§ Directed Graph

So far, we have been working with ~~graphs~~ graphs with undirected edges. A directed edge is an edge where the end points are distinguished - one is the head & one is the tail, In particular, a directed edge is specified as an ordered pair of vertices u, v & is denoted by (u,v) or $\underset{\text{Tail}}{u} \to \overset{\text{head}}{v}$

<u>Defn</u>: A directed graph $G = (V, E)$ consists of a non-empty set of nodes V & a set of directed edges E. Each edge e of E is specified by an ordered pair of vertices $u, v \in V$. A directed graph is simple if it has no loops and no multiple edges.

<u>Example</u>: $V = \{1,2,3\}$, $E = \{(1,2),(2,3),(3,1)\}$

<u>Remark</u>: ~~But both~~ Between any two distinct vertices u, v, there can be <u>atmost</u> one directed edge from u to v. (But both (u,v) & (v,u) are allowed, since they are in opposite direction.)

In a directed graph (digraph), we distinguish between two type of degrees because edges have a direction.

<u>Indegree</u>: The indegree of a vertex v, denoted by $\deg^-(v)$ is the ~~*~~ number of edges entering v.

$$\deg^-(v) = \left|\{(u,v) \in E : u \in v\}\right|$$

Out degree: The outdegree of a vertex v, denoted by $\deg^+(v)$ is the number of edges leaving v.

$$\deg^+(v) = |\{ (v,u) \in E : u \in V \}|$$

Total degree of v as: $\deg(v) = \deg^+(v) + \deg^-(v)$.

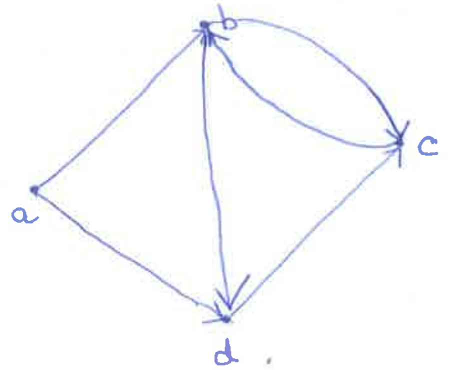


Fig-I

indegree (c) = 2
outdegree (c) = 1.

If a node has outdegree 0 it is called sink, if it has indegree 0, it is called source

node (a) is source

Defn: A directed walk (or more simply, a walk) in a directed graph G is a sequences of vertices $v_0, v_1, \ldots, v_k$ & edges

$$v_0 \to v_1,\ v_1 \to v_2,\ \ldots,\ v_{k-1} \to v_k$$

such that $v_{i-1} \to v_i$ is a edge of G.

- A directed path in a directed graph is a walk where the nodes in the walk are all different.

- A directed cycle in a directed graph is a closed walk where all the vertices v_i are different.

- ~~a→b, b→c, c→d~~ ~~a→b~~. c, b, a, d is not a walk in the graph Fig-I, since b→a is not an edge.

- a→b, b→c, c→b, b→d. – walk.

- a→b, b→d is a path

- b→d, d→c, c→b. is a cycle.

Def A path or cycle in a directed graph is said to be Hamiltonian if it visits every node in the graph.
$(a \to b, b \to d, d \to c)$ is the only Hamiltonian path in the graph (Fig-I). It has no Hamiltonian cycle.

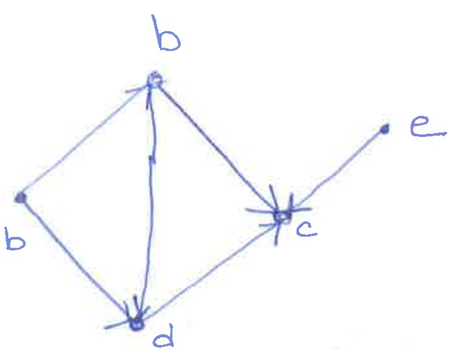
§ Strong Connectivity

A directed graph $G = (V, E)$ is said to be strongly connected if for every pair of nodes $u, v \in V$, there is a directed path from u to v.

(The Fig-I, is not strongly connected since there is no directed path from node b to node a.

