THE LNM INSTITUTE OF INFORMATION TECHNOLOGY
DEPARTMENT: MATHEMATICS
OPTIMIZATION (MTH3011)
EXAM TYPE: MID TERM

Time: 90 minutes | Date: 30/09/2019 | Maximum Marks: 25

1. A LNMIIT dietitian, Smart, has to develop a special diet using two foods F_1 and F_2. Each packet (containing 50 g) of food F_1 contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food F_2 contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires at least 240 units of calcium, at least 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimize the amount of vitamin A in the diet? What is the minimum amount of vitamin A? Justify whether your feasible region is convex hull or convex polyhedron. **[5]**

Solution: Let x and y be the number of packets of food F_1 and F_2 respectively used by Smart in the special diet. Then $x,\ y \geq 0$, O.F. is Minimize $z = 6x + 3y$ and the constrains will be $12x + 3y \geq 240$ for calcium requirement i.e $4x + y \geq 80$, $4x + 20y \geq 460$ for iron requirement i.e $x + 5y \geq 115$, $6x + 4y \leq 300$ for cholesterol requirement i.e $3x + 2y \leq 150$.

The feasible solutions space S determined by the constraints is shown in Figure 1.

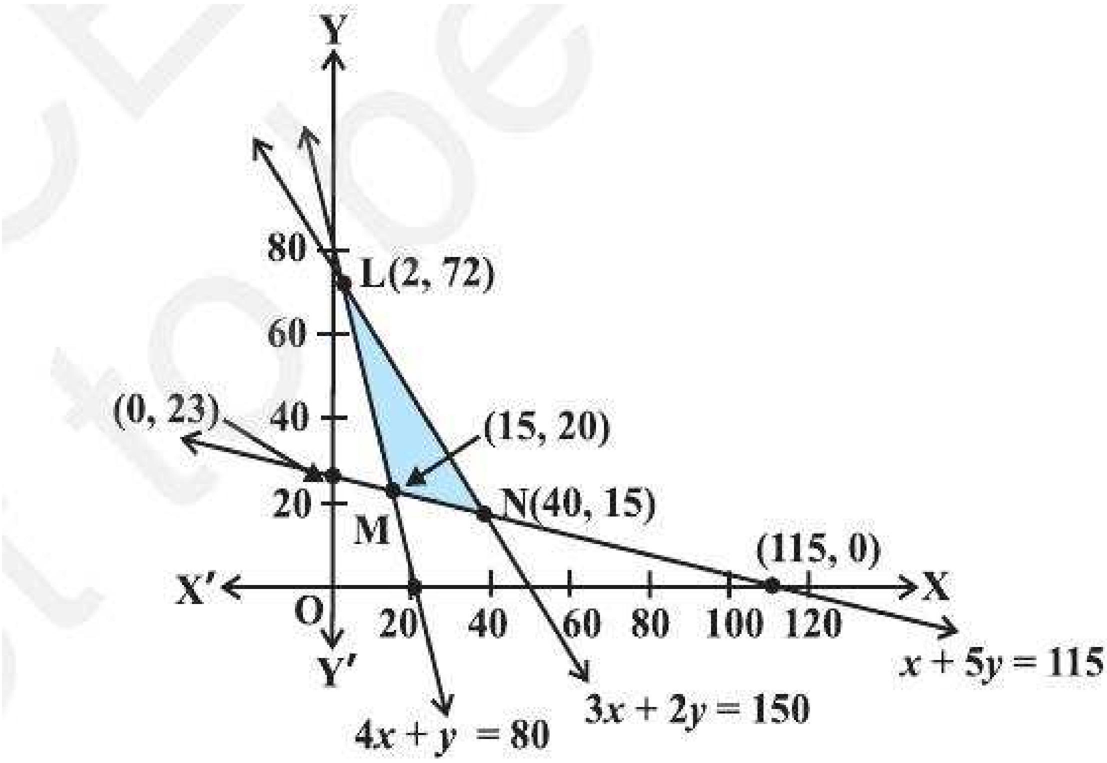


Figure 1: Feasible solution space of the LNMIIT dietitian model.

S is bounded by the line segments joining the points M, L, N. Hence the extreme points are M(15,20), L(2,72), N(40,15) with the values of O.F. as $z_M = 150$, $z_L = 285$, $z_N = 228$ repectively. The feasible region is bounded and hence the problem has a minimum value $z_{min} = 150$ at the extreme point M(15,20). Hence, the amount of vitamin A under the given constraints in the problem will be minimum having value $z_{min} = 150$ units if Smart uses 15 packets of food F_1 and 20 packets of food F_2 in the special diet.

S is a convex polyhedron as S is the convex combination of 3 points only.

Or

Prove or disprove: the following problems has infinite number of solutions.

(i) Maximize $z = 3x_1 + 2x_2$

Subject to $x_1 - x_2 \leq 1,$

$x_1 + x_2 \geq 2,$

$x_1,\ x_2 \geq 0.$ **[2.5]**

(ii) Maximize $z = 6x_1 + 10x_2$

Subject to $3x_1 + 5x_2 \leq 10,$

$x_1 + x_2 \leq 15,$

$x_1,\ x_2 \geq 0.$ **[2.5]**

Solution: Attached a hand written seperate sheet.

2. Solve the following problem using simplex table: **[5]**

Minimize $z = x_1 + x_2$

Subject to $2x_1 + x_2 \geq 4,$

$x_1 + 7x_2 \geq 7,$

$x_1,\ x_2 \geq 0.$

What will be a upper bound of number of basic feasible solutions of this L.P.P.?

Solution: To make the L.P.P in standard form, we first introduce surplus variables x_3 and x_4 in 1st and 2nd constraints respectively. Then to start with intial basis matrix as identity matrix I_2, we introduce two artificial variable x_5 and x_6 in the 1st and 2nd constraints respectively and will solve it by one of the following methods after converting the minimization problem to maximization problem.

Big-M Method:

$z' = -x_1 - x_2 + 0x_3 + 0x_4 - Mx_5 - Mx_6$, Sub. to $2x_1 + x_2 - x_3 + 0x_4 + x_5 + 0x_6 = 4$
$x_1 + 7x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 7$
$x_1,\ x_2, \cdots, x_6 \geq 0$. For initial B.F.S, $B = [\beta_1, \beta_2] = [\alpha_5, \alpha_6] = I_2,\ \underline{\mathbf{x_B}} = [x_5, x_6]^t = [4, 7]^t$, $\underline{\mathbf{c_B}} = [c_5, c_6]^t = [-M, -M]^t$. Then we can make following tables:

Table 1: First Simplex Table of Big-M Method

			$\underline{\mathbf{c_j}} \rightarrow$	-1	-1	0	0	$-M$	$-M$	$\frac{x_{B_i}}{y_{ij}},\ y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
$-M$	α_5	x_5	4	2	1	-1	0	1	1	4
$-M$	α_6	x_6	7	1	$\boxed{7}$	0	-1	0	1	$1 \Rightarrow$
$z' = -11M$			$(z_j$-$c_j) \rightarrow$	$-3M+1$	$-8M+1 \Uparrow$	M	M	0	0	Min. $\{\frac{x_{B_i}}{y_{ij}},\ y_{ij} > 0\}$=1

Since in Table 1, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_2 coming in basis and x_6 leaving the basis.

Since in Table 2, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_1 coming in basis and x_5 leaving the basis.

Since in Table 3, the optimality condition $z_j - c_j \geq 0$ for all j satisfies and no artificial variable in the final basis, the current solution is a optimal basic feasible solution (B.F.S). As $z_j - c_j > 0$ for non-basis vectors and the optimal solution is non-degerate basic solution, we get the unique basic optimal solution of the original maximization problem as $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ and $z'_{max} = -\frac{31}{13}$. Then the unique basic optimal solution of the original (minimization) problem as $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ and $z_{min} = -z'_{max} = \frac{31}{13}$.

Table 2: Second Simplex Table of Big-M Method

			$\underline{\mathbf{c_j}} \rightarrow$	-1	-1	0	0	$-M$	$-M$	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
$-M$	α_5	x_5	3	$\boxed{\frac{13}{7}}$	0	-1	$\frac{1}{7}$	1	-$\frac{1}{7}$	$\frac{21}{13} \Rightarrow$
-1	α_2	x_2	1	$\frac{1}{7}$	1	0	-$\frac{1}{7}$	0	$\frac{1}{7}$	7
$z' = -3M - 1$			$(z_j$-$c_j) \rightarrow$	$-\frac{13M}{7}+\frac{6}{7} \Uparrow$	0	M	$-\frac{M}{7}+\frac{1}{7}$	0	$\frac{8M}{7}-\frac{1}{7}$	Min. $\{\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$ $\}$=$\frac{21}{13}$

Table 3: Third Simplex Table of Big-M Method

			$\underline{\mathbf{c_j}} \rightarrow$	-1	-1	0	0	$-M$	$-M$	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
-1	α_1	x_1	$\frac{21}{13}$	1	0	-$\frac{7}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	-$\frac{1}{13}$	
-1	α_2	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	-$\frac{2}{13}$	-$\frac{1}{13}$	$\frac{2}{13}$	
$z' = -\frac{31}{13}$			$(z_j$-$c_j) \rightarrow$	0	0	$\frac{6}{13}$	$\frac{1}{13}$	M-$\frac{6}{13}$	M-$\frac{1}{13}$	Min. $\{\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$ $\}$=

Two Phase Method:

Phase I: $z'_1 = 0x_1 + x_2 + 0x_3 + 0x_4 - x_5 - x_6$, Sub. to $2x_1 + x_2 - x_3 + 0x_4 + x_5 + 0x_6 = 4$
$x_1 + 7x_2 + 0x_3 - x_4 + 0x_5 + x_6 = 7$
$x_1,\ x_2, \cdots, x_6 \geq 0$. For initial B.F.S, $B = [\beta_1, \beta_2] = [\alpha_5, \alpha_6] = I_2$, $\underline{\mathbf{x_B}} = [x_5, x_6]^t = [4, 7]^t$, $\underline{\mathbf{c_B}} = [c_5, c_6]^t = [-1, -1]^t$. Then we can make following tables:

Table 4: First Simplex Table of Two Phase Method: Phase I

			$\underline{\mathbf{c_j}} \rightarrow$	0	0	0	0	-1	-1	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
-1	α_5	x_5	4	2	1	-1	0	1	1	4
-1	α_6	x_6	7	1	$\boxed{7}$	0	-1	0	1	$1 \Rightarrow$
$z'_1 = -11$			$(z_j$-$c_j) \rightarrow$	-3	$-8 \Uparrow$	1	1	0	0	Min. $\{\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$ $\}$=1

Since in Table 4, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_2 coming in basis and x_6 leaving the basis.

Since in Table 5, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_1 coming in basis and x_5 leaving the basis.

Since in Table 6, the optimality condition $z_j - c_j \geq 0$ for all j satisfies and no artificial variable in the final basis, the current solution is a optimal basic feasible solution (B.F.S) for phase I and we move to phase II with $z'_2 = -x_1 - x_2 + 0x_3 + 0x_4$. Then we can make following tables (1st table similar to the last table of phase 1 with the change in c_i's and removing artificial columns):

Since in Table 7, the optimality condition $z_j - c_j \geq 0$ for all j satisfies, the current solution is a optimal basic feasible solution (B.F.S). As $z_j - c_j > 0$ for non-basis vectors and the

Table 5: Second Simplex Table of Two Phase Method: Phase I

			$\underline{\mathbf{c_j}} \rightarrow$	0	0	0	0	-1	-1	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
-1	α_5	x_5	3	$\boxed{\frac{13}{7}}$	0	-1	$\frac{1}{7}$	1	-$\frac{1}{7}$	$\frac{21}{13} \Rightarrow$
0	α_2	x_2	1	$\frac{1}{7}$	1	0	-$\frac{1}{7}$	0	$\frac{1}{7}$	7
$z_1' = -3$			(z_j-c_j) $\rightarrow$	$-\frac{13}{7}$ $\Uparrow$	0	1	$-\frac{1}{7}$	0	$\frac{8}{7}$	Min. $\{\frac{x_{B_i}}{y_{ij}}, y_{ij} > 0\}=\frac{21}{13}$

Table 6: Third Simplex Table of Two Phase Method: Phase I

			$\underline{\mathbf{c_j}} \rightarrow$	0	0	0	0	-1	-1	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	for $\alpha_j \uparrow$
0	α_1	x_1	$\frac{21}{13}$	1	0	-$\frac{7}{13}$	$\frac{1}{13}$	$\frac{7}{13}$	-$\frac{1}{13}$	
0	α_2	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	-$\frac{2}{13}$	-$\frac{1}{13}$	$\frac{2}{13}$	
$z_1' = 0$			(z_j-c_j) $\rightarrow$	0	0	0	0	1	1	Min. $\{\frac{x_{B_i}}{y_{ij}}, y_{ij} > 0\}$

