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The LNM Institute of Information Technology, Jaipur
Department of Mathematics
Mathematics-III MTH213
End Term Exam: Hints & Solutions

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1. (a) Let f be an entire function whose real part is $x^2 + xy - y^2$. Find $f'(z)$. [4]

   Solution: $f'(z) = u_x + iv_x$. Since $f(z)$ is entire, $v_x = -u_y$. Hence $u_x = 2x + y$ and $v_x = 2y - x$. Thus $f'(z) = 2x + y + i(2y - x)$.

   (b) Find all values of $z \in \mathbb{C}$, which satisfy the equation

   $$(1 - z)^{10} = 2^{10}$$

   [3]

   Solution: $2 = 2e^{i0}$. For a complex number z, $(1-z)^{10} = 2^{10}e^{i0}$. Hence $1-z = 2e^{i\frac{0+2k\pi}{10}}$, $k = 0, 1, \ldots, 9$. Thus $z = 1 - 2e^{i\frac{0+2k\pi}{10}}$, $k = 0, 1, \ldots, 9$.

   (c) Determine the region where the function $f(z) = e^{\overline{z}}$ is analytic. [3]

   Solution: Consider $f(z) = e^{\overline{z}} = e^x(\cos y - i \sin y)$.
   Then real and imaginary parts are $u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$.
   This imply $u_x(x, y) = e^x \cos y$ and $v_x(x, y) = -e^x \sin y$ and $u_y(x, y) = -e^x \sin y$ and $v_y(x, y) = -e^x \cos y$.
   If $f(z)$ is analytic, then the CR-equations must be satisfied and therefore

   $$u_x = v_y, \quad u_y = -v_x$$

   Now $u_x = v_y$ will be satisfied only if $\cos y = 0$. But for these values of y $u_y \neq -v_x$.
   hence $f(z)$ will not be analytic on $\mathbb{C}$.

2. (a) Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there do not exist any point c on the line $y = 1 - x$ joining z_1 and z_2 such that

   $$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(c).$$

   What do you infer from this? [4]

   Solution: $f'(z) = 3z^2$ and $\frac{f(z_1)-f(z_2)}{z_1-z_2} = i$. To find a complex number c such that $i = 3c^2$ on the line $x + y = 1$.
   $3c^2 = i \Rightarrow 3(x^2 - y^2) + i6xy = i \Rightarrow x^2 = y^2, \;\; 6xy = 1$. using $x^2 = y^2$ and (x, y) are on the line $x + y = 1$, we have $x = y = \frac{1}{2}$. But this does not satisfy $6xy = 1$. Thus we are done.
   This example implies that the Mean value theorem of real calculus does not extend to functions of complex variable.

   (b) Suppose f is entire and $|f(z)| \leq \alpha|z|, \forall z$ and for α a fixed positive number. Show that $f(z) = az$ where a is a complex number. [6]

Solution: Let z_0 be a complex number and $R > 0$ be any real number. Then on the circle $C : z = z_0 + Re^{i\theta},\ 0 \le \theta \le 2\pi$,
$|f(z_0 + Re^{i\theta})| \le \alpha|z_0 + Re^{i\theta}| \le \alpha(|z_0| + R)$. Then by Cauchy's inequality,

$$|f'(z_0)| \le \frac{\alpha(|z_0| + R)}{R}, \text{ for all } R > 0$$

Consequently, $|f'(z_0)| \le \alpha$. Since this is true for every $z_0 \in \mathbb{C}$, $f'(z)$ is a bounded function.
Also since $f(z)$ is entire, $f'(z)$ is also entire.
Thus by Liouville theorem, $f'(z)$ is constant, say $f'(z) = a$. Hence $f(z) = az + c$.
Since $|f(z)| \le \alpha|z|$, $f(0) = 0$. Thus $c = 0$.
$f(z) = az$.

