Solutions for Exercise Sheet 4

Handout: Oct 10 — Deadline: Oct 17, 4pm

Question 4.1 (0.25 marks) Say whether the following array is a Max-Heap (justify your answer):

34	20	21	16	14	11	3	14	17	13

Solution: It's not a Max-Heap because node 17 violates the max-heap property as it's larger than its parent (i.e., 16).

Question 4.2 (0.25 marks)

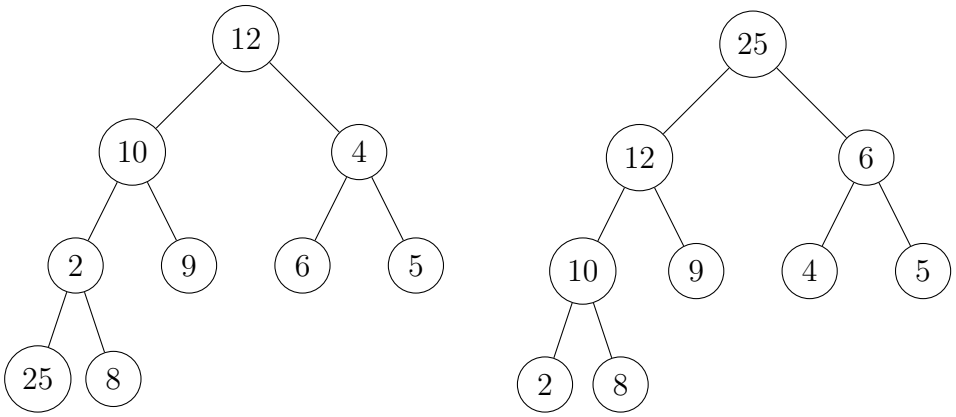
Consider the following input for HeapSort:

12	10	4	2	9	6	5	25	8

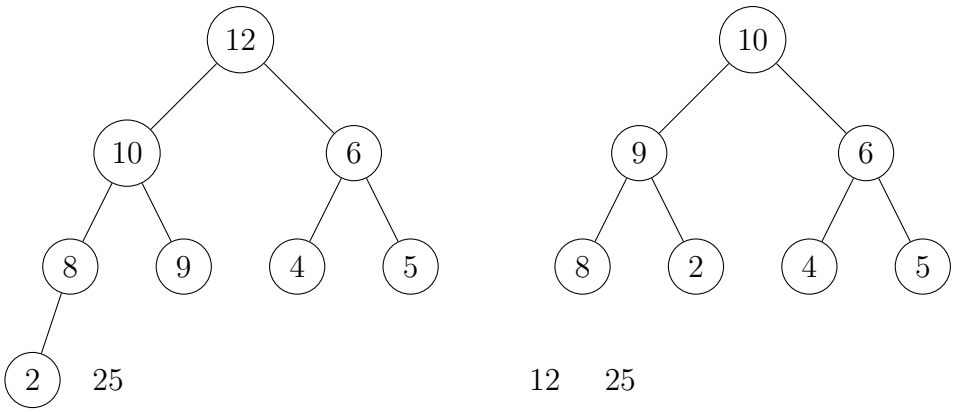
Create a heap from the given array and sort it by executing HeapSort. Draw the heap (the tree) after Build-Max-Heap and after every execution of Max-Heapify in line 5 of HeapSort. You don't need to draw elements extracted from the heap, but you can if you wish.

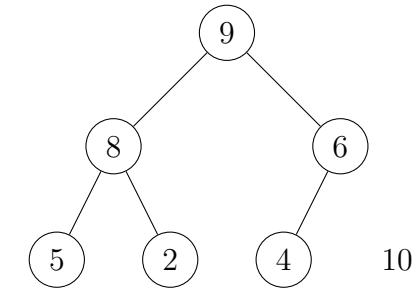
Solution:

Initial structure (for completeness, this wasn't asked for) and after Build-Max-Heap:

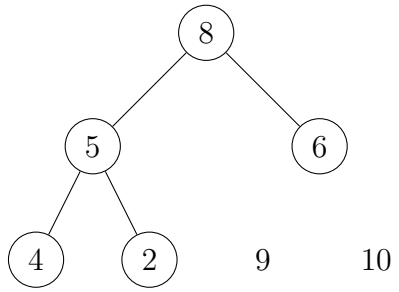


After each call of Max-Heapify:





12 25



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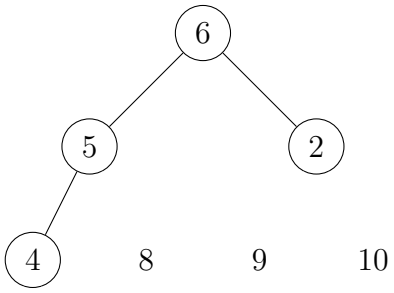

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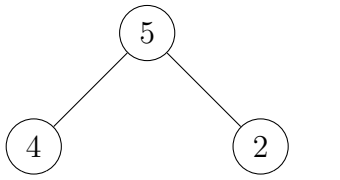

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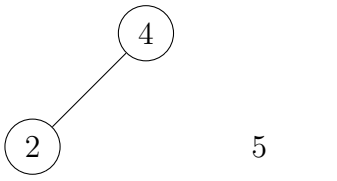

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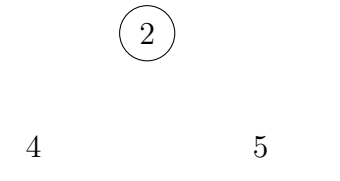

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Question 4.3 (0.5 marks)

1. Provide the pseudo-code of a MAX-HEAPIFY(A, i) algorithm that uses a WHILE loop instead of the recursion used by the algorithm shown at lecture.
2. Prove correctness of the algorithm by loop invariant.

