The LNM Institute of Information Technology
Jaipur, Rajasthan

Hint: Mid Semester Exam

MATH-II, 22^{nd} February 2016
Time: $1\frac{1}{2}$ Hours, Maximum Marks: 60

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1. (a) Let A, B be two square symmetric matrices of order n. Prove that AB is skew-symmetric matrix if and only if $AB = -BA$. **[4 marks]**

Ans. $(AB)^T = -AB \iff B^TA^T = -AB \iff BA = -AB \iff AB = -BA.$

(b) Let A be an $n\times n$ matrix. Then show that

(i) If A is an idempotent matrix then $\det(A)$ either 0 or 1.

(ii) If A is a nilpotent matrix then $\det(A) = 0$.

(iii) If A is an orthogonal matrix then $\det(A) = \pm 1$.

[2+2+2 marks]

Ans. (i) $A^2 = A \implies \det(A^2) = \det(A) \implies \det(A)\det(A) = \det(A) \implies \det(A)(\det(A)-1) = 0 \implies$ either $\det(A) = 0$ or 1.

(ii) Suppose A is a nilpotent matrix of order k, then $A^k = 0 \implies \det(A^k) = 0 \implies (\det(A))^k = 0 \implies \det(A) = 0$.

(iii) $AA^T = A^TA = I \implies \det(AA^T) = \det(A^TA) = \det(I) \implies \det(A)\det(A^T) = 1 \implies (\det(A))^2 = 1 \implies \det(A) = \pm 1$.

2. (a) Find a basis for the null space $N(A)$ of the matrix A given below:

$$A = \begin{pmatrix} 3 & 4 & 0 & 7 \\ 1 & -5 & 2 & -2 \\ -1 & 4 & 0 & 3 \\ 1 & -1 & 2 & 2 \end{pmatrix}$$

Also verify the rank-nullity theorem. **[6 marks]**

Ans. We know $N(A) = \{X \in \mathbb{R}^4 : AX = 0\}$. Therefore, applying EROs, we get an equivalent system $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x_1, x_2, x_3$ are basic variable and x_4 is free variable, so by setting $x_4 = r$, we get $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = r\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \{\begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\}$ is a basis for the $N(A) \Rightarrow$ nullity of A is 1. The number of non-zero rows in RREF(A) is 3 *i.e.*, rank$(A) = 3$, which verify the rank-nullity theorem.

(b) Find $\alpha \in \mathbb{R}$ such that the system

$$\begin{aligned} x + y - z &= 1, \\ 2x + 3y + \alpha z &= 3, \\ x + \alpha y + 3z &= 2, \end{aligned}$$

posses (i) no solution, (ii) infinitely many solutions, (iii) a unique solution. **[6 marks]**

Ans. The augmented matrix of the system is $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 2 & 3 & \alpha & 3 \\ 1 & \alpha & 3 & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \alpha+2 & 1 \\ 0 & \alpha-1 & 4 & 1 \end{array}\right)$ Now performing $R_3 \to R_3 - (\alpha-1)R_2$, we get $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & \alpha+2 & 1 \\ 0 & 0 & (\alpha-2)(\alpha+3) & \alpha-2 \end{array}\right) \implies$ if $\alpha = -3$, system has no solution. If $\alpha = 2$, system has infinite number of solutions $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5t \\ 1-4t \\ t \end{pmatrix}$, $t \in \mathbb{R}$. If $\alpha \neq 2$ and $\alpha \neq 3$, system has unique solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{\alpha+3} \\ \frac{1}{\alpha+3} \end{pmatrix}$.

3. (*a*) Let u, v, w be three linearly independent vectors in $\mathbb{R}^n$, where $n \geq 3$. For what real values of k, are the vectors $v - u$, $kw - v$ and $u - w$ linearly independent? **[6 marks]**

Ans. vectors $v-u$, $kw-v$ and $u-w$ are L.I. iff $\alpha(u-v)+\beta(kw-v)+\gamma(u-w) = 0_V \Longrightarrow \alpha = \beta = \gamma = 0$. Here, $\alpha(u-v)+\beta(kw-v)+\gamma(u-w) = 0_V \Longrightarrow (\alpha+\gamma)u + (-\alpha-\beta)v + (\beta k - \gamma)w = 0_V$, since vectors u, v, w are L.I. therefore $\alpha+\gamma = -\alpha-\beta = \beta k - \gamma = 0$. This homogeneous system of linear equations with unknown α, β, γ has trivial solution if $k \neq 1$.

(*b*) Consider the vector space $\mathbb{R}^2(\mathbb{R})$. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be two vectors in $\mathbb{R}^2$ such that $u_1v_1 + u_2v_2 = 0$ and $u_1^2 + u_2^2 = 1 = v_1^2 + v_2^2$. Determine whether the set $\{u, v\}$ form a basis for $\mathbb{R}^2$ or not? **[6 marks]**

Ans. With the given conditions,both vectors u and v are orthonormal to each other, therefore $u = (1,0)$ and $v = (0,1)$. Set $\{u,v\}$ is L.I. thus form a basis of $\mathbb{R}^2$.

