Production Function

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Syllabus: Production Function

- **Production analysis:**
- Profit maximization, Cost minimization, Returns to scale, Cobb-Douglas and ACMS production functions.

The Production Function

- **The Production Function:**
- A Production Function is a mathematical function which relates the quantities of inputs and the quantities of output within a production unit, which may be variously defined as an activity or process, a firm, an industry or even a national economy. It is usually regarded as a technical relationship between the quantities of inputs and the maximum amount of output which can be produced with a given set of inputs.
- We shall consider two factors of production – the labour input (L) and the capital input (K), with the production function being

$$Q = f(K, L), \qquad \ldots(01)$$

- where Q is the output.

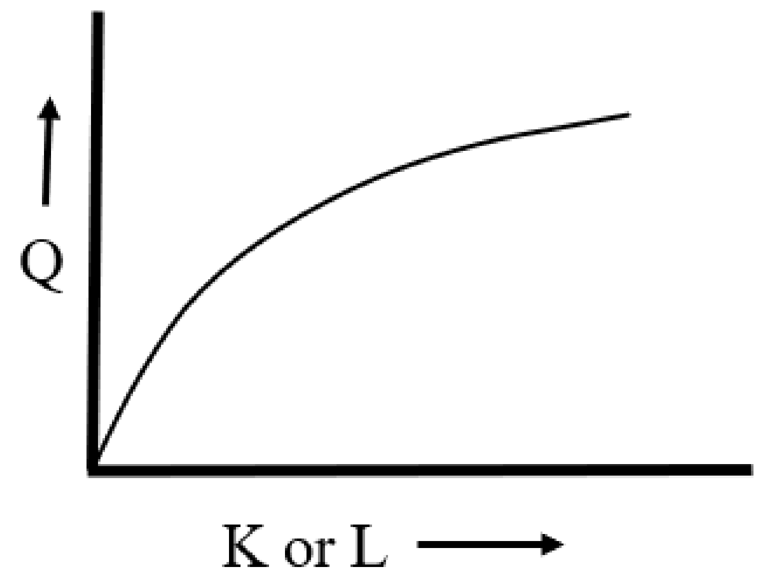
The General Shape of Production Function

- The classical theory of production assumes that the marginal products of labour and capital are positive, but they are diminishing so that the graph of output against capital or labour will have the general shape as illustrated below.

- It is assumed that (01) is a single valued continuous function which is twice differentiable so that the assumption of diminishing marginal products of capital and labour requires

$$\frac{\partial Q}{\partial K} = Q_K > 0, \frac{\partial Q}{\partial L} = Q_L > 0, \frac{\partial^2 Q}{\partial K^2} = Q_{KK} < 0, and \ \frac{\partial^2 Q}{\partial L^2} = Q_{LL} < 0, \ldots (02)$$

- where Q_K and Q_L are the marginal products of capital and labour.



Marginal Rate of Substitution

- A given level of output can be produced by different combinations of capital and labour and so

