Solutions

1. Let A be an $n \times n$ real matrix.

* For any vector $x \in \mathbb{R}^n$,

$$x^T (AA^T) x = (A^T x)^T (A^T x) = \|A^T x\|^2 \geq 0.$$

Ily $$x^T (A^T A) x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0.$$

* Let $A \in M_{n\times n}$ have SVD:

$$A = U \Sigma V^T, \text{ where } U \in M_{n\times n}, V \in M_{n \times n}$$
are orthogonal

, Σ is a diagonal Matrix.

Consequently,

$$AA^T = (U\Sigma V^T)(U\Sigma V^T)^T = (U\Sigma V^T)(V \Sigma^T U^T)$$
$$= (U \Sigma \underbrace{V^T V} \Sigma^T U^T)$$
$$= (U \underbrace{\Sigma \Sigma^T} U^T).$$
$$= U (\Sigma \Sigma^T) U^T$$

and

$$A^T A = (U\Sigma V^T)^T (U \Sigma V^T)$$
$$= V (\Sigma^T \Sigma) V^T.$$

Now $\Sigma\Sigma^T$ is an $\underline{n\times n}$ block diagonal Matrix whose non-zero entries are $\sigma_1^2, \ldots \sigma_r^2$ (threst zero, & $r = \text{rank}(A)$).

Ily $\Sigma^T\Sigma$ is an $n\times n$ block diagonal Matrix whose non-zero diagonal entries are $\sigma_1^2, \ldots, \sigma_p^2$.

There fore, the non-zero eigen-values of AA^T & A^TA are exactly $\sigma_1^2, \ldots \sigma_r^2$.

2. Let V be a real inner product space with norm

$$\|x\| = (\langle x,x\rangle)^{1/2}.$$

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$$

$$= \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$$

$$\Rightarrow \quad 2\langle x,y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2$$

$$\Rightarrow \quad \langle x,y\rangle = \frac{1}{2}\left(\|x+y\|^2 - \|x\|^2 - \|y\|^2\right)$$

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3. Let n be the number of vertices of this Graph. First try to create 3-regular graph

Then by Euler's Theorem we see:

$$2\cdot 35 = 3n$$

$$\Rightarrow \quad 3n = 70$$

$\sum \deg(v_i) = 2|E|$ & $\deg(v_i) \geq 3$.

$2|E| \geq 3n$

$\Rightarrow 35\cdot 2 \geq 3n$

$\Rightarrow n \leq \frac{70}{3}$.

There is no integral solution, so this graph is not possible.

We next try a graph where every vertex, but one, is degree 3. We see.

$$70 = 3(n-1) + 4 \Rightarrow \frac{66}{3} = n-1$$

$\Rightarrow$ ~~22+1=~~

$\Rightarrow n = 23$

Note that this means that there are 22 (an even no. of vertices) of degree 3.

So, $n = 23$.

4. Let G be a simple graph of order 9.

Assume, by contradiction G is not connected. Then G has at least two components. ~~Let us consider~~ Let the vertex sets of two distincts components be of order n_1 & n_2 with $n_1, n_2 \geq 1$ & $n_1+n_2 \leq 9$.

~~Let~~ Pick v_1 in the component of order n_1 & v_2 in a different component of order n_2.

Since, There are no edges between different components:

$$deg(v_1) \leq n_1 - 1$$

& $$deg(v_2) \leq n_2 - 1$$

Therefore, $$deg(v_1) + deg(v_2) \leq n_1+n_2-2$$

$$\leq 9-2 = 7.$$

$\Rightarrow$ $$deg(v_1) + deg(v_2) \leq 7.$$

This contradicts the hypothesis that every pair of distinct vertices satisfies $deg(u) + deg(v) \geq 8$.

5. T is a tree of order n

Let n_1 be the number of vertices of degree 1.

Let n_2 be the number of vertices of degree 3.

$n_1 + n_2 = n$. $\rightarrow (1)$

$$\sum_{v \in V(T)} deg(v) = 1 \cdot n_1 + 3 \cdot n_2 = n_1 + 3n_2$$

Now, Since T is a tree $\Rightarrow |E| = (n-1)$.

By Euler's Lemma

$\Rightarrow \quad 2(n-1) = n_1 + 3n_2 \quad \to (1)$

$n = n_1 + n_2 \quad \to (2)$

$2(n-1) - n = 2n_2$

$\Rightarrow \quad n-2 = 2n_2 \quad \Rightarrow \quad \boxed{n_2 = \frac{(n-2)}{2}.}$

$\Rightarrow$ T contains $\left(\frac{n-2}{2}\right)$ vertices of degree 3.

6. Let P and Q be paths of maximum length in a connected graph G.

Let $V(P)$, $V(Q)$ denotes the vertex set of P & Q respectively.

Let $P = p_0 p_1 \dots p_r$

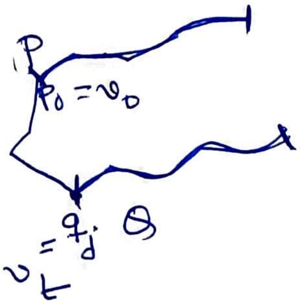
$Q = q_0 q_1 \dots q_s$.

where r & s denote the number of edges in P and Q respectevely. Since P & Q be paths of maximum length $\Rightarrow r = s$.

Assume $V(P) \cap V(Q) = \phi$

Choose p_0, since G is connected there is atleast one path from p_0 to Q, let q_j be the first vertex of Q reached along a shortest such path.

Let $p_0 = v_0 v_1 \dots, v_t = q_j$



be this shortest path from p_0 to Q.

~~By~~ None of the interval $[v_1, v_2] \dots [v_{t-2}, v_{t-1}]$, lie in P or in Q (Shortest path)

and P & Q ~~assume~~ are disjoint $\Rightarrow t \geq 1$.

New path

$$R := q_0 q_1 \cdots q_j (= v_t) v_{t-1} \cdots \underset{\substack{\shortparallel \\ p_0}}{v_0} p_1 \cdots p_r .$$

So R has $j+t+r$ edges. Since $t \geq 1$ & $j \geq 0$.

$\Rightarrow j+t+r \geq r+1 = s+1 > s$ $\rightarrow\leftarrow$ maximality of Q.