Problem Set 5

Q1: Let $\lambda(x) = \dfrac{f_1(x)}{f_0(x)}$.

Then

x	-4	-3	0	1	2	5
$\lambda(x)$	3	$\frac{3}{2}$	$\frac{1}{6}$	$\frac{1}{3}$	1	4

By NP lemma, the MP ϕ_0 will be of the form:

$$(*) — \phi_0(x) = \begin{cases} 1 & \text{if } \lambda(x) > K \\ \gamma & \text{if } \lambda(x) = K \\ 0 & \text{if } \lambda(x) < K \end{cases}$$ [we might need this to meet $(**)$]

and $$\underset{H_0}{E}\left[\phi_0(X)\right] = \alpha. \quad — (**)$$

The table indicates that one should first reject H_0 for $x=5$, then $x=-4$, $x=-3$, $x=2$, $x=1$ and $x=0$ in order.

So, $\lambda(x) > K \iff |x| > K_0$

Now, (i) if we choose $K_0 > 5$ then H_0 is accepted for all x [in the support of f_0 / f_1]

(ii) If $K_0 = 5$ then $\phi_0(x) = \begin{cases} \gamma & \text{if } x = 5 \\ 0 & \text{otherwise.} \end{cases}$

$$E_{H_0}\left[\phi_0(X)\right] = \gamma P_{f_0}(X=5) = \gamma \times 0.05.$$

If level $\alpha = 0.05$, then we must take $\gamma = 1$.

Thus, the test will be $\phi_0(x) = \begin{cases} 1 & \text{if } x = 5 \\ 0 & \text{otherwise.} \end{cases}$

[ANS]

If $\alpha = 0.075$ then no choice of $\gamma \in [0,1]$ satisfies $(**)$.

So, we need a lower choice of K_0.

(iii) For any $K_0 \in (4,5)$, no new point is included in the critical region (except $x = 5$).

For $K_0 = 4$, we have

$$\phi_0(x) = \begin{cases} 1 & \text{if} \quad x = 5 \\ \gamma & \text{if} \quad x = -4 \\ 0 & \text{ow.} \end{cases}$$

$$E_{H_0}[\phi(X)] = P_{f_0}(X=5) + \gamma\, P_{f_0}(X=-4)$$

$$= 0.05 + \gamma\, 0.05$$

For $\alpha = 0.075$, $(**)$ gives $\gamma = \dfrac{0.075 - 0.05}{0.05}$

$$= \frac{0.025}{0.05} = \frac{1}{2}$$

Thus, the optimal test is

$$\phi_0(x) = \begin{cases} 1 & \text{if} \quad x = 5 \\ \frac{1}{2} & \text{if} \quad x = -4 \\ 0 & \text{otherwise.} \end{cases}$$

[ANS]

Q2: As ϕ^* is MP for testing $H_0: \underset{\sim}{X} \sim f_0$ vs $H_1: \underset{\sim}{X} \sim f_1$ at level α with power β^*.

Then
$$\phi^*(\underset{\sim}{x}) = \begin{cases} 1 & \text{if} \quad \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} > K \\ 0 & \text{if} \quad \dfrac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} < K \end{cases} \qquad \text{--- } (*)$$

and
$$E_{f_0}\left[\phi^*(\underset{\sim}{X})\right] = \alpha. \qquad \text{--- } (**)$$

Also,
$$E_{f_1}\left[\phi^*(\underset{\sim}{X})\right] = \beta^* \qquad \text{--- } (***)$$

Now, $1 - \phi^*(\underset{\sim}{x}) = \phi^{**}(\underset{\sim}{x})$ takes the following form

$$\phi^{**}(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \frac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} < K, \text{ i.e., } \frac{f_0(\underset{\sim}{x})}{f_1(\underset{\sim}{x})} > \frac{1}{K} \\ 0 & \text{if } \frac{f_0(\underset{\sim}{x})}{f_1(\underset{\sim}{x})} < \frac{1}{K} \end{cases} \quad \text{(1)}$$

and it satisfies

$$\text{(2)} \quad E_{f_1}\left[\phi^{**}(\underset{\sim}{x})\right] = (1-\beta^*).$$

From (1) and (2), we have ϕ^{**} is MP for $H_0: \underset{\sim}{X} \sim f_1$ vs $H_1: \underset{\sim}{X} \sim f_0$ at level $(1-\beta^*)$.