$$\mathrm{f(K, L) = Contant}$$

- traces out the isoquants. For variation along any isoquant

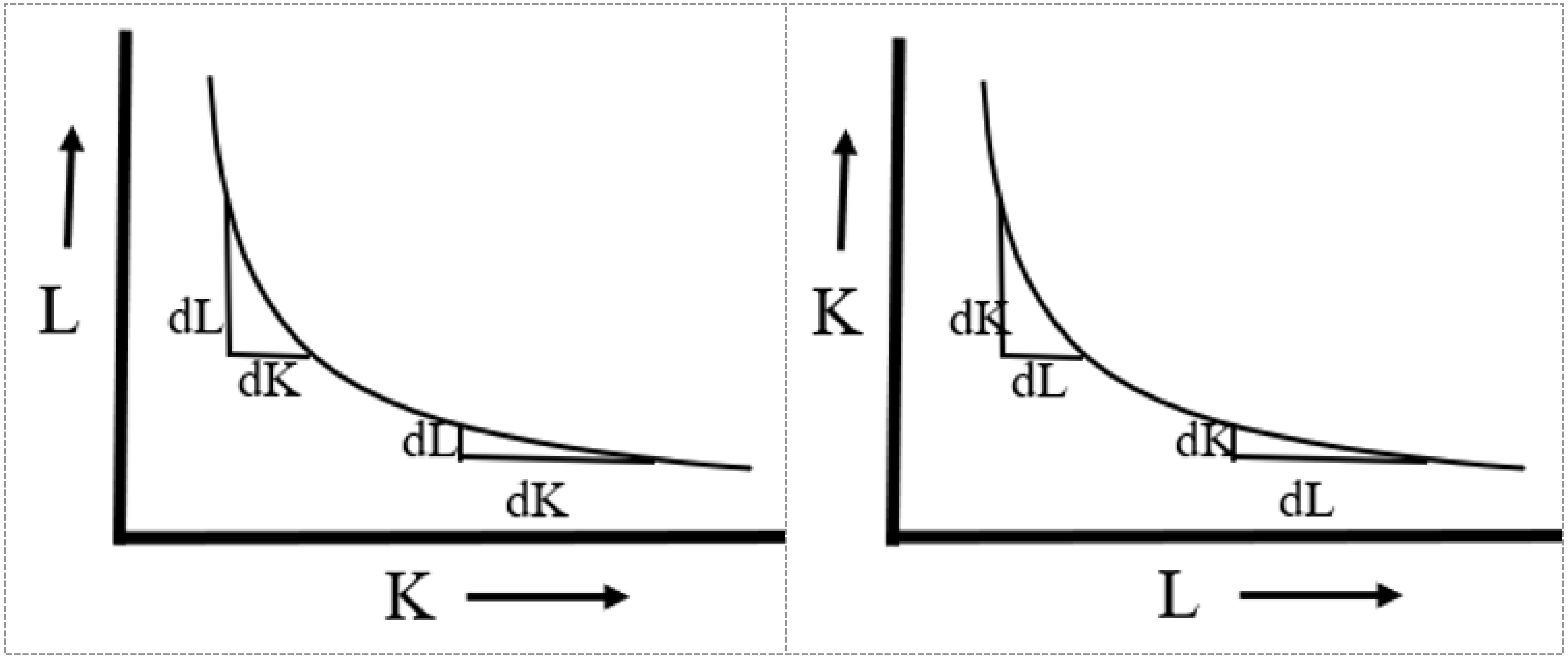
$$\frac{\partial \mathrm{f}}{\partial \mathrm{K}} \mathrm{dK} + \frac{\partial \mathrm{f}}{\partial \mathrm{L}} \mathrm{dL} = 0,$$

- or $\mathrm{Q_K\, dK + Q_L\, dL = 0.}$
- Hence,

$$-\frac{dK}{dL} = \frac{Q_L}{Q_K} = R > 0. \qquad \ldots(03)$$

- Therefore, the isoquants have a negative slope and the absolute value of this is the **marginal rate of substitution (R).**

Marginal Rate of Substitution



Marginal Rate of Substitution

- The Marginal Rate of Substitution measures the rate at which one input can be substituted for the other input. It is assumed that dK/dL is decreasing and so d^2K/dL^2 is positive. I.e., it is assumed that as the quantity of one of the inputs increases, the marginal rate of substitution decreases so that the reduction in the level of one variable made possible by increase in the level of the other variable become progressively smaller. Hence the isoquants are assumed to be convex to the origin.
- For a change from the point (K, L) along the constant product curve, d(K/L) measures the change in the use of K compared with L and dR represents the corresponding change in the marginal rate of substitution.

The Elasticity of Substitution

- A measure of the rate of change of R is given by **the elasticity of substitution** between the factors K and L which is defined as the proportionate change in the ratio K/L expressed as a fraction of the proportionate change in R

$$\sigma = \frac{d\left(Ln\left(\frac{K}{L}\right)\right)}{d\big(Ln(R)\big)} = \frac{d\left(\frac{K}{L}\right)/\left(\frac{K}{L}\right)}{dR/R}, \qquad \dots(04)$$

- where variation is along the constant product curve. It can be shown that alternative expressions for σ are

$$\sigma = \frac{R(LR+K)}{KL\left(R\frac{\partial R}{\partial K} - \frac{\partial R}{\partial L}\right)} = \frac{R(LR+K)}{KL\left(\frac{d^2K}{dL^2}\right)}, \qquad \dots(05)$$

Derivation

- The elasticity of substitution σ is inversely proportional to the curvature of the constant product curve $\left(\frac{d^2K}{dL^2}\right)$ and measures the ease of substitution of L for K. If σ = 0 then the substitution is impossible since $\left(\frac{d^2K}{dL^2}\right)$ is infinite. If $\sigma = \infty$ then the constant product curve is a straight line since $\left(\frac{d^2K}{dL^2}\right)$ is zero. Since R, L, K and $\left(\frac{d^2K}{dL^2}\right)$ are all positive or zero, (05) shows that $\sigma \geq 0$.
- **Derivation of the Expressions in (05):**
- From (04)

$$d\left(\frac{K}{L}\right) = \frac{LdK - KdL}{L^2}, \qquad \text{... (06)}$$

- since $$d\left(\frac{K}{L}\right) = \frac{\partial\left(\frac{K}{L}\right)}{\partial K}dK + \frac{\partial\left(\frac{K}{L}\right)}{\partial L}dL = \frac{1}{L}dK - \frac{K}{L^2}dL.$$

$$dR = \frac{\partial R}{\partial K}dK + \frac{\partial R}{\partial L}dL. \qquad \text{... (07)}$$

