Systems Analysis and Control

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Lecture 3: Linearization

Introduction

In this Lecture, you will learn:

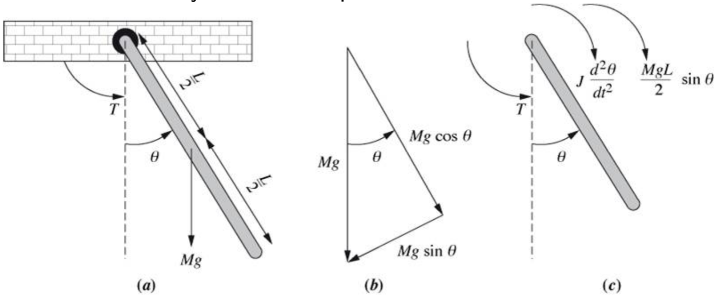
How to Linearize a Nonlinear System *System*.

- Taylor Series Expansion
- Derivatives
- L'hoptial's rule
- Multiple Inputs/ Multiple States

Lets **Start** with an Example

A Simple Pendulum

Consider the rotational dynamics of a pendulum:



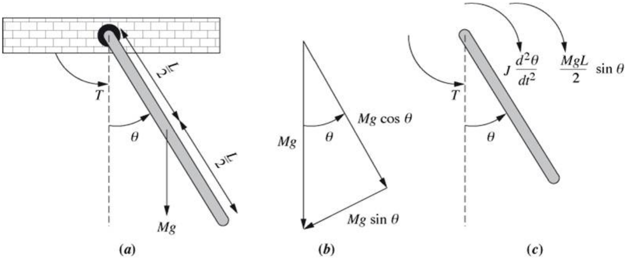
- The *input* is a motor-driven moment, $T(t)$.
- The *output* is the angle, $\theta(t)$.
- The moment of inertia about the pivot point is J.
- The force of gravity, Mg acts on the center of mass.
    - Force creates a moment about the pivot (See Figure b)):

$$N(t) = -Mg \sin\theta(t) \cdot \frac{l}{2}$$

A Simple Pendulum

The governing equation is Newton's law:

$$\ddot{\theta}(t) = \frac{N(t)}{J} + T(t)$$



Equations of Motion (EOM):

$$\ddot{\theta}(t) = -\frac{Mgl}{2J}\sin\theta(t) + \frac{T(t)}{J}$$
$$y(t) = \theta(t)$$

First-order form: Let $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{Mgl}{2J}\sin x_1(t) + \frac{T(t)}{J}$$
$$y(t) = x_1(t)$$

A Simple Pendulum

The Problem

First-order form:

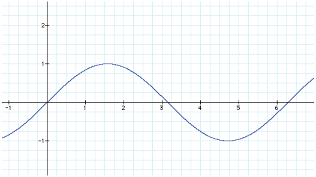
$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -\frac{Mgl}{2J}\sin x_1(t) + \frac{T(t)}{J}$$
$$y(t) = x_1(t)$$

Although we have the system in first-order form, it cannot be put in state-space because of the $\sin x_1$ term.

What to do???

Although $\sin x$ is nonlinear,
small sections look linear.

- **Near $x = 0$:** $\sin x \cong x$
- **Near $x = \pi/2$:** $\sin x \cong 1$
- **Near $x = \pi$:** $\sin x \cong \pi - x$



We must use these linear approximations *very carefully*!

Accuracy of the Small Angle Approximation

The approximation will only be accurate for a narrow band of x.

