Algorithm Design and Analysis(H)
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Assignment 6
NP-Completeness

Polynomial-time Reduction from 3-SAT to 3-Coloring

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1 3-Coloring

1.1 Problem Definition

Given an undirected graph $G = (V, E)$, a coloring of G is an assignment of colors to the vertices of G such that no two adjacent vertices have the same color. The 3-coloring problem is to determine whether a given graph G can be colored with at most 3 colors.

1.2 Proof of NP

A certificate for the 3-coloring problem is an assignment of colors to the vertices of G. We can verify this certificate in polynomial time by:

1. Checking each edge in E to ensure no two adjacent vertices have the same color, which takes $O(|E|) = O(|V|^2)$ time.
2. Counting the number of colors used in the certificate, which takes $O(|V|)$ time.

Thus, the verification algorithm runs in polynomial time, so the 3-coloring problem is in NP.

2 Polynomial-time Reduction from 3-SAT to 3-Coloring

2.1 Base Graph Construction

We first construct a base graph G_{base} with 3 initial vertices: Base, True, and False, which are connected in a cycle. Then for each variable v_i in the 3-SAT formula, we add two vertices v_i and $\overline{v_i}$ to G_{base}. We join each pair of v_i and $\overline{v_i}$ together and connect them to the Base vertex, which will form a triangle with the Base vertex, v_i, and $\overline{v_i}$.

An example of the base graph with 3 variables is shown in Figure 1.

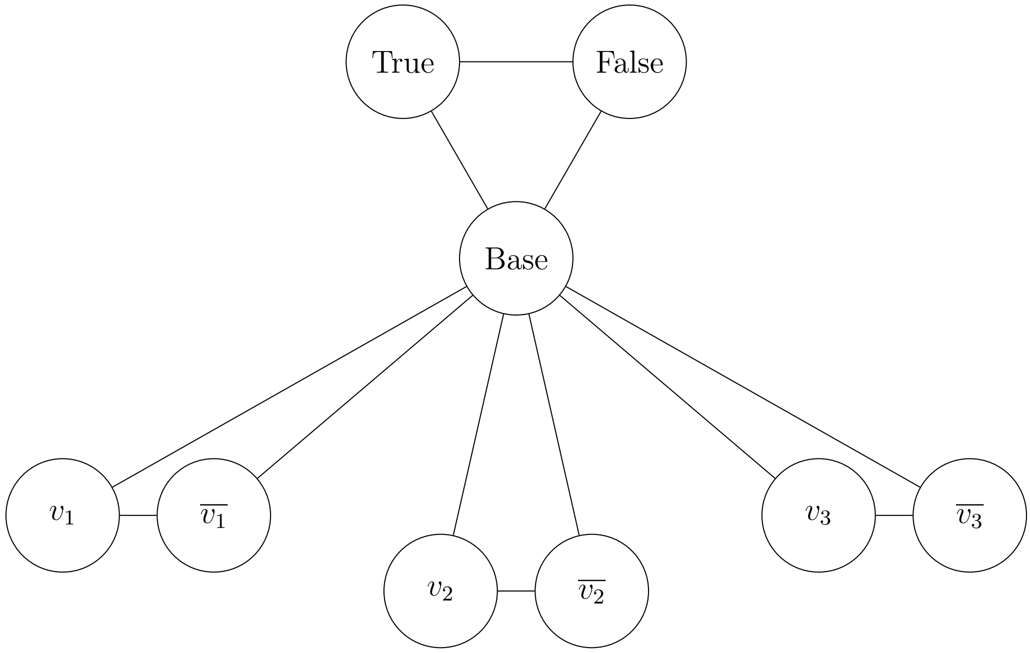

Figure 1: Base Graph G_{base} with 3 Variables

We can notice that the base graph G_{base} already obtains some important properties:

1. The graph can be 3-colored.
2. The True, False, and Base vertices will be colored with 3 different colors.
3. For all the variable vertices v_i and $\overline{v_i}$, one of them will be colored with the same color as the True vertex, and the other will be colored with the same color as the False vertex.

