Tutorial 2: Basic Topology

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This tutorial note is adapted from Chapter 2 of *Principles of Mathematical Analysis* by Walter Rudin. I am going to make the following clear:

- Countable and uncountable sets.
- Metric space, open and closed sets in a metric space.
- Compact sets in a metric space.

1 Countable and Uncountable Sets

Definition 1 (equivalence)**.** If there exists a bijection between A and B, we say that A and B can be put in **1-1 correspondence**, or that A and B have the same **cardinality**, or A and B are **equivalent**. We write $A \sim B$.

How do we measure the "size" of a set A, especially when it is infinite? Are $[0, 1)$ and $[0, \infty)$ of the same size? Are $\mathbb{Z}$ and $\mathbb{Q}$ of the same size? We make use of bijections and give the following definition.

Definition 2 (countability and uncountability)**.** For any positive integer n, let $[n]$ be the set $\{1, 2, ..., n\}$. Let $\mathbb{N}$ be the set of all positive integers. For any set A, we say
(i) A is **finite** if $A \sim [n]$ for some n;
(ii) A is **infinite** if it is not finite;
(iii) A is **countable** if $A \sim \mathbb{N}$;
(iv) A is **uncountable** if A is neither finite nor countable;
(v) A is **at most countable** if A is finite or countable.

Equivalently, we may say that the elements of any countable set can be arranged in a sequence with distinct items.

Exercise 1. Show that $\mathbb{Z}$ is countable. How do you give a bijection between $\mathbb{Z}$ and $\mathbb{N}$?

Exercise 2. (a) Show that $\mathbb{Q}$ is countable. How do you give a bijection between $\mathbb{Q}$ and $\mathbb{N}$?
(b) Show that if A and B are countable, then the set $A \times B = \{(a, b) \mid a \in A, b \in B\}$ is countable.

Theorem 1. Every infinite subset of a countable set A is countable.

Proof. Suppose $E \subset A$, and E is infinite. Since A is countable, arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Now construct a sequence $\{n_k\}$ as follows:
Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen $n_1, ..., n_{k-1}$, let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.
Keep doing this, we obtain an infinite sequence $\{x_{n_k}\}$ that arranges all the elements of E. Hence E is coutnable. $\square$

Now we introduce the notation of union and intersection of a collection of sets.

Definition 3. Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω, denoted by E_α. In this sense, we call A an **index set**.
The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall speak of a **collection of sets**, or a **family of sets**.
The **union** of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$S = \bigcup_{\alpha \in A} E_\alpha.$$

Note that the index set A is not necessarily countable. If it is countable, however, the usual notation is

$$S = \bigcup_{m=1}^{\infty} E_m.$$

The **intersection** of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. The notations are similar.

An important note: the ∞ sign above is only a notation. It has nothing to do with "taking a limit of finite unions" in any sense! I know a lot of non-math students have made this mistake, including myself. The following exercise tells you that these two interpretations are distinct.

Exercise 3. For any $n \in \mathbb{N}$, we associate it with $E_n = [0, 1/n)$.
(a) What is $\bigcap_{n=1}^{\infty} E_n$? Prove your answer.
(b) What is $\lim_{n\to\infty} (\bigcap_{i=1}^{n} E_n)$? I have not talked about limits but you must have learned it in calculus class.

Exercise 4. Let $A = (0, 1]$ be the index set. For any $\alpha \in A$, we associate it with $E_\alpha = [-1, \alpha]$. What is $\bigcap_{\alpha \in A} E_\alpha$? Prove your answer.

Exercise 5. Show that if A, B are countable sets, then $A \cup B$ is countable.

This exercise essentially shows that finite union of countable sets are countable. (From 2-union to finite union is trivial!) In fact, we can improve this fact to the following theorem:

Theorem 2. Let $\{E_n\}, n = 1, 2, 3, ...$ be a sequence of countable sets, and put

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable. In other words, countable union of countable sets is still countable.

There are, of course, examples of uncountable sets.

Theorem 3. Let A be the set of all the sequences consisting of 0 and 1. Then A is uncountable. (The elements of A are like 1001101011...)

