Problem Set 6

Q1. (a) $X \sim N(\mu, \sigma^2)$

$W = \left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$ [location - scale]

(b) $X \sim Exp(\lambda)$

$W = \lambda X \sim Exp(1)$ [scale]

(c) $X \sim U(-\theta, \theta)$

$W = \frac{X}{\theta} \sim U(-1, 1)$ [scale]

(d) $X \sim Gamma(n, \theta)$

$W = \theta X \sim Gamma(n, 1)$ [scale]

(e) $X \sim N(\theta, \theta^2)$

$W = \frac{X}{\theta} \sim N(1, 1)$ [Scale]

(f) $X \sim U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$

$W = X - \theta \sim U(-\frac{1}{2}, \frac{1}{2})$ [location]

Q2: (a) $X_i = W_i + \theta$ for $i = 1, \dots n$, $W_i \sim f$, free of θ.

$\therefore X_i - X_j = W_i - W_j = Y_{ij}$ is also free of θ. $\forall (i,j)$.

So, any function of Y_{ij} has a distribution free of θ.

(b) $X_i = W_i/\theta$ for $i=1,\dots,n$, $W_i \sim f$, free of θ.

$$\frac{X_i}{\theta} = W_i \; ; \; i=1,\dots,n$$

$$Y = \frac{W_1^2}{\sum_{j=1}^{n} W_j^2} \quad \text{and} \quad Z = \frac{W_{(1)}}{W_{(n)}}$$, being functions of W_i; $i=1,\dots,n$, have distributions free of θ.

Q3: Let $X_1,\dots,X_n \overset{\text{IID}}{\sim} f_\beta$, where

$$f_\beta(x) = \frac{\gamma}{\beta}\left(\frac{x}{\beta}\right)^{\gamma-1} \quad ; \quad 0 \le x \le \beta$$

$$f_\beta(\underset{\sim}{x}) = \frac{\gamma^n}{\beta^n} \frac{\left(\prod_{i=1}^{n} x_i\right)^{\gamma-1}}{\beta^{n(\gamma-1)}} \; \mathbb{I}\left(x_{(n)} \le \beta\right)$$

The log-likelihood function:

$$\ell_n(\beta) = n\log\gamma - n\log\beta + (\gamma-1)\sum_{i=1}^{n}\log x_i - n(\gamma-1)\log\beta + \log\left\{\mathbb{I}\left(x_{(n)} \le \beta\right)\right\}.$$

$$= n\log\gamma - n\gamma\log\beta + (\gamma-1)\sum_{i=1}^{n}\log x_i + \log\left\{\mathbb{I}\left(x_{(n)} \le \beta\right)\right\}.$$

Verify that $X_{(n)}$ is the MLE of β. — (1)

Further, observe that $Y = \frac{X}{\beta}$ then

$$f_Y(y) = \gamma\, y^{\gamma-1} \quad ; \quad 0 \le y \le 1$$

Thus, X belongs to a scale family.

So, any function of X_i/β, $i=1,\dots,n$, can be a pivot. — (2)

Combining ① and ②, we consider the pivot

$$T(\underset{\sim}{x}, \beta) = \frac{X_{(n)}}{\beta} = Y_{(n)}.$$

<u>Distribution of $T(\underset{\sim}{x}, \;)$</u>

CDF of $T(\underset{\sim}{x}, \beta)$:

$$\begin{aligned} P\left(T(\underset{\sim}{x}, \beta) \le t\right) &= \left[P\left(Y_1 \le t\right)\right]^n \\ &= \left(\gamma \int_0^t y^{\gamma-1} dy\right)^n \\ &= \left(\left[y^\gamma\right]_0^t\right)^n \\ &= t^{n\gamma} \quad — ③ \end{aligned}$$

