Spacecraft Dynamics and Control

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Lecture 13: The Effect of a Non-Spherical Earth

Introduction

In this Lecture, you will learn:

The Non-Spherical Earth

- The gravitational potential
- Expression in the R-T-N frame
- Perturbations
  - Periodic
  - Secular

Mission Planning

- Sun-Synchronous Orbits
- Frozen Orbits
- Critical Inclination

Recall The Perturbation Equations

$$\vec{F}_{disturbance} = R\hat{e}_R + T\hat{e}_T + N\hat{e}_N$$

Semi-major Axis

$$\dot{a} = 2\sqrt{\frac{a^3}{\mu(1-e^2)}}\left[eR\sin f + T(1+e\cos f)\right]$$

Eccentricity:

$$\dot{e} = \sqrt{\frac{a(1-e^2)}{\mu}}\left[R\sin f + T(\cos f + \cos E_{ecc})\right]$$

Inclination:

$$\frac{d}{dt}i = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\cos(\omega+f)}{1+e\cos f}$$

RAAN:

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}}\frac{N\sin(\omega+f)}{\sin i(1+e\cos f)}$$

Argument of Perigee:

$$\dot{\omega} = -\dot{\Omega}\cos i + \sqrt{\frac{a(1-e^2)}{e^2\mu}}\left(-R\cos f + T\frac{(2+e\cos f)\sin f}{1+e\cos f}\right)$$

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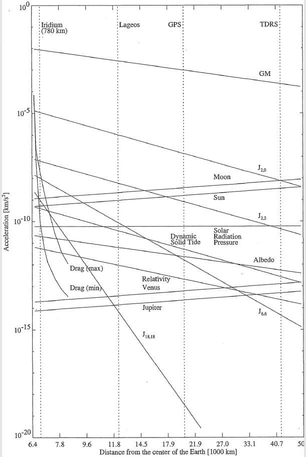
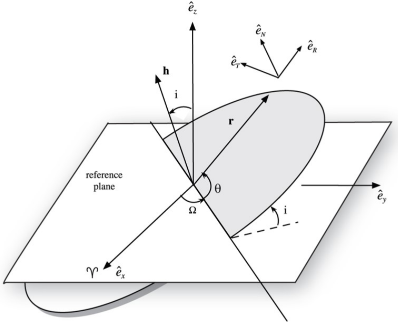
Drag is only significant in LEO. Above LEO, J_2 is more important (From Gil/Montenbruck).



$$\vec{F} = N\hat{e}_N + R\hat{e}_R + T\hat{e}_T$$

Satellite-Normal CS (R-T-N):

- $\hat{e}_R$ points along the earth $\rightarrow$ satellite vector.
- $\hat{e}_N$ points in the direction of $\vec{h}$
- $\hat{e}_T$ is defined by the RHR
  - $\hat{e}_T \cdot v > 0.$



Recall that gravity for a point mass is

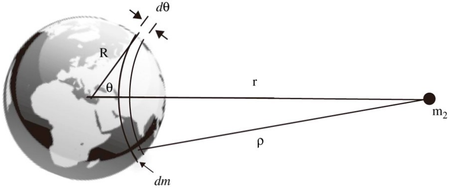
$$\vec{F} = -\mu \frac{\vec{r}}{\|\vec{r}\|^2}$$

Gravity force derives from the potential field.

$$\vec{F} = \nabla U$$

To find U, we integrate

$$dU = -\frac{2\pi R^2 G \sigma m_2 \sin\theta}{\rho} d\theta$$



For a uniform spherical mass,

- There is symmetry about the line $\vec{r}_{12}$.
- The point-mass approximation holds.

The Non-spherical Earth

A Distorted Potential Field

For a spherical earth, dU is symmetric

$$dU = -\frac{2\pi R^2 G\sigma m_2 \sin\theta}{\rho} d\theta$$

The actual gravity field

- Is not precisely spherical.
- density varies throughout the earth.

The result is a distorted potential field.

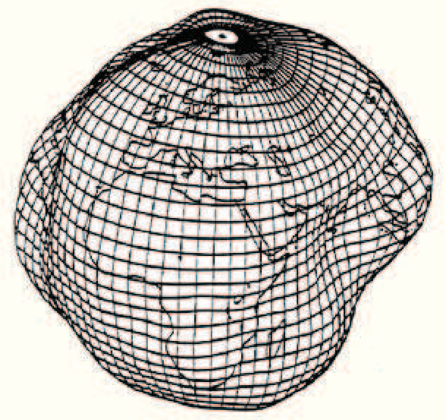


Figure: The geoid, 15000:1 scale

The Non-spherical Earth
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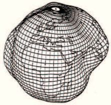
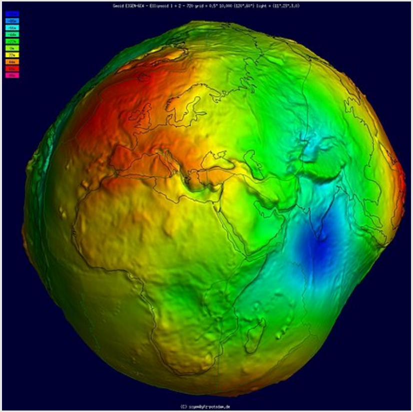


Figure: The geoid, 15000:1 scale

The Geoid is the surface of gravitational and centrifugal equipotential

- Describes the surface of the ocean if it covered the entire earth



Of course, for orbit perturbations, we exclude the centrifugal potential energy.

Socrates: So how do we derive the potential field?

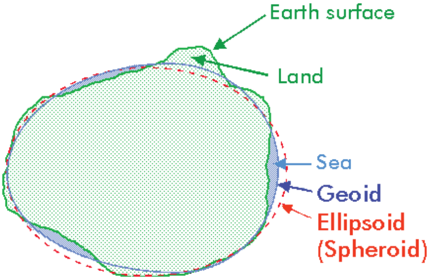
Tycho Brahe: We measure it!!!

Definition 1.

Physical Geodesy is the study of the gravitational potential field of the earth.

Definition 2.

The **Geoid** is equipotential surface which coincides with the surface of the ocean.



