The LNM Institute of Information Technology
Department: Mathematics
Optimization (MTH3011)
Exam Type: Mid Term

Time: 3 hours | Date: 07/12/2019 | Maximum Marks: 50

1. Is (1,1,2) (i) a feasible solution (ii) basic feasible solution of the following problem. Find all the basic solutions and basic feasible solutions of it.

Maximize $Z = 5x_1 + 10x_2 + 15x_3$. Subject to $x_1 + x_2 + x_3 = 4,\ 2x_1 + 5x_2 - 2x_3 = 3,$ $x_1,\ x_2,\ x_3 \geq 0$. **[5]**

Solution: (i) Here, $(1,1,2)$ satifies the constraints as $1+1+2=4$, $2\times1+5\times1-2\times2=3$ and $x_1,\ x_2,\ x_3 \geq 0$. So $(1,1,2)$ is a feasible solution. (ii) The LPP in standard form: $A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & -2 \end{pmatrix}$. Here rank$(A)$=2 i.e. columns of $x_1,\ x_2,\ x_3$ are linearly independent. Thus $(1,1,2)$ can't be a BFS of it.

In every BFS, number of basic variables will be 2. Here $m=2$, $n=3$. So possible maximum number of BFS is $\binom{n}{m} = 3$. Let $B_1 = \begin{pmatrix} 1 & 1 \\ 2 & 5 \end{pmatrix}$. Then $|B_1| = 3$, $\underline{\mathbf{x}}_{B_1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $B_1^{-1} = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}$. Then $B_1\underline{\mathbf{x}}_{B_1} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Or, $\underline{\mathbf{x}}_{B_1} = B_1^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 5 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 17 \\ -5 \end{pmatrix}$. So $x_1 = \frac{13}{3}$, $x_2 = -\frac{5}{3}$. Thus it is a basic solution but not a basic feasible solution.

Let $B_2 = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$. Then $|B_2| = -4$, $\underline{\mathbf{x}}_{B_2} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, $B_2^{-1} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}$. Then $B_2\underline{\mathbf{x}}_{B_2} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Or, $\underline{\mathbf{x}}_{B_2} = B_2^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -11 \\ -5 \end{pmatrix}$. So $x_1 = \frac{11}{4}$, $x_3 = \frac{5}{4}$. Thus it is a basic solution and also a basic feasible solution.

Let $B_3 = \begin{pmatrix} 1 & 1 \\ 5 & -2 \end{pmatrix}$. Then $|B_3| = -7$, $\underline{\mathbf{x}}_{B_2} = \begin{pmatrix} x_2 \\ x_3 \end{pmatrix}$, $B_3^{-1} = -\frac{1}{7}\begin{pmatrix} -2 & -1 \\ -5 & 1 \end{pmatrix}$. Then $B_3\underline{\mathbf{x}}_{B_3} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Or, $\underline{\mathbf{x}}_{B_2} = B_2^{-1}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -2 & -1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -11 \\ -17 \end{pmatrix}$. So $x_2 = \frac{11}{7}$, $x_3 = \frac{17}{7}$. Thus it is a basic solution and also a basic feasible solution.

Or

A canning company operates two canning plants. The growers are willing to supply fresh fruits in the following amounts: (i) S_1: 200 tonnes at \$ 11/tonne (ii) S_2: 310 tonnes at \$ 10/tonne (iii) S_3: 420 tonnes at \$ 9/tonne. The canned fruits are sold at \$50/tonne to the distributors. The company can sell at this price all they can produce. The objective is to find the best mixture of the quantities supplied by the three growers to the two plants so that the company maximises its profits. (a) Formulate the problem as a linear program. (b) What assumptions have you made in expressing the problem as a linear program.

Shipping costs in \$ per tonne				Plant capacities and labour costs		
From/To	Plant A	Plant B			Plant A	Plant B
S_1	3	3.5		Capacity	460 tonnes	560 tonnes
S_2	2	2.5		Labour cost	\$26/tonne	\$21/tonne
S_3	6	4				

Solution: Let x_{ij} be the number of tonnes supplied from grower i (i=1,2,3 for S_1, S_2 and S_3 respectively) to plant j (j=1 for Plant A and j=2 for Plant B) where $x_{ij} \geq 0, i = 1,2,3; j = 1,2$. Fruits cost, shipping costs from S_1, S_2 and S_3 are $11(x_{11}+x_{12})$, $3x_{11}+3.5x_{12}$; $10(x_{21}+x_{22})$, $2x_{21}+2.5x_{22}$; $9(x_{31}+x_{32})$, $6x_{31}+4x_{32}$ respectively. Labour costs in Plant A & Plant B are $26(x_{11}+x_{21}+x_{31})$, $21(x_{12}+x_{22}+x_{32})$. Then the objective function is Maximize $z = 50\sum_{i=1}^{3}\sum_{j=1}^{2} x_{ij} - 11(x_{11}+x_{12}) - (3x_{11}+3.5x_{12}) - 10(x_{21}+x_{22}) - (2x_{21}+2.5x_{22}) - 9(x_{31}+x_{32}) - (6x_{31}+4x_{32})$. The constraints are (i) grower availablity constraints: $x_{11}+x_{12} \leq 200$, $x_{21}+x_{22} \leq 310$, $x_{31}+x_{32} \leq 420$ (ii) Plant capacity constraint: $x_{11}+x_{21}+x_{31} \leq 420$, $x_{12}+x_{22}+x_{32} \leq 560$.

The basic assumptions are: (i) can ship from a grower any quantity we desire (ii) no loss in weight in processing at the plant (iii) no loss in weight in shipping (iv) can sell all we produce.

2. Use simplex method to solve the following problem: Minimize $Z = x_1 - 2x_2 - 3x_3$; Subject to $-2x_1 + x_2 + 3x_3 = 2$, $2x_1 + 3x_2 + 4x_3 = 1$, $x_1,\ x_2,\ x_3 \geq 0$. **[5]**

   Solution: The L.P.P is in standard form. To start with intial basis matrix as identity matrix I_2, we introduce two artificial variable x_4 and x_5 in the 1st and 2nd constraints respectively and will solve it by Big-M Method.