For the sake of simplicity, we call the color of the True vertex as True color, the color of the False vertex as False color, and the color of the Base vertex as Base color. For all the variable vertices v_i colored with the True color, we assign the value of v_i to be True. And for all the variable vertices v_i colored with the False color, we assign the value of v_i to be False.

We can observe that the properties of the base graph G_{base} help us maintain the consistency of the variable values. It guarantees that if v_i is True, then $\overline{v_i}$ must be False, and vice versa. Also, it ensures that each variable v_i will be assigned with either True or False.

2.2 Clause Graph Construction

With the base graph G_{base} constructed, we can now build the clause graph G_{clause} for each clause in the 3-SAT formula. For each clause $C_j = v_1 \vee v_2 \vee v_3$, we build a subgraph G^j_{clause} as shown in Figure 2.

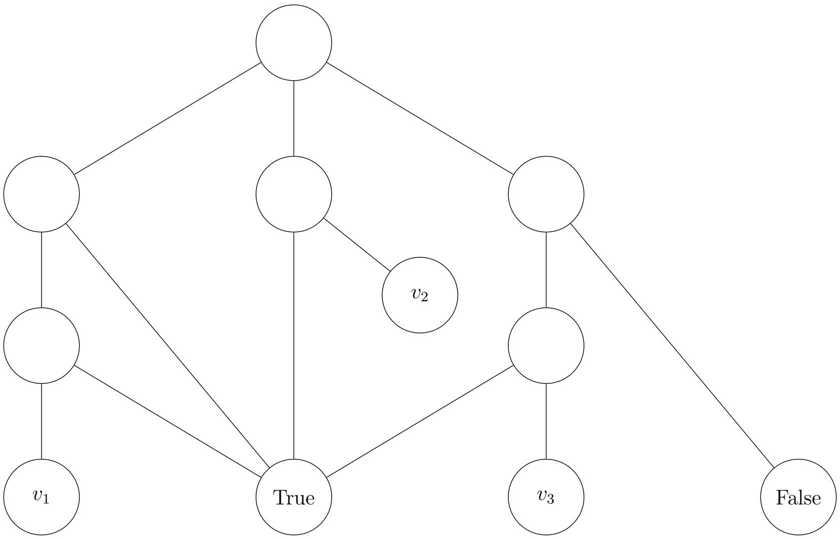

Figure 2: Clause Graph G^j_{clause} for Clause $C_j = v_1 \vee v_2 \vee v_3$

The clause graph G_{clause} has the property that if and only if all the variable vertices are colored with the False color, then there does not exist a 3-coloring for the clause graph.

Proof.

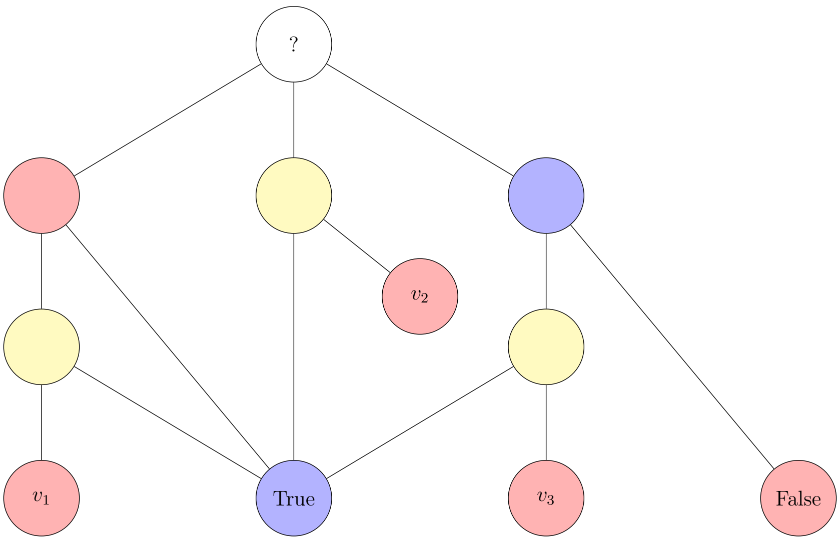

Figure 3: Clause Graph G^j_{clause} with All Variable Vertices Colored with the False Color

