# 714 project-Time-Reversible Markov Chain

Wenxin Zu-U75249292

## 1. Time-Reversibility:

A Markov chain is in steady state if the initial state of an irreducible [1] positive recurrent Markov chains is chosen according to the stationary probabilities.
Consider a positive recurrent Markov chain $\{X_n : n \geq 0\}$ is in steady state, which have transition probabilities $P_{ij}$ and stationary probabilities $\pi_i$. If time is reversed, i.e., suppose that we trace the sequence of states going backwards in time at some time, a Markov chain of the same distribution with transition probabilities $P_{ij}^*$ is produced. Which means, we consider a sequence of state starting at time n, $X_n, X_{n-1}, \ldots$.

Where, $P_{ij}^* = P\{X_m = j \mid X_{m+1} = i\}$

$$= P\{X_{m+1} = i \mid X_m = j\} P\{X_m = j\} \;/\; P\{X_{m+1} = i\}$$

$$= \frac{\pi_j P_{ji}}{\pi_i}$$

Since the Markov property is "given the current state, the future is independent of the past". It can be redefined as "given the current state, the past is independent of the future". We think a Markov chain $\{X_n : n \geq 0\}$, which present time as being time $m+1$. Then the present state is $X_{m+1}$, the past state $X_m$ and the future states $X_{m+2}, X_{m+3}, \ldots$ are independent. Which means:

$$P\{X_m = j \mid X_{m+1} = i, X_{m+2}, X_{m+3}, \ldots\} = P\{X_m = j \mid X_{m+1} = i\}$$

That proves the reversed process is a Markov chain with transition probabilities

$$P_{ij}^* = \frac{\pi_j P_{ji}}{\pi_i}$$

If $P_{ij}^* = P_{ij}$ for all $i, j$, then the Markov chain is said to be time reversible with condition:

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i, j \tag{1}$$

If there are some nonnegative numbers $x_i$, where $\sum_i x_i = 1$ and satisfies (1). Then it is a time reversible Markov chain with stationary probabilities $x_i$, which means:

$$x_i P_{ij} = x_j P_{ji} \quad \text{for all } i, j. \quad \sum_i x_i = 1 \tag{2}$$

Summing over $i$:

$$\sum_i x_i P_{ij} = x_j \sum_i P_{ji} = x_j, \quad \sum_i x_i = 1$$

Since the stationary probabilities $\pi_i$ is the only solution above, then $x_i = \pi_i$ *for all* $i$.
For any other Markov chain, there are no solutions for equation (2).
From (2):

$$x_i P_{ij} = x_j P_{ji}$$

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[1] Irreducible: A Markov chain is irreducible if in the transition graph there exists a path from every state to every other state, i.e., we wouldn't be stuck in a small group of nodes.

$$x_k P_{kj} = x_j P_{jk}$$

If $P_{ij}P_{jk} > 0$, then:

$$\frac{x_i}{x_k} = \frac{P_{ji}P_{kj}}{P_{ij}P_{jk}}$$

There exists a necessary condition for time reversibility:

$$P_{ik}P_{kj}P_{ji} = P_{ij}P_{jk}P_{ki} \qquad \text{for all } \ i,j,k \tag{3}$$

From the time reversibility, the rate of a sequence from $i\ to\ k\ to\ j\ to\ i$ must equal the rate of a sequence from $i\ to\ j\ to\ k\ to\ i$. So, we have:

$$\pi_i P_{ik}P_{kj}P_{ji} = \pi_i P_{ij}P_{jk}P_{ki}$$

We can conclude that:

$$P_{i,i_1}P_{i_1,i_2} \dots P_{i_{k,i}} = P_{i,i_k}P_{i_k,i_{k-1}} \dots P_{i_1,i} \quad \text{for all states } i, i_1, \dots, i_k \tag{4}$$

## 2. A Tandem Queue Model

Consider a Markov chain with state $(X_n, Y_n)$ at time $n$. Suppose $X_n = i \geq 0$. Then a new arrival came in with probability $p$, resulting in $X_{n+1} = i + 1$. Suppose any new arrival came in, which is placed onto the top of $X_n$. If there were no new arrival in a period and $i > 0$, then it will be processing the top of $X_n$ with probability α, resulting in $X_{n+1} = i - 1$. Thus:

If $X_n = i > 0$,

$$X_{n+1} = \begin{cases} i+1 & \text{with probability } p \\ i & \text{with probability } (1-p)(1-\alpha) \\ i-1 & \text{with probability } (1-p)\alpha \end{cases}$$

If $X_n = 0$,

$$X_{n+1} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

Assume $Y_n$ operates same as $X_n$, but $Y_n$ gets new arrival from $X_n$ completed one with service-completion probability β.

