End Sem Solutions :-

Q.1 (ii) Given $y' - Ay = -By^2$ (Bernoulli's Eqn.)

$n=2$, put $u = y^{1-2} = y^{-1}$. This gives

$$u' = -\frac{y'}{y^2} = -\left(\frac{Ay - By^2}{y^2}\right) = B - \frac{A}{y}$$

$$\Rightarrow \quad u' = B - Au$$

or $$u' + Au = B$$ [03 marks]

General solution is:

$$u(x) = ce^{-Ax} + \frac{B}{A}$$

$$\Rightarrow \quad y = \frac{1}{u} = \frac{A}{B + ce^{-Ax}}.$$ [02 marks]

(i) IVP: $y' = \sqrt{y}, \quad y(0) = 0$

$$f(x,y) = \sqrt{y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}$$

$$\Rightarrow \left|\frac{\partial f}{\partial y}\right| = \frac{1}{2}\frac{1}{|\sqrt{y}|}$$

which is not bounded in any neighbourhood of the origin. [02 marks]

$\Rightarrow$ f does not satisfy Lipschitz condition and hence the solution is not unique (By Uniqueness theorem).

Indeed, the two solutions of the given diff'al equation are:

$$y(x) \equiv 0 \quad \& \quad y(x) = \begin{cases} \frac{x^2}{4} & \text{if } x \geq 0 \\ -\frac{x^2}{4} & \text{if } x < 0 \end{cases}$$

[02 marks]

(iii) Given family $y^2 = 4c(x+c)$

Eliminating c, we get

$$(yy')^2 + 2xyy' - y^2 = 0. \quad \text{[02 marks]}$$

To get orthogonal trajectories we replace y' by $-\frac{1}{y'}$ to get

$$\frac{y^2}{y'^2} - \frac{2xy}{y'} - y^2 = 0$$

or, $$y^2 - 2xyy' - y^2 y'^2 = 0$$

or, $$(yy')^2 + 2xyy' - y^2 = 0$$

same as earlier. Hence the given family of curves are self orthogonal. [02 marks].

Q2 (i) ODE: $y'' + \left(-2 - \frac{2}{x}\right)y' + \frac{4}{x}y = 0$ — (1)

One solution $y_1(x) = e^{2x}$ & another solution is $y_2(x)$.

Since y_1 & y_2 are solutions of eq (1), we have

$$y_1'' + \left(-2 - \frac{2}{x}\right)y_1' + \frac{4}{x}y_1 = 0 \text{ — (2) } \times y_2$$

& $$y_2'' + \left(-2 - \frac{2}{x}\right)y_2' + \frac{4}{x}y_2 = 0 \text{ — (3) } \times y_1$$

(2) $\times y_2$ − (3) $\times y_1$ gives us

$$\left(y_1'' y_2 - y_1 y_2''\right) + \left(-2 - \frac{2}{x}\right)\left(y_1' y_2 - y_2' y_1\right) = 0$$

or, $$\left(y_1 y_2'' - y_1'' y_2\right) = \left(2 + \frac{2}{x}\right)\left(y_1 y_2' - y_1' y_2\right)$$

or, $W'(x) = \left(2+\frac{2}{x}\right) W(x)$ —— 4 [03 marks]

Solving eq 4, we get

$$\frac{W'(x)}{W(x)} = \left(2+\frac{2}{x}\right)$$

$$\Rightarrow \ln W(x) = 2x + 2\ln x + A$$

$$\Rightarrow W(x) = A x^2 e^{2x}$$ [02 marks]

Now the ODE

$$y_1(x)v' - y_1'(x)v = W(x)$$

takes the form

$$v' - 2v = Ax^2 \quad \text{—— } 5$$

we solve it by the method of undetermined co-efficients.

