Exercise Series 6 (Solutions)

Exercise 1 Using the hint, $X_1 = Y_1^{\theta}$, $X_2 = Y_2^{\theta}$, where $Y_1, Y_2 \overset{iid}{\sim} \text{Exp.}(1)$.

Now, $\log X_1 = \theta \log Y_1$, $\log X_2 = \theta \log Y_2$

$$\therefore \frac{\log X_1}{\log X_2} = \frac{\log Y_1}{\log Y_2}.$$

$Y_1, Y_2 \overset{iid}{\sim}$ Exponential (1) $\leftarrow$ free of θ

$\Rightarrow \log Y_1, \log Y_2$ are iid with their distribution free of θ.

[Try to find the pdf of $\log Y$]

$\therefore$ the dist. of $\dfrac{\log Y_1}{\log Y_2}$ is free of θ

$\Rightarrow \dfrac{\log X_1}{\log X_2} = \dfrac{\log Y_1}{\log Y_2}$ has dist. free of θ and hence is ancillary.

Exercise 2 (a) $T = \sum_{i=1}^{n} X_i \sim \text{Binomial}(n, p)$

$$\mathbb{E}_p[g(T)] = \sum_{t=0}^{n} g(t) \binom{n}{t} p^t (1-p)^{n-t}$$

$$= (1-p)^n \sum_{t=0}^{n} g(t) \binom{n}{t} \left(\frac{p}{1-p}\right)^t$$

$\therefore \mathbb{E}_p[g(T)] = 0 \quad \forall p \in (0,1) \iff \sum_{t=0}^{n} g(t) \binom{n}{t} \left(\frac{p}{1-p}\right)^t = 0 \quad \forall p \in (0,1)$

$$\iff \sum_{t=0}^{n} g(t) \binom{n}{t} z^t = 0 \quad \forall z \in (0, \infty) \quad \left[z = \frac{p}{1-p}\right]$$

The left side is a polynomial (of degree n) in z, which is zero for all z. Therefore, the coefficients of the polynomial must all be 0.

$\therefore\ g(t)\binom{n}{t} = 0 \quad \forall\ t = 0, 1, \dots, n.$

But $\binom{n}{t} \neq 0 \quad \forall\ t = 0, 1, \dots, n$

$\Rightarrow g(t) = 0 \quad \forall\ t = 0, 1, \dots, n.$

Since Binomial (n, p) is supported on $\{0, 1, \dots, n\}$, it follows that $\mathbb{P}_p[g(T) = 0] = 1 \quad \forall\ p \in (0,1)$

Therefore, $T = \sum_{i=1}^{n} X_i$ is complete

This can also be seen by noticing that

$$p_p(\underline{x}) = p^{\sum_{i=1}^{n} x_i}(1-p)^{n - \sum_{i=1}^{n} x_i}$$

$$= e^{\sum_{i=1}^{n} x_i \log p + (n - \sum_{i=1}^{n} x_i)\log(1-p)}$$

$$= e^{\sum_{i=1}^{n} x_i \log\left(\frac{p}{1-p}\right) + n\log(1-p)}$$

$$= e^{a(p)T(\underline{x}) - b(p)}\, c(\underline{x})$$

where $a(p) = \log\left(\frac{p}{1-p}\right)$, $b(p) = -n\log(1-p)$,

$T(\underline{x}) = \sum_{i=1}^{n} x_i$, $c(\underline{x}) = 1$

Member of the exponential family

$$\{a(p) : p \in (0,1)\} = \left\{\log\left(\frac{p}{1-p}\right) : p \in (0,1)\right\}$$

$$= \{\log(z) : z \in (0, \infty)\}$$

$$= (-\infty, \infty) = \mathbb{R} \text{ contains an open set}$$

Therefore, $T(\underline{X}) = \sum_{i=1}^{n} X_i$ is complete.

