Spacecraft and Aircraft Dynamics

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Lecture 11: Longitudinal Dynamics

In this Lecture we will cover:

Longitudinal Dynamics:

- Finding dimensional coefficients from non-dimensional coefficients
- Eigenvalue Analysis
- Approximate modal behavior
  - short period mode
  - phugoid mode

Review: Longitudinal Dynamics

Combined Terms

$$\begin{aligned}\Delta\dot{u} + \Delta\theta g\cos\theta_0 &= X_u\Delta u + X_w\Delta w + X_{\delta_e}\delta_e + X_{\delta_T}\delta_T\\ \Delta\dot{w} + \Delta\theta g\sin\theta_0 - u_0\Delta\dot{\theta} &= Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta\dot{\theta} + Z_{\delta_e}\delta_e + Z_{\delta_T}\delta_T\\ \Delta\ddot{\theta} &= M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta\dot{\theta} + M_{\delta_e}\delta_e + M_{\delta_T}\delta_T\end{aligned}$$

Force Coefficients

Force/Moment Coefficients can be found in Table 3.5 of Nelson

TABLE 3.5
Summary of longitudinal derivatives

$$X_u = \frac{-(C_{D_u} + 2C_{D0})QS}{mu_0} (\mathrm{s}^{-1}) \qquad X_w = \frac{-(C_{D_\alpha} - C_{L0})QS}{mu_0} (\mathrm{s}^{-1})$$

$$Z_u = \frac{-(C_{L_u} + 2C_{L0})QS}{mu_0} (\mathrm{s}^{-1})$$

$$Z_w = \frac{-(C_{L_\alpha} + C_{D0})QS}{mu_0} (\mathrm{s}^{-1}) \qquad Z_{\dot{w}} = C_{z\dot{\alpha}} \frac{c}{2u_0} QS/(u_0 m)$$

$$Z_\alpha = u_0 Z_w (\mathrm{ft/s^2}) \text{ or } (\mathrm{m/s^2}) \qquad Z_{\dot{\alpha}} = u_0 Z_{\dot{w}} \, (\mathrm{ft/s}) \text{ or } (\mathrm{m/s})$$

$$Z_q = C_{Z_q} \frac{c}{2u_0} QS/m \, (\mathrm{ft/s}) \text{ or } (\mathrm{m/s}) \qquad Z_{\delta_e} = C_{Z\delta_e} QS/m \, (\mathrm{ft/s^2})$$

$$\mathrm{M}_u = C_{m_u} \frac{(QSc)}{u_0 I_y} \left(\frac{1}{\mathrm{ft} \cdot \mathrm{s}}\right) \text{ or } \left(\frac{1}{\mathrm{m} \cdot \mathrm{s}}\right)$$

$$\mathrm{M}_w = C_{m_\alpha} \frac{(QS\bar{c})}{u_0 I_y} \left(\frac{1}{\mathrm{ft} \cdot \mathrm{s}}\right) \text{ or } \left(\frac{1}{\mathrm{m} \cdot \mathrm{s}}\right) \qquad M_{\dot{w}} = C_{m\dot{\alpha}} \frac{\bar{c}}{2u_0} \frac{QS\bar{c}}{u_0 I_y} (\mathrm{ft}^{-1})$$

$$\mathrm{M}_\alpha = u_0 \mathrm{M}_w \, (\mathrm{s}^{-2}) \qquad \mathrm{M}_{\dot{\alpha}} = u_0 \mathrm{M}_{\dot{w}} \, (\mathrm{s}^{-1})$$

$$\mathrm{M}_q = C_{m_q} \frac{\bar{c}}{2u_0} (QS\bar{c})/I_y \, (\mathrm{s}^{-1}) \qquad \mathrm{M}_{\delta_e} = C_{m_{\delta_e}} (QS\bar{c})/I_y \, (\mathrm{s}^{-2})$$

