Systems Analysis and Control

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Lecture 13: Root Locus Continued

Overview

In this Lecture, you will learn:

Review

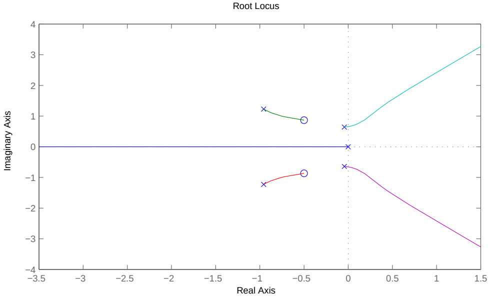
- Definition of Root Locus

Points on the Real Axis

- Symmetry
- Drawing the Real Axis

What Happens at High Gain?

- The effect of Zeros
- Asymptotes



Definition 1.

The **Root Locus** of $G(s)$ is the set of all poles of

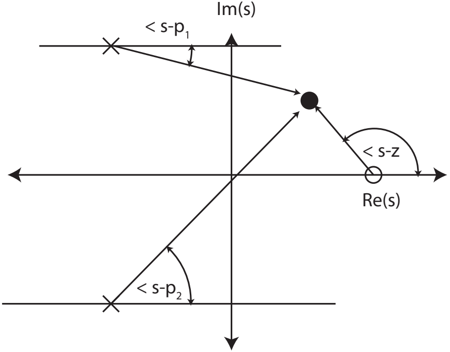
$$\frac{kG(s)}{1+kG(s)}$$

as k ranges from 0 to ∞

Root Locus

Review

$$G(s) = \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$



For a point on the root locus:

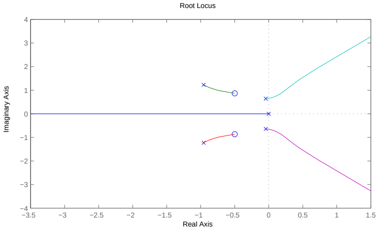
$$\angle G(s) = \sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = -180^\circ$$

Root Locus Demo 1

Wiley+ Root Locus Demo 1

Constructing the Root Locus

Symmetry



Our goal is to find all points on the root locus.

- Our path will be easier by using some basic properties

Symmetry:

- Complex roots come in pairs: $a \pm b\imath$.
- Points on the root locus are mirrored above/below the real axis

We can divide points on root locus into

- Points on the real axis
- Symmetric Pairs off the real axis.

Examine a point $s = a$ on the Real Axis.

- On the real axis, $s = a$ is a real number.
- For a point on the Root Locus,

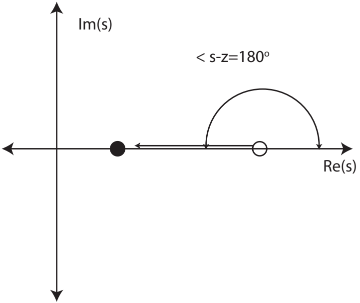
$$\sum_{i=1}^{m} \angle(a - z_i) - \sum_{i=1}^{n} \angle(a - p_i) = -180^\circ$$

Phase Contribution: The Zeros
$(a - z_i)$:

Case 1: z_i is Real

- If $a > z_i 0$, then $\angle(a - z_i) = 0^\circ$
- If $a < z_i$, then $\angle(a - z_i) = -180^\circ$

Contribution is 0° or 180°!



Case 2: z_i is Complex: Complex Roots some in pairs.

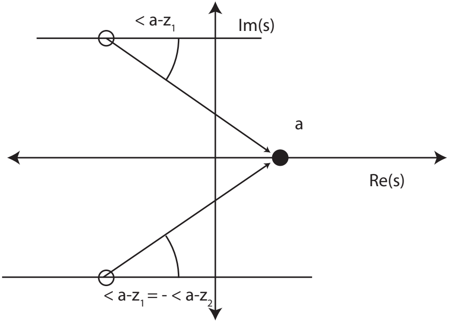
$$z_{1,2} = b \pm c\imath$$

- Zero 1: $a - z_1 = a - b + c\imath$

$$\angle(a - z_1) = \tan^{-1}\left(\frac{a-b}{c}\right)$$

- Zero 2: $a - z_2 = a - b - c\imath$

$$\angle(a - z_2) = \tan^{-1}\left(-\frac{a-b}{c}\right)$$
$$= -\angle(a - z_1)$$



Therefore, the total contribution to phase is 0°!