   Min. $z' = x_1 - 2x_2 - 3x_3 + Mx_4 + Mx_5$, Sub. to $-2x_1 + x_2 + 3x_3 + x_4 + 0x_5 = 2$
   $2x_1 + 3x_2 + 4x_3 + 0x_4 + x_5 = 1$
   $x_1,\ x_2, \cdots, x_5 \geq 0$. For initial B.F.S, $B = [\beta_1, \beta_2] = [\alpha_4, \alpha_5] = I_2$, $\underline{\mathbf{x_B}} = [x_4, x_5]^t = [2, 1]^t$, $\underline{\mathbf{c_B}} = [c_4, c_5]^t = [M, M]^t$. Then we can make following tables:

Table 1: First Simplex Table of Big-M Method

			$\underline{\mathbf{c_j}} \rightarrow$	1	-2	-3	M	M	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	for $\alpha_j \uparrow$
M	α_4	x_4	2	-2	1	3	1	0	$\frac{2}{3}$
M	α_5	x_5	1	2	$\boxed{3}$	4	0	1	$\frac{1}{4} \Rightarrow$
$z' = 3M$			(z_j-c_j) $\rightarrow$	-1	$4M+2$	$7M+3 \Uparrow$	0	0	Min. $\{\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0\}$=$\frac{1}{4}$

Since in Table 1, the optimality condition $z_j - c_j \leq 0$ for all j does not satisfy, then we can go to next table with x_3 coming in basis and x_5 leaving the basis.

Table 2: First Simplex Table of Big-M Method

			$\underline{\mathbf{c_j}} \rightarrow$	1	-2	-3	M	M	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	for $\alpha_j \uparrow$
M	α_4	x_4	$\frac{5}{4}$	-$\frac{7}{2}$	-$\frac{5}{4}$	0	1	-$\frac{3}{4}$	
-3	α_3	x_3	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	0	$\frac{1}{4}$	
$z' = \frac{5M-3}{4}$			(z_j-c_j) $\rightarrow$	$-\frac{7M+5}{2}$	-$\frac{5M+1}{4}$	0	0	$-\frac{3M+3}{4}$	Min. $\{\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0\}$

Since in Table 2, the optimality condition $z_j - c_j \leq 0$ for all j satisfies, but artificial variable x_4 remains in the basis with positive value ($x_4 = \frac{5}{4}$). Then the given LPP has no feasible solution.

3. The following table shows the transportation cost (in Rs. 100) of shipping one unit of a mobile product of a company from the source plants P_1, P_2, P_3 to the destination cities C_1, C_2, C_3, C_4. The demand of each destination and the supply of each source are also given in the table. How should mobile distributed to the cities such that the company minimizes cost of mobile supply? **[5]**

	C_1	C_2	C_3	C_4	Supply
P_1	1	2	-	3	20
P_2	7	1	2	3	12
P_3	-	6	5	4	12
Demand	16	7	11	10	

Solution: The T.P. is balanced as $\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 44$. Since the roots from P_1 to C_3 and P_3 to C_1 are closed, we put very large postive cost M in the corresponding cells to avoid allocation. For initial BFS, we will follow VAM method.

Table 3: First Table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply	Supply	Supply	Supply
P_1	$\boxed{16_I}$ 1	$\boxed{4_{IV}}$ 2	M	3	a_1=20 d_{rw1}=1	a_1=4 d_{rw1}=1	a_1=4 d_{rw1}=1	a_1=4
P_2	7	$\boxed{1_V}$ 1	$\boxed{11_{II}}$ 2	3	a_2=12 d_{rw2}=1	a_2=12 d_{rw2}=1	a_2=1 d_{rw2}=2	a_2=1
P_3	M	$\boxed{2_{VI}}$ 6	5	$\boxed{10_{III}}$ 4	a_3=12 d_{rw3}=1	a_3=12 d_{rw3}=1	a_3=12 d_{rw3}=2	a_3=2
Demand	b_1=16 d_{cl1}=6	b_2=7 d_{cl2}=1	b_3=11 d_{cl3}=3	b_4=10 d_{cl4}=0				
Demand	×	b_2=7 d_{cl2}=1	b_3=11 d_{cl3}=3	b_4=10 d_{cl4}=0				
Demand	×	b_2=7 d_{cl2}=1	×	b_4=10 d_{cl4}=0				
Demand	×	b_2=7 d_{cl2}=1	×	×				

We can see that the number of occupied cell is 6 ($= m + n - 1$) and they do not form a loop. Hence it is a intial B.F.S. For optimality test, we have to find u_i for $i = 1, 2, 3$ and v_j for $j = 1, 2, 3, 4$ satisfying $c_{ijc} = u_i + v_j$ for occupied cells. I.e. $u_1 + v_1 = 1$, $u_1 + v_2 = 2$, $u_2 + v_2 = 1$, $u_2 + v_3 = 2$, $u_3 + v_2 = 6$, $u_3 + v_4 = 4$. Since, we have to choose one u_i=0 or v_j=0, we choose v_2=0 (as it appears maximum times). Then $u_1 = 2$, $u_2 = 1$, $u_3 = 6$, $v_1 = -1$, $v_3 = 1$, $v_4 = 2$. Then we have calculated the cell evaluation $\Delta_{ij} = c_{ij} - u_i - v_j$ for unoccupied cells (i, j)th as $\Delta_{13} = M - 3$, $\Delta_{14} = 3$, $\Delta_{21} = 7$, $\Delta_{24} = 4$, $\Delta_{31} = M - 5$, $\Delta_{33} = -2$ (shown in Table 4).

Since, $\Delta_{33} = -2 < 0$, current B.F.S. is not minimum solution and for improvement, we allocate maximum possible unit to $(3, 3)$th cell and adjust this additional allotment such that the cell $(3, 2)$th become empty as shown Fig. 5

So, all the first modified allocated cells are given in Table 6

For optimality test, we have to find u_i for $i = 1, 2, 3$ and v_j for $j = 1, 2, 3, 4$ satisfying $c_{ijc} = u_i + v_j$ for occupied cells. I.e. $u_1 + v_1 = 1$, $u_1 + v_2 = 2$, $u_2 + v_2 = 1$, $u_2 + v_3 = 2$, $u_3 + v_3 = 5$, $u_3 + v_4 = 4$. Since, we have to choose one u_i=0 or v_j=0, we choose u_1=0 (as it appears as one of maximum times). Then $u_2 = -1$, $u_3 = 2$, $v_1 = 1$, $v_2 = 2$, $v_3 = 3$,

Table 4: Second Table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{16_I}$ 1	$\boxed{4_{IV}}$ 2	$M-3$ M	3 3	a_1=20
P_2	7 7	$\boxed{1_V}$ 1	$\boxed{11_{II}}$ 2	4 3	a_2=12
P_3	$M-5$ M	$\boxed{2_{VI}}$ 6	-2 5	$\boxed{10_{III}}$ 4	a_3=12
Demand	b_1=16	b_2=7	b_3=11	b_4=10	

