Problem Set 4

Q1: $P(X \geq 0) = 1$ implies X is non-neg with prob. 1.

By Markov inequality

$$\frac{1}{5} = P(X \geq 10) \leq \frac{E(X)}{10} \Rightarrow E(X) \geq 2.$$

Q2: $P(X - E(X) \leq -3) = 0.2$ and $P(X - E(X) \geq 3) = 0.3$

implies $P(-3 \leq X - E(X) \leq 3) = 0.5$.

By Chebyshev's inequality:

$$0.5 = P(|X - E(X)| \geq 3) \leq \frac{var(X)}{9} \Rightarrow var(X) \geq \frac{9}{2}.$$

Q3:

$$P(\mu - 2\sigma \leq \bar{X}_n \leq \mu + 2\sigma) \geq 0.99$$

$$\Rightarrow P(|\bar{X}_n - \mu| > 2\sigma) \leq 0.01$$

By Chebyshev's inequality

$$P(|\bar{X}_n - \mu| > 2\sigma) \leq \frac{var(\bar{X}_n)}{4\sigma^2} = \frac{1}{4n}$$

So, to ensure $P(|\bar{X}_n - \mu| > 2\sigma) \leq 0.01$ we need $0.01 = \frac{1}{4n}$,

i.e., $n = \frac{1}{4 \times 0.01} = \frac{100}{4} = 25.$

Taking a larger n would lead to a smaller bound than 0.01, so (≤ 0.01) holds trivially.

So, 25 ($= n$) is the smallest sample size which ensures the bound 0.01.

Q4: $E(Z_n) = n^2 \frac{1}{n} + 0 = n$, So $\lim_{n \to \infty} E(Z_n) = \infty$.

Fix $\varepsilon > 0$, $P(|Z_n| > \varepsilon) = \begin{cases} 0 & \text{if } \varepsilon > n \\ \frac{1}{n} & \text{if } \varepsilon \leq n \end{cases}$

$\rightarrow 0$ as $n \to \infty$.

So, $Z_n \xrightarrow{P} 0$.

Q5: X_i : RV indicating the residential area of the ith atendee of the concert, $i=1,\dots,n$.

$X_i \in \{ C(=1),\ S(=0) \}$. $\qquad P(X_i = 1) = p = 0.75$

Need to find $\quad P\left(n - \sum_{i=1}^{1200} X_i \le 270\right)$

$$= P\left(\bar{X}_n \geq 1 - \frac{270}{1200} \right)$$

$$= P\left(\frac{\sqrt{1200}\,(\bar{X}_n - 0.75)}{\sqrt{0.75 \times 0.25}} \geq \frac{\sqrt{1200}\left(\frac{31}{40} - 0.75\right)}{\sqrt{0.75\times 0.25}} \right)$$

Q6: By CLT

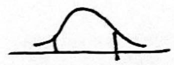
$$\frac{\sqrt{n}\,(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$$

So,

$$P\left(|\bar{X}_n - \mu| < 0.3 \right) = P\left(\frac{\sqrt{n}\,|\bar{X}_n - \mu|}{\sigma} < \frac{\sqrt{n}\ 0.3}{\sigma} \right)$$

$$\approx P\left(|Z| < \frac{\sqrt{n}}{10} \right) \quad \text{as } \sigma = 3$$

[when n is large]



$$= \Phi\left(\frac{\sqrt{n}}{10}\right) - \Phi\left(-\frac{\sqrt{n}}{10}\right)$$

$$= 2\Phi\left(\frac{\sqrt{n}}{10}\right) - 1 \qquad \text{— } (*)$$

Need to find n s.t.

$$2\Phi\left(\frac{\sqrt{n}}{10}\right) - 1 \geq 0.95 \qquad \text{— } (**)$$

$$\Rightarrow \Phi\left(\frac{\sqrt{n}}{10}\right) \geq \frac{1.95}{2} = 0.975$$

Given $\boxed{\Phi(1.96) = 0.975}$ if we take $\frac{\sqrt{n}}{10} = 1.96$, i.e., $n = \boxed{384.16} \approx 385$

then equality holds. Taking a larger n would make the probability in $(*)$ higher, thereby satisfying $(**)$ trivially.

So, $n = 385$ is the minimum sample size ensuring $(**)$.

Q7: Let X_i be the RV indicating the status of the ith item ; $i=1,\dots,n$

$X_i = 1 \Rightarrow$ ith item is defective. $\qquad P(X_i = 1) = 0.1$

To find n s.t. $\quad P\left(\frac{\sum_1 X_i}{n} \le 0.13 \right) \geq 0.99$

Do similarly

Q8: $X_i \overset{IID}{\sim} U(0,\theta)$

$T_n = X_{(n)}$. and $Z_n = n(T_n - \theta)$

Note that, F_Z is continuous everywhere.

