THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
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End Term Exam (Part-2) **Mathematics-I**

SOLUTIONS

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1. (a) Let S be a nonempty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$. If $\beta = \inf S$, then show that for every $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$. **[2 marks]**

Ans. i. Let $\epsilon > 0$ be given. Suppose there is no $x \in S$ such that $\beta + \epsilon > x$. **[1 marks]**

ii. Then $\beta < \beta + \epsilon \leq x$ for all $x \in S$. This contradicts that β is greatest lower bound. **[1 marks]**

(b) Let S be a nonempty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$ be a lower bound. If for every $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$ then show that $\beta = \inf S$ **[3 marks]**

Ans. Need to show that if m is any lower bound for S then $\beta \geq m$.

i. Assume contrary, Let m be a lower bound for S such that $\beta < m$. **[1 mark]**

ii. Then for $\epsilon = m - \beta > 0$, there is a $x \in S$ such that $\beta + (m - \beta) > x$. That is $m > x$. **[1 mark]**

iii. This contradicts that m is a lower bound for S. **[1 mark]**

(c) Let (a_k) be a real sequence such that (a_{2k}) and (a_{2k-1}) diverges to $+\infty$. Then Show that (a_k) diverges to $+\infty$. **[5 marks]**

Ans. i. Let $M \in \mathbb{R}$ be given. **[1 mark]**

ii. Then there exist positive integers k_1 and k_2 such that

$$\begin{aligned} k \geq k_1 &\implies a_{2k} \geq M \\ k \geq k_2 &\implies a_{2k-1} \geq M. \qquad \textbf{[1mark]} \end{aligned}$$

iii. Set $n_0 := \max\{2k_1, 2k_2 - 1\}$. **[1 mark]**

iv. Then we claim $n \geq n_0 \implies a_n \geq M$. **[2 marks]**

If $n \geq n_0$ is odd integer then $n = 2k - 1$ for some $k \in \mathbb{N}$. Also $n \geq n_0 \implies 2k - 1 \geq 2k_2 - 1 \implies k \geq k_2$. Hence $a_n = a_{2k-1} \geq M$

If $n \geq n_0$ is an even integer then $n = 2k$ for some $k \in \mathbb{N}$. Also $n \geq n_0 \implies 2k \geq 2k_1 \implies k \geq k_1$. Hence $a_n = a_{2k} \geq M$

2. (a) For $k \in \mathbb{N}$, let $a_{2k-1} := \dfrac{1}{4^k}$ and $a_{2k} := \dfrac{1}{9^k}$. Discuss the convergence/divergence of the series $\displaystyle\sum_{k=1}^{\infty} a_k$.

[3 marks]

Ans. i. Note that $|a_{2k-1}|^{\frac{1}{2k-1}} = \left(\dfrac{1}{2^{2k}}\right)^{\frac{1}{2k-1}} = \dfrac{1}{2}\left(\dfrac{1}{2}\right)^{\frac{1}{2k-1}}$, $\quad |a_{2k}|^{\frac{1}{2k}} = \left(\dfrac{1}{3^{2k}}\right)^{\frac{1}{2k}} = \dfrac{1}{3} \;\; \forall k \in \mathbb{N}$ **[1 mark]**

ii. Hence $|a_k|^{\frac{1}{k}} \leq \dfrac{1}{2}$ for all $k \geq 1$. **[1 mark]**

iii. By the root Test, $\displaystyle\sum_k a_k$ is absolutely convergent. **[1 mark]**

(b) Let D be a non-empty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous. Prove that f is continuous on D. **[2 marks]**

Ans. i. Let $\epsilon > 0$ be given and $c \in D$. **[1 mark]**

ii. By uniform continuity of f over D, there is $\delta > 0$ such that

$$x \in D \text{ and } |x - c| < \delta \implies |f(x) - f(c)| < \epsilon.$$

[1 mark]

(c) Let r be a nonnegative rational number and consider $a_n := \sum_{k=1}^{n} \frac{k^r}{n^{r+1}}$ for $n \in \mathbb{N}$. Determine the limit of the sequence (a_n) by expressing the nth term as a Riemann sum for a suitable function. **[5 marks]**

