Lecture-1

§ A Brief Review

Linear Transformation

- Let U & V be vector spaces over $\mathbb{K}$. A function $T: U \to V$ is called a Linear transformation if for all $x, y \in U$ & $\alpha \in \mathbb{K}$:

$$T(x+y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x)$$

- $T: U \to V$ a linear Transformation. Then

$$T(0_U) = 0_V. \quad \Big(T(x) = T(x + 0_U) = T(x) + T(0_U) \Rightarrow T(0_U) = 0_V \Big)$$

- Examples. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a Reflection relative x-axis

$$T(x_1, x_2) = (x_1, -x_2).$$

- $T: \mathbb{R}^3 \to \mathbb{R}^3$ is an orthogonal projection on th xy-plane

$$T(x_1, x_2, x_3) = (x_1, x_2, 0)$$

- $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a translation by the vector $u = (1,0)$

$$T(x_1, x_2) = (x_1 + 1, x_2) \; ? \quad \big(T(0,0) = (1,0) \big).$$

- $T: M_2(\mathbb{C}) \to \mathbb{C}$, $T(A) = tr(A)$

$$\Big(T(A+B) = tr(A+B) = tr(A) + tr(B) = T(A) + T(B) \Big)$$
$$T(\alpha A) = tr(\alpha A) = \alpha\, tr(A) = \alpha T(A).$$

Matrix Representation

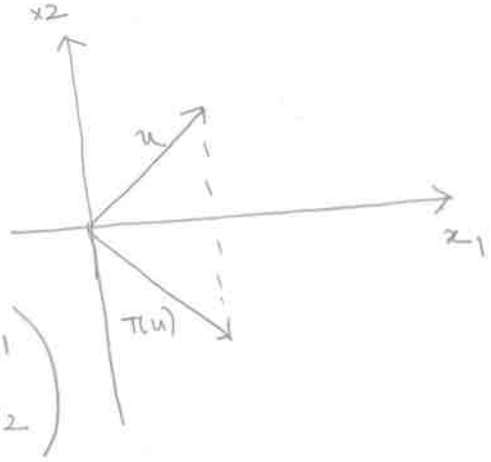
Let $T: \mathbb{K}^n \to \mathbb{K}^k$ be a linear transformation & let $E_n = (e_1, \ldots, e_n)$ be the ordered standard basis of $\mathbb{K}^n$ & $\mathcal{E}_k$ = the ordered standard basis of $\mathbb{K}^k$

Then we have; for $x = (x_1, x_2, \ldots, x_n)$

$$T(x) = T(x_1e_1 + x_2e_2 + \cdots + x_ne_n) = x_1T(e_1) + x_2T(e_2) + \cdots + x_nT(e_n)$$

Denoting by $[x]$ the vector column ~~vec~~ version of a vector x

$$[T(x)] = \underbrace{\Big[[T(e_1)] \mid [T(e_2)] \mid \cdots \mid T(e_n)\Big]}_{\text{Matrix associated with } T} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$



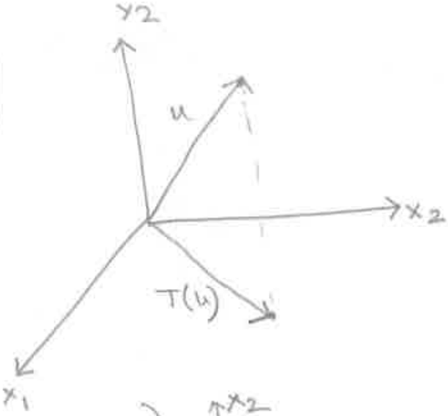
Example: $T(x_1, x_2) = (x_1, -x_2)$

$$T(e_1x_1 + e_2x_2) = x_1T(e_1) + x_2T(e_2)$$

$$= \Big[[T(e_1)] \mid [T(e_2)]\Big] \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$[T] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

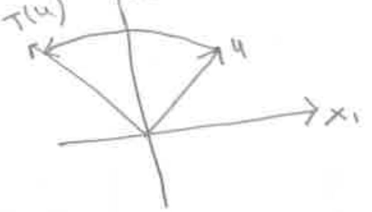


- $T(x_1, x_2, x_3) = (x_1, x_2, 0)$

$$= x_1T(e_1) + x_2T(e_2) + x_3T(e_3)$$

$$= \Big[T(e_1) \mid T(e_2) \mid T(e_3)\Big] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$



- $T(1,0) = (\cos\theta, \sin\theta)$, $T(0,1) = (-\sin\theta, \cos\theta)$

$$T(x) = [T]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{pmatrix}$$

Thm: Let U & V be vector spaces over $\mathbb{K}$ with $\dim U = n$, $\dim V = k$. Let $\mathcal{B}_1 = (b_1, b_2, \ldots, b_n)$ be a basis of U & $\mathcal{B}_2$ be a Basis of V & let $T: U \to V$ be a l.T. Then, there exists uniquely a $k\times n$ matrix $[T]_{\mathcal{B}_2,\mathcal{B}_1}$ s.t $\forall x \in U$,

$$[T(x)]_{\mathcal{B}_2} = [T]_{\mathcal{B}_2,\mathcal{B}_1} x_{\mathcal{B}_1}$$

Example: Let $T: \mathbb{P}_2 \to \mathbb{P}_1$ be the linear Transformation. $T(p) = Dp$, where Dp denotes the derivative of the polynomial p. Find the matrix associated with T relative to the standard. Basis of $\mathbb{P}_2$ & $\mathbb{P}_1$.

