4.4 IMPROVING A BASIC FEASIBLE SOLUTION.

We have seen earlier that if a linear programming problem has an optimal solution, then one of the basic feasible solutions will provide this optimal value for the objective function. Now although finite in number, there may be more than one basic feasible solutions of a linear programming problem. Hence our endeavour will be to find a new basic feasible solution from another basic feasible solution with an improved value for the objective function.

Now a basic feasible solution is characterised by the basis matrix associated with it. To get a new basic feasible solution from a given one, we are to change a vector from the given basis matrix, by another vector not in the basis, by the replacement method as given in our preliminary discussions.

Let $\mathbf{x}_B$ be a basic feasible solution of a linear programming problem :

Find $\mathbf{x} \geq \mathbf{0}$

subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$,

which maximizes $z = \mathbf{c}\mathbf{x}$.

Here $\mathbf{A} = \left[a_{ij} \right]_{m \times n} = \left[\mathbf{a}_1, \mathbf{a}_2, \ldots\ldots, \mathbf{a}_n \right]$

and $\mathbf{B}$ is the corresponding basis matrix given by

$$\mathbf{B} = \left[\beta_1, \beta_2, \ldots\ldots, \beta_m \right].$$

Then $\mathbf{a}_j$ which is not a vector of $\mathbf{B}$ can be expressed as a linear combination of the vectors of $\mathbf{B}$ as

$$\mathbf{a}_j = \sum_{i=1}^{m} y_{ij}\, \beta_i \,. \qquad \ldots (1)$$

Now, if some $y_{rj} \neq 0$, then we know that we can replace the vector β_r from $\mathbf{B}$ by $\mathbf{a}_j$ still maintaining the basis character of $\mathbf{B}$. If $y_{rj} \neq 0$ and if we replace β_r from $\mathbf{B}$ by $\mathbf{a}_j$, then we shall find this value of β_r in terms of $\mathbf{a}_j$ and the remaining vectors of $\mathbf{B}$ from (1) as

$$\beta_r = \frac{1}{y_{rj}}\, \mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}} \right) \beta_i \,. \qquad \ldots (2)$$

Now, for the basic feasible solution $\mathbf{x}_B$ of $\mathbf{A}\mathbf{x} = \mathbf{b}$, we have

$$\mathbf{B}\mathbf{x}_B = \mathbf{b}$$

or,
$$\sum_{i=1}^{m} x_{Bi}\, \beta_i = \mathbf{b}. \qquad \ldots (3)$$

If c_{Bi} be the price corresponding to the basic variable x_{Bi} and if z_B be the value of the objective function for the basic feasible solution $\mathbf{x}_B$, then we have

$$z_B = \mathbf{c}_B\, \mathbf{x}_B = \sum_{i=1}^{m} c_{Bi}\, x_{Bi} \,. \qquad \ldots (4)$$

Setting (2) in (3), we get

$$\sum_{\substack{i=1 \\ i \neq r}}^{m} x_{Bi}\, \beta_i + x_{Br} \left[\frac{1}{y_{rj}}\, \mathbf{a}_j - \sum_{\substack{i=1 \\ i \neq r}}^{m} \left(\frac{y_{ij}}{y_{rj}} \right) \beta_i \right] = \mathbf{b}$$

or,
$$\sum_{\substack{i=1 \\ i \neq r}}^{m} \left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br} \right) \beta_i + \frac{x_{Br}}{y_{rj}}\, \mathbf{a}_j = \mathbf{b} \qquad \ldots (5)$$

Thus a new basic solution of the problem is

$$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r}, \quad i = 1, 2, \ldots\ldots, (r-1), (r+1), \ldots\ldots, m \quad \ldots (6)$$

together with $\frac{x_{B_r}}{y_{rj}}$ and remaining $(n - m)$ zeros.

In order that this solution will be feasible in addition to being basic, we must have

$$x_{B_i} - \frac{y_{ij}}{y_{rj}} x_{B_r} \geq 0, \quad i \neq r \quad \ldots (7)$$

$$\text{and} \quad \frac{x_{B_r}}{y_{rj}} \geq 0. \quad \ldots (8)$$

If the conditions of *feasibility* be satisfied and this new solution gives an *improved value* of the objective function than that given by $\mathbf{x}_B$, then we shall desire this solution and discard $\mathbf{x}_B$.

To realise these two criterions, we have two arbitrary quantities to select and they are the suffixes r in β_r and j in $\mathbf{a}_j$ which are uptil now arbitrary, except that $y_{rj} \neq 0$.

If $x_{B_r} = 0$, then the conditions (7) and (8) are automatically satisfied and the solution is feasible.

If $x_{B_r} \neq 0$; then to satisfy (8), $y_{rj} > 0$, for $x_{B_r} > 0$.

Let $y_{rj} > 0$, then (7) is automatically satisfied if $y_{ij} = 0$ or $y_{ij} > 0$. Thus the feasibility conditions are to be satisfied only for those i for which $y_{ij} > 0$.

