2025 Level 1 - Quantitative Methods

Rates and Returns

a. interpret interest rates as required rates of return, discount rates, or opportunity costs and explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

b. calculate and interpret different approaches to return measurement over time and describe their appropriate uses

c. compare the money-weighted and time-weighed rates of return and evaluate the performance of portfolios based on these measures

d. calculate and interpret annualized return measures and continuously compounded returns, and describe their appropriate uses

e. calculate and interpret major return measures and describe their appropriate uses

Rates and Returns

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Interest rates (r) - can be thought of in 3 ways

1/ required rates of return → determining the FV of a PV

2/ discount rate → determining the PV of a FV

3/ opportunity cost → value forgone (current consumption vs. saving)

Determinants of Interest Rates

vary over time and continuously change

$r = r_f$ → real default risk-free rate (single period)

+ inflation premium → expected inflation over a period of time

+ default risk premium → compensates for credit risk

+ liquidity premium → risk of loss vs. fair value if an investment needs to be converted to cash quickly

+ maturity premium → compensation for greater price sensitivity from changes in rates

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- nominal risk-free rate:

$$(1 + r) = (1 + r_f)(1 + \pi^e) \quad \text{or} \quad r = r_f + \pi^e$$

Note: all rates are quoted on an annual basis

e.g. 3-mos. T-Bill @ 4% is $.04/4$ = 1% over 3 months

Rates of Return/

1/ HPR - holding period return

$$R = \frac{(P_t - P_0) + I}{P_0}$$

		e.g./
P_t	- ending price	105
P_0	- beginning price	100
I	- all income	Ø

$$HPR = \frac{(105 - 100)}{100} = 5\%$$

- multi-period HPR

$$R = [(1 + HPR_1)(1 + HPR_2)(1 + HPR_3)] - 1$$

2/ Arithmetic or Mean Return $\bar{R}_i = \frac{1}{T} \cdot \sum_{t=1}^{T} R_{it}$

3/ Geometric Mean Return $\bar{R}_{G_i} = [(1 + R_{i1})(1 + R_{i2}) \ldots (1 + R_{iT})]^{1/T} - 1$

b

e.g./ R_{i1} = -50%, R_{i2} = 35%, R_{i3} = 27%

$$[(.5)(1.35)(1.27)]^{1/3} - 1 = -5\%$$

- the growth rate or compounded return on an investment

Rates of Return/

$R_G \leq R_A$ - unless all observations are equal, then $R_G = R_A$

- as the variability in the data increases, difference between R_G and R_A increases

R_G: growth of $1

$PV(1 + R_G)^N = FV$

- use to estimate E(R) over multiple periods

R_A: avg. return over time

$PV(1 + R_A)^N \neq FV$

- use to estimate E(R) over one period

4/ Harmonic Mean

$$\bar{X}_H = \frac{n}{\Sigma(1/X)}$$

- arithmetic mean - all obs. have equal weight

$\longrightarrow$ obs. weight inversely proportional to its magnitude

∴ reduces the effect of outliers

b

- most often used with ratios (amount/unit)

e.g./ P/E 45, 15, 15

R_A = 25

$$\bar{X}_H = \frac{3}{\Sigma(1/45 + 1/15 + 1/15)} = 3/.1\dot{5} = 19.28$$

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Rates of Return/

4/ Harmonic Mean

- applied e.g. - dollar-cost averaging (typical DC strategy)

1,000/month for 2 months in a stock $P_0 = 10$ $P_1 = 15$

$\bar{X}_H = \frac{2}{(1/10 + 1/15)} = 2/.1\dot{6} = 12/\text{sh.}$

Proof: $1000/10 = 100$ sh. $1000/15 = 66.67$ sh.

$2000/166.67 \text{ sh.} = 12/\text{sh.}$

Relationship: $R_A \times \bar{X}_H = R_G^2$

5/ Others

removes outliers (on both sides)

a) trimmed mean (common with CPI)

- remove a %'age from both the largest and smallest

e.g. 100 obs., 8% trimmed = 84 obs. (8 highest, 8 lowest)

b b) winsorized mean - replacing values at both end with the cutoff value

e.g. 100 obs. obs. 1-8 replaced all = obs. 9
obs. 93-100 replaced all = obs. 92

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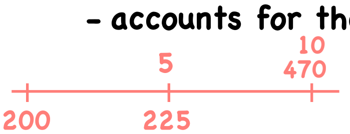
- Money-weighted Return (IRR, YTM)

- accounts for the timing and magnitude of investments



$CF_0 = -200$ $CF_1 = -220$ $CF_2 = 480$ CPT IRR = 9.39%

committed more money to a poor performance year (money weighted)

$HPR = \frac{25+5}{200} = 15\%$ $HPR = \frac{20+10}{450} = 6.67\%$ $R_A = 10.84\%$

- not comparable across investors/investments
- mwrr represents what 'your' money earned, not what \$1 could earn

- Time-weighted returns (= R_G)
  - the growth of \$1 over a given time period
  - comparable across investments

c → break investment period into holding periods (determined by any significant cash in/out-flows)

→ calculate each HPR, then compound the HPRs, express annually

- Time-weighted returns (= R_G)
  - previous example from mwrr

$$[(1.15)(1.06\dot{6})]^{1/2} - 1 = 10.75498\%$$

  - large funds, HPR = 1 day → $\underbrace{(1 + HPR_1)(1 + HPR_2) + \ldots + (1 + HPR_{365})}_{\text{for liquid underlyings with market prices}} - 1$

c

- Annualized Return/ – all rates/returns are quoted annually

$$R_{annual} = (1 + R_{period})^{365/period}$$

→

daily	$(1+R)^{365} - 1$	7d/wk. 365d/yr.
weekly	$(1+R)^{52} - 1$	52wk./yr.
monthly	$(1+R)^{12} - 1$	12mos./yr.
quarterly	$(1+R)^{4} - 1$	

250 trading days

5d/wk., 20d/m., 250d/yr.

period = 43 days

$$R_{annual} = (1 + R_{43})^{365/43} - 1$$

period = 540 days

$$R_{annual} = (1 + R_{540})^{365/540} - 1$$

- annualizing returns can be misleading – assumes that returns can be repeated

- Continuously Compounded Returns

d

$\ln\left({P_t}/{P_0}\right)$ e.g./ $r_c = \ln\left({105}/{100}\right) = 4.879$

or $\ln(1 + r)$ $\qquad r_c = \ln(1.05)$

$$e^{r_c} - 1 = r$$

$$e^{.04879} - 1 = 5\%$$

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• Gross and Net Return/

gross - return before deductions for mgmt. exp., custodial fees, taxes, etc. but after trading expenses (what the fund earns)

- appropriate measure for evaluating and comparing the investment skill of managers

net - what the investor earns

Pre-tax and After-tax Nominal Return/

- default is to report/state pre-tax return
- each investor's marginal tax rate may differ
- components of return may be reported e.g. 11% — 2% div. — 1% int. — 4% realized gains

Real Returns/

e $$1 + \text{real return} = \frac{1 + \text{nominal return}}{1 + \text{inflation rate}} \quad / \quad \frac{1 + \text{nominal return}}{1 + \text{risk} - \text{free rate}} = 1 + \text{risk premium}$$

(typically p-o-p CPI)

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Real Returns/

After-tax real return → investor measure of growth in purchasing power of portfolio

Leveraged Returns/

- leverage can be obtained through margin loans, derivatives, or collateralized loans (repos)

- if $R_P > r_d$, leverage enhances return

Value borrowed

$$R_L = R_P / P_E = \frac{R_P \times (V_E + V_B) - (V_B \times r_b)}{V_E} = R_P + \frac{V_B}{V_E}(R_P - r_d)$$

leveraged return

portfolio equity

$$\left[\frac{R_P V_E}{V_E} + \frac{R_P V_B}{V_E} - \frac{V_B r_d}{V_E} = R_P + \frac{V_B(R_P - r_d)}{V_E} \right]$$

e e.g./ 10m equity portfolio

$R_P = 8\%$

30% debt financed, $r_d = 5\%$

$$R_L = 8\% + {}^{3M}/_{7M}\,(8\% - 5\%) = 9.2857\%$$

Time Value of Money in Finance

a. calculate and interpret the present value (PV) of fixed-income and equity instruments based on expected future cash flows

b. calculate and interpret the implied return of fixed-income instruments and required return and implied growth of equity instruments given the present value (PV) and cash flows

c. explain the cash flow additivity principle, its importance for the no-arbitrage condition, and its use in calculating implied forward interest rates, forward exchange rates, and option values

The Time Value of Money in Finance

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- 3 rules of money
  - larger CFs are worth more
  - less risky CFs are worth more (lower discount rate)
  - CFs sooner are worth more (time value of money)

t=0	t=1	t=2	t=3
	risky 176	safe 213	risky 306

which one is best?

- calculate PV → value today - requires discounting at a rate that depends on the timing and type of CF

recall: $r_f + \pi^e$ - gov't. bonds

+ default
+ liquidity

} corporate/private debt

+ maturity - longer-term debt
+ equity - equity over debt

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$$FV = PV(1+r)^t \qquad \text{or} \qquad FV = PVe^{rT}$$

$$\underbrace{PV = \frac{FV}{(1+r)^t} = FV(1+r)^{-t} \qquad PV = FVe^{-rT}}_{\text{single cash flow.}}$$

Fixed Income/ debt instruments (bonds, loans, mortgages, etc.)

- ZCB - zero coupon bonds (i.e. T-Bills → up to 1 yr. mat.)
  - sold at a discount, mature at par - single CF at maturity
- Coupon bonds (Notes, Bonds)
  - investor receives a number of interest payments over time and par at maturity
- Fully ammortizing bonds (mortgage, auto loan)
  - investor receives level payments of both interest and principal

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ZCB/zero-coupon bond: PV = $^{FV}/_{(1+r)^T}$ r = discount rate, IRR, or YTM

e.g./
- 20-yr. ZCB, YTM = 6.7% → PV = $^{100}/_{(1.067)^{20}}$ = 27.33453
  (TVM keys FV = 100 $^I/_Y$ = 6.7 PMT = 0 N = 20 CPT PV)
- price in 3 years if YTM is unchanged?
  a) $FV_3 = PV(1+r)^3$ = $27.33453(1.067)^3$ = 33.20510591
  PV = -27.33453
  N = 3
  $^I/_Y$ = 6.7
  PMT = 0
  CPT FV
  or b) $PV_3 = {}^{FV}/_{(1+r)^{17}}$ = $^{100}/_{(1.067)^{17}}$ = 33.20510591
  FV = 0
  $^I/_Y$ = 6.7%
  PMT = 0
  N = 17
  CPT PV
- PV = 22.68224 → YTM = ?
  $22.68224 = {}^{100}/_{(1+r)^{20}}$ → PV = -22.68224
  N = 20
  PMT = 0
  FV = 100
  $(100/22.68224)^{1/20} - 1 = r$
  CPT $^I/_Y$ = 7.6999 ~ 7.7%
- r = -.05%, 10 yr. ZCB
  PV = $^{100}/_{(.9995)^{10}}$ = 100.50137   FV = 100 N = 10 $^I/_Y$ = -.05 PMT = 0
  CPT PV
- 6 yrs. later P_0 = 95.72, YTM = ? $95.72 = {}^{100}/_{(1+r)^4}$ → $(100/95.72)^{1/4} - 1$ = 1.09957%
  PV = -95.72 FV = 100 N = 4 PMT = 0 CPT $^I/_Y$

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- Coupon bond

$$PV = \frac{PMT_1}{(1+r)} + \frac{PMT_2}{(1+r)^2} + \cdots + \frac{FV + PMT_N}{(1+r)^N}$$