Derivation

- Again $$R = -\frac{dK}{dL} \Rightarrow dK = -RdL. \qquad \ldots(08)$$

$$\sigma = \frac{d\left(\frac{K}{L}\right)/\left(\frac{K}{L}\right)}{dR/R} = \left(\frac{LdK - KdL}{KL}\right)/\left(\frac{\frac{\partial R}{\partial K}dK + \frac{\partial R}{\partial L}dL}{R}\right)$$

$$= \left(\frac{R}{KL}\right)\frac{L(-RdL) - KdL}{\frac{\partial R}{\partial K}(-RdL) + \frac{\partial R}{\partial L}dL} = \left(\frac{R}{KL}\right)\frac{(LR + K)}{\left(\frac{\partial R}{\partial K}R - \frac{\partial R}{\partial L}\right)} = \frac{R(LR + K)}{KL\left(\frac{d^2K}{dL^2}\right)},$$

since $$\frac{d^2K}{dL^2} = \frac{d}{dL}\left(\frac{dK}{dL}\right) = \frac{d}{dL}(-R)$$

$$= -\frac{\left(\frac{\partial R}{\partial K}dK + \frac{\partial R}{\partial L}dL\right)}{dL} = \left(\frac{\partial R}{\partial K}R - \frac{\partial R}{\partial L}\right), \text{from (08).} \qquad \text{QED}$$

Another Expression for σ

- σ can also be written as

$$\sigma = \frac{Q_K Q_L (K Q_K + L Q_L)}{-KL(Q_K^2 Q_{LL} + Q_L^2 Q_{KK} - 2Q_K Q_L Q_{KL})}. \qquad \ldots (09)$$

- It clearly shows that σ is a symmetric function of marginal products and K & L.
- **Derivation:**

$$dR = \frac{\partial R}{\partial K} dK + \frac{\partial R}{\partial L} dL = \left(\frac{\partial \left(\frac{Q_L}{Q_K} \right)}{\partial K} \frac{dK}{dL} + \frac{\partial \left(\frac{Q_L}{Q_K} \right)}{\partial L} \right) dL$$

$$= \left\{ \frac{Q_K Q_{LK} - Q_L Q_{KK}}{Q_K^2} \left(-\frac{Q_L}{Q_K} \right) + \frac{Q_K Q_{LL} - Q_L Q_{KL}}{Q_K^2} \right\} dL.$$

$$\therefore \frac{dR}{R} = \frac{TdL}{Q_K^2 Q_L},$$

- where $T = Q_K^2 Q_{LL} + Q_L^2 Q_{KK} - 2Q_K Q_L Q_{KL}.$

Derivation (Continued)

$$d\left(\frac{K}{L}\right) = \frac{LdK - KdL}{L^2} = \left(\frac{L\frac{dK}{dL} - K}{L^2}\right)dL$$

$$= \left(\frac{L\left(-\frac{Q_L}{Q_K}\right) - K}{L^2}\right)dL = \frac{-(KQ_K + LQ_L)}{Q_K L^2}dL.$$

$$\frac{d\left(\frac{K}{L}\right)}{\frac{K}{L}} = \frac{-(KQ_K + LQ_L)}{Q_K LK}dL.$$

$$\therefore \sigma = \frac{d\left(\frac{K}{L}\right)/\frac{K}{L}}{dR/R} = \frac{Q_K Q_L (KQ_K + LQ_L)}{-KLT}. \qquad \text{QED.}$$

The Constant Returns to Scale Property

- **The Constant Returns to Scale Property:**
- Another property of the production function is whether it has constant returns to scale property. That is, whether increasing the level of all inputs by a factor c increases output by the same factor c. If this is so,

$$f(\lambda K, \lambda L) = \lambda f(K, L) = \lambda Q.$$

- The production function is then said to be a homogeneous function of order one. Applying Euler's theorem, it gives

$$Q = Q_K K + Q_L L,$$

- so that if the input factors are paid their marginal products, then the total product Q is exhausted between them.

