Simple linear regression

Overview

Simple linear regression

Inference for simple linear regression

- Errors and residuals
- Hypothesis tests for regression coefficients
- If there is time: confidence and prediction intervals

Announcements

Homework 5 has been graded

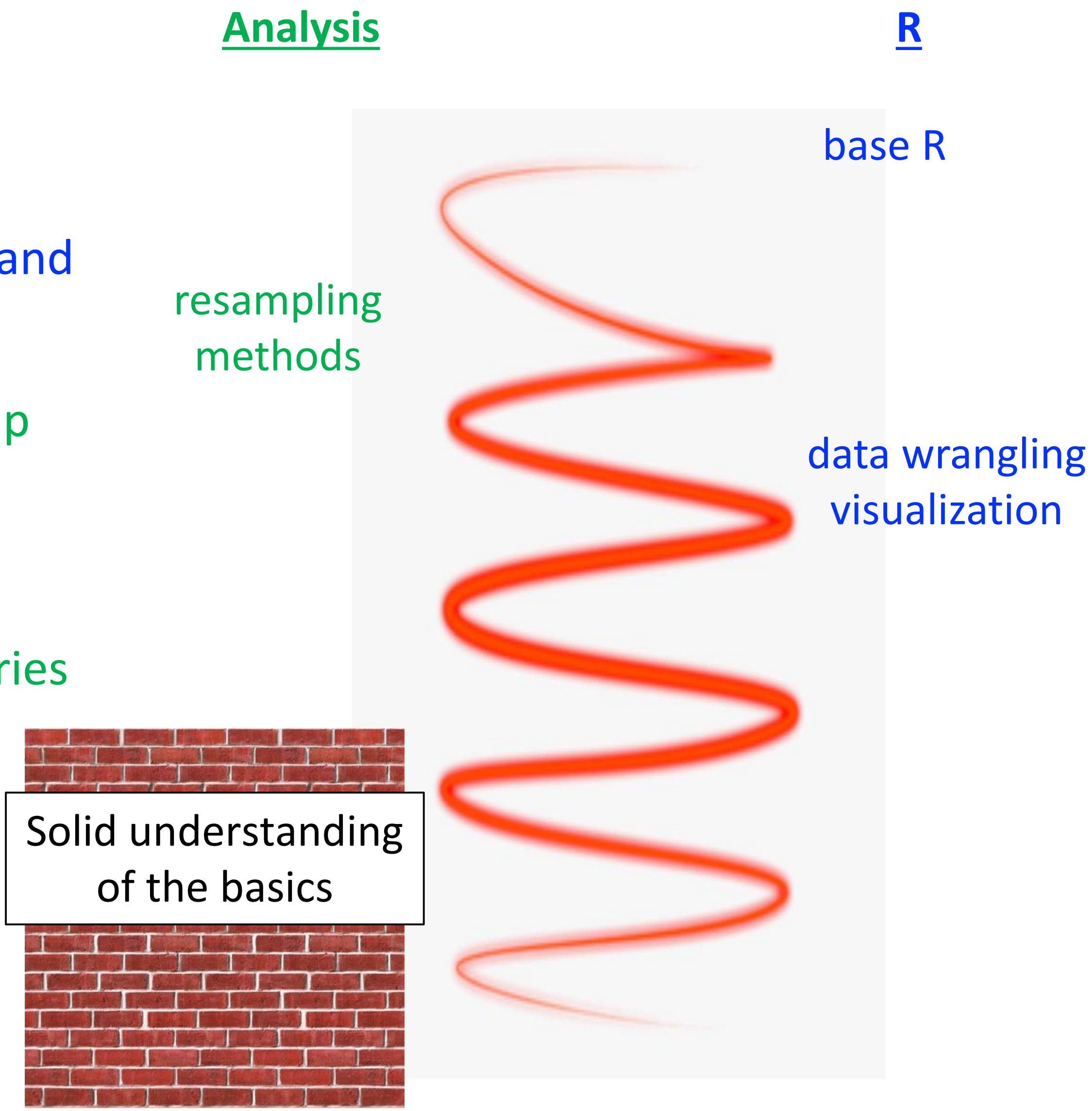
- You have a week for regrade requests
- Midterm exams should be graded soon

Homework 6 is out

- Due Sunday October 29th at 11pm

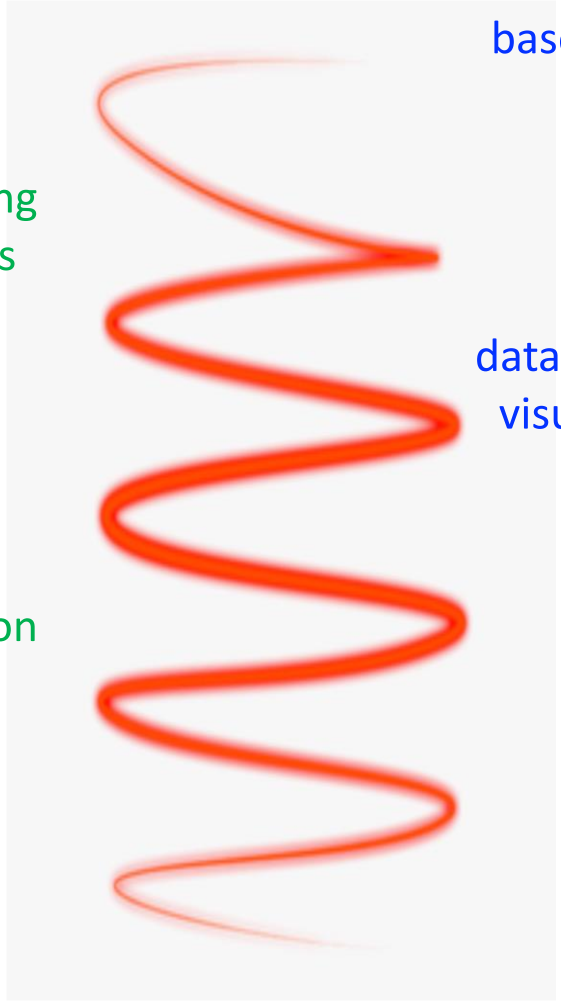
Where we are: completed

1	Aug 31	Course overview, introduction to R, descriptive statistics
2	Sep 5-7	Review of central statistical concepts and exploratory analysis using R
3	Sep 12-14	Confidence Intervals and the bootstrap
4	Sep 19-21	Review of hypothesis tests and permutation tests in R
5	Sep 26-28	Parametric, non-parametric and theories of hypothesis testing
6	Oct 3-5	Data manipulation and visualization
7	Oct 10-12	Review and midterm exam
8	Oct 17-19	Odds and ends, October break



Where we are: up next

9	Oct 24-26	Simple linear regression
10	Oct 31-Nov 2	Multiple regression
11	Nov 7-9	Model selection and logistic regression



Next: building linear models

We will use these models to:

1. Make accurate predictions
2. Understand the relationship between explanatory variables x_i's and a response variable y

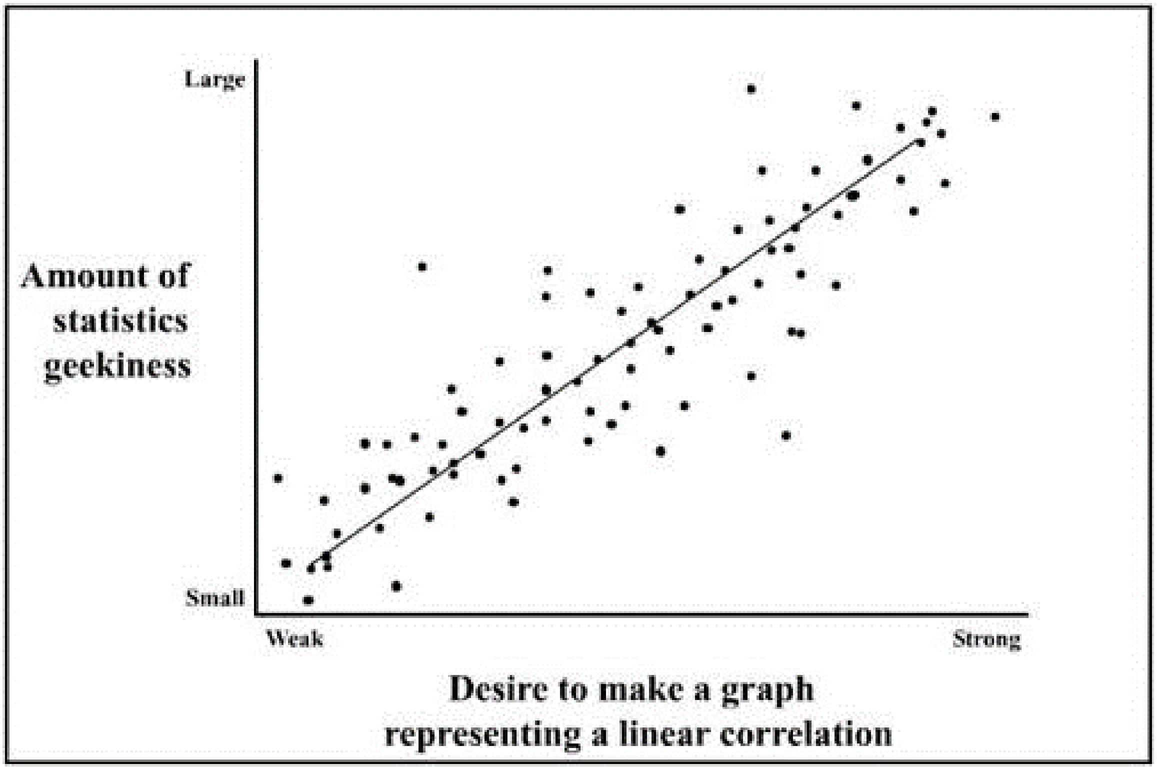
Linear regression

Regression is method of using one variable ***x*** to predict the value of a second variable ***y***

$$\hat{y} = f(x)$$

In **linear regression** we fit a line to the data, called the **regression line**

