The LNM Institute of Information Technology
Jaipur, Rajasthan

End Semester Exam

Part-B, Hints & Solutions

MATH-II, 26^{th} April 2016
Time: 3 Hours, Maximum Marks: 100

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1. (a) Prove or disprove: For an $n \times n$ matrix A, if the system $A^2x = 0$ has a non-trivial solution then the system $Ax = 0$ also has a non-trivial solution. **[4 marks]**

Ans. Assume that the system $Ax = 0$ has only the trivial solution. Then for $y \in R^n$, we have

$$\begin{aligned}
&\Rightarrow A^2y = 0\\
&\Rightarrow A(Ay) = 0\\
&\Rightarrow Ay = 0, \text{ since the system } Ax = 0 \text{ has only the trivial solution}\\
&\Rightarrow y = 0, \text{ since the system } Ax = 0 \text{ has only the trivial solution}\\
&\Rightarrow \text{the system } Ax = 0 \text{ has no non-trivial solution.}
\end{aligned}$$

(b) Let $\{u_1, u_2, \ldots, u_n\}$ be a basis for a vector space V. Show that the set $\{u_1+u_2, u_2+u_3, \ldots, u_n+u_1\}$ is also a basis for V if and only if n is an odd integer. **[6 marks]**

Ans. To check the linear dependency or independency of the set $\{u_1+u_2, u_2+u_3, \ldots, u_n+u_1\}$, we consider

$$\begin{aligned}
&\alpha_1(u_1 + u_2) + \alpha_2(u_2 + u_3) + \cdots + \alpha_n(u_n + u_1) = 0_V,\\
&\Rightarrow (\alpha_1 + \alpha_n)u_1 + (\alpha_1 + \alpha_2)u_2 + \cdots + (\alpha_{n-1} + \alpha_n)u_n = 0_V\\
&\Rightarrow \alpha_1 + \alpha_n = 0, \alpha_1 + \alpha_2 = 0, \cdots , \alpha_{n-1} + \alpha_n = 0.
\end{aligned}$$

The system of these equations has a non-zero solution if and only if

$$\begin{vmatrix}
1 & 0 & 0 & \cdots & 0 & 1\\
1 & 1 & 0 & \cdots & 0 & 0\\
0 & 1 & 1 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & & \vdots & \vdots\\
0 & 0 & 0 & \cdots & 1 & 0\\
0 & 0 & 0 & \cdots & 1 & 1
\end{vmatrix} = 1 + (-1)^{n+1} = 0.$$

Notice that $1 + (-1)^{n+1} = 0$ iff n is even. Thus the set $\{u_1 + u_2, u_2 + u_3, \ldots, u_n + u_1\}$ is linearly independent iff n is an odd integer. Since this set contains n vectors, we conclude that given set is a basis iff n is an odd integer.

2. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x, y, z) = (3x, x - y, 2x + y + z)$. Is T invertible? If so, find a rule for T^{-1} like the one which defines T. **[5 marks]**

Ans. Here, $N(T) = \{(x, y, z)|T(x, y, z) = (0, 0, 0)\} = \{(0, 0, 0)\} \Rightarrow T$ is one-one. By Rank-Nullity theorem $\dim(R(T)) = 3 \Rightarrow T$ is onto. Since T is one-one and onto, therefore T is invertible.
To find T^{-1}, consider $T^{-1}(a, b, c) = (x, y, z)$, where $(a, b, c) \in \mathbb{R}^3$.

$$\begin{aligned}
&\Rightarrow T(x, y, z) = (a, b, c)\\
&\Rightarrow (3x, x - y, 2x + y + z) = (a, b, c)\\
&\Rightarrow x = a/3,\ y = (a/3) - b,\ z = b + c - a,
\end{aligned}$$

Thus, $T^{-1}(a, b, c) = \left(\frac{a}{3}, \frac{a}{3} - b, b + c - a\right)$.

(b) (**Cauchy-Schwartz inequality**) Let V be an inner product space and $x, y \in V$. Then prove that

$$| < x, y > | \leq ||x||\, ||y||.$$

Let $V = \mathbb{R}^n$, with the usual dot product as inner product then give an equivalent expression of the Cauchy-Schwartz inequality in V. **[4+1 marks]**

Ans. Let $x, y \in V$. If $x = 0$ or $y = 0$ then the inequality holds. So let us assume that $x \neq 0$. For $\lambda \in \mathbb{F}$ consider the inner product

$$\begin{aligned}0 &\leq <\lambda x + y, \lambda x + y> = <\lambda x, \lambda x + y> + <y, \lambda x + y> \\ &= <\lambda x, \lambda x> + <\lambda x, y> + <y, \lambda u> + <y, y> \\ &= \lambda\bar{\lambda} <x, x> + \lambda <x, y> + \bar{\lambda} <y, x> + <y, y> \\ &= \lambda\bar{\lambda}||x||^2 + \lambda <x, y> + \bar{\lambda} <y, x> + ||y||^2.\end{aligned}$$

