1. $\{x_t \text{ for } t = \cdots, -2, -1, 0, 1, 2, \cdots\}$

a) When ① $E[x_t] = \mu$ is constant

② $Var(x_t) = \sigma^2$ is constant

③ $\gamma(t_1, t_2) = cov(t_1, t_2) = E\{[y_{t_1} - E(y_{t_1})][y_{t_2} - E(y_{t_2})]\} = \gamma(|t_1 - t_2|)$

we could say that x_t is weakly stationary

b) $x_t = \sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}$, $\varepsilon_{t'} \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$ $\forall t'$

$E[x_t] = E[\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{4} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(x_t) = Var(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{4} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \cdot \sum_{j=0}^{4} \alpha_i$ is constant

$\gamma(k) = cov(x_t, x_{t-k}) = cov(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}, x_{t-k})$

$= \sum_{j=0}^{4} \alpha_j\, cov(\varepsilon_{t-j}, x_{t-k})$

$= \sum_{j=0}^{4} \alpha_j\, cov(\varepsilon_{t-j}, \sum_{i=0}^{4} \alpha_i \varepsilon_{t-k-i})$

$= \sum_{j=0}^{4} \sum_{i=0}^{4} \alpha_j \alpha_i\, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } t-j = t-k-i \Rightarrow i = k+j \\ \sigma^2 & \text{if } t-j \neq t-k-i \Rightarrow i \neq k+j \end{cases}$$

$\gamma(k) = \sum_{j=0}^{4} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{4} \alpha_j \alpha_{j+k}$

$\Rightarrow$ x_t is weakly stationary

c) $x_t = \sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}$, $\varepsilon' \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$

$E[x_t] = E[\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{n} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(x_t) = Var(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{n} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \sum_{j=0}^{n} \alpha_j$ is constant

$\gamma(k) = cov(x_t, x_{t-k}) = cov(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}, x_{t-k})$

$$= \sum_{j=0}^{n} \sum_{i=0}^{n} \alpha_j \alpha_i \, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } \; t-j = t-k-i \quad \Rightarrow \quad i = k+j \\ \sigma^2 & \text{if } \; t-j \neq t-k-i \quad \Rightarrow \quad i \neq k+j \end{cases}$$

$$\gamma(k) = \sum_{j=0}^{n} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{n} \alpha_j \alpha_{j+k}$$

$\Rightarrow$ X_t is weakly stationary

d7. $X_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$, $\varepsilon_{t'} \overset{iid}{\sim} (0, \sigma_\varepsilon^2)$ $\forall t'$

$E[X_t] = E[\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}] = \sum_{j=0}^{\infty} \alpha_j E[\varepsilon_{t-j}] = 0$ is constant

$Var(X_t) = Var(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}) = \sum_{j=0}^{\infty} \alpha_j Var(\varepsilon_{t-j}) = \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \alpha_j < \sigma_\varepsilon^2 \cdot C < \infty$ is finite and constant

$\gamma(k) = Cov(X_t, X_{t-k}) = cov(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}, X_{t-k})$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_j \alpha_i \, cov(\varepsilon_{t-j}, \varepsilon_{t-k-i})$$

$$cov(\varepsilon_{t-j}, \varepsilon_{t-k-j}) = \begin{cases} 0 & \text{if } \; t-j = t-k-i \quad \Rightarrow \quad i = k+j \\ \sigma^2 & \text{if } \; t-j \neq t-k-i \quad \Rightarrow \quad i \neq k+j \end{cases}$$

$$\gamma(k) = \sum_{j=0}^{\infty} \alpha_j \alpha_{k+j} \sigma^2 = \sigma^2 \sum_{j=0}^{\infty} \alpha_j \alpha_{j+k}$$

$\Rightarrow$ X_t is weakly stationary if $\sum_{j=0}^{\infty} \alpha_j^2 \leq C < \infty$

2. AR(1) $y_t = 0.5 y_{t-1} + \varepsilon_t$, $\varepsilon_t \overset{iid}{\sim} (0,1)$

a) $y_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}$

$y_t = 0.5 y_{t-1} + \varepsilon_t$

$= 0.5(0.5 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$

$\vdots$

$= \sum_{j=0}^{\infty} 0.5^j \varepsilon_{t-j}$

$\Rightarrow a_j = 0.5^j$

b) $\mu = E[y_t] = E[0.5 y_{t-1} + \varepsilon_t] = 0.5 E[y_{t-1}] = 0.5\mu \Rightarrow \mu = 0$

c) $Var(y_t) = \gamma(0) = Var(0.5 y_{t-1} + \varepsilon_t) = \frac{1^2}{1-0.5^2} = \frac{4}{3}$

d) $cov(y_t, y_{t-k}) = \gamma(k) = cov(0.5 y_{t-1} + \varepsilon_t, y_{t-k})$

$= 0.5\, cov(y_{t-1}, y_{t-k})$

$= 0.5\, \gamma(k-1)$

$= 0.5^2\, \gamma(k-2)$

$\vdots$

$= 0.5^k\, \gamma(0)$

$= 0.5^k \cdot \frac{4}{3}$