LNMIIT/B.Tech/C/IC/2018-19/ODD/MTH213/MT

The LNM Institute of Information Technology, Jaipur
Mathematics-III
Mid Term

Duration: 90 mins. Max.Marks: 50

Name: ________________ Roll No.: ________________

NOTE: You should attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **Start a new question on a new page and answer all its parts in the same place.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format.

Question No.				
Page No.				

1. (a) For any two complex numbers z_1 and z_2, prove that $|z_1 + z_2| \leq |z_1| + |z_2|$. **[4]**

Proof.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\
&= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\
&= z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2} \\
&= |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_2|^2 \\
&= |z_1|^2 + (z_1\overline{z_2} + \overline{z_1\overline{z_2}}) + |z_2|^2 \\
&= |z_1|^2 + 2\ \mathrm{Re}(z_1\overline{z_2}) + |z_2|^2. \qquad ....(A)
\end{aligned}$$

Using Re $z \leq |\mathrm{Re}\ z| \leq |z|$.
we obtain

$$\begin{aligned}
|z_1 + z_2|^2 &= |z_1|^2 + 2\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2 \\
&\leq |z_1|^2 + 2|z_1\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||\overline{z_2}| + |z_2|^2 \\
&= |z_1|^2 + 2|z_1||z_2| + |z_2|^2 \\
&= (|z_1| + |z_2|)^2,
\end{aligned}$$

and so $|z_1 + z_2| \leq |z_1| + |z_2|$.

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(b) Find all the fifth roots of 32 and locate them geometrically. **[4]**

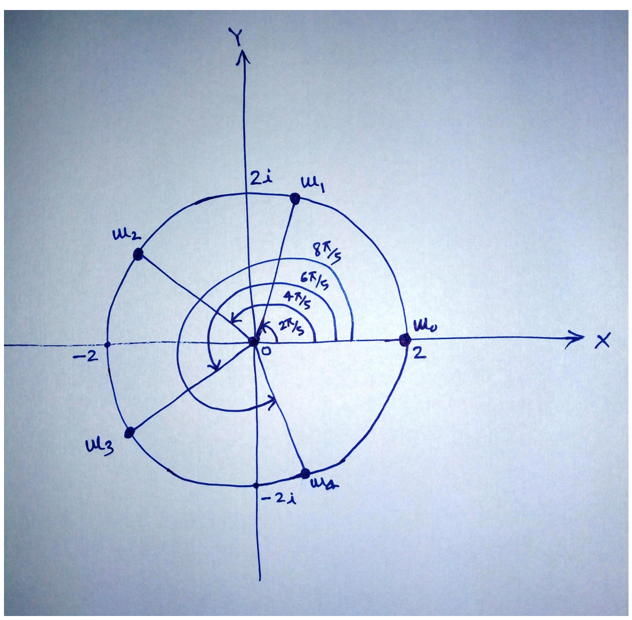
Solution. $32 = 32(\cos 0 + i \sin 0)$. Hence

$$\begin{aligned}\sqrt[5]{32} &= 32^{1/5}\left[\cos\left(\frac{0+2k\pi}{5}\right) + i\sin\left(\frac{0+2k\pi}{5}\right)\right], \quad k = 0,1,2,3,4.\\ &= 2\left[\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)\right], \quad k = 0,1,2,3,4.\end{aligned}$$

Thus the fifth roots of 32 are w_k, $k = 0,1,2,3,4$ given by

$$\begin{aligned}w_0 &= 2(\cos 0 + i\sin 0) = 2(1) = 2,\\ w_1 &= 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right),\\ w_2 &= 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right),\\ w_3 &= 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right),\\ w_4 &= 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right).\end{aligned}$$

These roots are located geometrically on the circle with radius 2 and centre O as shown in the picture.



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2. (a) Let $f(z) = x^3 + i(1-y)^3$. Find all the points where the function is differentiable. Also find the derivatives at all those points. For which value of z, $f(z)$ is analytic? **[3]**

Solution. Let $f(z) = x^3 + i(1-y)^3 = u(x,y) + iv(x,y)$, where $u(x,y) = x^3$ and $v(x,y) = (1-y)^3$. Observe that

$$u_x = v_y \Rightarrow 3x^2 = -3(1-y)^2 \Rightarrow x^2 + (1-y)^2 = 0 \Rightarrow x = 0 \text{ and } y = 1.$$

Also $u_y = -v_x \Rightarrow 0 = 0$.

