LMI Methods in Optimal and Robust Control

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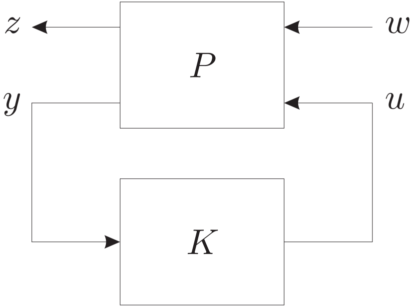
Lecture 10: An LMI for H_∞-Optimal Output Feedback Control

Optimal Output Feedback

Recall: Linear Fractional Transformation



Plant:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \qquad \text{where} \qquad P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Controller:

$$u = Ky \qquad \text{where} \qquad K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$$

Defining the System Variables

Choose K to minimize

$$\|P_{11} + P_{12}(I - KP_{22})^{-1}KP_{21}\|$$

Equivalently choose $\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ to minimize

$$\left\| \left[\begin{array}{c|c} \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & \begin{matrix} B_1 + B_2 D_K Q D_{21} \\ B_K Q D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} & D_{11} + D_{12} D_K Q D_{21} \end{array}\right] \right\|_{H_\infty}$$

where $Q = (I - D_{22}D_K)^{-1}$.

Representing the Closed-Loop System

Recall the Matrix Inversion Lemma:

Lemma 1.

$$
\begin{aligned}
&\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \\
&= \begin{bmatrix} A^{-1} + A^{-1}B(D - CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix} \\
&= \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1} \end{bmatrix}
\end{aligned}
$$

Closed Loop System is Nonlinear Function of A_K, B_K, C_K, D_K.

Recall that

$$\begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} = \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix}$$

where $Q = (I - D_{22} D_K)^{-1}$. Then

$$\begin{aligned} A_{cl} &:= \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I & -D_K \\ -D_{22} & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A & 0 \\ 0 & A_K \end{bmatrix} + \begin{bmatrix} B_2 & 0 \\ 0 & B_K \end{bmatrix} \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} A + B_2 D_K Q C_2 & B_2 (I + D_K Q D_{22}) C_K \\ B_K Q C_2 & A_K + B_K Q D_{22} C_K \end{bmatrix} \end{aligned}$$

Likewise

$$\begin{aligned} C_{cl} &:= \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} D_{12} & 0 \end{bmatrix} \begin{bmatrix} I + D_K Q D_{22} & D_K Q \\ Q D_{22} & Q \end{bmatrix} \begin{bmatrix} 0 & C_K \\ C_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} C_1 + D_{12} D_K Q C_2 & D_{12} (I + D_K Q D_{22}) C_K \end{bmatrix} \end{aligned}$$

A New Set of Decision Variables

Thus we have

$$\left[\begin{array}{cc|c} A+B_2D_KQC_2 & B_2(I+D_KQD_{22})C_K & B_1+B_2D_KQD_{21} \\ B_KQC_2 & A_K+B_KQD_{22}C_K & B_KQD_{21} \\ \hline \left[C_1+D_{12}D_KQC_2\right. & \left. D_{12}(I+D_KQD_{22})C_K\right] & D_{11}+D_{12}D_KQD_{21} \end{array}\right]$$

where $Q=(I-D_{22}D_K)^{-1}$.

- This is nonlinear in (A_K, B_K, C_K, D_K).
- Hence we make a change of variables (First of several).

$$\begin{aligned} A_{K2} &= A_K + B_KQD_{22}C_K \\ B_{K2} &= B_KQ \\ C_{K2} &= (I+D_KQD_{22})C_K \\ D_{K2} &= D_KQ \end{aligned}$$

A New parametrization of the closed-loop system.

This yields the system

$$\left[\begin{array}{c|c} \begin{bmatrix} A + B_2 D_{K2} C_2 & B_2 C_{K2} \\ B_{K2} C_2 & A_{K2} \end{bmatrix} & \begin{matrix} B_1 + B_2 D_{K2} D_{21} \\ B_{K2} D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 + D_{12} D_{K2} C_2 & D_{12} C_{K2} \end{bmatrix} & D_{11} + D_{12} D_{K2} D_{21} \end{array}\right]$$

Which is affine in $\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right]$.

Inverting the Variable Substitution

Recovering D_K

Hence we can optimize over our new variables.

