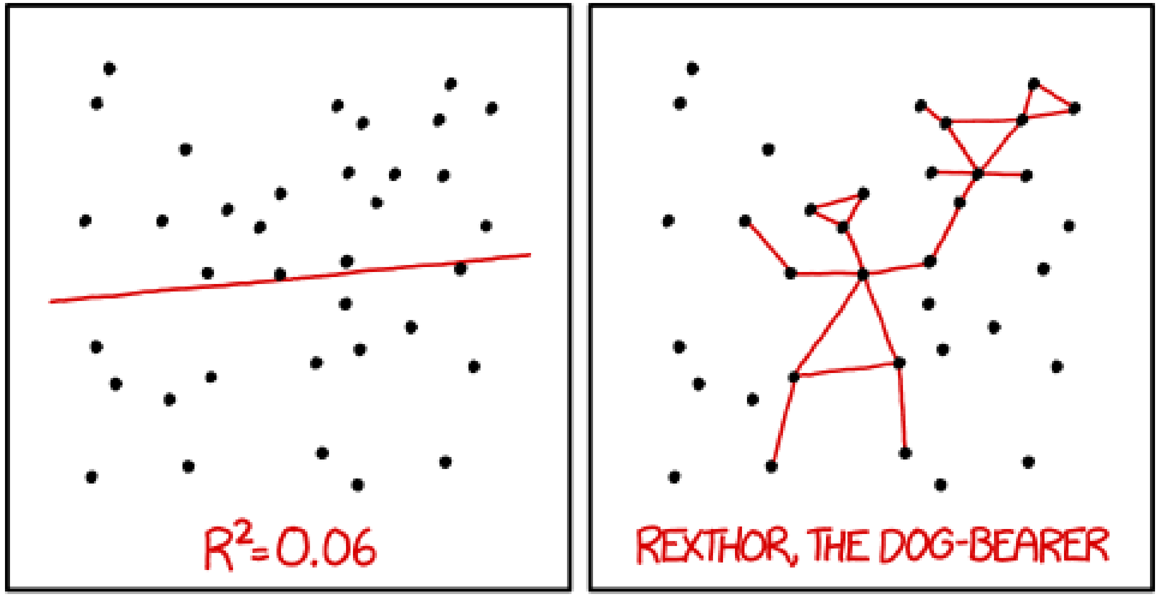
Multiple regression continued



I DON'T TRUST LINEAR REGRESSIONS WHEN IT'S HARDER TO GUESS THE DIRECTION OF THE CORRELATION FROM THE SCATTER PLOT THAN TO FIND NEW CONSTELLATIONS ON IT.

Overview

Quick review of what we have covered in multiple regression

Log transformations of the response variable y

Multicollinearity

Polynomial regression

Quick review

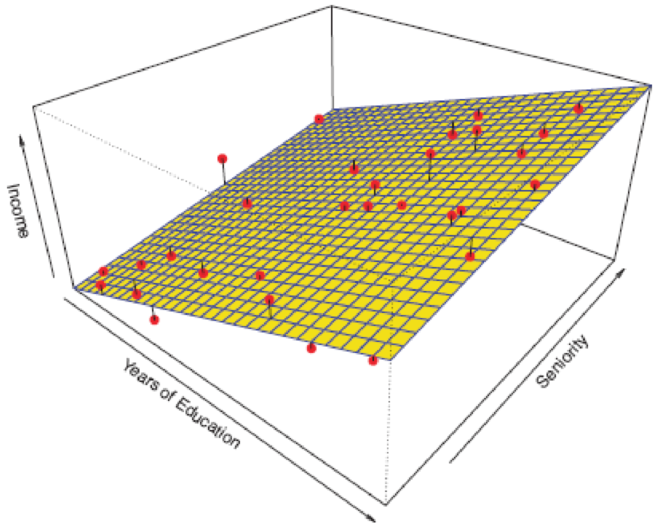
Multiple regression

In multiple regression we try to predict a quantitative response variable *y* using several predictor variables $x_1, x_2, \ldots, x_k$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

Goals:

- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)

