1. $\{x_t \text{ for } t = \cdots, -2, -1, 0, 1, 2, \cdots\}$

a). When ①. $E[x_t] = \mu$ is constant

②. $Var(x_t) = \sigma^2$ is constant

③. $\gamma(t_1, t_2) = cov(t_1, t_2) = E\{[y_{t_1} - E(y_{t_1})][y_{t_2} - E(y_{t_2})]\} = \gamma(|t_1 - t_2|)$

we could say that x_t is weakly stationary

b). $x_t = \sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}$, $\varepsilon_{t'} \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$ $\forall t'$

$E[x_t] = E[\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{4} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(x_t) = Var(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{4} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \cdot \sum_{j=0}^{4} \alpha_i$ is constant

$\gamma(k) = cov(x_t, x_{t-k}) = cov(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}, x_{t-k})$

$= \sum_{j=0}^{4} \alpha_j \, cov(\varepsilon_{t-j}, x_{t-k})$

$= \sum_{j=0}^{4} \alpha_j \, cov(\varepsilon_{t-j}, \sum_{i=0}^{4} \alpha_i \varepsilon_{t-k-i})$

$= \sum_{j=0}^{4} \sum_{i=0}^{4} \alpha_j \alpha_i \, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } t-j = t-k-i \Rightarrow i = k+j \\ \sigma^2 & \text{if } t-j \neq t-k-i \Rightarrow i \neq k+j \end{cases}$$

$\gamma(k) = \sum_{j=0}^{4} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{4} \alpha_j \alpha_{j+k}$

$\Rightarrow$ x_t is weakly stationary

c). $x_t = \sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}$, $\varepsilon' \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$

$E[x_t] = E[\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{n} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(x_t) = Var(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{n} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \sum_{j=0}^{n} \alpha_j$ is constant

$\gamma(k) = cov(x_t, x_{t-k}) = cov(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}, x_{t-k})$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n} \alpha_j \alpha_i \, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } t-j = t-k-i \Rightarrow i = k+j \\ \sigma^2 & \text{if } t-j \neq t-k-i \Rightarrow i \neq k+j \end{cases}$$

$$\gamma(k) = \sum_{j=0}^{n} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{n} \alpha_j \alpha_{j+k}$$

$\Rightarrow X_t$ is weakly stationary

d7. $X_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$, $\varepsilon_{t'} \overset{iid}{\sim} (0, \sigma_\varepsilon^2) \;\; \forall t'$

$E[X_t] = E[\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{\infty} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(X_t) = Var(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{\infty} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \alpha_j < \sigma_\varepsilon^2 \cdot C < \infty$ is finite and constant

$\gamma(k) = cov(X_t, X_{t-k}) = cov(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}, X_{t-k})$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_j \alpha_i \, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } t-j = t-k-i \Rightarrow i = k+j \\ \sigma^2 & \text{if } t-j \neq t-k-i \Rightarrow i \neq k+j \end{cases}$$

$$\gamma(k) = \sum_{j=0}^{\infty} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{\infty} \alpha_j \alpha_{j+k}$$

$\Rightarrow X_t$ is weakly stationary if $\sum_{j=0}^{\infty} \alpha_j^2 \leq C < \infty$

2. AR(1) $y_t = 0.5 y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} (0,1)$

a) $y_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$

$y_t = 0.5 y_{t-1} + \varepsilon_t$

$= 0.5(0.5 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$

$\vdots$

$= \sum_{j=0}^{\infty} 0.5^j \varepsilon_{t-j}$

$\Rightarrow a_j = 0.5^j$

b) $\mu = E[y_t] = E[0.5 y_{t-1} + \varepsilon_t] = 0.5 E[y_{t-1}] = 0.5\mu \Rightarrow \mu = 0$

c) $Var(y_t) = \gamma(0) = Var(0.5 y_{t-1} + \varepsilon_t) = \frac{1^2}{1-0.5^2} = \frac{4}{3}$

