对于自由表面采用直角坐标系

1.表示势函数

散射波含有三种 p 波和两种 s 波

$$\text{p 波：} \varphi_j^s(x,y)=\int_{-\infty}^{\infty} A_j^s(k)\, e^{ikx-\nu_{\alpha j}y}\, dk \quad (j=1,2,3) \text{（8a）}$$

$$\text{s 波：} \psi_{\mathrm{m}}^{\mathrm{s}}(x,y)=\int_{-\infty}^{\infty} B_m^s(k)\, e^{ikx-\nu_{bm}y}\, dk \quad (\mathrm{m}=1,2)(8\mathrm{b})$$

驻波含有三种 p 波和两种 s 波

$$\text{p 波：} \varphi_j^R(x,y)=\int_{-\infty}^{\infty} A_j^R(k)\, e^{ikx-\nu_{\alpha j}y}\, dk \quad (j=1,2,3) \text{（8a）}$$

$$\text{s 波：} \psi_{\mathrm{m}}^{\mathrm{R}}(x,y)=\int_{-\infty}^{\infty} B_m^R(k)\, e^{ikx-\nu_{bm}y}\, dk \quad (\mathrm{m}=1,2)(8\mathrm{b})$$

2.位移分量

固相

总P波位移分量：

$$u_x^{(p)}=u_x^{s(p)}+u_x^{R(p)}=\int_{-\infty}^{\infty} i\,k[A_j^s(k)+A_j^R(k)]e^{ikx-\nu_{aj}y}dk \quad (j=1,2,3)$$

$$u_y^{(p)}=u_y^{s(p)}+u_y^{R(p)}=\int_{-\infty}^{\infty}(-\nu_{aj})[A_j^s(k)+A_j^R(k)]e^{ikx-\nu_{aj}y}dk \quad (j=1,2,3)$$

总S波位移分量：

$$u_x^{(s)}=u_x^{s(s)}+u_x^{R(s)}=\int_{-\infty}^{\infty}(-\nu_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-\nu_{bm}y}dk \quad (m=1,2)$$

$$u_y^{(s)}=u_y^{s(s)}+u_y^{R(s)}=\int_{-\infty}^{\infty}(-ik)[B_m^s(k)+B_m^R(k)]e^{ikx-\nu_{bm}y}dk \quad (m=1,2)$$

总固相位移分量

$$\begin{aligned} u_x=u_x^{(p)}+u_x^{(s)}&=\sum_{j=1}^{3}\int_{-\infty}^{\infty} i\,k[A_j^s(k)+A_j^R(k)]e^{ikx-\nu_{aj}y}dk \\ &+\sum_{m=1}^{2}\int_{-\infty}^{\infty}(-\nu_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-\nu_{bm}y}dk \\ u_y=u_y^{(p)}+u_y^{(s)}&=\sum_{j=1}^{3}\int_{-\infty}^{\infty}(-\nu_{aj})[A_j^s(k)+A_j^R(k)]e^{ikx-\nu_{aj}y}dk \\ &+\sum_{m=1}^{2}\int_{-\infty}^{\infty}(-ik)[B_m^s(k)+B_m^R(k)]e^{ikx-\nu_{bm}y}dk \end{aligned}$$

总液相位移分量

$$u_x = u_x^{(p)} + u_x^{(s)} = \sum_{j=1}^{3} \alpha_j^L \int_{-\infty}^{\infty} i\,k[A_j^s(k) + A_j^R(k)]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \beta_m^L \int_{-\infty}^{\infty} (-v_{bm})[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

$$u_y = u_y^{(p)} + u_y^{(s)} = \sum_{j=1}^{3} \alpha_j^L \int_{-\infty}^{\infty} (-v_{aj})[A_j^s(k) + A_j^R(k)]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \beta_m^L \int_{-\infty}^{\infty} (-ik)[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

总冰相位移分量

$$u_x = u_x^{(p)} + u_x^{(s)} = \sum_{j=1}^{3} \alpha_j^I \int_{-\infty}^{\infty} i\,k[A_j^s(k) + A_j^R(k)]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \beta_m^I \int_{-\infty}^{\infty} (-v_{bm})[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

$$u_y = u_y^{(p)} + u_y^{(s)} = \sum_{j=1}^{3} \alpha_j^I \int_{-\infty}^{\infty} (-v_{aj})[A_j^s(k) + A_j^R(k)]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \beta_m^I \int_{-\infty}^{\infty} (-ik)[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

