

Systems Analysis and Control

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Lecture 4: The Laplace Transform

Introduction

In this Lecture, you will learn:

Simple ways to use state-space.

- How to find the output given an input.
- Linearity

Linear Systems.

- The Fourier Series
 - ▶ Representing signals as the sum of sinusoids.
 - ▶ Representing systems using response to sinusoids.
- The Laplace transform
 - ▶ Representing signals in the frequency domain.
 - ▶ Representing systems using response to sinusoids.

How to represent a system using a *Transfer Function*.

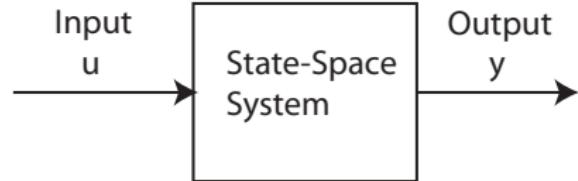
- How to find the output given an input.

Recall the state-space form

Find the output given the input

State-Space:

$$\begin{aligned}\dot{x} &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \quad x(0) = 0\end{aligned}$$



Basic Question: Given an input function, $u(t)$, what is the output?

Solution: Solve the differential Equation.

Example: The equation

$$\dot{x}(t) = ax(t), \quad x(0) = x_0$$

has solution

$$x(t) = e^{at}x_0,$$

But we are interested in **Matrices!!!**

Not a rule, but sometimes... If it works for scalars, it also works for matrices.

The Solution to State-Space

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$

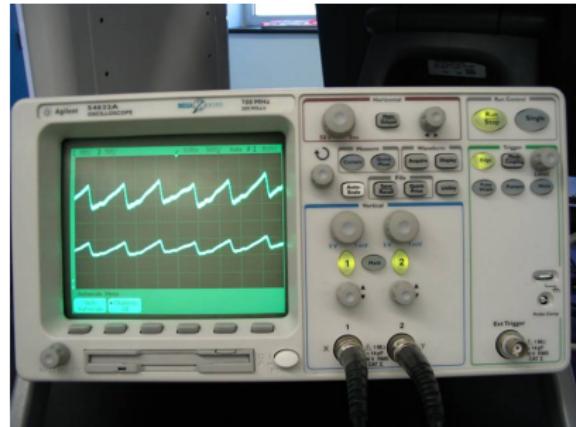
$$y(t) = Cx(t) + Du(t) \quad x(0) = 0$$

The equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0$$

has solution

$$x(t) = e^{At}x_0$$



The exponential for matrices is defined the same way as scalars

$$e^{At} = I + (At) + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots$$

Don't Worry! You will never have to calculate a matrix exponential by hand.

The important part is that

$$\dot{x}(t) = \frac{d}{dt}e^{At}x_0 = Ae^{At}x_0 = Ax(t), \quad \text{and} \quad x(0) = e^0x_0 = x_0$$

What happens when we add an input instead of an initial condition?

Find the output given the input

State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t) \quad x(0) = 0$$

The equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0$$

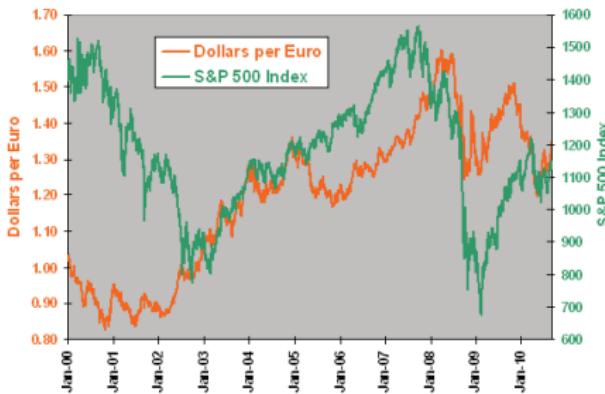
has solution

$$x(t) = \int_0^t e^{A(t-s)} Bu(s) ds$$

Proof.

