1 假設轉換後的間斷時間分配為韋伯分布

模擬資料為考慮三個應力水準(k=1,2,3)以及正常使用(無應力,k=0)，每一應力下有三個樣本($n_k = 3$)，每一個樣本去模擬45個失效時間點(m=45)，使用韋伯模型去生資料，並且使用標準化後的應力水準去做，即$x_0 = 0, x_1 = 0.111, x_2 = 0.556, x_3 = 1$
使用間斷時間分配為韋伯分配去生資料，
設定參數為$\theta = (a_0, a_1, b_0, b_1, \beta_0, \beta_1) = (3, 2, 5, 4, 20, -5)$

1.1 初步分析1000次模擬結果

表 1: 信賴區間涵蓋率與平均長度

Stress	Sample	$\hat{a}$		$\hat{b}$		$\hat{\beta}$	
		CP	AL	CP	AL	CP	AL
0.111C	1	0.944	0.394597	0.95	0.187295	0.946	9.421931
	2	0.943	0.393134	0.947	0.186164	0.941	9.421601
	3	0.932	0.394793	0.942	0.187496	0.964	9.343637
0.556C	4	0.944	0.57337	0.938	0.279985	0.95	8.316378
	5	0.935	0.582771	0.924	0.284585	0.946	8.234326
	6	0.936	0.577934	0.933	0.282269	0.947	8.286658
1C	7	0.937	0.805883	0.945	0.400389	0.948	7.234606
	8	0.938	0.807501	0.941	0.402016	0.949	7.20065
	9	0.948	0.808568	0.948	0.401743	0.946	7.246519

1			
	a (3.222)	b (5.444)	beta (19.445)
mean	3.23031	5.43976	20.47516
sd	0.10964	0.05154	2.47923
2.50%	3.03181	5.32528	16.17451
97.50%	3.48067	5.53901	26.02184
MSE	0.01208	0.00267	7.20165

4			
	a (4.112)	b (7.224)	beta (17.22)
mean	4.12407	7.21943	17.89902
sd	0.15708	0.07558	2.20508
2.50%	3.828	7.07743	14.36939
97.50%	4.433	7.36811	22.99132
MSE	0.0248	0.00573	5.31857

7			
	a (5)	b (9)	beta (15)
mean	5.01894	8.9935	15.67953
sd	0.21928	0.10825	1.90428
2.50%	4.63342	8.76937	12.56478
97.50%	5.47702	9.19742	20.01709
MSE	0.04839	0.01175	4.08441

2			
mean	3.23155	5.44042	20.38228
sd	0.10761	0.05192	2.59068
2.50%	3.03246	5.33428	16.32513
97.50%	3.45691	5.53937	26.23606
MSE	0.01166	0.0027	7.58186

5			
mean	4.12135	7.21924	18.04176
sd	0.15372	0.07501	2.19982
2.50%	3.83691	7.07145	14.32355
97.50%	4.45353	7.36591	22.73666
MSE	0.02369	0.00564	5.50965

8			
mean	5.02909	8.98679	15.74234
sd	0.2196	0.10875	1.92297
2.50%	4.64279	8.7566	12.48254
97.50%	5.51121	9.18168	20.10302
MSE	0.04901	0.01199	4.24444

3			
mean	3.22776	5.44159	20.44691
sd	0.10662	0.05093	2.61829
2.50%	3.02816	5.33576	16.06878
97.50%	3.45547	5.53789	26.16589
MSE	0.01139	0.0026	7.85243

6			
mean	4.11985	7.22146	18.08849
sd	0.15035	0.07259	2.21936
2.50%	3.84801	7.0784	14.26304
97.50%	4.43291	7.35665	22.98063
MSE	0.02264	0.00527	5.67489

9			
mean	5.02332	8.98828	15.77746
sd	0.21807	0.10677	1.89228
2.50%	4.6371	8.75898	12.477
97.50%	5.49983	9.18485	20.06205
MSE	0.04796	0.0115	4.17469

2 H-M 演算法

先做在正常應力下的估計，並且使用常態、均勻、伽瑪這些先驗分布去做。
在沒有應力的情況下，要估計的未知參數僅有θ的a_0, b_0, β_0部分，因為使用標準化應力水準$x_0 = 0$,因此真實參數為$\theta = (3, 0, 5, 0, 20, 0)$,令要估計的參數$\theta' = (a_0, b_0, \beta_0) = (3, 5, 20)$，其對數概似估計函數的公式如下:

$$
\begin{aligned}
logL(\theta') = {} & 135 \log \Gamma\left(\frac{1}{\beta_0} + 1\right) + 135 \log \beta_0 + 135 \log a_0 \\
& + (\beta_0 - 1) * \sum_{j=1}^{3} \sum_{i=1}^{45} \left[\log \left(\Gamma(\frac{1}{\beta_0} + 1) \frac{a_0}{b_0} (e^{b_0 t_{i,j0}} - e^{b_0 t_{i-1,j0}}) \right) \right] \\
& - \sum_{j=1}^{3} \sum_{i=1}^{45} \left[\Gamma\left(\frac{1}{\beta_0} + 1\right) \frac{a_0}{b_0} \left(e^{b_0 t_{i,j0}} - e^{b_0 t_{i-1,j0}}\right) \right]^{\beta_0} + b_0 \sum_{j=1}^{3} \sum_{i=1}^{45} t_{ij0}
\end{aligned}
$$

以下為給定先驗分布為常態分布的演算法過程。
演算法步驟如下:
步驟一: 給定初始值$(a_0^{(0)}, b_0^{(0)}, \beta_0^{(0)})$=(1,1,13)
步驟二: 產生$a_0^* \sim N(a_0^{(0)}, 0.1^2)$
步驟三: 產生$u \sim U(0,1)$， 並對其取對數 $u = \log(u)$。
步驟四:

$$
\begin{aligned}
\alpha_{a_0^{(0)}, a_0^*} &= \log \left[min \left(\frac{L(a_0^*, b_0^{(0)}, \beta_0^{(0)} | t_0, x_0) * exp(-\frac{(a_0^* - 3.00202)^2}{2 * 0.1^2})}{L(a_0^{(0)}, b_0^{(0)}, \beta_0^{(0)} | t_0, x_0) * exp(-\frac{(a_0^{(0)} - 3.00202)^2}{2 * 0.1^2})}, 1 \right) \right] \\
&= min\{135(\log a_0^* - \log a_0^{(0)}) \\
&\quad + (\beta_0^{(0)} - 1) * \left[\sum_{j=1}^{3} \sum_{i=1}^{45} \left(\log \left(\Gamma(\frac{1}{\beta_0^{(0)}} + 1) \frac{a_0^*}{b_0^{(0)}} (e^{b_0^{(0)} t_{i,j0}} - e^{b_0^{(0)} t_{i-1,j0}}) \right) \right) \right. \\
&\qquad \left. - \sum_{j=1}^{3} \sum_{i=1}^{45} \left(\log \left(\Gamma(\frac{1}{\beta_0^{(0)}} + 1) \frac{a_0^{(0)}}{b_0^{(0)}} (e^{b_0^{(0)} t_{i,j0}} - e^{b_0^{(0)} t_{i-1,j0}}) \right) \right) \right] \\
&\quad - \sum_{j=1}^{3} \sum_{i=1}^{45} \left[\Gamma\left(\frac{1}{\beta_0^{(0)}} + 1\right) \frac{a_0^*}{b_0^{(0)}} \left(e^{b_0^{(0)} t_{i,j0}} - e^{b_0^{(0)} t_{i-1,j0}}\right) \right]^{\beta_0^{(0)}} \\
&\quad + \sum_{j=1}^{3} \sum_{i=1}^{45} \left[\Gamma\left(\frac{1}{\beta_0^{(0)}} + 1\right) \frac{a_0^{(0)}}{b_0} \left(e^{b_0^{(0)} t_{i,j0}} - e^{b_0^{(0)} t_{i-1,j0}}\right) \right]^{\beta_0^{(0)}} \\
&\quad - \frac{(a_0^* - 3.00202)^2}{2 * 0.1^2} + \frac{(a_0^{(0)} - 3.00202)^2}{2 * 0.1^2}, 0\}
\end{aligned}
$$

$u < \alpha_{a_0^{(0)}, a_0^{(*)}}$，則$a_0^{(1)} = a_0^*$，否則$a_0^{(1)} = a_0^{(0)}$