This requires

$$\frac{x_{Bi}}{y_{ij}} - \frac{x_{Br}}{y_{rj}} \geq 0, \quad y_{ij} > 0$$

$$\text{or,} \quad \frac{x_{Br}}{y_{rj}} \leq \frac{x_{Bi}}{y_{ij}}, \quad y_{ij} > 0. \quad \ldots (9)$$

Thus, if we choose the vector β_r such that

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}} \left\{ \frac{x_{Bi}}{y_{ij}}, \ y_{ij} > 0 \right\} = \theta \text{ (say)}, \quad \ldots (10)$$

then (9) is satisfied and the solution (6) is feasible too.

What we need is simply to compute $\frac{x_{Br}}{y_{rj}}$ from (10). This tells us which column r of the basis matrix **B** is to be removed, to get a better value for z.

Now we are to choose j of $\mathbf{a}_j$ such that the new basic solution makes the objective function at least as great as the current basic solution. The price vector component c_{Br} changes to c_j as β_r is changed to $\mathbf{a}_j$. If z' be the new value of the objective function, then we have

$$z' = \sum_{\substack{i=1 \\ i \neq r}}^{m} c_{Bi}\left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br}\right) + c_j \frac{x_{Br}}{y_{rj}}$$

$$= \sum_{i=1}^{m} c_{Bi}\left(x_{Bi} - \frac{y_{ij}}{y_{rj}}\, x_{Br}\right) + c_j \frac{x_{Br}}{y_{rj}},$$

since the contribution of the term is zero for $i = r$

$$= \sum_{i=1}^{m} c_{Bi}\, x_{Bi} - \frac{x_{Br}}{y_{rj}} \cdot \sum_{i=1}^{m} c_{Bi}\, y_{ij} + \frac{x_{Br}}{y_{rj}}\, c_j$$

$$= z_B - \frac{x_{Br}}{y_{rj}}\left(\sum_{i=1}^{m} c_{Bi}\, y_{ij} - c_j\right), \text{ by (4)}$$

$$= z_B - \frac{x_{Br}}{y_{rj}}\,(z_j - c_j)\,. \qquad \ldots \qquad (11)$$

From (11), we observe that $z' > z_B$, if $(z_j - c_j)$ be negative for $\frac{x_{Br}}{y_{rj}} \geq 0$, by (8). Thus, by choosing any vector $\mathbf{a}_j$ for which $(z_j - c_j)$ is negative, we can improve the value of the objective function by

$$-\frac{x_{Br}}{y_{rj}}\,(z_j - c_j)\,.$$

Hence, if the problem be that of maximizing, then we shall select that $\mathbf{a}_j$ for β_r which will make

$$\frac{x_{Br}}{y_{rj}}\,(z_j - c_j)$$

minimum-most negative, that is to say, which will make $(z_j - c_j)$ minimum-most negative in order to minimize the computation involved.

To summarise, we shall determine the vector $\mathbf{a}_k$ to enter the basis as follows :

$$z_k - c_k = \underset{j}{\text{Min}} \left\{ z_j - c_j \mid z_j - c_j < 0 \right\} \quad \ldots \quad (12)$$

The subscript k in (12) indicates the vector to enter the basis.

For a minimizing problem, we are to choose that $\mathbf{a}_j$ which will make $(c_j - z_j)$ minimum-most negative.

The vector which is deleted from the basis is called the *departing vector* or *leaving vector* while the vector which is introduced in the basis is called the *entering vector.*

This process is clearly a monotonic process in which each value of the objective function becomes greater than its previous value. Hence this process is to be continued until there are no vectors $\mathbf{a}$ for which

$$z_j - c_j < 0 .$$

This method is used iteratively to get a basic feasible solution from another basic feasible solution with an improved value of the objective function so long an $\mathbf{a}_j$ is obtained with $z_j - c_j < 0$ and at least one

$$y_{ij} > 0 .$$

The quantities $(z_j - c_j)$ are called the *index numbers* corresponding to $\mathbf{a}_j$.

In the illustration of the previous section if we compute $(z_j - c_j)$, for $j = 2, 3$ (since $j = 1$ and $j = 4$ refer to the basic variables), then we get

$$z_2 - c_2 = 5 - (-4) = 9 \text{ and } z_3 - c_3 = -1 - 0 = -1 .$$

Thus the only $(z_j - c_j)$, $j = 2, 3$, that is negative is $(z_3 - c_3)$ and hence we shall choose $\mathbf{a}_3$ to enter the basis. To determine the departing vector, we use the criterion

$$\frac{x_{Br}}{y_{rj}} = \underset{i}{\text{Min}} \left\{ \frac{x_{Bi}}{y_{ij}} , \; y_{ij} > 0 \right\}$$

$$= \text{Min} \left\{ \frac{x_1}{y_{13}} , \frac{x_4}{y_{43}} \right\} .$$

But in the present case $y_{13} = -\frac{1}{2}$ and is negative.

$y_{43} = \frac{3}{2}$ being the only positive quantity, $\mathbf{a}_4$ will leave the basis.

