Exercise Series 5 (Solutions)

Exercise 1

(a) Joint pdf of $X_{(1)}, X_{(n)}$:

$$f_{(1,n)}(y,z) = \frac{n!}{(1-1)!\,(n-1-1)!\,(n-n)!} f_X(y) f_X(z)$$
$$[F_X(y)]^{1-1}[F_X(z)-F_X(y)]^{n-1-1}[1-F_X(z)]^{n-n},$$
$$y<z$$

For Uniform $(0,\theta)$:

$$f_X(t) = \begin{cases} \frac{1}{\theta} & \text{if } 0<t<\theta \\ 0 & \text{o.w} \end{cases}, \quad F_X(t) = \begin{cases} 0 & \text{if } t<0 \\ \frac{t}{\theta} & \text{if } 0\leq t \leq \theta \\ 1 & \text{o.w.} \end{cases}$$

$$\therefore f_{(1,n)}(y,z) = n(n-1)\frac{1}{\theta^2}\left(\frac{z-y}{\theta}\right)^{n-2}, \quad 0<y<z<\theta$$
$$= n(n-1)\frac{(z-y)^{n-2}}{\theta^n}, \quad 0<y<z<\theta$$

(b) Define $r = z-y$ and $v = \frac{y+z}{2}$, so that

$y = v - \frac{r}{2}$ and $z = v + \frac{r}{2}$.

$$0<y<z<\theta \iff 0< v-\frac{r}{2} < v+\frac{r}{2} < \theta, \quad 0<r<\theta$$
$$\iff \frac{r}{2} < v < \theta - \frac{r}{2}, \quad 0<r<\theta$$

Also, Jacobian of the transformation is

$$J = \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}+\frac{1}{2} = 1$$

Therefore the joint pdf of R & V is

$$f_{R,V}(r,v) = n(n-1)\frac{r^{n-2}}{\theta^n}, \quad 0<r<\theta, \quad \frac{r}{2}<v<\theta-\frac{r}{2}$$

(c) Marginal pdf of R is

$$f_R(r) = \int_{r/2}^{\theta-\frac{r}{2}} f_{R,V}(r,v)\,dv, \quad 0<r<\theta$$

$$= \frac{n(n-1)r^{n-2}}{\theta^n}\left(\theta-\frac{r}{2}-\frac{r}{2}\right), \quad 0<r<\theta$$

$$= \frac{n(n-1)r^{n-2}}{\theta^n}(\theta-r), \quad 0<r<\theta$$

$$= n(n-1)\left(\frac{r}{\theta}\right)^{n-2}\left(1-\frac{r}{\theta}\right)\frac{1}{\theta}, \quad 0<r<\theta$$

It is easy to check that the pdf of $\frac{R}{\theta}$ is

$$n(n-1)\,t^{n-2}(1-t), \quad 0<t<1$$

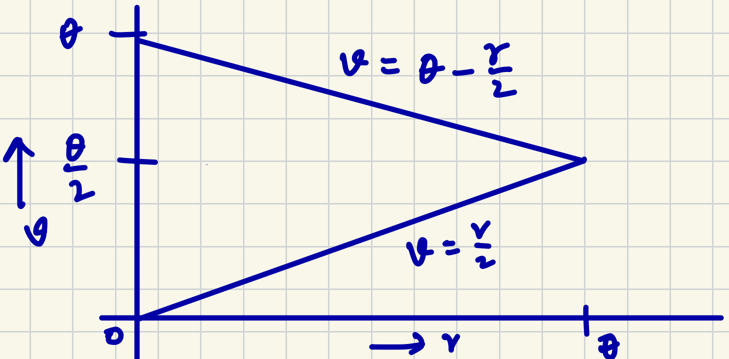
$$= n(n-1)\,t^{n-1-1}(1-t)^{2-1}, \quad 0<t<1$$

$$\therefore \frac{R}{\theta} \sim \text{Beta}(n-1, 2)$$

(d) For the pdf of V, we need to integrate $f_{R,V}(r,v)$ w.r.t. r over the support of the pdf.

