The Cobb-Douglas Production Function

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Introduction

- **The Cobb-Douglas Production Function:**
- The general form of the Cobb-Douglas production function for two factors is

$$Q = AK^{\alpha}L^{\beta}, \qquad \dots (19)$$

- where A, α and β are parameters. The value of A is determined partly by the units of measurement of Q, K and L and partly by the efficiency of the production process. The relevance of the efficiency of the process can be seen by considering two Cobb-Douglas production functions, which differ only in the value of A. For given levels of K and L, the function with the higher value of A will have the larger value of Q and so will be the more efficient process.

Properties of the CD Production Function

- **Properties of the CD Production Function:**

- **Property 1.** The function is single valued and continuous (for positive K and L) and the marginal products of capital and labour are

$$\frac{\partial Q}{\partial K} = Q_K = A\alpha K^{\alpha-1}L^{\beta} = \frac{\alpha Q}{K} > 0, \qquad \text{and } \frac{\partial Q}{\partial L} = Q_L = A\beta K^{\alpha}L^{\beta-1} = \frac{\beta Q}{L} > 0.$$

- This implies that $\alpha > 0$ and $\beta > 0$, since Q, K and L are positive and marginal products are assumed to be positive.

- The second order derivatives are

$$Q_{KK} = \alpha(\alpha-1)\frac{Q}{K^2}, \qquad \ldots (20)$$

$$Q_{LL} = \beta(\beta-1)\frac{Q}{L^2}. \qquad \ldots (21)$$

- $Q_{KK} < 0$ and $Q_{LL} < 0$ by the diminishing property of marginal products of capital and labour. This implies that

$$0 < \alpha < 1 \text{ and } 0 < \beta < 1. \qquad \ldots (22)$$

Properties of the CD Production Function

- **Property 2.** This function is homogeneous of degree $\alpha + \beta$.

  If $\alpha + \beta = 1$, we have constant returns to scale.

  If $\alpha + \beta < 1$, we have decreasing returns to scale.

  If $\alpha + \beta > 1$, we have increasing returns to scale.

- **Property 3.** The marginal rate of substitution is

$$R = \frac{Q_L}{Q_K} = \frac{\beta Q/L}{\alpha Q/K} = \frac{\beta K}{\alpha L}.$$

- To derive the elasticity of substitution we can write this equation as

$$\mathrm{Ln}(R) = \mathrm{Ln}\left(\frac{\beta}{\alpha}\right) + \mathrm{Ln}\left(\frac{K}{L}\right).$$

- Hence

$$\sigma = \frac{d\mathrm{Ln}\left(\frac{K}{L}\right)}{d\mathrm{Ln}(R)} = 1.$$

Properties of the CD Production Function

- **Property 4.** The cost minimization (or profit maximization for given Q is

$$\frac{f_L}{f_K} = \frac{w}{r} \Rightarrow \frac{\beta K}{\alpha L} = \frac{w}{r},$$

or

$$L = \frac{\beta}{\alpha} . \frac{r}{w} . K,$$

- and substituting into the production function gives

$$Q = AK^{\alpha+\beta}\left(\frac{\beta}{\alpha} . \frac{r}{w}\right)^{\beta},$$

- which, on rearranging, yields the cost minimizing input function

$$K = \left(\frac{Q}{A}\right)^{\frac{1}{n}}\left(\frac{\beta}{\alpha} . \frac{r}{w}\right)^{-\frac{\beta}{n}},$$

- where $n = \alpha + \beta$ is the degree of returns to scale.

Property 4 (Continued)

- Similarly for labour input

$$L = \left(\frac{Q}{A}\right)^{\frac{1}{n}} \left(\frac{\alpha}{\beta}.\frac{w}{r}\right)^{-\frac{\alpha}{n}}.$$

- The cost function is then obtained as follows.

$$\text{Total Cost} = rK + wL$$

$$= \left(\frac{Q}{A}\right)^{\frac{1}{n}} \left\{ r\left(\frac{\beta}{\alpha}.\frac{r}{w}\right)^{-\frac{\beta}{n}} + w\left(\frac{\alpha}{\beta}.\frac{w}{r}\right)^{-\frac{\alpha}{n}} \right\}$$

$$= kQ^{\frac{1}{n}} r^{\frac{\alpha}{n}} w^{\frac{\beta}{n}},$$

- where the constant k is given by

$$k = n\left(A\alpha^{\alpha}\beta^{\beta}\right)^{-\frac{1}{n}}.$$

- Thus, the cost function is log-linear in output and factor prices.