- System is linear in new variables and eliminates original variables.
- However, the change of variables must be **invertible**.

Now suppose we have D_{K2}. Then

$$D_{K2} = D_K Q = D_K (I - D_{22} D_K)^{-1}$$

implies that

$$D_K = D_{K2}(I - D_{22} D_K) = D_{K2} - D_{K2} D_{22} D_K$$

or

$$(I + D_{K2} D_{22}) D_K = D_{K2}$$

which can be inverted to get

$$D_K = (I + D_{K2} D_{22})^{-1} D_{K2}$$

Inverting the Variable Substitution

Recovering C_K

If we recall that

$$(I - QM)^{-1} = I + Q(I - MQ)^{-1}M$$

then we get

$$I + D_K Q D_{22} = I + D_K(I - D_{22}D_K)^{-1}D_{22} = (I - D_K D_{22})^{-1}$$

Examine the variable C_{K2}

$$\begin{aligned} C_{K2} &= (I + D_K(I - D_{22}D_K)^{-1}D_{22})C_K \\ &= (I - D_K D_{22})^{-1}C_K \end{aligned}$$

Hence, given D_K and C_{K2}, we can recover C_K as

$$C_K = (I - D_K D_{22})C_{K2}$$

Inverting the Variable Substitution

Once we have C_K and D_K, the other variables are easily recovered as

$$B_K = B_{K2}Q^{-1} = B_{K2}(I - D_{22}D_K)$$
$$A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$$

To summarize, the original variables can be recovered as

$$D_K = (I + D_{K2}D_{22})^{-1}D_{K2}$$
$$B_K = B_{K2}(I - D_{22}D_K)$$
$$C_K = (I - D_K D_{22})C_{K2}$$
$$A_K = A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K$$

Closed Loop System Parameters

$$\left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] := \left[\begin{array}{c|c} \begin{bmatrix} A + B_2 D_{K2} C_2 & B_2 C_{K2} \\ B_{K2} C_2 & A_{K2} \end{bmatrix} & \begin{matrix} B_1 + B_2 D_{K2} D_{21} \\ B_{K2} D_{21} \end{matrix} \\ \hline \begin{bmatrix} C_1 + D_{12} D_{K2} C_2 & D_{12} C_{K2} \end{bmatrix} & D_{11} + D_{12} D_{K2} D_{21} \end{array}\right]$$

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

Or

$$A_{cl} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$$

$$B_{cl} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}$$

$$C_{cl} = \begin{bmatrix} C_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_2 & 0 \end{bmatrix}$$

$$D_{cl} = \begin{bmatrix} D_{11} \end{bmatrix} + \begin{bmatrix} 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}$$

Optimal Output Feedback Control

However, if we apply the KYP Lemma, the result is bilinear in X and A_K, B_K, C_K, D_K

- Dual KYP Lemma is used for Controller Synthesis
- Primal KYP Lemma is used for observer Synthesis
- For Observer-Based Controller Synthesis, we need both Primal AND Dual forms....

Lemma 2 (Transformation Lemma).

Suppose that

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0$$

Then there exist X_2, X_3, Y_2, Y_3 such that

$$X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1} = Y^{-1} > 0$$

where $Y_{cl} = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ has full rank.

Optimal Output Feedback Control

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where $Y_{cl} = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ has full rank.

The primal variable is

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

The dual variable is

$$\begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

Proof.

- Since
$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0,$$
by the Schur complement $X_1 > 0$ and $X_1^{-1} - Y_1 < 0$. Since $I - X_1Y_1 = X_1(X_1^{-1} - Y_1)$, we conclude that $I - X_1Y_1$ is invertible.
- Choose any two square invertible matrices X_2 and Y_2 such that
$$X_2Y_2^T = I - X_1Y_1$$
- Because X_2 and Y_2 are invertible,
$$Y_{cl}^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \text{ and } X_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$$
are also non-singular.

□

Optimal Output Feedback Control

Proof.

- Since
$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0,$$
by the Schur complement $X_1 > 0$ and $X_1^{-1} - Y_1 < 0$. Since $I - X_1Y_1 = X_1(X_1^{-1} - Y_1)$, we conclude that $I - X_1Y_1$ is invertible.
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are also non-singular.

□

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

will be the half-dual transformation.