NASA's Geodesy Video

The first measurements of the earth were made by Eratosthenes

- Third Librarian of Library of Alexandria (240BC).
- Invented "Geography"
- Invented Latitude and Longitude
  - The difference in angle between high noon at two points on the earth.
  - Measured using deep wells
- Measured the circumference of the earth.
- May have starved to death.

Development of Geodesy

Eratosthenes of Cyrene(276-195 BC)

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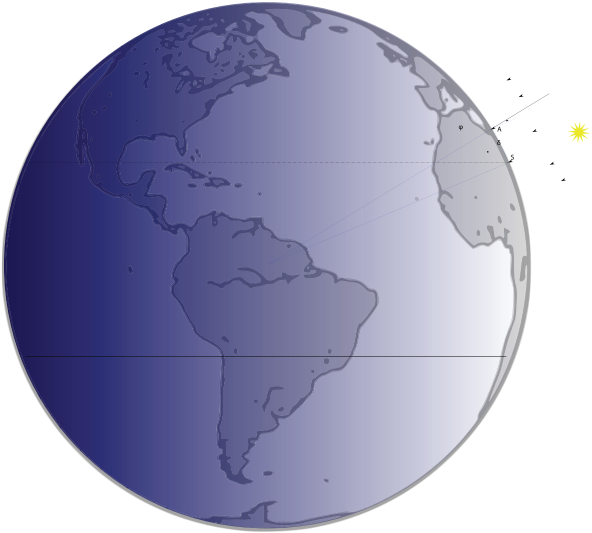
- Starved himself to death after going blind and therefore being unable to read.

Geometry of Eratosthenes



The Non-spherical Earth

A Distorted Potential Field

Question: So how do we measure the potential field of the earth?

LAGEOS: Laser Geodynamics Satellites

1. Precisely measure the trajectory of a satellite as it orbits the earth
2. Account for drag, third-body dynamics, etc.
3. Remaining perturbation must be causes by gravitational potential

The orbits of the LAGEOS satellites are measured precisely by laser reflection.

Note: Only measures potential along path of the orbit.

- We must observe for a long time to get comprehensive data.

$$a = 12,278km, \quad i = 109.8^\circ, 52.6^\circ,$$ Launch dates: 1976, 1992

The Non-spherical Earth
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$a = 12,278km, \quad i = 109.8°, 52.6°,$ Launch dates: 1976, 1992

- Does not measure potential field directly.
- Requires this field to be fit to the trajectory data.

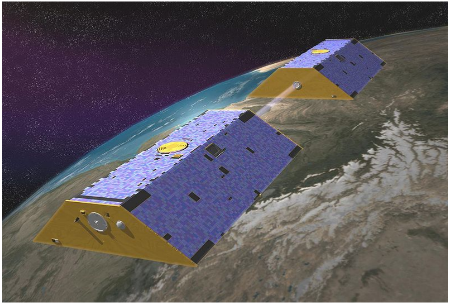
The Non-spherical Earth

Measuring satellite positions from earth is inaccurate.

- Atmospheric Distortion

GRACE (2002):

1. Measure the relative position of two adjacent satellites
2. Relative motion yields gradient of the potential field
3. Allows direct reconstruction of $U(\vec{r})$.



Less fancy methods:

- Survey markers
- Altimetry
- Ocean level variation

$a = 6700km$, $i = 90^\circ$

2025-04-10

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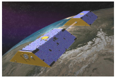

Less fancy methods:

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Data from GRACE

Ocean surface equivalent

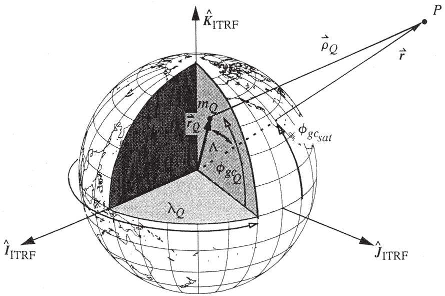
The Non-spherical Earth

Question: So what is $U(\vec{r})$? (Needed to compute $\vec{F} = \nabla U$)
Response: Too much data to write as a function.

In order to be useful, we match the data to a few basis functions.

Coordinates: Express position using ϕ_{gc}, λ, r.

- ϕ_{gc} is declination from equatorial plane.
- λ is right ascension, measured from Greenwich meridian.
- r is radius



We will have a function of form

$$U(\phi_{gc}, \lambda, r)$$

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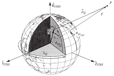


We will have a function of form

$$U(\phi_{gc}, \lambda, r)$$

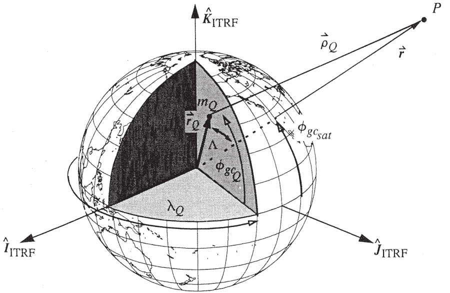
Note that U will be defined in ECEF coordinates.

- We will need to change to ECI and ultimately RTN coordinates in order to apply the orbit perturbation equations.
- This is one of those cases where RTN is not the natural coordinate system for the force.

The Harmonics

The potential has the form

$$U(\phi_{gc}, \lambda, r) = \frac{\mu}{r} + U_{zonal}(r, \phi_{gc}) + U_{sectorial}(r, \lambda) + U_{tesseral}(r, \phi_{gc}, \lambda)$$

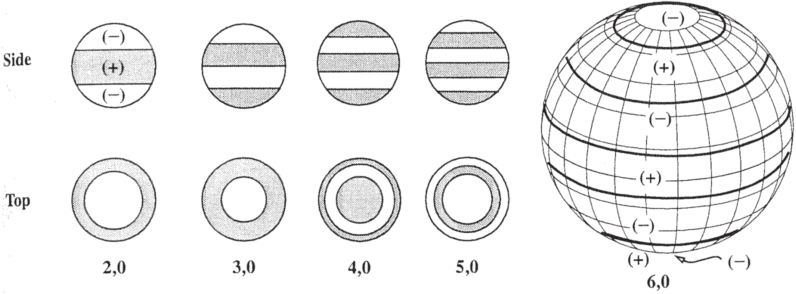


Actually, $U_{sectorial}$ varies with ϕ_{gc}, but not "harmoniously".

