LNMIIT/B.Tech/C/IC/2019-20/ODD/MTH102/Q

THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT OF MATHEMATICS
MATHEMATICS-1 & MTH102
QUIZ-I

Time: 35 minutes | Date: 05/09/2019 | Maximum Marks: 10

Note: You should attempt all questions.

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1. Let $A := \left\{ \frac{1}{3} + \frac{n}{3n+1} \,\middle|\, n \in \mathbb{N} \right\}$. Find $\sup A$. Justify your answer. **[3 marks]**

Solution: Note that the sequence $a_n := \frac{n}{3n+1}$ is increasing, because

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{3n+4} \times \frac{3n+1}{n} = \frac{3n^2+4n+1}{3n^2+4n} > 1. \ \ \forall n \geq 1.[1 marks]$$

Also $a_n < \frac{n}{3n} = \frac{1}{3}$ for each $n \geq 1$. Hence the supremum of A is the limit of the sequence $\frac{1}{3} + a_n$, which is $\frac{2}{3}$. [2 marks]

e Solution: Claim: $\sup A = \frac{2}{3}$. [0.5 marks]

(a) Since $\frac{1}{3} + \frac{n}{3n+1} < \frac{1}{3} + \frac{n}{3n} = \frac{2}{3}$ for each $n \geq 1$, hence $\frac{2}{3}$ is an upper bound for the set A. [0.5 marks]

(b) Let $\epsilon > 0$ be given.

i. If $\epsilon \geq \frac{1}{3}$, Then for each $x \in A$, we have $\frac{2}{3} - \epsilon \leq \frac{1}{3} < x$

ii. If $\epsilon < \frac{1}{3}$ Then by Archemidean property there exist $n \in \mathbb{N}$ such that $n > \frac{1-3\epsilon}{9\epsilon}$. This implies $9n\epsilon > 1 - 3\epsilon \implies 9n\epsilon + 3\epsilon > 1 \implies \epsilon > \frac{1}{3(3n+1)}$
Now

$$\frac{2}{3} - \epsilon = \frac{1}{3} + \left(\frac{1}{3} - \epsilon\right) < \frac{1}{3} + \left(\frac{1}{3} - \frac{1}{3(3n+1)}\right) = \frac{1}{3} + \frac{1}{3}\left(1 - \frac{1}{3n+1}\right) = \frac{1}{3} + \frac{n}{3n+1}$$

[2 marks]

2. Investigate the convergence/divergence of the following sequence
$x_n := \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}, \quad n \in \mathbb{N}$. **[3 marks]**

Solution Note that

$$\frac{1+2+\ldots+n}{n+n^2} \leq x_n \leq \frac{1+2+\ldots+n}{1+n^2}.[1 marks]$$

$$\frac{1}{2} = \frac{\frac{n(n+1)}{2}}{n+n^2} \leq x_n \leq \frac{\frac{n(n+1)}{2}}{1+n^2}.[1 marks]$$

By Sandwich theorem $x_n \to \frac{1}{2}$ [1 marks].

3. (a) Check the convergence of the series $\sum_{n=1}^{\infty} ne^{-n^2}$ using ratio test. **[2 marks]**

Solution

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)^2}}{ne^{-n^2}}[0.5marks]$$

$$= \lim_{n\to\infty} \frac{(n+1)e^{-n^2-2n-1}}{ne^{-n^2}} = \lim_{n\to\infty}(1+\frac{1}{n})e^{-2n-1} = 0.[1marks]$$

Since $L < 1$, the series converges. [0.5 marks]

(b) Check the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} log\left(1+\frac{1}{\sqrt{n}}\right)$. **[2 marks]**

Solution Take $a_n = \frac{1}{\sqrt{n}}log\left(1+\frac{1}{\sqrt{n}}\right)$ and $b_n = \frac{1}{n}$. Then

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}log\left(1+\frac{1}{\sqrt{n}}\right)}{\frac{1}{n}} = \lim_{x\to 0+} \frac{log(1+\sqrt{x})}{\sqrt{x}} = \lim_{x\to 0+} \frac{1}{1+\sqrt{x}}.[1.5marks]$$

Therefore $L = 1$ and since $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent, by limit comparison test, the given series is divergent. [0.5 marks]