1/12 (五)

$X_t \quad t=1,2,\cdots,T$

1. $E(X_t)=\mu$ for all t

2. $var(X_t)=\gamma(0)$ for all t

3. $cor(X_t, X_{t-k})=\gamma(k)$

$$\rho(k)=Corr(X_t, X_{t-k})=\frac{cov(X_t, X_{t-k})}{Var(X_t)}=\frac{\gamma(k)}{\gamma(0)}$$

$$\hat{\rho}(k)=\frac{\frac{1}{T}\sum_{t=k+1}^{T}(X_t-\bar{X})(X_{t-k}-\bar{X})}{\frac{1}{T}\sum_{t=1}^{T}(X_t-\bar{X})^2}$$

$$t=\frac{\hat{\rho}(k)-\rho(k)}{\sqrt{var(\rho(k))}}$$

$$\bar{X}=\frac{1}{T}\sum_{t=1}^{T}X_t$$

$H_0: \rho(k)=0$

$H_a: \rho(k)\neq 0$

$$Var(\hat{\rho}(k))=\frac{1}{T}+\frac{2}{T}\sum_{i=1}^{q}\rho^2(i)$$

$$t=\frac{\hat{\rho}(k)}{\sqrt{var\,\hat{\rho}(k)}}\sim N(0,1)$$

$$q=T^{\frac{1}{3}}$$

if $\rho(i)$ of all $i\neq k$ are equal to zero then : $Var(\hat{\rho}(k))=\frac{1}{T}$

$$\Rightarrow \quad t=\sqrt{T}\,\hat{\rho}(k)\sim N(0,1)$$

$H_0: \rho(1)=\rho(2)=\cdots=\rho(q)=0$

H_a: if $\rho(i)\neq 0$ for at least one $i\in\{1,\cdots,q\}$

$$Q(q)=T\,\hat{\rho}^2(1)+T\,\hat{\rho}(2)+\cdots+T\,\hat{\rho}(q)\sim\chi^2(q)$$

$$=T\sum_{i=1}^{q}\hat{\rho}^2(i)\sim\chi^2(q)$$

$= a + \sum_{i=0}^{\infty} b_i u_{t-i}$

$X_t = \mu + \sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}$ $\qquad \varepsilon_s \overset{iid}{\sim} (0, \sigma^2)$ for any s.

$$E[X_t] = E[\mu + \sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}] = E[\mu] + \sum_{i=0}^{\infty} \delta_i E[\varepsilon_{t-i}]$$

$$= \mu + 0$$

$$= \mu$$

$$Var(X_t) = Var(\mu + \sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}) = Var(\sum_{i=1}^{\infty} \delta_i \varepsilon_{t-i}) = \sum_{i=0}^{\infty} \delta_i \underbrace{Var(\varepsilon_{t-i})}_{\sigma^2} = \sigma^2 \underbrace{\sum_{i=0}^{\infty} \delta_i}_{\text{常数 } c}$$

$Var(a+w) = Var(w)$

$$Var(by + cz) = \underbrace{Var(by)}_{\downarrow} + \underbrace{Var(cz)}_{\downarrow} + \underbrace{2cor(by, cz)}_{\downarrow}$$

$$= b^2 Var(y) + c^2 Var(z) + 2bc\, cor(y, z)$$

$$Var(\sum_{i=0}^{\infty} b_i u_{t-i}) = \sum_{i=0}^{\infty} b_i^2 Var(u_{t-i}) + \underbrace{\sum_{i=0}^{\infty} \sum_{j \neq 0} b_i b_j cor(u_{t-i}, u_{t-j})}_{\to 0}$$

$\gamma(k) = cor(X_t, X_{t-k}) = cor(\mu + \sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}, X_{t-k})$ $\qquad X_t = a + \sum_{i=0}^{\infty} b_i u_{t-i}$

$= cor(\sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}, X_{t-k})$ $\qquad X_{t-k} = a + \sum_{i=0}^{\infty} b_i u_{t-k-i}$

$= \sum_{i=0}^{\infty} cov(\delta_i \varepsilon_{t-i}, X_{t-k})$ $\qquad cov(u_{t-i}, X_{t-k}) = cov(u_{t-i}, a + \sum_{j=0}^{\infty} b_j u_{t-k-j})$