$$[T]_{P_1,P_2} = \left[(D1)_{P_1}\ (Dt)_{P_1}\ (Dt^2)_{P_1}\right]$$

$$= \left[(0)_{P_1}, (1)_{P_1}, (2t)_{P_1}\right]$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \checkmark$$

$T(1) = 0 = 0.1 + 0.t$

$T(t) = 1 = 1.1 + 0.t$

$T(t^2) = 2t = 0.1 + 2.t$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Use this Matrix to Calculate the image of the polynomial

$p(t) = 1 - 2t + 3t^2$. ?

Null Space & Image

Defn: $T: U \to V$ be a Linear transformation.

The null space or Kernel $N(T)$ of the L.T T is the subspace of U:

$$N(T) = \{x \in U : T(x) = 0_V\}.$$

The image $Im(T) = \{T(x) \in V : x \in U\}$. ✓

Let $T: \mathbb{K}^n \to \mathbb{K}^k$ be a linear transformation. Then

$$n = null(T) + \underbrace{rank(T)}_{\dim Im(T)}. \quad ✓$$

Composition and Invertibility

$T: U \to V$ & $S: V \to W$ be linear transformations.

$$S \circ T : U \to W$$
$$x \longmapsto S(T(x))$$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[dr, "S \circ T"'] & V \arrow[d, "S"] \\ & W \end{tikzcd}$$

Propn: Let $T: U \to V$ & $S: V \to W$ be linear transformations.

Then the fn $ST: U \to W$ is a linear transformation.

$\dim U = n$, $\dim V = p$ & $\dim W = k$

& let $\mathcal{B}_U$, $\mathcal{B}_V$, & $\mathcal{B}_W$ be bases of U, V & W respectively. Let $A = [T]_{\mathcal{B}_V, \mathcal{B}_U}$, $B = [S]_{\mathcal{B}_W, \mathcal{B}_V}$ be the matrices of L.Ts relative to the fixed bases in U, V, W.

$$\left[(ST)(x)\right]_{\mathcal{B}_W} = \left[S(T(x))\right]_{\mathcal{B}_W}$$

$$= B\left[(T(x))\right]_{\mathcal{B}_V}$$

$$= \underline{BA}\,[x]_{\mathcal{B}_U}$$

Hence, the Matrix $[ST]_{\mathcal{B}_W, \mathcal{B}_U}$ of the L.T (ST) relative to the basis $\mathcal{B}_U$ in the domain & the Basis $\mathcal{B}_W$ in the co-domain is

$$[ST]_{\mathcal{B}_W, \mathcal{B}_U} = BA$$

Change of Basis

Let $T: \mathbb{K}^n \to \mathbb{K}^n$ be a linear transformation and let $\mathcal{B} = (b_1, b_2, \ldots, b_n)$ be a basis of $\mathbb{K}^n$.

Given a vector $x \in \mathbb{K}^n$, the co-ordinate vector of the image of x can be determined both using the matrix $A = [T]_{\mathcal{E}_n, \mathcal{E}_n}$ & $B = [T]_{\mathcal{B}, \mathcal{B}}$.

$$[T(x)]_{\mathcal{E}_n} = A\,[x]_{\mathcal{E}_n} \qquad [T(x)]_{\mathcal{B}} = B\,[x]_{\mathcal{B}}$$

$$
\begin{tikzcd}
{[x]_{\mathcal{E}_n}} \arrow[r, "A"] \arrow[d, "M_{\mathcal{B} \leftarrow \mathcal{E}_n}"'] \arrow[dr, phantom, "\circlearrowleft"] & {[T(x)]_{\mathcal{E}_n}} \\
{[x]_{\mathcal{B}}} \arrow[r, "B"] & {[T(x)]_{\mathcal{B}}} \arrow[u, "M_{\mathcal{E}_n \leftarrow \mathcal{B}}"']
\end{tikzcd}
$$

$$[T(x)]_{\mathcal{E}_n} = M^{-1}_{\mathcal{B} \leftarrow \mathcal{E}_n} [T(x)]_{\mathcal{B}}$$

$$= M^{-1}_{\mathcal{B} \leftarrow \mathcal{E}_n} B\,[x]_{\mathcal{B}}$$

$$= \underbrace{M^{-1}_{\mathcal{B} \leftarrow \mathcal{E}_n} B\, M_{\mathcal{B} \leftarrow \mathcal{E}_n}}\,[x]_{\mathcal{E}_n}$$

Hence $\underline{A = M^{-1} B M}$.

Problem: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection relative to the straight line whose equation is $y = 2x$.

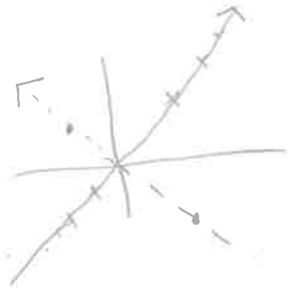
Find an analytic expression for T?

