Sum of squares for sampled-data systems

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Abstract

This article proposes a new approach to stability analysis of linear systems with sampled-data inputs. The method, based on a variation of the discrete-time Lyapunov approach, provides stability conditions using functional variables subject to convex constraints. These stability conditions can be solved using the sum of squares methodology with little or no conservatism in both the case of synchronous and asynchronous sampling. Numerical examples are included to show convergence.

Index Terms

Sampled-Data systems, Lyapunov function, Sum of squares.

I. Introduction

In recent years, much attention has been paid to Networked Control Systems (NCS) (see [1], [2]). These systems contain several distributed plants which are connected through a communication network. In such applications, a heavy temporary load of computation on a processor can corrupt the sampling period of a controller. On the other side, the sampling period can be included in the design in order to avoid this load. In both cases, the variations of the sampling period will affect the stability properties of the system. Another phenomenon, which has been widely investigated concerns stability under packet losses. In wireless networks, a transmission of data packets is not always guaranteed. The objective is to guarantee stability even if some packets are lost in the communication. It is thus an important issue to develop robust stability conditions with respect to the variations of sampling period.

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Sampled-data systems have extensively been studied in the literature [3]–[7] and the references therein. It is now reasonable to design controllers which guarantee the robustness of the solutions of the closed-loop system under periodic samplings. However in the case of asynchronous sampling, there are still several open problems. For example, the practical situation where the difference between two successive sampling instants is not constant but time-varying. Recently, several articles have addressed the problem of time-varying periods based on a discrete-time approach, [8]–[10]. Recent papers have considered the modeling of continuous-time systems with sampled-data control in the form of continuous-time systems with delayed control input. In [4], a Lyapunov-Krasovskii approach was introduced. Improvements were provided in [5], [11], using the small gain theorem, and in [12], based on the analysis of impulsive systems. These approaches dealt with time-varying sampling periods as well as with uncertain systems (see [4] and [12]). Nevertheless, these sufficient conditions are still conservative. This means that the sufficient conditions obtained by continuous time approaches are not able to guarantee asymptotic stability whereas the system is stable. Recently several authors [13]–[15] refined those approaches and obtained tighter conditions.

The key insight of this paper is that once we have developed the discrete-continuous Lyapunov conditions sufficient for stability, then these conditions can be verified computationally using recently developed algorithms for the optimization of polynomial functions. In particular, we use the machinery developed in [16] to reformulate the stability question as a convex optimization problem with polynomial variables. We then use the software package SOSTOOLS [17] to solve the optimization problem. As can be seen in the numerical examples, the result is a sequence of stability tests of increasing accuracy. Furthermore, in the numerical examples, the accuracy of the stability test approaches the analytical limit exponentially fast as a complexity of the algorithm increases.

This article is based on a Lyapunov approach introduced in [18]. This result is based on the discrete-time Lyapunov theorem and expressed with the continuous-time model of sampled-data systems. More precisely, this article analyzes the link between the discrete-time Lyapunov theorem employed, for instance in [8]–[10], and the continuous-time approach proposed in [4], [12], [13], [15]. Asymptotic stability criteria are provided for both synchronous and asynchronous samplings. Those criteria were expressed in terms of linear matrix inequalities. The main contribution of this paper is the use of sum of squares tools to provide larger upper-bounds of the maximum allowable sampling period than the existing ones (based on the continuous-time modeling).

This article is organized as follows. The next section formulates the problem. Section 3 presents a result on asymptotic stability of sampled-data systems. Section 4 presents several theorems on asymptotic stability of sampled-data systems expressed in terms of sum of squares. Some examples and simulations are provided in Section 6 and show the efficiency of the method.

Notation : Throughout the article, the sets $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ and $\mathbb{S}^n$ denote respectively the set of natural numbers, nonnegative real numbers, the set of n-dimensional real-valued vectors, the set of $n\times n$ real valued matrices and the subspace of $\mathbb{R}^{n\times n}$ of symmetric matrices. The superscript 'T' stands for the matrix transposition. The notation $P>0$ for $P\in\mathbb{S}^n$ means that P is positive definite. The symbols I and 0 represent the identity and the zero matrices of appropriate dimension.