步驟五: 產生$b_0^* \sim N(b_0^{(0)}, 0.1^2)$

步驟六:

$$\alpha_{b_0^{(0)},b_0^*} = \log\left[min\left(\frac{L(a_0^{(1)},b_0^*,\beta_0^{(0)}|t_0,x_0)*exp(-\frac{(b_0^*-4.9994)^2}{2*0.1^2})}{L(a_0^{(1)},b_0^{(0)},\beta_0^{(0)}|t_0,x_0)*exp(-\frac{(b_0^{(0)}-4.9994)^2}{2*0.1^2})},1\right)\right]$$

$$= min\{(\beta_0^{(0)}-1)*\left[\sum_{j=1}^{3}\sum_{i=1}^{45}\left(\log\left(\Gamma(\frac{1}{\beta_0^{(0)}}+1)\frac{a_0^{(1)}}{b_0^*}(e^{b_0^* t_{i,j0}}-e^{b_0^* t_{i-1,j0}})\right)\right)\right.$$
$$\left.-\sum_{j=1}^{3}\sum_{i=1}^{45}\left(\log\left(\Gamma(\frac{1}{\beta_0^{(0)}}+1)\frac{a_0^{(1)}}{b_0^{(0)}}(e^{b_0^{(0)} t_{i,j0}}-e^{b_0^{(0)} t_{i-1,j0}})\right)\right)\right]$$
$$-\sum_{j=1}^{3}\sum_{i=1}^{45}\left[\Gamma\left(\frac{1}{\beta_0}+1\right)\frac{a_0^{(1)}}{b_0^*}\left(e^{b_0^* t_{i,j0}}-e^{b_0^* t_{i-1,j0}}\right)\right]^{\beta_0^{(0)}}$$
$$+\sum_{j=1}^{3}\sum_{i=1}^{45}\left[\Gamma\left(\frac{1}{\beta_0^{(0)}}+1\right)\frac{a_0^{(1)}}{b_0^{(0)}}\left(e^{b_0^{(0)} t_{i,j0}}-e^{b_0^{(0)} t_{i-1,j0}}\right)\right]^{\beta_0^{(0)}}+(b_0^*-b_0^{(0)})\sum_{j=1}^{3}\sum_{i=1}^{45}t_{ij0}$$
$$-\frac{(b_0^*-4.9994)^2}{2*0.1^2}+\frac{(b_0^{(0)}-4.9994)^2}{2*0.1^2},0\}$$

$u < \alpha_{b_0^{(0)},b_0^{(*)}}$，則$b_0^{(1)} = b_0^*$，否則$b_0^{(1)} = b_0^{(0)}$

步驟七: 產生$\beta_0^* \sim N(\beta_0^{(0)}, 0.1^2)$

步驟八:

$$\alpha_{\beta_0^{(0)},\beta_0^*} = \log\left[min\left(\frac{L(a_0^{(1)},b_0^{(1)},\beta_0^*|t_0,x_0)*exp(-\frac{(\beta_0^*-22.83913)^2}{2*0.1^2})}{L(a_0^{(1)},b_0^{(1)},\beta_0^{(0)}|t_0,x_0)*exp(-\frac{(\beta_0^{(0)}-22.83913)^2}{2*0.1^2})},1\right)\right]$$

$$= min\{135\left[\log\Gamma\left(\frac{1}{\beta_0^*}+1\right)-\log\Gamma\left(\frac{1}{\beta_0^{(0)}}+1\right)\right]-135\left(\log\beta_0^*-\log\beta_0^{(0)}\right)$$
$$+(\beta_0^*-1)*\left[\sum_{j=1}^{3}\sum_{i=1}^{45}\left(\log\left(\Gamma(\frac{1}{\beta_0^*}+1)\frac{a_0^{(1)}}{b_0^{(1)}}(e^{b_0^{(1)} t_{i,j0}}-e^{b_0^{(1)} t_{i-1,j0}})\right)\right]\right.$$
$$\left.-(\beta_0^{(0)}-1)*\sum_{j=1}^{3}\sum_{i=1}^{45}\left(\log\left(\Gamma(\frac{1}{\beta_0^{(0)}}+1)\frac{a_0^{(1)}}{b_0^{(1)}}(e^{b_0^{(1)} t_{i,j0}}-e^{b_0^{(1)} t_{i-1,j0}})\right)\right)\right.$$
$$-\sum_{j=1}^{3}\sum_{i=1}^{45}\left[\Gamma\left(\frac{1}{\beta_0^*}+1\right)\frac{a_0^{(1)}}{b_0^{(1)}}\left(e^{b_0^{(1)} t_{i,j0}}-e^{b_0^{(1)} t_{i-1,j0}}\right)\right]^{\beta_0^*}$$
$$+\sum_{j=1}^{3}\sum_{i=1}^{45}\left[\Gamma\left(\frac{1}{\beta_0^{(1)}}+1\right)\frac{a_0^{(1)}}{b_0^{(1)}}\left(e^{b_0^{(1)} t_{i,j0}}-e^{b_0^{(1)} t_{i-1,j0}}\right)\right]^{\beta_0^{(0)}}$$
$$-\frac{(\beta_0^*-22.83913)^2}{2*0.1^2}+\frac{(\beta_0^{(0)}-22.83913)^2}{2*0.1^2},0\}$$

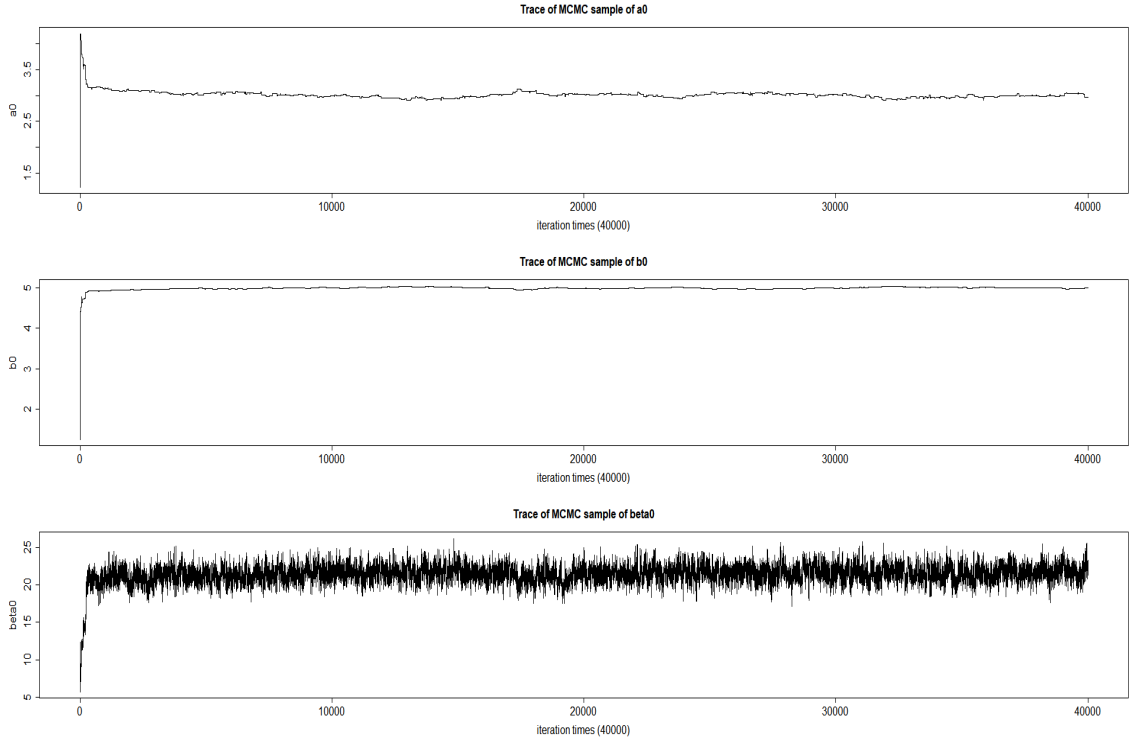
$u < \alpha_{\beta_0^{(0)},\beta_0^{(*)}}$，則$\beta_0^{(1)} = \beta_0^*$，否則$\beta_0^{(1)} = \beta_0^{(0)}$