Soln:

Idea: images of $\{T(1,0), T(0,1)\}$, i.e $[T]_{\mathcal{E}_2,\mathcal{E}_2}$ – not immediate –

However, there are vector whose images are particularly easy to find.

- $y = 2x$, (Reflection) $\Rightarrow$ $T(1,2) = (1,2)$



If one looks for a straight line passing through origin and perpendicular to $y=2x$.

i.e, vector $(-2,1)$, we have $T(-2,1) = (2,-1)$.

If we choose the Basis $\mathcal{B} = \{(1,2), (-2,1)\}$.

Then, $[T]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$T(1,2) = (1,2) = 1(1,2) + 0.(-2,1)$, $T(-2,1) = 0(1,2) - 1(-2,1)$.

It follows that

$$[T(x,y)]_{\mathcal{E}_2} = M_{\mathcal{E}_2 \leftarrow \mathcal{B}} [T]_{\mathcal{B}\mathcal{B}} M^{-1}_{\mathcal{E}_2 \leftarrow \mathcal{B}} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$= \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} -3x + 4y \\ 4x + 3y \end{bmatrix}$$

$$T(x,y) = \left(-\frac{3}{5}x + \frac{4}{5}y, \frac{4}{5}x + \frac{3}{5}y \right).$$

More Examples:

1. $T: \mathbb{R}^3 \to \mathbb{R}^2$ be a Linear transformation defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2, 2x_3)$$

$$N_T = \{ v \in \mathbb{R}^3 : T(v) = 0 \}.$$

$$T(x_1, x_2, x_3) = 0 \Rightarrow x_1 = x_2, x_3 = 0.$$

$$N_T = \text{Null Space} = \{ (x, x, 0) : x \in \mathbb{R} \}.$$

2. $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by.

$$T(f(x)) = 2f'(x) + \int_0^x 3f(t)\,dt.$$

$N_T = ?$ $\quad \beta = \{1, x, x^2\}$ is a Basis for $P_2(\mathbb{R})$.

$$\text{Range}(T) = \text{Span}\{T(1), T(x), T(x^2)\}.$$

$$= \text{span}\{3x, 2 + \tfrac{3}{2}x^2, 4x + x^3\}.$$

Clearly $\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$ is L. Indep $\Rightarrow$ Rank$(T) = 3$.

$\Rightarrow$ Rank-Nullity $\Rightarrow N(T) = 0$.

3. $T: P_3(\mathbb{R}) \to P_2(\mathbb{R})$, $T(f(x)) = f'(x)$

$\beta = \{1, x, x^2, x^3\}$, $\quad \beta' = \{1, x, x^2\}$.

$$T(1) = 0 = 0.1 + 0.x + 0.x^2$$

$$T(x) = 1 = 1.1 + 0.x + 0.x^2$$

$$T(x^2) = 2x = 0.1 + 2.x + 0.x^2$$

$$T(x^3) = 0.1 + 0.x + 3.x^2$$

$$[T]_{\beta, \beta'} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Examples:

1. The space of $m\times n$ matrices: Let $\mathbb{M}_{m\times n}(\mathbb{F})$ denotes the set of all matrices of order $m\times n$, $m,n \in \mathbb{N}$, over the field $\mathbb{F}$. For $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$. Define

- $(A+B)_{ij} = a_{ij} + b_{ij}$. $(\alpha A)_{ij} = \alpha a_{ij}$, $\alpha \in \mathbb{F}$

Then $\mathbb{M}_{m\times n}(\mathbb{F})$ is a vector space over the field $\mathbb{F}$. $m=n$, we denote $\mathbb{M}_{m\times n}(\mathbb{F})$ by $\underline{\underline{\mathbb{M}_n(\mathbb{F})}}$.

2. The space of functions: Let S be an arbitrary set (not necessarily a subset of $\mathbb{F}$).

$$\mathcal{F}^S := \{ f : S \to \mathbb{F} \text{ mapping} \}$$

Then $\mathcal{F}^S$ equipped with vector addition

- $(f+g)(s) := f(s) + g(s)$, $s \in S$, $f, g \in \mathcal{F}^S$
- $(\alpha f)(s) = \alpha f(s)$

$(\mathcal{F}, +, \cdot)$ forms a vector space.

Remark: When $S = \mathbb{N}$, $\mathcal{F}^{\mathbb{N}}$ is called the set of all sequences, it sometimes denoted as $\mathcal{F}^{\infty}$.

3. Let P be a fixed $m\times m$ matrix with entries in the field $\mathbb{F}$ and let Q be a fixed $n\times n$ matrix over $\mathbb{F}$.

$$T : \mathbb{M}_{m\times n}(\mathbb{F}) \to \mathbb{M}_{m\times n}(\mathbb{F}) , T(A) = PAQ.$$

$$T(\alpha A + B) = P(\alpha A + B)Q = (\alpha PA + PB)Q = \alpha PAQ + PBQ = \alpha(T(A)) + T(B)$$

4. $V = \{ f : f : \mathbb{R} \to \mathbb{R}, \text{ continuous} \}$. Define

$T : V \longrightarrow V$ by

$$(Tf)(x) = \int_0^x f(t)\,dt$$

$$(T(f+g))(x) = \int_0^x (f(t)+g(t))\,dt = T(f) + T(g).$$

5. $T : \mathbb{P}_3(\mathbb{R}) \longrightarrow \mathbb{P}_2(\mathbb{R})$, $T(f(x)) = f'(x)$.

