To lay a better foundation for the general theory of power series, we turn next to certain general questions related to the convergence and divergence of arbitrary series. We shall return to the subject of power series in Chapter 11.

### 10.9 Exercises

Each of the series in Exercises 1 through 10 is a telescoping series, or a geometric series, or some related series whose partial sums may be simplified. In each case, prove that the series converges and has the sum indicated.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{2}.$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^{n-1}} = 3.$

3. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{3}{4}.$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{6^n} = \frac{3}{2}.$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1.$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} = \frac{1}{4}.$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} = 1.$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+n^2+n}{2^{n+1}n(n+1)} = 1.$

9. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n+1)}{n(n+1)} = 1.$

10. $\displaystyle\sum_{n=2}^{\infty} \frac{\log\,[(1+1/n)^n(1+n)]}{(\log n^n)[\log\,(n+1)^{n+1}]} = \log_2 \sqrt{e}.$

Power series for $\log(1+x)$ and $\arctan x$ were obtained in Section 10.8 by performing various operations on the geometric series. In a similar manner, without attempting to justify the steps, obtain the formulas in Exercises 11 through 19. They are all valid at least for $|x| < 1$. (The theoretical justification is provided in Section 11.8.)

11. $\displaystyle\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$

12. $\displaystyle\sum_{n=1}^{\infty} n^2x^n = \frac{x^2+x}{(1-x)^3}.$

13. $\displaystyle\sum_{n=1}^{\infty} n^3x^n = \frac{x^3+4x^2+x}{(1-x)^4}.$

14. $\displaystyle\sum_{n=1}^{\infty} n^4x^n = \frac{x^4+11x^3+11x^2+x}{(1-x)^5}.$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n} = \log\frac{1}{1-x}.$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1} = \tfrac{1}{2}\log\frac{1+x}{1-x}.$

17. $\displaystyle\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$

18. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2!}x^n = \frac{1}{(1-x)^3}.$

19. $\displaystyle\sum_{n=0}^{\infty} \frac{(n+1)(n+2)(n+3)}{3!}x^n = \frac{1}{(1-x)^4}.$

20. The results of Exercises 11 through 14 suggest that there exists a general formula of the form

$$\sum_{n=1}^{\infty} n^k x^n = \frac{P_k(x)}{(1-x)^{k+1}},$$

where $P_k(x)$ is a polynomial of degree $k$, the term of lowest degree being $x$ and that of highest degree being $x^k$. Prove this by induction, without attempting to justify the formal manipulations with the series.

21. The results of Exercises 17 through 19 suggest the more general formula

$$\sum_{n=0}^{\infty} \binom{n+k}{k} x^n = \frac{1}{(1-x)^{k+1}}, \qquad \text{where} \quad \binom{n+k}{k} = \frac{(n+1)(n+2)\cdots(n+k)}{k!}$$

Prove this by induction, without attempting to justify the formal manipulations with the series.

22. Given that $\sum_{n=0}^{\infty} x^n/n! = e^x$ for all x, find the sums of the following series, assuming it is permissible to operate on infinite series as though they were finite sums.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n!}$. (b) $\displaystyle\sum_{n=2}^{\infty} \frac{n+1}{n!}$. (c) $\displaystyle\sum_{n=2}^{\infty} \frac{(n-1)(n+1)}{n!}$.

23. (a) Given that $\sum_{n=0}^{\infty} x^n/n! = e^x$ for all x, show that

$$\sum_{n=1}^{\infty} \frac{n^2 x^n}{n!} = (x^2 + x)e^x,$$

assuming it is permissible to operate on these series as though they were finite sums.

(b) The sum of the series $\sum_{n=1}^{\infty} n^3/n!$ is $ke$, where $k$ is a positive integer. Find the value of $k$. Do not attempt to justify formal manipulations.

24. Two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are called ***identical*** if $a_n = b_n$ for each $n \geq 1$. For example, the series

$$0 + 0 + 0 + \cdots \qquad \text{and} \qquad (1-1) + (1-1) + (1-1) + \cdots$$

are identical, but the series

$$1 + 1 + 1 + \cdots \qquad \text{and} \qquad 1 + 0 + 1 + 0 + 1 + 0 + \cdots$$

are not identical. Determine whether or not the series are identical in each of the following pairs:

(a) $1 - 1 + 1 - 1 + \cdots$ and $(2-1) - (3-2) + (4-3) - (5-4) + \cdots$.
(b) $1 - 1 + 1 - 1 + \cdots$ and $(1-1) + (1-1) + (1-1) + (1-1) + \cdots$.
(c) $1 - 1 + 1 - 1 + \cdots$ and $1 + (-1+1) + (-1+1) + (-1+1) + \cdots$.
(d) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ and $1 + (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{4}) + (\frac{1}{4} - \frac{1}{8}) + \cdots$.

