# BSDS (2024-2025 Semester II): Statistics II

## Quiz 2

| NAME : | ROLL : |
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**Time:** 75 minutes **Total attainable marks:** 30

*Please answer the questions in the space provided.* **NO** *extra page will be provided during the exam.*

1. Let $\{T_n\}$ be a sequence of random variables such that $T_n \xrightarrow{p} T$ as $n \to \infty$, where $T$ is another random variable. Prove or disprove (with a counter example) the following statements:

   A. $T_n - T \xrightarrow{p} 0$ as $n \to \infty$.

   B. $T_n^2 - T^2 \xrightarrow{p} 0$ as $n \to \infty$.

   C. $E(T_n - T) \to 0$ as $n \to \infty$.

   D. If $T$ is degenerate then $\text{var}(T_n) \to 0$ as $n \to \infty$. [$2.5 \times 4 = 10$]

A. True.

Proof: Let $W_n = T_n - T$

If $T_n \xrightarrow{P} T$ then for any $\varepsilon > 0$,

$$P(|T_n - T| > \varepsilon) \to 0 \quad \text{as } n \to \infty.$$

i.e., $P(|W_n - 0| > \varepsilon) \to 0$ as $n \to \infty$.

Thus, $W_n \xrightarrow{P} 0$.

B. True. $T_n \xrightarrow{P} T$ as $n \to \infty$

$\Rightarrow$ $g(T_n) \xrightarrow{P} g(T)$ for a continuous function $g$.

Choose $g(x) = x^2$, which is continuous.

Thus, $T_n^2 \xrightarrow{P} T^2$ as $n \to \infty$.

C. False.

Counter example:

Let $T_n$ be a two-point distribution with

$$P(T_n = 0) = 1 - \frac{1}{n} \quad \text{and} \quad P(T_n = n^2) = \frac{1}{n}.$$

Then $E(T_n) = n \to \infty$.

But $T_n \xrightarrow{P} T$ where $T$ is degenerate at 0.

[ To see this, take any $\varepsilon > 0$,

then $$P(|T_n| > \varepsilon) = \begin{cases} 0 & \text{if } \varepsilon > n^2 \\ \frac{1}{n} & \text{if } \varepsilon < n^2 \end{cases}$$

$\to 0$ as $n \to \infty$. ]

So, $E(T_n - T) = E(T_n) - E(T) = n \to \infty$ as $n \to \infty$.

D. False. Counter example.

In the above example, $T$ is degenerate at 0.

But $\operatorname{var}(T_n) = E(T_n^2) - E^2(T_n)$

$$= \frac{n^4}{n} - n^2 = n^2(n-1) \to \infty \text{ as } n \to \infty.$$

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2. Consider the problem of testing $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$ based on a random sample $X_1, \cdots, X_n$ from a distribution with pdf $f_\theta$. Consider the test function $\phi_0(\mathbf{x}) = \alpha$ for all $\mathbf{x}$, and $\alpha \in (0, 1)$.

A. Is $\phi_0$ a level-$\alpha$ test?

B. Draw the power function for $\phi_0$. [$3 + 2 = 5$]

A. $\phi_0(x) = \alpha$ for all $x$.

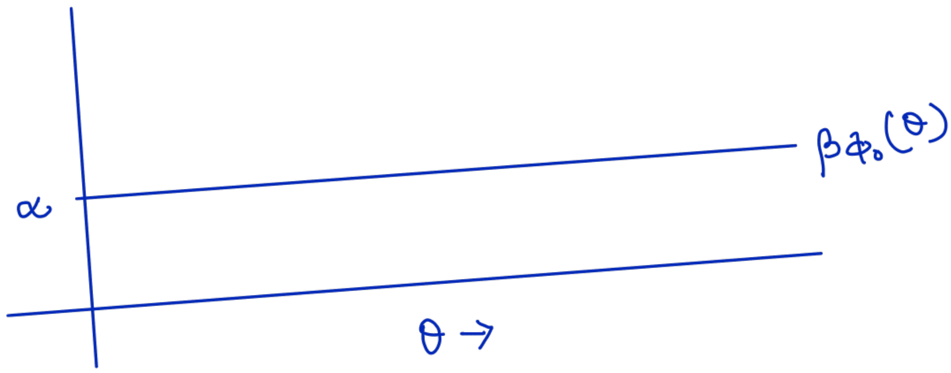
Size of $\phi_0$ is $\sup_{\theta \in \Theta_0} \beta_{\phi_0}(\theta)$

$$= \sup_{\theta \in \Theta_0} E_\theta\left[\phi_0(X)\right]$$

$$= \alpha$$

So, it is a level-$\alpha$ test.

