# LMI Methods in Optimal and Robust Control

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Lecture 14: LMIs for Robust Control in the LFT Framework

# Types of Uncertainty

In this Lecture, we will cover

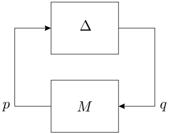
- **Unstructured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\|_{H_\infty} < 1\}$$
- **Structured, Static, norm-bounded:**
$$\mathbf{\Delta} := \{\text{diag}(\delta_1, \cdots, \delta_K, \Delta_1, \cdots \Delta_N) \,:\, |\delta_i| < 1,\ \bar{\sigma}(\Delta_i) < 1\}$$
- **Structured, Dynamic, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta_1, \Delta_2, \cdots \in \mathcal{L}(L_2) \,:\, \|\Delta_i\|_{H_\infty} < 1\}$$
- **Unstructured, Static, norm-bounded:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \leq 1\}$$
- **Parametric, Polytopic:**
$$\mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \Delta = \sum_i \alpha_i H_i,\ \alpha_i \geq 0,\ \sum_i \alpha_i = 1\}$$
- **Parametric, Interval:**
$$\mathbf{\Delta} := \{\sum_i \Delta_i \delta_i \,:\, \delta_i \in [\delta_i^-, \delta_i^+]\}$$

Each of these can be Time-Varying or Time-Invariant!

# Back to the Linear Fractional Transformation

The interval and polytopic cases rely on **Linearity** of the uncertain parameters.

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$




The Linear-Fractional Transformation, however

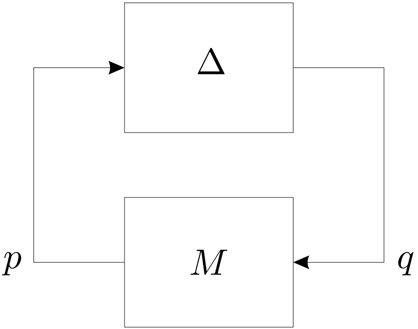
$$\begin{bmatrix}\dot{x}(t)\\ p(t)\end{bmatrix} = \bar{S}(P,\Delta)\begin{bmatrix}x(t)\\ q(t)\end{bmatrix} = (P_{22}+P_{21}\Delta(I-P_{11}\Delta)^{-1}P_{12})\begin{bmatrix}x(t)\\ q(t)\end{bmatrix}$$

is an arbitrary rational function.

We focus on two results:

- The S-Procedure for Unstructured Uncertainty Sets
- The Structured Singular Value for Structured Uncertainty Sets.

# Robust Stability




**Questions:**

- Is $\bar{S}(M, \Delta)$ stable for all $\Delta \in \mathbf{\Delta}$?
- Is $I - \Delta M_{11}$ invertible for all $\Delta \in \mathbf{\Delta}$?

# Redefine Robust and Quadratic Stability

Suppose we have the system
$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

**Definition 1.**

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if $(I - M_{11}\Delta)$ is invertible for all $\Delta \in \mathbf{\Delta}$.

Alternatively, if
$$\begin{bmatrix} \dot{x}(t) \\ z(t) \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$$

**Definition 2 (Continuous-Time).**

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if for some $\beta > 0$,
$M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12} + \beta I$ is Hurwitz for all $\Delta \in \mathbf{\Delta}$.

Alternatively, if
$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \bar{S}(M, \Delta) \begin{bmatrix} x_k \\ w_k \end{bmatrix}$$

**Definition 3 (Discrete-Time).**

The pair $(M, \mathbf{\Delta})$ is **Robustly Stable** if
$\rho(M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}) = \beta < 1$ for all $\Delta \in \mathbf{\Delta}$.

# Quadratic Stability - Parametric Uncertainty

**Focus** on the 1,1 block of $\bar{S}(M,\Delta)$:
If $$\dot{x}(t) = \bar{S}(M,\Delta)x(t),$$

**Definition 4 (Continuous Time).**

The pair $(M, \mathbf{\Delta})$ is **Quadratically Stable** if there exists a $P > 0$ such that

$$\bar{S}(M,\Delta)^T P + P\bar{S}(M,\Delta) < -\beta I \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

Alternatively, if $$x_{k+1} = \bar{S}(M,\Delta)x_k,$$

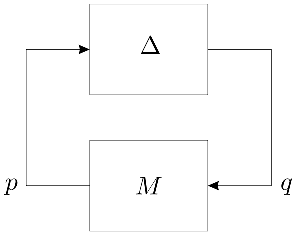
**Definition 5 (Discrete Time).**

The pair $(M, \mathbf{\Delta})$ is **Quadratically Stable** if there exists a $P > 0$ such that

$$\bar{S}(M,\Delta)^T P\bar{S}(M,\Delta) - P < -\beta I \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

for all $\Delta \in \mathbf{\Delta}$.

