INDIAN STATISTICAL INSTITUTE
Mathematics I: BSDS First Year
Semester I, Academic Year 2024-25
Midsem Exam

Date: 30/09/2024 **Full Marks: 50** **Duration: 3 Hours**

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **This is a closed-book exam. You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. (10 marks) Suppose $f : [0,1] \to [0,1]$ is a continuous function. Show that there exists $c \in [0,1]$ such that $f(c) = c$.

2. (10 marks) Show that the set

$$C = \left\{ \sqrt[3]{m} + 2n + 1 : m, n \in \mathbb{N} \right\}$$

is countable. Here $\sqrt[3]{m}$ denotes the real cube-root of m.

3. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers defined as follows: $a_1 = 1$ and for all $n \in \mathbb{N}$, $a_{n+1} = (5 + 4a_n)/(5 + a_n)$.

   (a) (10 marks) Show that there exists $\lambda \in (0,1)$ such that for all $n \geq 2$,

$$|a_{n+1} - a_n| \leq \lambda |a_n - a_{n-1}|.$$

   (b) (10 marks) Prove that $\lim_{n\to\infty} a_n$ exists and find its value.

4. Define the set

$$S = \{x \in \mathbb{Q} : x^3 < 2\}.$$

   (a) (5 marks) Find, with justification, whether $\sup S$ exsits. If it exists, calculate its value.

   (b) (5 marks) Find, with justification, whether $\inf S$ exsits. If it exists, calculate its value.