**INDIAN STATISTICAL INSTITUTE**
**Mathematics I: BSDS First Year**
**Semester I, Academic Year 2024-25**
**Final Exam**

**Date: 09/12/2024** **Full Marks: 50** **Duration: 3 Hours**

- **Show all your work and write explanations when needed. If you are using a result stated and/or proved in class, please quote it correctly.**
- **This is a closed-book exam. You are NOT allowed to use class notes, books, homework solutions, list of theorems, formulas etc.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function such that $f(0) = f'(0) = 0$ and $|f''(x)| \leq 1$ for all $x \in \mathbb{R}$.
   (a) (6 marks) Prove that $|f(x)| \leq 1/2$ for all $x \in [-1, 1]$.
   (b) (1 + 3 =4 marks) Show that $f$ is Riemann integrable on $[-1, 1]$ and
   $$\int_{-1}^{1} f(x)dx \in [-1, 1].$$

2. (a) For each $s \in (0, \infty)$, define a function $\phi_s : (0, \infty) \to (0, \infty)$ by
   $$\phi_s(x) = x^{-s}, \quad x \in (0, \infty).$$
   Show the following:
   - (3 marks) $\phi_s$ is Riemann integrable on $(0, 1]$ if $s < 1$.
   - (3 marks) $\phi_s$ is Riemann integrable on $[1, \infty)$ if $s > 1$.

   (b) (4 marks) Show that the function $g : (0, \infty) \to (0, \infty)$ defined by
   $$g(x) = \frac{1}{x^2 + \sqrt{x}}, \quad x \in (0, \infty)$$
   is Riemann intergrable on $(0, \infty)$.

**Please Turn Over**

3. Define a map $T : \mathbb{R}^4 \to \mathbb{R}^3$ by

$$T\big((x_1, x_2, x_3, x_4)^T\big) = (x_1 + 2x_2,\ x_3 + x_4 - x_2,\ x_1 + 2x_3 + 2x_4)^T$$

for all $(x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4$.

(a) (4 marks) Show that $T$ is a linear transformation.

(b) (4 marks) Calculate the matrix $A$ for the linear transformation $T$ (with respect to the standard bases).

(c) (2 marks) Compute, with justification, the rank of $A$, where $A$ is as in 3(b) above.

4. Suppose $V := \mathbb{R}^4$ and define

$$S_1 := \{(v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4 : 2v_1 - v_2 = 2v_3 - v_4 = 0\},$$
$$S_2 := \{(u_1, u_2, u_3, u_4)^T \in \mathbb{R}^4 : u_1 + u_2 = u_3 + u_4 = 0\}.$$

(a) (3+3=6 marks) Show that $S_1$ and $S_2$ are linear subspaces of $V$.

(b) (5 marks) Find, with full justification, a basis for $S_1$.

(c) (1 mark) Compute the dimension of $S_1$.

(d) (6 marks) Show that the map $\psi$ defined by

$$\psi\big((v_1, v_2, v_3, v_4)^T\big) = (2v_1,\ -v_2,\ 2v_3,\ -v_4)^T$$

for all $(v_1, v_2, v_3, v_4)^T \in S_1$ is a vector space isomorphism from $S_1$ onto $S_2$. Justify all the steps.

(e) (2 marks) Using 4(c) and 4(d) above (or otherwise), compute the dimension of $S_2$.