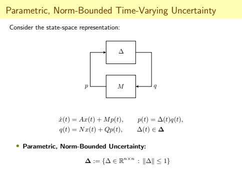
Consider the state-space representation:




$$\dot{x}(t) = Ax(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t), \qquad \Delta(t) \in \boldsymbol{\Delta}$$

- **Parametric, Norm-Bounded Uncertainty:**

$$\boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} \; : \; \|\Delta\| \leq 1\}$$




If we close the loop,

$$\dot{x}(t) = Ax(t) + Mq(t), \qquad q(t) = \Delta(t)(Nx(t) + Qq(t)),$$
$$q(t) = (I - \Delta(t)Q)^{-1}\Delta(t)Nx(t)$$
$$\dot{x}(t) = (A + M(I - \Delta(t)Q)^{-1}\Delta(t)N)x(t) = \bar{S}\left(\begin{bmatrix} A & M \\ N & Q \end{bmatrix}, \Delta\right)$$

But this is complicated, so we seek a simpler approach.

$$V(x) = x^T P x$$

$$\dot{V}(x) = x(t)^T P(Ax(t) + Mq(t)) + (Ax(t) + Mq(t))^T Px(t) < 0$$

for all $p, x$ such that

$$\|q\|^2 \le \|Nx + Qq\|^2$$

or

$$\begin{bmatrix} x \\ q \end{bmatrix}^T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \le \begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} N^T \\ Q^T \end{bmatrix} \begin{bmatrix} N & Q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} = \begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} N^T N & N^T Q \\ Q^T N & Q^T Q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix}$$

# Parametric, Norm-Bounded Uncertainty

**Quadratic Stability:** There exists a $P > 0$ such that

$$x^T P(Ax+Mq)+(Ax+Mq)^T Px < 0 \text{ for all } [x,q] \in \left\{x, q \,:\, q = \Delta p, \begin{matrix} p = Nx + Qq, \\ \Delta \in \mathbf{\Delta} \end{matrix}\right.$$

## Theorem 6.

*The system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Mq(t), & q(t) &= \Delta(t)p(t), \\ p(t) &= Nx(t) + Qq(t), & \Delta \in \mathbf{\Delta} &:= \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\} \end{aligned}$$

*is quadratically stable if and only if there exists some $P > 0$ such that*

$$\begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ M^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ q \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ q \end{bmatrix} \,:\, \begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \le 0 \right\}$$

## Parametric, Norm-Bounded Uncertainty

**Quadratic Stability:** There exists a $P > 0$ such that

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The quadratic stability condition is a conditional LMI

- Positive on a subset of $[x, q]$
- $[x, q]$ lies in an ellipsoid (a semialgebraic set).)
- Enforcing an LMI on a subset is usually hard.

# Parametric, Norm-Bounded Uncertainty

## Proof, If.

If

$$\begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} A^T P + PA & PM \\ M^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} < 0$$

$$\text{for all } \begin{bmatrix} x \\ q \end{bmatrix} \in \left\{ \begin{bmatrix} x \\ q \end{bmatrix} : \begin{bmatrix} x \\ q \end{bmatrix} \begin{bmatrix} -N^T N & -N^T Q \\ -Q^T N & I - Q^T Q \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \le 0 \right\}$$

then

$$x^T P(Ax + Mq) + (Ax + Mq)^T Px < 0$$

for all $x$, $q$ such that

$$\|q\|^2 \le \|Nx + Qq\|^2$$

Therefore, since $q = \Delta p$ implies $\|q\| \le \|p\|$, we have quadratic stability.
The *only if* direction is similar.

□

Quadratic stability here requires positivity of a matrix on a *subset*.

- This is Generally a very hard problem
- NP-hard to determine if $x^T F x \geq 0$ for all $x \geq 0$. (Matrix Copositivity)

S-procedure to the rescue!

