Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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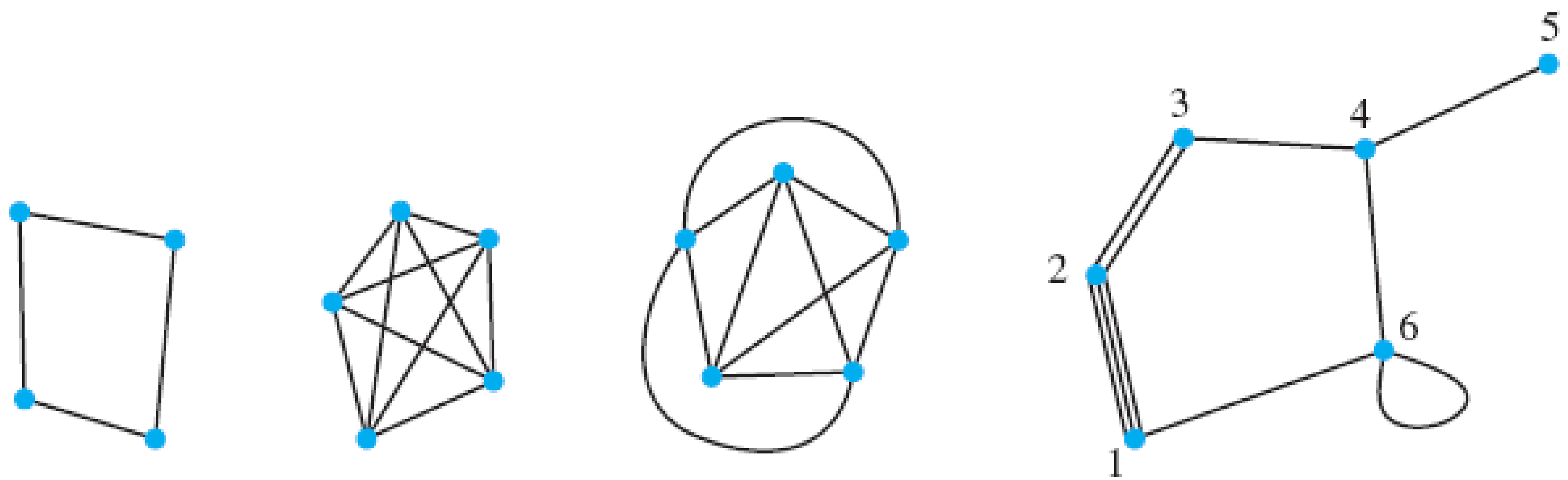
- **Definition**. A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints.



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Complete Graphs

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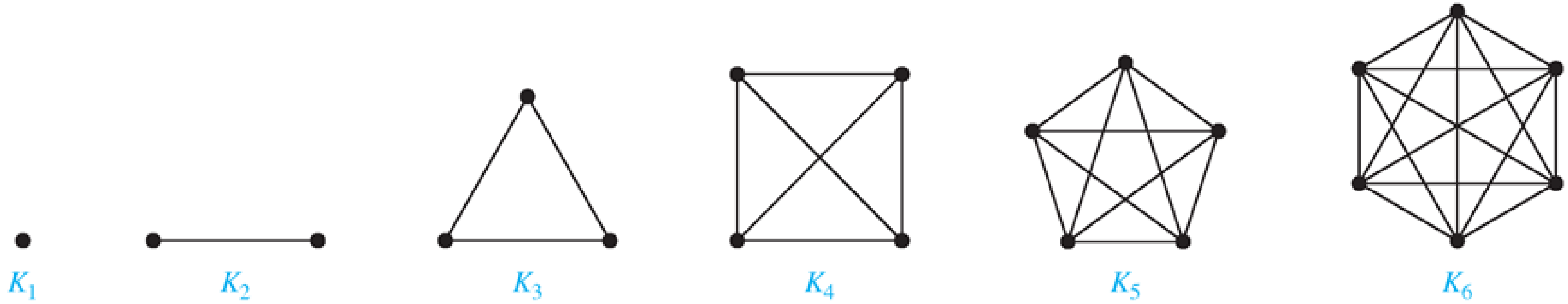

Cycles

- A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

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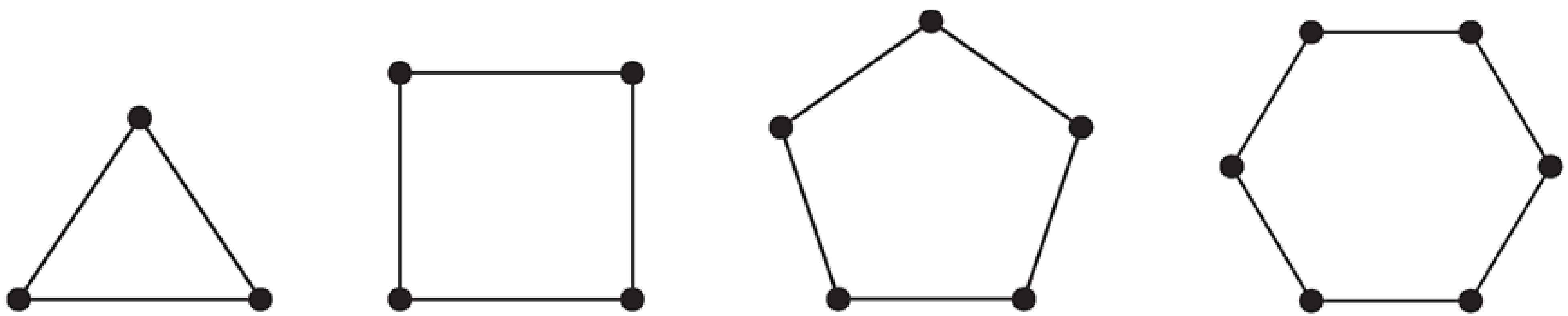

C_3 $\quad$ C_4 $\quad$ C_5 $\quad$ C_6

Wheels

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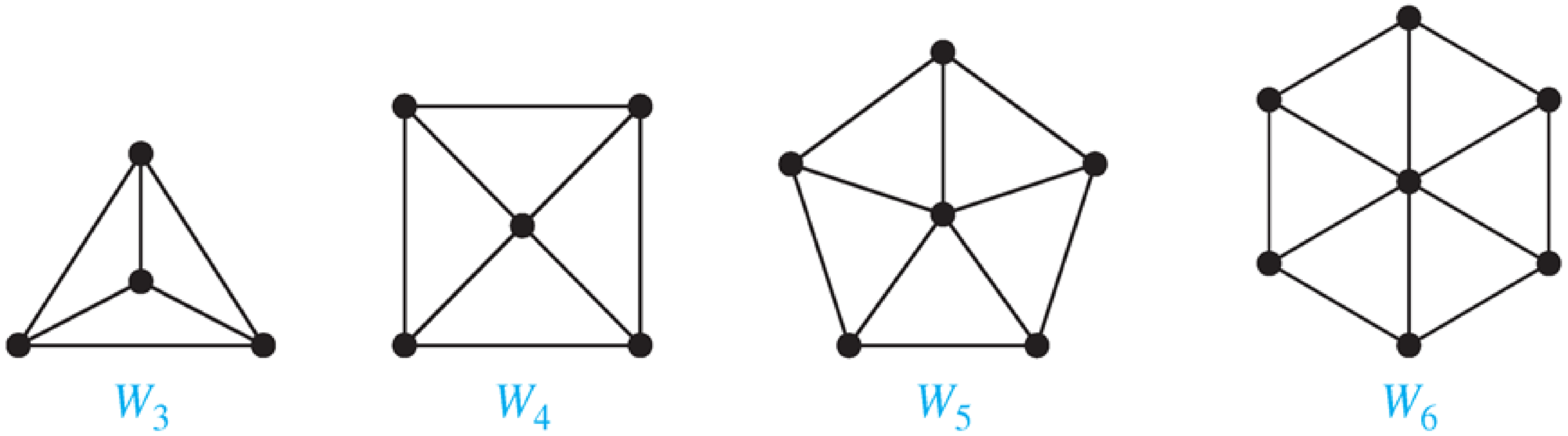

N-dimensional Hypercube

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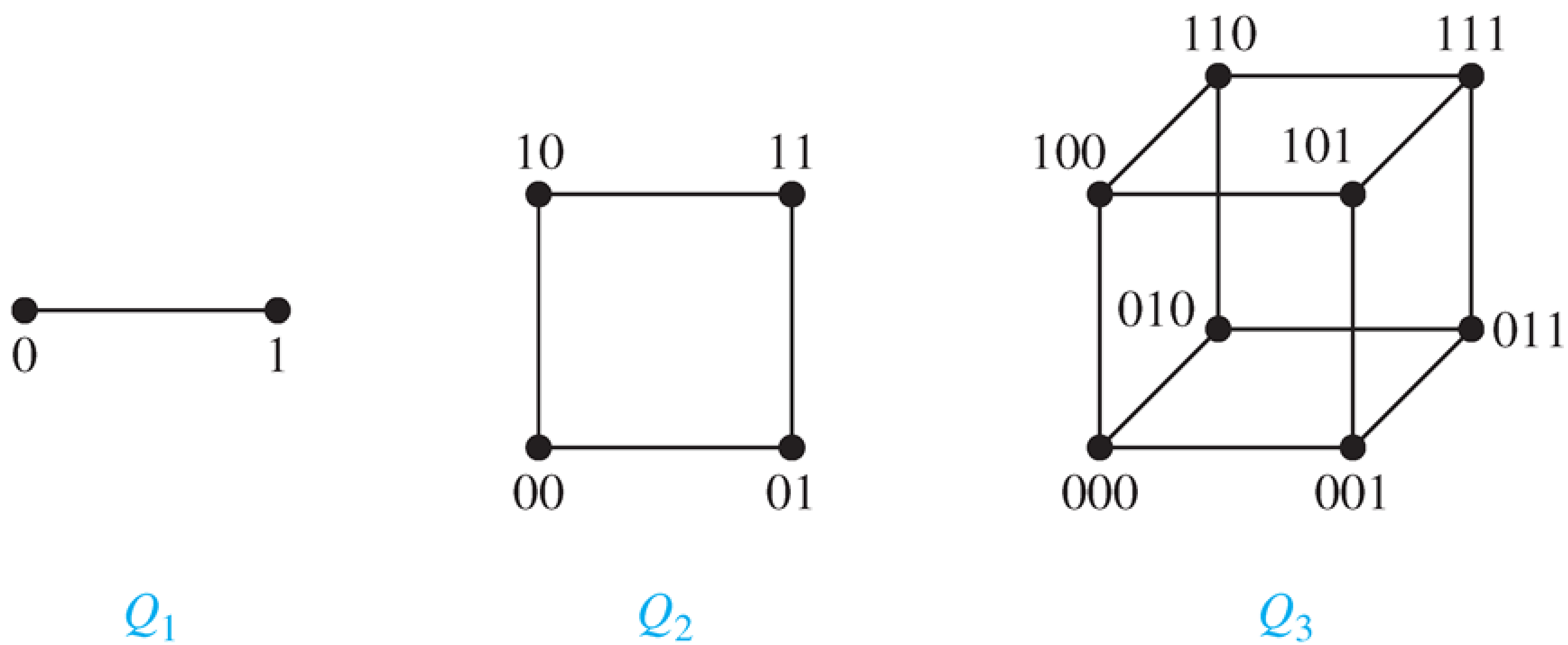