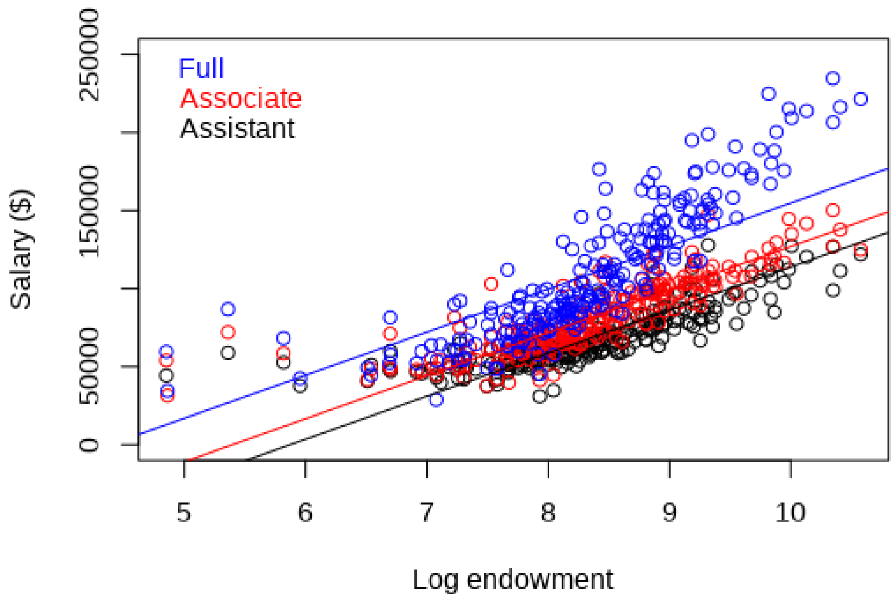


Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

Suppose we want to predict faculty salary *y* as a function of endowment x_1, with separate intercepts for faculty rank



$$x_{i1} = \log(\text{endowment})$$

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1}$$

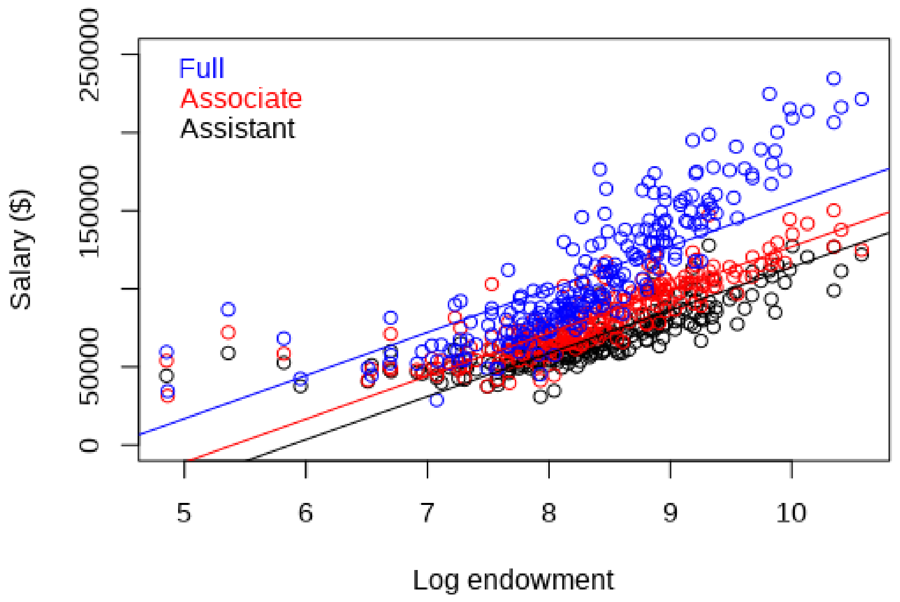
$$= \begin{cases} & \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} & \text{if full professor} \end{cases}$$

$$x_{i3} = \begin{cases} 1 & \text{if associate professor} \\ 0 & \text{otherwise} \end{cases}$$

Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it



Suppose we want to predict faculty salary as a function of endowment with separate intercepts for faculty rank

```
> summary(fit_prof_rank_offset)

Call:
lm(formula = salary_tot ~ log_endowment + rank_name, data = IPED_2)

Residuals:
   Min     1Q Median     3Q    Max
-52464 -10844  -2703   6936  74994

Coefficients:
                      Estimate Std. Error t value            Pr(>|t|)
(Intercept)          -120822.1     6713.9  -18.00 <0.0000000000000002 ***
log_endowment          27569.9      791.7   34.82 <0.0000000000000002 ***
rank_nameAssociate    -27855.4     1685.5  -16.53 <0.0000000000000002 ***
rank_nameAssistant    -40973.7     1685.5  -24.31 <0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 18370 on 707 degrees of freedom
Multiple R-squared:  0.7192,	Adjusted R-squared:  0.718
F-statistic: 603.7 on 3 and 707 DF,  p-value: < 0.00000000000000022
```