The S-procedure asks the question:

- Is $z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$?

**Corollary 7 (S-Procedure).**

*$z^T F z \geq 0$ for all $z \in \{x : x^T G x \geq 0\}$ if there exists a scalar $\tau \geq 0$ such that $F - \tau G \succeq 0$.*

Sufficiency is Obvious!

- The S-procedure is **Necessary** if $\{x : x^T G x > 0\}$ has an interior point.

# An LMI for Parametric, Norm-Bounded Uncertainty

**Theorem 8 (Dual Version).**

*The system*

$$\dot{x}(t) = Ax(t) + Mq(t), \qquad q(t) = \Delta(t)p(t),$$
$$p(t) = Nx(t) + Qq(t), \qquad \Delta \in \boldsymbol{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

*is quadratically stable if and only if there exists some $\mu \ge 0$ and $P > 0$ such that*

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}$$

Noting that the LMI can be written as

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & -\mu I \end{bmatrix} + \mu \begin{bmatrix} M \\ Q \end{bmatrix} \begin{bmatrix} M \\ Q \end{bmatrix}^T < 0$$

or

$$\begin{bmatrix} AP + PA^T & PN^T & M^T \\ NP & -\mu I & Q^T \\ M & Q & -\frac{1}{\mu} I \end{bmatrix} < 0$$

we see that this condition is simply an $H_\infty$ gain condition on the nominal system $\|\cdot\|_{H_\infty} < 1$.

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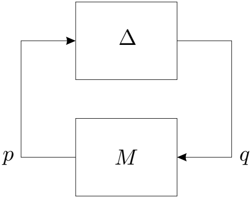
we see that this condition is simply an $H_\infty$ gain condition on the nominal system $\|\cdot\|_{H_\infty} < 1$.

- We skipped the Primal version, but it should be obvious.
- Set $\mu = 1$ and we have an LMI for $\|\cdot\|_{H_\infty} < 1$

# Necessity of the Small-Gain Condition

This leads to the interesting result:




If $\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\| \le 1\}$, then

- $\bar{S}(P, \Delta) \in H_\infty$ if **and only if** $\|M_{11}\|_{H_\infty} < 1$
- The small gain condition is necessary and sufficient for stability.
- Quadratic Stability is equivalent to stability.
- Holds for Dynamic and Parametric Uncertainty
  - Does this mean Quadratic and Robust Stability are Equivalent?

# Quadratic Stability and Equivalence to Robust Stability

Consider Quadratic Stability in Discrete-Time: $x_{k+1} = S_l(M, \Delta)x_k$.

**Definition 9.**

$(S_l, \mathbf{\Delta})$ is QS if

$$S_l(M, \Delta)^T P S_l(M, \Delta) - P < 0 \qquad \text{for all } \Delta \in \mathbf{\Delta}$$

**Theorem 10 (Packard and Doyle).**

*Let $M \in \mathbb{R}^{(n+m)\times(n+m)}$ be given with $\rho(M_{11}) \leq 1$ and $\sigma(M_{22}) < 1$. Then the following are equivalent.*

1. *The pair $(M, \Delta = \mathbb{R}^{m\times m})$ is quadratically stable.*
2. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is quadratically stable.*
3. *The pair $(M, \Delta = \mathbb{C}^{m\times m})$ is robustly stable.*

# Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

However, we can add controllers:

**Theorem 11.**

*The system with* $u(t) = Kx(t)$ *and*

$$\dot{x}(t) = Ax(t) + Bu(t) + Mq(t), \qquad q(t) = \Delta(t)p(t),$$
$$p(t) = Nx(t) + Qq(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} : \|\Delta\| \le 1\}$$

*is quadratically stable if and only if there exists some* $\mu \ge 0$ *and* $P > 0$ *such that*

$$\begin{bmatrix}(A+BK)P + P(A+BK)^T & P(N+D_{12}K)^T \\ (N+D_{12}K)P & 0\end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I\end{bmatrix} < 0\}$$

Of course, this is bilinear in $P$ and $K$, so we make the change of variables $Z = KP$.

