# Fundamentals of solving ODE

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## 1 Abstract

Differential equations are essential in science. For example, in physics, it is able to model capacitors, which are, intuitively, complex. We call differential equations that only have one independent variable, such as $x$, *ordinary differential equations*, often abbreviated as *ODE*. It is a important skill to be able to solve them. The aim of this article is to provide the basis to solving ODEs. We will begin with the simplest form of differential equations, and move on to harder ones thereafter.

## 2 Separable Differential Equations

Separable differential equations are one of the simplest ODE to solve. A separable differential equation is an ODE that can be written in the form of

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x) \cdot g(y) \tag{1}$$

Where $f$ and $g$ are polynomials that only consist of $x$ and $y$, respectively.
Equation (1) can be rearranged as

$$\frac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x \tag{2}$$

The general procedure to solve separable equations are:

1. Write the equation in the form of equation (1).

2. Rearrange (1) to (2)

3. Integrate both sides. Remember to add the constant of integration.

4. Evaluate $y$.

**Sample Problem 2.1.** Given

$$(\cos y + 2)\frac{dy}{dx} = 2x \quad y(1) = 0$$

find $x$ when $y = \pi$.

**Sample Solution.** Rewrite the given equation as,

$$(\cos y + 2)\,dy = 2x\,dx.$$

Integrate,

$$\int (\cos y + 2)\,dy = \int 2x\,dx$$

$$\sin y + 2y = x^2 + C.$$

Given $y(1) = 0$, we obtain,

$$\sin 0 + 2 \cdot 0 = 1^2 + C \Longrightarrow C = -1.$$

Therefore, when $y = \pi$,

$$x^2 = \sin \pi + 2\pi + 1 \Longrightarrow \boxed{x = \pm\sqrt{2\pi + 1}}.$$

There are a few variations of separable equations, for example, exponential models and logistic models. Let's discuss them below.

**Exponential Models**  Given a quantity, say, population $(P)$, that grows proportionally to its size. We can write

$$\frac{dP}{dt} = kP \tag{3}$$

where $k$ is a constant. Note equation (3) is separable, if we rearrange (3) as

$$\frac{1}{P}\,dP = k\,dt. \tag{4}$$

Recall that

$$\int \frac{1}{x} = \ln x$$

Integrate (4),

$$\int \frac{1}{P}\,dP = \int k\,dt$$

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$$\ln P = kt + C_1$$

we obtain

$$P = e^{kt+C_1} = Ce^{kt} \tag{5}$$

Equation (5) is the general solution of exponential models. Given initial conditions, we can find $C$ and obtain a *particular solution.*

**Logistic models**   Population, in reality, shouldn't grow exponentially without bounds. Instead, it should asymptote towards a particular maximum limit which the environment can load. Therefore, the logistic model, which is in the general form of

$$\frac{dN}{dt} = rN(1 - \frac{N}{k}) \tag{6}$$

better represent the growth of quantities that behave similar to population. In equation (6), $r$ is the rate of growth and $k$ is the maximum limit of quantity $N$.

From (6),

$$\frac{1}{N(1 - \frac{N}{k})} \frac{dN}{dt} = r \tag{7}$$