Tutorial 1: the Real Number System

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In this tutorial, I am going to make the following clear:

- Definition of field, ordered set, and ordered field.
- Definition of $\sup$, $\inf$, and the LUB/GLB property.
- Definition of the real field $\mathbb{R}$.

1 Ordered Sets

Definition 1 (order). *Let S be a set. An* ***order*** *on S is a relation, denoted by $<$, with the following two properties:*
(i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y,\ x = y,\ x > y$$

is true.
(ii) If $x, y, z \in S$, and if $x < y$ and $y < z$, then $x < z$.

If an order is defined on S, we say S is an **ordered set**. Moreover, for convenience we can write $y > x$ in place of $x < y$. The notation $x \leq y$ indicates that $x < y$ or $x = y$.

Definition 2 (upper bound). *Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is* ***bounded above****, and call β an* ***upper bound*** *of E.*

Of course, we can define the **lower bounds** in the same way.

Definition 3 (LUB). *Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:*
(i) α is an upper bound of E.
(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.
Then α is called the ***least upper bound*** *of E, or the* ***supremum*** *of E, and we write $\alpha = \sup E$.*

Likewise, we can define the **greatest lower bound**, or **infimum** of E, which is denoted by $\inf E$.

Exercise 1. *Let $A = \{q \in \mathbb{Q} \mid q^2 < 2\}$. Give an upper bound of A. Does A have a least upper bound?*

Exercise 2. *Let $B = \{x \in \mathbb{R} \mid x^2 < 2\}$. Give an upper bound of B. Does B have a least upper bound?*

Exercise 3. *Give a set $E \subset \mathbb{R}$, such that $\sup E$ exists in $\mathbb{R}$ but it is not in E.*

Definition 4 (LUB property). *An ordered set S is said to have the* ***least-upper-bound property*** *if the following is true: for any nonempty subset $E \subset S$, E is bounded above implies that $\sup E$ exists in S.*

Likewise, we can define the **greatest-lower-bound property**.

Exercise 4. *Does $\mathbb{Q}$ have LUB property? Does it have GLB property?*

Now, we shall state a theorem about a close relation between LUB and GLB. Prove it as an exercise.

Theorem 1. *Let S be an ordered set. Then S has LUB property if and only if it has GLB property.*

Hint. You only have to prove that LUB property implies GLB property (Why?). To this end, for any subset $E \subset S$ with a lower bound, we need to find $\inf E$. Now, consider the set of all the lower bounds of E. Can you claim something important for this set? $\square$

2 Fields

Definition 5. *A **field** is a set F with two closed operations $+$ and $*$ satisfying the "CANI-CANI-D" axioms:*
(C) $+$ is commutative.
(A) $+$ is associative.
(N) There is a neutral element for $+$, denoted by 0. This means $0 + x = x$ for every $x \in F$.
(I) There is an additive inverse for every $x \in F$, denoted by $-x$, with $x + (-x) = 0$.

(C) $*$ is commutative.
(A) $*$ is associative.
(N) There is a neutral element for $*$, denoted by 1. This means $1 * x = x$ for every $x \in F$.
(I) There is an multiplicative inverse for every $x \in F, x \neq 0$, denoted by x^{-1}, with $x * x^{-1} = 1$.

(D) The distributive law $x * (y + z) = x * y + x * z$ holds for all $x, y, z \in F$.

So many rules! But this is a good thing because we have enough properties to use, and therefore fields are almost the "best" algebraic structure among others. On the contrary, if only a part of the rules are available, things will become exquisite. For example, F is a group if "ANI"; it is an abelian group if "CANI"; it is a ring if "CANI-AN-D", and so on.

Exercise 5. *Show that in a field F, $x + z = y + z$ implies $x = y$, and $xz = yz, z \neq 0$ implies $x = y$. This is called the cancellation law.*

Definition 6. *An **ordered field** is a field F which is also an ordered set, such that*
(i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$.
(ii) $xy > 0$ if $x \in F, y \in F, x > 0, y > 0$.

3 The Real Field

Definition 7. *An ordered field with the least-upper-bound property containing $\mathbb{Q}$ as a subfield, denoted by $\mathbb{R}$, is called the **real field**.*