Derivation of the Formula for the Focus of a Parabola

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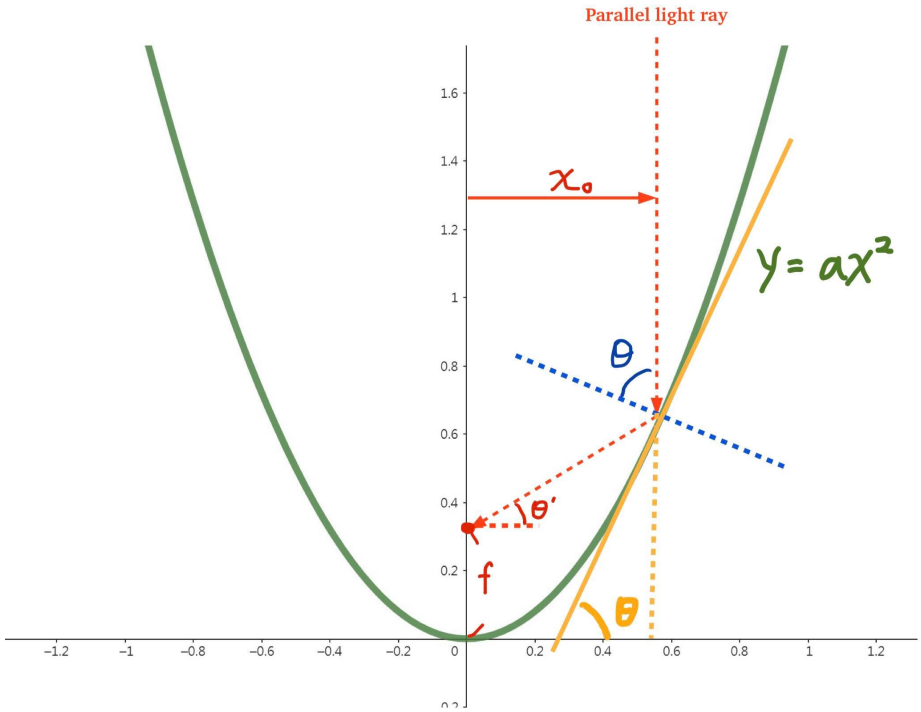
1 Introduction

The parabola is a geometric curve taught in junior high school after the students have learned the basics of quadratic equations. However, such knowledge of the shape is limited, as only the equation and properties related to its coefficients are discussed. Furthermore, only the general equation

$$y = ax^2 + bx + c$$

is well introduced, which one cannot find any relationship to the focus of parabolas. Hence few students know the fact that parabolas have a focus. However, when the methods of astronomical observation methods are presented in earth science classes [1], the students learn that *parabolic mirrors* does a better job at focusing light than a more intuitive alternative, a *spherical mirror*. The reason is often omitted. However, to prove that there exists a focus for a parabola is not so difficult. Therefore the aim of this paper is to provide a detailed derivation of such fact[2].

Figure 1: Parabola with parameters. The parabola shown here is $y = 2x^2$



2 Proof

Figure 1 shows a parabola whose formula we shall represent as

$$y = ax^2 \tag{1}$$

where $a > 0$ as indicated by the opening direction. Various variables are superimposed.

As shown in the figure, the apex of the parabola is at the origin $O(0, 0)$, with $a > 0$, the open side is up. There are two places marked as θ. One is shown in yellow and one in blue. There is also a red angle marked as θ'.

The parallel light ray is positioned at a distance of x_0 to the y axis which the light ray is parallel to.

The yellow line is the tangent line to the parabola which passes through the point

where the parallel light ray is incident. By Eq. (1), the point can be written as (x_0, ax_0^2) which we can later take advantage of to find the line of the reflected ray. The angle θ is the angle such tangent line makes with the x axis. Similarly, the angle θ' is the angle which the reflected light ray makes with the x axis.

We shall now begin our discussion. We first investigate θ. By taking derivatives we can find the slope of the parabola at position x as

$$y'(x) = \frac{\mathrm{d}}{\mathrm{dx}} y = \frac{\mathrm{d}}{\mathrm{dx}}(ax^2) = 2ax \tag{2}$$

We can therefore obtain the tangent value of θ as

$$\tan\theta = y'(x_0) = 2ax_0 \tag{3}$$

We then turn our focus to the reflected light ray. By geometry the blue θ and the yellow θ are identical, which we shall skip its detailed explanations as it is troublesome to present with text. To find θ', we notice that the light ray is parallel to the y axis. By drawing triangles we find the angle as

$$\theta' = \pi/2 - (\pi - 2\theta) = 2\theta - \pi/2$$

When finding the tangent value of it we may also implement a few trigonometry identities such as

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{4}$$

and

$$\sin(\phi - \pi/2) = -\cos(\phi) \qquad \cos(\phi - \pi/2) = \sin(\phi)$$

Using them we get the following:

$$\tan\theta' = \tan(\pi/2 - (\pi - 2\theta)) = \tan(2\theta - \pi/2)$$

$$= \frac{\sin(2\theta - \pi/2)}{cos(2\theta - \pi/2)} = \frac{-\cos(2\theta)}{\sin 2\theta} = -\frac{1}{\tan(2\theta)} \tag{5}$$

$$= -\frac{1 - \tan^2\theta}{2\tan\theta} = \frac{4a^2x_0^2 - 1}{4ax_0}$$

Note that the result obtained in Eq. (5) is in fact the slope of the reflected light ray. Because we know that such line passes through the incident point (x_0, ax_0^2), we can find the equation of the line by

$$\begin{aligned} y - y_0 &= \frac{4a^2x_0^2 - 1}{4ax_0}(x - x_0) \\ y - ax_0^2 &= \frac{4a^2x_0^2 - 1}{4ax_0}(x - x_0) \end{aligned} \tag{6}$$

One final step: our main focus is the point of intersection of the reflected light ray with the y axis. Therefore we substitute $x = 0$ and we find

$$\begin{aligned} y - ax_0^2 &= \frac{4a^2x_0^2 - 1}{4ax_0}(-x_0) \\ y &= ax_0^2 - \frac{4a^2x_0^2 - 1}{a} = ax_0^2 - ax_0^2 + \frac{1}{4a} = \frac{1}{4a} \end{aligned} \tag{7}$$

The final solution of y is *independent* of x_0. That is, whichever parallel light ray we choose, it will always pass $(0, 1/4a)$. Hence all light is centered, or *focused*, on that point. Therefore, we call

$$(0, \frac{1}{4a})$$

the *focus* of the parabola $y = ax^2$, which we set out to prove.

References

[1] 翰林出版社Han-lin Publishing House (2022) 高中選修地科*High School Elective Earth Science*

[2] Varsity Tutors *Focus of a Parabola* https://www.varsitytutors.com/hotmath/hotmath_help/topics/focus-of-a-parabola