Solution: 1)

```
MAX-HEAPIFY(A, i)
 1: while true do
 2:     ℓ = Left(i)
 3:     r = Right(i)
 4:     if ℓ ≤ A.heap-size and A[ℓ] > A[i] then
 5:         largest = ℓ
 6:     else
 7:         largest = i
 8:     if r ≤ A.heap-size and A[r] > A[i] then
 9:         largest = r
10:     if largest == i then
11:         return
12:     Swap A[i] with A[largest]
13:     i = largest
```

2) We set the following loop invariant: " Let ROOT be the node at which MAX-HEAPIFY(A, i) is called i.e., the initial i. At each iteration of the WHILE loop, the tree rooted in ROOT obtained by removing the two subtrees with roots Left(i) and Right(i) is a Max-Heap"

- **Initialisation:** Before the loop starts the considered tree consists only of the node ROOT which is trivially a Max-Heap.
- **Maintenance:** By loop invariant after i iterations the tree rooted in ROOT down to node i is a heap. In the current iteration we swap i with the largest between i, Left(i) and Right(i) ensuring that by the end of the iteration we have a heap of height $i + 1$.
- **Termination** the algorithm terminates when either: 1) i a leaf. Then the loop invariant says that the whole tree rooted in ROOT is a max-heap (because the two subtrees Left(i) and Right(i) are empty). 2) the heap-property is respected before a leaf is reached. Then the algorithm returns a Max-Heap because the loop invariant tells me that up to node i we have a Max-Heap and Left(i) and Right(i) are heaps too because the subtrees of ROOT where originally Max-Heaps and below i they are still untouched.

Question 4.4 (1.25 marks)

1. Show that each child of the root of an n-node heap is the root of a sub-tree of at most $(2/3)n$ nodes. (*HINT: consider that the maximum number of elements in a subtree happens when the left subtree has the last level full and the right tree has the last level empty. You might want to use the formula seen at lecture:* $\sum_{i=0}^{k-1} 2^i = 2^k - 1$).
2. As a consequence of (1) we can use the recurrence equation $T(n) \leq T(2n/3) + \Theta(1)$ to describe the runtime of Max-Heapify(A, n). Prove the runtime of Max-Heapify using the Master Theorem.

Solution: 1) The size of the whole tree is 1 (the root) + |Left subtree| + |right subtree|. Let the right subtree have height h and the left subtree be full at height $h+1$. Then:

$$|\text{Left subtree}| = 1 + 2 + 4 \cdots + 2^h + 2^{h+1} = \sum_{i=0}^{h+1} 2^i = 2^{h+2} - 1 \quad (1)$$

$$|\text{Right subtree}| = 1 + 2 + 4 \cdots + 2^h = \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$$

Then,

$$n = 1 + (2^{h+2} - 1) + (2^{h+1} - 1) = 1 + (2(2^{h+1}) - 1) + (2^{h+1} - 1) = 3(2^{h+1}) - 1$$

So $2^{h+1} = \frac{n+1}{3}$, and substituting into Eq (1),

$$|\text{Left subtree}| = 2^{h+2} - 1 = 2(2^{h+1}) - 1 = 2\frac{n+1}{3} - 1 = (2/3)n - 1/3 < (2/3)n$$

2) We apply the Master Theorem with $a = 1, b = 3/2, f(n) = \Theta(1)$, and watershed function $n^{\log_b a} = n^0 = 1$, because $\log_{1.5} 1 = 0$. Thus, Case 2 holds with $k = 0$ such that $n^{\log_b a} \log^k n = \Theta(1)$ and $T(n) = \Theta(n^0 \log^{k+1} n) = \Theta(\log n)$, and the runtime of Max-Heapify(A, n) is $O(\log n)$ since each subtree has *at most* $(2/3)n$ nodes.

Question 4.5 (1 mark)

Argue that the runtime of HeapSort on an already sorted array of distinct numbers is $\Omega(n \log n)$.

Solution: Build-Max-Heap will move the $\lfloor n/2 \rfloor$ smallest elements to the end of the array (to be all leaves). Then each of the first $\lfloor n/2 \rfloor$ Heapify operations will have to traverse at least $\log n - 1$ levels after the root of the tree is swapped with the last element of the heap to reconstruct a Max-Heap. Hence the runtime is

$$T(n) \geq (n/2 - 1)(\log n - 1) = \Omega(n \log n)$$

Question 4.6 (0.25 marks)

Implement HeapSort(A, n).