$$\angle(a - z_1) + \angle(a - z_2) = \angle(a - z_1) - \angle(a - z_1) = 0^\circ$$

The Poles

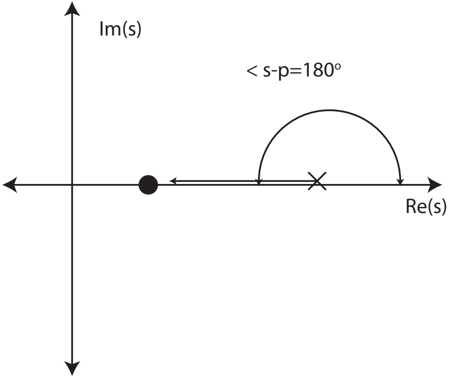
$$\sum_{i=1}^{m} \angle(a - z_i) - \sum_{i=1}^{n} \angle(a - p_i) = -180^\circ$$

Now lets do the *poles*.

Case 1: p_i is Real

- If $a - p_i > 0$, then $\angle(a - p_i) = 0^\circ$
- If $a - p_i < 0$, then $\angle(a - p_i) = -180^\circ$

Contribution is either 0° or 180°!



Same as for zeros!!!

Case 2: p_i **are Complex:** Complex Roots some in pairs.

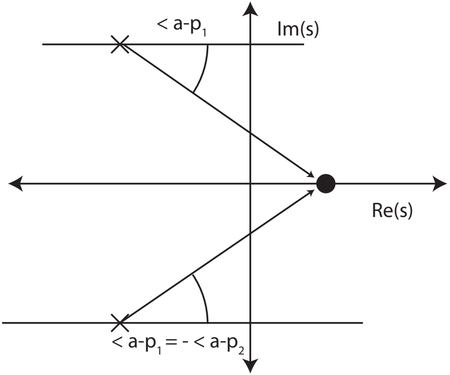
$$z_{1,2} = b \pm c\imath$$

- Pole 1: $a - p_1 = a - b + c\imath$

$$\angle(a - p_1) = \tan^{-1}\left(\frac{a-b}{c}\right)$$

- Pole 2: $a - p_2 = a - b - c\imath$

$$\angle(a - p_2) = -\angle(a - p_1)$$



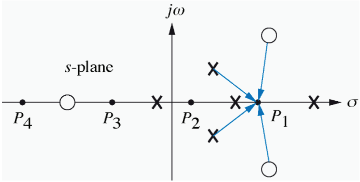
Again, the Total Contribution is $0°$!

Constructing the Root Locus: The Real Axis

ON-OFF

Summary: A point on the real axis: $s = a$

- Complex poles and zeros don't matter
- Real poles and zeros contribute 0° or 180°
  - 0° if the pole/zero is to the left of a
  - 180° if the pole/zero is to the right of a



The **PHASE** of $G(a)$ is

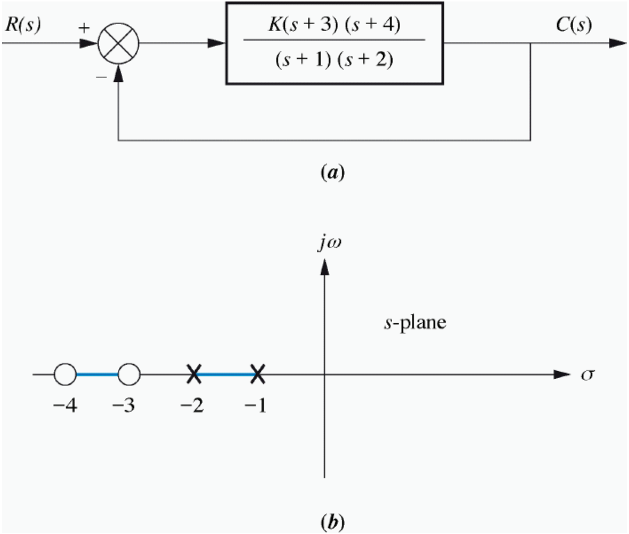
$$\angle G(a) = 180^\circ \cdot (\text{\# of poles and zeros to the right of } a)$$

A Simple Rule:

- If the # of poles and zeros to the right of a is EVEN.
  - We are **OFF** the root locus.
- If the # of poles and zeros to the right of a is ODD.
  - We are **ON** the root locus.

