Stabilization via LMIs

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Lecture 22: Stabilization via LMIs

Linear Matrix Inequalities are often a *Simpler* way to solve control problems.

Semidefinite Programming

$$\min c^T x :$$

$$Y_0 + \sum_i x_i Y_i > 0$$

Here the Y_i are symmetric matrices.

Linear Matrix Inequalities

$$\text{Find} x :$$

$$Y_0 + \sum_i x_i Y_i > 0$$

Commonly Takes the Form

$$\text{Find} X :$$

$$\sum_i A_i X B_i + Q > 0$$

LMIs unite time-domain and frequency-domain analysis

Theorem 1 (Lyapunov).

Suppose there exists a continuously differentiable function V for which $V(0) = 0$ and $V(x) > 0$ for $x \neq 0$. Furthermore, suppose $\lim_{\|x\| \to \infty} V(x) = \infty$ and

$$\lim_{h \to 0^+} \frac{V(x(t+h)) - V(x(t))}{h} = \frac{d}{dt} V(x(t)) < 0$$

for any x such that $\dot{x}(t) = f(x(t))$. Then for any $x(0) \in \mathbb{R}$ the system of equations

$$\dot{x}(t) = f(x(t))$$

has a unique solution which is stable in the sense of Lyapunov.

The Lyapunov Inequality

Lemma 2.

A is Hurwitz if and only if there exists a $P > 0$ such that

$$A^T P + PA < 0$$

Proof.

Suppose there exists a $P > 0$ such that $A^T P + PA < 0$.

- Define the Lyapunov function $V(x) = x^T P x$.
- Then $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$.
- Furthermore,

$$\begin{aligned} \dot{V}(x(t)) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\ &= x(t)^T A^T P x(t) + x(t)^T P A x(t) \\ &= x(t)^T \left(A^T P + PA \right) x(t) \end{aligned}$$

- Hence $\dot{V}(x(t)) < 0$ for all $x \neq 0$. Thus the system is globally stable.
- Global stability implies A is Hurwitz.

The Lyapunov Inequality

Proof.

For the other direction, if A is Hurwitz, let

$$P = \int_0^\infty e^{A^T s} e^{As} ds$$

- Converges because A is Hurwitz.
- Furthermore
$$\begin{aligned} PA &= \int_0^\infty e^{A^T s} e^{As} A ds \\ &= \int_0^\infty e^{A^T s} A e^{As} ds = \int_0^\infty e^{A^T s} \frac{d}{ds}\left(e^{As}\right) ds \\ &= \left[e^{A^T s} e^{As} \right]_0^\infty - \int_0^\infty \frac{d}{ds} e^{A^T s} e^{-As} \\ &= -I - \int_0^\infty A^T e^{A^T s} e^{-As} = -I - A^T P \end{aligned}$$
- Thus $PA + A^T P = -I < 0$.

Other Versions:

Lemma 3.

(A, B) *is controllable if and only if there exists a* $X > 0$ *such that*

$$A^T X + XA + BB^T \leq 0$$

Lemma 4.

(C, A) *is stabilizable if and only if there exists a* $X > 0$ *such that*

$$AX + XA^T + C^T C \leq 0$$

These also yield Lyapunov functions for the system.

The Static State-Feedback Problem

Lets start with the problem of stabilization.

Definition 5.

The **Static State-Feedback Problem** is to find a feedback matrix K such that

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ u(t) &= Kx(t)\end{aligned}$$

is stable

We have already solved the problem earlier using the Controllability Grammian.

- Find K such that $A + BK$ is Hurwitz.

Can also be put in LMI format:

$$\textbf{Find } X > 0, K \; :$$
$$X(A + BK) + (A + BK)^T X < 0$$

Problem: Bilinear in K and X.

The Static State-Feedback Problem

- The bilinear problem in K and X is a common paradigm.
- Bilinear optimization is not convex.
- To convexify the problem, we use a change of variables.

Problem 1:

$$\textbf{Find } X > 0, K \; :$$
$$X(A + BK) + (A + BK)^T X < 0$$

Problem 2:

$$\textbf{Find } P > 0, Z \; :$$
$$AP + BZ + PA^T + ZB^T < 0$$

Definition 6.