The Zonal Harmonics

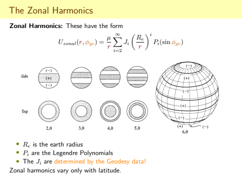
Zonal Harmonics: These have the form

$$U_{zonal}(r,\phi_{gc}) = \frac{\mu}{r}\sum_{i=2}^{\infty} J_i \left(\frac{R_e}{r}\right)^i P_i(\sin\phi_{gc})$$



- R_e is the earth radius
- P_i are the Legendre Polynomials
- The J_i are determined by the Geodesy data!

Zonal harmonics vary only with latitude.



Technically, the zonal harmonics are only the $P_i(\sin\phi_{gc})$ terms where the P_i are the Legendre polynomials

$$P_n(x) = P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

This is Rodrigues' formula

- This is a bit confusing, since, e.g.

$$P_1(\sin\phi) = \cos\phi$$

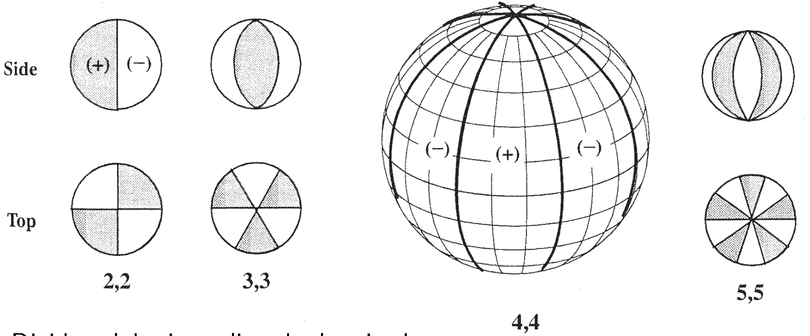
$$P_2(\sin\phi) = 3\cos^2\phi$$

- What is even more confusing is some texts (e.g. Curtis) measure $\phi = 90 - \phi_{gc}$.
- Then $\sin\phi_{gc}$ becomes $\cos\phi$.

The Sectorial Harmonics

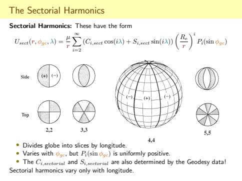
Sectorial Harmonics: These have the form

$$U_{sect}(r,\phi_{gc},\lambda)=\frac{\mu}{r}\sum_{i=2}^{\infty}\left(C_{i,sect}\cos(i\lambda)+S_{i,sect}\sin(i\lambda)\right)\left(\frac{R_e}{r}\right)^i P_i(\sin\phi_{gc})$$



- Divides globe into slices by longitude.
- Varies with ϕ_{gc}, but $P_i(\sin\phi_{gc})$ is uniformly positive.
- The $C_{i,sectorial}$ and $S_{i,sectorial}$ are also determined by the Geodesy data!

Sectorial harmonics vary only with longitude.



Many texts ignore the Sectorial and Tesseral Harmonics

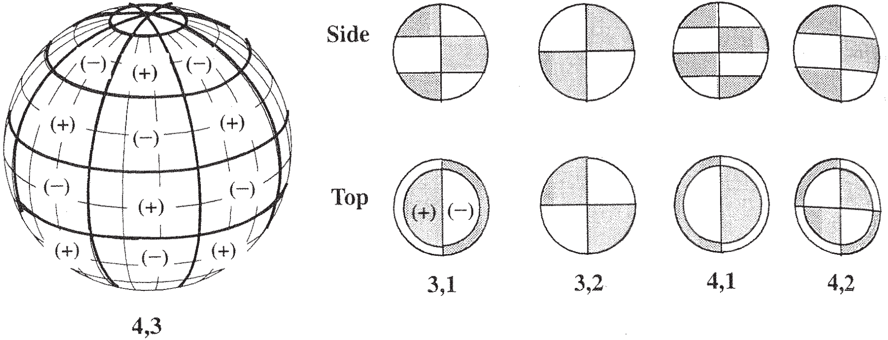
- The effect often appears random/hard to predict. Not much secular perturbation
- The exception to this is repeating ground tracks.

If interested, "Satellite Orbits" by Gil and Montenbruck has all the dynamics well-explained.

The Tesseral Harmonics

These have the form

$$U_{tesseral}(r, \phi_{gc}, \lambda) = \frac{\mu}{r} \sum_{i,j=2}^{\infty} \left(C_{i,j} \cos(i\lambda) + S_{i,j} \sin(i\lambda)\right) \left(\frac{R_e}{r}\right)^i P_{i,j}(\sin \phi_{gc})$$



- Divides globe into slices by longitude and latitude.
- The $C_{i,j}$ and $S_{i,j}$ are also determined by the Geodesy data!

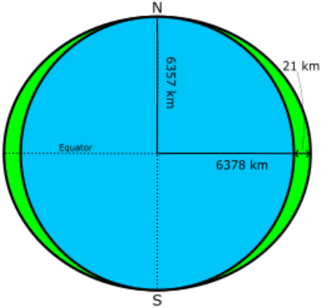
The J2 Perturbation

For simplicity, we ignore all harmonics except the first zonal harmonic.

$$\Delta U_{J2}(r, \phi_{gc}) = -\frac{\mu}{r} J_2 \left(\frac{R_e}{r}\right)^2 \left[\frac{3}{2}\sin^2(\phi_{gc}) - \frac{1}{2}\right]$$

This corresponds to a single band about the equator.

- The earth is 21 km wider than it is tall.
- A flattening ratio of $\frac{1}{300}$.
- $J_2 = .0010826$
- $J_3 = .000002532$
- $J_4 = .000001620$



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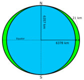

Image credit: `https://ai-solutions.com/_freeflyeruniversityguide/j2_perturbation.htm`

The J2 Perturbation

Defined in the Wrong Coordinate System

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- Expressed in the ECI Frame (same as ECEF here)
- Since $\sin \phi_{gc} = \frac{z}{r}$,

$$\Delta U_{J2}(r, \phi_{gc}) = -\frac{\mu}{r} \frac{J_2}{2} \left(\frac{R_e}{r}\right)^2 \left[\frac{3z^2}{r^2} - 1\right]$$

We now calculate the perturbation force as

$$\begin{aligned}\vec{F} &= -\frac{\partial U_{J2}}{\partial r}\hat{e}_R + \frac{\partial U_{J2}}{\partial z}\hat{e}_z \\ &= -\mu J_2 R_e^2 \left[\frac{3z}{r^5}\hat{e}_z + \left(\frac{3}{2r^4} - \frac{15z^2}{2r^6}\right)\hat{e}_R\right]\end{aligned}$$

But to use our perturbation equations, we need a force expressed in the R-T-N frame.