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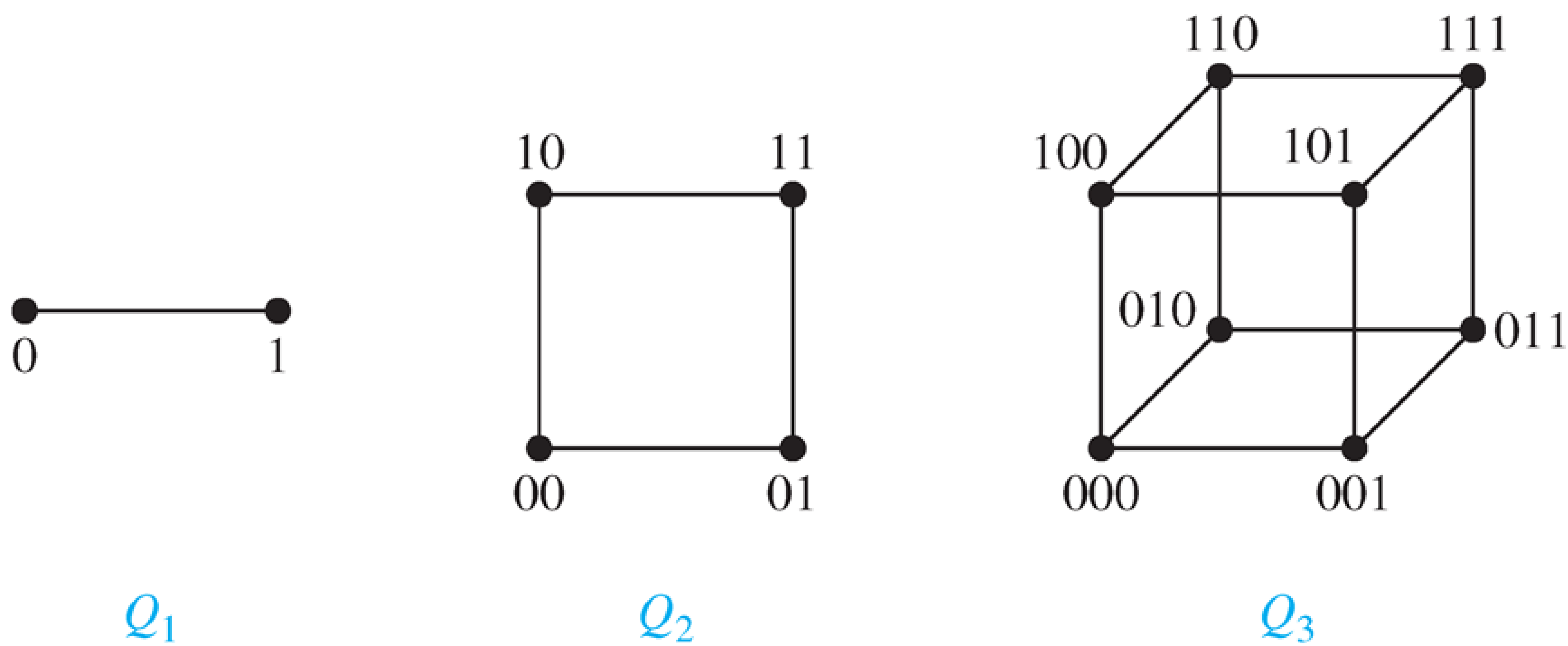

How many vertices? How many edges?

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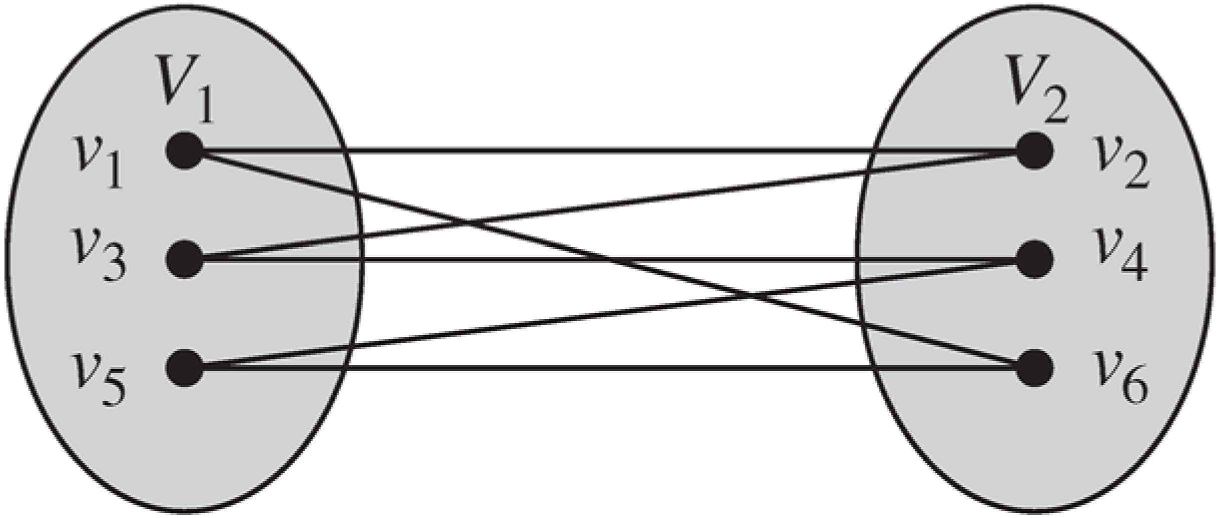

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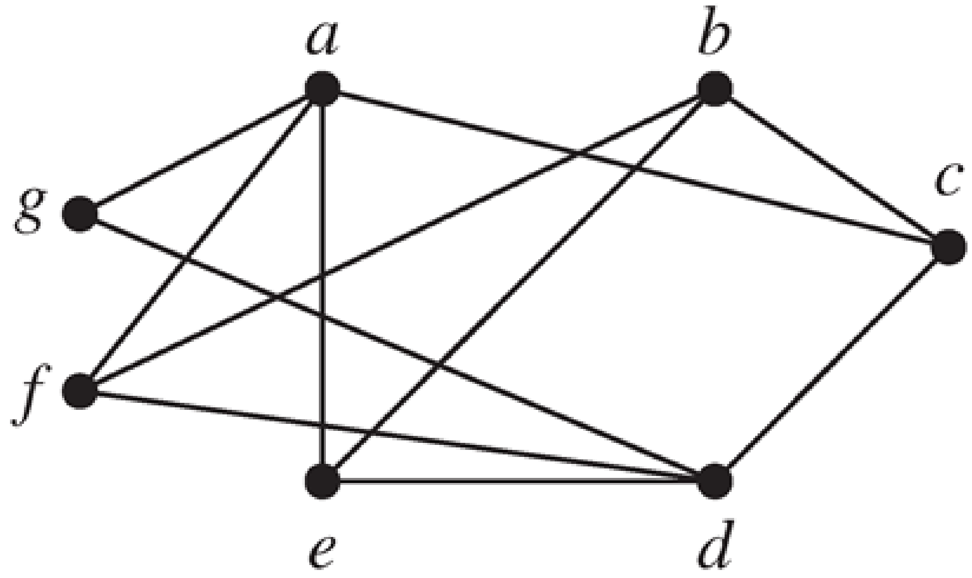

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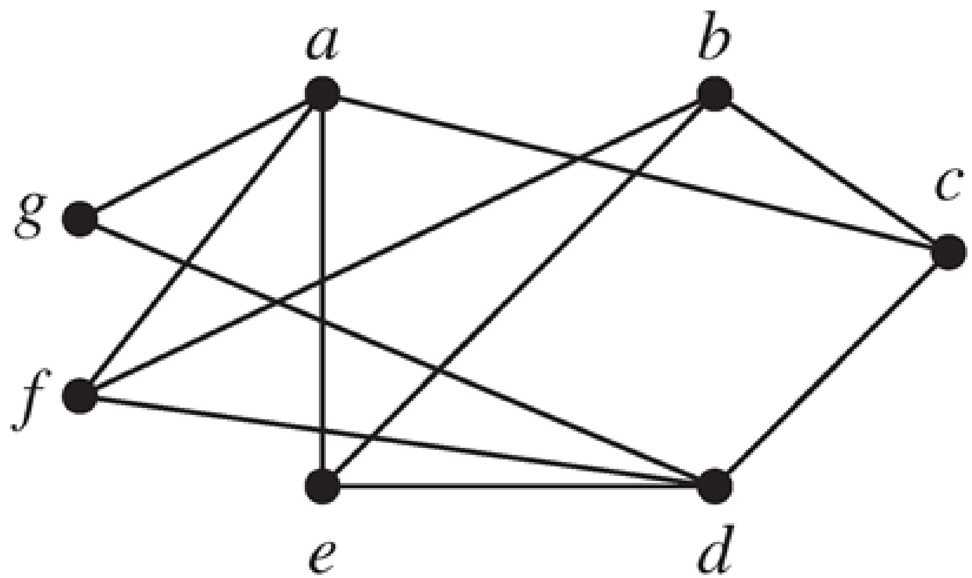
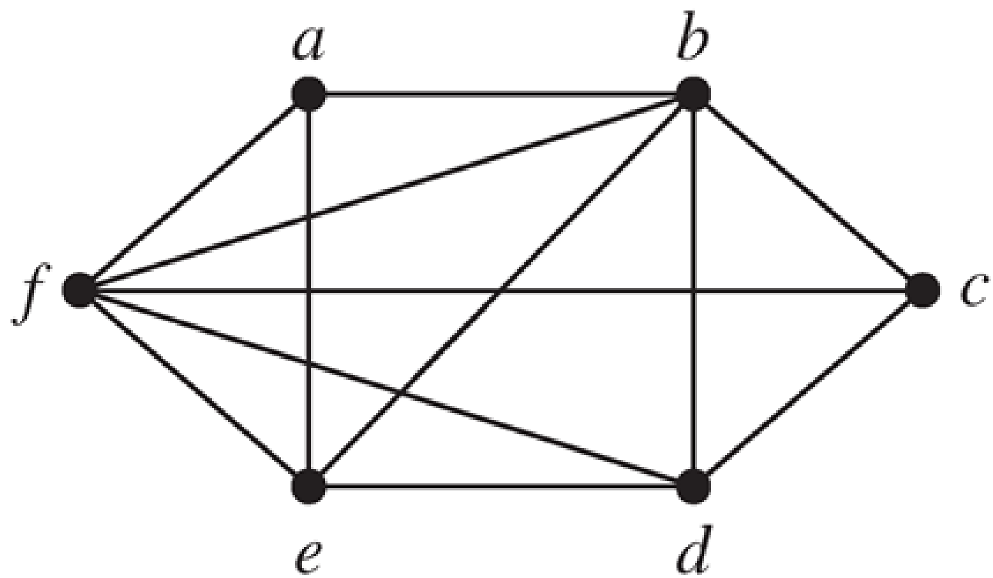
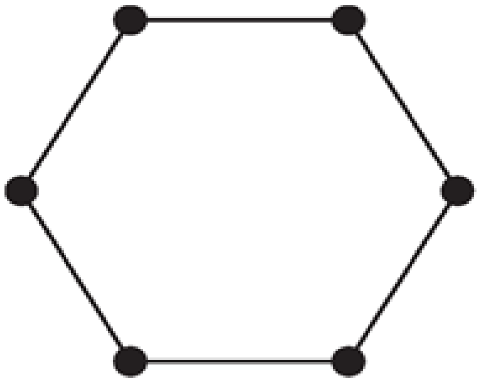




- **Example** Show that C_6 is bipartite.