Two optimization problems are equivalent if a solution to one will provide a solution to the other.

Theorem 7.

Problem 1 is equivalent to Problem 2.

The Dual Lyapunov Equation

Problem 1:

Find $X > 0,$:

$XA + A^T X < 0$

Problem 2:

Find $Y > 0,$:

$YA^T + AY < 0$

Lemma 8.

Problem 1 is equivalent to problem 2.

Proof.

First we show 1) solves 2). Suppose $X > 0$ is a solution to Problem 1. Let $Y = X^{-1} > 0$.

- If $XA + A^T X < 0$, then

$$X^{-1}(XA + A^T X)X^{-1} < 0$$

- Hence

$$X^{-1}(XA + A^T X)X^{-1} = AX^{-1} + X^{-1}A^T = AY + YA^T < 0$$

- Therefore, Problem 2 is feasible with solution $Y = X^{-1}$.

The Dual Lyapunov Equation

Problem 1:

Find $X > 0, :$

$XA + A^T X < 0$

Problem 2:

Find $Y > 0, :$

$YA^T + AY < 0$

Proof.

Now we show 2) solves 1) in a similar manner. Suppose $Y > 0$ is a solution to Problem 1. Let $X = Y^{-1} > 0$.

- Then

$$\begin{aligned} XA + A^T X &= X(AX^{-1} + X^{-1}A^T)X \\ &= X(AY + YA^T)X < 0 \end{aligned}$$

□

Conclusion: If $V(x) = x^T P x$ proves stability of $\dot{x} = Ax$,

- Then $V(x) = x^T P^{-1} x$ proves stability of $\dot{x} = A^T x$.

The Stabilization Problem

Thus we rephrase Problem 1

Problem 1:

Find $P > 0, K$:

$$(A + BK)P + P(A + BK)^T < 0$$

Problem 2:

Find $X > 0, Z$:

$$AX + BZ + XA^T + ZB^T < 0$$

Theorem 9.

Problem 1 is equivalent to Problem 2.

Proof.

We will show that 2) Solves 1). Suppose $X > 0$, Z solves 2). Let $P = X > 0$ and $K = ZP^{-1}$. Then $Z = KP$ and

$$\begin{aligned}(A + BK)P + P(A + BK)^T &= AP + PA^T + BKP + PK^T B^T \\ &= AP + PA^T + BZ + Z^T B^T < 0\end{aligned}$$

Now suppose that $P > 0$ and K solve 1). Let $X = P > 0$ and $Z = KP$. Then

$$AP + PA^T + BZ + Z^T B^T = (A + BK)P + P(A + BK)^T < 0$$

The Stabilization Problem

The result can be summarized more succinctly

Theorem 10.

(A, B) is static-state-feedback stabilizable if and only if there exists some $P > 0$ and Z such that

$$AP + PA^T + BZ + Z^T B^T < 0$$

with $u(t) = ZP^{-1}x(t)$.

Standard Format:

$$\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} P \\ Z \end{bmatrix} + \begin{bmatrix} P & Z^T \end{bmatrix} \begin{bmatrix} A^T \\ B^T \end{bmatrix} < 0$$

There are a number of general-purpose LMI solvers available. e.g.

- `SeDuMi` - Free
- `LMI Lab` - in Matlab Robust Control Toolbox (in Computer lab)
- `YALMIP` - A nice front end for several solvers (Free)

The Schur complement

Before we get to the main result, recall the Schur complement.

Theorem 11 (Schur Complement).

For any $S \in \mathbb{S}^n$, $Q \in \mathbb{S}^m$ and $R \in \mathbb{R}^{n \times m}$, the following are equivalent.

1. $\begin{bmatrix} M & R \\ R^T & Q \end{bmatrix} > 0$
2. $Q > 0$ *and* $M - RQ^{-1}R^T > 0$

A commonly used property of positive matrices.
Also Recall: If $X > 0$,

- then $X - \epsilon I > 0$ for ϵ sufficiently small.

The KYP Lemma (AKA: The Bounded Real Lemma)

The most important theorem in this class.