- Y_{cl}^T contains the top half of the dual Lyapunov variable.
- X_{cl} contains the bottom half of the primal Lyapunov variable.

Optimal Output Feedback Control

Proof.

- Now define X and Y as

$$X = Y_{cl}^{-T} X_{cl} \qquad \text{and} \qquad Y = X_{cl}^{-1} Y_{cl}^{T}.$$

Then

$$XY = Y_{cl}^{-1} X_{cl} X_{cl}^{-1} Y_{cl} = I$$

□

Note that

$$Y_{cl}^{T} X = X_{cl} \qquad \text{and} \qquad X_{cl} Y = Y_{cl}^{T}$$

Optimal Output Feedback Control

Proof.

- Now define X and Y as

$$X = Y_{cl}^{-T} X_{cl} \quad \text{and} \quad Y = X_{cl}^{-1} Y_{cl}^{T}.$$

Then

$$XY = Y_{cl}^{-1} X_{cl} X_{cl}^{-1} Y_{cl} = I$$

□

Note that

$$Y_{cl}^{T} X = X_{cl} \quad \text{and} \quad X_{cl} Y = Y_{cl}^{T}$$

We will be applying the half-dual transformation

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \overbrace{\begin{bmatrix} A^T X + XA & XB & C^T \\ B^T X & -\gamma I & D^T \\ C & D & -\gamma I \end{bmatrix}}^{\text{primal KYP lemma}} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} Y_{cl}^T A^T X Y_{cl} + Y_{cl}^T X A Y_{cl} & Y_{cl}^T X B & Y_{cl}^T C^T \\ B^T X Y_{cl} & -\gamma I & D^T \\ C Y_{cl} & D & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y_{cl}^T A^T X_{cl}^T + X_{cl} A Y_{cl} & X_{cl} B & Y_{cl}^T C^T \\ B^T X_{cl}^T & -\gamma I & D^T \\ C Y_{cl} & D & -\gamma I \end{bmatrix}$$

Since

$$Y_{cl}^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \text{ and } X_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix},$$

this will only work if Y_2 and X_2 are somehow eliminated from the expression.

Optimal Output Feedback Control

Lemma 3 (Converse Transformation Lemma).

Given $X = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} > 0$ *where* X_2 *has full column rank. Let*

$$X^{-1} = Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}$$

then

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} > 0$$

and $Y_{cl} = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$ *has full column rank.*

This result shows the Transformation Lemma is not conservative.

Optimal Output Feedback Control

Proof.

Since X_2 is full rank, $X_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$ also has full column rank. Note that $YX = I$ implies

$$Y_{cl}^T X = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} = X_{cl}.$$

Hence

$$Y_{cl}^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} Y = X_{cl} Y$$

has full column rank. Now, since $XY = I$ implies $X_1 Y_1 + X_2 Y_2^T = I$, we have

$$X_{cl} Y_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ X_1 Y_1 + X_2 Y_2^T & X_1 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix}$$

Furthermore, because Y_{cl} has full rank,

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_{cl} Y_{cl} = X_{cl} Y X_{cl}^T = Y_{cl}^T X Y_{cl} > 0$$

Optimal Output Feedback Control

Proof.

Since X_2 is full rank, $X_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix}$ also has full column rank. Note that $YX = I$ implies

$$Y_{cl}^T X = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} = X_{cl}.$$

Hence

$$Y_{cl}^T = \begin{bmatrix} Y_1 & Y_2 \\ I & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} Y = X_{cl} Y$$

has full column rank. Now, since $XY = I$ implies $X_1 Y_1 + X_2 Y_2^T = I$, we have

$$X_{cl} Y_{cl} = \begin{bmatrix} I & 0 \\ X_1 & X_2 \end{bmatrix} \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ X_1 Y_1 + X_2 Y_2^T & X_1 \end{bmatrix} = \begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix}$$

Furthermore, because Y_{cl} has full rank,

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_{cl} Y_{cl} = X_{cl} Y X_{cl}^T = Y_{cl}^T X Y_{cl} > 0$$

□

Note the relationship:

$$\begin{bmatrix} Y_1 & I \\ I & X_1 \end{bmatrix} = X_{cl} Y_{cl}$$

Note how both X_2 and Y_2 vanish?

Optimal Output Feedback Control

Theorem 4.

The following are equivalent.