Now, the support is $\left\{(r,v) : 0<r<\theta, \frac{r}{2}<v<\theta-\frac{r}{2}\right\}$

Therefore, the range of integration of r depends on whether $v \gtrless \frac{\theta}{2}$.



For $v \le \frac{\theta}{2}$, the range of integration is $(0, 2v)$

For $v > \frac{\theta}{2}$, the range of integration is $(0, 2(\theta - v))$

So, the pdf of V is

$$f_V(v) = \int_0^{2v} f_{R,V}(r,v)\,dr = \frac{n(n-1)}{\theta^n} \int_0^{2v} r^{n-2}\,dr$$

$$= \frac{n(2v)^{n-1}}{\theta^n}, \quad \text{if } 0 \le v \le \frac{\theta}{2}$$

$$f_V(v) = \int_0^{2(\theta-v)} f_{R,V}(r,v)\,dr = \frac{n(n-1)}{\theta^n} \int_0^{2(\theta-v)} r^{n-2}\,dr$$

$$= \frac{n\{2(\theta-v)\}^{n-1}}{\theta^n}, \text{ if } \frac{\theta}{2} < v \le \theta.$$

$$\therefore f_V(v) = \begin{cases} \frac{n\,2^{n-1}}{\theta^n} v^{n-1}, & 0 \le v \le \frac{\theta}{2} \\ \frac{n\,2^{n-1}}{\theta^n} (\theta - v)^{n-1}, & \frac{\theta}{2} < v \le \theta \\ 0, & \text{o.w.} \end{cases}$$

It is easy to check that the pdf is symmetric about $\frac{\theta}{2}$, i.e., $f_V\left(v - \frac{\theta}{2}\right) = f_V\left(v + \frac{\theta}{2}\right) \;\forall\, v \in \mathbb{R}$

Also, the pdf is decreasing in $\left|v - \frac{\theta}{2}\right|$. The max. is obtained at $v = \frac{\theta}{2}$. So, the mode is $\frac{\theta}{2}$.

(e) $\mathbb{E}(R)$ and $\mathbb{E}(R^2)$ can be directly calculated by computing the integrals $\int_0^\theta r f_R(r)\,dr$ and $\int_0^\theta r^2 f_R(r)\,dr$.

You can also use $\frac{R}{\theta} \sim \text{Beta}(n-1, 2)$ to compute

$$\mathbb{E}\left(\frac{R}{\theta}\right) = \frac{n-1}{n+1} \Rightarrow \mathbb{E}(R) = \frac{n-1}{n+1}\theta$$

$$\text{Var}\left(\frac{R}{\theta}\right) = \frac{2(n-1)}{(n+1)^2(n+2)} \Rightarrow \text{Var}(R) = \frac{2(n-1)}{(n+1)^2(n+2)}\theta^2.$$

For V, $\mathbb{E}(V) = \int_0^{\theta} f_V(v)\,dv = \int_0^{\theta/2} v f_V(v)\,dv + \int_{\theta/2}^{\theta} v f_V(v)\,dv$

$$= \frac{n2^{n-1}}{\theta^n}\int_0^{\theta/2} v\, v^{n-1}\,dv + \frac{n2^{n-1}}{\theta^n}\int_{\theta/2}^{\theta} v(\theta-v)^{n-1}\,dv$$

$$= \frac{n2^{n-1}}{\theta^n}\left[\frac{v^{n+1}}{n+1}\right]_0^{\theta/2} + \frac{n2^{n-1}}{\theta^n}\int_0^{\theta/2}(\theta-t)\,t^{n-1}\,dt \quad [t=\theta-v]$$

$$= \frac{n2^{n-1}}{\theta^n}\,\frac{\theta^{n+1}}{\cancel{2^{n+1}}(n+1)} + \frac{n2^{n-1}}{\theta^n}\left[\theta\,\frac{\theta^n}{n2^n} - \frac{\theta^{n+1}}{\cancel{2^{n+1}}(n+1)}\right]$$

$$= \frac{\theta}{2},$$

Alternatively, since $f_V(v)$ is symmetric about $\frac{\theta}{2}$, $\mathbb{E}(V) = \frac{\theta}{2}$.