Check the solution:

$$\begin{aligned}\dot{x}(t) &= e^0 Bu(t) + A \int_0^t e^{A(t-s)} Bu(s) ds \\ &= Bu(t) + Ax(t)\end{aligned}$$



Find the output given the input

Solution for State-Space

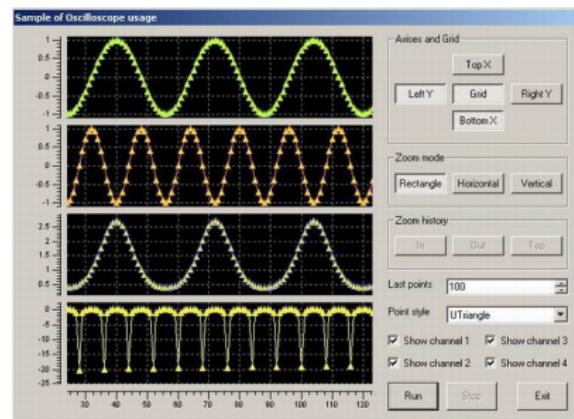
State-Space:

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t) \quad x(0) = 0$$

Now that we have $x(t)$, finding $y(t)$ is easy

$$\begin{aligned} y(t) &= Cx(t) + Du(t) \\ &= \int_0^t Ce^{A(t-s)} Bu(s) ds + Du(t) \end{aligned}$$



Conclusion: Given $u(t)$, only one integration is needed to find $y(t)$!

Note that the state, x , doesn't appear!!

We have solved the problem.

Calculating the Output

Numerical Example, $u(t) = \sin(t)$

State-Space:

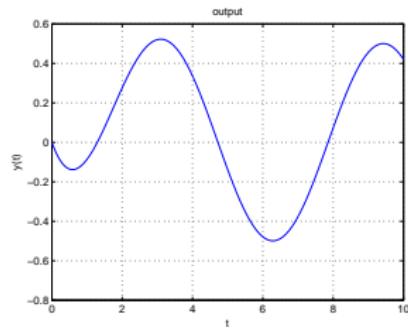
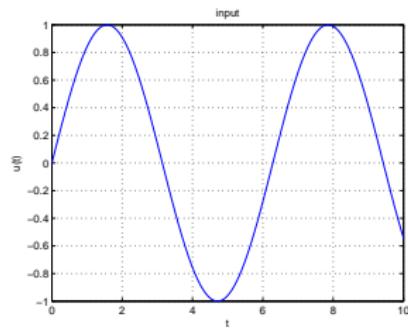
$$\dot{x} = -x(t) + u(t)$$

$$y(t) = x(t) - .5u(t) \quad x(0) = 0$$

$$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$$

Solution:

$$\begin{aligned} y(t) &= \int_0^t Ce^{A(t-s)} Bu(s)ds + Du(t) \\ &= e^{-t} \int_0^t e^s \sin(s) ds - \frac{1}{2} \sin(t) \\ &= \frac{1}{2} e^{-t} (e^s (\sin s - \cos s)|_0^t) - \frac{1}{2} \sin(t) \\ &= \frac{1}{2} e^{-t} (e^t (\sin t - \cos t) + 1) - \frac{1}{2} \sin(t) \\ &= \frac{1}{2} (e^{-t} - \cos t) \end{aligned}$$



Calculating Complicated Outputs

Linearity

Problem: Finding the integral of complicated functions is tough.

Solution: Use linear systems.

Lets call the system G .

G is a mechanism of converting inputs to outputs. For a state-space system,

$$y(t) = \int_0^t C e^{A(t-s)} B u(s) ds + Du(t)$$



Definition 1.

A **SYSTEM** is **Linear** if for any scalars, α and β , and any inputs u_1 and u_2 ,

$$G(\alpha u_1 + \beta u_2) = \alpha Gu_1 + \beta Gu_2$$

State-space systems are **LINEAR**.

Properties of Linearity

Properties: Multiplication

- $y = Gu$ implies $\alpha y = G(\alpha u)$.
- Amplification of input amplifies output equally.

Properties: Addition

- $y_1 = Gu_1$ and $y_2 = Gu_2$ implies $y_1 + y_2 = G(u_1 + u_2)$.
- Can calculate complicated outputs by addition.
 - ▶ Step 1: Decompose input into simple parts: $u(t) = u_1(t) + u_2(t)$
 - ▶ Step 2: Calculate simple outputs: y_1 and y_2 .
 - ▶ Step 3: Calculate the complicated output: $y(t) = y_1(t) + y_2(t)$

Example: Using previous system,

$$\begin{aligned}\dot{x} &= -x(t) + u(t) \\ y(t) &= x(t) - .5u(t) \quad x(0) = 0\end{aligned}$$

Let $u(t) = 2 \sin(t) - \cos(t)$.

