Statistical Inference

B. Statistical Data Science 2nd Year   Indian Statistical Institute

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Exercise Series 3 (Solutions)

Solution 1. 1(a) $p_\theta(\mathbf{x}) = p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}, \quad x_i \in \{0,1\}\,\forall i = 1,\ldots,n$. Therefore,

$$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = \frac{p^{\sum_{i=1}^n x_i}(1-p)^{n-\sum_{i=1}^n x_i}}{p^{\sum_{i=1}^n y_i}(1-p)^{n-\sum_{i=1}^n y_i}} = p^{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i}(1-p)^{\sum_{i=1}^n y_i - \sum_{i=1}^n x_i}.$$

This ratio is free of p if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ ($T(\mathbf{x}) = T(\mathbf{y})$ with $T(\mathbf{x}) = \sum_{i=1}^n x_i$). Therefore, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is minimal sufficient for p.

1(b) $p_\theta(\mathbf{x}) = e^{-n\lambda}\dfrac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}, \quad x_i \in \{0,1,2,\ldots\}\,\forall i = 1,\ldots,n,$. Therefore,

$$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}\frac{\prod_{i=1}^n y_i!}{\lambda^{\sum_{i=1}^n y_i}} = \lambda^{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i}\frac{\prod_{i=1}^n y_i!}{\prod_{i=1}^n x_i!}.$$

This ratio is free of λ if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. So, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is minimal sufficient for λ.

1(c) $p_\theta(\mathbf{x}) = p^n(1-p)^{\sum_{i=1}^n x_i - n}, \quad x_i \in \{1,2,\ldots\}\,\forall i = 1,\ldots,n$. So,

$$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = \frac{p^n(1-p)^{\sum_{i=1}^n x_i - n}}{p^n(1-p)^{\sum_{i=1}^n y_i - n}} = (1-p)^{\sum_{i=1}^n x_i - \sum_{i=1}^n y_i}.$$

This ratio is free of p if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Therefore, $T(\mathbf{X}) = \sum_{i=1}^n X_i$ is a minimal sufficient statistic for p.

1(d) For $\mathsf{Uniform}(\theta, 1)$, $\theta < 1$:

$p_\theta(\mathbf{x}) = \dfrac{1}{(1-\theta)^n}\mathbf{1}\{x_{(1)} > \theta\}\mathbf{1}\{x_{(n)} < 1\}$. Therefore,

$$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = \frac{\mathbf{1}\{x_{(1)} > \theta\}\mathbf{1}\{x_{(n)} < 1\}}{\mathbf{1}\{y_{(1)} > \theta\}\mathbf{1}\{y_{(n)} < 1\}}.$$

This ratio is free of θ if and only if $x_{(1)} = y_{(1)}$. So, $T(\mathbf{X}) = X_{(1)} = \min_{i=1,\ldots,n} X_i$ is minimal sufficient for θ.

For $\mathsf{Uniform}(\theta, \theta+1), \theta \in \mathbb{R}$:

$p_\theta(\mathbf{x}) = \mathbf{1}\{x_{(1)} > \theta\}\mathbf{1}\{x_{(n)} < \theta + 1\}$. $\dfrac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})}$ is free of θ if and only if $x_{(1)} = y_{(1)}$ and $x_{(n)} = y_{(n)}$, i.e., $(x_{(1)}, x_{(n)}) = (y_{(1)}, y_{(n)})$. So, $\left(X_{(1)}, X_{(n)}\right)$ is minimal sufficient for θ.

1(e) For $\mathsf{Normal}(0, \sigma^2), \theta = \sigma^2$:

$p_\theta(\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2}, \quad x_i \in \mathbb{R}, \forall i = 1, \ldots, n$. So, $\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2\right)}$ is free of the parameter if and only if $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2$. So, $\sum_{i=1}^n X_i^2$ is a minimal sufficient statistic for σ^2.