Properties of the CD Production Function

- **Property 5.** The profit maximization conditions ($r = pf_k$ & $w = pf_L$) gives

$$r = p\frac{\alpha Q}{K} \text{ and } w = p\frac{\beta Q}{L}.$$

- Thus

$$\alpha = \frac{rK}{pQ} \text{ and } \beta = \frac{wL}{pQ}$$

- can be interpreted as the respective shares of capital and labour in total input. The firm's optimal output and input levels result from solving these two equations together with

$$Q = AK^{\alpha}L^{\beta}.$$

- Writing in logarithmic form, these are

$$\mathrm{Ln}(Q) = \mathrm{Ln}(A) \qquad + \alpha\mathrm{Ln}(K) + \beta\mathrm{Ln}(L)$$

$$\mathrm{Ln}(Q) = \mathrm{Ln}\left(\frac{r}{p}\right) - \mathrm{Ln}(\alpha) + \mathrm{Ln}(K)$$

$$\mathrm{Ln}(Q) = \mathrm{Ln}\left(\frac{w}{p}\right) - \mathrm{Ln}(\beta) \qquad + \mathrm{Ln}(L)$$

Property 5 (Continued)

- Rearranging the equations in matrix form we get

$$\begin{pmatrix} 1 & -\alpha & -\beta \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} \mathrm{Ln}(Q) \\ \mathrm{Ln}(K) \\ \mathrm{Ln}(L) \end{pmatrix} = \begin{pmatrix} \mathrm{Ln}(A) \\ \mathrm{Ln}\left(\frac{r}{p}\right) - \mathrm{Ln}(\alpha) \\ \mathrm{Ln}\left(\frac{w}{p}\right) - \mathrm{Ln}(\beta) \end{pmatrix}.$$

- The determinant of the coefficient matrix is $1 - \alpha - \beta$. So, if $\alpha + \beta = 1$ then the matrix is singular and there is no solution. Output is indeterminate in the constant returns case. If we examine the second order conditions, we get

$$Q_{KK} = \alpha(\alpha - 1)\frac{Q}{K^2} < 0 \Rightarrow 0 < \alpha < 1,$$

$$Q_{LL} = \beta(\beta - 1)\frac{Q}{L^2} < 0 \Rightarrow 0 < \beta < 1,$$

and

$$Q_{KK}Q_{LL} - (Q_{KL})^2 = \frac{Q^2\alpha\beta}{K^2L^2}(1 - \alpha - \beta) > 0.$$

Property 5 (Continued)

- The conditions boil down to $\alpha + \beta < 1$ and decreasing returns are required. This may be an unreasonable assumption. So, we have to reject the perfectly competitive hypothesis.
- In the case where the product market is not competitive, we have

$$w = \frac{\lambda\beta Q}{L}, r = \frac{\lambda\alpha Q}{K}, \frac{w}{r} = \frac{\beta K}{\alpha L},$$

- and for $\eta \neq -1$, $\lambda = (\eta + 1)p$, so that

$$L = \frac{(\eta + 1)p\beta Q}{w} \text{ and } K = \frac{(\eta + 1)p\alpha Q}{r}.$$

- The second order condition gives

$$\frac{p(\eta + 1)\alpha\beta Q^{\alpha}}{K^2L^2}\{(\alpha + \beta)(\eta + 1) - 1\} < 0$$

or both $(\eta + 1) > 0$ and $(\alpha + \beta)(\eta + 1) < 1$.

- This condition requires $-1 < \eta < 0$ and hence the demand for the product must be inelastic. The second part limits the maximum value of $(\eta + 1)$ to $\frac{1}{(\alpha+\beta)}$.

The Statistical Model of Production Function

- **The Statistical Model of Production Function:**
- **1. Cross Section Data:**
- Further difficulties emerge when we formulate appropriate statistical models for testing. We consider cross-section studies first. The parameters α and β are assumed to be equal for all firms, as are the prices which they face. Disturbance terms are incorporated to account for variations in the data. This is done multiplicatively as

$$Q_i = AK_i^{\alpha} L_i^{\beta} u_{1i}.$$

- Most often u_{1i} is held to represent the technical efficiency of the ith entrepreneur based on his skill, knowledge, luck, technical advantages, efforts, and so forth. Random error terms are also introduced into the equations for the marginal productivity conditions.

$$\alpha \frac{Q_i}{K_i} = -\frac{r}{p} u_{2i}, \qquad \beta \frac{Q_i}{L_i} = -\frac{w}{p} u_{3i}.$$

