

LMI Methods in Optimal and Robust Control

Matthew M. Peet
Arizona State University

Lecture 13: LMIs for Optimal Control and Quadratic Stability with Affine
Polytopic and Interval Uncertainty

Types of Uncertainty

We will start with Time-Varying Parametric Uncertainty.

- **Unstructured, Dynamic, norm-bounded:**

$$\Delta := \{\Delta \in \mathcal{L}(L_2) : \|\Delta\|_{H_\infty} < 1\}$$

- **Structured, Static, norm-bounded:**

$$\Delta := \{\text{diag}(\delta_1, \dots, \delta_K, \Delta_1, \dots, \Delta_N) : |\delta_i| < 1, \bar{\sigma}(\Delta_i) < 1\}$$

- **Structured, Dynamic, norm-bounded:**

$$\Delta := \{\text{diag}(\Delta_1, \Delta_2, \dots) \in \mathcal{L}(L_2) : \|\Delta_i\|_{H_\infty} < 1\}$$

- **Unstructured, Parametric, norm-bounded:**

$$\Delta := \{\Delta \in \mathbb{R}^{n \times n} : \|\Delta\| \leq 1\}$$

- **Parametric, Polytopic:**

$$\Delta := \{\Delta \in \mathbb{R}^{n \times n} : \Delta = \sum_i \alpha_i H_i, \alpha_i \geq 0, \sum_i \alpha_i = 1\}$$

- **Parametric, Interval:**

$$\Delta := \{\sum_i \Delta_i \delta_i : \delta_i \in [\delta_i^-, \delta_i^+]\}$$

Each of these can be Time-Varying or Time-Invariant!

Additive Affine Time-Varying Interval and Polytopic Uncertainty

Stability Concepts

Recall the system with Affine Time-Varying uncertainty (No Input).

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

where

$$\Delta A(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$

where $\delta(t)$ lies in either the intervals

$$\delta_i(t) \in [\delta_i^-, \delta_i^+]$$

or the simplex

$$\delta(t) \in \{\delta : \sum_i \alpha_i = 1, \alpha_i \geq 0\}$$

Definitions: Use Robust Stability for Static Uncertainty

Definition 1.

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is **Robustly Stable** over Δ if $A_0 + \Delta$ is Hurwitz for all $\Delta \in \Delta$.

Note that Robust Stability DOES NOT imply stability if $\Delta(t)$ is time-varying.

- It implies that for any $\Delta \in \Delta$, there exists a $P(\Delta) > 0$ such that

$$(A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta$$

- For a fixed Δ , this implies stability using Lyapunov function $V(x) = x^T P(\Delta)x$.
- Does not imply stability for TV Δ because if $V(x, t) = x^T P(\Delta(t))x$,

$$\begin{aligned} \frac{d}{dt} V(x(t), t) &= x(t)^T \left((A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t)) \right) x(t) \\ &\quad + x(t)^T \left(\frac{d}{dt} P(\Delta(t)) \right) x(t) \\ &\leq x(t)^T \left(\frac{d}{dt} P(\Delta(t)) \right) x(t) \end{aligned}$$

Lecture 13

Definitions: Use Robust Stability for Static Uncertainty

Definition 1.

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is **Robustly Stable** over Δ if $A_0 + \Delta$ is Hurwitz for all $\Delta \in \Delta$.Note that Robust Stability DOES NOT imply stability if $\Delta(t)$ is time-varying.

- It implies that for any $\Delta \in \Delta$, there exists a $P(\Delta) > 0$ such that

$$(A + \Delta)^T P(\Delta) + P(\Delta)(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta$$

- For a fixed Δ , this implies stability using Lyapunov function

$$V(x) = x^T P(\Delta)x.$$

- Does not imply stability for TV Δ because if $V(x, t) = x^T P(\Delta(t))x$,

$$\begin{aligned} \frac{d}{dt}V(x(t), t) &= x(t)^T \left((A + \Delta(t))^T P(\Delta(t)) + P(\Delta(t))(A + \Delta(t)) \right) x(t) \\ &\quad + x(t)^T \left(\frac{d}{dt} P(\Delta(t)) \right) x(t) \\ &\leq x(t)^T \left(\frac{d}{dt} P(\Delta(t)) \right) x(t) \end{aligned}$$

Robust Stability is necessary and sufficient for static uncertainty.

