

Spacecraft and Aircraft Dynamics

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Lecture 6: The Orbital Plane

Introduction

In this Lecture, you will learn:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concepts

- The Earth-Centered Inertial reference frame
- The line of nodes

Practice

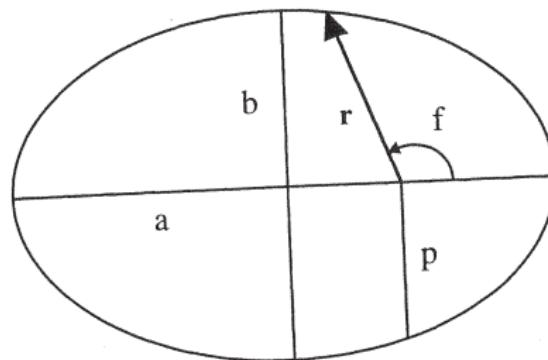
- How to construct all orbital elements from \vec{r} and \vec{v}
- A Numerical Illustration

The Orbital Elements

2D orbits

So far, all orbits are parameterized by 3 parameters

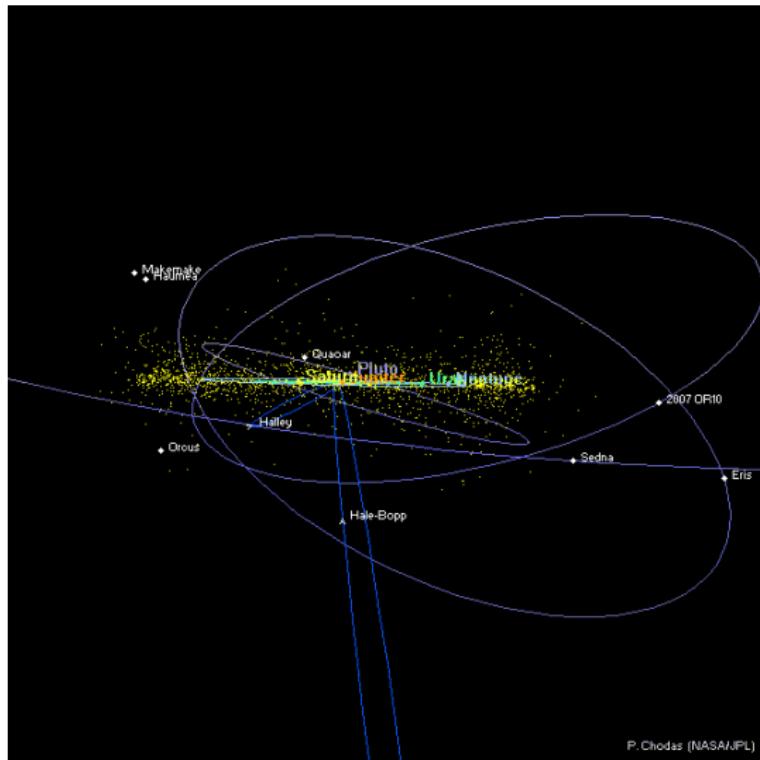
- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

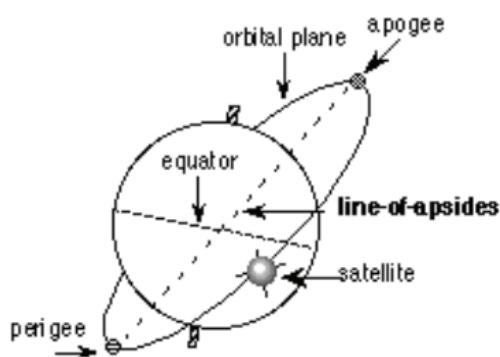
The Orbital Elements

But orbits are not 2-dimensional!



The Orbital Elements

Note: We have shown how to use a , e and f to find the scalars r and v .



Question: How do we find the vectors \vec{r} and \vec{v} ?

Answer: We have to determine how the orbit is oriented in space.