Example 8.11.3 *Solve by simplex method.*

$$\text{Maximize,} \quad z = x_1 - x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 &\leq 10 \\ 2x_1 \phantom{{}+x_2} - x_3 &\leq 2 \\ 2x_1 - 2x_2 + 3x_3 &\leq 0, \quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

[C.U.(P)'88]

Solution. This is a maximization problem.

$b_i \geq 0$ for all i and the constraints are involved with the sign "$\leq$". Introducing three slack variables x_4, x_5, x_6 one in each constraint, we get the following converted equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 10 \\ 2x_1 - x_3 + x_5 &= 2 \\ 2x_1 - 2x_2 + 3x_3 + x_6 &= 0 \end{aligned}$$

The adjusted objective function z is given by

$$z = x_1 - x_2 + 3x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6.$$

Here all slack vectors together constitute a unit basis matrix $B = I_3$ and then as

$$\mathbf{b} = [10, 2, 0], \quad \mathbf{x}_B = B^{-1}\mathbf{b} \geq \mathbf{0}$$

which gives a feasible solution.

Thus initial B.F.S. $= \mathbf{x}_B = B^{-1}\mathbf{b} = \mathbf{b} = [x_{B1}, x_{B2}, x_{B3}] = [x_4, x_5, x_6]$
$= [b_1, b_2, b_3] = [10, 2, 0]$.

Here the solution is degenerate.

[This problem can be solved by usual method; though Degeneracy occurs at the initial stage].

$$\mathbf{c} = (1, -1, 3, 0, 0, 0), \quad \mathbf{c}_B = (c_4, c_5, c_6) = (0, 0, 0) = \mathbf{0}$$

$$\mathbf{y}_j = B^{-1}\mathbf{a}_j = I_3^{-1}\mathbf{a}_j = \mathbf{a}_j [j = 1, 2, \cdots, 6]$$

$$z_B = \text{Value of the objective function} = \mathbf{c}_B\mathbf{x}_B = 0$$

$$z_j - c_j = \mathbf{c}_B\mathbf{y}_j - c_j = \mathbf{0}\mathbf{y}_j - c_j = -c_j.$$

With these data we shall construct the initial table.

Now without going details we shall solve the problem in a compact form.

Simplex tables:

		c	1	−1	3	0	0	0	
Basis	$\mathbf{c}_B$	b	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4(\mathbf{e}_1)$	$\mathbf{a}_5(\mathbf{e}_2)$	$\mathbf{a}_6(\mathbf{e}_3)$	Min. ratio
$\mathbf{a}_4$	0	10	1	1	1	1	0	0	$\frac{10}{1} = 10$
$\mathbf{a}_5$	0	2	2	0	−1	0	1	0	……
$\mathbf{a}_6^*$	0	0	2	−2	3^*	0	0	1	$\frac{0}{3} = 0^*$
$z_j - c_j$		0	−1	1	-3^*	0	0	0	
$\mathbf{a}_4^*$	0	10	$\frac{1}{3}$	$\frac{5}{3}^*$	0	1	0	$-\frac{1}{3}$	$10/\frac{5}{3} = 6^*$
$\mathbf{a}_5$	0	2	$\frac{8}{3}$	$-\frac{2}{3}$	0	0	1	$\frac{1}{3}$	……
$\mathbf{a}_3$	3	0	$\frac{2}{3}$	$-\frac{2}{3}$	1	0	0	$\frac{1}{3}$	……
$z_j - c_j$		0	1	-1^*	0	0	0	1	
$\mathbf{a}_2$		6							
$\mathbf{a}_5$		6							
$\mathbf{a}_3$		4							
$z_j - c_j$		6	$\frac{6}{5}$	0	0	$\frac{3}{5}$	0	$\frac{4}{5}$	

As all $z_j - c_j \geq 0$ $[j = 1, 2, \cdots, 6]$ in the third table, then the third table is the optimal table and we need not complete the third table. We require to calculate only the column, under the vector **b** which gives the optimal B.F.S. and the optimal value of z. Thus $\max z = 6$ at $x_2 = 6, x_5 = 6, x_3 = 4$, i.e., for $x_1 = 0$ (non-basic), $x_2 = 6, x_3 = 4$ the original problem attains its maximum.

Note. (1) In the first table $\frac{x_{B2}}{y_{23}}$ is not calculated as $y_{23} < 0$. Similarly in the second table $\frac{x_{B2}}{y_{22}}$ and $\frac{x_{B3}}{y_{32}}$ are not calculated as $y_{22}, y_{32} \leq 0$.