$$\hat{y}_i = \begin{cases} \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} + \boxed{\hat{\beta}_2} & \text{if assistant professor} \\ \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} + \boxed{\hat{\beta}_3} & \text{if associate professor} \\ \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} & \text{if full professor} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i3}$$

$$= -120,822 + 27,570 x_{i1} - 40,973 x_{i2} - 27,855 x_{i3}$$

Interaction terms

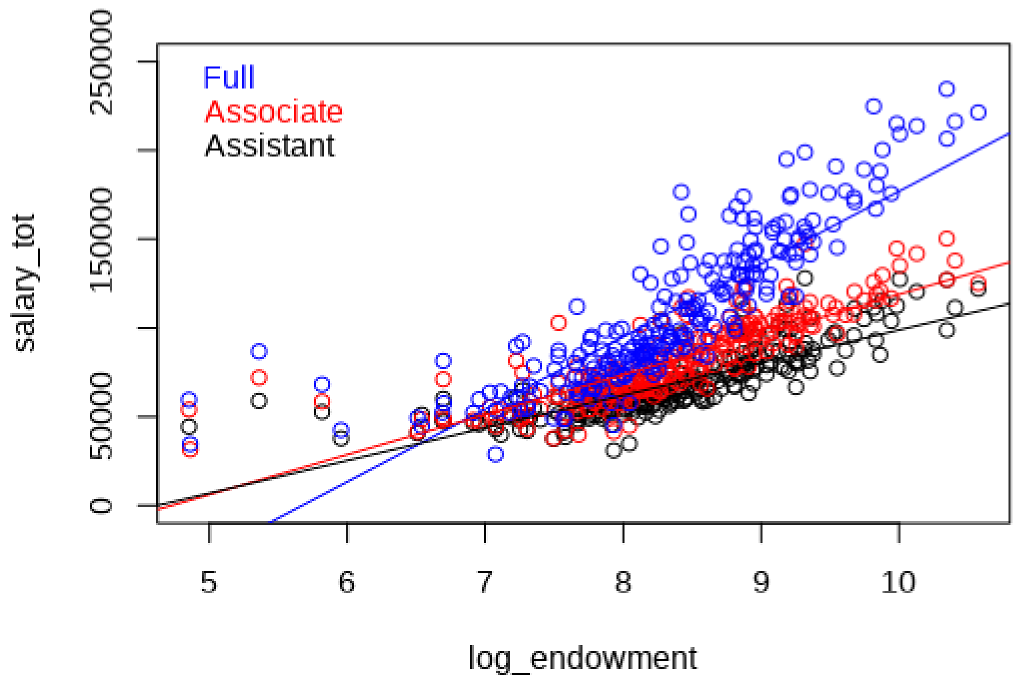
An ***interaction effect*** occurs when the response variable y is influenced by the levels of two or more predictors in a non-additive way

We can model this using an equation with an interaction term

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$

An interaction term between a quantitative and categorical variable corresponds to different slopes depending for the quantitative variable depending on the value of the categorical variable

Interaction terms



If Full Professor:

$$\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$$

If Assistant Professor:

$$\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$$

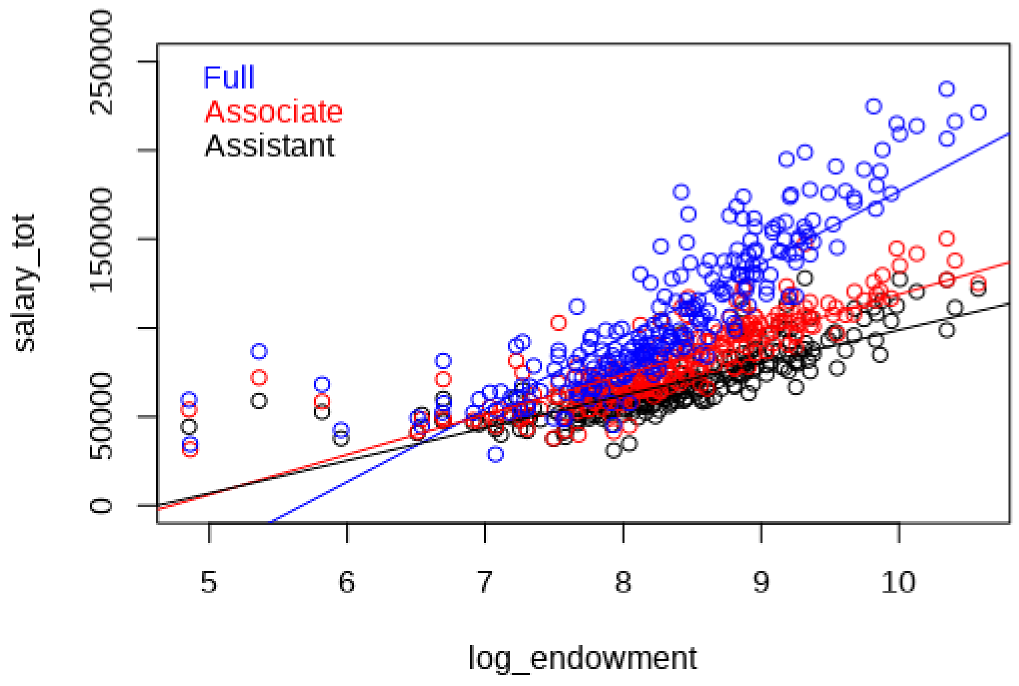
Modification to intercept if Assistant Professor

