Graph-Theory

§ Definitions and examples

Defn: A graph is a set of points, called vertices, together with a collection of lines, called edges, connecting some of the points. The set of vertices must not be empty.

If G is a graph we may write $V(G)$ and $E(G)$ for the set of vertices and the set of edges respectively. The number of vertices is written $|G|$, and the number of edges is written $e(G)$ (or $|E|$).

- A graph may have multiple edges, i.e., more than one edge between some pair of vertices, or loops, i.e., edges from a vertex to itself. A graph without multiple edges or loops is called simple. Many natural problems only make sense in the setting of simple graphs.

Defn: For each vertex v, the set of vertices which are adjacent to v is called the neighbourhood of v, written $\Gamma(v)$. A neighbour of v is any element of the neighbourhood.

Remark: Normally we do not include v in $\Gamma(v)$; it is only included if there is a loop.

Defn: The degree of a vertex v, written $d(v)$, is the number of ends of edges which connect to that vertex.

Example: In a simple graph, $d(v) = |\Gamma(v)|$

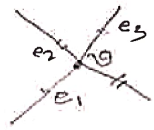
loop: As both ends of a loop go to the same vertex, each loop contributes 2 to the degree.

Qn: What happens if we add up the degrees of all the vertices?

Lemma (Euler's handshaking Lemma): The sum of the degrees of the vertices of a graph is equal to twice the number of edges.

Remark: We shall consider only locally-finite graphs i.e., graphs such that vertices occur ~~inf~~ in finitely many edge pairs; (each vertex is connected to only a finite number of edges).

Pf of Lemma:



Define a set of incidences:

$$\mathcal{I} := \{ (v,e) : v \in V, e \in E, v \text{ is an endpoint of } e \}.$$

Count $\mathcal{I}$

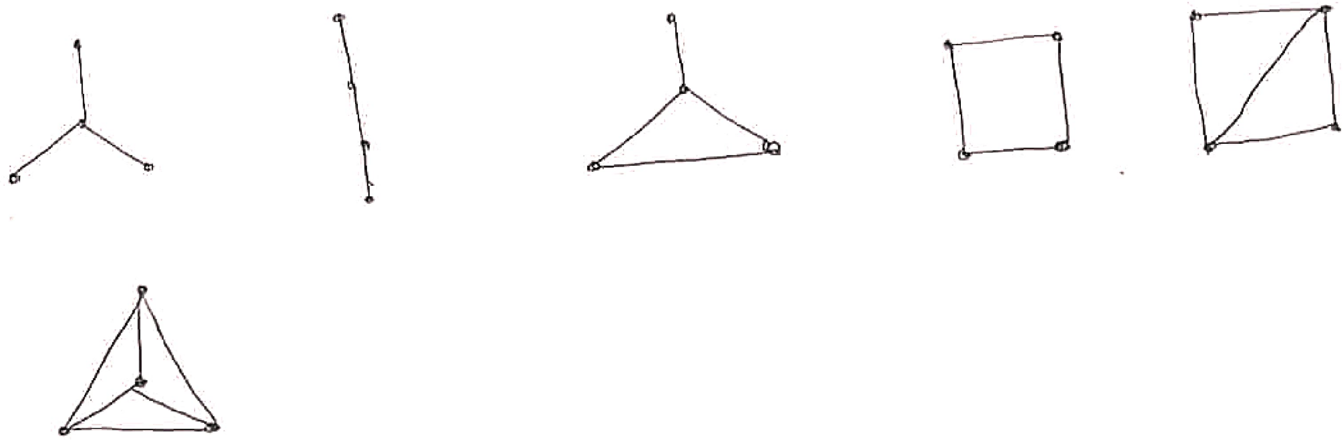
By vertices: Fix a vertex v. The number of edges ~~ind~~ incident to v is $\deg(v)$. Therefore, counting incidences by summing over vertices gives:

$$|\mathcal{I}| = \sum_{v \in V} \deg(v)$$

By edges: Each edge $e \in E$ has exactly two end points, so it contributes exactly 2 ordered incidences (u,e) & (w,e). Hence counting by edges gives

$$|I| = \sum_{e \in I} 2 = 2|E|$$

$\Rightarrow$ By equating $\boxed{2|E| = \sum_{v \in V} \deg(v)}$



Examples of graph (Real World)

1. (Facebook graph): V is the set of Facebook users and an edge is places between two vertices, if they are friends of each other.