Using the method of undetermined co-efficients a particular solⁿ of the non-homo. eqn is:

$$v_p(x) = a_0 + a_1 x + a_2 x^2$$

Substituting into 5, we get

$$a_1 + 2a_2 x - 2a_0 - 2a_1 x - 2a_2 x^2 = Ax^2$$

$$\Rightarrow \quad a_1 - 2a_0 = 0, \quad 2a_2 - 2a_1 = 0$$
$$2a_2 = -A$$

$$\Rightarrow \quad a_2 = -\frac{A}{2}, \quad a_1 = -\frac{A}{2}, \quad a_0 = -\frac{A}{4}$$

Thus, $\quad y_p(x) = -\frac{A}{4} - \frac{A}{2}x - \frac{A}{2}x^2$

Hence the required solution is:

$$y(x) = -\frac{A}{4}\left(1 + x + 2x^2\right)$$

(ii) suppose on contrary that $y_1(x)$ and $y_2(x)$ (two solutions of the given ODE) are LI.
Therefore,

$$W(y_1, y_2)(x) \neq 0 \quad \forall \ x \in I \quad \text{[02 marks]}$$

Now let $x = x_0$ be a point of maximum/minimum for y_1 & y_2 i.e.

$$y_1'(x_0) = 0 \quad \& \quad y_2'(x_0) = 0$$

$$\Rightarrow W(y_1, y_2)(x_0) = y_1(x_0)y_2'(x_0) - y_1'(x_0)y_2(x_0)$$
$$= 0$$

which is a contradiction.

Hence, y_1 & y_2 are linearly dependent.

[02 marks].

Q.3 The matrix associated with the given model is:

$$A = \begin{bmatrix} 0.50 & 1 \\ -2.25 & 0.5 \end{bmatrix}$$

Eigenvalues are the roots of

$$\det(A - \lambda I) = 0$$

Solving this, we get

$$\lambda^2 - \lambda + 2.5 = 0$$

$$\Rightarrow \lambda = \frac{1+3i}{2}, \quad \frac{1-3i}{2}$$ [02 marks]

Let us now find the solution of

$$\frac{dx}{dt} = Ax, \quad x = \begin{bmatrix} x \\ y \end{bmatrix}$$

corresponding to the eigenvalue $\lambda_1 = \frac{1+3i}{2}$.

Eigenvector for $\lambda_1 = \frac{1+3i}{2}$ is the solⁿ of:

$$(A - \lambda_1 I) \cdot X = 0$$

Solving, we get the eigenvector as

$$X_1 = \begin{bmatrix} 1 \\ 3i/2 \end{bmatrix}$$ [02 marks]

Thus, the solution is:

$$e^{\lambda_1 t} X_1 = e^{\frac{1+3i}{2}t} \begin{bmatrix} 1 \\ \frac{3i}{2} \end{bmatrix} = e^{\frac{t}{2}} \begin{bmatrix} e^{\frac{3it}{2}} \\ \frac{3i}{2} e^{\frac{3it}{2}} \end{bmatrix}$$

$$\text{Real part} = e^{\frac{t}{2}} \begin{bmatrix} \cos\frac{3t}{2} \\ -\frac{3}{2}\sin\frac{3t}{2} \end{bmatrix} = X_1^*$$

$$\text{Imaginary part} = e^{t/2} \begin{bmatrix} \sin\frac{3t}{2} \\ \frac{3}{2}\cos\frac{3t}{2} \end{bmatrix} = X_1^{**}$$

[02 marks]

The initial condition is given as:

$$X(0) = \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Note that X_1^* satisfies this initial condition. Therefore X_1^* is the solution to the IVP.

[01 mark].

Q.4 (i) Let $f(x)$ be any function which is atleast n-times continuously differentiable in $[-1,1]$.