- *There exists a* $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|\underline{S}(K,P)\|_{H_\infty} < \gamma$.
- *There exist* $X_1, Y_1, A_n, B_n, C_n, D_n$ *such that* $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

Moreover, such a controller is given by $D_K = (I + D_{K2} D_{22})^{-1} D_{K2}$,
$B_K = B_{K2}(I - D_{22} D_K)$, $C_K = (I - D_K D_{22}) C_{K2}$,
$A_K = A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K$ *where*

$$\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}$$

for any full-rank X_2 *and* Y_2 *such that*

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1}$$

Proof: If.

Suppose there exist $X_1, Y_1, A_n, B_n, C_n, D_n$ such that the LMI is feasible. Since

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0,$$

by the transformation lemma, there exist X_2, X_3, Y_2, Y_3 such that

$$X := \begin{bmatrix} X & X_2 \\ X_2^T & X_3 \end{bmatrix} = \begin{bmatrix} Y & Y_2 \\ Y_2^T & Y_3 \end{bmatrix}^{-1} > 0$$

where $Y_{cl} = \begin{bmatrix} Y & I \\ Y_2^T & 0 \end{bmatrix}$ has full row rank. Let $K = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ where

$$\begin{aligned} D_K &= (I + D_{K2} D_{22})^{-1} D_{K2} \\ B_K &= B_{K2}(I - D_{22} D_K) \\ C_K &= (I - D_K D_{22}) C_{K2} \\ A_K &= A_{K2} - B_K (I - D_{22} D_K)^{-1} D_{22} C_K. \end{aligned}$$

Optimal Output Feedback Control

Proof: If.

and where

$$\begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}.$$

Thus $\underline{S}\left(\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right], P \right) = \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array} \right]$, where

$$\begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

$$= \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix}$$

$$\begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix}$$

We now apply the KYP lemma to this system.

Optimal Output Feedback Control

Proof: If.

Since Y_{cl} is full rank, $\|\underline{\mathsf{S}}(K,P)\|_{H_\infty} < \gamma$ if and only if

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0$$

In the next several slides, we show that

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} =$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

$\square$

Proof: If.

First we note that

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X_{cl} & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} Y_{cl}^T A_{cl}^T X_{cl}^T + X_{cl} A_{cl} Y_{cl} & X_{cl} B_{cl} & Y_{cl}^T C_{cl}^T \\ B_{cl}^T X_{cl}^T & -\gamma I & D_{cl}^T \\ C_{cl} Y_{cl} & D_{cl} & -\gamma I \end{bmatrix}$$

By comparing terms, our goal is then to show that

$$\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X_{cl} A_{cl} Y_{cl} & X_{cl} B_{cl} \\ C_{cl} Y_{cl} & D_{cl} \end{bmatrix}$$
$$= \begin{bmatrix} AY_1 + B_2 C_n & A + B_2 D_n C_2 & B_1 + B_2 D_n D_{21} \\ A_n & X_1 A + B_n C_2 & X_1 + B_n D_{21} \\ C_1 Y_1 + D_{12} D_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} \end{bmatrix}$$

□

Optimal Output Feedback Control

Proof: If.

$$\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \right) \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

□

We now examine each of these terms separately.

Optimal Output Feedback Control

Proof: If.

First, we have

$$
\begin{aligned}
&\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 & 0 \\ X_1 & X_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} \begin{bmatrix} Y_1 & I & 0 \\ Y_2^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} I & 0 & 0 \\ X_1 & X_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} AY_1 & A & B_1 \\ 0 & 0 & 0 \\ C_1Y_1 & C_1 & D_{11} \end{bmatrix} \\
&= \begin{bmatrix} AY_1 & A & B_1 \\ X_1AY_1 & X_1A & X_1B_1 \\ C_1Y_1 & C_1 & D_{11} \end{bmatrix}
\end{aligned}
$$

□

Optimal Output Feedback Control

Proof: If.

Next, we observe that

$$\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} I & 0 & 0 \\ X_1 & X_2 & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \begin{bmatrix} Y_1 & I & 0 \\ Y_2^T & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B_2 \\ X_2 & X_1 B_1 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 & 0 \\ C_2 Y_1 & C_2 & D_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{bmatrix}$$

The obvious change of variables is now

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} + \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proof: If.