3. (a) Use Cauchy's residue theorem to evaluate the integral $\displaystyle\int_{C:|z|=\pi} \frac{e^{-z}}{z^2(z-1)(z-4)}dz$, where the contour C is taken in the counterclockwise direction. **[4 marks]**

Solution: Consider $f(z) = \dfrac{e^{-z}}{z^2(z-1)(z-4)}$. Note that $f(z)$ has two singularities 0 and 1 inside the contour $|z| = \pi$. Note that $z = 0$ is a pole of order 2 and $z = 1$ is a pole of order 1. Then by Cauchy residue theorem, $\dfrac{e^{-z}}{z^2(z-1)(z-4)}dz = 2\pi i(\text{Res}_{z=0}f(z) + \text{Res}_{z=1}f(z))$.
$\text{Rez}_{z=0}f(z) = \frac{1}{16}$ and $\text{Rez}_{z=1}f(z) = -\frac{1}{3}e^{-1}$. Thus $\displaystyle\int_{|z|=\pi} f(z)\, dz = 2\pi i(\frac{1}{16} - \frac{1}{3}e^{-1}))$

(b) Using contour integral, find $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$. [6]

Solution: Put $z = e^{i\theta}$. Then $\cos\theta = \dfrac{z+z^{-1}}{2}$, $dz = izd\theta$.
So $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i}\int_C \frac{dz}{z^2+4z+1}$ where C is the unit circle $|z| = 1$.
Now $f(z) = \dfrac{2}{i(z^2+4z+1)}$ has only one singularity $z = -2+\sqrt{3}$ inside C and is a simple pole.
$\text{Res}_{z=-2+\sqrt{3}}f(z) = \frac{1}{i\sqrt{3}}$. Thus by Cauchy residue theorem $I = 2\pi i \times \frac{1}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$.

4. Determine the linearity and order of the given partial differential equation and then find its general solution.

$$x(y^2 - z^2)z_x - y(z^2 + x^2)z_y = (x^2 + y^2)z$$

[1+1+8]

Ans. The equation is quasilinear, and of first order.
The auxiliary equation is

$$\frac{dx}{x(y^2 - z^2)} = -\frac{dy}{y(x^2 + z^2)} = \frac{dz}{z(y^2 + x^2)}.$$

Observe that

$$\frac{dx}{x(y^2 - z^2)} = -\frac{dy}{y(x^2 + z^2)} = \frac{dz}{z(y^2 + x^2)} = \frac{xdx + ydy + zdz}{0}.$$

Therefore $xdx + ydy + zdz = 0$. Which implies $x^2 + y^2 + z^2 = c_1$.
Also we have

$$\frac{dx/x - dy/y}{y^2 - z^2 + z^2 + x^2} = \frac{dz}{z(x^2 + y^2)},$$

which implies

$$\frac{dx}{x} - \frac{dy}{y} = \frac{dz}{z}.$$

Therefore $log\frac{yz}{x} = log(c_2)$, i.e. $\frac{yz}{x} = c_2$.
Hence the general solution is $z = \frac{x}{y}G(x^2 + y^2 + z^2)$ or $F(\frac{yz}{x}, x^2 + y^2 + z^2) = 0$, where F, G are arbitrary differentiable functions.

5. Let f and g be respectively C^2 and C^1 functions on $\mathbb{R}$ and $c > 0$, a real number. Determine whether the following equation is in canonical form and then solve it.

$$y_{tt} - c^2 y_{xx} = 0, -\infty < x < \infty, t > 0,$$
$$y(x, 0) = f(x), -\infty < x < \infty,$$
$$y_t(x, 0) = g(x) - \infty < x < \infty.$$

[2+8]