A directed graph is said to be weakly connected if the corresponding undirected graph is connected. (Fig-I) is weakly connected.

Defn: A directed graph is called a directed Acyclic graph (or DAG) if it does not contain any directed cycles.

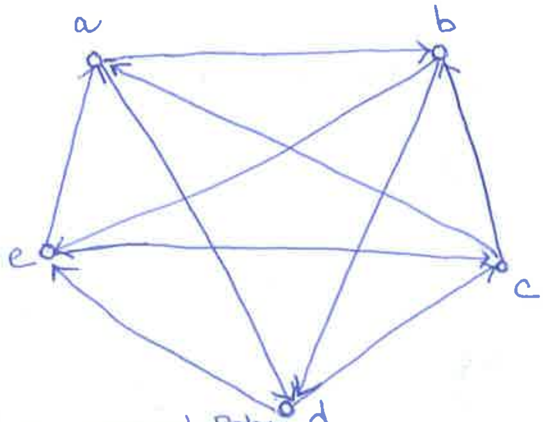


Remark: A directed, at a first glance don't appear to be particularly interesting & DAGs arise in many scheduling and optimization problems.

§ Tournament Graphs.

Suppose that n players compete in a round-robin tournament and that for every pair of players u & v, either u beats v or v beats u.

Interpreting the results of a round-robin tournament can be problematic - there might all sorts of cycles where x beats y & y beats z, yet z beats x. Who is the best player? Graph theory does not solve this problem but it can provide some interesting perspectives



5-node tournament Graph.

The results of a round-Robin tournament can be represented with a tournament graph. This is a directed graph in which the vertices represent players and the edges indicate the outcomes of the game. In particular, an edge from u to v indicates that player u defeated player v.

Theorem: Every tournament graph contains a directed Hamiltonian path.

Proof: Let $P(n)$ be the proposition that every tournament ~~Hamiltonian~~ graph with n-vertices contains a directed Hamiltonian path.

$P(1)$ is trivially true: every graph with a single vertex has a Hamiltonian path consisting of only that vertex.

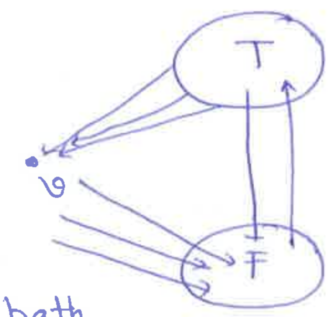
We assume $P(1), P(2), \ldots, P(n)$ are all true & prove that $P(n+1)$ is true.

Consider a tournament graph $G = (V, E)$ with $(n+1)$ players. Select one vertex "v" arbitrarily.

· Every other vertex in the tournament either has an edge to vertex v or an edge from vertex v.

· Thus we can partition the remaining vertices, into two corresponding sets, T & F, each containing atmost n-vertices, where $T = \{u : u \rightarrow v \in E\}$
$F = \{u : v \rightarrow u \in E\}$.

· The vertices in T together with edges that join them form a smaller tournament. Thus by induction hypothesis, there is a Hamiltonian path within T.



Similarly there is a Hamiltonian path within the tournament on the vertices in F.

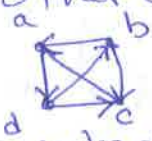
Joining the path in T to the vertex v followed by the path in F gives a Hamiltonian path through the whole tournament.

§ The King Chicken Theorem

Suppose that there are n-chickens in a farmland.
Chicken are rather aggresive birds that tends to establish dominance by pecking.
In particular, for each pair of distinct chickens, either the first pecks the second or the second pecks the first, but not both. We say that chicken u virtually pecks chicken v if either,

* Chicken u directly ~~pf~~ peaks chicken v, or
* Chicken u pecks some other chicken w who in turn pecks chicken v.

- A chicken that virtually pecks every other chicken is called a <u>king - Chicken</u>.



king $\{a, b, d\}$ are king.

(The vertices are chicken, and an edge $u \to v$ indicates that chicken u peaks chicken v).

<u>Theorem:</u> The chicken with the largest outdegree in an n-chicken tournament is a king.

