The LNM Institute of Information Technology
Department of Mathematics
Mathematics-1 & MTH102
Quiz-2

Time: 35 minutes | Date: 14/11/2019 | Maximum Marks: 10

Note: You should attempt all questions.

Q1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^4+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Check the continuity of f at $(0,0)$. (Justify your claim) [2 Marks]

- Solution: For continuity $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

  Now, check the existence of limit:

  $$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^4+y^2}}$$

  Take path $y = mx^2$, we get

  $$\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{x^4+m^2x^4}} = \frac{1}{\sqrt{1+m^2}}$$ [1.5 Marks]

  As limit depends on m, i.e., limit is not unique $\Rightarrow$ limit of $f(x,y)$ does not exist at $(0,0)$. [0.5 Marks]

Q2. Let $g : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$g(x,y) = \begin{cases} x\dfrac{x^2-y^2}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Using the definition of differentiability, prove or disprove that g is differentiable at $(0,0)$. [3 Marks]

- Ans: The definition of differentiability at $(0,0)$

  $$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f_x(0,0) - f_y(0,0) - f(0,0)}{\sqrt{h^2+k^2}}$$ [0.5 Marks]

  $$\lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$ [0.5 Marks]

  $$\lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0$$ [0.5 Marks]

  Now,

  $$\lim_{(h,k)\to(0,0)} \frac{f(h,k) - f_x(0,0) - f_y(0,0) - f(0,0)}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} h\frac{h^2-k^2}{\sqrt{h^2+k^2}\sqrt{h^2+k^2}}$$

$= \lim_{(h,k)\to(0,0)} h\dfrac{h^2-k^2}{h^2+k^2}$ [0.5 Marks]

Now, we will check the existence of limit $= \lim_{(h,k)\to(0,0)} h\dfrac{h^2-k^2}{h^2+k^2}$

As $|h\frac{h^2-k^2}{h^2+k^2} - 0| \leq |h| < \epsilon$ for any neighborhood $|h-o| < \delta \leq \epsilon; |k-o| < \delta \leq \epsilon$

$\Rightarrow \lim_{(h,k)\to(0,0)} h\dfrac{h^2-k^2}{h^2+k^2}$ exists. [0.5 Marks]

It shows that given function is differentiable at $(0,0)$ [0.5 Marks]

Q3. Test the convergence of the improper integral $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x^2}$. [2 Marks]

Answer: Take $f(x) = \frac{1-e^{-x}}{x^2}$ and $g(x) = \frac{1}{x^2}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} (1-e^{-x}) = 1.$$

[1 Marks]

Hence, Both $\displaystyle\int_1^\infty f(x)$ and $\displaystyle\int_1^\infty g(x)$ either converges or diverges. But, we know that $\displaystyle\int_1^\infty g(x) = \int_1^\infty \frac{1}{x^2}$ converges. So $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x^2}$ also converges. [1 Marks]

Q4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \dfrac{x}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$$

Determine the directional derivatives of f at $(0,0)$ in all possible direction. [3 Marks]

Answer: Let $u = (u_1, u_2)$ be a unit vector in R^2.

$D_u f(0,0) = \lim_{t\to 0} \frac{f(tu_1, tu_2) - f(0,0)}{t}$ [0.5 Marks]

$D_u f(0,0) = 0$ if $u_1 = 0$ [0.5 Marks]

$D_u f(0,0) = 0$ if $u_2 = 0$ [1 Marks]

$D_u f(0,0) = \lim_{t\to 0} \frac{u_1}{tu_2} = \infty$, does not exist. if $u_1 u_2 \neq 0$ [1 Marks]