# An LMI for Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

**Theorem 12.**

*There exists a* $K$ *such that the system with* $u(t) = Kx(t)$

$$\dot{x}(t) = Ax(t) + Bu(t) + Mq(t), \qquad q(t) = \Delta(t)p(t),$$
$$p(t) = Nx(t) + Qq(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}$$

*is quadratically stable if and only if there exists some* $\mu \ge 0$*,* $Z$ *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^TB^T & PN^T + Z^TD_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}.$$

*Then* $K = ZP^{-1}$ *is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the $H_\infty$ norm.

## An LMI for Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

**Theorem 12.**

*There exists a $K$ such that the system with $u(t) = Kx(t)$*

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*is quadratically stable if and only if there exists some $\mu \ge 0$, $Z$ and $P > 0$ such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \mu \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0\}.$$

*Then $K = ZP^{-1}$ is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the $H_\infty$ norm.

This is from Boyd page 101

# An LMI for $H_\infty$-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

**Theorem 13.**

*There exists a $K$ such that the system with $u(t) = Kx(t)$*

$$\begin{aligned}
&\dot{x}(t) = Ax(t) + Bu(t) + Mq(t) + B_2 w(t), \qquad q(t) = \Delta(t)p(t),\\
&p(t) = Nx(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n\times n} \,:\, \|\Delta\| \le 1\}\\
&z(t) = Cx(t) + D_{22}u(t)
\end{aligned}$$

*satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$ if there exists some $\mu \ge 0$, $Z$ and $P > 0$ such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^TB^T + B_2B_2^T + \mu MM^T & (CP + D_{22}Z)^T & PN^T + Z^TD_{12}^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ NP + D_{12}Z & 0 & -\mu I \end{bmatrix} < 0.$$

*Then $K = ZP^{-1}$ is the corresponding controller.*

An LMI for $H_\infty$-Optimal Quadratically Stabilizing Controllers with Parametric Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

**Theorem 13.**

*There exists a $K$ such that the system with $u(t) = Kx(t)$*

$$\dot{x}(t) = Ax(t) + Bu(t) + Mq(t) + B_2 w(t), \qquad q(t) = \Delta(t)p(t),$$
$$p(t) = Nx(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta} := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \le 1\}$$
$$z(t) = Cx(t) + D_{22}u(t)$$

*satisfies $\|z\|_{L_2} \le \gamma \|w\|_{L_2}$ if there exists some $\mu \ge 0$, $Z$ and $P > 0$ such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + \mu M M^T & (CP + D_{22}Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ NP + D_{12}Z & 0 & -\mu I \end{bmatrix} < 0.$$

*Then $K = ZP^{-1}$ is the corresponding controller.*

This is from Boyd page 110.
I believe it relies on the following alternative to the S-procedure [Xie, 1992] (See also Caverly Notes), which is similar to Finsler's Lemma

## Theorem 14.

*The following are equivalent*

1. $$Q + F\Delta E + E^T \Delta F^T > 0 \qquad \textit{for all } \|\Delta\| < 1$$
2. *There exists some $\epsilon > 0$ such that*
$$Q + \epsilon F F^T + \epsilon^{-1} E^T E > 0$$

Unfortunately, to put the LMI in the form of 1 requires us to eliminate the pass-through term $Q$.

# Structure, Norm-Bounded Uncertainty

For the case of structured parametric uncertainty, we define the structured set

$$\boldsymbol{\Delta} = \{\Delta = \mathrm{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) \,:\, \delta_i \in \mathbb{R},\, \Delta \in \mathbb{R}^{n_k \times n_k}\}$$

$$\Delta = \begin{bmatrix} \delta_1 I_{n1} & & & & & \\ & \ldots & & & & \\ & & \delta_s I_{ns} & & & \\ & & & \Delta_{s+1} & & \\ & & & & \ldots & \\ & & & & & \Delta_{s+f} \end{bmatrix}$$

- $\delta$ and $\Delta$ represent unknown parameters.
- $s$ is the number of scalar parameters.
- $f$ is the number of matrix parameters.

# The Structured Singular Value

For the case of structured parametric uncertainty, we define the structured singular value.

## Definition 15.

Given system $M \in \mathcal{L}(L_2)$ and set $\mathbf{\Delta}$ as above, we define the **Structured Singular Value** of $(M, \mathbf{\Delta})$ as

$$\mu(M, \mathbf{\Delta}) = \frac{1}{\inf_{\substack{\Delta \in \mathbf{\Delta} \\ I - M\Delta \text{ is singular}}} \|\Delta\|}$$

Of course, $\bar{S}(M, \mathbf{\Delta})$ is stable if and only if $\mu(M_{11}, \mathbf{\Delta}) < 1$.