Take $\lambda = -\frac{<y,x>}{||x||^2}$, then

$$\begin{aligned}0 &\leq \frac{<y,x>}{||x||^2}\frac{\overline{<y,x>}}{||x||^2}||x||^2 - \frac{<y,x>}{||x||^2} <x, y> - \frac{\overline{<y,x>}}{||x||^2} <y, x> + ||y||^2 \\ &\leq -\frac{<y,x>}{||x||^2} <x, y> + ||y||^2.\end{aligned}$$

Hence, we get

$$|<x, y>|^2 \leq ||x||^2||y||^2 \implies |<x, y>| \leq ||x||||y||.$$

If $V = \mathbb{R}^n$, then for $x, y, \in V$, $<x, y> = \sum_{i=1}^{n} x_i y_i$. Therefore the Cauchy-Schwartz inequality takes the form

$$\left|\sum_{i=1}^{n} x_i y_i\right| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2\right)^{1/2}.$$

3. (*a*) Find all initial conditions so that the initial value problem $(x^2 - 5x + 6)y' = (2x - 5)y$, with $y(x_0) = y_0$ has (i) no solution, (ii) more than one solution, and (iii) only one solution. Justify your answer. Moreover, discuss with reference to the existence and uniqueness theorem. **[5]**

Ans. The general solution of the given DE is $y = C(x - 2)(x - 3)$.
By substituting the initial condition we get $y_0 = C(x_0 - 2)(x_0 - 3)$. Therefore, the given euation has
(i) Unique solution if $x_0 \neq 2 \& x_0 \neq 3$
(ii) Infinite solution if $x_0 = 2, x_0 = 3 \& y_0 = 0$
(iii) No solution if $x_0 = 2, x_0 = 3 \& y_0 \neq 0$

Here $f(x, y) = \dfrac{(2x - 5)y}{(x - 2)(x - 3)}$ and $\dfrac{\partial f}{\partial y} = \dfrac{(2x - 5)}{(x - 2)(x - 3)}$. The existence and uniqueness theorem guarantees the existence of unique solution in the vicinity of (x, y) where f and $\frac{\partial f}{\partial y}$ is continuous. Thus existence of unique solution is guaranteed for all (x, y) for which $(x - 2)(x - 3) \neq 0$.

(*b*) Apply improved Euler method to compute $y(x)$ at 0.4 for the initial value problem: $y' = x + xy^3$, $y(0) = 1$ by taking the step size 0.2. **[4]**

Ans. Note that here $f(x, y) = x + xy^3$, step size $h = 0.2$, $x_0 = 0$ and $y_0 = y(x_0) = 1$.
So $x_1 = x_0 + h = 0.2$, $x_2 = x_1 + h = 0.4$. We need an approximation y_2 of solution y at $x_2 = 0.4$.
Using improved Euler formula we get
$\tilde{y}_1 = y_0 + h * f(x_0, y_0) = 1 + 0.2 * f(0, 1) = 1$
$y_1 = y_0 + h * \left(\frac{f(x_0,y_0)+f(x_1,\tilde{y}_1)}{2}\right) = 1.04$
Similarly, $\tilde{y}_2 = y_1 + h * f(x_1, y_1) = 1.04 + 0.2 * f(0.2, 1.04) = 1.085$
$y_2 = y_1 + h * \left(\frac{f(x_1,y_1)+f(x_2,\tilde{y}_2)}{2}\right) = 1.1736$
Hence the approximate solution at $x = 0.4$ is $y_2 = 1.1736$.

4. (*a*) One solution of the linear ODE

$$y'' + \left(-2 - \frac{2}{x}\right)y' + \frac{4}{x}y = 0$$

is $y_1(x) = e^{2x}$. Let $y_2(x)$ be another solution and $W(y_1, y_2)(x)$ be the Wronskian of y_1 and y_2. Then prove that

$$W'(x) = \left(2 + \frac{2}{x}\right)W(x).$$

Find a solution $W(x)$ of this differential equation? For this solution $W(x)$, find a particular solution of the first order linear ODE

