THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT: MATHEMATICS
OPTIMIZATION (MTH3011)
EXAM TYPE: QUIZ 2

Time: 30 minutes Date: 16/11/2019 Maximum Marks: 7.5

1. Show that if we add a fixed number k to each element of the pay-off matrix, then the optimal strategies remain unchanged while the value of the game is increased by k.
Find the value and the optimal strategies for the following two-person zero-sum game:

A \ B	B_I	B_{II}	B_{III}	B_{IV}
A_I	3	2	5	1
A_{II}	4	5	3	5
A_{III}	5	3	5	1
A_{IV}	1	5	1	9

[3.5 marks]

Solution:- Part1: Let the original pay-off matrix $P_1 = (a_{ij})_{m\times n}$ and $P_2 = (a_{ij} + k)_{m\times n}$ after adding a fixed quantity k to each element a_{ij} of P_1. Let E_1 and E_2 indicate the original and new expected pay-off function corresponding to a mixed strategy X of player A and a mixed strategy Y of player B. Then $E_1 = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j$, $E_2 = \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij} + k)x_iy_j = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j + k\sum_{i=1}^{m}\sum_{j=1}^{n} x_iy_j = E_1 + k\sum_{i=1}^{m} x_i\sum_{j=1}^{n} y_j = E_1 + k\sum_{i=1}^{m} x_i = E_1 + k$, as $\sum_{i=1}^{m} x_i = \sum_{j=1}^{n} y_j = 1$.

For adding a fixed quantity k to each element a_{ij} of P_1, the nture of pay-off function remains same and the new pay-off function can be obtained from the original pay-off function by adding k only. Thus the optimal strategies for both the matrix games with pay-off E_1 , E_2 are same and the value of the game is increased by k.

Part2: Let the pure strategies of player A are A_1, A_2, A_3, A_4 and player B are B_1, B_2, B_3, B_4. Then

A \ B	B_I	B_{II}	B_{III}	B_{IV}
A_I	3	2	5	1
A_{II}	4	5	3	5
A_{III}	5	3	5	1
A_{IV}	1	5	1	9

Here, A_{III} dominates A_I. Then, we get the following reduced matrix.

A \ B	B_I	B_{II}	B_{III}	B_{IV}
A_{II}	4	5	3	5
A_{III}	5	3	5	1
A_{IV}	1	5	1	9

Now, B_I is inferior to B_{III} for B. So, the reduced matrix will be

A \ B	B_{II}	B_{III}	B_{IV}
A_{II}	5	3	5
A_{III}	3	5	1
A_{IV}	5	1	9

Again, B_{II} is inferior to $\frac{1}{2}(B_{III}+B_{IV})$, so

		B: B_{III}	B_{IV}
A	A_{II}	3	5
	A_{III}	5	1
	A_{IV}	1	9

Since, $\frac{1}{2}(A_{III}+A_{IV})=A_{II}$, we get further reduced matrix as

		B: B_{III}	B_{IV}
A	A_{III}	5	1
	A_{IV}	1	9

Let $X=(x_1,x_2,x_3,x_4)$ and $Y=(y_1,y_2,y_3,y_4)$ are the optimal mixed strategy of A and B. Then $x_1=0=x_2, x_3=\frac{9-1}{(5+9)-(1+1)}=\frac{8}{12}=\frac{2}{3}$, $x_4=\frac{5-1}{12}=\frac{1}{3}$, $y_1=0=y_2, y_3=\frac{9-1}{(5+9)-(1+1)}=\frac{8}{12}=\frac{2}{3}$, $y_4=\frac{5-1}{12}=\frac{1}{3}$ and value of the game is $\gamma=\frac{45-1}{12}=\frac{11}{3}$.

Or

Solve the following game by matrix method. **[3.5 marks]**

		B: B_I	B_{II}	B_{III}
A	A_I	2	4	12
	A_{II}	9	6	3

Solution:- Let $B_1=\begin{pmatrix}2&4\\9&6\end{pmatrix}$. Then $Adj(B_1)=\begin{pmatrix}6&-4\\-9&2\end{pmatrix}$, $T_r=(1,1)$. Then $\bar{X}_r=(x_1,x_2)=\frac{T_rAdj(B_1)}{T_rAdj(B_1)T_r^t}=\frac{(1,1)\begin{pmatrix}6&-4\\-9&2\end{pmatrix}}{(1,1)\begin{pmatrix}6&-4\\-9&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}=\frac{(-3,-2)}{(-3,-2)\begin{pmatrix}1\\1\end{pmatrix}}=(\frac{3}{5},\frac{2}{5})$, all are positive. Then, $\bar{Y}_r=(y_1,y_2)=\frac{T_r(Adj(B_1))^t}{T_rAdj(B_1)T_r^t}=\frac{(1,1)\begin{pmatrix}6&-9\\-4&2\end{pmatrix}}{-5}=\frac{(2,-7)}{-5}=(-\frac{2}{5},\frac{7}{5})$ which is invalid as probabilty. Thus we reject B_1.