Calculation of z_B, elements of $z_j - c_j$ row and $\mathbf{x}_B$

$$y_{40} = z_B = \frac{\frac{5}{3} \times 0 - 10 \times (-1)}{\frac{5}{3}} = 6,$$

$$y_{41} = z_1 - c_1 = \frac{\frac{5}{3} \times 1 - \frac{1}{3} \times (-1)}{\frac{5}{3}} = \frac{6}{5},$$

$$y_{42} = z_2 - c_2 = 0,$$

$$y_{43} = z_3 - c_3 = 0,$$

$$y_{44} = z_4 - c_4 = \frac{\frac{5}{3} \times 0 - 1 \times (-1)}{\frac{5}{3}} = \frac{3}{5},$$

$$y_{45} = z_5 - c_5 = 0,$$

$$y_{46} = z_6 - c_6 = \frac{\frac{5}{3} \times 1 - (-\frac{1}{3}) \times (-1)}{\frac{5}{3}} = \frac{4}{5},$$

$$x_{B1} = \frac{10}{\frac{5}{3}} = 6,$$

$$x_{B2} = \frac{\frac{5}{3} \times 2 - 10 \times (-\frac{2}{3})}{\frac{5}{3}} = 6,$$

$$x_{B3} = \frac{\frac{5}{3} \times 0 - 10 \times (-\frac{2}{3})}{\frac{5}{3}} = 4.$$

We now solve a problem in a compact form.

Example 8.11.4 *Solve the L.P.P. by simplex method.*

$$\textit{Maximize,} \quad z = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 3x_1 + x_2 &\leq 15 \\ 3x_1 + 4x_2 &\leq 24, \quad x_1, x_2 \geq 0, \end{aligned}$$

Adding slack variables one to each constraint, the converted equations are

$$\begin{aligned} 3x_1 + x_2 + x_3 &= 15 \\ 3x_1 + 4x_2 + x_4 &= 24, \quad x_1, x_2, x_3, x_4 \geq 0: \end{aligned}$$

Simplex tables:

		c	4	3	0	0	
Basic	c_B	b	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3(e_1)$	$\mathbf{a}_4(e_2)$	Min. ratio
$\mathbf{a}_3^*$	0	15	3^*	1	1	0	$\frac{15}{3} = 5^*$
$\mathbf{a}_4$	0	24	3	4	0	1	$\frac{24}{3} = 8$
$z_j - c_j$		0	-4^*	-3	0	0	
$\mathbf{a}_1$	4	5	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{5}{1/3} = 15$
$\mathbf{a}_4^*$	0	9	0	3^*	-1	1	$\frac{9}{3} = 3^*$
$z_j - c_j$		20	0	$-\frac{5}{3}^*$	$\frac{4}{3}$	0	
$\mathbf{a}_1$	4	4					
$\mathbf{a}_2$	3	3					
$z_j - c_j$		25	0	0	$\frac{7}{9}$	$\frac{5}{9}$	

Final basis $B = (\mathbf{a}_1, \mathbf{a}_2)$ optimal value of $z = \max z = 25$ at B.F.S. $\mathbf{x}_B =$ $[x_{B1}, x_{B2}] = [x_1, x_2] = [4, 3]$, i.e., at $x_1 = 4$, and $x_2 = 3$.

▶ **Example 8.11.5** *Solve the L.P.P.*

$$\textit{Minimize,} \quad z = -2x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 - 5x_2 &\leq 7 \\ 4x_1 + x_2 &\leq 8 \\ 7x_1 + 2x_2 &\leq 16\,, \quad x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Solution. The problem is a problem of minimization.

Let $z' = -z$; then $\min z = -\max(-z) = -\max z'$. Hence the problem is a problem of maximization of $z' = -z = -(-2x_1 + 3x_2) = 2x_1 - 3x_2$ and finally $\min z = -\max(z')$ with the same solution set. $b_i \geq 0$ for all i and constraints are associated with the sign "$\leq$".

Introducing three slack variables x_3, x_4 and x_5 (one in each inequation) we get the following converted equations

$$\begin{aligned} 2x_1 - 5x_2 + x_3 \qquad\qquad &= 7 \\ 4x_1 + x_2 \qquad + x_4 \qquad &= 8 \\ 7x_1 + 2x_2 \qquad\qquad + x_5 &= 16. \end{aligned}$$

The adjusted objective function is $z' = 2x_1 - 3x_2 + 0.x_3 + 0.x_4 + 0.x_5$.

Here all the slack vectors are unit vectors which produce a unit basis. Initial

$$\begin{aligned} \text{B.F.S.} = \mathbf{x}_B &= [x_{B1}, x_{B2}, x_{B3}] = [x_3, x_4, x_5] = [7, 8, 16] \\ \mathbf{c}_B &= (c_{B1}, c_{B2}, c_{B3}) = (c_3, c_4, c_5) = (0, 0, 0) \\ z &= \mathbf{c}_B\mathbf{x}_B = 0 \text{ and } \mathbf{y}_j = B^{-1}\mathbf{a}_j = \mathbf{a}_j \\ z_j - c_j &= \mathbf{c}_B\mathbf{y}_j - c_j = \mathbf{0}\mathbf{y}_j - c_j = -c_j \end{aligned}$$

Now with the values of $z_j - c_j$ etc. we construct the initial table and solve accordingly.