Modification to slope if Assistant Professor

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i1} \cdot x_{i2}$$

Interaction terms




```
Call:
lm(formula = salary_tot ~ log_endowment + rank_name + log_endowment:rank_name,
    data = IPED_2)

Residuals:
   Min     1Q Median     3Q    Max
-46914  -9554  -2263   6233  99678

Coefficients:
                                Estimate Std. Error t value Pr(>|t|)
(Intercept)                      -231986       9989 -23.224   <2e-16 ***
log_endowment                      40888       1190  34.357   <2e-16 ***
rank_nameAssociate                125551      14289   8.786   <2e-16 ***
rank_nameAssistant                146880      14429  10.180   <2e-16 ***
log_endowment:rank_nameAssociate  -18369       1701 -10.800   <2e-16 ***
log_endowment:rank_nameAssistant  -22482       1717 -13.094   <2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 16260 on 705 degrees of freedom
Multiple R-squared:  0.7806,    Adjusted R-squared:  0.7791
F-statistic: 501.7 on 5 and 705 DF,  p-value: < 2.2e-16
```

Intercept for full professor

Slope for full professor

Modification to intercept for assistant prof

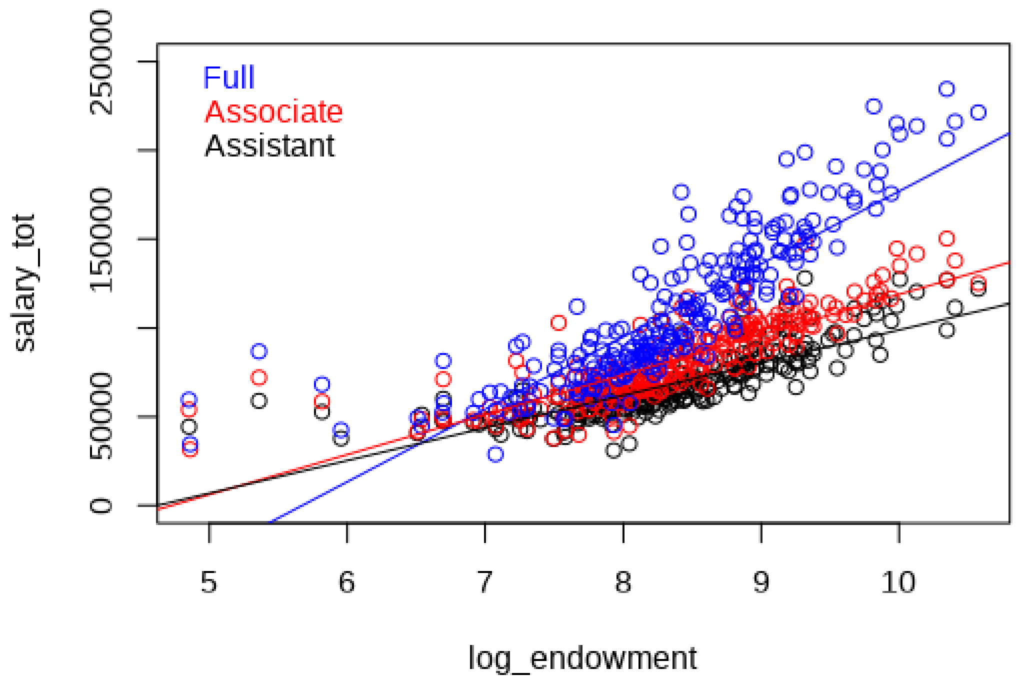
Modification to slope for assistant prof

x_{i1}: Log endowment (continuous)

x_{i2}: Assistant prof (indicator/dummy variable)

$$\hat{y}_i = \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} + \boxed{\hat{\beta}_2} x_{i2} + \boxed{\hat{\beta}_3} x_{i1} \cdot x_{i2}$$