Proof. It is proved by the famous *Cantor's diagonal argument*. Suppose that A is countable, which contains the sequences $s_1, s_2, ...$ Now, consider such a sequence s, with the following property:

if the 1-st digit of s_1 is 1/0, then the 1-st digit of s is 0/1;
if the 2-nd digit of s_2 is 1/0, then the 2-nd digit of s is 0/1;
...
if the n-th digit of s_n is 1/0, then the n-th digit of s is 0/1;
...

Clearly, s exists in A because s is a binary sequence. But s differs from any $s_n \in A$ in at least one place, hence $s \notin A$. This is a contradiction. $\square$

Corollary 1. $\mathbb{R}$ is uncountable.

Exercise 6. Remember that we can write all the numbers in $\mathbb{N}$ in the binary form, and therefore $\mathbb{N}$ becomes a set of many binary sequences. Can we say $\mathbb{N}$ is equivalent to A? (If yes, then Theorem 3 would say $\mathbb{N}$ is uncountable!)

2 Metric Spaces

Definition 4. A set X, whose elements we shall call **points**, is said to be a **metric space** if for any two points $p, q \in X$ there is a real number $d(p, q)$, called the distance from p to q, such that for any $p, q, r \in X$,
(M1) $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$;
(M2) $d(p, q) = d(q, p)$;
(M3) $d(p, r) \leq d(p, q) + d(q, r)$.
Any function d with these properties is called a **distance function**, or a **metric**.

Exercise 7. (a) Show that $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$ is a metric on $\mathbb{R}$.
(b) Show that $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|_2$ for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is a metric on $\mathbb{R}^n$. Here, for any vector $\boldsymbol{a}$,

$$\|\boldsymbol{a}\|_2 = \sqrt{a_1^2 + ... + a_n^2}.$$

(c) Show that for any nonempty set X, the function

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

is a metric on X. It is called the discrete metric.

Definition 5. Let X be a metric space. All the points and sets mentioned below are understood to be points and subsets of X.
(a) A **neighborhood**, or **open ball** of p of **radius** r, denoted by $N_r(p)$, consists of all q such that $d(p, q) < r$.
(b) A point p is a **limit point** of a set E if every neighborhood of p contains some $q \in E, q \neq p$.
(c) E is **closed** if every limit point of E is a point of E.
(d) A point p is an **interior point** of E if there exists some $r > 0$ such that $N_r(p) \subset E$.
(e) E is **open** if every point of E is an interior point of E.
(f) E is **dense** if every point of X is either a point of E, or a limit point of E (or both).

Exercise 8. Show that every neighborhood is an open set.

Theorem 4. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

Theorem 5. A set is open if and only if its complement is closed.

Theorem 6. The following is about the intersection/union of open/closed sets.
(a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.
(b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
(c) For any finite collection $G_1, G_2, ..., G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
(d) For any finite collection $F_1, F_2, ..., F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

Exercise 9. Note that (c) and (d) require finiteness. Can you give an example to show that the intersection of infinitely many open sets can be closed?

Definition 6. If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E, then the **closure** of E is $\overline{E} = E \cup E'$.

Theorem 7. If X is a metric space and $E \subset X$, then
(a) $\overline{E}$ is closed;
(b) $E = \overline{E}$ if and only if E is closed;
(c) For every closed set F which contains E, $\overline{E} \subset F$.

By (a) and (c), $\overline{E}$ is the smallest closed set that contains E.

Exercise 10. Let $E \subset \mathbb{R}$ be bounded above. Show that $\sup E \in \overline{E}$.

3 Compact Sets

Definition 7. A subset K of a metric space X is said to be **compact** if every open cover of K contains a finite subcover.
Here, by open cover of a set K we mean a collection $\{G_\alpha\}$ of open sets, such that $K \subset \bigcup_\alpha G_\alpha$. When K is compact, the definition says that there must be a finite subcollection $G_{\alpha_1}, G_{\alpha_2}, ..., G_{\alpha_n}$, such that $K \subset \bigcup_{i=1}^n G_{\alpha_i}$.

Theorem 8. Compact subsets of a metric space are closed.

Theorem 9 (Cantor's intersection)**.** If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

Corollary 2 (Nested compact sets)**.** Consider a sequence of nonempty compact sets $\{K_n\}$, such that $K_n \supset K_{n+1}$ for all n. Then, $\bigcap_{i=1}^\infty K_i$ is nonempty.

Theorem 10 (Heine-Borel)**.** In $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded.