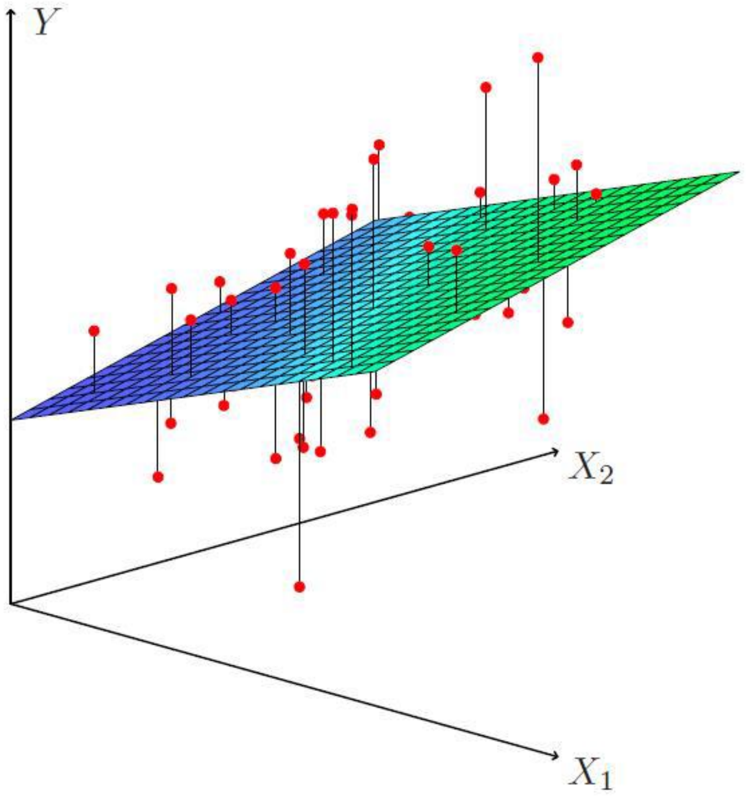
Multiple regression



Overview

Analysis of variance for regression

Multiple regression

- Basic ideas
- Categorical predictors

Bonus material if there is time

- Relating simple linear regression to multiple regression

Quick review of simple linear regression

Review: The process of building regression models

Choose the form of the model

Fit the model to the data

Assess how well the model describes the data

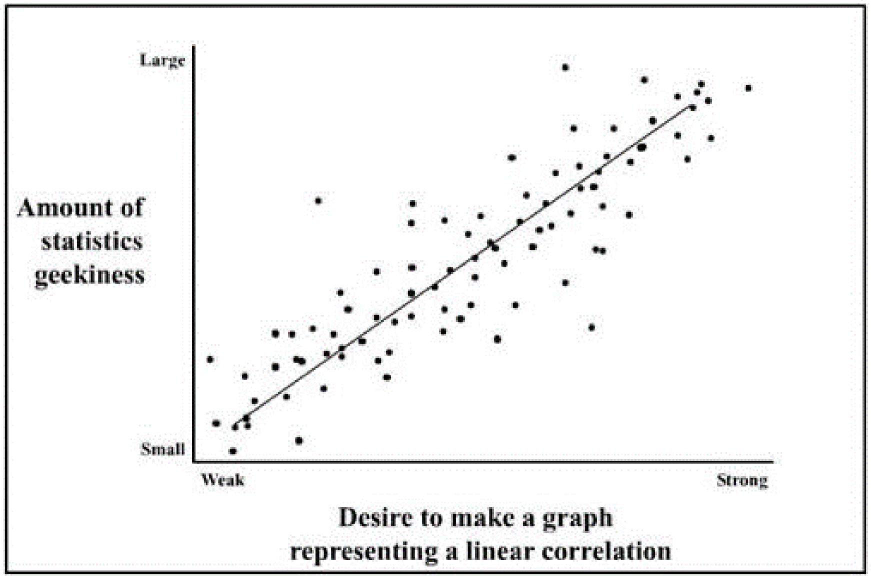
Use the model to address questions of interest

All models are wrong, but some models are useful

Analysis of Variance (ANOVA) for regression

In an analysis of variance, we break down the **total variability** in the **response variable y** into:

- 1. the variability explained by the model
- 2. the variability not explained by the model
    - i.e., the residuals