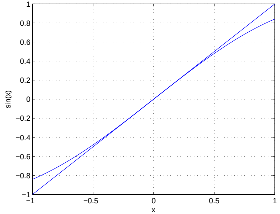


Figure: $\sin(x)$ and x near $x_0 = 0$

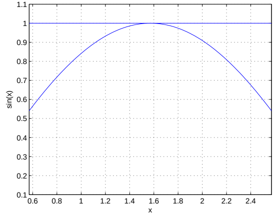


Figure: $\sin(x)$ and x near $x_0 = \frac{\pi}{2}$

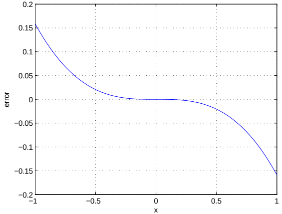


Figure: Error near $x = 0$

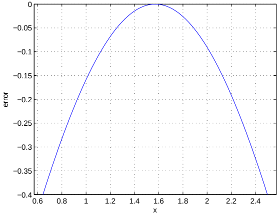


Figure: Error near $x = \frac{\pi}{2}$

- **80% Accuracy:** $x \in [-1.2, 1.2]$
- **95% Accuracy:** $x \in [-.7, .7]$

- **80% Accuracy:** $x \in [.9, 2.2]$
- **95% Accuracy:** $x \in [1.25, 1.9]$

Linear Approximation

We can use the tangent to approximate a nonlinear function near a point x_0.

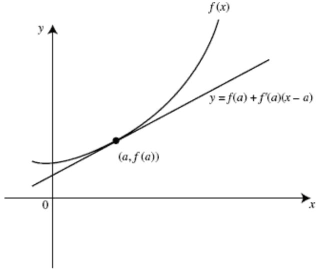
Key Point: The approximation is *tangent* to the function at the point x_0.

$$f(x) \cong ax + b$$

- The slope is given by
$$a = \frac{d}{dx} f(x)|_{x=x_0}$$
- The *y-intercept* is given by
$$b = f(x_0) - ax_0$$

The **linear approximation** is given by

$$f(x) \cong f(x_0) + \frac{d}{dx} f(x)|_{x=x_0}(x - x_0)$$



A General Method For Linear Approximation

Problem: Approximate the scalar function $f(x)$ near the point x_0 using

$$y(x) = ax + b$$

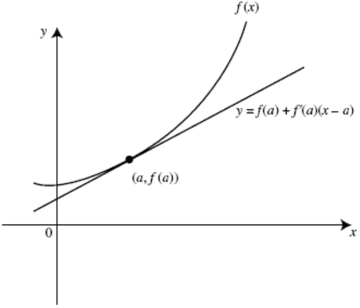


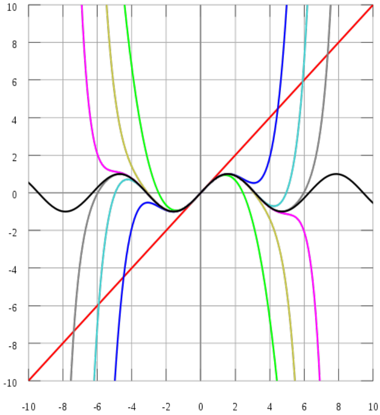
Figure 9.2-1

The **Linear Approximation** is given by

$$y(x) = f(x_0) + \frac{d}{dx} f(x)|_{x=x_0} (x - x_0)$$

Linear Approximation

Note: The Linear Approximation is just the first two terms in the Taylor Series representation.



$$f(x) = f(x_0) + \frac{d}{dx}f(x)|_{x=x_0}\frac{(x-x_0)}{1!} + \frac{d}{dx}f(x)|_{x=x_0}\frac{(x-x_0)^2}{2!} + \cdots$$

Example: Pendulum

Return to the dynamics of a pendulum:

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= -\frac{Mgl}{2J}\sin x_1(t) + \frac{1}{J}T(t) \\
y(t) &= x_1(t)
\end{aligned}
$$

The nonlinear term is $\sin x_1$

- We want to linearize $\sin x_1$.
- Choose an operating point, x_0!
  - Depends on what we want to do!
  - Options are limited.

Hanging Pendulum: $x_0 = 0$

Inverted Pendulum: $x_0 = \pi$

Tracking: $x_0 = ???$

Example: Balance an Inverted Pendulum

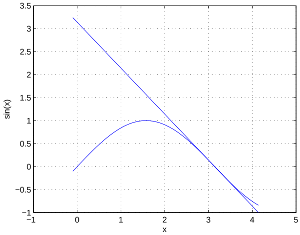
For the inverted pendulum: $f(x) = \sin x$ and $x_0 = \pi$.