(b) $f_\lambda = \text{Poisson}(\lambda)$, $\lambda > 0$, $T = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\lambda)$

$$\mathbb{E}_\lambda[g(T)] = \sum_{t=0}^{\infty} g(t)\, e^{-n\lambda} \frac{(n\lambda)^t}{t!}$$

$$= e^{-n\lambda} \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t}{t!}$$

$\therefore\ \mathbb{E}_\lambda[g(T)] = 0 \quad \forall \lambda > 0 \iff \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t}{t!} = 0 \quad \forall \lambda > 0.$

If $\lambda = 0$ was a possibility, then we would have gotten

$$- \; g(0) + \underbrace{\left[\sum_{t=1}^{\infty} g(t) \frac{(n\lambda)^t}{t!}\right]_{\lambda=0}}_{=0} = 0 \implies g(0) = 0.$$

$$- \left[\frac{d}{d\lambda} \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t}{t!}\right]_{\lambda=0} = \left[\sum_{t=1}^{\infty} g(t) \frac{n^t \cdot t\, \lambda^{t-1}}{t!}\right]_{\lambda=0}$$

$$= n\, g(1) + \left[\sum_{t=2}^{\infty} t\, g(t)\, n^t \frac{\lambda^{t-1}}{t!}\right]_{\lambda=0}$$

$$= n g(1) = 0 \implies g(1) = 0$$

$\vdots$

$$\left[\frac{d^k}{d\lambda^k} \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t}{t!}\right]_{\lambda=0} = 0 \implies g(k) = 0$$

$\vdots$

This gives us $g(0) = 0$, $g(1) = 0$, $g(2) = 0$, ...

Since Poisson dist. is supported on $\{0, 1, 2, \dots\}$, this gives us $\mathbb{P}_\lambda[g(T) = 0] = 1 \quad \forall \lambda \geq 0.$

Since in our case, 0 is not a possible value for λ, we take limit as $\lambda \to 0$. Notice that $\lambda \longmapsto \sum_{t=0}^{\infty} g(t) \frac{(n\lambda)^t}{t!}$ is a continuous (in fact infinitely many times differentiable function). Taking limit as $\lambda \to 0$, our previous argument works.

Therefore, $T = \sum_{i=1}^{n} X_i$ is complete. This can also be shown by showing that the Poisson distribution is a member of the exponential family. Verify that the required condition is fulfilled

(c) $f_p = \text{Geometric}(p)$, $p \in (0,1)$. $\quad T = \sum_{i=1}^{n} X_i$

$$f_p(x) = p\,(1-p)^x, \quad x = 0, 1, \ldots$$

$$\therefore\ p_p(\underline{x}) = p^n\,(1-p)^{\sum_{i=1}^{n} x_i}$$

$$= e^{\sum_{i=1}^{n} x_i \log(1-p) + n\log p}$$ is a member of the exponential family.

$$a(p) = \log(1-p)$$

$$p \in (0,1) \iff (1-p) \in (0,1) \iff \log(1-p) \in (-\infty, 0)$$

$\therefore \{\log(1-p) : p \in (0,1)\} = (-\infty, 0)$ contains an open set.

So, $T = \sum_{i=1}^{n} X_i$ is complete

(d) $\text{Uniform}(\theta, 1)$, $\theta < 1$. $\quad T = X_{(1)}$.

$$f_X(y) = \frac{1}{1-\theta}, \quad \theta \le y \le 1$$

$$F_X(y) = \frac{y-\theta}{1-\theta}, \quad \theta \le y \le 1 \quad [\,0 \text{ if } y < \theta,\ 1 \text{ if } y > 1\,]$$

$\therefore$ pdf of $X_{(1)}$ is

$$n\,\frac{1}{1-\theta}\left(1 - \frac{y-\theta}{1-\theta}\right)^{n-1}, \quad \theta \le y \le 1$$

$$= n\,\frac{(1-y)^{n-1}}{(1-\theta)^n}, \quad \theta \le y \le 1$$

$$\mathbb{E}_\theta[g(T)] = \frac{n}{(1-\theta)^n}\int_\theta^1 g(t)\,(1-t)^{n-1}\,dt$$

$\mathbb{E}_\theta[g(T)] = 0 \quad \forall\, \theta < 1 \iff \int_\theta^1 g(t)(1-t)^{n-1}\,dt = 0 \quad \forall\, \theta<1$

Take derivative w.r.t. θ to get

$$g(\theta)(1-\theta)^{n-1} = 0 \quad \forall\, \theta < 1$$

$$\Rightarrow g(\theta) = 0 \quad \forall\, \theta < 1.$$

Since the dist of $X_{(1)}$ is supported on $(\theta, 1)$, it follows that $\mathbb{P}_\theta[g(T)=0] = 1 \quad \forall\, \theta < 1$.

