Q1

Let, $A = \{a_1, a_2, \cdots a_n\}$

$p: \frac{1}{n}\sum a_i = m$

$q: \exists a_i \in A,\ a_i \leq m$

Proof by contradiction

Assume: $p \to \neg q$

$\neg q: \forall a_i \in A,\ a_i > m$

Then, $\sum_{i=0}^{n} a_i > nm$

$\frac{1}{n}\sum_{i=0}^{n} > m$

Therefore, $\neg q$ contradicts p, q must be true.

Q2(a)

By definition $p \leftrightarrow q$ is true if both p, q are true or false

p	q	$p \wedge q$	$\neg p \wedge \neg q$	$(p\wedge q) \vee (\neg p \wedge \neg q)$	$p \leftrightarrow q$
1	0	0	0	0	0
1	1	1	0	1	1
0	0	0	1	1	1
0	1	0	0	0	0

$\therefore p \leftrightarrow q \equiv (p\wedge q) \vee (\neg p \wedge \neg q)$

Q2(b)

p	q	r	$\neg(p \wedge q \wedge r)$	$\neg p \vee \neg q \vee \neg r$
0	0	0	1	1
0	0	1	1	1
0	1	0	1	1
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	0	0

$\therefore \neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$

Q3(a)

$P(x): \exists k \in \mathbb{N},\ x = 3k$

$Q(x): x > 15$

Verify statements:

$\forall x \in \{3\}, P(x)$ is true Since $\{3\} \subseteq D$, $\exists x \in D, P(x)$

$\forall y \in \{17, 19, 23\}, Q(y)$ is true Since $\{17, 19, 23\} \subseteq D$, $\exists y \in D, Q(y)$

Since no numbers in D meets $P(z) \wedge Q(z)$

$\neg(\exists z \in D, P(z) \wedge Q(z))$

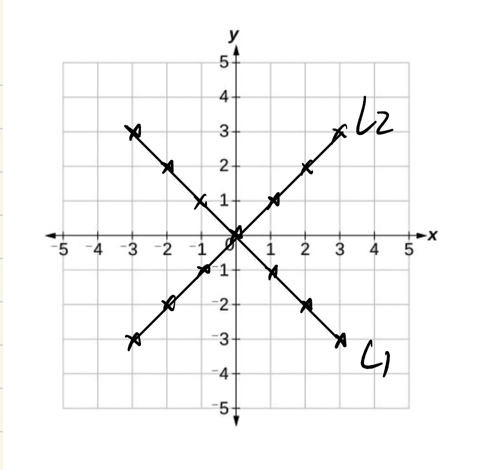
Q3(b)

$L_1: y = -x$

(3, −3)
(2, −2)
(1, −1)
(0, 0)
(−1, 1)
(−2, 2)
(−3, 3)

$L_2: y = x$

(3, 3)
(2, 2)
(1, 1)
(0, 0)
(−1, −1)
(−2, −2)
(−3, −3)



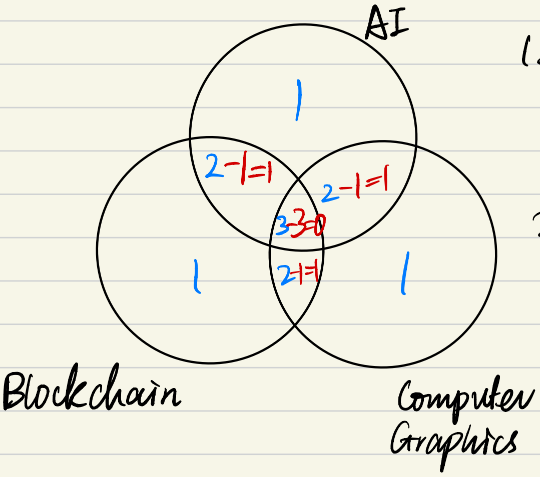
Solution to the equation is the common point of L_1 and L_2, which is the common element of sets A and B

$\therefore A = \{(x, -x) \mid x \in D\}$
$B = \{(x, x) \mid x \in D\}$

Solution to L_1 and L_2 : $A \cap B$

Q4(a)

By Counting Overlaps:

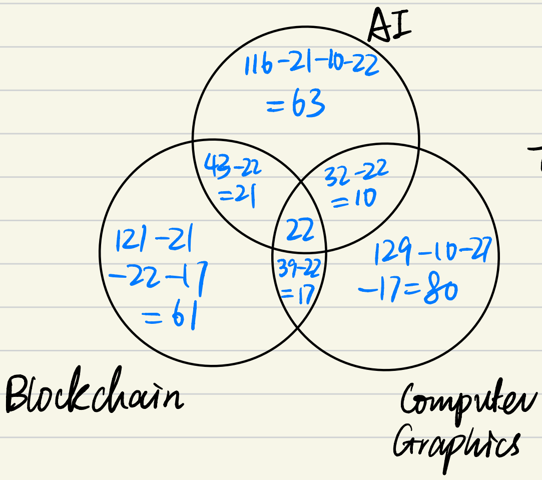


1. 116 + 121 + 129 = 366
Blue text shows the number of times each part is calculated.

2. 366 − 43 − 32 − 39 = 252
Red text shows the count after subtraction

3. Since students who took all 3 courses aren't counted:
252 + 22 = 274

By counting each section:



Total students = (63 + 61 + 80) + (21 + 10 + 17) + 22
= 274

Q4(b)

Based on the graphs from Q4(a),

Student with ≥ 2 electives, selected blockchain $= 21+22+17=60$

Q5(a) $(p \wedge (p \to q)) \to q$

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
F	F	T	F	T
F	T	T	F	T
T	F	F	F	T
T	T	T	T	T

Q5(b)

$(p \wedge q) \to p$ Simplification

$p \to (p \vee s)$ Addition

$((p \vee s) \wedge ((p \vee s) \to \neg r)) \to \neg r$ Modus poneus

$(\neg r \wedge (r \vee t)) \to t$ Disjunctive syllogism

Q5(c)

Let,

p : student is cheating

q : student pass

s : student is good.

We know, $S1: p \to \neg q$

$S2: s \to q$

(i) Equivalence: $p \to s$

By contradiction:

Assume $p \to s$

$((p \to s) \wedge (s \to q)) \to (p \to q)$

$(p \to q)$ contradicts $S1$

(i) is wrong

(iv) Equivalence: $\neg s \to \neg p$

By contrapositive: $(\neg s \to \neg p) \to (p \to s)$

$((p \to s) \wedge (s \to q)) \to (p \to q)$

$(p \to q)$ contradicts $S1$

(iv) is wrong.

(ii) Equivalence: $\neg p \to \neg s$

By contrapositive, $(\neg p \to \neg s) \to (s \to p)$

$((s \to p) \wedge (p \to \neg q)) \to (s \to \neg q)$

Contradicts $S2$

(ii) is wrong.

