PRACTICE FINAL SOLUTION

1. (a) In this case

$$y_t = 0.5 + 0.5y_{t-1} + 0.5y_{t-2} + u_t$$

By writing in lag operator format we have

$$(1 - 0.5L - 0.5L^2)y_t = 0.5 + u_t$$

To check for unit root of $1-0.5L-0.5L^2 = 0$, set $L = 1$ and we get $1-0.5-0.5 = 0$. Hence $L = 1$ is a root of $1 - 0.5L - 0.5L^2 = 0$.

(b) In this case

$$y_t = 0.5 + 0.8t + 0.1y_{t-1} + 0.1y_{t-2} + u_t$$

Let $x_t = y_t - t$. By subtracting t for the both of the sides of the above equation we have

$$y_t - t = 0.2 + 0.1(y_{t-1} - (t-1)) + 0.1(y_{t-2} - (t-2)) + u_t$$

So,

$$x_t = 0.2 + 0.1x_{t-1} + 0.1x_{t-2} + u_t.$$

By writing the process for x_t in lag operating format,

$$(1 - 0.1L - 0.1L^2)x_t = u_t.$$

The root of $1 - 0.1L - 0.1L^2$ are equals

$$L_{1,2} = \frac{0.1 \pm \sqrt{0.01 + 0.4}}{-.2}.$$

Since both of the roots are in absolute values greater than one the process is weakly stationary. Regarding mean of x_t, we have

$$\mathbb{E}(x_t) = 0.2 + 0.1\mathbb{E}(x_{t-1}) + 0.1\mathbb{E}(x_{t-2}) \Rightarrow \mathbb{E}(x_t) = \frac{0.2}{1 - 0.2} = 0.25$$

and hence $\mathbb{E}(y_t) = 0.25 + t$. For the variance of x_t, we have

$$\text{var}(x_t) = 0.01\text{var}(x_{t-1}) + 0.01\text{var}(x_{t-2}) + 0.02\text{cov}(x_{t-1}, x_{t-2}) + 1 \Rightarrow 0.98\gamma_x(0) = 0.02\gamma_x(1) + 1,$$

where $\gamma_x(1) = \text{cov}(x_{t-1}, x_{t-2})$ and $\gamma_x(0) = \text{var}(x_t)$. We also have

$$\gamma_x(1) = 0.1\gamma_x(0) + 0.1\gamma_x(1) \Rightarrow \gamma_x(1) = \frac{1}{9}\gamma_x(0).$$

Therefore,

$$0.98\gamma_x(0) = \frac{0.02}{9}\gamma_x(0) + 1 \Rightarrow \gamma_x(0) \approx 1.023.$$

So, $\text{var}(y_t) = \gamma_x(0) \approx 1.023$.

(c) When the process is weakly trend stationary the variance does not depends on time and it is finite while in the case of unit root process the conditional variance depends on time and can goes to infinity as t goes to infinity.

(d) See the section related to ADF test in the last lecture note.

2. (a) Since the root of $1-0.5L=0$ equals $L^*=2$ and it is outside of the unit circle, we can conclude that the AR component is weakly stationary and hence the ARMA process is weakly stationary.

   (b)

$$
\begin{aligned}
y^f_{t+1|t} &= \mathbb{E}(y_{t+1}|I_t) \\
&= \mathbb{E}(1+0.5y_t+u_{t+1}-0.5u_t|I_t) \\
&= 1+0.5\mathbb{E}[y_t|I_t]+0-0.5\mathbb{E}[u_t|I_t] \\
&= 1+0.5y_t-0.5u_t
\end{aligned}
$$

   (c) We have

$$
\begin{aligned}
e^f_{t+1|t} &= y_{t+1}-y^f_{t+1|t} \\
&= 1+0.5y_t+u_{t+1}-0.5u_t-y^f_{t+1|t} \\
&= 1+0.5y_t+u_{t+1}-0.5u_t-(1+0.5y_t-0.5u_t) \\
&= u_{t+1}
\end{aligned}
$$

   Therefore, $\text{var}\left(e^f_{t+1|t}\right)=1$

   (d) We know that $\lim_{\tau\to\infty}y^f_{t+\tau|t}=\mathbb{E}(y_t)$. And $\mathbb{E}(y_t)=1+0.5\mathbb{E}(y_{t-2})=1+0.5\mathbb{E}(y_t)$. Therefore, $\mathbb{E}(y_t)=2$. So, $\lim_{\tau\to\infty}y^f_{t+\tau|t}=2$

   (e) Since y_t is a weakly stationary process, $\lim_{\tau\to\infty}e^f_{t+\tau|t}=\text{var}(y_t)=0.25\text{var}(y_{t-1})+\text{var}(u_t)-0.25\text{var}(u_{t-1})=1$