<u>Pf.</u> By contradiction, Let u be a node in a tournament $G = (V, E)$ with maximum outdegree and suppose u is not a king.

Let $Y = \{v : u \to v \in E\}$. Then outdegree$(u) = |Y|$.

Since u is not a king, there is a chicken $x \notin Y$ and that is not pecked by any chicken in Y.

Since for any pair of chickens, one pecks the other $\Rightarrow$ x pecks u as well as every chicken in Y. $\Rightarrow$ outdeg $(x) = |Y| + 1$ $>$ outdeg(u)
$\rightarrow\leftarrow$.

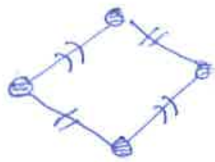
§. Matchings

Let $G = (V, E)$ be a graph.

Defn: A subset M of edges ($M \subseteq E$) is said to be Matching / Independent if no two edges are incident on any vertex or equivalently every vertex is contained in at most one edge. (no two edges in M share a common vertex).

Types of Matching

- A matching M is perfect if every vertex of G is incident with exactly one edge of M.



- Maximum Matching:
  A matching that contains the largest possible number of edges in the graph.

- Example:

$V = \{1, 2, 3, 4\}$, $E = \{\{1,2\}, \{2,3\}, \{3,4\}\}$

$M_1 = \{\{1,2\}\}$, $M_2 = \{\{2,3\}\}$.

- $M = \{\{1,2\}, \{3,4\}\}$.

It is also a maximum matching (size is 2).

* Bi-partite graph: Let G be bipartite Graph with $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, b_3\}$.

$E = \{\{a_1,b_1\}, \{a_1,b_2\}, \{a_2,b_2\}, \{a_3,b_3\}\}$

Possible Matchings

$M_1 = \{ \{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\} \}$

$M_2 = \{ \{a_1, b_2\}, \{a_3, b_3\} \}$.

- Alternatively, one can consider a matching of a graph M as a sub-graph of G such that $d_M(v) = 1$ for all $v \in V(M)$. A matching is complete if M is spanning. A vertex "$\underline{v}$" is said to be saturated if $v \in M$ and else unsaturated.

For a subset $S \subset V$, $N(S) = \bigcup_{v \in S} N(v)$. ($N$-denotes nbd of '$v$').

Theorem (Hall's marriage theorem; Hall, 1935).

Let G be a bi-partite graph with the two vertex ~~sets~~ sets V_1, V_2. Then there exists a complete matching on V_1 iff

$$|N(S)| \geq |S| \quad \text{for all } S \subset V_1.$$

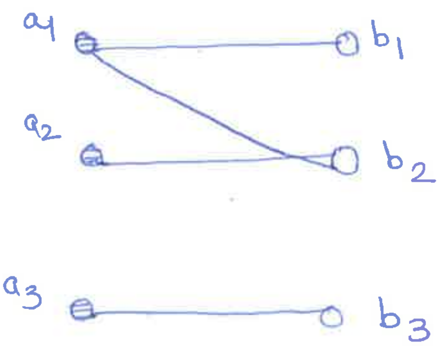
where $N(S) = \{ b \in V_2 : \exists\ a \in S \text{ with } (a,b) \in E \}$.

Example:

$\cancel{A =}$ $V_1 = \{a_1, a_2, a_3\}$, $V_2 = \{b_1, b_2, b_3\}$,

$E = \{(a_1, b_1), (a_1, b_2), (a_2, b_2), (a_3\ b_3)\}$

Now, let us check Hall's statment



$\cancel{\{a_1}$

$S = \{a_1\}$. , $N(S) = \{b_1, b_2\}$

$S = \{a_2\}$. , $N(S) = \{b_2\}$

$S = \{a_3\}$, $N(S) = \{b_3\}$

$S = \{a_1, a_2\}$, $N(S) = \{b_1, b_2\}$

$S = \{a_1, a_3\}$, $N(S) = \{b_1, b_2, b_3\}$

$S = \{a_1, a_2, a_3\}$, $N(S) = \{b_1, b_2, b_3\}$.

