Metric Space Properties of Graph Distance

Properties of a Metric Space

A function $d : X \times X \to \mathbb{R}$ is called a **metric** on a set X if it satisfies the following properties for all $x, y, z \in X$:

1. **Non-negativity**
   $d(x, y) \geq 0$, and $d(x, y) = 0 \iff x = y$.
2. **Symmetry**
   $d(x, y) = d(y, x)$.
3. **Triangle Inequality**
   $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is called a **metric space**.

Graph Distance as a Metric

Let $G = (V, E)$ be a connected graph. We define $\delta_G(u, v)$ as the length of the shortest path between any two vertices $u, v \in V$. We can show that (V, δ_G) is a metric space.

1. Non-negativity

The length of any path is non-negative, so $\delta_G(u, v) \geq 0$. The shortest path from a vertex to itself has zero length, so $\delta_G(u, u) = 0$. For any two distinct vertices $u \neq v$ in a connected graph, a path exists and has a positive length, so $\delta_G(u, v) > 0$. Thus, $\delta_G(u, v) \geq 0$ and $\delta_G(u, v) = 0 \iff u = v$.

2. Symmetry

The shortest distance from vertex u to v is the same as the shortest distance from v to u:

$$\delta_G(u, v) = \delta_G(v, u)$$

3. Triangle Inequality

For any three vertices $u, v, w \in V$, the path formed by concatenating the shortest path from u to v and the shortest path from v to w is one possible path from u to w. Its length is $\delta_G(u, v) + \delta_G(v, w)$. The actual shortest path from u to w must have a length less than or equal to this path:

$$\delta_G(u, w) \leq \delta_G(u, v) + \delta_G(v, w)$$

Conclusion

Since δ_G satisfies all three properties, (V, δ_G) is a metric space.