Lecture Note 2

- Stationary Process
- Autocorrelation function
- Linear time series models
- Autoregressive of order one, AR(1), process

Stationary Process

Definition 1 (Weak stationarity)**.** *A stochastic process $\{y_t, t \in \mathcal{T}\}$ is said to be* ***weakly stationary*** *if it has a* ***constant mean*** *and* ***constant variance*** *and its* ***covariance function****, $\gamma(t_1, t_2)$, defined by*

$$\gamma(t_1, t_2) = cov(y_{t_1}, y_{t_2}) = \mathbb{E}\{[y_{t_1} - \mathbb{E}(y_{t_1})][y_{t_2} - \mathbb{E}(y_{t_2})]\},$$

depends only on the ***absolute difference*** *$|t_1 - t_2|$, namely*

$$\gamma(t_1, t_2) = \gamma(|t_1 - t_2|).$$

Definition 2 (Weakly trend stationary)**.** *A stochastic process $\{y_t, t \in \mathcal{T}\}$ is said to be* ***weakly trend stationary*** *if $y_t = x_t + d_t$, where x_t is* ***weakly stationary*** *and d_t is* ***deterministic****.*

Definition 3. *(White noise process) The process $\{\varepsilon_t, t \in \mathcal{T}\}$ is said to be a white noise process if it has* ***mean zero****, a* ***constant variance****, and ε_t and ε_s are* ***uncorrelated*** *for all $s \neq t$.*

Autocorrelation function

• In Statistics, the correlation coefficient between two random variables X and Y is defined as

$$\rho_{X,Y} = \frac{\text{cov}(X, Y)}{\sqrt{\text{var}(X)\text{var}(Y)}} = \frac{\mathbb{E}\{[X - \mathbb{E}(X)][Y - \mathbb{E}(Y)]\}}{\sqrt{\mathbb{E}\{[X - \mathbb{E}(X)]^2\}\mathbb{E}\{[Y - \mathbb{E}(Y)]^2\}}}$$

• When the sample $\{(x_t, y_t)|t = 1, 2, \cdots, T\}$ is available, the correlation can be consistency estimated by its sample counterpart

$$\hat{\rho}_{X,Y} = \frac{\frac{1}{T}\sum_{t=1}^{T}[(x_t - \bar{x})(y_t - \bar{y})]}{\sqrt{\frac{1}{T}\sum_{t=1}^{T}(x_t - \bar{x})^2 \frac{1}{T}\sum_{t=1}^{T}(y_t - \bar{y})^2}},$$

where $\bar{x} = \frac{1}{T}\sum_{t=1}^{T} x_t$ and $\bar{y} = \frac{1}{T}\sum_{t=1}^{T} y_t$.

• The correlation coefficient is between -1 and 1, $-1 \leq \rho_{XY} \leq 1$.

• Let x_t be a weakly stationary process. The correlation coefficient between x_t and x_{t-k} is called lag-k autocorrelation of x_t and is commonly denoted by $\rho(k)$. Specifically, we define

$$\rho(k) = \frac{\text{cov}(x_t, x_{t-k})}{\sqrt{\text{var}(x_t)\text{var}(x_{t-k})}} = \frac{\gamma(t, t-k)}{var(x_t)} = \frac{\gamma(k)}{\gamma(0)}.$$

• $\rho(0) = 1$, $\rho(k) = \rho(-k)$, and $-1 \leq \rho(k) \leq 1$.

• $\hat{\rho}(k) = \frac{\frac{1}{T}\sum_{t=k+1}^{T}(x_t-\bar{x})(x_{t-k}-\bar{x})}{\frac{1}{T}\sum_{t=1}^{T}(x_t-\bar{x})^2}$ is a consistent estimator of $\rho(k)$. If $\{x_t\}_{t=1}^{T}$ is a sequence of independently identically distributed (iid) random variables and $\mathbb{E}(x_t)^2 < \infty$, then $\hat{\rho} \overset{a.s.}{\sim} \mathcal{N}(0, \frac{1}{T})$.

Generally, if x_t is a weakly stationary process satisfying, $x_t = \mu + \sum_{i=0}^{q}\delta_i\varepsilon_{t-i}$, where $\delta_0 = 1$ and $\{\varepsilon\}_i$ is a sequence of iid random variables with mean 0, then $\hat{\rho}(k) \overset{a.s.}{\sim} \mathcal{N}(\rho(k), \frac{1}{T} + \frac{2}{T}\sum_{i=1}^{q}\rho(i)^2)$.