- In *simple* linear regression, we use a single variable x, to predict y



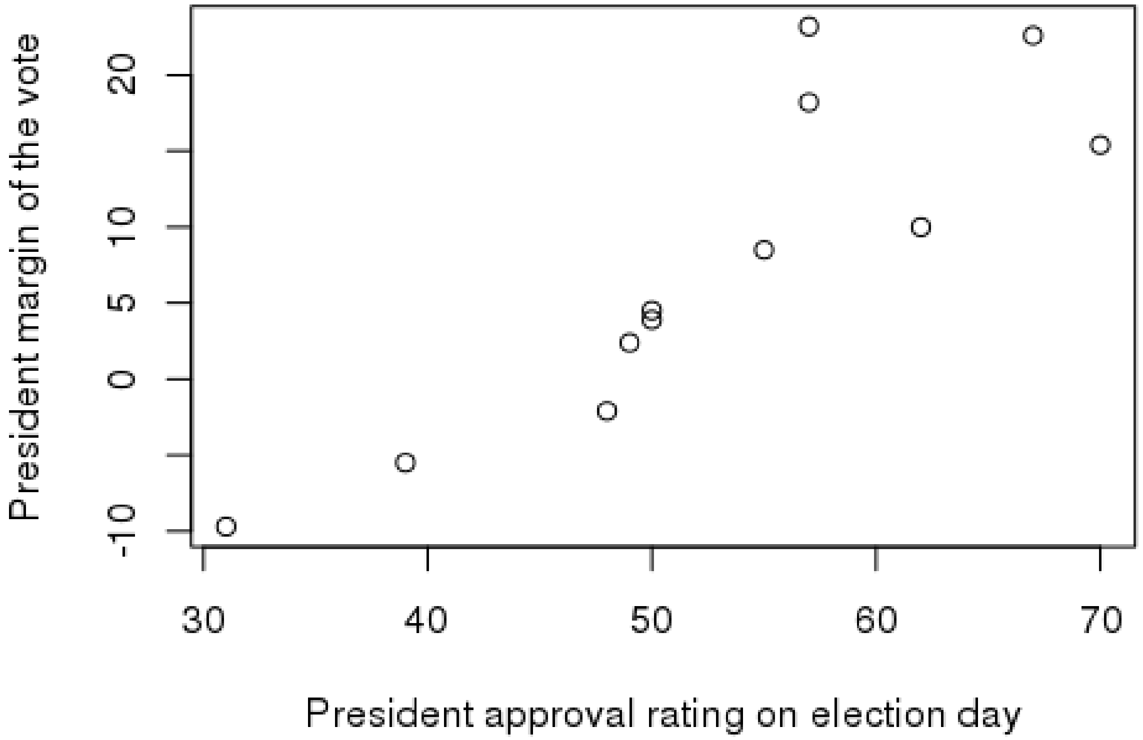
Motivation: Predicting the 2020 election

Predict the margin of the popular vote based on the president’s approval rating

Data from an article on the 2012 election on the [Five Thirty Eight website]

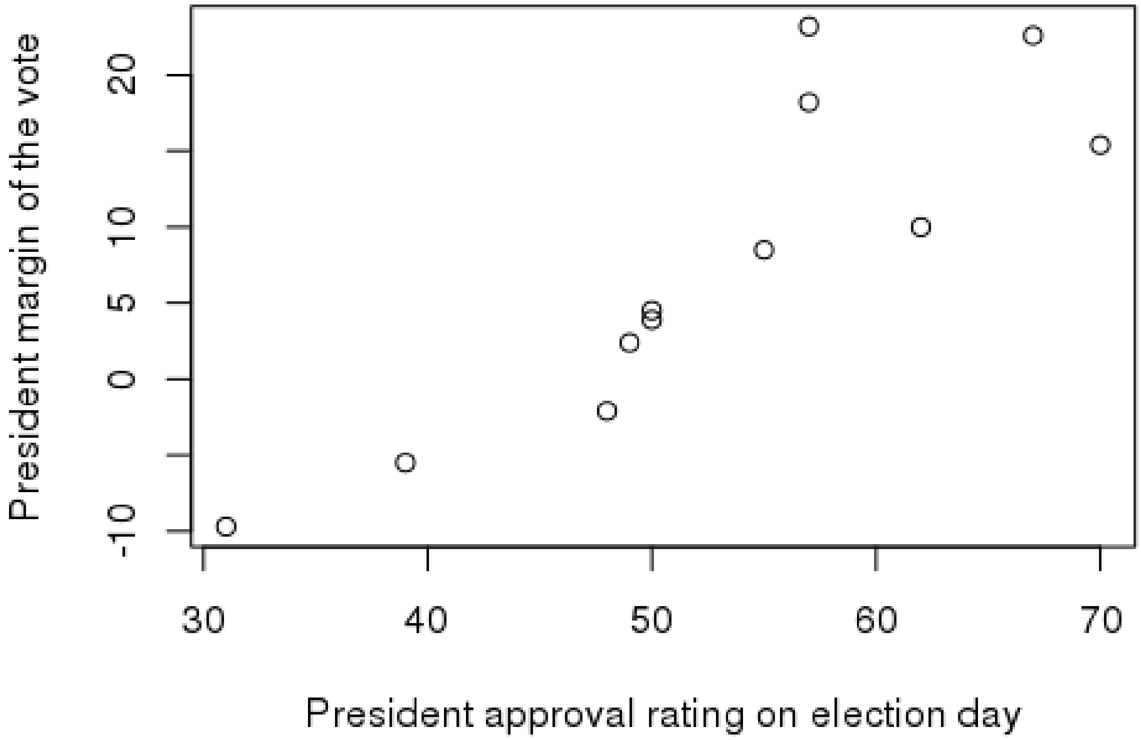
Approval rating vote margin regression line

From previous 12 US president's running for reelection



Approval rating vote margin regression line

From previous 12 US president's running for reelection



$\hat{y} = b_0 + b_1 \cdot x$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

R: `lm(y ~ x)`

$\hat{\beta}_0$ = -36.76

$\hat{\beta}_1$ = 0.84

$\hat{y}$ = -36.76 + 0.84 · x

Approval rating vote margin regression line

1. If a president had a 0% approval rating, what vote margin does this model predict the president would get?
   A: would have a margin of -36.76% of the vote

2. If a president's approval rating increased by 1%, how much would the president's margin of the vote be predicted to increase by?
   A: .84 increase in the margin of the vote

3. At what presidential approval level would there be an exactly even split of the vote?
   A: Margin of $\hat{y} = 0$, solving for x we get
   36.76/.84 = 43.76% approval rating

$\hat{y} = b_0 + b_1 \cdot x$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

R: `lm(y ~ x)`

$\hat{\beta}_0$ = -36.76

$\hat{\beta}_1$ = 0.84

$\hat{y} = -36.76 + 0.84 \cdot x$

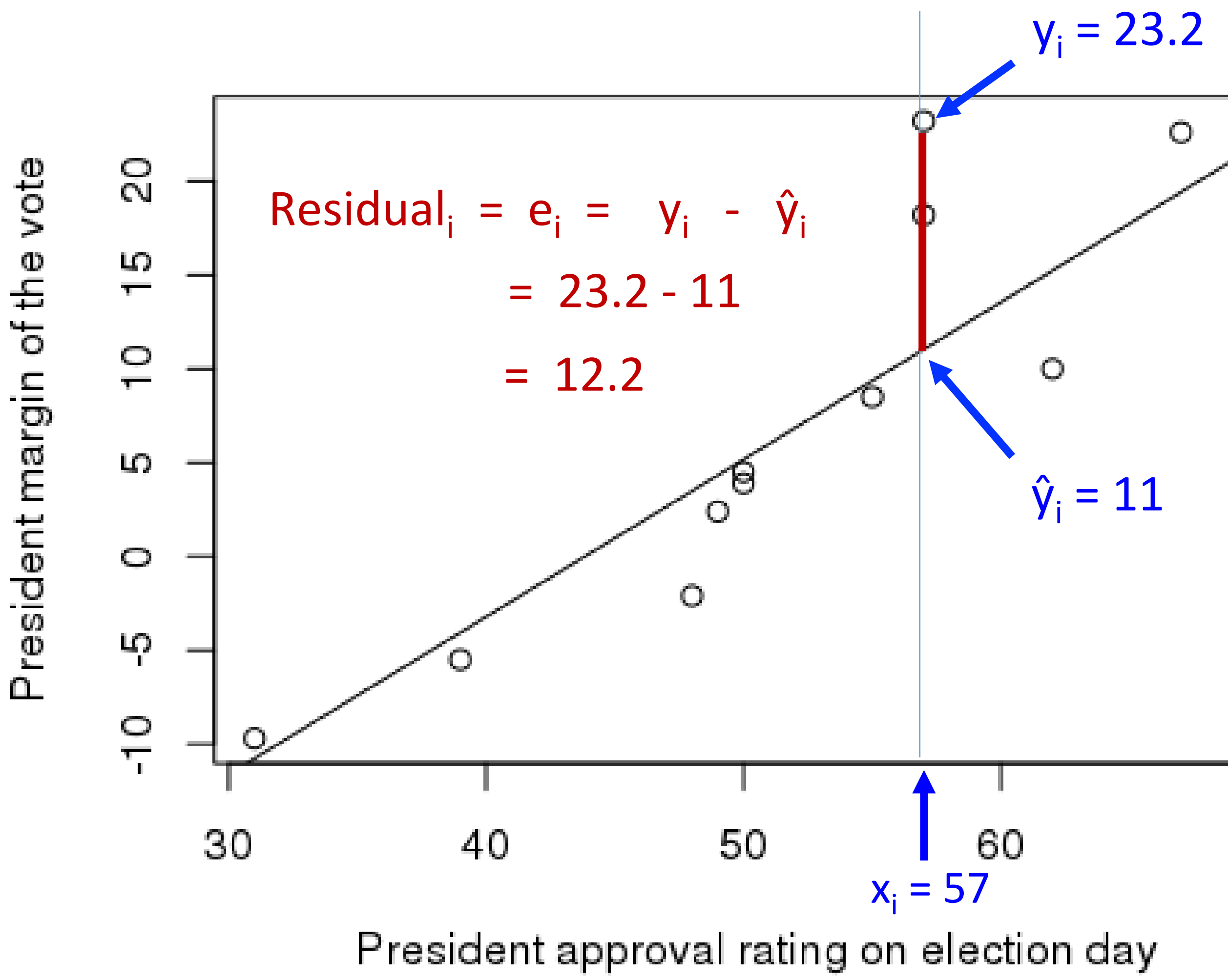
Residuals

The **residual** at a data value is the difference between the observed (y) and predicted value of the response variable