Table 5: First re-allocation of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{16}$	$\boxed{4}$			a_1=20
P_2		$\boxed{1}+2$	$\boxed{11}-2$		a_2=12
P_3		$\boxed{2}-2$	$\bullet+2$	$\boxed{10}$	a_3=12
Demand	b_1=16	b_2=7	b_3=11	b_4=10	

Table 6: First modified allocated cells after VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{16}$ 1	$\boxed{4}$ 2	M	3	a_1=20
P_2	7	$\boxed{3}$ 1	$\boxed{9}$ 2	3	a_2=12
P_3	M	6	$\boxed{2}$ 5	$\boxed{10}$ 4	a_3=12
Demand	b_1=20	b_2=20	b_3=50	b_4=60	

$v_4 = 2$. Then we have calculated the cell evaluation $\Delta_{ij} = c_{ij} - u_i - v_j$ for unoccupied cells (i, j)th as $\Delta_{13} = M - 3$, $\Delta_{14} = 1$, $\Delta_{21} = 7$, $\Delta_{24} = 2$, $\Delta_{31} = M - 3$, $\Delta_{32} = 2$ (shown in Table **??**).

Since, $\Delta_{ij} > 0$ for all unoccupied cells, current B.F.S. is minimum solution with basic variables $x_{11} = 16$ (i.e. 16 mobiles from plant P_1 to city C_1), $x_{11} = 4$ (i.e. 4 mobiles from plant P_1 to city C_2), $x_{22} = 3$ (i.e. 3 mobiles from plant P_2 to city C_2), $x_{23} = 9$ (i.e. 9 mobiles from plant P_2 to city C_4), $x_{33} = 2$ (i.e. 2 mobiles from plant P_3 to city C_4) and $x_{34} = 10$ (i.e. 10 mobiles from plant P_3 to city C_4) and the minimum transportation costs will be $16 \times 1 + 4 \times 2 + 3 \times 1 + 9 \times 2 + 2 \times 5 + 10 \times 4 = 95$ hundreds in rupees.

4. (a) Show that if a constant be added to any row and/or any column of the cost matrix of an assignment probelm, then the resulting assignment problem has the same optimal solution as the optimal problem. **[3]**

Solution: Let the cost matrix be $C = (c_{ij})_{n\times n}$ and suppose that we add α_i to row i, $i = 1, 2, \cdots, n$ and β_j to column j, $j = 1, 2, \cdots, n$. Then the new cost matrix is $\bar{C} = (\bar{c}_{ij})_{n\times n}$ where $\bar{c}_{ij} = c_{ij} + \alpha_i + \beta_j$. if we denote z and $\bar{z}$ the val-

Table 7: Fifth Table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{16}$ 1	$\boxed{4}$ 2	$M-3$ M	1 3	a_1=20
P_2	7 7	$\boxed{3}$ 1	$\boxed{9}$ 2	2 3	a_2=12
P_3	$M-3$ M	2 6	$\boxed{2}$ 5	$\boxed{10}$ 4	a_3=12
Demand	b_1=16	b_2=7	b_3=11	b_4=10	

ues of the objective function of the original and the new problems respectively, then $\bar{z} = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{c}_{ij}x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n}(c_{ij} + \alpha_i + \beta_j)x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i x_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_j x_{ij} = z + \sum_{i=1}^{n} \alpha_i \left(\sum_{j=1}^{n} x_{ij}\right) + \sum_{j=1}^{n} \beta_j \left(\sum_{i=1}^{n} x_{ij}\right) = z + \sum_{i=1}^{n} \alpha_i + \sum_{j=1}^{n} \beta_j$, as $\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{n} x_{ij} = 1$.

Thus we see that z and $\bar{z}$ differ by a constant which is independent of values of the variables x_{ij} and hence the optimal solution of the original problem must be the optimal solution of the new problem and conversely.

(b) Show that if all $c_{ij} \geq 0$ and we can find a set $x_{ij} = x_{ij}^*$ such that $\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}^* = 0$ for a minimization problem, then this solution is optimal. **[2]**

Solution: We have $c_{ij} \geq 0$ and $x_{ij} \geq 0$. Then $\sum_{i=1}^{m}\sum_{j=1}^{m} c_{ij}x_{ij}^* \geq 0$. Hence its minimum value is zero which attained for $x_{ij} = x_{ij}^*$. Thus the present solution is optimal.

Or

The processing times in hours for the jobs when allocated to the different machines are indicated below. When a job is not possible to be made in a particular manner, it is indicated as '-'. How would the jobs be allocated to the different machines such that the total time minimizes? **[5]**

	Machine				
Job	I	II	III	IV	V
A	5	-	10	-	10
B	6	9	17	20	10
C	10	14	-	-	14
D	7	7	10	5	8
E	12	14	17	12	-

Solution: Since, the Job A can't be possible to be made by Machine II & IV, Job B can't be possible to be made by Machine III & IV and Job E can't be possible to be made by Machine V, we put large cost M which is bigger than the other costs and then row reduction and column reduction has been done in following tables.

Table 8: Row reduction of Hungarian method of assignment problem

Depots/Towns	I	II	III	IV	V
A	0	M-5	5	M-5	5
B	0	3	11	14	4
C	0	4	M-10	M-10	4
D	2	2	5	0	3
E	0	2	5	0	M-12

Table 9: Column reduction of Hungarian method of assignment problem

Depots/Towns	I	II	III	IV	V
A	0	M-7	0	M-5	2
B	0	1	6	14	1
C	0	2	M-15	M-10	1
D	2	0	0	0	0
E	0	0	0	0	M-15

We can cover all the zeros by 4 lines as shown in Table

Table 10: Cover all zeros by Hungarian method of assignment problem

Depots/Towns	I	II	III	IV	V
A	0	M-7	0	M-5	2
B	0	1	6	14	1
C	0	2	M-15	M-10	1
D	2	0	0	0	0
E	0	0	0	0	M−15

Since the minimum number of lines to cover all the zeros is 4< 5, the order of the cost matrix, we choose the least cost 1 (say) at cell (2,2)th from the uncovered cells and subtract it from all the remaining uncovered cells and add it to the cell where both horizontal and vertical lines pass (no change at the cells where only horizontal or vertical line passes). Then we cover the zeros as shown in the Table 11

Table 11: Cover all zeros by Hungarian method of first modified matrix of assignment problem

Depots/Towns	I	II	III	IV	V
A	1	M-7	0	M-5	2
B	0	0	5	13	0
C	0	1	M-16	M-11	0
D	3	0	0	0	0
E	1	0	0	0	M−15

Now, the minimum number of lines to cover all the zeros is 5, which is equal to the order of the cost matrix as shown in Table ??. Then we can make job assignment $A \to III$, $B \to I$, $C \to V$, $D \to II$, $E \to IV$, as shown in Table 12 and the minumum total time will be $10 + 6 + 14 + 7 + 12 = 49$ hours.

