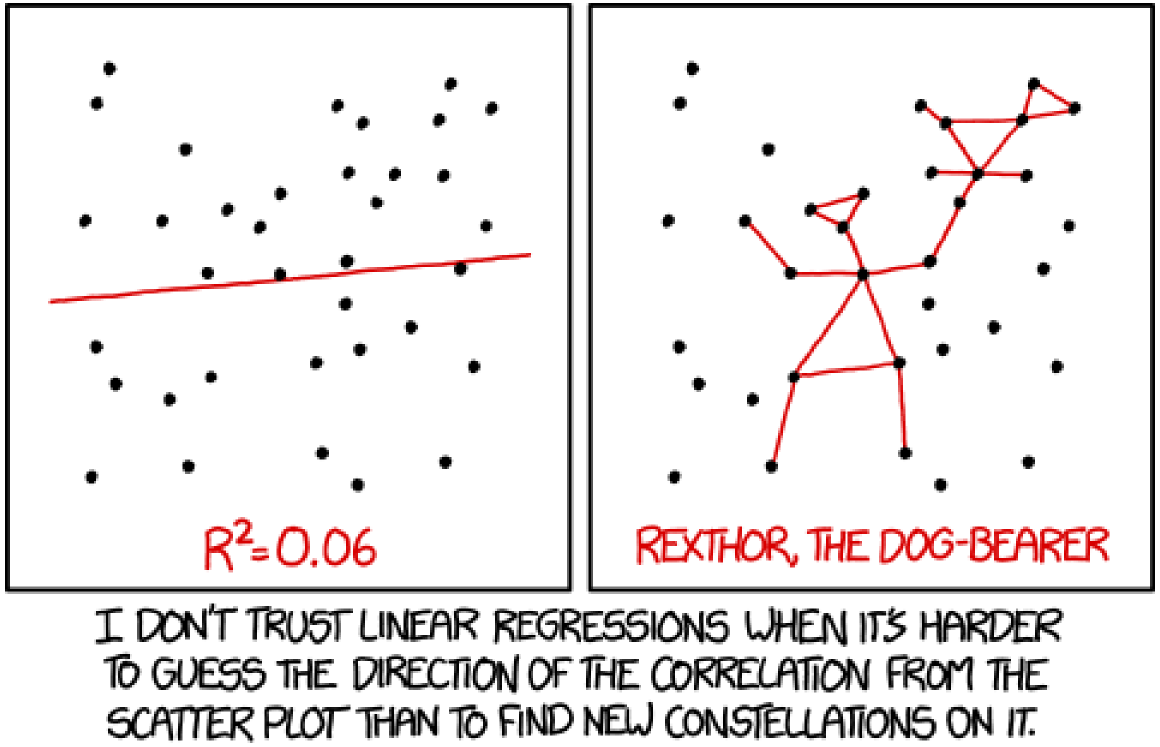
# Multiple regression continued



# Overview

Quick review of multiple regression with categorical offsets

Interaction effects

Log transformations of the response variable y

Multicollinearity

If there is time: Polynomial regression

# Quick review

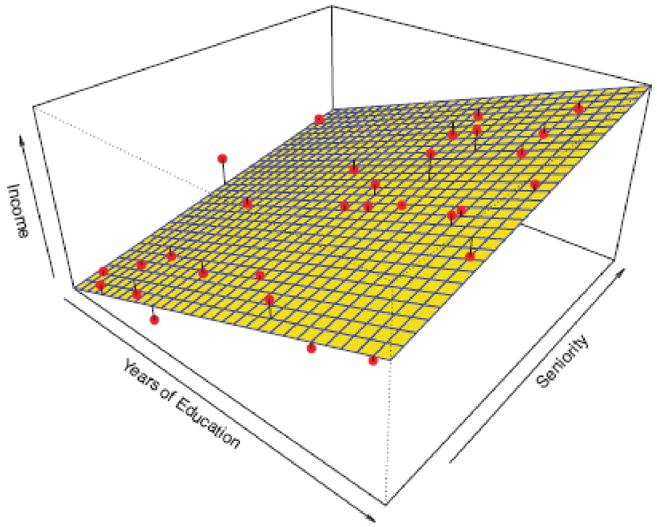
# Multiple regression

In multiple regression we try to predict a quantitative response variable *y* using several predictor variables $x_1, x_2, \ldots, x_k$

$$\hat{y} = \hat{\beta}_0 + \hat{\beta}_1 \cdot x_1 + \hat{\beta}_2 \cdot x_2 + \ldots + \hat{\beta}_k \cdot x_k$$

