

Grammians

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Lecture 20: Grammians

Lyapunov Equations

Proposition 1.

Suppose A is Hurwitz and Q is a square matrix. Then

$$X = \int_0^{\infty} e^{A^T s} Q e^{As} ds$$

is the unique solution to the Lyapunov Equation

$$A^T X + X A + Q = 0$$

Proposition 2.

Suppose $Q > 0$. Then A is Hurwitz if and only if there exists a solution $X > 0$ to the Lyapunov equation

$$A^T X + X A + Q = 0$$

Proposition 3.

Suppose A is Hurwitz and $X_1 \geq 0$ satisfies

$$A^T X_1 + X_1 A = -Q$$

Suppose X_2 satisfies

$$A^T X_2 + X_2 A < -Q.$$

Then $X_2 > X_1$.

Proof.

$$\begin{aligned} A^T(X_2 - X_1) + (X_2 - X_1)A &= (A^T X_2 + X_2 A) - (A^T X_1 + X_1 A) \\ &= A^T X_2 + X_2 A + Q = -Q' < 0 \end{aligned}$$

Since A is Hurwitz and $Q > 0$, by the previous Proposition $X_2 - X_1 > 0$



Recall From State-Space Systems:

- Controllable means we can do eigenvalue assignment.
- Observable means we can design an observer.
- Controllable and Observable means we can design an observer-based controller.

Questions:

- How difficult is the control problem?
- What is the effect of an input on an output?

To give quantitative answers to these questions, we use Grammians.

Definition 1.

For pair (C, A) , the **Observability Grammian** is defined as

$$Y = \int_0^{\infty} e^{A^T s} C^T C e^{As} ds$$

Definition 2.

The finite-time **Controllability Grammian** of pair (A, B) is

$$W := \int_0^{\infty} e^{As} B B^T e^{A^T s} ds$$

Grammians are linked to Observability and Controllability

Theorem 3.

For a given pair (C, A) , the following are equivalent.

- $\ker Y = 0$
- $\ker \Psi_T = 0$
- $\ker O(C, A) = 0$

Theorem 4.

For any $t \geq 0$,

$$R_t = C_{AB} = \text{Image}(W_t)$$

Observability Grammian

Recall the state-space system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Assume that A is Hurwitz.

Recall the Observability Operator $\Psi_o : \mathbb{R}^n \rightarrow L_2[0, \infty)$.

$$(\Psi_o x_0)(t) = \begin{cases} Ce^{At}x_0 & t \geq 0 \\ 0 & t \leq 0 \end{cases}$$

- $\Psi_o x_0 \in L_2$ because A is Hurwitz.
- When $u = 0$, this is also the solution.
- We would like to look at the “size” of the output produced by an initial condition.
 - ▶ Now we know how to measure the “size” of the output signal.

$$\|y\|_{L_2}^2 = \langle \Psi_o x_0, \Psi_o x_0 \rangle_{L_2} = \langle x_0, \Psi_o^* \Psi_o x_0 \rangle_{\mathbb{R}^n}$$

- How to calculate the adjoint $\Psi_o^* : L_2 \rightarrow \mathbb{R}^n$?

Observability Grammian

It can be easily confirmed that the adjoint of the observability operator is

$$\Psi_o^* z = \int_0^\infty e^{A^T s} C^T z(s) ds$$

Then

$$\Psi_o^* \Psi_o x_0 = \left[\int_0^\infty e^{A^T s} C^T C e^{As} ds \right] x_0$$

Which is simply the observability grammian

$$Y_o = \Psi_o^* \Psi_o = \int_0^\infty e^{A^T s} C^T C e^{As} ds$$

Recall from the HW: Y_o is the solution to

$$A^* Y_o + Y_o A + C^T C = 0$$

and $Y_o > 0$ if and only if (C, A) is observable.

Proposition 4.

Then (C, A) is observable if and only if there exists a solution $X > 0$ to the Lyapunov equation

$$A^T X + X A + C^T C = 0$$

Observability Grammian

Physical Interpretation

The physical interpretation is clear: how much does an initial condition affect the output in the L_2 -norm

$$\|y\|_{L_2} = x_0^T Y_o x_0$$

Since this is just a matrix, we can take this further by looking at which directions are most observable.

- Will correspond to $\bar{\sigma}(Y_o)$.

Definition 5.

The **Observability Ellipse** is

$$E_o := \left\{ x : x = Y_o^{1/2} x_0, \|x_0\| = 1 \right\}$$

Observability Grammian

Physical Interpretation

Definition 6.