Interaction terms



If Full Professor:

$$\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$$

If Assistant Professor:

$$\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$$

Modification to intercept if Assistant Professor

Modification to slope if Assistant Professor

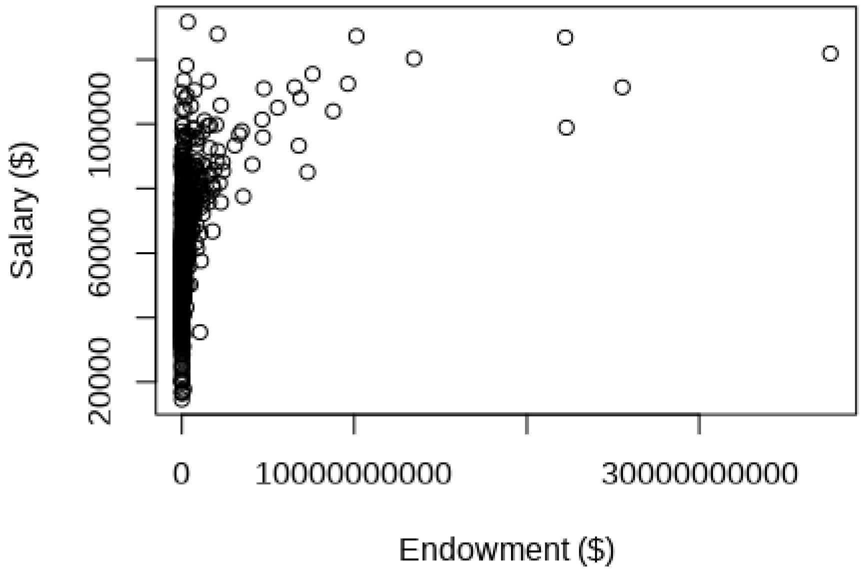
$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

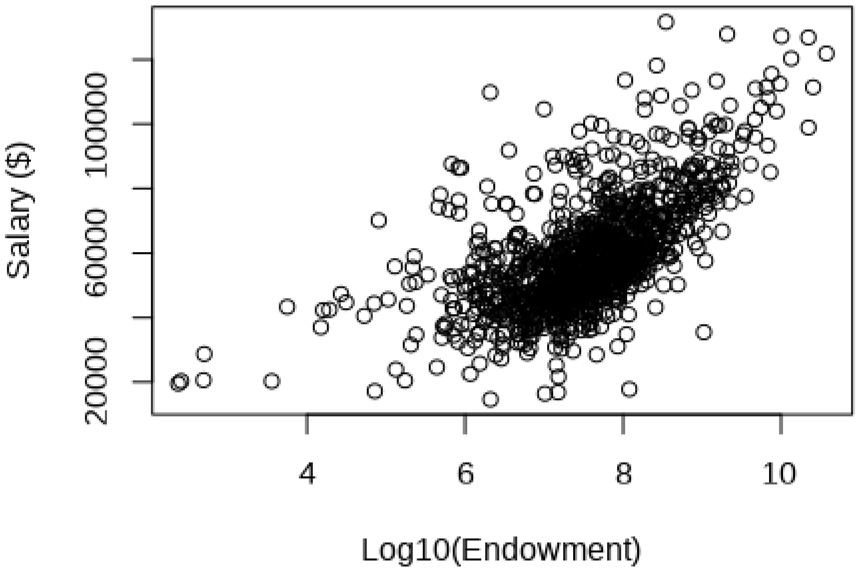
$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i1} \cdot x_{i2}$$

Questions?