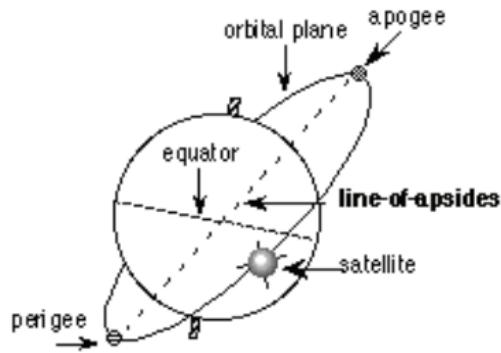
- Orientation is determined by vectors \vec{e} and \vec{h} .
- We need 3 new orbital elements
 - ▶ Orientation can be determined by 3 rotations.

The Coordinate System

Earth-Centered Inertial (ECI)

Question: How do we find the vectors \vec{r} and \vec{v} ?

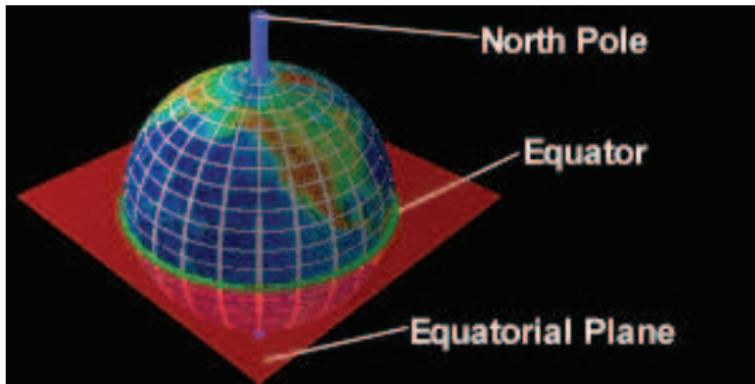
Response: In which coordinate system??



- The origin is the center of the earth
- We need to define the \hat{x} , \hat{y} , and \hat{z} vectors.

ECI: The Equatorial Plane

Defining the \hat{z} vector



- The \hat{z} vector is defined to be the vector parallel to the axis of rotation of the earth.
- Can apply to other planets
- Does not apply to Heliocentric Coordinates

Definition 1.

The **Equatorial Plane** is the set of vectors normal to the axis of rotation.

The Ecliptic Plane

Heliocentric Coordinates

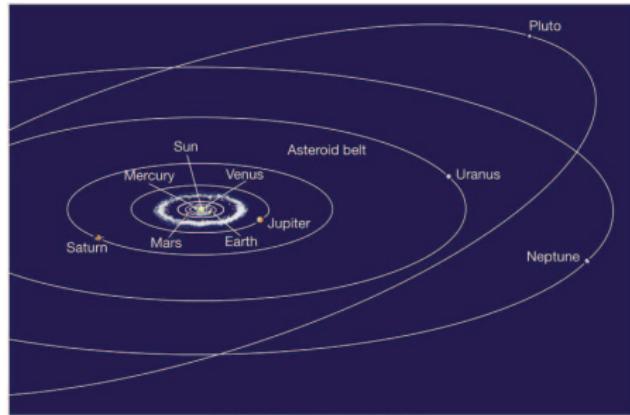
The rotation vector of the sun is unreliable.

- In heliocentric coordinates, the \hat{z} vector is normal to the ecliptic plane.

Definition 2.

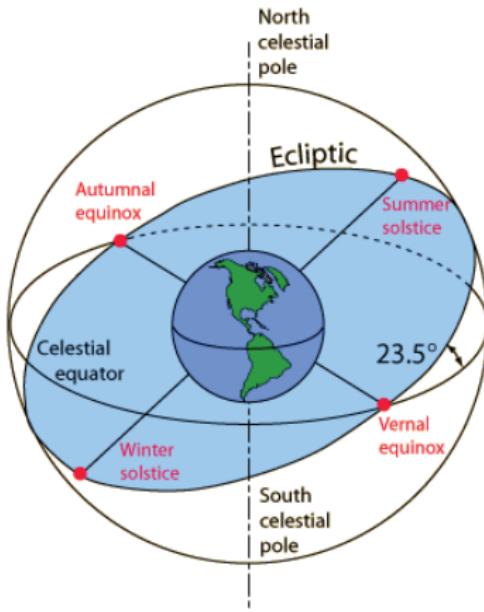
The **Ecliptic Plane** is the orbital plane of the earth in motion about the sun.

