Indian Statistical Institute

BSDS IInd Year

Academic Year 2025 - 2026: Semester I

Course: Probability II

Instructor: Antar Bandyopadhyay

Assignment # 1

Date Given: August 13, 2025 **Date Due: August 22, 2025**
Total Points: 10

6.5.4 Suppose X and Y are two standard normal variables. find an expression for $\mathbf{P}(x + 2Y \leq 3)$ in terms of the standard normal distribution function Φ,

(a) in case when X and Y are independent; and

(b) in case when X and Y have *bivariate normal* distribution with correlation $1/2$.

6.5.6 Let X and Y be independent standard normal variables.

(a) For a constant k, find $\mathbf{P}(X > kY)$.

(b) If $U = \sqrt{3}X + Y$ and $V = X - \sqrt{3}Y$, find $\mathbf{P}(U > kV)$.

(c) Find $\mathbf{P}\left(U^2 + V^2 < 1\right)$.

(d) find the conditional distribution of X given $V = v$.

6.5.10 Show that if V and W have a bivariate normal distribution then

(a) every linear combination $aV + bW$ has a normal distribution;

(b) every pair of linear combinations $(aV + bW, cV + dW)$ has a bivariate normal distribution;

(c) Find the parameters of the distributions obtained in (a) and (b) above in terms of the parameters of the joint distribution of V and W.

6.5.11 Show that for standard bivariate normal variables X and Y with correlation ρ,

$$\mathbf{E}\left[\max(X, Y)\right] = \sqrt{\frac{1-\rho}{\pi}}$$

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Assignment # 2

Date Given: August 20, 2025 | **Date Due: August 29, 2025**
Total Points: 10

Problem # 1 Use **R** software to present *contour plots* of the *bivariate normal density function* with five different choices of the mean vector and the variance-co-variance matrix. Printout of the plots preferably in colour should be submitted.

Problem # 2 Suppose (X, Y) be distributed as bivariate normal with means μ_X and μ_Y; variances σ_X^2 and σ_Y^2 respectively and correlation ρ. For $x, y \in \mathbb{R}$, find the conditional distribution of Y given $X = x$, and the conditional distribution of X given $Y = y$.

6.5.7(e) In the setup of the **Problem # 2** above, if $\mu_X = \mu_Y = 0$, then show that $X \cos\theta + Y \sin\theta$ and $Y \cos\theta - X \sin\theta$ are two independent random variables, if

$$\theta = \frac{1}{2} \cot^{-1} \left(\frac{\sigma_X^2 - \sigma_Y^2}{2\rho\sigma_X\sigma_Y} \right)$$

6.5.9 Suppose that $W \sim \text{Normal}\left(\mu, \sigma^2\right)$ distribution. Given that $W = w$, suppose Z has conditional distribution as $\text{Normal}\left(aw + b, \tau^2\right)$.

(a) Show that the joint distribution of W and Z is bivariate normal, and find its parameters.

(b) What is the marginal distribution of Z.

(c) What is the conditional distribution of W given $Z = z$.

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Assignment # 3

Date Given: August 27, 2025 **Date Due: September 05, 2025**
Total Points: 10

1. Suppose $X_1, X_2, \cdots, X_d$ be *i.i.d.* standard normal random variables. Let $\mathbf{X}$ be the random (column) vector with components $X_1, X_2, \cdots, X_d$ stacked as a column with d entries. Let A be a $d \times d$ *non-singular* matrix with real entries. Let $\mathbf{Y} := A\mathbf{X}$. Show that $\mathbf{Y}$ is a *multivariate normal* random vector with *mean vector* $\mathbf{0}$ and *variance co-variance matrix* AA^T.

2. In the setup of **Problem # 1** above, suppose $B_{k\times d}$ is a *rectangular* matrix with real entries which is *full row rank* (that is, rows are linearly independent). Let $\mathbf{W} := B\mathbf{X}$, then show that $\mathbf{W}$ is a *multivariate normal* random vector with *mean vector* $\mathbf{0}$ and *variance co-variance matrix* BB^T.

3. Show that if Σ is a *positive definite (p.d.)* matrix of order $d \times d$, then it is necessarily a variance-co-variance matrix of some random vector of dimension d. [**Hint:** Use Problem # 1 above].

4. Determine whether $\Sigma := ((\sigma_{ij}))_{1\leq i,j\leq d}$, where

$$\sigma_{ij} := \begin{cases} 1 & \text{if } i = j, \\ \rho & \text{otherwise} \end{cases}$$

   and $|\rho| < 1$, is a p.d. matrix. [**Hint:** Use Problem # 3 above].

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Assignment # 4

Date Given: September 03, 2025 **Date Due: September 12, 2025**
Total Points: 10

Problem # 1 Suppose $X_1, X_2, X_3, \cdots, X_n$ are *i.i.d.* Normal random variables with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$. Then show that the two *statistics*, $\bar{X}_n := \frac{1}{n}\sum\limits_{i=1}^{n} X_i$ and $s_n^2 := \frac{1}{n}\sum\limits_{i=1}^{n}\left(X_i - \bar{X}_n\right)^2$ are independent. Find the distributions of $\bar{X}_n$ and s_n^2.

Problem # 2 Give **five** different examples of non-negative integer valued discrete random variables with mean **infinity** and give **five** different examples of non-negative real valued continuous random variables with mean **infinity**.