Thus, $X_{(1)}$ is complete.

(e) Uniform $(\theta, \theta+1)$, $T = (X_{(1)}, X_{(n)})$

Here, $Y = X - \theta \sim \text{Unif}(0,1)$

$\therefore X_{(n)} - X_{(1)} = Y_{(n)} - Y_{(1)}$, where $Y_1 \dots Y_n \overset{iid}{\sim} \text{Unif}(0,1)$.

We already calculated $\mathbb{E}(Y_{(n)} - Y_{(1)}) = \underbrace{\frac{n-1}{n+1}}_{\text{free of } \theta}$ [Series 4, Exercise 2]

$$\therefore \mathbb{E}_\theta[X_{(n)} - X_{(1)}] = \frac{n-1}{n+1} \quad \forall\, \theta \in \mathbb{R}$$

$$\iff \mathbb{E}_\theta\left[X_{(n)} - X_{(1)} - \frac{n-1}{n+1}\right] = 0 \quad \forall\, \theta \in \mathbb{R}$$

But, $X_{(n)} - X_{(1)}$ has a continuous dist. (we derived its pdf). Therefore, $\mathbb{P}_\theta\left[X_{(n)} - X_{(1)} = \frac{n-1}{n+1}\right] = 0 \quad \forall\, \theta$

So, we have identified a function g, where

$$g(y,z) = z - y - \frac{n-1}{n+1}, \text{ such that}$$

$\mathbb{E}_\theta[g(T)] = 0 \quad \forall\, \theta$ but $\mathbb{P}_\theta[g(T)=0] \neq 1 \quad \forall\, \theta$.

So, $T = (X_{(1)}, X_{(n)})$ is not complete.

(h) Laplace $(0,\sigma)$, $\sigma > 0$. $\quad T = \sum_{i=1}^{n} |X_i|$

$$f_\sigma(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}} = e^{-\frac{|x|}{\sigma} - \log(2\sigma)}$$
$$= e^{a(\sigma)\tau(x) - b(\sigma)} c(x)$$

$a(\sigma) = -\frac{1}{\sigma}$, $\tau(x) = |x|$, $b(\sigma) = \log(2\sigma)$, $c(x) \equiv 1$

$\{a(\sigma) : \sigma > 0\} = \{-\frac{1}{\sigma} : \sigma > 0\} = (-\infty, 0)$ contains an open set.

$\therefore$ $T = \sum_{i=1}^{n} |X_i|$ is complete

(i) Normal (θ, θ^2)

$\mathbb{E}_\theta(X) = \theta$, $\mathbb{E}_\theta(X^2) = \mathrm{Var}_\theta(X) + \{\mathbb{E}_\theta(X)\}^2 = \theta^2 + \theta^2 = 2\theta^2$

$\therefore$ $\mathbb{E}_\theta\left(\sum_{i=1}^{n} X_i\right) = n\theta$, $\mathbb{E}_\theta\left(\sum_{i=1}^{n} X_i^2\right) = 2n\theta^2$

$\sum X_i \sim N(n\theta, n\theta^2)$

$\mathbb{E}_\theta\left(\left(\sum_{i=1}^{n} X_i\right)^2\right) = n\theta^2 + (n\theta)^2 = (n+n^2)\theta^2$

$\therefore$ $\mathbb{E}_\theta\left[\frac{1}{n+n^2}\left(\sum_{i=1}^{n} X_i\right)^2\right] = \theta^2$

$\mathbb{E}_\theta\left[\frac{1}{2n}\sum_{i=1}^{n} X_i^2\right] = \theta^2$

$\therefore$ $\mathbb{E}_\theta\left[\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n+1)}\left(\sum_{i=1}^{n} X_i\right)^2\right] = 0 \quad \forall\theta$

Verify that $\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - \frac{1}{n(n+1)}\left(\sum_{i=1}^{n} X_i\right)^2$ has a continuous dist. So, $T = \left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is not complete

