Complex Analysis, final exam

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Problem1. [$\wp(z)$]

We know that Weierstrass $\wp-$ function satisfies $(\wp')^2 = 4\wp^3 - g_2\wp - g_3$.

Derive $\wp^{(4)}(z)$

Problem2. [Conformal mapping]

Derive the image of the upper half plane under the conformal mapping

$$F(Z) = \int_0^z \frac{d\zeta}{\sqrt{\zeta(\zeta-1)(\zeta-\lambda)}}$$

for $\lambda \in \mathbb{R}$ and $\lambda \neq 1$

Problem3. [Liouville Theorem]

Suppose $f : \mathbb{C} \to \mathbb{C}$ is a holomorphic function such that the function $g(z) = f(z)f(1/z)$ is bounded on $\mathbb{C}$ $\{0\}$ Show that

(1) if $f(0) \neq 0$ then f is constant

(2) if $f(0) = 0$, then there exist $n \in \mathbb{N}$ and $a \in \mathbb{C}$ such that $f(z) = az^n$ for all $z \in \mathbb{C}$

Problem4. [Addition Formulae]

Consider the elliptic curve $y^2 = 4x^3 - 8x, \quad (x, y) \in \mathbb{C} \times \mathbb{C}$, and two points $(x_1, y_1) = (2, 4)$ and $(x_2, y_2) = (-1, 2)$ on the elliptic curve. Let $+$ be the addition. Find out $(2, 4) + (-1, 2)$

Problem5. [Differential Equation]

Consider equation $z^2(z + 1)\omega'' - z\omega' + \frac{1}{4}\omega = 0$. It has two solutions$z^{\alpha_1}g_1(z)$ and $z^{\alpha_2}g_2(z)$ around $z = 0$. Determine$\alpha_1$and α_2

Problem6. [Casorati-Weierstrass Theorem]

Let $g : \mathbb{C}\ \{0\} \to mathbbC\ \{0\}$ be an injective holomorphic function. Prove that g does not have a essential singularity at ∞

Problem7. [Representation of Elliptic Function]

For any given complex numbers a_k and b_k for $k = 1, 2...n$ in the fundamental parallelogram satisfying $\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b_i mod(m2\pi + ni)$ for some$m, n \in \mathbb{Z}$, there exists an elliptic function f that has a_i as zeros and b_i as poles.

Problem8. [Zeros]

Let $\Omega = \mathbb{C}\ (-\infty, 0]$ and let $log : \Omega \to \mathbb{C}$be the branch of the complex logarithm on Ω that is real on the positive real axis. Show that for $0 < t < \infty$, the number of solutions $z \in \Omega$ to the equation

$$logz = \frac{t}{z}$$

is finite and independent of t .

Problem9. [Riemann Mapping Theorem]

Let $\Omega_1 \subseteq \Omega_2$ be two bounded simply-connected domains in $\mathbb{C}$. We also assume that $0 \in \Omega$ Now suppose $f_1 \mathbb{D} \to \Omega_1$ and $f_2 \mathbb{D} \to \Omega_2$ are Riemann mappings, satisfying $f_1(0) = f_2(0) = 0$.

Show that

$$|f_1'(0)| \leq |f_2'(0)|$$

Problem10.

Evaluating Fresnel integral

$$\lim_{R\to\infty} \int_0^R sin(t^2)dt$$