AR(2):

1. $y_t = 0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t$, $u_t \sim iid(0,1)$

a. $y_t = 0.5 L\, y_t - 0.04 L^2\, y_t + u_t$

$(1 - 0.5L + 0.04L^2)\, y_t = u_t$

$1 - 0.5L + 0.04L^2 = 0 \Rightarrow L = \frac{5}{2}$ or $L = 10$

$|L| = \frac{5}{2} > 1$ or $|L| = 10 > 1$

$\Rightarrow$ This process is weakly stationary

b. $\mu = E[y_t] = E[0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t]$

$= E[0.5\, y_{t-1}] - E[0.04\, y_{t-2}] + E[u_t]$

$= 0.5\, E[y_{t-1}] - 0.04\, E[y_{t-2}] + 0$

$E[y_t] = E[y_{t-1}] = E[y_{t-2}] = \mu$

$\Rightarrow \mu = 0.5\mu - 0.04\mu \Rightarrow \mu = 0$

$\gamma(0) = Var(y_t) = cov(y_t, y_t) = cov(0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t,\ y_t)$

$= 0.5\, cov(y_{t-1}, y_t) - 0.04\, cov(y_{t-2}, y_t) + cov(u_t, y_t)$

$= 0.5\, cov(y_{t-1}, y_t) - 0.04\, cov(y_{t-2}, y_t) + Var(u_t)$

$\Rightarrow \gamma(0) = 0.5\,\gamma(1) - 0.04\,\gamma(2) + 1$

for $\gamma(k)$, $k = 1, 2, \cdots$ $\quad \gamma(k) = cov(y_t, y_{t-k}) = 0.5\, cov(y_{t-1}, y_{t-k}) - 0.04\, cov(y_{t-2}, y_{t-k})$

$= 0.5\,\gamma(k-1) - 0.04\,\gamma(k-2)$

$\Rightarrow \gamma(1) = 0.5\,\gamma(0) - 0.04\,\gamma(1) \Rightarrow 1.04\,\gamma(1) = 0.5\,\gamma(0) \Rightarrow \gamma(0) = 2.08\,\gamma(1)$

$\gamma(2) = 0.5\,\gamma(1) - 0.04\,\gamma(0) \Rightarrow \gamma(2) = 0.5\,\gamma(1) - 0.04 \times 2.08\,\gamma(1) = 0.4168\,\gamma(1)$

$\Rightarrow 2.08\,\gamma(1) = 0.5\,\gamma(1) - 0.04 \times 0.4186\,\gamma(1) + 1 \Rightarrow \gamma(1) \approx 0.63$

$\Rightarrow \gamma(0) = 2.08\,\gamma(1) = 2.08 \times 0.63 = 1.31$

c) $\gamma(k)$, $k = 1, 2, 3, \cdots$

$$\gamma(k) = cov(y_t, y_{t-k}) = 0.5\, cov(y_{t-1}, y_{t-k}) - 0.04\, cov(y_{t-2}, y_{t-k})$$

$$= 0.5\, \gamma(k-1) - 0.04\, \gamma(k-2)$$

d) $\rho(k) = \frac{\gamma(k)}{\gamma(0)} = \frac{0.5\,\gamma(k-1) - 0.04\,\gamma(k-2)}{0.5\,\gamma(1) - 0.04\,\gamma(2) + 1}$

for $k=0$ $\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1$; for $k=1$ $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{0.63}{1.31} = 0.48$

for $k=2$ $\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{0.4168 \times 0.63}{1.31} = 0.2$

e) $$\alpha(k) = \frac{cov(y_t, y_{t-k} \mid y_{t-1}, y_{t-2}, \cdots, y_{t-(k-1)})}{Var(y_t \mid y_{t-1}, y_{t-2}, \cdots, y_{t-(k-1)})}$$

for $k=1$, $cov(y_t, y_{t-1}) = cov(0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t,\ y_{t-1}) = 0.5\, Var(y_{t-1}) - 0.04\, cov(y_{t-1}, y_{t-2})$

$\Rightarrow \alpha(1) = 0.5 - 0.04\, \rho(1)$

for $k=2$, $cov(y_t, y_{t-2} \mid y_{t-1}) = cov(0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t,\ y_{t-2} \mid y_{t-1})$

$$= 0.5\, cov(y_{t-1}, y_{t-2} \mid y_{t-1}) - 0.04\, cov(y_{t-2}, y_{t-2} \mid y_{t-1})$$

$$= -0.04\, Var(y_{t-2} \mid y_{t-1})$$

$\Rightarrow \alpha(2) = -0.04$

for $k=3$, $cov(y_t, y_{t-3} \mid y_{t-1}, y_{t-2}) = cov(0.5\, y_{t-1} - 0.04\, y_{t-2} + u_t,\ y_{t-3} \mid y_{t-1}, y_{t-2})$

$$= \underbrace{0.5\, cov(y_{t-1}, y_{t-3} \mid y_{t-1}, y_{t-2})}_{=0} - 0.04\, cov(y_{t-2}, y_{t-3} \mid y_{t-1}, y_{t-2})$$

$\Rightarrow \alpha(3) = 0$

$\Rightarrow \alpha(k) = 0$ for $k \geq 3$

2. MA : $y_t = 1 + u_t + 2u_{t-3}$, $u_s \sim iid\ (0,1)$

a. $E[y_t] = E[1 + u_t + 2u_{t-3}]$

$= 1 + E[u_t] + 2E[u_{t-3}]$

$= 1 + 0 + 0 = 1$

b. $\gamma_y(0) = Var(y_t) = Var(1 + u_t + 2u_{t-3}) = Var(u_t) + 4Var(u_{t-3}) = 1 + 4 = 5$

c. $\gamma_y(3) = cov(y_t, y_{t-3}) = cov(1 + u_t + 2u_{t-3}, y_{t-3})$

$= cov(u_t, y_{t-3}) + 2\, cov(u_{t-3}, y_{t-3})$

$cov(u_t, y_{t-3}) = cov(u_t, y_{t-3}) = cov(u_t, 1 + u_{t-3} + 2u_{t-6}) = 0$

$cov(u_{t-3}, y_{t-3}) = cov(u_{t-3}, 1 + u_{t-3} + 2u_{t-6}) = 1$

$\Rightarrow \gamma(3) = 2$

for $k \neq 3$ $\gamma_y(k) = cov(y_t, y_{t-k}) = cov(1 + u_t + 2u_{t-3}, y_{t-k})$

$= cov(u_t, y_{t-k}) + 2\, cov(u_{t-3}, y_{t-k})$

$= 0$

d. $y_t = 1 + u_t + 2 \cdot L^3 u_t$

$(1 + 2L^3) u_t = -1 + y_t$

$1 + 2L^3 = 0 \Rightarrow L < 1$

$\Rightarrow$ this MA process is not invertible e7. $\rho_y(k) = \frac{\gamma(k)}{\gamma(0)}$

for $k=3$ $\quad \rho_y(k) = \frac{\gamma(3)}{\gamma(0)} = \frac{2}{5}$, for $k=0$ $\quad \rho_y(0) = \frac{\gamma(0)}{\gamma(0)} = 1$

for $k \neq 3$, $k>0$, $\rho_y(k) = \frac{\gamma(k)}{\gamma(0)} = 0$

f7. MA : $x_t = \mu + v_t + \theta v_{t-3}$, $v_s \sim iid(0, \sigma_v^2)$

$= \mu + v_t + \theta \cdot L^3 \cdot v_t$

Sorry my head blown up ...