Analytical Solution for Wave Scattering by a Cavity in a Half-Space

Antigravity Agent

December 27, 2025

1 Introduction

This document details the analytical solution to the problem of elastic wave scattering by a cylindrical cavity embedded in a half-space. The solution involves solving for the coefficients of the reflected and scattered waves by satisfying boundary conditions at both the free surface of the half-space and the surface of the cavity.

The problem effectively reduces to solving a system of linear equations for 10 unknown constants:

- A_{Rj} $(j = 1, 2, 3)$: Amplitude coefficients of the reflected P-waves (P1, P2, P3).
- B_{Rm} $(m = 1, 2)$: Amplitude coefficients of the reflected S-waves (SV1, SV2).
- A_{Sj} $(j = 1, 2, 3)$: Amplitude coefficients of the scattered P-waves from the cavity.
- B_{Sm} $(m = 1, 2)$: Amplitude coefficients of the scattered S-waves from the cavity.

2 Wave Potentials and Coordinate Systems

We employ two coordinate systems:

1. **Cartesian Coordinates** (x, z): Origin at the free surface. Used for the half-space boundary conditions.
2. **Cylindrical Coordinates** (r, θ): Origin at the center of the cavity, at depth h. Used for the cavity boundary conditions.

Transformation validity: $x = r \sin\theta$, $z = h + r\cos\theta$.

2.1 Free Surface Potentials

The reflected wave potentials are given by:

$$\phi_j^{(R)} = A_{Rj} \exp[ik_{\alpha,j}(x\sin\theta_{\alpha,j} - z\cos\theta_{\alpha,j})] \tag{1}$$

$$\psi_m^{(R)} = B_{Rm} \exp[ik_{\beta,m}(x\sin\theta_{\beta,m} - z\cos\theta_{\beta,m})] \tag{2}$$

2.2 Cavity Surface Potentials

The scattered wave potentials are expanded in terms of Hankel functions (representing outgoing waves):

$$\Phi_j^{(S)} = \sum_{n=-\infty}^{\infty} A_{Sj,n} H_n^{(1)}(k_{\alpha,j} r) e^{in\theta} \tag{3}$$

$$\Psi_m^{(S)} = \sum_{n=-\infty}^{\infty} B_{Sm,n} H_n^{(1)}(k_{\beta,m} r) e^{in\theta} \tag{4}$$

3 Boundary Conditions

3.1 Free Surface ($z = 0$)

At the free surface ($z = 0$), the stress-free conditions ($\sigma_{zz} = 0, \sigma_{xz} = 0$) and permeability conditions lead to the following 5 equations. These link the Reflection coefficients (A_R, B_R) to the Scattering coefficients (A_S, B_S) which must be transformed to the Cartesian basis.

The equations are derived as:

$$\sum_{j=1}^{3} k_{\alpha j}^2 A_{Rj} T_{1j}^P + \sum_{m=1}^{2} k_{\beta m}^2 B_{Rm} T_{1m}^S = F_1(\text{Incident}) \tag{5}$$

(Detailed terms omitted for brevity, see code implementation for full coefficients involving material parameters K, C_{ij}, μ).

3.2 Cavity Surface ($r = R$)

At the cavity surface, we have stress-free conditions ($\sigma_{rr} = 0, \sigma_{r\theta} = 0$) and fluid/thermal conditions. These allow us to express the scattering coefficients in terms of the reflected wave coefficients (which act as "incident" waves onto the cavity).

The system is written as a matrix equation:

$$[E^S]\mathbf{X}_S + [E^R]\mathbf{X}_R = 0 \tag{6}$$

The elements of the scattering matrix $[E^S]$ are explicitly derived. For example, the first row (Radial Stress) is:

$$E_{1j}^S = A_P k_{\alpha j}^2 H_n''(k_{\alpha j} R) + B_P \left(\frac{k_{\alpha j}}{R} H_n' - \frac{n^2}{R^2} H_n \right) \tag{7}$$

$$E_{1,3+m}^S = C_S \frac{in}{R} k_{\beta m} H_n' - D_S \frac{in}{R^2} H_n \tag{8}$$

where A_P, B_P, C_S, D_S are material constants derived from Lame parameters λ, μ and Biot coefficients.

4 Solution Method

The total system is a coupled 10×10 linear system $M\mathbf{X} = \mathbf{F}$. We solve this computationally using the following steps:

1. Define all material properties (λ, μ, K_l, etc.) and wavenumbers k.
2. Construct the 5×5 Cavity Matrix E^S using the Hankel function expressions.
3. Construct the 5×5 Free Surface Matrix M_{FF}.
4. Compute the transformation matrices $\Lambda(n)$ to relate the Cartesian and Cylindrical coefficients (using Graf's addition theorem).
5. Assemble the global matrix M and load vector F (from the incident plane wave).
6. Solve $X = M^{-1}F$ to obtain the 10 unknown coefficients.