Optimization

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1 Queuing Theory:

Example 1: John Macko is a student at Ozark U. He does odd jobs to supplement his income. Job requests come every 5 days on the average, but the time between requests is exponential. The time for completing a job is also exponential with mean 4 days. (a) What is the probability that John will be out of jobs? (b) If John gets about \$50 a job, what is his average monthly income? (c) If at the end of the semester, John decides to subcontract on the outstanding jobs at \$40 each. How much, on the average, should he expect to pay?

Solution of Example 1: $\frac{1}{\lambda} = 5$ days i.e. $\lambda = 0.2$/day. $\frac{1}{\mu} = 4$ days i.e. $\mu = 0.25$/day. Hence $\lambda_n = 0.2,\ n \geq 0$ and $\mu_n = 0.2,\ n \geq 1$. For steady state probability, $p_n = \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_n}p_0,\ n \geq 1$. It is a (M/M/1):(FCFS/$\infty$/$\infty$) model. Then, $p_1 = \frac{0.2}{0.25}p_0 = \frac{4}{5}p_0$, $p_2 = \frac{0.2\times0.2}{0.25\times0.25}p_0 = \frac{4^2}{5^2}p_0, \cdots$, $p_n = \frac{0.2\times0.2\times\cdots\times0.2}{0.25\times0.25\times\cdots\times0.25}p_0 = \frac{4^n}{5^n}p_0,\ n \geq 1$. Hence (a) the probability that John will be out of jobs is $p_0 = 1 - \rho = 1 - \frac{\lambda}{\mu} = 0.2$, (b) if John gets about \$50 a job, his average monthly income is $\mu \times month \times \$50 = \$0.25 \times 30 \times 50 = \$375$ (c) $L_S = \sum_{n=0}^{\infty} np_n = \dfrac{\rho}{1-\rho}$. $L_S - L_Q = \frac{\lambda_{eff.}}{\mu} = \frac{\lambda}{\mu} = \rho$. Then expected number of jobs awaiting for completion is $L_Q = L_S - \rho = \frac{\rho^2}{1-\rho} = \frac{16}{5} = 3.2$ jobs. Thus, at the end of the semester, if John decides to subcontract on the outstanding jobs at \$40 each, on the average, he should expect to pay as $L_Q \times \$40 = 3.2 \times \$40 = \$128$.

Example 2: A cafeteria can seat a maximum of 50 persons. Customers arrive in a Poisson stream at the rate of 10 per hour and are served (one at a time) at the rate of 12 per hour. (a) What is the probability that an arriving customer will not eat in the cafeteria because it is full? (b) Suppose that three customers (with random arrival times) would like to be seated together. What is the probability that their wish can be fulfilled? (Assume that arrangements

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can be made to seat them together as long as three seats are available.)

Solution of Example 2: $\lambda_n = 10,\ n \geq 0$ and $\mu_n = 12,\ n \geq 1$. It is a (M/M/1):(FCFS/50/∞) model. Then $p_n = \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{n-1}}{\mu_1\mu_2\mu_3\cdots\mu_n} p_0 = \frac{10^n}{12^n} p_0 = \rho^n p_0,\ 1 \leq n \leq 50,\ \rho = \frac{5}{6}$. Then $\sum_{n=0}^{50} p_n = 1$ gives $p_0 \sum_{n=0}^{50} \rho^n = 1$. Or, $p_0 \frac{1-\rho^{51}}{1-\rho} = 1$. Or, $p_0 = \frac{1-\rho}{1-\rho^{51}}$. Then $p_n = \frac{(1-\rho)\rho^n}{1-\rho^{51}},\ 1 \leq n \leq 50$. The probability, that an arriving customer will not eat in the cafeteria because it is full, is $p_{50} = \frac{(1-\rho)\rho^{50}}{1-\rho^{51}} = 1.831581371 \times 10^{-5} \simeq 0.00002$. (b) The probability that their wish can't be fulfilled, if 48 or more customers are already in cafeteria, is $p_{48} + p_{49} + p_{50} \simeq 0.00007$. Hence, the probability that their wish can be fulfilled is 0.99993.