Definitions: Quadratic Stability for Dynamic Uncertainty

Definition 2.

The system

$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

is **Quadratically Stable** over Δ if there exists a $P > 0$ such that

$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta.$$

Quadratic Stability **Implies Stability** of trajectories for any $\Delta(t)$ with $\Delta(t) \in \Delta$ for all $t \geq 0$.

- Use the Lyapunov function $V(x) = x^T P x$.

$$\frac{d}{dt} V(x(t)) = x(t)^T ((A + \Delta(t))^T P + P(A + \Delta(t))) x(t) < 0$$

Counterintuitive:

- Robust Stability does not imply stability!
- Stability does not imply quadratic stability!

Quadratic Stability is Conservative

Definition 3.

The system

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

is **Quadratically Stable** over Δ if there exists a $P > 0$ such that

$$(A + \Delta A)^T P + P(A + \Delta A) < 0 \quad \text{for all } \Delta A \in \Delta.$$

Quadratic Stability is CONSERVATIVE.

- There are Stable Systems which are not Quadratically Stable

$$\dot{x} = A(t)x,$$

$$A(t) = \delta_1(t) \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} + \delta_2(t) \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix}, \quad \delta_i \geq 0, \quad \delta_1 + \delta_2 = 1$$

- Use $V(x) = \max\{x^T P_1 x, x^T P_2 x\}$ where

$$P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Lecture 13

2022-06-07

Quadratic Stability is Conservative

Definition 3.

The system

$$\dot{x}(t) = (A_0 + \Delta A(t))x(t)$$

is **Quadratically Stable** over Δ if there exists a $P > 0$ such that

$$(A + \Delta A)^T P + P(A + \Delta A) < 0 \quad \text{for all } \Delta A \in \Delta.$$

Quadratic Stability is CONSERVATIVE.

• There are Stable Systems which are not Quadratically Stable

$$\dot{x} = A(t)x,$$

$$A(t) = \delta_1(t) \begin{bmatrix} -100 & 0 \\ 0 & -1 \end{bmatrix} + \delta_2(t) \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix}, \quad \delta_1 \geq 0, \quad \delta_1 + \delta_2 = 1$$

• Use $V(x) = \max\{x^T P_1 x, x^T P_2 x\}$ where

$$P_1 = \begin{bmatrix} 14 & -1 \\ -1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Quadratic Stability is sometimes referred to as an “infinite-dimensional LMI”

- Meaning it represents an infinite number of LMI constraints
- One for each possible value of $\Delta \in \Delta$
- Also called a parameterized LMI
- Such LMIs are not tractable.
- For polytopic sets, the LMI can be made finite.

Enforcing an LMI on the entire Polytope

Making an infinite-dimensional LMI finite dimensional

Theorem 4 (LMIs on the Polytope).

The following are equivalent for any H, L_i, R_i .

$$H + \sum_i L_i \Delta R_i > 0 \quad \text{for all } \Delta \in Co(\Delta_1, \dots, \Delta_k) \quad (1)$$

$$H + \sum_i L_i \Delta_j R_i > 0 \quad \text{for all } j = 1, \dots, k \quad (2)$$

Proof.

To show $1 \Rightarrow 2$, note that $\Delta_j \in Co(\Delta_1, \dots, \Delta_k)$ for each j . Next, show $2 \Rightarrow 1$.

$$\begin{aligned} H + \sum_i L_i \Delta R_i &= H + \sum_i L_i \left(\sum_j \alpha_j \Delta_j \right) R_i && \alpha_i \geq 0, \sum_j \alpha_j = 1 \\ &= \sum_j \alpha_j \left(H + \sum_i L_i \Delta_j R_i \right) \geq 0 \end{aligned}$$

□

An LMI for Polytopic Quadratic Stability

Definition 5.

