The LNM Institute of Information Technology, Jaipur, Rajasthan
OPTI-2019 (Odd Semester): Quiz1, 12th September

Maximum Marks: 7.5 Time: 30 minutes

Name: **Roll No.:**

1. Suppose Jaipur Traffic Police Department has the following requirements for traffic polices:

Period	Clock time (24 hour day)	Minimum number of traffic polices required
1	6 A.M.–10 A.M.	35
2	10 A.M.–2 P.M.	80
3	2 P.M.–6 P.M.	70
4	6 P.M.–10 P.M.	40
5	10 P.M.–2 A.M.	25
6	2 A.M.–6 A.M.	20

Traffic polices report to the headquarter at the begining of each period and work for eight consecutive hours. The Department wants to determine the minimum number of traffic polices so that there may be sufficient numer of traffic polices available for each period. Formulate this as an L.P.P. Comment upon the disadvantages you foresee of formulating and solving this problem as a linear program. **[2.5 marks]**

Answer: Let x_i number of traffic polices will join at $i-th$ period where $i = 1$ (for 6 A.M.-10 A.M), 2 (for 10 A.M.-2 P.M.), 3 (for 2 P.M.-6 P.M.), 4 (for 6 P.M.-10 P.M.), 6 (for 10 P.M.-2 A.M.), 6 (for 2 A.M.-6 A.M.). Then O.F. is Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$ and the constrains will be $x_6 + x_1 \geq 35$, $x_1 + x_2 \geq 80$, $x_2 + x_3 \geq 70$, $x_3 + x_4 \geq 40$, $x_4 + x_5 \geq 25$, $x_5 + x_6 \geq 20$, $x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6 \geq 0$ and $x_i \in \mathbb{Z}^+ \cup \{0\}$. **[2 marks]**

Some of the disadvantages of solving this problem as a linear program are: (i) really need variable values which are integer (ii) some traffic polices will always end up working at late night (iii) how do we choose the traffic polices to use, e.g. if $x_3 = 20$ which 20 traffic polices do we choose to begin their work at period $i = 3$ (for 2 P.M.-6 P.M.). (iv) what happens if traffic polices fail to report in (e.g. if they are sick), we may fall below the minimum number required etc. **[0.5 mark]**

2. If S be a set of all feasible solutions of $A\underline{\mathbf{x}} = \underline{\mathbf{b}},\ \underline{\mathbf{x}} \geq \underline{\mathbf{0}}$ and if $\underline{\mathbf{x}}^* \in S$ minimize the O.F. $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$, then $\underline{\mathbf{x}}^*$ also maximizes the function $z' = -\underline{\mathbf{c}}^t\underline{\mathbf{x}}$ over S. **[1.5 marks]**

Answer: Let $\underline{\mathbf{x}}$ ($\neq \underline{\mathbf{x}}^*$) be any other feasible solution of S. Since, $\underline{\mathbf{x}}^*$ minimizes $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$, we have $\underline{\mathbf{c}}^t\underline{\mathbf{x}}^* \leq \underline{\mathbf{c}}^t\underline{\mathbf{x}}$ for all $\underline{\mathbf{x}} \in S$ with $A\underline{\mathbf{x}}^* = \underline{\mathbf{b}},\ \underline{\mathbf{x}}^* \geq \underline{\mathbf{0}}$ and $A\underline{\mathbf{x}} = \underline{\mathbf{b}},\ \underline{\mathbf{x}} \geq \underline{\mathbf{0}}$. Then $-\underline{\mathbf{c}}^t\underline{\mathbf{x}}^* \geq -\underline{\mathbf{c}}^t\underline{\mathbf{x}}$ for all $\underline{\mathbf{x}} \in S$ with $A\underline{\mathbf{x}}^* = \underline{\mathbf{b}},\ \underline{\mathbf{x}}^* \geq \underline{\mathbf{0}}$ and $A\underline{\mathbf{x}} = \underline{\mathbf{b}},\ \underline{\mathbf{x}} \geq \underline{\mathbf{0}}$. From the definition of maximization, $\underline{\mathbf{x}}^*$ maximizes the O.F. $z' = -\underline{\mathbf{c}}^t\underline{\mathbf{x}}$ over S. Hence, $\underline{\mathbf{x}}^*$ also maximizes the function $z' = -\underline{\mathbf{c}}^t\underline{\mathbf{x}}$ over S if $\underline{\mathbf{x}}^* \in S$ minimize the O.F. $z = \underline{\mathbf{c}}^t\underline{\mathbf{x}}$ over S. **[1.5 marks]**

3. Solve the following LPP and also justify whether optimal solution will be unique if it exits : Maximize $z = 2x_1 - x_2 + 2x_3$
Sub. to $2x_1 + x_2 \leq 10$,
$x_1 + 2x_2 - 2x_3 \leq 20$,
$x_2 + 2x_3 \leq 5$,
$x_1,\ x_2,\ x_3 \geq 0$ **[3.5 marks]**

Answer: To make the L.P.P in standard form, we have to introduce slack variables x_4, x_5 and x_6 in 1st, 2nd and 3rd constraints respectively.