[ANS]

Further, observe that power of ϕ^{**},

$$E_{f_0}\left[\phi^{**}(\underset{\sim}{x})\right] = 1-\alpha.$$

Now, if ϕ^* is unbiased then $\alpha < \beta^*$,

$$\Rightarrow \underset{\substack{\Uparrow \\ \text{power of} \\ \phi^{**}}}{(1-\alpha)} > \underset{\substack{\Uparrow \\ \text{size of} \\ \phi^{**}}}{(1-\beta^*)}.$$

So, ϕ^{**} is also unbiased.

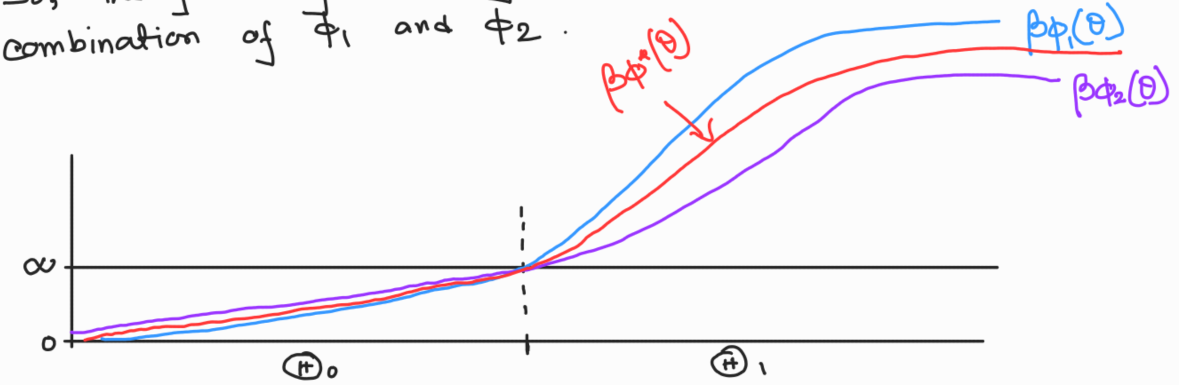
[ANS]

Q3: Let $\phi^* = c\phi_1 + (1-c)\phi_2$ for some $0 \le c \le 1$.

$$\therefore \beta_{\phi^*}(\theta) = E_\theta\left[\phi^*(\underset{\sim}{x})\right] = c\, E_\theta\left[\phi_1(\underset{\sim}{x})\right] + (1-c)\, E_\theta\left[\phi_2(\underset{\sim}{x})\right]$$

$$= c\,\beta_{\phi_1}(\theta) + (1-c)\,\beta_{\phi_2}(\theta).$$

So, the power function of ϕ^* will also be the same convex combination of ϕ_1 and ϕ_2.



Q4: $X_1, \dots, X_n \overset{iid}{\sim} N(\theta, 5^2)$.

$H_0: \theta \le 30$ vs $H_1: \theta > 30$.

$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \bar{x}_n \geq c \\ 0 & \text{ow} \end{cases}$$

$$\beta_\phi(\theta) = P_\theta(\bar{X}_n \geq c) = P_\theta\left(\frac{\bar{X}_n - \theta}{5/\sqrt{n}} \geq \frac{c-\theta}{5/\sqrt{n}}\right)$$

$$= 1 - \Phi\left(\frac{c-\theta}{5\sqrt{n}}\right).$$

Thus, we need to solve the following two equations for the two unknown qualities

$$1 - \Phi\left(\frac{c-30}{5/\sqrt{n}}\right) = 0.1$$

and $$1 - \Phi\left(\frac{c-35}{5/\sqrt{n}}\right) = 0.9.$$

Q5: $X_1, \dots, X_n \sim U(0, \theta)$

$H_0: \theta = 1$ vs $H_1: \theta \neq 1$.