Table 7: Frist Simplex Table of Two Phase Method: Phase II

			$\underline{\mathbf{c_j}} \rightarrow$	-1	-1	0	0	$\frac{x_{B_i}}{y_{ij}}$, $y_{ij} > 0$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	for $\alpha_j \uparrow$
-1	α_1	x_1	$\frac{21}{13}$	1	0	-$\frac{7}{13}$	$\frac{1}{13}$	
-1	α_2	x_2	$\frac{10}{13}$	0	1	$\frac{1}{13}$	-$\frac{2}{13}$	
$z_1' = -\frac{31}{13}$			(z_j-c_j) $\rightarrow$	0	0	$\frac{6}{13}$	$\frac{1}{13}$	Min. $\{\frac{x_{B_i}}{y_{ij}}, y_{ij} > 0\}$

optimal solution is non-degerate basic solution, we get the unique basic optimal solution of the original maximization problem as $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ and $z'_{max} = -\frac{31}{13}$. Then the unique basic optimal solution of the original (minimization) problem as $x_1 = \frac{21}{13}$, $x_2 = \frac{10}{13}$ and $z_{min} = -z'_{max} = \frac{31}{13}$.

3. (i) Given the L.P.P.:
Minimize $z = 2x_1 + x_2 + x_3$
Subject to $x_1 - 3x_2 + 4x_3 = 5$,
$x_1 - 2x_2 \leq 3$,
$2x_2 - x_3 \geq 4$,
$x_1,\ x_2 \geq 0$
and x_3 is unrestricted in sign. Formulate the dual of the L.P.P. **[3]**

(ii) If $\underline{\mathbf{u}}$ be any feasible solution to the primal maximization problem: Maximize $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$ Subject to $A\underline{\mathbf{x}} \leq \underline{\mathbf{b}}$, $\underline{\mathbf{x}} \geq \underline{\mathbf{0}}$ and $\underline{\mathbf{v}}$ be any feasible solution to the corresponding dual problem, then show that $\underline{\mathbf{c}}^t\underline{\mathbf{u}} \leq \underline{\mathbf{b}}^t\underline{\mathbf{v}}$ where superscript 't' indicates the transpose of a matrix. **[2]**

Solution: (i) Sine x_3 is unrestricted in sign, we can write $x_3 = x_{31} - x_{32}$; $x_{31}, x_{32} \geq 0$. The primal in standard form can be written as

Minimize $z = 2x_1 + x_2 + x_{31} - x_{32}$
Subject to $x_1 - 3x_2 + 4x_{31} - 4x_{32} \geq 5,$
$- x_1 + 3x_2 - 4x_{31} + 4x_{32} \geq -5,$
$- x_1 + 2x_2 + 0x_{31} + 0x_{32} \geq -3$
$0x_1 + 2x_2 - x_{31} + x_{32} \geq 4$
$x_1, x_2, x_{31}, x_{32} \geq 0$

Then dual will be in standard form as:
Maximize $w = 5y_{11} - 5y_{12} - 3y_2 + 4y_3$
Subject to $y_{11} - y_{12} - y_2 + 0y_3 \leq 2$
$- 3y_{11} + 3y_{12} + 2y_2 + 2y_3 \leq 1,$
$4y_{11} - 4y_{12} + 0y_2 - y_3 \leq 1,$
$- 4y_{11} + 4y_{12} + 0y_2 + y_3 \leq -1,$
$y_{11}, y_{12}, y_2, y_3 \geq 0$

Or, Maximize $w = 5y_1 - 3y_2 + 4y_3$
Subject to $y_1 - y_2 + 0y_3 \leq 2$
$- 3y_1 + 2y_2 + 2y_3 \leq 1,$
$4y_1 + 0y_2 - y_3 = 1,$
$y_2, y_3 \geq 0$ and $y_1 = y_{11} - y_{12}$ is unrestricted in sign.

(ii) Let $\underline{\mathbf{u}}$ be any feasible solution to the primal maximization problem: Maximize $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$ Subject to $A\underline{\mathbf{x}} \leq \underline{\mathbf{b}}$, $\underline{\mathbf{x}} \geq \underline{\mathbf{0}}$. Then

$$A\underline{\mathbf{u}} \leq \underline{\mathbf{b}} \tag{1}$$
$$\text{Or, } \underline{\mathbf{v}}^t A\underline{\mathbf{u}} \leq \underline{\mathbf{v}}^t\underline{\mathbf{b}} \tag{2}$$

Let $\underline{\mathbf{v}}$ be any feasible solution to the corresponding dual problem Minimize $w = \underline{\mathbf{b}}^t\underline{\mathbf{y}}$ Subject to $A^t\underline{\mathbf{y}} \geq \underline{\mathbf{c}}$, $\underline{\mathbf{y}} \geq \underline{\mathbf{0}}$. Then

$$A^t\underline{\mathbf{v}} \geq \underline{\mathbf{c}} \tag{3}$$
$$\text{Or, } \underline{\mathbf{u}}^t A^t\underline{\mathbf{v}} \geq \underline{\mathbf{u}}^t\underline{\mathbf{c}} \tag{4}$$
$$\text{Or, } \underline{\mathbf{u}}^t(\underline{\mathbf{v}}^t A)^t \geq \underline{\mathbf{u}}^t\underline{\mathbf{c}} \tag{5}$$
$$\text{Or, } \{(\underline{\mathbf{v}}^t A)\underline{\mathbf{u}}\}^t \geq \{\underline{\mathbf{c}}^t\underline{\mathbf{u}}\}^t \tag{6}$$
$$\text{Or, } (\underline{\mathbf{v}}^t A)\underline{\mathbf{u}} \geq \underline{\mathbf{c}}^t\underline{\mathbf{u}}. \tag{7}$$

From equation (2) and (7), we get $\underline{\mathbf{c}}^t\underline{\mathbf{u}} \leq \underline{\mathbf{v}}^t A\underline{\mathbf{u}} \leq \underline{\mathbf{v}}^t\underline{\mathbf{b}}$. Then $\underline{\mathbf{c}}^t\underline{\mathbf{u}} \leq \underline{\mathbf{b}}^t\underline{\mathbf{v}}$ as $\underline{\mathbf{v}}^t\underline{\mathbf{b}}$ is a scalar (i.e. $\underline{\mathbf{v}}^t\underline{\mathbf{b}} = \{\underline{\mathbf{v}}^t\underline{\mathbf{b}}\}^t = \underline{\mathbf{b}}^t\underline{\mathbf{v}}$).