- Obviously, $\mu(M, \mathbf{\Delta}) < \|M\|$
- For $\mathbf{\Delta} := \{\Delta \in \mathcal{L}(L_2) \,:\, \|\Delta\| \le 1\}$, $\mu(M, \mathbf{\Delta}) = \|M\|$
- $\mu(\alpha M, \mathbf{\Delta}) = |\alpha| \mu(M, \mathbf{\Delta})$
- Can increase $M$ by a factor $\frac{1}{\mu(M, \mathbf{\Delta})}$ before losing stability.
- In general, computing $\mu$ is NP-hard unless uncertainty is unstructured.

# Scalings and The Structured Singular Value

Suppose $\mathbf{\Theta} = \{\Theta : \Theta\Delta = \Delta\Theta \text{ for all } \Delta \in \mathbf{\Delta}\}$

- Then $\mu(M, \mathbf{\Delta}) = \inf_{\Theta \in \mathbf{\Theta}} \|\Theta M \Theta^{-1}\|$.
- $\Theta$ is the set of *scalings*.

# Scalings and The Structured Singular Value

$$\mathbf{\Delta} = \{\Delta = \mathrm{diag}(\delta_1 I_{n1}, \cdots, \delta_s I_{ns}, \Delta_{s+1}, \cdots, \Delta_{s+f}) \,:\, \delta_i \in \mathbb{R},\, \Delta \in \mathbb{R}^{n_k \times n_k}\}$$

Define the set of scalings

$$\mathbf{P\Theta} := \{\mathrm{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1} I, \cdots, \theta_{s+f} I \,:\, \Theta_i > 0,\, \theta_j > 0\}$$

**Theorem 16.**

*Suppose system $M$ has transfer function $\hat{M}(s) = C(sI - A)^{-1}B + D$ with $\hat{M} \in H_\infty$. The following are equivalent*

- *There exists $\Theta \in \mathbf{P\Theta}$ such that $\|\Theta M \Theta^{-1}\|^2 < \gamma$.*
- *There exists $\Theta \in \mathbf{P\Theta}$ and $X > 0$ such that*

$$\begin{bmatrix} A^T X + XA & XB \\ B^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C^T \\ D^T \end{bmatrix} \Theta \begin{bmatrix} C & D \end{bmatrix} < 0$$

**Note:** To minimize $\gamma$, you must use bisection.

# An LMI for Stability of Structured, Norm-Bounded Uncertainty

This allows us to generalize the S-procedure to structured uncertainty

**Theorem 17.**

*The system*

$$\dot{x}(t) = Ax(t) + Mp(t), \qquad p(t) = \Delta(t)q(t),$$
$$q(t) = Nx(t) + Qp(t), \qquad \Delta \in \mathbf{\Delta}, \;\; \|\Delta\| \le 1$$

*is quadratically stable if and only if there exists some* $\Theta \in \mathbf{P\Theta}$ *and* $P > 0$ *such that*

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0\}$$

This is an LMI in $\Theta$ and $P$.

- The constraint $\Theta \in \mathbf{P\Theta}$ is linear

$$\mathbf{P\Theta} := \{\mathrm{diag}(\Theta_1, \cdots, \Theta_s, \theta_{s+1}I, \cdots, \theta_{s+f}I) \,:\, \Theta_i > 0,\, \theta_j > 0\}$$

# An LMI for Stability with Structured, Norm-Bounded Uncertainty

To prove the theorem, we can take a closer look at the scalings:

Since $T\Delta = \Delta T$ for $T \in \mathbf{\Theta}$, the system can equivalently be written as

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + MT^{-1}q(t), & q(t) &= \Delta(t)p(t), \\
p(t) &= TNx(t) + TQT^{-1}q(t), & \Delta &\in \mathbf{\Delta},\ \|\Delta\| \le 1
\end{aligned}$$

for any $T \in \mathbf{\Theta}$. Then

$$\begin{bmatrix} AP + PA^T & PN^T \\ NP & 0 \end{bmatrix} + \begin{bmatrix} MM^T & MQ^T \\ QM^T & QQ^T - I \end{bmatrix} < 0$$

becomes

$$\begin{bmatrix} AP + PA^T & PN^T T^T \\ TNP & 0 \end{bmatrix} + \begin{bmatrix} MT^{-2}M^T & MT^{-2}Q^T T^T \\ TQT^{-2}M^T & TQT^{-2}Q^T T^T - I \end{bmatrix} < 0\}$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 \\ 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

