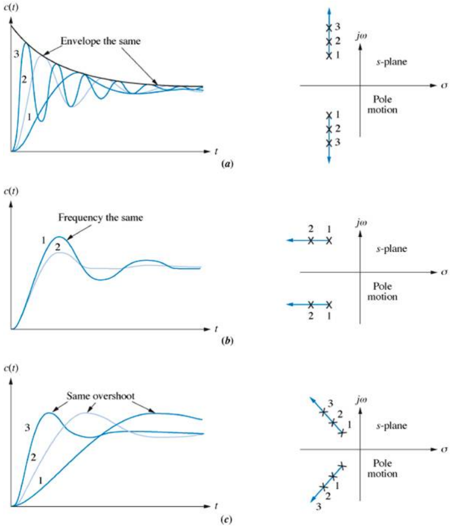
$$\frac{1.8}{\omega_n} = T_r < T_{r,desired}$$

Thus we require

$$\omega_n > \frac{1.8}{T_{r,desired}}$$

Recall that $\omega_n = \sqrt{\sigma^2 + \omega^2}$, so the geometric interpretation is a circle: $\|s\| > \frac{1.8}{T_{r,desired}}$.

Complex Poles



Mostly, we have several constraints

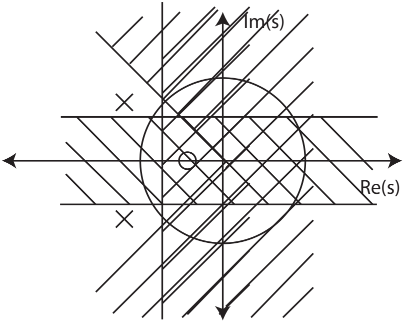
$$\omega_d > \frac{\pi}{T_{p,desired}}$$

$$\sigma < -\frac{4}{T_{s,desired}}$$

$$\sigma < \frac{\ln(M_{p,desired})}{\pi}\omega_d$$

$$\omega_n > \frac{1.8}{T_{r,desired}}$$

Any pole locations not prohibited are allowed.



Complex Poles

Multiple Constraints

High Performance Aircraft:

- **Overshoot:** Reduce overshoot to less than 5%.

$$M_{p,desired} = .05$$

$$\frac{\sigma}{\omega_d} < \frac{\ln(M_{p,desired})}{\pi} = -.9535$$

- An difficult requirement to meet?
- **Rise Time:** Quick response is critical. Limit Rise Time to 1s or less

$$T_{r,desired} = 1$$

$$\omega_n > \frac{1.8}{T_{r,desired}} = 1.8$$

- **Settling Time:** Limit settling time to $T_{s,desired} = 3.5s$.

$$T_{s,desired} = 3.5s$$

$$\sigma < -\frac{4.6}{T_{s,desired}} = -1.333$$

Complex Poles

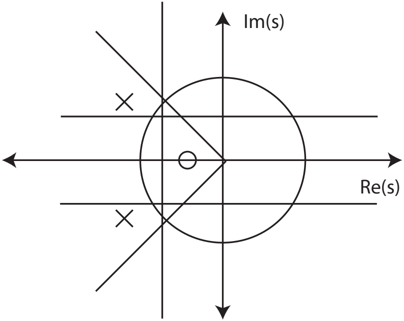
Multiple Constraints

We have the required
Overshoot: Along a line of about

$$\theta = \text{atan}\left(\frac{\omega_d}{\sigma}\right) = \text{atan}\left(\frac{1}{-.9535}\right) = 46^\circ$$

Which means a damping ratio of
$\zeta = \sin(90 - 46^\circ) = .69$.

- Roughly $\omega_d = \sigma$



To satisfy T_s, $\sigma < -1.333$, so lets try $\sigma = -1.5$.

- Then $\omega_d < 1.5$
- Choose $\omega_d = 1.4$

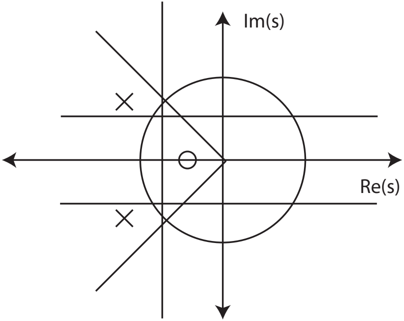
Complex Poles

Multiple Constraints

For T_r, need $\omega_n > 1.8$. However,

$$\omega_n = \sqrt{\omega_d^2 + \sigma^2} = 2.05$$

So rise time is already satisfied.



If we need to decrease rise time, increase omega, while staying on lines of constant overshoot

Complex Poles

Missile Example

Estimate Performance Specs:

Summary

What have we learned today?

In this Lecture, you will learn:

Characteristics of the Response

Complex Poles

- Rise Time
- Settling Time
- Percent Overshoot

Performance Specifications

- Geometric Pole Restrictions

Next Lecture: Designing Controllers