A Bayesian Model for Brain Network Functional Connectivity using PyMC3

Rui Wang
M.S. Defense Presentation

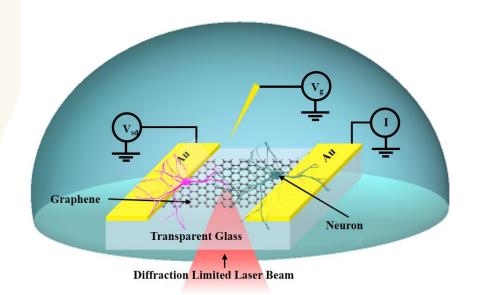
Department of Biostatistics

- Introduction
- Methods
 - Spatiotemporal Structure
 - Hierarchical Structure
 - Double Fusion
 - Prior Distribution
 - PyMC3 and NUTS
 - Optimization and Decomposition
- Simulation and Case Study

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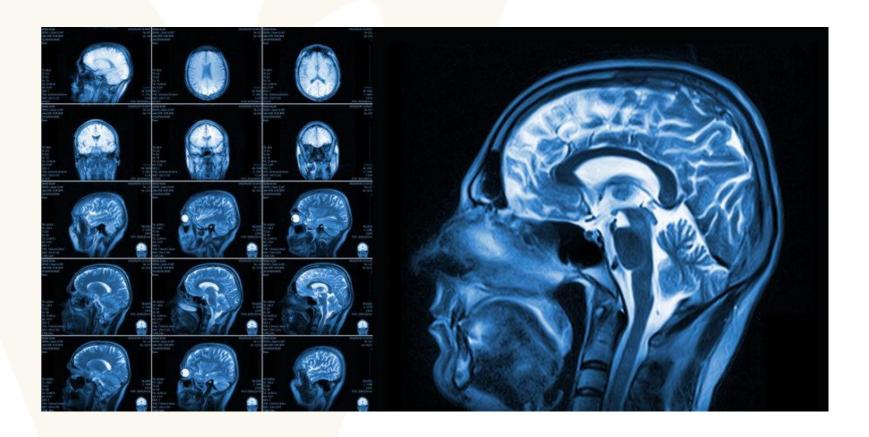
Previous Study

Precise Timing
High Electrical Sensitivity
High Throughput
High Spatial Accuracy
Long-term Duration

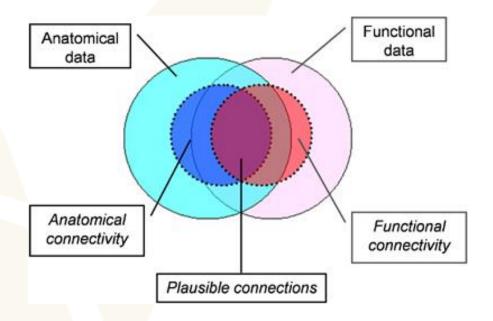


Wang, R et. al, *Nano Letter* (in review)

Brain Imaging

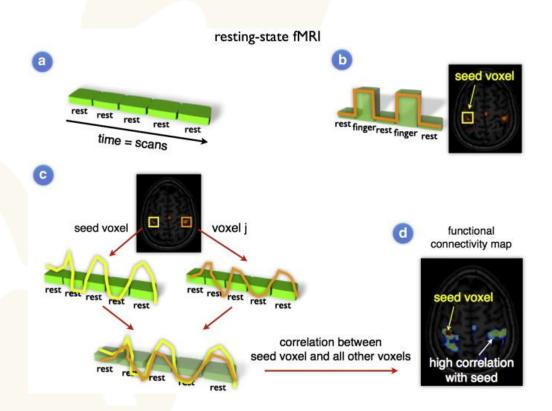


Brain Imaging



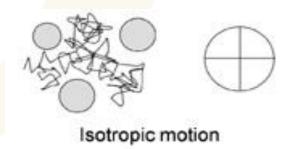
Rykhlevskaia et. al, *Psychophysiology* 45, 2 (2008)

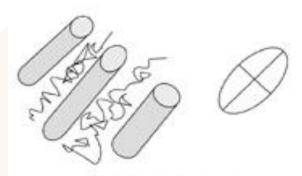
Functional Connectivity



van den Heuvel et. al, European Neuropsychopharmacology 20, 8 (2008)