Consider the integral

$$I = \int_{-1}^{1} f(x) P_n(x)\, dx$$

$$= \frac{1}{2^n \lfloor n} \int_{-1}^{1} f(x) \frac{d^n}{dx^n}(x^2-1)^n\, dx \quad \left\{ \begin{array}{l} \text{Using Rodrigues} \\ \text{formula} \end{array} \right\}$$

Integrating by parts n-times we obtain

$$I = \frac{(-1)^n}{2^n \lfloor n} \int_{-1}^{1} f^{(n)}(x)\,(x^2-1)^n\,dx$$

If $f(x) = P_n(x)$ then

$$f^{(n)}(x) = \frac{1}{2^n \lfloor n}\,\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = \frac{\lfloor 2n}{2^n \lfloor n} \cdot \quad \text{[02 marks]}$$

Case I:- $m \neq n$ then without loss of generality we take $m < n$ and $f(x) = P_m(x)$.

$$\Rightarrow f^{(n)}(x) = 0$$

and so $I = 0$ [02 marks]

Case II:- $m = n$ then

$$I = \frac{\lfloor 2n}{(2^n \lfloor n)^2} \int_{-1}^{1} (1-x^2)^n\,dx = \frac{2\lfloor 2n}{(2^n \lfloor n)^2} \int_0^1 (1-x^2)^n\,dx$$

put $x = \sin\theta$

$$\Rightarrow I = \frac{2\lfloor 2n}{(2^n \lfloor n)^2} \int_0^{\pi/2} \cos^{2n+2-1} d\theta$$

$$= \frac{2\lfloor 2n}{(2^n \lfloor n)^2} \cdot B\left(\frac{1}{2}, n+1\right)$$

$$= \frac{\lfloor 2n}{(2^n \lfloor n)^2} \cdot \frac{\Gamma\frac{1}{2}\,\Gamma n+1}{\Gamma n+3/2}$$

$$= \frac{2}{2n+1}. \quad \text{[03 marks]}$$

(ii) Let u & v be two eigenfunctions corresponding to an eigenvalue λ. Then

$$(pu')' + qu + \lambda \varrho u = 0 \quad \text{--- (1)}$$

$$\& \quad (pv')' + qv + \lambda \varrho v = 0 \quad \text{--- (2)}$$

(1) $\times v$ − (2) $\times u$ $\Rightarrow$

$$[p W(u, v)]' = 0 \quad \text{--- (3)}$$ [02 marks]

Now u & v satisfy the given B.C.'s i.e.

$u(a) \neq u(b)$, $u'(a) \neq u'(b)$

$\Rightarrow$ $u(a) \neq 0$ or $u(b) \neq 0$, $u'(a) \neq 0$ or $u'(b) \neq 0$

Also, $u(a) = 0 = u'(a)$ & $u(b) = 0 = u'(b)$ is not possible otherwise we will get only trivial solution (By uniqueness theorem).

Thus, we can write B.C. as:

$$\left.\begin{aligned} c_1 u(a) + c_2 u'(a) &= 0 \\ \& \quad d_1 u(b) + d_2 u'(b) &= 0 \end{aligned}\right\} \quad \text{--- (4)}$$

where c_1 or c_2 not equal to zero and d_1 or d_2 not equal to zero.

Similar, B.C are true for v. [02 marks]

Thus, we get

$$W(u,v) = 0 \quad \text{at} \quad x = a \;\&\; x = b.$$

Hence,

$$pW(u,v) \equiv 0$$

$$\Rightarrow W(u,v) \equiv 0$$

Hence, u & v are linearly dependent [01 mark].

(iii) Given ODE (Bessel's equation)

$$xy'' + y' + xy = 0$$

with $y(0) = 0$

Taking Laplace transform we get

$$\mathcal{L}[xy'' + y' + xy] = 0.$$

$$\Rightarrow -\frac{d}{ds}[s^2 Y(s) - sY(0)] + [sY(s) - Y(0)] - \frac{d}{ds}Y(s) = 0$$

$$\Rightarrow -\frac{d}{ds}[s^2 Y(s) - s] + [sY(s) - 1] - \frac{dY(s)}{ds} = 0$$

$$\Rightarrow (s^2+1)\frac{dY(s)}{ds} = -sY(s) \qquad \text{[03 marks]}$$

separating the variables, we get

$$\frac{dY(s)}{Y(s)} = \frac{-s\,ds}{s^2+1}$$

$$\Rightarrow \log Y(s) = -\frac{1}{2}\log(s^2+1) + c$$

$$\Rightarrow Y(s) = \frac{A}{\sqrt{s^2+1}}, \quad A = e^c \qquad \text{[01 mark]}$$