(iii) Equivalence: $s \to \neg p$

By contradiction, assume $s \to p$

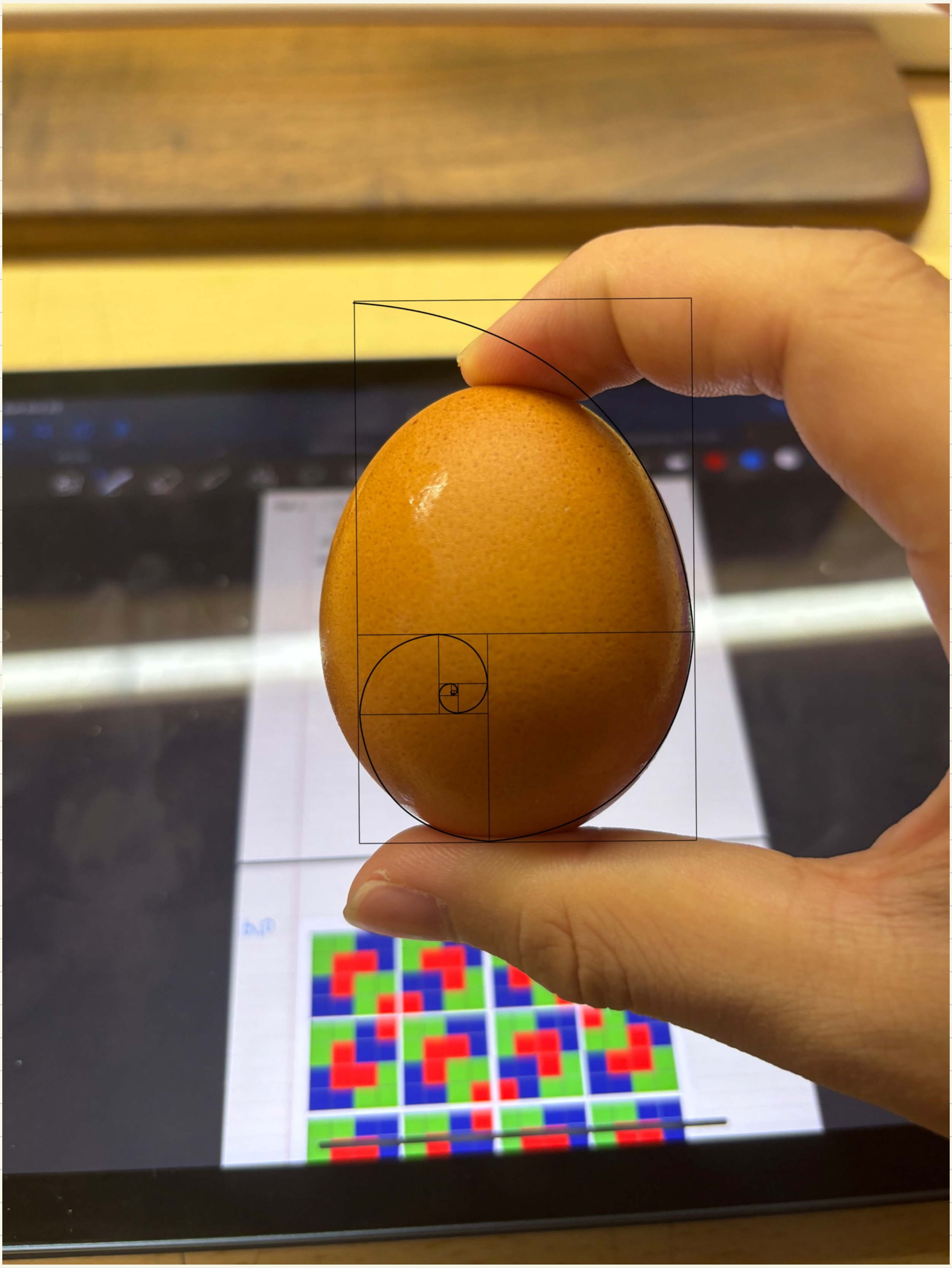
$((s \to p) \wedge (p \to \neg q)) \to s \to \neg q$

Contradicts with $S2$.

$\therefore$ $s \to \neg p$ is true.

(iii) is correct.

Q6:



Q7(a) $D^T E$

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 7+27+55 & 8+30+60 \\ 14+36+66 & 16+40+72 \end{bmatrix} = \begin{bmatrix} 89 & 98 \\ 116 & 128 \end{bmatrix}$$

Q7(b)

$$A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & -5 \\ 5 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 5 \\ -5 & -8 \end{bmatrix}$$

$$(A-B)(A+B) = \begin{bmatrix} -1 & 1 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -9 & -9 \end{bmatrix}$$

Therefore $A^2 - B^2 \neq (A-B)(A+B)$

Q7(c)

$$C^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$C^3 = C^2 \cdot C = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$C^3 - 6C^2 + 7C$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Therefore, $k = 2$

Q8(a)

IsPrime-A (n)

```
for integer i←2 to n-1
    if n mod i ==0 then
        return False
return True
```

Constant Factor	Frequency
c_1	$n-3$
c_2	$n-3$
c_3	1
c_4	1

IsPrime-B (n)

```
for integer i←2 to floor(sqrt(n))+1
    if n mod i ==0 then
        return False
return True
```

Constant Factor	Frequency
c_6	$\lfloor\sqrt{n}\rfloor - 1$
c_7	$\lfloor\sqrt{n}\rfloor - 1$
c_8	1
c_9	1

Their worst-case time complexity:

$IsPrime(A) = c_1(n-3) + c_2(n-3) + c_3 + c_4 = O(n)$

$IsPrime(B) = c_6(\lfloor\sqrt{n}\rfloor - 1) + c_7(\lfloor\sqrt{n}\rfloor - 1) + c_8 + c_9 = O(\sqrt{n})$

$\therefore$ IsPrime(B) is more efficient, because it has slower growth rate

Q(9) $f(n) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots - \frac{1}{2n}$