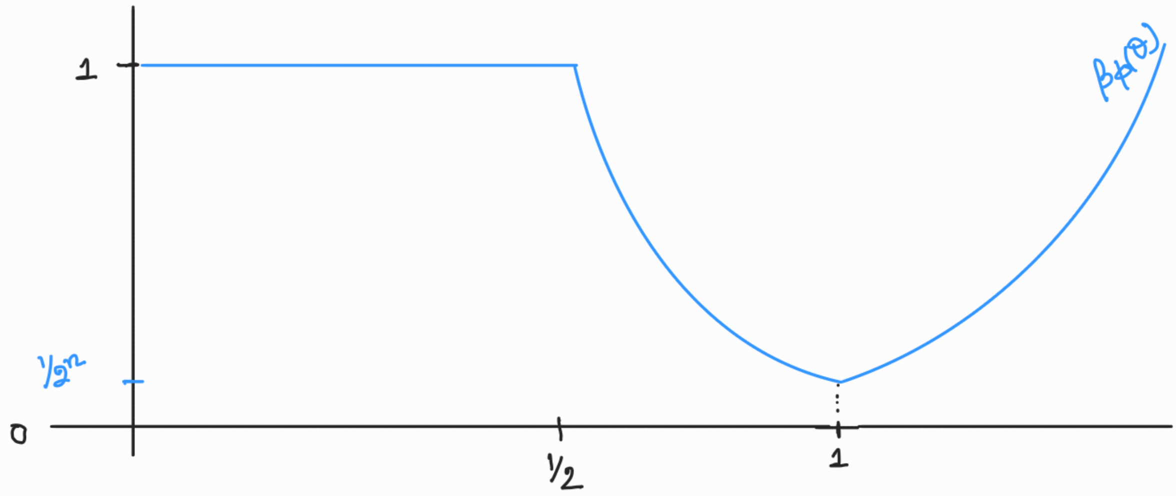
$$\phi(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } x_{(n)} \le 1/2 \text{ or } x_{(n)} \geq 1 \\ 0 & \text{ow}. \end{cases}$$

$$\beta_\phi(\theta) = P_\theta\left(X_{(n)} \le \tfrac{1}{2}\right) + P_\theta\left(X_{(n)} \geq 1\right)$$

$$= \begin{cases} P_\theta\left(X_{(n)} \le \theta\right) + 0 = 1 & \text{if } \theta < \frac{1}{2} \\ P_\theta\left(X_{(n)} \le \frac{1}{2}\right) + 0 = \left(\frac{1}{2\theta}\right)^n & \text{if } \frac{1}{2} \le \theta \le 1 \\ P_\theta\left(X_{(n)} \le \frac{1}{2}\right) + P_\theta\left(X_{(n)} \geq 1\right) = \left(\frac{1}{2\theta}\right)^n + 1 - \left(\frac{1}{\theta}\right)^n & \text{if } \theta > 1 \end{cases}$$

$$= \left(\frac{1}{\theta}\right)^n \left(\frac{1}{2^n} - 1\right) + 1$$

verify!



Q6: $X_1, \cdots, X_n \overset{IID}{\sim} N(0, \theta^2)$

$H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1 \ (>\theta_0)$

$$\lambda(\underset{\sim}{x}) = \frac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left\{-\frac{\sum_{i=1}^n x_i^2}{2}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right)\right\}$$

Thus, $\lambda(\underset{\sim}{x}) > k$

$$\Leftrightarrow \quad -\sum_{i=1}^n x_i^2 \left(\frac{1}{\theta_1^2} - \frac{1}{\theta_0^2}\right) > 2\left[\log k - n\log(\theta_0/\theta_1)\right]$$

$$\Leftrightarrow \quad \sum_{i=1}^n x_i^2 > \frac{2\left[\log k - n\log(\theta_0/\theta_1)\right]}{\left(1/\theta_0^2 - 1/\theta_1^2\right)} = k_0$$

as $\theta_1 > \theta_0$ the inequality remains unaltered.