Structural Connectivity





Anisotropic motion

Rykhlevskaia et. al, *Psychophysiology* 45, 2 (2008)



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Spatiotemporal Structure

$$Y_{cv}(t) = \beta_c + b_c(v) + d_c + \epsilon_{cv}(t)$$

- β_c : the grand mean
- $b_c(v)$: the zero-mean voxel-specific random effect
 - Local spatial dependency:

$$Cov(b_c(v), b_c(v')) = K_c(||v - v'||)$$

- d_c : the zero-mean ROI-specific random effect
- $\epsilon_{cv}(t)$: the noise
 - AR (1) temporal structure

Kernel Covariance Function

Constant	K(x,x')=c
Linear	$K(x,x') = x^T x'$
Gaussian noise	$K(x,x') = \sigma^2 \delta_{x,x'}$
Squared exponential	$K(x, x') = \exp(-\frac{\ x - x'\ ^2}{2l^2})$
Exponential	$K(x,x') = \exp(-\frac{\ x - x'\ }{l})$
Matérn	$K(x, x') = \frac{2^{1-v}}{\Gamma(v)} \left(\frac{\sqrt{2v} x - x' }{l} \right)^{v} B_{v} \left(\frac{\sqrt{2v} x - x' }{l} \right)$
Periodic	$K(x,x') = \exp\left(-\frac{2\sin^2(\frac{x-x'}{2})}{l^2}\right)$
Rational quadratic	$K(x, x') = (1 + x - x' ^2)^{-\alpha}, \alpha \ge 0$

Kernel Covariance Function

$$r = \|v - v'\|\varphi_c$$

Exponential (Matérn1/2):

$$\sigma_{b_c}^2 \exp(-r)$$

Gaussian or square exponential (Matérn∞):

$$\sigma_{b_c}^2 \exp(-\frac{1}{2}r^2)$$

Matérn5/2:

$$\sigma_{b_c}^2 (1 + \sqrt{5}r + \frac{5}{3}r^2) \exp(-\sqrt{5}r)$$

Matérn3/2:

$$\sigma_{b_c}^2(1+\sqrt{3}r)\exp(-\sqrt{3}r)$$

Temporal Correlation

AR (1) structure:

$$\epsilon_{cv}(t) = \delta_c + \phi_{cv} \epsilon_{cv}(t-1) + w(t)$$

- δ_c : the constant shift
- ϕ_{cv} : the coefficient with $|\phi_{cv}| < 1$
- w(t): the Gaussian random noise

$$E[\epsilon_{cv}(t)] = \frac{\delta_c}{1 - \phi_{cv}}$$

$$Var[\epsilon_{cv}(t)] = \frac{\sigma_{cv}^2}{1 - \phi_{cv}^2}$$

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Hierarchical Structure

$$Y_c(t) = \boldsymbol{\beta}_c + \boldsymbol{b}_c + \boldsymbol{d}_c + \boldsymbol{\epsilon}_c(t)$$

- $Y_c(t) = [Y_{c1}(t), Y_{c2}(t), ..., Y_{cV}(t)]^T$
- $\boldsymbol{\beta}_c = \beta_c \boldsymbol{J}_{(1 \times V)}$
- $b_c = [b_{c1}, b_{c2}, ..., b_{cV}]^T$
- $d_c = d_c J_{(1 \times V)}$
- $\epsilon_c(t) = [\epsilon_{c1}(t), \epsilon_{c2}(t), ..., \epsilon_{cV}(t)]^T$

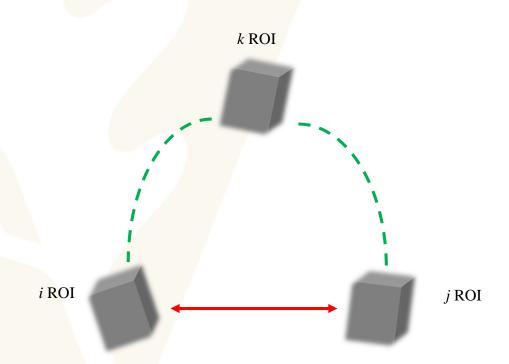
Hierarchical Structure

$$Y_c(t) = \boldsymbol{\beta}_c + \boldsymbol{b}_c + \boldsymbol{d}_c + \boldsymbol{\epsilon}_c(t)$$

- $\beta_c \sim N(0, \sigma_{\beta_c}^2)$
- $\boldsymbol{b}_c \sim N(0, \Sigma_{b_c})$
- $d_c \sim N(0, \Sigma_d)$
- $\epsilon_{cv}(t) \sim N(\frac{\delta_c}{1-\phi_{cv}}, \frac{\sigma_{cv}^2}{1-\phi_{cv}^2})$

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Double Fusion



Double Fusion

$$L_d(direct) = \lambda L_{sc} + (1 - \lambda) L_{nfc}$$

$$L_d(indirect) = M_{sc}\lambda L_{sc} + (1 - M_{sc}\lambda)L_{nfc}$$

$$L_d = \omega L_d(direct) + (1 - \omega)L_d(indirect)$$

$$\Sigma_d = L_d \times L_d^T$$

20

Double Fusion

$$\rho_d = \begin{pmatrix} 1 & \rho_{12} & \dots & \rho_{1n} \\ & 1 & \ddots & \vdots \\ & & 1 & \rho_{(n-1)n} \\ & & & 1 \end{pmatrix}_{n \times n}$$

$$[\rho_{12}, \dots, \rho_{1n}, \rho_{23}, \dots, \rho_{2n}, \dots, \rho_{(n-1)n}]_{n_vec}$$

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$$Y_{cv}(t) = \beta_c + b_c(v) + d_c + \epsilon_{cv}(t)$$

• $\beta_c \sim N(0, 100^2)$

$$Y_{cv}(t) = \beta_c + b_c(v) + d_c + \epsilon_{cv}(t)$$

$$Cov(b_c(v), b_c(v')) = \sigma_{b_c}^2 \exp(-\|v - v'\|\varphi_c)$$

- $\varphi_c \sim \text{Unif}(0,20)$
- σ_{b_c} ~ Unif (0, 100)

$$Y_{cv}(t) = \beta_c + b_c(v) + d_c + \epsilon_{cv}(t)$$

- $\lambda \sim \text{Beta}(1,1)$
- $\omega \sim \text{Beta}(1,1)$
- $log \sigma_{d_c} \sim Unif(-8, 8)$

$$Y_{cv}(t) = \beta_c + b_c(v) + d_c + \epsilon_{cv}(t)$$

$$\epsilon_{cv}(t) = \delta_c + \phi_{cv} \epsilon_{cv}(t-1) + w(t)\varphi_c$$

- $\phi_{cv} \sim \text{Unif}(0,1)$
- $\sigma_{cv} \sim \text{Unif}(0, 100)$

$$Y_{obs} \sim N(Y_{cv}, \sigma^2)$$

• $\sigma \sim \text{Unif}(0, 100)$

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PyMC3 and NUTS

$$Y \sim N(\mu, \sigma^2)$$

$$\mu = \alpha + \beta_1 X_1 + \beta_2 X_2$$

- $\alpha \sim N(0, 100)$
- β_1 or $\beta_2 \sim N(0, 20)$
- $\sigma \sim \text{HalfNormal}(0, 1)$

PyMC3 and NUTS

```
import pymc3 as pm
with pm.Model() as basic_model:
    # Priors for unknown model parameters
    alpha = pm.Normal('alpha', mu=0, sd=100)
    beta = pm.Normal('beta', mu=0, sd=20, shape=2)
    sigma = pm.HalfNormal('sigma', sd=1)
    # Expected value of outcome
    mu = alpha + beta[0]*X1 + beta[1]*X2
    # Likelihood (sampling distribution) of observations
    Y obs = pm.Normal('Y obs', mu=mu, sd=sigma, observed=Y)
with basic_model:
    # instatiate sampler
    step = pm.NUTS()
    # draw 1000 posterior samples and tune 500 as default
    trace = pm.sample(1000, step = step)
```

