H_2-Optimal Estimation of Linear Delayed and PDE Systems

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***Abstract*— The H_2 norm is a performance metric commonly used in design of estimators. However, H_2-optimal estimation for delayed and PDE systems is complicated by the lack of an equivalent time-domain characterization. To address this problem, we re-characterize the H_2-norm in terms of a map from initial condition to output. We then convert the delayed/PDE system to a Partial Integral Equation (PIE), characterize H_2 norm of the PIE using storage functions, propose a class of PIE-based observers, and formulate the associated H_2-optimal estimation problem as a convex optimization problem. The resulting observers are validated using numerical simulation.**

I. Introduction

Partial Differential Equations (PDEs) are used to describe the evolution of processes whose states are distributed over a spatial domain. For example, processes such as fluid flow [1], [2], vibroaccoustics [3], [4], chemical reaction networks [5], and time-delay systems [6], are modeled as PDEs whose corresponding distributed states are velocity profile, displacement, species concentration, and history. For such systems, it is often desirable to be able to track the evolution of the system using sensor measurements – either for the purpose of feedback control [7]–[11] or for monitoring and fault detection [12]–[14].

Unlike Ordinary Differential Equations (ODEs) and other such lumped-parameter systems, however, direct measurement of the system state of a PDE requires an uncountable number of sensors – a practical impossibility. Consequently, there has been significant interest in the development of observers wherein by tracking a finite set of measurements, we may infer real-time estimates of the entire distributed state. For ODEs, the problem of state estimation has been largely solved, with special cases including the Luenberger observer, the Kalman filter, and Linear Matrix Inequalities (LMIs) for H_∞-optimal observers and filters – methods that can be applied to state estimation for any linear ODE with state-space representation. For PDEs, however, the need to integrate boundary conditions and a distributed system state precludes the existence of a convenient and universal state-space representation. This means that most efforts to design estimators for such systems are ad hoc – requiring significant modification for even minor changes in the model [15]. As a result, most approaches to the estimation of the PDE state entail a reduction of the PDE state to finite dimensions, either through early or late lumping.

Early-lumping [16]–[18] entails the reduction of the PDE to a state-space ODE through Galerkin projection, spatial discretization or modal decomposition. Late-lumping [19]–[21], by contrast, formulates synthesis conditions using the distributed state but then enforces those conditions on a finite number of test functions. In both cases, neither the stability nor performance of the estimator can be proven unless the truncation error can be bounded through some auxiliary, ad-hoc process.

Recently, efforts have been made to synthesize observers for PDE systems without lumping through the use of a more convenient state-space representation of PDEs [22]–[25]. This method integrates the PDE evolution equation with the boundary conditions by defining the state as the highest spatial derivative of the distributed state and parameterizing the evolution of this state by means of integral operators with polynomial kernels. The integral operators used form an algebra of bounded linear operators, which can be represented using matrices and optimized using LMIs; the representation of a PDE using such operators is referred to as a Partial Integral Equation (PIE).

While H_∞-optimal observer design for PDE systems that admit a PIE representation has been considered in [26]–[28], and while previous results on LQG filters for delayed and PDE systems can be found in [**?**], [**?**] the problem of H_2-optimal estimation does not seem to have been previously considered.

Unlike H_∞-optimal observer synthesis, wherein a proxy for H_∞ performance is L_2-gain, the main technical difficulty for H_2-optimal estimation is the identification of a time-domain proxy for H_2 performance. To address this difficulty, we rely on an initial condition to output L_2-gain characterization of the H_2 metric as proposed in [29]. This allows us to extend classical LMIs for H_2-performance to LPI-type conditions to performance bounds on the error dynamics of the PIE-based observer.

This paper is structured as follows. First, Section II gives preliminary definitions from the PIEs framework, which is used to derive the LPI optimization problems. Then, Section III introduces a time-domain characterization of the H_2 norm and formulates the H_2-optimal observer synthesis problem. Then, Section IV gives an LPI to compute upper-bounds of the H_2-norm of a PIE and extends this result to an LPI condition for computing H_2-optimal observer gains. Finally, the observers are tested using numerical simulation in Section V.

Notation: $L_2^p[a,b]$ is the Hilbert space of *Lesbegue* square-integrable $\mathbb{R}^p$-valued functions on domain $s \in [a,b]$, endowed with the standard inner product. $\mathbb{R}L_2^{m,p}[a,b]$ denotes

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the Hilbert $\mathbb{R}^m \times L_2^p[a,b]$. Occasionally, we omit domain and simply write L_2^p or $\mathbb{R}L_2^{m,p}$. We use the bold font, (e.g. $\mathbf{x}$) to indicate scalar or vector-valued functions of a spatial variable. For Hilbert spaces X, Y, $\mathcal{L}(X,Y)$ denotes the set of bounded linear operators from X to Y with $\mathcal{L}(X) := \mathcal{L}(X,X)$. We use the calligraphic font (e.g. $\mathcal{A}$) to represent operators.

II. State Space and Convex Optimization: PIs, PIEs, and LPIs

In this section, we introduce the algebra of PI operators, the class of systems modeled using PIEs, and the class of convex optimization problems defined in LPI Inequality constraints. These concepts are key to understand the problem formulation and to motivate the proposed solutions.