II. Problem formulation

Consider the following sampled-data system

$$\forall t\in[t_k,\ t_{k+1}),\quad \dot{x}(t)=Ax(t)+Bu(t_k), \tag{1}$$

where $x\in\mathbb{R}^n$ and $u\in\mathbb{R}^m$ represent the state and the input vectors. Define the sampling times $\{t_k\}_{k\in\mathbb{N}}$ to be an increasing sequence of positive scalars such that $\bigcup_{k\in\mathbb{N}}[t_k,\ t_{k+1})=[0,\ +\infty)$. Suppose that the sampling intervals, T_k, are bounded so that there exist positive scalars $\mathscr{T}_1\leq\mathscr{T}_2$ such that

$$T_k:=t_{k+1}-t_k\in[\mathscr{T}_1,\ \mathscr{T}_2], quad\forall k\in\mathbb{N}. \tag{2}$$

The sequence $\{t_k\}_{k\in\mathbb{N}}$ represents the sampling instants of the controller. The matrices A and B are constant, known, and of appropriate dimension. The control law is chosen to be linear state feedback,

$$u=Kx \tag{3}$$

with a gain $K\in\mathbb{R}^{m\times n}$. Then the closed-loop system is governed by

$$\dot{x}(t)=Ax(t)+BKx(t_k)\quad\forall t\in[t_k,\ t_{k+1}). \tag{4}$$

The differential equation (4) with the control law (3) can be integrated over a sampling period. If we define $x_k := x(t_k)$ and the function

$$\forall s \in [0,\ \mathscr{T}_2], \quad \Gamma(s) = \left[e^{As} + \int_0^s e^{A(s-\theta)} d\theta BK \right],$$

we have the following discrete-time system

$$x_{k+1} = \Gamma(T_k) x_k, \tag{5}$$

where $x(t) = \Gamma(t - t_k) x_k$ for $t \in [t_k, t_{k+1}]$.

Notation: Taking a cue from time-delay systems theory, we denote the segment of solution on $t \in [t_k, t_{k+1}]$ by x_{Tk}, so that

$$x_{Tk}(s) = \Gamma(s) x(t_k) \qquad \text{for } s \in [0, T_k].$$

We use $\mathscr{K}^n$ to denote the space of continuous maps from $[0,\ \mathscr{T}_2] \to \mathbb{R}^n$, where recall $\mathscr{T}_2$ is the upper-bound on the T_k.

If the sampling period, T, is constant, the discrete dynamics become $x_{k+1} = \Gamma(T) x_k$, where T is the sampling period. A simple method to check the stability of the system is to ensure that $\Gamma(T)$ has all eigenvalues inside the unit circle. If the sampling period is time-varying, then we must verify that $\Gamma(T_k)$ has eigenvalues inside the unit circle for all $T_k \in [\mathscr{T}_1,\ \mathscr{T}_2]$ which is an infinite dimensional problem. Verifying such infinite-dimensional conditions is difficult. If the system is uncertain, the difficulty increases. However, several authors have investigated this approach to stability analysis [8], [9], [19].

Based on the observation that sampled-data systems are a special case of time-delay systems with time-varying delay, many authors treat sampled-data systems in a manner similar to time-delay systems and use similar approaches to the question of stability. For example, sufficient conditions for stability of sampled-data systems based were derived in [4] by analyzing stability of a class of systems with time-varying delay. However, these results were somewhat conservative in that they did not account for the unique structure of the variation in delay in a sampled-data system. In [12], the authors introduce a new type of Lyapunov-Krasovskii functional which depends more explicitly on the delay function. In particular, they use the fact the $\dot{\tau} = 1$ in their formulation. This led to improvement in the accuracy of the stability conditions. In the present article , we take a different approach which does not model the hold as a delay, but rather uses a new type of sampled-data Lyapunov Theorem introduced in [18] and inspired by [16]. The conditions are enforced using sum-of-squares optimization.

III. Asymptotic stability of sampled-data systems

A. Main theorem

In this section we introduce a new Lyapunov theorem which applies to general nonlinear sampled-data systems. This theorem accounts for the interaction between continuous and discrete element of a sampled-data system. A version of this result was introduced in [18] and was partially inspired by the concept of spacing functions introduced in [16]. Essentially, the theorem says that if there exists a Lyapunov function which has a net decrease over every sampling interval, then there exists a storage function which is continuously decreasing for all time. Consider the following system.

$$\dot{x}(t) = f(x(t), x(t_k)), \qquad t \in [t_k, t_{k+1}], \quad k = 1, \ldots, \infty. \tag{6}$$

We assume global existence and continuity of solutions.

Theorem 1: [18] For given positive scalars $\mathscr{T}_1 \leq \mathscr{T}_2$, suppose $V : \mathbb{R}^n \to \mathbb{R}^+$ satisfies the following for $\mu_1 > \mu_2 > 0$ and $p > 0$

$$\mu_1 |x|^p \leq V(x) \leq \mu_2 |x|^p, \qquad \text{for all } x \in \mathbb{R}^n. \tag{7}$$

Assume that the sampling interval, T_k, satisfies (2), then two following statements are equivalent.