$\beta = \{1, x, x^2, x^3\}$, $\beta' = \{1, x, x^2\}$.

6. $\ell^2 = \{ \{x_n\}_{n \geq 1} : \sum |x_n|^2 < \infty \}$.

$T : \ell^2 \to \ell^2$ by $T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots)$

- $\|x\|$ of a vector - Extends the Concept of length.

However, the length of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$ is not the only geometric concept which can be expressed algebraically.

- If (x_1, x_2, x_3) & $(y_1, y_2, y_3) \in \mathbb{R}^3$, then the angle θ, between them can be obtained using the "Scalar product" $\langle x, y\rangle = x_1y_1 + x_2y_2 + x_3y_3 = \|x\| \, \|y\| \cos\theta$

where $\|x\| = (x_1^2 + x_2^2 + x_3^2)^{1/2} = [\langle x, x\rangle]^{1/2}$

$\|y\| = (\langle y, y\rangle)^{1/2}$.

- The Scalar product is a very useful concept that we would like to extend it other spaces.

Defn: Let X be a real vector space. An inner product on X is a function $\langle \cdot, \cdot \rangle$: $X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$ & $\alpha, \beta \in \mathbb{R}$:

(a) $\langle x, x\rangle \geq 0$. b) $\langle x, x\rangle = 0$ iff $x = 0$.

(c) $\langle \alpha x + \beta y, z\rangle = \alpha \langle x, z\rangle + \beta \langle y, z\rangle$.

d) $\langle x, y\rangle = \langle y, x\rangle$.

$$\langle \vec{0}, \vec{0}\rangle = \langle 0\vec{0}, \vec{0}\rangle = 0 \, \langle \vec{0}, \vec{0}\rangle = 0.$$

- Scalar product in $\mathbb{R}^3$ is an inner product.

<u>Example:</u> $\langle \cdot , \cdot \rangle : \mathbb{R}^K \times \mathbb{R}^K \to \mathbb{R}$ defined by

$x = (x_1, x_2, \ldots, x_K)$, $y = (y_1, y_2, \ldots, y_K)$.

$$\langle x, y \rangle = \sum_{n=1}^{K} x_n y_n = y^T x$$

is an inner product. ~~Hero~~

(a) $\langle x, x \rangle = \underbrace{\sum_{n=1}^{K} x_n^2}_{} \geq 0$ ✓

b) $\langle x, x \rangle = 0 \Rightarrow \sum_{n=1}^{K} x_n^2 = 0 \Rightarrow x_n = 0 \quad \forall n = 1, \ldots k.$ ✓

c) $\langle \alpha x + \beta y, z \rangle = \sum_{n=1}^{K} (\alpha x_n + \beta y_n) \cdot z_n.$

$$= \alpha \sum_{n=1}^{k} x_n z_n + \beta \sum_{n=1}^{k} y_n z_n.$$

$$= \alpha . \langle x, z \rangle + \beta \langle y, z \rangle$$

<u>d)</u> $\langle x, y \rangle = \sum_{n=1}^{k} x_n y_n = \sum_{n=1}^{k} y_n x_n = \langle y, x \rangle.$

This called the "Standard inner" <u>product</u>.

<u>Remark:</u> We need suitable modifications to be made to define inner product on complex spaces.

For example, $x, y \in \mathbb{C}^3$

$\langle x, x \rangle = \underline{x_1^2} + \underline{x_2^2} + \underline{x_3^2}$ need not be Real,

in particular need not be positive.

Let X be a complex vector space. (vector space over complex)

An inner product on X is a function

$\langle \cdot , \cdot \rangle : X \times X \to \mathbb{C}$ such that for all

$x, y, z \in X$, $\alpha, \beta \in \mathbb{C}$,

(a) $\langle x, x \rangle \in \mathbb{R}$ & $\langle x, x \rangle \geq 0$.

b) $\langle x, x \rangle = 0$ iff $x = 0$.

c) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

d) $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Example: $\langle \cdot , \cdot \rangle : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}$ defined by

$\langle x, y \rangle = \sum_{n=1}^{k} x_n \overline{y_n}$ is an inner product on $\mathbb{C}^k$.

EXR

Defn: A real or complex vector space X with an inner product $\langle \cdot , \cdot \rangle$ is inner product space.

Example: Let X be a k-dimensional vector space with basis $\{e_1, e_2, \ldots, e_k\}$. Let $x, y \in X$ have the representation $x = \sum_{n=1}^{k} \lambda_n e_n$, $y = \sum_{n=1}^{k} \mu_n e_n$

$\langle x, y \rangle := \sum_{n=1}^{k} \lambda_n \overline{\mu_n}$ is an inner product on X.