- From the earth, the ecliptic plane is defined by the apparent motion of the sun about the earth.
 - ▶ Determined by the location of eclipses (hence the name).



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The Ecliptic Plane



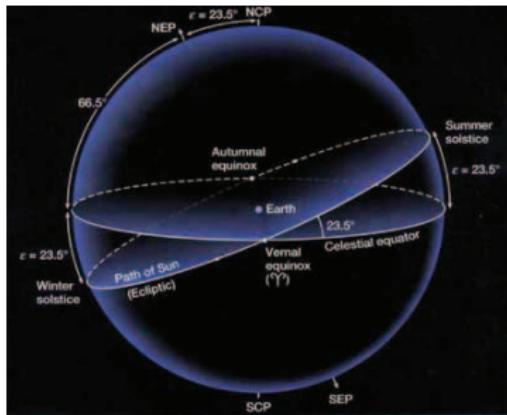
Definition 3.

The **Inclination to the Ecliptic** is the angle between the equatorial and ecliptic planes.

Currently, the inclination to the ecliptic is 23.5 deg.

ECI: The First Point of Aries

- To complete the ECI coordinate system, we will define an \hat{x} axis in the equatorial plane.
- The \hat{y} axis is then given by the right-hand rule.



A fixed location for \hat{x} is the intersection of the equatorial and ecliptic planes. But there are two such points, at the two equinoxes (vernal and autumnal)

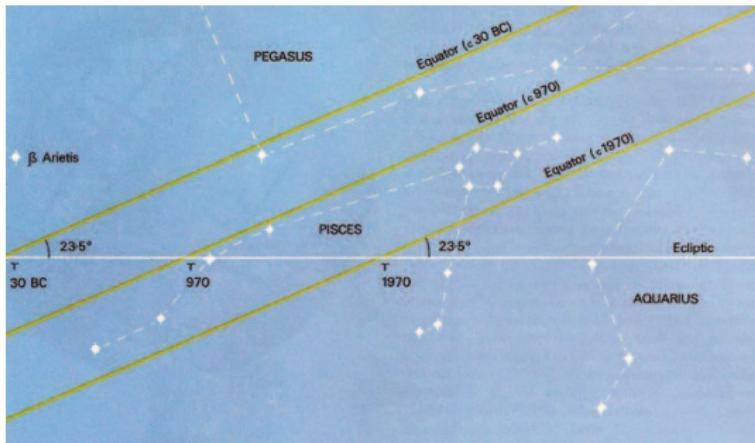
The **First Point of Aries** is the earth-sun vector at the vernal equinox.

Question Does the FPOA lie at the ascending or descending node?

Celestial Sphere

ECI: The First Point in Aries

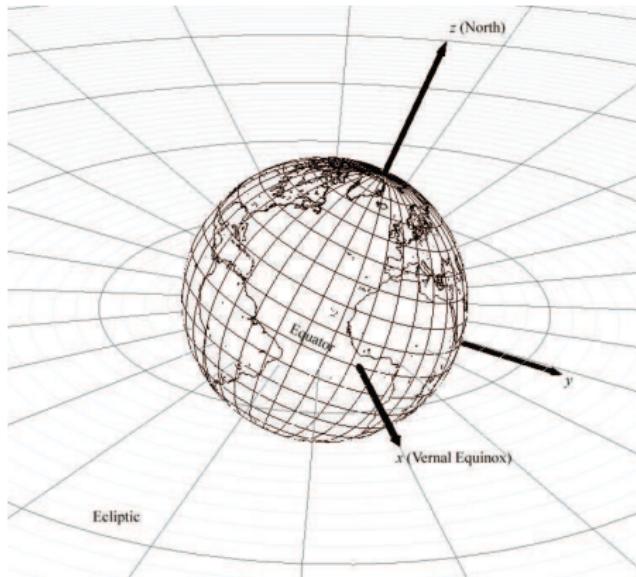
- The First Point of Aries is so named because this direction used to point towards the Constellation Aries.
- Precession of the earth's rotation vector means the FPOA now actually points toward Pisces.