(i) If x is a solution of Equation (6), then

$$V(x(t_{k+1})) - V(x(t_k)) < 0, \qquad \text{for all } k \geq 0.$$

(ii) There exist continuous functionals $Q_k : \mathbb{R} \times \mathscr{K} \to \mathbb{R}$, differentiable over $[t_k \; t_{k+1})$ which satisfy the following

$$Q_k(T_k, z) = Q_k(0, z), \qquad \text{for all } k \geq 0 \text{ and } z \in \mathscr{K}, \tag{8}$$

and such that if x is a solution of Equation (6), then

$$\frac{\mathrm{d}}{\mathrm{d}t} [V(x(t)) + Q_k(t - t_k, x_{Tk})] < 0, \qquad \text{for all } t \in [t_k, t_{k+1}]. \tag{9}$$

Moreover, if either of these statements is satisfied, then solutions of system (6) are asymptotically stable.

Proof: Assume (ii) is satisfied. Define the storage function

$$W(t) = [V(x(t)) + Q_k(t - t_k, x_{Tk})].$$

Then $\frac{\mathrm{d}}{\mathrm{d}t}W(t) < 0$ for $t \in [t_k, t_{k+1}]$ and

$$\begin{aligned}
V(x(t_{k+1})) - V(x(t_k)) &= \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}s} V(x(s))\mathrm{d}s \\
&= \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}s} V(x(s))\mathrm{d}s + Q_k(T_k, x_{Tk}) - Q_k(0, x_{Tk}) \\
&= \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}s} V(x(s))\mathrm{d}s + \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}s} Q_k(s - t_k, x_{Tk})\mathrm{d}s \\
&= \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}}{\mathrm{d}s} W(s)\mathrm{d}s < 0.
\end{aligned}$$

Hence (i) is satisfied.

Now assume (i) is satisfied. Define, for all functions $z \in \mathscr{K}$,

$$Q_k(s, z) := -V(z(s)) + \frac{s}{T_k}\left(V(z(T_k)) - V(z(0))\right).$$

Then the functional satisfies

$$Q_k(0, z) = -V(z(0)) \quad \text{and} \quad Q_k(T_k, z) = -V(z(T_k)) + (V(z(T_k)) - V(z(0))) = -V(z(0)).$$

Consequently, this leads to $Q_k(T_k, z) = Q_k(0, z)$ which ensures that condition (8) is satisfied. Furthermore, by considering $s = t - t_k$ and $z = x_{Tk}$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}[V(x(t)) + Q_k(t - t_k, x_{Tk})] &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[V(x(t)) - V(x_{Tk}(t - t_k)) + \tfrac{t - t_k}{T_k}\left(V(x_{Tk}(T_k)) - V(x_{Tk}(0))\right)\right] \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[V(x(t)) - V(x(t)) + \tfrac{t - t_k}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right)\right] \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{t - t_k}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right)\right] \\
&= \tfrac{1}{T_k}\left(V(x(t_{k+1})) - V(x(t_k))\right) < 0.
\end{aligned} \tag{10}$$

This proves the equivalence between (i) and (ii).

Now, from the discrete-time Lyapunov theorem, we have $\lim_{k\to\infty} x(t_k) = 0$. To show $\lim_{t\to\infty} x(t)$, we note that by assumption of the existence and continuity of solutions, the solution map is continuous and thus bounded on the interval $[0, \mathscr{T}_2]$. Therefore $\lim_{k\to\infty} \|x_{Tk}\|_\infty \to 0$ where $\|\cdot\|_\infty$ is the supremum norm. We conclude that $\lim_{t\to\infty} x(t)$. ■

There are several articles in the literature which use related approaches (see for instance [12], [13]). Typically, however, these results are expressed as positivity of a Lyapunov-Krasovskii functional which is positive definite. In the above result, positivity is relaxed through the use of the spacing functional Q.

The following sections show how the conditions of Theorem 1 can be enforced using sum-of-squares optimization for asymptotic and exponential stability in both the synchronous and asynchronous case. This is similar to the approach taken in [16].

B. Stability under synchronous sampling

Recall the sampled-data system.

$$\dot{x}(t) = Ax(t) + BKx(t_k), \qquad \text{for } t \in [t_k, t_k + T_k],\ \ k \geq 0. \tag{11}$$

The following theorem gives conditions for stability. The conditions of the theorem can be enforced using sum-of-squares, as will be described shortly.

Theorem 2: Consider system (11) with $T_k = T$ for $T > 0$. If there exist $P \in \mathbb{S}^n$ and a polynomial matrix, $M : [0,\ T] \to \mathbb{S}^{2n}$, of degree N, such that

$$P > 0, \qquad \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T M(0) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \qquad M(T) = 0, \tag{12}$$

and such that for all $\tau \in [0,\ T]$, the following inequality holds

$$\Psi(\tau) = \begin{bmatrix} 0 \\ I_n \end{bmatrix} P \begin{bmatrix} (BK)^T \\ A^T \end{bmatrix}^T + \begin{bmatrix} (BK)^T \\ A^T \end{bmatrix} P \begin{bmatrix} 0 \\ I_n \end{bmatrix}^T + \dot{M}(\tau) + M(\tau) \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T + \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T M(\tau) < 0. \tag{13}$$

Then the closed loop system is asymptotically stable for the constant sampling period T. Moreover the condition

$$\Gamma^T(T)\, P\, \Gamma(T) - P < 0$$

is satisfied for this given T.