If part An example of all the variable vertices colored with the False color is shown in Figure 3. We first give the variable vertices the False color, and color the True and False vertices with the True and False colors, respectively. The remaining part of the graph must be colored in this specific way. This is because the colored vertices constrain their neighboring vertices to pick the single remaining available colors, effectively determining the coloring for the rest of the graph.

As we can see, the top vertex in the clause graph G^j_{clause} cannot be colored with any of the 3 colors, since the second layer of vertices it connects to are colored with all 3 colors. This means for a clause graph G^j_{clause}, if all the variable vertices are colored with the False color, then there does not exist a 3-coloring for the clause graph.

Only If part If any of the variable vertices are colored with the True color, then there exists a 3-coloring for the clause graph. For the sake of simplicity here, we put the coloring of the clause graph for all other possible variable values in appendix A. □

2.3 Final Graph Construction

The final graph G_{final} is constructed by:

1. Constructing the base graph G_{base}.
2. For each clause $C_j = v_1 \lor v_2 \lor v_3$, constructing the clause graph G^j_{clause} and merge it with the base graph G_{base}.

The 3-SAT formula is satisfiable if and only if there exists a 3-coloring for the corresponding final graph G_{final}.

Proof.

If part If the 3-SAT formula is satisfiable, then there exists a truth assignment that satisfies all the clauses. We can assign the True color to the True vertex and the False color to the False vertex in the final graph G_{final}. Since each clause is satisfied, there exists some 3-color assignment for each clause graph as well. Thus, there exists a 3-coloring for the final graph G_{final}.

Only If part If there exists a 3-coloring for the final graph G_{final}, then we assign variable values based on the coloring of the variable vertices in the final graph. Since each clause graph has a 3-coloring, the corresponding clause is satisfied. Thus, the 3-SAT formula is satisfiable. □

Hence, for each 3-SAT formula, we can construct a corresponding final graph G_{final}. If we can determine whether there exists a 3-coloring for the final graph G_{final}, then we can solve the 3-SAT problem. Thus, the 3-SAT problem is polynomial-time reducible to the 3-coloring problem.

A Clause Graph Coloring

Some graph may have multiple 3-colorings. Here we only show one of the possible 3-colorings for each clause graph.

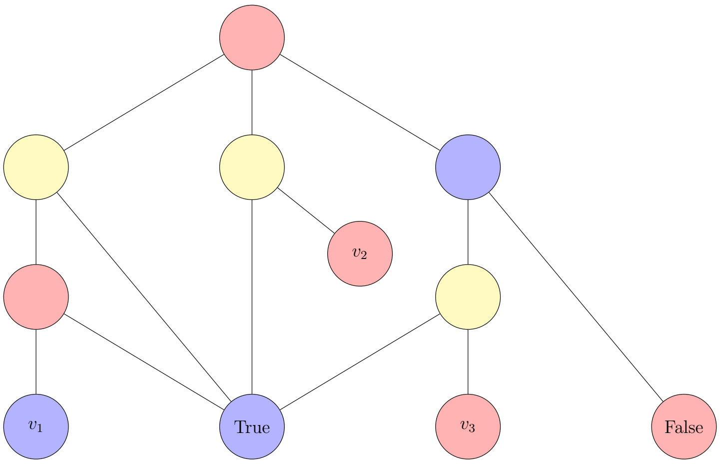

Figure 4: Clause Graph G^j_{clause} with $v_1 = \text{True}$, $v_2 = \text{False}$, and $v_3 = \text{False}$