- Since Motion of the FPOA is caused by precession, its motion is **Periodic**, not Secular.
 - ▶ The Period is about 26,000 years.
- The Coordinate System is not truly inertial.

Summary: The ECI frame

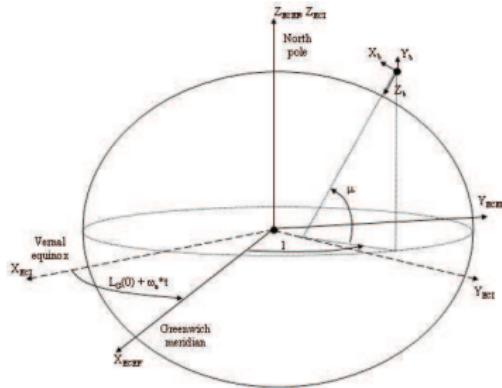
- \hat{z} - North Pole
- \hat{x} - FPOA
- \hat{y} - Right Hand Rule



Because the FPOA migrates with time, positions given in ECI must be referenced to a year

- **J2000** - frame as defined at 12:00 TT on Jan 1, 2000.
- **TOD** - True of Data: date is listed explicitly.

Other Reference Frames



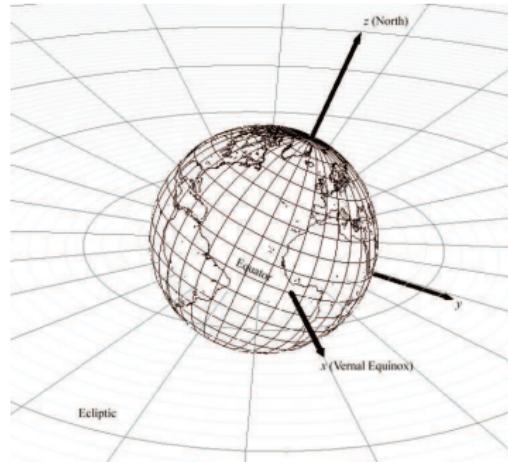
Note there are many other reference frames of interest

- Earth Centered Earth Fixed
- Topocentric Horizon
- Topocentric Equatorial
- Gaussian (Satellite Radial)
- Frenet System (Satellite Normal, Drag)
- Equinoctial Orbital Elements

We will return to some of these frames when necessary.

Orbital Elements

Now that we have our coordinate system,



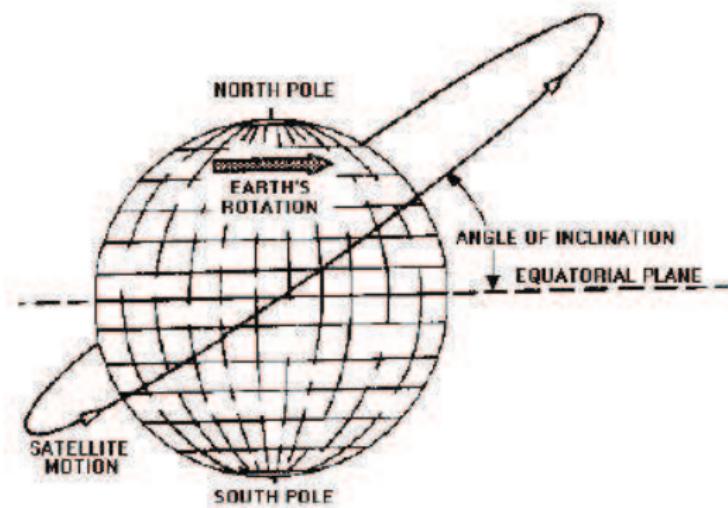
Question: Suppose we are given \vec{r} and \vec{v} in the ECI frame. How to describe the orientation of the orbit?