Proof: Consider the classical quadratic Lyapunov function for linear continuous-time systems. Define $V : \mathbb{R}^n \to \mathbb{R}^+$ as $V(x) = x^T P x$, where $P > 0$ is in $\mathbb{S}$. This function V satisfies condition (7) from Theorem 1. Now define the following function for all $s \in [0,\ T]$ and $z \in \mathscr{K}$

$$Q(s,z) = \begin{bmatrix} z(0) \\ z(s) \end{bmatrix}^T M(s) \begin{bmatrix} z(0) \\ z(s) \end{bmatrix}.$$

First, from (12), we note that

$$Q(0,z) = \begin{bmatrix} z(0) \\ z(0) \end{bmatrix}^T M(0) \begin{bmatrix} z(0) \\ z(0) \end{bmatrix} = z(0)^T \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T M(0) \begin{bmatrix} I_n \\ I_n \end{bmatrix} z(0) = 0.$$

Furthermore,

$$Q(T,z) = \begin{bmatrix} z(0) \\ z(T) \end{bmatrix}^T M(T) \begin{bmatrix} z(0) \\ z(T) \end{bmatrix} = 0.$$

Therefore, we have $Q(T,z) = Q(0,z) = 0$ and hence condition (8) is satisfied.

Computing the derivative term (9), we get

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] &= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t)^T P x(t) + \begin{bmatrix} x_{Tk}(0) \\ x_{Tk}(t-t_k) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x_{Tk}(0) \\ x_{Tk}(t-t_k) \end{bmatrix} \right] \\
&= \tfrac{\mathrm{d}}{\mathrm{d}t}\left[x(t)^T P x(t) + \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \right] \\
&= x^T(t) P \dot{x}(t) + \dot{x}^T(t) P x(t) + 2 \begin{bmatrix} 0 \\ \dot{x}(t) \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \tfrac{\mathrm{d}}{\mathrm{d}t} M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}.
\end{aligned}$$

Recalling that $\dot{x}(t) = Ax(t) + BKx(t_k)$, we get

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] &= \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \left(\begin{bmatrix} 0 \\ I_n \end{bmatrix}^T P \begin{bmatrix} BK & A \end{bmatrix} + \begin{bmatrix} K^T B^T \\ A^T \end{bmatrix} P \begin{bmatrix} 0 & I_n \end{bmatrix} \right) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&\quad + 2 \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T M(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} + \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \dot{M}(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix} \\
&= \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \Psi(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix},
\end{aligned}$$

for all $t \in [t_k, t_k + T]$. Thus by inequality (13), we have $\frac{\mathrm{d}}{\mathrm{d}t}\left[V(x(t)) + Q(t-t_k, x_{Tk})\right] < 0$ for all $t \in [t_k, t_k + T]$. By virtue of Theorem 1, then, the closed loop system is asymptotically stable for the constant sampling period T. ∎

We emphasize that Theorem 2 only guarantees the stability of the solutions of system (11) for a fixed sampling period, T. By virtue of Theorem 1, the previous theorem also ensures that $V(\Gamma(T)x(t_k)) - V(x(t_k)) < 0$, for all $x(t_k) \in \mathbb{R}^n$, which corresponds to the inequality:

$$\Gamma^T(T)P\Gamma(T) - P < 0.$$

for a given T. As mentioned in the introduction, the stability of sampled-data systems with a constant sampling period can also be verified numerically by checking if the eigenvalues of the matrix $\Gamma(T)$ are lie within the unit circle. However, this method fails if the system matrices (A,B) are uncertain. For example, if matrices (A,B) lie in a polytope, it is difficult to investigate the eigenvalues of the matrix $\Gamma(T)$. By contrast, in Theorem 2, the stability condition linearly depends on the matrices A and B. Therefore, it is relatively simple to to extend the previous results to the case of systems with polytopic uncertainty. This is a significant advantage of the proposed methodology.

Another remark concerns the choice of the functional Q introduced in Theorem 2. Note that, from the proof of Theorem 2, the condition $\Psi(\tau) < 0$ for all τ in $[0,\ T]$ is equivalent to the negativity of the derivative of the function $W(t) = V(x(t)) + Q(t - t_k, x_{Tk})$. This is not the case in [13] or [18] which can be interpreted as using an integral term of the form

$$\tilde{Q}(t - t_k, x_{KT}) = (t_{k+1} - t)\int_{t_k}^{t} \dot{x}^T(s)R\dot{x}(s)\mathrm{d}s,$$

so that $W = V + Q + \tilde{Q}$. Although this term has an important role in reducing the conservatism of the stability conditions in [13] and [18], it unavoidably leads to the use of the Jensen inequality to compute an upper bound of the derivative of W. This inequality introduces conservatism in the stability criteria. In the present paper, this term is unnecessary, yielding an exact condition for the derivative of $V + Q$. This implies that the stability criterion from Theorem 3 should be less conservative then the ones from [13] or [18]. The conservatism of the previous Theorem 2 only depends on the degree, N, of the polynomial matrix M.