Lemma 12.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- $\|G\|_{H_\infty} \le \gamma$.
- *There exists a* $X > 0$ *such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} < 0$$

Can be used to calculate the H_∞-norm of a system

- Originally used to solve LMI's using graphs. (Before Computers)
- Now used directly instead of graphical methods like Bode.

The feasibility constraints are linear

- Can be combined with other methods.

The KYP Lemma

Proof.

We will only show that ii) implies i). The other direction requires the Hamiltonian, which we have not discussed.

- We will show that if $y = Gu$, then $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$.
- From the 1 x 1 block of the LMI, we know that $A^T X + XA < 0$, which means A is Hurwitz.
- Because the inequality is strict, there exists some $\epsilon > 0$ such that

$$\begin{aligned}
&\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \\
&= \begin{bmatrix} A^T X + XA & XB \\ B^T X & -\gamma I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \epsilon I \end{bmatrix} < 0
\end{aligned}$$

- Let $y = Gu$. Then the state-space representation is

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
\dot{x}(t) &= Ax(t) + Bu(t) \qquad\qquad x(0) = 0
\end{aligned}$$

The KYP Lemma

Proof.

- Let $V(x) = x^T X x$. Then the LMI implies

$$\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \left[\begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \right] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} C^T \\ D^T \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$

$$= \begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T X + XA & XB \\ B^T X & -(\gamma - \epsilon) I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \frac{1}{\gamma} y^T y$$

$$= x^T (A^T X + XA) x + x^T X B u + u^T B^T X x - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= (Ax + Bu)^T X x + x^T X (Ax + Bu) - (\gamma - \epsilon) u^T u + \frac{1}{\gamma} y^T y$$

$$= \dot{x}(t)^T X x(t) + x(t)^T X \dot{x}(t) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2$$

$$= \dot{V}(x(t)) - (\gamma - \epsilon) \|u(t)\|^2 + \frac{1}{\gamma} \|y(t)\|^2 < 0$$

The KYP Lemma

Proof.

- Now we have $\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2 < 0$
- Integrating in time, we get

$$\int_0^T \left(\dot{V}(x(t)) - (\gamma - \epsilon)\|u(t)\|^2 + \frac{1}{\gamma}\|y(t)\|^2\right)dt$$

$$= V(x(T)) - V(x(0)) - (\gamma - \epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2\Big) dt < 0$$

- Because A is Hurwitz, $\lim_{t\to\infty} x(t) = 0$.
- Hence $\lim_{t\to\infty} V(x(t)) = 0$.
- Likewise, because $x(0) = 0$, we have $V(x(0)) = 0$.

□

The KYP Lemma

Proof.

- Since $V(x(0)) = V(x(\infty)) = 0$,

$$\lim_{T\to\infty}\left[\dot V(x(T)) - \dot V(x(0)) - (\gamma-\epsilon)\int_0^T \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^T \|y(t)\|^2\Big) dt\right]$$
$$= 0 - 0 - (\gamma-\epsilon)\int_0^\infty \|u(t)\|^2 dt + \frac{1}{\gamma}\int_0^\infty \|y(t)\|^2 dt$$
$$= -(\gamma-\epsilon)\|u\|_{L_2}^2 + \frac{1}{\gamma}\|y\|_{L_2}^2 dt < 0$$

- Thus

$$\|y\|_{L_2}^2 dt < (\gamma^2 - \epsilon\gamma)\|u\|_{L_2}^2$$

- By definition, this means $\|G\|_{H_\infty}^2 \le (\gamma^2-\epsilon\gamma) < \gamma^2$ or

$$\|G\|_{H_\infty} < \gamma$$

A Variation on the KYP lemma is the positive-real lemma

Lemma 13.

Suppose

$$\hat{G}(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right].$$

Then the following are equivalent.

- G *is passive. i.e.* $(\langle u, Gu \rangle_{L_2} \geq 0)$.
- *There exists a* $P > 0$ *such that*

$$\begin{bmatrix} A^T P + PA & PB - C^T \\ B^T P - C & -D^T - D \end{bmatrix} \leq 0$$