$\therefore$ PDF of $T(\underset{\sim}{x}, \beta)$:

$$f_T(t) = n\gamma\, t^{n\gamma - 1} \quad ; \quad 0 \le t \le 1.$$

<u>Finding confidence interval</u>

1) Find a, b such that

$$P\left(a \le T(\underset{\sim}{x}, \beta) \le b\right) = (1-\alpha).$$

We choose a, b symmetrically such that

$$P\left(T(\underset{\sim}{x}, \beta) \le a\right) = \frac{\alpha}{2} \text{ and } P\left(T(\underset{\sim}{x}, \beta) \ge b\right) = \frac{\alpha}{2}$$

$$\left.\begin{aligned} &\text{i.e., } \quad a^{n\gamma} = \frac{\alpha}{2} \iff a = (\alpha/2)^{1/n\gamma} \quad \text{from ③} \\ &\text{Similarly, } \quad 1 - b^{n\gamma} = \frac{\alpha}{2} \iff b = (1-\alpha/2)^{1/n\gamma}. \end{aligned}\right\} — ④$$

Thus,

$$P\left(\left[\alpha/2\right]^{1/n\gamma} \leq T(\underset{\sim}{x},\beta) \leq \left[1-\alpha/2\right]^{1/n\gamma}\right) = (1-\alpha).$$

2) Next we solve the equations w.r.t. β:

$$T(\underset{\sim}{x},\beta) = a = (\alpha/2)^{1/n\gamma} \quad \text{— (I)}$$

$$\iff \frac{X_{(n)}}{\beta} = (\alpha/2)^{1/n\gamma}$$

$$\iff \hat{\beta}_n(a) = \frac{X_{(n)}}{(\alpha/2)^{1/n\gamma}}$$

and $T(\underset{\sim}{x},\beta) = b = (1-\alpha/2)^{1/n\gamma}$ — (II)

$$\iff \hat{\beta}_n(b) = \frac{X_{(n)}}{(1-\alpha/2)^{1/n\gamma}}.$$

3) Finally, observe that $T(\underset{\sim}{x},\beta) = \frac{X_{(n)}}{\beta}$ is a strictly decreasing function of β.

So, $a \leq T(\underset{\sim}{x},\beta) \leq b$

$$\iff \hat{\beta}_n(b) \leq \beta \leq \hat{\beta}_n(a).$$

Thus, $$P\left(\frac{X_{(n)}}{(1-\alpha/2)^{1/n\gamma}} \leq \beta \leq \frac{X_{(n)}}{(\alpha/2)^{1/n\gamma}}\right) = (1-\alpha).$$

<u>Ans:</u> $\left[\frac{X_{(n)}}{(1-\alpha/2)^{1/n\gamma}}, \frac{X_{(n)}}{(\alpha/2)^{1/n\gamma}}\right]$ is a $(1-\alpha)$-confidence interval based on MLE of β.

Q4: (a) $X_i \overset{iid}{\sim} U(\theta - 1/2, \theta + 1/2)$; $i = 1, \dots, n$

(1) $W_i = (X_i - \theta) \sim U(-1/2, 1/2)$; $i = 1, \dots, n$

Any function of W_i; $i = 1, \dots, n$; is a valid pivot.

Observe that, $E(X_i) = \theta$

So, $\bar{X}_n$ is a good estimator of θ.

Therefore, we choose the pivot $\bar{W}_n = (\bar{X}_n - \theta)$

By CLT,

$$T(\underset{\sim}{x}, \theta) = \sqrt{n}\,\frac{\bar{X}_n - \theta}{\sqrt{\operatorname{var}(X_1)}} \xrightarrow{d} N(0,1)$$

where $\operatorname{var}(X_1) = \frac{1}{12}$ (verify)

We choose $T(\underset{\sim}{x}, \theta) = 2\sqrt{3n}\,(\bar{x}_n - \theta)$ as an appropriate pivot.

(2) We know that

$$P\left(-\tau_{\alpha/2} \le T(\underset{\sim}{x}, \theta) \le \tau_{\alpha/2}\right) \approx (1-\alpha)$$

by CLT, where $\tau_{\alpha/2}$ is the upper $\alpha/2$ point of $N(0,1)$ distribution.