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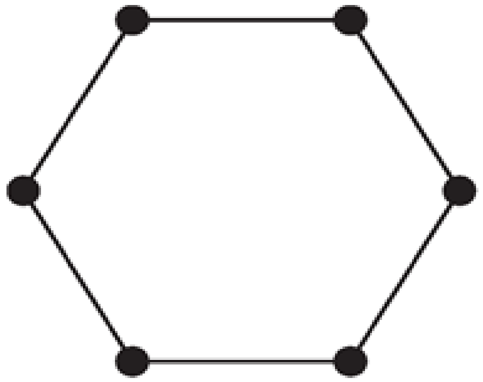

C_6

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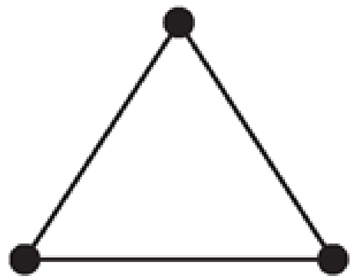

C_3

Complete Bipartite Graphs

- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

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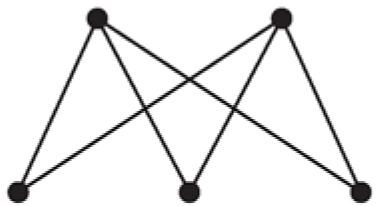
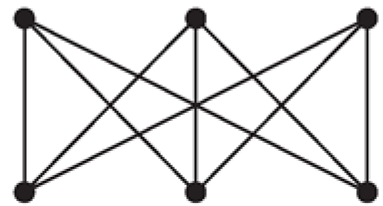


$K_{2,3}$



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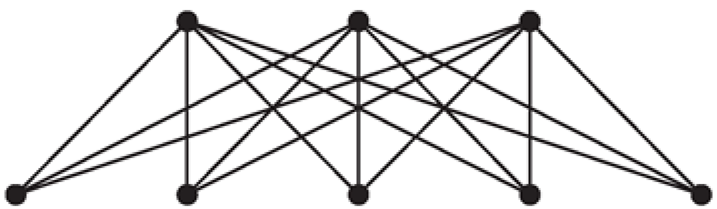
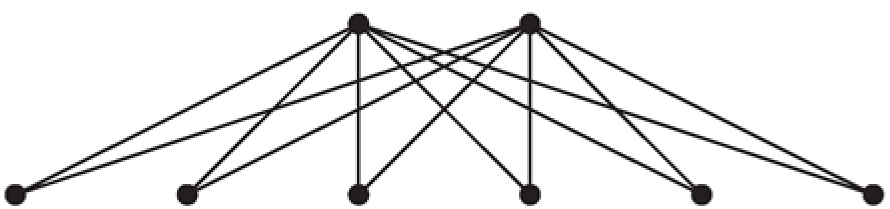


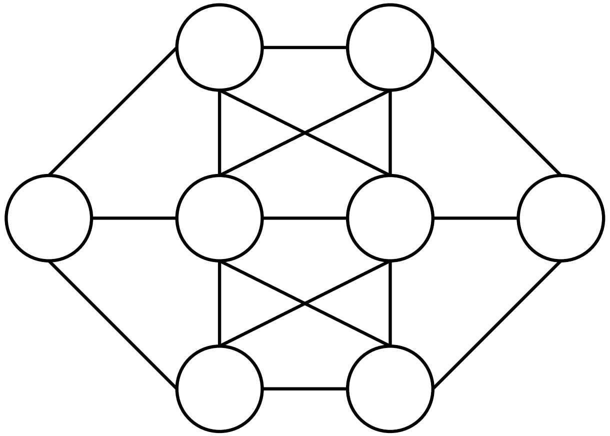
$K_{3,5}$



$K_{2,6}$

Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.



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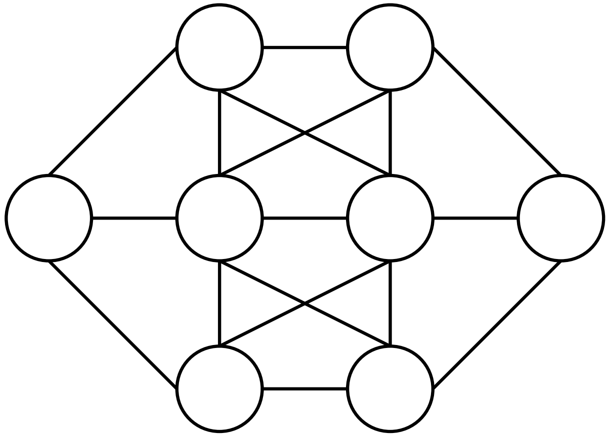

- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Bipartite Graphs and Matchings

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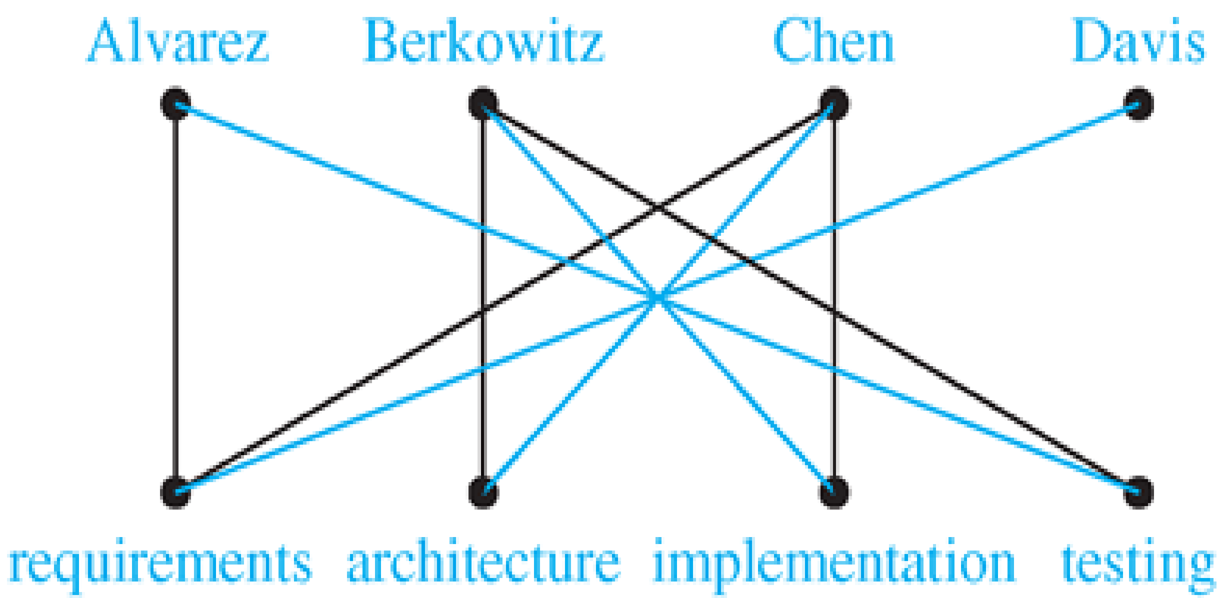

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Case (i): For all integers j with $1 \leq j \leq k$, the vertices in every set of j elements from W_1 are adjacent to at least $j + 1$ elements of W_2

Case (ii): For some integer j with $1 \leq j \leq k$, there is a subset W_1' of j vertices such that there are exactly j neighbors of these vertices in W_2

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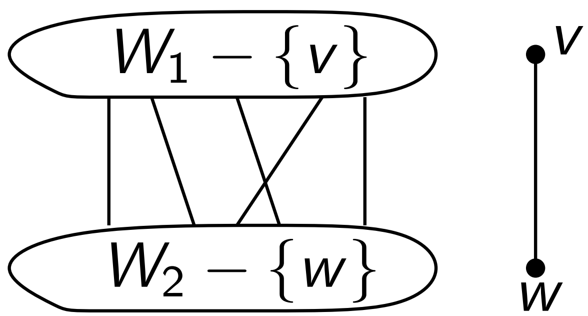

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If not, there is a subset B of t vertices with $1 \leq t \leq k + 1 - j$ s.t. $|N(B)| < t$.

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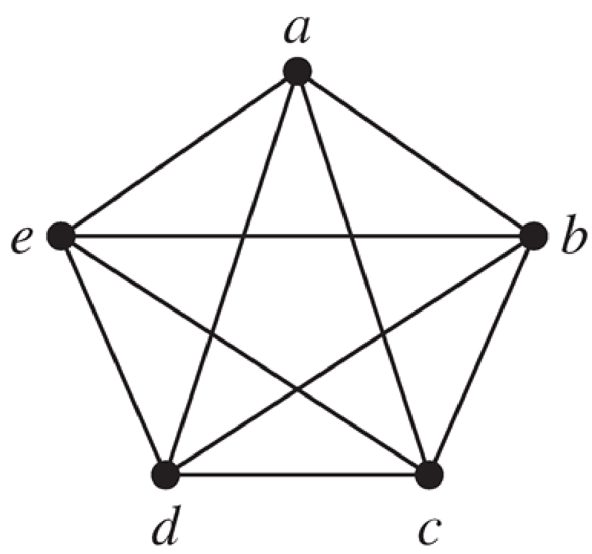

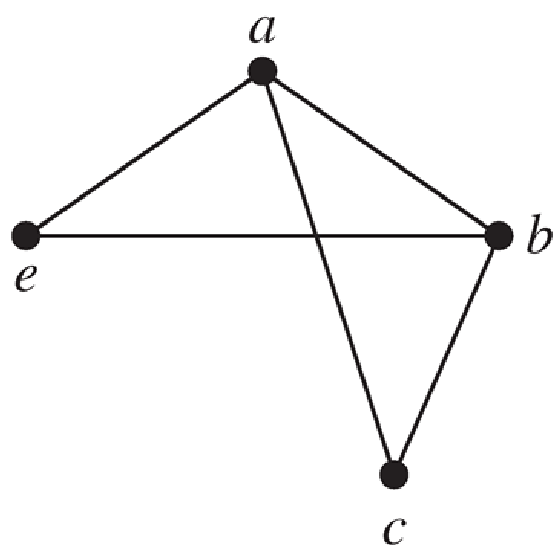

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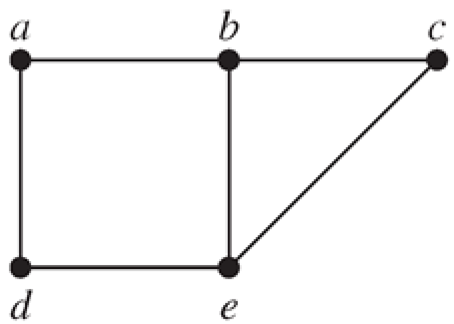
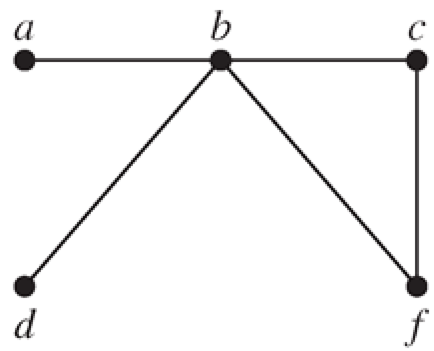


G_1



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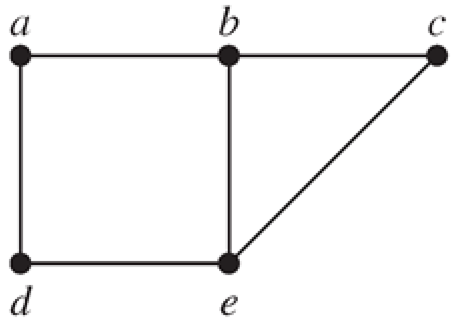

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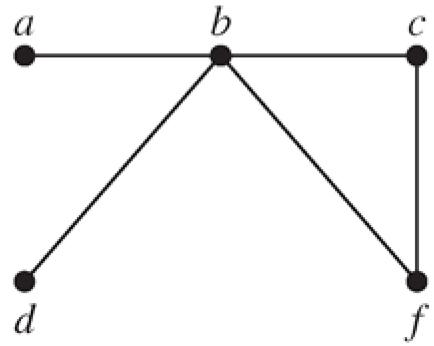

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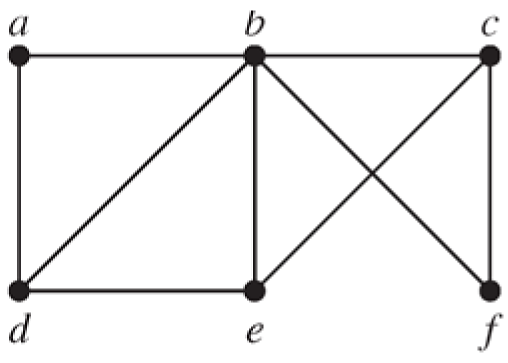

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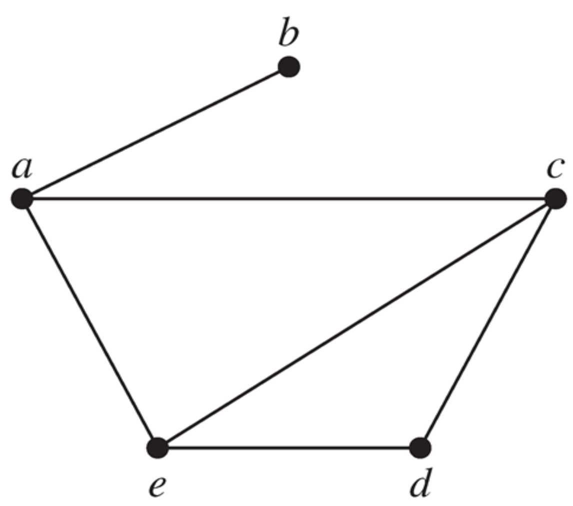