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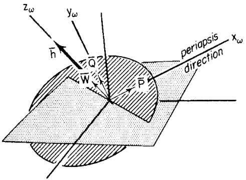
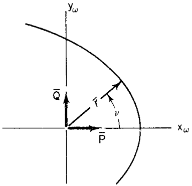
But to use our perturbation equations, we need a force expressed in the R-T-N frame.

These calculations are from the 1993 version of Prussing and Conway

Recall: Perifocal to ECI Transformation





To convert a PQW vector to ECI, we use

$$\vec{r}_{ECI} = R_3(\Omega)R_1(i)R_3(\omega)\vec{r}_{PQW} = R_{PQW\to ECI}\vec{r}_{PQW}$$

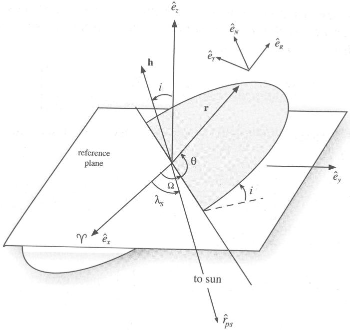
$$R_{PQW\to ECI} = \begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}\begin{bmatrix}\cos\omega & -\sin\omega & 0\\ \sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}\cos\Omega\cos\omega - \sin\Omega\sin\omega\cos i & -\cos\Omega\sin\omega - \sin\Omega\cos\omega\cos i & \sin\Omega\sin i\\ \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos i & -\sin\Omega\sin\omega + \cos\Omega\cos\omega\cos i & -\cos\Omega\sin i\\ \sin\omega\sin i & \cos\omega\sin i & \cos i\end{bmatrix}$$

The R-T-N to ECI Transformation

An additional rotation gives us the R-T-N frame.

$$R_{RTN\to ECI}$$
$$= \begin{bmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{bmatrix}$$
$$\begin{bmatrix}\cos(\omega+f) & -\sin(\omega+f) & 0\\ \sin(\omega+f) & \cos(\omega+f) & 0\\ 0 & 0 & 1\end{bmatrix}$$



$$R_{RTN\to ECI} =$$
$$\begin{bmatrix}\cos\Omega\cos\theta-\sin\Omega\sin\theta\cos i & -\cos\Omega\sin\theta-\sin\Omega\cos\theta\cos i & \sin\Omega\sin i\\ \sin\Omega\cos\theta+\cos\Omega\sin\theta\cos i & -\sin\Omega\sin\theta+\cos\Omega\cos\theta\cos i & -\cos\Omega\sin i\\ \sin\theta\sin i & \cos\theta\sin i & \cos i\end{bmatrix}$$

Where for brevity, we define $\theta=\omega+f$. This gives us the expression

$$\hat{e}_z = \sin i\sin(\omega+f)\hat{e}_R + \sin i\cos(\omega+f)\hat{e}_T + \cos i\hat{e}_N$$

The R-T-N to ECI Transformation

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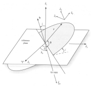

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$$\hat{e}_z = \sin i\sin(\omega+f)\hat{e}_R + \sin i\cos(\omega+f)\hat{e}_T + \cos i\hat{e}_N$$

Since the final rotation is just $R_3(f)$, we combine it with the $R_3(\omega)$ rotation so that

$$R_3(f)R_3(\omega) = R_3(f+\omega) = R_3(\theta)$$

similar to PC, page 200

Forces in the R-T-N Frame

$$\vec{F} = -\mu J_2 R_e^2 \left[\frac{3z}{r^5} \hat{e}_z + \left(\frac{3}{2r^4} - \frac{15z^2}{2r^6} \right) \hat{e}_R \right]$$

From the rotation matrices, we have that

$$\hat{e}_z = \sin i \sin(\omega + f)\hat{e}_R + \sin i \cos(\omega + f)\hat{e}_T + \cos i \hat{e}_N$$

and since

$$z = r \sin \phi_{gc} = r \sin i \sin(\omega + f),$$

this yields the disturbing force in the R-T-N frame:

$$\vec{F} = \frac{-3\mu J_2 R_e^2}{r^4} \left[\left(\frac{1}{2} - \frac{3\sin^2 i \sin^2 \theta}{2} \right) \hat{e}_R + \sin^2 i \sin\theta \cos\theta \hat{e}_T + \sin i \sin\theta \cos i \hat{e}_N \right]$$

$$= \frac{-3\mu J_2 R_e^2}{r^4} \begin{bmatrix} \frac{1}{2} - \frac{3\sin^2 i \sin^2 \theta}{2} \\ \sin^2 i \sin\theta \cos\theta \\ \sin i \sin\theta \cos i \end{bmatrix}_{RTN}$$

where again, for brevity, we use $\theta = \omega + f$

The J2 Perturbation

The primary effect of J_2 is on Ω and ω.

$$N = \frac{-3\mu J_2 R_e^2}{r^4} \sin i \sin(\omega + f) \cos i$$

We plug the force equations into the expressions for $\dot{\Omega}$ and $\dot{\omega}$

$$\dot{\Omega} = \sqrt{\frac{a(1-e^2)}{\mu}} \frac{N \sin(\omega + f)}{\sin i (1 + e \cos f)}$$

to get

$$\dot{\Omega} = -\frac{3\mu J_2 R_e^2}{h p^3} \cos i \sin^2(\omega + f) \left[1 + e \cos f\right]^3$$

This is the instantaneous rate of change.

- The angles $\theta = \omega + f$ and f will cycle from $0°$ to $360°$ over each orbit.
- We would like to know how much of that perturbation is **secular**?
- What is the average over θ?