Example $\ell^2 = \{ \{x_n\} \;\; \sum_1 |x_n|^2 < \infty \}$

If $a = \{a_n\}$, $b = \{b_n\} \in \ell^2$ then the Sequence $\{a_n \bar{b}_n\} \in \ell^2$.

$$\sum_1 a_n \bar{b}_n \le \left(\sum_1 |a_n|^2\right)^{1/2} \left(\sum_1 |\bar{b}_n|^2\right)^{1/2} < \infty.$$

Define inner product on ℓ^2 by

$$\cancel{\langle a_n, b} \quad \langle a, b \rangle = \sum_{n=1}^{\infty} a_n \bar{b}_n$$ is an inner ~~an~~ on ℓ^2.

Properties of Inner Product.

Lemma

Let X be an inner product space, $x, y, z \in X$ & $\alpha, \beta \in \mathbb{F}$.

Then

(a) $\langle 0, y \rangle = \langle x, 0 \rangle = 0$.

(b) $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$.

(c) $\langle \alpha x + \beta y, \alpha x + \beta y \rangle = |\alpha|^2 \langle x, x \rangle + \alpha \bar{\beta} \langle x, y \rangle + \beta \bar{\alpha} \langle y, x \rangle + |\beta|^2 \langle y, y \rangle$.

pf.

$\langle \bar{0}, y \rangle = \langle 0 \cdot 0, y \rangle = 0 \langle 0, y \rangle = 0$.

$\langle x, \bar{0} \rangle = \overline{\langle 0, x \rangle} = \bar{0} = 0$.

b) $\langle x, \alpha y + \beta z \rangle = \overline{\langle \alpha y + \beta z, x \rangle}$

$= \overline{\alpha \langle y, x \rangle + \beta \langle z, x \rangle}$

$= \bar{\alpha}\, \overline{\langle y, x \rangle} + \bar{\beta}\, \overline{\langle z, x \rangle} = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$.

Examples:

1. $A = (a_{ij})$, $B = (b_{ij})$ in $M_2(\mathbb{R})$ of the 2×2 real matrices, define the inner product:

$$\langle A, B\rangle = tr(B^T A) = \sum_{i,j=1}^{2} a_{ij} b_{ij}$$

Defn: Let V be a real inner product space with inner product $\langle \cdot, \cdot \rangle$ & let $x \in V$. The norm of x is defined by

$$\|x\| = [\langle x, x\rangle]^{1/2}$$

The distance $d(x,y)$ between $x, y \in V$ is defined by

$$d(x,y) = \|x-y\|$$

Propn: (i) $\|x\| \geq 0$ & $\|x\| = 0$ iff $x = 0$. (ii) $\|\alpha x\| = |\alpha| \|x\|$.

(iii) $\|x+y\| \leq \|x\| + \|y\|$

Pf: $\|x\| \geq 0$ $(\langle x, x\rangle \geq 0)$. $\langle x, x\rangle = 0$ iff $x = 0$ $\Rightarrow \|x\| = 0 \Leftrightarrow x = 0$.

$\|\alpha x\| = [\langle \alpha x, \alpha x\rangle]^{1/2} = (\alpha^2 \langle x, x\rangle)^{1/2} = |\alpha| (\langle x, x\rangle)^{1/2} = |\alpha| \|x\|$.

Exr: $d(x,y) \geq 0$, & $d(x,y) = 0$ iff $x = y$.

$d(x,y) = d(y,x)$

$d(x,z) \leq d(x,y) + d(y,z)$.

iii) $\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$

$\leq \|x\|^2 + \|y\|^2 + \underbrace{|2\langle x, y\rangle|}_{\leq 2 \|x\| \cdot \|y\|}$?

$= (\|x\| + \|y\|)^2$

$=$.

Cauchy - Schwartz inequality

Let V be a real inner product space & let $x, y \in V$.

Then

$$|\langle x,y\rangle| \leq \|x\| \cdot \|y\|.$$

& $\quad |\langle x,y\rangle| = \|x\|\,\|y\|$ iff $\{x,y\}$ is linearly dependent set.

Pf. Theorem holds, trivially if $y=0$. Suppose $y \neq 0$.

Then for $\alpha \in \mathbb{R}$ & $x, y \in V$,

$$0 \leq \langle x-\alpha y, x-\alpha y\rangle = \langle x,x\rangle - \alpha\langle x,y\rangle - \alpha\langle y,x\rangle + \alpha^2\langle y,y\rangle$$

$$= \langle x,x\rangle - \alpha\langle x,y\rangle - \alpha\left(\langle x,y\rangle - \alpha\|y\|^2\right)$$

set $\quad \alpha = \dfrac{\langle x,y\rangle}{\|y\|^2}$.