- *Slope:*

$$a = \frac{d}{dx} f(x)|_{x=x_0} = \cos(\pi) = -1$$

- $f(x_0) = f(\pi) = \sin(\pi) = 0$
- Thus (for $x \cong \pi$)

$$\begin{aligned}\sin(x) &\cong f(x_0) + a(x - x_0) = 0 - 1(x - \pi) \\ &= \pi - x\end{aligned}$$



This gives the first-order dynamics:

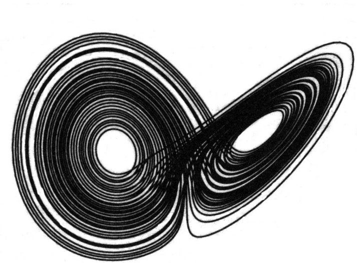
$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{Mgl}{2J} x_1(t) \underbrace{- \frac{Mgl}{2J}\pi}_{\text{Trouble}} + \frac{1}{J} T(t) \\ y(t) &= x_1(t)\end{aligned}$$

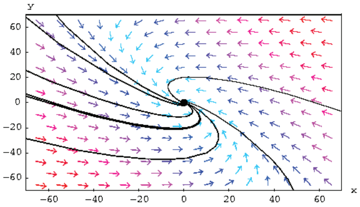
New Problem: How to eliminate the constant term

$$-\frac{Mgl}{2J}\pi$$

Solution: Measure Displacement from Equilibrium

Define the state variable as Displacement from Equilibrium





Definition 1.

x_0 is an **Equilibrium Point** of $\dot{x} = f(x)$ if $f(x_0) = 0$. (Then $\dot{x} = f(x_0) = 0$)

- Nonlinear systems may have *many* equilibrium points.
- Linear systems only have one equilibrium point ($x_0 = 0$).

A Change of Variables

Consider distance from equilibrium $\Delta x = x - x_0$

The pendulum has infinite equilibria.

- Down equilibria: $\theta_0 = 0 + 2\pi n$ for $n = 1, \cdots, \infty$
- Up equilibria: $\theta_0 = \pi + 2\pi n$ for $n = 1, \cdots, \infty$

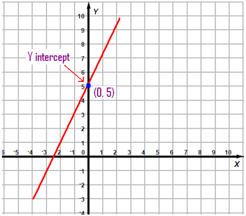
Lets choose $\theta_0 = \pi$ i.e. $x_0 = \begin{bmatrix} \pi & 0 \end{bmatrix}^T$:

$$\dot{x}_1(t) = x_2(t), \qquad\qquad \dot{x}_2(t) = \frac{Mgl}{2J}(x_1(t) - \pi)$$

Problem: Translate the equilibrium to $\Delta x_0 = 0$.

Solution: Define a new variable $\Delta x = x - x_0$ ($\Delta x_1 = x_1 - \pi$, $\Delta x_2 = x_2$)

- Then
  $$\Delta \dot{x}_1(t) = \dot{x}_1(t) = x_2(t) = \Delta x_2(t)$$
  $$\Delta \dot{x}_2(t) = \dot{x}_2(t) = \frac{Mgl}{2J}(x_1(t) - \pi) = \frac{Mgl}{2J}\Delta x_1(t)$$



- Thus $\Delta x_0 = 0$ is the equilibrium!!!

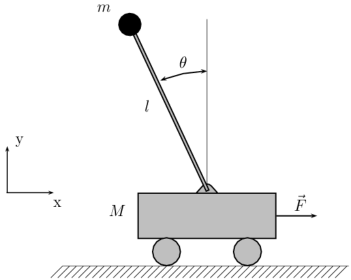
Measuring Displacement from Equilibrium

Pendulum Example

Summary:

- Linearize about equilibrium $x_1 = \pi$, $x_2 = 0$. This yields

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = \frac{Mgl}{2J}(x_1(t) - \pi) + \frac{1}{J}T(t)$$



- Define new states as displacement from equilibrium:

$$\Delta x_1(t) = x_1(t) - \pi$$
$$\Delta x_2(t) = x_2(t)$$

- Get the new dynamics:

$$\Delta\dot{x}_1(t) = \Delta x_2(t) \qquad \Delta\dot{x}_2(t) = \frac{Mgl}{2J}\Delta x_1(t) + \frac{1}{J}T(t)$$

Δx_1 is angle from the vertical.

Now we are ready for state-space.