B. The power function of $\phi_0$ is $\beta_{\phi_0}(\theta) = \alpha \ \forall \ \theta$:



3. Consider the problem of testing $H_0 : X \sim U(0, \theta_0)$ and $H_1 : X \sim f_{\theta_0}$ where

$$f_{\theta_0}(x) = \frac{2}{\theta_0^2}x, \qquad 0 < x \leq \theta_0, \quad \theta_0 > 0 \text{ is known.}$$

A. Draw the pdf of $X$ under $H_0$ as well as under $H_1$.

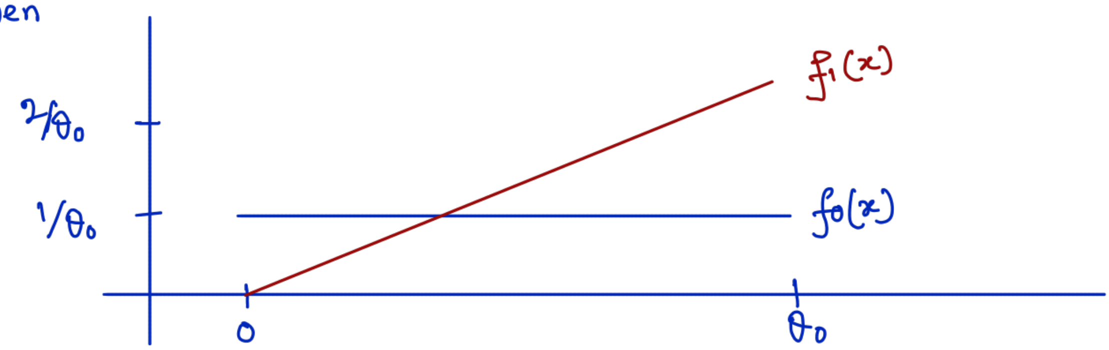
B. Find the most powerful (MP) test for testing $H_0$ against $H_1$ at level $\alpha$ based on a sample of size one (i.e., $n = 1$).

C. Find the power of the MP test.

D. Is the MP test unbiased? [2 + 3 + 2.5 + 2.5 = 10]

A. Let $f_0$ and $f_1$ be the PDFs of $X$ under $H_0$ and $H_1$ respectively.

Then




B. By NP lemma, the MP test is of the following form:

$$\phi(x) = \begin{cases} 1 & \text{if } \frac{f_1(x)}{f_0(x)} = \lambda(x) > K \\ 0 & \text{if } \lambda(x) < K, \end{cases}$$

and satisfies $E_{H_0}[\phi(X)] = \alpha$.

Here $\lambda(x) = \frac{2x}{\theta_0^2} \cdot \theta_0 > K$

$$\Rightarrow \quad x > \frac{k\theta_0}{2} = c, \text{ say.}$$

Thus, the MP test is of the form

$$\phi(x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c, \end{cases} \quad \text{— } (*)$$

[ Note that $P_{H_0}(X = c) = 0$ ]

satisfying $E_{H_0}\left[\phi(X)\right] = \alpha$

$$\Rightarrow \quad P_{H_0}(X \geq c) = \alpha$$

$$\Rightarrow \quad \int_c^{\theta_0} \frac{dx}{\theta_0} = \alpha$$

$$\Rightarrow \quad 1 - \frac{c}{\theta_0} = \alpha$$

$$\Rightarrow \quad c = \theta_0(1-\alpha).$$

The test in $(*)$ with $c = \theta_0(1-\alpha)$ is the MP test.

C. Power of the MP test is

$$P_{H_1}(X \geq c) = \int_c^{\theta_0} \frac{2x\,dx}{\theta_0^2} = 1 - \frac{c^2}{\theta_0^2}$$

$$= 1 - (1-\alpha)^2$$

$$= \alpha(2-\alpha).$$

D. The test is unbiased if size $\leq$ power,

i.e., if $\alpha \leq \alpha(2-\alpha)$

i.e., $\alpha \leq 1$, which is true.