$$y_1(x)v' - y_1'(x)v = W(x)$$

using the method of variation of parameter. **[7]**

Sol: $y_1(x) = e^{2x}$ and y_2 are solutions, they satisfy the given ODE

$$y_1'' + \left(-2 - \frac{2}{x}\right) y_1' + \frac{4}{x} y_1 = 0,$$

and

$$y_2'' + \left(-2 - \frac{2}{x}\right) y_2' + \frac{4}{x} y_2 = 0.$$

Multipy y_2 in first equation and y_1 with second and subtracting we get

$$\begin{aligned}
&(y_1''y_2 - y_2''y_1) + \left(-2 - \frac{2}{x}\right)(y_1'y_2 - y_2'y_1) = 0 \\
\Rightarrow \quad &(y_1''y_2 - y_2''y_1) = \left(2 + \frac{2}{x}\right)(y_1'y_2 - y_2'y_1) \\
\Rightarrow \quad &W'(x) = \left(2 + \frac{2}{x}\right) W(x).
\end{aligned}$$

Solving for $W(x)$, we get $W(x) = Cx^2e^{2x}$.

Now the ODE $y_1(x)v' - y_1'v = W(s)$ takes the form $v' - 2v = Cx^2$, which we solve by method of variation of parameter. The general solution being $v = Ae^{2x}$, by replacing the parameter A by $u(x)$ i.e. $v_p(x) = u(x)e^{2x}$.

Solving for $u(x)$, we get $u(x) = -\frac{C}{4}e^{-2x}(2x^2 + 2x + 1)$ and $v_p(x) = -\frac{C}{4}(2x^2 + 2x + 1)$.

(b) Find a general solution in terms of a series for the following second order equation known as Airy's equation:

$$y'' - xy = 0 \qquad x \in \mathbb{R}.$$

[6]

Sol: Note that here $P(x) = 0, Q(x) = -x$ which are analytic everywhere and hence every point is an ordinary point. The given DE admits a power series solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

By differentiating term by term twice we get

$$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n, \quad xy(x) = \sum_{n=0}^{\infty} a_n x^{n+1}.$$

Substituting into the DE, we get

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

By rewriting

$$2.1a_2 + \sum_{n=1}^{\infty} (n+1)(n+2)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n.$$

For this equation to be satisfied for all x in some interval, the coefficients of like powers of x must equal; hence $a_2 = 0$, and we obtain the following recurrence relation

$$(n+2)(n+1)a_{n+2} = a_{n-1} \forall n = 1, 2, 3...$$

.

Since $a_2 = 0$, $a_5 = a_8 = a_{11} = \cdots = 0$.

For the sequence $a_0, a_3, a_6, a_9, ...$ we set $n = 1, 4, 7, 10, ...$ in the recurrence relation to get

$$a_3 = \frac{a_0}{2 \cdot 3}, \qquad a_6 = \frac{a_3}{5 \cdot 6} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6}, \qquad a_9 = \frac{a_6}{8 \cdot 9} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9}, ...$$

.

This suggests the general formula

$$a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdots (3n-1)(3n)}, n \geq 4.$$

For the sequence $a_1, a_4, a_7, a_{10}, ...$ we set $n = 2, 5, 8, 11, ...$ in the recurrence relation to get

$$a_4 = \frac{a_1}{3 \cdot 4}, \qquad a_7 = \frac{a_4}{6 \cdot 7} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7}, \qquad a_{10} = \frac{a_7}{9 \cdot 10} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10}, ...$$

.

This suggests the general formula

$$a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)}, n \geq 4.$$

Thus the general solution for Airy's equation is

$$\begin{aligned} y &= a_0 \left[1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \cdots + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n)} + \cdots \right] \\ &\quad + a_1 \left[x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \cdots + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1)} + \cdots \right] \\ &= a_0 y_1 + a_1 y_2. \end{aligned}$$

By ratio test y_1 & y_2 converge for all x and since the ratio of y_1 & y_2 is a non-constant function, they are LI and hence forms a basis for Airy eqution.

5. (a) Prove the following orthogonality relation using Rodrigues' formula for Legendre polynomial $P_n(x)$:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{mn},$$

where, δ_{mn} represent 0 if $m \neq n$ otherwise it takes value 1.
(Hint: One may use here Rodrigues's formula $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2-1)^n]$) **[6]**

Sol: Let $f(x)$ be any function with at least n continuous derivatives in $[-1, 1]$. Consider the integral

$$I = \int_{-1}^{1} f(x) P_n(x) dx = \frac{1}{2^n n!} \int_{-1}^{1} f(x) \frac{d^n}{dx^n} (x^2-1)^n dx.$$

Repetition of integration by parts gives

$$I = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} f^n(x) (x^2-1)^n dx.$$

If $m \neq n$, without any loss of generality we take $f = P_m, m < n$ and then $f^n(x) = 0$ and $I = 0$.