The **Observability Ellipse** is

$$E_o := \left\{ x : x = Y_o^{1/2} x_0, \|x_0\| = 1 \right\}$$

Notes:

1. E_o is an ellipse.

$$E_o = \{x : x^T Y_o^{-1} x = 1\}$$

For a proof,

- ▶ let $x \in E_o$. Then there exists some $|x_0| = 1$ such that $x_0 = Y_o^{-1/2} x$.
 - ▶ Then $x^T Y_o^{-1} x = x^T Y_o^{-1/2} Y_o^{-1/2} x = |x_0|^2 = 1$.
 - ▶ Thus $E_o \subset \{x : x^T Y_o^{-1} x = 1\}$. The other direction is similar
2. The Principal Axes of E_o are the eigenvectors of $Y_o^{1/2}$, u_i .
 3. The lengths of the Principal Axes of E_o are $\sigma_i(Y_o)$.
 4. If $\sigma_i(Y_o) = 0$, the u_i is in the unobservable subspace.

Controllability Operator

Recall the Controllability Operator $\Psi_c : L_2(-\infty, 0] \rightarrow \mathbb{C}^n$

$$\Psi_c u = \int_{-\infty}^0 e^{-As} B u(s) ds$$

Which maps an input to a final state $x(0)$.

- Adjoint $\Psi_c^* : \mathbb{R}^n \rightarrow L_2(-\infty, 0]$

$$(\Psi_c^* x)(t) = B^* e^{-A^* t} x$$

Recall: The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is controllable if for any $x(0) \in \mathbb{R}^n$, there exists some $u \in L_2(-\infty, 0]$ such that

$$x(0) = \Psi_c u$$

Controllability Grammian

Definition

Definition 7.

The Controllability Grammian is

$$\begin{aligned} X_c &:= \Psi_c \Psi_c^* = \int_{-\infty}^0 e^{-As} B B^T e^{-A^T s} ds \\ &= \int_0^{\infty} e^{As} B B^T e^{A^T s} ds \end{aligned}$$

Recall

- X_c is the solution to

$$A X_c + X_c A^T + B B^T = 0$$

- $X_c > 0$ if and only if (A, B) is controllable.

Proposition 5.

Then (A, B) is controllable if and only if there exists a solution $X > 0$ to the Lyapunov equation

$$AX + XA^T + BB^T = 0$$

Proposition 6.

Suppose (A, B) is controllable. Then

1. X_c is invertible
2. Given x_0 , the solution to

$$\min_{u \in L_2(-\infty, 0]} \|u\|_{L_2}:$$

$$x_0 = \Psi_c u$$

is given by

$$u_{opt} = \Psi_c^* X_c^{-1} x_0$$

Proof.

The first is clear from $X_c > 0$. For the second part, we first show that u_{opt} is feasible. We then show that it is optimal.

- For feasibility, we note that

$$\begin{aligned}\Psi_c u_{opt} &= \Psi_c \Psi_c^* X_c^{-1} x_0 \\ &= X_c X_c^{-1} x_0 \\ &= x_0\end{aligned}$$

which implies feasibility



Proof.

Now that we know that u_{opt} is feasible, we show that for any other \bar{u} , if \bar{u} is feasible, then $\|\bar{u}\|_{L_2} \geq \|u_{opt}\|_{L_2}$.

- Define $P := \Psi_c^* X_c^{-1} \Psi_c$.

$$P^2 = \Psi_c^* X_c^{-1} \underbrace{\Psi_c \Psi_c^*}_{X_c} X_c^{-1} \Psi_c = \Psi_c^* X_c^{-1} \Psi_c = P$$

- Furthermore $P^* = P$.
- Thus P is a projection operator, which means

$$\langle Pu, (I - P)u \rangle = 0$$

- Thus for any \bar{u}

$$\|\bar{u}\|^2 = \|P\bar{u} + (I - P)\bar{u}\|^2 = \|P\bar{u}\|^2 + \|(I - P)\bar{u}\|^2.$$



Controllability Grammian

Proof.

- If \bar{u} is feasible, then

$$\begin{aligned}\|P\bar{u}\|^2 &= \|\Psi_c^* X_c^{-1} \Psi_c \bar{u}\|^2 \\ &= \|\Psi_c^* X_c x_0\|^2 \quad \text{since } \bar{u} \text{ is feasible} \\ &= \|u_{opt}\|^2\end{aligned}$$

- We conclude that

$$\|\bar{u}\|^2 = \|u_{opt}\|^2 + \|(I - P)\bar{u}\|^2 \geq \|u_{opt}\|^2$$

- Thus u_{opt} is optimal



This shows that u_{opt} is the minimum-energy input to achieve the final-state x_0 .