Table 12: Car assignment by Hungarian method of assignment problem

Depots/Towns	I	II	III	IV	V
A	1	M-7	$\boxed{0}_{i}$	M-5	2
B	$\boxed{0}_{ii}$	$\cancel{0}$	5	13	$\cancel{0}$
C	$\cancel{0}$	1	M-16	M-11	$\boxed{0}_{iii}$
D	3	$\boxed{0}_{iv}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$
E	1	$\cancel{0}$	$\cancel{0}$	$\boxed{0}_{v}$	M-15

5. A Indian Steel Company CEO has to visit five cities Jaipur, Varansi, Mumbai, Kolkata, Chennai for execute the plants in the citites. He does not want to visit any city twice before his completing tour of all the cities and wishes to return to the starting city. Costs (in Rs. 1000) of going from one city to another are given below in the usual manner. Determine the optimal cost route. **[5]**

From/To	Jaipur	Kolkata	Mumbai	Chennai	Varansi
Jaipur	∞	2	1	6	1
Kolkata	1	∞	4	4	2
Mumbai	5	3	∞	1	5
Chennai	4	7	2	∞	1
Varansi	2	6	3	6	∞

Solution:

6. Solve the game graphically whose pay-off matrix is given below: **[5]**

2	-2
4	6
0	7
5	2
3	3
-4	1

Solution:

7. Transform to L.P.P and solve the game problems whose pay-off matrices are given below: **[5]**

-1	2	1
1	-2	2
3	4	-3

Solution:

8. Visitors' parking at the LNMIIT, Jaipur is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked

car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following: (i) The probability, p_n of n cars in the system. (ii) The effective arrival rate for cars that actually use the lot (system). (iii) The average number of cars in the lot (system). (iv) The average time a car waits for a parking space inside the lot. (v) The average number of occupied parking spaces. (vi) The average utilization of the parking lot. **[5]**

Solution:

Or

For the queue model $(M/M/1) : (GD/\infty/\infty)$; find L_S, L_q, W_S, W_q, $\bar{c}$. **[5]**

9. As a project manager, you are faced with the activity network and estimated activity times shown in Figure 1. Determine the critical path for the project network.

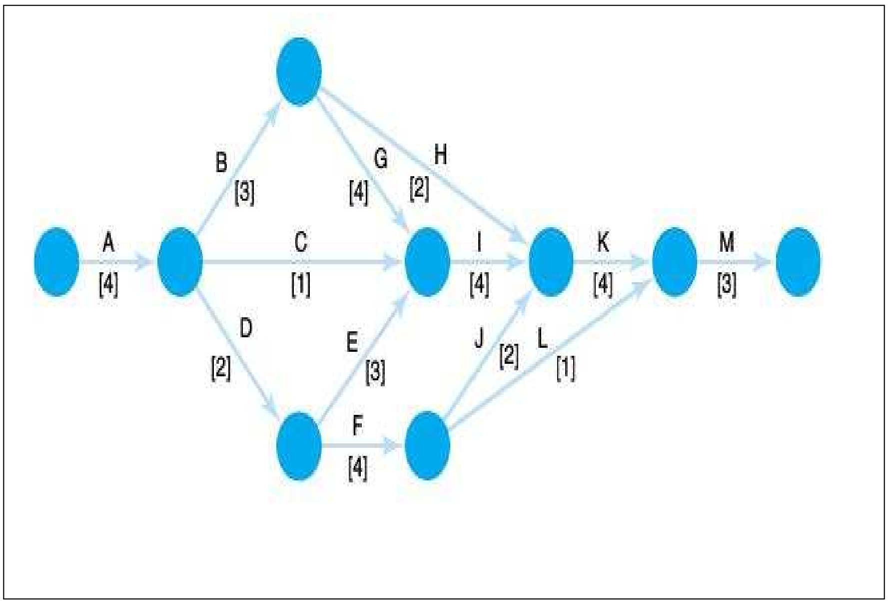


Figure 1: The project network.

Solution:

10. Jaipur development authority has installed 1000 street lights. Cost of individual replacement is Rs. 400. Failure probabilities are as given below. Find the number of lights will be replaced in each of the first 10 months. What will be the average cost associated with individual replacement?

Month	1	2	3	4	5
Failure probability	0.1	0.1	0.3	0.3	0.2

[5]

Solution:

Q.5

From/To	Jaipur	Kolkata	Mumbai	Chennai	Varansi
Jaipur	∞	2	1	6	1
Kolkata	1	∞	4	4	2
Mumbai	5	3	∞	1	5
Chennai	4	7	2	∞	1
Varansi	2	6	3	6	∞

$k = 4 < 5$

$1 \to 2 \to 1$ ✗

6. Solve the game graphically whose pay-off matrix is given below:

2	-2
4	6
0	7
5	2
3	3
-4	1

7. Transform to L.P.P and solve the game problems whose pay-off matrices are given below: [5]

-1	2	1
1	-2	2
3	4	-3

$1 \to 2 \to 3 \to 4 \to 5$

$z^*_{new} = 3$

$\therefore z^* = 2+4+1+1+2 = 10$

$1 \to 2 \to 5 \to 1$, $3 \to 4 \to 4$ ✗

$1 \to 3 \to 2 \to 4 \to 5 \to 1$ $\quad z^*_{new} = 3$

$z^* = 1+3+4+1+2 = 11$

8. Visitors' parking at the LNMIIT, Jaipur is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following: (i) The probability, p_n of n cars in the system. (ii) The effective arrival rate for cars that actually use the lot (system). (iii) The average number of cars in the lot (system). (iv) The average time a car waits for a parking space inside the lot. (v) The average number of occupied parking spaces. (vi) The average utilization of the parking lot. [5]

Or

For the queue model $(M/M/1) : (GD/\infty/\infty)$; find L_S, L_q, W_S, W_q, $\bar{c}$. [5]

9. As a project manager, you are faced with the activity network and estimated activity times shown in Figure 1. Determine the critical path for the project network.