Question: Can we compute the output without recalculating the integral???

Example of Linearity

Recall our simple state-space system:

$$A = -1; \quad B = 1; \quad C = 1; \quad D = -.5$$

- We found the that input $u_1(t) = \sin(t)$ produces output $y_1(t) = \frac{1}{2} (e^{-t} - \cos t)$.

We can also find that for input $u_2(t) = \cos(t)$, we have

$$\begin{aligned} y_2(t) &= \int_0^t Ce^{A(t-s)} Bu_2(s)ds + Du_2(t) \\ &= e^{-t} \int_0^t e^s \cos(s)ds - \frac{1}{2} \cos(t) \\ &= \frac{1}{2} (\sin t - e^{-t}) \end{aligned}$$

So for the **composite input**, $u(t) = 2u_1(t) - u_2(t)$, we find

$$\begin{aligned} y(t) &= 2y_1(t) - y_2(t) \\ &= (e^{-t} - \cos t) - \frac{1}{2} (\sin t - e^{-t}) \\ &= \frac{3}{2} e^{-t} - \cos t - \frac{1}{2} \sin t \end{aligned}$$

New Topic: Fourier Series

We calculated the response to a complicated signal

$$u(t) = 2 \sin(t) - \cos(t)$$

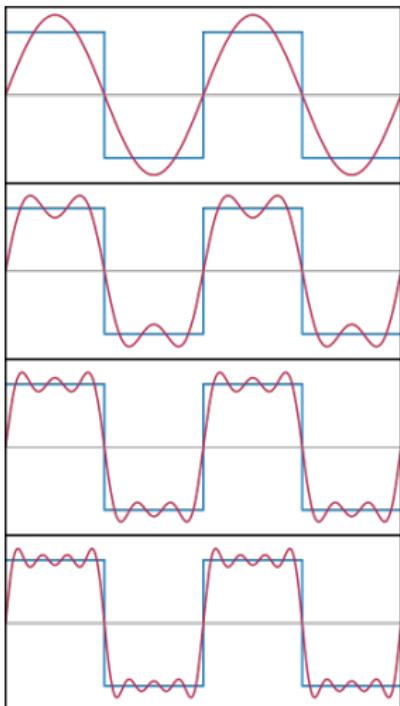
by adding the responses to the simple sinusoids:

$$u_1(t) = \sin(t) \quad \text{and} \quad u_2(t) = \cos(t)$$

So that $y(t) = y_1(t) + y_2(t)$

Question: What about more complicated signals???

Claim: We can recreate ANY signal $u(t)$ by adding up sinusoids.



Conclusion: Since we know the output when the input is sinusoid, we can calculate the solution for ANY input signal.

New Topic: Fourier Series

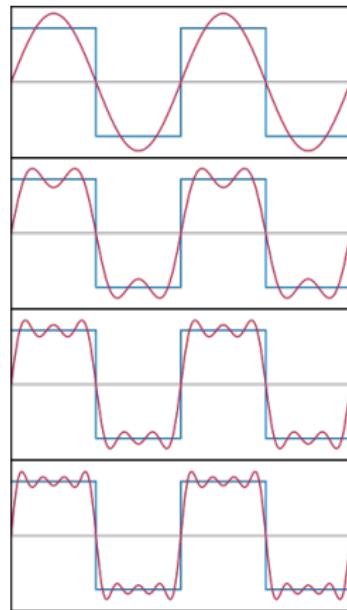
Question: How do we express our input using sines and cosines???

Answer: The Fourier Series.

Theorem 2 (Fourier Series).

Any input signal, u on the time interval $[-\pi, \pi]$ is the combination of sines and cosines:

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \sin(nt) + b_n \cos(nt)]$$



$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(s) \sin(ns) ds \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(s) \cos(ns) ds$$

The Fourier Series

Assume we know the output produced by sinusoids:

- $y_{1,n}(t)$ is the output from $u_{1,n}(t) = \sin(nt)$
- $y_{2,n}(t)$ is the output from $u_{2,n}(t) = \cos(nt)$

Then for the input

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \sin(nt) + b_n \cos(nt)]$$

The output is

$$y(t) = \frac{b_0}{2} y_{2,0}(t) + \sum_{n=1}^{\infty} [a_n y_{1,n}(t) + b_n y_{2,n}(t)]$$

Conclusion: The functions $y_n(t)$ make it very easy to calculate the output for any input.