A. The Algebra of Partial Integral Operators

We begin by defining the algebra of PI operators which will be used to parameterize PIEs in Subsection II-B.

Definition 1: Given a matrix P and polynomials Q_1, Q_2, R_0, R_1, a 4-PI operator $\mathcal{P} = \mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right] \subset \mathcal{L}(\mathbb{R}L_2^{m_1,n_1}, \mathbb{R}L_2^{m_2,n_2})$ is such that

$$\left(\mathcal{P}\begin{bmatrix} x \\ \mathbf{x}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(\theta)\mathbf{x}(\theta)d\theta \\ Q_2(s)x + \mathcal{R}\mathbf{x}(s)\end{bmatrix},$$

$$(\mathcal{R}\mathbf{x})(s) = R_0(s)\mathbf{x}(s) + \int_a^s R_1(s,\theta)\mathbf{x}(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)d\theta.$$

We refer to $\mathbf{\Pi}_4$ as the set of 4-PI operators. If $m_1 = m_2$ and $n_1 = n_2$, this set of PI operators is closed under composition, addition, and adjoint; explicit formulae for these operations can be obtained in terms of the polynomial matrices used to parameterize them [23].

As in Defn. 1, the notation $\mathit{\Pi}\left[\begin{array}{c|c} P & Q_1 \\ \hline Q_2 & \{R_i\}\end{array}\right]$ is used to indicate the 4-PI operator associated with the matrix P and polynomial parameters Q_i, R_j. The associated dimensions (m_1, n_1, m_2, n_2) are inherited from the dimensions of the constant matrix $P \in \mathbb{R}^{m_2 \times m_1}$ and polynomial matrices $Q_1(s) \in \mathbb{R}^{m_2 \times n_1}$, $Q_2(s) \in \mathbb{R}^{n_2 \times m_1}$, and $R_0(s), R_1(s,\theta), R_2(s,\theta) \in \mathbb{R}^{n_2 \times n_1}$. In the case where a dimension is zero, we use $\emptyset$ in place of the associated parameter with zero dimension.

B. Partial Integral Equations

It has been shown in, e.g. [23], that a large class of PDE coupled with ODEs, with sensed and regulated outputs, $y(t) \in \mathbb{R}^{n_y}$, $z(t) \in \mathbb{R}^{n_z}$, and in-domain disturbances, $w(t) \in \mathbb{R}^{n_w}$, may be equivalently represented using a partial integral equation (PIE) of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \qquad \mathcal{T}\mathbf{x}(0) = 0, \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad y(t) = \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21}w(t), \end{aligned} \tag{1}$$

where the parameters $\mathcal{A}, \mathcal{B}_1, \mathcal{C}_2$, etc., are all 4-PI operators and where the solution of the PIE, $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$ yields a solution to the PDE as $\mathcal{T}\mathbf{x}(t)$. The PIE state, $\mathbf{x}(t)$, combines the ODE state with a spatial derivative of the PDE state and admits no boundary conditions or continuity constraints.

The solution of this class of PIE is formally defined as follows, where $x \in L_{2e}^p[0,\infty)$ means $x(t) \in \mathbb{R}^p$ and $\int_0^T \|x(t)\|^2 dt$ is finite for all $T \geq 0$.

Definition 2 (PIE solution): Given PI operators $\mathcal{T}$, $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_{21}$ we say $\{\mathbf{x}, z, y\}$ is a solution to the PIE system for given initial condition $\mathbf{x}(0) \in \mathbb{R}L_2^{m,n}[a,b]$ and input $w \in L_{2e}^{n_w}[0,\infty)$, if $\mathcal{T}\mathbf{x}(t)$ is *Frechét* differentiable for all $t \in [0,\infty)$, and if $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$, $z \in L_{2e}^{n_z}[0,\infty)$, and $y \in L_{2e}^{n_y}[0,\infty)$ satisfy Eq. (1) for all $t \in [0,\infty)$.

C. Linear PI Operator Inequalities

Having defined PI operators and the PIE state-space representation, we are ready to show how these tools are used to derived computationally tractable optimization problems. As described in Subsection II-A, 4-PI operators of the form given in Defn. 1 constitute a composition algebra of bounded linear operators and are parameterized by polynomial matrices, which in turn can be parameterized by the coefficients of those polynomials. In this paper, we reformulate the problem of H_2-optimal estimator synthesis as an optimization problem where the decision variables are themselves PI operators and are subject to inequality constraints affine in those decision variables – See, e.g. Eqn. (12) in Thm. 8. Optimization problems in this form may be solved by using matrices to parameterize the coefficients of the polynomials that define the PI operator variables. Inequalities are enforced by using positive matrices to parameterize positive PI operators, as described in [22].

III. Problem Formulation

Having provided sufficient preliminary concepts, we next present the mathematical formulation of the problem. In this section we provide a time-domain characterization of the H_2 norm and use this characterization to define the problems of H_2 norm bounding and H_2-optimal estimation for systems that admit a PIE representation.