Goals:

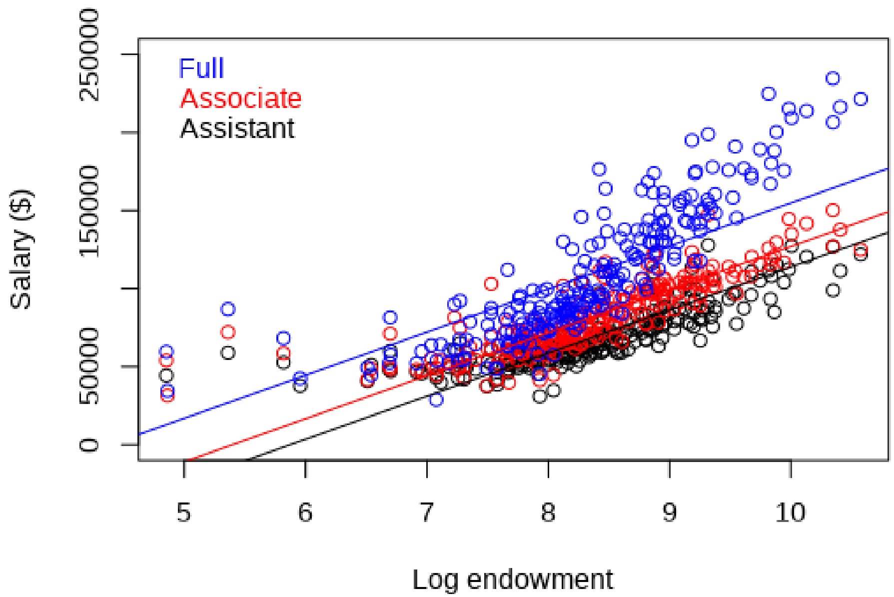
- To make predictions as accurately as possible
- To understand which predictors (x) are related to the response variable (y)



# Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it




Suppose we want to predict faculty salary *y* as a function of endowment $x_1$, with separate intercepts for faculty rank

$$x_{i1} = \log(\text{endowment})$$

$$x_{i2} = \begin{cases} 1 & \text{if assistant professor} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1}$$

$$= \begin{cases} \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} & \text{if full professor} \end{cases}$$

$$x_{i3} = \begin{cases} 1 & \text{if associate professor} \\ 0 & \text{otherwise} \end{cases}$$

# Categorical predictors

Predictors can be categorical as well as quantitative

- When a qualitative predictor has k levels, we need to use k -1 dummy variables to code it

Suppose we want to predict faculty salary *y* as a function of endowment $x_1$, with separate intercepts for faculty rank

