Section 8.17

Exercise 2(b)

Given $f(x,y) = e^{xy}\cos(xy^2)$, $X(t) = \cos t$, and $Y(t) = \sin t$. We need to find $F'(t)$ and $F''(t)$, where $F(t) = f\big(X(t), Y(t)\big)$.

First, find the derivatives of $X(t)$ and $Y(t)$:

$$X'(t) = -\sin t, \quad Y'(t) = \cos t$$

Next, find the partial derivatives of $f(x,y)$:

$$\begin{aligned}\frac{\partial f}{\partial x} &= ye^{xy}\cos(xy^2) - y^2e^{xy}\sin(xy^2)\\ \frac{\partial f}{\partial y} &= xe^{xy}\cos(xy^2) - 2xye^{xy}\sin(xy^2)\end{aligned}$$

Now, using the chain rule:

$$\begin{aligned}F'(t) &= \frac{\partial f}{\partial x}X'(t) + \frac{\partial f}{\partial y}Y'(t)\\ &= \left[(\sin t)e^{(\cos t)(\sin t)}\cos\big((\cos t)(\sin^2 t)\big) - (\sin^2 t)e^{(\cos t)(\sin t)}\sin\big((\cos t)(\sin^2 t)\big)\right](-\sin t)\\ &\quad+\left[(\cos t)e^{(\cos t)(\sin t)}\cos\big((\cos t)(\sin^2 t)\big) - 2(\cos t)(\sin t)e^{(\cos t)(\sin t)}\sin\big((\cos t)(\sin^2 t)\big)\right](\cos t)\\ &= -\sin^2 t\cdot e^{\cos t\sin t}\cos(\cos t\sin^2 t) + \sin^3 t\cdot e^{\cos t\sin t}\sin(\cos t\sin^2 t)\\ &\quad+\cos^2 t\cdot e^{\cos t\sin t}\cos(\cos t\sin^2 t) - 2\cos^2 t\sin t\cdot e^{\cos t\sin t}\sin(\cos t\sin^2 t)\\ &= (\cos^2 t - \sin^2 t)e^{\cos t\sin t}\cos(\cos t\sin^2 t) + (\sin^3 t - 2\cos^2 t\sin t)e^{\cos t\sin t}\sin(\cos t\sin^2 t)\\ &= e^{\cos t\sin t}[\cos(2t)\cos(\cos t\sin^2 t) - \sin t\cos(2t)\sin(\cos t\sin^2 t)]\end{aligned}$$

To find $F''(t)$, we would need to differentiate $F'(t)$ with respect to t, which involves further applications of the chain rule and product rule, leading to a complex expression.

Exercise 2(c)

Given $f(x,y) = \log\left(\frac{1+e^{x^2}}{1+e^{y^2}}\right)$, $X(t) = e^t$, and $Y(t) = e^{-t}$. We need to find $F'(t)$ and $F''(t)$, where $F(t) = f\big(X(t), Y(t)\big)$.

First, find the derivatives of $X(t)$ and $Y(t)$:

$$X'(t) = e^t, \quad Y'(t) = -e^{-t}$$

Next, find the partial derivatives of $f(x,y)$:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{1+e^{x^2}}\cdot e^{x^2}\cdot 2x = \frac{2xe^{x^2}}{1+e^{x^2}}\\ \frac{\partial f}{\partial y} &= \frac{-1}{1+e^{y^2}}\cdot e^{y^2}\cdot 2y = \frac{-2ye^{y^2}}{1+e^{y^2}}\end{aligned}$$

Now, using the chain rule:

$$\begin{aligned} F'(t) &= \frac{\partial f}{\partial x} X'(t) + \frac{\partial f}{\partial y} Y'(t) \\ &= \frac{2e^t e^{e^{2t}}}{1+e^{e^{2t}}} e^t + \frac{-2e^{-t} e^{e^{-2t}}}{1+e^{e^{-2t}}} (-e^{-t}) \\ &= \frac{2e^{2t} e^{e^{2t}}}{1+e^{e^{2t}}} + \frac{2e^{-2t} e^{e^{-2t}}}{1+e^{e^{-2t}}} \end{aligned}$$

To find $F''(t)$, we would need to differentiate $F'(t)$ with respect to t, which again involves further applications of the chain rule and quotient rule.

Exercise 3(b)

Given $f(x, y, z) = x^2 - y^2$ at a general point of the surface $x^2 + y^2 + z^2 = 4$ in the direction of the outward normal at that point.

The gradient of f is $\nabla f = (2x, -2y, 0)$. The gradient of $g(x, y, z) = x^2 + y^2 + z^2 - 4$ is $\nabla g = (2x, 2y, 2z)$, which is normal to the surface.