For $\mathsf{Normal}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$:

$p_\theta(\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^n x_i^2 + n\mu^2 - 2\mu\sum_{i=1}^n x_i\right\}}, \quad x_i \in \mathbb{R}, \forall i = 1, \ldots, n$. Therefore,

$$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})} = e^{-\frac{1}{2\sigma^2}\left\{\left(\sum_{i=1}^n x_i^2 - \sum_{i=1}^n y_i^2\right) - 2\mu\left(\sum_{i=1}^n x_i - \sum_{i=1}^n y_i\right)\right.}.$$

This ratio is free of the parameters if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2$. So, $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ is minimal sufficient for (μ, σ^2). Consequently, $\left(\bar{X}, S^2\right)$ is also minimal sufficient for (μ, σ^2). (Recall that $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right) \mapsto \left(\bar{X}, S^2\right)$ is a bijection)

1(f)

$$\begin{aligned} p_\theta(\mathbf{x}) &= \left(\frac{1}{\pi}\right)^n \prod_{i=1}^n \frac{\sigma}{\sigma^2 + (x_i - \mu)^2}, \quad x_i \in \mathbb{R}, \forall i = 1, \ldots, n \\ &= \left(\frac{1}{\pi}\right)^n \prod_{i=1}^n \frac{\sigma}{\sigma^2 + (x_{(i)} - \mu)^2}, \quad -\infty < x_{(1)} < x_{(2)} < \cdots < x_{(n)} < \infty. \end{aligned}$$

$\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})}$ is free of the parameters if and only if $x_{(i)} = y_{(i)}$ for all $i = 1, \ldots, n$. So, the order statistics $\left(X_{(1)}, \ldots, X_{(n)}\right)$ are minimal sufficient for θ irrespective of whether $\mu = 0$ or $\sigma = 1$.

Note: The sample $\mathbf{X}$ is not minimal sufficient. The order statistics are sufficient. But the sample $\mathbf{X}$ cannot be expressed as a function of the order statistics $\left(X_{(1)}, \ldots, X_{(n)}\right)$. This is because all possible permutations of the same sample leads to the same set of order statistics.

1(g)

$$\begin{aligned} p_\theta(\mathbf{x}) &= \frac{1}{(2b)^n} e^{-\frac{1}{b}\sum_{i=1}^n |x_i - a|}, \quad x_i \in \mathbb{R}, \forall i = 1, \ldots, n \\ &= \frac{1}{(2b)^n} e^{-\frac{1}{b}\sum_{i=1}^n |x_{(i)} - a|}, \quad -\infty < x_{(1)} < x_{(2)} < \cdots < x_{(n)} < \infty. \end{aligned}$$

If a is unknown, then the order statistics $\left(X_{(1)}, \ldots, X_{(n)}\right)$ are minimal sufficient. When $a = 0$, $T(\mathbf{X}) = \sum_{i=1}^n |X_i|$ is minimal sufficient for b.

1(h) $\mathsf{Normal}(\theta, \theta^2)$:

$p_\theta(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\theta^n} e^{-\left\{\frac{1}{2\theta^2}\sum_{i=1}^n x_i^2 + \frac{n}{2} - \frac{1}{\theta}\sum_{i=1}^n x_i\right\}}, \quad x_i \in \mathbb{R}, \forall i = 1, \ldots, n$. $\frac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})}$ is free of θ if and only if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ and $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2$. So, $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ is minimally sufficient for θ.

$\mathsf{Normal}(\theta, \theta)$:

$p_\theta(\mathbf{x}) = \dfrac{1}{(2\pi)^{n/2}} \dfrac{1}{\theta^{n/2}} e^{-\left\{\frac{1}{2\theta}\sum_{i=1}^n x_i^2 + \frac{n\theta}{2} - \sum_{i=1}^n x_i\right\}}, \quad x_i \in \mathbb{R}, \forall i = 1, \dots, n.$ $\dfrac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})}$ is free of θ if and only if $\sum_{i=1}^n x_i^2 = \sum_{i=1}^n y_i^2$. So, $\displaystyle\sum_{i=1}^n X_i^2$ is minimal sufficient for θ.