(3) Solve the equations:

$$T(\underset{\sim}{x},\theta) = \tau_{\alpha/2} \iff \hat{\theta}_n(b) = \bar{X}_n - \frac{\tau_{\alpha/2}}{2\sqrt{3n}}$$

and $$T(\underset{\sim}{x},\theta) = -\tau_{\alpha/2} \iff \hat{\theta}_n(a) = \bar{X}_n + \frac{\tau_{\alpha/2}}{2\sqrt{3n}}.$$

(4) Finally as $T(\underset{\sim}{x},\theta)$ is a strictly decreasing function of θ, we have

$$P_\theta\left(\bar{X}_n - \frac{\tau_{\alpha/2}}{2\sqrt{3n}} \le \theta \le \bar{X}_n + \frac{\tau_{\alpha/2}}{2\sqrt{3n}}\right) \approx (1-\alpha).$$

So, $\left[\bar{X}_n - \frac{\tau_{\alpha/2}}{2\sqrt{3n}}, \bar{X}_n + \frac{\tau_{\alpha/2}}{2\sqrt{3n}}\right]$ is an approximate confidence interval with coefficient $(1-\alpha)$.

<u>(b)</u> $X_i \overset{\text{IID}}{\sim} f_\theta(x)$ where $f_\theta(x) = \frac{2x}{\theta^2}$; $0<x<\theta$, $\theta>0$.
$i=1,\cdots,n$.

Let $W_i = \frac{X_i}{\theta}$; $i=1,\cdots,n$.

Then the pdf of W_i is:

$$f_W(w) = 2w, \quad 0<w<1,$$

which is fee of θ.

So, any function of W_i; $i=1,\cdots,n$; is a valid pivot.

Further, verify that, $X_{(n)}$ is the MLE of θ.

Combining the above two facts, we choose

$$T(\underset{\sim}{X}, \theta) = \frac{X_{(n)}}{\theta}.$$

[Complete the proceedure as in Q.3]

Q5. We know that $F_T(T;\theta) \sim U(0,1)$ distribution.

Let α_1, α_2 satisfies $\alpha_1 + \alpha_2 = \alpha$.

Then $\int_{\alpha_1}^{1-\alpha_2} dx = 1 - \alpha_1 - \alpha_2 = 1-\alpha$

i.e., $P\left(\alpha_1 \leq F_T(T;\theta) \leq \alpha_2\right) = (1-\alpha)$. — (1)

As $F_T(T; U(T)) = \alpha_1$, $U(T)$ is the solution of the equation $F_T(T;\theta) = \alpha_1$.

Similarly, $L(T)$ is the solution of $F_T(T;\theta) = \alpha_2$. — (2)

Finally, as $F_T(T;\theta)$ is stricty decreasing w.r.t. θ,

$$\{\alpha_1 \leq F_T(T;\theta) \leq \alpha_2\} \Leftrightarrow \{L(T) \leq \theta \leq U(T)\}$$

Hence, $P_\theta\left(L(T) \le \theta \le U(T) \right) = (1-\alpha)$.

Hence $[L(T), U(T)]$ is a confidence interval with coefficience $(1-\alpha)$.

Q7:

$$X \sim Beta(\theta, 1)$$

$$f_X(x) = \frac{\Gamma\theta+1}{\Gamma\theta} x^{\theta-1} \quad ; \quad 0<x<1$$

$$Y = -(\log X)^{-1} \iff X = e^{-1/y}$$

$$J = \left| \frac{\partial x}{\partial y} \right| = e^{-1/y} / y^2$$

$$f_Y(y) = \theta\, e^{-\frac{\theta-1}{y} - \frac{1}{y}} / y^2 \qquad 0<y<\infty$$

$$= \theta\, y^{-2} e^{-\theta/y} \quad ; \quad 0<y<\infty$$

The confidence coefficient of $[Y/2, Y]$ is :

$$P_\theta\left[\frac{Y}{2} \le \theta \le Y\right] = P_\theta\left[\theta \le Y \le 2\theta\right]$$

$$= \int_\theta^{2\theta} \frac{1}{\theta}\left(\frac{\theta}{y}\right)^2 e^{-\theta/y}\, dy$$

$\left[\begin{array}{l} \text{Here} \\ Z = \theta/y \end{array} \right]$

$$= \int_{1/2}^{1} e^{-z}\, dy = \underline{e^{-1/2} - e^{-1}}.$$

[ANS]

Q8: Let $X_i \overset{IID}{\sim} U(0,\theta)$; $i = 1, \ldots, n$

Given the pivot $T(\underset{\sim}{x},\theta) = \frac{X_{(n)}}{\theta}$,

choose a, b such that

$$\left.\begin{aligned} &P\left(T(\underset{\sim}{x},\theta) \le a\right) = \frac{\alpha}{2} \\ \text{and} \quad &P\left(T(\underset{\sim}{x},\theta) \ge b\right) = \frac{\alpha}{2} \end{aligned}\right\} \text{— (symmetry)}$$