FV = 100
N = yrs. × period
PMT = coupon/period
$^I/_Y$ = $^{YTM}/_{period}$

- 7 yr., 2% annual bond issued at YTM = 2%
  par bond   FV = 100 PMT = 2 $^I/_Y$ = 2 N = 7 CPT PV
- one year later, P = 93.091, YTM = ?
  FV = 100 N = 6 PMT = 2 PV = -93.091 CPT $^I/_Y$ = 3.287566
- 20 yr., 6.7% semi issued at YTM = 6.70% par bond
  - 7.7%? FV = 100 N = 20 × 2 = 40 PMT = $^{6.70}/_2$ = 3.35 $^I/_Y$ = $^{7.70}/_2$ = 3.85
    CPT PV = 89.8788
- 20 yr. ZCB assuming YTM = 6.7% semi
  FV = 100 N = 40 $^I/_Y$ = 3.35 PMT = 0 CPT PV = 26.7658
  vs. 27.33 annually

- **Perpetuity** (some bonds, preferred shares)

$PV = \frac{PMT}{r} \longrightarrow$ YTM

e.g./ 3.3% qtly. coupon, P = 97.03

$97.03 = \frac{.825}{r_4} \rightarrow r_4 = .8502\%$

$r = 3.401\%$

- **Annuity** (mortgage, car loans)

$\underset{pmt}{A} = \frac{r(PV)}{1-(1+r)^{-t}}$ or/ TVM keys → CPT PMT

e.g./ 800k, 30yr. - FRM @ 5.25%

N = 30 × 12 = 360
FV = 0 (fully amortizing)
PV = -800,000
I/Y = 5.25/12 = .4375 CPT PMT 4,417.63

Amortization Table:

	PMT	i	P	B
Month 1	4417.63	3500	917.63	799,082.37
2	4417.63	3495.99	921.64	798,160.73
		$B \times r/12$	PMT - i	$B_0 - P = B_1$

- **Equity** - pref. shares, common shares
  - constant dividend - perpetuity
  - growing dividend (constant rate) - growing perpetuity
  - growing dividend (non-constant rate)

Constant dividend/ - many REITs
- perpetuity

$PV = D/r$

D = 1.50 r = 15%

$PV = 1.50/.15 = 10$

Constant growth dividend - commercial real estate (to calculate property values)

$PV = \frac{D_0(1+g)}{r-g}$

for CRE: Prop. Value $= \frac{NOI_1}{r-g} \rightarrow$ cap rate

Variable growth dividend - growth moving to value
- 2 stage model

$$PV = \underbrace{\sum_{i=1}^{n} \frac{D_0(1+g_s)}{(1+r)^n}}_{\text{explicit discount period}} + \frac{\overbrace{\frac{D_0(1+g_s)^n(1+g_L)}{r-g}}^{\text{terminal value = perpetuity}}}{(1+r)^n}$$

e.g./ $D_0 = 1.50$ $g = 6\%$ $r = 15\%$ - growing perpetuity (typical of value stocks)

$$PV = \frac{1.50(1.06)}{.15-.06} = 1.59/.09 = 17.67$$

→ now assume $g_S = 6\%$ for 3 yrs., $g_L = 2\%$ thereafter

$$PV = \frac{1.50(1.06)}{1.15} + \frac{1.50(1.06)^2}{(1.15)^2} + \frac{1.50(1.06)^3}{(1.015)^3} + \frac{\frac{1.50(1.06)^3(1.02)}{.15-.02}}{(1.15)^3}$$

$$= 1.3826 + 1.2744 + 1.1747 + \frac{14.01734}{(1.15)^3}$$

$$= 13.04833$$

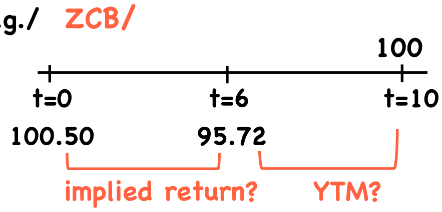
recall: $PV = \frac{FV}{(1+r)^T}$ → $PV(1+r)^T = FV$

$$(1+r)^T = FV/PV$$

$$1+r = (FV/PV)^{1/T}$$

$$r = (FV/PV)^{1/T} - 1$$

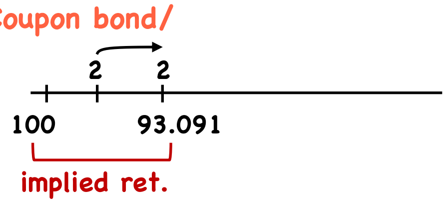
e.g./ ZCB/



$$r = (95.72/100.50)^{1/6} - 1 = -.8088\%$$

$$r = (100/95.72)^{1/4} - 1 = 1.10\%$$

Coupon bond/



$FV = ?$ $= 2(1.02) + 2 + 93.091$

$= 97.131$

$$r = (97.131/100)^{1/2} - 1 = -1.445\%$$

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Equity/ $PV = \frac{D_0(1+g)}{r-g}$ → $PV(r-g) = D_0(1+g)$

$r - g = \frac{D_0(1+g)}{PV}$

$r = \frac{D_0(1+g)}{PV} + g$ → GGM

div. yield / growth

e.g./ $P_0 = 63$ $D_1 = 1.76$ $g = 4\%$

$r = \frac{1.76}{63} + 4\% = 6.79\%$

- what if r = 7% → g?

$g = 7\% - {}^{1.76}/_{63} = 4.21\%$

$g = r - \frac{D_0(1+g)}{PV}$

- $PV = \frac{D_0(1+g)}{r-g}$ → divide both sides by E

$\frac{PV}{E} = \frac{{}^{D_0}/_{E}(1+g)}{r-g}$

PE ratio

dividend payout ratio - DPR

- given a ${}^{P}/_{E}$ and DPR - we can solve for r or g given the other
- compare the implied r or g to the required r or expected/estimated g

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forward P/E → ${}^{PV}/_{E_{t+1}} = \frac{{}^{D_{t+1}}/_{E_{t+1}}}{r-g}$

- if the forward div. is expected to ↑ or g is expected to ↑, stock/index will trade at a higher forward multiple
- a higher required return leads to a lower multiple

e.g./ 1/ forward PE = 28 E(DPR) = 70%

g = 4% r = ?

$28 = \frac{70\%}{r - .04}$ → 28r - 1.12 = .7

28r = 1.82

r = 1.82/28 = 6.5%

2/ forward PE = 19 E(DPR) = 60%

r = 8% g = ?

$19 = \frac{.60}{.08 - g}$ → 1.52 - 19g = .6

.92 = 19g

$g = {}^{.92}/_{19} = 4.84\%$

3/ forward PE = 28 E(DPR) = 70%

r = 9% g = 4.5% buy/sell stock?

$PE = \frac{.7}{.09 - .045} = 15.5\times$

- should be trading at 15.5× forward earnings, not 28 ∴ sell

- Cash flow additivity → principle of no arbitrage - 2 economically equivalent strategies should have the same price

$r = 6\%$
which one?

A	45	45	45
B	60	40	32.5
A - B	-15	5	12.50

- calculate PV of both, select the higher one

or/ take the difference in CFs (A - B)
if PV > 0, choose A, else B

PV = 0 → both equivalent

Proof/

$r = 6\%$

A	100
B	120
	-20

- B is clearly superior

PV(A) = 94.339
PV(B) = 113.2075
A - B -18.8679

$^{-20}/_{1.06} = -18.8679$

$r = 2.5\%$ (t=0 to t=1), $f_{1,1}$ (t=1 to t=2)
t=0 t=1 t=2
$r = 3.5\%$ (t=0 to t=2)

- invest for 2 yrs.
  - buy a 2yr. ZCB
  - buy a 1yr. ZCB, buy another in one year

spot — $f_{1,1}$ (implied forward rate)

$(1.035)^2 = (1.025)(1 + f_{1,1})$

$f_{1,1} = \frac{(1.035)^2}{(1.025)} - 1 = 4.5097\%$

- if $f_{1,1} = 5\%$ → lock-in rate with FRA → $(1.025)(1.05) = 1.07625$
vs.
$(1.035)^2 = 1.071225$
.005025

- would provide an arbitrage profit of .5025 per 100 of par

e.g./

	1yr.	2yr.
May 31	98.028	95.109
June 15	97.402	93.937

$\Delta f_{1,1} = ?$

$(100/98.028) - 1 = 2.011\%$

$(100/95.109)^{1/2} - 1 = 2.539\%$

$f_{1,1} = \frac{(1.02539)^2}{(1.02011)} - 1$
$= 3.069\%$

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e.g./

	1yr.	2yr.
May 31	98.028	95.109
June 15	97.402	93.937

$\Delta f_{1,1} = ?$

$f_{1,1} = 3.069\%$

$\rightarrow \left(100/97.402\right) - 1 = 2.6673\%$

$\left(100/93.937\right)^{1/2} - 1 = 3.1767\%$

$\frac{(1.031767)^2}{(1.026673)} - 1 = 3.6886$

$\Delta f_{1,1} = 3.6886\% - 3.069\% = 61.964$ bps

or/ 98.028/95.109 = 3.069%

97.402/93.937 = 3.688%

3.688% - 3.069%

forex - forward exchange rate $F_{f/d}$

1000 USD

FV $\rightarrow$ $1000e^{r_d T}$ $\quad$ $1000S_{f/d}e^{r_f T}$

$1000e^{r_d T} \cdot F_{f/d} = 1000S_{f/d}e^{r_f T}$

$F_{f/d} = S_{f/d} \cdot \frac{e^{r_f T}}{e^{r_d T}} \rightarrow S_{f/d}\left[\frac{1 + r\left(days/360\right)}{1 + r\left(days/360\right)}\right]$

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Forex/ e.g./ $S_{JPY/USD} = 134.40$ $\quad r_{USD} = 2\%$ $\quad r_{JPY} = .05\%$ $\quad$ 6 mos. $\quad F_{JPY/USD} = ?$

$F_{JPY/USD} = S_{JPY/USD} \cdot \frac{e^{r_{JPY}T}}{e^{r_{USD}T}} = 134.40\left(\frac{e^{.0005(.5)}}{e^{.02(.5)}}\right) = 133.0959$

e.g./

	r_f	r_d
May 31	2.012%	1.291%
June 15	2.667%	1.562%

$\rightarrow F_{f/d} = 1.2602 \cdot \frac{e^{.0212}}{e^{.01291}} = 1.269319$

$F_{f/d} = 1.2602 \cdot \frac{e^{.02667}}{e^{.01562}} = 1.274202$

$S_{f/d} = 1.2602$ $\quad$ 1 yr. $\rightarrow \Delta F_{f/d} = ?$

$\Delta F_{f/d} = \uparrow .004883$

or 48.83 pips

Option pricing/

hS, $-c$ $\rightarrow$ hS^+, c^+ / hS^-, c^-

select h such that both outcomes are equal

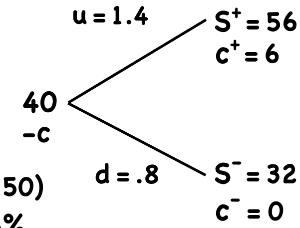
$hS^+ - c^+ = hS^- - c^-$

$hS^+ - hS^- = c^+ - c^-$

$h(S^+ - S^-) = c^+ - c^-$

$h = \frac{c^+ - c^-}{S^+ - S^-}$

Option pricing/



(X = 50)
r = 5%

$$h = \left|\frac{6-0}{56-32}\right| = {}^{6}/_{24} = .25$$

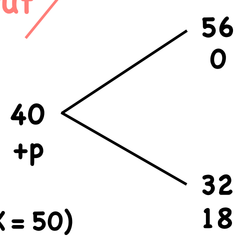
$hS^+ - c^+ = .25(56) - 6 = 8$
$hS^- - c^- = .25(32) - 0 = 8$

identical
∴ 0.25 S − c is a risk-free portfolio

$$hS_0 - c_0 = \frac{hS^+ - c^+}{1+r} \qquad .25(40) - c_0 = \frac{.25(56)-6}{1.05}$$

$$c_0 = 10 - {}^{8}/_{1.05} = 2.38095$$

Put/



(X = 50)
r = 5%

$$h = \left|\frac{0-18}{56-32}\right| = .75$$

$$hS_0 + p_0 = \frac{hS^- + p^-}{1+r}$$

$$.75(40) + p_0 = \frac{.75(32)+18}{1.05}$$

$$p_0 = {}^{42}/_{1.05} - 30$$

$$p_0 = 10$$

Statistical Measures of Asset Returns

a. calculate, interpret, and evaluate measures of central tendency and location to address an investment problem

b. calculate, interpret, and evaluate measures of dispersion to address an investment problem

c. interpret and evaluate measures of skewness and kurtosis to address an investment problem

d. interpret correlation between two variables to address an investment problem

Statistical Measures of Asset Returns

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- Measures of central tendency/
  - where the data are centered   i.e. the expected value
  - Arithmetic mean $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$
    - describes a representative possible outcome
    - sensitive to outliers (extreme values)
  - Median - middle value
    - not affected by outliers

    $\frac{(n+1)}{2}$ odd # of obs.