Matching is:

$M = \{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$.

Pf: Let $|V_1| = k$, and our proof will be by induction on k.

If $k = 1$, the proof is trivial.

Let $G = V_1 \cup V_2$ be such that the result holds for any graph with strictly smaller V_1.

Suppose that $|N(S)| \geq |S| + 1$. for all $S \subsetneq V_1$.

Then choose $(v, w) \in E \cap V_1 \times V_2$ & consider the induced subgraph $G' = \langle V - \{v, w\} \rangle$.

Since we have removed only w from V_2 and that

$|N(S)| \geq |S|+1$. $\forall \; S \subsetneq V_1$

we get $|N(S')| \geq |S'| \quad \forall \; S' \subset V_1 \setminus \{v\}$.

Thus there is a complete matching M on $V_1 \setminus \{v\}$ in G' by induction hypothesis & $M \cup \{(v,w)\}$ is a complete matching on V_1 in G as desired.

If the above is not true, there exists $A_\phi \subsetneq V_1$ such that $N(A) = B$. & $|A| = |B|$. Then by induction hypothesis, there is a complete ~~matin~~ matching M_0 on A in the induced subgraph $\langle A \cup B \rangle$. Trivially Hall's condition holds i.e., for all $S \subset A$, $|N(S) \cap B| = |N(S)| \geq |S|$.

Let $G' = G - \langle A \cup B \rangle$.

Let $S \subset V_1 \setminus A$.

Suppose if $|N'(S)| < |S|$ where $N'(S) = N(S) \cap (V_2 \setminus B)$

Then we have that

$$N(S \cup A) = [N(S) \cap (V_2 \setminus B)] \cup B.$$

and hence $|N(S \cup A)| \leq |N(S) \cap (V_2 \setminus B)| + |B|$
$=$
$< |S| + |A| \qquad (|A| = |B|)$

$(N(S \cup A) = N(S) \cup N(A))$

$|N(S \cup A)| \geq |S \cup A| = |S| + |A|$

$\rightarrow\leftarrow$.

Hence G' also satisfies Hall's condition and by induction hypothesis G' has a complete ~~matic~~ matching M' on $V_1 \setminus A$. Thus, we have a complete matching $M := M_0 \cup M'$ on V_1 in G.

<u>Proposition:</u> Let $d \geq 1$. Let G be a bipartite graph on $V_1 \sqcup V_2$ such that $|N(S)| \geq |S| - d$ for all $S \subset V_1$. Then G has a matching with atleast $|V_1| - d$ edges.

<u>Pf:</u> Set $V_2' := V_2 \cup \{\underbrace{1, 2, \dots d}_{\text{vertex}}\}$. Define $G' = V_1 \sqcup V_2'$ and edge set as $E(G) \cup (V_1 \times [d])$. Then it is easy to see that Hall's ($N(S) \geq |S|$) and hence there is a complete matching M of V_1 in G'. Now, if we remove the edges in M incident on $\{1, 2, \dots d\}$, we get a matching with atleast ~~$|V_1|$ edge~~ $|V_1| - d$ edges as required.

Defn: (Independent sets and covers):

An $\boxed{\text{independent set}}$ of vertices is $S \subset V$ such that no two vertices in S are adjacent.
A subset of vertices $S (\subset V)$ is a $\boxed{\text{vertex cover}}$ if every edge in G is incident to atleast one vertex in S. An $\boxed{\text{edge cover}}$ is a set of edges $E' \subseteq E$ such that every vertex is contained in atleast one in E'.

—

Definition: (Independence number and cover number)

$$\alpha(G) = \max\{ |S| : S \text{ independent vertex set} \}$$

$$\alpha'(G) = \max\{ |M| : M \text{ independent edge set} \}$$

or (matching)

$$\beta(G) = \min\{ |S| : S \text{ vertex cover} \}.$$

$$\beta'(G) = \min\{ |E'| : E' \text{ edge cover} \}.$$

Aim: We first derive some trivial relations between the four quantities.