The pair $(A + \Delta, \Delta)$ is **Quadratically Stable** over Δ if there exists a $P > 0$ such that

$$(A + \Delta)^T P + P(A + \Delta) < 0 \quad \text{for all } \Delta \in \Delta.$$

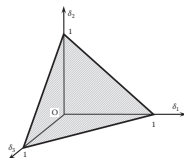
Theorem 6.

$(A + \Delta, \Delta)$ is quadratically stable over $\Delta := \text{Co}(A_1, \dots, A_k)$ if and only if there exists a $P > 0$ such that

$$(A + A_i)^T P + P(A + A_i) < 0 \quad \text{for } i = 1, \dots, k$$

The theorem says the LMI only needs to hold at the **EXTREMAL POINTS** or **VERTICES** of the polytope.

- In Fact, Quadratic Stability **MUST** be expressed as an LMI
- There is **NO** Ricatti Eqn. Equivalent.



An LMI for Interval Quadratic Stability

Recall the system with Affine Time-Varying uncertainty.

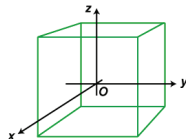
$$\dot{x}(t) = (A_0 + \Delta(t))x(t)$$

where

$$\Delta(t) = A_1\delta_1(t) + \cdots + A_k\delta_k(t)$$

where $\delta_i(t) \in [-1, 1]$. Note: $\delta(t)$ lies in the *hypercube*.

Interval Stability is a Kind of Polytopic Uncertainty.



The vertices of the hypercube define the vertices of the uncertainty set

$$V := \left\{ A_0 + \sum_i A_i \delta_i, \delta_i \in \{-1, 1\} \right\}$$

Theorem 7 (Quadratic Stability using 2^k LMI constraints!).

$(A + \Delta, \Delta)$ is quadratically stable over $\Delta := Co(V)$ if and only if there exists a $P > 0$ such that

$$\left(A_0 + \sum_i A_i \delta_i \right)^T P + P \left(A_0 + \sum_i A_i \delta_i \right) < 0 \quad \text{for every } \delta \in \{-1, 1\}^k$$

An LMI for Quadratic Polytopic Stabilization

Controller Synthesis is a simple application of the previous theorem:

Theorem 8.

There exists a K such that

$$\dot{x}(t) = (A + \Delta_A + (B + \Delta_B)K)x(t)$$

*is **quadratically stable** for $(\Delta_A, \Delta_B) \in Co((A_1, B_2), \dots, (A_k, B_k))$ if and only if there exists some $P > 0$ and Z such that*

$$(A + A_i)P + P(A + A_i)^T + (B + B_i)Z + Z^T(B + B_i)^T < 0 \quad \text{for } i = 1, \dots, k.$$

with $K = ZP^{-1}$.

Note that here the controller doesn't depend on Δ !

- If you want K to depend on Δ , the problem is harder.
- But this would require sensing Δ in real-time.

An LMI for Quadratic D-Stabilization

Lemma 9 (An LMI for Quadratic D-Stabilization).

Suppose there exists $X > 0$ and Z such that

$$\begin{bmatrix} -rP & AP + BZ \\ (AP + BZ)^T & -rP \end{bmatrix} + \begin{bmatrix} 0 & A_iP + B_iZ \\ (A_iP + B_iZ)^T & 0 \end{bmatrix} < 0,$$
$$AP + BZ + (AP + BZ)^T + A_iP + B_iZ + (A_iP + B_iZ)^T + 2\alpha P < 0, \quad \text{and}$$
$$\begin{bmatrix} AP + BZ + (AP + BZ)^T & c(AP + BZ - (AP + BZ)^T) \\ c((AP + BZ)^T - (AP + BZ)) & AP + BZ + (AP + BZ)^T \end{bmatrix}$$
$$+ \begin{bmatrix} A_iP + B_iZ + (A_iP + B_iZ)^T & c(A_iP + B_iZ - (A_iP + B_iZ)^T) \\ c((A_iP + B_iZ)^T - (A_iP + B_iZ)) & A_iP + B_iZ + (A_iP + B_iZ)^T \end{bmatrix} < 0$$