Simplex tables:

Basis	c_B	c / b	2 / a_1	−3 / a_2	0 / $a_3(e_1)$	0 / $a_4(e_2)$	0 / $a_5(e_3)$	Min. ratio
a_3	0	7	2	−5	1	0	0	$\frac{7}{2}$
a_4^*	0	8	4^*	1	0	1	0	$\frac{8}{4} = 2^*$
a_5	0	16	7	2	0	0	1	$\frac{16}{7}$
$z_j - c_j$		0	-2^*	3	0	0	0	
a_3	0	3	0	$-\frac{11}{2}$	1	$-\frac{1}{2}$	0	
a_1	2	2	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	
a_5	0	2	0	$\frac{1}{4}$	0	$-\frac{7}{4}$	1	
$z_j - c_j$		4	0	$\frac{7}{2}$	0	$\frac{1}{2}$	0	

As none of $z_j - c_j < 0$, therefore the solution is optimal.

Hence $\max z' = 4$.

Now $\min z = -\max z' = -4$. Hence the minimum value of z is -4 corresponding to the optimal basic feasible solution.

$\mathbf{x}_B = [x_3, x_1, x_5] = [3, 2, 2]$, i.e., for $x_1 = 2$, $x_2 = 0$, the objective function of the original problem attains its minimum [x_2 is a non-basic variable.].

▸ **Example 8.11.6** *Solve the L.P.P.*

$$\textit{Maximize,} \quad z = x_1 + x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &\le 10 \\ 3x_2 + 2x_3 &\le 8 \\ x_2 + 3x_3 &\le 15, \quad x_1 \ge 0, x_2 \ge 0 \text{ and } x_3 \ge 0. \end{aligned}$$

Solution. $b_i \ge 0$ for all i.

Introducing three slack variables x_4, x_5 and x_6, one to each constraint we get the following equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 10 \\ 0 \cdot x_1 + 3x_2 + 2x_3 + x_5 &= 8 \\ 0 \cdot x_1 + x_2 + 3x_3 + x_6 &= 15 \end{aligned}$$

The adjusted objective function z is given by

$$z = x_1 + x_2 + 3x_3 + 0 \cdot x_4 + 0 \cdot x_5 + 0 \cdot x_6.$$

The slack vectors constitute a basis matrix which is a unit matrix. But in this problem the column vector $\mathbf{a}_1$, associated with the variable x_1 is also a unit vector ($\mathbf{e}_1$). Hence in this problem unit basis matrix is not unique. But as the problem is a problem of maximization and the coefficient of x_1 in the objective function is a positive quantity, the initial basis matrix may be selected in such a way, that x_1 is a basic variable, i.e., the column vector $\mathbf{a}_1(\mathbf{e}_1)$ associated with x_1 be included in the initial unit basis. And due to this selection the problem may be solved quickly. Unit vector $\mathbf{e}_1$, associated with the variable x_4 be kept outside the basis matrix, i.e., x_4 is to be considered as a non-basic variable.

Therefore, initial B.F.S.

$$
\begin{aligned}
\mathbf{x}_B &= [x_1, x_5, x_6] = [10, 8, 15] \\
\mathbf{c}_B &= (c_1, c_5, c_6) = (1, 0, 0), \mathbf{y}_j = \mathbf{a}_j \\
z = \mathbf{c}_B \mathbf{x}_B &= 1 \times 10 + 0 \times 8 + 0 \times 15 = 10 \\
z_1 - c_1 &= z_5 - c_5 = z_6 - c_6 = 0 \\
z_2 - c_2 &= (1, 0, 0)[2, 3, 1] - 1 = 1 \\
z_3 - c_3 &= (1, 0, 0)[-1, 2, 3] - 3 = -4 \\
z_4 - c_4 &= (1, 0, 0)[1, 0, 0,] - 0 = 1.
\end{aligned}
$$

Simplex tables:

		c	1	1	3	0	0	0	
Basis	$\mathbf{c}_B$	$\mathbf{b}$	$\mathbf{a}_1(\mathbf{e}_1)$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4(\mathbf{e}_1)$	$\mathbf{a}_5(\mathbf{e}_2)$	$\mathbf{a}_6(\mathbf{e}_3)$	Min. ratio
$\mathbf{a}_1$	1	10	1	2	-1	1	0	0	......
$\mathbf{a}_5^*$	0	8	0	3	2^*	0	1	0	$\frac{8}{2} = 4^*$
$\mathbf{a}_6$	0	15	0	1	3	0	0	1	$\frac{15}{3} = 5$
$z_j - c_j$		10	0	1	-4^*	1	0	0	
$\mathbf{a}_1$	1	14	1	$\frac{7}{2}$	0	1	$\frac{1}{2}$	0	
$\mathbf{a}_3$	3	4	0	$\frac{3}{2}$	1	0	$\frac{1}{2}$	0	
$\mathbf{a}_6$	0	3	0	$-\frac{7}{2}$	0	0	$-\frac{3}{2}$	1	
$z_j - c_j$		26	0	7	0	1	2	0	

As none of $z_j - c_j < 0$, therefore the solution set is optimal and the optimal value of z is 26 for the B.F.S. $\mathbf{x}_B = [x_1, x_3, x_6] = [14, 4, 3]$, i.e., for $x_1 = 14$, $x_2 = 0$ and $x_3 = 4$ the original objective function attains its maximum [x_2 is a non-basic variable.].