Answer: 3 new orbital elements.

- Inclination
- Right Ascension
- Argument of Periapse

The Orbital Plane

Inclination, i



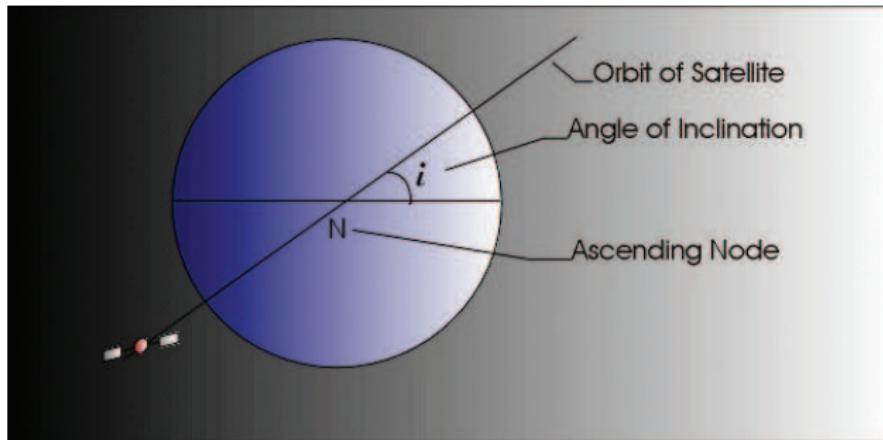
Angle the orbital plane makes with the reference plane.

The orbit is

- **Prograde** if $0 < i < 90^\circ$.
- **Retrograde** if $90 < i < 180^\circ$.

The Orbital Plane

Inclination, i

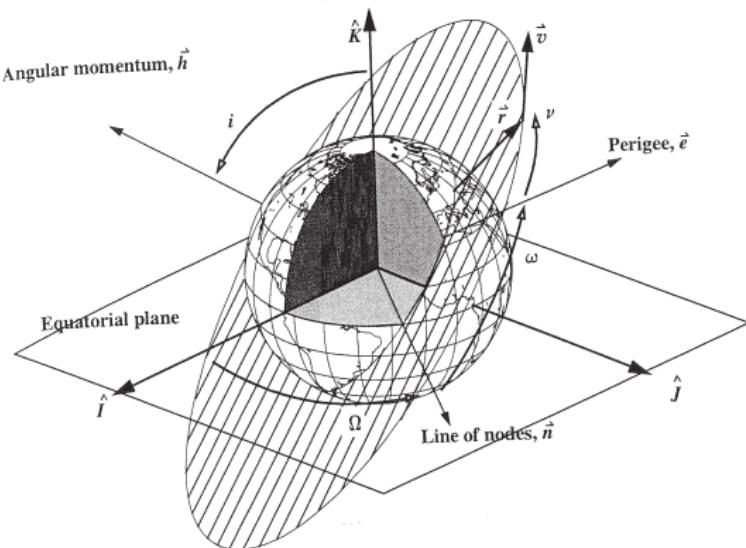


Inclination can be found from \vec{h} as

$$\vec{h} \cdot \hat{z} = h \cos i.$$

- If \vec{h} is defined in ECI, then $i = \cos^{-1} \frac{h_3}{h}$.
- No quadrant ambiguity because by definition, $i \leq 180$ deg