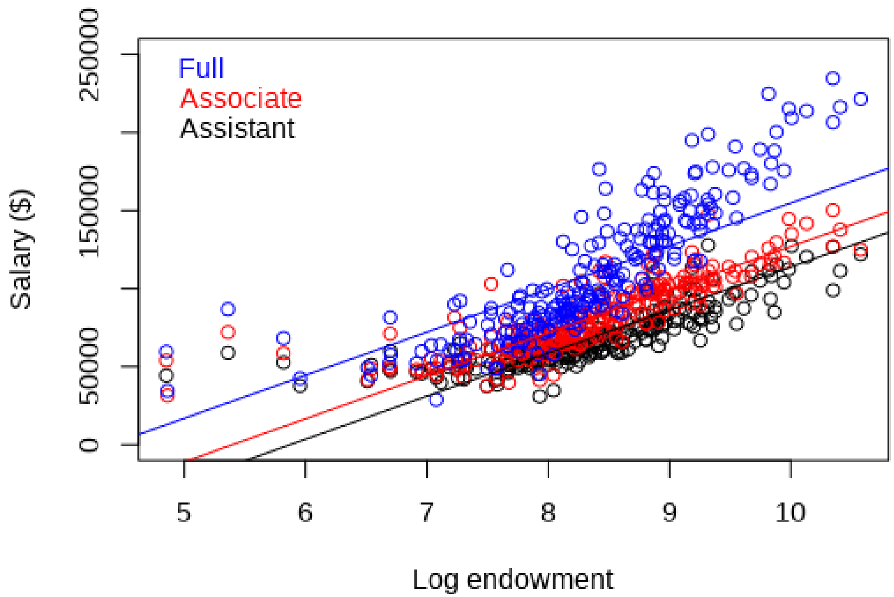
```
> summary(fit_prof_rank_offset)

Call:
lm(formula = salary_tot ~ log_endowment + rank_name, data = IPED_2)

Residuals:
   Min     1Q Median     3Q    Max 
-52464 -10844  -2703   6936  74994 

Coefficients:
                      Estimate Std. Error t value          Pr(>|t|)    
(Intercept)          -120822.1     6713.9  -18.00 <0.0000000000000002 ***
log_endowment          27569.9      791.7   34.82 <0.0000000000000002 ***
rank_nameAssociate    -27855.4     1685.5  -16.53 <0.0000000000000002 ***
rank_nameAssistant    -40973.7     1685.5  -24.31 <0.0000000000000002 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 18370 on 707 degrees of freedom
Multiple R-squared:  0.7192,	Adjusted R-squared:  0.718 
F-statistic: 603.7 on 3 and 707 DF,  p-value: < 0.00000000000000022
```




$$\hat{y}_i = \begin{cases} \boxed{\hat{\beta}_0} + \boxed{\hat{\beta}_1} x_{i1} + \boxed{\hat{\beta}_2} & \text{if assistant professor} \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \boxed{\hat{\beta}_3} & \text{if associate professor} \\ \hat{\beta}_0 + \hat{\beta}_1 x_{i1} & \text{if full professor} \end{cases}$$

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i1} + \hat{\beta}_2 x_{i2} + \hat{\beta}_3 x_{i3}$$

$$= -120,822 + 27,570 x_{i1} - 40,973 x_{i2} - 27,855 x_{i3}$$

# Interaction terms

The models we have looked at the relationship between the response and the predictors has been ***additive*** and ***linear***

$$y = \beta_1 x_1 + \beta_2 x_2 + ... + \beta_k x_k + \epsilon$$

These models assume that each predictor acts independently on the response y and that the relationship is linear

We can relax both of these assumptions

# Interaction terms

An ***interaction effect*** occurs when the response variable y is influenced by the levels of two or more predictors in a non-additive way

For example, a professor's salary might be more effected by the size of a school's endowment depending on the number of students who attend the school

We can model this using an equation with an interaction term

$$y = \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_3 (x_1 \cdot x_2) + \epsilon$$

# Interaction terms: categorical predictors

An interaction between a categorical and a quantitative variable corresponds to different slopes for the quantitative variable depending on the value of the categorical variable

- e.g., professor's salary might be more effected by the size of a school's endowment depending whether she is an Assistant or a Full Professor

If Full Professor: $\text{salary} \approx \beta_0 + \beta_1 \cdot \text{endowment}$

If Assistant Professor: $\text{salary} \approx (\beta_0 + \beta_2) + (\beta_1 + \beta_3) \cdot \text{endowment}$

Additive term if Assistant Professor

Change in slope if Assistant Professor

# Interaction terms

$$
\begin{aligned}
\text{salary} \approx \beta_0 &+ \beta_1 \cdot \text{endowment} \\
&+ \beta_2 \cdot \text{assistant\_rank\_dummy} \\
&+ \beta_3 \cdot (\text{assistant\_rank\_dummy} \cdot \text{endowment})
\end{aligned}
$$