Model Diagnostics

Gelman Rubin statistics:

$$\hat{R} = \frac{\hat{V}}{W}$$

Effective sample size:

$$\hat{n}_{eff} = \frac{mn}{1 + 2\sum_{t=1}^{T} \hat{\rho}_t}$$

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Optimization and Decomposition

- Vectorization
- Cholesky decomposition

$$X \sim N(\mu, \Sigma)$$

$$\Sigma = U^T U$$

$$X = \mu + U^T Z, Z \sim N(0, 1)$$

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Simulation Study

- Generate time-series data with a length of T = 128 scans using AR (1) (coefficient: 0.6) at 5 ROIs and each ROI contains 100 voxels
- Imposed correlation using a multivariate normal distribution

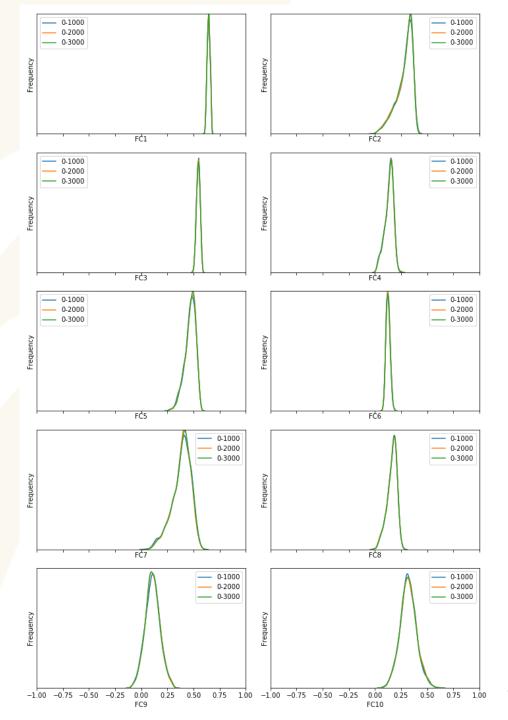
$$\rho_d = \begin{pmatrix} 1 & 0.6 & 0 & 0.5 & 0 \\ & 1 & 0.2 & 0.1 & 0 \\ & & 1 & 0 & 0.1 \\ & & & 1 \end{pmatrix}$$

• $SC \sim W_p(6, \rho_d)$

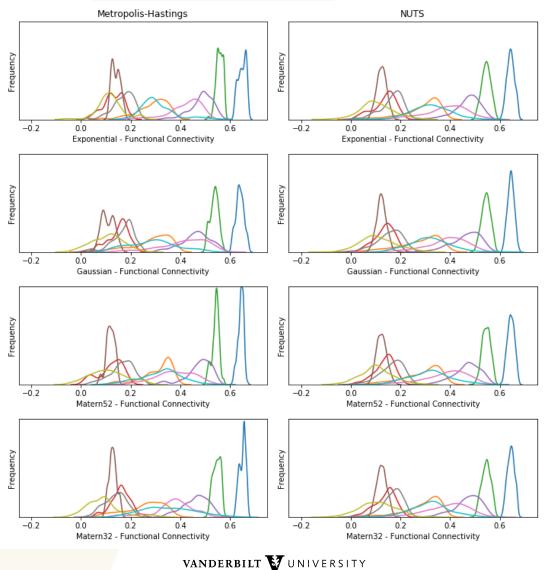
Simulation Study

		Bayesian correct SC				Bayesian independence				
FC	Correct	Median (SD)	[2.5% 97.5%]	R	\widehat{n}_{eff}	Median (SD)	[2.5% 97.5%]	R	\widehat{n}_{eff}	
$ ho_1$	0.6	0.645 (0.014)	[0.617 0.669]	0.999	1675.438	0.531 (0.125)	[0.170 0.652]	1.002	979.951	
ρ_2	0.0	0.310 (0.080)	[0.074 0.378]	0.999	1110.936	0.305 (0.078)	[0.091 0.379]	0.999	1303.385	
ρ_3	0.5	0.546 (0.017)	[0.513 0.576]	0.999	1877.821	0.463 (0.108)	[0.151 0.572]	0.999	1033.009	
$ ho_4$	0.0	0.145 (0.042)	[0.039 0.204]	0.999	1341.864	0.146 (0.042)	[0.040 0.204]	0.999	1034.929	
$ ho_5$	0.2	0.478 (0.053)	[0.340 0.547]	0.999	1152.252	0.432 (0.090)	[0.197 0.537]	1.001	1285.429	
ρ_6	0.1	0.123 (0.020)	[0.088 0.165]	1.000	1721.880	0.017 (0.106)	[-0.194 0.216]	1.000	1218.262	
ρ_7	0.0	0.399 (0.083)	[0.180 0.522]	0.999	1102.607	0.419 (0.098)	[0.174 0.558]	0.999	1381.804	
ρ_8	0.0	0.173 (0.049)	[0.047 0.235]	0.999	1293.770	0.158 (0.068)	[0.016 0.283]	1.000	1251.840	
ρ_9	0.1	0.105 (0.070)	[-0.026 0.254]	0.999	1417.283	0.057 (0.063)	[-0.066 0.177]	1.000	1458.304	
ρ_{10}	0.2	0.316 (0.077)	[0.165 0.484]	1.000	1524.805	0.348 (0.118)	[0.054 0.532]	0.999	1095.445	

Convergence



Bayesian Correct SC



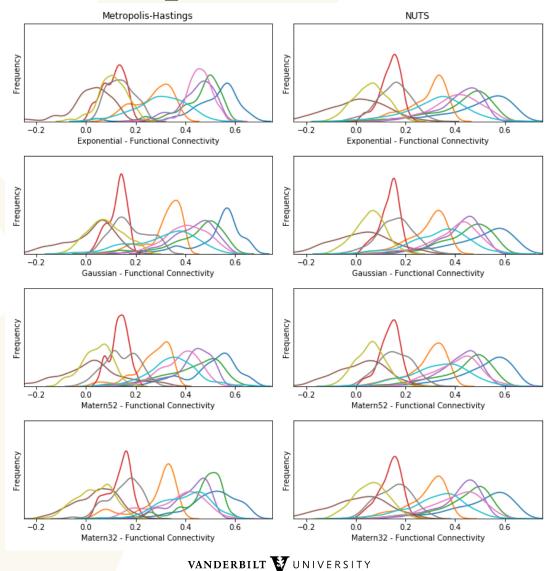
Bayesian Correct SC

	Metropolis-Hastings Median (SD)					NUTS Median (SD)			
FC	Correct	Exponential	Gaussian	Matérn52	Matérn32	Exponential	Gaussian	Matérn52	Matérn32
ρ_1	0.6	0.651(0.016)	0.640(0.016)	0.645(0.016)	0.654(0.016)	0.651(0.015)	0.640(0.015)	0.645(0.015)	0.654(0.015)
ρ_2	0.0	0.303(0.061)	0.305(0.061)	0.333(0.061)	0.273(0.061)	0.303(0.077)	0.305(0.077)	0.333(0.077)	0.273(0.077)
ρ_3	0.5	0.559(0.014)	0.540(0.014)	0.544(0.014)	0.548(0.014)	0.559(0.018)	0.540(0.018)	0.544(0.018)	0.548(0.018)
$ ho_4$	0.0	0.129(0.040)	0.160(0.040)	0.132(0.040)	0.162(0.040)	0.129(0.046)	0.160(0.046)	0.132(0.046)	0.162(0.046)
ρ_5	0.2	0.492(0.046)	0.469(0.046)	0.486(0.046)	0.470(0.046)	0.492(0.052)	0.469(0.052)	0.486(0.052)	0.470(0.052)
ρ_6	0.1	0.139(0.017)	0.107(0.017)	0.119(0.017)	0.128(0.017)	0.139(0.019)	0.107(0.019)	0.119(0.019)	0.128(0.019)
ρ_7	0.0	0.438(0.068)	0.430(0.068)	0.378(0.068)	0.379(0.068)	0.438(0.088)	0.430(0.088)	0.378(0.088)	0.379(0.088)
ρ_8	0.0	0.179(0.043)	0.182(0.043)	0.170(0.043)	0.142(0.043)	0.179(0.047)	0.182(0.047)	0.170(0.047)	0.142(0.047)
ρ_9	0.1	0.116(0.053)	0.111(0.053)	0.108(0.053)	0.086(0.053)	0.116(0.066)	0.111(0.066)	0.108(0.066)	0.086(0.066)
$ ho_{10}$	0.2	0.293(0.069)	0.293(0.069)	0.336(0.069)	0.350(0.069)	0.293(0.077)	0.293(0.077)	0.336(0.077)	0.350(0.077)