The directional derivative is given by $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$, where $\mathbf{u}$ is a unit vector in the direction of ∇g. So, $\mathbf{u} = \frac{\nabla g}{\|\nabla g\|} = \frac{(2x,2y,2z)}{\sqrt{4x^2+4y^2+4z^2}} = \frac{(x,y,z)}{\sqrt{x^2+y^2+z^2}} = \frac{(x,y,z)}{2}$.

Therefore,

$$\begin{aligned} D_{\mathbf{u}} f &= (2x, -2y, 0) \cdot \frac{(x, y, z)}{2} \\ &= \frac{2x^2 - 2y^2 + 0}{2} \\ &= x^2 - y^2 \end{aligned}$$

Section 8.22

Exercise 2

The substitution $u = (x - y)/2$ and $v = (x + y)/2$ changes $f(u, v)$ into $F(x, y)$. We want to express $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ in terms of $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$.

We have:

$$\begin{aligned} x &= u + v \\ y &= v - u \end{aligned}$$

Then, by the chain rule:

$$\begin{aligned}\frac{\partial F}{\partial x} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial f}{\partial u}\left(\frac{1}{2}\right) + \frac{\partial f}{\partial v}\left(\frac{1}{2}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)\\ \frac{\partial F}{\partial y} &= \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial f}{\partial u}\left(-\frac{1}{2}\right) + \frac{\partial f}{\partial v}\left(\frac{1}{2}\right) = \frac{1}{2}\left(-\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\right)\end{aligned}$$

Exercise 3

The equations $u = f(x, y)$, $x = X(s, t)$, and $y = Y(s, t)$ define u as a function of s and t, say $u = F(s, t)$.

(a) Express $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ in terms of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial X}{\partial s}, \frac{\partial X}{\partial t}, \frac{\partial Y}{\partial s}$, and $\frac{\partial Y}{\partial t}$.

By the chain rule:

$$\begin{aligned}\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s}\\ \frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t}\end{aligned}$$

(b) If $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$, show that

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + 2\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s}\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2$$

Differentiating $\frac{\partial F}{\partial s}$ with respect to s:

$$\begin{aligned}\frac{\partial^2 F}{\partial s^2} &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s}\right)\\ &= \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\right)\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial y}\right)\frac{\partial Y}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2}\\ &= \left(\frac{\partial^2 f}{\partial x^2}\frac{\partial X}{\partial s} + \frac{\partial^2 f}{\partial y\,\partial x}\frac{\partial Y}{\partial s}\right)\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \left(\frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial X}{\partial s} + \frac{\partial^2 f}{\partial y^2}\frac{\partial Y}{\partial s}\right)\frac{\partial Y}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2}\\ &= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + \frac{\partial^2 f}{\partial y\,\partial x}\frac{\partial Y}{\partial s}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2 + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2}\end{aligned}$$

Since $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}$, we have:

$$\frac{\partial^2 F}{\partial s^2} = \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s^2} + \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial s}\right)^2 + 2\frac{\partial X}{\partial s}\frac{\partial Y}{\partial s}\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s^2} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial s}\right)^2$$

(c) Find similar formulas for $\frac{\partial^2 F}{\partial s\,\partial t}$ and $\frac{\partial^2 F}{\partial t^2}$.

$$
\begin{aligned}
\frac{\partial^2 F}{\partial s\,\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial s}\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t}\right) \\
&= \frac{\partial^2 f}{\partial x^2}\frac{\partial X}{\partial s}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial s\,\partial t} + \frac{\partial^2 f}{\partial y\,\partial x}\frac{\partial Y}{\partial s}\frac{\partial X}{\partial t} + \frac{\partial^2 f}{\partial x\,\partial y}\frac{\partial X}{\partial s}\frac{\partial Y}{\partial t} + \frac{\partial^2 f}{\partial y^2}\frac{\partial Y}{\partial s}\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial s\,\partial t}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t}\right) \\
&= \frac{\partial^2 f}{\partial x^2}\left(\frac{\partial X}{\partial t}\right)^2 + \frac{\partial f}{\partial x}\frac{\partial^2 X}{\partial t^2} + 2\frac{\partial X}{\partial t}\frac{\partial Y}{\partial t}\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial Y}{\partial t}\right)^2 + \frac{\partial f}{\partial y}\frac{\partial^2 Y}{\partial t^2}
\end{aligned}
$$

Exercise 4(a)

Solve Exercise 3 in the special case: $X(s,t) = s + t, Y(s,t) = st$.