$\sum_{i=1}^{2} a_i (\sum_{j=1}^{2} b_j)$ $\qquad = \sum_{i=0}^{\infty} \delta_i cov(\varepsilon_{t-i}, X_{t-k})$ $\qquad = \sum_{j=0}^{\infty} cov(u_{t-i}, b_j u_{t-k-j})$

$= a_1[b_1 + b_2] + a_2[b_1 + b_2]$

$= a_1 b_1 + a_1 b_2 + a_2 b_1 + a_2 b_2$ $\qquad = \sum_{i=0}^{\infty} \delta_i [\sum_{j=0}^{\infty} \delta_j cov(\varepsilon_{t-i}, \varepsilon_{t-k-j})]$ $\leftarrow$ $\qquad = \sum_{j=0}^{\infty} b_j cov(u_{t-i}, u_{t-k-j})$

$= \sum_{i=1}^{2} \sum_{j=1}^{2} a_i b_j$

$\longrightarrow \; = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta_i \delta_j \, cov(\varepsilon_{t-i}, \varepsilon_{t-k-j})$

ε_s are independent across "s" $\Rightarrow$ $cov(\varepsilon_t, \varepsilon_{t'}) = \begin{cases} 0 & \text{if } t \neq t' \\ \sigma^2 & \text{if } t = t' \end{cases}$

$cov(\varepsilon_{t-i}, \varepsilon_{t-k-j}) \neq 0$? $\quad t - i = t - k - j \Rightarrow j = i + k$

if $j = i + k \Rightarrow cov(\varepsilon_{t-i}, \varepsilon_{t-k-j}) = \sigma^2$

$j \neq i + k \Rightarrow cov(\varepsilon_{t-i}, \varepsilon_{t-k-j}) = 0$

$\gamma(k) = \sum_{i=0}^{\infty} \delta_i \delta_{i+k} \sigma^2 \Rightarrow \gamma(k) = \sigma^2 \sum_{i=0}^{\infty} \delta_i \delta_{i+k}$

AR(1) $x_t = b_0 + b_1 x_{t-1} + u_t$ $u_t \overset{iid}{\sim} (0, \sigma^2)$

assume $\{x_t\}_{t=-\infty}^{\infty}$ is a weakly stationary process.

$E[x_t] = \mu \Rightarrow \mu = E[b_0 + b_1 x$

1/19(五)

AR(1): $y_t = a + b_{t-1} + u_t$

AR(P): $y_t = a + b_1 y_{t-1} + b_2 y_{t-2} + \cdots + b_p y_{t-p} + u_t$ $u_s \overset{iid}{\sim} (0, \sigma^2)$

Lag operator

AR(1): $y_t = a + b_1 L y_t + u_t$

$\Rightarrow y_t - b_1 L y_t = a + u_t$

$\Rightarrow (1 - b_1 L) y_t = a + u_t$

AR(P): $y_t = a + b_1 L y_t + b_2 L^2 y_t + \cdots + b_p L^P y_t + u_t$

$y_t - b_1 L y_t - b_2 L^2 y_t \cdots + b_p L^P y_t = a + u_t \Rightarrow (1 - b_1 L - b_2 L^2 - \cdots - b_p L^P) y_t = a + u_t$

$1 - b_1 L = 0 \Rightarrow L^* = \frac{1}{b_1}$ if AR(1) is stationary then $|L^*| > 1 \Rightarrow \frac{1}{|b_1|} > 1 \Rightarrow$
$|b_1| < 1$

AR(1): $y_t = a + b_1 y_{t-1} + u_t$

$(1 - b_1 L) y_t = a + u_t$

$f(L) = C_0 + C_1 L + C_2 L^2 + C_3 L^3 + \cdots$

we want to choose $C_0, C_1, C_2, \cdots$ such that $f(L)(1 - b_1 L) = 1$

$(C_0 + C_1 L + C_2 L^2 + \cdots)(1 - b_1 L) = C_0 - C_0 b_1 L + C_1 L - C_1 b_1 L^2 + C_2 L^2 - C_2 b_1 L^3 + C_3 L^3 - C_3 b_1 L^4 + \cdots$

$= C_0 + (C_1 - C_0 b_1) \cdot L + (C_2 - C_1 b_1) \cdot L^2 + (C_3 - C_2 b_1) L^3 + \cdots$