4. (*a*) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that $T(1,0,0) = (1,0,0), T(1,1,0) = (1,1,1)$ and $T(1,1,1) = (1,1,0)$. Find (*i*) $T(x,y,z)$, (*ii*) $N(T)$, (*iii*) $R(T)$. **[8 marks]**

Ans. Since vectors $(1,0,0)$, $(1,1,0)$, $(1,1,1)$ are L.I., therefore form a basis for $\mathbb{R}^3$. Let $(x,y,z) \in \mathbb{R}^3$ then $(x,y,z) = \alpha(1,0,0)+\beta(1,1,0)+\gamma(1,1,1) \Longrightarrow \alpha = (x-y), \beta = (y-z), \gamma = z \Longrightarrow T(x,y,z) = T((x-y)(1,0,0)+(y-z)(1,1,0)+z(1,1,1)) = (x-y)T(1,0,0)+(y-z)T(1,1,0)+zT(1,1,1) = (x,y,y-z)$. $N(T) = \{(x,y,z)\mathbb{R}^3 : T(x,y,z) = 0_V\} = \{(0,0,0)\}$. By rank-nullity theorem $R(T) = \mathbb{R}^3$.

(*b*) Let $T : U \to V$ be a linear transformation. Show that T is one-one iff null space $N(T)$ is the zero subspace, $\{0_U\}$ of U. **[6 marks]**

Ans. Suppose T is one-one. Then $T(u) = T(v) \Rightarrow u = v$. If $u \in N(T)$, then $T(u) = 0_V = T(0_U) \Rightarrow u = 0_U$. This means that $N(T) = \{0_U\}$.
Conversely suppose $N(T) = \{0_U\}$. Also let $T(u) = T(v)$, then $T(u-v) = T(u) - T(v) = 0_V$. So $u - v \in N(T) = \{0_U\}$. So $u - v = 0_U$, *i.e.*, $u = v$. This implies that T is one-one.

5. (*a*) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation defined by $T(x,y,z) = (x+y,\ y-z)$. Let $B_1 = \{(1,0,1),(0,1,1),(1,1,1)\}$ and $B_2 = \{(1,2),(-1,1)\}$ be bases for $\mathbb{R}^3$ and $\mathbb{R}^2$ respectively, then find the matrix of T with respect to bases B_1 and B_2. **[6 marks]**

Ans. We have $T(1,0,1) = (1,-1) = a_{11}(1,2) + a_{21}(-1,1) = 0(1,2) + (-1)(-1,1)$
$T(0,1,1) = (1,0) = a_{12}(1,2) + a_{22}(-1,1) = \frac{1}{3}(1,2) + (-\frac{2}{3})(-1,1)$
$T(1,1,1) = (2,0) = a_{13}(1,2) + a_{23}(-1,1) = \frac{2}{3}(1,2) + (-\frac{4}{3})(-1,1)$
Thus matrix of T with respect to bases B_1 and B_2 is $\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} \\ -1 & -\frac{2}{3} & -\frac{4}{3} \end{pmatrix}$

(*b*) Verify that the mapping define by $\langle u,v\rangle = 10u_1v_1+3u_1v_2+3u_2v_1+2u_2v_2+u_2v_3+u_3v_2+u_3v_3$, where, $u = (u_1,u_2,u_3)$, $v = (v_1,v_2,v_3)$ is an inner product on $\mathbb{R}^3(\mathbb{R})$. Find the angle between the vectors $(1,1,1)$ and $(2,-5,2)$. **[6 marks]**

Ans. $\langle u, v+w\rangle = 10u_1(v_1+w_1)+3u_1(v_2+w_2)+3u_2(v_1+w_1)+2u_2(v_2+w_2)+u_2(v_3+w_3)+u_3(v_2+w_2)+u_3(v_3+w_3) = 10u_1v_1+3u_1v_2+3u_2v_1+2u_2v_2+u_2v_3+u_3v_2+u_3v_3+10u_1w_1+3u_1w_2+3u_2w_1+2u_2w_2+u_2w_3+u_3w_2+u_3w_3 = \langle u,v\rangle + \langle u,w\rangle$
Similarly, one can easily prove that $\langle \alpha u, v\rangle = \alpha\langle u,v\rangle$, $\alpha \in \mathbb{R}$. Since $u, v \in \mathbb{R}^3$ therefore it is easy to show $\overline{\langle u,v\rangle} = \langle v,u\rangle$.
Also by definition of $\langle , \rangle$, $\langle u,u\rangle \geq 0$ and $\langle u,u\rangle = 0$ iff $u = 0_V$.