    $\frac{\left(\frac{n}{2}\right) + \left(\frac{n+2}{2}\right)}{2}$ even # of obs. - average the middle 2 obs.
  - mode - most frequently occurring value in a dataset
    - a dataset can have more than one mode, or $\underbrace{\text{no mode}}_{\text{all obs. are different}}$
    - single mode → unimodal
    - two modes → bimodal

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Return Bin (%)	Absolute Frequency	Relative Frequency (%)	Cumulative Absolute Frequency	Cumulative Relative Frequency (%)
−5.0 to −4.0	1	0.08	1	0.08
−4.0 to −3.0	7	0.56	8	0.64
−3.0 to −2.0	23	1.83	31	2.46
−2.0 to −1.0	77	6.12	108	8.59
−1.0 to 0.0	470	37.36	578	45.95
0.0 to 1.0	555	44.12	1,133	90.06
1.0 to 2.0	110	8.74	1,243	98.81
2.0 to 3.0	13	1.03	1,256	99.84
3.0 to 4.0	1	0.08	1,257	99.92
4.0 to 5.0	1	0.08	1,258	100.00
	count	%	$\sum$count	$\sum$%

Outliers/
- do nothing - if values are legitimate and correct
  - eliminates judgment
- Delete → trimmed mean
- Replace → winsorized mean

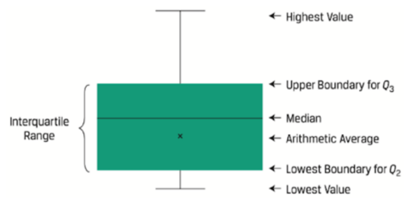
- compare mean with and without outliers

Measures of location – quantiles

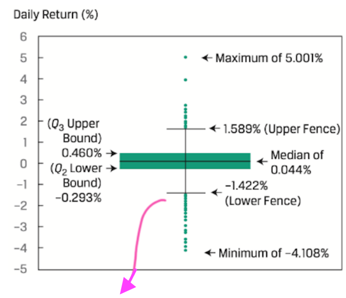
IQR – interquartile range

- quartiles 25, 50, 75
- quintiles 20, 40, 60, 80
- deciles 10, 20 … 90
- percentiles 1, 2 … 99

- Box and whisker plot



Box & whisker plot



upper fence = (1.5 x IQR) + upper bound

lower fence = lower bound – (1.5 x IQR)

uses/
- rank performance of portfolios and investment managers in terms of percentile/quartile in which they fall
- investment research → bottom return decile → short
  → top return decile → long

long/short HF

- Dispersion – variability around the central tendency
  – a measure of risk or uncertainty

- range: max. value – min. value
  – no information about the shape of the distribution
  – sensitive to outliers

- mean absolute deviation $\text{MAD} = \dfrac{\sum |X_i - \bar{X}|}{n}$

- sample variance $S^2 = \dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}$ pop: $\sigma^2 = \dfrac{\sum_{i=1}^{n} (X - \mu)^2}{n}$

  in units squared
  – can be added/subtracted

  $n-1$ ↓ degrees of freedom
  (n – 1 independent obs.)

- sample standard deviation $S = \sqrt{S^2} = \sqrt{\dfrac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}}$ pop $\sigma = \sqrt{\sigma^2}$
  – in the same units as the data itself

- **Downside deviation**

  e.g. target semideviation: $S_{target} = \sqrt{\dfrac{\sum_{i=1}^{n}(X_i - B)^2}{n-1}}$ $\quad \forall X_i \leq B$

  full sample n

  some minimum level

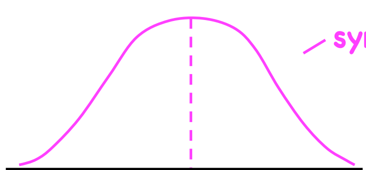
  - as B ↑ S_{target} ↑

- coefficient of variation $CV = S/\bar{X}$ - a measure of relative dispersion

  e.g. - for returns, CV measures the risk per unit of return

  - lower = better (less uncertainty, tighter distribution)

- **Normal distribution - most common**



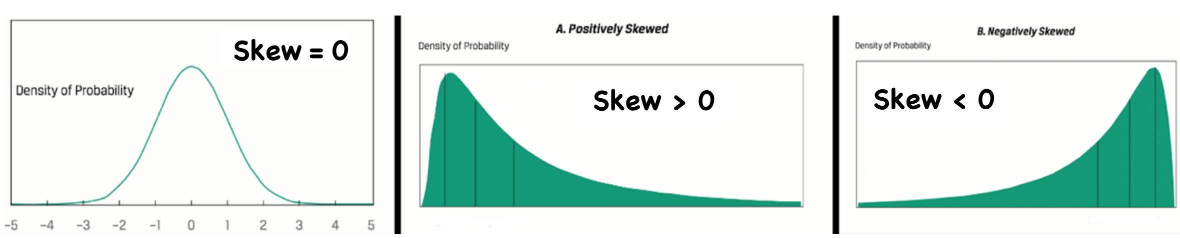
symmetrical

mean = median = mode

- completely described by 2 parameters: μ , σ^2

$X \sim N(\mu, \sigma^2)$

Skew

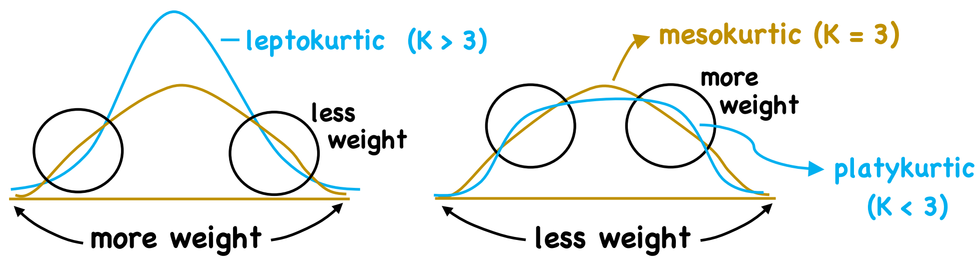


mean = median = mode | mean > median > mode | mean < median < mode

$$\text{Skew} \approx \frac{1}{n}\frac{\sum_{i=1}^{n}(X_i - \bar{X})^3}{S^3} \quad \text{for } n > 100$$

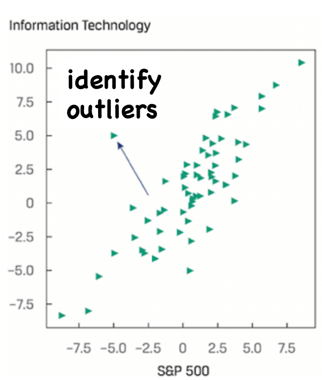
Kurtosis $K_E = \left[\frac{1}{n}\frac{\sum_{i=1}^{n}(X_i - \bar{X})^4}{S^4}\right] - 3$

- measures the combined weight of the tails relative to the rest of the distribution



Lepto → $K_e > 0$ - ow head, uw shoulders, ow tails
Meso → $K_e = 0$ - normal dist.
Platy → $K_e < 0$ - uw head, ow shoulders, uw tails

Scatter Plot



- used to visualize the joint variation in 2 numerical values
  - may be no relationship, a linear or non-linear relationship
- scatter plot matrix
  - assess for pairwise association among many variables

Covariance → the joint variability of 2 random variables
→ expressed in the same units as the variables

$$S_{XY} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{n - 1}$$

$S_{XY} > 0$ when they covary together
$(X_i - \bar{X}) > 0$ when $(Y_i - \bar{Y}) > 0$
and $(X_i - \bar{X}) < 0$ when $(Y_i - \bar{Y}) < 0$

Page 9

Correlation → measures the linear association between 2 variables

$$r_{XY} = \frac{S_{XY}}{S_X S_Y}$$

• $Cov(XY)$ determines the sign of r_{XY}

Properties:

1/ $-1 \leq r \leq 1$

2/ $r = 0$ implies no linear relationship → maximum diversification

3/ $r = 1$ → perfect positive correlation → perfect replication

4/ $r = -1$ → perfect negative correlation → perfect hedge

Limitations/
- linear association only
- unreliable when outliers are present
- correlation does not imply causation
  - spurious correlation → chance relationship
    - a third variable

calculation

$X/a \;\overline{\;\; r_{XY} \;\;}\; Y/a$

$a \rightarrow X$, $a \rightarrow Y$ → r_{XY}

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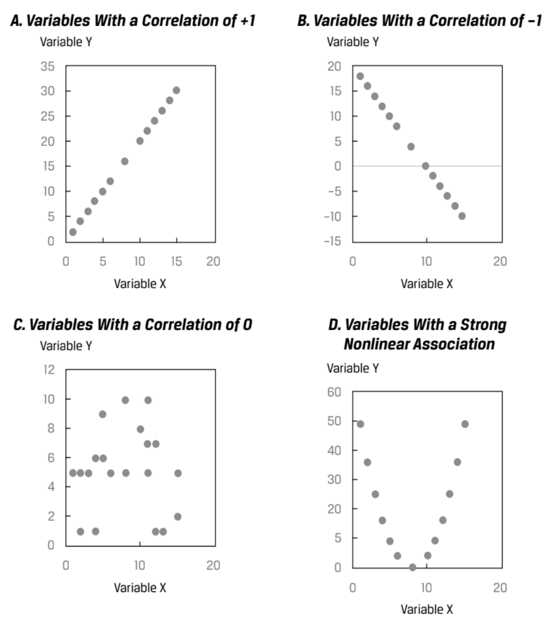
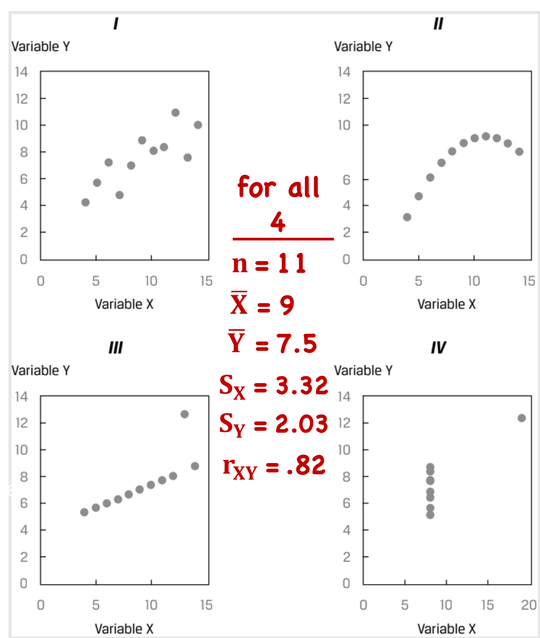




for all 4

$n = 11$

$\bar{X} = 9$

$\bar{Y} = 7.5$

$S_X = 3.32$

$S_Y = 2.03$

$r_{XY} = .82$

∴ visual inspection important

Probability Trees and Conditional Expectations

a. calculate expected values, variances, and standard deviations and demonstrate their application to investment problems

b. formulate an investment problem as a probability tree and explain the use of conditional expectations in investment application

c. calculate and interpret an updated probability in an investment setting using Bayes' formula