The Statistical Model of Production Function

- Any departure from the $u_{2i} = 1$ or $u_{3i} = 1$ implies failure to achieve profit maximization. These error terms are usually taken to reflect the firm's economic or commercial efficiency. These arises due to misallocation of inputs and hence are also termed as allocative inefficiency.
- Linearizing the system, we now have

$$\begin{aligned}
&\mathrm{Ln}(Q_i) - \alpha\mathrm{Ln}(K_i) - \beta\mathrm{Ln}(L_i) = \mathrm{Ln}(A) + \mathrm{Ln}(u_{1i})\\
&\mathrm{Ln}(Q_i) - \quad \mathrm{Ln}(K_i) \qquad\qquad\quad = \mathrm{Ln}\left(\frac{r}{p}\right) - \mathrm{Ln}(\alpha) + \mathrm{Ln}(u_{2i})\\
&\mathrm{Ln}(Q_i) \qquad\qquad\qquad - \ \mathrm{Ln}(L_i) = \mathrm{Ln}\left(\frac{w}{p}\right) - \mathrm{Ln}(\beta) + \mathrm{Ln}(u_{3i}).
\end{aligned}$$

- Notice that even when $\alpha + \beta \neq 1$, so that the coefficient matrix (B) is non-singular, the production function is not identified. This follows from the rank and order conditions. A number of approaches to this problem have been suggested.

The Statistical Model of Production Function

- From the order condition we know that identification can be achieved by adding exogenous variables to one of the other equations and asserting that the variable do not appear in the production function.
- Consider the marginal productivity of capital condition. Including the stock of capital at the beginning of the period using partial-adjustment type considerations seems reasonable, since investment does not occur instantaneously. However, this suggestion takes us beyond the cross-section framework and raises time-series problems.
- Alternatively, the constant price assumption can be dropped. This can be done by, say, allowing wages and other input prices to vary regionally.
- More systematically, we might abandon our assumptions of perfectly elastic input supply (and output demand) with some prices becoming endogenous variables in an imperfectly competitive world.
- Identification of the production function can also be achieved if technical efficiency is not correlated with economic efficiencies.

The Statistical Model of Production Function

- **The Statistical Model of Production Function:**
- **2. Time Series Data on a Firm:**
- If time-series data are available, say, successive annual observations on the same firm, problems addition to those associated with a single observation arise.
- Relative prices change over time and hence the optimal combinations of the factor inputs will also change. Observations are therefore much more likely to refer to disequilibrium positions because of lags in the adjustment process.
- However, the major problem with time-series data is technical progress.
- The result is that factor input rates, substitutability between the factors, efficiency parameters and the economies of scale behaviour can all change. Thus, the parameters, and even the mathematical form, of the production function change over time.

Technical Progress

- Two methods of allowing for this have been suggested. One assumes that technical progress can be measured by adding a time trend, t, to the production function to give

$$Q_t = f(t, K_t, L_t),$$

- and, for example, in the case of the Cobb-Douglas function

$$Q_t = Ae^{\theta t}K_t^{\alpha}L_t^{\beta}.$$

- Here θ is the exponential rate of technical progress and implies that output rises at a rate of $100 \times \theta$ percent per annum independently of changes in the factor inputs, and in particular, independent of new investment. This type of technical progress is not associated with the measures of capital or labour and is known as disembodied technical progress.

Embodied Technical Progress

- The second approach to technical progress assumes that new investment embodies technical advances and that only after investment occurs can technical progress have any effect. This is known as embodied technical progress and gives rise to vintage model because a different production function applies for each vintage (or age) of capital. For example,

$$Q_t(\nu) = f\big(\nu, L_t(\nu), K_t(\nu)\big),$$

- where

$Q_t(\nu) =$ Output at time t for machines of vintage ν,

$L_t(\nu) =$ Labour used at time t on machines of vintage ν,

$K_t(\nu) =$ Capital of vintage ν in use at time t.

Embodied Technical Progress (Continued)

- Technical progress can be introduced through the term ν. For example, with the Cobb-Douglas function

$$Q_t(\nu) = Ae^{\theta\nu}\left(K_t(\nu)\right)^{\alpha}\left(L_t(\nu)\right)^{\beta}.$$

- The production function at time t is

$$Q_t = f(L_t, K_t),$$

- with

$$Q_t = \int_{-\infty}^{t} Q_t(\nu)\,d\,\nu, \qquad L_t = \int_{-\infty}^{t} L_t(\nu)\,d\,\nu, \qquad \text{and } K_t = \int_{-\infty}^{t} K_t(\nu)\,d\,\nu$$

- Being the measures of total output, labour input and capital input at time t. Some progress can be made with these expressions, if assumptions are made about the rate of depreciation of capital.
- [An Introduction to Applied Econometric Analysis: R.F. Wynn and K. Holden, 1974, The Macmillan Press Ltd., London.]

Thank You