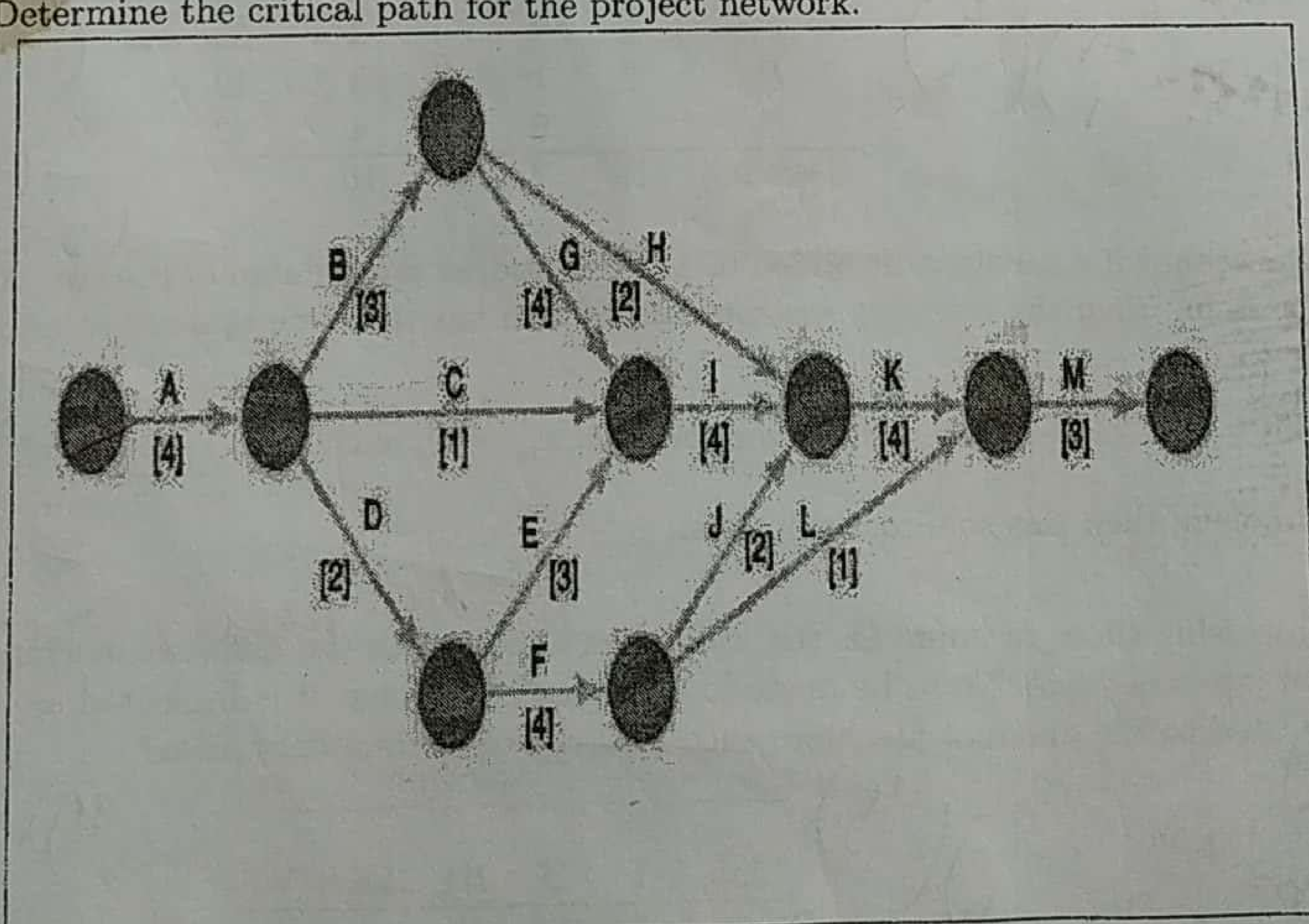
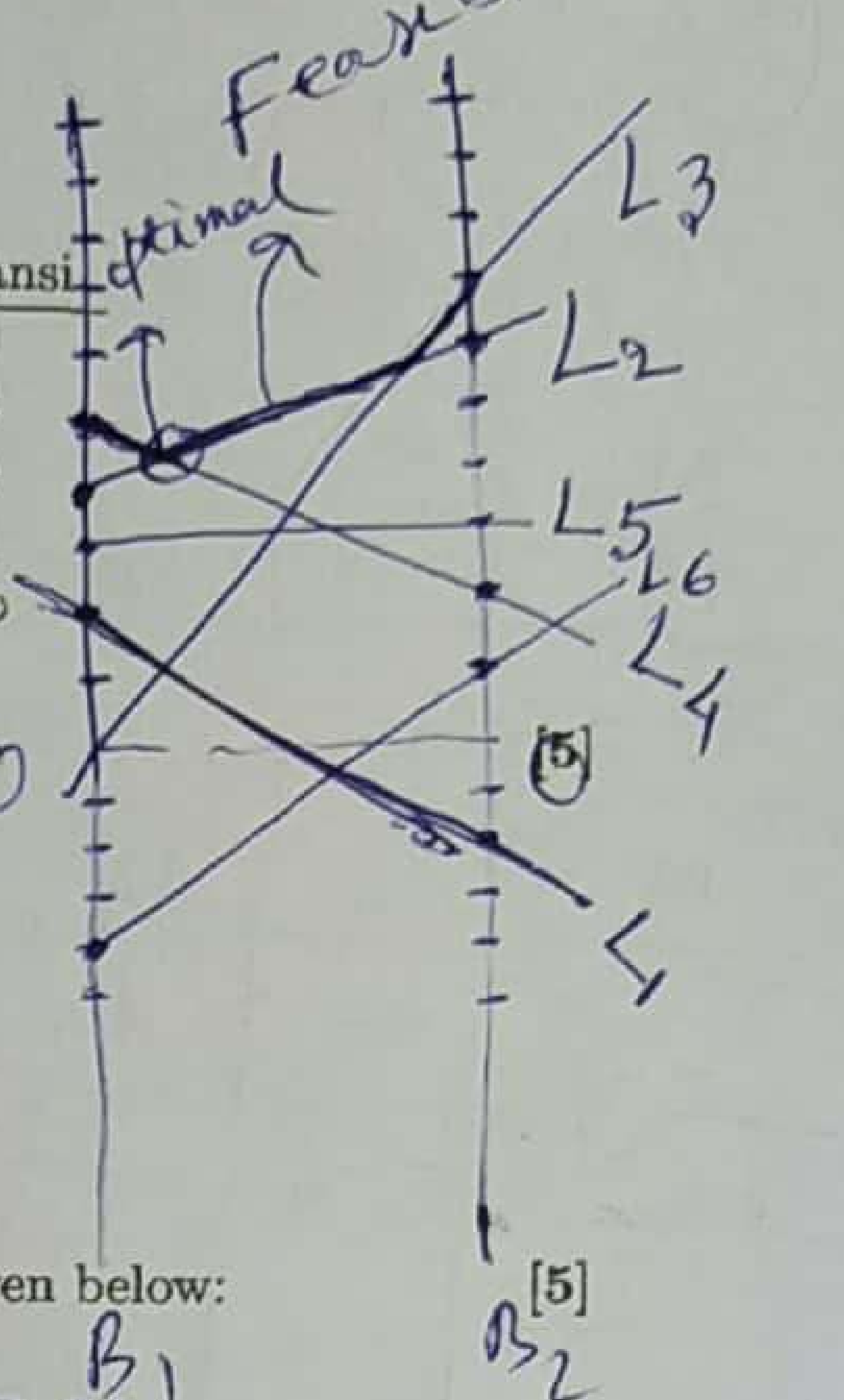


Figure 1: The project network.