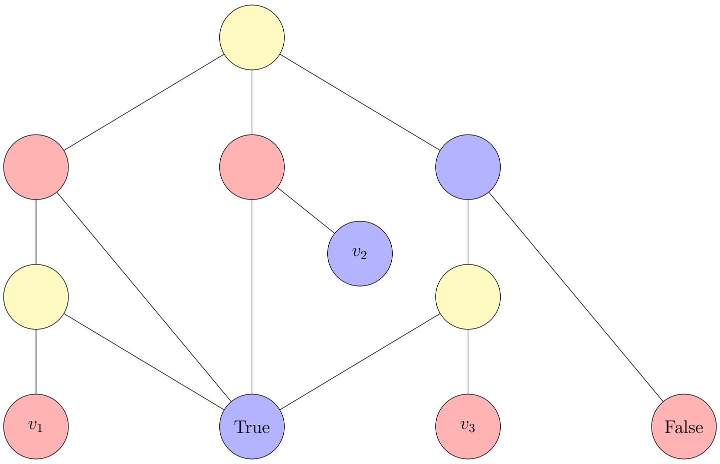

Figure 5: Clause Graph G^j_{clause} with $v_1 = \text{False}$, $v_2 = \text{True}$, and $v_3 = \text{False}$

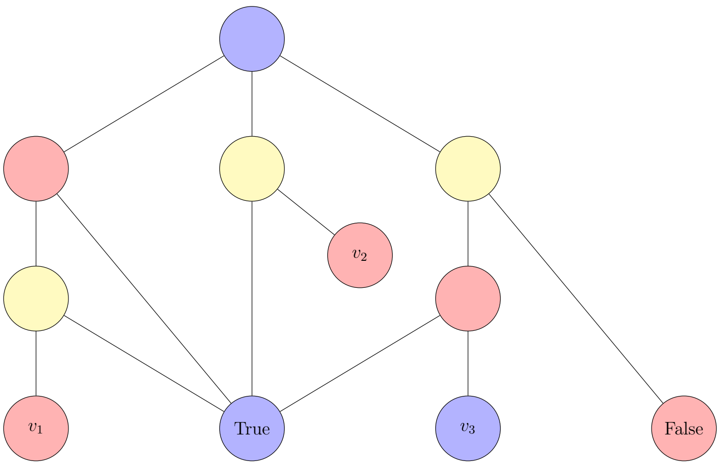

Figure 6: Clause Graph G^j_{clause} with $v_1 = \text{False}$, $v_2 = \text{False}$, and $v_3 = \text{True}$

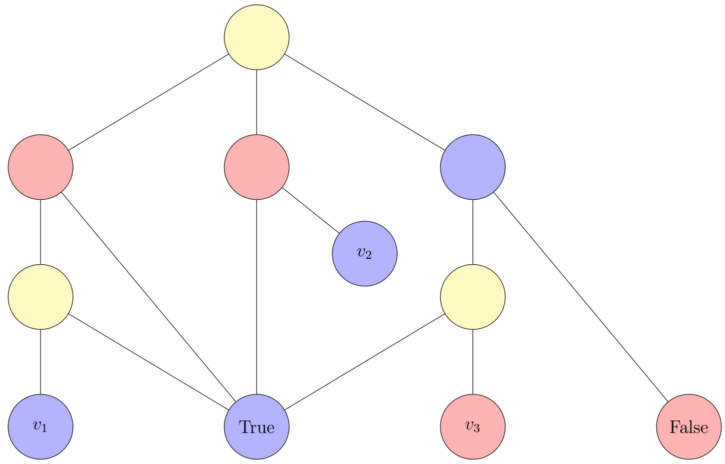

Figure 7: Clause Graph G^j_{clause} with $v_1 = \text{True}$, $v_2 = \text{True}$, and $v_3 = \text{False}$

