Solution of Quiz-1:

1. $T : V \to V$, V is a complex inner product space.

s.t $\langle Tv, v\rangle = 0 \;\; \forall\, v \in V$.

$$0 = \langle T(v+w), v+w\rangle = \langle Tv + Tw, v+w\rangle$$

$$= \underbrace{\langle Tv, v\rangle}_{=0} + \langle Tv, w\rangle + \langle Tw, v\rangle + \underbrace{\langle Tw, w\rangle}_{=0}$$

$$= \langle Tv, w\rangle + \langle Tw, v\rangle$$

$\Rightarrow$ $\langle Tv, w\rangle + \langle Tw, v\rangle = 0. \;\; \forall\, v, w \in V \to (1)$

Replacing w by $\underline{iw}_{\in V}$ in (1) yields

$$\langle Tv, iw\rangle + \langle T(iw), v\rangle = 0.$$

$$\bar{i}\,\langle Tv, w\rangle + i\,\langle Tw, v\rangle = 0$$

$$i\left(-\langle Tv, w\rangle + \langle Tw, v\rangle\right) = 0$$

$\Rightarrow$ $-\langle Tv, w\rangle + \langle Tw, v\rangle = 0 \to (2)$.

Adding (1) + (2) $\Rightarrow$ $2\langle Tw, v\rangle = 0$ $\Rightarrow$ $\langle Tw, v\rangle = 0 \;\forall\, v, w \in V$

letting $v = Tw \Rightarrow$ $\langle Tw, Tw\rangle = 0$ $\Rightarrow$ $Tw = 0 \;\forall\, w \in V$

i.e $T \equiv 0$.

For the real inner product spaces the preceding conclusion is false.

$T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x) := (-x_2, x_1) \;\; \forall\; x = (x_1, x_2) \in \mathbb{R}^2$

$$\langle Tx, x\rangle = \langle (-x_2, x_1), (x_1, x_2)\rangle \overset{\text{standard}}{=} -x_2 x_1 + x_1 x_2 = 0.$$

$\forall\, x \in \mathbb{R}^2$

2. Orthogonally diagonalize $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Soln:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & -2 \\ -2 & \lambda-1 \end{vmatrix}$$

$$= (\lambda-1)^2 - 4 = (\lambda-3)(\lambda+1)$$

For the eigenvalue $\lambda_1 = 3$, the system $(\lambda_1 I - A)u = 0$.

E_{λ_1} = Eigenspace w.r.t. λ_1 = span $\{(1,1)\}$.

Normalize eigen vector for $u = (1,1)$ is $\frac{1}{\sqrt{2}}(1,1)$

$$= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

For the eigenvalue $\lambda_2 = -1$, the homogeneous sys $(\lambda_2 I - A)v = 0$

E_{λ_2} = span $\{(-1,1)\}$

Normalize eigenvector $= \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$P = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$. Then $P^T A P = \frac{1}{2}\begin{pmatrix} 6 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$.

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3. ExR in the Class..