A. The H_2 norm of a PIE

For this subsection, we restrict our consideration to the characterization of the H_2 norm of a system represented by a PIE of the form

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathcal{T}\mathbf{x}(0) = 0, \end{aligned} \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}L_2^{m,n}[a,b]$ is the state, $w(t) \in \mathbb{R}^{n_w}$ is a disturbance, and $z(t) \in \mathbb{R}^{n_z}$ is the output, $\mathcal{T}, \mathcal{A}, \mathcal{B}_1$, and $\mathcal{C}_1$ are 4-PI operators as defined in Section II. Specifically, in Definition. 3, we define the H_2 norm of this system as L_2-gain of initial condition to output of an auxiliary system with no disturbance. While non-standard, we will see that this characterization of H_2 performance is equivalent in a certain sense to the standard definition of H_2 norm.

Definition 3: Consider solutions of the auxiliary PIE

$$\begin{aligned} \partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t), \\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad \mathcal{T}\mathbf{x}(0) = \mathcal{B}_1 x_0. \end{aligned} \tag{3}$$

We define the H_2 norm of System (2) as

$$\sup_{\substack{z,\mathbf{x} \text{ satisfy } (3) \\ \|x_0\|=1}} \|z\|_{L_2}.$$

To see the relationship between the definition of H_2 norm in Definition 3 and the standard definition, recall the usual state-space representation of an ODE given by

$$\begin{bmatrix}\dot{x}(t)\\ z(t)\end{bmatrix} = \begin{bmatrix}A & B\\ C & 0\end{bmatrix}\begin{bmatrix}x(t)\\ w(t)\end{bmatrix},\ t\geq 0, \tag{4}$$

where $x(t)\in\mathbb{R}^n$. Then if A is Hurwitz, and we define the transfer function as $\hat{G}(s) = C(sI-A)^{-1}B$, the standard definition of H_2 norm of System (4), denoted G, is given as

$$\begin{aligned}\|G\|_{H_2}^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\left(G^*(i\omega)G(i\omega)d\omega\right)\\ &= \text{trace}\left(B_1^T\int_0^\infty e^{A^T\tau}C_1^TC_1e^{A\tau}d\tau B_1\right),\end{aligned}$$

where we have used the inverse Laplace transform to obtain the time-domain characterization [30].

Corollary 4: Suppose A is Hurwitz and $\hat{G}(s) = C(sI-A)^{-1}B$. Consider solutions of the auxiliary ODE

$$\dot{x}(t) = Ax(t),\quad z(t) = Cx(t),\quad x(0) = Bx_0, \tag{5}$$

Then

$$\sup_{\substack{z \text{ satisfies } (5)\\ \|x_0\|=1}} \|z\|_{L_2} \leq \|G\|_{H_2} \leq \sqrt{n_w}\sup_{\substack{z \text{ satisfies } (5)\\ \|x_0\|=1}} \|z\|_{L_2}.$$

Proof: Suppose $\{x,z\}$ satisfy (5) with initial condition $x(0) = Bx_0$. Then $x(t) = e^{At}Bx_0$ and hence if $\|x_0\|=1$, we have

$$\begin{aligned}\|z\|_{L_2}^2 &= \int_0^\infty x(\tau)^TC^TCx(\tau)d\tau\\ &= \int_0^\infty x_0^TB^Te^{A^T\tau}C^TCe^{A\tau}Bx_0d\tau\\ &\leq \bar{\sigma}\left(\int_0^\infty B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &\leq \text{trace}\left(\int_0^\infty B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right) = \|G\|_{H_2}^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|G\|_{H_2}^2 &= \text{trace}\left(\int_0^\infty B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &\leq n_w\bar{\sigma}\left(\int_0^\infty B^Te^{A^T\tau}C^TCe^{A\tau}Bd\tau\right)\\ &= n_w\sup_{\|x_0\|=1}\int_0^\infty x_0^TB^Te^{A^T\tau}C^TCe^{A\tau}Bx_0d\tau\\ &= n_w\sup_{\|x_0\|=1}\|z\|_{L_2}^2.\end{aligned}$$

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Clearly, if the PIE has a single input, the proposed definition of H_2 norm coincides with the typical definition. Alternatively, in the case of multiple inputs, our time-domain characterization of H_2 norm would coincide with an alternative definition of H_2 norm given by

$$\|\hat{G}\|_{H_2}^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{\sigma}\left(G^*(i\omega)G(i\omega)d\omega\right).$$

Having defined the H_2-norm, we proceed to formulate the H_2-optimal estimator synthesis problem.

B. H_2-Optimal Estimators

Our goal is to design observers for the class of coupled PDE systems which admit a PIE representation of form

$$\begin{aligned}\partial_t(\mathcal{T}\mathbf{x}(t)) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1w(t), \qquad \mathcal{T}\mathbf{x}(0) = 0,\\ z(t) &= \mathcal{C}_1\mathbf{x}(t), \qquad y(t) = \mathcal{C}_2\mathbf{x}(t) + D_{21}w(t),\end{aligned} \tag{6}$$

where, as discussed in Section II, the state of the original PDE is obtained from the solution of the PIE as $\mathcal{T}\mathbf{x}(t)$, the signal $y(t)$ are measurements of the PDE and $z(t)$ represents those parts of the state by which we will measure the performance of our estimator. Our estimator dynamics are then assumed to have the Luenberger observer structure