$$\mathbb{E}(V^2) = \frac{n2^{n-1}}{\theta^n}\int_0^{\theta/2} v^2 v^{n-1}\,dv + \frac{n2^{n-1}}{\theta^n}\int_{\theta/2}^{\theta} v^2(\theta-v)^{n-1}\,dv$$

$$= \frac{n2^{n-1}}{\theta^n}\left[\frac{v^{n+2}}{n+2}\right]_0^{\theta/2} + \frac{n2^{n-1}}{\theta^n}\int_0^{\theta/2}(\theta-t)^2\,t^{n-1}\,dt$$

$$= \frac{n2^{n-1}}{\theta^n}\,\frac{\theta^{n+2}}{2^{n+2}(n+2)} + \frac{n2^{n-1}}{\theta^n}\left[\theta^2\,\frac{\theta^n}{2^n n} - 2\theta\,\frac{\theta^{n+1}}{2^{n+1}(n+1)} + \frac{\theta^{n+2}}{2^{n+2}(n+2)}\right]$$

$$= \frac{n}{8(n+2)}\theta^2 + n\theta^2\left[\frac{1}{2n} - \frac{1}{2(n+1)} + \frac{1}{8(n+2)}\right]$$

$$= n\theta^2\left\{\frac{1}{n(n+1)} + \frac{1}{4(n+2)}\right\}$$

(f) For $X \sim \text{Uniform}(a,b)$, $X-a \sim \text{Uniform}(0, b-a)$.

Therefore, $X_1, \dots, X_n \overset{iid}{\sim} \text{Uniform}(a,b)$

$$\Updownarrow$$

$$\underbrace{X_1-a}_{Y_1}, \dots, \underbrace{X_n-a}_{Y_n} \overset{iid}{\sim} \text{Uniform}(0,b-a)$$

$$\left.\begin{aligned} \tilde{R} &= X_{(n)} - X_{(1)} = Y_{(n)} - Y_{(1)} = R \\ \tilde{V} &= \frac{X_{(1)} + X_{(n)}}{2} = \frac{Y_{(1)} + Y_{(n)}}{2} + a = V + a \end{aligned}\right]$$

where R, V are based on $Y_1 \dots Y_n \overset{iid}{\sim} \text{Uniform}(0,b-a)$

The jt. pdf of $\tilde{R}, \tilde{V}$ is

$$f_{\tilde{R},\tilde{V}}(r,v) = n(n-1)\frac{r^{n-2}}{(b-a)^n}, \quad 0<r<b-a, \quad a+\frac{r}{2} < v < b - \frac{r}{2}$$

Marginal pdf's of $\tilde{R}$ & $\tilde{V}$ are

$$f_{\tilde{R}}(r) = n(n-1)\frac{r^{n-2}(b-a-r)}{(b-a)^n}, \quad 0<r<b-a$$

$$f_{\tilde{V}}(v) = \begin{cases} \frac{n2^{n-1}}{(b-a)^n}(a+v)^{n-1}, & a \le v \le \frac{a+b}{2} \\ \frac{n2^{n-1}}{(b-a)^n}(b-v)^{n-1}, & \frac{a+b}{2} < v \le b \\ 0, & \text{o.w.} \end{cases}$$

$$\mathbb{E}(\tilde{R}) = \mathbb{E}(R) = \frac{n-1}{n+1}(b-a)$$

$$V(\tilde{R}) = V(R) = \frac{2(n-1)}{(n+1)^2(n+2)}(b-a)^2$$

$$\mathbb{E}(\tilde{V}) = \mathbb{E}(V) + a = a + \frac{b-a}{2} = \frac{a+b}{2}$$

[Also, pdf of $\tilde{V}$ is symmetric about $\frac{a+b}{2}$]

$$V(\tilde{V}) = V(V) = n\theta^2\left\{\frac{1}{n(n+1)} + \frac{1}{4(n+2)}\right\}$$