Ans. For this equation $R = 1, S = 0, T = -c^2$. Hence $S^2 - 4RT = 4c^2 > 0$. Therefore the equation is hyperbolic. Since in the canonical form for the hyperbolic equations the coefficients of y_{tt} and y_{xx} must vanish therefore it is not in canonical form.
We introduce the characteristic variables $\xi = x - ct$ and $\eta = x + ct$. Then the equation transforms to

$$\frac{\partial^2 y}{\partial\xi\partial\eta} = 0.$$

Therefore

$$y(x, t) = F(\xi) + G(\eta) = F(x - ct) + G(x + ct).$$

Here F, G are arbitrary C^2 functions. After inserting the initial conditions we get,

$$F(x) + G(x) = f(x),$$

$$-cF'(x) + cG'(x) = g(x).$$

Solving for $F(x)$ and $G(x)$ we get,

$$F(x) = \frac{1}{2c}[cf(x) - \int_{x_0}^{x} g(x)dx],$$

$$G(x) = \frac{1}{2c}[cf(x) + \int_{x_0}^{x} g(x)dx].$$

Therefore $y(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$.

6. Let F be a continuous function in x, t, and f is a C^2 function in x only. Then solve the following initial value problem.

$$u_{tt} - c^2 u_{xx} = F(x,t), -\infty < x < \infty, t > 0,$$
$$u(x,0) = f(x), u_t(x,0) = 0, -\infty < x < \infty.$$

[10 marks]

Ans. Let p satisfies

$$p_{tt} - c^2 p_{xx} = F(x,t), -\infty < x < \infty, t > 0, \tag{1}$$
$$p(x,0) = 0, p_t(x,0) = 0, -\infty < x < \infty,$$

and q satisfies

$$q_{tt} - c^2 q_{xx} = 0, -\infty < x < \infty, t > 0, \tag{2}$$
$$q(x,0) = f(x), q_t(x,0) = 0, -\infty < x < \infty,$$

Then note that $u = p + q$. We consider the function $v(x,t,\tau)$ which satisfies the following equation.

$$v_{tt} - c^2 v_{xx} = 0 \qquad -\infty < x < \infty,\ t > \tau > 0.$$
$$v(x,\tau,\tau) = 0, v_t(x,\tau,\tau) = F(x,\tau).$$

The solution to this problem is

$$v(x,t,\tau) = \frac{1}{2c}\int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)ds.$$

Consider $p(x,t) = \int_0^t v(x,t,\tau)d\tau$. Then clearly $p(x,0) = 0$.

$$p_t = v(x,t,t) + \int_0^t v_t(x,t,\tau)d\tau = \int_0^t v_t(x,t,\tau)d\tau.$$

Since $v(x,t,t) = 0$ we have $p_t(x,0) = 0$.

$$p_{tt} = v_t(x,t,t) + \int_0^t v_{tt}(x,t,\tau)d\tau = \int_0^t v_{tt}(x,t,\tau)d\tau + F(x,t).$$

Also we have $p_{xx} = \int_0^t v_{xx}(x,t,\tau)d\tau$. Therefore we get p satisfies (1). From arguments of question 5 we obtain

$$q(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]$$

since $g = 0$ in this case. Hence

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} F(s,\tau)dsd\tau.$$

7. An insulated rod of length l has its ends A and B maintained at $0°\ C$ and $100°\ C$ respectively until steady state conditions prevails. If B is suddenly reduced to $0°\ C$ and maintained at $0°\ C$, find the temperature at distance x from A at time t. **[10 marks]**
(**Hint.** Using the steady state condition, first find the initial temperature $u(x,0)$ and then solve the corresponding heat conduction problem)

Ans. We know the heat conduction is $u_t = ku_{xx}$. Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (steady state condition)). When u depends only on x, heat equation reduces to $u_{xx} = 0 \Rightarrow u = ax + b$. Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, $\Rightarrow b = 0,\ a = 100/l$.
Thus we get initial condition $u(x, 0) = \frac{100}{l}x$. Thus exactly, we have to solve the following IBVP:

$$\begin{aligned} u_t - ku_{xx} &= 0, \quad 0 < x < l,\ t > 0, \\ u(0,t) = u(l,t) &= 0, \quad t \geq 0 \\ u(x,0) &= \frac{100}{l}x, \quad 0 \leq x \leq l. \end{aligned}$$