$$\partial_t\left(\mathcal{T}\tilde{\mathbf{x}}(t)\right) = \mathcal{A}\tilde{\mathbf{x}}(t) + \mathcal{L}\left(\mathcal{C}_2\tilde{\mathbf{x}}(t) - y(t)\right),\ \mathcal{T}\tilde{\mathbf{x}}(0) = 0, \tag{7}$$

which mirrors the dynamics of the observed system, but without the disturbance, which is unknown. The term, $\mathcal{C}_2\tilde{\mathbf{x}}(t) - y(t)$, reflects the difference between the predicted and measured output from the PDE. This term is weighted by the observer gain, $\mathcal{L}:\mathbb{R}^{n_y}\to\mathbb{R}L_2^{m,n}$ which is taken to be a PI operator. By combining the observer in Eqn. (7) with the measured output of a PDE, real-time estimates of the PDE state can be obtained as $\mathcal{T}\tilde{\mathbf{x}}(t)$ and used in conjunction with state-feedback controllers or fault detection algorithms.

The H_2-optimal estimation problem, then, is to choose $\mathcal{L}$ which minimizes the H_2-norm of the map from disturbance w to error in the regulated output, which we define as $e_z(t) = \mathcal{C}_1\tilde{\mathbf{x}}(t) - z(t)$. This map can likewise be represented as a PIE with state $\mathbf{e}(t) = \tilde{\mathbf{x}}(t) - \mathbf{x}(t)$, where $\tilde{\mathbf{x}}$ satisfies Eqn. (7) and $\mathbf{x}$ satisfies Eqn. (6) so that

$$\begin{aligned}\partial_t\left(\mathcal{T}\mathbf{e}(t)\right) &= (\mathcal{A}+\mathcal{L}\mathcal{C}_2)\mathbf{e}(t) - \left(\mathcal{B}_1 + \mathcal{L}D_{21}\right)w(t),\\ e_z(t) &= \mathcal{C}_1\mathbf{e}(t), \qquad \mathcal{T}\mathbf{e}(0) = 0.\end{aligned} \tag{8}$$

We see that the system presented herein (8) is of the form in Eqn. (2) with $\mathcal{A}\mapsto\mathcal{A}+\mathcal{L}\mathcal{C}_2$ and $\mathcal{B}_1\mapsto-(\mathcal{B}_1+\mathcal{L}\mathcal{D}_{21})$. Thus we can formulate the H_2-optimal synthesis problem using the auxiliary PIE from Defn. 3

$$\begin{aligned}\partial_t\left(\mathcal{T}\mathbf{e}(t)\right) &= (\mathcal{A}+\mathcal{L}\mathcal{C}_2)\mathbf{e}(t),\\ e_z(t) &= \mathcal{C}_1\mathbf{e}(t), \quad \mathcal{T}\mathbf{e}(0) = -\left(\mathcal{B}_1+\mathcal{L}\mathcal{D}_{21}\right)x_0,\end{aligned} \tag{9}$$

as

$$\min_{\mathcal{L}\in\mathbf{\Pi}}\sup_{\substack{z,\mathbf{e}\text{ satisfy } (9)\\ \|x_0\|=1}} \|e_z\|_{L_2}. \tag{10}$$

In Section IV-B, we will reformulate the H_2-optimal estimation problem as an LPI. First, however, we need to address the problem of how to use LPIs to compute the H_2 norm of a PIE.

IV. Main Results

At this point, we are ready to derive an LPI for the estimation problem. First, we adapt the widely used Schur Complement lemma to PI operators. With this result, we extend the condition proposed in [29] for computing optimal upper-bounds on the H_2 norm of PDE systems. The new condition allow us to finally derive an LPI to solve the H_2-optimal estimation problem.

A. An LPI for the H_2 norm

In this section, we show how to use LPIs to compute the H_2 norm of a PIE. We begin by reformulating the following result from [29].

Theorem 5: Suppose System (2) is defined by 4-PI operators $\mathcal{T}, \mathcal{A}, \mathcal{C}_1, \mathcal{B}_1 \in \mathbf{\Pi}_4$ of appropriate dimensions. If there exists a constant γ, a positive constant $\epsilon > 0$ and a self-adjoint 4-PI operator $\mathcal{P}$ such that $\mathcal{P} \succeq \epsilon I$,

$$trace(\mathcal{B}_1^* \mathcal{P} \mathcal{B}_1) < \gamma^2,$$
$$\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} + \mathcal{C}_1^* \mathcal{C}_1 \preceq -\epsilon I, \tag{11}$$

then the H_2 norm of System (2) is upper bounded by γ.

We now use an extension of the Schur complement to obtain an LPI for bounding the H_2 norm which will be used for estimator design in Section IV-B. This reformulation, however, requires us to define vertical and horizontal concatenation of $\mathbf{\Pi}_4$ operators such that the concatenated operator is in $\mathbf{\Pi}_4$ (See Lemmas 39 and 40 from [23]). This definition separately concatenates the real and distributed portions of the operator so that if, e.g. $\mathcal{P} \in \mathcal{L}(\mathbb{R}L_2^{n,m})$ and $\mathcal{Q} \in \mathcal{L}(\mathbb{R}L_2^{p,q})$, then

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \in \mathcal{L}(\mathbb{R}^{n+p} \times L_2^{m+q}).$$

In proof of the following lemma, we do not re-order rows and columns. However, the result holds for the standard definition of concatenation since inequalities are preserved under symmetric reordering of rows and columns.