$\therefore$ the MP test is of the form

$$① - \quad \phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^n x_i^2 > k_0 \\ 0 & \text{otherwise.} \end{cases}$$

and it satisfies

$$E_{H_0}\left[\phi_0(\underset{\sim}{x})\right] = \beta_{\phi_0}(\theta_0) = \alpha. \quad — ②$$

Now $\quad \beta_{\phi_0}(\theta) = P_\theta\left(\sum_{i=1}^{n} x_i^2 > K_0\right)$

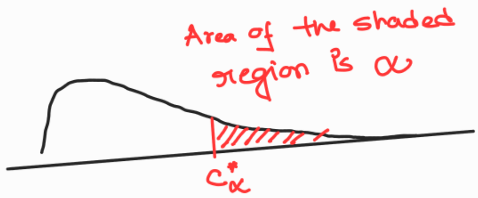
$$\left[\text{Under } \theta, \quad \sum_{i=1}^{n} x_i^2/\theta^2 \sim \chi^2_{(n)}\right]$$

$$= P_\theta\left(\sum_{i=1}^{n} x_i^2/\theta^2 > K_0/\theta^2\right)$$

$$= P_\theta\left(T_n > \frac{K_0}{\theta^2} \,\middle|\, T_n \sim \chi^2_{(n)}\right) \quad \text{--- } (*)$$

To solve ②, we use some statistical software to find the value, say, c^*_α, in the $\chi^2_{(n)}$ distributn, such that above $P\left(T_n \geq c^*_\alpha \mid T_n \sim \chi^2_{(n)}\right) = \alpha$

Next equate



$$c^*_\alpha = K_0/\theta_0^2$$

$$\Rightarrow K_0 = \theta_0^2 c^*_\alpha$$

$$\therefore \quad \phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i^2 \geq \theta_0^2 c^*_\alpha \\ 0 & \text{otherwise.} \end{cases}$$

– is the MP test.

$\boxed{\text{Generalization}}$ If $\boxed{H_1 : \theta > \theta_0}$ i.e., $\theta \in \underbrace{(\theta_0, \infty)}_{\Theta_1}$

then for any choice of $\theta_1 \in \Theta_1$,

similar calculation as above will show that

$$\lambda(\underset{\sim}{x}) > K \iff \sum_{i=1}^{n} x_i^2 > K_0 = \frac{2\left[\log K - n \log(\theta_0/\theta_1)\right]}{\left(1/\theta_0^2 - 1/\theta_1^2\right)}$$

[Note that for two different choices of θs in Θ_1, say, θ_1 and θ_1', the relation between K and K_0 may change. For eg, let $\theta_0 = 1$ and $\theta_1 = 2$

$$K_0 = 8\left(\log K + n \log 2\right)/3$$

and for $\theta_1 = 3$, $K_0 = 9(\log k + n\log 3)/4$.]

Nevertheless, for any choice of $\theta_1 \in \Theta_1$, the MP test be of the form

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i^2 \geq K_0 \\ 0 & \text{o/w} \end{cases}$$

for both the alternatives.

Further, K_0 will be determined by eqn. (2).

Again, (2) requires K_0 such that

$$P_{\theta_0}\left(T_n > K_0/\theta_0^2 \,\middle|\, T_n \sim \chi^2_{(n)}\right) = \alpha,$$

which provides the same soln. $K_0 = \theta_0^2 c_\alpha^*$.

So, the MP test for testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1 (> \theta_0)$ is also UMP for testing $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$.