Log transformation of the response variable y

As we've seen, we can take a log transformation of an *explanatory x* variable to make a non-linear relationship more linear

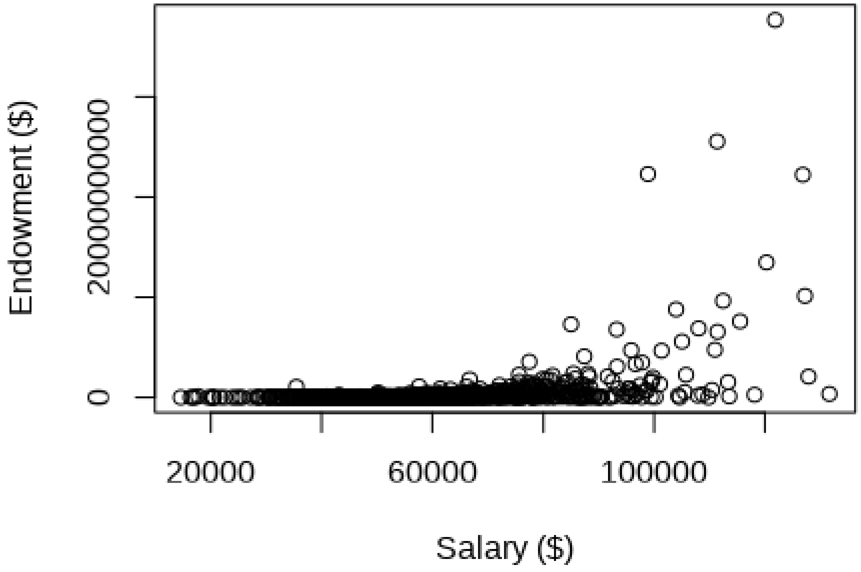


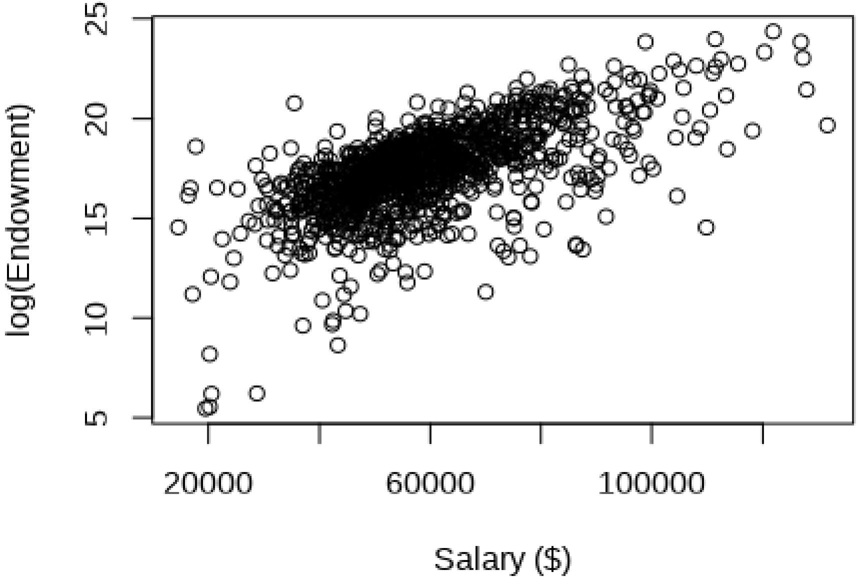


Log transformation of the response variable y

Often, it can be useful to take log transformation of a *response variable* *y* to make the relationship more linear

- This can also be useful to deal with heteroskedasticity





Log transformation of the response variable y

How can we interpret the regression coefficients when we have taken a log transformation of the response variable y?

$$log(\hat{y}) = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

If we exponentiate both sides we get:

$$\hat{y} = e^{\hat{\beta}_0 + \hat{\beta}_1} \qquad = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x}$$

If we increase x by 1, we multiply the previous predicted value of ŷ by $e^{\hat{\beta}_1}$

$$\hat{y} = \hat{f}(x+1) = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x + 1} = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x} \cdot e^{\hat{\beta}_1} = \hat{f}(x) \cdot e^{\hat{\beta}_1}$$

Log transformation of the response variable y

Side note: Often the natural (base e) log of y is used because for small values of $\hat{\beta}$

$$e^{\hat{\beta}} \approx 1 + \hat{\beta}$$

This is used as a justification for using the natural log, since this allows one to directly see what $e^{\hat{\beta}}$ approximately is from just looking at $\hat{\beta}$