步驟九: 重複n=20000+2000*10迭代數，得各參數的2000個後驗樣本。

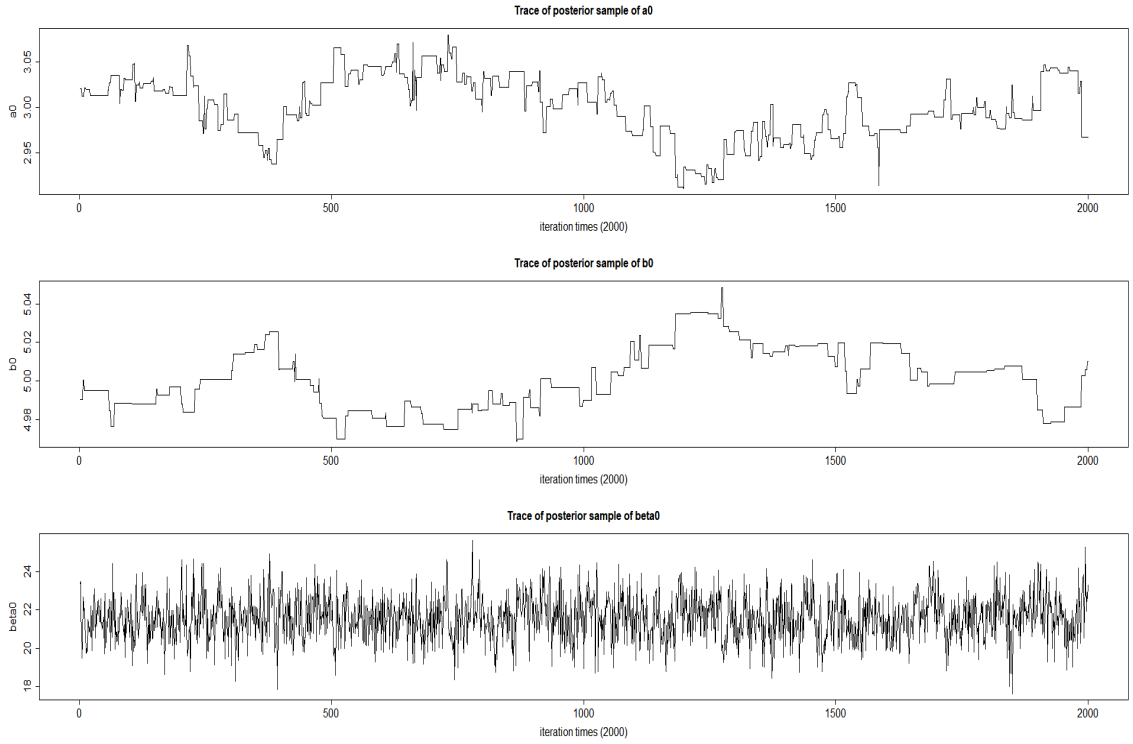
2.1 設定初始值為(1,1,13)

	a_0	b_0	β_0
initial	1	1	13
proposal density:N(,sd=)	1	1	1
prior:N(mle,sd=)	0.1	0.1	2
Acceptance rate	0.014975	0.007525	0.7046
true value	3	5	20
mean	3.00015	5.00082	21.59158
mle	2.98998	5.00629	21.64919
sd	0.03400	0.01606	1.14638
sd of mle	0.04687	0.02192	1.44977
2.50%	2.92679	4.97486	19.35885
97.50%	3.05687	5.03508	23.94733

