Lecture 15: L_p Spaces and Inequalities

L_p Spaces

Definition 1 (L_p Space)**.** *Let $p \geq 1$. The L_p space is defined as:*

$$L_p = \{X : \Omega \to \mathbb{R} : E[|X|^p] < \infty\}$$

L_p is a linear space / vector space.

If $X_n \in L_p$, then we say $X_n \xrightarrow{L_p} X$, where $X \in L_p$, if $E[|X_n - X|^p] \to 0$ as $n \to \infty$.

The L_p Norm

We define a 'norm' $||\cdot||_p : L_p \to [0, \infty)$

$$X \mapsto ||X||_p = (E[|X|^p])^{\frac{1}{p}}$$

Properties:

1. $||X||_p \geq 0$
2. $||c \cdot X||_p = |c| \cdot ||X||_p$
3. $||X||_p = 0 \iff X = 0$ a.s. [Warning : It is not X=0]

Triangle Inequality

Is the following true?

$$||X + Y||_p \leq ||X||_p + ||Y||_p$$

Inequalities

1 Holder's Inequality

Let $p \geq 1$ and $q \geq 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

Holder pair (p,q) conjugate $q = \frac{p}{p-1}$	(p, q)	Note: $p = 2 \to L_2$ $q = 2$ unique s.t. when $p = q$.

Then, $E[|X \cdot Y|] \leq ||X||_p \cdot ||Y||_q$ Proof take $p = q = 2$.

$$\text{corr. } |E[X \cdot Y]| \leq s.E(X) \cdot s.E(Y)$$
$$|E[X \cdot Y]| \leq \sqrt{E[X^2]E[Y^2]}$$

Cauchy Schwarz Inequality

Inequality 1 (Minkowski's Inequality)**.** *Let $p \geq 1$, then*

$$||X+Y||_p \leq ||X||_p + ||Y||_p$$

Corollary 1. *The triangle inequality for L_p ($||\cdot||_p$) holds whenever $p \geq 1$.*

Proof of Minkowski's Inequality

Step 1: There is nothing to prove if $||X||_p = \infty$ or $||Y||_p = \infty$.

Step 2: It is enough to prove the case when $||X||_p < \infty$ and $||Y||_p < \infty$. Note that if $X, Y \in L_p$, then $||X+Y||_p < \infty$.

Furthermore, without loss we can assume $X, Y \geq 0$. Why? Because $||X+Y||_p = (E[|X+Y|^p])^{\frac{1}{p}}$, and we know $||X||_p$ are dealt with absolute values. If we can prove the inequality for the non-negative random variables, consider $|X|$ and $|Y|$ etc:

$$||\,|X|+|Y|\,|| \leq ||\,|X|\,|| + ||\,|Y|\,||$$

so we have to just show:

$$(E[(|X|+|Y|)^p])^{\frac{1}{p}} \leq (E[|X|^p])^{\frac{1}{p}} + (E[|Y|^p])^{\frac{1}{p}}$$

Then, since $E[|X+Y|^p] \leq E[(|X|+|Y|)^p]$, the result follows.

So, let's assume $X, Y \geq 0$ and prove:

$$(E[(X+Y)^p])^{\frac{1}{p}} \leq (E[X^p])^{\frac{1}{p}} + (E[Y^p])^{\frac{1}{p}}$$

Consider $E[(X+Y)^p]$:

$$\begin{aligned} E[(X+Y)^p] &= E[(X+Y)(X+Y)^{p-1}] \\ &= E[X(X+Y)^{p-1} + Y(X+Y)^{p-1}] \\ &= E[X(X+Y)^{p-1}] + E[Y(X+Y)^{p-1}] \end{aligned}$$

Apply Hölder's Inequality to each term. Let $V = (X+Y)^{p-1}$. We need the conjugate exponent q for p: $\frac{1}{p} + \frac{1}{q} = 1 \implies q = \frac{p}{p-1}$. Note that $(p-1)q = p$.

$$E[X \cdot V] \leq ||X||_p \cdot ||V||_q$$
$$E[Y \cdot V] \leq ||Y||_p \cdot ||V||_q$$

Summing them:

$$E[(X+Y)^p] \leq ||X||_p \cdot ||V||_q + ||Y||_p \cdot ||V||_q$$
$$E[(X+Y)^p] \leq (||X||_p + ||Y||_p) \cdot ||V||_q$$

Now, let's substitute $||V||_q$:

$$||V||_q = (E[|V|^q])^{\frac{1}{q}} = \left(E[((X+Y)^{p-1})^q]\right)^{\frac{1}{q}} = (E[(X+Y)^p])^{\frac{1}{q}}$$

So,

$$E[(X+Y)^p] \leq (||X||_p + ||Y||_p)\,(E[(X+Y)^p])^{\frac{1}{q}}$$

Since $\frac{1}{q} = 1 - \frac{1}{p}$:

$$E[(X+Y)^p] \leq (||X||_p + ||Y||_p)\,(E[(X+Y)^p])^{1-\frac{1}{p}}$$

If $E[(X+Y)^p] = 0$, the inequality holds. If not, we can divide:

$$\frac{E[(X+Y)^p]}{(E[(X+Y)^p])^{1-\frac{1}{p}}} \leq ||X||_p + ||Y||_p$$

$$(E[(X+Y)^p])^{1-(1-\frac{1}{p})} \leq ||X||_p + ||Y||_p$$

$$(E[(X+Y)^p])^{\frac{1}{p}} \leq ||X||_p + ||Y||_p$$

$$||X+Y||_p \leq ||X||_p + ||Y||_p \quad \square$$

Proof of Hölder's Inequality

To show: Take $p \geq 1$,
$E[|XY|] \leq ||X||_p \cdot ||Y||_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.
Let $A = ||X||_p$ and $B = ||Y||_q$.
Claim: WLOG, we can assume $0 < A, B < \infty$. Why?

- If $A = 0$, then $X = 0$ a.s., so $XY = 0$ a.s., and similarly B is also trivial.
- If $A = \infty$ or $B = \infty$, the RHS is ∞, and the inequality holds trivially.

Without loss we can assume X,Y$\geq$ 0,why?
We can just use |X|,|Y| otherwise. Note |XY|=|X|.|Y|
To show $E[X \cdot Y] \leq (E[X^p])^{\frac{1}{p}} \cdot (E[Y^q])^{\frac{1}{q}}$ when $p \geq 1$, & $\frac{1}{p} + \frac{1}{q} = 1$.

& furthermore $0 < E[X^p], E[Y^q] < \infty$

Lemma 1 (From Analysis)**.** $p \geq 1$ $a, b \geq 0$

(Digression: Convex Function) A function $f : \mathbb{R} \to \mathbb{R}$ *is called* ***CONVEX*** *if* $\forall x_1, x_2$ *and* $\alpha \in [0, 1]$, $\forall x_2 \geq x_1$*:*

$$f(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha f(x_1) + (1 - \alpha) f(x_2)$$