The process $\{X_n\}$ is a birth-death chain with downward transition probability $q_\alpha$, where $q_\alpha = (1-p)\alpha$, and a stationary distribution $\pi_\alpha$ exists if $p < q_\alpha$. Since we have $\pi_\alpha(i)p = \pi_\alpha(i+1)q_\alpha$ or $\frac{\pi_\alpha(i+1)}{\pi_\alpha(i)} = \frac{p}{q_\alpha}$, then we can get:

$$\pi_{\alpha(i)} = \left(\frac{p}{q_\alpha}\right)^i \left(1 - \frac{p}{q_\alpha}\right) \quad \text{for } i = 0,1,\dots$$

The process $\{Y_n\}$ is also a birth-death chain with downward transition probability $q_\beta$, where $q_\beta = (1-p)\beta$, and a stationary distribution $\pi_\beta$ exists if $p < q_\beta$, where,

$$\pi_\beta(i) = \left(\frac{p}{q_\beta}\right)^i \left(1 - \frac{p}{q_\beta}\right) \quad \text{for } i = 0,1,\dots$$

Now assume $X_0 \sim \pi_\alpha$. Since $\{X_n\}$ is a stationary birth-death process, which is time-reversible. Define $A_n = \begin{cases} 1, & if\ X_n\ has\ an\ arrival\ at\ time\ n \\ 0, & otherwise \end{cases}$, that is $A_n = 1$ occurs when $X_n = X_{n-1} + 1$. Define $D_n = \begin{cases} 1, & if\ X_n\ has\ a\ departure\ at\ time\ n \\ 0, & otherwise \end{cases}$, that is $D_n = 1$ occurs when $X_n = X_{n-1} - 1$. Consider $k$ is the present time, the present queue size $X_k$ is independent of the future arrivals $A_{k+1}, A_{k+2}, \ldots$. In the reversed process, if $k$ represents the present time, then the future arrivals correspond to the departures $D_k, D_{k-1}, \ldots$. Therefore, the departures $(D_1, D_2, \ldots, D_k)$ are independent of the queue size $X_k$. According to the reversibility, arrivals in the reversed process have the same probabilistic behavior as arrivals in the forward process, so the departures $D_1, D_2, \ldots$ are *i.i.d.* Bernoulli($p$). Which means, the output process of $\{X_n\}$ is the same probabilistically as the input process of its queue.

Assume that, at each time, if there is work in $\{X_n\}$ and not received a new arrival then denote that there is a coin and flip it get a head with probability α. Same as $\{Y_n\}$ with probability β.

Since we know that $\{X_n\}$ is independent of the departures till time *n*, which is the arrivals till time *n* in $\{Y_n\}$. Then we have $\{X_n\}$ is independent of $\{Y_n\}$.

Assuming that $(X_0, Y_0) \sim \pi$. Since $\pi_\alpha$ and $\pi_\beta$are stationary for $\{X_n\}$ and $\{Y_n\}$, then we have $X_1 \sim \pi_\alpha$ and $Y_1 \sim \pi_\beta$.

Let $A_k^X$ denote the indicator of an arrival to $\{X_n\}$ at time $k$, and

$$S_k^X = \begin{cases} 1, if \text{ the service completion coin flip and get a head at time } k \\ 0, \quad otherwise \end{cases}.$$

Let $A_k^Y$ denote the indicator of an arrival to $\{Y_n\}$ at time $k$, and

$$S_k^Y = \begin{cases} 1, if \text{ the service completion coin flip and get a head at time } k \\ 0, \quad otherwise \end{cases}$$

Assume the random variables $X_0, Y_0, A_1^X, S_1^X, A_1^Y$, and $S_1^Y$are independent. Then we write $X_1$ and $A_1^Y$ as function of $(X_0, A_1^X, S_1^X)$:

$$X_1 = \begin{cases} X_0 + 1 & \text{if } A_1^X = 1 \\ X_0 & \text{if } A_1^X = 0 \text{ and } S_1^X = 0 \\ X_0 - 1 & \text{if } A_1^X = 0 \text{ and } S_1^X = 1 \text{ and } X_0 > 0 \\ 0 & \text{if } A_1^X = 0 \text{ and } X_0 = 0 \end{cases}$$

$$A_1^Y = \begin{cases} 1 & \text{if } X_0 > 0 \text{ and } A_1^X = 0 \text{ and } S_1^X = 1 \\ 0 & otherwise \end{cases}$$

Since $Y_0$ is independent of $(X_1, A_1^Y)$ and $X_1$ is independent of $A_1^Y$, then random variables $Y_0$, $X_1$, $A_1^Y$ are independent. Since $S_1^Y$ is independent of $(Y_0, X_1, A_1^Y)$, then random variables $Y_0$, $X_1$, $A_1^Y$, and $S_1^Y$ are independent. Further, since $Y_1$ is a function of $(Y_0, A_1^Y, S_1^Y)$, then $X_1$ and $Y_1$ are independent.