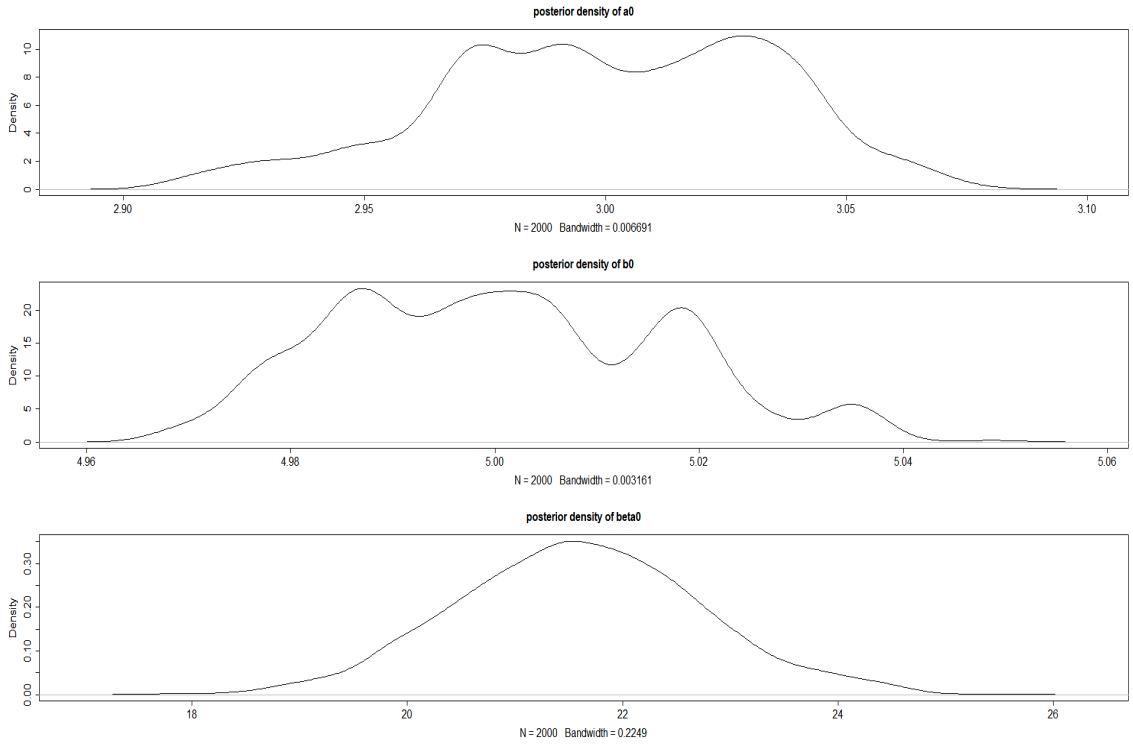
2.1.1 MCMC樣本路徑圖



2.1.2 後驗樣本路徑圖



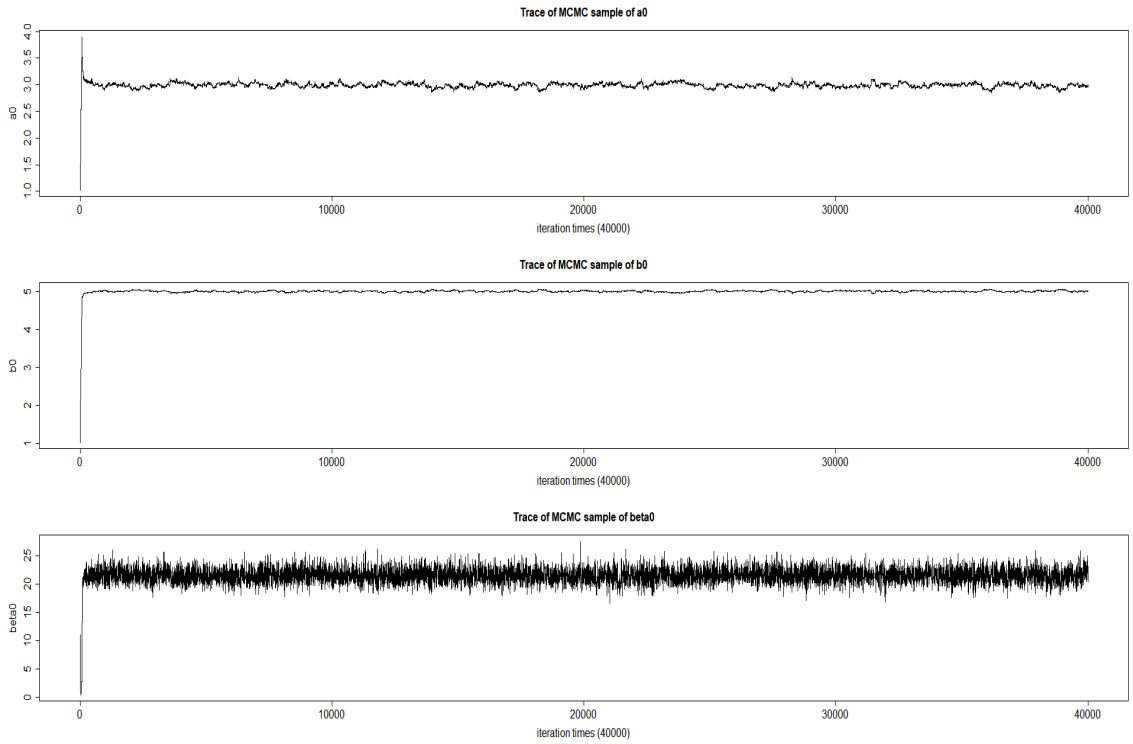
2.1.3 後驗樣本機率密度圖



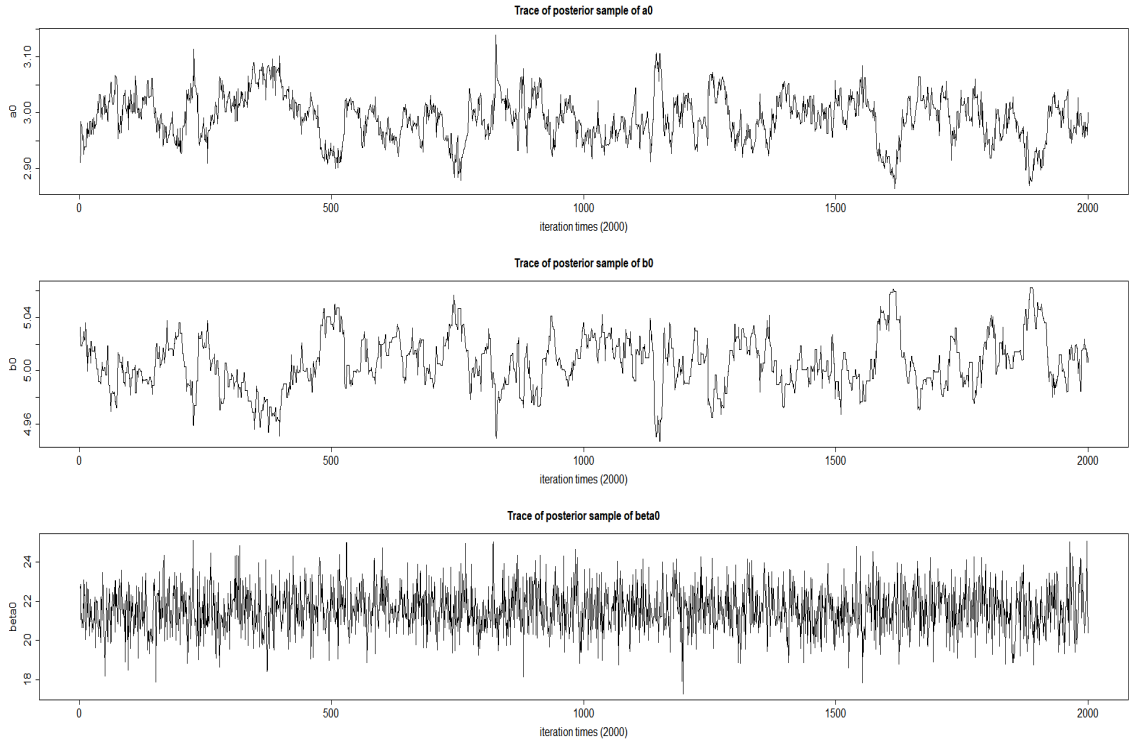
2.2 調整建議分配的標準差

	a_0	b_0	β_0
initial	1	1	13
proposal density:N(,sd=)	0.1	0.1	3
prior:N(mle,sd=)	0.1	0.1	2
Acceptance rate	0.149375	0.07215	0.37615
true value	3	5	20
mean	2.99080	5.00602	21.58530
mle	2.98998	5.00629	21.64919
sd	0.04128	0.01978	1.17091
sd of mle	0.04687	0.02192	1.44977
2.50%	2.91064	4.96850	19.39204
97.50%	3.07423	5.04661	23.97417