C. Asynchronous sampling

The case of asynchronous sampling, where T_k is unknown but bounded in some range, is clearly more realistic than the synchronous case in a networked control scenario. However, the stability conditions for this case are not significantly more complex than for the synchronous case. We simply allow the function M to vary with the sampling period, T_k.

Theorem 3: Consider system (6). For given $0 < \mathscr{T}_1 < \mathscr{T}_2 < \infty$, if there exist $P \in \mathbb{S}^n$, positive definite and a bi-polynomial matrix $\bar{M} : [0,\ \mathscr{T}_2] \times [\mathscr{T}_1,\ \mathscr{T}_2] \to \mathbb{S}^{2n}$ such that for all $T \in [\mathscr{T}_1,\ \mathscr{T}_2]$,

$$P > 0, \quad \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T \bar{M}(0,T) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \quad \bar{M}(T,T) = 0, \tag{14}$$

and such that for all $s \in [0,\ T]$, the following inequality holds

$$\begin{aligned} \bar{\Psi}(\tau,T) = \begin{bmatrix} 0 \\ I_n \end{bmatrix} P \begin{bmatrix} (BK)^T \\ A^T \end{bmatrix}^T + \begin{bmatrix} (BK)^T \\ A^T \end{bmatrix} P \begin{bmatrix} 0 \\ I_n \end{bmatrix}^T + \tfrac{d}{ds}\bar{M}(s,T) \\ +\bar{M}(s,T) \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ BK & A \end{bmatrix}^T \bar{M}(s,T) < 0. \end{aligned} \tag{15}$$

Then if $T_k \in [\mathscr{T}_1, \mathscr{T}_2]$ for all $k \geq 0$, the closed loop system is asymptotically stable. Moreover the condition

$$\Gamma^T(T)P\Gamma(T) - P < 0$$

is satisfied for all $T \in [\mathscr{T}_1, \mathscr{T}_2]$

Proof: The proof is a trivial extension of Theorem 2. ■

IV. Exponential stability of sampled-data systems

In this section we introduce an extension of the previous theorems 2 and 3 to cope with the problem of exponential stability. The objective is here to ensure a guaranteed decay rate α of the solutions of the sampled-data system.

A. Synchronous sampling

The following theorem gives conditions for exponential stability. The conditions of the theorem can be enforced using sum-of-squares, as will be described shortly.

Theorem 4: Consider system (11) with $T_k = T$ for some given $T > 0$ and $\alpha > 0$. If there exist $P \in \mathbb{S}^n$, positive definite and a polynomial matrix, of degree N, $M : [0,\ T] \to \mathbb{S}^{2n}$ such that

$$P > 0, \quad \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T M(0) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \quad M(T) = 0, \tag{16}$$

and such that for all $\tau \in [0,\ T]$, the following inequality holds

$$\Psi_\alpha(\tau) = \Psi(\tau) + 2\alpha \left[\begin{bmatrix} 0 \\ I_n \end{bmatrix} P \begin{bmatrix} 0 \\ I_n \end{bmatrix}^T + M(\tau) \right] < 0. \tag{17}$$

Then closed loop system is exponentially stable with a guaranteed rate α for the constant sampling period T. Moreover it implies the following inequality

$$\Gamma^T(T)P\Gamma(T) - e^{-2\alpha T}P < 0$$

Proof: The proof follows the line of the one from Theorem 2. Consider the same quadratic Lyapunov function and the same functional Q as in Theorem 2. we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}[V(x(t)) + Q(t-t_k, x_{Tk})] + 2\alpha(V(x(t)) + Q(t-t_k, x_{Tk})) = \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix}^T \Psi_\alpha(t-t_k) \begin{bmatrix} x(t_k) \\ x(t) \end{bmatrix},$$

for all $t \in [t_k, t_k + T]$. Thus if there exists a solution of inequality (13), integrating the previous inequality leads to

$$\begin{aligned} V(x(t)) + Q(t-t_k, x_{Tk}) &\le e^{-2\alpha(t-t_k)}[V(x(t_k)) + Q(t_k - t_k, x_{Tk})] \\ V(x(t)) + Q(t-t_k, x_{Tk}) &\le e^{-2\alpha(t-t_k)}V(x(t_k)), \end{aligned}$$

In particular, taking $t = t_{k+1}$, we get $V(x(t_{k+1})) \le e^{-2\alpha T}V(x(t_k))$. This ensures the exponential stability of the discrete-time system. Finally the argument on the existence and continuity of the solutions allows to conclude the proof. ■