2. V is the set of cities and an edge represents roads/trains/air between cities

3. Collaboration graph V is the set of all mathematicians who have published articles and an edge represents that two mathematicians have collaborated on a paper together.

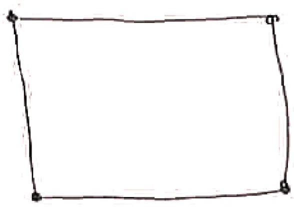
Defn: (Weighted graphs) A graph G with a weight function $w: E \to \mathbb{R}$.

Ex: G is the Road network with the weight w denoting average traffic in a day.

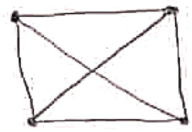
A **complete graph** is a graph in which every pair of distinct vertices is connected by an edge.

* If the graph has n vertices, the complete is is denoted by K_n
* K_n has exactly $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.
* Every vertex in K_n has degree "$n-1$"

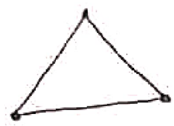
Example:



Not complete



Complete



Example:

* Intersection Graph : Let $\underbrace{S_1, S_2, \ldots S_n}_{=\mathcal{F}}$ be subsets of a set S.

Define G as follows :

* Each vertex of G corresponds to a set $S_i \in \mathcal{F}$.
* Two vertices S_i and S_j are connected by edge iff S_i & S_j intersect, i.e., $S_i \cap S_j \neq \phi$.

G is complete iff every two distinct sets in $\mathcal{F}$ intersect.

Pf: Suppose G is complete. By defn. of completeness every pair of distinct vertices v_i, v_j is joined by an edge. An edge between v_i, v_j means exactly $S_i \cap S_j = \phi$.

Conversely, $\overset{\forall i \neq j}{S_i \cap S_j} \neq \phi \Rightarrow$ then for every pair of distinct vertices v_i, v_j we have an edge between them.
$\Rightarrow G$ is complete.

Delaunay graph: $P \subseteq \mathbb{R}^N, N \geq 1$ - finite set of points.

For $y \in \mathbb{R}^N$, $d(y, P) = \min_{x \in P} |x-y|$.

Define $C_x := \{ y : d(y, P) = |x-y| \}, x \in P$.

Delaunay graph is the intersection graph on P with intersecting sets C_x, $x \in P$. C_x is called as the Voronoi cell of x.

" C_x " := is the set of all points points in $\mathbb{R}^d$ that are at least as close to x to any other point in P.

$G(P) = (P, E)$ where $(x, x') \in E$ iff $C_x \cap C_{x'} \neq \phi$.

Euclidean Lattices: Let $B_r(x)$ be the closed ball of radius r centred at x. The N- dimensional integer lattice is the intersection graph formed with $\mathbb{Z}^N$ as vertex set & $B_{1/2}(z)$, $z \in \mathbb{Z}^N$ as the intersecting sets.

Construction:

Step-1: Take N- dimensional lattice integer:

$$\mathbb{Z}^N := \{ (n_1, n_2, .., n_N) : n_i \in \mathbb{Z}^N \}$$

Step-2: For each lattice point $z \in \mathbb{Z}^N$, consider the closed Ball:

$$B_{1/2}(z) = \{ y \in \mathbb{R}^N : |y - z| \leq 1/2 \}.$$

Step-3: $G = (V, E)$ as follows

- Vertex $V = \mathbb{Z}^N$.
- Two vertices $z, z' \in \mathbb{Z}^N$ are adjacent (join by a edge) if their closed Ball intersects:

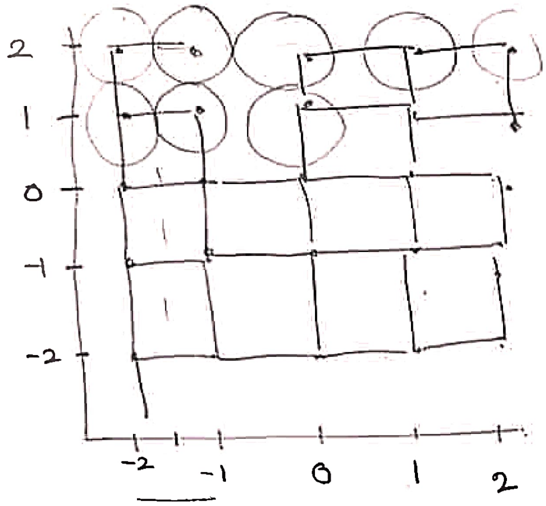
$$(z, z') \in E \iff B_{1/2}(z) \cap B_{1/2}(z') \neq \phi.$$

Step-4: Characterization

- Two Balls $B_{1/2}(z)$ & $B_{1/2}(z')$ intersect iff

$$|z-z'| \leq 1.$$

- Since both z, z' are in $\mathbb{Z}^N$, this condition means they differ in exactly one co-ordinate by ± 1 are equal in all others.

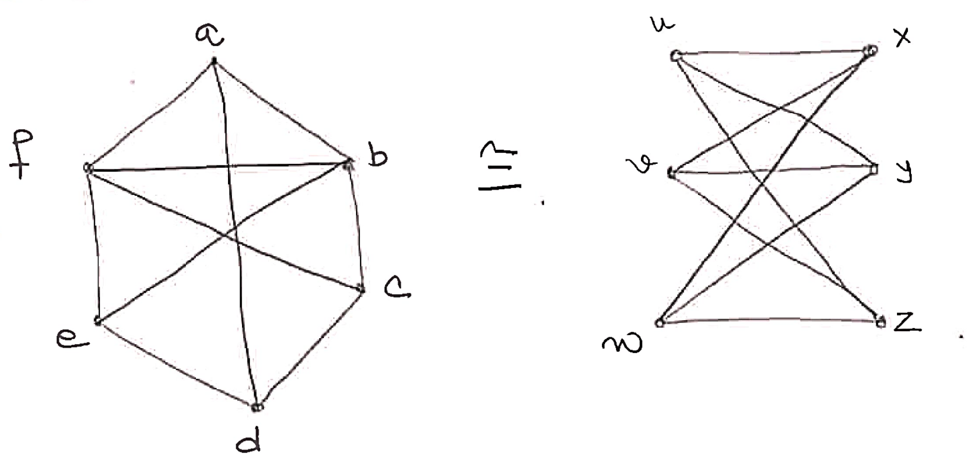


Definition: If G and H are two graphs then an isomorphism between G and H is a bijection

$$\phi : V(G) \to V(H)$$

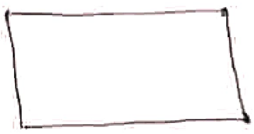
such that for any $v, w \in V(G)$ the number of edges between v and G in G is the same as the numbers of edges between $\phi(v)$ & $\phi(w)$ in H. If such a bijection exists we say that G and H are isomorphic & write $G \cong H$.

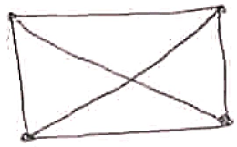
Example



Defn: Let G and H be graphs. We say that H is a subgraph of G ("G" contains "H") if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$ and $E(H) \subseteq E(G)$ then H is a spanning subgraph.

Ex:





G is always a subgraph of itself. A proper subgraph of G is any subgraph other than itself.

Defn: A graph is connected if it cannot be written as a disjoint union of two graphs, (there does-not exist a partition of its vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$ s.t no edge connect V_1 with V_2).

* Regular graph: A regular graph is one in which all the vertices have the same degree.

Examples:

$1 \to 4$ $(1 \to 2 \to 3 \to 4)$

$V = \{1, 2, 3, 4\}$

$E = \{\{1,2\}, \{2,3\}, \{3,4\}\}$.

1 2 3 4.

disconnected

$V = \{1, 2, 3, 4\}$.