Ans. i. $a_n = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^r = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^r\left(\frac{i}{n} - \frac{i-1}{n}\right)$ **[2 marks]**

ii. Consider $f : [0,1] \to \mathbb{R}$ defined by $f(x) := x^r$ and let $P_n := \left\{0, \frac{1}{n}, \cdots, \frac{n}{n}\right\}$ and $s_{n,i} := \frac{i}{n}$ for $n \in \mathbb{N}$ and $i = 1, \cdots, n$, then $a_n = S(P_n, f)$. **[2 marks]**

iii. Since $\mu(P_n) = \frac{1}{n} \to 0$, so we have $a_n = S(P_n, f) \to \int_0^1 x^r dx = \frac{1}{1+r}$ as $n \to \infty$ **[1 mark]**

3. (*a*) *Viviani's curve* is the intersection of the cylinder $(x-\alpha)^2 + y^2 = \alpha^2$ and the sphere $x^2+y^2+z^2 = 4\alpha^2$ and has parametric equation: **[5 Marks]**

$$\alpha : [0, 4\pi] \longrightarrow \mathbb{R}^3 : t \mapsto \alpha\left(1 + \cos t, \sin t, 2\sin\frac{t}{2}\right).$$

Show that the curvature and torsion of this curve are given by

$$\kappa(t) = \frac{\sqrt{13 + 3\cos t}}{\alpha(3 + \cos t)^{3/2}}, \qquad \tau(t) = \frac{6\cos\frac{t}{2}}{\alpha(13 + 3\cos t)}.$$

Ans. The curve in the parametric form is: $r(t) = \alpha\left(1 + \cos t, \sin t, 2\sin\frac{t}{2}\right)$.
The curvature (κ) and the torsion (τ) are given by

$$\kappa = \frac{||v \times a||}{||v||^3}, \qquad \tau = \frac{\begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix}}{||v \times a||^2}.$$ **[1 Mark]**

Here, $v = \frac{dr}{dt} = \alpha\left(-\sin t, \cos t, \cos\frac{t}{2}\right), a = \frac{dv}{dt} = \alpha\left(-\cos t, -\sin t, -\frac{1}{2}\sin\frac{t}{2}\right)$. **[1 Mark]**
Solving we get

$$v \times a = \alpha^2\left[\left(-\frac{1}{2}\cos t \sin\frac{t}{2} + \sin t \cos\frac{t}{2}\right)i + \left(\frac{1}{2}\sin t \sin\frac{t}{2} + \cos t \cos\frac{t}{2}\right)j + k\right].$$

$$\implies ||v \times a|| = \frac{\alpha^2\sqrt{13 + 3\cos t}}{2\sqrt{2}}, \quad ||v|| = \frac{\alpha\sqrt{3 + \cos t}}{\sqrt{2}}, \quad \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \\ \dddot{x} & \dddot{y} & \dddot{z} \end{vmatrix} = \frac{3}{4}\alpha^3\cos\frac{t}{2}.$$ **[2 Marks]**

Therefore, we obtain $\kappa = \frac{\sqrt{13+3\cos t}}{\alpha(3+\cos t)^{3/2}}$ and $\tau = \frac{6\cos\frac{t}{2}}{\alpha(13+3\cos t)}$. **[1 Mark]**

(*b*) Let $f(x, y, z) = x^2 + y^2 + z^2$. Prove f is differentiable at $(1, 1, 1)$ with linear transformation $T(x, y, z) = 2x + 2y + 2z$ as its derivative. **[3 Marks]**
(*Hint: One possibility is to show that the error function $\epsilon(H)$ tends to zero as $||H||$ tends to zero.*)

Ans. To prove f is differentiable with total derivative T as described we need to show that

$$\epsilon(H) = \frac{f(X_0 + H) - f(X_0) - T(H)}{||H||} \to 0 \quad \text{as} \quad ||H|| \to 0.$$

Here, $X_0 = (1, 1, 1)$ and $H = (h_1, h_2, h_3)$. **[1 Mark]**
Note that

$$\begin{aligned} f(X_0 + H) - f(X_0) - T(H) &= f(1 + h_1, 1 + h_2, 1 + h_3) - f(1, 1, 1) - (2h_1 + 2h_2 + 2h_3) \\ &= (1 + h_1)^2 + (1 + h_2)^2 + (1 + h_3)^2 - 3 - 2h_1 - 2h_2 - 2h_3 \\ &= h_1^2 + h_2^2 + h_3^2 = ||H||^2. \qquad [\textbf{1 Mark}] \end{aligned}$$