2. $p_\theta(x_1, x_2, x_3) = \mathbf{1}\{\min\{x_1+1, x_2, x_3-1\} > \theta\}\mathbf{1}\{\max\{x_1, x_2-1, x_3-2\} < \theta\}$. So, $\dfrac{p_\theta(x_1, x_2, x_3)}{p_\theta(y_1, y_2, y_3)}$ is free of θ if and only if $\min\{x_1+1, x_2, x_3-1\} = \min\{y_1+1, y_2, y_3-1\}$ and $\max\{x_1, x_2-1, x_3-2\} = \max\{y_1, y_2-1, y_3-2\}$. Therefore, $(\min\{X_1+1, X_2, X_3-1\}, \max\{X_1, X_2-1, X_3-2\})$ is minimal sufficient for θ.

3.

$$\begin{aligned}
p_\theta(\mathbf{x}) &= \begin{cases} \dfrac{1}{(2\theta)^n n!} & \text{if } -\theta < \dfrac{x_i}{i} - 1 < \theta, \forall i = 1, \dots, n, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{1}{(2\theta)^n n!} & \text{if } \left|\dfrac{x_i}{i} - 1\right| < \theta, \forall i = 1, \dots, n, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \dfrac{1}{(2\theta)^n n!} & \text{if } \max_{i=1,\dots,n} \left|\dfrac{x_i}{i} - 1\right| < \theta, \\ 0 & \text{otherwise,} \end{cases} \\
&= \frac{1}{(2\theta)^n n!} \mathbf{1}\left\{\max_{i=1,\dots,n} \left|\frac{x_i}{i} - 1\right| < \theta\right\}.
\end{aligned}$$

$\dfrac{p_\theta(\mathbf{x})}{p_\theta(\mathbf{y})}$ is free of θ if and only if $\displaystyle\max_{i=1,\dots,n} \left|\frac{x_i}{i} - 1\right| = \max_{i=1,\dots,n} \left|\frac{y_i}{i} - 1\right|$. So, a minimal sufficient statistic for θ is $\displaystyle\max_{i=1,\dots,n} \left|\frac{X_i}{i} - 1\right|$.

4. $p_{\boldsymbol\theta}(\mathbf{x}, \mathbf{y}) = \dfrac{1}{(\theta_3 - \theta_1)^n (\theta_4 - \theta_2)^n} \mathbf{1}\{x_{(1)} > \theta_1\}\mathbf{1}\{x_{(n)} < \theta_3\}\mathbf{1}\{y_{(1)} > \theta_2\}\mathbf{1}\{y_{(n)} < \theta_4\}$. Therefore, $\dfrac{p_{\boldsymbol\theta}(\mathbf{x}, \mathbf{y})}{p_{\boldsymbol\theta}(\mathbf{x}', \mathbf{y}')}$ is free of $\boldsymbol\theta$ if and only if $x_{(1)} = x'_{(1)}$, $y_{(1)} = y'_{(1)}$, $x_{(n)} = x'_{(n)}$ and $y_{(n)} = y'_{(n)}$. So, $X_{(1)}, Y_{(1)}, X_{(n)}, Y_{(n)}$ are jointly minimal sufficient for $\theta_1, \theta_2, \theta_3, \theta_4$.