New Dynamics:

$$\Delta\dot{x}_1(t) = \Delta x_2(t)$$
$$\Delta\dot{x}_2(t) = \frac{Mgl}{2J}\Delta x_1(t) + \frac{1}{J}T(t)$$

State-Space Form:

$$\Delta\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{2J} & 0 \end{bmatrix}\Delta x(t) + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} T(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\Delta x(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{2J} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \end{bmatrix}$$

Example: Balance an Inverted Pendulum

Applications: Walking robots.

Example: Balance an Inverted Pendulum

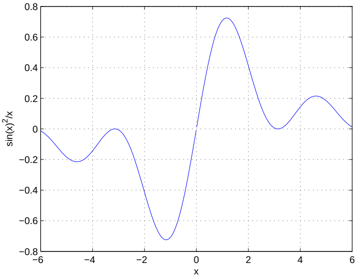
Applications: Segway.

Numerical Example: Using l'Hôpital's rule

Occasionally you will encounter a system such as

$$\ddot{x}(t) = -\dot{x}(t) + \frac{\sin^2(x(t))}{x(t)}$$

where you want to linearize about the **zero equilibrium**.



The nonlinear term is $\frac{\sin^2 x}{x}$ with equilibrium point $x_0 = 0$.
Recall the formula:

$$f(x) \cong f(x_0) + f'(x_0)(x - x_0)$$

Thus we must calculate $f(x_0)$ and $f'(x_0)$.

Lets start with $f(x_0)$. Initially, we see that $f(0) = \frac{0}{0}$, which is indeterminate.
To help, we use L'hopital's Rule.

Theorem 2 (l'Hôpital's Rule).

If $g(0) = 0$ *and* $h(0) = 0$*, then*

$$\lim_{x\to 0} \frac{g(x)}{h(x)} = \lim_{x\to 0} \frac{g'(x)}{h'(x)}$$

Lets linearize $f(x) = \dfrac{\sin^2 x}{x}$ about $x_0 = 0$. We need to find $f(x_0)$ and $f'(x_0)$.

$$f(0) = \lim_{x\to 0} \frac{\overbrace{\sin^2(x)}^{g(x)}}{\underbrace{x}_{h(x)}} = \lim_{x\to 0} \frac{\overbrace{2\sin x \cos x}^{g'(x)}}{\underbrace{1}_{h'(x)}} = \frac{0}{1} = 0$$

which is as expected. Now,

$$f'(x) = \frac{2\sin x \cos x}{x} - \frac{\sin^2 x}{x^2} = \frac{\overbrace{2x\sin x \cos x - \sin^2 x}^{g(x)}}{\underbrace{x^2}_{h(x)}}$$

So unfortunately,

$$f'(0) = \frac{0}{0}$$

Example Continued

So once more we apply L'hopital's rule:

$$f'(0) = \lim_{x\to 0} f'(x) = \lim_{x\to 0} \frac{2\sin x\cos x + 2x\cos^2 x - 2x\sin^2 x - 2\sin x\cos x}{2x}$$

$$= \lim_{x\to 0} \frac{\overbrace{2x(\cos^2 x - \sin^2 x)}^{g'(x)}}{\underbrace{2x}_{h'(x)}} = \frac{0}{0}$$

Ooops, we must apply l'Hôpital's rule AGAIN:

$$\lim_{x\to 0} \frac{2x(\cos^2 x - \sin^2 x)}{2x} = \lim_{x\to 0} \frac{\overbrace{2(\cos^2 x - \sin^2 x) - 8x\cos x\sin x}^{g''(x)}}{\underbrace{2}_{h''(x)}} = \frac{2}{2} = 1$$

Which was a lot of work for such a simple answer (easier way?). We have the linearized equation of motion:

$$\ddot{x}(t) = -\dot{x}(t) + 0 + 1\cdot(x(t) - 0)$$

For state-space, let $x_1 = x$, $x_2 = \dot{x}$. Then $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} x(t)$.

Inverted Pendula Videos

Double Inverted Pendulum

Triple Inverted Pendulum

This is actually a switched controller!

1. Linearize about hanging position
   1.1 Design Energy *Maximizing Controller*
   1.2 Will cause pendulum to swing up.
2. Linearize about inverted position
   2.1 Switch controllers when near inverted position

Summary

What have we learned today?

How to Linearize a Nonlinear System *System*.

- Taylor Series Expansion
- Derivatives
- L'hoptial's rule
- Multiple Inputs/ Multiple States

Next Lecture: Laplace Transform