10. Jaipur development authority has installed 1000 street lights. Cost of individual replacement is Rs. 400. Failure probabilities are as given below. Find the number of lights will be replaced in each of the first 10 ... individual replacement?

From/To	Jaipur	Kolkata	Mumbai	Chennai	Varansi
Jaipur	∞	2	1	6	1
Kolkata	1	∞	4	4	2
Mumbai	5	3	∞	1	5
Chennai	4	7	2	∞	1
Varansi	2	6	3	6	∞



6. Solve the game graphically whose pay-off matrix is given below:

2	-2
4	6
0	7
5	2
3	3
-4	1

$x_2 = \frac{2-5}{6-11} = \frac{3}{5}$, (L_2, L_4)

$x_4 = \frac{4-6}{-5} = \frac{2}{5} x_2$

$y_1 = \frac{2-6}{-5} = \frac{4}{5}, x_4$

	y_1	y_2	
	4	6	4
	5	2	2
	5	6	

$v = 4$, $\bar{v} = 5$

7. Transform to L.P.P and solve the game problems whose pay-off matrices are given below: [5]

-1	2	1	-1
1	-2	2	-2
3	4	-3	-3
3	4	2	$\bar{v}=2$

$y_2 = \frac{4-5}{-5} = \frac{1}{5}$, $v = \frac{8-30}{-5} = \frac{22}{5}$

$v = -1$ $\therefore -1 \le v \le 2$

We add 2 to make $v' > 0$ in new game

	y_1	y_2	y_3
x_1	1	4	3
x_2	3	0	4
x_3	5	6	-1

$X_i = \frac{x_i}{v}$, $Y_j = \frac{y_j}{v}$

For B:

max $\frac{1}{v} = Y_1 + Y_2 + Y_3$

$Y_1 + 4Y_2 + 3Y_3 \le 1,$

$3Y_1 + 4Y_3 \le 1,$

$5Y_1 + 6Y_2 - Y_3 \le 1,$

$Y_i \ge 0.$

For A:

min $\frac{1}{v} = X_1 + X_2 + X_3$

$X_1 + 3X_2 + 5X_3 \ge 1,$

$4X_1 + 6X_3 \ge 1,$

$3X_1 + 4X_2 - X_3 \ge 1,$

$X_i \ge 0.$

8. Visitors' parking at the LNMIIT, Jaipur is limited to five spaces only. Cars making use of this space arrive according to a Poisson distribution at the rate of six cars per hour. Parking time is exponentially distributed with a mean of 30 minutes. Visitors who cannot find an empty space on arrival may temporarily wait inside the lot until a parked car leaves. That temporary space can hold only three cars. Other cars that cannot park or find a temporary waiting space must go elsewhere. Determine the following: (i) The probability, p_n of n cars in the system. (ii) The effective arrival rate for cars that actually use the lot (system). (iii) The average number of cars in the lot (system). (iv) The average time a car waits for a parking space inside the lot. (v) The average number of occupied parking spaces. (vi) The average utilization of the parking lot. [5]

Or

For the queue model $(M/M/1) : (GD/\infty/\infty)$; find L_S, L_q, W_S, W_q, $\bar{c}$. [5]

$n_8 = 0.2 \times 321 + 0.3 \times 373 + 0.3 \times 332 + 0.1 \times 220 + 0.1 \times 295$

$n_9 = 0.2 \times 373 + 0.3 \times 332 + 0.3 \times 220 + 0.1 \times 283 + 0.1 \times 323$

$= 74.6 + 99.6 + 66 + 28.3 + 32.3 \approx 301$

$n_{10} = 0.2 \times 332 + 0.3 \times 220 + 0.3 \times 283 + 0.1 \times 323 + 0.1 \times 301 \approx 284$

9. As a project manager, you are faced with the activity network and estimated activity times shown in Figure 1. Determine the critical path for the project network.