Or

Use duality to solve: Maximize $z = 2x_1 + 3x_2$ **[5]**
Subject to $-x_1 + 2x_2 \leq 4,$
$x_1 + x_2 \leq 6,$
$x_1 + 3x_2 \leq 9,$
$x_1,\ x_2 \geq 0.$

Solution: The dual will be in standard form as:
Minimize $w = 4y_1 + 6y_2 + 9y_3$
Subject to $-y_1 + y_2 + y_3 \geq 2$
$2y_1 + y_2 + 3y_3 \geq 3,$
$y_1, y_2 \geq 0.$

To make this L.P.P. (the dual) in standard form, we first introduce surplus variables y_4 and y_5 in 1st and 2nd constraints respectively. Then to start with intial basis matrix as identity matrix I_2, we introduce two artificial variable y_6 and y_7 in the 1st and 2nd constraints respectively and will solve it by Big-M Method.

$z' = 4y_1+6y_2+9y_3+0y_4+0y_5+My_6+My_7$, Sub. to $-y_1+y_2+y_3-y_4+0y_5+y_6+0y_7 = 2$
$2y_1 + y_2 + 3y_3 + 0y_4 - y_5 + 0y_6 + y_7 = 3$
$y_1,\ y_2, \cdots, y_7 \geq 0$. For initial B.F.S, $B = [\beta_1, \beta_2] = [\alpha_6, \alpha_7] = I_2$, $\underline{\mathbf{y}}_{\mathbf{B}} = [y_6, y_7]^t = [2, 3]^t$, $\underline{\mathbf{c}}_{\mathbf{B}} = [c_6, c_7]^t = [M, M]^t$. Then we can make following tables:

Table 8: First Simplex Table of Big-M Method for Dual

			$\underline{\mathbf{c}}_{\mathbf{j}} \rightarrow$	4	6	9	0	0	M	M	$\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0$
$\underline{\mathbf{c}}_{\mathbf{B}}$	B	$\underline{\mathbf{x}}_{\mathbf{B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	for $\alpha_j \uparrow$
M	α_6	y_6	2	-1	1	1	-1	0	1	0	2
M	α_7	y_7	3	2	1	$\boxed{3}$	0	-1	0	1	$1 \Rightarrow$
$z' = 5M$			$(z_j$-$c_j) \rightarrow$	$M-4$	$2M-6$	$4M-9 \Uparrow$	$-M$	$-M$	0	0	Min. $\{\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0\}$=1

Since in Table 8, the optimality condition $z_j - c_j \leq 0$ for all j does not satisfy, then we can go to next table with y_3 coming in basis and y_7 leaving the basis.

Table 9: Second Simplex Table of Big-M Method for Dual

			$\underline{\mathbf{c}}_{\mathbf{j}} \rightarrow$	4	6	9	0	0	M	M	$\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0$
$\underline{\mathbf{c}}_{\mathbf{B}}$	B	$\underline{\mathbf{x}}_{\mathbf{B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	for $\alpha_j \uparrow$
M	α_6	y_6	1	-$\frac{5}{3}$	$\boxed{\frac{2}{3}}$	0	-1	$\frac{1}{3}$	1	-$\frac{1}{3}$	$\frac{3}{2} \Rightarrow$
9	α_3	y_3	1	$\frac{2}{3}$	$\frac{1}{3}$	1	0	-$\frac{1}{3}$	0	$\frac{1}{3}$	3
$z' = M+9$			$(z_j$-$c_j) \rightarrow$	$-\frac{5M}{3}+2$	$\frac{2M}{3}-3 \Uparrow$	0	$-M$	$\frac{M}{3}-3$	0	$3-\frac{4M}{3}$	Min. $\{\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0\}$=$\frac{3}{2}$

Since in Table 9, the optimality condition $z_j - c_j \leq 0$ for all j does not satisfy, then we can go to next table with y_2 coming in basis and y_6 leaving the basis.

Table 10: Third Simplex Table of Big-M Method for Dual

			$\underline{\mathbf{c}}_{\mathbf{j}} \rightarrow$	4	6	9	0	0	M	M	$\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0$
$\underline{\mathbf{c}}_{\mathbf{B}}$	B	$\underline{\mathbf{x}}_{\mathbf{B}}$	$\underline{\mathbf{b}}$	α_1	α_2	α_3	α_4	α_5	α_6	α_7	for $\alpha_j \uparrow$
6	α_2	y_2	$\frac{3}{2}$	-$\frac{5}{2}$	1	0	-$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	-$\frac{1}{2}$	
9	α_3	y_3	$\frac{1}{2}$	$\frac{3}{2}$	0	1	$\frac{1}{2}$	-$\frac{1}{2}$	-$\frac{1}{2}$	$\frac{1}{2}$	
$z' = \frac{27}{2}$			$(z_j$-$c_j) \rightarrow$	$-\frac{11}{2}$	0	0	$-\frac{9}{2}$	$-\frac{3}{2}$	$\frac{9}{2} - M$	$\frac{3}{2} - M$	Min. $\{\frac{y_{B_i}}{y_{ij}},\ y_{ij} > 0\}$

Since in Table 10, the optimality condition $z_j - c_j \leq 0$ for all j satisfies and no artificial variable in the final basis, the current solution is a optimal basic feasible solution (B.F.S). As $z_j - c_j < 0$ for non-basis vectors and the optimal solution is non-degerate basic solution, we get the unique basic optimal solution of the original maximization problem as $y_2 = \frac{3}{2}$, $y_3 = \frac{1}{2}$ and $z'_{min} = \frac{27}{2}$. Then the basic optimal solution of the primal (maximization) problem is given by the row of $z_j - c_j$ and column α_6, α_7 respectively as $x_1 = \frac{9}{2}$, $x_2 = \frac{3}{2}$ and $z_{max} = \frac{27}{2}$ after setting $M = 0$.