d) $cov(y_t, y_{t-k}) = \gamma(k) = cov(0.5 y_{t-1} + \varepsilon_t, y_{t-k})$

$= 0.5\, cov(y_{t-1}, y_{t-k})$

$= 0.5\, \gamma(k-1)$

$= 0.5^2\, \gamma(k-2)$

$\vdots$

$= 0.5^k\, \gamma(0)$

$= 0.5^k \cdot \frac{4}{3}$

PS1Q3

Xiang Li

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```
library("quantmod")
```

##(a) Compile quarterly data for the U.S. real gross private domestic investment (DI) from 1947Q1 to 2023Q4.

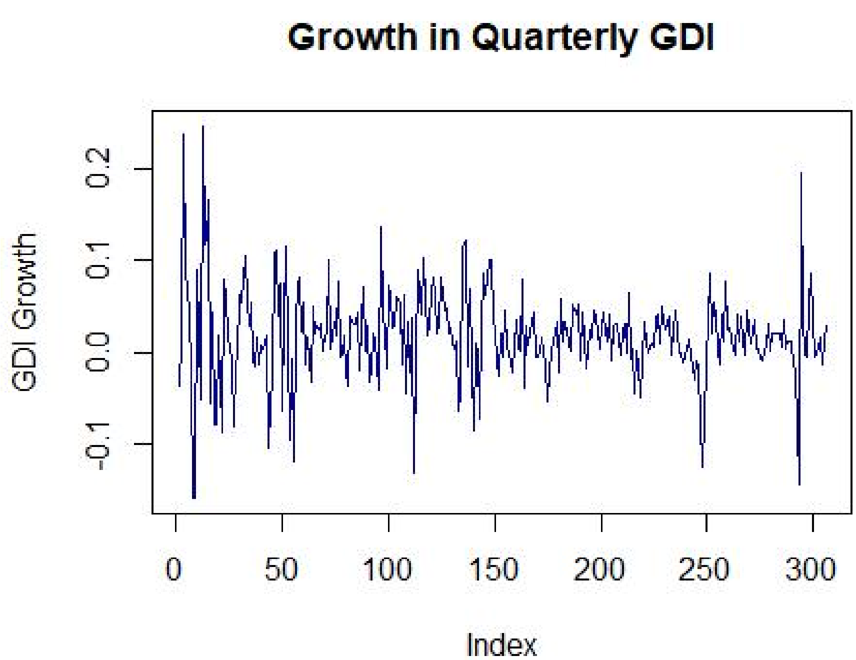
```
getSymbols(Symbols ="GPDI",src = "FRED", from = '1947/01/01')

## [1] "GPDI"

qua_di <- as.matrix(GPDI[,1])
qua_di_date = as.Date(row.names(qua_di))
n_obs_qua_di = length(qua_di_date)
```

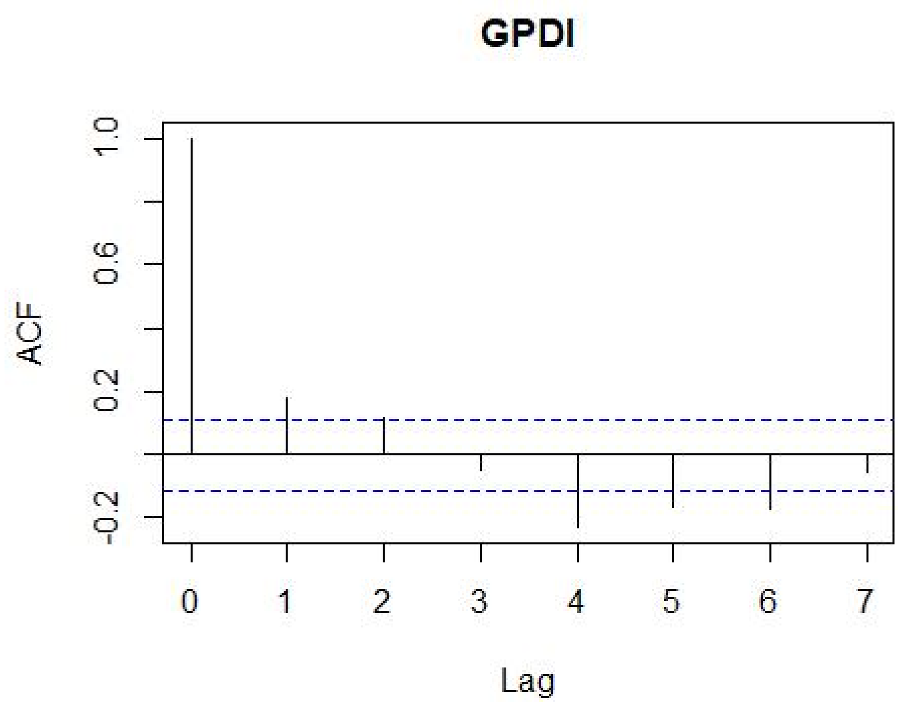
##(b) Compute growth in quarterly DI (GDI), provide its summary statistics and plot the data.