From the CDF of $T(\underset{\sim}{x},\theta)$ we get

$$a = \left(\alpha/2\right)^{1/n} \quad \text{and} \quad b = \left(1-\alpha/2\right)^{1/n}$$

By solving $T(\underset{\sim}{x},\theta) = a$ and $T(\underset{\sim}{x},\theta) = b$; and as $T(\underset{\sim}{x},\theta)$ is a decreasing function of θ,

we get

$$P_\theta\Bigg[\underbrace{\frac{X_{(n)}}{\left(1-\alpha/2\right)^{1/n}}}_{L(\underset{\sim}{x})} \le \theta \le \underbrace{\frac{X_{(n)}}{\left(\alpha/2\right)^{1/n}}}_{U(\underset{\sim}{x})}\Bigg] = (1-\alpha). \qquad \longrightarrow \text{I}$$

Next, consider $\theta' \ne \theta$.

Let us find the prob.

$$P_\theta\left[L(\underset{\sim}{x}) \le \theta' \le U(\underset{\sim}{x})\right]$$

$$= P_\theta\left[\theta'\left(\frac{\alpha}{2}\right)^{1/n} \le X_{(n)} \le \theta'\left(1-\frac{\alpha}{2}\right)^{1/n}\right] \quad - (1)$$

Let $\theta' < \theta$ then

$$(1) = P\left[X_{(n)} \le \theta'\left(1-\frac{\alpha}{2}\right)^{1/n}\right] - P\left[X_{(n)} < \theta'\left(\frac{\alpha}{2}\right)^{1/n}\right]$$

$$= \left(\frac{\theta'}{\theta}\right)^n\left(1-\frac{\alpha}{2}\right) - \left(\frac{\theta'}{\theta}\right)^n\frac{\alpha}{2}$$

$$= \left(\frac{\theta'}{\theta}\right)^n(1-\alpha) < (1-\alpha). \quad - (2)$$

Let $\theta' > \theta$ then

$$(1) = P_\theta\left[\theta'(\alpha/2)^{1/n} \le X_{(n)} \le \min\left\{\theta'\left(1-\frac{\alpha}{2}\right)^{1/n}, \theta\right\}\right]$$

$$= \min\left\{1, \left(\frac{\theta'}{\theta}\right)^n\left(1-\frac{\alpha}{2}\right)\right\} - \left(\frac{\theta'}{\theta}\right)^n\left(\frac{\alpha}{2}\right)$$

$$= \min\left\{1-\left(\frac{\theta'}{\theta}\right)^n\frac{\alpha}{2}, \left(\frac{\theta'}{\theta}\right)^n(1-\alpha)\right\}$$

Now, $1-\left(\frac{\theta'}{\theta}\right)^n\frac{\alpha}{2} \le 1-\alpha$ for sufficiently large n. So, $(1) \le (1-\alpha). \quad - (3)$

Combining (2) and (3) we get that the confidence interval is unbiased.

Q9. (a) $X_i \overset{\text{IID}}{\sim} N(\mu_0, \sigma^2)$; $i=1,\dots,n$, which belongs to the exponential family.

From the joint distribution

$$f_{\underset{\sim}{X}}(\underset{\sim}{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2\right\}$$

we get that $T(\underset{\sim}{x}) = \sum_{i=1}^{n}(x_i-\mu_0)^2$ is a complete sufficient statistic (CSS). [Note that μ_0 is known]

Observe that,

$$\left(\frac{X_i-\mu_0}{\sigma}\right) \overset{\text{IID}}{\sim} N(0,1); \quad i=1,\dots,n$$

$$\Rightarrow \sum_{i=1}^{n}\left(\frac{X_i-\mu_0}{\sigma}\right)^2 = \frac{T(\underset{\sim}{X})}{\sigma^2} \sim \chi^2_{(n)} \text{ distn.}$$

$$\therefore \quad E(T(\underset{\sim}{X})) = n\sigma^2$$

$$\Rightarrow \quad E\left(\frac{1}{n}T(\underset{\sim}{X})\right) = \sigma^2.$$

So, $\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu_0)^2 = S_0^2$ is the UMVUE of σ^2.

