Indian Statistical Institute

BSDS Ist Year

Academic Year 2024 - 2025: Semester I

Course: Probability Theory I

Instructor: Antar Bandyopadhyay

Final Examination

Date: December 11, 2024 — **Total Points:50**
Time: 10:30 AM - 01:30 PM — **Duration: 180 minutes**

Note: There are 8 problems each carrying 8 points and thus with a total of 64 points. Solve as many as you can. Maximum you can score is 50. Show all your works. Use only the pages provided. The rough works must be done on the pages provided for the same. Do not forget to write your name and roll number. Also circle the location/centre you are attending. Good luck!

Name: ______________________ **Roll No.:** ______________________

Location/Centre: Kolkata Delhi Bangalore

Please DO NOT write any thing below

For Instructor/TA use only

Problems	1	2	3	4	5	6	7	8	Total
Points									

[Please Only Turn at 10:30 AM]

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1. An urn contains 50 green balls, 25 red balls and 25 yellow balls. Balls are being selected out of the urn **with replacement**. The sampling is done till 10 balls of color red are obtained. Let X be the total number of sample and Y be the number of green balls in the sample.

   (a) Find $\mathbf{E}[X]$. [4]

(b) What is the conditional distribution of Y given $X = 50$. [4]

2. Suppose a random number U is generated out of Uniform $(0, 1)$ distribution. Once U is observed, say $U = u$, then we toss a coin with probability of head u, repeatedly till we get a head. Let X be the number of tosses before the first head.

   (a) Find the *marginal distribution* of X. [6]

(b) Using (a) or otherwise compute $\mathbf{E}[X]$. [2]

3. Suppose X and Y are independent and identically distributed Normal $(0, 1)$ random variables. Find the *probability density function* and the *cumulative distribution function* of the random variable

$$Z = \frac{X}{|Y|}.$$

$[4 + 4 = 8]$

4. Balls are being drawn at random and *without replacement* from a box containing n balls which are numbers $1, 2, \ldots, n$. Let X_i be the number on the ball drawn at the i^{th}-draw.

   (a) Find the distribution of X_1. [2]

   (b) Find $\operatorname{Corr}(X_1, X_2)$. [3]

(c) Find $\mathbf{Var}\left(X_1 + X_2 + \cdots + X_{n-1}\right)$. [3]

5. A chestnut drawer has two drawers. 10 different pairs of socks are randomly placed in the two drawers.

   (a) Let N be the number of complete pairs of socks in the first drawer. Find the distribution of N. [4]

(b) What is the probability that each drawer has at least one complete pair of socks? [4]

6. Suppose X and Y are independent random variable with distribution Exponential (λ), where $\lambda > 0$, Find the *probability density function (pdf)* of $Z = X - Y$. [8]

7. Suppose (X, Y) is a pair of random variables with joint density given by

$$f(x, y) = \begin{cases} c\,x\,y\,(1 - x - y) & \text{if } 0 < x,\ y,\ x + y < 1; \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find c. [2]

(b) Find the marginal distribution of X. [2]

(c) Find $\mathbf{E}\left[Y \,\middle|\, X = x\right]$ where $0 < x < 1$. [4]

8. Find the *probability density function (pdf)* of the random variable X, such that, $\log X \sim \text{Normal}\,(0, 1)$. [8]

Rough Work Page

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