• Testing individual Autocorrelation Coefficient Function (ACF):

$$\mathcal{H}_0 : \rho(k) = 0, \text{ v.s. } \mathcal{H}_a : \rho(k) \neq 0 \longrightarrow \text{t-ratio} = \frac{\hat{\rho}(k)}{\sqrt{\frac{1}{T} + \frac{2}{T}\sum_{i=1}^{q}\hat{\rho}(i)^2}} \overset{a.s.}{\sim} \mathcal{N}(0, 1).$$

Or if we assume that $\rho_i = 0$ for all $i \neq k$ then:

$$\text{t-ratio} = \sqrt{T}\hat{\rho}(k) \overset{a.s.}{\sim} \mathcal{N}(0, 1).$$

• Joint test for several autocorrelations:

$$\mathcal{H}_0 : \rho(1) = \rho(2) = \cdots = \rho(m) = 0, \text{ v.s. } \mathcal{H}_a : \rho(i) \neq 0 \text{ for at least one } i \in \{1, 2, \cdots, m\},$$

Box and Pierce(1970): $Q^*(m) = T\sum_{\ell=1}^{m}\hat{\rho}(\ell)^2 \overset{a.s.}{\sim} \chi^2(m)$,

Ljung and Pierce(1978): $Q(m) = T(T+2)\sum_{\ell=1}^{m}\frac{\hat{\rho}(\ell)^2}{T-\ell} \overset{a.s.}{\sim} \chi^2(m)$.

• Choice of m: $m \approx T^{1/3}$ or $m \approx \ln(T)$.

Linear time series models

Definition 4. *A time series x_t is said to be linear if it can be written as:*

$$x_t = \mu + \sum_{i=0}^{\infty}\delta_i\varepsilon_{t-i}, \text{ where } \varepsilon_t \overset{iid}{\sim} (0, \sigma^2). \tag{1}$$

Q: What is the expected value, variance, and autocovariance function of x_t generated by (1)?

A: $\mathbb{E}(x_t) = \mu + \sum_{i=0}^{\infty} \delta_i \mathbb{E}(\varepsilon_{t-i}) = \mu$.

$\text{var}(x_t) = \sum_{i=0}^{\infty} \delta_i^2 \text{var}(\varepsilon_{t-i}) = \sigma^2 \sum_{i=0}^{\infty} \delta_i^2$.

$\gamma(\ell) = cov(x_t, x_{t-\ell}) = \mathbb{E}(x_t - \mu)(x_{t-\ell} - \mu) = \mathbb{E}\left[\left(\sum_{i=0}^{\infty} \delta_i \varepsilon_{t-i}\right)\left(\sum_{j=0}^{\infty} \delta_j \varepsilon_{t-\ell-j}\right)\right]$

$= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta_i \delta_j \mathbb{E}\left(\varepsilon_{t-i}\varepsilon_{t-\ell-j}\right) = \sum_{i=0}^{\infty} \delta_i \delta_{i-\ell} \mathbb{E}(\varepsilon_{t-i}^2) = \sigma^2 \sum_{i=0}^{\infty} \delta_i \delta_{i-\ell}$

Q: Given the answer to the previous question, can we conclude that x_t is a weakly stationary process?

A: For x_t to be weakly stationary, we need its variance to exist, i.e. $\gamma(0) = \text{var}(x_t) < \infty$. So, we need $\sum_{i=0}^{\infty} \delta_i^2 < \infty$. We also need the autocovariance function to exist, i.e. $\gamma(\ell) = \text{cov}(x_t, x_{t-\ell}) < \infty$ for all ℓ. But note that, $\sum_{i=0}^{\infty} \delta_i \delta_{i-\ell} \leq \left(\sum_{i=0}^{\infty} \delta_i^2\right)^{1/2} \left(\sum_{i=0}^{\infty} \delta_{i-\ell}^2\right)^{1/2}$. So, if $\sum_{i=0}^{\infty} \delta_i^2 < \infty$, then $\gamma(\ell) < \infty$ for all ℓ. Assuming that $\sum_{i=0}^{\infty} \delta_i^2 < \infty$, x_t have constant mean and variance, and its autocovariance function only depends on the absolute value of the time distance. So, we can conclude that x_t is a weakly stationary process.