Lemma 6 (Schur Complement): Suppose $\mathcal{P}, \mathcal{Q}, \mathcal{R} \in \mathbf{\Pi}_4$. Then the following are equivalent.

1) $\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \succeq \epsilon I$ for some $\epsilon > 0$.
2) $R - Q^* P^{-1} Q \succeq \epsilon I$ and $P \succeq \epsilon I$ for some $\epsilon > 0$.

Proof: In this proof, there is no rearrangement of rows or columns. Now, mirroring the standard proof of the Schur complement, suppose that 1) is true. Then, we have

$$\langle \mathbf{x}, \mathcal{P}\mathbf{x} \rangle = \left\langle \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ 0 \end{bmatrix} \right\rangle \geq \epsilon \left\| \mathbf{x} \right\|^2,$$

which implies that $\mathcal{P}$ is invertible. Now note that

$$\begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^* P^{-1} \mathcal{Q} \end{bmatrix} = \begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} I & -\mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix},$$

and hence

$$\langle \mathbf{x}, (\mathcal{R} - \mathcal{Q}^* P^{-1} \mathcal{Q}) \mathbf{x} \rangle = \left\langle \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^* P^{-1} \mathcal{Q} \end{bmatrix} \begin{bmatrix} 0 \\ \mathbf{x} \end{bmatrix} \right\rangle$$
$$= \left\langle \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix} \right\rangle$$
$$\geq \epsilon \left\| \begin{bmatrix} -\mathcal{P}^{-1}\mathcal{Q}\mathbf{x} \\ \mathbf{x} \end{bmatrix} \right\|^2 \geq \epsilon \left\| \mathbf{x} \right\|^2.$$

For the converse, suppose 2) is true. Then

$$\begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} = \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \mathcal{P} & 0 \\ 0 & \mathcal{R} - \mathcal{Q}^* P^{-1} \mathcal{Q} \end{bmatrix} \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix},$$

which implies

$$\left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\rangle \geq \epsilon \left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2.$$

Now, define $\left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix}^{-1} \right\|_{\mathcal{L}(\mathbb{R}L_2)} = \delta$. Then

$$\left\| \begin{bmatrix} I & \mathcal{P}^{-1}\mathcal{Q} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2 \geq \delta \left\| \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2,$$

and hence

$$\left\langle \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}, \begin{bmatrix} \mathcal{P} & \mathcal{Q} \\ \mathcal{Q}^* & \mathcal{R} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\rangle \geq \epsilon\delta \left\| \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right\|^2,$$

as desired. ∎

Theorem 7: Suppose System (2) is defined by 4-PI operators $\mathcal{T}, \mathcal{A}, \mathcal{C}_1, \mathcal{B}_1 \in \mathbf{\Pi}_4$ of appropriate dimensions. If there exists a constant γ, a positive constant $\epsilon > 0$, a symmetric matrix $W \in \mathbb{R}^{n_w \times n_w}$, and a self-adjoint 4-PI operator $\mathcal{P}$ such that $P \succeq \epsilon I$,

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^* P \mathcal{A} + \mathcal{A}^* P \mathcal{T} \end{bmatrix} \preceq -\epsilon I, \tag{12}$$
$$\begin{bmatrix} W & \mathcal{B}_1^* \mathcal{P} \\ \mathcal{P}\mathcal{B}_1 & \mathcal{P} \end{bmatrix} \succeq \epsilon I, \tag{13}$$
$$trace(W) \leq \gamma, \tag{14}$$

then the H_2 norm of System (2) is upper bounded by γ.

Proof: Take γ and $\mathcal{P}$ as stated above. Then, Inequality (12) combined with Lemma 6 implies

$$\mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{P} \mathcal{A} + \frac{1}{\gamma} \mathcal{C}^* \mathcal{C} \preceq -\epsilon I.$$

Likewise, Inequality (13) combined with Lemma 6 implies

$$W - \mathcal{B}^* \mathcal{P} \mathcal{P}^{-1} \mathcal{P} \mathcal{B} = W - \mathcal{B}^* \mathcal{P} \mathcal{B} > 0.$$

Now W and $\mathcal{B}^* \mathcal{P} \mathcal{B}$ are matrices and hence $\text{trace}(\mathcal{B}^* \mathcal{P} \mathcal{B}) < \text{trace}\, W \preceq \gamma$. Define $\hat{\mathcal{P}} = \gamma \mathcal{P}$ so that $\mathcal{P} = \frac{1}{\gamma} \hat{\mathcal{P}}$ and hence

$$\mathcal{A}^* \hat{\mathcal{P}} \mathcal{T} + \mathcal{T}^* \hat{\mathcal{P}} \mathcal{A} + \mathcal{C}^* \mathcal{C} \preceq -\gamma \epsilon I, \qquad \text{trace}(\mathcal{B}^* \hat{\mathcal{P}} \mathcal{B}) < \gamma^2,$$

which implies the conditions of Thm 5 are satisfied. ∎

In next subsection, we use this LPI for the H_2 norm to synthesize observers which minimize a bound on the H_2 norm of the error dynamics.