From here, we get

$$Y(s) = \frac{A}{s}\cdot\left(1+\frac{1}{s^2}\right)^{-\frac{1}{2}}$$

$$= \frac{A}{s}\left\{1 - \frac{1}{2}\cdot\frac{1}{s^2} + \frac{1}{\lfloor 2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{s^4} - \cdots \right.$$
$$\left. \cdots + \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n \lfloor n}\,\frac{(-1)^n}{s^{2n}} + \cdots\right\}$$

Taking the inverse Laplace transform on both sides, we get

$$y(x) = A\left\{1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right\}$$

Now $y(0)=1 \Rightarrow A = 1$ and we get

$$y(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots$$

$$y(x) = J_0(x).$$ [02 marks].

Q. 5 (i) $f(x) = \frac{x^2}{2}, \quad -\pi < x < \pi$

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx).$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = (-1)^n\frac{2}{n^2}, \quad n=1,2,\cdots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \quad \forall\, n\geq 1.$$ [02 marks]

Thus,

$$f(x) = \frac{\pi^2}{6} - 2\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + \cdots\right)$$ [01 mark]

At $x=0$, we get

$$f(0) = \frac{\pi^2}{6} - 2\left(1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots\right)$$

but $f(0) = 0$, so

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots = \frac{\pi^2}{12}.$$ [01 mark]

(ii) $$f(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

$$f(x) = \int_0^\infty \{A(w)\cos wx + B(w)\sin wx\}\, dw$$ [01 mark]

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(w)\cos wx\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{1} \cos wx\, dx = \frac{2}{\pi w}\sin w$$ [01 mark]

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(w)\sin wx\, dx$$

$$= \frac{1}{\pi}\int_{-1}^{1} \sin wx\, dx = 0$$ [01 mark]

Thus,

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos wx \sin w}{w}\, dw$$ [01 mark]

At $x=0$, we get

$$f(0) = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\, dw$$

but $f(0)=0$, so

$$\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}.$$

[01 mark]

Q.6 (i) $u_t + 2uu_x = -3u$, $t>0$, $-\infty<x<\infty$

$u(x,0) = b\sin x$ $\quad -\infty<x<\infty$

Characteristic equations are:

$$\frac{dt}{1} = \frac{dx}{2u} = \frac{du}{-3u}$$

From 2nd & 3rd we get

$$u = -\frac{3}{2}x + c_1$$

or, $u + \frac{3}{2}x = c_1 = U$

Also, from 01st & 03rd we get

$$dt = -\frac{1}{3u}\,du$$

$$\Rightarrow \quad t = -\frac{1}{3}\ln u + c_2$$

$$\Rightarrow \quad t + \frac{1}{3}\ln u = c_2 = V$$

[03 marks]

The complete integral is:

$$U = F(V)$$

i.e. $$u + \frac{3}{2}x = F\left(t + \frac{1}{3}\ln u\right) \quad \text{—(1)}$$

Now using $u(x,0) = b\sin x$, we get

$$\Rightarrow b\sin x + \frac{3}{2}x = F\left(0 + \frac{1}{3}\ln(b\sin x)\right)$$

put $\frac{1}{3}\ln(b\sin x) = w$

$$\Rightarrow \quad x = \sin^{-1}\left(\frac{1}{b}e^{3w}\right)$$

Thus,

$$e^{3w} + \sin^{-1}\left(\frac{1}{b}e^{3w}\right) = F(w) \qquad \text{[01 mark]}$$

