Quiz 1

1. Let $X_1, \dots, X_n \overset{iid}{\sim} U(\theta, 1)$ where $\theta < 1$.
   (a) Find MLE of θ.
   (b) Find c and d such that $c + d\hat{\theta}_{MLE}$ is an unbiased estimator of θ.

2. Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, 1)$, $\mu \in \mathbb{R}$, and $T(X)$ be an unbiased estimator of $\psi(\mu) = \mu^2$.
   (a) Show that Cramer Rao Lower Bound for $Var(T(X))$ is $\frac{4\mu^2}{n}$.
   (b) Show that $T(X) = \bar{X}_n^2 - \frac{1}{n}$ is the UMVUE of $\psi(\mu)$.
   (c) Find variance of $T(X)$. (Hint: $E(\bar{X}_n^4) = \mu^4 + \frac{6\mu^2}{n} + \frac{3}{n^2}$)

3. Let $X_1, \dots, X_n$ be a random sample from

$$f_\theta(x) = \begin{cases} \frac{\theta}{x^{\theta+1}} & \text{if } x > 1; \theta > 0, \\ 0 & \text{o.w.} \end{cases}$$

   (a) Check whether it belongs to an exponential family, i.e. show that, $f_\theta(x) = \exp\{h(x) + c(\theta) + \theta T(x)\}$, for some functions h, c and T. Find $T(X)$.
   (b) Based on this T, find the UMVUE of $\frac{1}{\theta}$.

4. Let $Y \sim$ Exponential(1). Derive the pdf (say f_X) for $X = (\frac{Y}{\lambda})^k$, where $\lambda > 0$.
   (a) Provide a method to simulate from the distribution of X, starting from uniform(0, 1) random variables.

Mid-semester Examination

1. Let $X_1, \ldots, X_n$ be a random sample from exponential distribution with pdf
$$f_\lambda(x) = \lambda \exp\{-\lambda x\}, \quad x > 0, \lambda > 0.$$
Consider the parameter of interest $\psi(\lambda) = \frac{\lambda}{1+\lambda}$.
   (a) Find an MLE of $\psi(\lambda)$, say $T_{MLE,n}$.
   (b) Show that S_n/n is consistent for $1/\lambda$.
   (c) Is $T_{MLE,n}$ consistent for $\psi(\lambda)$?
   (d) Consider another estimator of $\psi(\lambda)$ as $W_n = \frac{1}{n}\sum_{i=1}^{n} \exp\{-X_i\}$.
      i. Show that W_n is unbiased for $\psi(\lambda)$.
      ii. Is W_n consistent for $\psi(\lambda)$?
      iii. Find sequences of real numbers $\{a_n\}$ and $\{b_n\}$ such that $\frac{W_n - a_n}{b_n} \xrightarrow{d} X$, where $X \sim N(0,1)$ distribution. [Hint: Apply CLT]

2. Suppose the random measurements of a quantity is modeled as $Gamma(\alpha, \beta)$ with pdf
$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma\alpha} x^{\alpha-1} \exp\{-\beta x\}, \quad x > 0, \alpha, \beta > 0.$$
Let the true measurement is α/β. If n independent measurements are taken, find the Uniformly Minimum Variance Unbiased Estimator (UMVUE)?

3. Let $X \sim$ normal$(0,1)$. Define $Y = -X\mathbb{I}(|X| \leq 1) + X\mathbb{I}(|X| > 1)$. Find the distribution of Y.

4. A factory producers electronic components, and the number of defective components produced by each machine per day follows a Poisson distribution with a mean of 4 defectives. If 50 machines are in operation, what is the probability that the total number of defectives produced by all machines in a day is between 180 and 200? [You may use the fact $\Phi(\sqrt{2}) = 0.9214$.]

Quiz 2

1. Let $\{T_n\}$ be a sequence of random variables such that $T_n \xrightarrow{P} T$ as $n \to \infty$, where T is another random variable. Prove or disprove (with a counter example) the following statements:

   (a) $T_n - T \xrightarrow{P} 0$ as $n \to \infty$.

   (b) $T_n^2 - T^2 \xrightarrow{P} 0$ as $n \to \infty$.

   (c) $E(T_n - T) \to 0$ as $n \to \infty$.

   (d) If T is degenerate then $var(T_n) \to 0$ as $n \to \infty$.

2. Consider the problem of testing $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$ based on a random sample $X_1, \ldots, X_n$ from a distribution with pdf f_θ. Consider the test function $\phi_0(x) = \alpha$ for all x, and $\alpha \in (0, 1)$.

   (a) Is ϕ_0 a level-α test?

   (b) Draw the power function for ϕ_0.

3. Consider the problem of testing $H_0 : X \sim U(0, \theta_0)$ and $H_1 : X \sim f_{\theta_0}$ where

$$f_{\theta_0}(x) = \frac{2}{\theta_0^2}x, \quad 0 < x \leq \theta_0, \quad \theta_0 > 0 \text{ is known.}$$

   (a) Draw the pdf of X under H_0 as well as under H_1.

   (b) Find the most powerful (MP) test for testing H_0 against H_1 at level α based on a sample of size one (i.e., $n = 1$).

   (c) Find the power of the MP test.