Probability Trees and Conditional Expectations

- Key point - investment decisions are made under uncertainty
  - the expected value of a random variable → E(X) → is a probability-weighted average of the possible outcomes

E(X) — forecast of future value
E(X) — estimate of the 'true' population mean based on a sample

$$E(X) = P(X_1)X_1 + P(X_2)X_2 + \dots + P(X_n)X_n = \sum_{i=1}^{n} P(X_i)X_i$$

one outcome only

- Variance of a random variable

$$\sigma^2(X) = P(X_1)[X_1 - E(X)]^2 + \dots + P(X_n)[X_n - E(X)]^2 = \sum_{i=1}^{n} P(X_i)[X_i - E(X)]^2$$

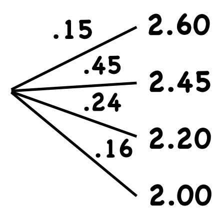
$$\sigma(X) = \sqrt{\sigma^2(X)}$$

e.g./

P	EPS (X)
.15	2.60
.45	2.45
.24	2.20
.16	2.00

E(X) = .15(2.60) + .45(2.45) + .24(2.20) + .16(2.00)

= 2.3405



$$\sigma^2(X) = .15(2.60 - 2.3405)^2 + .45(2.45 - 2.3405)^2 + .24(2.20 - 2.3405)^2 + .16(2.00 - 2.3405)^2$$

$$= .038785$$

$$\sigma(X) = \sqrt{.038785} = .196939$$

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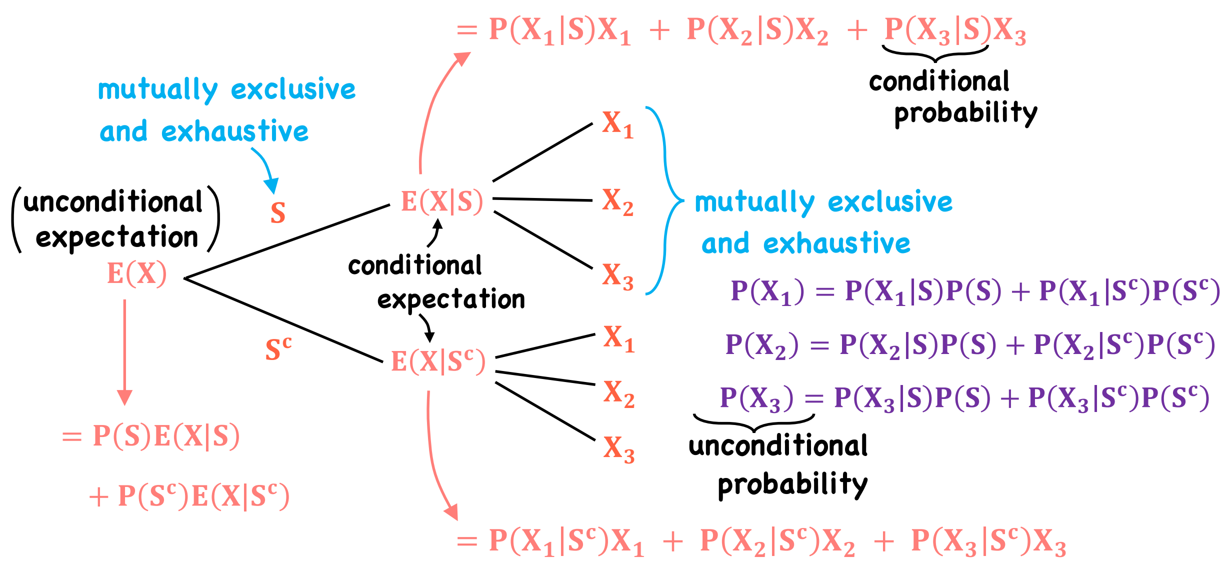

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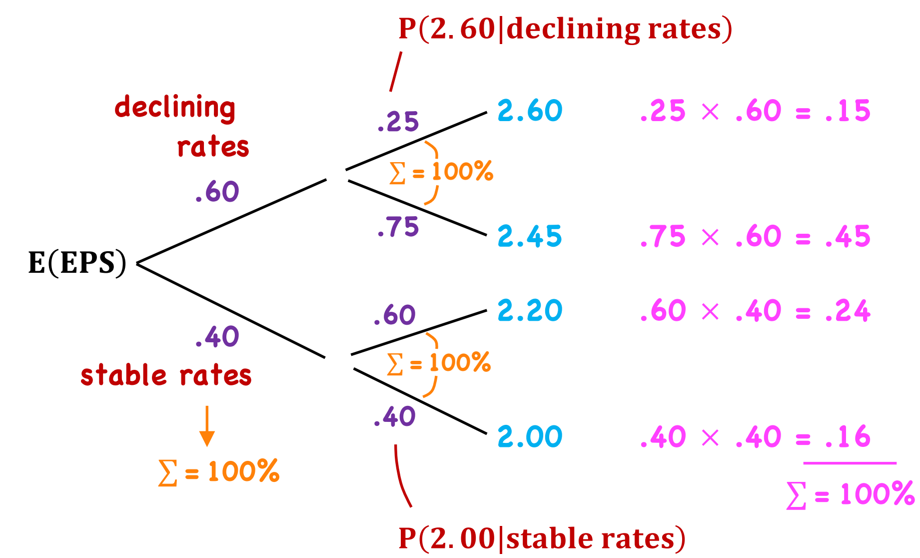


$$E(EPS) = P(X_1|S)P(S)X_1 + P(X_2|S)P(S)X_2 + P(X_3|S^c)P(S^c)X_3 + P(X_4|S^c)P(S^c)X_4$$

(.25)(.60)2.60 + (.75)(.60)2.45 + (.60)(.40)2.20 + (.40)(.40)(2.00)

- **Bayes' Formula:** a method for updating prior probabilities based on new information

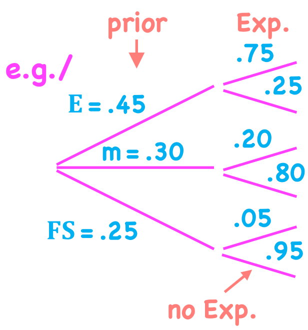
Recall: Total Probability Rule

$$P(E) = P(E|S_1)P(S_1) + P(E|S_2)P(S_2) + \ldots + P(E|S_n)P(S_n)$$

Q: given that we observe E, what is $P(S_n)$? → $P(S_n|E)$

$$P(S_n|E) = \frac{P(E|S_n)P(S)}{P(E)} = \frac{P(E|S_n)}{P(E)} \bullet P(S)$$

e.g./



$P(E) =$
.45(.75) (.3375)
+
.30(.20) (.06)
+
.25(.05) (.0125)
41%

What is $P(\text{exceeds}|\text{expands})$

$$= \frac{(.75)(.45)}{.41} = \frac{.75}{.41} \bullet .45 = .8231$$

$$= \frac{P(\text{Exp.}|\text{Exc.})P(\text{Exc.})}{P(\text{Exp.})} = \frac{P(\text{Exp.}|\text{Exc.})}{P(\text{Exp.})} \bullet P(\text{Exc.})$$

Portfolio Mathematics

a. calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns

b. calculate and interpret the covariance and correlation of portfolio returns using the joint probability function for returns

c. define shortfall risk, calculate the safety-first ratio, and identify an optimal portfolio using Roy's safety-first criterion

Portfolio Mathematics

Page 1

calculate and interpret the expected value, variance, standard deviation, covariances, and correlations of portfolio returns

$\Longrightarrow$ Portfolio Returns ($E(R_P)$, σ_{R_P}, Cov_{ij}, ρ_{ij})

1/ $E(R_P) = E(W_1R_1 + W_2R_2 + \cdots + W_nR_n)$

also a random variable

possible value of R_1

$E(R_1) = \underbrace{P(R_{11})}_{\text{probability}}R_{11} + P(R_{12})R_{12} + \cdots + P(R_{1n})R_{1n}$

Page 2

e.g./

	W	$E(R_i)$
SnP500	.50	13%
Corp. bonds	.25	6%
MSCI EAFE	.25	15%

$E(R_P) = .5(13\%) + .25(6\%) + .25(15\%) = 11.75\%$

measure of expected reward

2/ $\sigma^2(R_P) = \sum_{i=1}^{n}\sum_{j=1}^{n} W_i W_j Cov(R_iR_j)$

to calculate portfolio variance, need:

1/ all $E(R_i)$

2/ all $Cov(R_i, R_j)$

(Exhibit #11)

$E(R_i) = \sum^{n} P(R_i)R_i$

$$Cov(R_iR_j) = \frac{\sum_{t=1}^{n}(R_{it} - \bar{R}_i)(R_{jt} - \bar{R}_j)}{n-1}$$

• assume 3 assets → R_1, R_2, R_3

$$\sigma^2(R_P) = W_1^2\sigma^2(R_1) + W_2^2\sigma^2(R_2) + W_3^2\sigma^2(R_3) + 2W_1W_2Cov(R_1R_2) + 2W_1W_3Cov(R_1R_3) + 2W_2W_3Cov(R_2R_3)$$

- $\sigma^2(R_P)$ = f(variances, covariances)

always > 0 (variances)

can be < 0 or > 0 (covariances)

- major point - by selecting assets with zero or negative covariance, portfolio risk is lowered

- for n securities (or asset classes) → n variances
  $n^2 - n$ covariances
  $(n^2 - n)/2$ distinct covariances

n = 5
5 vars.
25-5= 20 Covars.
$^{20}/_{2}$ = 10 unique Covars.

3/ Correlation → $\rho_{ij} = \dfrac{Cov(R_iR_j)}{\sigma_{R_i} \cdot \sigma_{R_j}}$ → $Cov(R_iR_j) = \rho_{ij} \cdot \sigma_{R_i} \cdot \sigma_{R_j}$

Exhibit 12 Covariance Matrix

	S&P 500	US Long-Term Corporate Bonds	MSCI EAFE
S&P 500	400	45	189
US long-term corporate bonds	45	81	38
MSCI EAFE	189	38	441

Exhibit 13 Correlation Matrix of Returns

S&P 500	US Long-Term Corporate Bonds	MSCI EAFE
1.00	0.25	0.45
0.25	1.00	0.20
0.45	0.20	1.00

$$\frac{189}{400^{1/2} \cdot 441^{1/2}} = \frac{189}{20.21} = 0.45$$

Ex #13

calculate and interpret the covariance and correlation of portfolio returns using a joint probability function for returns

Recall:

$$Cov(R_AR_B) = \frac{\sum(R_{A_i} - \bar{R}_A)(R_{B_i} - \bar{R}_B)}{n-1}$$

$$= \frac{1}{n-1}(R_{A_1} - \bar{R}_A)(R_{B_1} - \bar{R}_B) + \frac{1}{n-1}(R_{A_2} - \bar{R}_A)(R_{B_2} - \bar{R}_B) + \cdots + \frac{1}{n-1}(R_{A_n} - \bar{R}_A)(R_{B_n} - \bar{R}_B)$$

weights

probabilities?