$Z = 2x_1 - x_2 + 2x_3 + 0x_4 + 0x_5 + 0x_6$, Sub. to $2x_1 + x_2 + x_4 = 10$
$x_1 + 2x_2 - 2x_3 + x_5 = 20$
$x_2 + 2x_3 + x_6 = 5$
$x_1,\ x_2, \cdots, x_6 \geq 0$. For initial B.F.S, $B = [\underline{\mathbf{a_4}}, \underline{\mathbf{a_5}}, \underline{\mathbf{a_6}}]$, $\underline{\mathbf{x_B}} = [x_4, x_5, x_6]^t = [10, 20, 5]^t$, $\underline{\mathbf{c_B}} = [c_4, c_5, c_6]^t = [0, 0, 0]^t$. Then we can make following tables:

Table 1: First Simplex Table

			$c_j \rightarrow$	2	-1	2	0	0	0	$y_{ij} > 0$, $\frac{x_{B_i}}{y_{ij}}$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	$\underline{\mathbf{a_1}}$	$\underline{\mathbf{a_2}}$	$\underline{\mathbf{a_3}}$	$\underline{\mathbf{a_4}}$	$\underline{\mathbf{a_5}}$	$\underline{\mathbf{a_6}}$	for $\underline{\mathbf{a_j}} \uparrow$
0	$\underline{\mathbf{a_4}}$	x_4	10	2	1	0	1	0	0	$-$
0	$\underline{\mathbf{a_5}}$	x_5	20	1	2	-2	0	1	0	$-$
0	$\underline{\mathbf{a_6}}$	x_6	5	0	1	$\boxed{2}$	0	0	1	$\frac{5}{2} \Rightarrow$
$z = 0$			(z_j-c_j) $\rightarrow$	-2	1	-2 $\Uparrow$	0	0	0	Min. $\{\frac{x_{B_i}}{y_{ij}},\ y_{ij} > 0\ \}$=2.5

[1 mark]

Since in Table 1, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_3 entering the basis and x_6 leaving the basis after choosing $(z_3 - c_3)$ as most negative (another alternative will be $(z_1 - c_1)$).

Table 2: Second Simplex Table

			$c_j \rightarrow$	2	-1	2	0	0	0	$y_{ij} > 0$, $\frac{x_{B_i}}{y_{ij}}$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	$\underline{\mathbf{a_1}}$	$\underline{\mathbf{a_2}}$	$\underline{\mathbf{a_3}}$	$\underline{\mathbf{a_4}}$	$\underline{\mathbf{a_5}}$	$\underline{\mathbf{a_6}}$	for $\underline{\mathbf{a_j}} \uparrow$
0	$\underline{\mathbf{a_4}}$	x_4	10	$\boxed{2}$	1	0	1	0	0	$5 \Rightarrow$
0	$\underline{\mathbf{a_5}}$	x_5	25	1	3	0	0	1	1	25
2	$\underline{\mathbf{a_6}}$	x_3	$\frac{5}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	$-$
$z = 5$			(z_j-c_j) $\rightarrow$	-2 $\Uparrow$	2	0	0	0	1	Min. $\{\frac{x_{B_i}}{y_{ij}},\ y_{ij} > 0\ \}$=5

[1 mark]

Since in Table 2, the optimality condition $z_j - c_j \geq 0$ for all j does not satisfy, then we can go to next table with x_1 entering the basis and x_4 leaving the basis.