Problem: We need an infinite number of a_n and $y_n(t)$.

The Fourier Transform

An alternative to the Fourier Series is the **Fourier Transform**.

Instead of a **Sum** of sinusoids, we express the signal as an **Integral** of sinusoids:

$$u(t) = \int_0^{\infty} a(\omega) \sin(\omega t) d\omega + \int_0^{\infty} b(\omega) \cos(\omega t) d\omega$$

Alternatively, (using $\sin(\omega t) = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}$ and $\cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2}$):

$$u(t) = \int_{-\infty}^{\infty} \hat{u}(i\omega) e^{i\omega t} d\omega$$

Here $\hat{u}(i\omega)$ is the **Fourier Transform** of the signal $u(t)$ and is computed as follows:

Theorem 3.

The Fourier Transform of the signal u can be found as

$$\hat{u}(i\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} u(t) dt$$

Inverse Fourier Transform

Recall: \hat{u} is the Fourier Transform of u if

$$\hat{u}(\omega) = \int_{-\infty}^{\infty} e^{-\imath\omega t} u(t) dt$$

In order to recover u from \hat{u} , we can use the following:

Theorem 4 (Inverse Fourier Transform).

Given \hat{u} , the Fourier Transform of u , we can find u as

$$u(t) = \int_{-\infty}^{\infty} e^{\imath\omega t} \hat{u}(\omega) d\omega$$

The Fourier Transform

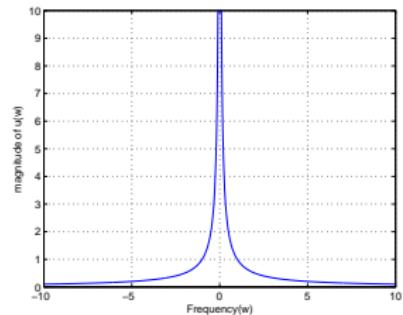
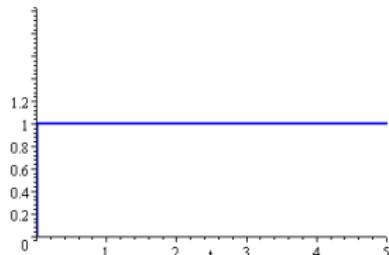
Example: Step Function

Consider the input signal:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

The Fourier transform is

$$\begin{aligned}\hat{u}(\omega) &= \int_0^{\infty} e^{-i\omega t} dt \\ &= \frac{1}{-\omega} [e^{-i\omega t}]_0^{\infty} = \frac{1}{-\omega}\end{aligned}$$



The Fourier Transform

Example

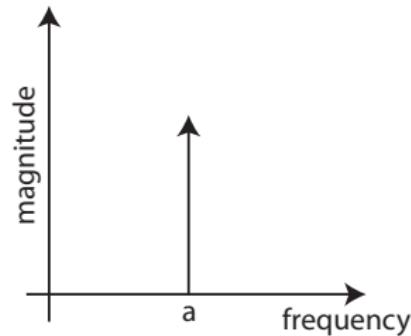
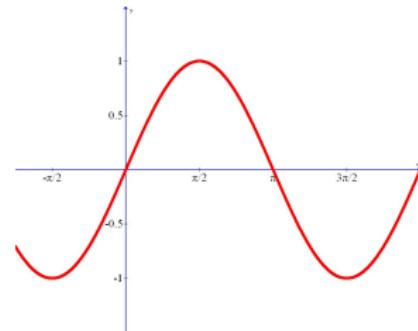
For the signal Consider the input signal:

$$u(t) = e^{iat}$$

The Fourier transform is

$$\hat{u}(i\omega) = \begin{cases} \infty & \omega = a \\ 0 & \omega \neq a \end{cases} = \delta(\omega - a)$$

δ is the Dirac Delta function.



We'll do many more examples next lecture.