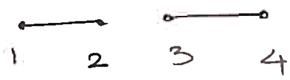
$E = \{\{1,2\}, \{3,4\}\}$.

$G = \{1,2\} \cup \{3,4\}$.

No path between

Defn: G is **k-Regular** if every vertex has degree k.

- A 0-regular graph is just a collection of isolated vertices.
- A 1-regular graph is a collection of disjoint edges.



- $\# |E| = \frac{nk}{2}$ (For k-regular graph).

(The sum of degrees of all vertices in a k-regular graph with n vertices is nk. By the Handshaking Lemma this is equal to $2|E|$ (2 times number of edges)).

$\Rightarrow \quad 2|E| = nk. \quad \Rightarrow \quad |E| = \frac{nk}{2}$.

Bipartite Graphs:

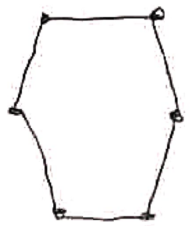
Defn: A graph is bipartite if we can colour the vertices red and blue in such a way that each edge connects a red vertex to a blue vertex.

* A bipartite graph whose vertex set can be divided into two disjoint subsets V_1 & V_2 such that
* Every edge of the graph joins a vertex in V_1 to a vertex in V_2.
* No edge exists between two vertices within the same subset.

$G = (V, E)$ if there exists a partition of the vertex set $V = V_1 \cup V_2$, $V_1 \cap V_2 = \phi$

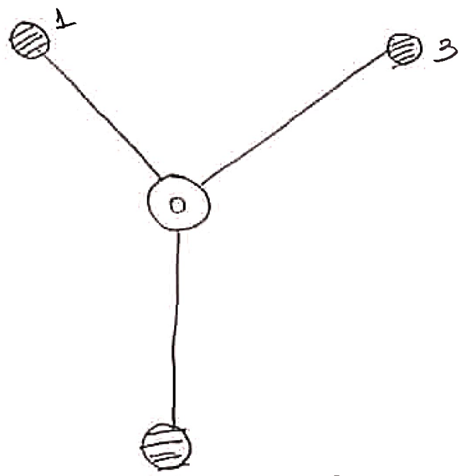
$$E \subseteq \{ (u, v) ; u \in V_1, v \in V_2 \}.$$

Example:

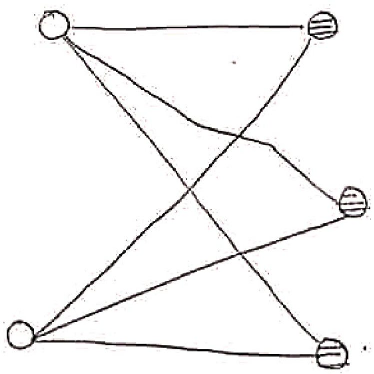


Hexagon (C_6).

Not a bipartite graph (C_3).

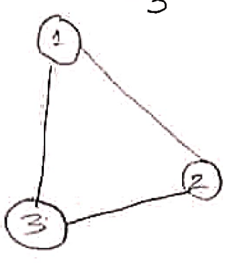


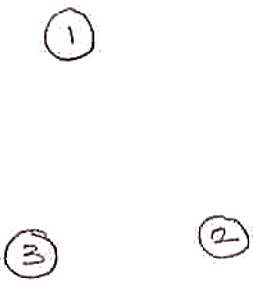
Star Graph



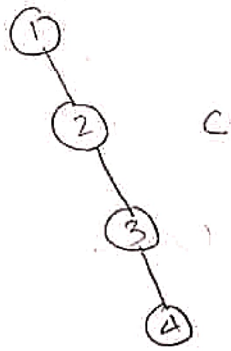
The Complement Graph: Let G be a simple graph. The complement of G, written $\overline{G}$ or G^c, is the simple graph with the same vertex set as G such that two vertices are adjacent in $\overline{G}$ iff they are not adjacent in G. The complement of the complement is the original graph, i.e., $\overline{(\overline{G})} = G$.

C_3





complement of C_3.



P_4

?

