Lecture Note 3

- Lag Operators
- Autoregressive of order 2, AR(2)
- Autoregressive of order p, AR(p)

Lag Operators

Definition 1. *If $x_t = x(t)$ is a function of time, the lag operator, L, is defined by*

$$Lx_t = x_{t-1}.$$

Definition 2. *Powers of the lag operator are defined as successive application of L, that is*

$$L^2 x_t = L(Lx_t) = Lx_{t-1} = x_{t-2},$$

and, in general, for any integer k,

$$L^k x_t = x_{t-k}.$$

The lag operator satisfies,

(i) $L(a + bx_t) = a + bLx_t = a + bx_{t-1}$.

(ii) $(a_1 L^p + a_2 L^q)x_t = a_1 L^p x_t + a_2 L^q x_t = a_1 x_{t-p} + a_2 x_{t-q}$.

Definition 3. *A lag polynomial is defined by*

$$a(L) = a_0 + a_1 L + a_2 L^2 + \cdots$$

Lag polynomials satisfy the following properties,

(i)

$$\begin{aligned} a(L)b(L) =& (a_0 + a_1 L + a_2 L^2 + \cdots)(b_0 + b_1 L + b_2 L^2 + \cdots) \\ =& a_0 b_0 + (a_1 b_0 + a_0 b_1)L + (a_2 b_0 + a_1 b_1 + a_0 b_2)L^2 + \cdots . \end{aligned}$$

(ii) $a(L)b(L) = b(L)a(L)$.

(iii) $(a(L))^2 = a(L)a(L)$.

(iv) The lag polynomial $a(L)$ can be factorized as follows:

$$a(L) = (1 - \lambda_1 L)(1 - \lambda_2 L)(1 - \lambda_3 L) \cdots ,$$

where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are coefficients.

(v) The inverse of the Lag polynomial $a(L)$ is

$$a(L)^{-1} = (1-\lambda_1 L)^{-1}(1-\lambda_2 L)^{-1}(1-\lambda_3 L)^{-1}\cdots.$$

Example 1. *Suppose x_t follows the following AR(1) process:*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \sim iid(0,\sigma^2)$. We can write

$$x_t = \phi_0 + \phi_1 L x_t + \varepsilon_t \Rightarrow (1-\phi_1 L)x_t = \phi_0 + \varepsilon_t \Rightarrow \phi(L)x_t = \phi_0 + \varepsilon_t.$$

To investigate whether the inverse of $\phi(L)$, $\phi(L)^{-1}$, exists, allow $\phi(L)^{-1} = a_0 + a_1 L + a_2 L^2 + \cdots$. Since $\phi(L)^{-1}\phi(L) = 1$, we can right

$$\begin{aligned}
&(a_0 + a_1 L + a_2 L^2 + \cdots)(1-\phi_1 L) = 1 \Rightarrow \\
&a_0 = 1, \\
&a_1 - a_0\phi_1 = 0 \Rightarrow a_1 = \phi_1, \\
&a_2 - a_1\phi_1 = 0 \Rightarrow a_2 = \phi_1^2, \\
&\vdots \\
&a_i = \phi_1^i \text{ for } i = 0,1,2,\cdots.
\end{aligned}$$

So for the inverse to exists we need $|\phi_1| < 1$. Then

$$\begin{aligned}
x_t =& \phi(L)^{-1}\phi_0 + \phi(L)^{-1}\varepsilon_t \\
=& (1 + \phi_1 + \phi_1^2 + \cdots)\phi_0 + (1 + \phi_1 L + \phi_1^2 L^2 + \cdots)\varepsilon_t \\
=& \frac{\phi_0}{1-\phi_1} + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_1^2\varepsilon_{t-2} + \cdots.
\end{aligned}$$

Therefore,

$$x_t = \frac{\phi_0}{1-\phi_1} + \sum_{i=0}^{\infty}\phi_1^i\varepsilon_{t-i}.$$

Autoregressive of order two, AR(2)

Definition 4. *A time series process x_t is said to be Autoregressive of order two, AR(2), if it can be written as*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t, \text{ where } \varepsilon_t \sim iid(0, \sigma^2). \tag{1}$$

• We can further write (1) as $(1 - \phi_1 L - \phi_2 L^2)x_t = \phi_0 + \varepsilon_t$.

