4.4 IMPROVING A BASIC FEASIBLE SOLUTION.

We have seen earlier that if a linear programming problem has an optimal solution, then one of the basic feasible solutions will provide this optimal value for the objective function. Now although finite in number, there may be more than one basic feasible solutions of a linear programming problem. Hence our endeavour will be to find a new basic feasible solution from another basic feasible solution with an improved value for the objective function.

Now a basic feasible solution is characterised by the basis matrix associated with it. To get a new basic feasible solution from a given one, we are to change a vector from the given basis matrix, by another vector not in the basis, by the replacement method as given in our preliminary discussions.

Let $\mathbf{x}_B$ be a basic feasible solution of a linear programming problem :

Find $\mathbf{x} \geq \mathbf{0}$

subject to $\mathbf{Ax} = \mathbf{b}$,

which maximizes $z = \mathbf{cx}$.

Here $\mathbf{A} = \left[a_{ij} \right]_{m \times n} = \left[\mathbf{a}_1, \mathbf{a}_2, \ldots\ldots, \mathbf{a}_n\right]$

and $\mathbf{B}$ is the corresponding basis matrix given by

$$\mathbf{B} = \left[\beta_1, \beta_2, \ldots\ldots, \beta_m\right].$$

Then $\mathbf{a}_j$ which is not a vector of $\mathbf{B}$ can be expressed as a linear combination of the vectors of $\mathbf{B}$ as

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij}\, \beta_i \,. \qquad \ldots (1)$$

Now, if some $y_{rj} \neq 0$, then we know that we can replace the vector β_r from $\mathbf{B}$ by $\mathbf{a}_j$ still maintaining the basis character of $\mathbf{B}$. If $y_{rj} \neq 0$ and if we replace β_r from $\mathbf{B}$ by $\mathbf{a}_j$, then we shall find this value of β_r in terms of $\mathbf{a}_j$ and the remaining vectors of $\mathbf{B}$ from (1) as

$$\beta_r = \frac{1}{y_{rj}}\, \mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}} \right) \beta_i \,. \qquad \ldots (2)$$

Now, for the basic feasible solution $\mathbf{x}_B$ of $\mathbf{Ax} = \mathbf{b}$, we have

$$\mathbf{Bx}_B = \mathbf{b}$$

or,

$$\sum_{i=1}^{m} x_{Bi}\, \beta_i = \mathbf{b}. \qquad \ldots (3)$$

If c_{Bi} be the price corresponding to the basic variable x_{Bi} and if z_B be the value of the objective function for the basic feasible solution $\mathbf{x}_B$, then we have

$$z_B = \mathbf{c}_B\, \mathbf{x}_B = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} \,. \qquad \ldots (4)$$

Setting (2) in (3), we get

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} x_{Bi}\, \beta_i + x_{Br} \left[\frac{1}{y_{rj}}\, \mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}} \right) \beta_i \right] = \mathbf{b}$$

or,

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br} \right) \beta_i + \frac{x_{Br}}{y_{rj}}\, \mathbf{a}_j = \mathbf{b} \qquad \ldots (5)$$

Thus a new basic solution of the problem is

$$x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br}, \quad i = 1, 2, \ldots\ldots, (r-1), (r+1), \ldots\ldots, m \qquad \ldots (6)$$

together with $\frac{x_{Br}}{y_{rj}}$ and remaining $(n - m)$ zeros.

In order that this solution will be feasible in addition to being basic, we must have

$$x_{Bi} - \frac{y_{ij}}{y_{rj}} x_{Br} \geq 0, \quad i \neq r \qquad \ldots (7)$$

$$\text{and} \quad \frac{x_{Br}}{y_{rj}} \geq 0. \qquad \ldots (8)$$

If the conditions of *feasibility* be satisfied and this new solution gives an *improved value* of the objective function than that given by $\mathbf{x}_B$, then we shall desire this solution and discard $\mathbf{x}_B$.

To realise these two criterions, we have two arbitrary quantities to select and they are the suffixes r in β_r and j in $\mathbf{a}_j$ which are uptil now arbitrary, except that $y_{rj} \neq 0$.

If $x_{Br} = 0$, then the conditions (7) and (8) are automatically satisfied and the solution is feasible.

If $x_{Br} \neq 0$; then to satisfy (8), $y_{rj} > 0$, for $x_{Br} > 0$.

Let $y_{rj} > 0$, then (7) is automatically satisfied if $y_{ij} = 0$ or $y_{ij} > 0$. Thus the feasibility conditions are to be satisfied only for those i for which $y_{ij} > 0$.