Nondimensional Force Coefficients

Nondimensional Force/Moment Coefficients can be found in Table 3.3 of Nelson

TABLE 3.3
Equations for estimating the longitudinal stability coefficients

	X-force derivatives	Z-force derivatives	Pitching moment derivatives
u	$C_{X_u} = -[C_{D_u} + 2C_{D_0}] + C_{T_u}$	$C_{z_u} = -\dfrac{M^2}{1-M^2}C_{L_0} - 2C_{L_0}$	$C_{m_u} = \dfrac{\partial C_m}{\partial M}M_0$
α	$C_{X_\alpha} = C_{L_0} - \dfrac{2C_{L_0}}{\pi e}\dfrac{C_{L_\alpha}}{AR}$	$C_{Z_\alpha} = -(C_{L_\alpha} + C_{D_0})$	$C_{m_\alpha} = C_{L_{\alpha_w}}\left(\dfrac{X_{cg}}{\bar{c}} - \dfrac{X_{ac}}{\bar{c}}\right) + C_{m_{\alpha \text{fus}}} - \eta V_H C_{L_{\alpha_t}}\left(1 - \dfrac{d\varepsilon}{d\alpha}\right)$
$\dot{\alpha}$	0	$C_{Z_{\dot{\alpha}}} = -2\eta C_{L_{\alpha_t}} V_H \dfrac{d\varepsilon}{d\alpha}$	$C_{m_{\dot{\alpha}}} = -2\eta C_{L_{\alpha_t}} V_H \dfrac{l_t}{c}\dfrac{d\varepsilon}{d\alpha}$
q	0	$C_{Z_q} = -2\eta C_{L_{\alpha_t}} V_H$	$C_{m_q} = -2\eta C_{L_{\alpha_t}} V_H \dfrac{l_t}{c}$
α_e	0	$C_{Z_{\delta_e}} = -C_{L_{\delta_e}} = -\dfrac{S_t}{S}\eta\dfrac{dC_{L_t}}{d\delta_e}$	$C_{m_{\delta_e}} = -\eta V_H \dfrac{dC_{L_t}}{d\delta_e}$

Subscript 0 indicates reference values and M is the Mach number.

- AR Aspect ratio
- C_{D_0} Reference drag coefficient
- C_{L_0} Reference lift coefficient
- C_{L_α} Airplane lift curve slope
- $C_{L_{\alpha_w}}$ Wing lift curve slope
- $C_{L_{\alpha_t}}$ Tail lift curve slope
- $\bar{c}$ Mean aerodynamic chord
- e Oswald's span efficiency factor
- l_t Distance from center of gravity to tail quarter chord
- V_H Horizontal tail volume ratio
- M Flight mach number
- S Wing area
- S_t Horizontal tail area
- $\dfrac{d\varepsilon}{d\alpha}$ Change in downwash due to a change in angle of attack
- η Efficiency factor of the horizontal tail

State-Space

From the homework, we have a state-space representation of form

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{\theta} \\ \Delta\dot{q} \end{bmatrix} = [A] \begin{bmatrix} u \\ w \\ \theta \\ q \end{bmatrix} + [B] \begin{bmatrix} \delta_e \\ \delta_T \end{bmatrix}$$

Where we get A and B from X_u, Z_u, etc.

Recall:

- Eigenvalues of A define stability of $\dot{x} = Ax$.
- A is 4×4, so A has 4 eigenvalues.
- Stable if eigenvalues all have negative real part.

Natural Motion

We mentioned that A is

- *Stable* if eigenvalues all have negative real part.

Now we say more: Eigenvalues have the form

$$\lambda = \lambda_R \pm \lambda_I \imath$$

If we have a **pair** of complex eigenvalues, then we have two more concepts:

1. **Natural Frequency:**
$$\omega_n = \sqrt{\lambda_R^2 + \lambda_I^2}$$
2. **Damping Ratio:**
$$d = -\frac{\lambda_R}{\omega_n}$$

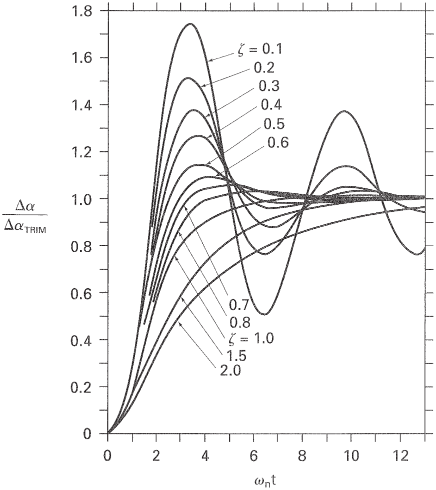
Natural frequency is how fast the the motion oscillates.