# An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

**Theorem 18.**

*There exists a* $K$ *such that the system with* $u(t) = Kx(t)$

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Mq(t), \qquad & q(t) &= \Delta(t)p(t), \\ p(t) &= Nx(t) + Qq(t) + D_{12}u(t), \qquad & \Delta &\in \mathbf{\Delta},\ \|\Delta\| \le 1 \end{aligned}$$

*is quadratically stable if there exists some* $\Theta \in \mathbf{P\Theta}$, $P > 0$ *and* $Z$ *such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0.$$

*Then* $K = ZP^{-1}$ *is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the $H_\infty$ norm.

# An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

## An LMI for Stabilizing State-Feedback Controllers with Structured Norm-Bounded Uncertainty

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*is quadratically stable if there exists some $\Theta \in \mathbf{P\Theta}$, $P > 0$ and $Z$ such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T & PN^T + Z^T D_{12}^T \\ NP + D_{12}Z & 0 \end{bmatrix} + \begin{bmatrix} M\Theta M^T & M\Theta Q^T \\ Q\Theta M^T & Q\Theta Q^T - \Theta \end{bmatrix} < 0.$$

*Then $K = ZP^{-1}$ is a quadratically stabilizing controller.*

We can also extend this result to optimal control in the $H_\infty$ norm.

This is from Boyd, page 102

Using $\Theta = \mu I$, we recover the LMI for unstructured uncertainty.

# An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

In this case, we set $Q = 0$.

**Theorem 19.**

*There exists a $K$ such that the system with $u(t) = Kx(t)$*

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) + Mq(t) + B_2 w(t), \qquad q(t) = \Delta(t)p(t),\\
p(t) &= Nx(t) + D_{12}u(t), \qquad \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\\
z(t) &= Cx(t) + D_{22}u(t)
\end{aligned}$$

*satisfies $\|y\|_{L_2} \le \gamma \|u\|_{L_2}$ if there exists some $\Theta \in \mathbf{P\Theta}$, $Z$ and $P > 0$ such that*

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + M\Theta M^T & (CP + D_{22}Z)^T & PN^T + Z^T D_{12}^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ NP + D_{12}Z & 0 & -\Theta \end{bmatrix} < 0.$$

*Then $K = ZP^{-1}$ is the corresponding controller.*

# An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Using the equivalent scaled system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + MT^{-1}q(t) + B_2 w(t), \qquad q(t) = \Delta(t)p(t), \\ p(t) &= TNx(t) + TD_{12}u(t), \qquad \Delta \in \mathbf{\Delta}, \ \|\Delta\| \le 1 \\ z(t) &= Cx(t) + D_{22}u(t) \end{aligned}$$

we get

$$\begin{bmatrix} AP + BZ + PA^T + Z^T B^T + B_2 B_2^T + MT^{-2}M^T & (CP + D_{22}Z)^T & PN^T T^T + Z^T D_{12}^T T^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ TNP + TD_{12}Z & 0 & -I \end{bmatrix} < 0.$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

## An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

An LMI for Optimal State-Feedback Controllers with Structured Norm-Bounded Uncertainty

Using the equivalent scaled system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + MT^{-1}q(t) + B_2w(t), \qquad q(t) = \Delta(t)p(t),\\ p(t) &= TNx(t) + TD_{12}u(t), \qquad \Delta \in \mathbf{\Delta},\ \|\Delta\| \le 1\\ z(t) &= Cx(t) + D_{22}u(t)\end{aligned}$$

we get

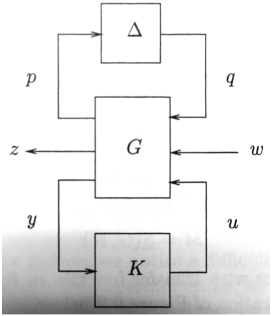
$$\begin{bmatrix} AP + BZ + PA^T + Z^TB^T + B_2B_2^T + MT^{-2}M^T & (CP + D_{22}Z)^T & PN^TT^T + Z^TD_{12}^TT^T \\ CP + D_{22}Z & -\gamma^2 I & 0 \\ TNP + TD_{12}Z & 0 & -I \end{bmatrix} < 0.$$

Pre- and Post-multiplying by $\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & T^{-1} \end{bmatrix}$, and using $\Theta = T^{-2} \in \mathbf{P\Theta}$, we recover the LMI condition.