2. $X_1, \dots, X_n \overset{iid}{\sim}$ Uniform$(0, \theta)$, $\theta > 0$

(a) Jt. pdf of $X_{(1)}, X_{(n)}$ is

$$n(n-1)\frac{(z-y)^{n-2}}{\theta^n}, \quad 0 \le y < z \le \theta$$

Define $u = \frac{y}{z}$, $v = z$, so that $y = uv$, $z = v$

Here $0 \le u \le 1$, $0 \le v \le \theta$

Jacobian of the transformation is

$$J = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

So, jt. pdf of U and V is

$$f_{U,V}(u,v) = n(n-1)\frac{(v-uv)^{n-2}}{\theta^n} v, \quad 0 \le u \le 1, \ 0 \le v \le \theta$$

$$= n(n-1) v^{n-1} (1-u)^{n-2} \frac{1}{\theta^n}, \quad 0 \le u \le 1, \ 0 \le v \le \theta$$

(b) The marginal distributions are

$$f_U(u) = n(n-1)(1-u)^{n-2}\frac{1}{\theta^n}\int_0^\theta v^{n-1}\,dv, \quad 0 \le u \le 1$$

$$= n(n-1)(1-u)^{n-2}\frac{1}{\theta^n}\frac{\theta^n}{n}, \quad 0 \le u \le 1$$

$$= (n-1)(1-u)^{n-2}, \quad 0 \le u \le 1.$$

$$f_V(v) = n(n-1) v^{n-1}\frac{1}{\theta^n}\int_0^1 (1-u)^{n-2}\,du, \quad 0 \le v \le \theta$$

$$= n(n-1)\frac{v^{n-1}}{\theta^n}\int_0^1 t^{n-2}\,dt, \quad 0 \le v \le \theta$$

$$= n(n-1)\frac{v^{n-1}}{\theta^n}\frac{1}{n-1}, \quad 0 \le v \le \theta$$

$$= n\frac{v^{n-1}}{\theta^n}, \quad 0 \le v \le \theta.$$

Now, $f_{U,V}(u,v) = n(n-1)(1-u)^{n-2}\dfrac{v^{n-1}}{\theta^n}, \quad 0 \le u \le 1, 0 \le v \le \theta$

$$= (n-1)(1-u)^{n-2}\, n\frac{v^{n-1}}{\theta^n}, \quad 0 \le u \le 1, 0 \le v \le \theta$$

$$= f_U(u)\, f_V(v), \quad 0 \le u \le 1, 0 \le v \le \theta$$

$\therefore U = \dfrac{X_{(1)}}{X_{(n)}}$ & $V = X_{(n)}$ are independent.

(c) The pdf of $U = \dfrac{X_{(1)}}{X_{(n)}}$ is

$$f_U(u) = (n-1)(1-u)^{n-2}, \quad 0 \le u \le 1$$

which is free of θ. Therefore, $\dfrac{X_{(1)}}{X_{(n)}}$ is ancillary.

(d) Jt. pdf of $X_{(i)}$ & $X_{(j)}$ is

$$\frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left(\frac{y}{\theta}\right)^{i-1}\left(\frac{z-y}{\theta}\right)^{j-i-1}\left(1-\frac{z}{\theta}\right)^{n-j}\frac{1}{\theta^2}, \quad 0 \le y < z \le \theta$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\,\frac{y^{i-1}(z-y)^{j-i-1}(\theta-z)^{n-j}}{\theta^n}, \quad 0 \le y < z \le \theta$$

$$= \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\,\frac{\left(\frac{y}{z}\right)^{i-1}\left(1-\frac{y}{z}\right)^{j-i-1} z^{j-2}(\theta-z)^{n-j}}{\theta^n}, \quad 0 \le y < z \le \theta$$

Define $u = \dfrac{y}{z}$, $v = z \iff y = uv$, $z = v$. $J = v$.

$$f_{U,V}(u,v) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\; u^{i-1}(1-u)^{j-i-1}\,\frac{v^{j-2}(\theta-v)^{n-j}}{\theta^n}\cdot v$$

$$0 \le u \le 1, \quad 0 \le v \le \theta$$

$$= \frac{(j-1)!}{(i-1)!\,(j-i-1)!} u^{i-1} (1-u)^{j-i-1}$$

$$\frac{n!}{(j-1)!\,(n-j)!} \left(\frac{v}{\theta}\right)^{j-1} \left(1-\frac{v}{\theta}\right)^{n-j}, \quad \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le \theta . \end{array}$$