From eq. ①, solution is:

$$u + \frac{3}{2}x = e^{3(t + \frac{1}{3}\ln u)} + \sin^{-1}\left(\frac{1}{b}e^{3(t+\frac{1}{3}\ln u)}\right)$$

$$u + \frac{3}{2}x = e^{3t}\cdot u + \sin^{-1}\left(\frac{1}{b}(e^{3t}\cdot u)\right)$$

$$\Rightarrow \quad \frac{1}{b}e^{3t}u = \sin\left(x - \frac{2}{3}u(e^{3t}-1)\right). \quad \text{[01 mark]}$$

(ii) $$x u_{xx} + 2x u_{xy} + (x-1)u_{yy} = 0$$

$R = x, \quad S = 2x, \quad T = (x-1)$

$S^2 - 4RT = 4x$

$\Rightarrow$ Given pde is hyperbolic if $x > 0$. [01 mark]

To find the canonical form, we consider the characteristic equation

$$R\left(\frac{dy}{dx}\right)^2 - S\left(\frac{dy}{dx}\right) + T = 0$$

$$\Rightarrow \quad x\left(\frac{dy}{dx}\right)^2 - 2x\left(\frac{dy}{dx}\right) + (x-1) = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2x \pm \sqrt{4x}}{2x}$$

For $x>0$, we get

$$\frac{dy}{dx} = 1 \pm \frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad y - (x \pm 2\sqrt{x}) = 0$$

So, let
$$\xi = y - (x + 2\sqrt{x})$$
$$\eta = y - (x - 2\sqrt{x}) \qquad \text{[01 marks]}$$

$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x = -\left(1+\frac{1}{\sqrt{x}}\right)u_\xi - \left(1-\frac{1}{\sqrt{x}}\right)u_\eta$$

$$u_y = u_\xi\, \xi_y + u_\eta\, \eta_y = u_\xi + u_\eta$$

$$u_{xx} = \frac{1}{2x^{3/2}}u_\xi + \left(1+\frac{1}{\sqrt{x}}\right)^2 u_{\xi\xi} + \left(1-\frac{1}{x}\right)u_{\xi\eta} - \frac{1}{2x^{3/2}}u_\eta$$
$$+ \left(1-\frac{1}{\sqrt{x}}\right)^2 u_{\eta\eta} + \left(1+\frac{1}{x}\right)u_{\xi\eta}$$

$$u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = -\left(1+\frac{1}{\sqrt{x}}\right)u_{\xi\xi} - \left(1-\frac{1}{\sqrt{x}}\right)u_{\xi\eta} - \left(1+\frac{1}{\sqrt{x}}\right)u_{\xi\eta} - \left(1-\frac{1}{\sqrt{x}}\right)u_{\eta\eta}$$

Substituting into the original equation, we get

$$\cancel{x\left(1+\frac{1}{\sqrt{x}}\right)^2 u_{\xi\xi}} + \cancel{2x\, u_{\xi\eta}} + \cancel{x\left(1-\frac{1}{\sqrt{x}}\right)^2 u_{\eta\eta}} + \frac{1}{2\sqrt{x}}(u_\xi - u_\eta)$$
$$- \cancel{2x\left(1+\frac{1}{\sqrt{x}}\right)u_{\xi\xi}} - \cancel{4x\, u_{\xi\eta}} - \cancel{2x\left(1-\frac{1}{\sqrt{x}}\right)u_{\eta\eta}} + \cancel{(x-1)u_{\xi\xi}}$$
$$+ 2(x-1)u_{\xi\eta} + \cancel{(x-1)u_{\eta\eta}} = 0$$

$$\Rightarrow \quad -2u_{\xi\eta} + \frac{1}{2\sqrt{x}}(u_\xi - u_\eta) = 0$$

$$\Rightarrow \quad u_{\xi\eta} + \frac{1}{\xi - \eta}(u_\xi - u_\eta) = 0. \qquad \text{[03 marks]}$$

Q.7 Let us write

$$p(x,t) = F(x)\,G(t)$$

so that the given pde becomes

$$\frac{F''}{F} = \frac{1}{c^2}\frac{G''}{G}$$

Since LHS is a f^n of x & RHS is a f^n of y, they are equal only if

$$\frac{F''}{F} = \frac{1}{c^2}\frac{G''}{G} = \lambda \quad (\lambda = \text{constant})$$