(j) Normal (θ, θ)

$$f_\theta(x) = \frac{1}{\sqrt{2\pi\theta}} \, e^{-\frac{1}{2\theta}(x-\theta)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{1}{2\theta}(x^2 - 2\theta x + \theta^2) - \frac{1}{2}\log\theta}$$

$$= \frac{1}{\sqrt{2\pi}} \, e^{-\frac{x^2}{2\theta} - \frac{\theta}{2} - \frac{\log\theta}{2}} \cdot e^{x}$$

$$= e^{a(\theta)\, z(x) - b(\theta)} \, c(x)$$

$$a(\theta) = -\frac{1}{2\theta}, \quad z(x) = x^2, \quad b(\theta) = \frac{1}{2}(\theta + \log\theta)$$

$$c(x) = \frac{1}{\sqrt{2\pi}} e^{x}$$

$\therefore$ f_θ belongs to the one-parameter exponential family

Also, $\{a(\theta) : \theta > 0\} = \left\{-\frac{1}{2\theta} : \theta > 0\right\} = (-\infty, 0)$

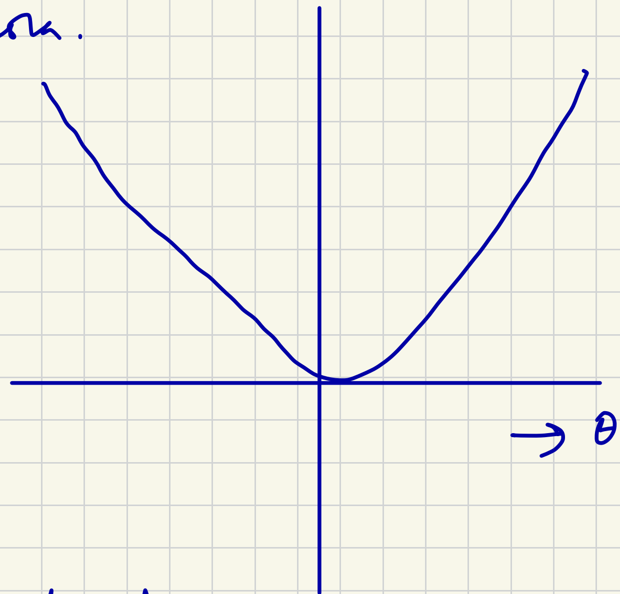
$\therefore$ $T = \sum_{i=1}^{n} X_i^2$ is complete

The cases of (k), (l), (m), (n) can be worked out by considering the exponential family structure in every case. Carefully examine whether the condition of the result stated in the class is satisfied.

Exercise 3 (a) This is similar to $N(\theta, \theta^2)$. Use direct calculation to show that $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ is minimal sufficient. Now, $(\bar{X}, S^2)$ is a 1-1 function of $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$

Therefore, $(\bar{X}, S^2)$ is also minimal sufficient.

(b) The parameter space $\{(\theta, a\theta^2) : \theta \in \mathbb{R}^2\}$ looks like a parabola.



Clearly it does not contain any open subset of $\mathbb{R}^2$.

(c) This is similar to Exercise 2 (i).

$$\bar{X} \sim N\left(\theta, \frac{a\theta^2}{n}\right), \quad \frac{(n-1)S^2}{a\theta^2} \sim \chi^2_{n-1}$$

$$\therefore \mathbb{E}_\theta(\bar{X}^2) = \frac{a}{n}\theta^2 + \theta^2 = \left(\frac{a}{n}+1\right)\theta^2$$

$$\mathbb{E}_\theta\left(\frac{(n-1)S^2}{a\theta^2}\right) = n-1 \implies \mathbb{E}_\theta(S^2) = a\theta^2$$

$$\therefore \mathbb{E}_\theta\left[\frac{n}{a+n}\bar{X}^2 - \frac{S^2}{a}\right] = 0 \quad \forall \theta$$

$\bar{X}, S^2$ have continuous dist. and they are independent.

Therefore $\mathbb{P}_\theta\left[\frac{n}{a+n}\bar{X}^2 - \frac{S^2}{a} = 0\right] = 0 \quad \forall \theta$

$\therefore (\bar{X}, S^2)$ is not complete.