If $f(x) = P_n(x)$, then

$$f^n(x) = \frac{1}{2^n n!} \frac{d^{2n}}{dx^{2n}} (x^2-1)^n = \frac{(2n)!}{2^n n!}.$$

Thus

$$I = \frac{(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1} (1-x^2)^n dx = \frac{2(2n)!}{2^{2n}(n!)^2} \int_{0}^{1} (1-x^2)^n dx.$$

Substituting $x = \sin\theta$, we get

$$I = \frac{2(2n)!}{2^{2n}(n!)^2}\int_0^{\pi/2}\cos^{2n+1}\theta d\theta = \frac{2(2n)!}{2^{2n}(n!)^2}I_n.$$

Since

$$\int \cos^{2n+1}\theta d\theta = \frac{1}{2n+1}\cos^{2n}\theta\sin\theta + \frac{2n}{2n+1}\int\cos^{2n-1}\theta d\theta,$$

we find

$$I_n = \frac{2n}{2n+1}I_{n-1} = \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}I_0.$$

Now

$$I_0 = \int_0^{\pi/2}\cos\theta d\theta = 1.$$

Hence,

$$I_n = \frac{2^n n!}{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)} = \frac{2^{2n}(n!)^2}{(2n)!(2n+1)}.$$

Thus, we get the desired result.

(b) Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$y'' + \lambda y = 0$$
$$y(0) = 0 = y(1).$$

Further, for any piecewsie continuous function $f(x)$ find coefficients a_n such that

$$f(x) = \sum_{n\geq 1} a_n y_n(x),$$

where $y_n(x)$ are eigen functions of the given problem. **[6]**

Sol: The characteristic equation is $m^2 + \lambda = 0$. For $\lambda \leq 0$, this BVP has only trivial solution $y = 0$. For $\lambda > 0$, take $\lambda = n^2$. The general solution is

$$y(x) = A\cos nx + B\sin nx$$

$$y(0) = 0 \implies A = 0 \text{ and } y(1) = 0 \implies B\sin n = 0.$$

Further, $B = 0$ implies $y = 0$. So assume $B \neq 0$ and then $\sin n = 0$, which implies

$$n = k\pi \qquad \forall k = 0, \pm1, \pm2, ...$$

The eigen values are $\lambda = n^2 = k^2\pi^2$ for all $k = 1, 2, 3, ...$, i.e. $\pi^2, 4\pi^2, 9\pi^2, \cdots$.

The corresponding eigen functions are $y_k(x) = B_k\sin(k\pi x)$ for all $k = 1, 2, 3, \cdots$.
Multiplying y_m both sides of $f(x) = \sum_{n\geq 1} a_n y_n(x)$ and integrating from 0 to 1, we get

$$\int_0^1 f(x)y_m(x)dx = \sum_{n\geq 1} a_n\int_0^1 y_n(x)y_m(x)dx$$

Using orthogonality of eigen functions w.r.t. weight function $q(x)$ which is 1 for this problem, we get

$$\int_0^1 f(x)y_m(x)dx = a_m\int_0^1 y_m^2(x)dx$$

and hence

$$a_n = \frac{\int_0^1 f(x)y_m(x)dx}{\int_0^1 y_m^2(x)dx}.$$

(c) The current $I(t)$ in a circuit involving resistance, conductance and capacitance is governed by the following IVP:

$$I'' + 4I = f(t), \qquad I(0) = 0,\ I'(0) = 0.$$

where

$$f(t) = \begin{cases} 0, & 0 \le t \le 5, \\ (t-5)/5, & 5 \le t \le 2\pi, \\ 1, & t \ge 10. \end{cases}$$

Using Laplace transform, find the current as a function of time *i.e.*, find $I(t)$. **[6]**

Sol: Note that

$$f(t) = \frac{t-5}{5}u(t-5) - \frac{t-10}{5}u(t-10)$$

and

$$\mathcal{L}\{f(t)\} = \frac{e^{-5s} - e^{-10s}}{5s^2}.$$

Assume $\mathcal{L}\{I(t)\} = G(s)$. By taking Laplace trasformation both sides of the DE and applying the Boundary conditions we get

$$(s^2+4)G(s) = \frac{e^{-5s} - e^{-10s}}{5s^2}.$$

Solving for $G(s)$ and applying partial fraction we get

$$G(s) = \frac{e^{-5s} - e^{-10s}}{5s^2(s^2+4)} = \frac{1}{20}\left[\frac{e^{-5s}}{s^2} - \frac{e^{-10s}}{s^2+4}\right].$$

using second shifting theorem we get inverse Laplace transform as

$$I(t) = (1/5)[u(t-5)h(t-5) - u(t-10)h(t-10)]$$

where

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t.$$