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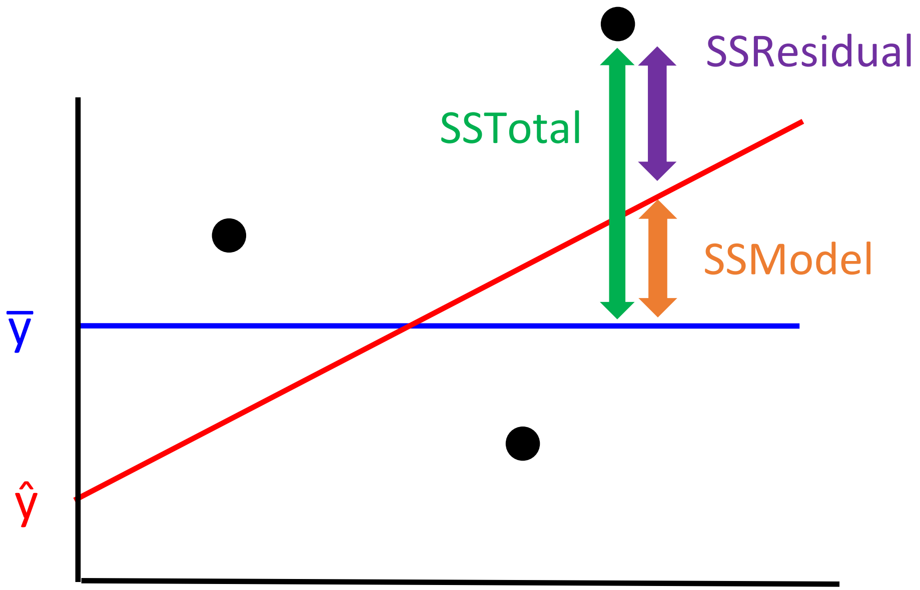


We can express this as:

- SSTotal = SSModel + SSResidual

$$y - \bar{y} = (\hat{y} - \bar{y}) + (y - \hat{y})$$

Added and subtracted ŷ

This equal 0 (proof via algebra)

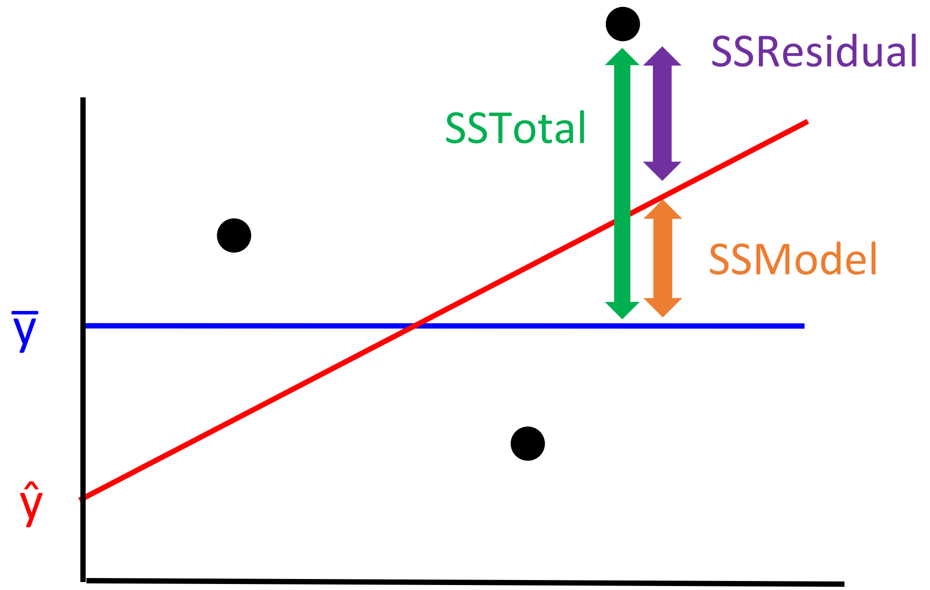
$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2 + (y_i - \hat{y}_i)^2 + \cancel{2(y_i - \hat{y}_i)(\hat{y}_i - \bar{y})}$$