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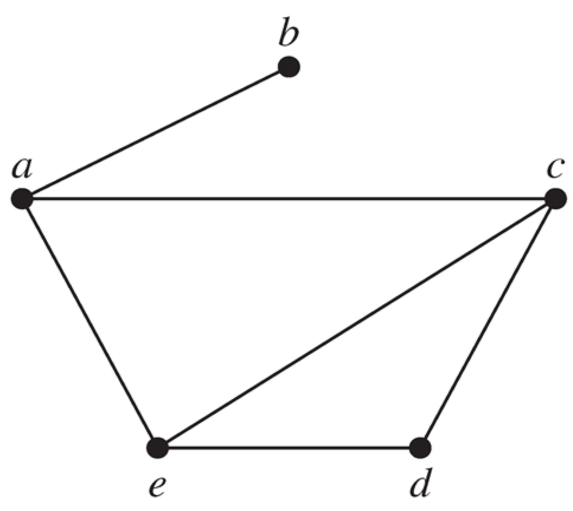

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

Representation of Graphs

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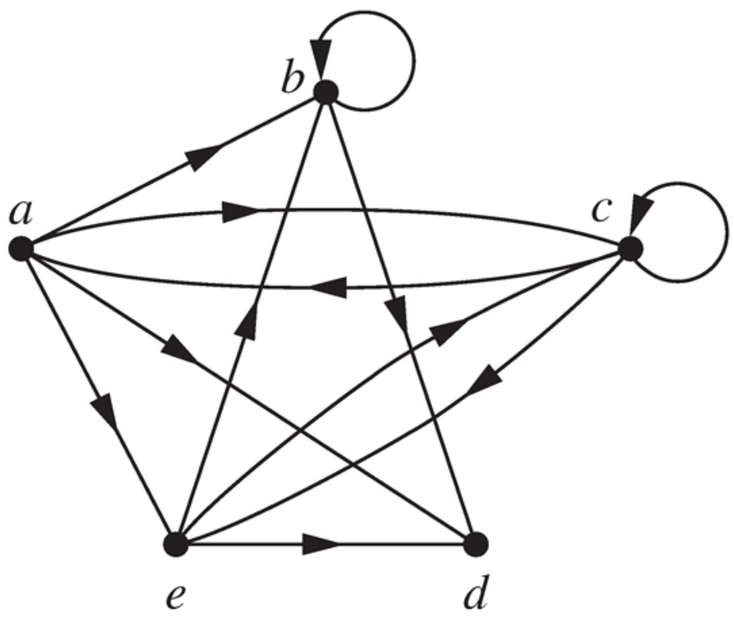

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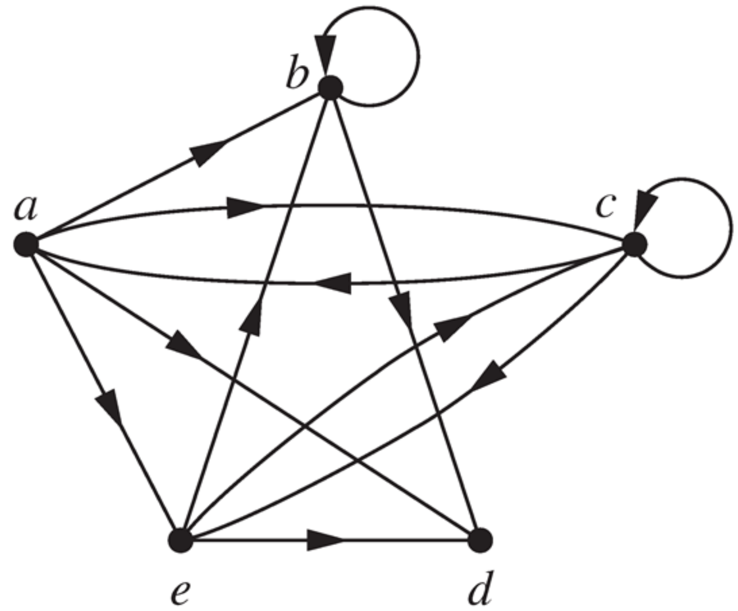

TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Adjacency Matrices

- **Definition** Suppose that $G = (V, E)$ is a simple graph with $|V| = n$. Arbitrarily list the vertices of G as $v_1, v_2, \ldots, v_n$. The *adjacency matrix* $\mathbf{A}_G$ of G, is the $n \times n$ zero-one matrix with 1 as its (i, j)-th entry when v_i and v_j are adjacent, and 0 as its (i, j)-th entry when they are not adjacent.

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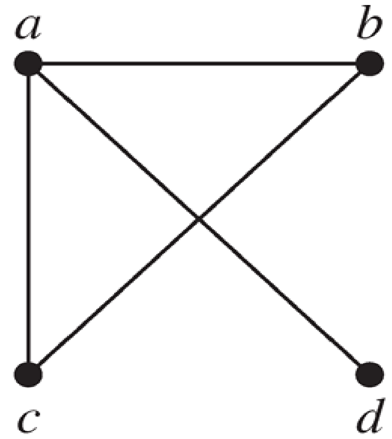

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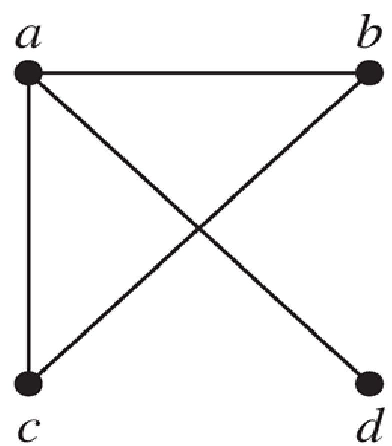

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrices

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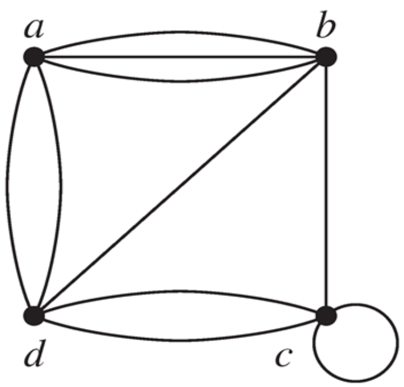

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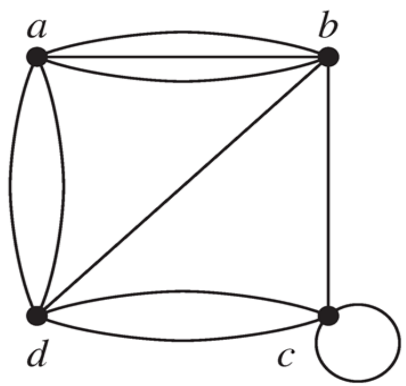

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Incidence Matrices

- **Definition** Let $G = (V, E)$ be an undirected graph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. The *incidence matrix* with respect to the ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]_{n\times m}$, where

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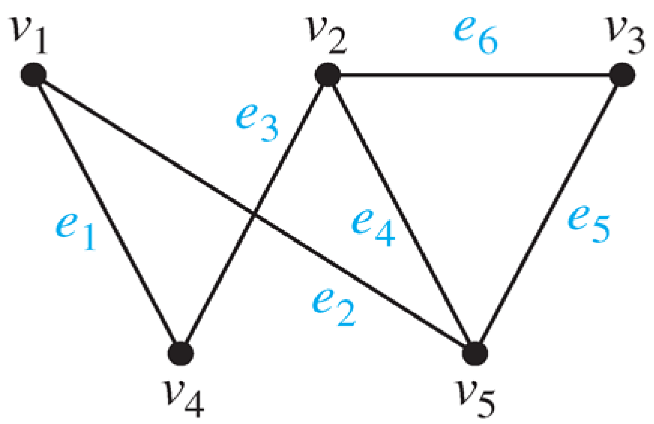

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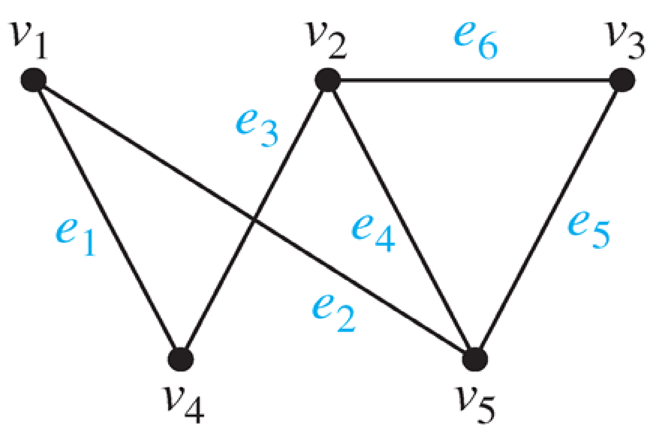

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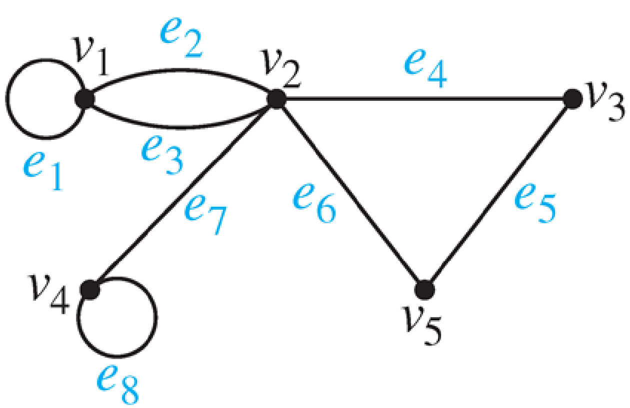

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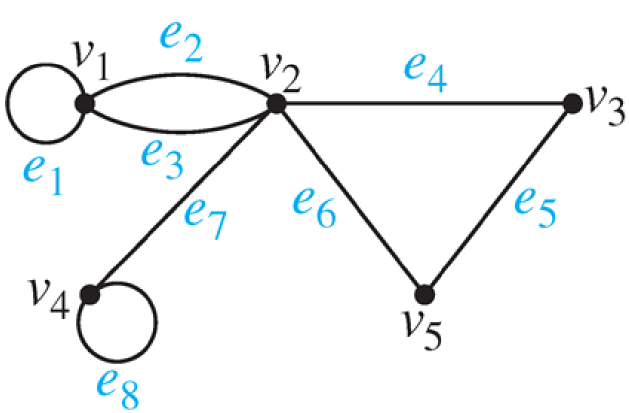

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Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an *isomorphism*.

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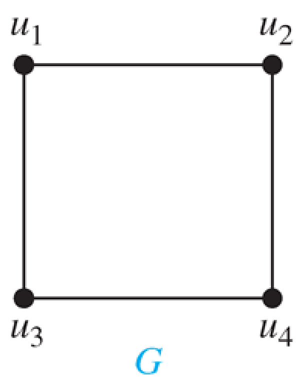
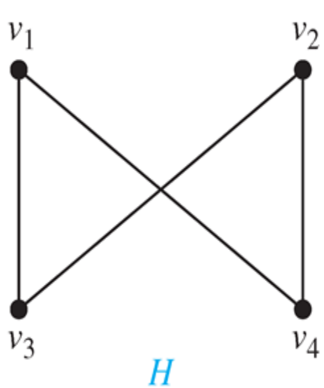


Are the two graphs isomorphic?



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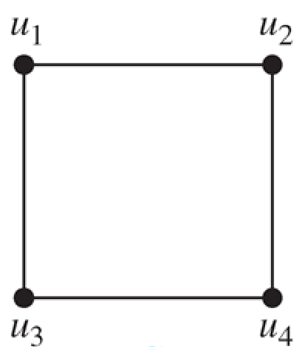
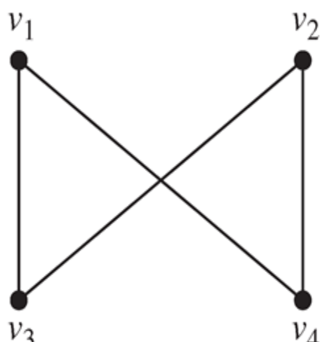


G



H

Are the two graphs isomorphic?