This requires

$$\frac{x_{Bi}}{y_{ij}} - \frac{x_{Br}}{y_{rj}} \geq 0, \quad y_{ij} > 0$$

$$\text{or,} \quad \frac{x_{Br}}{y_{rj}} \leq \frac{x_{Bi}}{y_{ij}}, \quad y_{ij} > 0. \qquad \ldots (9)$$

Thus, if we choose the vector β_r such that

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}} \left\{ \frac{x_{Bi}}{y_{ij}}, \; y_{ij} > 0 \right\} = \theta \text{ (say)}, \qquad \ldots (10)$$

then (9) is satisfied and the solution (6) is feasible too.

What we need is simply to compute $\dfrac{x_{Br}}{y_{rj}}$ from (10). This tells us which column r of the basis matrix $\mathbf{B}$ is to be removed, to get a better value for z.

Now we are to choose j of $\mathbf{a}_j$ such that the new basic solution makes the objective function at least as great as the current basic solution. The price vector component c_{Br} changes to c_j as β_r is changed to $\mathbf{a}_j$. If z' be the new value of the objective function, then we have

$$z' = \sum_{\substack{i=1\\ i\neq r}}^{m} c_{Bi}\left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br}\right) + c_j\,\frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br}\right) + c_j\,\frac{x_{Br}}{y_{rj}},$$

since the contribution of the term is zero for $i = r$

$$= \sum_{i=1}^{m} c_{Bi}\, x_{Bi} - \frac{x_{Br}}{y_{rj}} \cdot \sum_{i=1}^{m} c_{Bi}\, y_{ij} + \frac{x_{Br}}{y_{rj}}\, c_j$$

$$= z_B - \frac{x_{Br}}{y_{rj}}\left(\sum_{i=1}^{m} c_{Bi}\, y_{ij} - c_j\right), \text{ by (4)}$$

$$= z_B - \frac{x_{Br}}{y_{rj}}\,(z_j - c_j)\,. \qquad \ldots \qquad (11)$$

From (11), we observe that $z' > z_B$, if $(z_j - c_j)$ be negative for $\dfrac{x_{Br}}{y_{rj}} \geq 0$, by (8). Thus, by choosing any vector $\mathbf{a}_j$ for which $(z_j - c_j)$ is negative, we can improve the value of the objective function by

$$-\frac{x_{Br}}{y_{rj}}\,(z_j - c_j)\,.$$

Hence, if the problem be that of maximizing, then we shall select that $\mathbf{a}_j$ for β_r which will make

$$\frac{x_{Br}}{y_{rj}}\,(z_j - c_j)$$

minimum-most negative, that is to say, which will make $(z_j - c_j)$ minimum-most negative in order to minimize the computation involved.

To summarise, we shall determine the vector $\mathbf{a}_k$ to enter the basis as follows :

$$z_k - c_k = \underset{j}{\text{Min}} \left\{ z_j - c_j \mid z_j - c_j < 0 \right\} \qquad \dots \quad (12)$$

The subscript k in (12) indicates the vector to enter the basis.

For a minimizing problem, we are to choose that $\mathbf{a}_j$ which will make $(c_j - z_j)$ minimum-most negative.

The vector which is deleted from the basis is called the *departing vector* or *leaving vector* while the vector which is introduced in the basis is called the *entering vector.*

This process is clearly a monotonic process in which each value of the objective function becomes greater than its previous value. Hence this process is to be continued until there are no vectors $\mathbf{a}$ for which

$$z_j - c_j < 0 .$$

This method is used iteratively to get a basic feasible solution from another basic feasible solution with an improved value of the objective function so long an $\mathbf{a}_j$ is obtained with $z_j - c_j < 0$ and at least one

$$y_{ij} > 0 .$$

The quantities $(z_j - c_j)$ are called the *index numbers* corresponding to $\mathbf{a}_j$.

In the illustration of the previous section if we compute $(z_j - c_j)$, for $j = 2, 3$ (since $j = 1$ and $j = 4$ refer to the basic variables), then we get

$$z_2 - c_2 = 5 - (-4) = 9 \text{ and } z_3 - c_3 = -1 - 0 = -1 .$$

Thus the only $(z_j - c_j)$, $j = 2$, 3, that is negative is $(z_3 - c_3)$ and hence we shall choose $\mathbf{a}_3$ to enter the basis. To determine the departing vector, we use the criterion

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}} \left\{ \frac{x_{Bi}}{y_{ij}} , \; y_{ij} > 0 \right\}$$

$$= \text{Min} \left\{ \frac{x_1}{y_{13}} , \frac{x_4}{y_{43}} \right\} .$$

But in the present case $y_{13} = -\frac{1}{2}$ and is negative.

$y_{43} = \frac{3}{2}$ being the only positive quantity, $\mathbf{a}_4$ will leave the basis.