Exercise 4 $\mathbb{P}_\theta(X=0)=\theta,\ \mathbb{P}_\theta(X=1)=3\theta,\ \mathbb{P}_\theta(X=2)=1-4\theta,$
$0<\theta<\frac{1}{4}$

$$\mathbb{E}_\theta[g(X)] = \theta g(0) + 3\theta g(1) + (1-4\theta) g(2)$$
$$= \theta\{g(0)+3g(1)-4g(2)\} + g(2).$$

$$\mathbb{E}_\theta[g(X)] = 0 \quad \forall\ 0<\theta<\frac{1}{4}$$
$$\Leftrightarrow \theta\{g(0)+3g(1)-4g(2)\} + g(2) = 0 \quad \forall\ 0<\theta<\frac{1}{4}$$
$$\Leftarrow g(2)=0 \quad \text{and} \quad g(0)+3g(1)-4g(2)=0$$
$$\Leftrightarrow g(2)=0 \quad \text{and} \quad g(0)+3g(1)=0$$
$$\Leftrightarrow g(2)=0 \quad \text{and} \quad g(0)=-3g(1).$$

There are many such functions.
Eg. $g(0)=-3,\ g(1)=1,\ g(2)=0$ satisfies
$\mathbb{E}_\theta[g(X)]=0\ \forall\theta$ but $\mathbb{P}_\theta[g(X)=0]=1-4\theta \neq 1$

$\therefore$ X is not complete.

When $\mathbb{P}_\theta(X=0)=\theta,\ \mathbb{P}_\theta(X=1)=\theta^2,\ \mathbb{P}_\theta(X=2)=1-\theta-\theta^2,$
$0<\theta<\frac{1}{2}.$

$$\mathbb{E}_\theta[g(X)] = \theta g(0) + \theta^2 g(1) + (1-\theta-\theta^2) g(2)$$
$$= \theta\{g(0)-g(2)\} + \theta^2\{g(1)-g(2)\} + g(2)$$

$$\therefore\ \mathbb{E}_\theta[g(X)] = 0 \quad \forall\ 0<\theta<\frac{1}{2}$$
$$\Rightarrow \theta\{g(0)-g(2)\} + \theta^2\{g(1)-g(2)\} + g(2) = 0 \quad \forall\ 0<\theta<\frac{1}{2}$$
$$\Rightarrow g(0)-g(2) = g(1)-g(2) = g(2) = 0$$
$$\Rightarrow g(0)=g(1)=g(2)=0$$
$$\Rightarrow \mathbb{P}_\theta[g(X)=0]=1$$

So, X is complete

Exercise 5

$X \sim \text{Poisson}(\theta), \quad \theta \in \{1, 2\}$

$$\mathbb{E}_\theta\{g(X)\} = \sum_{x=0}^{\infty} g(x)\, e^{-\theta} \frac{\theta^x}{x!} = e^{-\theta} \sum_{x=0}^{\infty} g(x) \frac{\theta^x}{x!}$$

$$\mathbb{E}_\theta\{g(X)\} = 0 \quad , \quad \theta = 1, 2$$

$$\iff e^{-1} \sum_{x=0}^{\infty} g(x) \frac{1}{x!} = 0 \text{ and } e^{-2} \sum_{x=0}^{\infty} g(x) \frac{2^x}{x!} = 0$$

$$\iff \sum_{x=0}^{\infty} g(x) \frac{1}{x!} = 0 \quad \text{and} \quad \sum_{x=0}^{\infty} \frac{2^x g(x)}{x!} = 0$$

Remember, we only need to find one g that violates the condition for completeness.

Take g s.t. $g(2) = g(3) = \cdots = 0$

So, the above equations become

$$g(0) + g(1) = 0 \quad \text{and} \quad g(0) + 2g(1) = 0$$

This does not give us what we want.

Consider the first three terms now.

$$g(0) + g(1) + \frac{g(2)}{2} = 0, \quad g(0) + 2g(1) + 2g(2) = 0$$

$$\iff 2g(0) + 2g(1) + g(2) = 0, \quad g(0) + 2g(1) + 2g(2) = 0$$

$$\Leftarrow g(0) = g(2), \quad g(1) = -\frac{3}{2} g(0)$$

Take, eg. $g(0) = 4$, $g(1) = -6$, $g(2) = 4$, $g(x) = 0 \ \forall x = 3, 4, \ldots$

Verify that $\mathbb{E}_\theta[g(X)] = 0$ for $\theta = 1, 2$.

$4 - 6 + \frac{4}{2} = 0$

$4 - 2 \times 6 + \frac{2^2 \cdot 4}{2!}$

$= 4 - 12 + 8 = 0$

But clearly $\mathbb{P}_\theta[g(X) = 0] \neq 1$.

So, X is not complete in this case.