- the concept of joint probability

$$Cov(R_AR_B) = \sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{P\left(R_{A_i}R_{B_j}\right)}_{\text{probability}} \underbrace{\left(R_{A_i} - \bar{R}_A\right)\left(R_{B_j} - \bar{R}_B\right)}_{\text{value of cross product}}$$

where i & j = 1 to n are scenarios

if returns are independent:

→ $P(R_AR_B) = P(R_A)P(R_B)$

→ since independence is a stronger property than uncorrelatedness, this property holds for uncorrelated random variables

Exhibit #6/7

- Safety first rules focus on shortfall risk - the risk a portfolio value (or return) will fall below some minimum acceptable level over some time horizon

Let R_L = minimum acceptable level of return

z-value → $\text{SFRatio} = \dfrac{E(R_P) - R_L}{\sigma_P}$

- objective is to maximize this ratio
- optimal portfolio minimizes N(-SFRatio)

e.g./ R_L = 2%

Portfolio	R_P	σ
1	12%	15%
2	14%	16%

$\text{SFRatio}_1 = \dfrac{12-2}{15} = 0.6\dot{6}$ = NORM.S.DIST(-.667,1) = 0.2525

$\text{SFRatio}_2 = \dfrac{14-2}{16} = 0.75$ = NORM.S.DIST(-.75,1) = 0.227

Note: if $R_L = R_f$ → SFRatio = Sharpe Ratio

Example 3

Simulation Methods

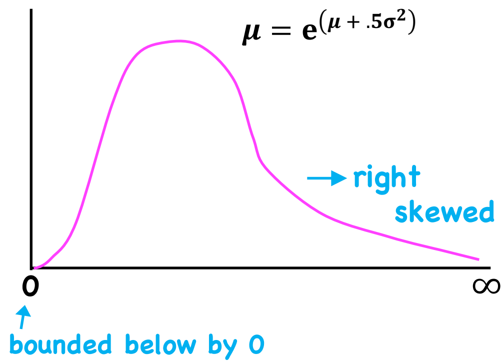
a. explain the relationship between normal and lognormal distributions and why the lognormal distribution is used to model asset prices when using continuously compounded asset returns

b. describe Monte Carlo simulation and explain how it can be used in investment applications

c. describe the use of bootstrap resampling in conducting a simulation based on observed data in investment applications

Common Probability Distributions

Page 1



- commonly used to model the probability distribution of asset prices

- a variable Y follows a lognormal distribution if LN(Y) is normally distributed

- completely described by 2 parameters → the μ and σ^2 of its associated normal distribution

$S_T / S_0 = 1 + R_H$ where R_H = holding period return

i.e./ $\dfrac{S_T - S_0}{S_0} = R_H$

$S_T/S_0 - S_0/S_0 = R_H \rightarrow S_T/S_0 - 1 = R_H \rightarrow S_T/S_0 = 1 + R_H$

Page 2

e.g. $S_T = 34.50$ $S_0 = 30$

$S_T/S_0 = 34.50/30 = 1.15 = 1 + R_H$ ∴ $R_H = 15\%$

$\ln\left(S_T/S_0\right) = r$ where r = continuously compounded return

$r = \ln\left(34.50/30\right) = \ln(1.15) = 0.13976$ and $r \sim N(\mu T, \sigma^2 T)$

∴ $34.50 = 30e^{.13976}$

- more generally: $S_T = S_0 e^r$ $\left(S_T/S_0 = e^r \text{ and } \ln\left(S_T/S_0\right) = r\right)$

- to assume returns are normally distributed, we assume returns are
  1/ independent
  2/ identically distributed (μ and σ^2 do not change from period to period)

- so, while $S_T = S_0(1 + R_H)^T$
  with cont. comp: $S_T = S_0 e^{rT}$

Volatility → annualized sd of the continuously compounded daily returns of the underlying asset

- since $r \sim N(\mu T, \sigma^2 T)$, sd = $\sigma\sqrt{T}$

  - so both the mean and variance of r scale linearly with time, but the s.d. scales linearly with the square root of time

e.g. if daily vol. = .01, annualized vol. = $.01\sqrt{250} = 15.81\%$ **example #1**

Monte Carlo Simulation/

Step 1: Specify the quantity of interest

e.g. MV_p in 10 years

Step 2: Specify a time grid → K sub-periods with Δt increment for the full time horizon

e.g. 20 sub-periods, Δt = 6 months

Step 3: Specify distributional assumptions for the key risk factors

e.g. $E(R_P)$ = 6.4% σ = 12%

$MV_t = MV_{t-1}(1 + R_{P_1})$

Step 4: Draw standard normal random numbers for each key risk factor over each K sub-periods.

- random number generator → produces a distribution of random numbers from 0 to 1, all equally likely

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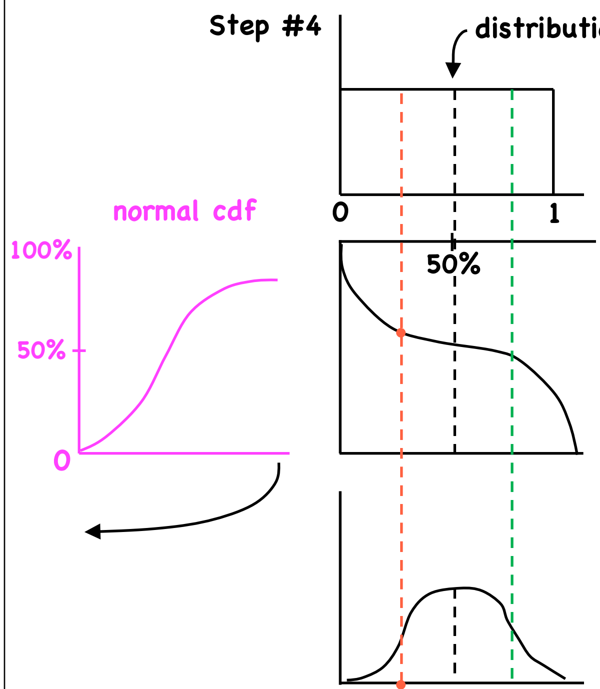
Monte Carlo Simulation/

Step #4



#1 → 0.32

#2 → 0.64

#20

1000 runs

Step #5:

#1 → 0.31561 on cdf

#2 → 0.7673 on cdf

#20

Step #6:

#1 → -0.48 z-value

$E(R_P) = 6.4\% - 0.48(12\%) = 0.64$

$MV_1 = MV_0(1.0032)$

#2 → 0.73 z-value

$E(R_P) = 6.4\% + .73(12) = 15.16\%$

$MV_2 = MV_1(1.0758)$

#20

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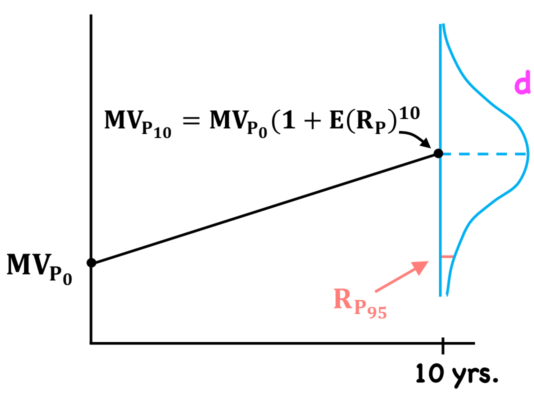
Monte Carlo Simulation/



distribution of possible $MV_{P_{10}}$

Objective: $MV_{P_{10}}$ = $2M with 95% PoS.

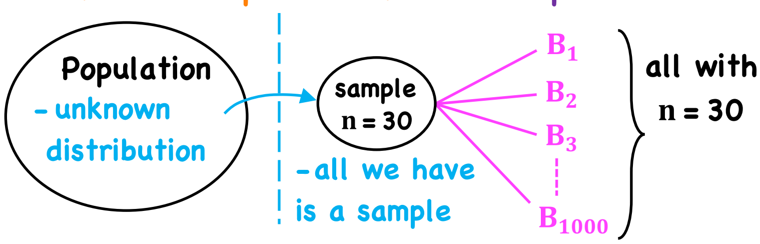
$2M / (1 + R_{P_{95}})^{10}$ = Beginning Capital

- Monte Carlo Simulation → provides only statistical estimates, not exact results
  → does not support cause and effect conclusions

Page 7

Resampling → repeatedly draw samples from an original data sample in order to estimate population parameters

1/ Bootstrap method/ uses computer simulation



- B_1
  - draw 1 observation, record, replace
  - draw another obs., record, replace

  n times

- rather than estimate the distribution, this method creates the distribution
- can also find SE of an estimator even when no analytical formula is available (e.g. median)

$$S_{\bar{X}} = \sqrt{\frac{\sum_{i=1}^{B}\left(\hat{\theta}_i - \bar{\theta}\right)^2}{B-1}}$$

standard error

Estimation and Inference

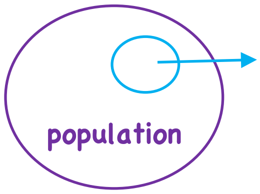
a. compare and contrast simple random, stratified random, cluster, convenience, and judgmental sampling and their implications for sampling error in an investment problem

b. explain the central limit theorem and its importance for the distribution and standard error of the sample mean

c. describe the use of resampling (bootstrap, jackknife) to estimate the sampling distribution of a statistic

Sampling and Estimation

Page 1



sample → a method of obtaining information about a population's parameters (μ & σ) through sample statistics ($\overline{X}$ & S)

A/ Probability sampling → every member of a population has an equal chance of being selected

∴ samples will be more representative of the population

1/ Simple Random Sampling → a subset of a larger population such that each element has an equal probability of being selected

e.g. population $n = 500$ → random number generator selects 50 numbers between 1 and 500 → sample size $n = 50$

- useful when data are homogeneous

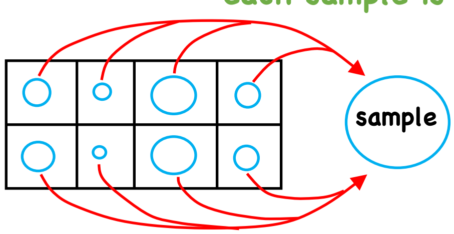
Page 2

2/ Systematic Sampling → when the population is too large to code

- select every K^{th} element until the desired sample size is reached

3/ Stratified Random Sampling - population is sub-divided into sub-populations based on one or more classifications

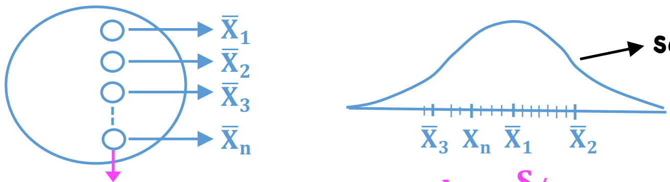
- simple random samples are then drawn from each sub-pop.
- each sample is then pooled to form the main sample



- each sub-sample is proportionate to the size of its sub-population
- guarantees that population sub-divisions are represented in the sample
- statistics will be more precise

- sample statistics are estimates of population parameters
  - not exact, subject to error