$$Residual_i \quad = \quad Observed_i \quad - \quad Predicted_i$$

$$e_i \quad = \quad y_i \quad - \quad \hat{y}_i$$

Approval rating vote margin regression line



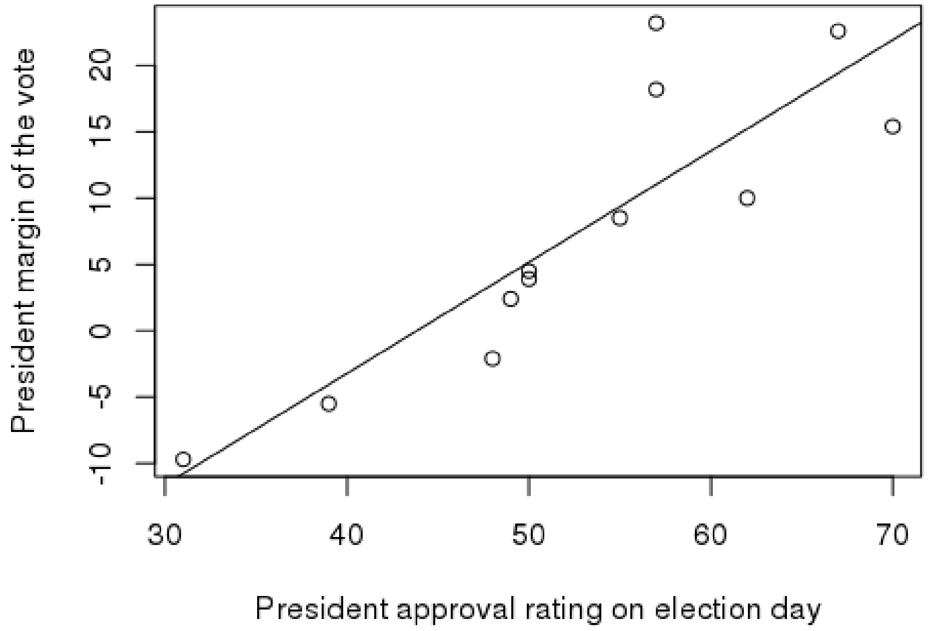
Approval rating vote margin regression line

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

Approval ***x***	**Margin obs** ***y***	**Margin pred** $\hat{y}$	**Residuals** $e = y - \hat{y}$
62	10	15.23	-5.23
50	4.5	5.17	-0.67
70	15.4	21.94	-6.54
67	22.6	19.43	3.17
57	23.2	11.04	12.16
48	-2.1	3.49	-5.59
31	-9.7	-10.76	1.06
57	18.2	11.04	7.16

Line of 'best fit'

The **least squares line**, also called **'the line of best fit'**, is the line which <u>minimizes the sum of squared residuals</u>



[Try to find the line of best fit]()

Approval rating vote margin regression line

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$

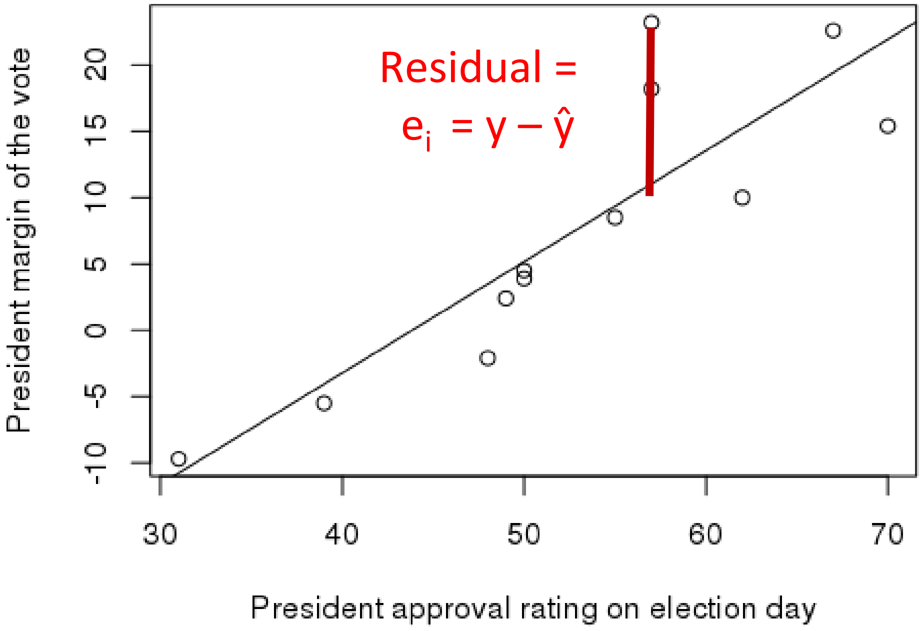
Approval x	**Margin obs** y	**Margin pred** $\hat{y}$	**Residuals** $e = y - \hat{y}$	**Residuals²** $e^2 = (y - \hat{y})^2$
62	10	15.23	-5.23	27.40
50	4.5	5.17	-0.67	0.45
70	15.4	21.94	-6.54	42.81
67	22.6	19.43	3.17	10.07
57	23.2	11.04	12.16	147.84
48	-2.1	3.49	-5.59	31.29
31	-9.7	-10.76	1.06	1.13
57	18.2	11.04	7.16	51.25

Q: Why do we minimize the sum of ***squared*** residuals rather than just the sum of residuals?

Minimizing the sum of the squared residuals to find the regression coefficients

To find the regression coefficients $\hat{\beta}_0$ and $\hat{\beta}_1$ we minimize the **sum of squared residuals**

- We will use the notation **SSRes** to denote the some of squared residuals
  - (The residual sum of squares is also called the **"error sum of squares" (SSE)**)



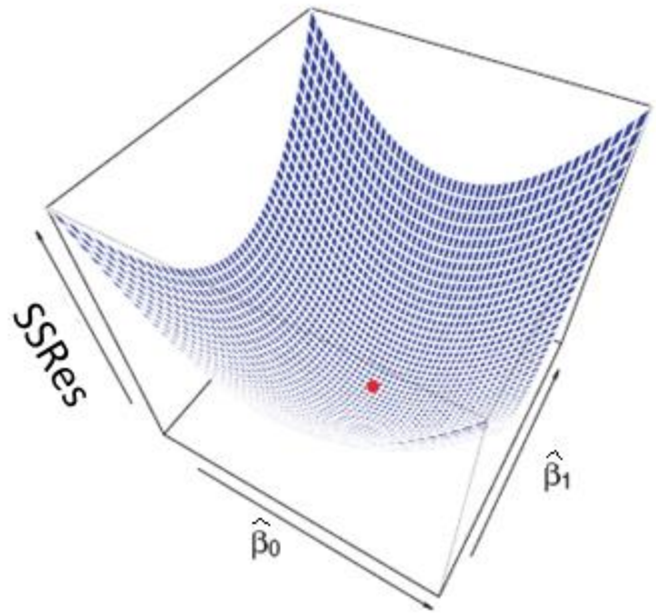
$$residual = e_i = y_i - \hat{y_i}$$

$$SSRes = \sum_{i=1}^{n} {e_i}^2 = \sum_{i=1}^{n} (y_i - \hat{y_i})^2$$

$$= \sum_{i=1}^{n} (y_i - \hat{f}(x_i))^2 = \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i))^2$$

R: `lm(y ~ x)`

How do we minimize the SSE?



$$SSRes \quad = \quad \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 x))^2$$

How do we find $\hat{\beta}_0, \hat{\beta}_1$?