B. An LPI for H_2-optimal Estimator

Now, we are ready to consider the problem of designing the estimator gain $\mathcal{L} \in \mathbf{\Pi}_4$ which minimizes a bound on the H_2 norm of the error dynamics defined in Subsection III-B.

Theorem 8: Consider 4-PI operators $\mathcal{T}, \mathcal{A}, \mathcal{B}_1, \mathcal{C}_1, \mathcal{C}_2$, and matrix D_{21} of appropriate dimensions. If there exists a constant γ, positive constants $\epsilon > 0$, a symmetric matrix W, a 4-PI operator $\mathcal{Z}$ and a self-adjoint 4-PI operator $\mathcal{P}$ such that $\mathcal{P} \succeq \epsilon I$,

$$\begin{bmatrix} -\gamma I & \mathcal{C}_1 \\ \mathcal{C}_1^* & \mathcal{T}^* \mathcal{P} \mathcal{A} + \mathcal{A}^* \mathcal{P} \mathcal{T} + \mathcal{T}^* \mathcal{Z} \mathcal{C}_2 + \mathcal{C}_2^* \mathcal{Z}^* \mathcal{T} \end{bmatrix} \preceq -\epsilon I,$$
$$\begin{bmatrix} W & -(\mathcal{B}_1^* \mathcal{P} + D_{21}^T \mathcal{Z}^*) \\ -(\mathcal{P}\mathcal{B}_1 + \mathcal{Z} D_{21}) & \mathcal{P} \end{bmatrix} \succeq \epsilon I,$$
$$\text{trace}(W) \leq \gamma,$$

then the H_2-norm of the System (8), defined by $\{\mathcal{T}, \mathcal{A}, \mathcal{C}_1, \mathcal{C}_2, D_{21}, \mathcal{L}\}$, where $\mathcal{L} = \mathcal{P}^{-1} \mathcal{Z}$, is upper bounded by γ.

Proof: Let $\mathcal{Z}=\mathcal{P}\mathcal{L}$. Then

$$\begin{bmatrix}-\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A}+\mathcal{L}\mathcal{C}_2\right)+\left(\mathcal{A}+\mathcal{L}\mathcal{C}_2\right)^*\mathcal{P}\mathcal{T}\end{bmatrix}$$
$$=\begin{bmatrix}-\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\left(\mathcal{A}+\mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right)+\left(\mathcal{A}+\mathcal{P}^{-1}\mathcal{Z}\mathcal{C}_2\right)^*\mathcal{P}\mathcal{T}\end{bmatrix}$$
$$=\begin{bmatrix}-\gamma I & \mathcal{C}_1\\ \mathcal{C}_1^* & \mathcal{T}^*\mathcal{P}\mathcal{A}+\mathcal{A}^*\mathcal{P}\mathcal{T}+\mathcal{T}^*\mathcal{Z}\mathcal{C}_2+\mathcal{C}_2^*\mathcal{Z}^*\mathcal{T}\end{bmatrix}\preceq-\epsilon I.$$

Likewise,

$$\begin{bmatrix}W & -(\mathcal{B}_1^*\mathcal{P}+D_{21}^T\mathcal{Z}^*)\\ -(\mathcal{P}\mathcal{B}_1+\mathcal{Z}D_{21}) & \mathcal{P}\end{bmatrix}$$
$$=\begin{bmatrix}W & -\left(\mathcal{B}_1+\mathcal{L}D_{21}\right)^*\mathcal{P}\\ -\mathcal{P}\left(\mathcal{B}_1+\mathcal{L}D_{21}\right) & \mathcal{P}\end{bmatrix}\succeq\epsilon I.$$

Finally, $\mathrm{trace}(W)\leq\gamma$. Now, by applying Theorem 7, the above equations imply that γ is an upper bound on the H_2-norm of the PIE system defined by $\{\mathcal{T},(\mathcal{A}+\mathcal{L}\mathcal{C}_2),-(\mathcal{B}_1+\mathcal{L}D_{21}),\mathcal{C}_1\}$ as in Eq. 8. ∎

C. Estimator Gain Reconstruction

Assuming that we have obtained $\mathcal{P},\mathcal{Z},W$ which satisfy Thm. 8, our next step is to construct the observer gain $\mathcal{L}=\mathcal{P}^{-1}\mathcal{Z}$ and use this gain in combination with the PIE estimator in (7) to track the state of a PDE.

First, if $\mathcal{P}=\mathit{\Pi}\left[\begin{array}{c|c}P & Q\\ \hline Q^T & \{R_i\}\end{array}\right]$, then the inverse $\mathcal{P}^{-1}$ can be computed using, e.g. Lem. 17 in [31] and numerically approximated by a PI operator

$$\mathcal{P}^{-1}\approx\hat{\mathcal{P}}:=\mathit{\Pi}\left[\begin{array}{c|c}\hat{P} & \hat{Q}\\ \hline \hat{Q}^T & \{\hat{R}_i\}\end{array}\right].$$

