where $v_0, v_1, \ldots, v_n$ are any vectors in $\mathbf{R}^n$ such that $v_0 = 0$ and $v_n = u$. We choose these vectors so they satisfy the recurrence relation $v_k = v_{k-1} + u_k e_k$. That is, we take

$$v_0 = 0, \quad v_1 = u_1 e_1, \quad v_2 = u_1 e_1 + u_2 e_2, \quad \ldots, \quad v_n = u_1 e_1 + \cdots + u_n e_n .$$

Then the kth term of the sum in (8.13) becomes

$$f(a + \lambda v_{k-1} + \lambda u_k e_k) - f(a + \lambda v_{k-1}) = f(b_k + \lambda u_k e_k) - f(b_k) ,$$

where $b_k = a + \lambda v_{k-1}$. The two points b_k and $b_k + \lambda u_k e_k$ differ only in their kth component. Therefore we can apply the mean-value theorem of differential calculus to write

$$f(b_k + \lambda u_k e_k) - f(b_k) = \lambda u_k \, D_k f(c_k), \tag{8.14}$$

where c_k lies on the line segment joining b_k to $b_k + \lambda u_k e_k$. Note that $b_k \to a$ and hence $c_k \to a$ as $\lambda \to 0$.

Using (8.14) in (8.13) we obtain

$$f(a + v) - f(a) = \lambda \sum_{k=1}^{n} D_k f(c_k) u_k .$$

But $\nabla f(a) \cdot v = \lambda$ V'(u) . $u = \lambda \sum_{k=1}^{n} D_k f(a) u_k$, so

$$f(a + v) - f(a) - \nabla f(a) \cdot v = \lambda \sum_{h=1}^{n} \{D_k f(c_k) - D_k f(a)\} u_k = \|v\| \, E(a, v),$$

where

$$E(a, v) = \sum_{k=1}^{n} \{D_k f(c_k) - D_k f(a)\} u_k .$$

Since $c_k \to a$ as $\|v\| \to 0$, and since each partial derivative $D_k f$ is continuous at a, we see that $E(a, v) \to 0$ as $\|v\| \to 0$. This completes the proof.

8.14 Exercises

1. Find the gradient vector at each point at which it exists for the scalar fields defined by the following equations :
   (a) $f(x, y) = x^2 + y^2 \sin(xy)$.
   (b) $f(x, y) = e^x \cos y$.
   (c) $f(x, y, z) = x^2 y^3 z^4$.
   (d) $f(x, y, z) = x^2 - y^2 + 2z^2$.
   (e) $f(x, y, z) = \log(x^2 + 2y^2 - 3z^2)$.
   (f) $f(x, y, z) = x^{y^z}$.
2. Evaluate the directional derivatives of the following scalar fields for the points and directions given :
   (a) $f(x, y, z) = x^2 + 2y^2 + 3z^2$ at (1, 1, 0) in the direction of i -j + $2k$.
   (b) $f(x, y, z)$ = (x/y)" at (1, 1, 1) in the direction of $2i + j - k$.
3. Find the points (x, y) and the directions for which the directional derivative of $f(x, y) = 3x^2 + y^2$ has its largest value, if (x, y) is restricted to be on the circle $x^2 + y^2 = 1$.
4. A differentiable scalar fieldfhas, at the point (1, 2), directional derivatives +2 in the direction toward (2, 2) and -2 in the direction toward (1, 1). Determine the gradient vector at (1, 2) and compute the directional derivative in the direction toward (4, 6).
5. Find values of the constants a, b, and c such that the directional derivative of $f(x, y, z) = axy^2 + byz + cz^2x^3$ at the point $(1, 2, -1)$ has a maximum value of 64 in a direction parallel to the z-axis.

6. Given a scalar field differentiable at a point $\boldsymbol{a}$ in $\mathbf{R}^2$. Suppose that $f'(\boldsymbol{a}; \boldsymbol{y}) = 1$ and $f'(\boldsymbol{a};$ z) = 2, where $\boldsymbol{y} = 2i$ + 3j and $\boldsymbol{z} = i$ + j. Make a sketch showing the set of all points (x, y) for which $f'(\boldsymbol{a};$ xi + $y\boldsymbol{j}) = 6$. Also, calculate the gradient V'(a).
7. Let f andg denote scalar fields that are differentiable on an open set S. Derive the following properties of the gradient:
   (a) grad f = 0 iff is constant on S.
   (b) grad $(f + g)$ = gradf + gradg .
   (c) grad (cf) = c gradfif c is a constant.
   (d) grad $(fg) = f \operatorname{grad} g + g \operatorname{grad} f$.
   (e) grad $\dfrac{f}{g} = \dfrac{g \operatorname{grad} f - f \operatorname{grad} g}{g^2}$ at points at which $g \neq 0$.
8. In $\mathbf{R}^3$ let $\boldsymbol{r}(x, y, z) = xi + y\boldsymbol{j} + zk$, and let $r(x, y, z) = \|\boldsymbol{r}(x, y, z)\|$.
   (a) Show that $\nabla r(x, y,$ z) is a unit vector in the direction of $\boldsymbol{r}(x, y,$ z).
   (b) Show that $\nabla(r^n) = nr^{n-2}\boldsymbol{r}$ if n is a positive integer. *[Hint:* Use Exercise 7(d).]
   (c) Is the formula of part (b) valid when n is a negative integer or zero?
   (d) Find a scalar field f such that $\nabla f = \boldsymbol{r}$.
9. Assume f is differentiable at each point of an n-ball $B(\boldsymbol{a})$. If f '(x; $\boldsymbol{y}$) = 0 for n independent vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$ and for every $\boldsymbol{x}$ in B(a), prove that f is constant on B(a).
10. Assume f is differentiable at each point of an n-ball B(a).
   (a) If ∇f (x) = 0 for every $\boldsymbol{x}$ in B(u), prove that f is constant on B(a).
   (b) If $f(\boldsymbol{x}) \leq f(\boldsymbol{a})$ for all $\boldsymbol{x}$ in B(u), prove that $\nabla f(\boldsymbol{a}) = 0$.
11. Consider the following six statements about a scalar field f: S $\to$ $\mathbf{R}$, where S $\subseteq \mathbf{R}^n$ and $\boldsymbol{a}$ is an interior point of S.
   (a) f is continuous at a.
   (b) f is differentiable at a.
   (c) f '(a; y) exists for every y in $\mathbf{R}^n$.
   (d) All the first-order partial derivatives off exist in a neighborhood of $\boldsymbol{a}$ and are continuous at $\boldsymbol{a}$.
   (e) ∇f *(a)* = *0.*
   (f) $f(\boldsymbol{x}) = \|\boldsymbol{x} - \boldsymbol{a}\|$ for all $\boldsymbol{x}$ in $\mathbf{R}^n$.

In a table like the one shown here, mark T in the appropriate square if the statement in row (x) always implies the statement in column (y). For example, if (a) always implies (b), mark T in the second square of the first row. The main diagonal has already been filled in for you.

	a	b	c	d	e	f
a	T					
b		T				
c			T			
d				T		
e					T	
f						T

8.15 A chain rule for derivatives of scalar fields

In one-dimensional derivative theory, the chain rule enables us to compute the derivative of a composite function g(t) $=f[r(t)]$ by the formula

$$g'(t) = f\,'[r(t)] \cdot r'(t).$$