BSDS (2024, Semester II): Statistics II
End-semester (Backpaper) Examination

NAME: Time: 180 minutes
ROLL: Total attainable marks: 50

Instructions:

1. Answer all questions.
2. Show all your work clearly.
3. You may use the provided facts/hints where applicable.

1. Let $X_1, ..., X_n$ be a random sample from uniform$(0, \theta)$ distribution. The highest order statistics $X_{\{n\}} = \max\{X_1, ..., X_n\}$ is a complete and sufficient statistic for this class of distributions: Find the UMVUE of θ.

   - $F_X(x) = \frac{x}{\theta}$, $0 < x < \theta$

   [6]

2. Suppose the lifetime of mobile batteries (in years) manufactured by company A is known to follow an exponential distribution with probability density function $f_\mu(x) = \frac{1}{\mu} \exp\{-x/\mu\}$, $x > 0$, $\mu > 0$. To estimate μ, company A tests 160 batteries and reports whether it has lasted for at least 1.25 years (15 months) or not. The outcome of the test shows that among the 160 batteries, only 64 lasted for at least 15 months. Based on this data, find an MLE of μ.
   [Hint: Use the invariance property of MLE.] [6]

3. Let $X_1, \cdots, X_n$ be iid random variables with continuous CDF F_X, and suppose $E(X_1) = \mu$. Define the random variables $Y_1, \cdots, Y_n$ as follows:

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the large sample distribution of $Z_n = \sum_{i=1}^{n} Y_i$, i.e., find $\{a_n\}$ $\{b_n\}$ such that

$$\frac{Z_n - a_n}{b_n} \to Z,$$

   where Z has a non-degenerate distribution. (b) Suppose it is known that $F_X(\mu) = 1/2$ for a sample of size 64 approximate the probability that Z_n lies in the interval $[16, 48]$. [3+3]

4. Suppose that a random sample of size n is to be taken from a distribution F with mean and the standard deviation is $\sigma = 3$. Use the central limit theorem to determine approximately the smallest value of n for which the following relation will be satisfied: $P(|\overline{X}_n - \mu| < 0.3) \geq 0.95$.
[You may use the following facts: Let τ_α be the upper-α point of $N(0,1)$ distribution, then $\tau_{0.025} = 1.96$ and $\tau_{0.05} = 1.64$] [6]

5. The lifetime of an equipment is normally distributed with mean and standard distribution 5. For testing the null hypothesis $H_0 : \theta \leq 30$ against the alternative hypothesis $H_A : \theta > 30$, a random sample of size n is chosen. Determine n and the cutoff c such that the test

$$\phi(x) = 1, \text{ if } \overline{x} \geq c$$
$$\phi(x) = 0, \text{ if } \overline{x} < c$$

has power function values 0.1 and 0.9 at the points $\theta = 30$ and $\theta = 35$ respectively. Draw the power function of the resultant test. [3+3+2]

6. The independent random variables $X_1, ..., X_n$ have common distribution

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (x/\beta)^\gamma & \text{if } 0 < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$$

(a) Find an MLE of β, say $\hat{\beta}_{\text{MLE}}$ when γ is known. (b) Find a $(1-\alpha)100\%$ confidence interval for β based on $\hat{\beta}_{\text{MLE}}$. [5+5]

7. A factory uses two different machines (Machines A and B) to produce metal rods. The management wants to check if the consistency (variation in rod length) is the same for both machines.

   (a) From Machine A, they take a sample of 16 rods and find a sample variance of 0.03 cm^2.

   (b) From Machine B, they take a sample of 25 rods and find a sample variance of 0.06 cm^2.

   Is there a significant difference between the variances of the two machines?

   [Write explicitly the modeling assumptions (if any), hypotheses to be tested, test statistic, test function and the conclusion obtained after testing. You may use the facts:

$$F_{0.025;15,25} = 2.38 \quad F_{0.025;15,24} = 2.44 \quad F_{0.975;15,25} = 0.38 \quad F_{0.975;15,24} = 0.37$$
$$F_{0.005;15,25} = 3.15 \quad F_{0.005;15,24} = 3.24 \quad F_{0.995;15,25} = 0.28 \quad F_{0.995;15,24} = 0.26$$

   where $F_{\alpha;n,m}$ is the upper-α point of $F_{n,m}$ distribution.] **[6]**

8. Let $X_1, ..., X_n$ be a random sample from normal(μ, σ^2) distribution, σ^2 is unknown. Consider the problem of testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. (a) Write the test procedure and test statistic for testing H_0 against H_1. Derive the distribution of the test statistic under H_0. (b) Derive the large sample distribution of the test statistic. (c) Consider an appropriate function of the test statistic, say $T(X, \mu)$, as a pivot. Starting from any two points (a, b) satisfying $P(a \leq T(X, \mu) \leq b) = (1 - \alpha)$, find an $(1 - \alpha)$-confidence interval for μ. (d) Show that, among all points (a, b) satisfying $P(a \leq T(X, \mu) \leq b) = (1 - \alpha)$, the width of the resultant confidence interval is minimum if $P(T(X, \mu) \leq a) = P(T(X, \mu) \geq b) = \alpha/2$. [4+4+4+4]