LNMIIT/B.Tech/C/IC/2017-18/ODD/MTH102/MT

The LNM Institute of Information Technology, Jaipur
Mathematics-I
Mid Term

Duration: 60 mins. Max.Marks: 20

Name: ________________ Roll No.: ________________

PART-B

NOTE: You should attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **Start a new question on a new page and answer all its parts in the same place.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format.

Question No.				
Page No.				

1. (a) State Archimedean Property. If three real numbers a, x, and y satisfy the inequalities

$$a \leq x \leq a + \frac{y}{n}$$

for every integer $n \geq 1$, using Archimedean property show that $x = a$. [2]

Solution: Archimedean Property [**1**]
Given any real numbers x and y, $x > 0$ in $\mathbb{R}$, there is positive integer n, such that $nx > y$.
Assume $x > a$, then the Archimedean Property tells us that there is a positive integer n satisfying

$$n(x-a) > y$$

which is a contradiction to the given inequality. Hence we cant have $x > a$ and thus $x = a$.[**1**]

(b) If $0 < \alpha < 1$ and (x_n) is a sequence satisfying

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n| \text{ for all } n \in \mathbb{N},$$

then show that (x_n) is a Cauchy sequence. [3]

Solution: Note that

$$|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n| \leq \alpha^2 |x_n - x_{n-1}| \leq \ldots \leq \alpha^n |x_2 - x_1|.[\mathbf{1}]$$

For $n > m$,

$$\begin{aligned} |x_n - x_m| &\leq \alpha^{n-2} + \alpha^{n-3} + \ldots + \alpha^{m-1} |x_2 - x_1| [\mathbf{1}] \\ &\leq \frac{\alpha^m}{1-\alpha} |x_2 - x_1| \to 0 \quad \text{ax} \quad m \to \infty.[\mathbf{1}] \end{aligned}$$

Thus (x_n) satisfies the Cauchy criterion.

2. (a) Establish the convergence/divergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$. [2]

Solution: Here $a_n = \left(\frac{n}{n+1}\right)^{n^2}$. Applying root test: we have $(a_n)^{\frac{1}{n}} = \dfrac{1}{(1+\frac{1}{n})^n} \to \dfrac{1}{e} < 1 \Rightarrow$ [**1**] $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$ is convergent.[**1**]

(b) Let (a_n) and (b_n) be two sequences such that

$$a_n = b_{n+1} - b_n \text{ for all } n \geq 1,$$

then show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\lim_{n\to\infty} b_n$ exists and

$$\sum_{n=1}^{\infty} a_n = -b_1 + \lim_{n\to\infty} b_n.$$

[3]

Solution: We note that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if (s_n) converges.[**1**] Now

$s_1 = a_1 = b_2 - b_1$

$s_2 = a_1 + a_2 = b_3 - b_1$

$\vdots$

$s_n = a_1 + a_2 + \cdots + a_n = b_{n+1} - b_1$.[**1**]

Thus, we see that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\lim_{n\to\infty} b_n$ exists and

$$\sum_{n=1}^{\infty} a_n = -b_1 + \lim_{n\to\infty} b_n.$$

[**1**]

3. (a) Suppose the third derivative of f is less than 3 in absolute value. Write the second degree Taylor polynomial $P_2(x)$ for f with center 0. If we approximate $f(x)$ by $P_2(x)$, then estimate the error in the approximation when $|x| \leq 0.1$. [2]

Solution: $P_2(x) = f(0) + \dfrac{f'(0)}{1!}x + \dfrac{f''(0)}{2!}x^2$.**[1]**
The error in approximation of Taylor polynomial of degree 2 is
$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3$ for some c between 0 and x.
Given $|f^{(3)}(x)| < 3$ and $|x| \leq 0.1$, so
$|R_2(x)| = \left|\dfrac{f^{(3)}(c)}{3!}x^3\right| \leq \dfrac{3}{3!}(0.1)^3$**[1]**

(b) Let $f(x)$ and $g(x)$ be continuous on $[a, b]$, differentiable on (a, b) and let $f(a) = f(b) = 0$. Then show that $g'(c)f(c) + f'(c) = 0$ for some $c \in (a, b)$. [3]

Solution: Define $h(x) = f(x)e^{g(x)}$.**[1]**
Thus $h(x)$ is continuous on $[a, b]$, differentiable on (a, b).
Also $h(a) = h(b) = 0$.
By Rolle's Theorem, we have $h'(c) = 0$ for some $c \in (a, b)$.**[1]**
So $h'(c) = e^{g(c)}[f'(c) + g'(c)f(c)] = 0$.
Since $e^{g(c)} \neq 0$, $f'(c) + g'(c)f(c) = 0$.**[1]**

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4. (a) Define lower and upper Riemann integrals of a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ from a closed interval $[a,b]$ to $\mathbb{R}$. Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. Show that $f(x)$ is not Riemann integrable. [2]

Solution: $\underline{\int_a^b} f = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}$ **[0.5]**

$\overline{\int_a^b} f = \inf\{U(P,f) : P \text{ is a partition of } [a,b]\}$**[0.5]**

Let $P = \{0, x_0, x_1, \ldots, x_n = 1\}$ be any partition of $[0,1]$. By density of rational and irrationals $m_i = 0$ and $M_i = 1$ where $m_i = \inf\{f(x) : x_i \leq x \leq x_{i+1}\}$ and $M_i = \sup\{f(x) : x_i \leq x \leq x_{i+1}\}$. $L(P,f) = 0$ and $U(P,f) = 0$. **[0.5]** Thus $\underline{\int_a^b} f = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\} = 0$ and $\overline{\int_a^b} f = \inf\{U(P,f) : P \text{ is a partition of } [a,b]\} = 1$.**[0.5]**

(b) Let $f(x) = \dfrac{x^2-4}{x-1}$, for $x \neq 1$. Find the intervals where the function f is decreasing / increasing. Find local maxima/minima (if any). Find the intervals where the graph of the function is concave up / concave down (convex / concave). Find the points of inflection (if any). [3]

Solution: $f(x) = \dfrac{x^2-4}{x-1} = x + 1 - \dfrac{3}{x-1}; \quad x \neq 1.$
$f'(x) > 0$ for all $x \neq 1$. So function is increasing on $(-\infty, 1) \cup (1, \infty)$**[1]**
So there is no point of local maxima.**[0.5]**
$f''(x) > 0$ for $x = 1$, the function $f(x)$ is convex on $(-\infty, 1)$ and $f''(x) < 0$ for all $x > 1$, the function is concave on $(1, \infty)$.**[1]**
At $x = 1$ concavity changes, but the function is not defined at $x = 1$, so there is no point of inflection.**[0.5]**