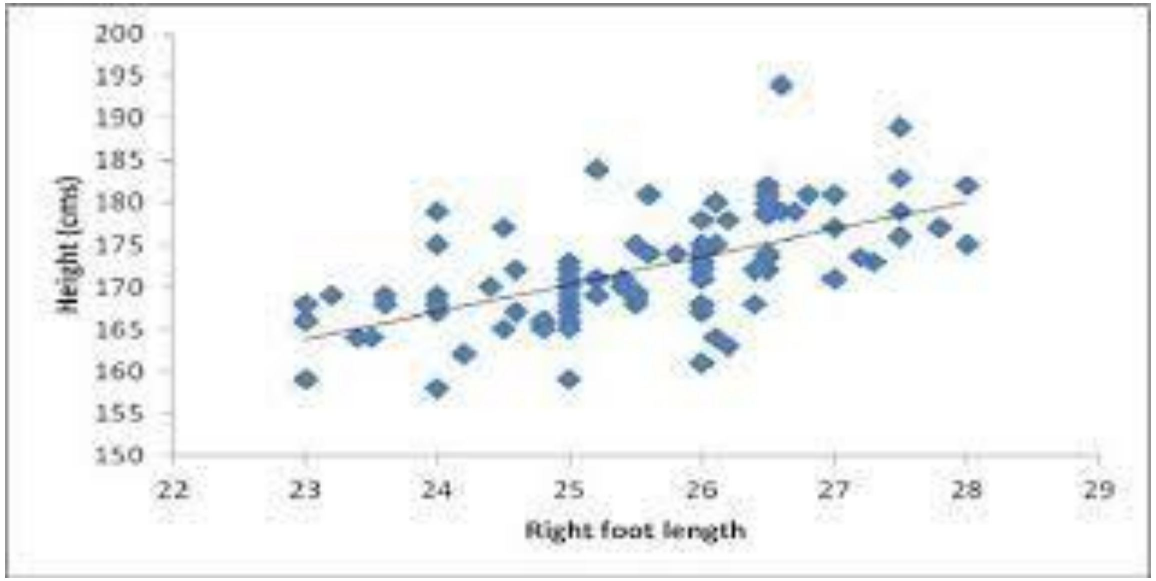
- Although it's not very hard to use the exp() on the regression coefficients in R

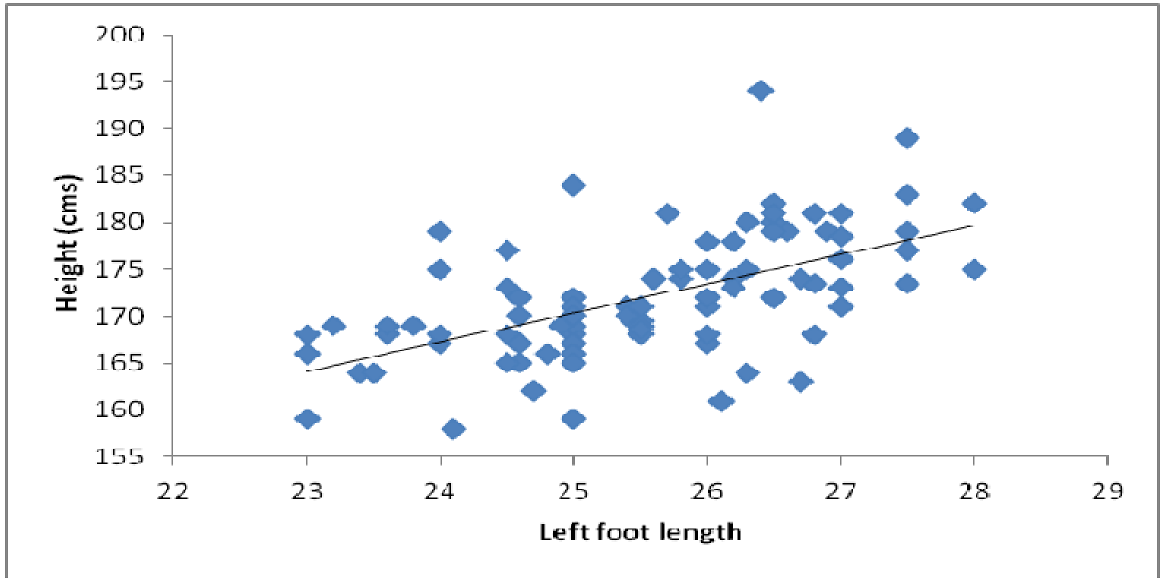
Let's try it in R…

Multicollinearity

Multicollinearity occurs when two or more variables are closely related to each other