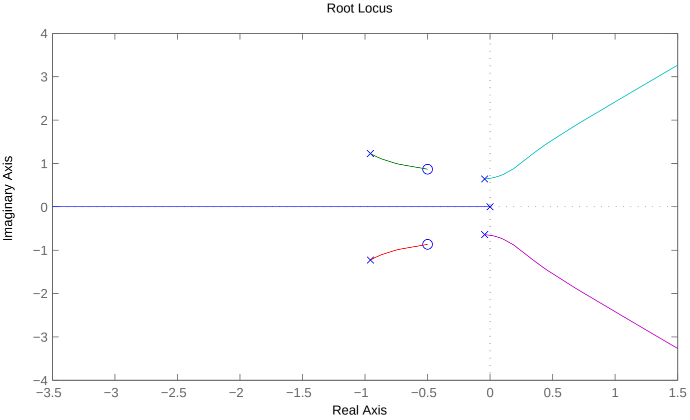
Constructing the Root Locus

Examples



Constructing the Root Locus

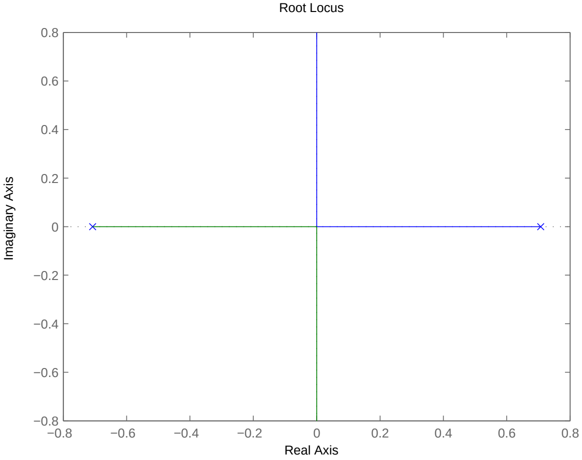
Examples



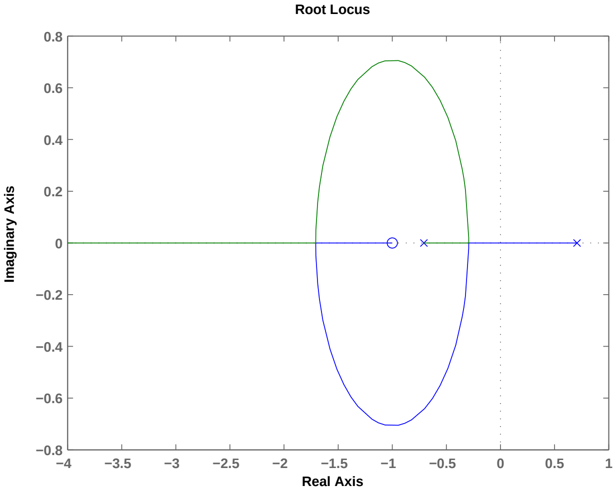
Constructing the Root Locus

Examples

$$G(s) = \frac{1}{s^2 - \frac{1}{2}}$$



The inverted pendulum with some derivative feedback: $\hat{K}(s) = k(1+s)$



Constructing the Root Locus

Large Gain

Now lets look at what happens when gain increases.

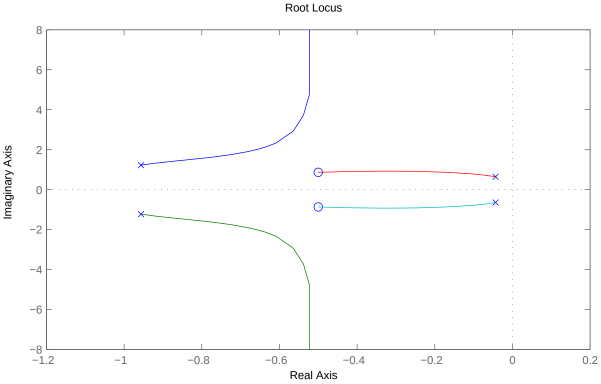


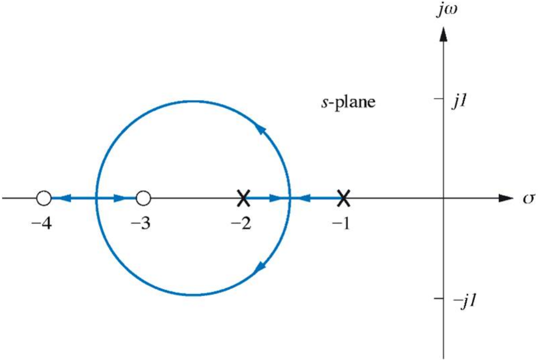
Figure: Suspension Problem

Conclusion: Stable poles oscillate more.