If M is a maximal matching, then to cover each edge we need distinct vertices and hence the vertex cover should have size atleast M

pf: Assume S is independent. Take any edge $(u,v) \in E$. Since u, v are adjacent, they can-not both lie in the independent set S. Thus atleast one of u or v is not in S, i.e atleast one of u, v lies in S^c. Because this holds for every edge, every edge has an end point in S^c. Hence S^c is a vertex cover.

Conversely, Assume S^c is a vertex cover. Suppose assume S is not independent. Then there exist $u, v \in S$ with $(u,v) \in E$. But as S^c ~~comp~~ is vertex cover, every edge has an ~~ede~~ end point in S^c

$\Rightarrow$ u, v - end points to ~~S~~ outside S, not possible.

$$\Rightarrow \alpha(G) + \beta(G) = n.$$

Theorem: (König, Egervary, 1931). For a $\boxed{\text{Bi partite graph}}$

$$\alpha'(G) = \beta(G)$$

pf: We will show that for a minimal vertex cover Q, there exists a matching of size atleast $|Q|$.

Partition Q into $A := Q \cap V_1$ and $B := Q \cap V_2$.

Let H & H' be induced subgraphs on $\langle A \cup (V_2 - B) \rangle$ and $(V_1 - A) \cup B$ respectively. If we ~~show~~ show that there is a complete matching on A in H and a complete matching on B in H', we have a

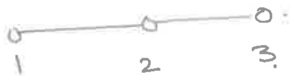
$\alpha'(G) \le \beta(G) \lesssim 2\alpha'(G)$

$\alpha(G) \le \beta'(G)$. (to cover vertices of an independent set, we need distinct edges).

<u>Lemma:</u> Let G be a graph. $S(\subseteq V)$ is an independent set iff S^c is a vertex cover. As a Corollary, we get $\alpha(G) + \beta(G) = n = |V|$.

<u>Example:</u>

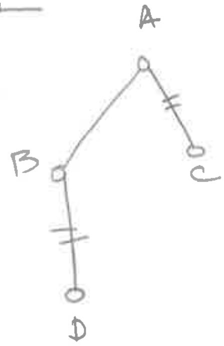
Let $G = (V, E)$, where $V = \{1,2,3\}$, $E = \{(1,2), (2,3)\}$



1. Independent set of vertices : $\{1,3\}$, $\{1\}$, $\{2\}$, $\{3\}$
2. Vertex Cover : $\{2\}$, ← min vertex cover. $\{1,2\}$, $\{2,3\}$ $\{1,2,3\}$.
3. Edge cover : $\{(1,2), (2,3)\}$.

<u>Observe:</u> $\left(\overset{\text{Maximum}}{\text{Independent set}}\right)^c$ = vertex cover.

<u>Example:</u>



$V = \{A, B, C, D\}$. $E = \{AB, AC, BD\}$.

Independent set: $\{C, D\}$, $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ $\{A, D\}$, $\{B, C\}$

Vertex Cover: $\{A, B\}$, $\{A, D\}$

Edge Cover: $\{AB, AC, BD\}$

$\{AC, BD\}$

$\beta'(G) = 2$.

matching of size atleast $|A|+|B| = |Q|$.

- Also, note that it suffices to show that there is a complete matching on A in H because we can reverse the roles of A & B, and apply the same argument to B as well.

Since $A \cup B$ is a vertex cover, there can not be an edge between $V_1 \setminus A$ & $V_2 \setminus B$. Suppose for some $S \subseteq A$ we have that $|N_H(S)| < |S|$. Since $N_H(S)$

$\langle A \cup (V_2 - B) \rangle$

covers all edges from S that are not incident on B,

$Q' = (Q \setminus S) + \underline{N_H(S)}$ is also a vertex cover.

By choice of S, $(|N_H(S)| < S)$ Q' is a smaller vertex cover than Q, contradicting the minimality of Q. Hence $|N_H(S)| \geq |S| \Rightarrow$ Hall's criterion satisfies $\Rightarrow$ there is a complete matching for A in H.

$\Rightarrow$ Matching is of size atleast $|A|$. This completes the proof.

Qn: Relation between Matching & Edge cover?