B. Asynchronous sampling

As for the case of asymptotic stability, the case of asynchronous sampling is addressed in this section. The sampling period T_k is unknown but bounded in some range. Following the same procedure as in section III, the following theorem is derived

Theorem 5: Consider system (6). For given $0 \le \mathscr{T}_1 < \mathscr{T}_2 < \infty$ and $\alpha > 0$, if there exist $P \in \mathbb{S}^n$, positive definite and a bi-polynomial matrix $\bar{M} : [0,\ \mathscr{T}_2] \times [\mathscr{T}_1,\ \mathscr{T}_2] \to \mathbb{S}^{2n}$ such that for all $T \in [\mathscr{T}_1,\ \mathscr{T}_2]$,

$$P > 0, \quad \begin{bmatrix} I_n \\ I_n \end{bmatrix}^T \bar{M}(0,T) \begin{bmatrix} I_n \\ I_n \end{bmatrix} = 0, \quad \bar{M}(T,T) = 0, \tag{18}$$

and such that for all $s \in [0,\ T]$, the following inequality holds

$$\bar{\Psi}_\alpha(\tau,T) = \bar{\Psi}(\tau,T) + 2\alpha \left[\begin{bmatrix} 0 \\ I_n \end{bmatrix} P \begin{bmatrix} 0 \\ I_n \end{bmatrix}^T + \bar{M}(\tau,T) \right] < 0. \tag{19}$$

Then if $T_k \in [\mathscr{T}_1,\ \mathscr{T}_2]$ for all $k \geq 0$, the closed loop system is exponentially stable with the decay rate α. Moreover it implies that

$$\forall T \in [\mathscr{T}_1,\ \mathscr{T}_2], \quad \Gamma^T(T)P\Gamma(T) - e^{-2\alpha T}P < 0.$$

Proof: The proof is a trivial extension of Theorems 3 and 4. ■

V. Sum of Squares as Algorithmic Tool

A. General presentation of SOS

The methodology we use to implement the conditions of Theorems 2 and 3 is based on the sum-of-squares decomposition of positive polynomials. When applying this methodology we assume that all matrix functions are polynomial, can be approximated by polynomials, or there is a change of coordinates that renders them polynomial.

Denote by $\mathbb{R}[y]$ the ring of polynomials in $y = (y_1, \ldots, y_n)$ with real coefficients. Denote by Σ_s the cone of polynomials that admits a SOS decomposition, i.e., those $p \in \mathbb{R}[y]$ for which there exist $h_i \in \mathbb{R}[y], i = 1, \ldots, M$ so that

$$p(y) = \sum_{i=1}^{M} h_i^2(y).$$

If $p(y) \in \Sigma_s$, then clearly $p(y) \geq 0$ for all y. The converse is not always true, although the converse does hold for univariate matrix-valued polynomials. The advantage of SOS is that the problem of testing if $p(y) \geq 0$ is known to be NP-hard, whereas testing if $p(y) \in \Sigma_s$ is equivalent to an SDP([20]), and hence is worst-case polynomial-time verifiable. SOS results apply to matrix-valued polynomials as well as scalars, although in this case the inequality means positive semidefinite. The SDPs related to SOS can be formulated efficiently and the solution can be retrieved using SOSTOOLS([17]), which interfaces with semidefinite solvers such as SeDuMi([21]).

Consider now the conditions in Theorem 2 which take the form.

$$L(s) \leq 0, \quad s \in \mathscr{S}, \tag{20}$$

where $L(s) \in \mathbb{R}^{n\times\ n}$ and $\mathscr{S}$ is a semialgebraic set described by polynomial inequalities:

$$\mathscr{S} = \{s \in R \quad | \quad g_i(s) \geq 0, \quad i = 1,\ldots,M\},$$

where $g_i(s)$ are polynomial functions. In order to test condition (20), we can apply Positivstellensatz results such as [22] which allow us to test positivity on a semialgebraic set using SOS. Specifically, Condition (20) holds if there exists SOS polynomials $P_i(s,y)$, such that

$$L(s) + \sum_{i=1}^{M} g_i(s) P_i(s,y) = P_0(s).$$

Intuitively, the above condition guarantees that when $s \in \mathscr{S}$, we have $L(s) \leq -\sum_{i=1}^{M} g_i(s) p_i(s,y) \leq 0$ since $g_i \geq 0$ and $p_i \geq 0$, and therefore $L(s) \leq 0$ for those s.

B. *Application to the stability theorem*

In this brief subsection, we identify the functions g_i's corresponding to sets $[0,T]$ and $[\mathscr{T}_1, \mathscr{T}_2]$ used in theorems 2 and 3. The function for Theorem 2 is

$$g_1(s) = -(T-s)s,$$

which represents $s \in [0,T]$ and for Theorem 3, we use

$$g_1(s,T) = -(T-s)s \text{ and } g_2(T) = -(\mathscr{T}_2 - T)(T - \mathscr{T}_1).$$

where g_2 represents $T \in [\mathscr{T}_1, \mathscr{T}_2]$.

