THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
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Mid Semester Exam (Solution)

Time: 90 Minutes — 12^{th} September 2016

Mathematics-1 — Maximum Marks: 50

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1. Show that between any two distinct real numbers there is a rational number. **[6]**

Ans. Suppose $x, y \in \mathbb{R}$, $x < y$ i.e $y - x > 0$. We have to find two integers m and n, $n \neq 0$ and $gcd(m, n) = 1$ such that

$$x < \frac{m}{n} < y \text{ or } x < \frac{m}{n} < x + (y - x).$$

Now by the Archimedean property on $y - x > 0$ and 1 there exists a positive integer n such that $n(y - x) > 1$.

Then we can find an integer m lying between nx and ny such that $nx < m < ny$ it implies $x < \frac{m}{n} < y$. Hence the result is proved.

2. Using Sandwich theorem, discuss the convergence of the following sequence: **[6]**

$$x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right), n \in \mathbb{N}.$$

Ans. Clearly $0 \leq x_n$. $x_n \leq (\sqrt{2} - 1)^n$ as $\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right) \leq \left(\sqrt{2} - 1\right)$ for all $n \in \mathbb{N}$.

Hence $0 \leq x_n \leq (\sqrt{2} - 1)^n)$ for all $n \in \mathbb{N}$.

Since $(\sqrt{2} - 1)^n \to 0$ and by Sandwich theorem x_n is convergent and converges to 0.

3. Using Ratio test for sequences, discuss the convergence/divergence of the following sequence:

[6]

$$x_n = \frac{n^n}{(n+1)(n+2)...(n+n)}, n \in \mathbb{N}.$$

Ans.

$$\frac{x_{n+1}}{x_n} = \frac{(n+1)(n+1)^{n+1}}{n^n(2n+1)(2n+2)}.$$

$$\frac{x_{n+1}}{x_n} = \frac{(1+\frac{1}{n})^{n+1}}{2(2+\frac{1}{n})}$$

Takeing limit as $n \to \infty$. So

$$limit_{n\to\infty}\left(\frac{x_{n+1}}{x_n}\right) = \frac{e}{4} < 1.$$

Hence by Ratio test the series is Convergent.

4. Using Root test, find whether the series **[6]**

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + ...$$

is convergent or divergent.

Ans. We have

$$x_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$$

$$(x_n)^{\frac{1}{n}} = \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-1}$$

$$(x_n)^{\frac{1}{n}} = \left(\frac{n+1}{n}\right)^{-1}\left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$(x_n)^{\frac{1}{n}} = \left(1+\frac{1}{n}\right)^{-1}\left[\left(1+\frac{1}{n}\right)^{n} - 1\right]^{-1}.$$

Takeing limit as $n \to \infty$. So

$$limit_{n\to\infty}\,(x_n)^{\frac{1}{n}} = 1.(e-1)^{-1} = \frac{1}{e-1} < 1.$$

So by Root test the series is convergent.

5. Investigate the convergence of the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n-\log n}$. **[6]**

Ans. Here $a_n = \frac{1}{n-\log n}$. Now $\frac{d(a_n)}{dn} = \frac{-(1-\frac{1}{n})}{(n-\log n)^2}$.

Clearly $\frac{d(a_n)}{dn} \leq 0$ for all $n \geq 1$. So a_n is a decreasing.

$\lim_{n\to\infty}\left(\frac{1}{n-\log n}\right) = \lim_{n\to\infty}\left(\frac{1}{n(1-\frac{\log n}{n})}\right) = 0.$

So by Leibnitz Test, series is convergent.

6. Check the continuity of $f(x) = \sin x, x \in \mathbb{R}$ by using $\epsilon - \delta$ definition. **[6]**

Ans. Let $\epsilon > 0$ and $x_0 \in \mathbb{R}$ be any arbitrary point.

$$|f(x) - f(x_0)| = |\sin x - \sin x_0| = |2\sin\frac{(x-x_0)}{2}.\cos\frac{(x-x_0)}{2}| \leq |2\frac{(x-x_0)}{2}|.$$

$$|f(x) - f(x_0)| \leq |(x - x_0)|.$$

Take $\delta = \epsilon$. therefore,

$$\text{whenever } |(x - x_0)| < \delta \text{ it implies } |f(x) - f(x_0)| < \epsilon.$$

So function is continuous at x_0.

Since x_0 is arbitrary point in $\mathbb{R}$. Hence continuous in $\mathbb{R}$.

7. Find the local extremum of the function $f(x) = \frac{1}{x^4-2x^2+7}$. **[7]**

Ans. It can be written as $f(x) = \frac{1}{(x^2-1)^2+6}$. Then $f(x) = \frac{-4(x^3-x)}{(x^4-2x^2+7)^2} = \frac{-4x(x-1)(x+1)}{(x^4-2x^2+7)^2}$.
For local extremum $f'(x) = 0$. $x = -1, 0, 1$ are the stationary points.
f has a local minimum at $x = 0$ as $f(x)$ changes sign from negative to positive at $x = 0$. f has a local maximum at $x = -1$ as $f(x)$ changes sign from positive to negative at $x = -1$. Also f has a local maximum at $x = 1$ as $f(x)$ changes sign from positive to negative at $x = 1$.

8. Show that a polynomial of odd degree has at least one real root. **[7]**

Ans. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$, $a_n \neq 0$ and n be odd.

$$f(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + ... + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right).$$

if $a_n > 0$, then $f(x) \to +\infty$ as $x \to +\infty$ and $f(x) \to -\infty$ as $x \to -\infty$.
Thus by the IVP, there exists x_0 such that $f(x_0) = 0$.

if $a_n < 0$, then $f(x) \to -\infty$ as $x \to +\infty$ and $f(x) \to +\infty$ as $x \to -\infty$.
Thus by the IVP, there exists x_0 such that $f(x_0) = 0$.

Hence every polynomial of odd degree has at least one real root.