5. Without any additional assumption, the joint pmf of the data is $p_{\mathtt{A}}^{N_{\mathtt{A}}} p_{\mathtt{B}}^{N_{\mathtt{B}}} p_{\mathtt{AB}}^{N_{\mathtt{AB}}} p_{\mathtt{O}}^{500 - N_{\mathtt{A}} - N_{\mathtt{B}} - N_{\mathtt{AB}}}$ (recall that $N_{\mathtt{A}} + N_{\mathtt{B}} + N_{\mathtt{AB}} + N_{\mathtt{O}} = 500$). Therefore, the ratio of the pmfs at two different samples is

$$p_{\mathtt{A}}^{N_{\mathtt{A}} - N'_{\mathtt{A}}} p_{\mathtt{B}}^{N_{\mathtt{B}} - N'_{\mathtt{B}}} p_{\mathtt{AB}}^{N_{\mathtt{AB}} - N'_{\mathtt{AB}}} p_{\mathtt{O}}^{N'_{\mathtt{A}} - N_{\mathtt{A}} + N'_{\mathtt{B}} - N_{\mathtt{B}} + N'_{\mathtt{AB}} - N_{\mathtt{AB}}}.$$

Here, $N'_{\mathtt{A}}, N'_{\mathtt{B}}, N'_{\mathtt{AB}}$ are based on the second sample. The ratio is free of the parameters $p_{\mathtt{A}}, p_{\mathtt{B}}, p_{\mathtt{AB}}, p_{\mathtt{O}}$ if and only if $N_{\mathtt{A}} = N'_{\mathtt{A}}$, $N_{\mathtt{B}} = N'_{\mathtt{B}}$, and $N_{\mathtt{AB}} = N'_{\mathtt{AB}}$. So, a minimal sufficient statistic for the parameters $(p_{\mathtt{A}}, p_{\mathtt{B}}, p_{\mathtt{AB}}, p_{\mathtt{O}})$ is $(N_{\mathtt{A}}, N_{\mathtt{B}}, N_{\mathtt{AB}})$.

Note: $(N_{\mathtt{A}}, N_{\mathtt{B}}, N_{\mathtt{O}})$, $(N_{\mathtt{B}}, N_{\mathtt{AB}}, N_{\mathtt{O}})$, $(N_{\mathtt{A}}, N_{\mathtt{AB}}, N_{\mathtt{O}})$ are also minimal sufficient.

With the additional information/modeling assumption in part (c), the updated joint pmf is

$$q_{\mathtt{A}}^{N_{\mathtt{A}} + N_{\mathtt{AB}}} q_{\mathtt{B}}^{N_{\mathtt{B}} + N_{\mathtt{AB}}} (1 - q_{\mathtt{A}})^{500 - N_{\mathtt{A}} - N_{\mathtt{AB}}} (1 - q_{\mathtt{B}})^{500 - N_{\mathtt{B}} - N_{\mathtt{AB}}}.$$

In this case, $(N_{\mathtt{A}} + N_{\mathtt{AB}}, N_{\mathtt{B}} + N_{\mathtt{AB}})$ is minimal sufficient.

Note: Notice that $N_{\mathtt{A}}+N_{\mathtt{AB}}$ is the total number of individuals with antigen A present and $N_{\mathtt{B}}+N_{\mathtt{AB}}$ is the total number of individuals with antigen B present. With the additional modeling assumption, we only need to consider these total numbers, which is similar to the Bernoulli case.

Solution 2. (a)

$$p_{\alpha,\beta}(\mathbf{x}) = \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n \prod_{i=1}^n \left[x_i^{\alpha-1}(1-x_i)^{\beta-1}\mathbf{1}\{x_i \in (0,1)\}\right]$$
$$= \left(\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} \left(\prod_{i=1}^n (1-x_i)\right)^{\beta-1} \mathbf{1}\{0 < x_{(1)} < x_{(n)} < 1\}.$$

$\dfrac{p_{\alpha,\beta}(\mathbf{x})}{p_{\alpha,\beta}(\mathbf{y})}$ is free of (α,β) if and only if $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$ and $\prod_{i=1}^n (1-x_i) = \prod_{i=1}^n (1-y_i)$. So, $\left(\prod_{i=1}^n X_i, \prod_{i=1}^n (1-X_i)\right)$ is minimal sufficient for (α,β).