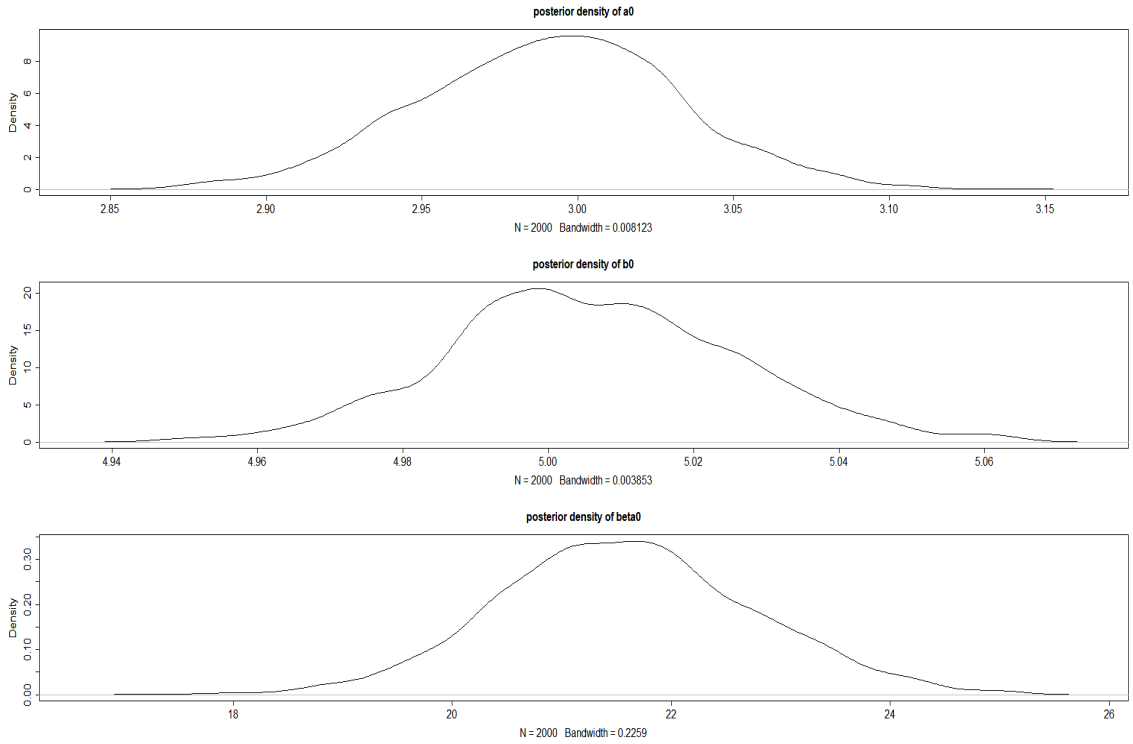
2.2.1 MCMC樣本路徑圖



2.2.2 後驗樣本路徑圖



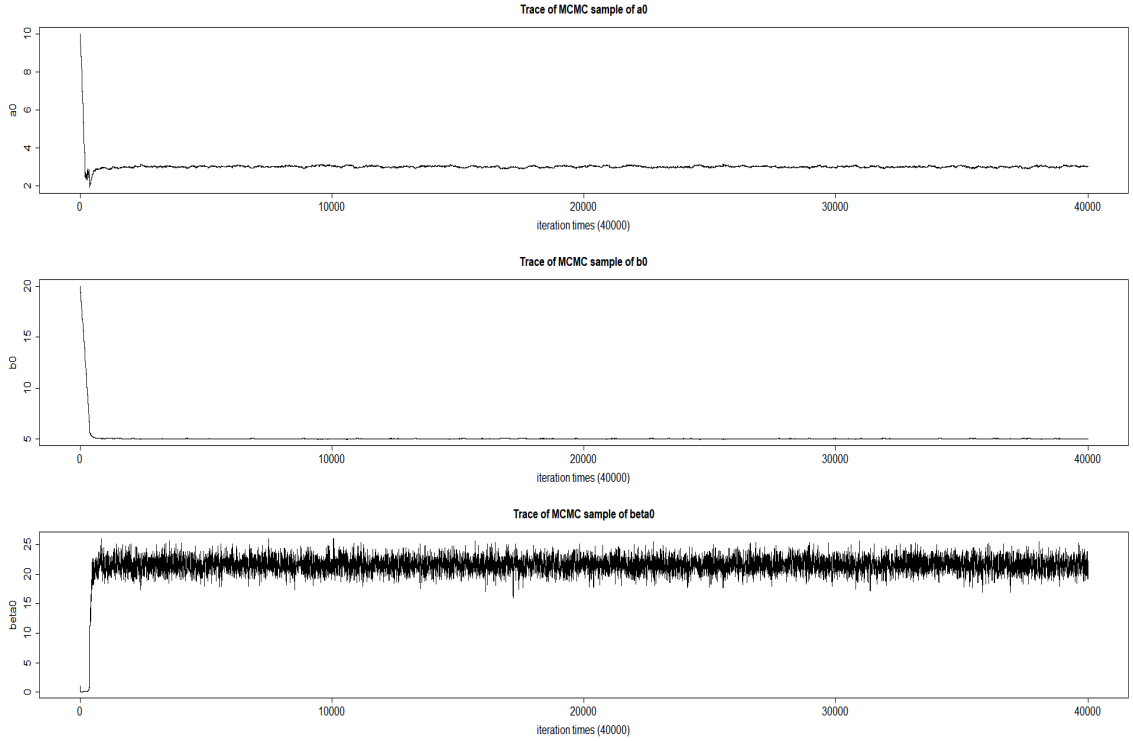
2.2.3 後驗樣本機率密度圖



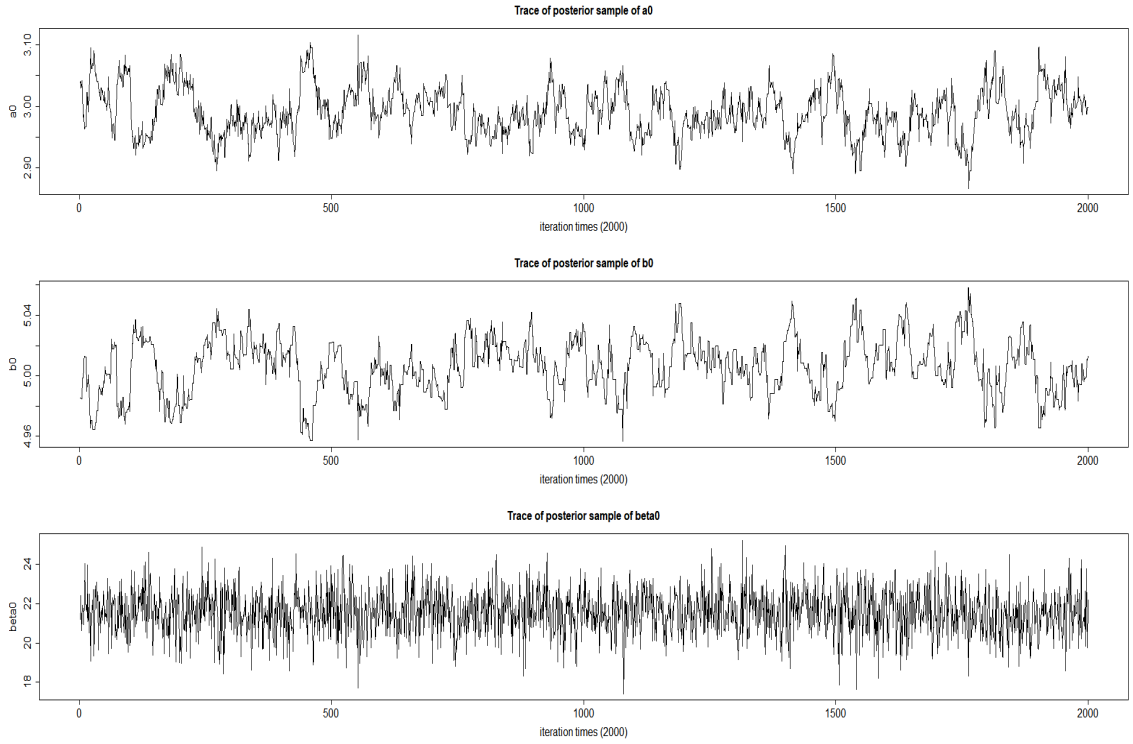
2.3 使用不同初始值(10,20,1)

	a_0	b_0	β_0
initial	10	20	1
proposal density:N(,sd=)	0.1	0.1	3
prior:N(mle,sd=)	0.1	0.1	2
Acceptance rate	0.154025	0.074925	0.371325
true value	3	5	20
mean	2.99095	5.00593	21.56118
mle	2.98998	5.00629	21.64919
sd	0.03874	0.01815	1.14212
sd of mle	0.04687	0.02192	1.44977
2.50%	2.91902	4.96995	19.27277
97.50%	3.06954	5.03909	23.79958