Complement $\Longrightarrow$

Defn: A walk in a graph is a sequence of the form $v_1 e_1, v_2, e_2, \ldots, v_r$ for some $r \geq 1$, where the v_i are vertices and e_i is an edge from v_i to v_{i+1} for each $1 \leq i < r$.

Remark: If $r = 1$ the walk consists of a single vertex and no-edges.
(A walk in a graph is an alternating sequence of vertices and edges, starting and ending with vertices). $A \to B \to C \to A \to B$ (vertices & edges Repeat)

Defn: A path is a walk in which all vertices (& hence all edges) are distinct, i.e., $P = v_1 \cdots v_p$ is said to be a path from v_1 to v_p if (v_i, v_{i+1}) are distinct for all i. A walk is closed if $v_p = v_1$. $A \to B \to C$ ($A \to B \to C \to A$ is not a path but closed walk or cycle).

A path is simple (also called as self-avoiding walk) if $v_i \neq v_j$ for all $i \neq j$. A walk or path is said to be closed if $v_k = v_1$ and else open. A closed path is called circuit. A circuit $C = v_1 \ldots v_{k-1} v_1$ with no repetition of intermediate vertices is called a cycle, $v_1 \ldots v_{k-1}$ are distinct.

Lemma: Let G be a simple graph with atleast two vertices. Then G must contain at least two vertices with the same degree.

Pf:

Let $G = (V, E)$ be a simple graph with n (≥ 2) vertices & m- edges.

Suppose no two vertices have the same degree, i.e., $\forall\, u, v \in V$, $deg(u) \neq deg(v)$.

A G is simple, the highest possible degree of any particular vertex can at most be connected to every other vertex in G.

So, for any $v \in V$, $0 \leq deg(v) \leq n-1$, and therefore there are n-many choices of degree that can be assigned to a particular vertex.

As there are n- vertices & n- (distinct) possible degrees, and as G cannot contain two vertices with same degree, then there must be an one-to-one correspondence between

$$V = \{v_1, v_2, \ldots, v_n\} \Leftrightarrow D = \{0, 1, \ldots, n-1\}$$

This implies, one vertex in G has degree 0, and another vertex has degree $n-1$, which is impossible (why)

(degree '0' means isolated point $\Rightarrow$ cannot have degree '$n-1$').

If G does-not have a vertex with degree 0 or $n-1$, then there are fewer elements in D than can be assigned to those in V, and by pigeon-hole principle, at least two element in V must be map to same degree.

Qn: A connected graph and path ?

Theorem: A graph G is connected iff there is a path between every pair of vertices.

- pf - Complete it! (Exe)

Trees:

We say a graph has a cycle if it has a subgraph isomorphic to C_n for some n.

Defn: A **tree** is a connected graph with no cycles.

A **forest** is a graph with no-cycles.

Remark: A tree must be a simple graph.
If G is a tree, we refer to a vertex of G with degree 1 as a leaf.

Theorem: A connected graph with n-vertices has atleast $(n-1)$ edges.

Pf: When $n=1$ there is nothing to prove.

Induction hypothesis: Assume the statement is true for any connected graph with k-vertices $(k\geq 1)$.

- Considers a connected graph with $k+1$ vertices.
- Remove a vertex v and all the edges incident to v.

Let the remaining graph be G'.

Case-I: If G' is connected.

By the induction hypothesis, G' has atleast $k-1$ edges.

To reconnect v to G', we need atleast one edge connecting v to some vertex in G'. Total edges in $G \geq k-1+1 = k$
$= (k+1) - 1$.

Case-II: G' is disconnected.

- Let G' split into components $C_1, C_2, \ldots, C_m$, $m \geq 2$.

Since G was connected, v must have had atleast one edge to each component.

- Each C_i has n_i ($n_i \leq k$) vertices, and by induction hypothesis, atleast $(n_i - 1)$ edges.

Adding v and the connecting edges, total

$$\text{edges} \geq \sum_{i=1}^{m}(n_i - 1) + m$$

$$= \sum_{i=1}^{m} n_i - \cancel{m} + \cancel{m}$$

$$= \sum_{i=1}^{m} n_i = k. = (k+1)-1.$$

$e(G) \geq n-1$

Qn: Does there exists a connected graph (with n-vertices) for which $e(G) = n-1$.