Closely related is the

Definition 1.

The **Period** is the time take to complete one oscillation

$$\tau = \frac{2\pi}{\omega_n}$$



Damping Ratio

Damping ratio is how much amplitude decays per oscillation.

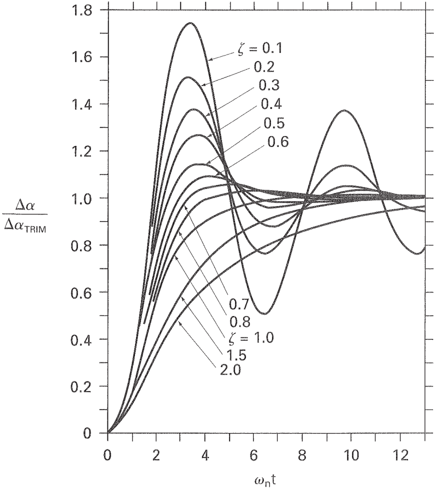
- Even if d is large, may decay slowly is ω_n is small

Closely related is

Definition 2.

The **Half-Life** is the time taken for the amplitude to decay by half.

$$\gamma = \frac{.693}{|\lambda_R|}$$



C172: $V_0 = 132kt$, $5000ft$.

$$\begin{bmatrix} \Delta\dot{u} \\ \Delta\dot{w} \\ \Delta\dot{q} \\ \Delta\dot{\theta} \end{bmatrix} = \begin{bmatrix} -.0442 & 18.7 & 0 & -32.2 \\ -.0013 & -2.18 & .97 & 0 \\ .0024 & -23.8 & -6.08 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ w \\ q \\ \theta \end{bmatrix}$$

Using the Matlab command `[u,V] = eigs(A)`, we find the eigenvalues as

Phugoid (Long-Period) Mode

$$\lambda_{1,2} = -.0209 \pm .18\imath$$

and Eigenvectors

$$\nu_{1,2} = \begin{bmatrix} -.1717 \\ -.0748 \\ .9131 \\ -.1038 \end{bmatrix} \pm \begin{bmatrix} .2826 \\ .1685 \\ 0 \\ .1103 \end{bmatrix} \imath$$

Short-Period Mode

$$\lambda_{3,4} = -4.13 \pm 4.39\imath$$

and Eigenvectors

$$\nu_{3,4} = \begin{bmatrix} 1 \\ -.0002 \\ .001 \\ -.0008 \end{bmatrix} \pm \begin{bmatrix} 0 \\ .0000001 \\ .0000011 \\ .0055 \end{bmatrix} \imath$$

Notice that this is hard to interpret. Lets scale u and q by equilibrium values.

After scaling the state by the equilibrium values, we find the eigenvalues unchanged (Why?) as

Phugoid (Long-Period) Mode

$$\lambda_{1,2} = -.0209 \pm .18\imath$$

but clearer Eigenvectors

$$\nu_{1,2} = \begin{bmatrix} -.629 \\ .0218 \\ -.0016 \\ .138 \end{bmatrix} \pm \begin{bmatrix} .0213 \\ .0007 \\ .0001 \\ .765 \end{bmatrix} \imath$$

Natural Frequency: $\omega_n = .181 rad/s$

Damping Ratio: $d = .115$

Period: $\tau = 34.7s$

Half-Life: $\gamma = 33.16s$

Motion dominated by variables u and θ.