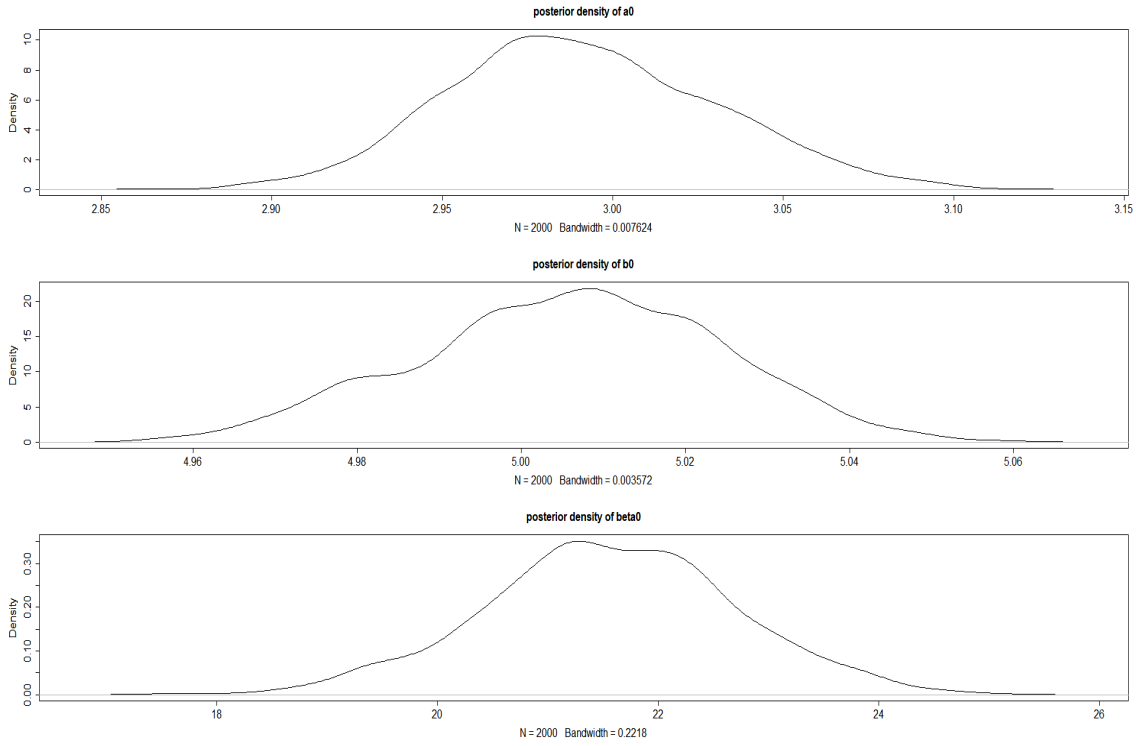
2.3.1 MCMC樣本路徑圖



2.3.2 後驗樣本路徑圖



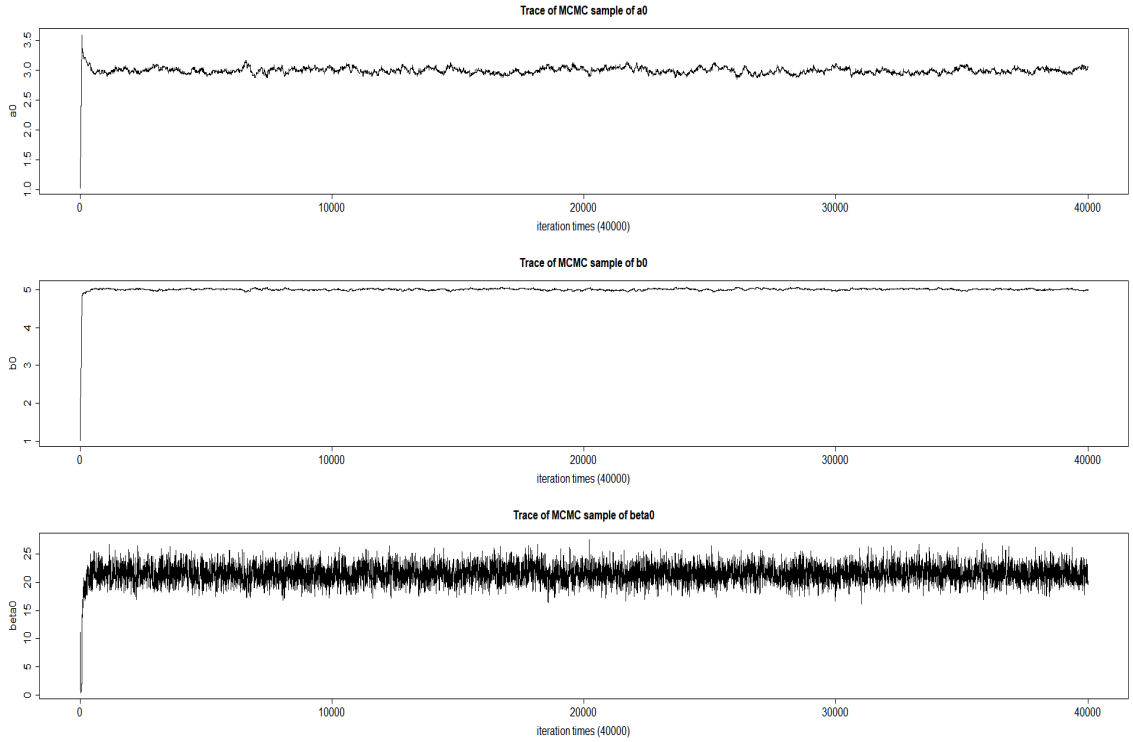
2.3.3 後驗樣本機率密度圖



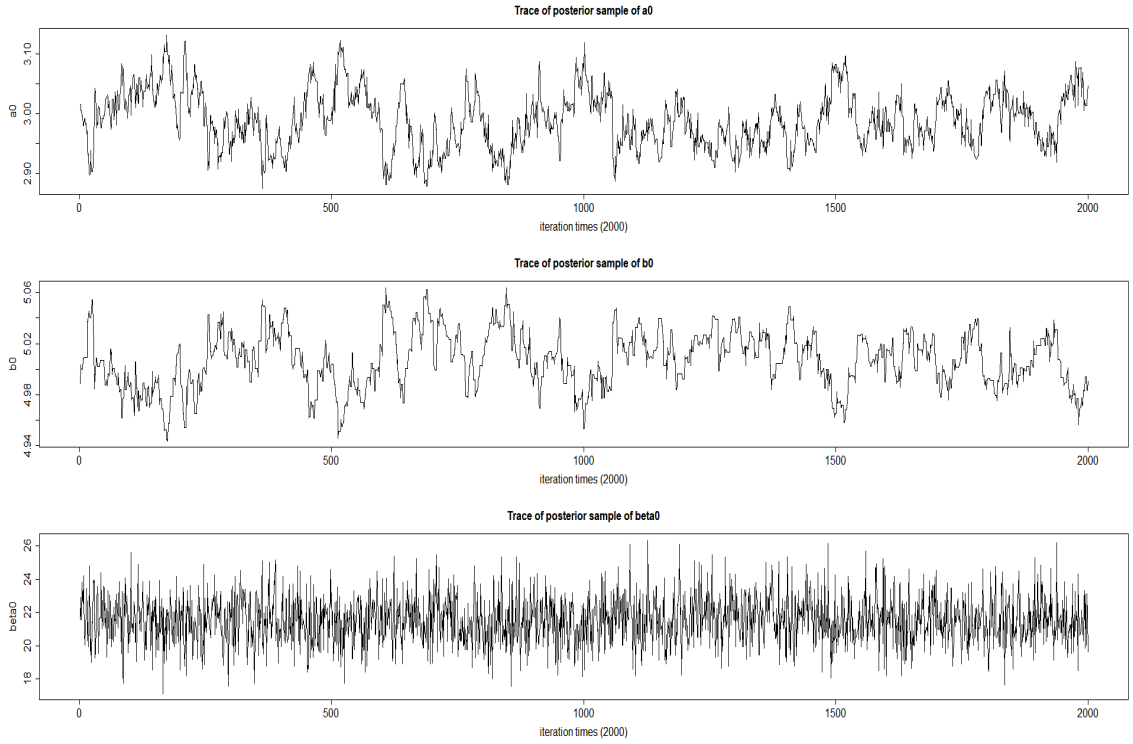
2.4 使用uniform prior

	a_0	b_0	β_0
initial	1	1	13
proposal density:N(,sd=)	0.1	0.1	3
Acceptance rate	0.151075	0.073175	0.417125
true value	3	5	20
mean	2.988644	5.007202	21.537052
mle	2.98998	5.00629	21.64919
sd	0.04679	0.02175	1.42992
sd of mle	0.04687	0.02192	1.44977
2.50%	2.90292	4.96511	18.81834
97.50%	3.08248	5.04777	24.50029

