10.4 Exercises

In Exercises 1 through 22, a sequence {f(n)} is defined by the formula given. In each case, (a) determine whether the sequence converges or diverges, and (b) find the limit of each convergent sequence. In some cases it may be helpful to replace the integer n by an arbitrary positive real x and to study the resulting function of x by the methods of Chapter 7. You may use formulas (10.9) through (10.13) listed at the end of Section 10.2.

1. $f(n) = \dfrac{n}{n+1} - \dfrac{n+1}{n}$.
2. $f(n) = \dfrac{n^2}{n+1} - \dfrac{n^2+1}{n}$.
3. $f(n) = \cos \dfrac{n\pi}{2}$.
4. $f(n) = \dfrac{n^2+3n-2}{5n^2}$.
5. $f(n) = \dfrac{n}{2^n}$.
6. $f(n) = 1 + (-1)^n$.
7. $f(n) = \dfrac{1+(-1)^n}{n}$.
8. $f(n) = \dfrac{(-1)^n}{n} + \dfrac{1+(-1)^n}{2}$.
9. $f(n) = 2^{1/n}$.
10. $f(n) = n^{(-1)^n}$.
11. $f(n) = \dfrac{n^{2/3}\sin(n!)}{n+1}$.
12. $f(n) = \dfrac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}$.
13. $f(n) = \sqrt{n+1} - \sqrt{n}$.
14. $f(n) = na^n$, where $|a| < 1$.
15. $f(n) = \dfrac{\log_a n}{n}$, $a > 1$.
16. $f(n) = \dfrac{100{,}000n}{1+n^2}$.
17. $f(n) = \left(1 + \dfrac{2}{n}\right)^n$.
18. $f(n) = 1 + \dfrac{n}{n+1}\cos\dfrac{n\pi}{2}$.
19. $f(n) = \left(1 + \dfrac{i}{2}\right)^{-n}$.
20. $f(n) = e^{-\pi i n/2}$.
21. $f(n) = \dfrac{1}{n}e^{-\pi i n/2}$.
22. $f(n) = ne^{-\pi i n/2}$.

Each of the sequences $\{a_n\}$ in Exercises 23 through 28 is convergent. Therefore, for every preassigned $\epsilon > 0$, there exists an integer N (depending on ϵ) such that $|a_n - L| < \epsilon$ if $n \geq N$, where $L = \lim_{n\to\infty} a_n$. In each case, determine a value of N that is suitable for each of the following values of ϵ: $\epsilon = 1, 0.1, 0.01, 0.001, 0.0001$.

23. $a_n = \dfrac{1}{n}$.
24. $a_n = \dfrac{n}{n+1}$.
25. $a_n = \dfrac{(-1)^{n+1}}{n}$.
26. $a_n = \dfrac{1}{n!}$.
27. $a_n = \dfrac{2n}{n^3+1}$.
28. $a_n = (-1)^n\left(\dfrac{9}{10}\right)^n$.

29. Prove that a sequence cannot converge to two different limits.
30. Assume $\lim_{n\to\infty} a_n = 0$. Use the definition of limit to prove that $\lim_{n\to\infty} a_n^2 = 0$.
31. If $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$, use the definition of limit to prove that we have $\lim_{n\to\infty} (a_n + b_n) = A + B$, and $\lim_{n\to\infty} (ca_n) = cA$, where c is a constant.
32. From the results of Exercises 30 and 31, prove that if $\lim_{n\to\infty} a_n = A$ then $\lim_{n\to\infty} a_n^2 = A^2$. Then use the identity $2a_nb_n = (a_n + b_n)^2 - a_n^2 - b_n^2$ to prove that $\lim_{n\to\infty}(a_nb_n) = AB$ if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

33. If α is a real number and n a nonnegative integer, the binomial coefficient $\binom{\alpha}{n}$ is defined by the equation

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)}{n!}.$$

(a) When $\alpha = -\frac{1}{2}$, show that

$$\binom{\alpha}{1} = -\frac{1}{2}, \quad \binom{\alpha}{2} = \frac{3}{8}, \quad \binom{\alpha}{3} = -\frac{5}{16}, \quad \binom{\alpha}{4} = \frac{35}{128}, \quad \binom{\alpha}{5} = -\frac{63}{256}.$$

(b) Let $a_n = (-1)^n \binom{-1/2}{n}$. Prove that $a_n > 0$ and that $a_{n+1} < a_n$.

34. Let f be a real-valued function that is monotonic increasing and bounded on the interval $[0, 1]$. Define two sequences $\{s_n\}$ and $\{t_n\}$ as follows:

$$s_n = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right), \qquad t_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

(a) Prove that $s_n \le \int_0^1 f(x)\,dx \le t_n$ and that $0 \le \int_0^1 f(x)\,dx - s_n \le \frac{f(1)-f(0)}{n}$.

(b) Prove that both sequences $\{s_n\}$ and $\{t_n\}$ converge to the limit $\int_0^1 f(x)\,dx$.

(c) State and prove a corresponding result for the interval $[a, b]$.

35. Use Exercise 34 to establish the following limit relations:

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2 = \frac{1}{3}.$

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n+k} = \log 2.$

(c) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \frac{\pi}{4}.$

(d) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}} = \log(1+\sqrt{2}).$

(e) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sin\frac{k\pi}{n} = \frac{2}{\pi}.$

(f) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n}\sin^2\frac{k\pi}{n} = \frac{1}{2}.$

10.5 Infinite series

From a given sequence of real or complex numbers, we can always generate a *new* sequence by adding together successive terms. Thus, if the given sequence has the terms

$$a_1, a_2, \ldots, a_n, \ldots,$$

we may form, in succession, the "partial sums"

$$s_1 = a_1, \qquad s_2 = a_1 + a_2, \qquad s_3 = a_1 + a_2 + a_3,$$

and so on, the partial sum s_n of the first n terms being defined as follows:

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k. \tag{10.14}$$