critical path

$A \to B \to G \to I \to K \to M$

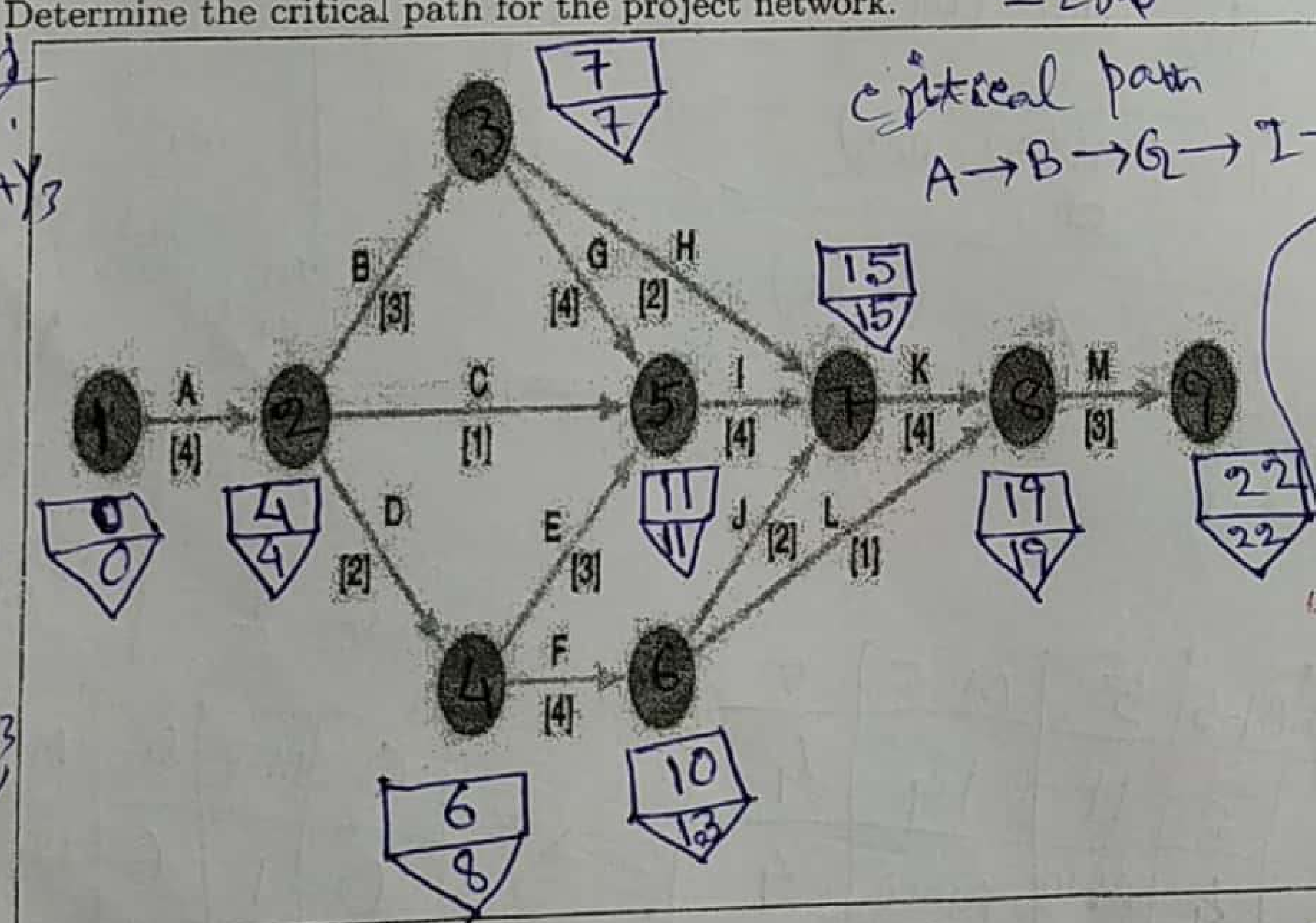


Figure 1: The project network.

$+0.3 \times 321 + 0.1 \times 373 + 0.1 \times 332$

$= 20 + 33 + 96.3 + 37.3 + 33.2 \approx 220,$

$n_7 = 0.2 \times 110 + 0.3 \times 321 + 0.3 \times 373 + 0.1 \times 332 + 0.1 \times 220$

$= 22 + 96.3 + 111.9 + 33.2 + 22 \approx 285$

10. Jaipur development authority has installed 1000 street lights. Cost of individual replacement is Rs. 400. Failure probabilities are as given below. Find the number of lights will be replaced in each of the first 10 months. What will be the average cost associated with individual replacement? [5]

Month	1	2	3	4	5
Failure probability	0.1	0.1	0.3	0.3	0.2

$\bar{n} = \frac{1000}{1 \times 0.1 + 2 \times 0.1 + 3 \times 0.3 + 4 \times 0.3 + 5 \times 0.2} = \frac{1000}{3.4} \approx 294$

$n_0 = 1000$, $n_1 = 0.1 \times 1000 = 100$, $n_2 = 0.1 \times 1000 + 0.1 \times 100 = 110$, $n_3 = 0.3 \times 1000 + 0.1 \times 110 + 0.1 \times 100 = 330 + 11 \approx 321$

$n_4 = 0.3 \times 1000 + 0.3 \times 100 + 0.1 \times 110 + 0.1 \times 321 = 300 + 30 + 11 + 32.1 \approx 373$

$n_5 = 0.2 \times 1000 + 0.3 \times 100 + 0.3 \times 110 + 0.1 \times 321 + 0.1 \times 373 = 200 + 30 + 33 + 32.1 + 37.3 \approx 332$, $n_6 = 0.2 \times 100 + 0.3 \times 110 + \ldots$

-1	2	1	-1
1	-2	2	-2
3	4	-3	-3
3	4	2	

$\gamma = -1$

$\gamma = 2$

$B = (a_{ij} + 2)$

$$Q = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 4 & 3 \\ x_2 & 3 & 0 & 4 \\ x_3 & 5 & 6 & -1 \end{array}$$

$Y_i = \frac{y_i}{\gamma}$ | $X_i = \frac{x_i}{\gamma}$

Max $\frac{1}{\gamma} = \Sigma Y_i$) Sub. to $Y_1 + 4Y_2 + 3Y_3 \le 1$, $3Y_1 + 4Y_3 \le 1$, $5Y_1 + 6Y_2 - Y_3 \le 1$, $Y_1, Y_2, Y_3 \ge 0$.

Standard form

Max $\frac{1}{\gamma} = Y_1 + Y_2 + Y_3 + 0Y_4 + 0Y_5 + 0Y_6$

Sub. to. $Y_1 + 4Y_2 + 3Y_3 + Y_4 = 1$, $3Y_1 + \cancel{4}4Y_3 + Y_5 = 1$, $5Y_1 + 6Y_2 - Y_3 + Y_6 = 1$, $Y_1, Y_2, \ldots, Y_6 \ge 0$.

C_B	B	Y_B	c_j / b	1 a_1	1 a_2	1 a_3	0 a_4	0 a_5	0 a_6	
0	a_4	Y_4	1	1	4	3	1	0	0	Min{1, 1/3, 1/5}
0	a_5	Y_5	1	3	0	4	0	1	0	
0	a_6	Y_6	1	$\boxed{5}$	6	-1	0	0	1	⇒
		$z_j - c_j$		-1↑	-1	-1	0	0	0	$R_3 \to R_3/5$
0	a_4	Y_4	4/5	0	14/5	16/5	1	0	-1/5	$R_2 \to R_2 - \frac{3}{5}R_3$
0	a_5	Y_5	2/5	0	-18/5	$\boxed{23/5}$	0	1	-3/5 ⇒	$R_1 \to R_1 - 1/5 R_3$
1	a_1	Y_1	1/5	1	6/5	-1/5	0	0	1/5	Min$\{\frac{4}{5}\times\frac{5}{16}, \frac{2}{5}\times\frac{5}{23}\} = \frac{2}{23}$
		$z_j - c_j$		0	1/5	-6/5↑	0	0	1/5	
0	a_4	Y_4	12/23	0	$\boxed{122/23}$	0	1	-16/23	5/23 ⇒	$R_2 \to R_2 \times \frac{5}{23}$
1	a_3	Y_3	2/23	0	-18/23	1	0	5/23	-3/23	$R_1 \to R_1 - \frac{16}{23}R_2$
1	a_1	Y_1	5/23	1	24/23	0	0	1/23	4/23	
		$z_j - c_j$		0	-17/23↑	0	0	6/23	1/23	
1	a_2	Y_2	6/61	0	1	0	23/122	-8/61	$\frac{5}{122}$	
1	a_3	Y_3	10/61	0	0	1	9/61	7/61	$-\frac{6}{61}$	
1	a_1	Y_1	7/61	1	0	0	-12/61	$\frac{11}{61}$	$\frac{8}{61}$	
		$z_j - c_j$		0	0	0	17/122	$\frac{10}{61}$	$\frac{9}{122}$	