Define a one-to-one correspondence:
$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, and $f(u_4) = v_2$

Isomorphism of Graphs

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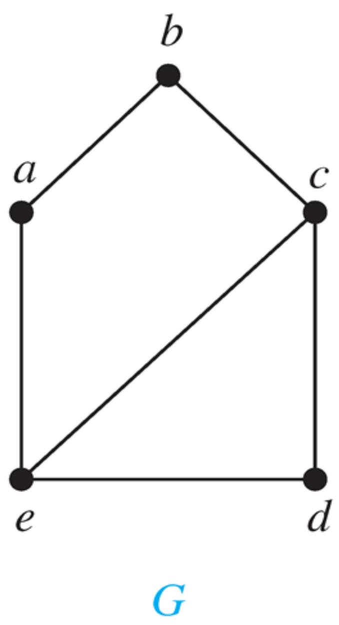
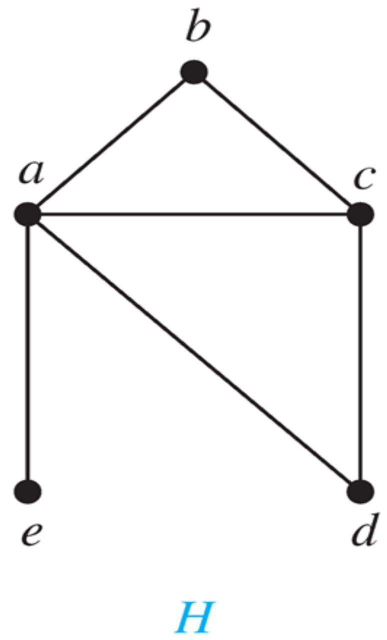
- Useful graph invariants include the number of vertices, number of edges, degree sequence, etc.

- **Example** Determine whether these two graphs are isomorphic.





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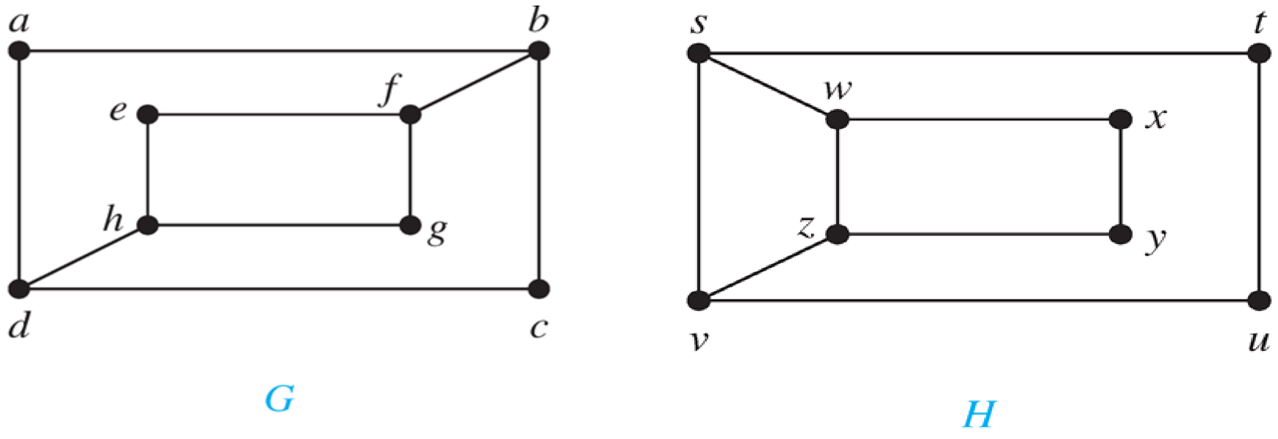

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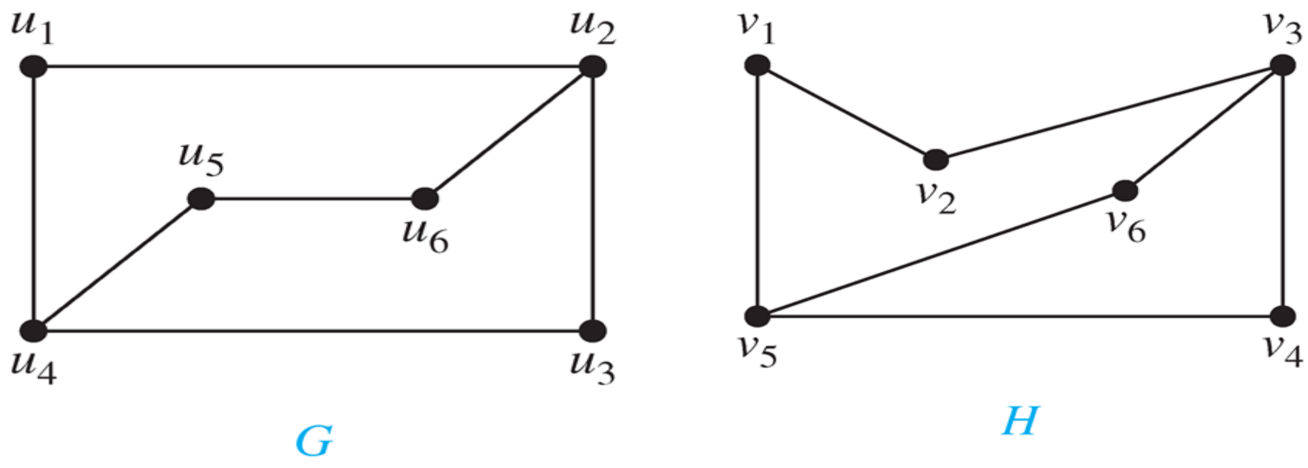

Path

- **Definition** Let n be a nonnegative integer and G an undirected graph. A *path of length n* from u to v in G is a sequence of n edges $e_1, e_2, \ldots, e_n$ of G for which there exists a sequence $x_0 = u, x_1, \ldots, x_{n-1}, x_n = v$ of vertices such that e_i has the endpoints x_{i-1} and x_i for $i = 1, \ldots, n$. The path is a *circuit* if it begins and ends at the same vertex, i.e., if $u = v$ and has length greater than zero. A path or circuit is *simple* if it does not contain *the same edge* more than once.

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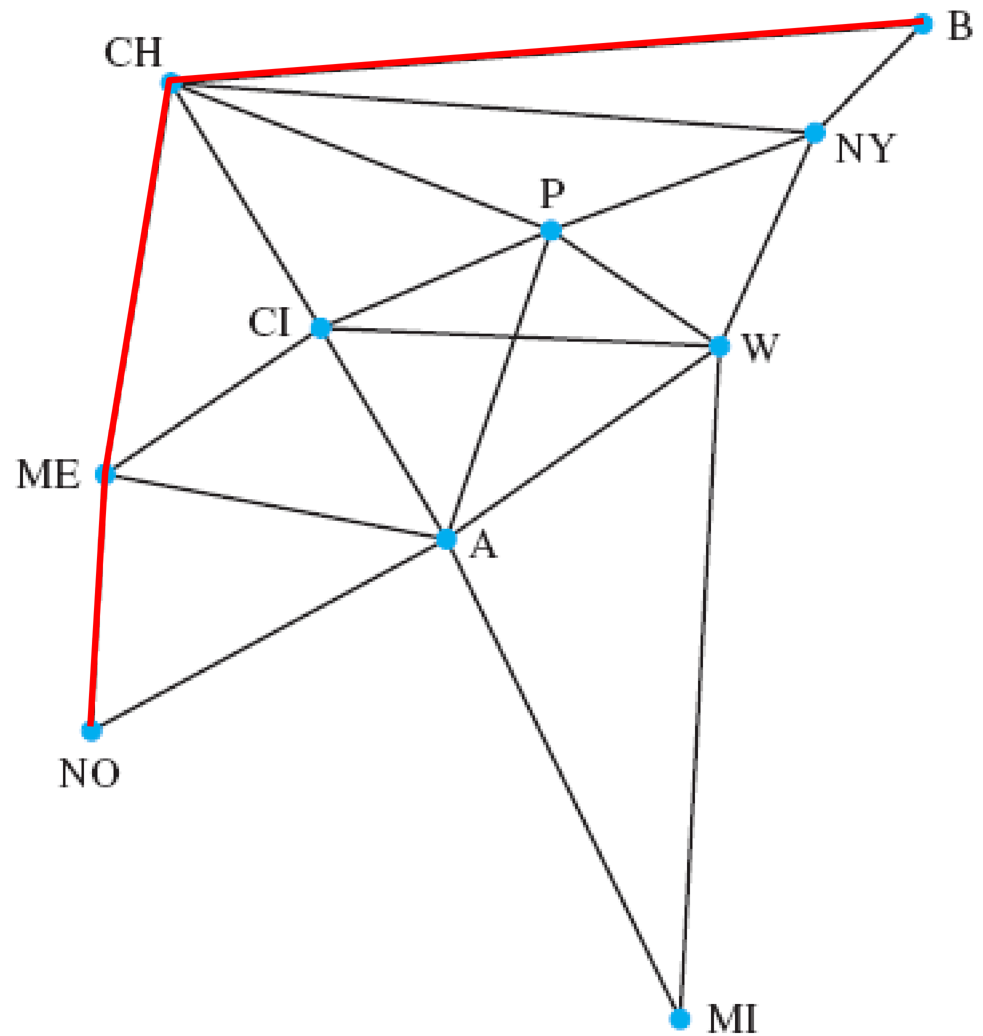
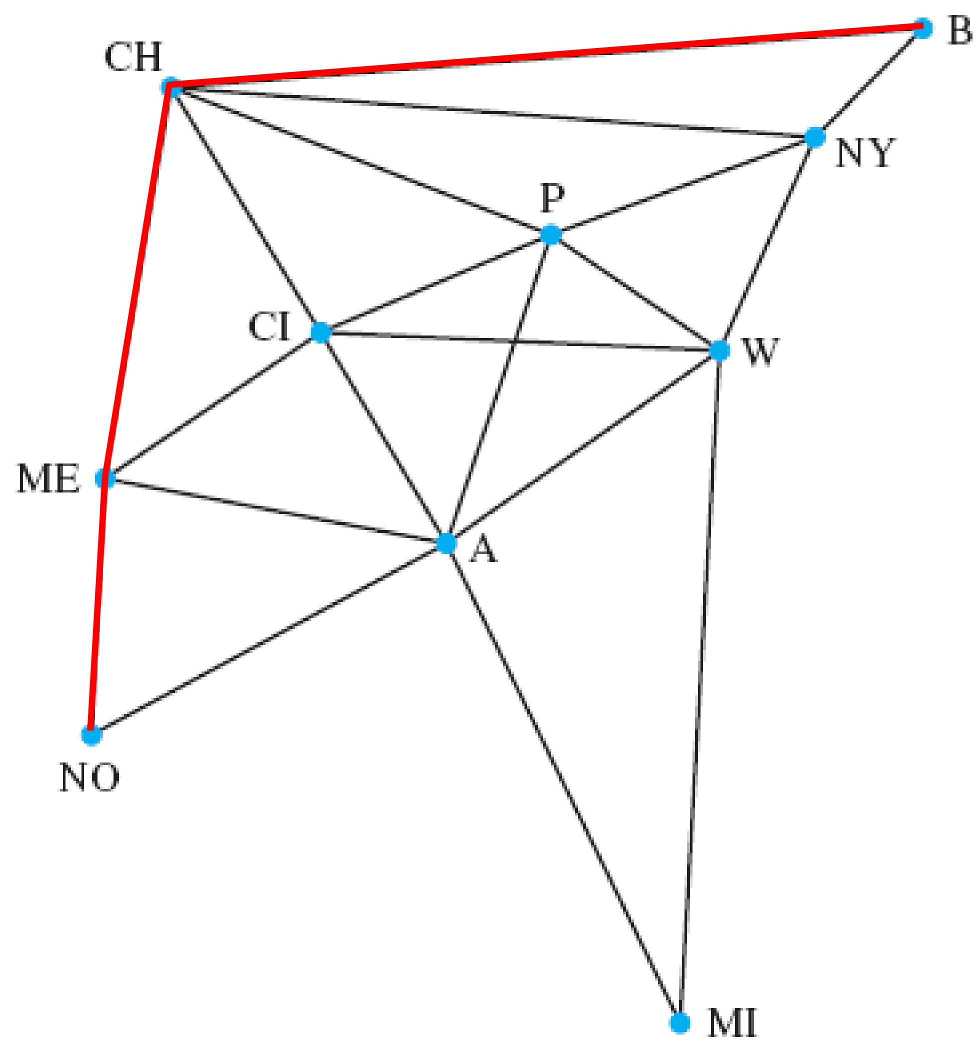
Length of a path $=$ # of edges on path