Next if $\boxed{H_0: \theta \leq \theta_0}$ is consider instead of $\theta = \theta_0$

Observe, from (*), that for the MP test ϕ_0,

power fn $\beta_{\phi_0}(\theta) = P_\theta\left(T_n \geq K_0/\theta^2 \,\middle|\, T_n \sim \chi^2_{(n)}\right)$

$= 1 - F_{T_n}\left(K_0/\theta^2\right)$ where F_{T_n} is the CDF of $\chi^2_{(n)}$ distr..

As F_{T_n} is non-decreasing, it is easy to see that as θ increases, $\beta_{\phi_0}(\theta)$ also increases.

$$\text{So, size} = \sup_{\theta \leq \theta_0} \beta_{\phi_0}(\theta) = \beta_{\phi_0}(\theta_0)$$

This implies that the equation size $= \alpha$

$$\Leftrightarrow \beta_{\phi_0}(\theta_0) = \alpha,$$

provides the same solution of $K_0 = \theta_0^2 c_\alpha^*$.

So, the MP test for $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1 (> \theta_0)$ is also

Q7: Power function of the proposed test, say, ϕ,

$$\beta_\phi(\theta) = P_\theta\left(\prod_{i=1}^{n} X_i \geq \frac{1}{2}\right)$$

$$= P_\theta\left(\sum_{i=1}^{n} \log X_i \geq -\log 2\right) \quad -①$$

Observe that, for $Y = -\log X$, the pdf of Y is

$$f_Y(y) = \theta \left(e^{-y}\right)^{\theta-1} e^{-y}, \qquad 0<y<\infty$$

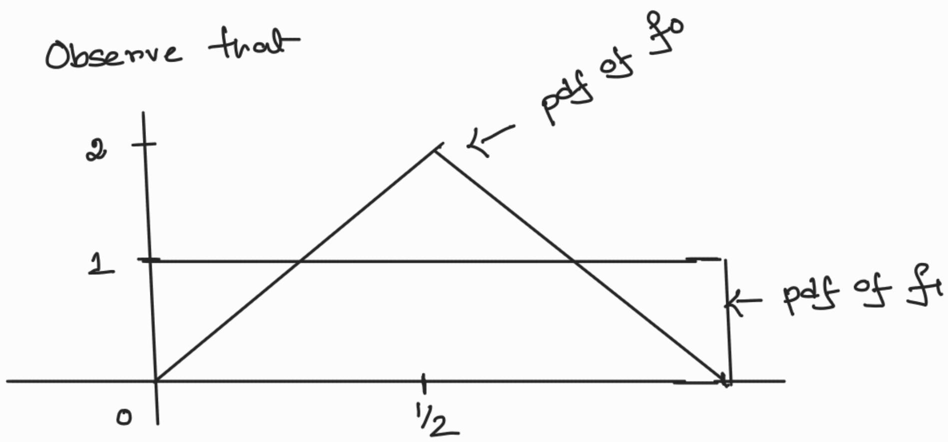
$$= \theta e^{-\theta y}; \quad 0<y<\infty.$$

$\therefore Y \sim$ exponential (θ)

So, $\sum_{i=1}^{n} Y_i \sim$ gamma (n, θ).

Thus, from ①, $\beta_\phi(\theta) = P_\theta\left(T \leq \log 2 \,\middle|\, T \sim \text{gamma}(n,\theta)\right)$

Q8: Observe that



Both f_0 and f_1 are symmetric about $1/2$.

Thus, $\lambda(\underset{\sim}{x}) = \dfrac{f_1(x)}{f_0(x)} > K$ iff $|x - \frac{1}{2}| > K_0$.

$$P_{f_0}\left(|X - \tfrac{1}{2}| > K_0\right) = P_{f_0}\left(X > K_0 + \tfrac{1}{2}\right) + P_{f_0}\left(X < \tfrac{1}{2} - K_0\right)$$

$$= \int_0^{\frac{1}{2}-K_0} 4x\,dx + \int_{\frac{1}{2}+K_0}^{1} 4(1-x)\,dx$$

$$= 4\left(\tfrac{1}{2}-K_0\right)^2 = 1 - 4K_0 + 4K_0^2$$

Thus, the size $=\alpha$ condition suggests

$$4K_0^2 - 4K_0 + 1 = \alpha$$

$$K_0 = \frac{4 \pm \sqrt{16 - 4\times 4\times(1-\alpha)}}{8}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{\alpha}$$

$K_0 = \frac{1}{2} - \frac{1}{2}\sqrt{\alpha}$ provides a feseable choice.