3. 散度

总**固相位移散度**：$\nabla \cdot \mathbf{u} = -\sum_{j=1}^{3} k_{aj}^2\, \varphi_j(x,y) = -\sum_{j=1}^{3} k_{aj}^2 \left(\int_{-\infty}^{\infty} \left[A_j^s(k) + A_j^R(k)\right] e^{ikx - v_{aj}y} dk \right)$

总**液相位移散度**：$\nabla \cdot \mathbf{u}^f = -\sum_{j=1}^{3} \alpha_j^l\, k_{aj}^2\, \varphi_j(x,y) = -\sum_{j=1}^{3} \alpha_j^l\, k_{aj}^2 \left(\int_{-\infty}^{\infty} \left[A_j^s(k) + A_j^R(k)\right] e^{ikx - v_{aj}y} dk \right)$

总**冰相位移散度**：$\nabla \cdot \mathbf{u}^i = -\sum_{j=1}^{3} a_j^i\, k_{aj}^2\, \varphi_j(x,y) = -\sum_{j=1}^{3} a_j^i\, k_{aj}^2 \left(\int_{-\infty}^{\infty} \left[A_j^s(k) + A_j^R(k)\right] e^{ikx - v_{aj}y} dk \right)$

4. 应变张量

固相

正应变ε_{xx}

$$\varepsilon_{xx} = \sum_{j=1}^{3} \int_{-\infty}^{\infty} (-k^2)\left[A_j^s(k) + A_j^R(k)\right]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \int_{-\infty}^{\infty} (-ikv_{bm})[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

正应变ε_{yy}

$$\varepsilon_{yy} = \sum_{j=1}^{3} \int_{-\infty}^{\infty} v_{aj}^2 \left[A_j^s(k) + A_j^R(k)\right]e^{ikx - v_{aj}y} dk$$
$$+ \sum_{m=1}^{2} \int_{-\infty}^{\infty} (ikv_{bm})[B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

剪应变ε_{xy}

$$\varepsilon_{xy} = \sum_{j=1}^{3} \int_{-\infty}^{\infty} (-ikv_{aj})\left[A_j^s(k) + A_j^R(k)\right]e^{ikx - v_{aj}y} dk$$
$$+ \frac{1}{2} \sum_{m=1}^{2} \int_{-\infty}^{\infty} [-(2k^2 - k_{\beta m}^2)][B_m^s(k) + B_m^R(k)]e^{ikx - v_{bm}y} dk$$

液相

正应变ε_{xx}^{l}

$$\varepsilon_{xx}^{l}=\sum_{j=1}^{3}\alpha_j^l\int_{-\infty}^{\infty}(-k^2)[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\sum_{m=1}^{2}\beta_m^l\int_{-\infty}^{\infty}(-ikv_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

正应变ε_{yy}^{l}

$$\varepsilon_{yy}^{l}=\sum_{j=1}^{3}\alpha_j^l\int_{-\infty}^{\infty}v_{aj}^2\,[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\sum_{m=1}^{2}\beta_m^l\int_{-\infty}^{\infty}(ikv_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

剪应变ε_{xy}^{l}

$$\varepsilon_{xy}^{l}=\sum_{j=1}^{3}\alpha_j^l\int_{-\infty}^{\infty}(-ikv_{aj})[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\frac{1}{2}\sum_{m=1}^{2}\beta_m^l\int_{-\infty}^{\infty}[-(2k^2-k_{\beta m}^2)][B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

冰相

正应变ε_{xx}^{i}

$$\varepsilon_{xx}^{i}=\sum_{j=1}^{3}\alpha_j^i\int_{-\infty}^{\infty}(-k^2)[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\sum_{m=1}^{2}\beta_m^i\int_{-\infty}^{\infty}(-ikv_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

正应变ε_{yy}^{i}

$$\varepsilon_{yy}^{i}=\sum_{j=1}^{3}\alpha_j^i\int_{-\infty}^{\infty}v_{aj}^2\,[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\sum_{m=1}^{2}\beta_m^i\int_{-\infty}^{\infty}(ikv_{bm})[B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