for $i = 1, \dots, k$. Then if $K = ZP^{-1}$, the pole locations, $z \in \mathbb{C}$ of $A(\Delta) + B(\Delta)K$ satisfy $|x| \leq r$, $\operatorname{Re} x \leq -\alpha$ and $z + z^* \leq -c|z - z^*|$ for all $\Delta \in Co(\Delta_1, \dots, \Delta_k)$.

Of course, if Δ is time-varying, eigenvalues are meaningless.

An LMI for Quadratic Polytopic H_∞ -Optimal State-Feedback Control

Recall the closed-loop in state feedback is:

$$\underline{S}(P, K) = \left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array} \right]$$

Now add uncertainty to system matrices A, B_1, B_2, C_1, D_{12} and D_{11} .

Theorem 10.

There exists an F such that $\|\underline{S}(P(\Delta), K(0, 0, 0, F))\|_{H_\infty} \leq \gamma$ for all $\Delta \in Co(\Delta_1, \dots, \Delta_k)$ if there exist $Y > 0$ and Z such that

$$\left[\begin{array}{cc} Y(A + A_i)^T + (A + A_i)Y + Z^T(B_2 + B_{2,i})^T + (B_2 + B_{2,i})Z & *^T & *^T \\ (B_1 + B_{1,i})^T & -\gamma I & *^T \\ (C_1 + C_{1,i})Y + (D_{12} + D_{12,i})Z & D_{11} + D_{11,i} & -\gamma I \end{array} \right] < 0 \quad i = 1, \dots, k$$

Then $F = ZY^{-1}$.

$$\underline{S}(P(\Delta), K) = \left[\begin{array}{c|c} A + B_2 F & B_1 \\ \hline C_1 + D_{12} F & D_{11} \end{array} \right] + \Delta \quad \Delta \in Co(\Delta_1, \dots, \Delta_k)$$

$$\Delta_i = \left[\begin{array}{c|c} A_i + B_{2,i} F & B_{1,i} \\ \hline C_{1,i} + D_{12,i} F & D_{11,i} \end{array} \right]$$

An LMI for Quadratic Polytopic H_∞ -Optimal State-Feedback Control

$$\tilde{S}(P, K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix}$$

Theorem 10.

There exists an F such that $\|\tilde{S}(P(\Delta), K)(0, 0, 0, F)\|_{H_\infty} \leq \gamma$ for all $\Delta \in \mathcal{C}(\Delta_1, \dots, \Delta_k)$ if there exist $Y > 0$ and Z such that

$$\begin{bmatrix} \gamma(A + A_1)^T + (A + A_1)Y + Y^T(A_2 + A_3)^T + (A_2 + A_3)Y & Y^T & Y^T \\ Y(A + A_1) + (A + A_1)Y + Y^T(A_2 + A_3)^T + (A_2 + A_3)Y & -Y & Y^T \\ Y(A + A_1) + (A + A_1)Y + Y^T(A_2 + A_3)^T + (A_2 + A_3)Y & Y & -Y \end{bmatrix} \leq 0 \quad i = 1, \dots, k$$

Then $F = ZY^{-1}$.

$$\tilde{S}(P(\Delta), K) = \begin{bmatrix} A + B_2 F & B_1 \\ C_1 + D_{12} F & D_{11} \end{bmatrix} + \Delta \quad \Delta \in \mathcal{C}(\Delta_1, \dots, \Delta_k)$$

$$\Delta_k = \begin{bmatrix} A_k + B_{2,k} F & B_{1,k} \\ C_{1,k} + D_{12,k} F & D_{11,k} \end{bmatrix}$$