4. In a balanced transportation problem, show that (i) the solution is never unbounded (ii) the number of basic variables is at most $(m+n-1)$. **[5]**

Solution: We have the T.P. as Optimize $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ subject to $x_{ij} \geq 0$ and

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ for } i = 1, 2, \cdots, m; \tag{8}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ for } j = 1, 2, \cdots, n. \tag{9}$$

Each varible x_{ij} appears exactly in two constraints: one in equation (8) and other in equation ((9) with a coefficient 1. We can see that x_{ij} is bounded satisfying $0 \leq x_{ij} \leq \max_{i,j}(a_i, b_j)$. Then no x_{ij} can assume an arbitrary large value as a_i and b_j are finite. Thus T.P. must have a B.F.S. as it has a F.S. Again, z will be bounded from below and above as c_{ij} and x_{ij} are finite.

Thus the O.F. will have some optimal value. Hence, there will be B.F.S. to the problem which will be optimal.

(ii) We have the T.P. as Optimize $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}} = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij}$ subject to $x_{ij} \geq 0$ and

$$\sum_{j=1}^{n} x_{ij} = a_i \text{ for } i = 1, 2, \cdots, m; \tag{10}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \text{ for } j = 1, 2, \cdots, n, \tag{11}$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \text{ as T.P. is balanced} \tag{12}$$

The sum of m constraints in equation (10) and $n-1$ constraints in equation (11) gives respectively

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \text{ by (12)} \tag{13}$$

$$\text{and } \sum_{j=1}^{n-1}\sum_{i=1}^{m} x_{ij} = \sum_{j=1}^{n-1} b_j \tag{14}$$

$$\text{Or, } \sum_{i=1}^{m}\sum_{j=1}^{n-1} x_{ij} = \sum_{j=1}^{n-1} b_j \tag{15}$$

From (13)$-$(15), we have

$$\sum_{i=1}^{m}(\sum_{j=1}^{n} x_{ij} - \sum_{j=1}^{n-1} x_{ij}) = \sum_{j=1}^{n} b_j - \sum_{j=1}^{n-1} b_j \tag{16}$$

$$\text{Or, } \sum_{i=1}^{m} x_{in} = b_n. \tag{17}$$

This is the $n-$th constraint of equation (10). Thus one of the $m+n$ constraints is redundant and hence a basis consists of at most $m+n-1$ variables.

Or

(i) If the optimal solution to a L.P.P. occurs at more than one extreme point of the feasible solutions, then show that every convex combinations of these extreme points also gives the optimal value of the objective functions. How many optimal solutions you will get here? Will these generated optimal solutions be extreme points/basic feasible solutions? **[3]**

(ii) Show that a hyperplane is a convex set. **[2]**

Solution: (i) Let the L.P.P. is Optimize $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$ subject to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i \text{ for } i = 1, 2, \cdots, r; \tag{18}$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i \text{ for } i = r+1, r+2, \cdots, r+s; \tag{19}$$
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \text{ for } i = r+s+1, r+s+2, \cdots, m; \tag{20}$$
$$x_i \geq 0 \text{ for } i = 1, 2, \cdots, n. \tag{21}$$

Let the O.F. z assume its optimal value z^* at the extreme points $\underline{\mathbf{x}}_1, \underline{\mathbf{x}}_2, \cdots, \underline{\mathbf{x}}_k$. Then

$$z^* = \underline{\mathbf{c}}^t\underline{\mathbf{x}}_1 = \underline{\mathbf{c}}^t\underline{\mathbf{x}}_2 = \cdots = \underline{\mathbf{c}}^t\underline{\mathbf{x}}_k. \tag{22}$$

Let $\underline{\mathbf{y}}$ be the convex combinations of these extreme points, then

$$\underline{\mathbf{y}} = \lambda_1\underline{\mathbf{x}}_1 + \lambda_2\underline{\mathbf{x}}_2 + \cdots + \lambda_k\underline{\mathbf{x}}_k \tag{23}$$
$$\text{with } 0 \leq \lambda_i \leq 1,\ i = 1, 2, \cdots, k \text{ and } \sum_{i=1}^{k} \lambda_i = 1. \tag{24}$$

We can show that $\underline{\mathbf{y}}$ is also a Feasible solution. Then $\underline{\mathbf{c}}^t\underline{\mathbf{y}} = \underline{\mathbf{c}}^t(\lambda_1\underline{\mathbf{x}}_1 + \lambda_2\underline{\mathbf{x}}_2 + \cdots + \lambda_k\underline{\mathbf{x}}_k)$ by (23) $= \sum_{i=1}^{k} \lambda_i\underline{\mathbf{c}}^t\underline{\mathbf{x}}_i = \sum_{i=1}^{k} \lambda_i z^*$ by (22) $= z^* \sum_{i=1}^{k} \lambda_i = z^*$ by (24).

Hence the result.

The result tells us that either optimal solution is unique or infinite in number.

(ii) Let S be the set of points on the hyperplane $p = \underline{\mathbf{c}}^t\underline{\mathbf{x}} = d_1x_1 + d_2x_2 + \cdots + d_nx_n$ where n dimentional variable $\underline{\mathbf{x}} \in \mathbb{R}^{\ltimes}$, constant vector $\underline{\mathbf{c}} \in \mathbb{R}^{\ltimes}$ and constant $d \in \mathbb{R}$. Let $\underline{\mathbf{X}}_1, \underline{\mathbf{X}}_2 \in S$. Then $p = \underline{\mathbf{c}}^t\underline{\mathbf{X}}_1 = \underline{\mathbf{c}}^t\underline{\mathbf{X}}_2$. Let $\underline{\mathbf{y}} = \lambda\underline{\mathbf{X}}_1 + (1-\lambda)\underline{\mathbf{X}}_2, 0 \leq \lambda \leq 1$ be any point on the convex combination of $\underline{\mathbf{X}}_1, \underline{\mathbf{X}}_2$. Then $\underline{\mathbf{c}}^t\underline{\mathbf{y}} = \underline{\mathbf{c}}^t\lambda\underline{\mathbf{X}}_1 + \underline{\mathbf{c}}^t(1-\lambda)\underline{\mathbf{X}}_2 = \lambda\underline{\mathbf{c}}^t\underline{\mathbf{X}}_1 + (1-\lambda)\underline{\mathbf{c}}^t\underline{\mathbf{X}}_2 = p\lambda + (1-\lambda) = p$. Thus $\underline{\mathbf{y}} \in S$. Hence, S i.e. the hyperplane is a convex set.