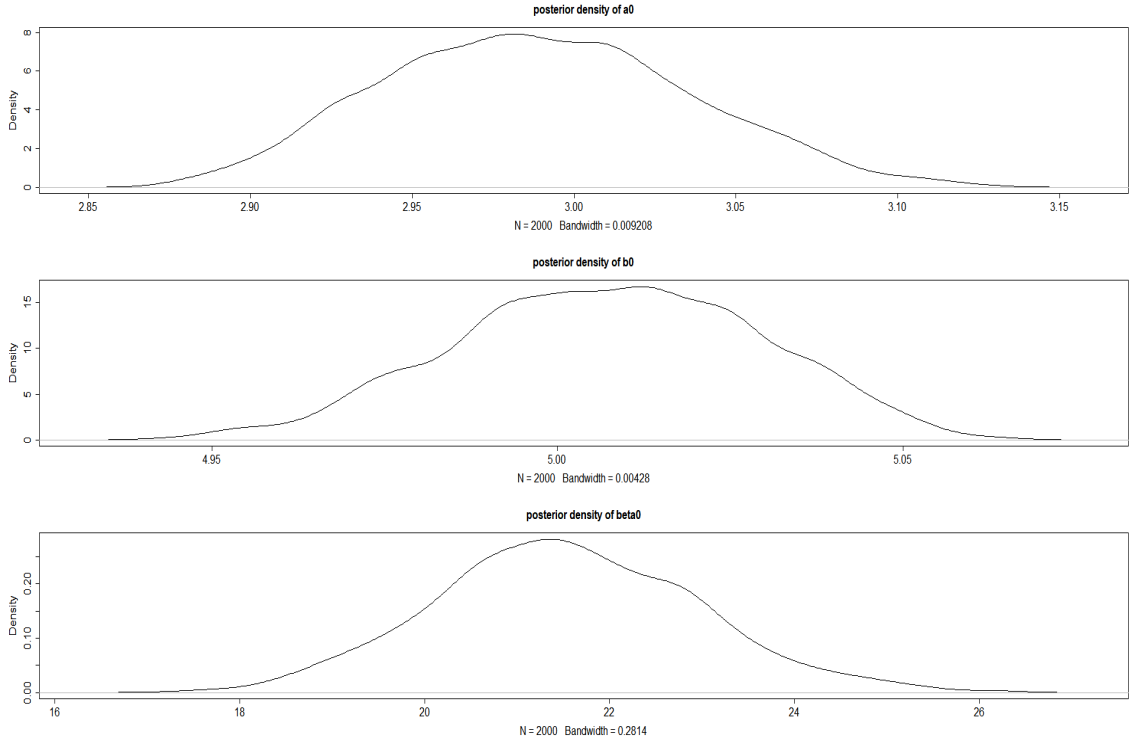
2.4.1 MCMC樣本路徑圖



2.4.2 後驗樣本路徑圖



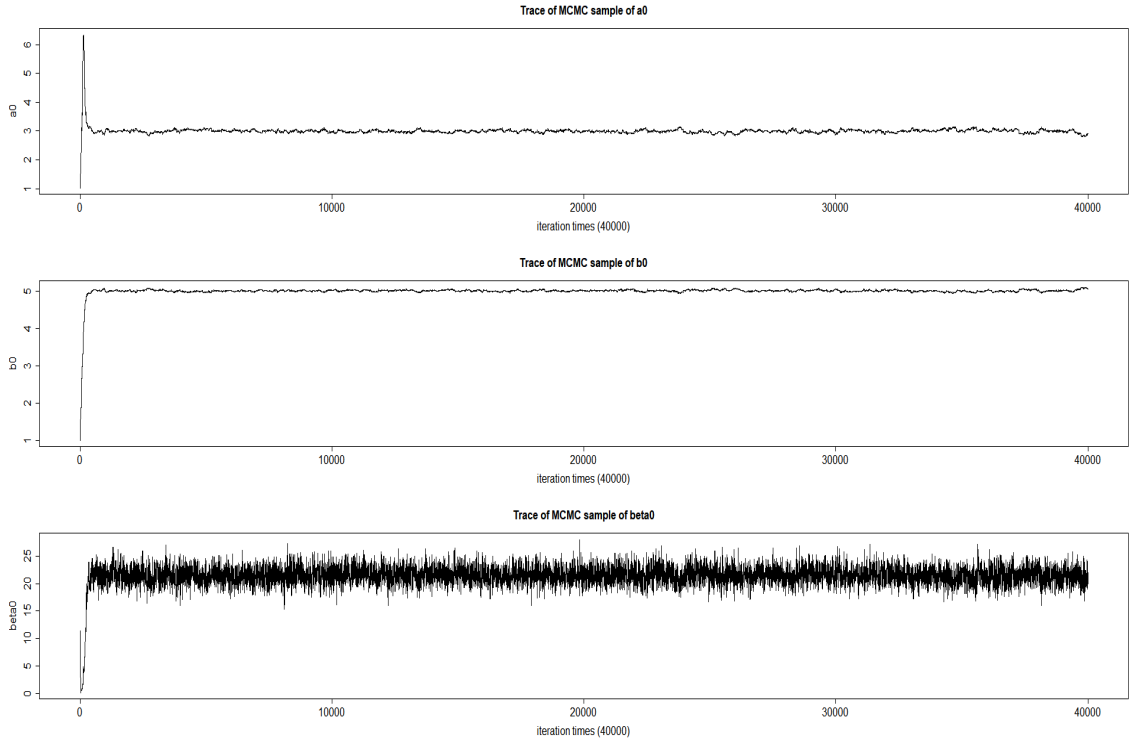
2.4.3 後驗樣本機率密度圖



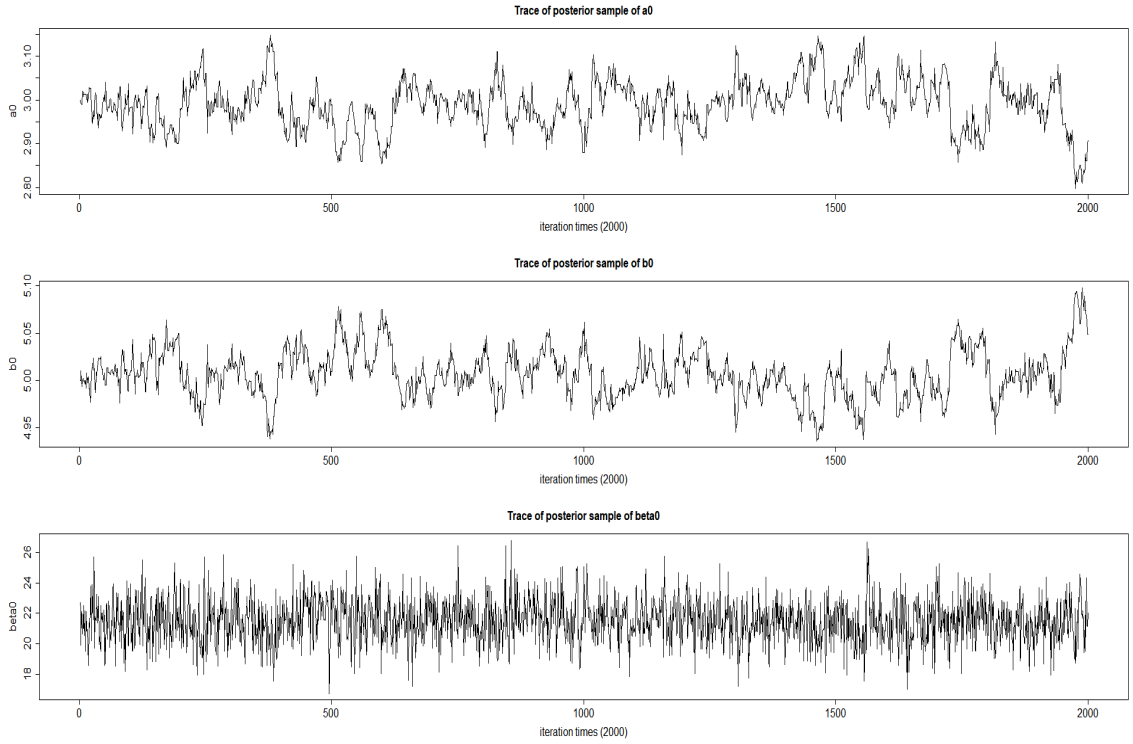
2.5 使用先驗分配為均勻分配並調整建議分布標準差

	a_0	b_0	β_0
initial	1	1	13
proposal density:N(,sd=)	0.1	0.05	2.5
Acceptance rate	0.1527	0.1459	0.474225
true value	3	5	20
mean	2.990312	5.00666	21.491087
mle	2.98998	5.00629	21.64919
sd	0.05556	0.02571	1.44990
sd of mle	0.04687	0.02192	1.44977
2.50%	2.87582	4.95769	18.63715
97.50%	3.10544	5.06058	24.33514