The coefficient of determination r^2

The **percentage of the total variability explained by the model** is given by

$$r^2 = \frac{SSModel}{SSTotal} = 1 - \frac{SSResidual}{SSTotal}$$

Coefficient of determination



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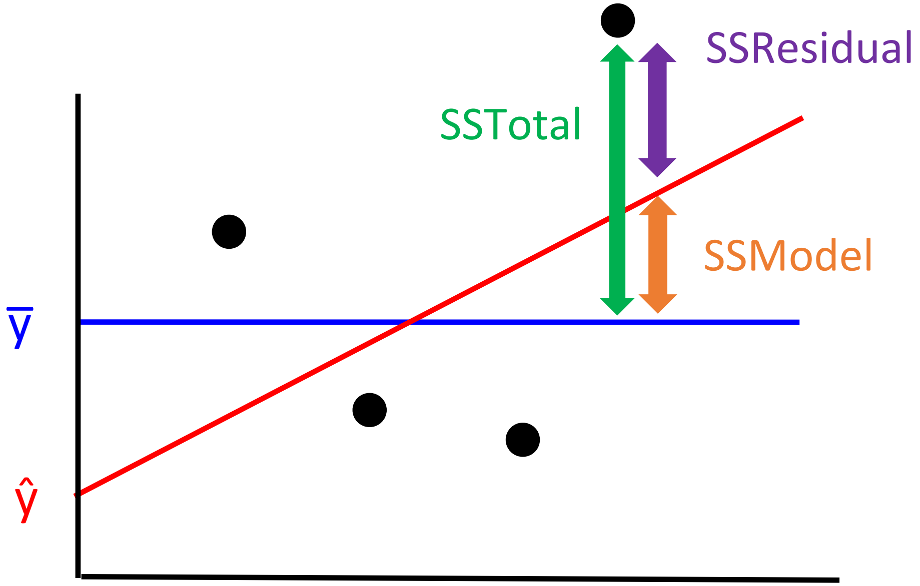
Hypothesis test based on ANOVA for regression

$$F = \frac{SSModel/df_{model}}{SSResidual/df_{error}}$$

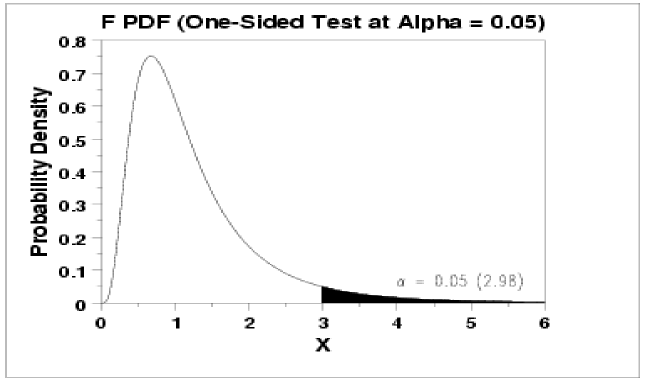
$df_{model} = 1$

$df_{error} = n - 2$



If the null hypothesis is true that $\beta_1 = 0$:

- Both the numerator and denominator are estimates of σ^2
- F comes from an F-distribution with df_{model}, df_{error} degrees of freedom
- For simple linear regression, this gives the same results as running a t-test.
  - $F = t^2$



Analysis of Variance (ANOVA) for regression in R

You can create an ANOVA table for regression relationships in R using:

- anova(lm_fit)

```
lm_fit <- lm(salary_tot ~ log_endowment, data = assistant_data)

anova(lm_fit)

```
```

SSModel

SSResidual

F
```

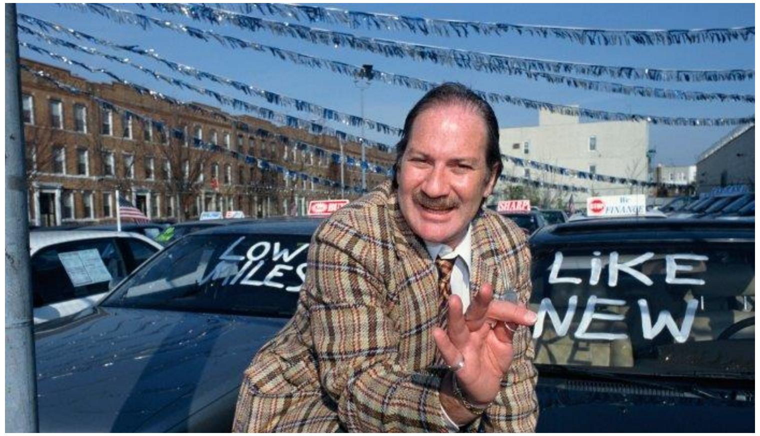
Analysis of Variance (ANOVA) for regression in R

You can create an ANOVA table for regression relationships in R using:

- anova(lm_fit)

We can check that the ANOVA relationships holds: SSTotal = SSModel + SSResidual using:

- The original data y values
- lm_fit$residuals
- lm_fit$fitted.values



You can also check that $F = t^2$ by comparing anova(lm_fit) and summary(lm_fit) values

Homework 7!

Questions?