Bayesian Correct SC

	Metropolis-Hastings			NUTS				
MSE	Exponential	Gaussian	Matérn52	Matérn32	Exponential	Gaussian	Matérn52	Matérn32
Total FC	0.043	0.042	0.040	0.037	0.040	0.039	0.040	0.041
Zero FC	0.083	0.084	0.075	0.066	0.076	0.073	0.076	0.078
Low FC	0.024	0.020	0.025	0.024	0.023	0.022	0.023	0.024
High FC	0.003	0.002	0.002	0.003	0.002	0.002	0.002	0.002

Bayesian Independence



Bayesian Independence

	Metropolis-Hastin <mark>gs</mark> Median (SD)					NUTS Median (SD)			
FC	Correct	Exponential	Gaussian	Matérn52	Matérn32	Exponential	Gaussian	Matérn52	Matérn32
ρ_1	0.6	0.547(0.016)	0.56(0.016)	0.529(0.016)	0.517(0.016)	0.547(0.015)	0.560(0.015)	0.529(0.015)	0.517(0.015)
ρ_2	0.0	0.283(0.061)	0.339(0.061)	0.283(0.061)	0.317(0.061)	0.283(0.077)	0.339(0.077)	0.283(0.077)	0.317(0.077)
ρ_3	0.5	0.488(0.014)	0.481(0.014)	0.453(0.014)	0.493(0.014)	0.488(0.018)	0.481(0.018)	0.453(0.018)	0.493(0.018)
$ ho_4$	0.0	0.116(0.040)	0.134(0.040)	0.136(0.040)	0.145(0.040)	0.116(0.046)	0.134(0.046)	0.136(0.046)	0.145(0.046)
ρ_5	0.2	0.461(0.046)	0.449(0.046)	0.443(0.046)	0.434(0.046)	0.461(0.052)	0.449(0.052)	0.443(0.052)	0.434(0.052)
ρ_6	0.1	0.032(0.017)	0.047(0.017)	0.025(0.017)	0.053(0.017)	0.032(0.019)	0.047(0.019)	0.025(0.019)	0.053(0.019)
ρ_7	0.0	0.456(0.068)	0.407(0.068)	0.391(0.068)	0.391(0.068)	0.456(0.088)	0.407(0.088)	0.391(0.088)	0.391(0.088)
ρ_8	0.0	0.136(0.043)	0.162(0.043)	0.139(0.043)	0.167(0.043)	0.136(0.047)	0.162(0.047)	0.139(0.047)	0.167(0.047)
ρ_9	0.1	0.108(0.053)	0.079(0.053)	0.048(0.053)	0.042(0.053)	0.108(0.066)	0.079(0.066)	0.048(0.066)	0.042(0.066)
$ ho_{10}$	0.2	0.307(0.069)	0.339(0.069)	0.356(0.069)	0.411(0.069)	0.307(0.077)	0.339(0.077)	0.356(0.077)	0.411(0.077)

Bayesian Independence

	Metropolis-Hastings			NUTS				
MSE	Exponential	Gaussian	Matérn52	Matérn32	Exponential	Gaussian	Matérn52	Matérn32
Total FC	0.118	0.119	0.108	0.118	0.111	0.113	0.113	0.113
Zero FC	0.080	0.081	0.068	0.076	0.078	0.079	0.079	0.080
Low FC	0.080	0.081	0.081	0.090	0.077	0.080	0.077	0.078
High FC	0.268	0.272	0.242	0.255	0.247	0.251	0.255	0.250