Now we solve a problem and observe how much the method be able to save time and labour.

Example 8.11.7 *Solve the L.P. problem by simplex method.*

$$\textit{Maximize,} \quad z = 3x_1 + 5x_2 + 4x_3$$

subject to

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15, \quad x_1, x_2, x_3 \ge 0.$$

[Meerut M.Sc.(Math)'84]

Solution. $\mathbf{b} = [8, 10, 15] \ge \mathbf{0}$. Thus introducing three slack variables, x_4, x_5 and x_6, one to each constraint and taking initial basis $B = (\mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6) = I_3$, we can start the initial simplex table and then solve in a compact table as shown below.

		c	3	5	4	0	0	0	
Basis	$\mathbf{c}_B$	$\mathbf{b}$	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4(\mathbf{e}_1)$	$\mathbf{a}_5(\mathbf{e}_2)$	$\mathbf{a}_6(\mathbf{e}_3)$	Min. ratio
$\mathbf{a}_4^*$	0	8	2	3*	0	1	0	0	$\frac{8}{3}$*
$\mathbf{a}_5$	0	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
$\mathbf{a}_6$	0	15	3	2	4	0	0	1	$\frac{15}{2}$
$z_j - c_j$		0	-3	-5*	-4	0	0	0	
$\mathbf{a}_2$		$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	......
$\mathbf{a}_5^*$		$\frac{14}{3}$	$-\frac{4}{3}$	0	5*	$-\frac{2}{3}$	1	0	$\frac{14}{3}/5 = \frac{14}{15}$*
$\mathbf{a}_6$		$\frac{29}{3}$	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{29}{3}/4 = \frac{29}{12}$
$z_j - c_j$		$\frac{40}{3}$	$\frac{1}{3}$	0	-4*	$\frac{5}{3}$	0	0	
$\mathbf{a}_2$		$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}/\frac{2}{3} = 4$
$\mathbf{a}_3$		$\frac{14}{15}$	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	......
$\mathbf{a}_6^*$		$\frac{89}{15}$	$\frac{41}{15}$*	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	$\frac{89}{15}/\frac{41}{15} = \frac{89}{41}$*
$z_j - c_j$		$\frac{256}{15}$	$-\frac{11}{15}$*	0	0	$\frac{17}{15}$	$\frac{4}{5}$	0	
$\mathbf{a}_2$		$\frac{50}{41}$							
$\mathbf{a}_3$		$\frac{62}{41}$							
$\mathbf{a}_1$		$\frac{89}{41}$							
$z_j - c_j$		$\frac{765}{41}$	0	0	0	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	

In the fourth table, all $z_j - c_j \ge 0$. Thus the fourth table is the optimal table. We now only calculate the elements under the column vector **b** which gives B.F.S.

and the value of the objective function. We need not complete the table.

$$\max z = \frac{765}{41} \quad \text{at} \quad x_1 = \frac{89}{41}, \quad x_2 = \frac{50}{41} \quad \text{and} \quad x_3 = \frac{62}{41}.$$

This method is extremely helpful for three or more than three constraints and for fractional cost coefficients. But here the final basis inverse will not be available. Of course, the value of the objective function needs not to be calculated in each table. It may be calculated at the optimal table only by using the formula $z_B = \mathbf{c}_B \mathbf{x}_B$.

Problem having Multiple Optimal Solutions

Example 8.11.8 *Use simplex method to solve the following L.P.P.*

$$\textit{Maximize,} \quad z = 5x_1 + 2x_2$$

subject to

$$\begin{aligned} 6x_1 + 10x_2 &\leq 30 \\ 10x_1 + 4x_2 &\leq 20, \quad x_1, x_2 \geq 0. \end{aligned}$$

[C.U. M.Com.'85]

Is the solution unique? If not, write down the convex combination of the alternative optima.

Solution: The constraints, after the addition of slack variables x_3 and x_4, one to each, are

$$\begin{aligned} 6x_1 + 10x_2 + x_3 &= 30 \\ 10x_1 + 4x_2 + x_4 &= 20, \quad x_j \geq 0, j = 1, 2, \cdots, 4. \end{aligned}$$

The adjusted objective function $z = 5x_1 + 2x_2 + 0 \cdot x_3 + 0 \cdot x_4$.