Finally, we state that the property of stationary distribution of $\{(X_k, Y_k)\}$ with the stationary distribution π that, the two queue sizes $X_k\ and\ Y_k$ are independent, where $\pi(i,j) = \pi_\alpha(i)\pi_\beta(j)$.

## 3. Random Walk:

Consider a finite connected graph with $n \geq 2$ nodes and positive weights $w_{ij} = w_{ji} > 0$ for any pair of nodes $i, j$. Define a Markov chain random walk on the nodes of the graph with probability $P_{ij} = \frac{w_{ij}}{\sum_j w_{ij}}$, where $w_{ij} = 0$, if $(i, j)$ is not an edge of the graph. We can say this Markov chain is irreducible and time reversible, then the time-reversibility equations became: $\frac{\pi_i}{\sum_k w_{i,k}} = \frac{\pi_j}{\sum_k w_{j,k}}$ for each pair $i, j$, there exists a constant $C$: $\frac{\pi_i}{\sum_k w_{i,k}} = C$, for all $i$, or, $\pi_i = C \sum_k w_{i,k}$, for all $i$. Since $\sum_i \pi = 1$, then

$$C = \left[\sum_i \sum_k w_{i,k}\right]^{-1}$$

Then we can get the stationary probabilities:

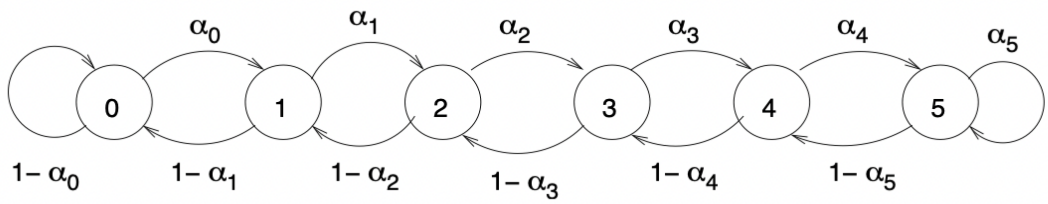
$$\pi_i = \frac{\sum_k w_{i,k}}{\sum_i \sum_k w_{i,k}}$$

## Examples:

(1) Consider a Markov chain with states $0,1, \dots, M$ and $P_{ij}$:

$$P_{i,i+1} = \alpha_i = 1 - P_{i,i+1}, \quad i = 1, \dots, M-1$$

$$P_{0,1} = \alpha_0 = 1 - P_{0,0}$$
$$P_{M,M} = \alpha_M = 1 - P_{M,M-1}$$




This is a time-reversible Markov chain, since between every 2 transitions from $i$ to $i+1$, there must be a transition from $i+1$ to $i$; and between every 2 transitions from $i+1$ to $i$, there must be a transition from $i$ to $i+1$. Then we can get $\pi_i P_{i,i+1} = \pi_{i+1} P_{i+1,i}$

Since:

$$\pi_0 \alpha_0 = \pi_1 (1 - \alpha_1)$$
$$\pi_1 \alpha_1 = \pi_2 (1 - \alpha_2)$$
$$\dots$$
$$\pi_i \alpha_i = \pi_{i+1} (1 - \alpha_{i+1}), \qquad i = 0,1, \dots, M-1$$

In terms of $\pi_0$:

$$\pi_1 = \frac{\alpha_0}{1 - \alpha_1} \pi_0$$

$$\pi_2 = \frac{\alpha_1}{1-\alpha_2}\pi_1 = \frac{\alpha_1\alpha_0}{(1-\alpha_2)(1-\alpha_1)}\pi_0$$
$$\dots$$

Then we can get:

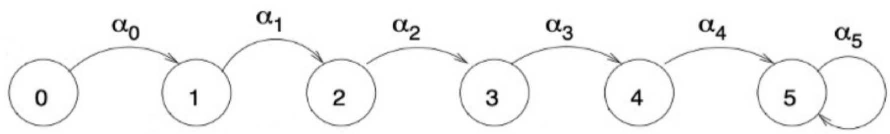
$$\pi_i = \frac{\alpha_{i-1}\alpha_{i-2}\dots\alpha_0}{(1-\alpha_i)(1-\alpha_{i-1})\dots(1-\alpha_1)}\pi_0 \quad \text{for } i = 1,2,\dots,M$$

