Lecture Note 5

- Moving average (MA) models
- Autoregressive moving average (ARMA) models
- Model Checking

Moving average (MA) models

Definition 1. *A time series process x_t is said to be moving average of order q, MA(q), if it can be written as*

$$x_t = c_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q},$$

or

$$x_t = c_0 + (1 - \theta_1 L - \theta_2 L^2 - \cdots - \theta_q L^q)\varepsilon_t,$$

where $\varepsilon_t \sim iid(0, \sigma^2)$.

Q: Is MA(q) is weakly stationary?

A: Note that MA(q) is a special case of linear time series models, $x_t = \mu + \sum_{j=0}^{\infty} \delta_j \varepsilon_{t-j}$, where $\delta_0 = 1$, $\delta_j = -\theta_j$ for $j = 1, 2, \cdots, q$ and $\delta_j = 0$ for $j = q+1, q+2, \cdots$. We know that a linear time series model is weakly stationary if $\sum_{j=0}^{\infty} \delta_j^2$ be a finite number. In case of MA(q) models we have $\sum_{j=0}^{\infty} \delta_j^2 = 1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2$ and hence it is finite. Therefore, MA(q) processes are weakly stationary.

Q: Compute autocovariance and autocorrelation functions of the following MA(1) process:

$$x_t = c_0 + \varepsilon_t - \theta \varepsilon_{t-1}.$$

A:

$$\gamma(0) = \text{var}(x_t) = (1 + \theta^2)\sigma^2.$$

$$\gamma(1) = \text{cov}(x_t, x_{t-1}) = \text{cov}(c_0 + \varepsilon_t - \theta\varepsilon_{t-1}, x_{t-1}) = \text{cov}(\varepsilon_t, x_{t-1}) - \theta\text{cov}(\varepsilon_{t-1}, x_{t-1}) = -\theta\sigma^2.$$

$$\gamma(2) = \text{cov}(x_t, x_{t-2}) = \text{cov}(c_0 + \varepsilon_t - \theta\varepsilon_{t-1}, x_{t-2}) = \text{cov}(\varepsilon_t, x_{t-2}) - \theta\text{cov}(\varepsilon_{t-1}, x_{t-2}) = 0.$$

We can show that $\gamma(k) = 0$ for all $k = 2, 3, \cdots$. So, $\rho(0) = 1$, $\rho(1) = -\frac{\theta}{1+\theta^2}$, and $\rho(k) = 0$ for all $k = 2, 3, \cdots$.

Q: Compute autocovariance function of the following MA(q) process:

$$x_t = c_0 + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q}.$$

A:

$$\gamma(0) = \text{var}(x_t) = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma^2.$$

$$\gamma(1) = (-\theta_1 + \theta_2\theta_1 + \theta_3\theta_2 + \cdots + \theta_q\theta_{q-1})\sigma^2.$$

$$\gamma(2) = (-\theta_2 + \theta_3\theta_1 + \theta_4\theta_2 + \cdots + \theta_q\theta_{q-2})\sigma^2.$$

$$\vdots$$

$$\gamma(q) = -\theta_q\sigma^2.$$

$$\gamma(k) = 0 \text{ for all } k = q+1, q+2, \cdots.$$

• Only the first q orders of the autocovariance and autocorrelation functions of MA(q) process can be different from zero.

• **Invertibility:** An MA(q) process is invertible if all the roots of $1-\theta_1 L-\theta_2 L^2-\cdots-\theta_q L^q = 0$ are outside of the unit circle. Therefore, an MA(1) process is invertible, if the root of $1-\theta L = 0$ is outside of the unit circle, i.e. $|L| > 1$, and hence $|\theta| < 1$.

Q: Is it possible to have two different MA(1) processes to have the same autocorrelation function?

A: The answer is yes. Note that for any MA(1) processes we have $\rho(0) = 1$ and $\rho(k) = 0$ for $k = 2, 3, \cdots$. So, for two MA(1) processes to have the same autocorrelation function we only need their $\rho(1)$ to be the same. So, let's find the values of θ such that $\rho_1 = -\frac{\theta}{1+\theta^2} = a$. We have

$$a\theta^2 + \theta + a = 0.$$

By Solving the above equation, we have

$$\theta_1^* = \frac{-1+\sqrt{1-4a^2}}{2a} \text{ and } \theta_2^* = \frac{-1-\sqrt{1-4a^2}}{2a}.$$

So the follow two MA(1) process have the same autocorrelation function:

$$X_t = \theta_0 + \varepsilon_t - \theta_1^*\varepsilon_{t-1} \text{ and } X_t = \theta_0 + \varepsilon_t - \theta_2^*\varepsilon_{t-1}.$$

Note that $\theta_1^*\theta_2^* = 1$. So, $\theta_1^* = \frac{1}{\theta_2^*}$. Therefore, if $|\theta_1^*| > 1$, then $|\theta_2^*| < 1$. So, if an MA(1) process has a coefficient θ such that $|\theta| > 1$ and hence it is not invertible, we can write a corresponding invertible MA(1) process with the same autocorrelation function.

Autoregressive moving average (ARMA) models

Definition 2. *A time series process x_t is said to be Autoregressive moving average of orders p and q, ARMA(p,q), if it can be written as*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q},$$

or

$$(1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p) x_t = \phi_0 + (1 - \theta_1 L - \theta_2 L^2 - \cdots - \theta_q L^q) \varepsilon_t,$$

where $\varepsilon_t \sim iid(0, \sigma^2)$.

Theorem 1. *An Autoregressive moving average of orders p and q, ARMA(p,q), process*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + \varepsilon_t - \theta_1 \varepsilon_{t-1} - \theta_2 \varepsilon_{t-2} - \cdots - \theta_q \varepsilon_{t-q},$$

is weakly stationary if all the roots of

$$\lambda^p - \phi_{p-1} \lambda^{p-1} - \phi_{p-2} \lambda^{p-2} - \cdots - \phi_p = 0,$$

are inside the unit circle, or all the roots of

$$1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p = 0$$

are outside of the unit circle.

Model checking

Remember that our goal of introducing linear time series models were to capture correlations across time. In order to make sure that our model is actually doing the job, we need to check the residuals to see if there is any serial correlation remains. We can apply Ljung and Box

(1978) approach to test for autocorrelation among the residuals.

(i) Estimate the coefficients of the model and compute the residuals denoted by $\hat{\varepsilon}_t$ for $t = 1, 2, \cdots, T$.

(ii) Apply Ljung and Box (1978) approach for joint test of several autocorrelations among the residuals.

$$\mathcal{H}_0 : \rho(1) = \rho(2) = \cdots = \rho(m) = 0,$$

$$\mathcal{H}_a : \rho_i \neq 0 \text{ for at least one } i.$$

$$Q(m) = T(T+2) \sum_{\ell=1}^{m} \frac{\hat{\rho}(\ell)^2}{T-\ell} \overset{a.s.}{\sim} \chi^2(m),$$

where

$$\hat{\rho}(\ell) = \frac{\frac{1}{T}\sum_{t=\ell+1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}_{t-\ell}}{\frac{1}{T}\sum_{t=1}^{T} \hat{\varepsilon}_t^2}.$$