Modal Illustration

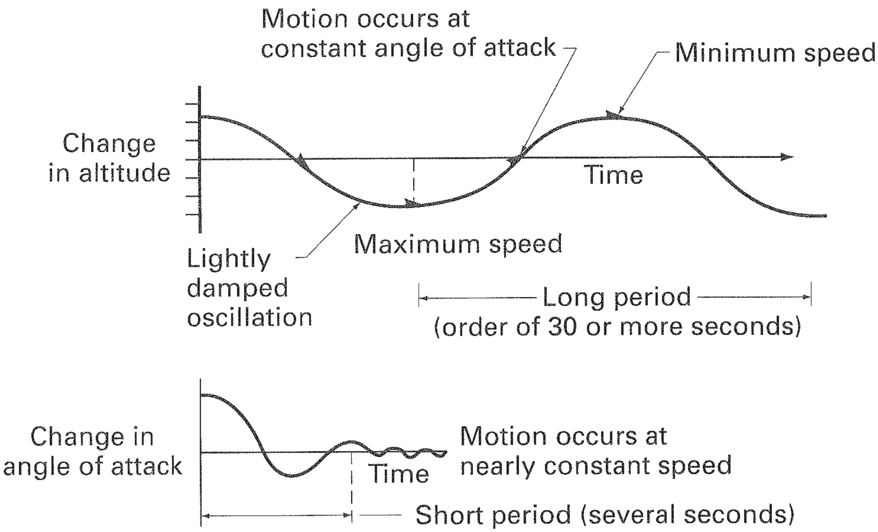


FIGURE 4.10

Using the Matlab command `[u,V] = eigs(A)`, we find the eigenvalues as
Short-Period Mode

$$\lambda_{3,4} = -4.13 \pm 4.39\imath$$

and Eigenvectors

$$\nu_{3,4} = \begin{bmatrix} -.0049 \\ -.655 \\ -.396 \\ -.006 \end{bmatrix} \pm \begin{bmatrix} .004 \\ .409 \\ .495 \\ .0423 \end{bmatrix} \imath$$

Natural Frequency: $\omega_n = 6.03 rad/s$
Damping Ratio: $d = .685$
Period: $\tau = 1.04s$
Half-Life: $\gamma = .167s$
Motion dominated by variables w and q.

Modal Illustration

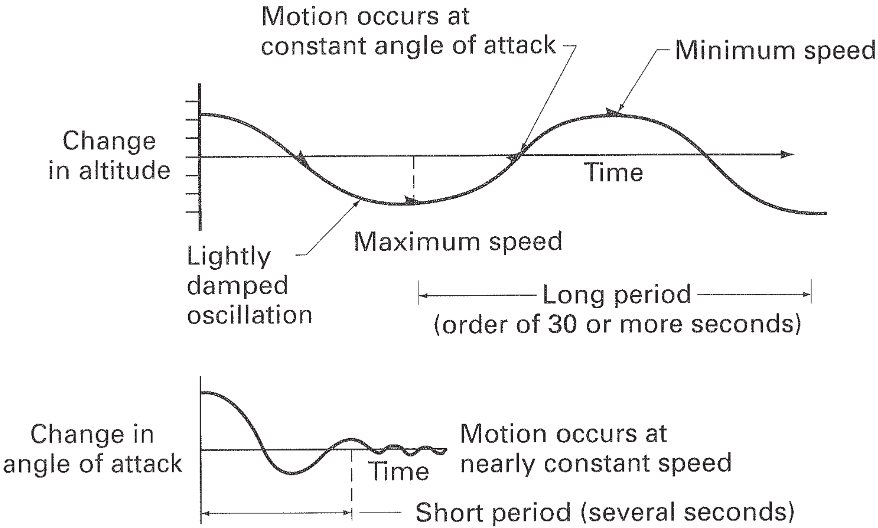


FIGURE 4.10

Now that we know that longitudinal dynamics have two modes:

- Short Period Mode
- Phugoid Mode (Long-Period Mode)

Short Period Mode:

- fix $u = 0$ and $w = 0$.
- study variation in θ and q.
- Similar to Static Longitudinal Stability

Long Period Mode:

- fix $q = 0$ and $w = 0$.
- study variation in θ and u.

Now we develop some simplified expressions to study these modes.