A Result

- **Result 1:** If Q is homogeneous of degree 1, then

$$Q = KQ_K + LQ_L,$$

and $$Q_K = KQ_{KK} + Q_K + LQ_{LK} \Rightarrow KQ_{KK} + LQ_{LK} = 0.$$

$$Q_L = LQ_{LL} + Q_L + KQ_{KL} \Rightarrow LQ_{LL} + KQ_{KL} = 0,$$

- and it can be shown that

$$\sigma = \frac{Q_K Q_L}{Q Q_{KL}}.$$

- **Note: (Euler′s Theorem):** If $f(x_1, x_2, \ldots, x_n)$ is homogeneous of order h, i.e.,
- $f(cx_1, cx_2, \ldots, cx_n) = c^h f(x_1, x_2, \ldots, x_n)$, then

$$\sum \frac{\partial f}{\partial x_i} x_i = hf(x_1, x_2, \ldots, x_n).$$

The Economic Model

- **The Economic Model:**
- Let us now examine a simple economic model which incorporates a production function.
- The technical relation is one constraint embedded in a model of firm behaviour and the observed values of price, capital, labour, and output are generated by a set of simultaneous relationships. Hence capital and labour cannot be treated as exogenous variables determining output in a single relationship – they are jointly determined.
- The simplest case is given by assuming perfect competition in both factor and product markets, which means that prices of output (p), capital (r) and labour (w) are predetermined – the firm is a price-taker.

Cost Minimization

- **Cost Minimization:**
- We first look at the cost minimization problem. For a given level of output Q, determined exogenously by demand conditions, say, the firm must choose its input levels K and L so as to minimize total costs

$$\text{Total Cost} = \mathrm{rK} + \mathrm{wL}\ \text{subject to}\ \mathrm{Q}_0 = \mathrm{f}(\mathrm{K}, \mathrm{L}).$$

- The Lagrangean is

$$\mathrm{C} = \mathrm{rK} + \mathrm{wL} - \lambda\big(\mathrm{Q}_0 - \mathrm{f}(\mathrm{K}, \mathrm{L})\big).$$

- The first order conditions are

$$\frac{\partial \mathrm{C}}{\partial \mathrm{K}} = \mathrm{r} + \lambda \mathrm{f}_{\mathrm{K}} = 0, \qquad \frac{\partial \mathrm{C}}{\partial \mathrm{L}} = \mathrm{w} + \lambda \mathrm{f}_{\mathrm{L}} = 0.$$

so, $$\frac{\mathrm{f}_{\mathrm{L}}}{\mathrm{f}_{\mathrm{K}}} = \frac{\mathrm{w}}{\mathrm{r}}, \text{and } \frac{\partial \mathrm{C}}{\partial \lambda} = 0 \text{ and } \mathrm{Q}_0 = \mathrm{f}(\mathrm{K}, \mathrm{L}).$$

- Thus, the optimum solution occurs when the marginal rate of substitution is equal to the factor price ratio. Solving the above two conditions gives the cost-minimizing input levels in terms of their prices and the given output level. Substituting these into the expression for C gives the minimum cost function, or just the cost function (the dual of the production function) in terms of r, w and Q.

Profit Maximization

- **Profit Maximization:**
- The profit maximization problem is to choose the levels of output and inputs which maximize profits. The profit maximizing values of Q, K and L can be obtained by maximizing

$$\pi = pQ - rK - wL$$

- subject to Q = f(K, L). The Lagrangean is

$$\pi = pQ - rK - wL - \lambda(Q - f(K, L))$$

- and the first order conditions are

$$\frac{\partial \pi}{\partial Q} = p - \lambda = 0 \text{ and } \frac{\partial \pi}{\partial K} = -r + \lambda f_K = 0 \;\Rightarrow r = pf_K.$$

$$\frac{\partial \pi}{\partial Q} = p - \lambda = 0 \text{ and } \frac{\partial \pi}{\partial L} = -w + \lambda f_L = 0 \;\Rightarrow w = pf_L.$$

- The two equations on the right are marginal productivity conditions: for each input, the values of its marginal product must equal its price, and solving these togethers with Q = f(K, L) yields the profit maximizing levels of Q, K and L.

Profit Maximization: An Alternative Way

- Alternatively, tackling the profit maximization problem directly and eliminating the constraint by substitution, we wish to maximize

$$\pi = pf(K, L) - rK - wL$$

- with respect to K and L. Since the second and third terms are linear in K and L, the second order conditions on π are equivalent to those on the first term on the right-hand side only.