Whose solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} a_n \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right).$$

where

$$\begin{aligned} a_n &= \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin\frac{n\pi x}{l} dx \\ &= \frac{200}{n\pi}(-1)^{n+1} \end{aligned}$$

Hence,

$$u(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \exp\left(-\frac{n^2\pi^2 kt}{l^2}\right) \sin\left(\frac{n\pi x}{l}\right).$$

8. Show that the solution $u(x,t)$ of the following heat conduction problem

$$u_t - ku_{xx} = F(x,t), \qquad 0 < x < l,\ t > 0,$$

$$u(x,0) = f(x), \quad 0 \leq x \leq l \ \text{ and } \ u(0,t) = u(l,t) = 0,\ t \geq 0,$$

is unique. **[10 marks]**

Ans. Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions of this problem. Then $v = (u_1 - u_2)$ satisfies

$$\begin{aligned} v_t - ku_{xx} &= 0, \quad 0 < x < l,\ t > 0, \\ v(0,t) = v(l,t) &= 0, \quad t \geq 0 \\ v(x,0) &= 0, \quad 0 \leq x \leq l. \end{aligned}$$

Let us define a function

$$E(t) = \frac{1}{2k}\int_0^l v^2(x,t)dx.$$

Therefore $E \geq 0$. On differentiating this function with respect to t, we get

$$\begin{aligned} \frac{dE}{dt} &= \frac{1}{k}\int_0^l vv_t dx, \\ &= \int_0^l vv_{xx} dx \\ &= vv_x|_0^l - \int_0^l v_x^2 dx \\ &= -\int_0^l v_x^2 dx \leq 0 \quad (\text{since } v(0,t) = v(l,t) = 0). \end{aligned}$$

Therefore E is a decreasing function of t. From the condition $v(x,0) = 0$, we have $E(0) = 0$. Therefore $E(t) \leq 0$ for all $t > 0$. But $E(t)$, by definition, is non-negative. Therefore

$$E(t) \equiv 0, \; \forall \, t > 0 \Rightarrow v(x,t) \equiv 0 \text{ in } 0 \leq x \leq l, \; t \geq 0.$$

9. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Using method of separation of variables, solve the following BVP:

$$u_{xx} + u_{yy} = 0, \quad (x,y) \in \Omega$$

with $u(0,y) = u(\pi,y) = u(x,\pi) = 0$ and $u(x,0) = \sin^2 x$. **[10 marks]**

Ans. By the method of separation of variable, let $u(x,y) = X(x)Y(y)$. This yields

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda.$$

The boundary conditions on X $[X(0) = X(\pi) = 0]$ imply $\lambda = -n^2$
and $F_n(x) = C_n \sin nx, \; n = 1, 2, \cdots$. Thus Y satisfies

$$Y'' = n^2 Y$$

$$\Rightarrow Y_n(y) = D_n e^{ny} + E_n e^{-ny}$$

Now B.C. $u(x,\pi) = 0 \Rightarrow Y_n(\pi) = 0$ as $X_n(x) \neq 0$. Hence,

$$Y_n(\pi) = D_n e^{n\pi} + E_n e^{-n\pi} = 0 \Rightarrow E_n = -D_n \frac{e^{n\pi}}{e^{-n\pi}}$$