Next, we note that by definition

$$\begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1}.$$

and hence

$$\begin{aligned} &\begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} \\ &= \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right] \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix}^{-1} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} \\ &= \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

□

Optimal Output Feedback Control

Proof: If.

We conclude that

$$\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \left(\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I & 0 & 0 \\ 0 & C_2 & D_{21} \end{bmatrix}$$

$$= \begin{bmatrix} B_2 C_n & B_2 D_n C_2 & B_2 D_n D_{21} \\ A_n - X_1 A Y_1 & B_n C_2 & B_n D_{21} \\ D_{12} C_n & D_{12} D_n C_2 & D_{12} D_n D_{21} \end{bmatrix}$$

□

Optimal Output Feedback Control

Proof: If.

In summary, we have

$$\begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{cl} & B_{cl} \\ C_{cl} & D_{cl} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} X_{cl}A_{cl}Y_{cl} & X_{cl}B_{cl} \\ C_{cl}Y_{cl} & D_{cl} \end{bmatrix}$$

$$= \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & 0 \\ C_1 & 0 & D_{11} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$+ \begin{bmatrix} X_{cl} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & B_2 \\ I & 0 \\ 0 & D_{12} \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} 0 & I & 0 \\ C_2 & 0 & D_{21} \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 \\ 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} AY_1 & A & B_1 \\ X_1AY_1 & X_1A & X_1B_1 \\ C_1Y_1 & C_1 & D_{11} \end{bmatrix} + \begin{bmatrix} B_2C_n & B_2D_nC_2 & B_2D_nD_{21} \\ A_n - X_1AY_1 & B_nC_2 & B_nD_{21} \\ D_{12}C_n & D_{12}D_nC_2 & D_{12}D_nD_{21} \end{bmatrix}$$

$$= \begin{bmatrix} AY_1 + B_2C_n & A + B_2D_nC_2 & B_1 + B_2D_nD_{21} \\ A_n & X_1A + B_nC_2 & X_1 + B_nD_{21} \\ C_1Y_1 + D_{12}D_n & C_1 + D_{12}D_nC_2 & D_{11} + D_{12}D_nD_{21} \end{bmatrix}$$

as desired. □

Proof: If.

We therefore conclude that

$$\left\| \underline{\mathsf{S}}\left(\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right], P \right) \right\|_{H_\infty} = \left\| \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array}\right] \right\|_{H_\infty} < \gamma$$

since

$$\begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} Y_{cl}^T A_{cl}^T X_{cl}^T + X_{cl} A_{cl} Y_{cl} & X_{cl} B_{cl} & Y_{cl}^T C_{cl}^T \\ B_{cl}^T X_{cl}^T & -\gamma I & D_{cl}^T \\ C_{cl} Y_{cl} & D_{cl} & -\gamma I \end{bmatrix} =$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

Optimal Output Feedback Control

Proof: Only If.

Likewise, if

$$\left\| \underline{\mathsf{S}}\left(\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array} \right], P \right) \right\|_{H_\infty} = \left\| \left[\begin{array}{c|c} A_{cl} & B_{cl} \\ \hline C_{cl} & D_{cl} \end{array} \right] \right\|_{H_\infty} < \gamma$$

then there exists $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} < 0.$$

Define

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_2^T & Y_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}^{-1} = X^{-1} \qquad \text{and} \qquad Y_{cl} = \begin{bmatrix} Y_1 & I \\ Y_2^T & 0 \end{bmatrix}$$

where because the inequalities are strict, we can assume that X_2 has full row rank. Then, according to the converse transformation lemma, Y_{cl} has full row rank and

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0.$$

Optimal Output Feedback Control

Proof: Only If.

Now define

$$\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} = \begin{bmatrix} X_2 & X_1 B_2 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{K2} & B_{K2} \\ C_{K2} & D_{K2} \end{bmatrix} \begin{bmatrix} Y_2^T & 0 \\ C_2 Y_1 & I \end{bmatrix} + \begin{bmatrix} X_1 A Y_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$A_{K2} = A_K + B_K (I - D_{22} D_K)^{-1} D_{22} C_K \qquad B_{K2} = B_K (I - D_{22} D_K)^{-1}$$

$$C_{K2} = (I + D_K (I - D_{22} D_K)^{-1} D_{22}) C_K \qquad D_{K2} = D_K (I - D_{22} D_K)^{-1}$$