Thus the Cauchy-Riemann equations are satisfied only when $x = 0$ and $y = 1$. That is, they hold only when $z = i$. Hence the given function can not be differentiable at any point $z \neq i$, and so it can not be analytic also at the points $z \neq i$.
Since the first order partial derivatives of u and v, i.e. u_x, u_y, v_x, v_y exist in a neighborhood of $z = i$ and these partial derivatives are continuous at $(0,1)$ and satisfy the Cauchy-Riemann equations at $z = i$, hence $f'(i)$ exists.

$$f'(i) = u_x(0,1) + iv_x(0,1) = 0 + i0 = 0.$$

Since f is differentiable only at $z = i$ and not at all points in some neighborhood of i, it is not analytic at $z = i$. Thus $f(z)$ is not analytic at any point z.

(b) Let $u(x,y) = 2x(1-y)$. Show that $u(x,y)$ is harmonic in some domain and find its harmonic conjugates. **[4]**

Solution. $u(x,y) = 2x(1-y) \Rightarrow u_x = 2(1-y) \Rightarrow u_{xx} = 0$. Also $u_y = -2x \Rightarrow u_{yy} = 0$. Thus $u_{xx} + u_{yy} = 0$, and so u is harmonic.
Let $v(x,y)$ be harmonic conjugate of $u(x,y)$. Then $u_x = v_y \Rightarrow v_y = 2(1-y) \Rightarrow v(x,y) = 2y - y^2 + \phi(x)$, where $\phi(x)$ is a function of x.
Now $u_y = -v_x \Rightarrow -2x = -\phi'(x) \Rightarrow \phi'(x) = 2x$, and so $\phi(x) = x^2 + c$, where c is an arbitrary constant. Consequently,

$$v(x,y) = 2y - y^2 + x^2 + c,$$

where c is an arbitrary constant.

3. (a) Write all possible Laurent series expansion of $f(z) = \frac{1}{(z-1)^2(z-3)}$ in powers of $(z-1)$. **[5]**

Solution. If $0 < |z-1| < 2$, then $|\frac{z-1}{2}| < 1$. Hence

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^2(z-3)} \\ &= \frac{1}{(z-1)^2(z-1-2)} \\ &= \frac{1}{(z-1)^2(-2)\left(1-\frac{z-1}{2}\right)} \\ &= \frac{-1}{2(z-1)^2}\sum_{k=0}^{\infty}\frac{(z-1)^k}{2^k} \\ &= \sum_{k=0}^{\infty}\frac{-(z-1)^{k-2}}{2^{k+1}}. \end{aligned}$$

If $2 < |z-1| < \infty$, then $|\frac{2}{z-1}| < 1$. Hence

$$\begin{aligned} f(z) &= \frac{1}{(z-1)^2(z-3)} \\ &= \frac{1}{(z-1)^2(z-1-2)} \\ &= \frac{1}{(z-1)^2(z-1)\left(1-\frac{2}{z-1}\right)} \\ &= \frac{1}{(z-1)^3}\sum_{k=0}^{\infty}\frac{2^k}{(z-1)^k} \\ &= \sum_{k=0}^{\infty}\frac{2^k}{(z-1)^{k+3}}. \end{aligned}$$

(b) For any two complex numbers z_1 and z_2, prove that

$$2\sin z_1 \cos z_2 = \sin(z_1+z_2) + \sin(z_1-z_2).$$

[2]

Proof.

$$\begin{aligned} 2\sin z_1\cos z_2 &= 2\left(\frac{e^{iz_1}-e^{-iz_1}}{2i}\right)\left(\frac{e^{iz_2}+e^{-iz_2}}{2}\right) \\ &= \frac{1}{2i}\left[e^{i(z_1+z_2)}+e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}-e^{-i(z_1+z_2)}\right] \\ &= \frac{e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}}{2i}+\frac{e^{i(z_1-z_2)}-e^{-i(z_1-z_2)}}{2i} \\ &= \sin(z_1+z_2)+\sin(z_1-z_2). \end{aligned}$$