Drawbacks:

- Don't have infinite time.
- Open-loop

Controllability Grammian

Physical Interpretation

The controllability Grammian tells us the minimum amount of energy required to reach a state.

$$\|u_{opt}\|_{L_2}^2 = x_0^T X_c^{-1} x_0$$

Definition 8.

The **Controllability Ellipse** is the set of states which are reachable with 1 unit of energy.

$$\{\Psi_c u : \|u\|_{L_2} \leq 1\}$$

Proposition 7.

The following are equivalent

1. $\{\Psi_c u : \|u\|_{L_2} \leq 1\}$
2. $\{X_c^{1/2} x : \|x\| \leq 1\}$
3. $\{x : x^T X_c^{-1} x \leq 1\}$

Controllability Grammian

Finite-Time Grammian

Because we don't always have infinite time:

- What is the optimal way to get to x in time T

Finite-Time Controllability Operator: $\Psi_T : L_2[0, T] \rightarrow \mathbb{R}^n$.

$$\Psi_T u := \int_0^T e^{A(T-s)} B u(s) ds$$

Finite-Time Controllability Grammian

$$X_T := \Psi_T \Psi_T^* = \int_0^T e^{As} B B^T e^{A^T s} ds$$

Note: $X_T \geq X_s$ for $t \geq s$.

Controllability Grammian

Finite-Time Grammian

Cannot be found by solving the Lyapunov equation.

Must be found by numerical integration of the matrix-differential equation:

$$\dot{X}_T(t) = AX_T(t) + X_T(t)A^T + BB^T$$

from $t = 0$ to $t = T$.

- X_c is the steady-state solution.

Proposition 8.

Suppose (A, B) is controllable. Then

1. X_T is invertible
2. The solution to

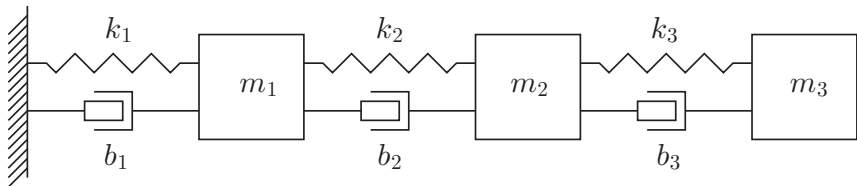
$$\min_{u \in L_2[0, T]} \|u\|_{L_2} :$$
$$x_f = \Psi_T u$$

is given by

$$u_{opt} = \Psi_T^* X_T^{-1} x_f$$

Finite-Time Grammian

Example



Consider the Spring-mass system ($k_i = m_i = 1$, $b_i = .8$)

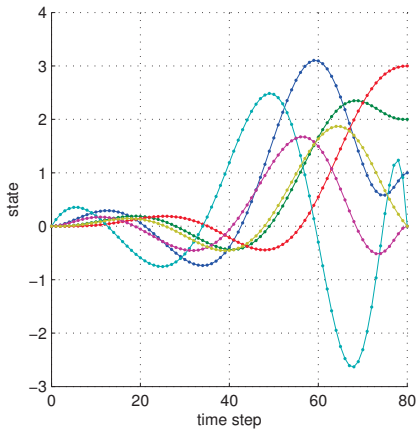
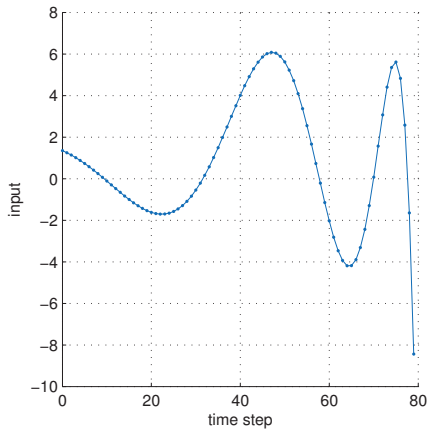
$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & -1.6 & .8 & 0 \\ 1 & -2 & 1 & .8 & -1.6 & .8 \\ 0 & 1 & -1 & 0 & .9 & -.8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u(t)$$

with desired final state

$$x_f = [1 \quad 2 \quad 3 \quad 0 \quad 0 \quad 0]^T$$

Finite-Time Grammian

Example



$$x_f = [1 \quad 2 \quad 3 \quad 0 \quad 0 \quad 0]^T$$