Calculus and linear algebra:

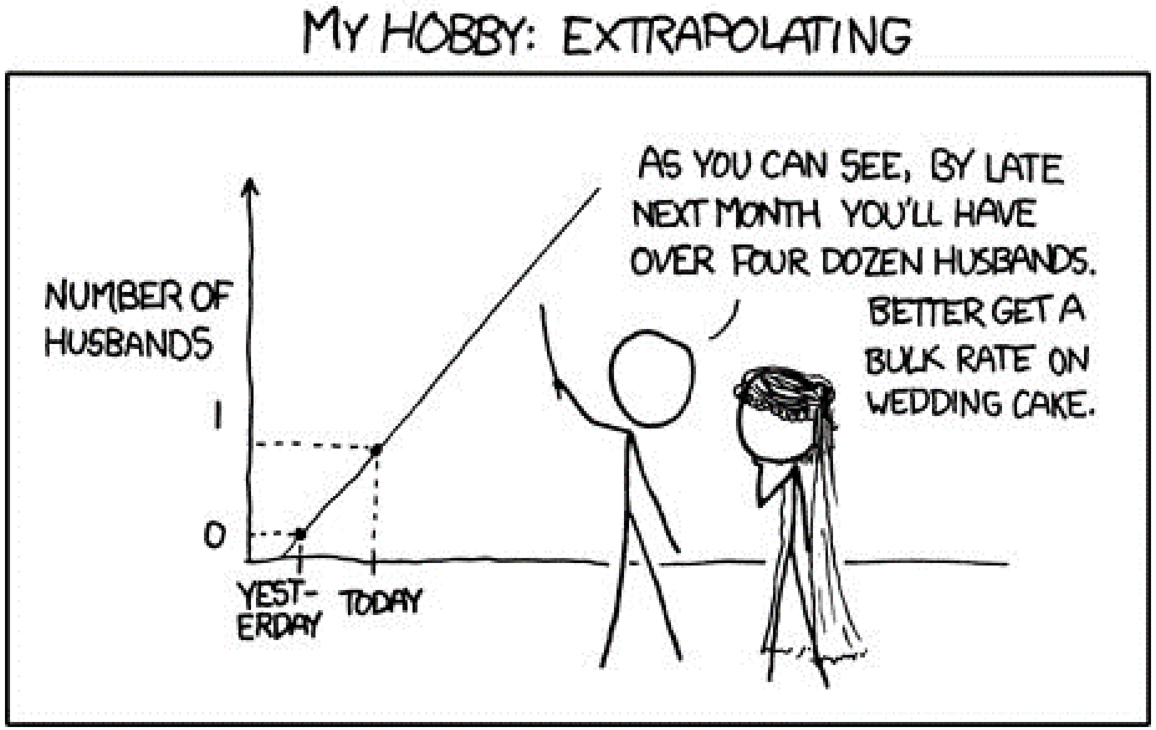
- Take the derivative, set to 0 and solve
- This mathematical convenience is why the squared loss is so commonly used

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} \qquad \hat{\beta}_1 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2}$$

Basic regression caution # 1

Avoid trying to apply the regression line to predict values far from those that were used to create the line.

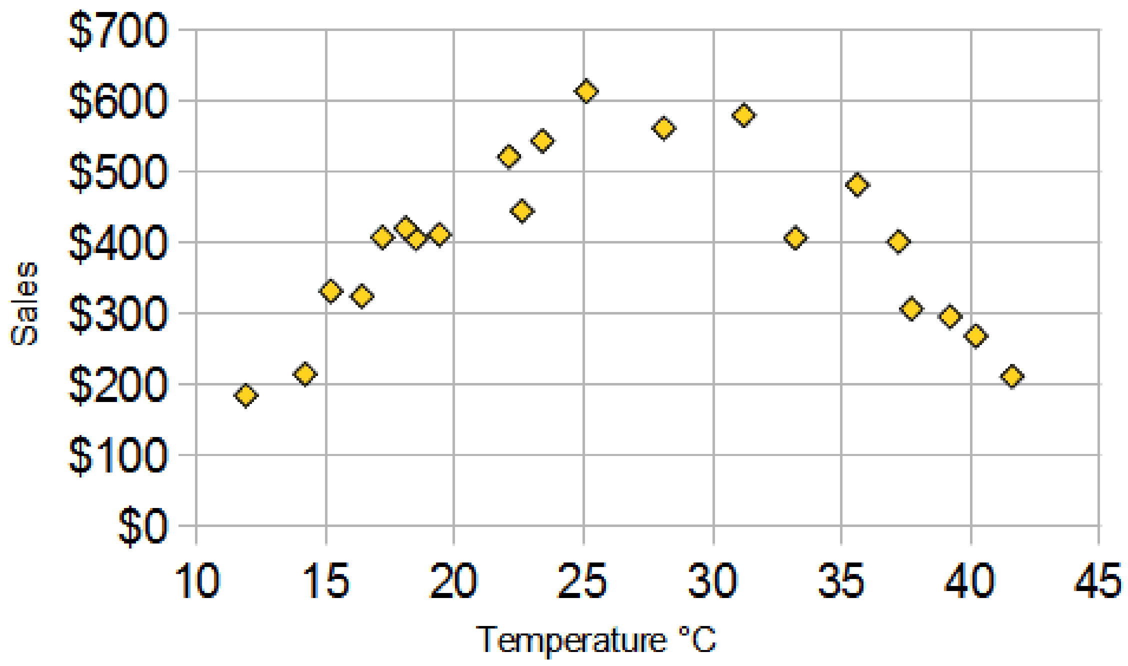
- i.e., do not extrapolate too far



Basic regression caution # 2

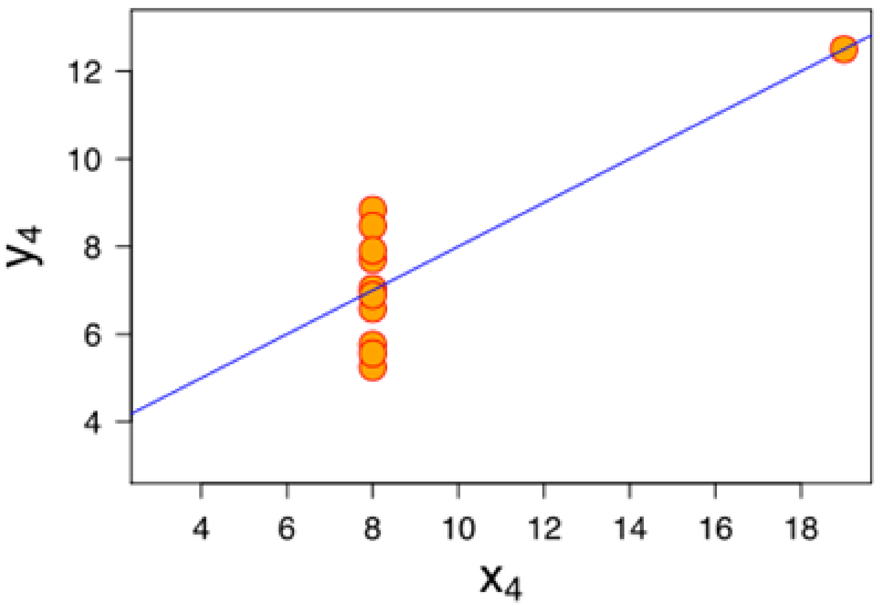
Plot the data! Linear regression is only appropriate when there is a linear trend in the data.

- We will discuss a set of checks on the appropriateness of using linear models soon



Basic regression caution #3

Be aware of outliers and high leverage points. They can have a large effect on the regression line.



Outlier: big $| y_i - \bar{y} |$

Leverage: big $| x_i - \bar{x} |$

Influential point: big outlier and leverage

There are statistics that quantify/describe influential points

- We will discuss these soon as well

Let’s try simple linear regression in R…

Faculty salaries

- Predict faculty salaries based on the size of a university's endowment

Inference for simple linear regression

Warning: there is a minefield of poor/misleading terminology out there

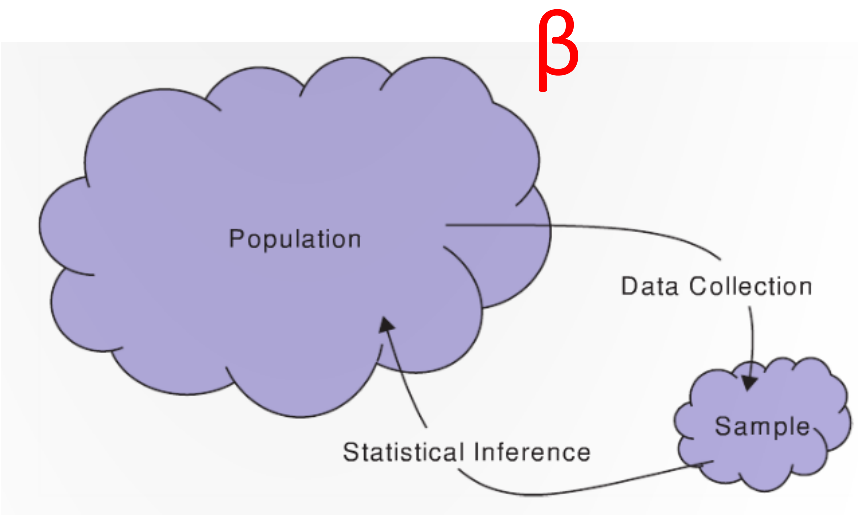
- So be careful when reading material related to inference on regression models

I will try to help you navigate this…

Inference for simple linear regression

The letter **b** or $\hat{\beta}$ is typically used to denote the slope ***of the sample***