剪应变ε_{xy}^{i}

$$\varepsilon_{xy}^{i}=\sum_{j=1}^{3}\alpha_j^i\int_{-\infty}^{\infty}(-ikv_{aj})[A_j^s(k)+A_j^R(k)]e^{ikx-v_{aj}y}dk+\frac{1}{2}\sum_{m=1}^{2}\beta_m^i\int_{-\infty}^{\infty}[-(2k^2-k_{\beta m}^2)][B_m^s(k)+B_m^R(k)]e^{ikx-v_{bm}y}dk$$

5. 代入本构方程

$$\left(K_2+C_{12}+C_{23}\right)\nabla^2\varphi_u^L+\left(K_1+C_{12}+C_{13}-\frac{2}{3}\mu_1-\frac{1}{3}\mu_{13}\right)\nabla^2\varphi_u^S+\left(2\mu_1+\mu_{13}\right)\left(\frac{\partial^2\varphi_u^S}{\partial z^2}+\frac{\partial^2\psi_u^S}{\partial z\partial x}\right)+\left(2\mu_3+\mu_{13}\right)\left(\frac{\partial^2\varphi_u^I}{\partial z^2}+\frac{\partial^2\psi_u^I}{\partial z\partial x}\right)+\left(K_3+C_{13}+C_{23}-\frac{2}{3}\mu_3-\frac{1}{3}\mu_{13}\right)\nabla^2\varphi_u^I=0 \qquad (20j)$$

$$\left(\mu_1+\frac{1}{2}\mu_{13}\right)\left(2\frac{\partial^2\varphi_u^S}{\partial x\partial z}+\frac{\partial^2\psi_u^S}{\partial x^2}-\frac{\partial^2\psi_u^S}{\partial z^2}\right)+\left(\mu_3+\frac{1}{2}\mu_{13}\right)\left(2\frac{\partial^2\varphi_u^I}{\partial x\partial z}+\frac{\partial^2\psi_u^I}{\partial x^2}-\frac{\partial^2\psi_u^I}{\partial z^2}\right)=0 \qquad (20k)$$

$$\frac{\partial\left(C_{12}\nabla^2\varphi_u^S+K_2\nabla^2\varphi_u^L+C_{23}\nabla^2\varphi_u^I\right)}{\partial z}=0 \qquad (20l)$$

这三个式子的本构方程势函数带入为

$(20j)$完整形式

$$
\begin{aligned}
&(K_2 + C_{12} + C_{23})\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^L\, k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)\\
&+\left(K_1 + C_{12} + C_{13} - \frac{2}{3}\mu_1 - \frac{1}{3}\mu_{13}\right)\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)\\
&+(2\mu_1+\mu_{13})\left(\sum_{j=1}^{3}\int_{-\infty}^{\infty}\nu_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk+\sum_{m=1}^{2}\int_{-\infty}^{\infty}(-ik\nu_{bm})\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk\right)\\
&+(2\mu_3+\mu_{13})\left(\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^I\,\nu_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk+\sum_{m=1}^{2}\int_{-\infty}^{\infty}\beta_m^I\,(-ik\nu_{bm})\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk\right)\\
&+\left(K_3 + C_{13} + C_{23} - \frac{2}{3}\mu_3 - \frac{1}{3}\mu_{13}\right)\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^I\, k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)=0
\end{aligned}
$$

方程$(20k)$完整形式

$$
\begin{aligned}
&\left(\mu_1+\frac{1}{2}\mu_{13}\right)\left(2\sum_{j=1}^{3}\int_{-\infty}^{\infty}(-ik\nu_{aj})\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right.\\
&\quad\left.+\sum_{m=1}^{2}\int_{-\infty}^{\infty}(-k^2)\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk-\sum_{m=1}^{2}\int_{-\infty}^{\infty}\nu_{bm}^2\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk\right)\\
&+\left(\mu_3+\frac{1}{2}\mu_{13}\right)\left(2\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^I\,(-ik\nu_{aj})\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right.\\
&\quad\left.+\sum_{m=1}^{2}\int_{-\infty}^{\infty}\beta_m^I\,(-k^2)\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk-\sum_{m=1}^{2}\int_{-\infty}^{\infty}\beta_m^I\,\nu_{bm}^2\left[B_m^s(k)+B_m^R(k)\right]e^{ikx-\nu_{bm}y}dk\right)=0
\end{aligned}
$$