Let $B_2=\begin{pmatrix}2&12\\9&3\end{pmatrix}$. Then $Adj(B_2)=\begin{pmatrix}3&-12\\-9&2\end{pmatrix}$, $T_r=(1,1)$. Then $\bar{X}_r=(x_1,x_2)=\frac{T_rAdj(B_2)}{T_rAdj(B_2)T_r^t}=\frac{(1,1)\begin{pmatrix}3&-12\\-9&2\end{pmatrix}}{(1,1)\begin{pmatrix}3&-12\\-9&2\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}=\frac{(-6,-10)}{(-6,-10)\begin{pmatrix}1\\1\end{pmatrix}}=(\frac{3}{8},\frac{5}{8})$, all are positive. Then, $\bar{Y}_r=(y_1,y_3)=\frac{T_r(Adj(B_2))^t}{T_rAdj(B_2)T_r^t}=\frac{(1,1)\begin{pmatrix}3&-9\\-12&2\end{pmatrix}}{-16}=\frac{(-9,-7)}{-16}=(\frac{9}{16},\frac{7}{16})$. Then possible value of the game is $\gamma=\frac{det(B_2)}{T_rAdj(B_2)T_r^t}=\frac{-102}{-16}=\frac{51}{8}$. Now, for player A; $2x_1+9x_2=2\times\frac{3}{8}+9\times\frac{5}{8}=\frac{51}{8}=\gamma$, $4x_1+6x_2=4\times\frac{3}{8}+6\times\frac{5}{8}=\frac{42}{8}<\gamma$, a contradiction as it should be $\geq\gamma$. Thus we reject B_2.

Let $B_3=\begin{pmatrix}4&12\\6&3\end{pmatrix}$. Then $Adj(B_3)=\begin{pmatrix}3&-12\\-6&4\end{pmatrix}$, $T_r=(1,1)$. Then $\bar{X}_r=(x_1,x_2)=\frac{T_rAdj(B_2)}{T_rAdj(B_3)T_r^t}=\frac{(1,1)\begin{pmatrix}3&-12\\-6&4\end{pmatrix}}{(1,1)\begin{pmatrix}3&-12\\-6&4\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}}=\frac{(-3,-8)}{(-3,-8)\begin{pmatrix}1\\1\end{pmatrix}}=(\frac{3}{11},\frac{8}{11})$, all are positive. Then,

$\bar{Y}_r = (y_2, y_3) = \frac{T_r(Adj(B_3))^t}{T_r Adj(B_3) T_r^t} = \frac{(1,1)\begin{pmatrix} 3 & -6 \\ -12 & 4 \end{pmatrix}}{-11} = \frac{(-9,-2)}{-11} = (\frac{9}{11}, \frac{2}{11})$, all are positive. Then possible value of the game is $\gamma = \frac{det(B_3)}{T_r Adj(B_3) T_r^t} = \frac{-60}{-11} = \frac{60}{11}$. Now, for player A; $2x_1 + 9x_2 = 2 \times \frac{3}{11} + 9 \times \frac{8}{11} = \frac{75}{11} > \gamma$, $4x_1 + 6x_2 = 4 \times \frac{3}{11} + 6 \times \frac{8}{11} = \frac{60}{11} = \gamma$ and for player B; $4y_2 + 12y_3 = 4 \times \frac{9}{11} + 12 \times \frac{2}{11} = \frac{60}{11} = \gamma$, $6y_2 + 3y_3 = 6 \times \frac{9}{11} + 3 \times \frac{2}{11} = \frac{60}{11} = \gamma$. Thus all the constrains satisfied for players A and B. Hence, $X = (\frac{3}{11}, \frac{8}{11})$ and $Y = (0, \frac{9}{11}, \frac{2}{11})$ are optimal strategies for A and B and value of the game is $\gamma = \frac{60}{11}$.

2. There are 5 jobs, each of which must go through machines M_1, M_2, M_3 in the order $M_1 M_2 M_3$. Processing times are given in the following Table. Determine the optimal sequence that minimizes the total elapsed time required to complete the following tasks and the corresponding time.

Machine\Job	1	2	3	4	5
M_1	16	14	13	19	15
M_2	18	10	20	15	16
M_3	12	11	15	19	16

[4 marks]

Solution:- $\min_i M_{i1} = 13 < \max_i M_{i2} = 20$, and $\min_i M_{i3} = 11 < \max_i M_{i2} = 20$. Hence, we can't convert it equivalently a n jobs 2 machines problem. We solve it by heuristic method (Campbell, Dudek, Smith). For the first set of n jobs 2 machines, let $A_1 = M_1$ and $B_1 = M_3$.

Then

Machine\Job	1	2	3	4	5
A_1	16	14	$(13)_{III}$	$(19)_V$	$(15)_{IV}$
B_1	$(12)_{II}$	$(11)_I$	15	19	16

and job sequence will be

3	5	4	1	2

Then time schedule will be

	Machine 1			Machine 2			Machine 3	
Job	In	Out	In	Out	Idle	In	Out	Idle
3	0	13	13	33	13	33	48	33
5	13	28	33	49	0	49	65	1
4	28	47	49	64	0	65	84	0
1	47	63	64	82	0	84	96	0
2	63	77	82	92	0	96	107	0

For the last set of n jobs 2 machines, let $A_2 = M_1 + M_2$ and $B_2 = M_2 + M_3$.

Then

Machine\Job	1	2	3	4	5
A_2	34	24	$(33)_{IV}$	$(34)_V$	$(31)_{III}$
B_2	$(30)_{II}$	$(21)_I$	35	34	32

and job sequence will be

5	3	4	1	2

Then time schedule will be

	Machine 1			Machine 2			Machine 3	
Job	In	Out	In	Out	Idle	In	Out	Idle
5	0	15	15	31	15	31	47	31
3	15	28	31	51	0	51	66	4
4	28	47	51	66	0	66	85	0
1	47	63	66	84	0	85	97	0
2	63	77	84	94	0	97	108	0

Thus the optimal job scedule according to this method is

3	5	4	1	2

and the corresponding total elapsed time will be 107 unit.