Therefore,

$$\epsilon(H) = \frac{||H||^2}{||H||} = ||H|| \to 0 \quad \text{as} \quad ||H|| \to 0.$$

Hence, f is differentiable at $(1, 1, 1)$. **[1 Mark]**

(*c*) A mosquito is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The mosquito is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly? **[2 Marks]**

Ans. Given that the temperature is $T(x, y, z) = x^2 + y^4 + 2z^2$. We know that the direction of the maximum increase in the temperature at $X_0 = (1, 1, 1)$ is $\frac{\nabla T}{||\nabla T||}\Big|_{X_0}$. **[1 Mark]**
Therefore, the desired direction is

$$u = \frac{(2x, 4y^3, 4z)}{\sqrt{4x^2 + 64y^6 + 64z^2}}\Bigg|_{(1,1,1)} = \frac{1}{\sqrt{33}}(1, 2, 1). \qquad [\textbf{1 Mark}]$$

4. (*a*) Many airlines require that carry-on luggage have a linear distance (sum of length, width, height) of no more than 45 inches with an additional requirement of being able to slide under the seat in front of you. Assuming that the carry-on is to have the shape of a rectangular box and one dimension is half of one of the other dimensions (to insure "slide under seat" is possible), what dimensions of the carry-on will lead to maximum storage (i.e., maximum volume)? **[6 Marks]**
[Hint: *Apply Lagrange's multiplier method with more than one constraints*]

Ans. If we let x, y and z denote length, width, and height, respectively of the box. Then our goal is to maximize the volume

$$V(x, y, z) = xyz \tag{1}$$

subject to the constraints

$$f(x, y, z) = x + y + z - 45 = 0 \tag{2}$$

$$\text{and} \quad g(x, y, z) = y - 2x = 0. \qquad [\textbf{1 Mark}] \tag{3}$$

Using the method of Lagrange's multiplier we have

$$\nabla V = \lambda \nabla f + \mu \nabla g. \tag{4}$$

Solving (4) for λ and μ, we get

$$yz = \lambda - 2\mu, \quad xz = \lambda + \mu, \quad xy = \lambda. \qquad [\textbf{2 Marks}]$$

Combining the last two equations yields $xz = xy + \mu$, so that the first equation becomes

$$yz = xy - 2(xz - xy) \quad \text{or} \quad yz = 3xy - 2xz.$$

Since $y = 2x$, this becomes

$$2xz = 6x^2 - 2xz \quad \text{or} \quad 4xz = 6x^2.$$

Since $x = 0$ leads to a zero volume, we must have $2z = 3x$, or $z = 1.5x$. **[1 Mark]**
Substituting into (2) yields

$$x + 2x + 1.5x = 45 \Longrightarrow x = 10.$$

If $x = 10$, then $y = 2x = 20$ and $z = 1.5x = 15$, so that the critical point is $(10, 20, 15)$. **[1 Mark]**
Since x, y and z must all be in $[0, 45]$, we are seeking the extrema of the volume over a closed set (in particular, the closed box $[0, 45] \times [0, 45] \times [0, 45]$) and the volume is zero on the boundary. Thus, the maximum volume must occur, and the only place left for it to occur is at the critical point $(10, 20, 15)$. **[1 Mark]**

(*b*) Suppose S is a "*light-bulb-shaped region*" shown in the Figure. Imagine a light-bulb cut off at the

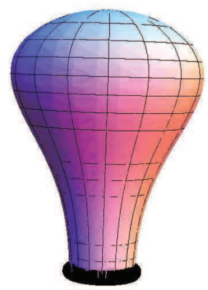

Figure 1: Bulb-shaped region

base so that its boundary is the unit circle $x^2 + y^2 = 1$, oriented with the outward-pointing normal. Suppose $F = (xe^{z^2-2z}, 1 + y + \sin(xyz), e^{z^2}\sin(z^2))$. Compute the flux integral $\int\int_S curl\ F \cdot d\sigma$ using Stokes' theorem. **[4 Marks]**