Notice that the second part is the marginal pdf of $X_{(j)}$. Therefore, the first part must be the marginal pdf of $\frac{X_{(i)}}{X_{(j)}}$. You may also verify this by directly calculating the integrals.

$$\therefore \quad f_U(u) = \frac{(j-1)!}{(i-1)!\,(j-i-1)!} u^{i-1} (1-u)^{j-i-1}, \quad 0 \le u \le 1$$

$$f_V(v) = \frac{n!}{(j-1)!\,(n-j)!} \left(\frac{v}{\theta}\right)^{j-1} \left(1-\frac{v}{\theta}\right)^{n-j}, \quad 0 \le v \le \theta$$

And, $$f_{U,V}(u,v) = f_U(u)\, f_V(v), \quad \begin{array}{l} 0 \le u \le 1 \\ 0 \le v \le \theta \end{array}$$

So, $U = \frac{X_{(i)}}{X_{(j)}}$ and $V = X_{(j)}$ are independent.

Notice that $\frac{X_{(i)}}{X_{(j)}} \sim \text{Beta}(i, j-i)$

and $\frac{X_{(j)}}{\theta} \sim \text{Beta}(j, n-j+1)$

This also shows that $\frac{X_{(i)}}{X_{(j)}}$ is ancillary.

3. The joint pdf of $X_{(i)}$ and $X_{(j)}$ is

$$f_{(i,j)}(y,z) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \{F_X(y)\}^{i-1} \{F_X(z) - F_X(y)\}^{j-i-1} \{1 - F_X(z)\}^{n-j} f_X(y) f_X(z), \quad y < z$$

The marginal pdf of $X_{(j)}$ is

$$f_{(j)}(z) = \frac{n!}{(j-1)!\,(n-j)!} \{F_X(z)\}^{j-1} \{1 - F_X(z)\}^{n-j} f_X(z)$$

$$\therefore f_{(i|j)}(y|z) = \frac{f_{(i,j)}(y,z)}{f_{(j)}(z)}$$

$$= \frac{(j-1)!}{(i-1)!\,(j-i-1)!} \frac{\{F_X(y)\}^{i-1} \{F_X(z) - F_X(y)\}^{j-i-1}}{\{F_X(z)\}^{j-1}} f_X(y), \quad y < z$$

$$= \frac{(j-1)!}{(i-1)!\,(j-i-1)!} \left\{\frac{F_X(y)}{F_X(z)}\right\}^{i-1} \left\{1 - \frac{F_X(y)}{F_X(z)}\right\}^{j-i-1} \frac{f_X(y)}{F_X(z)}, \quad y < z$$

Since $y < z$, $F_X(y) \leq F_X(z)$, so $0 \leq \frac{F_X(y)}{F_X(z)} \leq 1$.

If we transform $y \longmapsto F_X(y)$, then the Jacobian of this transformation is $\frac{1}{f_X(y)}$. Using this, verify that the conditional distribution of $\frac{F_X(X_{(i)})}{F_X(X_{(j)})}$ given $X_{(j)}$ is

Beta $(i, j-i)$

4. From Exercise 1,

$$f_{R,V}(r,v) = n(n-1)\frac{r^{n-2}}{\theta^n}, \quad 0 \le r \le \theta, \ \frac{r}{2} \le v \le \theta - \frac{r}{2}.$$

and

$$f_V(v) = \begin{cases} \frac{n\,2^{n-1}}{\theta^n} v^{n-1}, & 0 \le v \le \frac{\theta}{2} \\ \frac{n\,2^{n-1}}{\theta^n}(\theta - v)^{n-1}, & \frac{\theta}{2} < v \le \theta. \end{cases}$$

The conditional pdf of R given V is obtained by taking the ratio of the above pdf's. But, be careful about the range of the conditional pdf. Recall the range of r depending on v.