Therefore,

$$\Rightarrow Y_n(y) = D_n e^{ny} - D_n \frac{e^{n\pi}}{e^{-n\pi}} e^{-ny} = \frac{D_n}{e^{-n\pi}} \left(e^{n(y-\pi)} - e^{n(\pi-y)} \right) = \frac{D_n}{e^{-n\pi}} \left(e^{-n(\pi-y)} - e^{n(\pi-y)} \right)$$

$$= -\frac{2D_n}{e^{-n\pi}} \left(\frac{e^{n(\pi-y)} - e^{-n(\pi-y)}}{2} \right) = D_n^* \sinh(n(\pi - y)), \quad \text{where } \; D_n^* = -\frac{2D_n}{e^{-n\pi}}$$

Hence,

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y) = \sum_{n=1}^{\infty} (C_n \sin nx)(D_n^* \sinh(n(\pi - y)))$$

$$= \sum_{n=1}^{\infty} B_n^* \left[\sinh(n(\pi - y))\right] \sin nx, \quad \text{where } \; B_n^* = C_n D_n^*.$$

Now $u(x,0) = \sin^2 x$ gives

$$\sin^2 x = \sum_{n=1}^{\infty} \left[B_n^* \sinh(n\pi)\right] \sin nx$$

$$\Rightarrow B_n^* = \frac{2}{\pi \sinh(n\pi)} \int_0^{\pi} \sin^2 x \sin nx dx$$

$$= \begin{cases} -\dfrac{8}{n\pi(n^2 - 4)\sinh(n\pi)}, & \text{for odd } n \\ 0, & \text{for even } n \end{cases}$$

Hence the solution is

$$u(x,y) = -\frac{8}{\pi} \sum_{n=\text{odd}} \frac{\sin nx \cdot \sinh(n(\pi - y))}{(\sinh(n\pi)) \cdot n(n^2 - 4)}.$$

10. (**Maximum principle**) Let $u(x,y) \in C^2(\Omega) \cap C(\bar{\Omega})$ be a solution of Laplace's equation:

$$\nabla^2 u(x,y) := u_{xx} + u_{yy} = 0,$$

in a bounded region Ω with boundary $\partial\Omega$. Then show that the maximum values of u attains on the boundary $\partial\Omega$. **[10 marks]**

Ans. Since u is continuous in $\bar{\Omega}$, it attains its maximum either in Ω or on $\partial\Omega$. Suppose u achieves its maximum on $\bar{\Omega}$ at some point $(x_0, y_0) \in \Omega$. let

$$u(x_0,y_0) = \max_{\bar{\Omega}} u(x,y) = \max_{\Omega} u(x,y) = M_0 > M_b, \quad \text{where,} \ \ M_b = \max_{\partial\Omega} u(x,y)$$

Consider the function

$$v(x,y) = u(x,y) + \varepsilon[(x-x_0)^2 + (y-y_0)^2] \tag{3}$$

for some $\varepsilon > 0$. Note that $v(x_0,y_0) = u(x_0,y_0) = M_0$ and

$$\max_{\partial\Omega} v(x,y) \leq M_b + \varepsilon d^2,$$

where, d is the maximum distance of the boundary $\partial\Omega$ from the point (x_0,y_0).
For such $\varepsilon(0 < \varepsilon < (M_0 - M_b)/d^2)$, the maximum of v can not occur on $\partial\Omega$ because

$$M_0 = v(x_0,y_0) > \max_{\partial\Omega} v(x,y).$$

This implies there may be points in Ω where $v > M_0$. Let

$$v(x_1,y_1) = \max_{\Omega} v(x,y).$$

At (x_1,y_1), we must have

$$v_{xx} \leq 0 \quad \text{and } v_{yy} \leq 0 \Rightarrow v_{xx} + v_{yy} \leq 0. \tag{4}$$

On the other hand, from Eq. (3), we have

$$v_{xx} + v_{yy} = u_{xx} + u_{yy} + 2\varepsilon + 2\varepsilon = 4\varepsilon > 0,$$

This lead to a contradiction to Eq. (4). Thus,

$$\max_{\bar{\Omega}} v(x,y) \neq \max_{\Omega} v(x,y),$$

so the maximum of u attains on $\partial\Omega$.