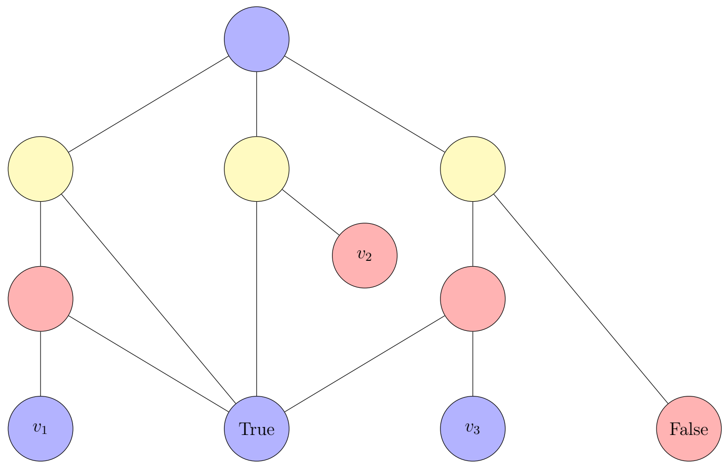

Figure 8: Clause Graph G^j_{clause} with $v_1 = \text{True}$, $v_2 = \text{False}$, and $v_3 = \text{True}$

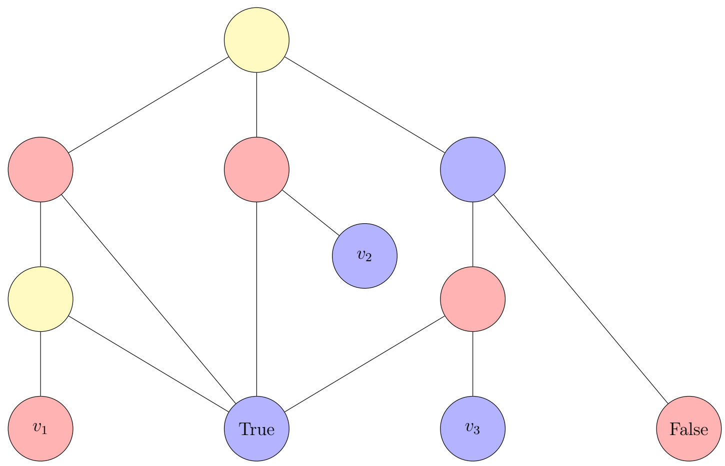

Figure 9: Clause Graph G^j_{clause} with $v_1 = \text{False}$, $v_2 = \text{True}$, and $v_3 = \text{True}$

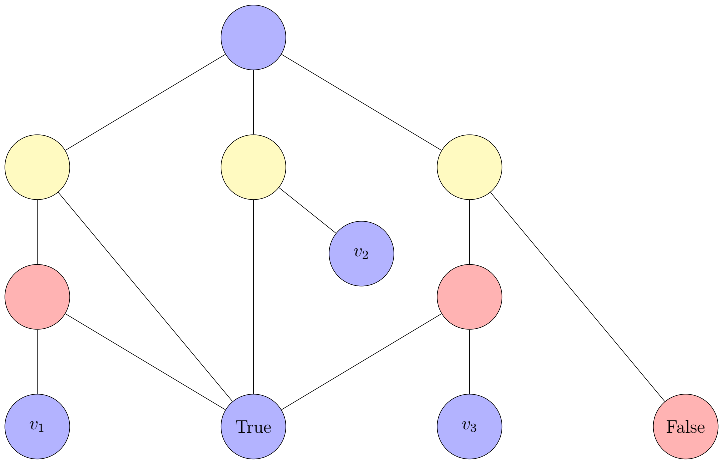

Figure 10: Clause Graph G_{clause}^{j} with $v_1 = \text{True}$, $v_2 = \text{True}$, and $v_3 = \text{True}$