Complex Analysis, midterm exam

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Problem1. [Cauchy-Riemann Equation]

Given u(x, y) as follows, find v(x, y) such that u+iv is an analytic function

$$u(x,y) = \frac{y}{x^2+y^2}, u(x,y) = x - xy$$

Problem2. [Rouché Theorem]

Let f be a holomorphic function on an open set U and d be an open disc whose closure is in U. Suppose $|f|$ is constant on∂D. Prove that f has at least one zero in D(hint: consider $g(z) = f(z) - f(z_0)$ for $z_0 \in D$)

Problem3. [Möbius Transformation]

It's known that the automorphism group Aut($\mathbb{H}$) of the upper half plane $\mathbb{H}$ is the group $\mathrm{PSL}_2\mathbb{R} = \{\frac{az+b}{cz+d}, a, b, c, d \in R, ad - bc = 1\}/\{\pm id\}$ Prove that an element in Aut($\mathbb{H}$) maps the positive y-axis to either a vertical line or a half circle intersecting the x-axis orthogonally.

Problem4. [Residue Theorem]

Evaluate the integral using residue theorem

$$\int_0^{+\infty} \frac{cosx}{x^2+a^2}\mathrm{d}x, a \in R$$

Problem5. [Schwarz Lemma]

Prove that a bijective conformal mapping from the disk to itself is a fractional linear transformation, using Schwarz Lemma.

Problem6. [Aut($\mathbb{H}$)]

Let$\phi \in \mathrm{Aut}(\mathbb{H})$. Suppose $\phi(i) = i$. Prove that ϕ has the form $\dfrac{zcos\theta - sin\theta}{zsin\theta + cos\theta}, \theta \in [0, 2\pi)$

Problem7. [Uniformly Convergence]

Prove the series $\sum_{n=1}^{+\infty} \dfrac{nz^n}{1-z^n}$ converge uniformly within every closed disk$\{|z| < r\}$ for each $r < 1$

Problem8. [Conformal Mapping]

Find a conformal mapping that sends the strip $|Imz| < \dfrac{\pi}{2}$ onto the disk $\mathbb{D} = \{|z| < 1\}$

Problem9. [Singularities]

Suppose f is meromorphic on $\mathbb{C}$ but not entire, and let $g = e^f$. Prove that g is not meromorphic on $\mathbb{C}$

Problem10.

Let Ω be the simply connected domain $\mathbb{C}\ (-\infty, 0]$, and define log on it. Evaluate the limits

$$\lim_{y \to 0^+} (\log_\Omega(x + yi) - \log_\Omega(x - yi))$$

for $x > 0$ and $x < 0$