Base case: $LHS = 1 - \frac{1}{2} = \frac{1}{2}$, $RHS = \frac{1}{2}$, $LHS = RHS$

Inductive Step:

Assume statement is true for $n = k$

That is, $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k}$

Proof: statement holds when $n = k+1$:

$LHS = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2k-1} - \frac{1}{2k}\right) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$

$RHS = \left(\frac{1}{k+2} + \frac{1}{k+3} + \dots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2k+2}\right)$

$= \left(\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}\right)$

$= \left(\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k}\right) + \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right)$

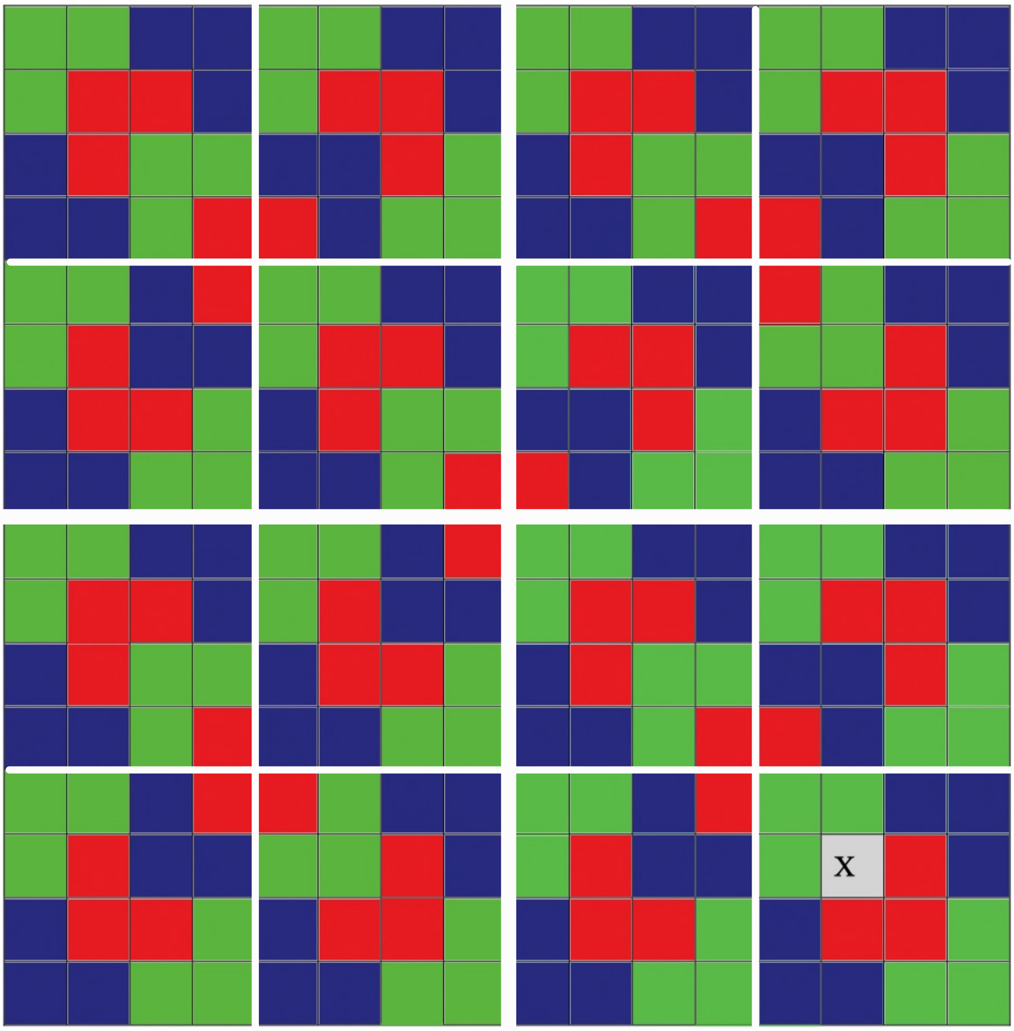
$\because \left(\frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k}\right) = \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2k-1} - \frac{1}{2k}\right)$ according to assumption

$\therefore LHS = RHS$

Therefore, By mathematical Induction we proved $\forall n \in \mathbb{Z}^+$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \dots \frac{1}{2n}$ is true.

Q10



A triomino looks like one of the followings.