5. A Rajasthan Car Company has three plants P_1, P_2, P_3 located throughout a state with production capacity 50, 75 and 25 units. Each day the firm must furnish its four retail shops located in cities C_1, C_2, C_3, C_4 with at least 20, 20 , 50, and 60 units respectively. The transportation costs , c_{ij}, in thousand rupees, between plants and cities are given in the following Table. The model seeks the minimum-cost shipping schedule x_{ij} between plant i and city j (i= 1, 2, 3; j= 1, 2, 3,4).

	C_1	C_1	C_3	C_4	Supply
P_1	3	5	7	6	50
P_2	2	5	8	2	75
P_3	3	6	9	2	25
Demand	20	20	50	60	

How should car be distributed to the cities such that the car company minimizes cost of car supply? **[5]**

Solution: The T.P. is balanced as $\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j = 150$

Table 11: First table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{20}$ 3	5	7	6	a_1=50 ($d_{row1} = 2$)
P_2	2	5	8	2	a_2=75 ($d_{row2} = 0$)
P_3	3	6	9	2	a_3=25 ($d_{row3} = 1$)
Demand	b_1=20 ($d_{col1} = 1$)	b_2=20 ($d_{col2} = 0$)	b_3=50 ($d_{col3} = 1$)	b_4=60 ($d_{col4} = 0$)	

Table 12: Second table of VAM method of trasportation problem

	C_2	C_3	C_4	Supply
P_1	5	7	6	a_1=30 ($d_{row1} = 1$)
P_2	5	8	2	a_2=75 ($d_{row2} = 3$)
P_3	6	9	$\boxed{25}$ 2	a_3=25 ($d_{row3} = 4$)
Demand	b_2=20 ($d_{col2} = 0$)	b_3=50 ($d_{col3} = 1$)	b_4=60 ($d_{col4} = 0$)	

Table 13: Third table of VAM method of trasportation problem

	C_2	C_3	C_4	Supply
P_1	5	7	6	a_1=30 ($d_{row1} = 1$)
P_2	5	8	$\boxed{35}$ 2	a_2=75 ($d_{row2} = 3$)
Demand	b_2=20 ($d_{col2} = 0$)	b_3=50 ($d_{col3} = 1$)	b_4=35 ($d_{col4} = 4$)	

Table 14: Fourth table of VAM method of trasportation problem

	C_2	C_3	Supply
P_1	5	7	a_1=30 ($d_{row1} = 1$)
P_2	$\boxed{20}$ 5	8	a_2=40 ($d_{row2} = 3$)
Demand	b_2=20 ($d_{col2} = 0$)	b_3=50 ($d_{col3} = 1$)	

Table 15: Fifth table of VAM method of trasportation problem

	C_3	Supply
P_1	$\boxed{30}$ 7	a_1=30
P_2	$\boxed{20}$ 8	a_2=20
Demand	b_3=50	

Hence all the allocated cells are given in Table 16.

[**Note:** We can make all allocation in same table also like Table 17.]

We can see that the number of occupied cell is 6= $m + n - 1$ and they do not form a loop. Hence it is a intial B.F.S. For optimality test, we have to find u_i for $i = 1, 2, 3$ and

Table 16: Sixth table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{20}$ 3	5	$\boxed{30}$ 7	6	a_1=50
P_2	2	$\boxed{20}$ 5	$\boxed{20}$ 8	$\boxed{35}$ 2	a_2=75
P_3	3	6	9	$\boxed{25}$ 2	a_3=25
Demand	b_1=20	b_2=20	b_3=50	b_4=60	

Table 17: Seventh Table of VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply	Supply	Supply	Supply	Supply
P_1	$\boxed{20_1}$ 3	5	$\boxed{30_5}$ 7	6	a_1=50 d_{rw1}=2	a_1=30 d_{rw1}=1	a_1=30 d_{rw1}=1	a_1=30 d_{rw1}=1	a_1=30
P_2	2	$\boxed{20_4}$ 5	$\boxed{20_6}$ 8	$\boxed{35_3}$ 2	a_2=75 d_{rw2}=0	a_2=75 d_{rw2}=3	a_2=75 d_{rw2}=3	a_2=40 d_{rw2}=3	a_2=20
P_3	3	6	9	$\boxed{25_2}$ 2	a_3=25 d_{rw3}=1	a_3=25 d_{rw3}=4	×	×	×
Demand	b_1=20 d_{cl1}=1	b_2=20 d_{cl2}=0	b_3=50 d_{cl3}=1	b_4=60 d_{cl4}=0					
Demand	×	b_2=20 d_{cl2}=0	b_3=50 d_{cl3}=1	b_4=60 d_{cl4}=0					
Demand	×	b_2=20 d_{cl2}=0	b_3=50 d_{cl3}=1	b_4=35 d_{cl4}=4					
Demand	×	b_2=20 d_{cl2}=0	b_3=50 d_{cl3}=1	×					
Demand	×	×	b_3=50	×					

v_j for $j = 1, 2, 3, 4$ satisfying $c_{ijc} = u_i + v_j$ for occupied cells. I.e. $u_1 + v_1 = 3$, $u_1 + v_3 = 7$, $u_2 + v_2 = 5$, $u_2 + v_3 = 8$, $u_2 + v_4 = 2$, $u_3 + v_4 = 2$. Since, we have to choose one u_i=0 or v_j=0, we choose u_2=0 (as it appears maximum times). Then $u_1 = -1$, $u_3 = 0$, $v_1 = 4$, $v_2 = 5$, $v_3 = 8$, $v_4 = 2$. Then we have calculated the cell evaluation $\Delta_{ij} = c_{ij} - u_i - v_j$ for unoccupied cells (i, j)th as $\Delta_{12} = 11$, $\Delta_{14} = 5$, $\Delta_{21} = -2$, $\Delta_{31} = -1$, $\Delta_{32} = 1$, $\Delta_{33} = 1$ (shown in Table 18).