Thm. (Gallai, '1959). If G is a graph without isolated vertices, then $\alpha'(G) + \beta'(G) = n = |V|$.

Pf. Suppose M is a maximal matching. Then

$S = V \setminus V(M)$ is also an independent vertex set.

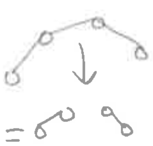
Indeed, If there are edges between vertices of S, then such edges can be added to M and one can obtain a larger matching. Hence S is independent.

Construct a edge cover as follows: Add all the edges in M to Q and for each $v \in S$, add one of its adjacent to Q. Thus $|Q| = |M| + |S|$ and since $V(M) \sqcup S = V$, we can derive that

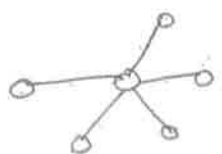
$$\alpha'(G) + \beta'(G) \leq |M| + |Q|$$
$$= 2|M| + |S| = n$$

$$\Rightarrow \alpha'(G) + \beta'(G) \leq n.$$

Let Q be a minimal edge cover. Then Q cannot contain a path of length more than 2. Else, by removing the middle edge in a path of length atleast 3, we can obtain a smaller edge cover.



(using a result, if G does not contain a path of length more than 2, then its connected components are all star graph)



$K_{1,4}$

$\Rightarrow$ Q is a graph consisting of star components.

If $C_1, \dots, C_k$ are the components of Q. Then $V(C_1) \cup \dots \cup V(C_k) = V$

and $E(C_1) \cup \dots \cup E(C_k) = Q$. Now choose a matching

$M = \{e_1, e_2, \dots, e_k\}$ by selecting one edge from

each component. (Since C_i's are disjoint), M is a

matching, we can derive

$$\alpha'(G) + \beta'(G) \geq |M| + |Q|$$
$$= k + \sum_{i=1}^{k} |E(C_i)|$$
$$= \sum_{i=1}^{k} |V(C_i)| = n$$

$(E(G) + k = n)$

Algorithm: Gale-Shapley Algorithm.

In 1962, Gale-Shapley proposed an algorithm to achieve Stable matching and this probably the best known of such algorithm. Along with Roth, Lloyd Shapley was awarded Noble prize in economics in 2012. (stable allocation pb).

Please see D.B. West (Section ~ 3).

§ Graph coloring

<u>Defn</u>: (Coloring of a graph) Let $G = (V, E)$ be a g

A graph is k-colorable if $\exists\ c : V \to \{1, 2, 3, \ldots, k\}$ such that $c(u) \neq c(v)$ if $(u, v) \in E$.

The chromatic number $\chi(G)$ is defined as

$$\chi(G) := \inf\{k : G \text{ is } k\text{-colorable}\}$$

<u>Examples</u>:

1. Empty Grap. No edges. All vertices can have the same color.

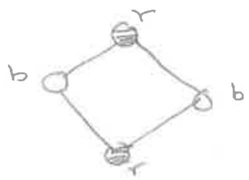
$\chi(G) = 1$.

2. Complete Graph K_n:

Every pair of vertices is adjacent.

$\chi(G) = n$.

3. Cyclic Graph : C_n



- If n is even, $\chi(C_n) = 2$
- If n is odd $\chi(C_n) = 3$.

4. Bipartite Graph:

Vertices: $V = V_1 \sqcup V_2$

$\chi(G) = 2$ (unless it has no edges, in which $\chi(G) = 1$).

Chromatic Polynomials

Let G be a graph with n-vertices. Let $P(G, q)$, $q \in \mathbb{N}$ be the number of ways of coloring (properly) a graph with q colours.

$$P(G, q) := \left| \{ c : V \to \{1, 2, \ldots, q\} : c(u) \neq c(v) \ \forall u \sim v \} \right|$$

If G is a graph & $e = (u, v)$ is an edge of G, then

- $G - e$ denote the graph with the edge "e" removed.
- G/e denote the graph with e contracted. merge the two end points u & v into a single vertex, (remove any loops that maybe created.) But will not matter to us as proper colorings of a multigraph and its corresponding simple graph are the same.