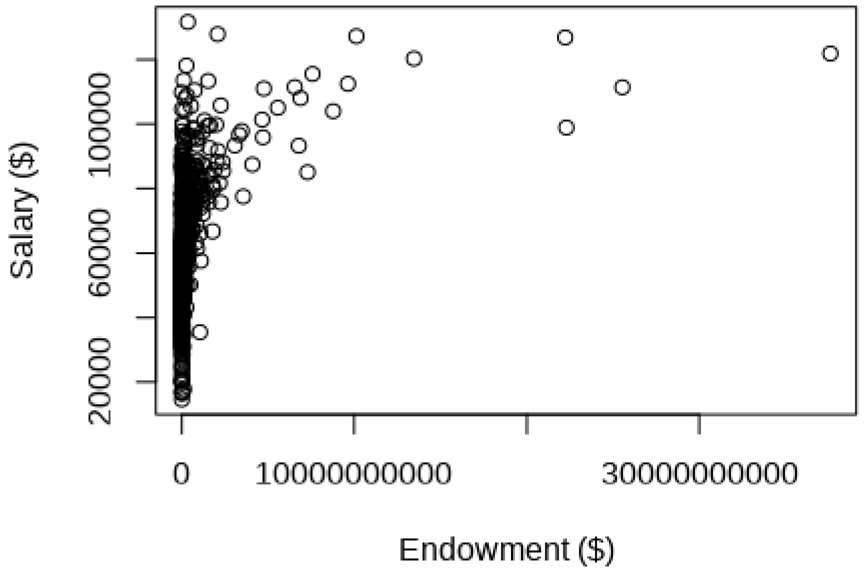
Questions?

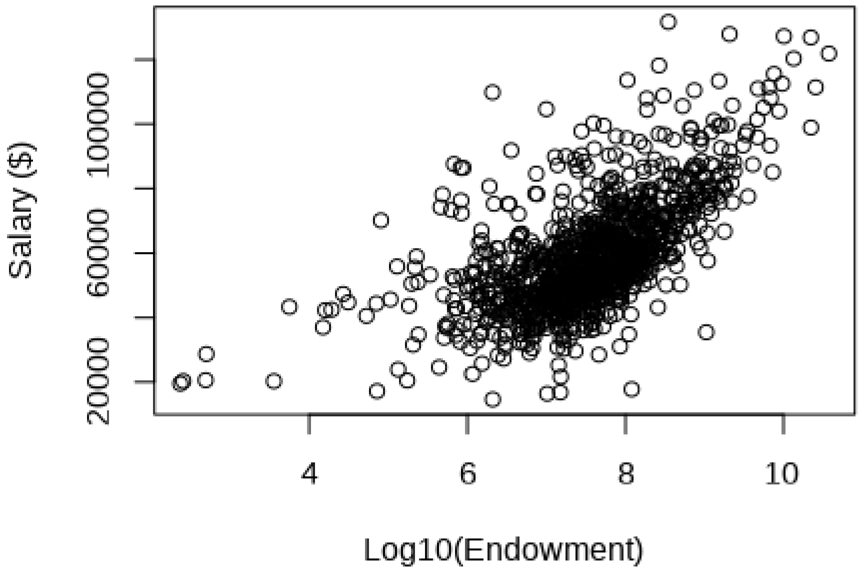
Let’s try it in R…

# Transformations of the response variable (y)

# Log transformation of the response variable y

As we've seen, we can take a log transformation of an *explanatory x* variable to make a non-linear relationship more linear