Path



Path

Path from Boston to New Orleans is B, CH, ME, NO



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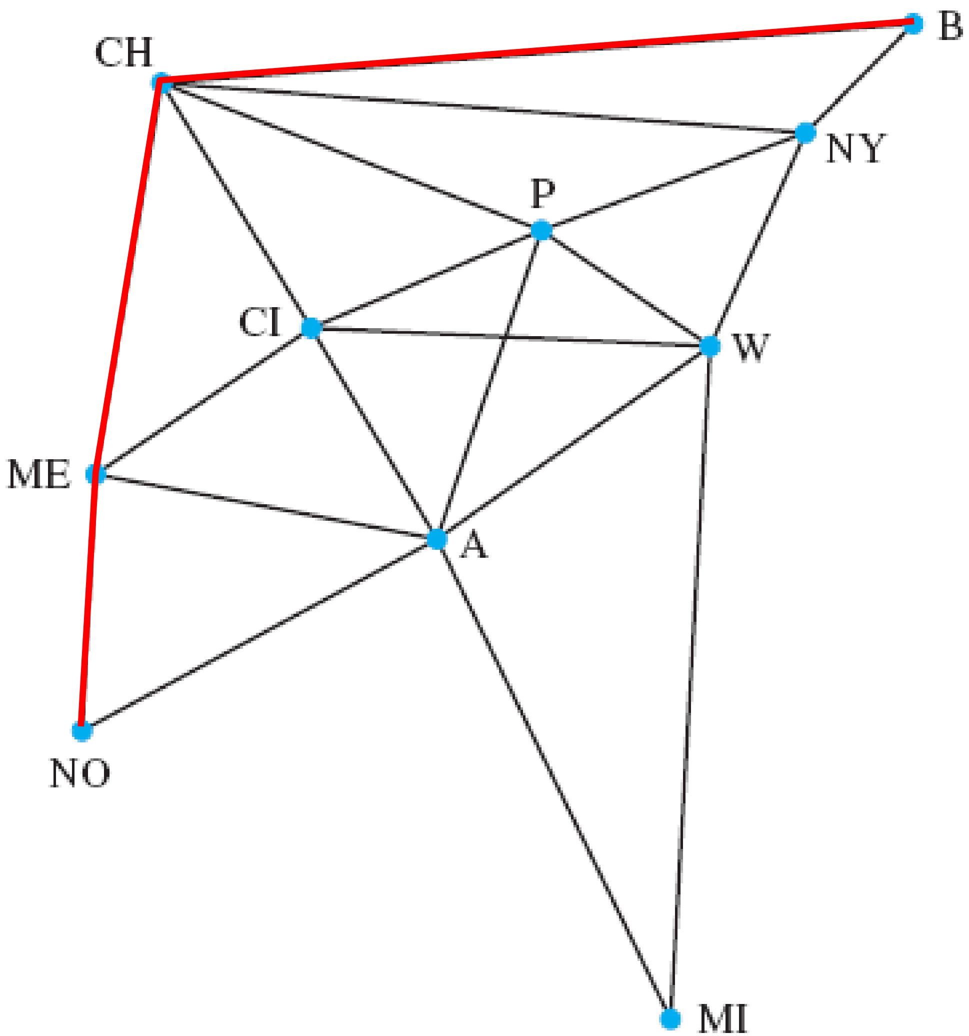
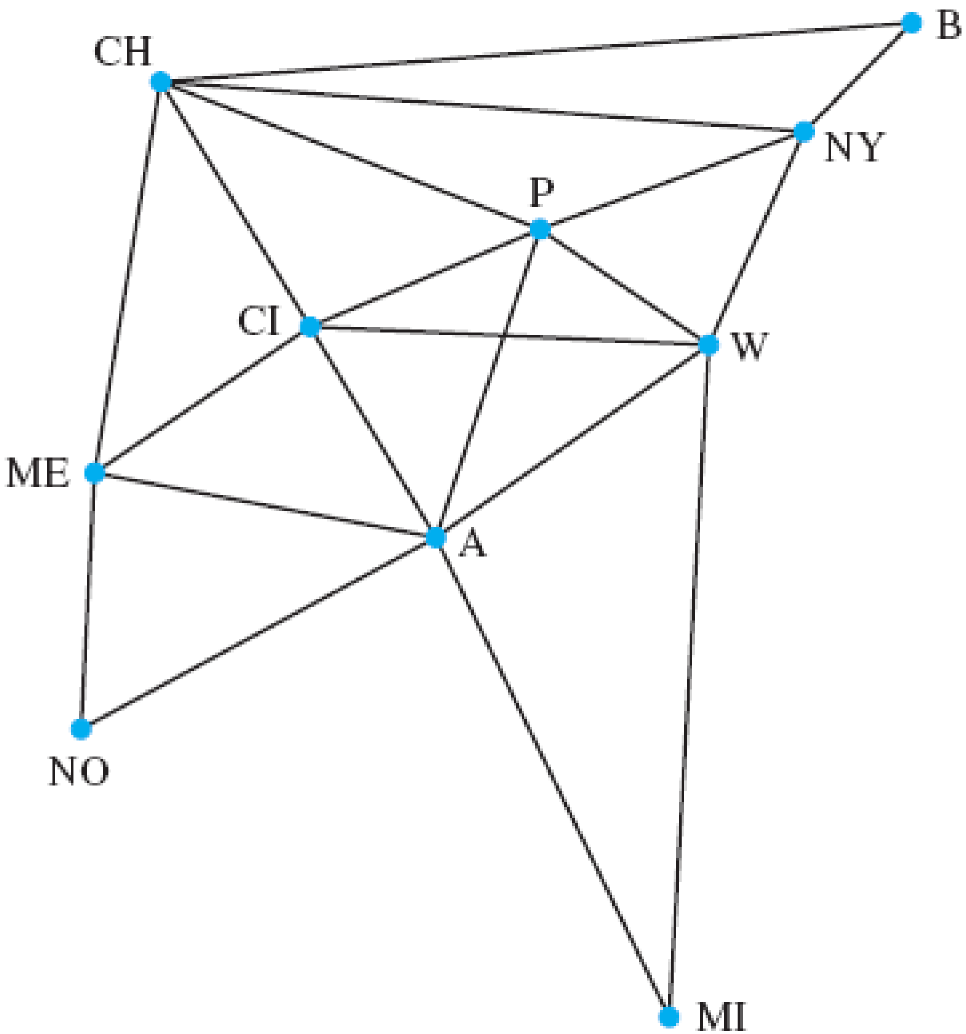
This path has length 3.



Connectivity



Company decides to lease only minimum number of communication lines it needs to be able to send a message from any city to any other city by using any number of intermediate cities.

What is the minimum number of lines it needs to lease?

Connectivity

- Choosing 10 edges?

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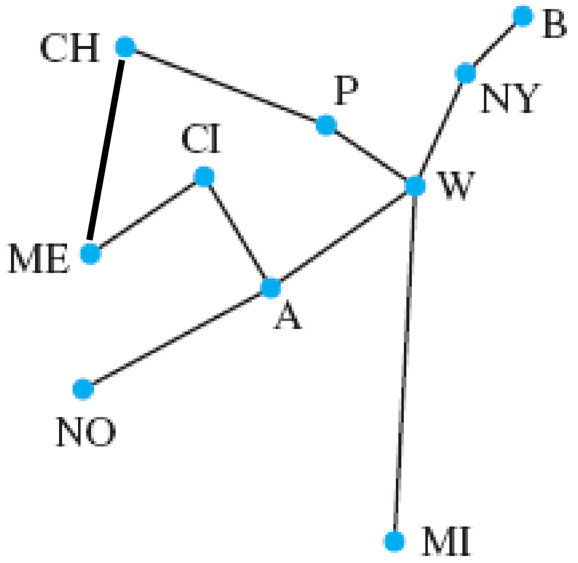

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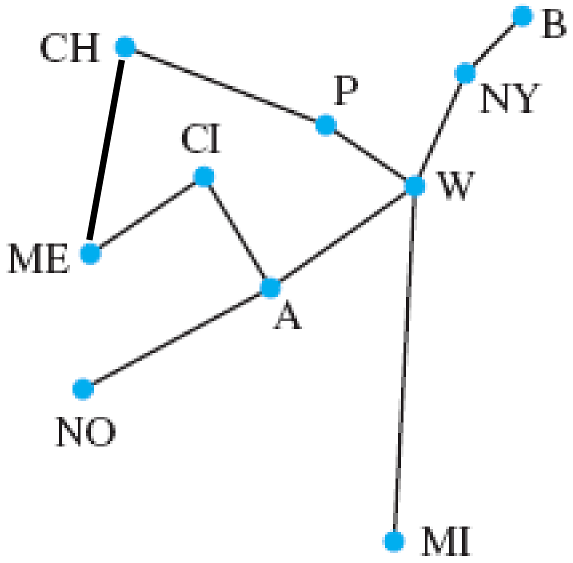

Too many.

Could throw away edge CI, A, and still have a solution.

Connectivity

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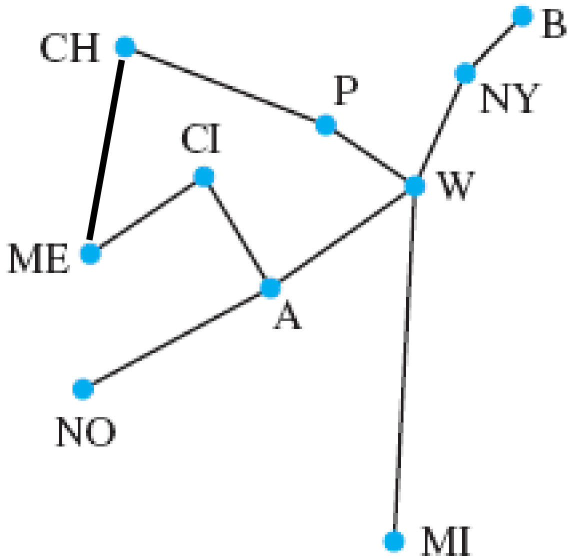

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Choosing 8 edges?

Connectivity

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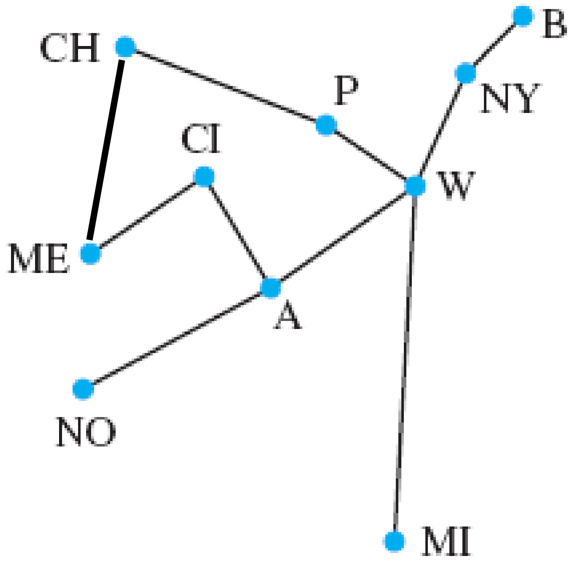

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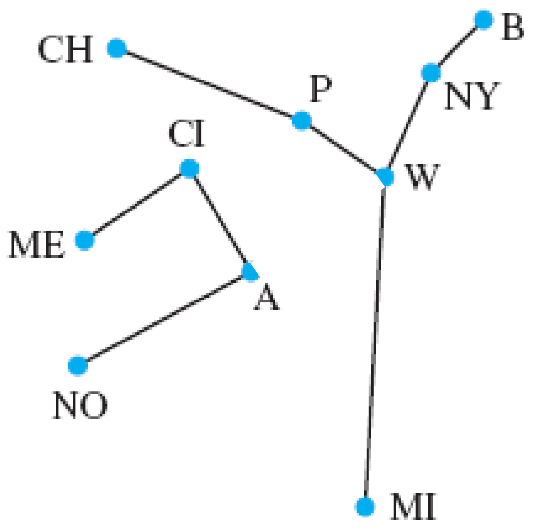