In this case, the uncertain system is:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + (B_1 + \Delta B_1(t))w(t) + (B_2 + \Delta B_2(t))u(t)$$

$$y(t) = (C_1 + \Delta C_1(t))x(t) + (D_{11} + \Delta D_{11}(t))w(t) + (D_{12} + \Delta D_{12}(t))u(t)$$

where

$$\Delta A(t) = A_1 \delta_1(t) + \dots + A_k \delta_k(t)$$

$$\Delta B_1(t) = B_{1,1} \delta_1(t) + \dots + B_{1,k} \delta_k(t)$$

$$\Delta B_2(t) = B_{2,1} \delta_1(t) + \dots + B_{2,k} \delta_k(t)$$

$$\Delta C_1(t) = C_{1,1} \delta_1(t) + \dots + C_{1,k} \delta_k(t)$$

$$\Delta D_{11}(t) = D_{11,1} \delta_1(t) + \dots + D_{11,k} \delta_k(t)$$

$$\Delta D_{12}(t) = D_{12,1} \delta_1(t) + \dots + D_{12,k} \delta_k(t)$$

An LMI for Quadratic Polytopic H_2 -Optimal State-Feedback Control

Similarly

Theorem 11.

There exists an F such that $\|\underline{S}(P(\Delta), K(0, 0, 0, F))\|_{H_2}^2 \leq \gamma$ for all $\Delta \in Co(\Delta_1, \dots, \Delta_k)$ if there exist $X > 0$ and Z such that

$$\begin{bmatrix} AX + B_2 Z + X A^T + Z^T B_2^T & B_1 \\ B_1^T & -I \end{bmatrix} + \begin{bmatrix} A_i X + B_{2,i} Z + X A_i^T + Z^T B_{2,i}^T & B_{1,i} \\ B_{1,i}^T & 0 \end{bmatrix} < 0 \quad i = 1, \dots, k$$

$$\begin{bmatrix} X & (C_1 X + D_{12} Z)^T \\ C_1 X + D_{12} Z & W \end{bmatrix} + \begin{bmatrix} 0 & (C_{1,i} X + D_{12,i} Z)^T \\ C_{1,i} X + D_{12,i} Z & 0 \end{bmatrix} > 0 \quad i = 1, \dots, k$$

$$\text{Trace} W < \gamma$$

Then $F = ZY^{-1}$.

Similar Steps can be taken for robust estimator design, using the LMIs in Duan.

- However, I am not aware of a robust version of the general optimal output feedback LMI for polytopic uncertainty.

An LMI for Quadratic Schur Stabilization

State Equations: Let $u(k) = Fx(k)$ In this case, the uncertain system is:

$$\begin{aligned}x(k+1) &= (A + \Delta A(k))x(k) + (B + \Delta B(k))u(k) \\ &= (A + \Delta A(k) + BF + \Delta B(k)F)x(k)\end{aligned}$$

where

$$\Delta A(k) = A_1\delta_1(k) + \cdots + A_m\delta_m(k)$$

$$\Delta B(k) = B_1\delta_1(k) + \cdots + B_m\delta_m(k)$$

Theorem 12.

There exists a F such that

$$x_{k+1} = (A + \Delta_A + (B + \Delta_B)F)x_k$$

is quadratically stable for $(\Delta_A, \Delta_B) \in Co((A_1, B_1), \dots, (A_m, B_m))$ if and only if there exists some $X > 0$ and Z such that

$$\begin{bmatrix} X & AX + BZ \\ (AX + BZ)^T & X \end{bmatrix} + \begin{bmatrix} 0 & A_iX + B_iZ \\ (A_iX + B_iZ)^T & 0 \end{bmatrix} > 0 \quad \text{for } i = 1, \dots, m.$$

In this case, we have $F = ZP^{-1}$.