Constructing the Root Locus

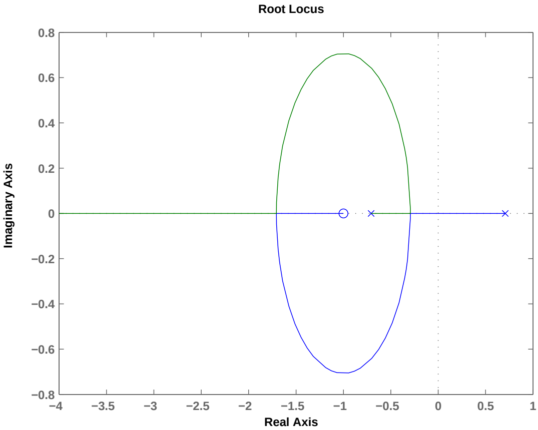
Large Gain

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$



Conclusion: Nothing Happens.

Again, Inverted Pendulum with derivative feedback.

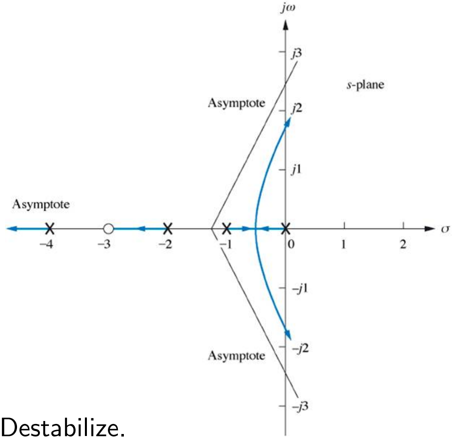


Conclusion: Poles get **More Stable**.

Constructing the Root Locus

Large Gain

$$G(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$



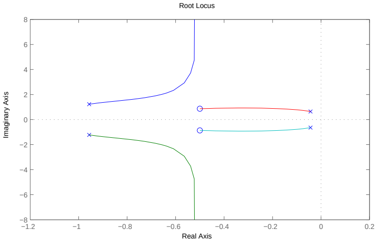
Conclusion: Poles Destabilize.

Notice the **Asymptotes**.

So what happens when k is large?

Logically, there are **TWO CASES:**

- Poles can remain small.
- Poles can get big.



Lets start with **Small Poles** ($\|s\| < \infty$).

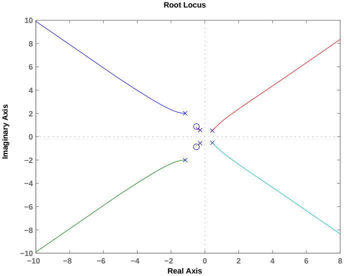
$$G(s) = \frac{n(s)}{d(s)}$$

- OL zeros are roots of $n(s) = 0$.
- OL poles are roots of $d(s) = 0$.

Now, closed loop:

$$\frac{kG(s)}{1 + kG(s)} = \frac{kn(s)}{d(s) + kn(s)}$$

CL poles are roots of
$$d(s) + kn(s) = 0$$



Constructing the Root Locus

Large Gain

At high gain, small CL poles are roots of

$$d(s) + kn(s) = 0$$

- If k is large:
- And s is small,

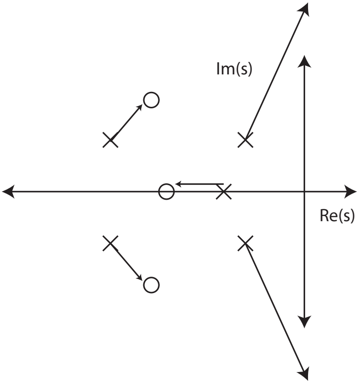
$$d(s) + kn(s) \cong kn(s)$$

As $k \to \infty$, small poles satisfy

$$d(s) + kn(s) \cong kn(s) = 0$$

which means $n(s) = 0$!!!

- $n(s) = 0$ means s is an OL zero!



At high gain, small CL poles are attracted by OL Zeros.

Constructing the Root Locus

Asymptotics

Now consider the other possibility:
Very Large Poles: still satisfy

$$d(s) + kn(s) = 0$$

In this case, $d(s)$ is not small.