Multiple regression

In multiple regression we try to predict a quantitative response variable *y* using several predictor variables $x_1, x_2, \ldots, x_k$

For multiple linear regression, the underlying model is:

$$y = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \ldots \beta_k \cdot x_k + \epsilon$$

We estimate coefficients using a data set to make predictions ŷ

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

Multiple regression

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

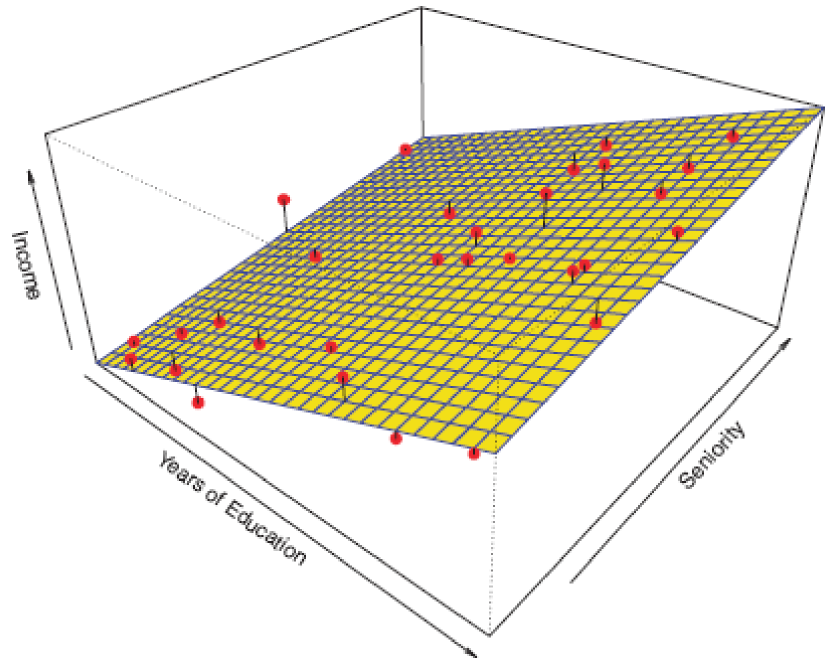
There are many uses for multiple regression models including:

- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

Multiple regression

$$\text{salary} \;=\; \hat{\beta}_0 \;+\; \hat{\beta}_1 \cdot \text{f(endowment)} \;+\; \hat{\beta}_2 \cdot \text{g(enrollment)}$$

Let's continue by exploring this in R…



Nested model comparison

We can also assess whether a particular subset of *q* parameters is 0

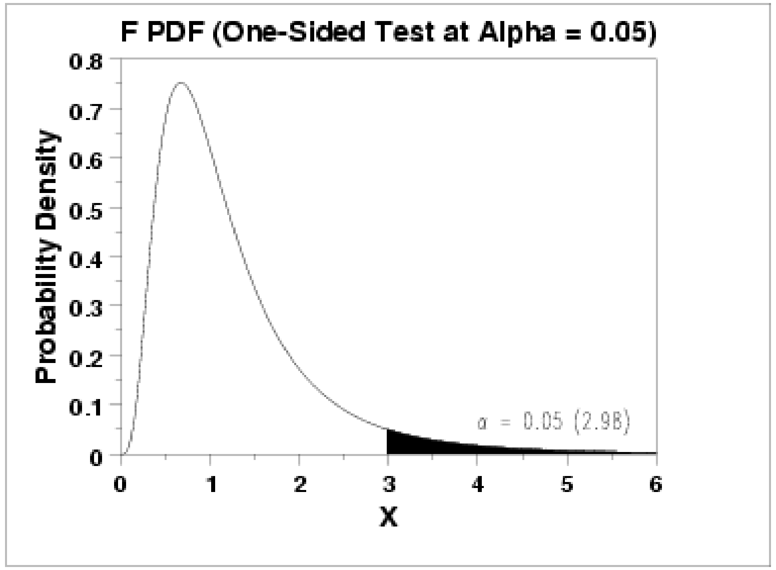
$$H_0: \beta_h = \beta_i = \ldots = \beta_g = 0$$

To do this we:

1. Fit the model without these features
2. Calculate the $SSRes_{Reduced}$ for the model without these predictors
3. Compare it to the full model $SSRes_{Full}$ with an F-statistic:

$$F = \frac{(SSRes_{Reduced} - SSRes_{Full})/q}{SSRes_{Full}/(n-k-1)}$$

where q is the number of additional terms in the full model



$df_1 = df_{Reduced} - df_{Full}$
$df_2 = df_{Full}$

Categorical predictors and interactions

Categorical predictors

Predictors can be categorical as well as quantitative

If a predictor only has two levels, we can use a single 'dummy variable' to encode these two levels:

- E.g., Assistant or Full Professor

$$x_i = \begin{cases} 1 & \text{if Assistant Professor} \\ 0 & \text{if Full Professor} \end{cases}$$

Assistant Professors have an additional value added β_1 to their y-intercepts

$$y_i = \beta_0 + \beta_1 x_1 + \epsilon_i = \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if Assistant Professor} \\ \beta_0 + \epsilon_i & \text{if Full Professor} \end{cases}$$

Categorical predictors

When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

- e.g., we would need two dummy variables to have different intercepts for Assistant, Associate and Full Professors

$$x_{i1} = \begin{cases} 1 & \text{if Assistant Professor} \\ 0 & \text{if Full Professor} \end{cases} \qquad x_{i2} = \begin{cases} 1 & \text{if Associate Professor} \\ 0 & \text{if Full Professor} \end{cases}$$

$$y_i \;=\; \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \epsilon_i \;=\; \begin{cases} \beta_0 + \beta_1 + \epsilon_i & \text{if Assistant Professor} \\ \beta_0 + \beta_2 + \epsilon_i & \text{if Associate Professor} \\ \beta_0 + \epsilon_i & \text{if Full Professor} \end{cases}$$

Let’s try it in R…

Interaction terms

The models we have looked at the relationship between the response and the predictors has been ***additive*** and ***linear***

$$y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + \epsilon$$

These models assume that each predictor acts independently on the response y and that the relationship is linear

We can relax both of these assumptions

Interaction terms

An ***interaction effect*** occurs when the response variable y is influenced by the levels of two or more predictors in a non-additive way

For example, a professor’s salary might be more effected by the size of a school’s endowment depending on the number of students who attend the school

We can model this using an equation with an interaction term

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$

Interaction terms: categorical predictors

When using categorical variables, the interaction corresponds to different slopes for the quantitative variable depending on the value of the categorical variable

- e.g., professor's salary might be more effected by the size of a school's endowment depending whether she is an Assistant or a Full Professor

If Full Professor: $\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$

If Assistant Professor: $\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$

Additive term if Assistant Professor

Change in slope if Assistant Professor

Interaction terms

$$\begin{aligned} \text{salary} \approx \beta_0 &+ \beta_1 \cdot \text{endowment} \\ &+ \beta_2 \cdot \text{assistant\_rank\_dummy} \\ &+ \beta_3 \cdot (\text{assistant\_rank\_dummy} \cdot \text{endowment}) \end{aligned}$$

Let's try it in R…

Bonus material: relating simple and multiple regression coefficients

Relating simple and multiple regression

Suppose we fit both a simple and multiple regression models to the same data.

simple linear regression coefficient for x_1

Simple regression model: $\hat{y} = \hat{\beta}_{0(1)} + \hat{\beta}_{1(1)} \cdot x_1$

multiple linear regression coefficient for x_1

Multiple regression model: $\hat{y} = \hat{\beta}_{0(2)} + \hat{\beta}_{1(2)} \cdot x_1 + \hat{\beta}_{2(2)} \cdot x_2$

Question: How are the coefficients $\hat{\beta}_{1(1)}$ and $\hat{\beta}_{1(2)}$ related?

Relating simple and multiple regression

Question: How are the simple regression coefficients $\hat{\beta}_{1(1)}$ and the multiple regression coefficient $\hat{\beta}_{1(2)}$ (for a predictor x_1) related?

We can view the multiple regression coefficient $\hat{\beta}_{1(2)}$ as the change in y with a unit increase in x_1 when we **set the predictor x_2 to fixed a value**.

We can view the simple regression coefficient $\hat{\beta}_{1(1)}$ as the change in y with a unit increase in x_1 **when we let the other predictor x_2 change with the value of x_1**.

Relating simple and multiple regression

If the predictor x_1 is correlated with x_2, then changing x_1 will be associated with changes in x_2

- This change in x_2 will also be associated with changes in y

We can assess the association between x_1 and x_2, using regression:

$$x_2 = \hat{\delta}_0 + \hat{\delta}_1 \cdot x_1$$

We can then relate the change in *y* with the change in x_1 in the simple regression coefficient to the multiple regression coefficients as:

$$\hat{\beta}_{1(1)} \cdot x_1 = \hat{\beta}_{1(2)} \cdot x_1 + \hat{\beta}_{2(2)} \cdot \hat{\delta}_1 \cdot x_1$$

Let's explore this in R…