We have:

$$
\frac{\partial X}{\partial s} = 1, \quad \frac{\partial X}{\partial t} = 1, \quad \frac{\partial Y}{\partial s} = t, \quad \frac{\partial Y}{\partial t} = s
$$

$$
\frac{\partial^2 X}{\partial s^2} = 0, \quad \frac{\partial^2 X}{\partial t^2} = 0, \quad \frac{\partial^2 Y}{\partial s^2} = 0, \quad \frac{\partial^2 Y}{\partial t^2} = 0
$$

Then, from Exercise 3:

$$
\begin{aligned}
\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(t) = \frac{\partial f}{\partial x} + t\frac{\partial f}{\partial y} \\
\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(s) = \frac{\partial f}{\partial x} + s\frac{\partial f}{\partial y}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial s^2} &= \frac{\partial^2 f}{\partial x^2}(1)^2 + 2(1)(t)\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}(t)^2 = \frac{\partial^2 f}{\partial x^2} + 2t\frac{\partial^2 f}{\partial x\,\partial y} + t^2\frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 F}{\partial t^2} &= \frac{\partial^2 f}{\partial x^2}(1)^2 + 2(1)(s)\frac{\partial^2 f}{\partial x\,\partial y} + \frac{\partial^2 f}{\partial y^2}(s)^2 = \frac{\partial^2 f}{\partial x^2} + 2s\frac{\partial^2 f}{\partial x\,\partial y} + s^2\frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

Exercise 5

The introduction of polar coordinates changes $f(x,y)$ into $\varphi(r,\theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$. Express the second-order partial derivatives $\frac{\partial^2\varphi}{\partial r^2}, \frac{\partial^2\varphi}{\partial r\,\partial\theta}$, and $\frac{\partial^2\varphi}{\partial\theta\,\partial r}$ in terms of the partial derivatives of f.

We have:

$$
\begin{aligned}
\frac{\partial x}{\partial r} &= \cos\theta, \quad \frac{\partial x}{\partial\theta} = -r\sin\theta \\
\frac{\partial y}{\partial r} &= \sin\theta, \quad \frac{\partial y}{\partial\theta} = r\cos\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\varphi}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \\
\frac{\partial\varphi}{\partial\theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\theta} = -\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta
\end{aligned}
$$

Now we find the second derivatives:

$$
\begin{aligned}
\frac{\partial^2\varphi}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) \\
&= \frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\varphi}{\partial r\,\partial\theta} &= \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial\theta}\right) = \frac{\partial}{\partial r}\left(-\frac{\partial f}{\partial x}r\sin\theta + \frac{\partial f}{\partial y}r\cos\theta\right) \\
&= -\frac{\partial f}{\partial x}\sin\theta - r\sin\theta\left(\frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial y\,\partial x}\sin\theta\right) + \frac{\partial f}{\partial y}\cos\theta + r\cos\theta\left(\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin\theta\right) \\
&= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta - r\sin\theta\cos\theta\frac{\partial^2 f}{\partial x^2} - r\sin^2\theta\frac{\partial^2 f}{\partial y\,\partial x} + r\cos^2\theta\frac{\partial^2 f}{\partial x\,\partial y} + r\cos\theta\sin\theta\frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\varphi}{\partial\theta\,\partial r} &= \frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial r}\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta\right) \\
&= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta + \cos\theta\left(-\frac{\partial^2 f}{\partial x^2}r\sin\theta + \frac{\partial^2 f}{\partial x\,\partial y}r\cos\theta\right) + \sin\theta\left(-\frac{\partial^2 f}{\partial y\,\partial x}r\sin\theta + \frac{\partial^2 f}{\partial y^2}r\cos\theta\right) \\
&= -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta - r\sin\theta\cos\theta\frac{\partial^2 f}{\partial x^2} + r\cos^2\theta\frac{\partial^2 f}{\partial x\,\partial y} - r\sin^2\theta\frac{\partial^2 f}{\partial y\,\partial x} + r\sin\theta\cos\theta\frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

Exercise 6

The equations $u = f(x,y,z)$, $x = X(r,s,t)$, $y = Y(r,s,t)$, and $z = Z(r,s,t)$ define u as a function of r, s, and t, say $u = F(r,s,t)$. Use an appropriate form of the chain rule to express the partial derivatives $\frac{\partial F}{\partial r}, \frac{\partial F}{\partial s}$, and $\frac{\partial F}{\partial t}$ in terms of partial derivatives of f, X, Y, and Z.

By the chain rule:

$$
\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial X}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial Z}{\partial r} \\
\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial X}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial Z}{\partial s} \\
\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}\frac{\partial X}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial Y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial Z}{\partial t}
\end{aligned}
$$

Exercise 7(a)

Solve Exercise 6 in the special case: $X(r,s,t) = r + s + t$, $Y(r,s,t) = r - 2s + 3t$, $Z(r,s,t) = 2r + s - t$.