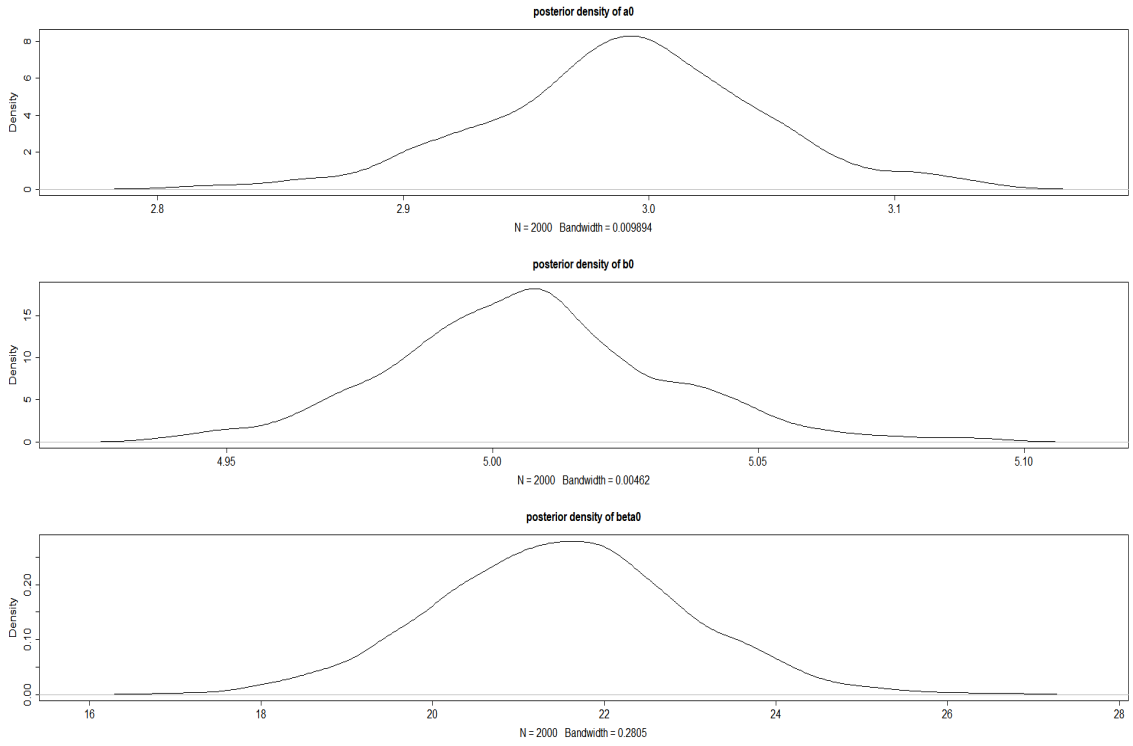
2.5.1 MCMC樣本路徑圖



2.5.2 後驗樣本路徑圖



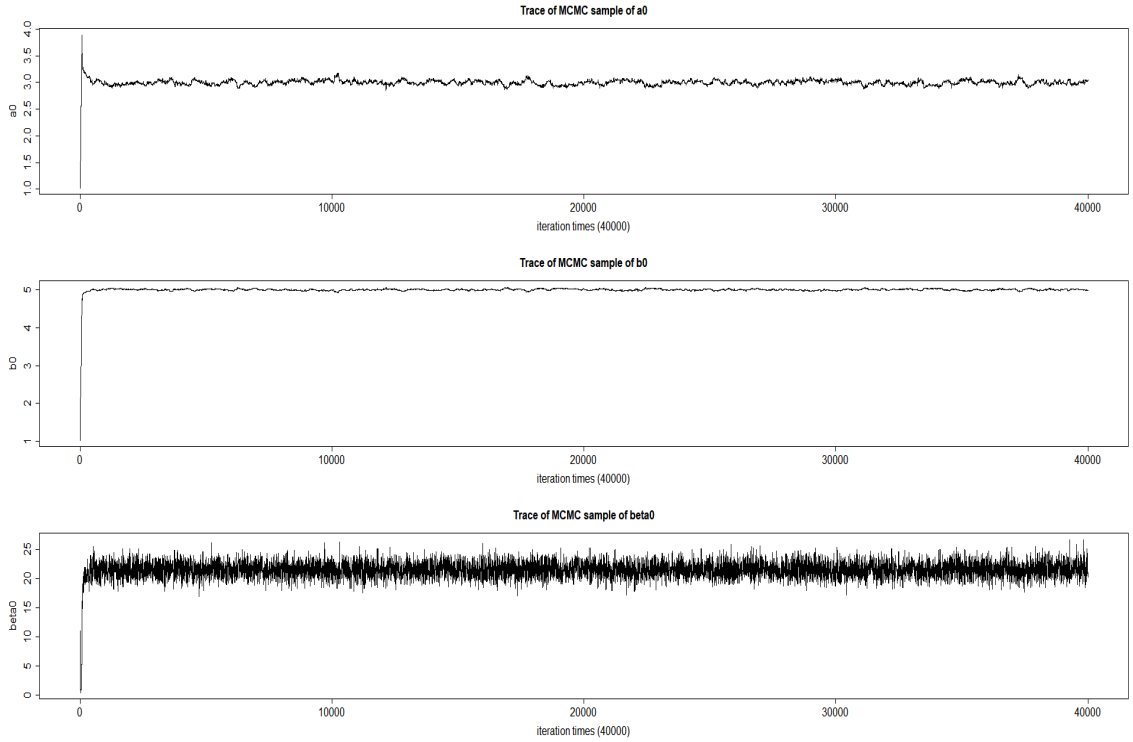
2.5.3 後驗樣本機率密度圖



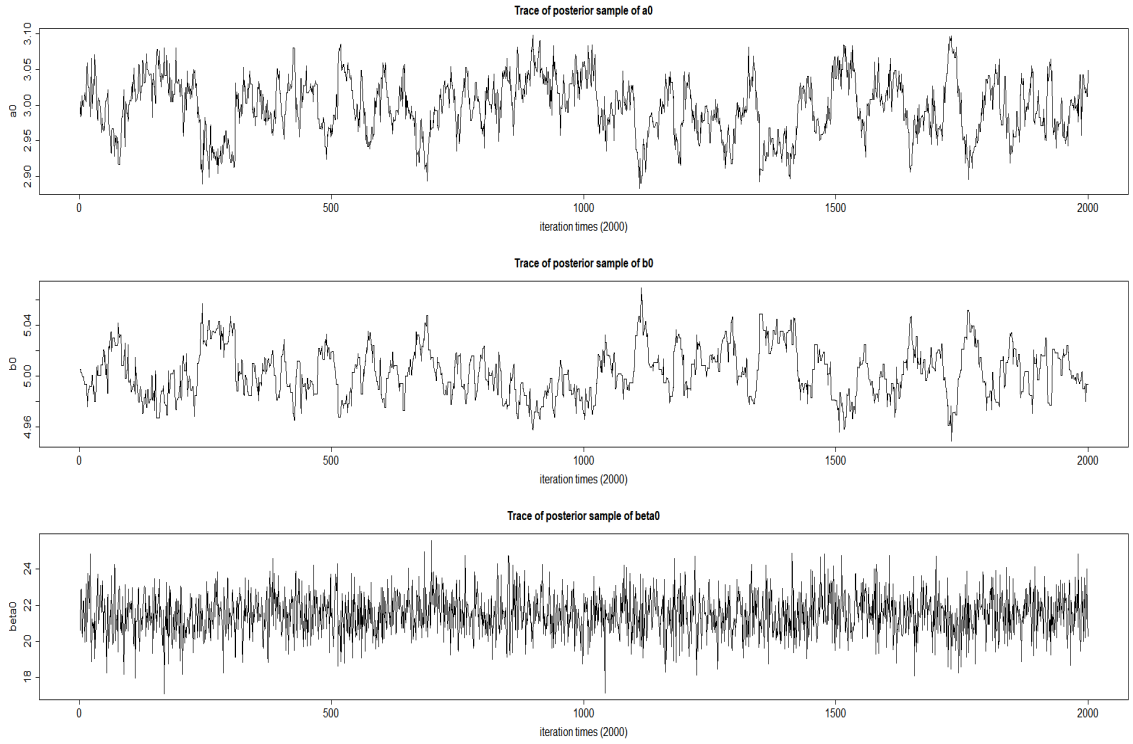
2.6 使用先驗分配為伽瑪分配

	a_0	b_0	β_0
initial	1	1	13
proposal density:N(,sd=)	0.1	0.1	3
prior:Gamma(alpha=,)	893.9956	2506.296	117.17184
prior:Gamma(,beta=)	0.003345	0.001997	0.1847644
prior:Gamma_variance	0.01	0.01	4
Acceptance rate	0.14985	0.07228	0.37178
true value	3	5	20
mean	2.99750	5.00294	21.50872
mle	2.98998	5.00629	21.64919
sd	0.04031	0.01916	1.18154
sd of mle	0.04687	0.02192	1.44977
2.50%	2.91792	4.96958	19.25499
97.50%	3.07263	5.04178	23.83358

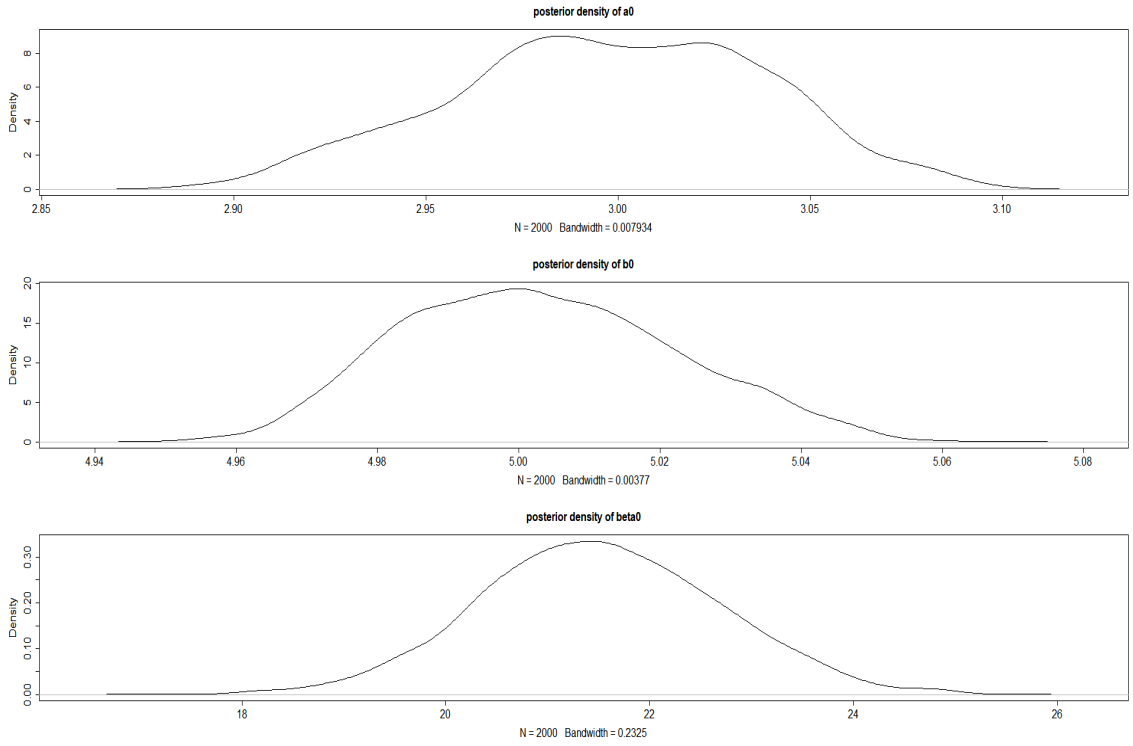
2.6.1 MCMC樣本路徑圖



2.6.2 後驗樣本路徑圖



2.6.3 後驗樣本機率密度圖



3 結論

- 上周對9個樣本的初步分析結果，可以看出隨著應力的上升，β的標準差有變小的趨勢；以及對於trend function的參數a_0, b_0的標準差與MSE相比於分配的參數β_0皆較小。
- 這周我先估計正常應力下的3個參數，並使用常態、均勻、伽瑪分配當作先驗分配去跑H-M演算法，從結果來看，不論哪一個先驗分配，其後驗樣本平均值皆與mle值很靠近。