Exercise 6 $\quad f_\theta(x) = e^{-(x-\theta)}, \; x > \theta, \; \theta \in \mathbb{R}$

$T = X_{(1)}$. $\quad f_X(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta \\ 0, & \text{o.w.} \end{cases}$

$F_X(y) = \begin{cases} 0 & \text{if } y \le \theta \\ 1 - e^{-(y-\theta)} & \text{if } y > \theta \end{cases}$ $\quad \int_\theta^y e^{-(x-\theta)} dx = \int_0^{y-\theta} e^{-z} dz = 1 - e^{-(y-\theta)}$

$\therefore$ the pdf of $T = X_{(1)}$ is

$$f_T(t) = n e^{-(t-\theta)} \left[1 - \left\{1 - e^{-(t-\theta)}\right\}\right]^{n-1}, \; t > \theta$$

$$= n e^{-n(t-\theta)}, \; t > \theta$$

Now, $\mathbb{E}_\theta[g(T)] = \int_\theta^\infty g(t)\, n e^{-n(t-\theta)} dt$

$$= n e^{n\theta} \int_\theta^\infty g(t) e^{-nt} dt$$

$\therefore \mathbb{E}_\theta[g(T)] = 0 \quad \forall \theta \in \mathbb{R}$

$\Rightarrow \int_\theta^\infty g(t) e^{-nt} dt = 0 \quad \forall \theta \in \mathbb{R}$

Taking derivative w.r.t. θ on both sides, we get

$-g(\theta) e^{-n\theta} = 0 \quad \forall \theta \in \mathbb{R}$

$\Rightarrow g(\theta) = 0 \quad \forall \theta \in \mathbb{R}$

$\therefore \mathbb{P}_\theta[g(T) = 0] = 1 \quad \forall \theta \in \mathbb{R}$

So, $T = X_{(1)}$ is complete.

The fact that it is sufficient is easy to verify

For example, we can write the joint pdf as

$$p_\theta(\underline{x}) = \prod_{i=1}^{n} e^{-(x_i-\theta)} \mathbb{1}\{x_i > \theta\}$$

$$= e^{-\left(\sum_{i=1}^{n} x_i - n\theta\right)} \mathbb{1}\{x_{(1)} > \theta\}$$

$$= e^{n\theta} \mathbb{1}\{x_{(1)} > \theta\} \; e^{-\sum_{i=1}^{n} x_i}$$

Therefore, factorization theorem tells us that $X_{(1)}$ is suff.

(b) $$S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 = \frac{1}{2n(n-1)} \sum_{i,j=1}^{n} (X_i - X_j)^2$$

Now, it is easy to check that if X has pdf $f_\theta(x) = e^{-(x-\theta)}$, $x > \theta$, then $Y = X - \theta$ has pdf $f(y) = e^{-y}$, $y > 0$, which is free of θ.

Therefore,

$$S^2 = \frac{1}{2n(n-1)} \sum_{i,j=1}^{n} (X_i - X_j)^2 = \frac{1}{2n(n-1)} \sum_{i,j=1}^{n} (Y_i - Y_j)^2$$

where $Y_1, \ldots, Y_n \overset{iid}{\sim} \underbrace{f}_{\text{free of } \theta}$, $f(y) = e^{-y}$, $y > 0$.

$\therefore$ the dist. of S^2 is free of θ. That is, S^2 is ancillary.

$$\left.\begin{array}{l} X_{(1)} \text{ is minimal sufficient} \\ S^2 \text{ is ancillary} \end{array}\right\} \xRightarrow{\text{Basu's Theorem}} X_{(1)} \text{ \& } S^2 \text{ are independent}$$

Exercise 7 (a) $f_\theta(x) = \dfrac{\theta^\alpha}{\Gamma(\alpha)} e^{-\theta x} x^{\alpha-1}, \quad x>0.$

[α is known]

$$\therefore\ p_\theta(\underline{x}) = \frac{\theta^{n\alpha}}{\{\Gamma(\alpha)\}^n} e^{-\theta \sum_{i=1}^n x_i} \left(\prod_{i=1}^n x_i\right)^{\alpha-1}, \quad x_i>0,\ i=1,\dots,n$$

$$= \underbrace{\theta^{n\alpha} e^{-\theta\sum_{i=1}^n x_i}}_{g_\theta(T(\underline{x}))} \underbrace{\frac{1}{\{\Gamma(\alpha)\}^n}\left(\prod_{i=1}^n x_i\right)^{\alpha-1}}_{h(\underline{x})}$$

$\therefore\ T = T(\underline{X}) = \sum_{i=1}^n X_i$ is sufficient.

