The LNM Institute of Information Technology
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MATH-II ■ Solutions Mid Sem-II

Q1. (i) The eigenvalues of A are given by $det(A - \lambda I) = 0$, I is the identity matrix of order 2. Solving this we get $\lambda = \cos\theta \pm i \sin\theta$ [02] marks

The eigenvector corresponding to the eigen value $\lambda_1 = \cos\theta + i\sin\theta$ is $E(\lambda_1) = [(i, 1)]$ and the eigenvector corresponding to the eigen value $\lambda_2 = \cos\theta - i\sin\theta$ is $E(\lambda_1) = [(1, i)]$. [02] marks

(ii) Given equation is a Legendre equation of order $n = 4$ (even integer) since it is of the form

$$(1 - x^2)y'' - 2xy' + 4(4 + 1)y = 0.$$

Therefore a polynomial solution to this equation is:

$$y(x) = a_0 \left(1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4\right) \qquad \text{[03] marks}$$

with $n = 4$ it becomes

$$y(x) = a_0 \left(1 - 10x^2 + \frac{35}{3}x^4\right). \qquad \text{[01] mark}$$

Now given that $y(1) = 10$, therefore $a_0 = \frac{8}{3}$ and thus

$$y(x) = \frac{8}{3}\left(1 - 10x^2 + \frac{35}{3}x^4\right). \qquad \text{[01] mark}$$

Q2. (i) Let $u = x + 1$, then the given series becomes

$$1 + u + 2u^2 + \ldots + nu^n + \ldots$$

Here $a_n = n$, using ratio test we get radius of convergence $R = 1$ and the interval of convergence as $|u| < 1$ or $-2 < x < 0$. [02] marks

(ii) Let $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$. Substituting into the governing equations and simplifying it we get

$$r^2 a_0 + \sum_{n=1}^{\infty} [(n + r)^2 a_n - a_{n-1}]x^n = 0. \qquad \text{[02] marks}$$

Since $a_0 \neq 0 \Longrightarrow r = 0, 0 \Longrightarrow n^2 a_n = a_{n-1}$, $n = 1, 2, 3, \ldots$.

Solving for a_n gives $a_1 = a_0$, $a_2 = \frac{a_0}{(2!)^2}$.

Thus

$$y = a_0 \left[1 + \frac{x}{1!} + \frac{x^2}{(2!)^2} + \dots\right]. \qquad \text{[02] marks}$$

(iii) Let α and β be two consecutive positive zeros of J_{p+1}. Let $f(x) = x^{p+1}J_{p+1}$. Then $f(\alpha) = f(\beta) = 0$. Thus there exists $c \in (\alpha, \beta)$ such that $f'(c) = 0$. Taking $\gamma = p+1$ in $[x^\gamma J_\gamma(x)]' = x^\gamma J_{\gamma-1}$, we see that $J_p(c) = 0$. Thus there exists a zero ofJ_p between consecutive zeros of J_{p+1}. [02] marks

Similarly taking $\gamma = p$ in $[x^{-\gamma} J_\gamma(x)]' = -x^{-\gamma} J_{\gamma+1}$, we conclude that there exists a zero of J_{p+1} between consecutive positive zeros of J_p. [01] mark

To prove uniqueness, let there exist two zero of J_p between consecutive zeros α and β of J_{p+1}. This implies that there exist a zero of J_{p+1} between α and β, which contradicts the fact that α and β are consecutive zeroes. [02] marks

Q3. (i) Compare with the equation $y'' + y = 0$. [02] marks

Now $k + 2\sin(x + \frac{\pi}{4}) > 1$ for $x \in [0, 5\pi]$. [02] marks

Since $\sin x$ has 6 zeros in $[0, 5\pi]$, by Sturm comparison theorem $\phi(x)$ must have atleast 5 zeros in $[0, 5\pi]$. [02] marks

(ii) Suppose $\lambda = p^2$ where $p \neq 0$. Now

$$y = A\cos px + B\sin px \qquad \text{[01] mark}$$

$y(0) = 0 = y'(0) = 0 \Longrightarrow A = 0, p = (n + 1/2)\ n = 0, \pm 1, \pm 2, \pm 3 \dots$ [01] mark

Hence eigenvalues $\lambda_n = (n + 1/2)^2$, $n = 1, 2, 3 \dots$ and the corresponding eigenfunctions are $y_n(x) = \sin(n + 1/2)x$. [02] marks

Q4. (i) Taking Laplace transform on both sides of

$$\frac{d^2I}{dt^2} + 2\frac{dI}{dt} - 3I = 5u(t-1)$$

we get

$$I(s) = \frac{5e^{-s}}{s(s^2 + 2s - 3)} + \frac{8s}{(s^2 + 2s - 3)} + \frac{16}{(s^2 + 2s - 3)}. \qquad \text{[02]marks}$$

Taking inverse Laplace transform we get

$$\begin{aligned} I(t) &= L^{-1}\left\{\frac{5e^{-s}}{s(s^2 + 2s - 3)} + \frac{8s}{(s^2 + 2s - 3)} + \frac{16}{(s^2 + 2s - 3)}\right\} \\ &= L^{-1}\left\{\frac{5e^{-s}}{s(s-1)(s+3)} + \frac{8s}{(s-1)(s+3)} + \frac{16}{(s-1)(s+3)}\right\} \\ &= 5\left(-\frac{1}{3} + \frac{e^{(t-1)}}{4} + \frac{e^{3(t-1)}}{12}\right)u(t-1) - 2(e^t + 3e^{-3t}). \qquad \text{[03]marks} \end{aligned}$$

(ii) We can write $f(t) = u(t-a) - u(t-b)$. [01] mark

The given equation becomes

$$y(t) + \int_0^t y(\tau)d\tau = u(t-a) - u(t-b).$$

Taking Laplace transform on both sides, we get

$$Y(s) = \frac{e^{-as}}{s+1} - \frac{e^{-bs}}{s+1}.$$ [02]marks

Taking inverse Laplace transform, we get

$$y(t) = e^{-(t-a)}u(t-a) - e^{-(t-b)}u(t-b).$$ [02]marks