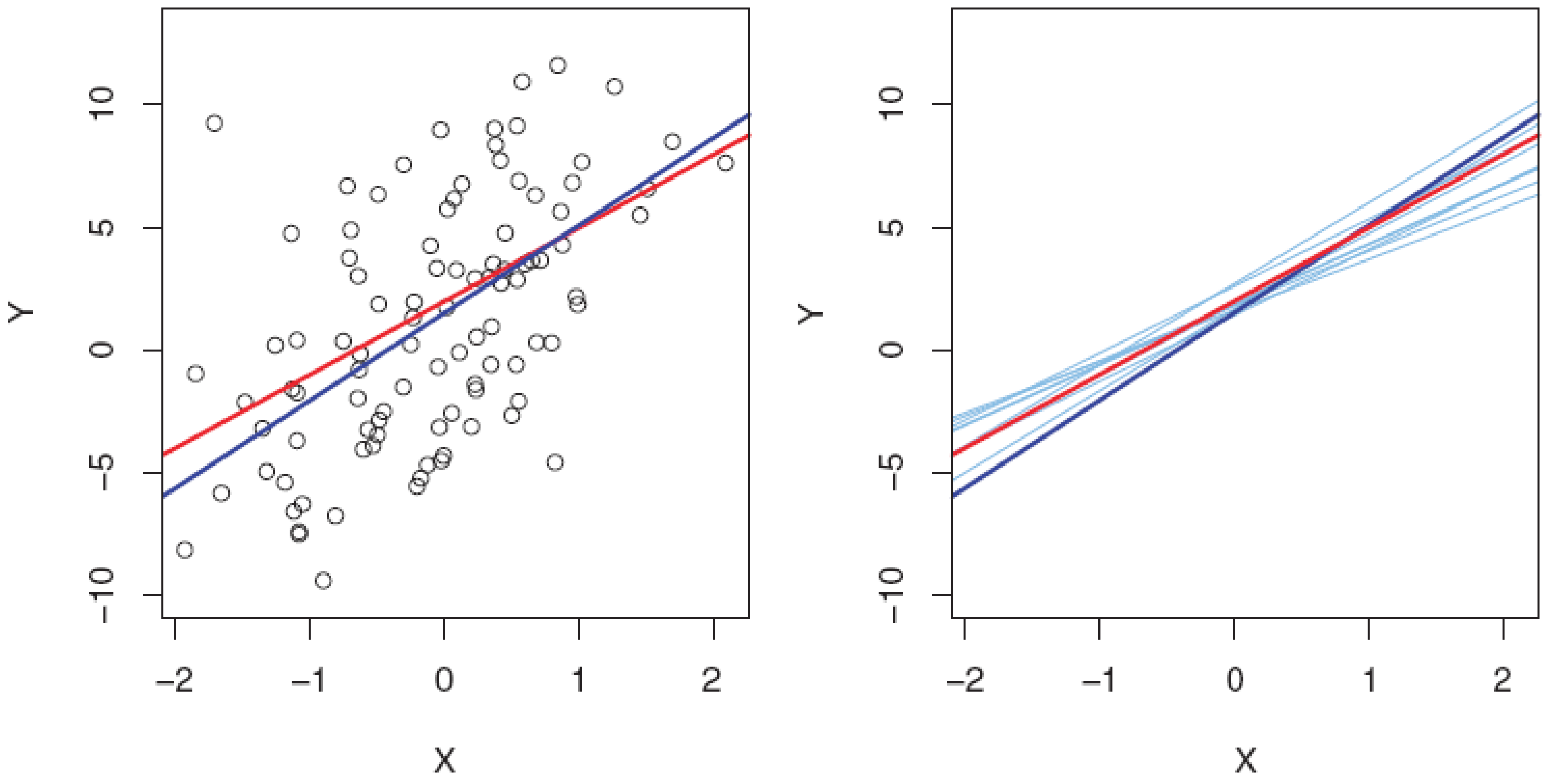
The Greek letter **β** is used to denote the slope ***of the population***



b $\hat{\beta}$

Population: β

Sample estimates: b $\hat{\beta}$



Linear regression underlying model

Intercept Slope } *Parameters*

True regression line: $$\mu_Y(x) = \beta_0 + \beta_1 x$$

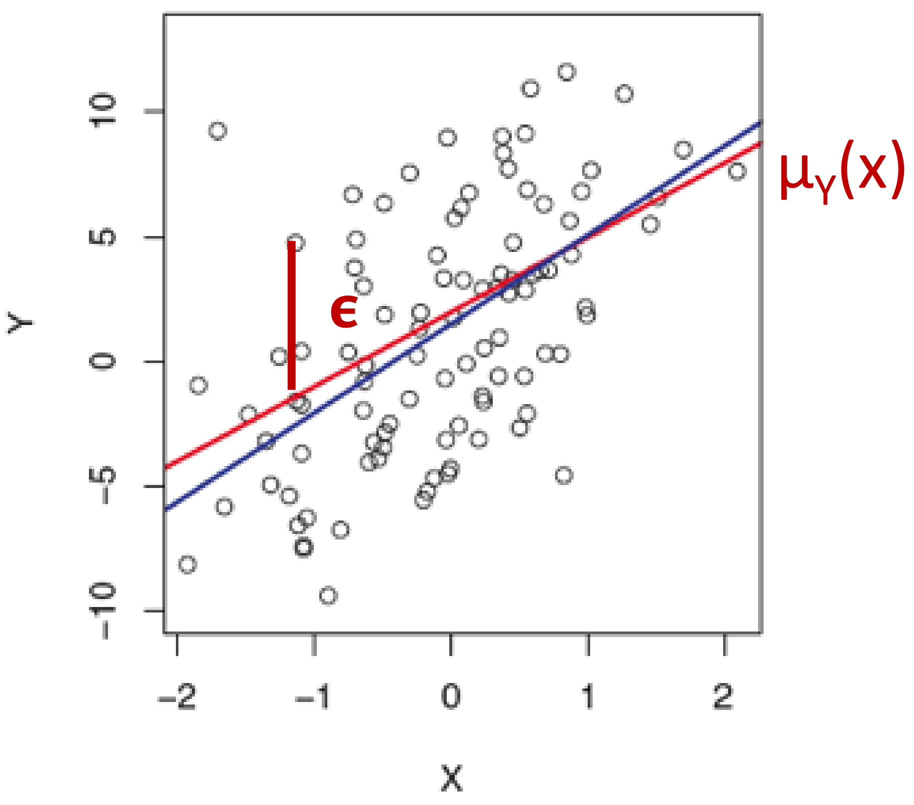
Error

Observed data point: $$Y = \beta_0 + \beta_1 x + \epsilon$$

$$= \mu_Y(x) + \epsilon$$

Errors ϵ_i are the difference between the **true regression line** $\mu_Y(x_i)$ and observed data points Y_i

- $\epsilon_i = Y_i - \mu_Y(x_i)$



Linear regression underlying model

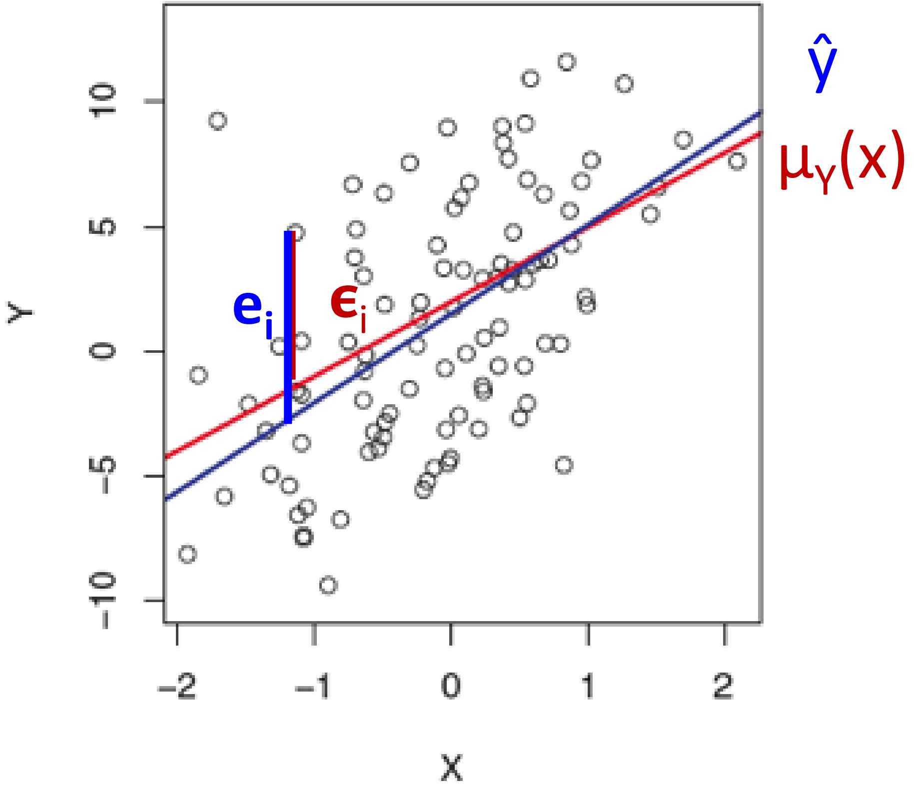
Intercept Slope } *Parameters*

True regression line: $$\mu_Y(x) = \beta_0 + \beta_1 x$$

Error

Observed data point: $$Y = \beta_0 + \beta_1 x + \epsilon$$

Estimated regression line: $$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 x$$



Errors ϵ_i are the difference between the **true regression line** $\mu_Y(x_i)$ and observed data points Y_i

- $\epsilon_i = Y_i - \mu_Y(x_i)$

Residuals e_i are the difference between the **estimated regression line** $\hat{y}_i$ and observed data points Y_i

- $e_i = Y_i - \hat{y}_i$

Linear regression underlying model

Intercept   Slope   }   *Parameters*

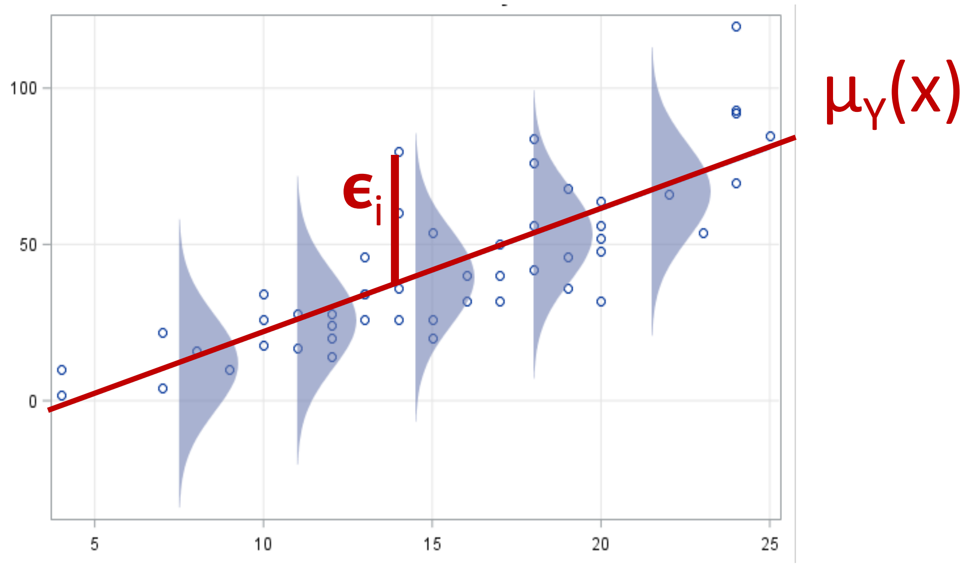
True regression line: $\mu_Y(x) = \beta_0 + \beta_1 x$