VI. Examples

Consider system (1) with several matrix definitions

- Example 1 from [4], [12]:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ -0.375 & -1.15 \end{bmatrix},$$

- Example 2 from [13]:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, BK = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix},$$

- and Example 3 from [23], [24]:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 0.1 \end{bmatrix}, BK = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Theorems	T_2 for Ex.1	T_2 for Ex.2
[4]	0.869	0.99
[12]	1.113	1.99
[13]	1.695	2.03
[14]	1.695	2.53
[18]	1.723	2.62
Th.2 $N=1$	0.702	2.319
Th.2 $N=3$	1.729	3.219
Th.2 $N=5$	1.729	3.269
Th.3 $N=1$	0.701	2.310
Th.3 $N=3$	1.729	3.218
Th.3 $N=5$	1.729	3.269

TABLE I: Maximum allowable sampling period T_2 for examples 1, 2, with $T_1 = 0$.

Theorems for Ex.3	
[18]	$[0.201,\ 1.623]$
Th.2 $N=1$	$\varnothing$
Th.2 $N=3$	$[0.2007,\ 2.016]\bigcup[2.606,\ 3.055]$
Th.2 $N=5$	$[0.2007,\ 2.020]\bigcup[2.470,\ 3.694]$
[18]	$[0.400,\ 1.251]$
Th.3 $N=1$	$\varnothing$
Th.3 $N=3$	$[0.4,\ 1.820]$ or $[2.680,\ 3.005]$
Th.3 $N=5$	$[0.4,\ 1.828]$ or $[2.520,\ 3.550]$

TABLE II: Interval of allowable asynchronous samplings of the form $[T_1,\ T_2]$ for example 3.

Tables I and II summarize the results obtained in the literature and using the theorems provided in the present paper for examples $1,2$ and 3. One can see that the obtained results are less conservative then existing ones. More precisely, for each example, Theorem 3 guaranteeing the stability of the solutions of the sampled-data systems with the same Lyapunov matrix P.

In Figure 1, we use our algorithm to examine the effect of sampling period on the decay rate of the system. Our results show that the decay rate is fixed at $\alpha = .5$ until the sampling period reaches approximately $T = 1.3$, after which the decay rate decreases until instability is

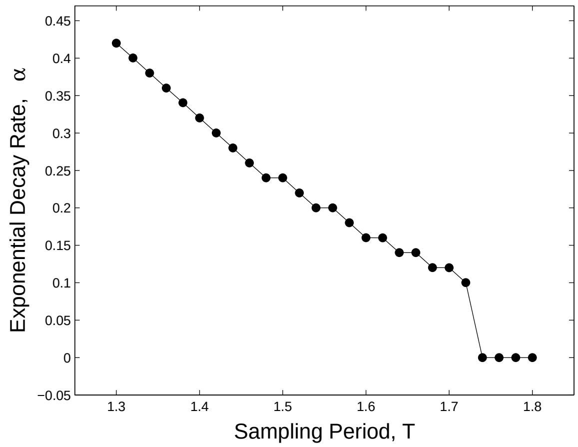

Fig. 1: Bound on Decay Rate vs. Synchronous Sampling Period, T for Example 2 with $N = 3$

encountered at $T = 1.729$.

Another important remark deals with Example 3. This system is well known in the time-delay literature because the delay has a stabilizing effect. This means that its solutions are not stable for sufficiently small delay but become stable for sufficiently large delay. The method proposed in this article is able to take into account this phenomena and is also able to isolate several intervals of possible values for the length of the sampling interval where the system is stable for asynchronous and synchronous sampling. Figure 2 illustrates the intervals of stability and exponential decay rate for synchronous sampling.

Note that our analysis of Example 3 indicates that Theorem 3 (with $N = 5$) can be used to proves stability for asynchronous sampling in $[0.4,\ 1.828]$ **OR** $[2.520,\ 3.550]$, but not over both simultaneously. This means that stability is not guaranteed if the sampling switches from one interval to the other. This recalls the classical behavior of switched systems: A system which switches between two stable subsystems is not necessary stable.