We have:

$$\frac{\partial X}{\partial r} = 1, \quad \frac{\partial X}{\partial s} = 1, \quad \frac{\partial X}{\partial t} = 1$$
$$\frac{\partial Y}{\partial r} = 1, \quad \frac{\partial Y}{\partial s} = -2, \quad \frac{\partial Y}{\partial t} = 3$$
$$\frac{\partial Z}{\partial r} = 2, \quad \frac{\partial Z}{\partial s} = 1, \quad \frac{\partial Z}{\partial t} = -1$$

Then, from Exercise 6:

$$\begin{aligned}
\frac{\partial F}{\partial r} &= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(1) + \frac{\partial f}{\partial z}(2) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + 2\frac{\partial f}{\partial z} \\
\frac{\partial F}{\partial s} &= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(-2) + \frac{\partial f}{\partial z}(1) = \frac{\partial f}{\partial x} - 2\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \\
\frac{\partial F}{\partial t} &= \frac{\partial f}{\partial x}(1) + \frac{\partial f}{\partial y}(3) + \frac{\partial f}{\partial z}(-1) = \frac{\partial f}{\partial x} + 3\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}
\end{aligned}$$

Exercise 12

Let $h(\mathbf{x}) = f[\mathbf{g}(\mathbf{x})]$, where $\mathbf{g} = (g_1, \dots, g_n)$ is a vector field differentiable at $\mathbf{a}$, and f is a scalar field differentiable at $\mathbf{b} = \mathbf{g}(\mathbf{a})$. Use the chain rule to show that the gradient of h can be expressed as a linear combination of the gradient vectors of the components of $\mathbf{g}$, as follows:

$$\nabla h(\mathbf{a}) = \sum_{k=1}^{n} D_k f(\mathbf{b}) \nabla g_k(\mathbf{a})$$

Let $\mathbf{x} = (x_1, \dots, x_p)$. Then $h(\mathbf{x}) = f(g_1(\mathbf{x}), \dots, g_n(\mathbf{x}))$. We want to find $\nabla h(\mathbf{a})$. The j-th component of $\nabla h(\mathbf{a})$ is $\frac{\partial h}{\partial x_j}(\mathbf{a})$.

By the chain rule:

$$\frac{\partial h}{\partial x_j}(\mathbf{a}) = \sum_{k=1}^{n} \frac{\partial f}{\partial g_k}(\mathbf{g}(\mathbf{a})) \frac{\partial g_k}{\partial x_j}(\mathbf{a}) = \sum_{k=1}^{n} D_k f(\mathbf{b}) \frac{\partial g_k}{\partial x_j}(\mathbf{a})$$

Thus, $\nabla h(\mathbf{a}) = \left(\frac{\partial h}{\partial x_1}(\mathbf{a}), \dots, \frac{\partial h}{\partial x_p}(\mathbf{a})\right)$. The k-th component of $\nabla g_k(\mathbf{a})$ is $\frac{\partial g_k}{\partial x_j}(\mathbf{a})$. Therefore,

$$\nabla h(\mathbf{a}) = \sum_{k=1}^{n} D_k f(\mathbf{b}) \nabla g_k(\mathbf{a})$$

Exercise 13

(a) If $\mathbf{f}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, prove that the Jacobian matrix $D\mathbf{f}(x, y, z)$ is the identity matrix of order 3.

The Jacobian matrix is given by:

$$Df(x,y,z) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{bmatrix}$$

Here, $f_1(x,y,z) = x$, $f_2(x,y,z) = y$, and $f_3(x,y,z) = z$. Therefore,

$$Df(x,y,z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

(b) Find all differentiable vector fields $\mathbf{f}: \mathbb{R}^3 \to \mathbb{R}^3$ for which the Jacobian matrix $D\mathbf{f}(x,y,z)$ is the identity matrix of order 3.

If $Df(x,y,z) = I$, then we must have:

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= 1, \quad \frac{\partial f_1}{\partial y} = 0, \quad \frac{\partial f_1}{\partial z} = 0 \\ \frac{\partial f_2}{\partial x} &= 0, \quad \frac{\partial f_2}{\partial y} = 1, \quad \frac{\partial f_2}{\partial z} = 0 \\ \frac{\partial f_3}{\partial x} &= 0, \quad \frac{\partial f_3}{\partial y} = 0, \quad \frac{\partial f_3}{\partial z} = 1 \end{aligned}$$

Integrating these equations, we get:

$$\begin{aligned} f_1(x,y,z) &= x + c_1 \\ f_2(x,y,z) &= y + c_2 \\ f_3(x,y,z) &= z + c_3 \end{aligned}$$

where c_1, c_2, c_3 are constants. Thus, the vector field $\mathbf{f}$ is given by:

$$\mathbf{f}(x,y,z) = (x + c_1)\mathbf{i} + (y + c_2)\mathbf{j} + (z + c_3)\mathbf{k}$$