

PROBLEM SET 3

Due: Tuesday February 6th

1. Consider the following AR(2) process for y_t , $t = 1, 2, \dots$,

$$y_t = 0.5y_{t-1} - 0.04y_{t-2} + u_t,$$

where $u_t \sim iid(0, 1)$.

- (a) Show that the process is weakly stationary.
 - (b) Find the mean and variance of y_t .
 - (c) Find the auto-covariance function of y_t , $\gamma(k)$, $k = 1, 2, 3, \dots$.
 - (d) Find the auto-correlation function of y_t , $\rho(k)$, $k = 1, 2, 3, \dots$.
 - (e) Find the partial auto-correlation function of y_t , $\alpha(k)$, $k = 1, 2, 3, \dots$.
2. Consider the following moving average (MA) process:

$$y_t = 1 + u_t + 2u_{t-3},$$

where u_s , for all s , are identically independently distributed (iid) with mean zero and variance one.

- (a) Show that $\mathbb{E}(y_t) = 1$.
- (b) Show that $\gamma_y(0) = \text{var}(y_t) = 5$.
- (c) Show that $\gamma_y(k) = \text{cov}(y_t, y_{t-k}) = 2$ when $k = 3$ and $\gamma_y(k) = 0$ for all the other values of $k > 0$.
- (d) Show that the MA process is not invertible.
- (e) Find $\rho_y(k) = \frac{\gamma_y(k)}{\gamma_y(0)}$ for all $k \geq 0$.
- (f) Consider another MA process as

$$x_t = \mu + v_t + \theta v_{t-3},$$

where v_s , for all s are iid with mean 0 and variance σ_v^2 . Find a value of θ such that the process is invertible and $\rho_x(k) = \rho_y(k)$ for all $k \geq 0$.

3. In this question we are going to analyze quarterly growth in the U.S. Gross National Product (GNP) between 1960Q1 and 2023Q4.
- (a) Compile quarterly data for the U.S. GNP between 1947Q1 and 2023Q1.
 - (b) Compute quarterly GNP growth and filter the data to only include observations between 1960Q1 and 2019Q4. Provide summary statistics of the filtered quarterly GNP growth and plot the data.
 - (c)
 - i. Compute and plot empirical autocorrelation function (ACF). Given the plot, do you expect any time-series correlation among the observations? Explain why?
 - ii. Set the maximum number of lags to the integer closest to the number of observations to the power one over third. Perform a test of joint autocorrelation in GNP growth and report your result. Is your finding consistent with that of the ACF plot? Explain why?
 - (d)
 - i. Compute and plot partial autocorrelation function (PACF). Given the plot, how many lags you would select for an AR process? Explain why?
 - ii. Use AIC and BIC to determine the number of lags for an AR process. Compare your findings with that of the PACF plot.
 - (e)
 - i. Consider an AR model with the number of lags selected by AIC. Estimate the coefficients of the AR model and compute the theoretical autocorrelation functions. Compare your findings with that of Part 3c.
 - ii. Perform a joint test of autocorrelation among the residuals of the estimated model and report your result.