The Line Of Nodes



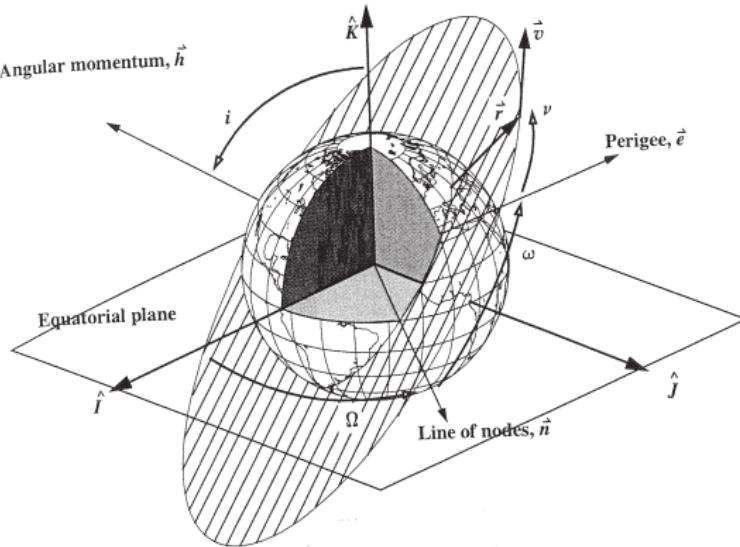
An important vector in defining the orbit is the line of nodes.

Definition 4.

The **Line of Nodes** is the vector pointing to where the satellite crosses the equatorial plane from the southern to northern hemisphere.

$$\vec{n} = \hat{z} \times \vec{h}$$

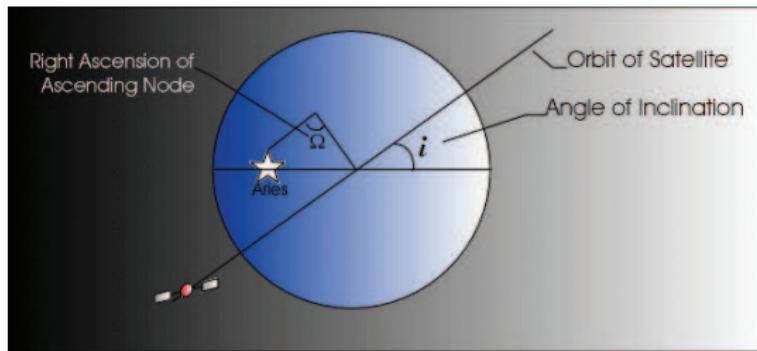
The Line Of Nodes



- Lies at the intersection of the equatorial and orbital planes.
- Points toward the **Ascending Node**.
- Zero for equatorial orbits ($i = 0$).

The Orbital Plane

Right Ascension of Ascending Node, Ω



The Angle measured from reference direction, \hat{x} in the reference plane to *ascending node*.

- Defined to be $0 \leq \Omega \leq 360$
- Undefined for equatorial orbits ($i = 0$).

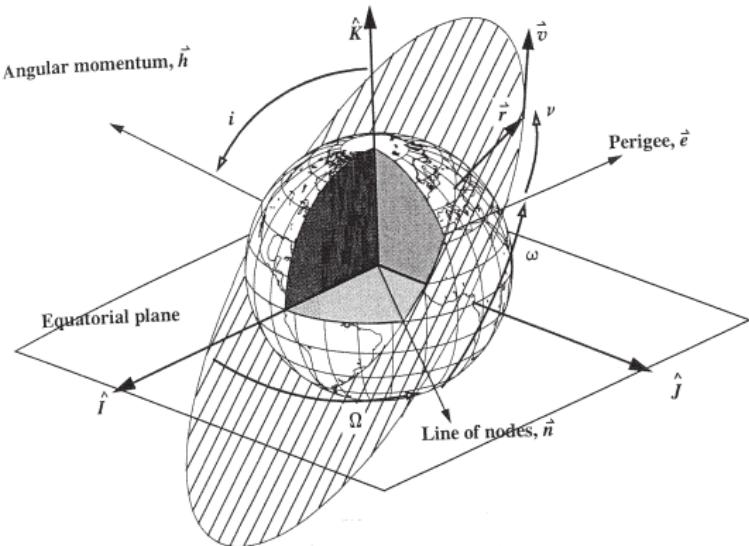
The Orbital Plane

Right Ascension of Ascending Node, Ω

RAAN can be found from the line of nodes as

$$\cos(\Omega) = \frac{\hat{x} \cdot \vec{n}}{\|\vec{n}\|}$$

Must resolve quadrant ambiguity.