25. (a) Use (10.26) to prove that

$$1 + 0 + x^2 + 0 + x^4 + \cdots = \frac{1}{1-x^2} \qquad \text{if } |x| < 1.$$

Note that, according to the definition given in Exercise 24, this series is not identical to the one in (10.26) if $x \neq 0$.

(b) Apply Theorem 10.2 to the result in part (a) and to (10.25) to deduce (10.27).

(c) Show that Theorem 10.2 when applied directly to (10.25) and (10.26) does not yield (10.27). Instead, it yields the formula $\sum_{n=1}^{\infty} (x^n - x^{2n}) = x/(1-x^2)$, valid for $|x| < 1$.

EXAMPLE 1. The integral test enables us to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{converges if and only if} \quad s > 1 .$$

Taking $f(x) = x^{-s}$, we have

$$t_n = \int_1^n \frac{1}{x^s}\,dx = \begin{cases} \dfrac{n^{1-s} - 1}{1 - s} & \text{if} \quad s \neq 1 , \\ \log n & \text{if} \quad s = 1 . \end{cases}$$

When $s > 1$ the term $n^{1-s} \to 0$ as $n \to \infty$ and hence $\{t_n\}$ converges. By the integral test, this implies convergence of the series for $s > 1$.

When $s \leq 1$, then $t_n \to \infty$ and the series diverges. The special case $s = 1$ (the *harmonic series*) was discussed earlier in Section 10.5. Its divergence was known to Leibniz.

EXAMPLE 2. The same method may be used to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^s} \quad \text{converges if and only if} \quad s > 1 .$$

(We start the sum with $n = 2$ to avoid $n$ for which $\log n$ may be zero.)

The corresponding integral in this case is

$$t_n = \int_2^n \frac{1}{x(\log x)}\,dx = \begin{cases} \dfrac{(\log n)^{1-s} - (\log 2)^{1-s}}{1 - s} & \text{if} \quad s \neq 1 , \\ \log(\log n) - \log(\log 2) & \text{if} \quad s = 1 . \end{cases}$$

Thus $\{t_n\}$ converges if and only if $s > 1$, and hence, by the integral test, the same holds true for the series in question.

## 10.14 Exercises

Test the following series for convergence or divergence. In each case, give a reason for your decision.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(4n - 3)(4n - 1)}$ . d
2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{2n-1}\,\log(4n+1)}{n(n+1)}$ . c
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{2^n}$ . c
4. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ . c
5. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin nx|}{n^2}$ . c
6. $\displaystyle\sum_{n=1}^{\infty} \frac{2 + (-1)^n}{2^n}$ . c
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}$ . c
8. $\displaystyle\sum_{n=2}^{\infty} \frac{\log n}{n\sqrt{n+1}}$ . c

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}}$. d

10. $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{(n+1)^3-1}$. c

11. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^s}$.

12. $\displaystyle\sum_{n=1}^{\infty} \frac{|a_n|}{10^n}$, $|a_n| < 10$. c

13. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1000n+1}$. d

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n\cos^2(n\pi/3)}{2^n}$.

15. $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\log n\,(\log\log n)^s}$.

16. $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$.

17. $\displaystyle\sum_{n=1}^{\infty} \int_0^{1/n} \frac{\sqrt{x}}{1+x^2}\,dx$.

18. $\displaystyle\sum_{n=1}^{\infty} \int_n^{n+1} e^{-\sqrt{x}}\,dx$.

19. Assume $f$ is a nonnegative increasing function defined for all $x \geq 1$. Use the method suggested by the proof of the integral test to show that

$$\sum_{k=1}^{n-1} f(k) \leq \int_1^n f(x)\,dx \leq \sum_{k=2}^{n} f(k).$$

Take $f(x) = \log x$ and deduce the inequalities

$$e\,n^n e^{-n} < n! < e\,n^{n+1}e^{-n}. \tag{10.41}$$

These give a rough estimate of the order of magnitude of $n!$. From (10.41), we may write

$$\frac{e^{1/n}}{e} < \frac{(n!)^{1/n}}{n} < \frac{e^{1/n}\,n^{1/n}}{e}.$$

Letting $n \to \infty$, we find that

$$\frac{(n!)^{1/n}}{n} \to \frac{1}{e} \quad \text{or} \quad (n!)^{1/n} \sim \frac{n}{e} \quad \text{as } n \to \infty.$$

## 10.15 The root test and the ratio test for series of nonnegative terms

Using the geometric series $\sum x^n$ as a comparison series, Cauchy developed two useful tests known as the *root test* and the *ratio test.*

If $\sum a_n$ is a series whose terms (from some point on) satisfy an inequality of the form

$$0 \leq a_n \leq x^n, \quad \text{where} \quad 0 < x < 1, \tag{10.42}$$

a direct application of the comparison test (Theorem 10.8) tells us that $\sum a_n$ converges. The inequalities in (10.42) are equivalent to

$$0 \leq a_n^{1/n} \leq x; \tag{10.43}$$

hence the name *root test.*