and we have

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y_{cl}^T A_{cl}^T X_{cl}^T + X_{cl} A_{cl} Y_{cl} & X_{cl} B_{cl} & Y_{cl}^T C_{cl}^T \\ B_{cl}^T X_{cl}^T & -\gamma I & D_{cl}^T \\ C_{cl} Y_{cl} & D_{cl} & -\gamma I \end{bmatrix}$$

$$= \begin{bmatrix} Y_{cl}^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} & C_{cl}^T \\ B_{cl}^T X & -\gamma I & D_{cl}^T \\ C_{cl} & D_{cl} & -\gamma I \end{bmatrix} \begin{bmatrix} Y_{cl} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0. \square$$

Conclusion

To solve the H_∞-optimal output-feedback problem, we solve

$$\min_{\gamma, X_1, Y_1, A_n, B_n, C_n, D_n} \gamma \qquad \text{such that}$$

$$\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$$

$$\begin{bmatrix} AY_1 + Y_1 A^T + B_2 C_n + C_n^T B_2^T & *^T & *^T & *^T \\ A^T + A_n + [B_2 D_n C_2]^T & X_1 A + A^T X_1 + B_n C_2 + C_2^T B_n^T & *^T & *^T \\ [B_1 + B_2 D_n D_{21}]^T & [X_1 B_1 + B_n D_{21}]^T & -\gamma I & *^T \\ C_1 Y_1 + D_{12} C_n & C_1 + D_{12} D_n C_2 & D_{11} + D_{12} D_n D_{21} & -\gamma I \end{bmatrix} < 0$$

Conclusion

Then, we construct our controller using

$$
\begin{aligned}
D_K &= (I + D_{K2}D_{22})^{-1}D_{K2}\\
B_K &= B_{K2}(I - D_{22}D_K)\\
C_K &= (I - D_K D_{22})C_{K2}\\
A_K &= A_{K2} - B_K(I - D_{22}D_K)^{-1}D_{22}C_K.
\end{aligned}
$$

where

$$
\left[\begin{array}{c|c} A_{K2} & B_{K2} \\ \hline C_{K2} & D_{K2} \end{array}\right] = \begin{bmatrix} X_2 & X_1B_2 \\ 0 & I \end{bmatrix}^{-1} \left[\begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} - \begin{bmatrix} X_1AY_1 & 0 \\ 0 & 0 \end{bmatrix}\right] \begin{bmatrix} Y_2^T & 0 \\ C_2Y_1 & I \end{bmatrix}^{-1}.
$$

and where X_2 and Y_2 are any matrices which satisfy $X_2Y_2^T = I - X_1Y_1$.

- e.g. Let $Y_2 = I$ and $X_2 = I - X_1Y_1$.
- The optimal controller is NOT uniquely defined.
- Don't forget to check invertibility of $I - D_{22}D_K$

Conclusion

The H_∞-optimal controller is a dynamic system.

- Transfer Function $\hat{K}(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$

Minimizes the effect of external input (w) on external output (z).

$$\|z\|_{L_2} \leq \|\underline{\mathsf{S}}(P,K)\|_{H_\infty} \|w\|_{L_2}$$

- Minimum Energy Gain

Alternative Formulation

Theorem 5.

Let N_o and N_c be full-rank matrices whose images satisfy

$$\textit{Im } N_o = \textit{Ker } \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$$
$$\textit{Im } N_c = \textit{Ker } \begin{bmatrix} B_2^T & D_{12}^T \end{bmatrix}$$

Then the following are equivalent.

- *There exists a* $\hat{K} = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]$ *such that* $\|S(K,P)\|_{H_\infty} < 1$.
- *There exist* X_1, Y_1 *such that* $\begin{bmatrix} X_1 & I \\ I & Y_1 \end{bmatrix} > 0$ *and*

$$\begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} X_1 A + A^T X_1 & *^T & *^T \\ B_1^T X_1 & -I & *^T \\ C_1 & D_{11} & -I \end{bmatrix} \begin{bmatrix} N_o & 0 \\ 0 & I \end{bmatrix} < 0$$

$$\begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AY_1 + Y_1 A & *^T & *^T \\ C_1 Y_1 & -I & *^T \\ B_1^T & D_{11}^T & -I \end{bmatrix} \begin{bmatrix} N_c & 0 \\ 0 & I \end{bmatrix} < 0$$