Major Depressive Disorder

	Control (n=23)	MDD (n=18)	Wilcoxon Rank Sum Tests
	Mean (SD)	Mean (SD)	
Age(years)	31.78 (10.16)	32.06 (8.55)	t = -0.512, p = 0.608
Sex (% female)	65%	50%	t = 0.828, p = 0.408
Education (years)	15.78 (1.73)	16.28 (1.90)	t = -0.512, p = 0.608
Beck Depression Inventory (BDI)	1.90 (2.62)	22.11 (9.38)	t = -4.085, p < 0.001
Montgomery-Asberg	0.70 (1.06)	25.29 (3.20)	t = -5.438, p < 0.001
Depression Rating Scale (MADRS)	0.70 (1.00)	23.29 (3.20)	t = -5.456, p < 0.001
Processing Speed Domain	0.36 (0.67)	0.20 (0.61)	t = 0.841, p = 0.401
Working Memory Domain	0.10 (0.88)	0.02 (0.81)	t = 0.158, p = 0.875
Episodic Memory Domain	0.23 (0.55)	0.07 (0.75)	t = 0.578, p = 0.563
Executive Function Domain	0.20 (0.55)	0.23 (0.59)	t = -0.053, p = 0.958

Convergence

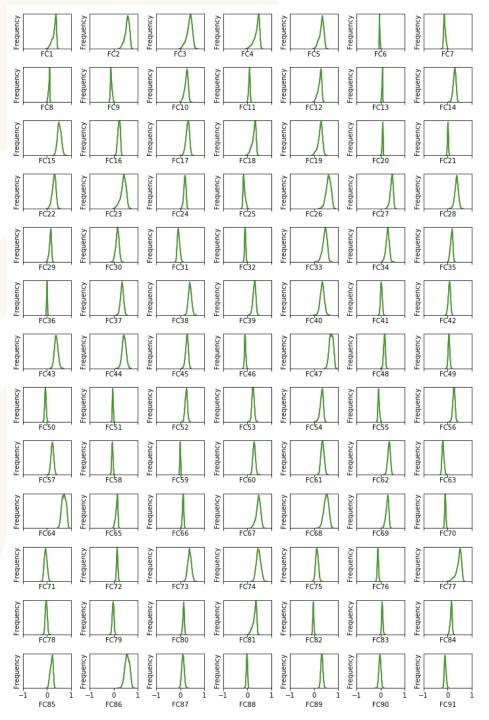
T = 150 time-series scans

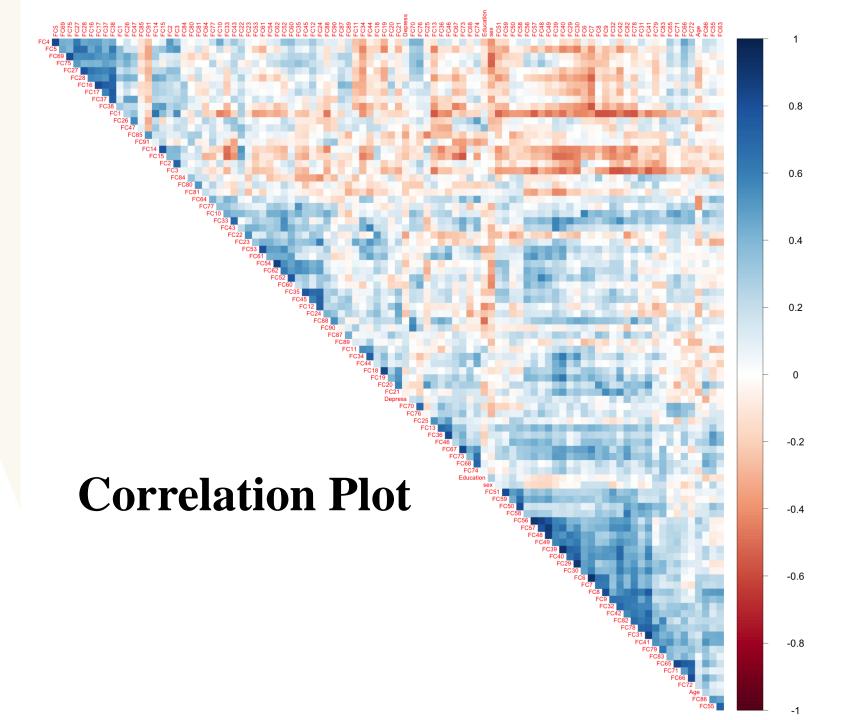
n = 14 ROIs

m = 300 voxels

For each subject:

~30 times computational costs





Case Study

- Cognitive domain
 - processing speed domain
 - working memory domain
 - episodic memory domain
 - executive function domain
- Correlation test under FDR (False Discovery Rate with 0.1 threshold):
 - For executive function domain, "FC80" in control group
- Cognitive domain \sim Age + Sex + Education + FC_i + Depress + FC_i * Depress
 - Interaction term FC_i * Depress: "FC80" in executive function domain



Variable Selection

	Processing Speed Domain	Working Memory Domain	Episodic Memory Domain	Executive Function Domain
Exhaustive	FC4, FC27, FC28,	FC26, FC29, FC62,	FC6, FC9, FC42,	FC6, FC7, FC26,
	FC48, FC57, FC69	FC64, FC69, FC71	FC57, FC78, FC79	FC27, FC50. FC79
Forward	FC10, FC20, FC29,	FC20, FC26, FC33,	FC6, FC9, FC11,	FC18, FC20, FC26,
	FC44, FC51, FC58	FC62, FC69, FC85	FC26, FC35, FC65	FC43, FC50, FC77
Backward	FC2, FC4, FC6, FC12,	FC7, FC18, FC19,	FC7, FC9, FC11,	FC3, FC6, FC8, FC11,
	FC20, FC24	FC25, FC26, FC33	FC22, FC29, FC30	FC13, FC26
Sequential	FC10, FC11, FC17,	FC26, FC29, FC62,	FC1, FC5, FC6, FC9,	FC18, FC20, FC26,
	FC28, FC29, FC78	FC64, FC69, FC71	FC11, FC35	FC43, FC50, FC77
Lasso	FC10, FC11, FC26,	FC20, FC26, FC62,	FC6, FC25, FC34,	FC11, FC26, FC58,
	FC29, FC70, FC85	FC64, FC69, FC84	FC57, FC77, FC86	FC70, FC77, FC85

Documentation

wangruinju minor change		Latest commit 304fe04 15 hours ago
■ accre	add examples	7 days ago
	add html	7 days ago
	minor change	15 hours ago
	minor change	15 hours ago
environment.yml	revise	7 days ago
i model.py	update	21 hours ago
slides.pdf	kernel function	21 hours ago

EE README.md

Double-Fusion

This repository documentation is used to explain the model in the papar by Kang, Hakmook, et al. "A bayesian double fusion model for resting-state brain connectivity using joint functional and structural data." Brain connectivity 7.4 (2017): 219-227.

Since GitHub doest not render the equation in Markdown, you can read the Readme in HTML or slides.

Introduction

Our brain network, as a complex integrative system, consists of many different regions that have each own task and function and simultaneously share structural and functional information. With the developed imaging techniques such as functional magnetic resonance imaging (fMRI) and diffusion tensor imaging (DTI), researchers can investigate the underlying brain functions related to human behaviors and some diseases or disorders in the nervous system such as major depressive disorder (MDD).

We developed a Bayesian hierarchical spatiotemporal model that combined fMRI and DTI data jointly to enhance the estimation of resting-state functional connectivity. Structural connectivity from DTI data was utilized to construct an informative prior for functional connectivity based on resting-state fMRI data through the Cholesky decomposition in a mixture model. The analysis took the advantages of probabilistic programming package as PyMC3 and next-generation Markov Chain Monte Carlo (MCMC) sampling algorithm as No-U-Turn Sampler (NUTS). PyMC3 is new, open-source

Future Work

- Other kernel covariance functions
- 200 ~ 300 subjects
 - Machine learning methods
 - MDD classification

Acknowledgement

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Dr. Hakmook Kang (Advisor)

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Dr. Jeffrey Blume

Dr. Christopher Fonnesbeck

Dr. Warren Taylor

Sandya Lakkur

David Schlueter

Ya-Chen Lisa Lin

and all the faulty and fellow students in the department

Thank you!