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{\dot{\theta}} = \frac{\dot{\Omega}}{h/r^2}$$

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- We would like to know how much of that perturbation is **secular**?
- What is the average over θ?

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{\dot{\theta}} = \frac{\dot{\Omega}}{h/r^2}$$

- Recall $\dot{\theta} = h/r^2$ comes from equal area - equal time. $\dot{A} = \frac{1}{2}\dot{\theta}r^2 = h/2$.
- We use the polar equation $r = \frac{p}{1+e\cos f}$ to eliminate r.

Averaging the J2 Perturbation

Starting with

$$\frac{d\Omega}{d\theta} = \frac{\dot{\Omega}}{h/r^2} = -3J_2 \left(\frac{R_e}{p}\right)^2 \cos i \sin^2\theta \left[1 + e\cos(\theta - \omega)\right]$$

Then the average change over an orbit is

$$\left.\frac{d\Omega}{d\theta}\right|_{AV} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\Omega}{d\theta} d\theta = -\frac{3J_2}{2\pi}\left(\frac{R_e}{p}\right)^2 \cos i \int_0^{2\pi} \sin^2\theta \left[1 + e\cos(\theta - \omega)\right] d\theta$$

Now we use $\cos(\theta - \omega) = \cos\omega\cos\theta + \sin\omega\sin\theta$ to get

$$\int_0^{2\pi} \sin^2\theta \left[1 + e\cos(\theta - \omega)\right] d\theta = \int_0^{2\pi} \sin^2\theta d\theta + e\int_0^{2\pi} \sin^2\theta\cos(\theta - \omega) d\theta$$

$$= \pi + e\cos\omega \int_0^{2\pi} \sin^2\theta\cos\theta d\theta + e\sin\omega \int_0^{2\pi} \sin^3\theta d\theta$$

$$= \pi + 0 + 0 = \pi$$

Thus, we have

$$\boxed{\left.\frac{d\Omega}{d\theta}\right|_{AV} = -\frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \cos i}$$

Averaging the J2 Perturbation

Given

$$\left.\frac{d\Omega}{d\theta}\right|_{AV} = -\frac{3}{2} J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

we can use the fact that

$$n = \frac{d\theta}{dt}|_{AV}$$

to get the final expression

$$\dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

J2 Nodal Regression

Physical Explanation

The ascending node migrates opposite the direction of flight

$$\dot{\Omega}_{J2,av} = -\frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \cos i$$

The equatorial bulge produces extra pull in the equatorial plane

- Creates an averaged torque on the angular momentum vector
- Like gravity, the torque causes $\vec{h}$ to precess.
- Only depends on inclination
  - Also a and e...

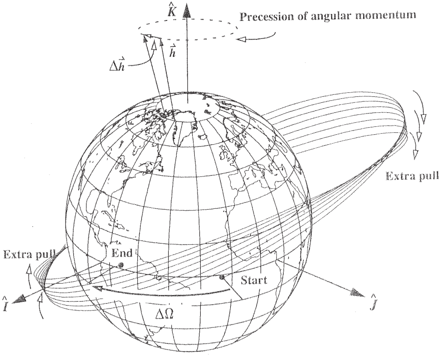


Image credit: Vallado

The nodal regression rate is often large. **Cannot Be Neglected!!!.**

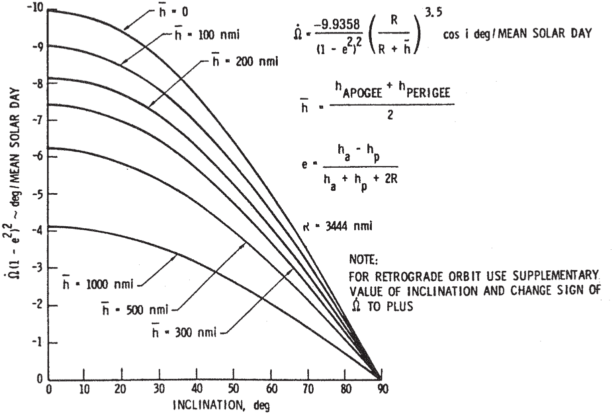


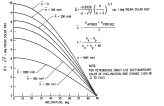
Fig. 10.2 Regression rate due to oblateness vs inclination for various values of average altitude.

Figure: Magnitude of Regression Rate vs. inclination and altitude

J2 Nodal Regression

Magnitude

The nodal regression rate is often large. **Cannot Be Neglected!!!.**



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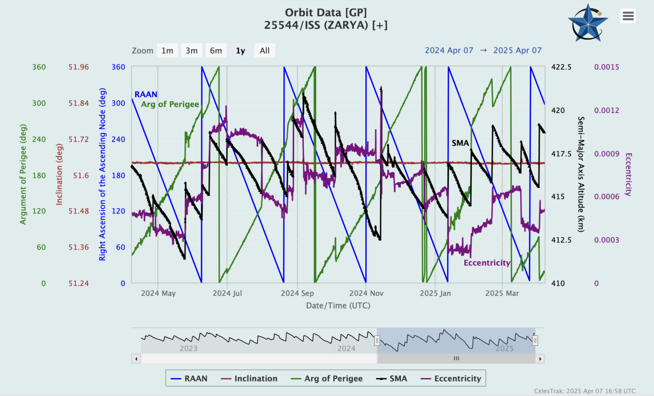


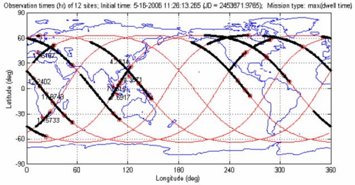
Repeating Ground Tracks

$\dot{\Omega}$ has a large effect on the design of *Repeating Ground Tracks*.

- The rotation of the earth over an orbit is

$$\Delta L_1 = -2\pi \frac{T}{T_E} = -2\pi \frac{2\pi\sqrt{\frac{a^3}{\mu}}}{T_E}$$

$T_E = 23.9345 hrs$ (1 sidereal day)



- The change in Ω over an orbit is

$$\Delta L_2 = -\frac{3\pi J_2 R_e^2 cos(i)}{a^2(1-e^2)^2}$$

- For a ground track to repeat, we require

$$j\left|\Delta L_1 + \Delta L_2\right| = \boxed{j\left|-2\pi\frac{2\pi\sqrt{\frac{a^3}{\mu}}}{T_E} - \frac{3\pi J_2 R_e^2 cos(i)}{a^2(1-e^2)^2}\right| = k2\pi}$$

  for some integers j and k.