So, $\mathcal{C}$, the set of continuity points of F_Z is $\mathbb{R}$.

Now, let F_n be the CDF of Z_n. Then

$$F_n(z) = P(n(T_n-\theta) \le z) = P(T_n \le \tfrac{z}{n}+\theta)$$

$$= \begin{cases} 0 & \text{if } z \le -n\theta \\ (1+z/n\theta)^n & \text{if } -n\theta < z \le 0 \\ 1 & \text{if } z > 0 \end{cases}$$

Also, for a fixed z in $-n\theta < z \le 0$,

$$F_n(z) = \left(1+\frac{z}{n\theta}\right)^{\frac{n\theta}{z}\cdot z/\theta} \longrightarrow e^{z/\theta} \text{ as } n\to\infty.$$

So, $F_n(z) \longrightarrow F_Z(z)$ for all $z \in \mathbb{R}\ (\equiv \mathcal{C})$

So, $Z_n \xrightarrow{d} Z$.

Q9: By Chebyshev's inequality.

$$P\left(\bar{X}_n \in [\mu-1, \mu+1]\right)$$

$$= P\left(|\bar{X}_n-\mu| \leq 1\right) \leq 1 - \frac{E\left(|\bar{X}_n-\mu|^r\right)}{1^r}$$

For $r=2$, $E\left((\bar{X}_n-\mu)^2\right) = \frac{\mu_2}{n} = \frac{5}{4\times 20} = \frac{1}{16} = 0.0625$

For $r=4$,

$$E\left[(\bar{X}_n-\mu)^4\right] = E\left[\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)\right\}^4\right]$$

$$= \frac{1}{n^4} E\left[\left\{\sum_{i=1}^{n}(x_i-\mu)\right\}^4\right]$$

$$= \frac{1}{n^4}\left[\sum_{i=1}^{n} E(x_i-\mu)^4 + \sum_{i\neq j} E\left((x_i-\mu)^2\right) E\left((x_j-\mu)^2\right)\right]$$

[other terms are ommitted as those would involve $E(x_i-\mu)=0$]

$$= \frac{\mu_4}{n^3} + \frac{n(n-1)\,\mu_2^2}{n^4}$$

$$\approx \frac{25\cancel{0}}{20\times20\times2\cancel{0}} + \frac{25}{20\times 16\times 20} \;\cancel{= \frac{25}{4\times 20}\left(\frac{1}{20}+\frac{1}{64}\right) = \frac{9\times 25}{20\times4\times25}}\; \frac{9}{80}$$

$$= \frac{\cancel{25}}{\cancel{400}_{16}}\left(\frac{1}{2}+\frac{1}{16}\right) = \frac{1}{32}\left(1+\frac{1}{8}\right) = \frac{9}{8\times 32} \approx 0.0352$$

Q10: By WLLN we have $\bar{X}_n \xrightarrow{P} \mu$.

By continuous mapping $\bar{X}_n^2 \xrightarrow{P} \mu^2$.

By CLT $\cancel{\frac{\sqrt{n}(\bar{X}_n-\mu)}{\sigma} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x}$

$$\frac{\sum x_i^2 - n(\sigma^2+\mu^2)}{\sqrt{n\,\mathrm{var}(x_i^2)}} \xrightarrow{d} Z \sim N(0,1)$$

$$\therefore \frac{\frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i^2 - \sqrt{n}(\sigma^2+\mu^2)}{\sqrt{\mathrm{var}(x_i^2)}} \xrightarrow{d} Z$$

$$\therefore \frac{1}{\sqrt{n}}\sum_{i=1}^{n} x_i^2 - \sqrt{n}(\sigma^2+\mu^2) \xrightarrow{d} \sqrt{\mathrm{var}(x_i^2)}\; Z$$

$$\left[Var(x_i^2) = E(x_i^4) - (\mu^2+\sigma^2)^2 = \mu_4 - (\mu^2+\sigma^2)^2 \right]$$

By Slutsky's lemma,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i^2 + \bar{x}_n^2 - \sqrt{n}(\sigma^2+\mu^2) \xrightarrow{d} \sqrt{\mu_4 - (\mu^2+\sigma^2)}\, Z + \mu^2.$$

$$\therefore\ a_n = \sqrt{n}(\sigma^2+\mu^2) \quad \text{and} \quad W = \sqrt{\mu_4 - \mu^2 - \sigma^2}\, Z + \mu^2$$

$$\sim N(\mu^2,\ \mu_4 - \mu^2 - \sigma^2)$$