Note: Equivalently, the *geometric means* $\overline{X}_{\mathrm{G}} = \left(\prod_{i=1}^n X_i\right)^{1/n}$ and $\overline{(1-X)}_{\mathrm{G}} = \left(\prod_{i=1}^n (1-X_i)\right)^{1/n}$ are minimal sufficient for (α,β).

(b)

$$p_{\alpha,\lambda}(\mathbf{x}) = \left(\frac{\lambda^\alpha}{\Gamma(\alpha)}\right)^n \prod_{i=1}^n \left[x_i^{\alpha-1} e^{-\lambda x_i} \mathbf{1}\{x_i > 0\}\right]$$
$$= \left(\frac{\lambda^\alpha}{\Gamma(\alpha)}\right)^n \left(\prod_{i=1}^n x_i\right)^{\alpha-1} e^{-\lambda \sum\limits_{i=1}^n x_i} \mathbf{1}\{x_{(1)} > 0\}.$$

The ratio $\dfrac{p_{\alpha,\lambda}(\mathbf{x})}{p_{\alpha,\lambda}(\mathbf{y})}$ is free of (α,λ) if and only if $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Thus, $\left(\prod_{i=1}^n X_i, \sum_{i=1}^n X_i\right)$ is minimal sufficient for (α,λ).

Note: The arithmatic mean (AM) and the geometric mean (GM) are jointly sufficient for the unknown parameters.

(c)

$$p_{\mu,\alpha}(\mathbf{x}) = \prod_{i=1}^n \left[\frac{\alpha\mu^\alpha}{x_i^{\alpha+1}} \mathbf{1}\{x_i \geq \mu\}\right] = \alpha^n \mu^{n\alpha} \left(\frac{1}{\prod_{i=1}^n x_i}\right)^{\alpha+1} \mathbf{1}\{x_{(1)} \geq \mu\}.$$

$\dfrac{p_{\mu,\alpha}(\mathbf{x})}{p_{\mu,\alpha}(\mathbf{y})}$ is free of (μ,α) if and only if $\prod_{i=1}^n x_i = \prod_{i=1}^n y_i$ and $x_{(1)} = y_{(1)}$. So, a minimal sufficient statistic for (μ,α) is $\left(\prod_{i=1}^n X_i, X_{(1)}\right)$.

(d)

$$p_{\lambda,k}(\mathbf{x}) = \prod_{i=1}^n \left[\frac{\lambda}{k}\left(\frac{x_i}{\lambda}\right)^{k-1} e^{-\left(\frac{x_i}{\lambda}\right)^k} \mathbf{1}\{x_i > 0\}\right] = \frac{\lambda^{n(2-k)}}{k^n} \left(\prod_{i=1}^n x_i\right)^{k-1} e^{-\frac{1}{\lambda^k}\sum_{i=1}^n x_i^k} \mathbf{1}\{x_{(1)} > 0\}.$$

A minimal sufficient statistic for this model is $(X_{(1)},\ldots,X_{(n)})$.

Note: It may be tempting to conclude that $\left(\prod_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^k\right)$ is minimal sufficient. But, $\sum_{i=1}^{k} X_i^k$ is NOT a statistic, as it involves unknown parameter k. This is not the case if k is known, in which case the only unknown parameter is λ. In that case, $\sum_{i=1}^{n} X_i^k$ is minimal sufficient for λ.