Subject to $\sum_{i=0}^{M}\pi_i = 1$:

$$\pi_0 = \frac{1}{\left[1+\sum_{j=1}^{M}\frac{\prod_{i=0}^{j-1}\alpha_i}{\prod_{i=1}^{j}(1-\alpha_i)}\right]}$$

$$\pi_i = \frac{\alpha_{i-1}\alpha_{i-2}\dots\alpha_0}{(1-\alpha_i)(1-\alpha_{i-1})\dots(1-\alpha_1)}\pi_0, \qquad \text{for } i = 1,2,\dots,M$$

(2) Consider a Markov chain with states $0,1,\dots,M$ and $P_{ij}$:




In this case, there exist two states such that $P_{ij} > 0$ but $P_{ji} = 0$, then this chain is not time- reversible Markov chain.

(3) Consider a negative drift simple random walk, restricted to be non-negative, in which $P_{0,1} = 1$ and otherwise $P_{i,i+1} = p < 0.5$, $P_{i,i-1} = 1-p > 0.5$. In this case, the chain can only make a transition of magnitude $\pm 1$, so for each state $i \geq 0$, the rate from $i$ to $i+1$ equals the rate from $i+1$ to $i$. The time-reversibility equations are:

$$\pi_0 = (1-p)\pi_1, \qquad p\pi_i = (1-p)\pi_{i+1}, \qquad i \geq 1$$

yielding $\pi_1 = \frac{\pi_0}{(1-p)}, \pi_2 = \frac{p\pi_0}{(1-p)^2}, \dots, \pi_n = \frac{p^{n-1}\pi_0}{(1-p)^n}, n \geq 1$ . Since $\sum_n \pi_n = 1$ always hold, so we can get:

$$\pi_0 1 + (1-p)^{-1}\sum_{n\geq 0}\left[\frac{p}{1-p}\right]^n = 1$$

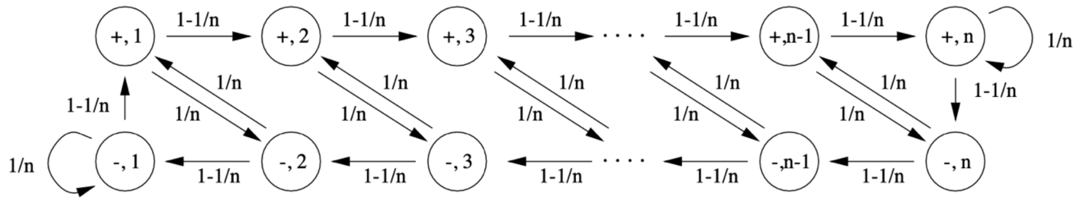
Since $\frac{p}{1-p} < 1$, the geometric series converges, and we can get the stationary distribution:

$$\pi_0 = \left(1+\frac{1}{1-2p}\right)^{-1}, \qquad \pi_n = (1-p)^{-1}\left[\frac{p}{1-p}\right]^{n-1}\pi_0, \qquad n \geq 1$$

Which we can simplifies to:

$$\pi_0 = \frac{1-2p}{2(1-p)}$$
$$\pi_n = \left(\frac{1}{2}-p\right)\left[\frac{p}{1-p}\right]^{n-1}, n \geq 1$$

(4) Consider a Markov chain switches between copies at rate $\frac{1}{n}$, and label the upstairs states $(+,1), (+,2), \ldots, (+,n)$ and the downstairs states $(-,1), (-,2), \ldots, (-,n)$.




The transition probabilities are:

$$P\big((+,x),(+,x+1)\big) = 1-\frac{1}{n} \quad \text{for } 1 \leq x < n, \qquad P\big((+,n),(-,n)\big) = 1-\frac{1}{n}$$
$$P\big((+,x),(-,x+1)\big) = \frac{1}{n} \quad \text{for } 1 \leq x < n, \qquad P\big((+,n),(+,n)\big) = \frac{1}{n}$$
$$P\big((-,x),(-,x-1)\big) = 1-\frac{1}{n} \quad \text{for } 1 \leq x < n, \qquad P\big((-,n),(+,n)\big) = 1-\frac{1}{n}$$
$$P\big((-,x),(+,x-1)\big) = \frac{1}{n} \quad \text{for } 1 \leq x < n, \qquad P\big((-,n),(-,n)\big) = \frac{1}{n}$$

This is a non-reversible walk, the arrival and departure weights of all states is 1, the transition matrix is stochastic, thus the stationary distribution of this chain is uniform on the new state space, with all states having probability $\frac{1}{2n}$. Therefore, the marginal distribution of the second component of state is also uniform.