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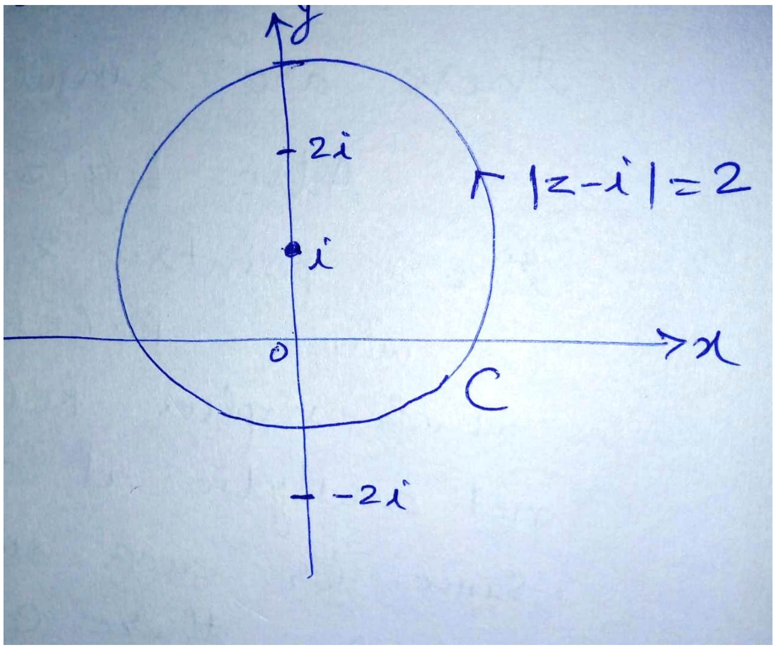
4. (a) Using Cauchy integral formula, evaluate the contour integrals $\int_C \frac{1}{(z^2+4)^2}dz$, where C is the circle $|z-i| = 2$ in positive directions. **[3]**

Solution. Let $f(z) = \frac{1}{(z^2+4)^2}$. Then $f(z)$ is not analytic at those points z such that $z^2 + 4 = 0$, i.e. at $z = 2i$ and $z = -2i$. But only $z = 2i$ lies inside C.



Hence applying the Cauchy integral formula, we have

$$\begin{aligned}\int_C \frac{1}{(z^2+4)^2}dz &= \int_C \frac{1}{(z-2i)^2(z+2i)^2}dz \\ &= \int_C \frac{\frac{1}{(z+2i)^2}}{(z-2i)^{1+1}}dz \\ &= \frac{2\pi i}{1!}\left[\frac{d}{dz}\frac{1}{(z+2i)^2}\right]_{z=2i} \\ &= 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} \\ &= \frac{-4\pi i}{(4i)^3} = \frac{-4\pi i}{-64i} = \frac{\pi}{16}.\end{aligned}$$

(b) Find all the singular points of $f(z) = \frac{\text{Log}(z+2)}{(z-4)(z-5)}$. Classify them as non-isolated, isolated, poles, removable and essential singularity. **[3]**

Proof $z = 4, 5$ are isolated singularities and simple poles. The points $z = x + iy$, where $x \leq -2$ and $y = 0$ are non-isolated singularities.

Detail Solution. Clearly the function $f(z) = \frac{\text{Log}(z+2)}{(z-4)(z-5)}$ is not defined at $z = 4$ and $z = 5$. Hence $f(z)$ is not analytic at $z = 4$ and $z = 5$, and so these are singular points of $f(z)$, which are clearly simple poles.

Also $\text{Log}(z+2)$ is not analytic at all those points z on the negative real axis for which $\text{Re}(z+2) \leq 0$, i.e. at the points $z = x + iy$, where $x \leq -2$ and $y = 0$. Since for each of these points $z = x$, where $x \leq -2$, in any neighborhood of $z = x$, $\text{Log}(z+2)$ is not analytic at the points on the negative real axis lying in that neighborhood. Hence these points are non-isolated singular points.

Now, for $z = 4$ and $z = 5$, there are deleted neighborhoods $0 < |z-4| < \frac{1}{2}$ and $0 < |z-5| < \frac{1}{2}$ in which $f(z)$ is analytic. Hence $z = 4$ and $z = 5$ are isolated singular points.