Very Large solutions of

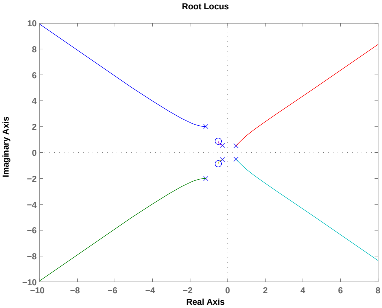
$$1 + kG(s)$$

are called asymptotics.

- asymptotics increase forever with k
  - $\lim_{k\to\infty} \|s\| = \infty$

Questions:

- Do asymptotes exist?
- Where do they go?



Recall that a point is on the root locus if

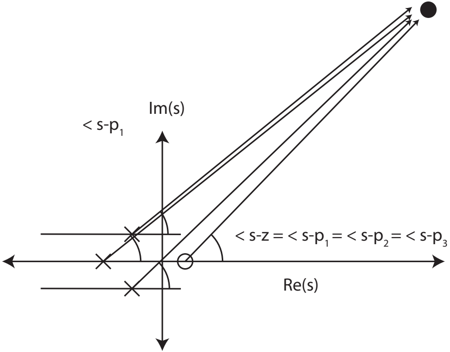
$$\angle G(s) = 180^\circ$$

Which means:

$$\sum_{i=1}^{m} \angle(s - z_i) - \sum_{i=1}^{n} \angle(s - p_i) = -180^\circ$$

However, when $\|s\| \to \infty$,

All Angles are the Same!!!



$$\begin{aligned}\angle(s - z_1) &= \angle(s - z_2) = \ldots = \angle(s - z_m) \\ &= \angle(s - p_1) = \angle(s - p_2) = \ldots = \angle(s - p_n) = \angle_\infty\end{aligned}$$

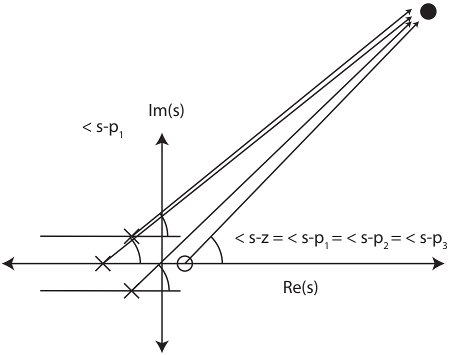
This makes life easier.

- just solve for one angle, $\angle_\infty$.

$$\angle_\infty \cdot (m-n) = 180^\circ$$

where

- m is the number of OL zeros
- n is the number of OL poles



So asymptotics occur at

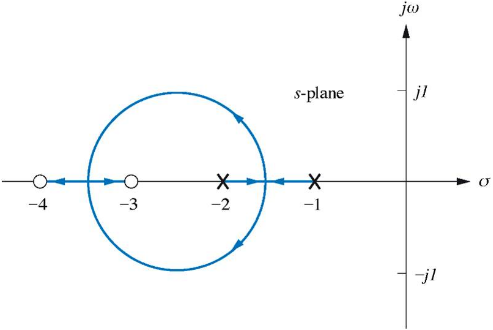
$$\angle_\infty = \frac{1}{m-n}\left(180^\circ + 360^\circ l\right)$$

for integers $l = 0, 1, 2, \cdots$.

Asymptotes

Case 1: $n - m = 0$

$$G(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$



Count: 2 zeros, 2 poles.

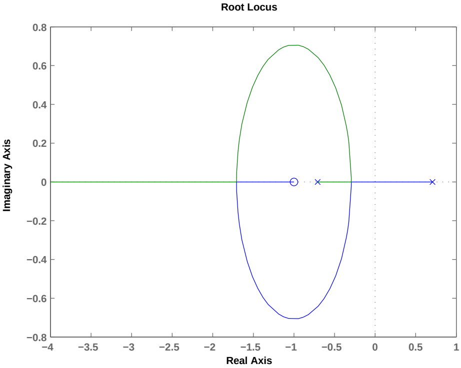
$$m - n = 0 \qquad \angle_\infty = \frac{1}{0}180^\circ = \infty$$

No Asymptotes.

Asymptotes

Case 2: $n - m = 1$

$$G(s) = \frac{1}{s^2 - \frac{1}{2}} \qquad\qquad K(s) = k(1+s)$$



Count: 1 zeros, 2 poles.