So, MP test

$$\phi_0(x) = \begin{cases} 1 & \text{if } |x-\frac{1}{2}| > \frac{1}{2} - \frac{1}{2}\sqrt{\alpha} \\ 0 & \text{ow.} \end{cases}$$

Q9:

$$\lambda(\underset{\sim}{x}) = \frac{f_1(\underset{\sim}{x})}{f_0(\underset{\sim}{x})} = \prod_{i=1}^{n} x_i^{-3/2} \times 4^n$$

$$\therefore\ \lambda(\underset{\sim}{x}) > K \iff -\frac{3}{2}\sum_{i=1}^{n} \log x_i > \log K - n\log 4$$

$$\iff \sum_{i=1}^{n} \log x_i < \frac{n\log 4 - \log K}{3/2}$$

The rest of the solution is similar to that of Q.7.

Q10: First we find the MP for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1 \ (> \theta_0)$.

$$\lambda(\underset{\sim}{x}) = \left(\frac{\theta_0}{\theta_1}\right)^n \exp\left\{ \sum_{i=1}^{n} (x_i - \mu) \left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) \right\}$$

$$\lambda(\underset{\sim}{x}) > K \iff \sum_{i=1}^{n} (x_i - \mu) > K_0 \quad \text{as} \quad \theta_0 < \theta_1$$

Thus, MP test is of the form

$$① \quad - \quad \phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} (x_i - \mu) \geq K_0 \\ 0 & \text{ow} \end{cases}$$

and it satisfies

$$\beta_{\phi_0}(\theta_0) = \alpha. \quad — ②$$

Now, $$\beta_{\phi_0}(\theta) = P_\theta\left(\sum_{i=1}^{n} (x_i - \mu) \geq K_0 \right) \quad —(*)$$

Let $y_i = (x_i - \mu)$ then $y_i \overset{IID}{\sim}$ exponential with mean θ.

$$f_{y,\theta}(y) = \frac{1}{\theta} e^{-y/\theta} \quad ; \quad y > 0$$

Define $Z_i = y_i/\theta$ then $Z_i \sim$ exponential (1).

$$T_n = \sum_{i=1}^{n} Z_i = \frac{1}{\theta} \sum_{i=1}^{n} (x_i - \theta) \sim \text{Gamma}(n, 1).$$

From $(*)$ $$\beta_{\phi_0}(\theta) = P\left(T_n \geq \frac{K_0}{\theta} \,\middle|\, T_n \sim \text{Gamma}(n,1) \right)$$

Thus, the choice of K_0 satisfying ② can be obtained for specific choice of α and using a statistical software, and inserting that choice in ① the MP test is obtained.

<u>Generalization:</u> (1) For $H_0: \theta = \theta_0$ against $H_1: \theta = \theta_1$ for any $\theta_1 > \theta_0$, the same MP test ϕ_0 is obtained.

Thus, ϕ_0 is UMP for testing $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$.

(2) Consider the test $H_0; \theta \leq \theta_0$ against $H_1: \theta > \theta_0$.

Observe that the power fn.

$$\beta_{\phi_0}(\theta) = 1 - P\left(T_n \leq K_0/\theta \mid T_n \sim \text{Gamma}(n,1)\right)$$

is an increasing fn. of θ.

[if $\theta \uparrow$ then $K_0/\theta \downarrow$, $P(T_n \leq K_0/\theta) \downarrow$ and

$1 - P(T_n \leq K_0/\theta) \uparrow$.]

Thus, $\text{size} = \sup_{\theta \leq \theta_0} \beta_{\phi_0}(\theta) = \beta_{\phi_0}(\theta_0)$, which

implies that $\text{size} = \alpha$ condition provides the same choice of K_0 as in the MP test.