5. (a) Suppose $f(z)$ is an entire function such that and $|f(z)| \leq |z|$. Using Cauchy's inequality prove that $|f'(z)|$ is bounded. Then prove that $f'(z)$ is constant. If $f(1) = 1$ and $f(i) = 2$, Find $f(z)$. **[4]**

Proof Let z_0 be a fixed complex number and R is any positive real number. Consider the circle $z = z_0 + Re^{it}$, $0 \leq t \leq 2\pi$. Then if z is on the circle

$$|f(z)| = |f(z_0 + Re^{it})| \leq |z_0 + Re^{it}| \leq |z_0| + R = M_R(say)$$

Then by Cauchy's inequality $|f'(z_0)| \leq \dfrac{|z_0| + R}{R}$ for any R.
taking $R \to \infty$, we get $|f'(z_0)| \leq 1$. Since z_0 was arbitrary, $|f(z)| \leq 1$ for all z. Hence $f'(z)$ is bounded.
Since $f(z)$ is analytic, $f'(z)$ is also analytic.
By Liouville's theorem $f'(z)$ is constant.
Let $f'(z) = a$, a is constant.
Then $f(z) = az + b$, b is constant.
Eliminating a and b by given equations, we get $f(z) = \dfrac{1+i}{2}z + \dfrac{1-i}{2}$.

(b) If $f(z)$ is real-valued and analytic function defined on a domain, then prove that $f(z)$ is constant. **[3]**

Proof let $f(z) = u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are real valued functions.
Since $f(z)$ is real valued $v(x, y) = 0$. So $v_x(x, y) = 0$ and $v_y(x, y) = 0$. Since $f(z)$ is analytic, we have $u_x(x, y) = v_y(x, y) = 0$ for all points in the domain. So at the points in the domain, we have

$$f'(z) = u_x(x, y) + iv_x(x, y) = 0$$

. So $f'(z) = 0$ on the domain
So $f(z)$ is constant.

6. (a) Using M-L Inequality, find an upper bound of

$$\left|\oint_C \frac{z^2 e^{(z+1)}}{z+1} dz\right|$$

where C is the circle $|z| = 4$. **[4]**

Proof $L = 8\pi$
$|f(z)| = \left|\dfrac{z^2 e^{z+1}}{z+1}\right|$
$\leq \dfrac{|z^2||e^{z+1}|}{|z| - 1}$
$= \dfrac{16.e^{x+1}}{3}$
$\leq \dfrac{16e^5}{3} = M.$
So $\left|\oint_C \frac{z^2 e^{(z+1)}}{z+1} dz\right| \leq \dfrac{128e^5\pi}{3}.$

(b) Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} \frac{(z-4-3i)^k}{5^{2k}}$$

[3]

Proof $R = 25$

7. (a) Find $\int_C ze^{-\frac{1}{(z-2)}}$ where C is any positively oriented closed contour with $z = 2$ inside it. **[3]**

Solution Here $z = 2$ is the only essential singularity of $f(z) = ze^{-\frac{1}{(z-2)}}$. The Laurent series expansion of $ze^{-\frac{1}{(z-2)}}$ is
$(z-2)e^{-\frac{1}{(z-2)}}+2e^{-\frac{1}{(z-2)}} = (z-2)\left[1 - \frac{1}{z-2} + \frac{1}{2!(z-2)^2} - \frac{1}{3!(z-2)^3} + \ldots\right]+2\left[1 - \frac{1}{z-2} + \frac{1}{2!(z-2)^2} - \frac{1}{3!(z-2)^3} + \ldots\right]$.
So the coefficient of $\frac{1}{z-2}$ is $b_1 = Res_{z=2}f(z) = \frac{-3}{2}$. So $\int_C ze^{-\frac{1}{(z-2)}} = -3\pi i$.

(b) Evaluate $I = \int_0^{2\pi} \frac{1}{(2+\cos\theta)^2} d\theta$. **[5]**

Solution Put $z = e^{i\theta}$.
$\cos\theta = \frac{1}{2}(z + \frac{1}{z})$ and $d\theta = \frac{dz}{iz}$.
$I = \dfrac{1}{i}\displaystyle\int_C \frac{4z}{(z^2+4z+1)^2}dz$, where C is the unit circle with counterclockwise direction.
Let $f(z) = \dfrac{4z}{(z^2+4z+1)^2}dz$.
$f(z)$ has two poles at $z_1 = -2+\sqrt{3}$ and $z_2 = -2-\sqrt{3}$ both of order 2.
Only $z = -2+\sqrt{3}$ lies inside the unit circle C.
So $Res_{z=z_1}f(z) = \dfrac{2\sqrt{3}}{9}$.
So $I = \dfrac{1}{i} \times 2\pi i \times \dfrac{2\sqrt{3}}{9} = \dfrac{4\sqrt{3}\pi}{9}$.