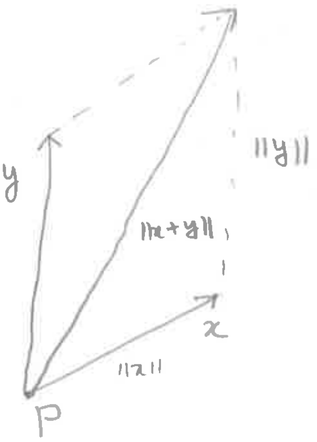
then,

$$0 \leq \langle x,x\rangle - \frac{\langle x,y\rangle^2}{\|y\|^2} - \frac{\langle x,y\rangle}{\|y\|^2}\left(\langle x,y\rangle - \frac{\langle x,y\rangle}{\|y\|^2}\cdot\|y\|^2\right)$$

$$= \langle x,x\rangle - \frac{\{\langle x,y\rangle\}^2}{\|y\|^2}$$

$$\Rightarrow \quad \langle x,y\rangle^2 \leq \|x\|^2\,\|y\|^2$$

$$|\langle x,y\rangle| \leq \|x\| \cdot \|y\|.$$



Now,

$$|\langle x,y\rangle| = \|x\| \cdot \|y\|$$

If $x = \alpha y$ or $y = \alpha x$. Then it holds.

$|\langle x,y\rangle| = \|x\| \cdot \|y\|$

$\hookrightarrow \langle x-\alpha y, x-\alpha y\rangle = 0 \Rightarrow x - \dfrac{\langle x,y\rangle}{\|y\|^2}\, y = 0 \Rightarrow \{x,y\}$ are L. dependent.

Gram Matrix:

Let V be a real inner product space, & let $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ be a basis for V. For $x, y \in V$ such that coordinate vectors of x & y relative to $\mathcal{B}$ are.

$$x_{\mathcal{B}} = (\alpha_1, \alpha_2, \ldots, \alpha_n), \quad y_{\mathcal{B}} = (\beta_1, \beta_2, \ldots, \beta_n).$$

$$\langle x, y\rangle = \langle \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_n b_n, \beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_n b_n\rangle$$

$$= \beta_1 \langle b_1, b_1\rangle \alpha_1 + \beta_1 \langle b_2, b_1\rangle \alpha_2 + \cdots +$$
$$\beta_2 \langle b_1, b_2\rangle \alpha_1$$

$$+ \beta_n \langle b_1, b_n\rangle \alpha_1 +$$

$$= [\beta_1 \;\; \beta_2 \;\; \cdots \;\; \beta_n] \begin{bmatrix} \langle b_1, b_1\rangle & \langle b_2, b_1\rangle & \cdots & \langle b_n, b_1\rangle \\ \langle b_1, b_2\rangle & & & \vdots \\ \vdots & & & \\ \langle b_1, b_n\rangle & & & \langle b_n, b_n\rangle \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \\ \alpha_n \end{bmatrix}$$

Given an inner product on a real vector space V with $\dim V = n$ and a basis $\mathcal{B}$ of V, it is possible to find an $n \times n$ real matrix

$$G = \begin{bmatrix} \langle b_1, b_1\rangle & & \\ \langle b_1, b_2\rangle & & \\ \vdots & & \langle b_n, b_n\rangle \end{bmatrix} \quad \text{s.t.} \quad \langle x, y\rangle = y_{\mathcal{B}}^T G x_{\mathcal{B}}.$$

$$(g_{ij}) = \langle b_i, b_j\rangle$$

<u>Defn:</u> A real symm matrix of order "n" is said to be a positive definite matrix if, for all non-zero vectors $x \in \mathbb{R}^n$,

$$x^T A x > 0.$$

<u>Example:</u> $\mathbb{R}^2$ with standard basis $\{e_1, e_2\} = \mathcal{E}_2$. Find the Gram matrix of $\mathcal{E}_2$, $G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

<u>Exr:</u> $\langle (x_1, x_2), (y_1, y_2)\rangle = \frac{1}{9} x_1 y_1 + \frac{1}{4} x_2 y_2$. Find G?

Propn: Let A be a real matrix of order n. The following are equivalent:

(i) The expression
$$\langle x, y\rangle = y^T A x$$
defines an inner product on $\mathbb{R}^n$.

(ii) A is a real positive definite matrix.

Pf: (i) ⇒ (ii)

$\langle e_i, e_j\rangle = e_j^T A e_i = a_{ji}$

$\langle e_j, e_i\rangle = e_i^T A e_j = a_{ij}$

$\Rightarrow a_{ij} = a_{ji} \Rightarrow A$ is symm.

$x^T A x = \langle x, x\rangle \geq 0$

$= 0$ iff $x = 0$

ii) ⇒ (i) exR

Corr: A real symmetric matrix is positive iff its eigen-value are positive numbers.

Pf: Suppose A is positive definite Matrix. Let λ be an e.v. of A & let x be associated e.v. Then

$0 < x^T A x = \lambda x^T x = \lambda \|x\|^2 \Rightarrow \lambda > 0$

Converse ?

§ Orthogonal and Unitary Diagonalisation

Defn: A matrix S in $\mathbb{M}_n(\mathbb{R})$ is said to be an orthogonal matrix if $SS^T = I$.