$\frac{4}{5}$ 0 14/5 16/5 1 0

$-\frac{2}{5}\times\frac{16}{23}$ 0 $\frac{18}{5}\times\frac{16}{23}$ $-\frac{16}{5}$ 0 $-\frac{16}{23}$

$\frac{60}{23\times5}$ 0 $\frac{610}{5\times23}$ 0 1 -16/23

$R_3 \to R_3 + \frac{1}{23}R_2$

$\frac{1}{5}$ 1 6/5 23/5 0 0 1/5

$\frac{1}{23}\times\frac{2}{5}$ 0 $-\frac{18}{5}\times\frac{1}{23}$ $-\frac{23}{5}$ 0 1/23 $-\frac{3}{5}\times\frac{1}{23}$

5/23 1 $\frac{120}{5\times23}$ 0 0 1/23 $\frac{20}{5\times23}$

$\frac{3}{5}\times\frac{16}{23}$

$\frac{25}{5\times23}$

Min$\{\frac{12}{23}\times\frac{23}{122}, \frac{5}{23}\times\frac{23}{22}\} = \frac{6}{61}$

$R_1 \to R_1 / \frac{123}{23}$

$R_2 \to R_2 + \frac{9}{61}R_1$

2/23 0 -18/23 1 0 5/23 -3/23

$+\frac{12}{23}\times\frac{9}{61}$ 0 $\frac{122}{23}\times9/61$ 0 9/61 $-16/23\times9/61$ $9/61\times$

230/23×61 0 0 1 9/61 161/23×61 -138/61×23

$\therefore \frac{1}{\gamma} = \Sigma Y_i = \frac{23}{61}$ $\therefore \gamma = \frac{61}{23}$ $\therefore y_i = \gamma Y_i \Rightarrow y_1 = \frac{7}{61}\times\frac{61}{23} = \frac{7}{23}$

$y_2 = \frac{6}{61}\times\frac{61}{23} = \frac{6}{23}$, $y_3 = \frac{10}{61}\times\frac{61}{23} = \frac{10}{23}$. Using duality theo, $X_1 = 17/122$, $X_2 = \frac{10}{61}$, $X_3 = \frac{9}{122}$ & $x_i = \gamma X_i$

$\therefore x_1 = \frac{17}{46}$, $x_2 = \frac{10}{23}$, $x_3 = \frac{9}{46}$

& γ in origin A is $\gamma - 2 = \frac{15}{23}$

$R_3 \to R_3 - \frac{12}{61}R_1$

5/23 1 24/23 0 1/23 4/23

$\frac{12}{23}\times\frac{12}{61}$ 0 -122/23×12/61 0 -12/61 16/23×12/61 $-5/23\times\frac{12}{61}$

161/23×61 1 0 0 -12/61 253/23×61 184/23×61

Q.8) Model: $(M/M/5):(FCFS/8/\infty)$

$c = 5,\ n = 5+3 = 8.$ $(5+3)$, L_q. $\mu = 2$ car/hour per parking slot.

$\lambda_n = 6$ car/hour, $0 \le n \le 8$.

$$\mu_n = \begin{cases} n\,\frac{60}{30}/\text{hour} = 2n/\text{hour} & \text{if } n \le 5 \\ 10 \text{ car/hour} & \text{if } n = 6,7,8 \end{cases}$$

$$p_n = \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}\,p_0, \quad n \le 8.$$

$$= \begin{cases} \frac{6\cdot 6 \cdots 6}{2\cdot 4 \cdots 10}\,p_0, & n \le 5 \\ = \frac{3^n}{n!}, & \cancel{n \le 5} \\ \frac{3^n}{5!\,5^{n-5}}\,p_0, & n = 6,7,8 \end{cases}$$

$\therefore \sum_{i=0}^{8} p_i = 1 \Rightarrow p_0 + p_0\left\{\frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right.$
$\left. + \frac{3^6}{5!\,5} + \frac{3^7}{5!\,5^2} + \frac{3^8}{5!\,5^3}\right\} = 1$

or, $p_0 = 0.04812$. $p_1 = 0.14436$, $p_2 = 0.21654$,
$p_3 = 0.21654$, $p_4 = 0.1624$, $p_5 = 0.09744$, $p_6 = 0.05847$,
$p_7 = 0.03508$, $p_8 = 0.02105$.

$\therefore \lambda_{lost} = \lambda p_8 = 6 \times 0.02105 = 0.1263.$

$\therefore \lambda_{eff} = \lambda - \lambda_{lost} = 6 - 0.1263 = 5.8737.$

$\therefore L_s = \sum_0^{\infty} n p_n = 3.1286,$ ~~$W_q = W_s$~~. $L_s = W_s \lambda_{eff}$.

$\Rightarrow W_s = \frac{3.1286}{5.8737} = 0.5326$ hour,

$W_q = W_s - \frac{1}{\mu} = 0.5326 - \frac{1}{2} = 0.03265$ hour.

$\bar{c} = L_s - L_q = \frac{\lambda_{eff}}{\mu} = \frac{5.8737}{2} = 2.9368.$

$\frac{\bar{c}}{c} = \frac{2.9368}{5} = 0.58736$

Q.8 Alternative

$\lambda_n = \lambda,\ \mu_n = \mu\ \forall n.$ $p_n = \frac{\lambda_0\lambda_1 \cdots \lambda_{n-1}}{\mu_1\mu_2\cdots\mu_n}p_0.$

$\therefore p_n = \rho^n p_0,\ \rho = \frac{\lambda}{\mu}.$ $\sum_0^{\infty} p_n = 1 \Rightarrow p_0 = (1-\rho).$

$p_n = (1-\rho)\rho^n,\ n \ge 1.$ $L_s = \sum_0^{\infty} n p_n = \sum_1^{\infty} n(1-\rho)\rho^n = (1-\rho)\rho\sum_{n=1} n\rho^{n-1}$

$= (1-\rho)\rho\frac{d}{d\rho}\sum_0^{\infty}\rho^n = (1-\rho)\rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}.$ $W_q = W_s - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu(1-\rho)}$

$\lambda = \lambda_{eff}$; $L_s = \lambda_{eff} W_s \Rightarrow W_s = \frac{1}{\mu-\lambda}$, $L_q = \lambda_{eff} W_q = \frac{\rho^2}{1-\rho}$, $\bar{c} = L_s - L_q = \frac{\rho}{1-\rho} - \frac{\rho^2}{1-\rho} = \rho.$