Theorem: Let G be a connected graph with n-vertices. Then G is a tree iff $e(G) = n-1$. (ExR)

Pf: If G is a tree.

- For $n = 1$, the tree has no edges, so $e(G) = 0 = n-1$.
- Assume every tree with k-vertices has exactly $k-1$ edges.
- Consider a tree with $k+1$ vertices.
- Since T is a finite tree, it has atleast one leaf (a vertex of degree 1).
- Remove that leaf & its incident edge.

The Remaining graph T' is still a tree with k-vertices.

~~By ind~~

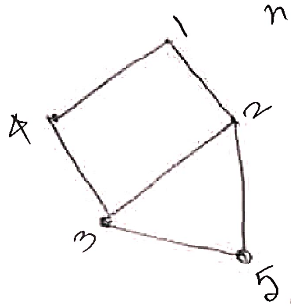
By induction $e(T') = k-1$.

- Then adding back the removed edge gives with n-vertices.
  $e(T) = k$. Thus, a tree has exactly $n-1$ edges.

If G is connected $e(G) = n-1$, then G is a tree

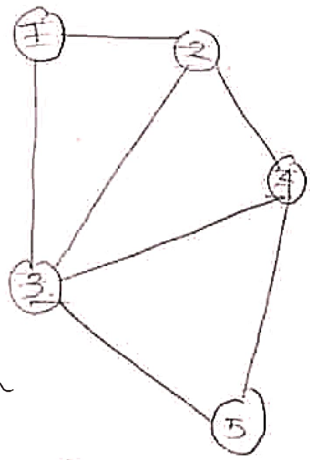
Suppose, for contradiction, G contains a cycle.

- Remove one edge ~~of~~ from this cycle.
- The graph remains connected (Since there is still an alt; path).
- For a connected graph with n-vertices, there is atleast $(n-1)$ edges (but here there $n-2$ edges $\rightarrow \leftarrow$).

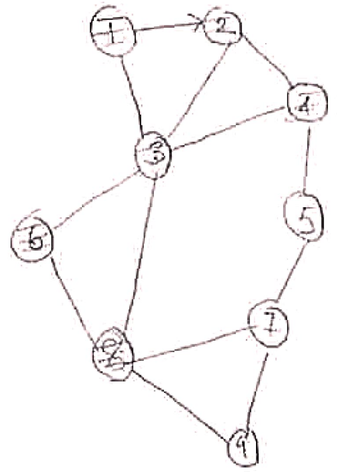


1. Walk: Edges can be repeated. & Vertices can be repeated.
   $(1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 2 \rightarrow 1 \rightarrow 3)$.

2. Trail: Edges cannot be Repeated, but vertices can be repeated.
   $1 \rightarrow 3 \rightarrow 8 \rightarrow 6 \rightarrow 3 \rightarrow 2$.
   $(1-3-8-6-3-2-1)$ Closed trail



3. Closed Trail is called circuit.
   Edges cannot be repeated, but vertices can be.



4. Path: Vertices & Edges both are not repeated.

5. Cycle: Closed Path is a cycle.

Theorem: Every finite tree with atleast two vertices contains atleast two leaves (vertices of degree 1).

Pf. Take a longest path P in T: Suppose the path is

$$v_1 - v_2 - \cdots - v_k. \quad , k \geq 2$$

Claim: v_1 & v_k are leaves.

- If v_1 had degree atleast 2, then there would be some neighbour $w \neq v_2$, then extending the path upto w, contradicting the maximality of P.

* Same with v_k.

Remark: For infinite tree, this is not the case.

e.g. $\mathbb{Z}$.

§ Euler's theorem and König's theorem on bi-partite graphs

<u>Defn</u> (Eulerian graph). A circuit (a closed trail) in a graph visiting every edge exactly once and every vertex is called an Eulerian circuit and a graph that has an <u>Eulerian circuit</u> is called an <u>Eulerian graph</u>.

<u>Theorem (Characterization)</u>: A finite connected graph is Eulerian iff every vertex has even degree.

