THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
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IInd Mid Semester Exam

MATH-III, OCTOBER 17, 2013

Part-2

TIME: 40 MINUTES, MAXIMUM MARKS: 26

Instructions: You should attempt all questions. Your writing should be legible and neat. Marks awarded are shown next to the question. **Start a new question on a new page and answer all its parts in the same place.** Please make an index showing the question number and page number on the front page of your answer sheet in the following format, otherwise you may be penalized by the deduction of **2 marks**.

Question No.				
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1. (a) Find and sketch the image of the rectangular region $a \leq x \leq b, c \leq y \leq d$ under the transformation $w = \exp z$. **[3 marks]**

   (b) Show that the transformation $w = \cosh z$ can be written as:

   $$Z = iz + \frac{\pi}{2}, \quad w = \sin Z.$$

   Prove that the transformation $w = \cosh z$ maps the semi-infinite strip $x \geq 0$, $0 \leq y \leq \frac{\pi}{2}$ in the $z-$plane onto the first quadrant $u \geq 0$, $v \geq 0$ of the $w-$plane. **[6 marks]**

2. (a) Let γ denote the upper half of the circle $|z| = r (r > 2)$, taken in the counterclockwise direction. Show that

   $$\int_\gamma \frac{2z^2 - 1}{z^4 + 5z + 4} \leq \frac{\pi r(2r^2 + 1)}{(r^2 - 1)(r^2 - 4)}.$$

   Hence or otherwise deduce that the value of the integral tends to zero as r tends to infinity. **[2+1 marks]**

   (b) (**Cauchy Integral Formula**) Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then prove that

   $$\int_C \frac{f(z)}{z - z_0}\, dz = 2\pi i f(z_0).$$

   Specialize this formula to a circle of radius a centered at z_0. Hence show that $|f(z_0)| \leq \max |f(z_0 + ae^{i\theta})|$. **[4+1+1 marks]**

3. (a) Obtain the first four non-zero terms of the Laurent series expansion for the function $f(z) = \frac{e^z}{z^3(z^2-4)}$ in the region $0 < |z| < 2$. **[3 marks]**

   (b) Use calculus of residues to compute the integral $\int_\gamma \frac{\sin^6 z + (z^2 - \frac{\pi^2}{36})^4}{(z - \frac{\pi}{6})^3}$, where γ is the unit circle traversed in the anticlockwise direction. **[5 marks]**