So, the MP test for test $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1\ (>\theta_0)$ is UMP for testing $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$.

<u>Q.11</u> First consider

$$H_0: \theta = \theta_0 \quad \text{against} \quad H_1: \theta = \theta_1\ (>\theta_0)$$

$$\lambda(\underset{\sim}{x}) = \frac{f_{\theta_1}(\underset{\sim}{x})}{f_{\theta_0}(\underset{\sim}{x})} = e^{-n(\theta_1 - \theta_0)} \left(\frac{\theta_1}{\theta_0}\right)^{\sum x_i} > K$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} x_i > K_0 \quad \text{as} \quad \theta_1 > \theta_0$$

MP is of the form

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} x_i > K_0 \\ \gamma & \text{if } \quad = \\ 0 & \text{if } \quad < \end{cases}$$

and it satisfies

$$\beta_{\phi_0}(\theta_0) = P_{\theta_0}\left(\sum_{i=1}^{n} X_i > K_0\right) + \gamma\, P_{\theta_0}\left(\sum_{i=1}^{n} X_i = K_0\right) = \alpha$$

The exact choice of (K_0, γ) can be obtained from using a statistical software.

<u>Generalization:</u> (i) Using similar arguments as in Q.6 and Q.10, one can generalize the test for $H_0: \theta = \theta_0$ against $H_1: \theta > \theta_0$.

(ii) To generalize for $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$, consider the power function of ϕ_0;

$$\beta_{\phi_0}(\theta) = P_\theta\left(\sum_{i=1}^{n} X_i > K_0\right) + \gamma\, P_\theta\left(\sum_{i=1}^{n} X_i = K_0\right)$$

One can show that $\beta_{\phi_0}(\theta)$ is an increasing function of θ. (the detail is given below, you may skip the details for now)

Thus, $$\text{size} = \sup_{\theta \le \theta_0} \beta_{\phi_0}(\theta) = \beta_{\phi_0}(\theta_0)$$

Hence, size $= \alpha$ condition provides the same solutn. of K_0.

Thus the MP test for testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1 (> \theta_0)$ is UMP for testing $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$.

[* Why $\beta_{\phi_0}(\theta)$ is increasing fn. of θ ?

Observe that, given $T_n = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$ under θ,

$$\beta_{\phi_0}(\theta) = P_\theta(T_n > k_0) + \gamma \left\{ P(T_n > k_0 - 1) - P(T_n > k_0) \right\}$$

$$= (1-\gamma) P_\theta(T_n > k_0) + \gamma P(T_n > k_0 - 1)$$

To show, $\beta_{\phi_0}(\theta)$ is increasing fn. of θ, it is now enough to show that $P_\theta(T_n > t_0)$ is an increasing fn. of θ, for any integer t_0.

$$\frac{\partial}{\partial\theta} P_\theta(T_n > t_0)$$

$$= \frac{\partial}{\partial\theta} \left[\sum_{t=t_0+1}^{\infty} e^{-n\theta} \frac{(n\theta)^t}{t!} \right]$$

$$= \sum_{t=t_0+1}^{\infty} \left[-n e^{-n\theta} \frac{(n\theta)^t}{t!} + e^{-n\theta} \frac{t\, n^t \theta^{t-1}}{t!} \right]$$

$$= -n \sum_{t=t_0+1}^{\infty} e^{-n\theta} \frac{(n\theta)^t}{t!} + n \sum_{t=t_0+1}^{\infty} e^{-n\theta} \frac{(n\theta)^{t-1}}{(t-1)!}$$

$$= -n \left[P_\theta(T_n \ge t_0+1) - P_\theta(T_n \ge t_0) \right]$$

$$= n P_\theta(T_n = t_0) > 0.$$

Thus, $P_\theta(T_n > t_0)$ is increasing fn. θ, and the result follows.]

Q12: This is similar to Q.11.