```
di_return <- diff(qua_di)/qua_di[1:n_obs_qua_di-1,1]
di_return_date = qua_di_date[2:n_obs_qua_di]
n_obs_qua_return = length(di_return_date)
plot(di_return, type = "l", col = "darkblue", main = "Growth in Quarterly GDI", ylab = "GDI Growth")
```



##(c) Compute and plot empirical autocorrelation function. Given the plot, do you expect any time-series correlation among the observations? Explain why?

```
acf(di_return,lag=round(n_obs_qua_return^(1/3)))
```




```
ACF=acf(di_return,lag=round(n_obs_qua_return^(1/3)), plot = FALSE)
ACF$acf

## , , 1
##
##             [,1]
## [1,]  1.00000000
## [2,]  0.18230712
## [3,]  0.11675470
## [4,] -0.04992628
## [5,] -0.22861728
## [6,] -0.16254377
## [7,] -0.17190977
## [8,] -0.05641716
```

Yes I do expect some time-series correlation among the observations. Because there are some lags where the bars extend beyond the dotted lines.

##(d) Set the maximum number of lags to the integer closest to the number of observations to the power one-third. Perform a test for joint autocorrelation in GDI and report your result. Does your finding consistent with that of Part 3c? Explain why?

```
t_ratio <- ACF$acf[2]*sqrt(n_obs_qua_return)
t_ratio

## [1] 3.189072

Box.test(di_return, lag = round(n_obs_qua_return^(1/3)), type = "Ljung-Box")

##
##  Box-Ljung test
##
## data:  di_return
## X-squared = 50.143, df = 7, p-value = 1.354e-08

Box.test(di_return, lag = round(n_obs_qua_return^(1/3)), type = "Box-Pierce")

##
##  Box-Pierce test
##
## data:  di_return
## X-squared = 49.199, df = 7, p-value = 2.074e-08
```

##(e) Consider an AR(1) model and compute the theoretical autocorrelation function. Compare your findings with that of Part 3c.

```
lag_di_return = rbind(NA, as.matrix(di_return[1:(n_obs_qua_return-1),1]))
intercept = matrix(1,n_obs_qua_return)
X = cbind(intercept,lag_di_return)
y = di_return
reg_result = ols(X[2:n_obs_qua_return,],as.matrix(y[2:n_obs_qua_return,1]))
1 - sum(reg_result$u_hat^2)/sum(y^2)

## [1] 0.1381177

beta_hat = reg_result$beta_hat
beta_hat

##             [,1]
## [1,] 0.01449087
## [2,] 0.18233574

var_beta_hat = reg_result$var_beta_hat
test_result = t_test(beta_hat,var_beta_hat)
test_result$t_stat

##           [,1]
## [1,] 4.769901
## [2,] 3.244819
```

```r
test_result$p_value
```

```
##               [,1]
## [1,] 1.843163e-06
## [2,] 1.175253e-03
```

```r
ar_coeff <- as.numeric(beta_hat[2])
ma_coeff <- 0
TACF <- ARMAacf(ar_coeff, ma_coeff, lag.max = round(n_obs_qua_return^(1/3)))
plot(c(0:round(n_obs_qua_return^(1/3))),ACF$acf,type='l',xlab='Lag',ylab='ACF
',ylim=c(-0.1,1))
lines(0:round(n_obs_qua_return^(1/3)),TACF,lty=2)
grid(nx = 4, ny = 4)
```