Ans. The point of this problem is to use Stokes' theorem to avoid computing the flux integral over S (whatever confusing surface that could be) and instead compute the line integral over the unit circle C in the $xy-$plane. We use the parameterization $r(t) = (\cos t, \sin t, 0)$, $0 \leq t \leq 2\pi$ so that

$$F(r(t)) = (\cos t, 1 + \sin t, 0) \quad \text{and} \quad dr = (-\sin t, \cos t, 0)dt. \qquad \textbf{[2 Marks]}$$

Therefore,

$$F.dr = \cos t dt \qquad \textbf{[1 Mark]}$$

Therefore, by Stokes theorem

$$\int\int_S curl\ F \cdot d\sigma = \int_C F \cdot dr = \int_0^{2\pi} \cos t\, dt = 0. \qquad \textbf{[1 Mark]}$$

5. (*a*) Compute the integral

$$\int_C \left(5y + \sqrt{1 + x^5}\right) dx + \left(5x - e^{y^2}\right) dy,$$

where C is a circle of radius 5 centered at the origin. **[3 Marks]**

Ans. We apply Green's theorem to compute the line integral. Here, $M(x, y) = 5y + \sqrt{1 + x^5}$, $N = 5x - e^{y^2}$. Therefore,

$$\frac{\partial M}{\partial y} = 5, \qquad \frac{\partial N}{\partial x} = 5. \qquad \textbf{[1 Mark]}$$

By Green's theorem,

$$\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy, \qquad \text{[1 Mark]}$$

Thus,

$$\int_C \left(2y + \sqrt{1+x^5}\right) dx + \left(5x - e^{y^2}\right) dy = \iint_R (5-5) dx dy = 0 \times Area(R) = 0. \qquad \text{[1 Mark]}$$

(b) In this problem S is the surface given by the quarter of the right-circular cylinder centered on the $z-$axis, of radius 2 and height 4, which lies in the first octant. A vector field $F(x,y,z) = y\hat{j}$ is defined on S.
(i) Compute the flux integral $\int\int_S F \cdot \hat{n}\, d\sigma$.
(*Hint: Use the normal which points outward from S, i.e. on the side away from the $z-$axis.*)
(ii) D be the solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $F = y\hat{j}$ out of D. **[4+3 Marks]**

Ans. (i) The surface S is given in Figure 2 and defined by

$$S : \{(x,y,z) : x^2 + y^2 = 4, \quad 0 \le z \le 4\}.$$

In parametric form

$$S : r(\theta, z) = (2\cos\theta, 2\sin\theta, z) \quad 0 \le \theta \le \pi/2, \quad 0 \le z \le 4. \qquad \text{[1 Mark]}$$

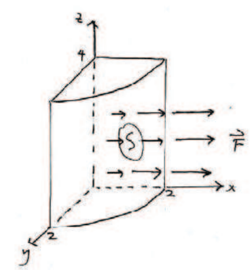

Figure 2: Surface S

[1 Mark]

Therefore,

$$\hat{n} = \frac{r_\theta \times r_z}{||r_\theta \times r_z||} = (\cos\theta, \sin\theta, 0) \quad \& \quad F = (0, 2\sin\theta, 0).$$

Moreover on S, we have $d\sigma = ||r_\theta \times r_z|| d\theta dz = 2 d\theta dz$ and $F.\overrightarrow{n} = 2\sin^2\theta$. **[1 Mark]**
Hence,

$$\iint_S F \cdot \hat{n}\, d\sigma = \int_0^4 \int_0^{\frac{\pi}{2}} 2\sin^2\theta\, 2 d\theta dz = 2 \int_0^4 dz \int_0^{\frac{\pi}{2}} (1 - \cos 2\theta) d\theta$$

$$= 8 \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{\frac{\pi}{2}} = 4\pi. \qquad \text{[1 Mark]}$$

(ii) From the divergence theorem we know that the flux of the field going out of D is given by

$$\iint_S F \cdot \hat{n}\, d\sigma = \iiint_D divF\, dV. \qquad \textbf{[1 Mark]}$$

Now, $divF = \nabla \cdot F = 1$. Therefore,

$$\text{Flux} = \iiint_D 1\, dV = \text{Volume } (D) = \frac{1}{4}\pi \times 2^2 \times 4 = 4\pi. \qquad \textbf{[2 Marks]}$$

✠ **End** ✠