- j is the # of orbits before repeat.
- k is the # of days (sidereal) before repeat.

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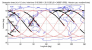

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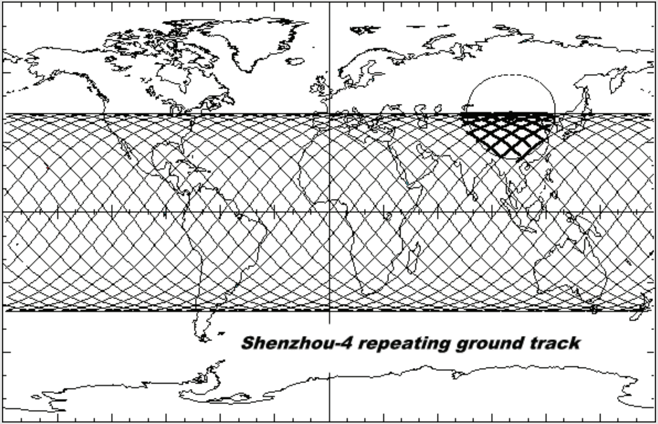

Figure: SZ-4 Repeating ground track (Sven's Space Place)

J2 Apsidal Rotation

Recall the Argument of Perigee Equation:

$$\dot{\omega} = -\dot{\Omega}\cos i + \sqrt{\frac{a(1-e^2)}{e^2\mu}}\left(-R\cos f + T\frac{(2+e\cos f)\sin f}{1+e\cos f}\right)$$

$$R = \frac{-3\mu J_2 R_e^2}{r^4}\left(\frac{1}{2} - \frac{3\sin^2 i \sin^2\theta}{2}\right), \qquad T = \frac{-3\mu J_2 R_e^2}{r^4}\sin^2 i \sin\theta\cos\theta$$

The argument of perigee (ω) is linked to RAAN (Ω). The average value is

$$\frac{d\omega}{d\theta} = -\frac{d\Omega}{d\theta}\cos i + \frac{3J_2R_e^2}{2p^2}\left[1 - \frac{3}{2}\sin^2 i\right]$$

where

$$\frac{d\Omega}{d\theta}\cos i = -\frac{3}{2}J_2\left(\frac{R_e}{p}\right)^2\cos^2 i$$
$$= -\frac{3}{2}J_2\left(\frac{R_e}{p}\right)^2(1-\sin^2 i)$$

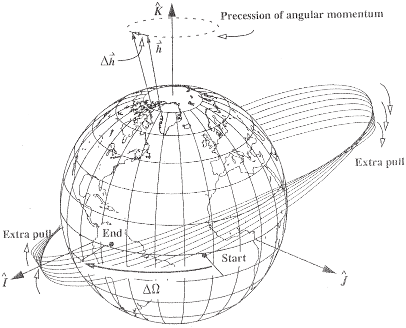

Image credit: Vallado

J2 Apsidal Rotation

Recall the Argument of Perigee Equation:

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$$R = \frac{-3\mu J_2 R_e^2}{r^4}\left(\frac{1}{2} - \frac{3\sin^2 i \sin^2\theta}{2}\right), \qquad T = \frac{-3\mu J_2 R_e^2}{r^4}\sin^2 i \sin\theta\cos\theta$$

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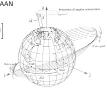


Image credit: Vallado

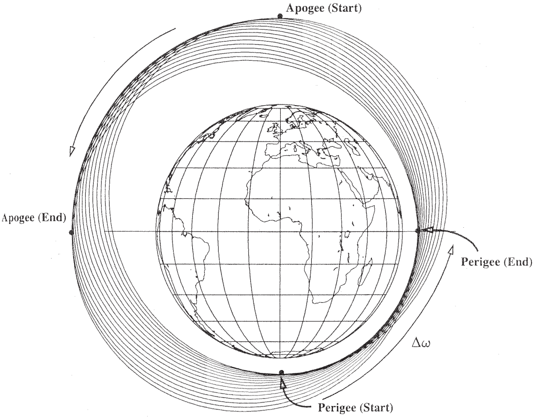
There are 3 parts acting here

- If the perigee were fixed in space, $\dot{\Omega}$ would shorten the angle to this point.
- A tangential component advances perigee
- A radial component pull perigee forward in the orbit.

J2 Apsidal Rotation

Similar to nodal regression, but perigee moves forward or backward, depending on inclination.

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$



The apsidal rotation rate is often large.

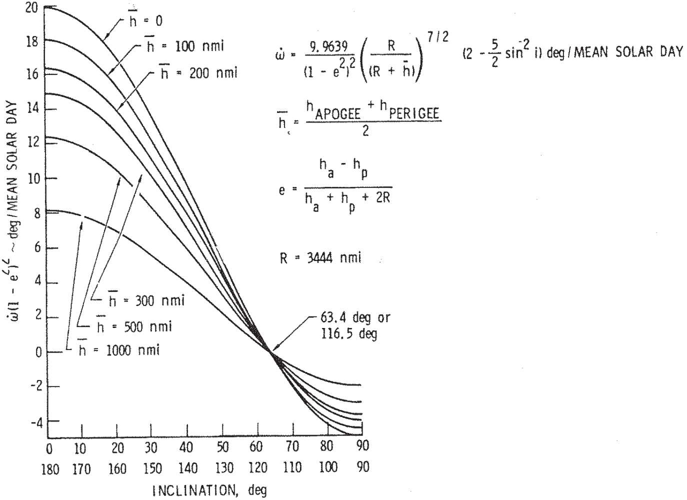


Figure: Magnitude of Regression Rate vs. inclination and altitude

The J_2 effect on other elements is usually minor. $\dot{a} \cong 0$.