   (d) Is the MP test unbiased?

4. Consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 (> \theta_0)$, based on a random sample X of size one from a distribution with pdf f_θ where

$$f_\theta(x) = \theta x^{-(\theta+1)}, \quad x > 1, \theta > 0.$$

   (a) Find the MP test for testing H_0 against H_1.

   (b) Find the power function of the MP test derived in part A.

   (c) Find the size of the proposed test when H_0 is generalized to $H_0 : \theta \leq \theta_0$.

End-semester Examination

1. Let $X_1, \dots, X_n$ be a random sample from a one-parameter beta distribution with parameter θ and the probability density function (PDF) as follows
$$f_\theta(x) = \frac{\Gamma(\theta+1)}{\Gamma(\theta)}(1-x)^{\theta-1}, \quad 0 < x \le 1, \quad \theta > 0.$$
   (a) Based on the n samples, find the MLE of $1/\theta$.
   (b) Is the MLE same as the UMVUE of $1/\theta$?

2. Suppose it is known that the number of items produced in a factory during a week is a random variable with mean 50.
   (a) Provide an upperbound on the probability that this week's production will exceed 75?
   (b) If the variance of a week's production is known to equal 25, then can you have a better upperbound compared to that in part (a)?
   (c) Further, provide a lowerbound of the probability that this week's production will be between 40 and 60?

3. Historical data indicate that 4% of the components produced at a certain manufacturing facility are defective. A labor dispute has recently been started, and management is curious about whether it will result in any change in this figure of 4%. If a random sample of 500 items indicated 25 defectives, is this significant evidence to conclude that products quality is now depreciated? [Write explicitly the modeling assumptions (if any), hypotheses to be tested, test statistic, test function and the conclusion obtained after testing. You may use the facts: $z_{0.05} = 1.64, z_{0.01} = 2.32$, where z_α is the upper α point of $N(0,1)$ distribution.]

4. Let $X_1, \dots, X_n$ be a random sample from uniform$(0,\theta)$ distribution. Consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 (\theta_1 > \theta_0)$.
   (a) Draw the graph of $\lambda(x) = f_1(x)/f_0(x)$ with respect to the statistic $X_{(n)}$, where f_0 and f_1 are joint pdfs of $\{X_1, \dots, X_n\}$ under H_0 and H_1, respectively. Hence or otherwise, determine if $\lambda(x)$ is monotone with respect to $X_{(n)}$. [It is enough to consider the behavior of $\lambda(x)$ in the space of x where at least one of $f_0(x)$ or $f_1(x)$ is positive.]
   (b) Consider the following two tests for testing H_0 against H_1:
$$\phi_0(x) = \begin{cases} 1 & \text{if } x_{(n)} \ge k, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \phi_1(x) = \begin{cases} 1 & \text{if } x_{(n)} \ge \theta_0 \text{ or } x_{(n)} \le c, \\ 0 & \text{otherwise,} \end{cases}$$
   Find k and c such that both ϕ_0 and ϕ_1 are size α tests.
   (c) Find the power functions of the tests ϕ_0 and ϕ_1. Which one has higher power under H_1.

(d) Verify if ϕ_0 remains a size-α test for the choice of k obtained in part (b) if the null hypothesis is modified to $H_0' : \theta \le \theta_0$.

5. Let $X_1, \ldots, X_n$ be a random sample from normal$(0, \sigma^2)$ distribution.

(a) Find the UMVUE of σ^2, say $\hat{\sigma}_n^2$.

(b) Using a function of $\hat{\sigma}_n^2$ as pivot, find a symmetric $(1-\alpha)$-confidence interval for σ^2. [Note: By symmetry we indicate the following: If $T(X, \sigma^2)$ is the pivot, then start with the points (a,b) such that $P(T(X,\sigma^2) < a) = \alpha/2$ and $P(T(X,\sigma^2) > b) = \alpha/2$.]

(c) Find the large sample distribution of the pivot $T_n = T(X, \sigma^2)$, i.e., find sequence of real numbers $\{a_n\}$ and $\{b_n\}$ such that

$$\frac{T_n - a_n}{b_n} \xrightarrow{d} Z, \quad \text{as } n \to \infty,$$

where Z is a non-degenerate distribution.

(d) Based on the large sample distribution obtained in (c) derive an approximate symmetric $(1-\alpha)$-confidence interval for σ^2.

(e) For $\alpha = 0.05$ and $n = 25$ compare the two confidence intervals obtained in parts (b) and (d) in terms of the ratio of the upper confidence limit (UCL) and the lower confidence limit (LCL). Which one provides a sharper confidence interval? You may use the following facts: Let z_α and $\chi^2_{\alpha,r}$ be the upper-α points of $N(0,1)$ and χ^2_r distributions. Then

α	z_α	$\chi^2_{\alpha,24}$	$\chi^2_{\alpha,25}$	$\chi^2_{1-\alpha,24}$	$\chi^2_{1-\alpha,25}$
0.025	1.96	39.36	40.64	12.40	13.12
0.05	1.64	36.42	37.65	13.85	14.61

(f) Find the minimum sample size n_0 ensuring that the ratio of UCL and LCL obtained in part (d) is at most 1.1.