Table 18: Cell evaluation of VAM method of trasportation problem

		v_1=4	v_2=5	v_3=8	v_4=2	
		C_1	C_2	C_3	C_4	Supply
u_1=-1	P_1	$\boxed{20}$ 3	1 5	$\boxed{30}$ 7	5 6	a_1=50
Let u_2=0	P_2	-2 2	$\boxed{20}$ 5	$\boxed{20}$ 8	$\boxed{35}$ 2	a_2=75
u_3=0	P_3	-1 3	1 6	1 9	$\boxed{25}$ 2	a_3=25
	Demand	b_1=20	b_2=20	b_3=50	b_4=60	

Since, $\Delta_{21} = -2 < 0$ and $\Delta_{31} = -1 < 0$, current B.F.S. is not minimum solution and for improvement, we allocate maximum possible unit to $(2, 1)$th cell and adjust this additional allotment such that the cell $(1, 1)$th and $(2, 2)$th become empty. Now, we can see that the number of occupied cell is 5$< m + n - 1 = 6$ and so we make a dummy allocation ϵ (a very small postive real number) at cell $(1, 2)$ (unallocated lowest cost cell) such that they do not form a loop (we ignore the cell $(1, 1)$ as it has been unallocated).

Table 19: First re-allocation of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	$\boxed{20}$ - $\boxed{20}$		$\boxed{30}$ + $\boxed{20}$		a_1=50
P_2	• + $\boxed{20}$	$\boxed{20}$	$\boxed{20}$ - $\boxed{20}$	$\boxed{35}$	a_2=75
P_3				$\boxed{25}$	a_3=25
Demand	b_1=20	b_2=20	b_3=50	b_4=60	

So, all the first modified allocated cells are given in Table 20

Table 20: First modified allocated cells after VAM method of trasportation problem

	C_1	C_2	C_3	C_4	Supply
P_1	3	$\boxed{\epsilon}$ 5	$\boxed{50}$ 7	6	a_1=50
P_2	$\boxed{20}$ 2	$\boxed{20}$ 5	8	$\boxed{35}$ 2	a_2=75
P_3	3	6	9	$\boxed{25}$ 2	a_3=25
Demand	b_1=20	b_2=20	b_3=50	b_4=60	

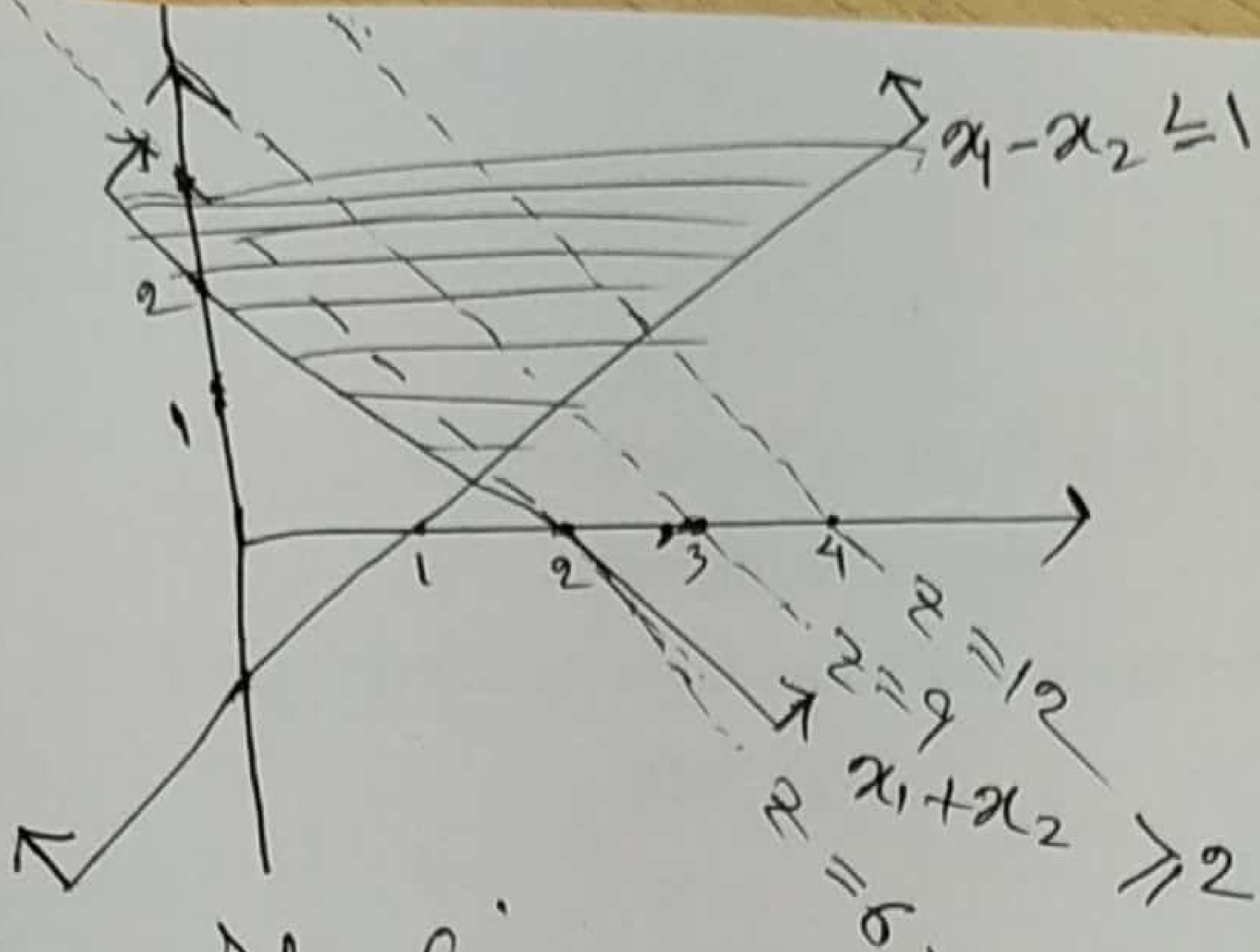
For optimality test, we have to find u_i for $i = 1, 2, 3$ and v_j for $j = 1, 2, 3, 4$ satisfying $c_{ijc} = u_i + v_j$ for occupied cells. I.e. $u_1 + v_2 = 5$, $u_1 + v_3 = 7$, $u_2 + v_1 = 2$, $u_2 + v_2 = 5$, $u_2 + v_4 = 2$, $u_3 + v_4 = 2$. Since, we have to choose one u_i=0 or v_j=0, we choose u_2=0 (as it appears maximum times). Then $u_1 = 0$, $u_3 = 0$, $v_1 = 2$, $v_2 = 5$, $v_3 = 7$, $v_4 = 2$. Then we have calculated the cell evaluation $\Delta_{ij} = c_{ij} - u_i - v_j$ for unoccupied cells (i, j)th as $\Delta_{11} = 11$, $\Delta_{14} = 4$, $\Delta_{23} = 1$, $\Delta_{31} = 1$, $\Delta_{32} = 1$, $\Delta_{33} = 2$ (shown in Table 21).