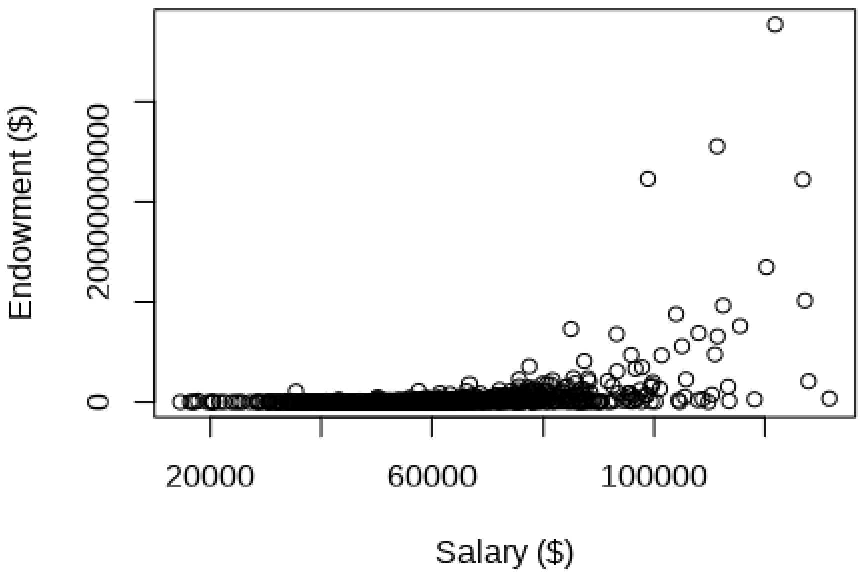


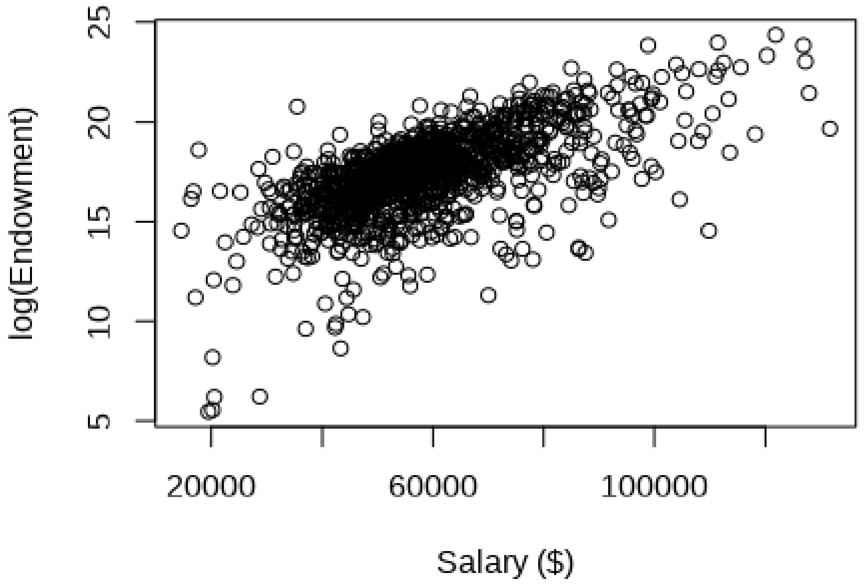



# Log transformation of the response variable y

Often, it can be useful to take log transformation of a *response variable y* to make the relationship more linear

- This can also be useful to deal with heteroskedasticity



# Log transformation of the response variable y

How can we interpret the regression coefficients when we have taken a log transformation of the response variable y?

$$log(\hat{y}) = \hat{\beta}_0 + \hat{\beta}_1 x_1$$

If we exponentiate both sides we get:

$$\hat{y} = e^{\hat{\beta}_0 + \hat{\beta}_1} \qquad = e^{\hat{\beta}_0} \cdot e^{\hat{\beta}_1 x}$$

If we increase x by 1, we multiply the previous predicted value of ŷ by $e^{\hat{\beta}_1}$

$\hat{y}$

# Log transformation of the response variable y

Side note: Often the natural (base e) log of y is used because for small values of $\hat{\beta}$

$$e^{\hat{\beta}} \approx 1 + \hat{\beta}$$

This is used as a justification for using the natural log, since this allows one to directly see what $e^{\hat{\beta}}$ approximately is from just looking at $\hat{\beta}$

- Although it's not very hard to use the exp() on the regression coefficients in R