• sampling error → difference between observed values of a statistic and population parameters as a result of using just a subset of the population



→ sampling distribution of the sample means

all equal n

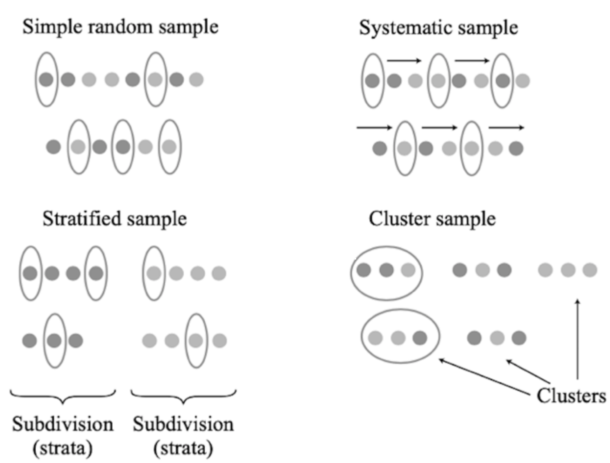
$sd_{\bar{X}} = {}^{S}/_{\sqrt{n}}$

<u>example #1</u>

4/ Cluster sampling - pop. is divided into clusters each of which is a mini representation of the population
- certain clusters are then selected as a whole using simple random sampling → one-stage cluster sampling

4/ Cluster sampling - if sub-samples are selected from each cluster → two-stage cluster sampling

• usually results in lowest precision since a cluster may not be representative of the population
- is both cost and time efficient however



B/ Non-probability sampling - depends on factors such as judgment or convenience (in terms of access to data)

- risk that samples may be non-representative

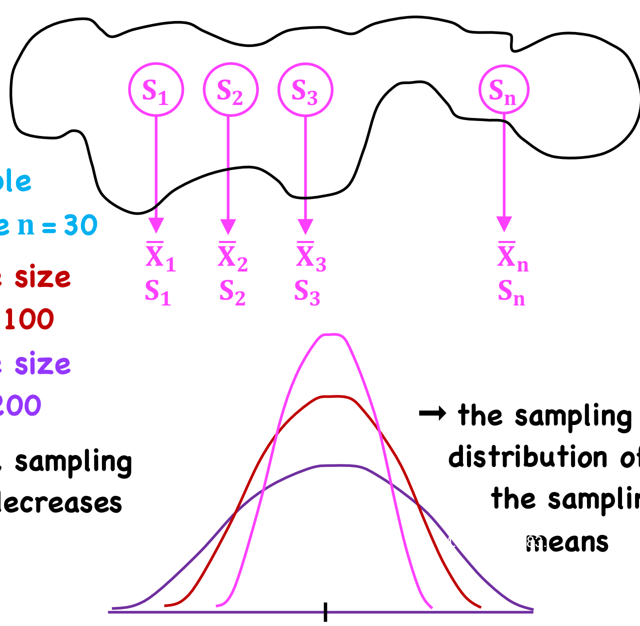
5/ Convenience sampling - observations are selected that are easy to obtain or are accessible

• not necessarily representative, but low cost

6/ Judgmental sampling → select observations based on experience and knowledge

→ useful when there is a time constraint and/or the specialty of the researcher would result in better representation

example 2, 3, 4



population with any distribution with μ and finite σ^2

sample size $n = 30$

sample size $n = 100$

sample size $n = 200$

- as $n\uparrow$, sampling error decreases

→ the sampling distribution of the sampling means

$\bar{X}_{\bar{X}}$ → best estimate of μ

$$S_{\bar{X}}^2 = S^2/n$$

$$\text{SE} \longrightarrow S_{\bar{X}} = S/\sqrt{n}$$

Standard Error

$S/\sqrt{n}$ or $\sigma/\sqrt{n}$ if we know σ

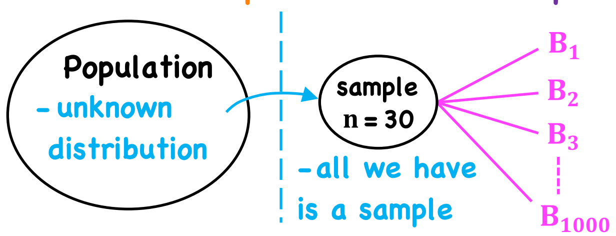
Note: sd $\neq$ SE

sd = dispersion from the mean (data description)

SE = sampling error (data inference)

Resampling → repeatedly draw samples from an original data sample in order to estimate population parameters

1/ Bootstrap method/ uses computer simulation



all with $n = 30$

- B_1
  - draw 1 observation, record, replace
  - draw another obs., record, replace
  - ⋮
  - n times

- rather than estimate the distribution, this method creates the distribution
- can also find SE of an estimator even when no analytical formula is available (e.g. median)

$$S_{\bar{X}} = \sqrt{\frac{\sum_{i=1}^{B}\left(\hat{\theta}_i - \bar{\theta}\right)^2}{B-1}}$$

standard error

2/ Jackknife method/ - omit one observation from a sample, one at a time

e.g./ $n = 30$

J_1 $n = 29$, omit X_1

J_2 $n = 29$, omit X_2

⋮

J_{30} $n = 29$, omit X_{30}

- will produce similar results from sample to sample (bootstrap may not)

Hypothesis Testing

a. explain hypothesis testing and its components, including statistical significance, Type I and Type II errors, and the power of a test.

b. construct hypothesis tests and determine their statistical significance, the associated Type I and Type II errors, and power of the test given a significance level

c. compare and contrast parametric and nonparametric tests, and describe situations where each is the more appropriate type of test

Hypothesis Testing

Statistical Inference → the process of making judgments about a larger group (pop.) based on a smaller group (sample)

e.g./ hypothesis testing – test to see whether a sample statistic is likely to come from a population with the hypothesized value of the population parameter

i.e. Does $\overline{X} = \mu_0$?

Hypothesis → a statement about one or more populations that are tested using sample statistics

Process: Step 1: State the hypothesis
2: Identify the appropriate test statistic
3: Specify the level of significance
4: State the decision rule
5: Collect data and calculate the test statistic
6: Make a decision

Step #1: State the hypothesis

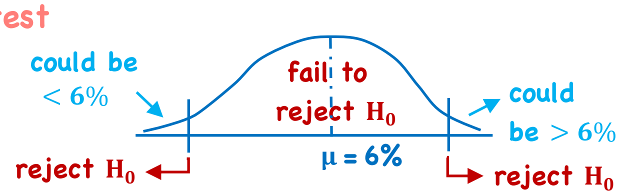
null → H_0 → assumed to be true unless we can reject

alternative → H_a – typically want to reject H_0

• Two-sided (two-tailed) test

e.g. $H_0 : \mu = 6\%$

vs. $H_a : \mu \neq 6\%$

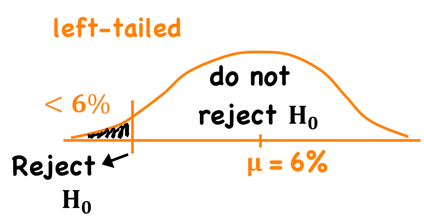


• One-sided (left or right tailed) test

e.g. $H_0 : \mu \leq 6\%$ or/ $H_0 : \mu \geq 6\%$

$H_a : \mu > 6\%$ $H_a : \mu < 6\%$

→ right-tailed ← left-tailed



- the null (H_0) always contains the equality sign

$$H_0 : \bar{X} = \mu_0 \quad H_0 : \bar{X} \le \mu_0 \quad H_0 : \bar{X} \ge \mu_0$$

- testing H_0 is always done at equality

Test Statistic:
(Step #2)

pop. σ^2 is known

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}}$$

distributed normally

pop. σ^2 is unknown

$$t = \frac{\bar{X} - \mu_0}{S / \sqrt{n}}$$

t-distributed

What We Want to Test	Test Statistic	Probability Distribution of the Statistic	Degrees of Freedom
Test of a single mean	$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$	t-Distributed	$n - 1$
Test of the difference in means	$t = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}}$	t-Distributed	$n_1 + n_2 - 2$
Test of the mean of differences	$t = \frac{\bar{d} - \mu_{d0}}{s_{\bar{d}}}$	t-Distributed	$n - 1$
Test of a single variance	$\chi^2 = \frac{s^2(n-1)}{\sigma_0^2}$	Chi-square distributed	$n - 1$
Test of the difference in variances	$F = \frac{s_1^2}{s_2^2}$	F-distributed	$n_1 - 1, n_2 - 1$
Test of a correlation	$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$	t-Distributed	$n - 2$
Test of independence (categorical data)	$\chi^2 = \sum_{i=1}^{m} \frac{(O_{ij} - E_{ij})^2}{E_{ij}}$	Chi-square distributed	$(r - 1)(c - 1)$

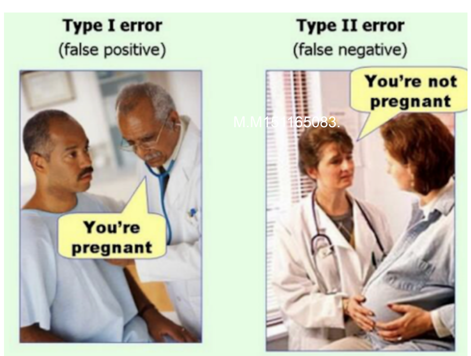
Step 3: Specify the Level of Significance

- level of sig. depends on the seriousness of making a mistake

	H_0 = true	H_0 = false
fail to reject	Correct $(1 - \propto)$ confidence level	Type II error β
reject	Type I error $\propto$ level of sig.	Correct $(1 - \beta)$ Power of a test

- as $\propto \downarrow$, $\beta \uparrow$
- only way to decrease both is to increase n

e.g.: H_0: not pregnant
H_a: pregnant



$$t = \frac{\bar{X} - \mu_0}{S/\sqrt{n}}$$

→ as $n \uparrow$, denom. $\downarrow$, t-stat $\uparrow$

Step #4: State the Decision Rule

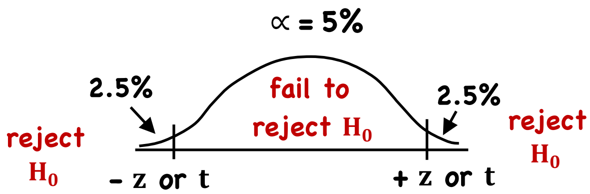
2-tail: Reject H_0 when $|\text{test} - \text{statistic}| > |\text{critical value}|$

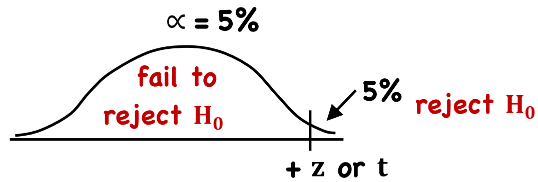
t or z ... $t_{\alpha/2}$ or $z_{\alpha/2}$

right tail: Reject H_0 when test-statistic $>$ critical value

left tail : Reject H_0 when test-statistic $<$ critical value

$(t_\alpha$ or $z_\alpha)$





Cut-off for . . .	Excel	Python	R
Right tail, 2.5%	NORM.S.INV(0.975)	norm.ppf(.975)	qnorm(.025,lower.tail=FALSE)
Left tail, 2.5%	NORM.S.INV(0.025)	norm.ppf(.025)	qnorm(.025,lower.tail=TRUE)
Right tail, 5%	NORM.S.INV(0.95)	norm.ppf(.95)	qnorm(.05,lower.tail=FALSE)
Left tail, 5%	NORM.S.INV(0.05)	norm.ppf(.05)	qnorm(.05,lower.tail=TRUE)

or T.INV (p,df)

Parametric Testing

- sample stats to test pop. parameters ($\bar{X}$ or S^2 for μ or σ^2)
- distributional assumptions

Non-parametric testing

- no parameters tested
- no distributional assumptions

1/ when data do not meet distributional assumptions

i.e. $n < 30$, pop. is non-normal

2/ when there are outliers

- test of median instead of mean

3/ when data are given in ranks or use an ordinal scale

NO IR

ordered

categorical

4/ hypothesis do not concern a parameter

e.g. Is a sample random?