Table 21: Second Cell evaluation after VAM method of trasportation problem

		v_1=2	v_2=5	v_3=7	v_3=2	
		C_1	C_2	C_3	C_4	Supply
u_1=0	P_1	1 3	$\boxed{\epsilon}$ 5	$\boxed{50}$ 7	5 6	a_1=50
Let u_2=0	P_2	$\boxed{20}$ 2	$\boxed{20}$ 5	1 8	$\boxed{35}$ 2	a_2=75
u_3=0	P_3	1 3	1 6	2 9	$\boxed{25}$ 2	a_3=25
	Demand	b_1=20	b_2=20	b_3=50	b_4=60	

Since, $\Delta_{ij} > 0$ for all unoccupied cells, current B.F.S. is minimum solution with basic variables $x_{13} = 50$ (i.e. plant P_1 to city C_3), $x_{21} = 20$ (i.e. plant P_2 to city C_1), $x_{22} = 20$ (i.e. plant P_2 to city C_2), $x_{24} = 35$ (i.e. plant P_2 to city C_4), $x_{34} = 25$ (i.e. plant P_3 to city C_4) and $x_{12} = \epsilon$ ($\epsilon \to 0$) (i.e. plant P_1 to city C_2) units of cars and the minimum transportation costs will be $50 \times 7 + 20 \times 2 + 20 \times 5 + 35 \times 2 + 25 \times 2 = 610$ thousands in rupees.

1). or. (i)



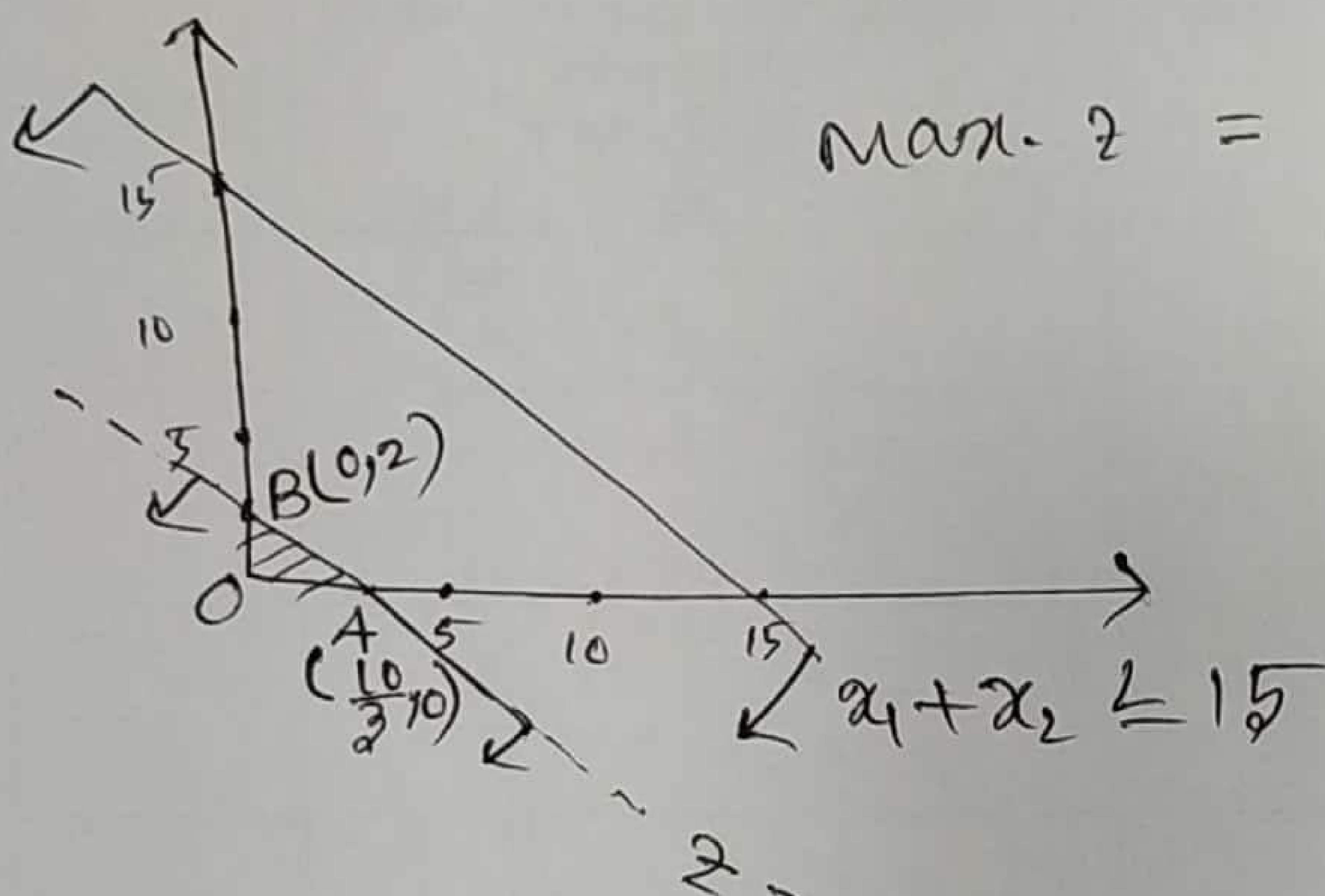
max $z = 3x_1 + 2x_2$
$x_1 - x_2 \ge 2$
$x_1 + x_2 \ge 0$
$x_1, x_2 \ge 0$

$z = 3x_1 + 2x_2$,

(a) $z = 6 \Rightarrow 3x_1 + 2x_2 = 6$
(b) $z = 9 \Rightarrow 3x_1 + 2x_2 = 9$
(c) $z = 12 \Rightarrow 3x_1 + 2x_2 = 12$

cost line moving away from the origin (i.e z increasing infinitely). Hence it has unbounded solution.

(ii)



max. $z = 6x_1 + 10x_2$,
$3x_1 + 5x_2 \le 10$,
$x_1 + x_2 \le 15$

$z_A = 20$, $z_O = 0$,
$z_B = 20$.

Since maximum value of z occurs at the points $A(\frac{10}{3}, 0)$, $B(0, 2)$. Any convex combination of A, B i.e on the any point on the line segment AB gives the maximum value $z_{max} = 20$. Hence it has infinite number of ~~infinite~~ optimal solutions.