# Let's try it in R…

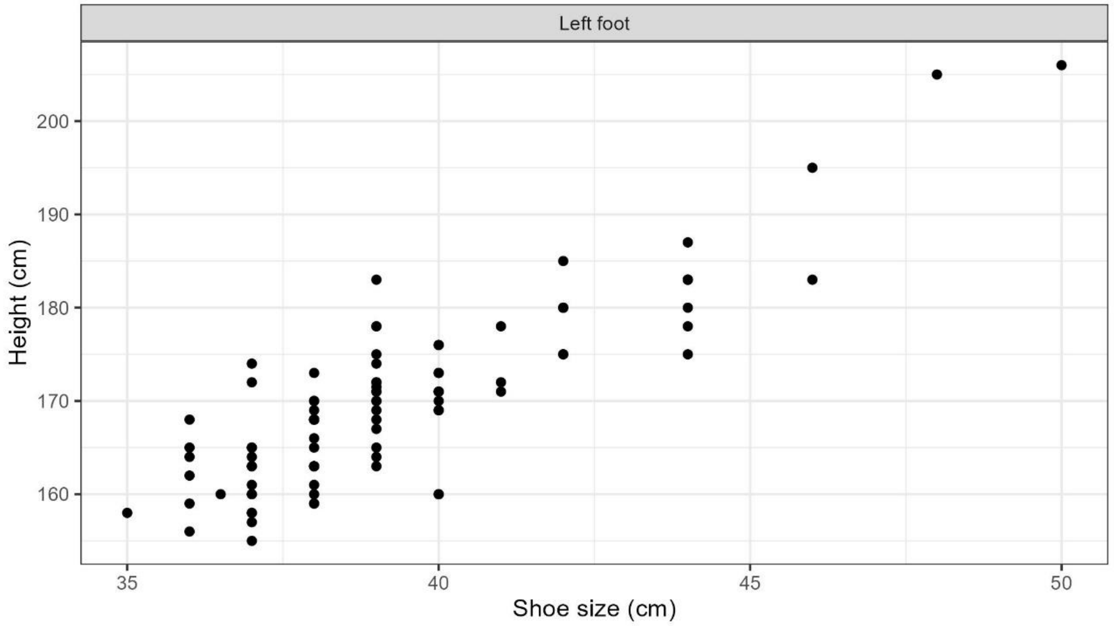
# Multicollinearity

# Multicollinearity

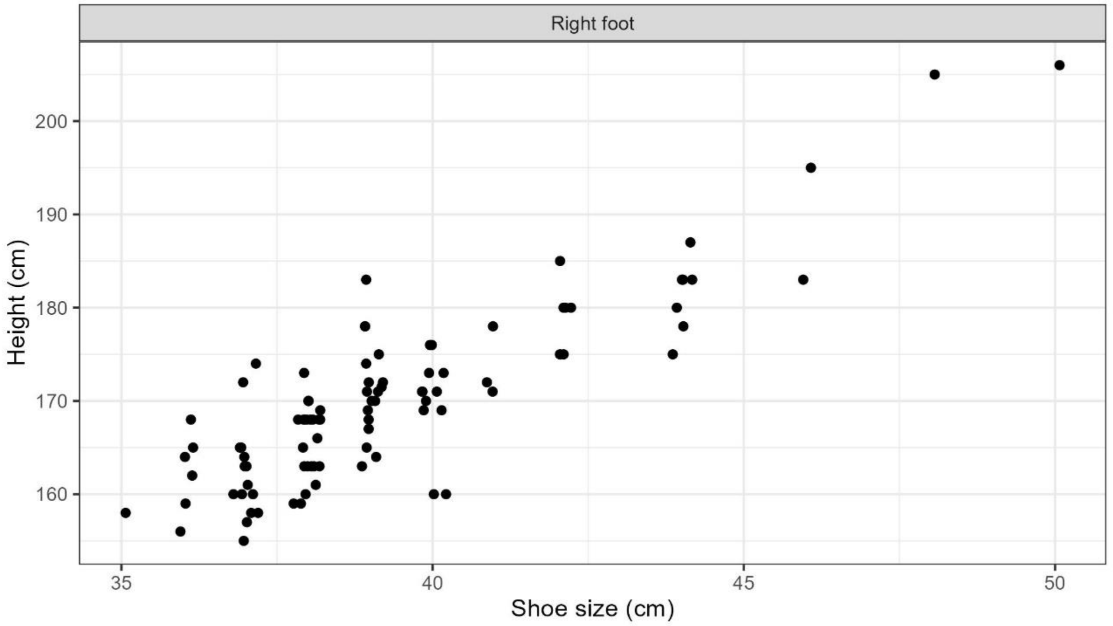
**Multicollinearity** occurs when two or more variables are closely related to each other

