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Section 7.1

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P218 Q5

(a)

$$E(X) = 1 \times P\{X = 1\} + 2 \times P\{X = 2\} = \theta + 2 \times (1 - \theta) = 2 - \theta$$

因此有:

$$\theta = 2 - E(X)$$

使用 $\bar{X}$ 代替 $E(X)$，有:

$$\begin{aligned}\hat{\theta} =& 2 - \bar{X} \\ =& 2 - \frac{1+2+2}{3} \\ =& \frac{1}{3}\end{aligned}$$

(b)

最大似然函数为:

$$L(\theta) = \prod P\{X = x_i\} = \prod \theta^{2-x_i}(1-\theta)^{x_i-1} = \theta^{2n-\sum x_i}(1-\theta)^{\sum x_i - n}$$

(c)

最大似然估计为:

$$\frac{d \ln L(\theta)}{d\theta} = \frac{2n - \sum x_i}{\theta} - \frac{\sum x_i - n}{1-\theta} = 0$$

带入 $n = 3$，$\sum x_i = 5$，解得 $\theta = \frac{1}{3}$。

补充 1

矩估计为:

$$\begin{aligned} E(X) &= \int_0^\theta x \cdot f(x;\theta) dx \\ &= \int_0^\theta x \cdot \frac{2}{\theta^2}(\theta - x) dx \\ &= \frac{2}{\theta^2}\int_0^\theta (\theta x - x^2) dx \\ &= \frac{2}{\theta^2}\left(\frac{1}{2}\theta x^2 - \frac{1}{3}x^3\right)\Big|_0^\theta \\ &= \frac{\theta}{3} \end{aligned}$$

因此有:

$$\theta = 3E(X)$$

使用 $\bar{X}$ 代替 $E(X)$，有:

$$\hat{\theta} = 3\bar{X}$$

补充 2

(1)

最大似然函数为:

$$L(\theta) = \prod \frac{\theta^x}{x!} e^{-\theta} = \frac{\theta^{\sum x_i}}{\prod x_i!} e^{-n\theta}$$

对数似然函数为:

$$\ln L(\theta) = \ln\theta \sum x_i - \ln \prod x_i! - n\theta$$

最大似然估计为:

$$\frac{d \ln L(\theta)}{d\theta} = \frac{\sum x_i}{\theta} - n = 0$$

解得 $\hat{\theta} = \frac{\sum x_i}{n}$。

(2)

最大似然函数为:

$$L(\theta) = \prod \theta\alpha x^{\alpha-1} e^{-\theta x^\alpha} = \theta^n \alpha^n \prod x_i^{\alpha-1} e^{-\theta \sum x_i^\alpha}$$

对数似然函数为:

$$\ln L(\theta) = n\ln\theta + n\ln\alpha + (\alpha-1)\sum \ln x_i - \theta \sum x_i^\alpha$$

最大似然估计为：

$$\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}-\sum x_i^\alpha=0$$

解得 $\hat{\theta}=\frac{n}{\sum x_i^\alpha}$。

补充 3

矩估计

矩估计为：

$$\begin{aligned}E(X)=&\int_0^1 x\cdot f(x;\theta)dx\\=&\int_0^1 x\cdot\theta(1-x)^{\theta-1}dx\\=&-x(1-x)^\theta\Big|_0^1+\int_0^1(1-x)^\theta dx\\=&\frac{1}{\theta+1}\end{aligned}$$

因此有：

$$\theta=\frac{1}{E(X)}-1$$

使用 $\bar{X}$ 代替 $E(X)$，有：

$$\hat{\theta}=\frac{1}{\bar{X}}-1\text{ 且}\bar{X}=\frac{\sum X_i}{n}$$

最大似然估计

最大似然函数为：

$$L(\theta)=\prod\theta(1-X_i)^{\theta-1}=\theta^n\prod(1-X_i)^{\theta-1}$$

对数似然函数为：

$$\ln L(\theta)=n\ln\theta+(\theta-1)\sum\ln(1-X_i)$$

最大似然估计为：

$$\frac{d\ln L(\theta)}{d\theta}=\frac{n}{\theta}+\sum\ln(1-X_i)=0$$

解得 $\hat{\theta}=-\frac{n}{\sum\ln(1-X_i)}$。