- The requirement for maximum is that the principal minors of the (Hessian) matrix of the second order derivatives alternate in sign. This provides

$$Q_{KK} < 0, Q_{LL} < 0 \text{ and } Q_{KK}Q_{LL} - (Q_{KL})^2 > 0. \quad \dots (11)$$

When Product Market is not Competitive

- **When Product Market is not Competitive:**
- Perfect competition is unlikely to occur in the product market and so some other assumption is necessary. If a constant elasticity demand curve is assumed and

$$p = bQ^{\eta},$$

- where $1/\eta$ is the elasticity of demand, p is the product price and b is a constant, the total revenue is

$$TR = pQ = bQ^{\eta+1}. \qquad \dots (11)$$

- Two special cases are worth noting. If $\eta = 0$ then price p is a constant and the entrepreneur can sell all of his output at this price. This is the case of perfect competition. If $\eta = -1$, then total revenue is constant and is unaffected by changes in p or Q. In this case, maximization of π is effectively the minimization of total cost. However, the minimum total cost occurs when L = 0, K = 0, Q = 0, so that if $\eta = -1$, the level of output must take on some predetermined value Q^0 (say) so that Q is not a variable. The problem is to find the least cost combination of K and L which gives output Q^0.

Maximization of Profit Function

- The Lagrangean of the profit function can be written as

$$\pi = bQ^{\eta+1} - rK - wL - \lambda\big(Q - f(K, L)\big).$$

- The necessary conditions for maximization are

$$\frac{\partial\pi}{\partial L} = -w + \lambda f_L = 0, \frac{\partial\pi}{\partial K} = -r + \lambda f_K = 0, \frac{\partial\pi}{\partial \lambda} = f(K, L) - Q = 0,$$

and when Q is a variable, *we have* $\frac{\partial\pi}{\partial Q} = (\eta + 1)bQ^{\eta} - \lambda = 0,$

which gives $w = \lambda f_L, r = \lambda f_K$ and hence $\frac{w}{r} = \frac{f_L}{f_K}$. ... (12)

If $\eta \neq -1$, then $\lambda = (\eta + 1)bQ^{\eta}, Q = f(K, L)$ and
if $\eta = -1$, then $Q^0 = f(K, L)$. ... (13)

- Thus (12) states that the marginal rate of substitution equals the ratio of factor prices and (13) gives the values of the marginal products.

The Second Order Conditions

- The second order conditions for maximization are

$$\begin{vmatrix} \pi_{LL} & \pi_{LK} & f_L \\ \pi_{KL} & \pi_{KK} & f_K \\ f_L & f_K & 0 \end{vmatrix} > 0, \qquad \dots (14)$$

- which becomes

$$D = 2f_{KL}f_Lf_K - f_{LL}f_K^2 - f_{KK}f_L^2 > 0, \qquad \dots (15)$$

- and when Q is a variable

$$\begin{vmatrix} \pi_{LL} & \pi_{LK} & \pi_{LQ} & f_L \\ \pi_{KL} & \pi_{KK} & \pi_{Kq} & f_K \\ \pi_{LQ} & \pi_{Kq} & \pi_{QQ} & -1 \\ f_L & f_K & -1 & 0 \end{vmatrix} < 0. \qquad \dots (16)$$

The Second Order Conditions (Continued)

- which reduces to

$$bD(\eta + 1)Q^{\eta-1} - \lambda\left(f_{KK}f_{LL} - f_{KL}^2\right) < 0, \qquad \ldots (17)$$

- where D is given by (15). The case of perfect competition ($\eta = 0$) simplifies (17) to

$$f_{KK}f_{LL} - f_{KL}^2 > 0. \qquad \ldots (18)$$

- This discussion of profit maximization has assumed that a ‘local’ maximum occurs, and profits decline once output passes the optimum value. If this is not the case, then profit can always be increased by increasing the level of output and the maximum occurs when output is infinite. Then, however the assumption of fixed values of r and w eventually becomes invalid, and the framework of the analysis will need to be changed. The behaviour of other entrepreneurs will also become important.

Appendix

- **Appendix:**
- For the Lagrangian

$$F = f + \lambda g,$$

- the maximization will occur if

$$|H_2| > 0, |H_3| < 0, |H_4| > 0, \ldots$$

- and the minimization will occur if

$$|H_2| < 0, |H_3| < 0, |H_4| < 0, \ldots$$

- where

$$H_K = \begin{pmatrix} F_{11} & F_{11} & \ldots & F_{1K} & g_1 \\ F_{21} & F_{22} & \ldots & F_{2K} & g_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ F_{K1} & F_{K2} & \ldots & F_{KK} & g_K \\ g_1 & g_2 & \ldots & g_K & 0 \end{pmatrix}.$$

Thank You