Short Period Approximation

For the short period mode, we have the following dynamics:

$$\begin{bmatrix} \frac{\dot{w}}{u_0} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \frac{Z_\alpha}{u_0} & 1 \\ M_\alpha + \frac{M_{\dot{\alpha}} Z_\alpha}{u_0} & M_q + M_{\dot{\alpha}} \end{bmatrix} \begin{bmatrix} \frac{w}{u_0} \\ q \end{bmatrix} + \begin{bmatrix} \frac{Z_{\delta_e}}{u_0} \\ M_{\delta_e} + \frac{M_{\dot{\alpha}} + Z_{\delta_e}}{u_0} \end{bmatrix} \delta_e$$

$$= A_{sp} \begin{bmatrix} \theta \\ q \end{bmatrix} + B_{sp} \delta_e$$

- To understand stability, we need the eigenvalues of A_{sp}.
- Eigenvalues are solutions of $\det(\lambda I - A_{sp}) = 0$.

Thus we want to solve

$$\det(\lambda I - A_{sp}) = \lambda^2 - (M_q + \frac{Z_\alpha}{u_0} + M_{\dot{\alpha}})\lambda + (\frac{Z_\alpha M_q}{u_0} - M_\alpha) = 0$$

We use the quadratic formula (Lecture 1):

$$\lambda_{3,4} = \frac{1}{2}(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}) \pm \frac{1}{2}\sqrt{(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0})^2 - 4(m_q \frac{Z_\alpha}{u_0} - M_\alpha)}$$

$$\lambda_{3,4} = \frac{1}{2}(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}) \pm \frac{1}{2}\sqrt{(M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0})^2 - 4(m_q \frac{Z_\alpha}{u_0} - M_\alpha)}$$

This leads to the *Approximation Equations*:

- **Natural Frequency:**

$$\omega_{sp} = M_q \frac{Z_\alpha}{u_0}$$

- **Damping Ratio:**

$$d_{sp} = -\frac{1}{2}\frac{M_q + M_{\dot{\alpha}} + \frac{Z_\alpha}{u_0}}{\omega_{sp}}$$

Short Period Mode Approximations

Example: C127

Approximate Natural Frequency:

$$\omega_{sp} = \sqrt{\frac{-481 * -431}{219} + 27.7} = 6.10 rad/s$$

True Natural Frequency:

$$\omega_{sp} = 6.03 rad/s$$

Approximate Damping Ratio:

$$d_{sp} = -\frac{4.32 - 2.20 - 1.81}{2 * 6.10} = .683 rad/s$$

True Damping Ratio:

$$d_{sp} = .685 rad/s$$

So, generally good agreement.

Long Period Approximation

Long period motion considers only motion in u and θ.

$$\begin{bmatrix}\dot{u}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix} X_u & -g \\ -\frac{Z_u}{u_0} & 0 \end{bmatrix} \begin{bmatrix} u \\ \theta \end{bmatrix}$$

This time, we must solve the simple expression

$$\det(\lambda I - A) = \lambda^2 - X_u \lambda - \frac{Z_u}{u_0} g = 0$$

Using the quadratic formula, we get

$$\lambda_{1,2} = \frac{X_u \pm \sqrt{X_u^2 + 4\frac{Z_u}{u_0} g}}{2}$$

Long Period Approximation

$$\lambda_{1,2} = \frac{X_u \pm \sqrt{X_u^2 + 4\frac{Z_u}{u_0}g}}{2}$$

This leads to the *Approximation Equations*:

- **Natural Frequency:**

$$\omega_{lp} = \sqrt{-\frac{Z_u g}{u_0}}$$

- **Damping Ratio:**

$$d_{lp} = -\frac{X_u}{2\omega_{lp}}$$

Long Period Mode Approximations

Example: C127

Approximate Natural Frequency:

$$\omega_{lp} = .181 rad/s$$

True Natural Frequency:

$$\omega_{lp} = .208 rad/s$$

Approximate Damping Ratio:

$$d_{lp} = .115 rad/s$$

True Damping Ratio:

$$d_{lp} = .106 rad/s$$

So, generally good, but not as good. Why?

Conclusion

In this lecture, we covered:

- How to find and interpret the eigenvalues and eigenvectors of a state-space matrix
  - Natural Frequency
  - Damping Ratio
- How to identify
  - Long Period Eigenvlaues/Motion
  - Short Period Eigenvalues/Motion
- Modal Approximations
  - Phugoid and Short-Period Modes
  - Formulas for natural frequency
  - Formulae for damping ratio