Question 2(c) Solution

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1 Theoretical Mean

Let X be random variable: $X \sim N(\mu, \sigma^2)$. Then, e^X would be random variable conforming to log-normal distribution.

To compute the expectation (mean) of a log-normal distribution $E(e^X)$, we integrate over $\int e^x f(x)$, where $f(x)$ is the probability density function (PDF) of a Gaussian distribution.

$$E(e^X) = \int_{-\infty}^{+\infty} e^x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $z = x - \mu$, $dz = 1 \cdot dx$:

$$\begin{aligned}
E(e^X) &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{x-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{(z+\mu)-\frac{z^2}{2\sigma^2}} dz \\
&= e^{\mu} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{z-\frac{z^2}{2\sigma^2}} dz
\end{aligned}$$

Now, consider the exponent separately:

$$\begin{aligned}
z - \frac{z^2}{2\sigma^2} &= \frac{2\sigma^2 z - z^2}{2\sigma^2} \\
&= \frac{\sigma^4 - (\sigma^2 - z)^2}{2\sigma^2} \\
&= \frac{\sigma^2}{2} - \frac{(z-\sigma^2)^2}{2\sigma^2}
\end{aligned}$$

Substituting this result back into the integral, we get

$$\begin{aligned}
E(e^X) &= e^{\mu} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{\sigma^2}{2}-\frac{(z-\sigma^2)^2}{2\sigma^2}} dz \\
&= e^{\mu+\frac{\sigma^2}{2}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\sigma^2)^2}{2\sigma^2}} dz
\end{aligned}$$

Observing the integral, we can tell that the inner function is of same form as the PDF of a Gaussian distribution whose mean and variance are both σ^2. Since the integral of a PDF $= 1$, the equation can be simplified to

$$E(e^X) = e^{\mu+\frac{\sigma^2}{2}}$$

Substituting in the mean of 0.02 and standard deviation of 0.05 from the original question, we get

$$
\begin{aligned}
E(e^X) &= e^{0.02+\frac{0.05^2}{2}} \\
&\approx 1.021451
\end{aligned}
$$

2 Theoretical Variance

To derive the variance of the log normal distribution, we can follow the König–Huygens theorem

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

Utilizing the theorem, we get

$$
\begin{aligned}
\mathrm{Var}(e^X) &= E(e^{2X}) - E(e^X)^2 \\
&= e^{2\mu+4\sigma^2} - E(e^X)^2 \\
&\approx 0.002665
\end{aligned}
$$

3 Empirical Mean and Variance

3.1 Code Snippet

```r
sizes <- c(10, 100, 1000, 10000, 100000)
for (n in sizes) {
    samples <- exp(rnorm(n, mu, sd))
    cat("Sample Size:", n, "- Var:", var(samples), "| Mean:", mean(
    samples), "\n")
}
```

3.2 Code Output

```
Sample Size: 10 - Var: 0.002932168 | Mean: 1.010555
Sample Size: 100 - Var: 0.002977596 | Mean: 1.018695
Sample Size: 1000 - Var: 0.002567438 | Mean: 1.023125
Sample Size: 10000 - Var: 0.002627022 | Mean: 1.020898
Sample Size: 1e+05 - Var: 0.002610527 | Mean: 1.021684
```

3.3 Conclusion

Since the simulation results approximates the theoretical mean and variance fairly well, our theoretical results have been justified.