- E.g., if they have a high correlation

Left foot




Right foot

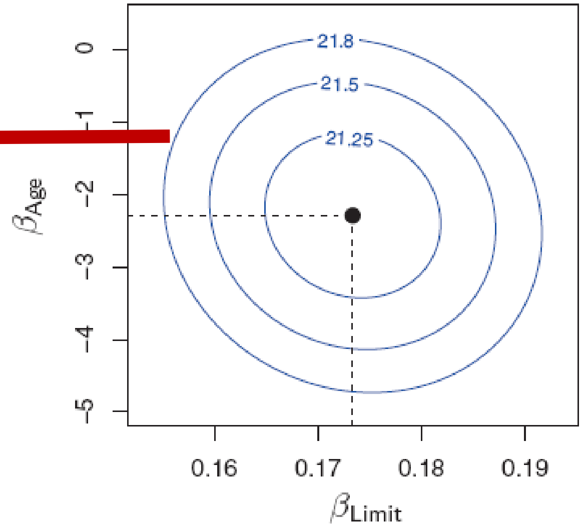


# Multicollinearity

Multicollinearity can make our estimate of the regression coefficients unstable

- i.e., a large range of coefficient β-hat values give the same SSResidual and $\hat{\sigma}_e$

Contours of equal $\hat{\sigma}_e$ value




This increases our estimate of the variance of the coefficients we measure and hence can decrease the power to detect a statistically significant predictor

# Multicollinearity

The **variance inflated factor** is a statistic that can be computed to test for multicollinearity for the $j^{th}$ explanatory variable:

$$VIF_j = \frac{1}{1-R_j^2}$$

where $R_j^2$ is the coefficient of determination for a model to predict $x_j$ using the other explanatory variables in the model $(x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_p)$

- i.e., the $R^2$ value for this model:

$$\hat{x}_j = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \ldots + \hat{\beta}_{j-1} x_{j-1} + \hat{\beta}_{j+1} x_{j+1} + \ldots + \hat{\beta}_p x_p$$

<u>Rule of thumb</u>: suspect multicollinearity for VIF > 5

car::vif(lm_fit)

# Are any of the predictors $x_i$ related to y?

We can set this up as a hypothesis test:

$H_0$: $\beta_1 = \beta_2 = \ldots = \beta_p = 0$

$H_A$: At least one $\beta_j \neq 0$

We can run a parametric hypothesis test based on an F statistic to test this hypothesis

# summary(lm_fit)

## Left foot

```
Call:
lm(formula = height ~ left_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.0750  -3.1323  -0.1036   2.6320  13.8964 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  53.2219     7.1640   7.429 5.19e-11 ***
left_shoe     2.9713     0.1818  16.343  < 2e-16 ***
```

## Right foot

```
Call:
lm(formula = height ~ right_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.6462  -3.2368   0.0896   2.3655  14.1697 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)  53.0717     7.1283   7.445  4.8e-11 ***
right_shoe    2.9734     0.1808  16.446  < 2e-16 ***
```

## Left and right foot

```
lm(formula = height ~ left_shoe + right_shoe, data = height_shoe)

Residuals:
     Min       1Q   Median       3Q      Max 
-12.9453  -3.3197   0.1906   2.3335  14.3130 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   53.141      7.165   7.416 5.78e-11 ***
left_shoe     -1.573      4.591  -0.343    0.733    
right_shoe     4.544      4.586   0.991    0.324    
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.808 on 92 degrees of freedom
Multiple R-squared:  0.7445,	Adjusted R-squared:  0.7389 
F-statistic:   134 on 2 and 92 DF,  p-value: < 2.2e-16
```

Neither coefficient is significant

Overall $H_0$: $\beta_1 = \beta_2 = 0$ is highly significant

This can happen when there is multicolinearity

Let's try it in R version 4.3.2…

# Polynomial regression

*Polynomial regression* extends linear regression to non-linear relationships by including nonlinear transformations of predictors

$$\text{salary} = \beta_0 + \beta_1 \cdot \text{endowment} + \beta_2 \cdot (\text{endowment})^2 + \beta_3 \cdot (\text{endowment})^3 + \varepsilon$$

Still a linear equation but non-linear in original predictors

# Polynomial regression

*Polynomial regression* extends linear regression to non-linear relationships by including nonlinear transformations of predictors

We can compare model fits by:

- Assessing if higher order terms are statistically significant
- Looking at the $r^2$ values
- Running hypothesis tests comparing nested models
- Etc.

# Let’s try it in R…