- E.g., if they have a high correlation





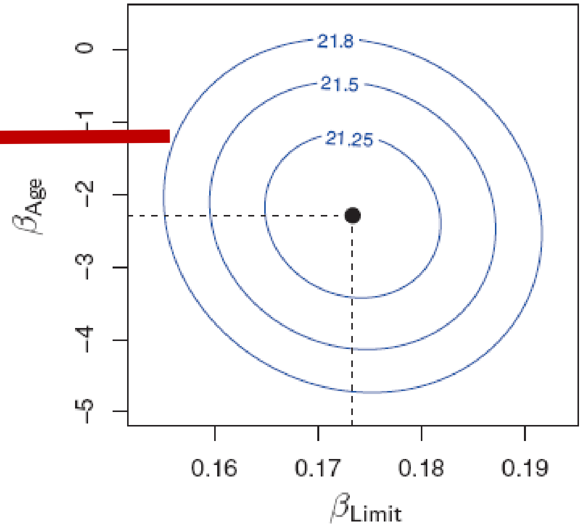
Multicollinearity

Multicollinearity can make our estimate of the regression coefficients unstable

- i.e., a large range of coefficient β-hat values give the same SSResidual and $\hat{\sigma}_\varepsilon$

Contours of equal

$\hat{\sigma}_\varepsilon$ value



This increases our estimate of the variance of the coefficients we measure and hence can decrease the power to detect a statistically significant predictor

Multicollinearity

The **variance inflated factor** is a statistic that can be computed to test for multicollinearity for the j^{th} explanatory variable:

$$VIF_j = \frac{1}{1-R_j^2}$$

where R_j^2 is the coefficient of determination for a model to predict x_j using the other explanatory variables in the model $(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$

- i.e., the R^2 value for this model:

$$\hat{x}_j = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \ldots + \hat{\beta}_{j-1} x_{j-1} + \hat{\beta}_{j+1} x_{j+1} + \ldots + \hat{\beta}_p x_p$$

<u>Rule of thumb</u>: suspect multicollinearity for VIF > 5

car::vif(lm_fit)

Are any of the predictors x_i related to y?

We can set this up as a hypothesis test:

H_0: $\beta_1 = \beta_2 = \ldots = \beta_p = 0$

H_A: At least one $\beta_j \neq 0$

We can run a parametric hypothesis test based on an F statistic to test this hypothesis

```
Call:
lm(formula = R ~ X1B + X2B + X3B + HR + BB + X1Bn + X2Bn + X3Bn +
    XHRn + XBBn, data = team_batting2)

Residuals:
    Min      1Q  Median      3Q     Max
-78.695 -15.457  -0.798  15.480  76.092

Coefficients:
               Estimate  Std. Error t value             Pr(>|t|)
(Intercept)  -574.88241    20.89696 -27.510 <0.0000000000000002 ***
X1B            -0.08976     0.43995  -0.204                0.838
X2B             1.70203     1.36050   1.251                0.211
X3B            -0.20163     4.71591  -0.043                0.966
HR              1.19258     1.47183   0.810                0.418
BB              0.24157     0.65658   0.368                0.713
X1Bn         3930.66847  2443.75215   1.608                0.108
X2Bn        -4839.59898  7517.51009  -0.644                0.520
X3Bn         8493.67060 26119.44048   0.325                0.745
XHRn         2061.44301  8146.72963   0.253                0.800
XBBn          588.32226  3628.53349   0.162                0.871
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 23.61 on 1140 degrees of freedom
Multiple R-squared:  0.9297,	Adjusted R-squared:  0.929
F-statistic:  1507 on 10 and 1140 DF,  p-value: < 0.00000000000000022
```

summary(lm_fit)

None of the coefficients are significant at the $\alpha = 0.05$ level

Overall H_0: $\beta_1 = \beta_2 = \ldots = \beta_k = 0$ is highly significant

This can happen when there is multicolinearity

Polynomial regression

Polynomial regression extends linear regression to non-linear relationships by including nonlinear transformations of predictors

$$
\begin{aligned}
\text{salary} = \;& \beta_0 + \beta_1 \cdot \text{endowment} \\
& + \beta_2 \cdot (\text{endowment})^2 + \\
& + \beta_3 \cdot (\text{endowment})^3 + \varepsilon
\end{aligned}
$$

Still a linear equation but non-linear in original predictors

Polynomial regression

Polynomial regression extends linear regression to non-linear relationships by including nonlinear transformations of predictors

We can compare model fits by:

- Assessing if higher order terms are statistically significant
- Looking at the r^2 values
- Running hypothesis tests comparing nested models
- Etc.

Let’s try it in R…