Propn: Let S be a matrix in $\mathbb{M}_n(\mathbb{R})$. Then the following are equivalent:

(i) S is an orthogonal matrix

(ii) $S^T S = I$

(iii) $SS^T = I$

iv) The Columns of S are an orthonormal Basis of $\mathbb{R}^n$.

v) The rows of S are an orthonormal Basis of $\mathbb{R}^n$

Pf:

$$(S^T S)_{ij} = c_i^T c_j = \langle c_i, c_j \rangle$$

where c_i, c_j are, respectively, the columns i & j of S.

$\langle c_i, c_j \rangle = 0$ for $i \neq j$ (From (ii)).

$\langle c_i, c_i \rangle = \|c_i\|^2 = 1$.

Defn: "A" real matrix is said to be <u>orthogonally diagonalisable</u>, if there exist a diagonal matrix D and an orthogonal matrix S such that

$$D = S^T A S$$

If a real matrix A is orthogonally diagonalisabble, then

$$A^T = (SDS^T)^T = SDS^T = A.$$

$\Rightarrow$ A is a Real symmetric matrix.

Qn: Converse ?

A is a real symmetric matrix, then A is orthogonally diagonalisable?

Lemma: Let A be real symmetric matrix & let x_1, x_2 be eigenvectors of A associated with distinct eigenvalues λ_1, λ_2 respectively. Then $\langle x_1, x_2 \rangle = 0$.

pf. Let λ_1, λ_2 & x_1, x_2 as above. Then.

$$\langle \underline{x_1}, A\underline{x_2} \rangle = (Ax_2)^T x_1 = x_2^T A^T x_1 = x_2^T A x_1$$
$$= \langle Ax_1, x_2 \rangle$$
$$= \lambda_1 \langle x_1, x_2 \rangle.$$

Since, $\langle x_1, Ax_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$ we have:

$$\Rightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle x_1, x_2 \rangle = 0 \Rightarrow \langle x_1, x_2 \rangle = 0.$$

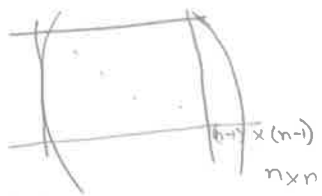
Theorem: "A" real square matrix is orthogonally diagonalisable iff it is symmetric.

pf. It remains to prove that, if A is symmetric, then A is orthogonally diagonalisable.

Idea: Use induction:

- If A is 1×1, the result holds trivially.
- Suppose A is an $n \times n$ - matrix, with $n \geq 2$, & that the assertion holds for all square matrix of order $(n-1)$.

Let λ be an eigen-value of A & with $\|x\| = 1$ be an eigen-vector.



Let $\mathcal{B} = \{x\} \cup B_\perp$ be an orthonormal Basis ~~for~~ of $\mathbb{R}^n$, where $\mathcal{B}_\perp$ is a Basis of span $\{x\}^\perp$. Then, given a vector $\vec{y}$ in $B_\perp$,

$$\langle Ay, x\rangle = x^T Ay = x^T A^T y = (Ax)^T y = \lambda (x)^T y$$
$$\langle Ay, x\rangle = x^T(Ay)$$
$$= \lambda \langle x, y\rangle = 0.$$

Matrix "S_1" whose columns consists of vectors of $\mathcal{B}$ is orthogonal & such that

$$S_1 = [\vec{x}, \vec{y}_1, \vec{y}_2, \ldots, \vec{y}_{n-1}] \quad (\vec{x}, \vec{y}_1 \in \mathcal{B})$$

$$S_1^T A S_1 = \begin{bmatrix} \lambda & 0 \\ 0 & M \end{bmatrix}.$$

$$S_1^T = \begin{pmatrix} \vec{x} \\ \vec{y}_1 \\ \vdots \\ \vec{y}_{n-1} \end{pmatrix}$$

Since A is symmetric. $\Rightarrow$ M is an $(n-1)\times(n-1)$ symmetric matrix.

$$S_1^T A S_1 = \begin{pmatrix} \vec{x}^T \\ \vec{y}_1^T \\ \vdots \\ y_{n-1} \end{pmatrix} (a_{ij}) (\vec{x}, \vec{y}_1, \ldots, \vec{y}_{n-1}) = \begin{pmatrix} \overset{\lambda \|x\|^2}{x^T A x} & & 0 \\ & y_1^T A y_1 & \\ 0 & & y_{n-1}^T A y_{n-1} \end{pmatrix}.$$

$\Rightarrow$ By induction hypothesis, there exists an $(n-1)\times(n-1)$ orthogonal matrix N such that

$$M = N \underline{D_1} N^T$$

$$\Rightarrow \quad \underline{S_1^T A S_1 = \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}} \begin{bmatrix} \lambda & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}^T.$$

Note that $\begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}$ is an orthogonal matrix.

It follows that $A = S_1 \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix} \begin{bmatrix} \lambda & 0 \\ 0 & D_1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}^T S_1^T$.

where $S = S_1 \begin{bmatrix} 1 & 0 \\ 0 & N \end{bmatrix}$ is an orthogonal matrix.

Corr: Let A be a real (symm). positive definite matrix. Then, there exists a non-singular matrix B s.t $A = BB^T$.