Error

Observed data point: $Y = \beta_0 + \beta_1 x + \epsilon$ $\epsilon \sim N(0, \sigma_\epsilon)$

Errors ϵ_i are the difference between the **true regression line** $\mu_Y(x_i)$ and observed data points Y_i

- $\epsilon_i = Y_i - \mu_Y(x_i)$



We will *assume* that the errors ϵ_i are **normally distributed**

- This is needed for inference using parametric methods
    - e.g., to use t-distributions and F-distributions

Recap: Errors vs. residuals

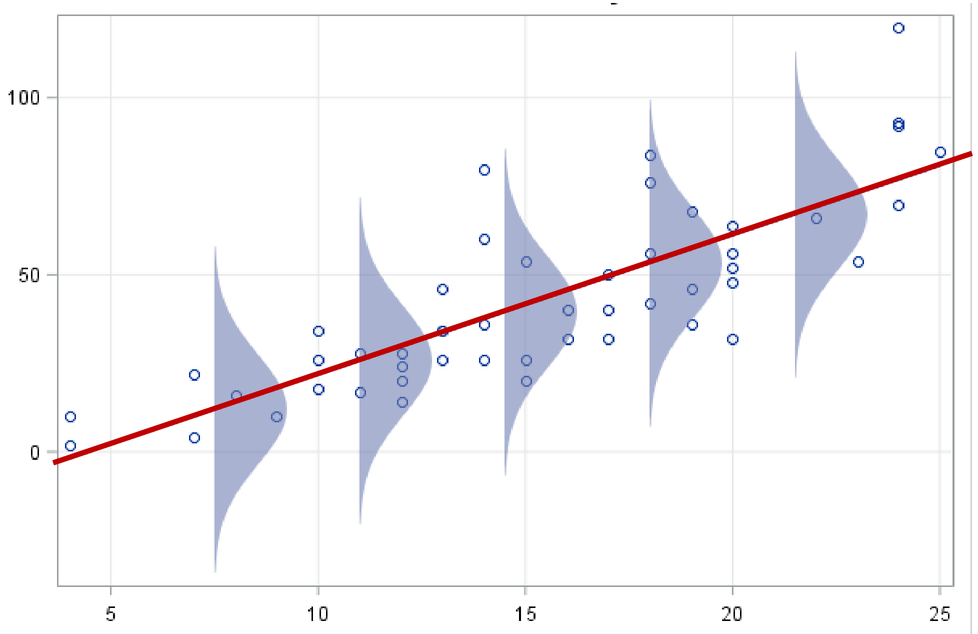
The data: $$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i \qquad \epsilon_i \sim N(0, \sigma_\epsilon)$$

"True" model: $\mu_Y(x_i) = \beta_0 + \beta_1 x_i$

- Errors: $\epsilon_i = Y_i - \mu_Y(x_i)$

Estimated model: $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$

- Residuals: $e_i = Y_i - \hat{y}_i$

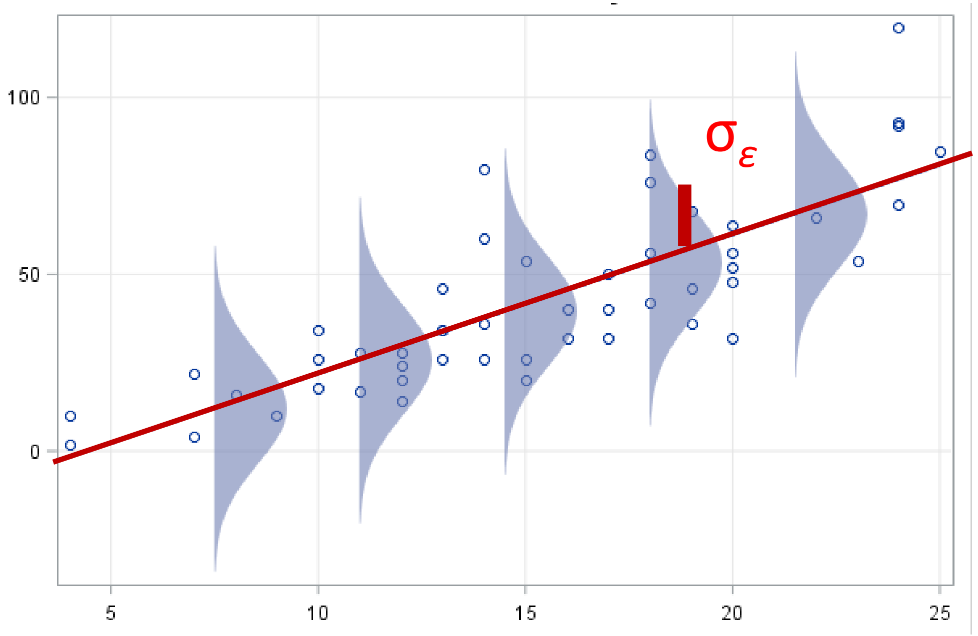


Standard deviation of the errors: σ_ε

The standard deviation of the errors is denoted $\boldsymbol{\sigma_\varepsilon}$

We can use the **standard deviation of residuals** $\hat{\sigma}_\varepsilon$ as an estimate standard deviation of the errors σ_ε

- $\hat{\sigma}_\varepsilon$ often called the ~~"residual standard error"~~
- $\hat{\sigma}_\varepsilon$ we will call it the "residual standard deviation"



$$\hat{\sigma}_\epsilon = \sqrt{\frac{1}{n-2} SSRes}$$

$$= \sqrt{\frac{1}{n-2} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2}$$

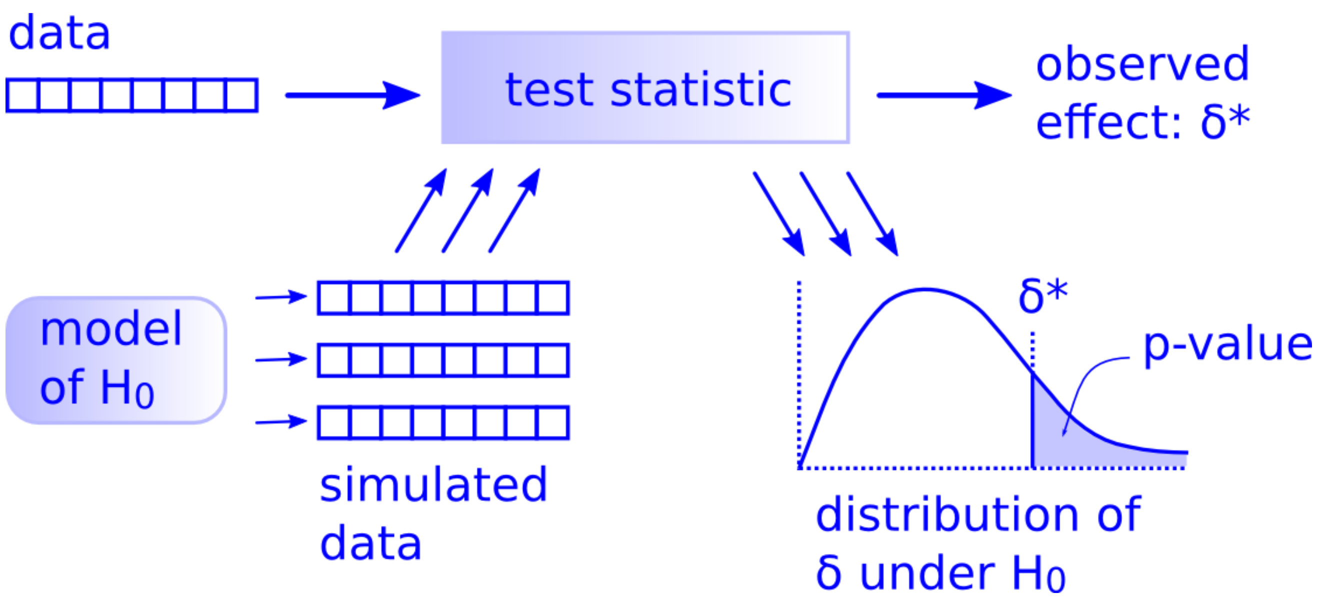
How are we feeling?

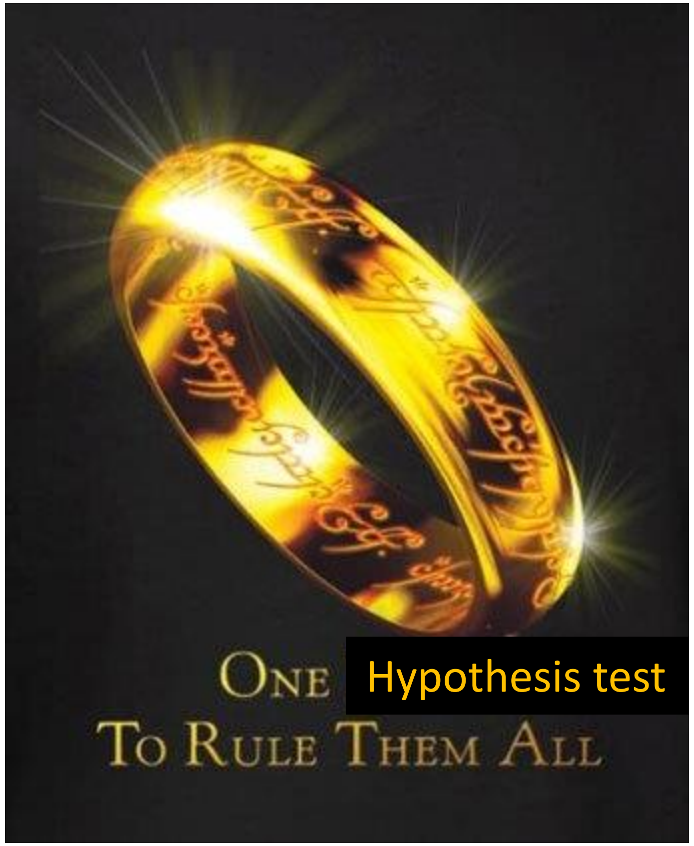
Let's quickly try this in R...