$$\mathbf{c} = (5, 2, 0, 0)$$

$$\mathbf{b} = \begin{bmatrix} 30 \\ 20 \end{bmatrix} \geq \mathbf{0} \quad \text{and} \quad B = (\mathbf{a}_3, \mathbf{a}_4) = I_2 \text{ is a unit matrix.}$$

$$\mathbf{x}_B = B^{-1}\mathbf{b} = \mathbf{b} \geq \mathbf{0}.$$

Thus with the initial basis B we can start the problem

$$\mathbf{y}_j = B^{-1}\mathbf{a}_j = \mathbf{a}_j, \mathbf{c}_B = (0, 0).$$

		c	5	2	0	0	
c_B	Basis	b	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3(\mathbf{e}_1)$	$\mathbf{a}_4(\mathbf{e}_2)$	Min. ratio
0	$\mathbf{a}_3$	30	6	10	1	0	$\frac{30}{6} = 5$
0	$\mathbf{a}_4^*$	20	10^*	4	0	1	$\frac{20}{10} = 2^*$
$z_j - c_j$		0	-5^*	-2	0	0	
0	$\mathbf{a}_3^*$	18	0	$\frac{38}{5}^*$	1	$-\frac{3}{5}$	$\frac{18}{38/5} = \frac{90}{38}^*$
5	$\mathbf{a}_1$	2	1	$\frac{2}{5}$	0	$\frac{1}{10}$	$\frac{2}{2/5} = 5$
$z_j - c_j$		10	0	0^*	0	$\frac{1}{2}$	
2	$\mathbf{a}_2$	$\frac{45}{19}$	0	1	$\frac{5}{38}$	$-\frac{3}{38}$	
5	$\mathbf{a}_1$	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{38}$	
$z_j - c_j$		10	0	0	0	$\frac{1}{2}$	

Here in the second table all $z_j - c_j \geq 0$. Hence the optimal solution has been obtained $\max z = 10$ at $x_3 = 18, x_1 = 2$, i.e., for $x_1 = 2, x_2 = 0$ (non-basic), the problem attains its maximum. But here $z_2 - c_2 = 0$ corresponding to a non-basic vector $\mathbf{a}_2$. Thus the solution is not unique. Using $\mathbf{a}_2$ as a vector to enter in the next basis we have $\max z = 10$ remains same but the optimal solution will change which has been shown from the table 3. Other optimal basic solution is $x_1 = \frac{20}{19}, x_2 = \frac{45}{19}$. We know that if there are more than one optimal solution then there exist infinite optimal solutions which will be obtained from the convex combination of the optimal solutions $\mathbf{x}_1 = [2, 0]$, $\mathbf{x}_2 = [\frac{20}{19}, \frac{45}{19}]$. Any optimal solution $\mathbf{x}$ is given by [Alternative optima]

$$\begin{aligned} \mathbf{x} &= \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2, 0 \leq \lambda \leq 1 \\ &= \lambda[2, 0] + (1-\lambda)\left[\frac{20}{19}, \frac{45}{19}\right] \end{aligned}$$

For example, if we take $\lambda = \frac{1}{2}$ then $\mathbf{x} = [1\frac{10}{19}, \frac{45}{38}]$ which is also an alternative optimal solution.

Note. $\mathbf{x}_1 = [x_1 = 2, x_2 = 0]$

Problem having an Unbounded Solution

Example 8.11.9 *Use the simplex method to solve the L.P.P.*

$$\textit{Maximize,} \quad 2x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + x_2 - 2x_3 &\leq 7 \\ -3x_1 + x_2 + 2x_3 &\leq 3, \quad x_1, x_2 \textit{ and } x_3 \geq 0 \end{aligned}$$ **[C.U.(H)'89]**

Solution: Adding two slack variables x_4 and x_5, one to each constraint, the constraints are

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 \quad &= 7 \\ -3x_1 + x_2 + 2x_3 \quad + x_5 &= 3, \quad x_j \geq 0, j = 1, \cdots, 5. \end{aligned}$$

and the objective function is $0x_1 + 2x_2 + x_3 + 0 \cdot x_4 + 0x_5$, $\mathbf{b} = [7, 3] \geq 0$ and $B = [\mathbf{a}_4, \mathbf{a}_5] = I_2$ will be the initial unit basis.

Simplex tables

		c	0	2	1	0	0	
Basis	$\mathbf{c}_B$	b	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4(\mathbf{e}_1)$	$\mathbf{a}_5(\mathbf{e}_2)$	Min. ratio
$\mathbf{a}_4$	0	7	1	1	−2	1	0	$\frac{7}{1} = 7$
$\mathbf{a}_5^*$	0	3	−3	1*	2	0	1	$\frac{3}{1} = 3^*$
$z_j - c_j$		0	0	−2*	−1	0	0	
$\mathbf{a}_4^*$	0	4	4*	0	−4	1	−1	$\frac{4}{4} = 1^*$
$\mathbf{a}_2$	2	3	−3	1	2	0	1	......
$z_j - c_j$		6	−6*	0	3	0	2	
$\mathbf{a}_1$	0	1	1	0	−1	$\frac{1}{4}$	$-\frac{1}{4}$	
$\mathbf{a}_2$	2	6	0	1	−1	$\frac{3}{4}$	$\frac{1}{4}$	
$z_j - c_j$		12	0	0	−3	$\frac{3}{2}$	$\frac{1}{2}$	

Note. (1) After completing the $z_j - c_j$ of the third table, we have seen that $z_3 - c_3 = -3 < 0$. Thus we require to complete the table. After completing, we have seen that $y_{i3} \leq 0$ for $i = 1, 2$ for which $z_3 - c_3$ is negative. Then the only conclusion is that the problem has no finite optimal value and the problem is said to have unbounded solution.

