Statistical Inference

B. Statistical Data Science 2nd Year Indian Statistical Institute

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Exercise Series 3

Exercise 1. Check whether the following distributions belong to the exponential family. If so, then identify a minimal sufficient statistic for the unknown parameters using the result proved in the class. Is the minimal sufficient statistic same as the one you obtained earlier?

(a) $f_\theta = \mathsf{Bernoulli}(p)$, $\theta = p$.

(b) $f_\theta = \mathsf{Poisson}(\lambda)$, $\theta = \lambda$.

(c) $f_\theta = \mathsf{Geometric}(p)$, $\theta = p$.

(d) $f_\theta = \mathsf{Uniform}(\theta, 1)$, $\theta < 1$. $f_\theta = \mathsf{Uniform}(\theta, \theta + 1)$, $\theta \in \mathbb{R}$.

(e) $f_\theta = \mathsf{Normal}(0, \sigma^2)$, $\theta = \sigma^2$. $f_\theta = \mathsf{Normal}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$.

(f) $f_\theta = \mathsf{Cauchy}(\theta, 1)$, $\theta \in \mathbb{R}$. $f_\theta = \mathsf{Cauchy}(0, \theta)$, $\theta > 0$. $f_\theta = \mathsf{Cauchy}(\mu, \sigma)$, $\theta = (\mu, \sigma)$.

(g) $f_\theta = \mathsf{Laplace}(\theta, 1)$, $\theta \in \mu$. $f_\theta = \mathsf{Laplace}(0, \theta)$, $\theta > 0$. $f_\theta = \mathsf{Laplace}(a, b)$, $\theta = (a, b)$.

(h) $f_\theta = \mathsf{Normal}(\theta, \theta^2)$. $f_\theta = \mathsf{Normal}(\theta, \theta)$.

(i) $\mathsf{Beta}(\alpha, \beta)$, $\theta = (\alpha, \beta)$.

(j) $\Gamma(\alpha, \lambda)$, $\theta = (\alpha, \lambda)$.

(k) $\mathsf{Pareto}(\mu, \alpha)$, $\mu > 0, \alpha > 0$, $\theta = (\mu, \alpha)$.

(l) $\mathsf{Weibull}(\lambda, k)$, $\lambda > 0, k > 0$, $\theta = (\lambda, k)$.

(m) $\mathsf{Multinomial}(n; p_1, \ldots, p_k)$.

Hint: $\mathsf{Multinomial}(n; p_1, \ldots, p_k)$ *is a discrete multivariate distribution with pmf*

$$\mathbb{P}(X_1 = x_1, \ldots, X_k = x_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}, \quad 0 \le x_1, \ldots, x_k \le n, x_1 + x_2 + \cdots + x_k = n.$$

(n) $\mathsf{BVN}(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \rho)$.

Exercise 2. Let $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \mathsf{Uniform}(0, \theta)$, where $\theta > 0$.

(a) Find $\mathbb{E}(X_{(1)})$ and $\mathbb{E}(X_{(n)})$.

(b) Define $R_n = X_{(n)} - X_{(1)}$ to be the range of the sample $X_1, \ldots, X_n$. Find $\mathbb{E}(R_n)$.

(c) What can you say about R_n as $n \to \infty$?

Exercise 3. Find the pdf of the sample median for the following cases. You may assume the number of observations n is odd, so that the sample median is $\widetilde{X} = X_{(\frac{n+1}{2})}$. If possible, calculate the expectation and variance of the sample median. Is it unbiased for the population median?

(a) $\mathsf{Uniform}(0, \theta)$.

(b) $\mathsf{Uniform}(\theta, \theta + 1)$.

(c) $\mathsf{Uniform}(-\theta, \theta)$.

(d) $\mathsf{Normal}(\mu, 1)$. $\mathsf{Normal}(\mu, \sigma^2)$.

(e) $\mathsf{Exponential}(\lambda)$.

(f) $\mathsf{Cauchy}(\mu, 1)$. $\mathsf{Cauchy}(\mu, \sigma)$.