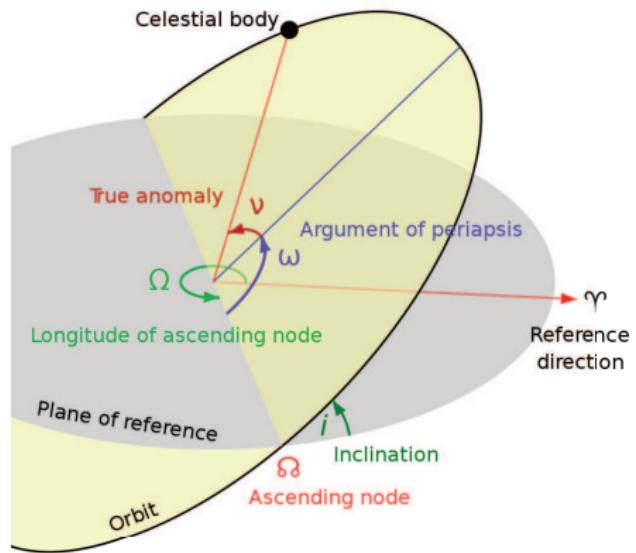


Quadrant Ambiguity: Calculators assume Ω is in quadrant 1 or 2. Correct as

$$\Omega = \begin{cases} \Omega & \hat{y} \cdot \vec{n} \geq 0 \\ 360 - \Omega & \hat{y} \cdot \vec{n} < 0 \end{cases}$$

Argument of Periapse, ω

- Undefined for *Circular* Orbits.
- define so $0 \leq \omega < 360$ deg



Definition 5.

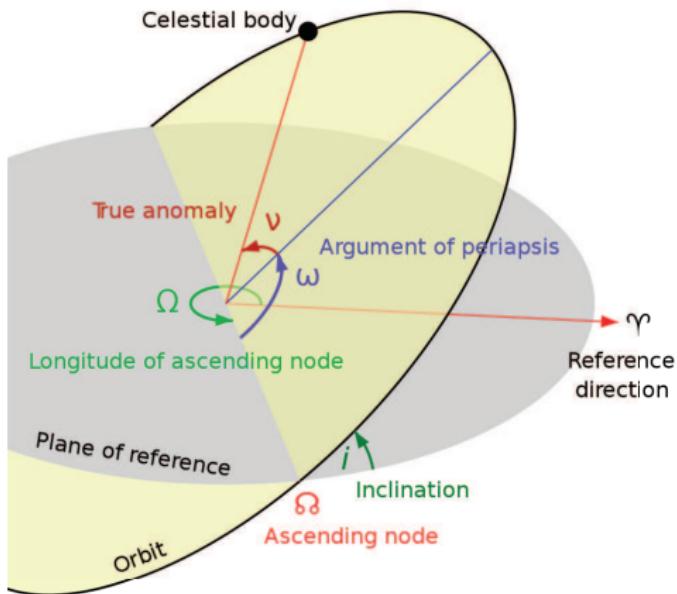
The **Argument of Periapse** is the angle from line of nodes to the point of periapse.

Argument of Periapse, ω

Can be calculated from

$$\cos(\omega) = \frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\| \|\vec{e}\|}$$

Must resolve quadrant ambiguity



Quadrant Ambiguity: Calculators assume ω is in quadrant 1 or 2. Correct as

$$\omega = \begin{cases} \omega & \hat{z} \cdot \vec{e} \geq 0 \\ 360 - \omega & \hat{z} \cdot \vec{e} < 0 \end{cases}$$

Summary: Visualization

Example: Finding Orbital Elements

Problem: Suppose we observe an object in the ECI frame at position

$$\vec{r} = \begin{bmatrix} 6524.8 \\ 6862.8 \\ 6448.3 \end{bmatrix} \text{ km} \quad \text{moving with velocity} \quad \vec{v} = \begin{bmatrix} 4.901 \\ 5.534 \\ -1.976 \end{bmatrix} \text{ km/s}$$

Determine the orbital elements.