[02 marks]

From the B.C.

$$p_x(0,t) = 0 \quad \& \quad p(l,t) = 0$$

we get

$$F'(0)\,G(t) = 0 \quad \& \quad F(l)\,G(t) = 0$$

For a non-trivial solution, we must have

$$F'(0) = 0 \quad \& \quad F(l) = 0.$$

Thus, we get

$$F'' - \lambda F = 0 \; ; \quad F'(0) = 0 = F(l)$$

Solving, we get

$$F_n(x) = B_n \cos\left(\frac{2n-1}{2}\cdot\frac{\pi x}{l}\right), \quad \lambda_n = \frac{(2n-1)^2\pi^2}{4l^2}$$

$n = 1, 2, \ldots$

[03 marks]

Also, we have

$$G'' - \lambda c^2 G = 0$$

Sol^n is:

$$G_n(t) = \alpha_n \cos(c\sqrt{\lambda_n}\,t) + \beta_n \sin(c\sqrt{\lambda_n}\,t)$$

Thus,

$$p_n(x,t) = F_n(x)\,G_n(t)$$

or, $p_n(x,t) = \left\{A_n \cos\left(\frac{2n-1}{2l}\pi ct\right) + B_n \sin\left(\frac{2n-1}{2l}\pi ct\right)\right\}$
$\times \cos\left(\frac{2n-1}{2}\pi \frac{x}{l}\right)$ [02 marks]

To get a solution that satisfies I.C., we consider

$$p(x,t) = \sum_{n=1}^{\infty} p_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left\{A_n \cos\left(\frac{2n-1}{2l}\pi ct\right) + B_n \sin\left(\frac{2n-1}{2l}\pi ct\right)\right\} \cos\left(\frac{2n-1}{2l}\pi x\right)$$

[01 mark]

$p(x,0) = f(x)$ gives

$$A_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{2n-1}{2l}\pi x\right) dx$$ [01 mark]

& $p_t(x,0) = g(x)$ gives

$$\frac{2n-1}{2l}\pi c B_n = \frac{2}{l}\int_0^l g(x)\cos\left(\frac{2n-1}{2l}\pi x\right) dx$$

$$\Rightarrow B_n = \frac{4}{(2n-1)\pi c}\int_0^l g(x)\cos\left(\frac{2n-1}{2l}\pi x\right) dx.$$

[01 mark]

Q. 8 We know that the solution to the heat equation with the given boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

where $A_n = 2\int_0^1 f(x)\sin n\pi x \, dx$

[02 marks]

Therefore,

$$A_n = 0 + 2\int_{\frac{1}{2}-\frac{\alpha}{2}}^{\frac{1}{2}+\frac{\alpha}{2}} \frac{u_0}{\alpha}\sin(n\pi x)\,dx + 0$$

$$= u_0 \cdot \frac{\cos\left(\frac{n\pi}{2}(1-\alpha)\right) - \cos\left(\frac{n\pi}{2}(1+\alpha)\right)}{\alpha n\pi/2}$$

$$A_n = \frac{4u_0}{\alpha n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi\alpha}{2}$$

$\Rightarrow$ For $n = 2m$ (even), we get $A_{2m} = 0$

For $n = 2m+1$ (odd), we get

$$A_{2m+1} = 2u_0(-1)^{m+1}\frac{\sin\left((2m+1)\pi\alpha/2\right)}{(2m+1)\pi\alpha/2}$$