Pf: A is orthogonally diagonalisable

$\Rightarrow$ $A = S\underline{D}S^T$ Setting "$D^{1/2}$" square root entries.

$$= SD^{1/2} D^{1/2} S^T = \underbrace{(SD^{1/2})}_{B}(SD^{1/2})^T$$

Remark

If A is orthogonally diagonalisable, then

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

where the diagonal entries of D are the n eigenvalues of A, if $S = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]$ is the diagonalising orthogonal matrix, then

$$A = SDS^T = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T$$

Thm: Let A be a real symm. matrix. Then A has a spectral decomposition: $A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T$

How to do a spectral decomposition.

Let A be a $n \times n$ - real symm. matrix.

1. Find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of A, possibly repeated.
2. Find the corresponding orthonormal set of eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.
3. A spectral decom. of A is

$$A = \lambda_1 u_1 u_1^T + \cdots + \lambda_n u_n u_n^T.$$

Example: Find the spectral decomposition of
(EXR)

$$A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{pmatrix}.$$

§ Singular Value Decomposition

If A is a $k \times n$ real matrix, then $(A^T A)$ is an $n \times n$ real symmetric matrix.

$\Rightarrow$ Then, the spectrum of $(A^T A)$, besides being a non-empty subset of Real numbers, consists only of non-negative numbers. $\left(0 \leq \langle Av, Av \rangle = (Av)^T Av = \lambda v^T v = \lambda \|v\|^2 \right)$

$\Rightarrow \underline{\lambda \geq 0}$

Suppose that $\{v_1, v_2, \dots, v_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigen vectors of $\underline{\underline{(A^T A)}}$. Then given $i \in \{1, 2, \dots, n\}$

$$\|Av_i\| = \left[\langle Av_i, Av_i \rangle \right]^{1/2} = \sqrt{\lambda_i \|v_i\|^2} = \sqrt{\lambda_i}.$$

where $\lambda_i \in \sigma(A^T A)$ is the eigenvalue associated with v_i.

Given $i, j \in \{1, 2, \dots, n\}$ with $i \neq j$

$$\langle Av_i, Av_j \rangle = v_j^T \underbrace{A^T A v_i}_{} = \lambda_i \langle v_i, v_j \rangle = 0.$$

$$\left(= v_j^T \lambda_i v_i \right.$$

Define, for all $i \in \{1, 2, \dots, n\}$, the non-negative real number $\sigma_i = \sqrt{\lambda_i}$, called a singular value of the matrix A.

Let $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r \geq 0$, be all the non-zero singular values, repeated as many times as the corresponding algebraic multiplicities.

Setting, for all $i \in \{1, 2, \dots, r\}$

$$u_i = \frac{1}{\|Av_i\|} Av_i = \frac{1}{\sigma_i} Av_i.$$

$\Rightarrow \langle u_i, u_j \rangle = 0$ for $i \neq j$, $\|u_i\| = 1 \Rightarrow \{u_1, u_2, \dots, u_r\}$ is an orthogonal subset of the column space $\underline{\underline{C(A)}}$ of matrix A.

$$C(A) = \operatorname{span}\left(\{Av_i : i = 1,2,\dots,r\} \cup \{Av_i : i = r+1,\dots,n\}\right)$$

$$= \operatorname{span}\left(\{Av_i : i = 1,2,\dots,r\} \cup \{0\}\right).$$

$$= \operatorname{span}\{Av_i : i = 1,2,\dots,r\}.$$

$\Rightarrow$ $\{u_1, \dots, u_r\}$ is an orthonormal Basis of $C(A)$.

Let $\{u_1, \dots, u_r, u_{r+1}, \dots, u_k\}$ be an orthonormal Basis of $\mathbb{R}^k$ Containing the orthonormal basis of $C(A)$. Then,

$$A[v_1\ v_2\ \cdots\ v_n] = [u_1\ u_2\ \cdots\ u_k]\underbrace{\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma}$$

where $D = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \sigma_r \end{bmatrix}$

$$A = [u_1\ u_2\ \cdots\ u_k]\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}[v_1\ v_2\ \cdots\ v_n]^T.$$

$$\boxed{A = U \Sigma V^T}$$

<u>Thm:</u> (SVD-decomposition): Let A be a $k \times n$ real matrix. Then there exist a $k \times k$ orthogonal matrix U & an $n \times n$ orthogonal matrix V s.t $\underline{A = U \Sigma V^T}$

<u>Example:</u> $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$

The Eigen values of $A^T A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\lambda_1 = 3, \lambda_2 = 2$.

Hence the singular value: $\sigma_1 = \sqrt{3}$, $\sigma_2 = \sqrt{2}$.

$\sigma_1 > \sigma_2$.

$v_1 = (0,1)$, $v_2 = (1,0)$

eiv. of $A^T A$.

$u_1 := \frac{1}{\|Av_1\|} Av_1 = \frac{1}{\sqrt{3}}(-1,1,1)$,

$u_2 = \frac{1}{\|Av_2\|} Av_2 = \frac{1}{\sqrt{2}}(1,0,1)$

- A norm one vector orthogonal to u_1, u_2 is
$$u_3 = \frac{1}{\sqrt{6}}(-1,-2,1).$$

$$A = \begin{pmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$