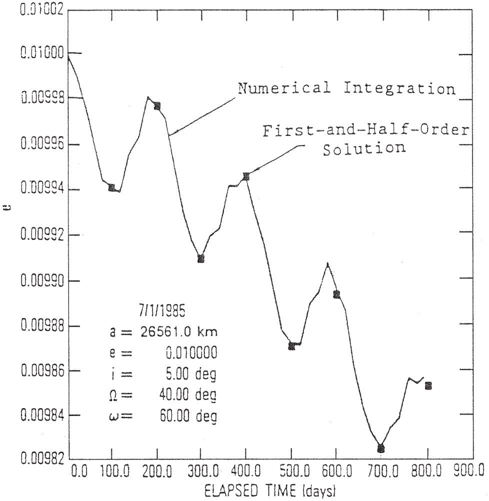


Figure: Eccentricity Change for Low-Inclination Orbit

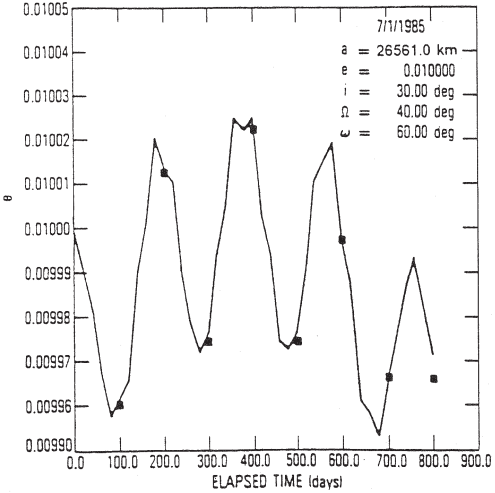


Figure: Eccentricity Change for Moderate-Inclination Orbit

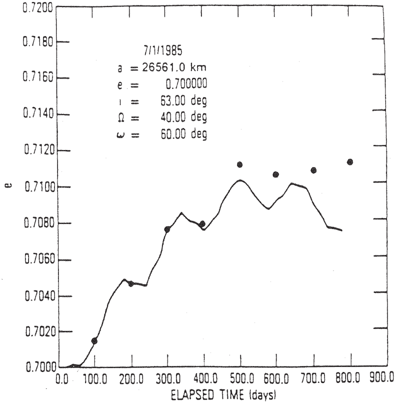


Figure: Eccentricity Change for High-Inclination Orbit

"Frozen Orbits" can be designed to minimize changes in eccentricity

- Use the J_3 perturbation (Not covered here)
- Require particular choices of e and ω

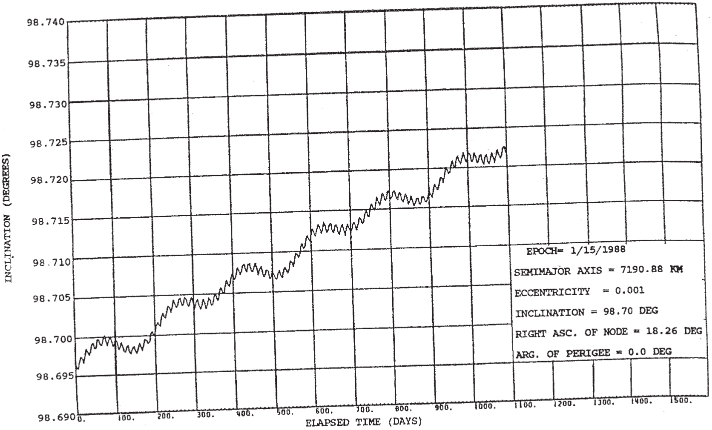


Fig. 10.6 Inclination variation without correction (5:30 orbit).

Figure: Inclination Change for Eccentric and Circular Orbits

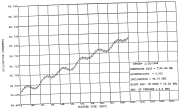


Fig. 10.6 Inclination variation without correction (5:30 orbit).

Figure: Inclination Change for Eccentric and Circular Orbits

To illustrate relative magnitude of these perturbations, for Gallileo satellites (T=14hr)

Source	acceleration $(10^{-9} m/s^2)$
Direct SRP (solar panels*)	122.0
Direct SRP (rotating bus)	9.1
Albedo	0.0–1.5
Infrared earth radiation	0.7–1.4
Antenna thrust	1.4
Thermal efects	0.1–0.7
Earth oblateness	37,600
Lunar acceleration	3300
Solar acceleration	1700
Venus accelerations	0.2
Jupiter accelerations	0.03
Higher-degree geoid potential	240
Solid earth tides	0.7
Ocean tides	0.08
General relativity (Schwarzschild)	0.3883

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$

Definition 3.

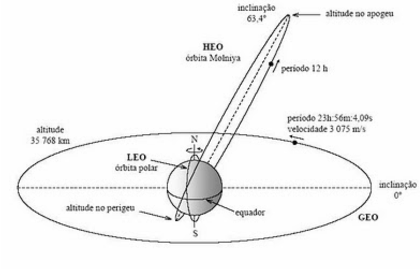
A **Critically Inclined Orbit** is one where $\dot{\omega} = 0$

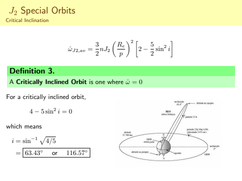
For a critically inclined orbit,

$$4 - 5\sin^2 i = 0$$

which means

$$\begin{aligned} i &= \sin^{-1}\sqrt{4/5} \\ &= \boxed{63.43^\circ \quad \text{or} \quad 116.57^\circ} \end{aligned}$$





J_2 Special Orbits
Critical Inclination

$$\dot{\omega}_{J2,av} = \frac{3}{2} n J_2 \left(\frac{R_e}{p}\right)^2 \left[2 - \frac{5}{2}\sin^2 i\right]$$

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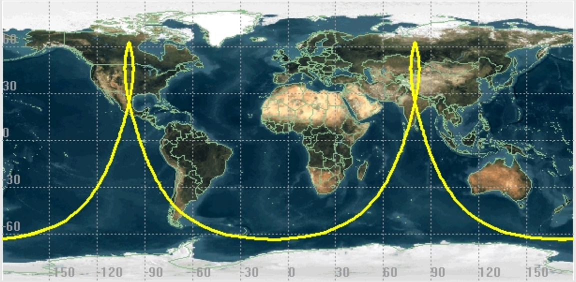


Figure: Molniya Orbit

Figure: Tundra Orbit

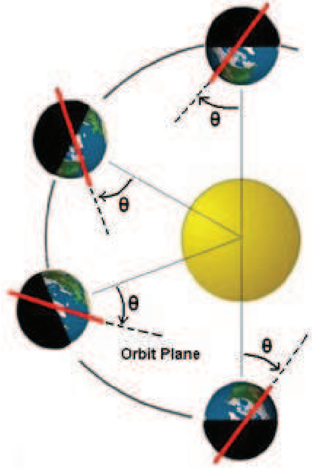
J_2 Special Orbits

Sun-Synchronous Orbits

Sun-Synchronous orbits maintain the same orientation of the orbital plane with respect to the sun.