Then, if $\mathcal{Z}=\mathit{\Pi}\left[\begin{array}{c|c}Z_1 & \emptyset\\ \hline Z_2 & \{\emptyset\}\end{array}\right]$, we have, by the 4-PI composition formula [23], that $\mathcal{L}=\mathit{\Pi}\left[\begin{array}{c|c}L_1 & \emptyset\\ \hline L_2 & \{\emptyset\}\end{array}\right]$, where

$$L_1=\hat{P}Z_1+\int_a^b\hat{Q}(s)Z_2(s)ds,$$
$$L_2(s)=\hat{Q}(s)^TZ_1+\hat{R}_0(s)Z_2(s)$$
$$+\int_a^s\hat{R}_1(s,\theta)Z_2(\theta)d\theta+\int_s^b\hat{R}_2(s,\theta)Z_2(\theta)d\theta.$$

L_1 represents the correction to the ODE state and L_2 represents a correction to the distributed state. In the following section, we test observers designed in this manner by numerical integration of a PIE estimator using the output from the numerical integration of the PDE it is observing.

V. Numerical Examples

In this section, we validate the proposed algorithm for observer synthesis by constructing the H_2-optimal observer gains and numerically integrating the estimator dynamics using the output from numerical integration of the associated PDEs subject to disturbances. Our illustration uses a PDE, specifically an unstable non-homogeneous reaction-diffusion equation, and a Delay Differential Equation (DDE) examples.:

In both cases, the command-line PDE input option of PIETOOLS [32] is used to obtain the PIE representation of the PDE. Solution of the LPI in Thm. 8, operator inversion, and estimator gain reconstruction is likewise performed using PIETOOLS and the semi-definite programming solver MOSEK. Numerical integration of both the PIE estimator and PDE plant are performed using a Galerkin projection with Chebyshev bases order up to 8, and as implemented in PIESIM [33]. In each case, we plot both the evolution of the performance metric being minimized (e_z) as well as the error in the estimate of the distributed PDE state.

Example 1: In this example, we consider the unstable, non-homogeneous reaction-diffusion PDE with both sensor and process noise where sensor measurements are taken at the boundary.

$$\dot{\xi}(t,s)=3\xi(t,s)+(s^2+0.2)\partial_s^2\xi(t,s)-\frac{s^2}{2}w(t),$$
$$z(t)=\int_0^1\xi(t,\theta)d\theta,\quad y(t)=\xi(t,1)+w(t),$$
$$\xi(t,0)=\partial_s\xi(t,1)=0. \tag{15}$$

This PDE is entered directly into PIETOOLS using the command-line interface to obtain the PIE parameters corresponding to the state, $\mathbf{x}(t)=\partial_s^2\xi(t)$,

$$\mathcal{T}=\mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{0,-\theta,-s\}\end{array}\right],\ \mathcal{B}_1=\mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline -0.5s^2 & \{\emptyset\}\end{array}\right],$$
$$\mathcal{A}=\mathit{\Pi}\left[\begin{array}{c|c}\emptyset & \emptyset\\ \hline \emptyset & \{s^2+0.2,-2\theta,-3s\}\end{array}\right],$$
$$\mathcal{C}_1=\mathit{\Pi}\left[\begin{array}{c|c}\emptyset & 0.5s^2-s\\ \hline \emptyset & \{\emptyset\}\end{array}\right],\ \mathcal{C}_2=\mathit{\Pi}\left[\begin{array}{c|c}\emptyset & -s\\ \hline \emptyset & \{\emptyset\}\end{array}\right],\ \mathcal{D}_{21}=1.$$

In Fig. 1, we plot simulations of the PDE and H_2-optimal estimator with $w(t)=sin(100t)$, for $t\geq0$, and PDE initial condition $\xi(0,s)=s$. Moreover, the errors in both the estimated state of the PDE and the regulated output decay quickly despite instability in the PDE and persistent high-frequency excitation. A time step of $0.002s$ is used in the simulation.

Example 2: Consider the following time-delay system from [34]:

$$\dot{x}(t)=A_0x(t)+A_dx(t-\tau)+B_1w(t)+B_2u(t),$$
$$y(t)=C_2x(t)+C_dx(t-\tau)+D_{21}w(t),$$
$$z(t)=C_1x(t), \tag{16}$$

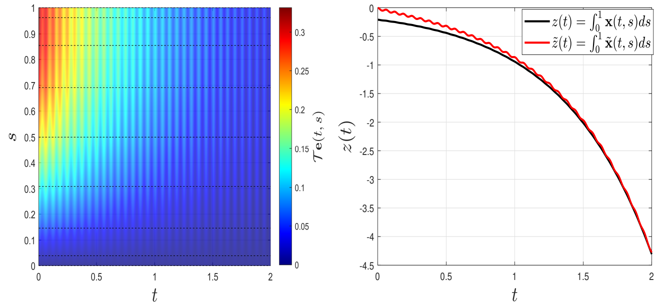
where $x(t),w(t),z(t)\in\mathbb{R}^2$, $u(t),y(t)\in\mathbb{R}$, and

$$A_0=\begin{bmatrix}-2 & 1\\ 0 & -1\end{bmatrix},\quad A_d=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}, C_1=\mathrm{I}_2, C_2=\begin{bmatrix}0\\1\end{bmatrix}$$
$$B_1=\begin{bmatrix}0.2 & 0\\ 0.2 & 0\end{bmatrix}, B_2=\begin{bmatrix}1\\1\end{bmatrix}, C_d=\begin{bmatrix}1\\1\end{bmatrix}, D_{21}=\begin{bmatrix}0 & 0.5\end{bmatrix},$$