Q: Suppose that x_t given by (1) is a weakly stationary process. Compute its mean, variance and autocovariance function.

A: $\mathbb{E}(x_t) = \phi_0 + \phi_1 \mathbb{E}(x_{t-1}) + \phi_2 \mathbb{E}(x_{t-2}) = \phi_0 + (\phi_1 + \phi_2)\mathbb{E}(x_t) \Rightarrow \mathbb{E}(x_t) = \frac{\phi_0}{1-\phi_1-\phi_2}$.

$$\begin{aligned} \text{var}(x_t) =& \text{cov}(x_t, x_t) = \phi_1 \text{cov}(x_{t-1}, x_t) + \phi_2 \text{cov}(x_{t-2}, x_t) + \text{cov}(\varepsilon_t, x_t) \\ =& \phi_1 \text{cov}(x_{t-1}, x_t) + \phi_2 \text{cov}(x_{t-2}, x_t) + \text{var}(\varepsilon_t) \Rightarrow \end{aligned}$$

or

$$\gamma(0) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2. \tag{2}$$

$$\begin{aligned} \text{cov}(x_t, x_{t-\ell}) =& \text{cov}(\phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t, x_{t-\ell}) \\ =& \phi_1 \text{cov}(x_{t-1}, x_{t-\ell}) + \phi_2 \text{cov}(x_{t-2}, x_{t-\ell}) \end{aligned}$$

for all $\ell = 1, 2, \cdots$. or

$$\gamma(\ell) = \phi_1 \gamma(\ell - 1) + \phi_2 \gamma(\ell - 2) \text{ for all } \ell = 1, 2, \cdots.$$

By substituting, $\ell = 1$ and $\ell = 2$, we get,

$$\gamma(1) = \phi_1 \gamma(0) + \phi_2 \gamma(1), \tag{3}$$

and

$$\gamma(2) = \phi_1 \gamma(1) + \phi_2 \gamma(0). \tag{4}$$

From equations (2), (3) and (4), we can find the values of $\gamma(0)$, $\gamma(1)$, and $\gamma(2)$ in terms of variance of ε_t, σ^2, and the parameters ϕ_1 and ϕ_2. Given the value of $\gamma(0)$, $\gamma(1)$, and $\gamma(2)$, we can compute $\gamma(\ell)$ for $\ell = 3, 4, \cdots$ in terms of variance of ε_t, σ^2, and the parameters ϕ_1 and ϕ_2 by

$$\gamma(\ell) = \phi_1\gamma(\ell-1) + \phi_2\gamma(\ell-2).$$

Theorem 1. *An Autoregressive of order two, AR(2), process*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t, \text{ where } \varepsilon_t \sim iid(0, \sigma^2),$$

is weakly stationary if all the roots of

$$\lambda^2 - \phi_1\lambda - \phi_2 = 0,$$

are inside the unit circle, or all the roots of

$$1 - \phi_1 L - \phi_2 L^2 = 0$$

are outside of unit circle.

Proof. By using Lag operator, we have,

$$(1 - \phi_1 L - \phi_2 L^2)x_t = \phi_0 + \varepsilon_t.$$

Let $\phi(L) = 1 - \phi_1 L - \phi_2 L^2$ and suppose $\phi(L)^{-1} = a_0 + a_1 L + a_2 L^2 + \cdots$ is the inserve of $\phi(L)$. So, a_i for $i = 1, 2, \cdots$ should take values such $\phi(L)^{-1}\phi(L) = 1$. Therefore,

$$(a_0 + a_1 L + a_2 L^2 + \cdots)(1 - \phi_1 L - \phi_2 L^2) = 1.$$

Hence,

$$
\begin{aligned}
&a_0 = 1, \\
&a_1 - a_0\phi_1 = 0 \Rightarrow a_1 = \phi_1, \\
&a_2 - a_1\phi_1 - a_0\phi_2 = 0, \\
&\vdots \\
&a_i - a_{i-1}\phi_1 - a_{i-2}\phi_2 = 0 \text{ for } i = 2, 3, \cdots .
\end{aligned}
$$