In fact, using the result proved in the class, you can show that $\sum_{i=1}^n X_i$ is minimal sufficient.

(b) Characteristic function of X is

$$\varphi_X(t) = \mathbb{E}\left(e^{itX}\right) = \frac{\theta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{itx} e^{-\theta x} x^{\alpha-1}\,dx$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha)}\int_0^\infty e^{-(\theta-it)x} x^{\alpha-1}\,dx$$

$$= \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha)}{(\theta-it)^\alpha} = \left(\frac{\theta}{\theta-it}\right)^\alpha, \quad t\in\mathbb{R}. \quad \dots (*)$$

$$\therefore\ \varphi_{X_1+\dots+X_n}(t) = \varphi_{X_1}(t)\cdots\varphi_{X_n}(t) \quad \text{[independence]}$$

$$= \{\varphi_X(t)\}^n \quad \text{[identical dist.]}$$

$$= \left(\frac{\theta}{\theta-it}\right)^{n\alpha}, \quad t\in\mathbb{R}$$

$\uparrow$ chf of Gamma$(n\alpha, \theta)$ [see $(*)$]

Due to uniqueness of characteristic functions,

$$X_1 + \cdots + X_n \sim \text{Gamma}(n\alpha, \theta)$$

(c) $$\mathbb{E}_\theta[g(T)] = \frac{\theta^{n\alpha}}{\Gamma(n\alpha)} \int_0^\infty g(t)\, e^{-\theta t}\, t^{n\alpha-1}\, dt$$

$$\therefore \mathbb{E}_\theta[g(T)] = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow \underbrace{\int_0^\infty g(t)\, t^{n\alpha-1}\, e^{-\theta t}\, dt}_{\text{one-sided Laplace transform}} = 0 \quad \forall\ \theta > 0$$

$$\Rightarrow g(t)\, t^{n\alpha-1} = 0 \quad \forall\ t > 0$$

$$\Rightarrow g(t) = 0 \quad \forall\ t > 0$$

$$\therefore \mathbb{P}_\theta[g(T) = 0] = 1 \quad \forall\ \theta > 0$$

So, $T = \sum_{i=1}^n X_i$ is complete.

(d) $$f_\theta(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} e^{-\theta x} x^{\alpha-1}, \quad x > 0.$$

Define $Y = \theta X$. Verify that the pdf of Y is

$$f_Y(y) = \frac{1}{\Gamma(\alpha)} e^{-y} y^{\alpha-1}, \quad y > 0.$$

Free of θ.

Moreover, $f_\theta(x) = \theta f_Y(\theta x)$.

Therefore, f_θ belongs to the scale family.

(e) Define $Y = \theta X$, so that $X = Y/\theta$.

Now, $X_{(i)} = \dfrac{Y_{(i)}}{\theta}$ and $X_i = \dfrac{Y_i}{\theta}$

$$\therefore \frac{X_{(i)}}{\sum_{i=1}^{n} X_i} = \frac{Y_{(i)}}{\sum_{i=1}^{n} Y_i}, \text{ where } Y_1 \ldots Y_n \overset{iid}{\sim} \text{Gamma}(1, \alpha).$$

free of θ.

Therefore, $\dfrac{X_{(i)}}{\sum_{i=1}^{n} X_i}$ has dist. free of θ

$\Rightarrow \dfrac{X_{(i)}}{T}$ is ancillary $\left[T = \sum_{i=1}^{n} X_i\right]$

(f) T is complete sufficient, $\dfrac{X_{(i)}}{T}$ is ancillary.

So, by Basu's Theorem, T & $\dfrac{X_{(i)}}{T}$ are independent.