where I_2 represents the identity matrix of order 2. As shown in [25], this system has an equivalent PDE representation

$$\dot{x}(t)=Ax(t)+A_d\phi(t,-1)+B_1w(t)+B_2u(t),$$
$$y(t)=Cx(t)+C_d\phi(t,-1)+D_{21}w(t),\ t\geq0,$$
$$\partial_t\phi(s,t)=I_\tau\partial_s\phi(s,t),\ \forall s\in[-1,0),\text{and }\phi(0,t)=x(t),$$

where $\phi(s,t):=x(t+\tau s)$ and $I_\tau=1/\tau\mathrm{I}_2$. PIE representa-



(a) (b)

Fig. 1: Numerical estimation of an H_2-optimal estimator for an unstable reaction-diffusion equation (Eq. (15)) using measurement at the boundary along with process and sensor disturbance $w(t) = sin(100t)$ and PDE initial condition $\xi(0,s) = -s^2/2$. (a): Evolution of error in estimate of the PDE state $\mathcal{T}\mathbf{e}(t) = \mathcal{T}\tilde{\mathbf{x}}(t) - \xi(t)$. (b): Evolution of the regulated output ($z(t)$) of both estimator and PDE.

TABLE I: This table lists the bound on H_2-norm of the estimator for system in Ex. 2 obtained from [34] and Thm. 8 ($\epsilon = 10^{-10}$) for different delays, τ

τ	0.1	0.3	0.5	0.7
Suh, et al. [34]	0.1342	0.1559	0.1792	0.2059
Thm. 8	0.1326	0.1546	0.1771	0.2009

tion of this PDE, as detailed in [25], is given by

$$\mathcal{T} = \Pi\left[\begin{array}{c|c} \mathrm{I}_2 & 0 \\ \hline \mathrm{I}_2 & \{0,0,-\mathrm{I}_2\}\end{array}\right], \qquad \mathcal{B}_1 = \Pi\left[\begin{array}{c|c} B_1 & \emptyset \\ \hline 0 & \{\emptyset\}\end{array}\right],$$

$$\mathcal{A} = \Pi\left[\begin{array}{c|c} A_0 + A_d & -A_d \\ \hline 0 & \{I_\tau,0,0\}\end{array}\right], \quad \mathcal{C}_1 = \Pi\left[\begin{array}{c|c} C_1 & 0 \\ \hline \emptyset & \{\emptyset\}\end{array}\right],$$

$$\mathcal{C}_2 = \Pi\left[\begin{array}{c|c} C_2 + C_d & -C_d \\ \hline \emptyset & \{\emptyset\}\end{array}\right], \qquad \mathcal{B}_2 = \Pi\left[\begin{array}{c|c} B_2 & \emptyset \\ \hline 0 & \{\emptyset\}\end{array}\right],$$

where the additional input $u(t)$ is considered in System (1).

Fig. 2 depicts a numerical simulation of the system states and H_2-optimal estimation when the system is subjected to non-zero initial conditions, a unit step as the control input, process and measurement disturbances. In this simulation we used a time step of $0.001s$. Furthermore, Table I compare the obtained bounds on the closed-loop H_2-norm using Thm. 8, with $\epsilon = 10^{-10}$, and the bounds of the reference [34].

VI. Conclusion

The H_2 norm is a commonly used performance metric in the estimation of linear state-space systems. However, minimization of the H_2 norm in the estimation of the state of PDE systems is complicated by the lack of state-space and transfer function representations for these systems. To address this problem, we have utilized the PIE state-space representation of systems of linear PDEs coupled with ODEs to parameterize a class of PIE-based observers and used an initial condition to output characterization of the H_2 norm. Based on this characterization, we have reformulated the problem of H_2-optimal estimation as a convex optimization problem defined by Linear PI Inequalities (LPIs). By solving

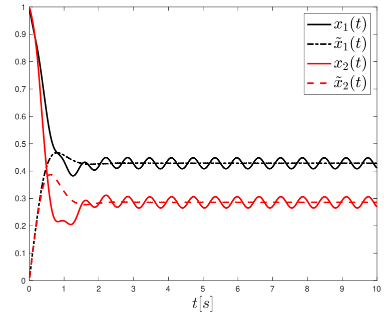


Fig. 2: Estimation of system state (Eq. (16)) using an H_2-optimal estimator. System is simulated using unit step control input $u(t)$, disturbances $w(t) = \begin{bmatrix} sin(10t) & sin(100t)\end{bmatrix}^T$, for $t \geq 0$, and initial conditions $x_1(t) = x_2(t) = 1$, for $t \in [-1, 0]$. Dashed lines show the estimator state and solid the system state.

this convex optimization problem, we construct observers that accurately track the distributed state of a PDE using only boundary measurement subject to both process and sensor noise. These observers were validated using numerical simulation of a time-delay system and an unstable non-homogeneous reaction-diffusion equation. Automation of the proposed algorithms in PIETOOLS allows for the efficient development of estimators for a large class of speculative and data-based models of PDEs coupled with ODEs.

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