Now, suppose that $a_i = c\lambda^i$ is the solution for $a_i - a_{i-1}\phi_1 - a_{i-2}\phi_2 = 0$. Then,

$$c\lambda^i - \phi_1 c\lambda^{i-1} - \phi_2 c\lambda^{i-2} = 0 \Rightarrow \lambda^2 - \phi_1\lambda - \phi_2 = 0.$$

The roots of the above equation are

$$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2} \text{ and } \lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$$

So, $a_i = c_1\lambda_1^i$ is one root of $a_i - a_{i-1}\phi_1 - a_{i-2}\phi_2 = 0$ and $a_i = c_2\lambda_2^i$ is the other one. So, in general all a_i such that

$$a_i = c_1\lambda_1^i + c_2\lambda_2^i \tag{5}$$

is the solution for $a_i - a_{i-1}\phi_1 - a_{i-2}\phi_2 = 0$ for all $i = 2, 3, \cdots$. We can determine the value of c_1 and c_2 such that $a_i = c_1\lambda_1^i + c_2\lambda_2^i$ to be also the solution for $i = 0, 1$. For $i = 0$, we have $a_1 = c_1 + c_2 = 1$ so $c_2 = 1 - c_1$. Also for $i = 1$, we have $a_2 = c_1\lambda_1 + c_2\lambda_2 = \phi_1$. Setting $c_2 = 1 - c_1$, we can further write

$$c_1(\lambda_1 - \lambda_2) + \lambda_2 = \phi_1 \Rightarrow c_1 = \frac{\phi_1 - \lambda_2}{\lambda_1 - \lambda_2}.$$

From (5), we can conclude that the inverse exists if the roots λ_1 and λ_2 are inside the unit circle, $|\lambda_j| < 1$ for $j = 1, 2$.

$$x_t = \phi(L)^{-1}\phi_0 + \phi(L)^{-1}\varepsilon_t \Rightarrow x_t = \phi_0\sum_{j=0}^{\infty} a_j + \sum_{j=0}^{\infty} a_j\varepsilon_{t-j},$$

where $a_i = c_1\lambda_1^i + c_2\lambda_2^i$. Since λ_1 and λ_2 are inside the unit circle, we can show that $\sum_{j=0}^{\infty} a_j < \infty$ and $\sum_{j=0}^{\infty} a_j^2 < \infty$. Hence the process is weakly stationary.

Note that since $\lambda^2 - \phi_1\lambda - \phi_2 = 0$, we have $1 - \phi_1\frac{1}{\lambda} - \phi_2\frac{1}{\lambda^2} = 0$. Setting $L = \frac{1}{\lambda}$, we get $1 - \phi_1 L - \phi_2 L^2 = 0$. Since the roots λ_1 and λ_2 should be inside the unit circle, so that the process be weakly stationary and $L = \frac{1}{\lambda}$, we need the roots of $1 - \phi_1 L - \phi_2 L^2 = 0$, L_1 and L_2, to be outside unit circle. □

Autoregressive of order p, AR(p)

Definition 5. *A time series process x_t is said to be Autoregressive of order p, AR(p), if it can be written as*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t, \text{ where } \varepsilon_t \sim iid(0, \sigma^2). \tag{6}$$

- We can further write (6) as $(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^P)x_t = \phi_0 + \varepsilon_t$.

Theorem 2. *An Autoregressive of order p, AR(p), process*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t, \text{ where } \varepsilon_t \sim iid(0, \sigma^2).$$

is weakly stationary if all the roots of

$$\lambda^p - \phi_{p-1}\lambda^{p-1} - \phi_{p-2}\lambda^{p-2} - \cdots - \phi_p = 0,$$

are inside the unit circle, or all the roots of

$$1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p = 0$$

are outside of the unit circle.