Table 3: Second Simplex Table

			$c_j \rightarrow$	2	-1	2	0	0	0	$y_{ij} > 0$, $\frac{x_{B_i}}{y_{ij}}$
$\underline{\mathbf{c_B}}$	B	$\underline{\mathbf{x_B}}$	$\underline{\mathbf{b}}$	$\underline{\mathbf{a_1}}$	$\underline{\mathbf{a_2}}$	$\underline{\mathbf{a_3}}$	$\underline{\mathbf{a_4}}$	$\underline{\mathbf{a_5}}$	$\underline{\mathbf{a_6}}$	for $\underline{\mathbf{a_j}} \uparrow$
2	$\underline{\mathbf{a_1}}$	x_1	5	1	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	
0	$\underline{\mathbf{a_5}}$	x_5	20	0	$\frac{5}{2}$	0	-$\frac{1}{2}$	1	1	
2	$\underline{\mathbf{a_3}}$	x_3	$\frac{5}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	
$z = 15$			(z_j-c_j) $\rightarrow$	0	3	0	1	0	1	

Since in the Table 3, $z_j - c_j \geq 0$ for all j, the current solution is a optimal basic feasible solution (B.F.S). As $z_j - c_j > 0$ for non-basis vectors and the optimal solution is non-degerate basic solution, we get the unique basic optimal solution as $x_1 = 5$, $x_2 = 0$, $x_3 = \frac{5}{2}$, $x_4 = 0$, $x_5 = 20$, $x_6 = 0$, and hence the optimal solution of the original problem is $x_1 = 5$, $x_2 = 0$, $x_3 = \frac{5}{2}$ and $z_{max} = 15$. **[1.5 marks]**

OR

Show that a B.F.S $\underline{\mathbf{x}}_{\mathbf{B}}$ of Maximize $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$, subject to $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$, $\underline{\mathbf{x}} \geq \underline{\mathbf{0}}$ where $A = (a_{ij})_{m\times n}$, $\underline{\mathbf{x}} = [x_1, x_2, \cdots, x_n]^t$, $\underline{\mathbf{b}} = [b_1, b_2, \cdots, b_m]^t$, rank($A$)=$m$; corresponds to an extreme point of the convex set of feasible solutions.

Answer: Without loss of generality, let first m columns of $A = (\alpha_1\ \alpha_2\ \cdots\ \alpha_n)$ are linearly independent such that

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = \underline{\mathbf{b}} \tag{1}$$

and we assume that $\underline{\mathbf{y}} = [x_1, x_2, \cdots, x_m, 0, 0, \cdots, 0]^t$ be a BFS of $A\underline{\mathbf{x}} = \underline{\mathbf{b}}$. If possible, let $\underline{\mathbf{y}}$ is not a extreme point of the convex set of feasible solutions S. Since $\underline{\mathbf{y}} \in S$, we have $\underline{\mathbf{y}} = \lambda\underline{\mathbf{u}} + (1-\lambda)\underline{\mathbf{v}}$ with $0 < \lambda < 1$ and $\underline{\mathbf{u}}(\neq \underline{\mathbf{y}}), \underline{\mathbf{v}}(\neq \underline{\mathbf{y}}) \in S$. As $\underline{\mathbf{u}}, \underline{\mathbf{v}} \geq \underline{\mathbf{0}}$ and $0 < \lambda < 1$, $\underline{\mathbf{u}}, \underline{\mathbf{v}}$ will be in the form $\underline{\mathbf{u}} = [u_1, u_2, \cdots, u_m, 0, 0, \cdots, 0]^t$, $\underline{\mathbf{v}} = [v_1, v_2, \cdots, v_m, 0, 0, \cdots, 0]^t$. Since $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in S$, we have

$$u_1\alpha_1 + u_2\alpha_2 + \cdots + u_m\alpha_m = \underline{\mathbf{b}} \tag{2}$$

$$v_1\alpha_1 + v_2\alpha_2 + \cdots + v_m\alpha_m = \underline{\mathbf{b}} \tag{3}$$

Subtracting equation (2) from equation (1), we get

$$(x_1 - u_1)\alpha_1 + (x_2 - u_2)\alpha_2 + \cdots + (x_m - u_m)\alpha_m = \underline{\mathbf{0}}. \tag{4}$$

As $\alpha_1\ \alpha_2\ \cdots\ \alpha_m$ are linearly independent, we have $x_1 - u_1 = 0,\ x_2 - u_2 = 0,\ \cdots\ x_m - u_m = 0$. Similarly from equations (1) and (3), we have $x_1 - v_1 = 0,\ x_2 - v_2 = 0,\ \cdots\ x_m - v_m = 0$. Then $x_1 = u_1 = v_1,\ x_2 = u_2 = v_2, \cdots, x_m = u_m = v_m$. So, $\underline{\mathbf{y}} \in S$ can't be written as convex combination of two other points $\underline{\mathbf{u}}, \underline{\mathbf{v}} \in S$. Thus $\underline{\mathbf{y}}$ is extreme point of S. **[3 marks]**