Propn: For any edge $e = (u, v) \in G$,

$$P(G, q) := P(G-e, q) - P(G/e, q), \ q \in \mathbb{N}.$$

pf: Observe that q-coloring of G is a proper q-coloring of $G-e$ in which v & w receive distinct colours and any coloring of $G-e$ in which in which v & w receive same colours is a proper q-coloring of G/e.

Remark: If $E = \phi$, then $P(G, q) = q^n$. We shall show that $P(G, q)$ is a polynomial in q. Hence we shall define $P(G, x)$, $x \in \mathbb{R}$ to be the polynomial such that

$P(G,q)$ is the number of proper-q-coloring of G,
$q \in \mathbb{N}$

Lemma:

1. $P(G,x)$ is a monic polynomial of degree n.
2. $\chi(G) = \min\{k \in \mathbb{N} : P(G,k) > 0\}$.
3. Now, let us assume that $P(G,x) = \sum_{i=1}^{n} a_i x^i$ (as a_0 is trivially 0).

Then we have that

$\sum_{i=1}^{n} a_i = 0$ or $P(G,x) = x^n$.

$a_{n-1} = -|E|$, $a_{n-i} = (-1)^i |a_{n-i}|$

Pf: Let m be the number of edges in G.

$m=0$; G has no edge $\Rightarrow$ $P(G,x) = x^n$.

This is a polynomial of degree n, and leading coefficient 1.

Induction hypothesis: Fix n(= number of vertices). Assume the claim holds for all graph on "n" vertices having fewer than m-edges.

Inductive Step: Let G be a graph with $m \geq 1$ edges.

Pick any edge $e = (u,v)$ of G. Then

$$P(G;x) = P(G-e;x) - P(G/e;x).$$

- $G-e$ has the same vertex set as G, hence has n-vertices with "$m-1$" edges. By induction hypotheses $P(G-e;x)$ is a polynomial of degree n & monic.

* G/e is the graph with atmost $(n-1)$ vertices. Two cases

a) If the contraction produces a loop at the merged vertex, then G/e has no proper colorings for any x & thus $P(G/e; x) \equiv 0$. So it does-not effect the leading term of $P(G; x)$.

b). If no loop is created, then G/e is a simple graph on exactly $(n-1)$ vertices; by applying the induction hypothesis, (on the number of vertices) we know that $P(G/e; x)$ is a poly. of degree $n-1$. with leading co-efficient 1.

$$P(G; x) = \left(x^n + a_{n-1}x^{n-1} + \cdots\right) - \left(x^{n-1} + \cdots\right)$$

$$= \text{monic poly.}$$

2. Two polynomials are equal if they agree at 1, . . .

3. <u>ExR</u>.

<u>Lemma</u>: If G has k-components, $G_1, \ldots G_k$, then

$$P(G; x) = \prod_{i=1}^{k} P(G_i; x).$$

and further $a_0 = \cdots = a_{k-1} = 0$, $|a_k| > 0$

Lemma: A graph G with n-vertices is a tree iff

$$P(G;x) = x(x-1)^{n-1}$$

Pf: Let T be a tree on "n" vertices.

- Pick any vertex as root. Let us colour it first; there are x choices.

- Then colour its neighbour: each neighbour can-not take the root's colour $\rightarrow$ $x-1$ choices.

- Continue recursively down the tree: each new vertex has $(x-1)$ choices because it is adjacent to only one already-coloured vertex.

$$\Rightarrow \quad P(T;x) = x(x-1)^{n-1}$$

Conversely, suppose G has n-vertices and

$$P(G;x) = x(x-1)^{n-1}$$

$\Rightarrow \deg(P(G;x)) = n$, coefficient of x^{n-1} is $-(n-1)$.

Observe that: For a Chromatic poly.

$$P(G;x) = x^n - m x^{n-1} + \cdots$$

where m = number of edges of G.

$$\Rightarrow \quad x(x-1)^{n-1} = x^n - (n-1)x^{n-1} + \cdots$$

$\Rightarrow$ G has exactly $(n-1)$ edges $\Rightarrow$ G is a Tree.