Parametric and Non-Parametric Tests of Independence

a. explain parametric and nonparametric tests of the hypothesis that the population correlation coefficient equals zero, and determine whether the hypothesis is rejected at a given level of significance

b. explain tests of independence based on contingency table data

Hypothesis Testing

Page 1

• Tests of Correlation

1/ Parametric test

			left	right
2-sided	$H_0: p = 0$	one-sided	$H_0: p \geq 0$	$H_0: p \leq 0$
	$H_a: p \neq 0$		$H_a: p < 0$	$H_a: p > 0$

- recall $r_{xy} = \dfrac{Cov(x,y)}{S_x S_y}$ → called a Pearson correlation or a Bivariate correlation

test-statistic: $t_{n-2} = \dfrac{r\sqrt{n-2}}{\sqrt{1-r^2}}$

- in testing r, as $n\uparrow$, H_0 rejected for even small correlations
  - big data sets, almost any r will be significant

e.g./ $r = .02$
$n = 10{,}000$

$t_{9,998} = \dfrac{.02\sqrt{9,998}}{\sqrt{1-.02^2}} \sim 2$ vs. critical $t = 1.96$

ex. #1

Page 2

• Tests of Correlation

2/ Non-parametric test

- if normality assumption for X or Y violated, or outliers are present

  • Spearman rank correlation coefficient
    • basically a correlation, but calculated on rank values and not the values of the observations of X or Y

1/ Rank all X from largest to smallest

- assign a rank: 1 = largest … n = smallest
- for a tie: assign all the same → average rank

  e.g. 3 tied for 6th (6 + 7 + 8)/3 = 7
  - each get a rank of 7

- repeat for Y

- **Tests of Correlation/**

  2/ Non-parametric test

  2/ On original data set (pre-ranked):

  calculate $d_i^2 = (\text{rank } X_i - \text{rank } Y_i)^2$ – where X and Y are original paired obs.

  3/ $r_S = 1 - \left[\frac{6\left(\sum_{i=1}^{n} d_i^2\right)}{n(n^2-1)}\right]$

  test r_S, if $n > 30$ $\quad t_{n-2} = \frac{r_S\sqrt{n-2}}{\sqrt{1-r_S^2}}$

  - white text example
  - Example #3

- **Tests of Independence/**

  – test if classification types are independent

  e.g./ Are growth stocks equally likely to be any size or are they more likely to be large-cap stocks?

- **Tests of Independence/** Contingency Table (2-way)

observed

Investment Type	Small	Medium	Large	Total
	Size Based on Market Capitalization			
Value	50	110	343	503
Growth	42	122	202	366
Blend	56	149	520	725
Total	148	381	1,065	1,594

A. Expected Frequency of ETFs by Size and Investment Type

Investment Type	Small	Medium	Large
	Size Based on Market Capitalization		
Value $E_{SV} \rightarrow$	46.703	120.228	336.070
Growth	33.982	87.482 $\rightarrow E_{MG}$	244.536
Blend	67.315	173.290	484.395
Total	148.000	381.000	1,065.000

B. Scaled Squared Deviation for Each Combination of Size and Investment Type

Investment Type	Small	Medium	Large
	Size Based on Market Capitalization		
Value	0.233	0.870	0.143
Growth	1.892	13.620	7.399
Blend	1.902	3.405	2.617

- non-parametric test of indep.

$$\chi^2 = \sum_{i=1}^{m} \frac{\left(O_{ij} - E_{ij}\right)^2}{E_{ij}}$$

df = (r – 1)(c – 1) (right-tailed)

m = # of cells (3 x 3 = 9)

O_{ij} = observed value in each cell

E_{ij} = expected value in each cell

$$E_{ij} = \frac{(\text{row}_i \text{ total}) \times (\text{column}_j \text{ total})}{\text{Overall total}}$$

$$E_{SV} = (503 \times 148)/1594 = 46.703$$

$$E_{MG} = (366 \times 381)/1594 = 87.482$$

$$\therefore \frac{\left(O_{ij} - E_{ij}\right)^2}{E_{ij}} SV = \frac{(50 - 46.703)^2}{46.703} = .2328$$

Page 5

• Tests of Independence/

H_0 : size and type are independent

H_a : size and type are not independent

B. Scaled Squared Deviation for Each Combination of Size and Investment Type

	Size Based on Market Capitalization		
Investment Type	Small	Medium	Large
Value	0.233	0.870	0.143
Growth	1.892	13.620	7.399
Blend	1.902	3.405	2.617

$$\sum_{i=1}^{m} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} = 32.08025$$

= CHISQ.INV(0.95,4) = 9.4877

$\therefore$ reject H_0

standardized residuals $= \dfrac{O_{ij} - E_{ij}}{\sqrt{E_{ij}}}$ → > 0 means more obs. than expected if categories were independent

< 0 - opposite (i.e. fewer obs.)

e.g./

$$SR_{SV} = \frac{50 - 46.703}{\sqrt{46.703}} = 0.48$$

$$SR_{MG} = \frac{122 - 87.482}{\sqrt{87.482}} = 3.69$$

more than expected if independent

example #4

Simple Linear Regression

a. describe a simple linear regression model, how the least squares criterion is used to estimate regression coefficients, and the interpretation of these coefficients

b. explain the assumptions underlying the simple linear regression model, and describe how residuals and residual plots indicate if these assumptions may have been violated

c. calculate and interpret measures of fit and formulate and evaluate tests of fit and of regression coefficients in a simple linear regression

d. describe the use of analysis of variance (ANOVA) in regression analysis, interpret ANOVA results, and calculate and interpret the standard error of estimate in a simple linear regression

e. calculate and interpret the predicted value for the dependent variable, and a prediction interval for it, given an estimated linear regression model and a value for the independent variable

f. describe different functional forms of simple linear regressions

Simple Linear Regression

LOS a (2.5p) Simple Linear Regression - describe

LOS b (8.5p) Estimating Parameters - describe

LOS c (7p) Assumptions of SLR - explain, describe

LOS d, e (5.5p) Analysis of Variance - calculate, interpret, describe

LOS f (8p) Hypothesis Testing → Coefficients - formulate, determine

LOS g (4p) Prediction & Prediction Intervals - calculate, interpret

LOS h (7p) Functional Forms of LR - describe

Page 1
LOS a
- describe

- Simple Linear Regression (LR) → one IV

DV - dependent variable - Y - the variable we are seeking to explain

IV - independent variable - X the explanatory variable

LR assumes a linear relationship between the DV and the IV

Variation of $Y = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$ → SST or total sum of squares

- best guess for Y is $\bar{Y}$, thus if X gives a more accurate estimate of Y than $\bar{Y}$, we say X helps explain Y

LOS b
- describe

$$\underbrace{Y_i = b_0 + b_1 X_i}_{\text{regression coefficients}} + \varepsilon_i$$

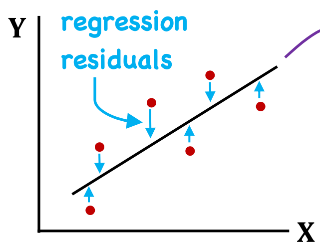
intercept (b_0) slope coefficient (b_1)

regression coefficients

(Y is regressed on X)

ε → error term (residual)
- the portion of the DV that cannot be explained by the IV

Page 2
LOS b
- describe



compute a line of best fit that minimizes the sum of the squared deviations between the observed values of Y and the predicted values (the regression line)

i.e. min. $\sum_{i=1}^{n}(\underset{\text{DV}}{Y_i} - \underbrace{\hat{b}_0 - \hat{b}_1 X_i}_{\text{predicted values of DV }(\hat{Y})})^2 = \sum_{i=1}^{n} \varepsilon_i^2$ → SSE - sum of the squares error (a.k.a. residual sum of squares)

$\begin{pmatrix} E(\varepsilon) = 0 \\ \sum \varepsilon_i = 0 \end{pmatrix}$

- Note: $\varepsilon = Y_i - \hat{Y}$ implies the residual is in the same units of measurement as the DV (Y)

$(\bar{X}, \bar{Y})$ lies on the regression line
↓
$\hat{b}_0 = \bar{Y} - \hat{b}_1 \bar{X}$

$$\hat{b}_1 = \frac{\mathrm{Cov}(X,Y)}{\sigma_x^2} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

→ denominator can never be negative, ∴ sign of $\hat{b}_1$ is determined solely by $\mathrm{Cov}(X,Y)$
- if $\hat{b}_1 > 0, r_{XY} > 0$

Page 3
LOS b
- describe

- Interpreting $\hat{b}_0$ and $\hat{b}_1$

$\hat{Y} = b_0$ if $X_i = 0$ → only makes sense if the IV has meaning at $X = 0$

$\hat{b}_1$ → the change in Y for a one unit change in X

e.g. ROA(%) = 4.875% + 1.25 • CAPEX(%) → ROA = 4.875% if CAPEX = 0
→ if CAPEX ↑ 1 unit (i.e. 1%), then ROA ↑ 1.25%

Data/
- cross-sectional - many observations on X & Y for the same time period
- time-series - many observations on Y (and sometimes X) from different time periods

example #2/3

Page 4
LOS c
- explain
- describe

• Assumptions/

1/ Linearity → the relationship between X & Y is linear in the parameters b_0 and b_1 → neither is multiplied or divided by another regression parameter

→ implies the IV must not be random - if so, there would be no linear relation between X & Y

2/ Homoskedasticity → $\mathrm{Var}(\varepsilon)$ is the same for all observations

(vs. heteroskedastic) - a violation indicates the data series may come from 2 different populations (CS) or regimes (TS)

3/ Independence → the pairs (X, Y) are independent of each other

∴ ε is uncorrelated across observations (no serial correlation)

- needed to correctly estimate the variances of b_0 and b_1

Page 5
LOS c
- explain
- describe

• Assumptions/

4/ Normality → ε is normally distributed

- required to conduct valid tests of the values of the regression coefficients

example #4

LOS d
- calculate
- interpret

• Analysis of Variance/

Total sum of squares (SST)

$$\sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

• total variance

sum of squared errors (SSE)

$$\sum_{i=1}^{n}(Y_i - \hat{Y})^2$$

• unexplained variance

Regression sum of squares (SSR)

$$\sum_{i=1}^{n}(\hat{Y} - \bar{Y})^2$$

• explained variance

∴ SST = SSE + SSR

or/ total SS = unexplained SS + explained SS

Page 6
LOS d
- calculate
- interpret

- Analysis of Variance/
  - Coefficient of Determination - measures the fraction of the total variation in the DV that is explained by the IV (goodness of fit measure)