Applications:

- Mapping
- Solar-Powered
- Shadow-evading
- Time-of-Day Apps



The earth rotates $360°$ about the sun every 365.25 days.

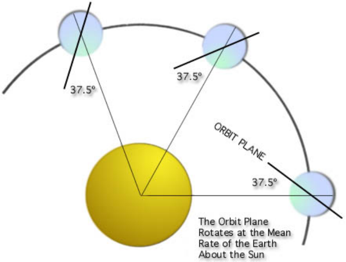
Definition 4.

A **Sun-Synchronous Orbit** is one where $\dot{\Omega} = .9855°/day = 1.992 \cdot 10^{-7} rad/s$.

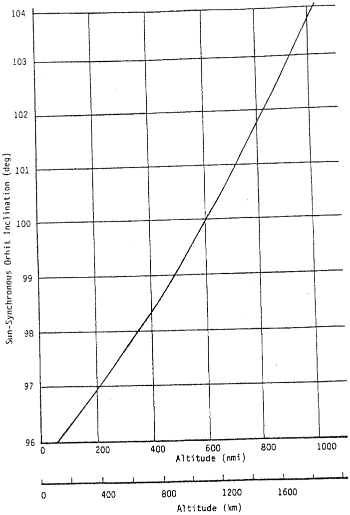
Thus

$$\boxed{\cos i = -1.992 \cdot 10^{-7} \left(\frac{p}{R_e}\right)^2 \frac{2}{3nJ_2}}$$

- The orbital plane rotates once every year.



Unlike critically inclined orbits, sun-synchronous orbits depend on altitude.



Problem: Design a sun-synchronous orbit with $r_p = R_e + 695km$ and $r_a = R_e + 705km$.

Solution: The desired inclination for a sun-synchronous orbit is given by

$$i = \cos^{-1}\left(1.992 \cdot 10^{-7}\left(\frac{p}{R_e}\right)^2 \frac{2}{3nJ_2}\right)$$

For this orbit $a = R_e + 700km = 7078km$. The eccentricity is

$$\boxed{e = 1 - \frac{r_p}{a} = .00071}$$

Thus $p = a(1-e^2) = 6999.65km$. $n = \sqrt{\frac{\mu}{a^3}} = .0011$. Finally, $J_2 = .0010826$. Thus the required inclination is

$$\boxed{i = 1.716rad = 98.33^\circ}$$

Problem: Molniya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 24 hours (actually Tundra orbit) and which precesses at $\dot{\Omega} = -.2^\circ/day$.

Solution: We can first use the period to solve for a. From

$$n = \sqrt{\frac{\mu}{a^3}} = 7.27 \cdot 10^{-5}$$

and $n = 2\pi/T = 2rad/day$ we have

$$a = \sqrt[3]{\frac{\mu}{n^2}} = 42,241km$$

Now the critical inclination for $\dot{\omega} = 0$ is $i = 63.4^\circ$ or $i = 116.6^\circ$. Since $\dot{\Omega} < 0$, we must choose $i = 63.4^\circ$. To achieve $\dot{\Omega} = -.2^\circ/day$, we use

$$\dot{\Omega} = -\frac{3nJ_2R_e^2}{2a^2(1-e^2)^2}\cos i$$

Numerical Example

Molniya Orbit

Problem: Molniya Orbits are usually designed so that perigee always occurs over the same latitude. Design a critically inclined orbit with a period of 24 hours (actually Tundra orbit) and which precesses at $\dot{\Omega} = -.2^\circ/day$.

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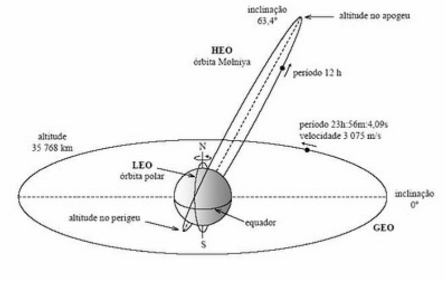
- Northern Molniya orbits have an argument of perigee of $+270^\circ$.
- Used for sensing and communication.
- Geosynchronous orbits cannot communicate well with or observe locations at high latitude.
- Molniya orbits launched from high latitude do not require large inclination changes after launch, unlike geosynchronous orbits.
- Provides continuous coverage with 3 satellites.
- Also used for US-observing spy sats and early-warning sats.
- Example of a semi-synchronous frozen tundra orbit with repeating ground track.

Since a is already fixed, we must use e. We can solve for e as

$$\boxed{e = \sqrt{1 - \sqrt{-\frac{3nJ_2R_e^2}{2\dot{\Omega}a^2}\cos i}} = .7459.}$$



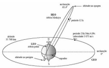
Note: Make sure the units of a and n match those of R_e and $\dot{\Omega}$, respectively.

Numerical Example

Molnaya Orbit, continued

Since a is already fixed, we must use e. We can solve for e as

$$\boxed{e = \sqrt{1 - \sqrt{-\frac{3nJ_2R_e^2}{2\dot{\Omega}a^2}\cos i}} = .7459.}$$



Note: Make sure the units of a and n match those of R_e and $\dot{\Omega}$, respectively.

Summary

This Lecture you have learned:

How to account for perturbations to Earth gravity

- Gravity Mapping
- Harmonic Functions
- J_2 Perturbation
  - Effect on Ω
  - Effect on ω
  - Minor effect (e, i)

How to design specialized orbits

- Critically - Inclined Orbit.
- Sun-Synchronous Orbit.
- Applications

Next Lecture: Interplanetary Mission Planning.