Internal Kinetic Energy Problem from the 2023 Taiwan Physics Olympiad 1st Round Qualifiers

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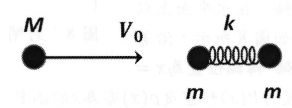
Problem

(12) A spring-mass system of two point masses m and spring constant k is placed on a horizontal surface. Another point mass M of speed V_0, parallel to the line connecting the two masses m, collides elastically with one point mass m. Neglecting any rotational motion possible, find the maximum elastic potential energy the system may obtain in terms of the parameters provided.



Solution

The problem can be solved quickly with the help of the concept of *internal kinetic energy.*

Concept. Consider the kinetic energy of a two-mass system

$$K = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

We may introduce the velocity of the center of mass

$$V_C = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Then the total energy can be rewritten as (see Appendix)

$$K = \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)(v_r)^2 + \frac{1}{2}(m_1 + m_2)V_C^2 = K_{int} + K_C$$

where $v_r = v_1 - v_2$ is the relative velocity between the masses.

The first term is the *internal kinetic energy* K_{int}. As one may notice, the kinetic energy of a perfectly inelastic collision is K_C. Hence, the maximum kinetic energy that a spring-mass system can transfer to its elastic potential energy is K_{int}.

Back to the problem. Immediately after the collision, the left mass m receives a velocity v according to the elastic collision formula

$$v = \frac{2M}{M+m}V_0$$

M moves forward with a velocity of

$$v_M = \frac{M-m}{M+m}V_0 < v$$

This tells us that worrying about M chasing after the spring-mass system is unnecessary. Since the right m is stationary immediately after collision, the relative velocity is $v_r = v - 0 = v$, and the internal kinetic energy can be easily calculated as

$$K_{int} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(v_r)^2 = \frac{1}{2}\left(\frac{m}{2}\right)\left(\frac{2M}{M+m}V_0\right)^2$$

which is the maximum elastic potential energy possible. Therefore, the above is our desired answer, simplifying yields $\boxed{m\left(\frac{M}{M+m}V_0^2\right)^2.}$

Appendix

$$K_{int} + K_C = \frac{1}{2}\mu v_r^2 + \frac{1}{2}(m_1+m_2)V_C^2$$

$$= \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)(v_1-v_2)^2 + \frac{1}{2}(m_1+m_2)\left(\frac{m_1v_1+m_2v_2}{m_1+m_2}\right)^2$$

$$= \frac{1}{2(m_1+m_2)}\left(m_1m_2v_1^2 - 2m_1m_2v_1v_2 + m_1m_2v_2^2m_1^2v_1^2 + 2m_1m_2v_1v_2 + m_2^2v_2^2\right)$$

$$= \frac{1}{2(m_1+m_2)}\left(m_1(m_1+m_2)v_1^2 + m_2(m_1+m_2)v_2^2\right)$$

$$= \boxed{\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2.}$$