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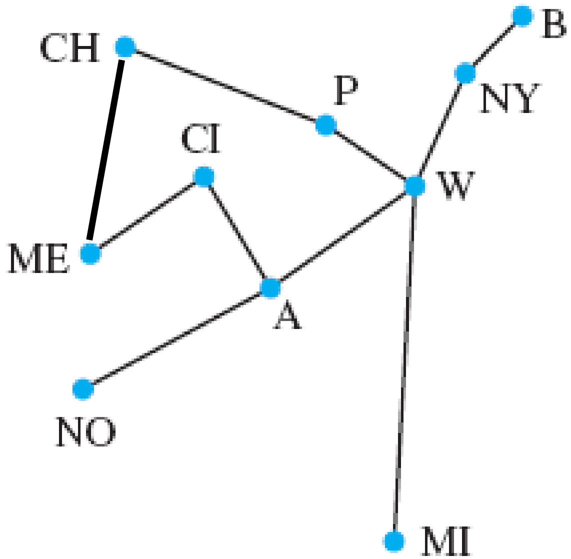

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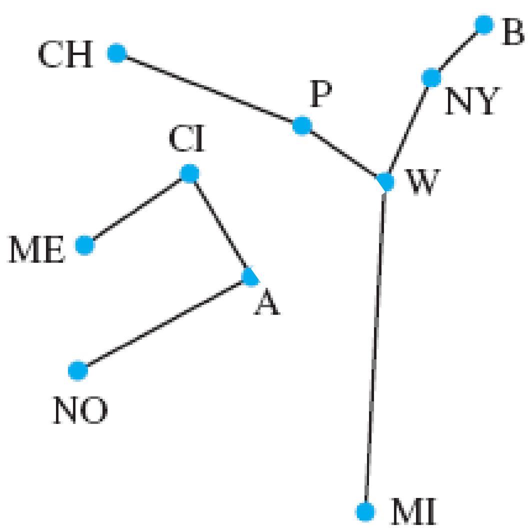

Not enough.

There is no path from, e.g., NO to B.

Connectivity

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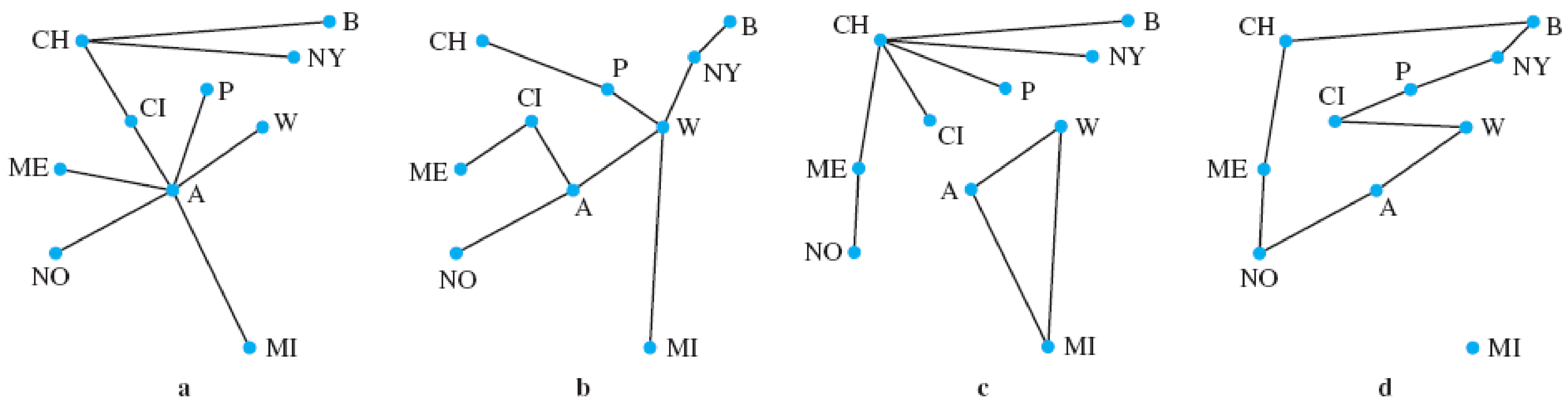

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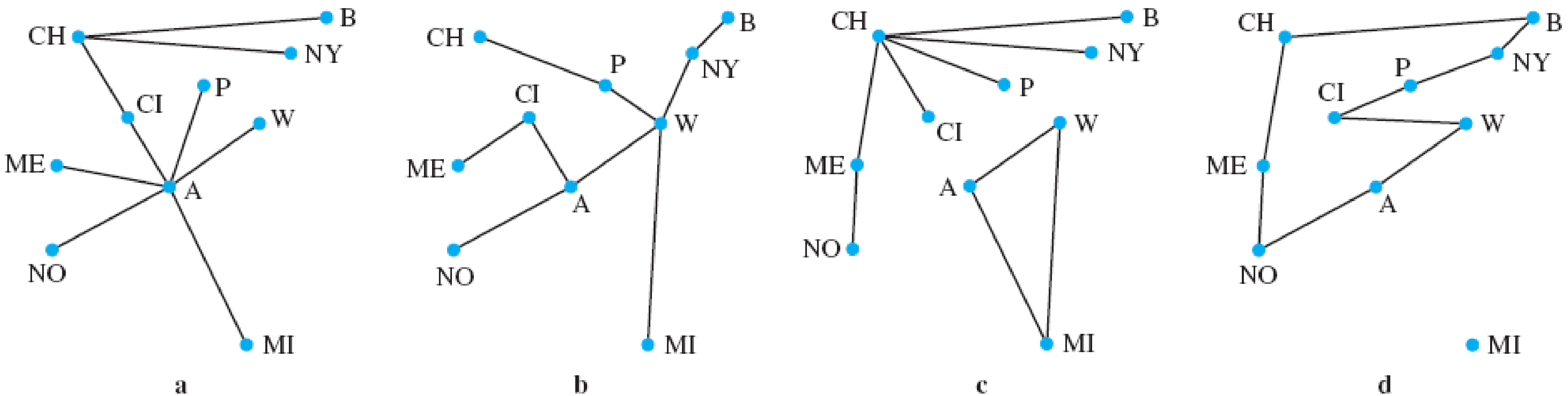

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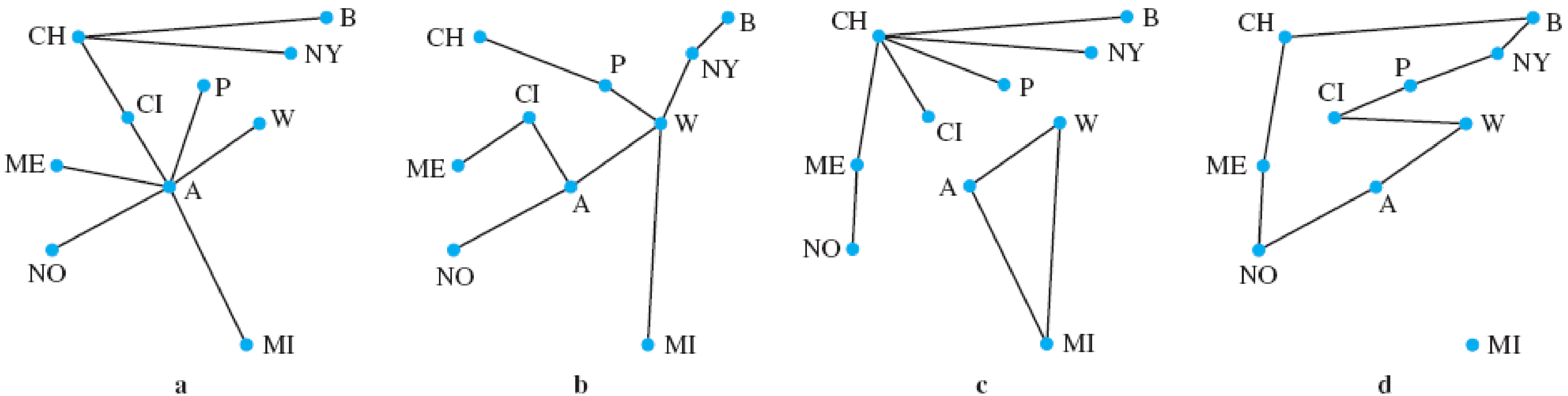

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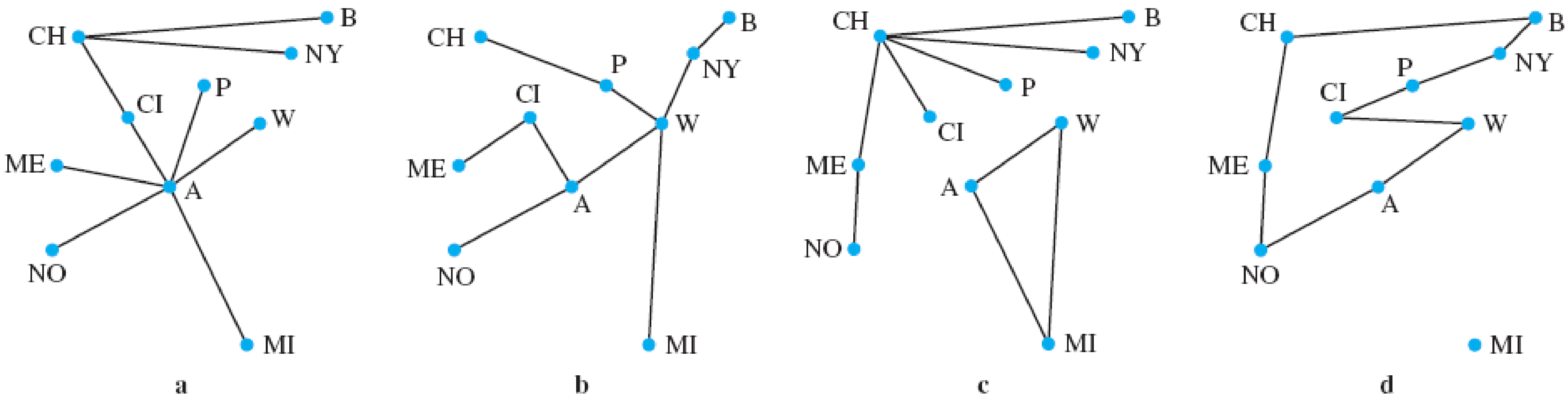

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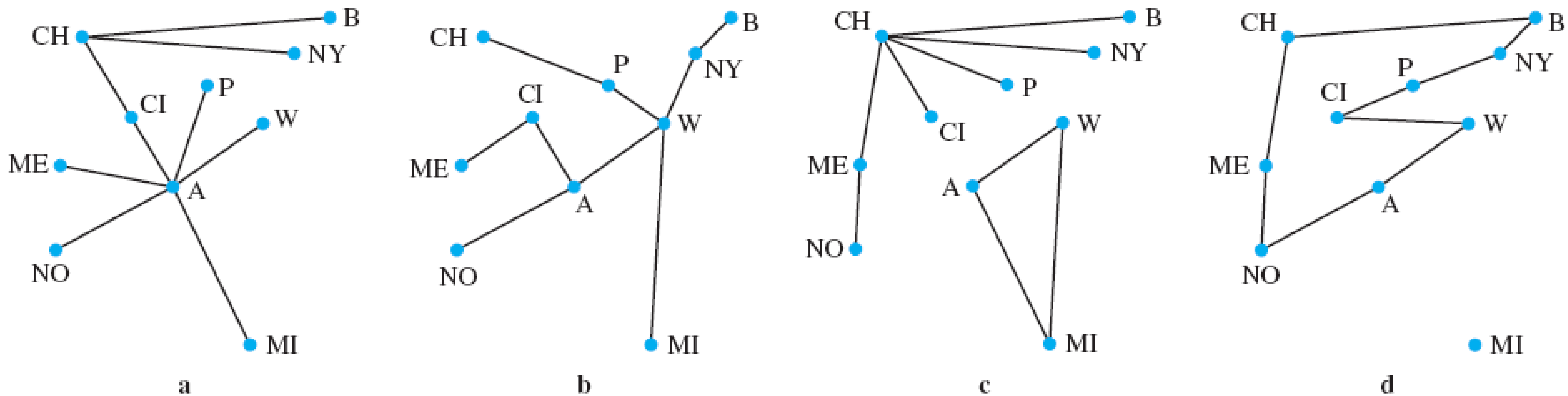

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- **Lemma** If there is a path between two distinct vertices x and y of a graph G, then there is a simple path between x and y in G.