Solution 3. The joint pdf of $(X_1, Y_1), \ldots, (X_n, Y_n)$ is

$$
\begin{aligned}
p_\theta(\mathbf{x}, \mathbf{y}) &= \prod_{i=1}^{n} \left[\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \right. \\
&\qquad \left. \exp\left\{ -\frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x_i-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x_i-\mu_x}{\sigma_x}\right)\left(\frac{y_i-\mu_y}{\sigma_y}\right) + \left(\frac{y_i-\mu_y}{\sigma_y}\right)^2 \right\}\right\}\right] \\
&= \left(\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\right)^n \times \\
&\qquad \exp\left\{ -\frac{1}{2(1-\rho^2)} \left\{ \sum_{i=1}^{n} \frac{x_i^2}{\sigma_x^2} + \sum_{i=1}^{n} \frac{y_i^2}{\sigma_y^2} - 2\mu_x \sum_{i=1}^{n} \frac{x_i}{\sigma_x^2} - 2\mu_y \sum_{i=1}^{n} \frac{y_i}{\sigma_y^2} \right.\right. \\
&\qquad\qquad \left.\left. -2\rho \sum_{i=1}^{n} \frac{x_i y_i}{\sigma_x\sigma_y} + \frac{2\rho}{\sigma_x\sigma_y}\left(\mu_x \sum_{i=1}^{n} y_i + \mu_y \sum_{i=1}^{n} x_i\right) \right\}\right\}
\end{aligned}
$$

(a) All the parameters are unknown: $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_iY_i\right)$ is minimal sufficient for $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

(b) $\sigma_x = \sigma_y = 1$: $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_iY_i\right)$ is minimal sufficient for (μ_x, μ_y, ρ).

Note: We cannot ignore the $\sum_{i=1}^{n} x_i^2$ and $\sum_{i=1}^{n} y_i^2$ terms since they are multiplied with $\dfrac{1}{(1-\rho^2)}$.

(c) $\mu_x = \mu_y = 0$: $\left(\sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_iY_i\right)$ is minimal sufficient for $(\sigma_x^2, \sigma_y^2, \rho)$.

Note: The terms $\sum_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} y_i$ vanish since $\mu_x = \mu_y = 0$. This won't be the case if μ_x and μ_y are known non-zero constants.

(d) $\rho = 0$: $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2\right)$ is sufficient for $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2)$.

(e) $\mu_x = \mu_y = \mu$, $\mu \in \mathbb{R}$ unknown: $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} X_i^2, \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_iY_i\right)$ is sufficient for $(\mu, \sigma_x^2, \sigma_y^2, \rho)$.

(f) $\mu_x = \mu_y = \mu$, $\sigma_x^2 = \sigma_y^2 = \sigma^2$, $\mu \in \mathbb{R}, \sigma > 0$ unknown: $\left(\sum_{i=1}^{n} X_i + \sum_{i=1}^{n} Y_i, \sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} Y_i^2, \sum_{i=1}^{n} X_iY_i\right)$ is minimal sufficient for (μ, σ^2, ρ).

Note: The exponent in this case is

$$
-\frac{1}{2(1-\rho^2)}\left\{ \frac{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}{\sigma^2} - \frac{2\mu(1-\rho)}{\sigma^2}\left(\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} y_i\right) - \frac{2\rho}{\sigma^2}\sum_{i=1}^{n} x_i y_i \right\}.
$$

(g) $\mu_x = \mu_y = \theta$, $\sigma_x = \sigma_y = \theta^2$, $\theta \in \mathbb{R}$ unknown: $\left(\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i, \sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i\right)$ is minimal sufficient for (θ, ρ).

(h) $\mu_x = \mu_y = \theta$, $\sigma_x^2 = \sigma_y^2 = \theta$, $\theta > 0$ unknown: $\left(\sum_{i=1}^n X_i + \sum_{i=1}^n Y_i, \sum_{i=1}^n X_i^2 + \sum_{i=1}^n Y_i^2, \sum_{i=1}^n X_i Y_i\right)$ is minimal sufficient for (θ, ρ).

Note: It may be tempting to think that if there are k unknown parameters in a problem, then a minimal sufficient statistic must be k-dimensional. While this is true in some cases, but not in general, as can be seen from the above examples. For bivariate (and, more generally, multivariate) distributions, the dependence (here, correlation) between the components can be extremely important. Notice that if $\rho = 0$ in the above example, then you will get further reduction in some of the cases.