6.3.24 A box contains four tickets, numbered $0, 1, 1$ and 2. Let S_n be the sum of the numbers obtained from n draws at random with replacement from the box.

(a) Display the distribution of S_n in a suitable table.

(b) Find $\mathbf{P}\left(S_{50} = 50\right)$ approximately.

(c) Find an exact formula for $\mathbf{P}\left(S_n = k\right)$ for $k = 0, 1, 2, \cdots$.

6.3.25 **Equality in Chebychev's Inequality:** Let μ, σ and k be three numbers, with $\sigma > 0$ and $k \geq 1$. Let X be a random variable with the following distribution:

$$\mathbf{P}\left(X = x\right) = \begin{cases} \frac{1}{2k^2} & \text{if } x = \mu + k\sigma \text{ or } x = \mu - k\sigma \\ 1 - \frac{1}{k^2} & \text{if } x = \mu \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the histogram of this distribution for $\mu = 0, \sigma = 10, k = 1, 2, 3$.

(b) Show that $\mathbf{E}\left[X\right] = \mu$, $\mathbf{Var}\left(X\right) = \sigma^2$ and $\mathbf{P}\left(|X - \mu| \geq k\sigma\right) = \frac{1}{k^2}$. (So there is an equality in Chebychev's inequality for this distribution of X. This means Chebychev's inequality cannot be improved without additional hypothesis on the distribution of X.)

(c) Show that if Y has $\mathbf{E}\left[Y\right] = \mu$, $\mathbf{Var}\left(Y\right) = \sigma^2$ and $\mathbf{P}\left(|Y - \mu| < \sigma\right) = 0$, then Y has the same distribution as X described above for $k = 1$.

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Assignment # 5

Date Given: September 10, 2025 **Date Due: September 19, 2025**
Total Points: 10

1. Suppose $A_1, A_2, A_3, \cdots, A_n, \cdots$ are events which are occurring *almost surely*. Then show that the events $\bigcap\limits_{i=1}^{n} A_i$ also occur *almost surely* for all $n \geq 1$. Further, determine whether the event $\bigcap\limits_{n=1}^{\infty} A_n$ also occur *almost surely*.

2. Give an example of a collection of events $\{A_\alpha\}_{\alpha \in \mathcal{I}}$, with some indexing set $\mathcal{I}$, such that, for each $\alpha \in \mathcal{I}$, the event A_α occurs almost surely, but $\bigcap\limits_{\alpha \in \mathcal{I}} A_\alpha = \emptyset$.

3. Suppose $\varepsilon_1, \varepsilon_2, \varepsilon_3, \cdots, \varepsilon_n$ be *i.i.d. standard normal* random variables. Define
$$X_1 = \varepsilon_1; \text{ and } X_{i+1} = X_i + \phi\, \varepsilon_{i+1}, i \geq 1$$
for some $\phi \in \mathbb{R}$. Find the joint distribution of $(X_1, X_2, X_3, \cdots X_n)$.

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Assignment # 6

Date Given: September 17, 2025 **Date Due: September 26, 2025**
Total Points: 10

Suppose $A_1, A_2, A_3, \cdots, A_n, \cdots$ are events and B be the event that $[A_n \text{ i.o.}]$ and C is the event that $[A_n \text{ eventually}]$. Then show that

1. $C \subseteq B$

2. $\mathbf{P}(B) = \lim\limits_{N \to \infty} \mathbf{P}\left(\bigcup\limits_{n=N}^{\infty} A_n\right)$, where the limit is *decreasing*.

3. $\mathbf{P}(C) = \lim\limits_{N \to \infty} \mathbf{P}\left(\bigcap\limits_{n=N}^{\infty} A_n\right)$, where the limit is *increasing*.

4. If f_n be the indicator function of the event A_n, then $\limsup\limits_{n \to \infty} f_n = \mathbf{1}_B$, and $\liminf\limits_{n \to \infty} f_n = \mathbf{1}_C$.

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Assignment # 7

Date Given: September 24, 2025 **Date Due: October 07, 2025**
Total Points: 10

1. Suppose $X_1, X_2, \ldots, X_n$ are i.i.d. outcomes of an *unbiased coin toss*. Show that

$$\mathbf{P}\left(X_n = H, X_{n+1} = H, \ldots, X_{n+99} = H \text{ i.o. }\right) = 1.$$

   In other words, probability of having 100 heads in succession infinitely often from repeated but independent unbiased coin toss is one.

2. For $(A_n)_{n\geq 1}$ a sequence of events show that

$$\mathbf{P}\left(A_n \text{ eventually }\right) \leq \liminf_{n\to\infty} \mathbf{P}\left(A_n\right) \leq \limsup_{n\to\infty} \mathbf{P}\left(A_n\right) \leq \mathbf{P}\left(A_n \text{ i.o. }\right)$$

3. Suppose we have a printed book with very large number of pages. Let X_n be the number of misprints in the book till page n. Show that there exists a constant $c > 0$, such that,

$$\frac{X_n}{n} \longrightarrow c \text{ a.s. as } n \to \infty$$

   State all your assumptions clearly and interpret the constant c.

4. Let $(U_n)_{n\geq 1}$ be i.i.d. Uniform $(0, 1)$ and $M_n := \max\left(U_1, U_2, \ldots, U_n\right)$. Show that $M_n \xrightarrow{\mathbf{P}} 1$ as $n \to \infty$. Hence or otherwise show that, in fact, $M_n \longrightarrow 1$ a.s. as $n \to \infty$.