The Fourier Transform

The Fourier transform can be applied to samples of music to understand the frequency composition.

Application: The Graphic Equalizer:

The Fourier Transform

Application

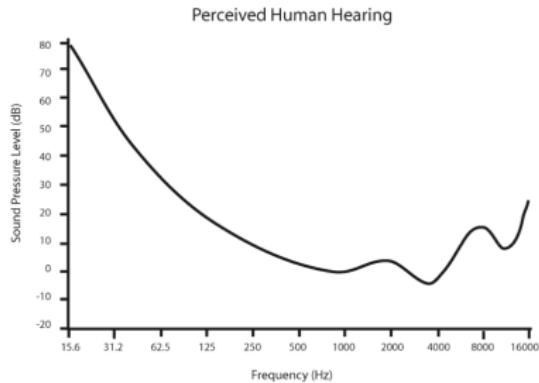


Figure: Equal Loudness Test

Audio Compression

- Sound is communicated in compression waves
- Humans can only hear sound in the range 20Hz-20kHz
- Music signals are usually simpler in frequency content than in time.
- e.g. The .mp3 standard.

Transfer Functions

Question: How to use the Fourier Transform to understand systems??

Suppose we know the output of a system for every possible input of the form $e^{-i\omega t}$.

- Input $u_\omega(t) = e^{-i\omega t}$ produces output $g(\omega, t)$

Then for any input with Fourier Transform $\hat{u}(i\omega)$, we can calculate the output y as

$$y(t) = \int_{-\infty}^{\infty} \hat{u}(i\omega) g(\omega, t) d\omega$$

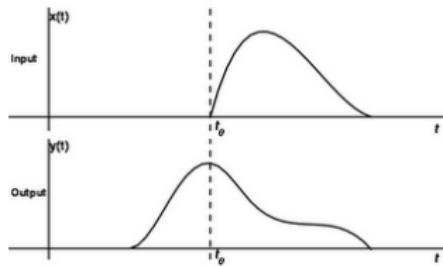
Problem: How to find $y(\omega, t)$?

- Need to calculate response to every frequency.

For causal Linear Time-Invariant Systems, the solution is easy.

- But we need to use the Laplace Transform.

Causality



Definition 5.

In a **Causal System**, a change in input at a later time $u(t + \tau)$ cannot affect the present output $y(t)$.

Every physical system in the universe is causal.

Causal:

$$\dot{x}(t) = x(t)$$

Non-Causal Systems are usually **artificial**.

Non-Causal

$$\dot{x}(t) = x(t + 1)$$

- Noise filtering

- ▶ Averages window of future and past.
- ▶ Playback must be delayed due to causality

Conclusion: We only consider causal systems in this class.

The Laplace Transform

Transfer Functions

For causal systems, we can use the Laplace Transform:

Definition 6.

The **Laplace Transform** of the signal u can be found as

$$\hat{u}(s) = \int_0^{\infty} e^{-st} u(t) dt$$

Essentially the same as the Fourier Transform, we have the following property.

Theorem 7.

For any causal, bounded, linear, time-invariant system, there exists a function $\hat{G}(s)$ such that

$$y(t) = \int_{-\infty}^{\infty} \hat{G}(i\omega) \hat{u}(i\omega) e^{-i\omega t} d\omega.$$

The function, \hat{G} is called the **Transfer Function** of the system.

The Laplace Transform

Transfer function example

Recall our old state-space system

$$\begin{aligned}\dot{x} &= -x(t) + u(t) \\ y(t) &= x(t) - .5u(t) \quad x(0) = 0\end{aligned}$$

This has a transfer function

$$\hat{G}(s) = \frac{1}{s+1} - .5$$

Take my word for it!

Before we can find these transfer functions ourselves:

- We need more practice with the Laplace transform
- We need to study more properties of the Laplace transform.
- Next Lecture

Summary

What have we learned today?

Simple ways to use state-space.

- How to find the output given an input.
- Linearity

Linear Systems.

- The Fourier Series
 - ▶ Representing signals as the sum of sinusoids.
 - ▶ Representing systems using response to sinusoids.
- The Laplace transform
 - ▶ Representing signals in the frequency domain.
 - ▶ Representing systems using response to sinusoids.

Next Lecture: Calculating the Laplace Transform