Solution for Problem Set 3

1. (a) The roots of $1 + 0.3L - 0.04L^2 = 0$ are $L_1 = 2.5$ and $L_2 = 10$. Since $|L_1| > 1$ and $|L_2| > 1$, the process is stationary.

   (b) The roots of $1 - 3.1L + 0.03L^2 = 0$ are $L_1 \approx 0.32$ and $L_2 \approx 103.01$. Since $|L_1| < 1$, the process is not stationary.

   (c) The roots of $1 - 2.5L + 2L^2 - 0.5L^3 = 0$ are $L_1 = 1$ and $L_2 = 1$ and $L_3 = 2$. Since $|L_1| = |L_2| = 1$, the process is not stationary.

2. (a) $\mathbb{E}(y_t) = \mathbb{E}(0.5 + 0.6y_{t-3} + v_t) = 0.5 + 0.6\mathbb{E}(y_{t-3}) = 0.5 + 0.6\mathbb{E}(y_t)$. Therefore

   $$0.4\mathbb{E}(y_t) = 0.5 \Rightarrow \mathbb{E}(y_t) = 1.25.$$

   (b) $\text{var}(y_t) = \text{var}(0.5+0.6y_{t-3}+v_t) = 0.36\text{var}(y_{t-3})+1 = 0.36\text{var}(y_t)+1$. Therefore

   $$0.64\text{var}(y_t) = 1 \Rightarrow \text{var}(y_t) = \frac{1}{0.64} = 1.5625.$$

   (c) $\gamma(1) = \text{cov}(y_t, y_{t-1}) = \text{cov}(0.5+0.6y_{t-3}+v_t, y_{t-1}) = 0.6\text{cov}(y_{t-3}, y_{t-1}) = 0.6\gamma(2)$.

   (d) $\gamma(2) = \text{cov}(y_t, y_{t-2}) = \text{cov}(0.5+0.6y_{t-3}+v_t, y_{t-2}) = 0.6\text{cov}(y_{t-3}, y_{t-2}) = 0.6\gamma(1)$.

   (e) By substituting $\gamma(2)$ from $\gamma(2) = 0.6\gamma(1)$ into $\gamma(1) = 0.6\gamma(2)$, we get

   $$\gamma(1) = 0.36\gamma(1) \Rightarrow 0.64\gamma(1) = 0 \Rightarrow \gamma(1) = 0.$$

   Furthermore, since $\gamma(1) = 0$, we have $\gamma(2) = 0.6\gamma(1) = 0$.

   (f) We have $\rho(k) = \frac{\gamma(k)}{\gamma(0)}$ and $\gamma(k) = 0.6\gamma(k-3)$, Therefore,

   $$\rho(k) = 0.6\rho(k-3)$$

   So, $\rho(3) = 0.6\rho(0) = 0.6 = 0.6^{3/3}$, $\rho(4) = 0.6\rho(1) = 0$, and $\rho(5) = 0.6\rho(2) = 0$. Similarly, $\rho(6) = 0.6\rho(3) = 0.6^2 = 0.6^{6/3}$,

$\rho(7) = 0.6\rho(4) = 0$, and $\rho(8) = 0.6\rho(5) = 0$. By repeating this we can conclude that

$$\rho(k) = \begin{cases} 0.6^{k/3} & \text{for } k = 3, 6, 9, \cdots, \\ 0 & otherwise, \end{cases}$$

3. (a) $\mathbb{E}(y_t) = \mathbb{E}[2(u_t - \varepsilon_t) + 0.5u_{t-1}] = 2\mathbb{E}(u_t) - 2\mathbb{E}(\varepsilon_t) + 0.5\mathbb{E}(u_{t-1}) = 0.$

   (b) $\text{var}(y_t) = \text{var}[2(u_t - \varepsilon_t) + 0.5u_{t-1}] = 4\text{var}(u_t) + 4\text{var}(\varepsilon_t) - 8\text{cov}(u_t, \varepsilon_t) + 0.25\text{var}(u_{t-1}) = 12.25.$

   (c) $\text{cov}(y_t, y_{t-1}) = \text{cov}[2(u_t - \varepsilon_t) + 0.5u_{t-1}, 2(u_{t-1} - \varepsilon_{t-1}) + 0.5u_{t-2}] = \text{cov}[u_{t-1}, u_{t-1} - \varepsilon_{t-1}] = \text{var}(u_{t-1}) - \text{cov}(u_{t-1}, \varepsilon_{t-1}) = 1.5.$

   (d) $\text{cov}(y_t, y_{t-2}) = \text{cov}[2(u_t - \varepsilon_t) + 0.5u_{t-1}, 2(u_{t-2} - \varepsilon_{t-2}) + 0.5u_{t-3}] = 0.$

   (e) $\text{cov}(y_t, y_{t-k}) = 0$ for $k = 3, 4, \cdots$.

   (f) Yes, since the process has a constant mean and variance and its autocovariance function depends only on the time distance.