Inference for linear regression: hypothesis tests

Hypothesis test for regression coefficients

There is only one hypothesis test!





Hypothesis test for regression coefficients

We can run hypothesis tests to assess whether there is a relationship between y and x, and calculate p-values

- H_0: $\beta_1 = 0$ (slope is 0, so no relationship between x and y
- H_A: $\beta_1 \neq 0$

One type of hypothesis test we can run is based on a t-statistic: $t = \frac{\hat{\beta}_1 - 0}{\hat{SE}_{\hat{\beta}_1}}$

- The t-statistic comes from a t-distribution with n - 2 degrees of freedom

$$\hat{SE}_{\hat{\beta}_1} = \frac{\hat{\sigma}_\epsilon}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}} \qquad \hat{SE}_{\hat{\beta}_0} = \hat{\sigma}_\epsilon \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

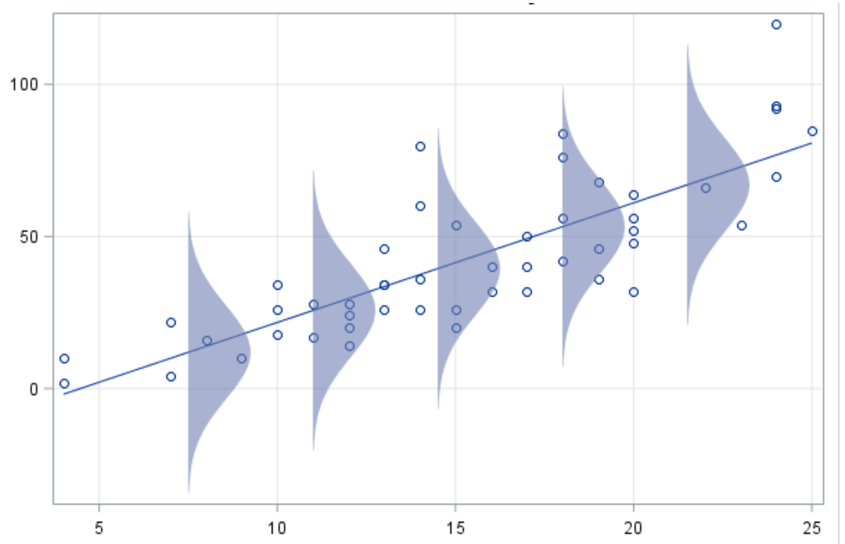
Inference using parametric methods

When using parametric methods, we make the following (LINE) assumptions:

- **Linearity**: A line can describe the relationship between x and y
- **Independence**: each data point is independent from the other points
- **Normality**: errors are normally distributed
- **Equal variance (homoscedasticity)**: constant variance of errors over the whole range of x values

$$Y_i = \beta_0 + \beta_1 x_i + \epsilon_i$$

$$\epsilon_i \sim N(0, \sigma_\epsilon)$$



These assumptions are usually checked after the models are fit using 'regression diagnostic' plots.

Let’s look at inference for simple linear regression in R

Back to faculty salaries…

Inference for linear regression: confidence intervals

We can estimate three types of intervals for a regression:

1. Confidence intervals for the regression coefficients: β_0 and β_1

2. Confidence intervals for the full line $\mu_Y(x)$

3. Prediction intervals where most of the data is expected

Confidence intervals for regression coefficients

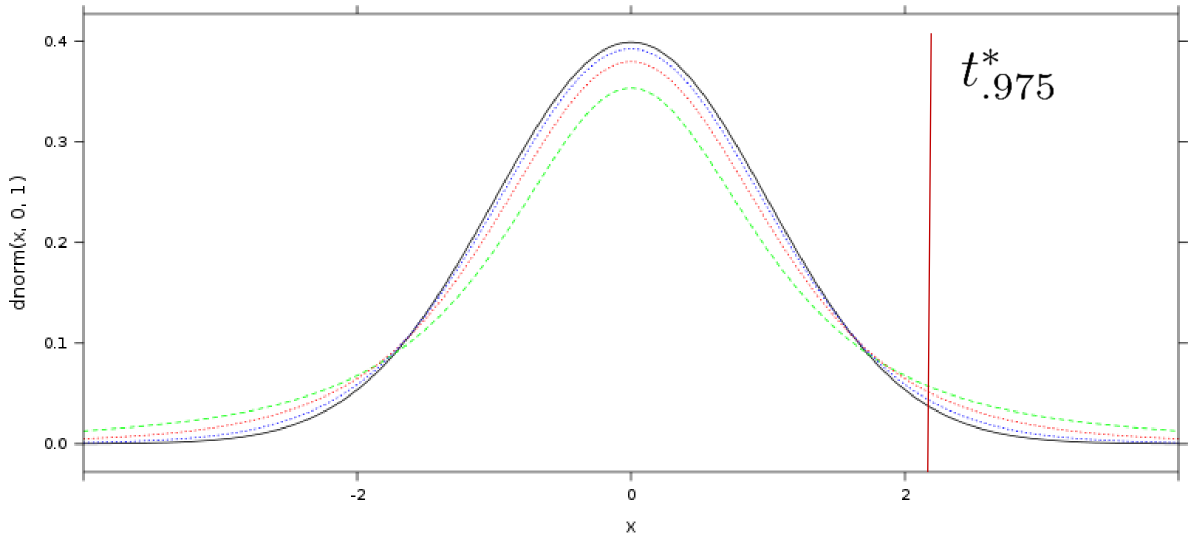
For the slope coefficient , the confidence interval is: $\hat{\beta}_1 \pm t^* \cdot SE_{\hat{\beta}_1}$

Where: $SE_{\hat{\beta}_1} = \frac{\sigma_\epsilon}{\sqrt{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}}$

t* is a quantile value from a t-distribution with n-2 degrees of freedom obtain to get a desired confidence level

qt(.975, df)

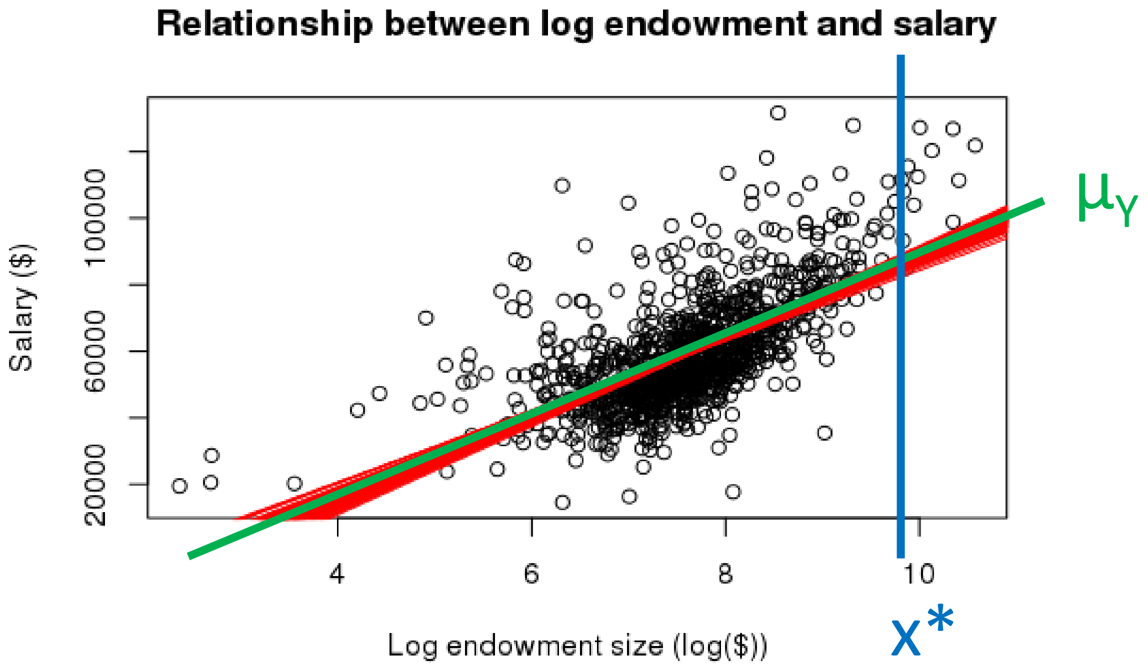
N(0, 1)
df = 2
df = 5
df = 15



Confidence intervals for the regression line μ_Y

A confidence interval for the mean response for the ***true regression line*** μ_Y when X = x* is:

$$\hat{y} \ \pm \ t^* \cdot SE_{\hat{\mu}} \quad \text{where} \quad SE_{\hat{\mu}} = \sigma_\epsilon \sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^n (x_i - \bar{x})^2}}$$



Note:

- There is more uncertainty at the ends of the regression line
- The confidence interval for the regression line μ_Y is different than the confidence interval for slope β_1