So, the test is unbiased.

4. Consider the problem of testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1 (> \theta_0)$, based on a random sample $X$ of size one from a distribution with pdf $f_\theta$ where

$$f_\theta(x) = \overset{\theta}{\cancel{\frac{1}{\theta}}} x^{-(\theta+1)}, \qquad x > 1, \quad \theta > 0.$$

A. Find the MP test for testing $H_0$ against $H_1$.

B. Find the power function of the MP test derived in part A.

C. Find the size of the proposed test when $H_0$ is generalized to $H_0 : \theta \leq \theta_0$. $[4 + 3 + 3 = 10]$

A. By NP lemma the MP test is of the form

$$\phi(x) = \begin{cases} 1 & \text{if} \quad \lambda(x) = \dfrac{f_{\theta_1}(x)}{f_{\theta_0}(x)} > k \\ 0 & \text{if} \quad \lambda(x) < k. \end{cases}$$

satisfying $\quad E_{\theta_0}(\phi(X)) = \alpha$

Here $\quad \lambda(x) = \dfrac{f_{\theta_1}(x)}{f_{\theta_0}(x)} = \left(\dfrac{\theta_1}{\theta_0}\right) x^{(\theta_0 - \theta_1)} > k$

$$\iff (\theta_0 - \theta_1)\log x > \log k - \log(\theta_1/\theta_0)$$

$$\iff \log x < \frac{\log k - \log(\theta_1/\theta_0)}{(\theta_0 - \theta_1)}$$

$$\iff x < c_\alpha \quad \text{where} \quad c_\alpha = \left(\frac{k\theta_1}{\theta_0}\right)^{\frac{1}{\theta_0 - \theta_1}}.$$

Therefore,

$$\phi(x) = \begin{cases} 1 & \text{if} \quad x \leq c_\alpha \\ 0 & \text{if} \quad x > c_\alpha, \end{cases} \qquad — (*)$$

[ Note that $P_{\theta_0}(X = c_\alpha) = 0$ ]

satisfying $\quad E_{\theta_0}\left[\phi(X)\right] = \alpha$

$$\Rightarrow \quad P_{\theta_0}\left(X \le c_\alpha\right) = \alpha$$

$$\Rightarrow \quad \int_1^{c_\alpha} \theta_0 \, x^{-(\theta_0+1)} \, dx = \alpha$$

$$\Rightarrow \quad \left[-x^{-\theta_0}\right]_1^{c_\alpha} = \alpha$$

$$\Rightarrow \quad 1 - c_\alpha^{-\theta_0} = \alpha$$

$$\Rightarrow \quad (1-\alpha) = c_\alpha^{-\theta_0}$$

$$\Rightarrow \quad c_\alpha = \left(\frac{1}{1-\alpha}\right)^{1/\theta_0}$$

The MP test is of the form $(*)$ with $c_\alpha = \left(\frac{1}{1-\alpha}\right)^{1/\theta_0}$.

<u>B.</u> The power function of the MP test derived in part A. is

$$\beta_\phi(\theta) = E_\theta\left[\phi(X)\right] = \int_1^{c_\alpha} \theta \, x^{-(\theta+1)} \, dx$$

$$= \left[-x^{-\theta}\right]_1^{c_\alpha}$$

$$= 1 - c_\alpha^{-\theta}$$

$$= 1 - (1-\alpha)^{\theta/\theta_0}$$

c. Observe that, $\beta_\phi(\theta)$ is monotonically increasing function of $\theta$.

[To see this, first observe that $-\frac{\theta}{\theta_0}\log(1-\alpha)$ is monotonicaly increasing w.r.t. $\alpha$, and $\beta_\phi(\theta)$ is a monotonicaly increasing transformation of $-\frac{\theta}{\theta_0}\log(1-\alpha)$]

Thus,

$$\sup_{\theta \le \theta_0} \beta_\phi(\theta) = \beta_\phi(\theta_0)$$

So, the size of the proposed test under the generalized hypothesis $H_0' : \theta \le \theta_0$ is same as the size of that under $H_0 : \theta = \theta_0$.

Consequently, the size $= \alpha$ condition leads to the same solution here.

So, the test can be generalized.

**Rough work**