[03 marks]

a) The temperature at the mid point i.e. at $x = \frac{1}{2}$ of the rod for $t = \frac{1}{\pi^2}$ is:

$$u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) = \sum_{m=0}^{\infty} 2u_0(-1)^{m+1}\frac{\sin\left((2m+1)\pi\alpha/2\right)}{(2m+1)\pi\alpha/2}\sin\left((2m+1)\frac{\alpha}{2}\right)\cdot e^{-(2m+1)^2}$$

$$= \sum_{m=0}^{\infty}\frac{2u_0}{e^{(2m+1)^2}}\frac{\sin\left((2m+1)\frac{\pi\alpha}{2}\right)}{(2m+1)\frac{\pi\alpha}{2}}$$

$$\Rightarrow u\left(\frac{1}{2}, \frac{1}{\pi^2}\right) \simeq u_1\left(\frac{1}{2}, \frac{1}{\pi^2}\right) = \frac{2u_0}{e}\left(\frac{\sin\pi\alpha/2}{\pi\alpha/2}\right)$$

[03 marks]

(b) To distinguish between the pulse with $\alpha = \frac{1}{1000}$ & $\alpha = \frac{1}{2000}$ we see that

$$\lim_{\alpha \to 0} \frac{\sin \pi\alpha/2}{\pi\alpha/2} = 1$$

So, for smaller values of α, $u\left(\frac{1}{2}, \frac{1}{\pi^2}\right)$ gets closer and closer to $\frac{2u_0}{e}$.

In particular,

$$u\left(\tfrac{1}{2}, \tfrac{1}{\pi^2}\right)\Big|_{\alpha = \frac{1}{1000}} - u\left(\tfrac{1}{2}, \tfrac{1}{\pi^2}\right)\Big|_{\alpha = \frac{1}{2000}}$$

$$= \frac{2u_0}{e}\left(\frac{\sin(\pi/2000)}{\pi/2000} - \frac{\sin(\pi/4000)}{\pi/4000}\right)$$

$$\simeq -\frac{2u_0}{e} \times 3.1 \times 10^{-7}$$ [02 marks].

Q⑨ (i) Given that $f(x,y)$ & $f^2(x,y)$ are harmonic, so we have

$$f_{xx} + f_{yy} = 0 \quad \text{—— ①}$$

$$\& \quad (f^2)_{xx} + (f^2)_{yy} = 0 \quad \text{—— ②}$$ [01 mark]

$$\Rightarrow (2ff_x)_x + (2ff_y)_y = 0$$

$$\Rightarrow ff_{xx} + f_x^2 + ff_{yy} + f_y^2 = 0$$

$$\Rightarrow f(f_{xx} + f_{yy}) + f_x^2 + f_y^2 = 0$$

From ①, we get

$$f_x^2 + f_y^2 = 0$$

$$\Rightarrow \quad f_x = 0 \quad \& \quad f_y = 0$$

$$\Rightarrow \quad f \text{ is a constant.}$$ [02 marks]

(ii) We know that the potential inside a sphere of radius R with $u(R,\phi) = f(\phi)$ is given by

$$u(r,\phi) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\phi), \quad (r \le R)$$

where,

$$A_n = \frac{2n+1}{2R^n} \int_0^{\pi} f(\phi) P_n(\cos\phi) \sin\phi \, d\phi$$

$n = 0, 1, 2, \cdots$ [02 marks]

Here, $R = 1$, $f(\phi) = \cos\phi$.

Therefore, we can write

$$f(\phi) = \cos\phi = P_1(\cos\phi) = w, \quad \text{if } w = \cos\phi$$

Thus, $a_0 = 0$, $a_1 = 1$, $a_n = 0 \quad \forall\, n \geq 2$

So, the potential inside the sphere is:

$$u(r,\phi) = A_1 r P_1(\cos\phi)$$

$$= r\cos\phi$$ [03 marks]

At North pole, $\phi = 0$, $r = R = 1$, so

$$u(1,0) = 1 \cdot \cos 0 = 1$$ [01 mark]

At South pole $\phi = \pi$, $r = R = 1$, so

$u(1,\pi) = 1 \cdot \cos\pi = -1$ [01 mark]

At Equator $\phi = \pi/2$, $r = R = 1$, so

$u(1, \pi/2) = 1 \cdot \cos\pi/2 = 0$ [01 mark].