This is not from Boyd, but should be

# Output-Feedback Robust Controller Synthesis

How to Solve the Output Feedback Case???




$$\inf_{K} \sup_{\Delta \in \mathbf{\Delta}} \|\underline{\mathsf{S}}(\bar{S}(G,\Delta),K)\|_{H_\infty}$$

Finally, we mention a Heuristic for Output-Feedback Controller synthesis.

**Initialize:** $\Theta = I$.
**Define:**

$$\hat{G}_\Theta(s) = \left[\begin{array}{c|cc} A & B_1\Theta^{-\frac{1}{2}} & B_2 \\ \hline \Theta^{\frac{1}{2}}C_1 & \Theta^{\frac{1}{2}}D_{11}\Theta^{-\frac{1}{2}} & \Theta^{\frac{1}{2}}D_{12} \\ C_2 & D_{21}\Theta^{-\frac{1}{2}} & 0 \end{array}\right]$$

**Step 1:** Fix $\Theta$ and solve

$$\inf_K \|\underline{S}(G_\Theta, K)\|_{H_\infty}$$

**Step 2:** Fix $K$ and minimize $\gamma$ such that there exists $\Theta \in \mathbf{P\Theta}$ ( or $\Theta \in \mathbf{P\Theta} \times I$ if you include the regulated output channel.) and $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} \\ B_{cl}^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2} \begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix} \Theta \begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

where $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ define $\underline{S}(G_I, K)$. (Requires Bisection).

**Step 3:** `GOTO` Step 1

## D-K Iteration

### A Heuristic for Dynamic Output Feedback Synthesis

Finally, we mention a Heuristic for Output-Feedback Controller synthesis.

**Initialize:** $\Theta = I$.
**Define:**

$$\hat{G}_\Theta(s) = \left[\begin{array}{c|cc} A & B_1\Theta^{-\frac{1}{2}} & B_2 \\ \hline \Theta^{\frac{1}{2}}C_1 & \Theta^{\frac{1}{2}}D_{11}\Theta^{-\frac{1}{2}} & \Theta^{\frac{1}{2}}D_{12} \\ C_2 & D_{21}\Theta^{-\frac{1}{2}} & 0 \end{array}\right]$$

**Step 1:** Fix $\Theta$ and solve

$$\inf_K \|\underline{S}(G_\Theta, K)\|_{H_\infty}$$

**Step 2:** Fix $K$ and minimize $\gamma$ such that there exists $\Theta \in \mathbf{P\Theta}$ ( or $\Theta \in \mathbf{P\Theta} \times I$ if you include the regulated output channel.) and $X > 0$ such that

$$\begin{bmatrix} A_{cl}^T X + X A_{cl} & X B_{cl} \\ B_{cl}^T X & -\Theta \end{bmatrix} + \frac{1}{\gamma^2}\begin{bmatrix} C_{cl}^T \\ D_{cl}^T \end{bmatrix}\Theta\begin{bmatrix} C_{cl} & D_{cl} \end{bmatrix} < 0$$

where $A_{cl}, B_{cl}, C_{cl}, D_{cl}$ define $\underline{S}(G_I, K)$. (Requires Bisection).

**Step 3:** `GOTO` Step 1

As with most heuristics, there are many variations on the D-K iteration. The one presented here is the simplest, and probably will not work well.

# A Word on D-K Iteration with Static Uncertainty

## A Heuristic for Dynamic Output Feedback Synthesis

The D-K iteration outlined in this lecture is only valid for *Dynamic Uncertainty*: $\Delta(t)$.

- Our Scalings $\Theta$ are time-invariant.

For Static uncertainties, we should search for *Dynamic Scaling Factors*

- $\Theta(s)$ is a *Transfer Function*
- This is much harder to represent as an LMI (Or by any other method!).
- Matlab has built-in functionality, but it is hard to use.

We will return to $\mu$ analysis for static uncertainties when we consider more advanced forms of optimization.