- if only 1 IV, square the correlation between IV and DV
- if multiple IVs: total variation in $Y = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$

explained variation in $Y = \sum_{i=1}^{n}(\hat{Y} - \bar{Y})^2$

$$R^2 = \frac{SSR}{SST} = \frac{\text{explained var.}}{\text{total var.}} = \frac{\sum_{i=1}^{n}(\hat{Y} - \bar{Y})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

measures fit but is not a statistical test

R^2 → Coeff. of Determination

∴ statistical test = F-test

$H_0: b_1 = 0$
$H_a: b_1 \neq 0$

multiple IVs
$H_0: b_1 = b_2 = \cdots = b_k = 0$
H_a: at least one $b_k \neq 0$

Page 7
LOS d
- calculate
- interpret

- Analysis of Variance/

$$F = \frac{MSR}{MSE} = \frac{SSR/df}{SSE/df} = \frac{SSR/k}{SSE/n-(k+1)}$$

$df_1 = k$
$df_2 = n - k - 1$
mean

k = slope coefficients
$k + 1$ = regression coefficients

- ANOVA table/

LOS e
- describe
- calculate
- interpret

Source	Sum of Squares	Degrees of Freedom	Mean Square	F-Statistic
Regression	$SSR = \sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2$	1	$MSR = \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}{1}$	$F = \frac{MSR}{MSE} = \frac{\frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}{1}}{\frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-2}}$
Error	+ $SSE = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2$	$n - 2$	$MSE = \frac{\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}{n-2}$ → $\sqrt{MSE} = SEE$	
Total	= $SST = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$	$n - 1$		

Page 8
LOS e
- describe
- calculate
- interpret

- **Standard Error of the Estimate (SEE)** - a measure of the s.d. of $\hat{\varepsilon}_i$

$$SEE = \sqrt{MSE} = \left(\frac{\sum_{i=1}^{n}(Y_i - \hat{Y})^2}{n-2}\right)^{1/2} = \left(\frac{\sum_{i=1}^{n}\varepsilon_i^2}{n-2}\right)^{1/2}$$

the smaller the SEE, the more accurate the regression

(a.k.a. the standard error of the regression or the root mean square error)

e.g./

Source	Sum of Squares	Degrees of Freedom	Mean Square	F-Statistic
Regression	191.625	1	191.625	16.0104
Error	47.875	4	11.96875	
Total	239.50	5		

$\hookrightarrow SEE = \sqrt{11.96875}$

= F.INV(.95,1,4)
= 7.71

Example #5

Page 9
LOS f
- formulate
- determine

1/ Hypothesis Tests of $\hat{b}_1$:

test statistic: $t = \dfrac{\hat{b}_1 - b_1}{S_{\hat{b}_1}}$ ← hypothesized value (b_1)

df = $n-(k+1)$

$\rightarrow$ standard error of $\hat{b}_1$ = $\dfrac{SEE}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}}$

e.g./ $b_1 = 1.25$ $\sum_{i=1}^{n}(X_i - \bar{X})^2 = 122.64$

Source	Sum of Squares	Degrees of Freedom		Mean Square	F-Statistic
Regression	191.625	1	k	191.625	16.0104
Error	47.875	4	n-(k+1)	11.96875	
Total	239.50	5	n-1		

$H_0: b_1 = 0$
$H_a: b_1 = \emptyset$ at $\propto = 5\%$

= T.INV(.05,4) = 2.776
↓
$n-(k+1)$

$$S_{\hat{b}_1} = \frac{\sqrt{11.9678}}{\sqrt{122.64}} = 0.312398 \rightarrow t = \frac{1.25 - 0}{.312398} = 4.00131 \rightarrow \text{Reject } H_0$$

$(t^2 = F,\ 4.00131^2 = 16.0104)$ ← SLR only

Note: $H_0: p = 0$
$H_a: p \neq 0$

$$t = \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \rightarrow \frac{0.8945\sqrt{4}}{\sqrt{1-.8945^2}} = 4.00131 \rightarrow \text{Reject } H_0$$

df $= n-2$

Page 10
LOS f
- formulate
- determine

2/ Hypothesis Tests of $\hat{b}_0$:

$$t = \frac{\hat{b}_0 - B_0}{S_{\hat{b}_0}} \quad \text{and} \quad S_{\hat{b}_0} = \sqrt{\frac{1}{n} + \frac{\bar{X}^2}{\sum_{i=1}^{n}(X_i - \bar{X})^2}}$$

df = $n - (k+1)$

3/ Hypothesis Tests of $\hat{b}_1$ if IV is an $\underbrace{\text{indicator variable}}_{\text{takes on a value of 0 or 1}}$ (dummy var.)

- same process as a test of $\hat{b}_1$

$Y = b_0 + b_1 IND$
- if $IND = 0$, $Y = b_0$ — b_0 = avg. of all 0 obs.
- if $IND = 1$, $Y = b_0 + b_1$ — $b_0 + b_1$ = avg. of all 1 obs.

S_1 → all 0 obs. $\bar{X}_0 = b_0$

S_2 → all 1 obs. $\bar{X}_1 = b_0 + b_1$

difference in means

$$\therefore (\bar{X}_1 - \bar{X}_0) = (b_0 + b_1) - b_0 = b_0 - b_0 + b_1 = b_1$$

∴ test-stat. for b_1 = test-stat. of differences in means

Page 11
LOS f
- formulate
- determine

• Level of Significance and p-values/

- most software output → $\propto = 5\%$, H_0: parameter = 0

recall: =(1 - T.DIST(+t,df,1)) × 2   example #6

LOS g
- calculate
- interpret

• Prediction interval (or CI) for $\hat{Y}$:

$$\hat{Y} = \hat{b}_0 + \hat{b}_1 X$$

$\hat{b}_0$, $\hat{b}_1$ → estimated with error

$(Y_i - \hat{Y}) = \varepsilon$ → estimated with error

2 sources of error

- recall for CI : $\hat{Y} \ {}^{+}\!/\!_{-} \ t_c \bullet S_{\hat{Y}}$

- since there are 2 sources of error and not just one → adjust $S_{\hat{Y}}$

forecast value of the IV

adjusted $S_{\hat{Y}}$: $$S_f^2 = \underbrace{S_e^2}_{\text{SEE}}\left[1 + \frac{1}{n} + \frac{(X_f - \bar{X})^2}{\underbrace{(n-1)S_x^2}_{\sum_{i=1}^{n}(X_i - \bar{X})^2}}\right] \quad \text{and} \quad S_f = \sqrt{S_f^2}$$

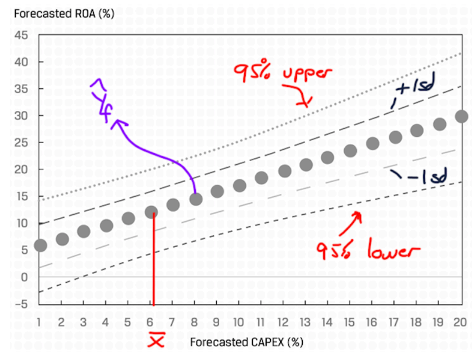
Page 12
LOS g
- calculate
- interpret

- **Prediction interval (or CI) for $\hat{Y}$:**

$$S_f^2 = S_e^2\left[1 + \frac{1}{n} + \frac{(X_f - \overline{X})^2}{(n-1)S_x^2}\right]$$

① ② ③

1. the better the fit of the regression model → lower S_e^2 → lower S_f^2
2. larger n = smaller S_f^2
3. close X_f is to $\overline{X}$ → smaller S_f^2 →



Steps/ Determine $\hat{Y}$
Select α
Determine t_c
Determine S_f
Determine $\hat{Y}_f \ ^{+}/\!- \ t_c \cdot S_f$

example #7

Page 13
LOS h
- describe

1/ Log-lin model

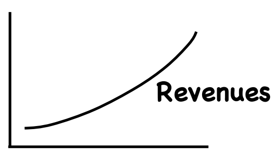
→ $Y = e^{b_0 + b_1 X_i}$

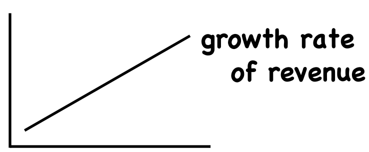
- take the log of both sides

$\ln Y_i = b_0 + b_1 X_i$

↓

relative change in Y for absolute change in X





2/ Lin-Log model

$Y = b_0 + b_1 \cdot \ln(X_i)$

↓

absolute change in Y for relative change in X

- when Y and X are significantly different in scale

exh. #35/36

e.g./ Y = percent X = billions of \$ in Revenue
(transform X with $\ln(X)$)

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LOS h
- describe

3/ Log-Log model $\ln Y_i = b_0 + b_1 \bullet \ln(X_i)$

the relative change in Y_i for a relative change in X_i

- exh. #37/38

- selecting a model depends on goodness of fit
  - R^2
  - F-stat
  - SEE (S_c)

- a plot of the residuals should show randomness and the distribution should be normal
  - if not, consider transforming the DV, IV, or both.

Introduction to Big Data Techniques

a. describe aspects of "fintech" that are directly relevant for the gathering and analyzing of financial data.

b. describe Big Data, artificial intelligence, and machine learning

c. describe applications of Big Data and Data Science to investment management

Intro. to Big Data Techniques

Page 1

Fintech – technological innovation in the design and delivery of financial services and products

- analysis of large datasets
  - traditional data
  - alternative data – social media, sensor networks
- analytical tools – for extremely large datasets → AI → very useful for complex non-linear relationships

Big Data/ data generated from traditional/alternative sources

Characteristics

quality vs. quantity

- Volume – millions to billions of data points
- Velocity – speed of recording/transmitting/generating has accelerated (real time/near real-time)
- Variety – text, image, speech, video
- Veracity – credibility/reliability of different sources

Page 2

semi

- structured – can be organized in tables, database entries
- unstructured – cannot be represented in tabular format (voice, images)

(all data must be structured before use)

Sources/

- financial markets – price/volume
- businesses – filings, press releases
- governments – economic data
- individuals – web footprints, transactional data
- sensors – images, traffic volume
- Internet of Things – smart buildings, appliances

3 main sources of alternative data:

- Individuals (text, photos, audio, website clicks/engagement)
- Business processes (sales, credit card data, corporate exhaust)
  - supply chain info.
  - PoS scanner data
  - etc...
- Sensor data (smartphones, RFID chips)
  → IoT – array of physical devices

Page 3

- Ethical/Legal issues → use of personal info./data scraped from web data
- Challenges
  - quality
  - volume
  - appropriateness
  - selection bias
  - missing data
  - outliers
  - data cleansing/organizing
- AI – artificial intelligence – systems that exhibit cognitive and decision-making ability comparable or superior to humans
  - from expert systems to neural networks
    (if-then rules) (based on how brains learn and process info.)
- Machine learning – seek to extract knowledge from large amounts of data without assumptions of the data's underlying probability distribution
  - learn from known examples to determine underlying structure in the data – generate structure without help from a human

Page 4

- requires large training datasets

training data	validation data	test data

test data → test ability to predict on new data

training data ↓ labelled data – inputs, outputs

$\underbrace{\text{labelled data – inputs, outputs}}_{\text{supervised learning}}$

validation data ↓ model validated and tuned

- over/under-fit – will not perform well out-of-sample
- Unsupervised learning – only inputs, no outputs
- Deep learning – multi-stage, non-linear data processing (supervised or unsupervised)
- Data Science – comp. science + statistics to extract info. from big data

Page 5

Data processing methods/

- Capture - low latency systems (very little delay) vs. high latency
- Curation - ensuring data quality/accuracy
  - detect errors and make adjustments for missing data
- Storage - record/archive/access in databases
- Search - how data are queried
- Transfer - movement of data from storage to analytical application

Data Visualization/

- how data is formatted/displayed/summarized in graphical format
  (e.g. heat maps, tree diagrams, network graphs)
- tag clouds (text data) - words are sized/displayed based on frequency

Page 6

- Text analytics - use large unstructured voice or text datasets
- Natural language processing (NLP) - comp. science + AI + linguistics
  - analyze/interpret human language
    (e.g. translation, speech recognition)