# Suborbital Projectile Motion

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## 1 Abstract

This paper is dedicated to the motion of a suborbital projectile, primarily focusing on its maximum height. This problem has shown significance on the 學科能力競賽and the Chiayi Senior High School Class of science's 10th grade exam.

## 2 Problem

A projectile of mass $m$ is launched upwards making an angle $\theta$ with the normal to the ground. Fine the maximum height $h$ to which the projectile can achieve during its flight. Consider Earth as a sphere of radius $R$ and mass $M$.

## 3 Solution

As the height may be significant compared with $R$, we may not approximate the gravitational potential energy using $U = mgh$ where

$$g = \frac{GM}{R^2}$$

hence we will use the energy conservation per unit mass as

$$\epsilon = \frac{v^2}{2} - \frac{GM}{r} = \text{const.}$$

where $v$ and $r$ are the velocity and distance to earth's center at a particular point in the projectile's orbit, respectively.

We can also use the conservation of angular momentum per unit mass as

$$\frac{L}{m} := \mathbf{R} \times \mathbf{v} = (R)(v \sin\theta) = (R+h)(v')$$

where $v'$ is the velocity at the apoapsis (farthest point to Earth's center) of the orbit, which must be parallel to the ground (tangent plane to the sphere).

Solving the above equation we get

$$v' = \frac{Rv\sin\theta}{R+h}$$

We can then substitute it into

$$\epsilon = \frac{v^2}{2} - \frac{GM}{R} = \frac{v'^2}{2} - \frac{GM}{R+h}$$

to find

$$\frac{v^2}{2} - \frac{GM}{R} = \frac{R^2 v^2 \sin^2\theta}{2(R+h)^2} - \frac{GM}{R+h}$$

Not much can be done about this relationship. However problems like this tend to give

$$v = \sqrt{\alpha g R}$$

for simplification. We first rewrite it as

$$v = \sqrt{\frac{\alpha GM}{R}}$$

Substituting that back we find, neatly

$$\left(\frac{\alpha}{2} - 1\right)\frac{1}{R} = \frac{R\alpha\sin^2\theta}{2(R+h)^2} - \frac{1}{R+h}$$

which should be easier to approach after turning it into

$$\left(\frac{\alpha}{2} - 1\right)(R+h)^2 + R(R+h) - \frac{\alpha\sin^2\theta}{2}R^2 = 0$$

We obtained a quadratic equation solving for $h$ in terms of $R$. As an example, we take $\alpha = 1$ and $\sin\theta = \frac{4}{5}$. Simplification yields

$$25(R+h)^2 - 50R(R+h) + 16R^2 = 0$$

giving

$$R + h = \frac{8}{5}R \text{ or } \frac{3}{5}R$$

The positive root $h = \frac{3}{5}R$ is what we seek to find, while the $h = -\frac{3}{5}R$ solution is the periapsis (closest point to Earth's center during orbit).