<u>pf:</u> Let $G = (V, E)$ be a connected graph. If G has only one vertex, the statement of the theorem follows trivially. So we assume that $|V| \geq 2$. Since G is connected $|E| \geq 1$.

Suppose that $C := v_1v_2\, v_2v_3 \ldots, v_nv_1$ is an Eulerian circuit of G. If we travel through C departing from v_1, then we will pass through every edge of G exactly once. Moreover, if m_i is the number of times we pass by v_i for $i \neq 1$, then $\deg(v_i) = 2m_i$ as in order to touch any of these vertices we have to travel through two incident edges we haven't encountered before. A similar argument shows that $\deg v_1 = 2m_1$ if m_1 is the number of times we visit v_1 after departure, as the last edge of G that is used to arrive to v_1 for last time canbe matched with the first edge, the one we used to depart from v_1.

For the converse, suppose that every vertex of G has even degree. We proceed by induction on the number n of edges. The implication trivially holds when $n=0$ (or $n=3$). Suppose $|E| = n > 3$ and also that the statement of the theorem holds for any connected graph with less than n edges. Choose a vertex $v_1 \in V$ and start travelling from v_1 through consecutive edges of G without repeating any of them until encountering a vertex from which we cannot continue travelling because all of the edges connected to this vertex has been already used. This will give a trail $T := v_1 v_2, v_2 v_3, \ldots v_\ell v_{\ell+1}$.
If $v_{\ell+1} \in V \setminus \{v_1\}$, then $v_{\ell+1}$ would have been connected to an odd number of edges in T, and, given $\deg v_{\ell+1}$ is even, we would have been able to continue travelling.
Thus $v_{\ell+1} = v_1$. We have proved that G contains a closed Trail. Let C be one with maximum number of edges.

We Claim that C is an Eulerian Circuit. Suppose, by way Contradiction that this is not the case.

Consider the subgraph $G' := (V, E')$ of G, where $E' = E \setminus C$. Since C is not an Eulerian Circuit

$\Rightarrow E'$ is non-empty. Take $e \in E'$ & let H be the connected component of G' containing e. Since H is connected & $|E'| < |E|$ it follows from the induction hypothesis that H has an Eulearian Circuit C'. Observe that one of the edges of C must be connected to a vertex in x in H, as otherwise G would be disconnected. Now notice that the closed trail that results from concatenating C & C' via x has more edges than C. This contradicts, however the maximality of C. As a consequence C must be an Eulerian circuit.

Theorem: A graph is bipartite iff it has no odd cycles.

pf: Suppose G is bipartite with partition $V = V_1 \cup V_2$. Let $v_0 v_1 \ldots v_k v_0$ be a cycle. W.L.O.G, we may assume that $v_0 \in V_1$. By bi-partiteness, $v_i \in V_2$ for $1 \le i \le k$ and i-odd and $v_i \in V_1$ for $1 \le i \le k$ & i-even. Since $v_k \sim v_0$, $v_k \in V_2$ & hence k is odd. Thus, the length of the cycle $k+1$ is even.

If part: Let $v_0 \in V$. Set $v \in V_1$ if $\text{dist}(v_0, v_1)$ is even or else $v \in V_2$.

(no. of shortest path / length of bet. v_0, v_1)

W.L.O.G, suppose $v \overset{\text{edge}}{\sim} w$ for $v, w \in V_1$. Then Let P, P' be respectively the shortest paths from v_0 to v, w respectively. Let P^* be the inversed path from v to v_0. Since $v, w \in V_1$

P^*, P' have even lengths &

so $P' \underline{wv} P^*$ is a closed walk starting at v & has odd length. $\Rightarrow$ it has an odd length cycle a contradiction. Thus there are no edges between vertices in V_1 or V_2 respectively, i.e G is bipartite.

(Remark: Every closed odd length walk contains an odd length cycle).

The problem (Starting Point of Graph Theory, solved by Euler 1736).

In the city of Königsberg (now Kaliningrad), the river Pregel divided the city into two islands connected to each other and the mainland by seven bridges.

Qn: Is it possible to start at some point, walk across every bridge exactly one, and return to the starting point? Negative By Euler.