(b) As, $\frac{T(\underset{\sim}{X})}{\sigma^2} = \frac{nS_0^2}{\sigma^2} \sim \chi^2_{(n)}$, which is free of σ^2, it serves as an appropriate pivot for σ^2 based on UMVUE.

Q.10 (a) $X_i \overset{IID}{\sim} \text{Exp}(\theta)$, $i = 1, \ldots, n$; with common pdf $f_\theta(x) = \theta e^{-\theta x}$; $x > 0$.

The transformation $W_i = \theta X_i$ yeilds common pdf of W_i as

$$f_W(w) = e^{-w} \;;\; w > 0,$$

which is free of θ.

(b) $W_1 \sim \text{Exp}(1)$ is itself a pivot.

Consider another pivot based on UMVUE of $1/\theta$;

$$T(\underset{\sim}{x}, \theta) = \sum_{i=1}^{n} W_i = \theta \sum_{i=1}^{n} X_i$$ as $\frac{1}{n}\sum_{i=1}^{n} X_i$ is the UMVUE of $1/\theta$.

Note that, $\sum_{i=1}^{n} W_i \sim \text{Gamma}(n, 1)$.

(c) i. <u>$(1-\alpha)$ symmetric confidence interval based on W_1</u>

Let (a, b) be such that

$$P(W_1 \leq a) = \frac{\alpha}{2} \quad \text{and} \quad P(W_1 \geq b) = \frac{\alpha}{2}$$

$$\Rightarrow \int_0^a e^{-w}\, dw = \left[-e^{-w}\right]_0^a = 1 - e^{-a} = \frac{\alpha}{2}$$

$$\Rightarrow 1 - \frac{\alpha}{2} = e^{-a} \quad \Rightarrow a = -\log\left(1 - \frac{\alpha}{2}\right)$$

Similarly, $b = -\log(\alpha/2)$.

$$\therefore \quad P_\theta\left(-\log\left(1-\frac{\alpha}{2}\right) \leq W_1 \leq -\log(\alpha/2)\right) = (1-\alpha)$$

$$\Rightarrow \quad P_\theta\left(-\frac{\log(1-\alpha/2)}{X_1} \leq \theta \leq -\frac{\log(\alpha/2)}{X_1}\right) = (1-\alpha).$$

$\therefore \left[-\frac{\log(1-\alpha/2)}{X_1}, -\frac{\log(\alpha/2)}{X_1}\right]$ is confidence interval of θ with coefficient $(1-\alpha)$.

ii. <u>$(1-\alpha)$-symmetric confidence interval for $\sum_{i=1}^{n} W_i$</u>

Find (a, b) such that

$$P\left(\sum_{i=1}^{n} W_i \leq a\right) = \frac{\alpha}{2} \quad \text{and} \quad P\left(\sum_{i=1}^{n} W_i \geq b\right) = \frac{\alpha}{2}.$$

$$\Rightarrow \gamma_{1-\alpha/2;\, n, 1} = a \quad \text{and} \quad b = \gamma_{\alpha/2;\, n, 1}$$

Thus, $$P_\theta\left(\gamma_{1-\alpha/2;\, n,1} \leq \sum_{i=1}^{n} W_i \leq \gamma_{\alpha/2;\, n,1}\right) = (1-\alpha)$$

$$\Rightarrow P_\theta\left(\gamma_{1-\alpha/2;\, n,1} \leq \theta \sum_{i=1}^{n} X_i \leq \gamma_{\alpha/2;\, n,1}\right) = (1-\alpha)$$

$$\Rightarrow P_\theta\left(\frac{\gamma_{1-\alpha/2;\, n,1}}{\sum_{i=1}^{n} X_i} \leq \theta \leq \frac{\gamma_{\alpha/2;\, n,1}}{\sum_{i=1}^{r} X_i}\right) = (1-\alpha)$$

Thus, $\left[\frac{\gamma_{1-\alpha/2;\, n-1}}{\sum_{i=1}^{n} X_i}, \frac{\gamma_{\alpha/2;\, n,1}}{\sum_{i=1}^{n} X_i}\right]$ is a $(1-\alpha)$-confidence interval for θ.