$= C_0 + \sum_{i=0}^{\infty} (C_i - C_{i-1} b_1) \cdot L^i = 1$

$C_0 = 1$

$C_i - C_{i-1} b_1 = 0$ for all $i \geq 1$

$C_1 = C_0 b_1 = b_1$

$C_2 = C_1 b_1 = b_1 \cdot b_1 = b_1^2 \quad , \quad i=2$

$C_3 = C_2 b_1 = b_1^2 \cdot b_1 = b_1^3 \quad , \quad i=3$

$\vdots$

$C_i = b_1^i$ for all $i \geq 1$

$\Rightarrow f(L) = 1 + b_1 L + b_1^2 L^2 + b_1^3 L^3 + \cdots$

if $|b_1| > 1 \Rightarrow f(L)$ 不存在

So suppose $|b_1| \leq 1 \Rightarrow \underbrace{f(L)(1-b_1L)}_{=1} y_t = f(L)(a+u_t) = f(L)\cdot a + f(L)\cdot u_t$.

if $|b_1| \leq 1$

$\Rightarrow y_t = f(L)\, a + f(L)\, u_t$.

$f(L)\, a = (1 + b_1 L + b_1^2 L^2 + b_1^3 L^3 + \cdots)\, a$

$= a + b_1 L a + b_1^2 L^2 a + \cdots$

$= a + b_1 a + b_1^2 a + \cdots$

$= a(1 + b_1 + b_1^2 + \cdots)$

$f(L)\, u_t = (1 + b_1 L + b_1^2 L^2 + b_1^3 L^3 + \cdots)\, u_t$

$= u_t + b_1 L u_t + b_1^2 L^2 u_t + b_1^3 L^3 u_t + \cdots$

$= u_t + b_1 u_{t-1} + b_1^2 u_{t-2} + b_1^3 u_{t-3} + \cdots = \sum_{i=0}^{\infty} b_1^i u_{t-i}$

$\Rightarrow y_t = a(1 + b_1 + b_1^2 + \cdots) + \sum_{i=0}^{\infty} b_1^i u_{t-i}$

if $b_1 = 1 \Rightarrow f(L)\cdot a = \infty$ and $\sum_{i=0}^{\infty} b_1^i u_{t-i} = \infty$

$\Rightarrow y_t$ is not weakly stationary if $|b_1| < 1 \Rightarrow f(L)\cdot a = \frac{a}{1-b_1}$ is finite

$\sum_{i=0}^{\infty} b_1^i = 1 + b_1^2 + b_1^4 + \cdots = \frac{1}{1-b_1^2} \Rightarrow y_t$ is weakly stationary

Make a guess that $C_i = k\lambda^i$ (根据蓝色2个公式)

$i = 0 \Rightarrow C_0 = k\lambda^0 = k \Rightarrow k = 1$

$i \geq 1 \Rightarrow C_1 - C_{i-1} b_1 = 0 \Rightarrow k\lambda^i - k\lambda^{i-1}\cdot b_1 = 0$

$\Rightarrow \lambda^i(1 - \frac{b_1}{\lambda}) = 0$

we know $\lambda \neq 0 \Rightarrow 1 - \frac{b_1}{\lambda} = 0$

$b_1 = \lambda$

Denote $\frac{1}{\lambda} = L^* \Rightarrow 1 - b_1 L^* = 0 \Rightarrow L^* = \frac{1}{b_1}$ or $\lambda = b_1$

$f(L) = 1 + \lambda L + \lambda^2 L + \lambda^3 L + \cdots$

for $f(L)$ to exist I need $|\lambda| \leq 1 \Rightarrow |L^*| \geq 1$

to get stationary we further need $|\lambda| < 1 \Rightarrow |L^*| > 1$

Example.

AR(2): $y_t = 1 + 0.4 y_{t-1} + 0.6 y_{t-2} + u_t$

$= 1 + 0.4 L y_t + 0.6 L^2 y_t + u_t$

$(1 - 0.4L - 0.6L^2) y_L = 1 + u_t$

$1 - 0.4L - 0.6L^2 = 0$

$L = 1$ or $L = -\frac{5}{3}$

$|L| = 1$ or $|L| = \frac{5}{3}$

有一个根的绝对值没有大于1

$6L^2 + 4L - 10 = 0$

$3L^2 + 2L - 5 = 0$

$\begin{matrix} 1 & -1 \\ 3 & 5 \end{matrix}$

$\Rightarrow$ the process is not weakly stationary