If $0 \le v \le \frac{\theta}{2}$, then $0 \le r \le 2v$

If $\frac{\theta}{2} < v \le \theta$, then $0 \le r \le 2(\theta - v)$.

$$\therefore f_{R|V}(r|v) = \begin{cases} (n-1)\frac{r^{n-2}}{2^{n-1}v^{n-1}}, & 0 \le r \le 2v \text{ if } 0 \le v \le \frac{\theta}{2}, \\ (n-1)\frac{r^{n-2}}{2^{n-1}(\theta-v)^{n-1}}, & 0 \le r \le 2(\theta - v) \text{ if } \frac{\theta}{2} < v \le \theta. \end{cases}$$

Verify that this is a valid pdf. That is

$$\int f_{R|V}(r|v)\,dr = 1 \quad \text{for all } v \in (0,\theta)$$

5. (a) $F_\theta(x) = \int_{-\infty}^{x} f_\theta(t)\,dt$

$$= \int_{-\infty}^{x} \frac{1}{\theta} f\left(\frac{t}{\theta}\right) dt$$

$$= \int_{-\infty}^{x/\theta} \frac{1}{\theta} f(z)\, \theta\, dz$$

$\frac{t}{\theta} = z \iff t = \theta z$

$dt = \theta\, dz$

t	$-\infty$	x
z	$-\infty$	x/θ

$$= \int_{-\infty}^{x/\theta} f(z)\, dz = F\left(\frac{x}{\theta}\right).$$

(b) Let $X \sim f_\theta$, where $f_\theta(x) = \frac{1}{\theta} f\left(\frac{x}{\theta}\right)$

Define $Y = \frac{X}{\theta}$. Then, the pdf of Y is

$\frac{1}{\theta} f(y) \cdot \theta = f(y)$

So, $Y = \frac{X}{\theta} \sim f$. Thus, $X \sim f_\theta \iff \frac{X}{\theta} \sim f$.

Now, $X_1, \ldots, X_n \overset{iid}{\sim} f_\theta \iff \underbrace{\frac{X_1}{\theta}}_{Y_1}, \ldots, \underbrace{\frac{X_n}{\theta}}_{Y_n} \overset{iid}{\sim} f$

$$\frac{X_j}{X_i} = \frac{\theta Y_i}{\theta Y_j} = \frac{Y_i}{Y_j}.$$

Since the jt. dist. of Y_i, Y_j is free of θ, it follows that the dist. of $\frac{Y_j}{Y_i}$ is free of θ.

That is $\frac{X_j}{X_i} = \frac{Y_j}{Y_i}$ is ancillary.

A similar argument can be used for part (c) since the jt. dist of $(Y_1 \ldots Y_n)$ is free of θ and

$\frac{X_{(j)}}{X_{(i)}} = \frac{Y_{(j)}}{Y_{(i)}}$. Therefore the dist. of $\frac{X_{(j)}}{X_{(i)}}$ is determined by the jt. dist of $Y_1 \cdots Y_n$, which is free of θ. Therefore, $\frac{X_{(j)}}{X_{(i)}}$ is ancillary.

(d) A similar argument can be made here as well.

$$\left(\frac{X_1}{X_n}, \frac{X_2}{X_n}, \cdots, \frac{X_{n-1}}{X_n}\right) \stackrel{\mathcal{D}}{=} \left(\frac{Y_1}{Y_n}, \frac{Y_2}{Y_n}, \cdots, \frac{Y_{n-1}}{Y_n}\right)$$

Therefore, the distribution depends on the jt. dist of $(Y_1, \cdots, Y_n)$, which is free of θ.

So, $\left(\frac{X_1}{X_n}, \cdots, \frac{X_{n-1}}{X_n}\right)$ is ancillary.

Note: In the above, we have used the notation $\stackrel{\mathcal{D}}{=}$, which means the random elements on the two sides have the same distribution. This is because all we have used is $X \sim f_\theta \iff Y = \frac{X}{\theta} \sim f$. That is, the distribution of X and θY are the same ($X \stackrel{\mathcal{D}}{=} \theta Y$). For our arguments, it is enough to have equality of the distributions and not necessarily of the random variables.