(2) For a problem having unbounded solution we cannot trace it before completing the final table.

► **Example 8.11.10** *Solve the L.P.P. by simplex method and prove that alternative optimal solutions exist. Find two optimal solutions.*

$$\textit{Maximize,} \quad z = 2x_1 - x_2 + 3x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + 5x_4 &\leq 12 \\ 3x_1 + 2x_2 + x_3 + 4x_4 &\leq 15, \quad x_1, x_2, x_3 \text{ and } x_4 \geq 0. \end{aligned}$$

Simplex table

		c	2	−1	3	1	0	0	
Basis	c_B	b	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5(\mathbf{e}_1)$	$\mathbf{a}_6(\mathbf{e}_2)$	Min. ratio
$\mathbf{a}_5^*$	0	12	2	1	3*	5	1	0	$\frac{12}{3} = 4^*$
$\mathbf{a}_6$	0	15	3	2	1	4	0	1	$\frac{15}{1} = 15$
$z_j - c_j$		0	−2	1	−3*	−1	0	0	......
$\mathbf{a}_3$	3	4	$\frac{2}{3}$	$\frac{1}{3}$	1	$\frac{5}{3}$	$\frac{1}{3}$	0	$\frac{4}{2/3} = 6$
$\mathbf{a}_6^*$	0	11	$\frac{7}{3}^*$	$\frac{5}{3}$	0	$\frac{7}{3}$	$-\frac{1}{3}$	1	$\frac{11}{7/3} = \frac{33}{7}^*$
$z_j - c_j$		12	0*	2	0	4	1	0	
$\mathbf{a}_3$	3	$\frac{6}{7}$							
$\mathbf{a}_1$	2	$\frac{33}{7}$							
$z_j - c_j$		12	0	2	0	4	1	0	

In the second table, all $z_j - c_j \geq 0$. Therefore, we reach at the optimal stage. Then $\max z = 12$ at $x_1 = 0$, $x_2 = 0$, $x_3 = 4$ and $x_4 = 0$. Now in the table $z_1 - c_1 = 0$ corresponding to a non-basis vector $\mathbf{a}_1$. Hence alternative optimal solutions exist. Thus taking $\mathbf{a}_1$ to vector enter in the basis we get another optimal solution which is $x_1 = \frac{33}{7}$, $x_2 = 0$, $x_3 = \frac{6}{7}$, $x_4 = 0$ and $\max z = 12$.

▸ **Example 8.11.11** *Solve the L.P.P. by simplex method*

$$\textit{Maximize,} \quad z = 2x_1 - 3x_2 - 2x_3 + 6x_4$$

subject to

$$\begin{aligned} 5x_1 - x_2 + 2x_3 + 6x_4 &\leq 20 \\ 2x_1 + 3x_2 + 4x_3 - 5x_4 &\leq 16 \\ x_1 + 2x_2 - 3x_3 + x_4 &\leq 2, \quad x_1, x_2, x_3 \text{ and } x_4 \geq 0. \end{aligned}$$

Simplex tables

		c	2	−3	−2	6	0	0	0	
Basis	c_B	b	a_1	a_2	a_3	a_4	$a_5(e_1)$	$a_6(e_2)$	$a_7(e_3)$	Min. ratio
a_5	0	20	5	−1	2	6	1	0	0	$\frac{20}{6}=\frac{10}{3}$
a_6	0	16	2	3	4	−5	0	1	0	......
a_7^*	0	2	1	2	−3	1^*	0	0	1	$\frac{2}{1}=2^*$
$z_j - c_j$		0	−2	3	2	-6^*	0	0	0	
a_5^*	0	8	−1	−13	20^*	0	1	0	−6	$\frac{8}{20}=\frac{2}{5}^*$
a_6	0	26	7	13	−11	0	0	1	5	......
a_4	6	2	1	2	−3	1	0	0	1	......
$z_j - c_j$		12	4	15	-16^*	0	0	0	6	
a_3	−2	$\frac{2}{5}$								
a_6	0	$\frac{152}{5}$								
a_4	6	$\frac{16}{5}$								
$z_j - c_j$		$\frac{92}{5}$	$\frac{16}{5}$	$\frac{23}{5}$	0	0	$\frac{4}{5}$	0	$\frac{6}{5}$	

All $z_j - c_j \geq 0$ in the third table. Thus we reach at the optimal stage. Then

$$\max z = \frac{92}{5} \quad \text{at} \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = \frac{2}{5} \quad \text{and} \quad x_4 = \frac{16}{5},$$

Verification of the result.

$$\max z = -2 \times \frac{2}{5} + 0 \times \frac{152}{5} + 6 \times \frac{16}{5} = \frac{92}{5}.$$

By using Duality theory

$$\max z = 20 \times \frac{4}{5} + 0 \times 16 + 2 \times \frac{6}{5} = \frac{92}{5}$$

Thus the correctness of the solution has been verified.