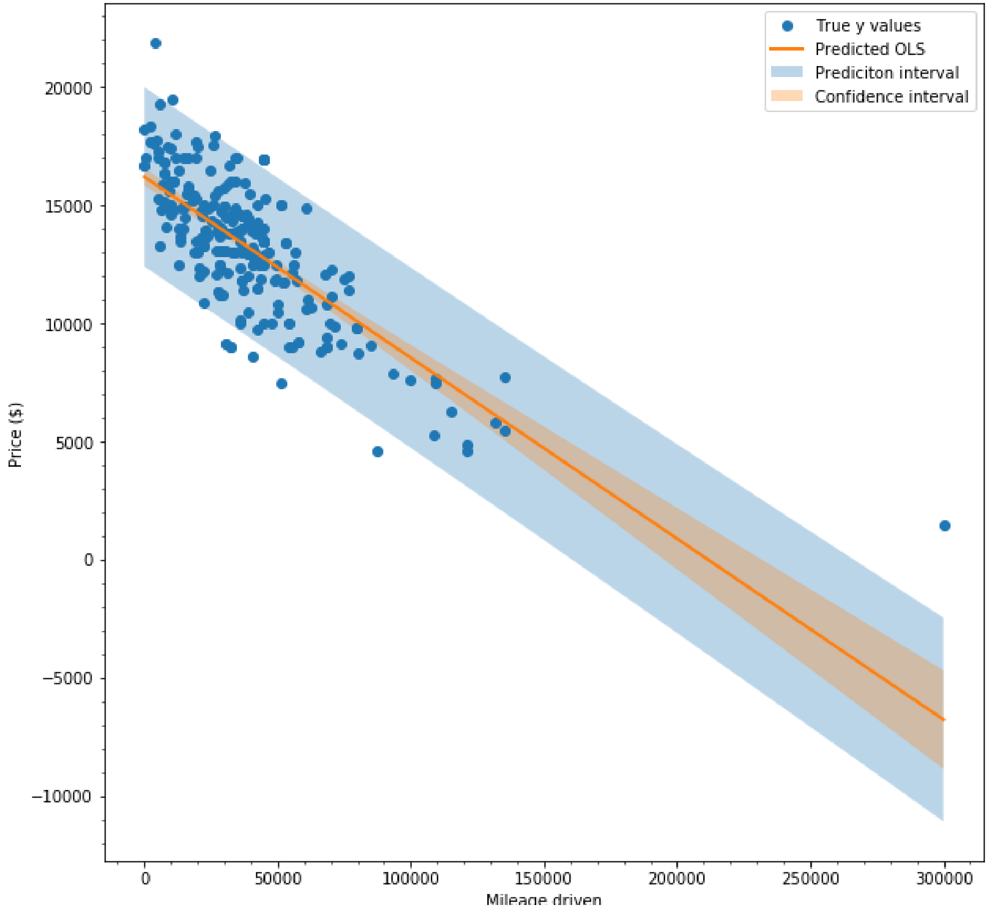
Prediction intervals

Confidence intervals give us a measure of uncertain about the true relationship between x and y for:

- The true regression slope β_1
- The true regression line μ_Y

Prediction intervals give us a range of plausible values for y

- i.e., 95% of our y's with be within this range



Prediction intervals

A **prediction intervals** for the y can be calculated using:

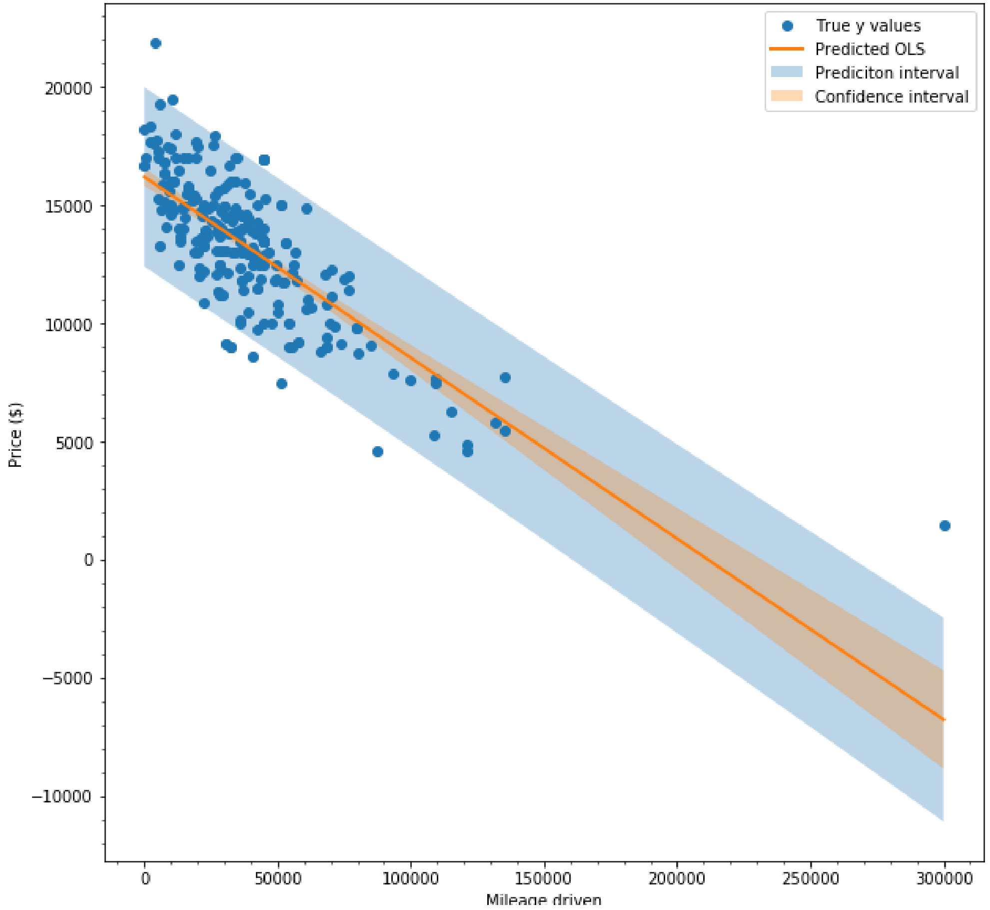
$$\hat{y} \ \pm \ t^* \cdot SE_{\hat{y}}$$

where

$$SE_{\hat{y}} = \sigma_\epsilon \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}}$$

Due to y's scattering around the true regression line

Due to uncertainty in where the true regression line is



Summary of confidence and prediction intervals

1. CI for Slope β $\hat{\beta}_1 \pm t^* \cdot SE_{\hat{\beta}_1}$ $SE_{\hat{\beta}_1} = \sigma_\epsilon \sqrt{\frac{1}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}}$

2. CI for regression line μ_Y at point x*

$$\hat{y} \ \pm \ t^* \cdot SE_{\hat{\mu}} \qquad SE_{\hat{\mu}} = \sigma_\epsilon \sqrt{\frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}}$$

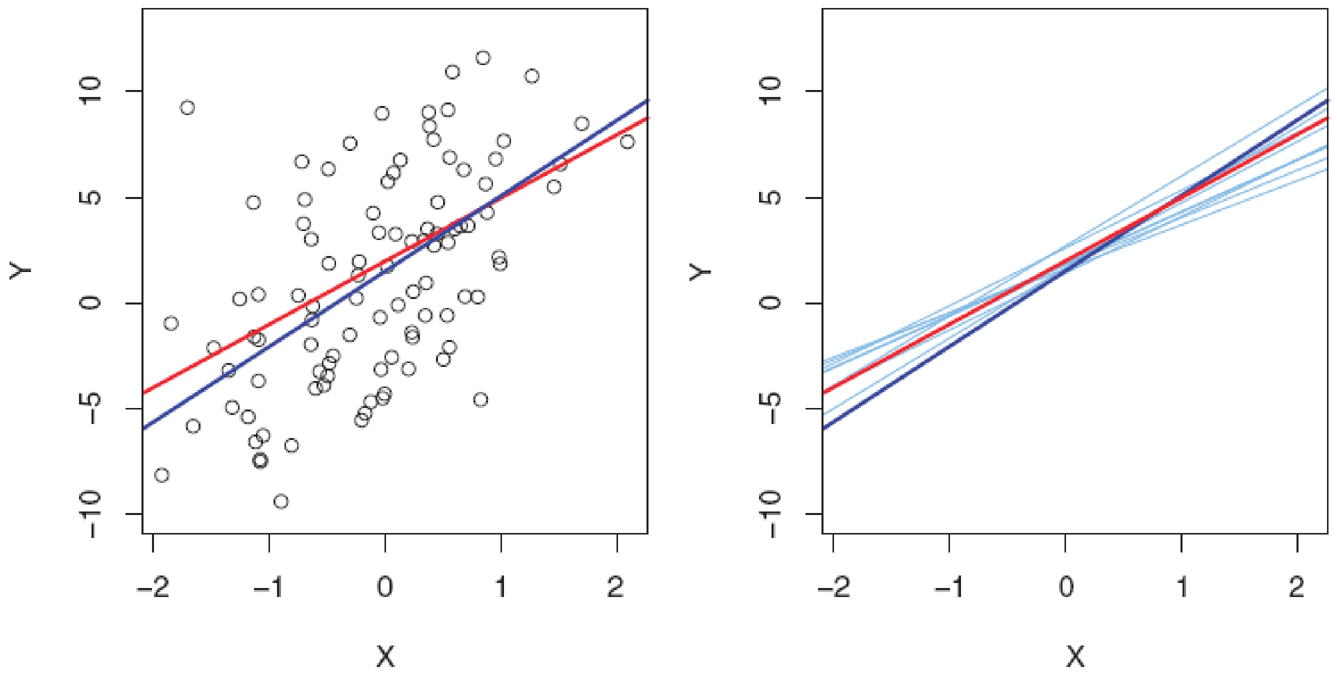
3. Prediction interval y

$$\hat{y} \ \pm \ t^* \cdot SE_{\hat{y}} \qquad SE_{\hat{y}} = \sigma_\epsilon \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\Sigma_{i=1}^{n}(x_i - \bar{x})^2}}$$

Resampling methods for inference in regression

We can also use resampling methods to estimate run hypothesis tests and create confidence intervals for the regression coefficients

- Bootstrap
- Permutation test



Let's look at inference for simple linear regression in R

More faculty salary data…