• The autocorrelation function of x_t is as follows:

$$\rho(\ell) = \frac{\gamma(\ell)}{\gamma(0)} = \frac{\sum_{i=0}^{\infty} \delta_i \delta_{i-\ell}}{\sum_{i=0}^{\infty} \delta_i^2}$$

Autoregressive of order one, AR(1), process

Definition 5. *A time series process x_t is said to be Autoregressive of order one, AR(1), if it can be written as*

$$x_t = \phi_0 + \phi_1 x_{t-1} + \varepsilon_t, \text{ where } \varepsilon_t \overset{iid}{\sim} (0, \sigma^2). \tag{2}$$

Q: Suppose that x_t given by (2) is a weakly stationary process. Compute its mean, variance and autocovariance function.

A: $\mathbb{E}(x_t) = \phi_0 + \phi_1 \mathbb{E}(x_{t-1}) = \phi_0 + \phi_1 \mathbb{E}(x_t) \Rightarrow \mathbb{E}(x_t) = \frac{\phi_0}{1-\phi_1}$.

$\text{var}(x_t) = \phi_1^2 \text{var}(x_{t-1}) + \text{var}(\varepsilon_t) = \phi_1^2 \text{var}(x_t) + \sigma^2 \Rightarrow \text{var}(x_t) = \frac{\sigma^2}{1-\phi_1^2}$.

$\gamma(\ell) = \text{cov}(x_t, x_{t-\ell}) = \text{cov}(\phi_0 + \phi_1 x_{t-1} + \varepsilon_t, x_{t-\ell}) = \phi_1 \text{cov}(x_{t-1}, x_{t-\ell}) = \phi_1 \gamma(\ell - 1)$. By substituting, $\gamma(\ell - 1) = \phi_1 \gamma(\ell - 2)$, we can further write, $\gamma(\ell) = \phi_1^2 \gamma(\ell - 2)$. By repeating this for ℓ times we can get $\gamma(\ell) = \phi_1^\ell \gamma(0) = \phi_1^\ell \frac{\sigma^2}{1-\phi_1^2}$.

Q: Find the condition(s) under which x_t given by (2) is a weakly stationary process.

A: By substituting $x_{t-1} = \phi_0 + \phi_1 x_{t-2} + \varepsilon_{t-1}$ into the equation for x_t, we get

$$x_t = \phi_0 + \phi_0\phi_1 + \phi_1^2 x_{t-2} + \varepsilon_t + \phi_1\varepsilon_{t-1}.$$

By substituting $x_{t-2} = \phi_0 + \phi_1 x_{t-3} + \varepsilon_{t-2}$, we can further write

$$x_t = \phi_0(1 + \phi_1 + \phi_1^2) + \phi_1^3 x_{t-3} + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_1^2\varepsilon_{t-2}.$$

Repeating this for n times, we get

$$x_t = \phi_0(1 + \phi_1 + \phi_1^2 + \cdots + \phi_1^n) + \phi_1^n x_{t-n} + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_1^2\varepsilon_{t-2} + \cdots + \phi_1^n\varepsilon_{t-n}.$$

If $|\phi_1| < 1$, as $n \to \infty$, we get

$$x_t = \phi_0(1 + \phi_1 + \phi_1^2 + \cdots) + \varepsilon_t + \phi_1\varepsilon_{t-1} + \phi_1^2\varepsilon_{t-2} + \cdots.$$

Since $|\phi_1| < 1$, $1 + \phi_1 + \phi_1^2 + \cdots = \frac{1}{1-\phi}$. Hence,

$$x_t = \frac{\phi_0}{1-\phi} + \sum_{i=0}^{\infty} \phi_1^i \varepsilon_{t-i}.$$

So, if $|\phi_1| < 1$, we can present $AR(1)$ process as a linear time series process where $\mu = \frac{\phi_0}{1-\phi}$ and $\delta_i = \phi_1^i$. We know that a linear time series process is weakly stationary if $\sum_{i=0}^{\infty} \delta_i^2 < \infty$. Here we have $|\phi_1| < 1$, therefore $\sum_{i=0}^{\infty} \delta_i^2 = \sum_{i=0}^{\infty} (\phi_1^2)^i = \frac{1}{1-\phi_1^2} < \infty$. So, we can conclude that a $AR(1)$ process is weakly stationary if $|\phi_1| < 1$.