方程$(20l)$完整形式

$$
\begin{aligned}
\frac{\partial}{\partial y}\Bigg(&C_{12}\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)\\
&\left(+K_2\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^L\, k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)\right)\\
&+C_{23}\left(-\sum_{j=1}^{3}\int_{-\infty}^{\infty}\alpha_j^I\, k_{aj}^2\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk\right)\Bigg)=0
\end{aligned}
$$

对y求偏导后： $\displaystyle\sum_{j=1}^{3}\int_{-\infty}^{\infty}\left[C_{12}+K_2\alpha_j^L+C_{23}\alpha_j^I\right]k_{aj}^2\nu_{aj}\left[A_j^s(k)+A_j^R(k)\right]e^{ikx-\nu_{aj}y}dk=0$

$$\frac{\partial\varphi_u^S}{\partial z}+\frac{\partial\psi_u^S}{\partial x}=\frac{\partial\varphi_u^I}{\partial z}+\frac{\partial\psi_u^I}{\partial x}\qquad(20\text{h})$$

$$\frac{\partial\varphi_u^S}{\partial x}-\frac{\partial\psi_u^S}{\partial z}=\frac{\partial\varphi_u^I}{\partial x}-\frac{\partial\psi_u^I}{\partial z}\qquad(20\text{i})$$

方程$(20h)$完整形式

$$
\begin{aligned}
&\left\{\sum_{j=1}^{3}\left[\int_{-\infty}^{\infty}A_j^s(k)(-\nu_{aj})e^{ikx-\nu_{aj}y}dk+\int_{-\infty}^{\infty}A_j^R(k)(-\nu_{aj})e^{ikx-\nu_{aj}y}dk\right]\right.\\
&\left.+\sum_{m=1}^{2}\left[\int_{-\infty}^{\infty}B_m^s(k)(ik)e^{ikx-\nu_{bm}y}dk+\int_{-\infty}^{\infty}B_m^R(k)(ik)e^{ikx-\nu_{bm}y}dk\right]\right\}\\
=&\left\{\sum_{j=1}^{3}\alpha_j^I\left[\int_{-\infty}^{\infty}A_j^s(k)(-\nu_{aj})e^{ikx-\nu_{aj}y}dk+\int_{-\infty}^{\infty}A_j^R(k)(-\nu_{aj})e^{ikx-\nu_{aj}y}dk\right]\right.\\
&\left.+\sum_{m=1}^{2}\beta_m^I\left[\int_{-\infty}^{\infty}B_m^s(k)(ik)e^{ikx-\nu_{bm}y}dk+\int_{-\infty}^{\infty}B_m^R(k)(ik)e^{ikx-\nu_{bm}y}dk\right]\right\}
\end{aligned}
$$

方程$(20i)$完整形式

$$
\begin{aligned}
&\left\{\sum_{j=1}^{3}\left[\int_{-\infty}^{\infty} A_j^s(k)(ik)e^{ikx-\nu_{aj}y}dk+\int_{-\infty}^{\infty} A_j^R(k)(ik)e^{ikx-\nu_{aj}y}dk\right]\right.\\
&\left.-\sum_{m=1}^{2}\left[\int_{-\infty}^{\infty} B_m^s(k)(-\nu_{bm})e^{ikx-\nu_{bm}y}dk+\int_{-\infty}^{\infty} B_m^R(k)(-\nu_{bm})e^{ikx-\nu_{bm}y}dk\right]\right\}\\
=&\left\{\sum_{j=1}^{3}\alpha_j^I\left[\int_{-\infty}^{\infty} A_j^s(k)(ik)e^{ikx-\nu_{aj}y}dk+\int_{-\infty}^{\infty} A_j^R(k)(ik)e^{ikx-\nu_{aj}y}dk\right]\right.\\
&\left.-\sum_{m=1}^{2}\beta_m^I\left[\int_{-\infty}^{\infty} B_m^s(k)(-\nu_{bm})e^{ikx-\nu_{bm}y}dk+\int_{-\infty}^{\infty} B_m^R(k)(-\nu_{bm})e^{ikx-\nu_{bm}y}dk\right]\right\}
\end{aligned}
$$

对于自由表面的矩阵方程是：

$$
\begin{bmatrix} A_1^R \\ A_2^R \\ A_3^R \\ B_1^R \\ B_2^R \end{bmatrix} = [S]_{5\times5}^{-1}\cdot[T]_{5\times5}\cdot\begin{bmatrix} A_1^S \\ A_2^S \\ A_3^S \\ B_1^S \\ B_2^S \end{bmatrix}
$$