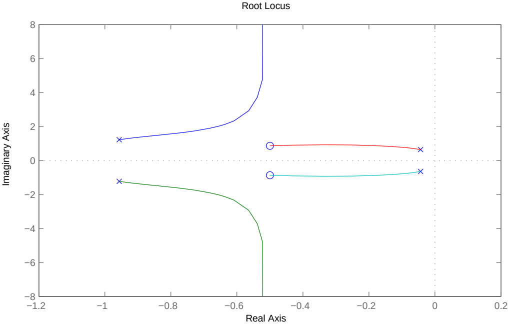
$$m - n = -1 \qquad\qquad \angle_\infty = -180^\circ$$

1 asymptote at -180°.

The suspension system.

Count: 2 zeros, 4 poles.

$$m - n = -2$$



$$\angle_{\infty} = -\frac{1}{2}\left(180^\circ + 360^\circ l\right) = -90^\circ - 180^\circ l = -90^\circ, -270^\circ$$

2 vertical asymptotes at 90° and 270°.

Poles **MAY** destabilize at large gain.

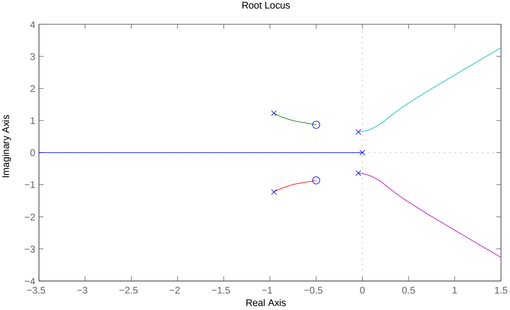
Asymptotes

Case 4: $n - m = 3$

The suspension system with integral Feedback.

Count: 2 zeros, 5 poles.

$$m - n = -3$$



$$\angle_\infty = -\frac{1}{3}(180° + 360° l) = -60° - 120° l = -60°, -180°, -300°$$

3 asymptotes at $60°$, $180°$ and $300°$.

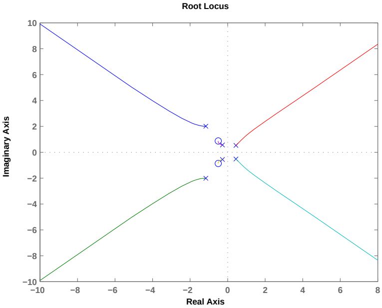
Poles **WILL** destabilize at large gain.

$n(s)$ is degree 2, $d(s)$ is degree 6.

$$G(s) = \frac{s^2 + s + 1}{s^6 + 2s^5 + 5s^4 - s^3 + 2s^2 + 1}$$

Count: 2 zeros, 6 poles.

$$m - n = -4$$



$$\angle_\infty = -\frac{1}{4}\left(180^\circ + 360^\circ l\right) = -45^\circ - 90^\circ l = -45^\circ, -135^\circ, -225^\circ, -315^\circ$$

4 asymptotes at 45°, 135°, 225° and 315°.

Poles **WILL** destabilize at large gain.

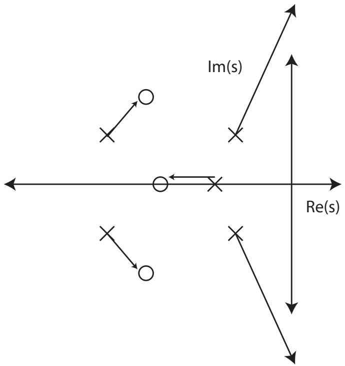
Asymptotics

Summary

Asymptotes depend only on relative number of poles and zeros.

- Location of poles/zeros doesn't matter
  - At least not for the angle

One pole goes to each zero.



When there are more poles than zeros:

Cases:

- $n - m = 0$ - No Asymptotes
- $n - m = 1$ - Asymptote at $180°$
- $n - m = 2$ - Asymptotes at $\pm 90°$
- $n - m = 3$ - Asymptotes at $180°$, $\pm 60°$
- $n - m = 4$ - Asymptotes at $\pm 45°$ and $\pm 135°$

Summary

What have we learned today?

Review

- Definition of Root Locus

Points on the Real Axis

- Symmetry
- Drawing the Real Axis

What Happens at High Gain?

- The effect of Zeros
- Asymptotes

Next Lecture: Centers of Asymptotes, Break Points and Departure Angles