VII. Conclusion

In this article, a novel analysis of continuous linear systems under asynchronous sampling is provided. This approach is based on the discrete-time Lyapunov Theorem applied to the continuous-time model of the sampled-data systems. Numerical results compare favorably with result in the literature. Perhaps the most important feature of the method presented in this paper is

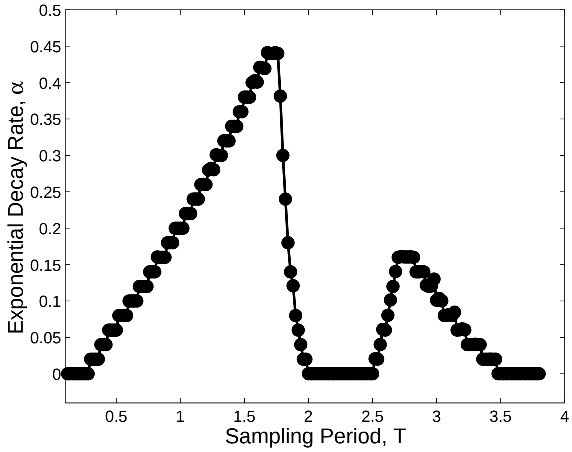

Fig. 2: Bound on Decay Rate vs. Synchronous Sampling Period, T for Example 3 with $N = 3$

that it is expressed using the sum-of-squares framework and is thus easily extended to nonlinear systems and systems with parametric uncertainty.

References

[1] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[2] S. Zampieri, "A survey of recent results in Networked Control Systems," in *Proc. of the* 17th *IFAC World Congress*, Seoul, Korea, July 2008.

[3] T. Chen and B. Francis, *Optimal sampled-data control systems*. Berlin, Germany: Springer-Verlag, 1995.

[4] E. Fridman, A. Seuret, and J.-P. Richard, "Robust sampled-data stabilization of linear systems: An input delay approach," *Automatica*, vol. 40, no. 8, pp. 1141–1446, 2004.

[5] H. Fujioka, "Stability analysis of systems with aperiodic sample-and-hold devices," *Automatica*, vol. 45, no. 3, pp. 771–775, 2009.

[6] W. Zhang and M. Branicky, "Stability of networked control systems with time-varying transmission period," in *Allerton Conf. Communication, Control, and Computing*, October 2001.

[7] W. Zhang, M. Branicky, and S. Phillips, "Stability of networked control systems," *IEEE Control Systems Magazine*, no. 21, 2001.

[8] Y. Suh, "Stability and stabilization of nonuniform sampling systems," *Automatica*, vol. 44, no. 12, pp. 3222–3226, 2008.

[9] Y. Oishi and H. Fujioka, "Stability and stabilization of aperiodic sampled-data control systems: An approach using robust linear matrix inequalities," in *Joint* 48th *IEEE Conference on Decision and Control and* 28th *Chinese Control Conference*, December 16-18 2009.

[10] L. Hetel, J. Daafouz, and C. Iung, "Stabilization of arbitrary switched linear systems with unknown time-varying delays," *IEEE Trans. on Automatic Control*, vol. 51, no. 10, pp. 1668–1674, Oct. 2006.

[11] L. Mirkin, "Some remarks on the use of time-varying delay to model sample-and-hold circuits," *IEEE Trans. on Automatic Control*, vol. 52, no. 6, pp. 1009–1112, 2007.

[12] P. Naghshtabrizi, J. Hespanha, and A. Teel, "Exponential stability of impulsive systems with application to uncertain sampled-data systems," *Systems and Control Letters*, vol. 57, no. 5, pp. 378–385, 2008.

[13] E. Fridman, "A refined input delay approach to sampled-data control," *Automatica*, vol. 46, no. 2, pp. 421–427, 2010.

[14] K. Liu and E. Fridman, "Discontinuous Lyapunov functionals for linear systems with sawtooth delays," in *Proc. of the* 8^{th} *IFAC Workshop on Time-Delay Systems*, September 2009.

[15] A. Seuret, "Stability analysis for sampled-data systems with a time-varying period," in *Proc. of the* 48^{th} *IEEE Conference on Decision and Control*, December 2009.

[16] M. Peet, A. Papachristodoulou, and S. Lall, "Positive forms and stability of linear time-delay systems," *SIAM Journal on Control and Optimization*, vol. 47, no. 6, pp. 3227–3258, 2009.

[17] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOSTOOLS: a general purpose sum of squares programming solver," *Proceedings of the IEEE Conference on Decision and Control*, 2002.

[18] A. Seuret, "A novel stability analysis of linear systems under asynchronous samplings," *Accepted in Automatica*, 2011.

[19] L. Hetel, "Robust stability and control of switched linear systems," PhD, Institut National Polytechnique de Lorraine - INPL, Nancy France, 2007.

[20] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, 2000.

[21] J. F. Sturm, "Using SeDuMi 1.02, a Matlab Toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11-12, pp. 625–653, 1999.

[22] M. Putinar, "Positive polynomials on compact semi-algebraic sets," *Indiana Univ. Math. J.*, vol. 42, no. 3, pp. 969–984, 1993.

[23] K. Gu, V.-L. Kharitonov, and J. Chen, *Stability of time-delay systems*. Birkhauser, 2003.

[24] W. Michiels, S.-I. Niculescu, and L. Moreau, "Using delays and time-varying gains to improve the static output feedback stabilizability of linear systems : a comparison," *IMA Journal of Mathematical Control and Information*, vol. 21, no. 4, pp. 393–418, 2004.