Using the hint, $\mathbb{E}\left(\dfrac{X_{(i)}}{T}\right) = \dfrac{\mathbb{E}(X_{(i)})}{\mathbb{E}(T)}$ $\left[\begin{array}{l}\text{Use } X_{(i)} = X \\ T = Y\end{array}\right]$

$$\text{Now, } \mathbb{E}(X_{(i)} \mid T) = \mathbb{E}\left(\frac{X_{(i)}}{T} \cdot T \,\Big|\, T\right)$$

$$= T \cdot \mathbb{E}\left(\frac{X_{(i)}}{T} \,\Big|\, T\right)$$

$$= T \cdot \mathbb{E}\left(\frac{X_{(i)}}{T}\right) \quad \left[\because \frac{X_{(i)}}{T} \text{ is indep. of } T\right]$$

$$= T \cdot \frac{\mathbb{E}(X_{(i)})}{\mathbb{E}(T)}, \text{ as required.}$$

<u>Note</u>: Since T is complete sufficient, $\mathbb{E}(X_{(i)} \mid T)$ is the UMVUE of $\mathbb{E}(X_{(i)})$. Therefore, $\dfrac{\mathbb{E}(X_{(i)})}{\mathbb{E}(T)} \cdot T$ is the UMVUE of $\mathbb{E}(X_{(i)})$. We will explore this later.

Exercise 8 $\quad f_\theta(x,y) = e^{-(\theta x + y/\theta)}, \; x>0, y>0, \theta>0.$

$$T = \sqrt{\frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n Y_i}}, \quad U = \sqrt{\sum_{i=1}^n X_i \sum_{i=1}^n Y_i}$$

(a) $p_\theta(\underline{x},\underline{y}) = e^{-\theta \sum_{i=1}^n x_i - \frac{1}{\theta}\sum_{i=1}^n y_i}$

Notice That $TU = \sum_{i=1}^n X_i$ and $\frac{U}{T} = \sum_{i=1}^n Y_i$

$$\therefore p_\theta(x,y) = \underbrace{e^{-\theta t u - \frac{1}{\theta}\frac{u}{t}}}_{g_\theta(t,u)}, \quad h(\underline{x},\underline{y}) \equiv 1.$$

So, by factorization Theorem, (T,U) is jointly sufficient for θ.

[$\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right)$ is jointly minimally sufficient for θ.
(T,U) is a 1-1 function of $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i\right)$]

(b) From the joint pdf of (X,Y),

$$f_\theta(x,y) = e^{-\theta x} e^{-y/\theta}, \quad x>0, y>0, \theta>0$$
$$= f_\theta^X(x)\, f_\theta^Y(y), \quad x>0, y>0, \theta>0.$$

where $f_\theta^X(x) = e^{-\theta x}, \quad x>0, \theta>0$

$f_\theta^Y(y) = e^{-y/\theta}, \quad y>0, \theta>0.$

Verify That $f_\theta^X(\cdot)$, $f_\theta^Y(\cdot)$ are valid pdf's

$\therefore$ X & Y are independent.

In fact, $X \sim \text{Exponential}\left(\text{mean} = \frac{1}{\theta}\right)$

$Y \sim \text{Exponential}\left(\text{mean} = \theta\right)$

$$\therefore \ \mathbb{E}_\theta(X) = \frac{1}{\theta}, \quad \mathbb{E}_\theta(Y) = \theta$$

$$\therefore \ \mathbb{E}_\theta\left(\sum_{i=1}^n X_i\right) = \frac{n}{\theta}, \quad \mathbb{E}_\theta\left(\sum_{i=1}^n Y_i\right) = n\theta$$

$$\therefore \ \mathbb{E}_\theta\left(\sum_{i=1}^n X_i \sum_{i=1}^n Y_i\right) = \mathbb{E}_\theta\left(\sum_{i=1}^n X_i\right) \mathbb{E}_\theta\left(\sum_{i=1}^n Y_i\right) = n^2$$

$$\Rightarrow \ \mathbb{E}_\theta\left(\frac{1}{n}\sum_{i=1}^n X_i \frac{1}{n}\sum_{i=1}^n Y_i - 1\right) = 0 \quad \forall \theta$$

Now, $\sum_{i=1}^n X_i = TU$, $\sum_{i=1}^n Y_i = \frac{U}{T}$

$$\therefore \ \sum_{i=1}^n X_i \sum_{i=1}^n Y_i = U^2$$

$$\Rightarrow \ \mathbb{E}_\theta\left(\left(\frac{U}{n}\right)^2 - 1\right) = 0. \quad \forall \theta$$

But, U has a continuous dist.

$$\therefore \ \mathbb{P}_\theta\left(\frac{U^2}{n} - 1 = 0\right) = 0 \quad \forall \theta$$

$\Rightarrow$ U and hence (T, U) is not complete