Solution: Although not necessary, as per your homework, lets first convert to canonical units ($1ER = 6378.14\text{km}$, $1TU = 806.3\text{s}$).

$$\vec{r}' = \frac{\vec{r}}{6378.14\text{km}} = \begin{bmatrix} 1.023 \\ 1.076 \\ 1.011 \end{bmatrix}$$

$$\vec{v}' = \vec{v} \frac{806.8\text{s}}{6378.14\text{km}} = \begin{bmatrix} .62 \\ .7 \\ -.25 \end{bmatrix}$$

First, lets construct angular momentum, \vec{h} , the line of nodes, \vec{n} and the eccentricity vector, \vec{e} .

Example: Finding Orbital Elements

Continued

We construct \vec{h} , \vec{n} and \vec{e} .

$$\vec{h} = \vec{r} \times \vec{v} = \begin{bmatrix} -.9767 \\ .882 \\ .049 \end{bmatrix} \frac{ER^2}{TU}$$

Since \vec{r} and \vec{v} are in ECI coordinates,

$$\vec{n} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \vec{h} = \begin{bmatrix} -.882 \\ -.9767 \\ 0 \end{bmatrix} \frac{ER^2}{TU}.$$

$$\vec{e} = \frac{1}{\mu} \vec{v} \times \vec{h} - \frac{\vec{r}}{r} = \begin{bmatrix} -.315 \\ -.385 \\ .668 \end{bmatrix}$$

where recall $\mu = 1$ in canonical units.

Example: Finding Orbital Elements

Continued

Now we begin solving for orbital elements.

$$e = \|\vec{e}\| = .8328$$

Use energy to calculate a .

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -.088$$

$$a = -\frac{\mu}{2E} = 5.664ER$$

$$p = \frac{h^2}{\mu} = 1.735ER$$

We can now calculate our three new orbital elements as indicated. Start with inclination

$$i = \cos^{-1} \left(\frac{\vec{h}}{h} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = 87.9 \text{ deg}$$

No quadrant ambiguity by definition.

Example: Finding Orbital Elements

Continued

Continue with RAAN, we want the angle between \hat{x} and \vec{n} .

$$\Omega = \cos^{-1} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot \frac{\vec{n}}{\|\vec{n}\|} \right) = \pm 132.10 \text{ deg}$$

Because cos has quadrant ambiguity, we must check the quadrant. Specifically, we need the sign of

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \vec{n} = -0.9767 < 0$$

Therefore, \vec{n} is in the *third* quadrant, and we need to correct

$$\Omega = 360 - 132.10 = 227.9 \text{ deg}$$

Example: Finding Orbital Elements

Continued

Next, the argument of perigee is the angle between \vec{e} and \vec{n} .

$$\omega = \cos^{-1} \left(\frac{\vec{n} \cdot \vec{e}}{\|\vec{n}\| \|\vec{e}\|} \right) = \pm 53.4 \text{ deg}$$

We resolve the quadrant ambiguity by checking

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \cdot \vec{e} = .668 > 0$$

so we are in the right quadrant

$$\omega = 53.4 \text{ deg}$$

Example: Finding Orbital Elements

Continued

Finally, we solve for true anomaly. But this is simply the angle between \vec{r} and \vec{e} , so we can use

$$f = \cos^{-1} \left(\frac{\vec{r} \cdot \vec{e}}{r_e} \right) = \pm 92.3 \text{ deg}$$

We resolve the quadrant ambiguity by checking

$$\vec{r} \cdot \vec{v} > 0$$

So we are in the right quadrant

$$f = 92.3 \text{ deg}$$

Summary

This Lecture you have learned:

The Orbital Plane

- Inclination
- Right Ascension
- Argument of Periapse

New Concepts

- The Earth-Centered Inertial reference frame
- The line of nodes

Practice

- How to construct all orbital elements from \vec{r} and \vec{v}
- A Numerical Illustration