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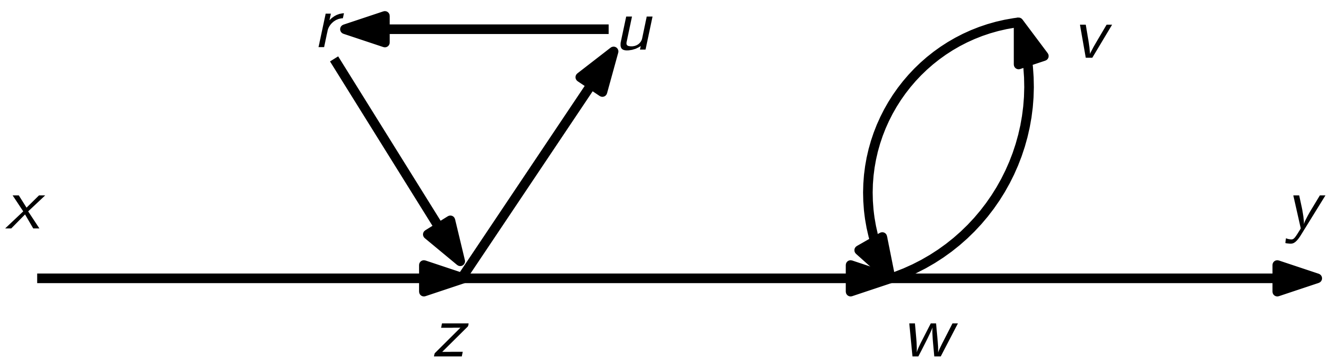

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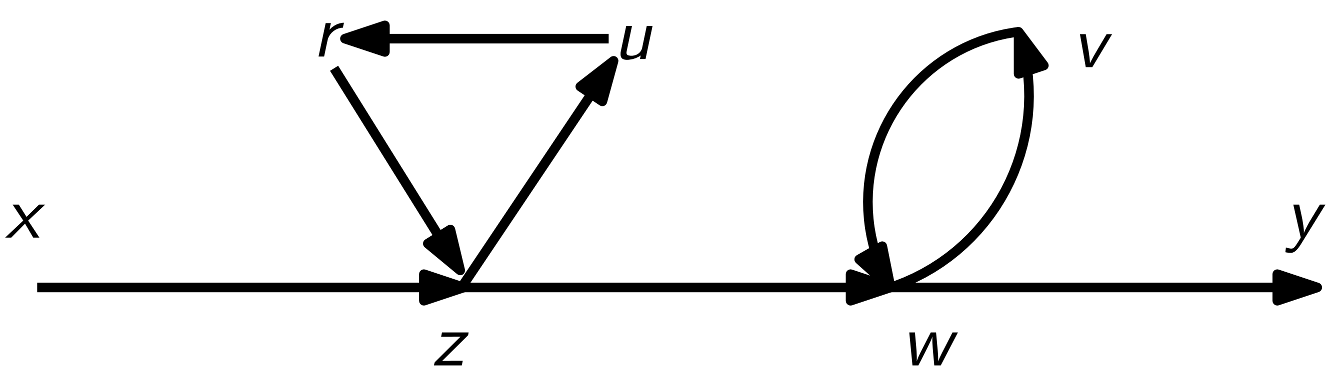

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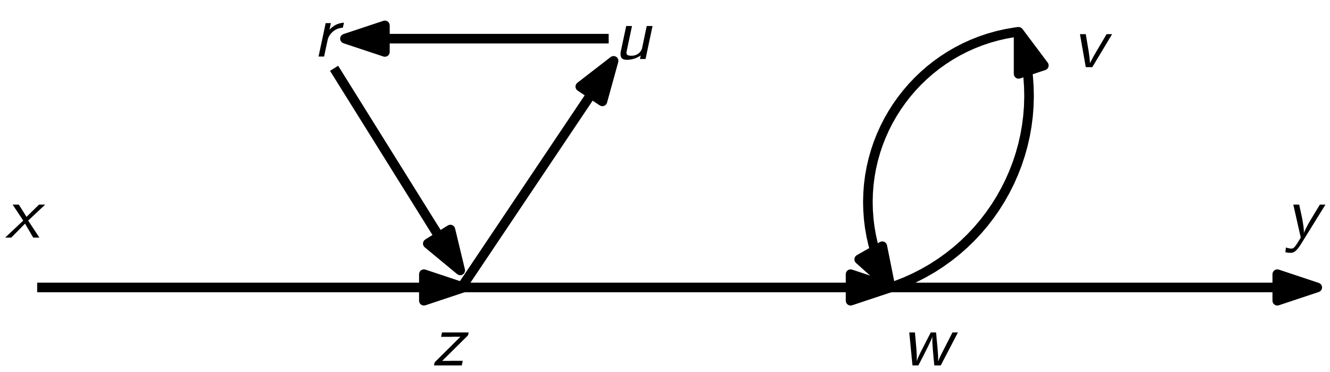

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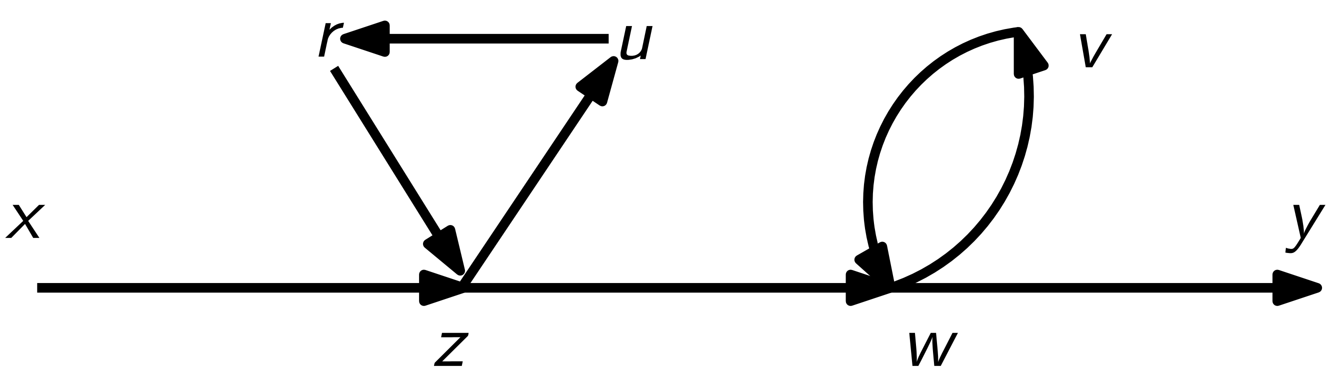

Path from x to y → Path from x to y

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Theorem There is a simple path between every pair of distinct vertices of a connected undirected graph.

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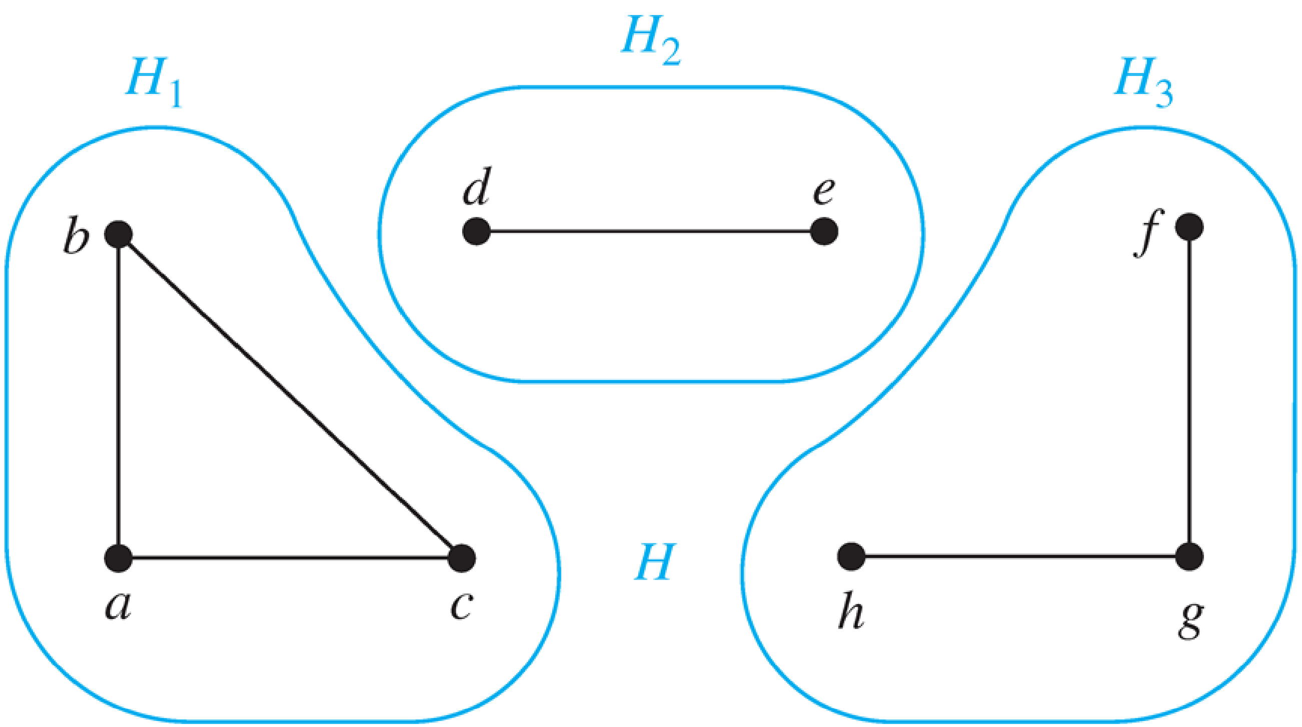

Connectedness in Directed Graphs

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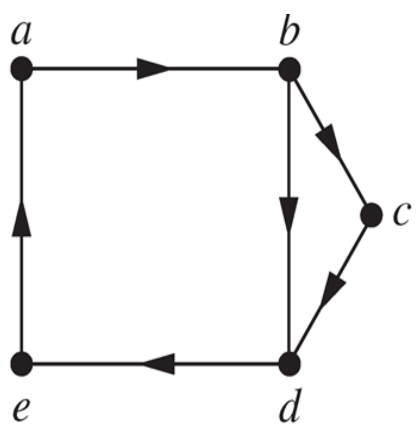

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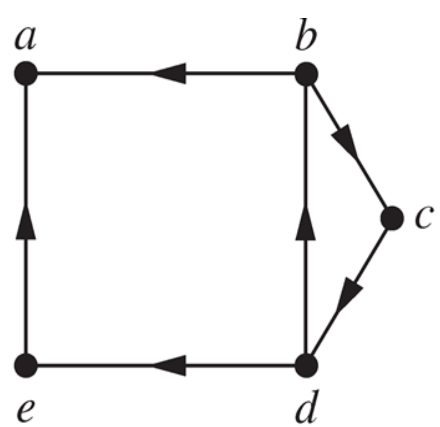

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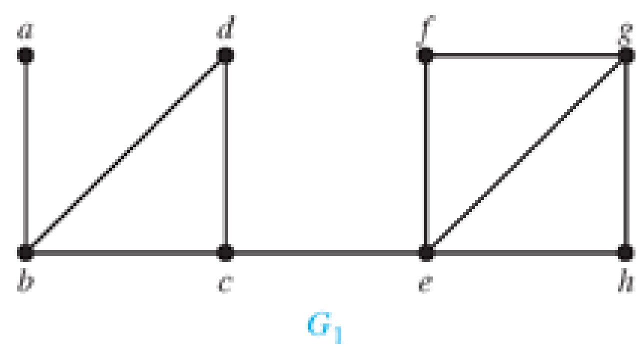

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- The existence of a simple circuit of length k is isomorphic invariant. In addition, paths can be used to construct mappings that may be isomorphisms.

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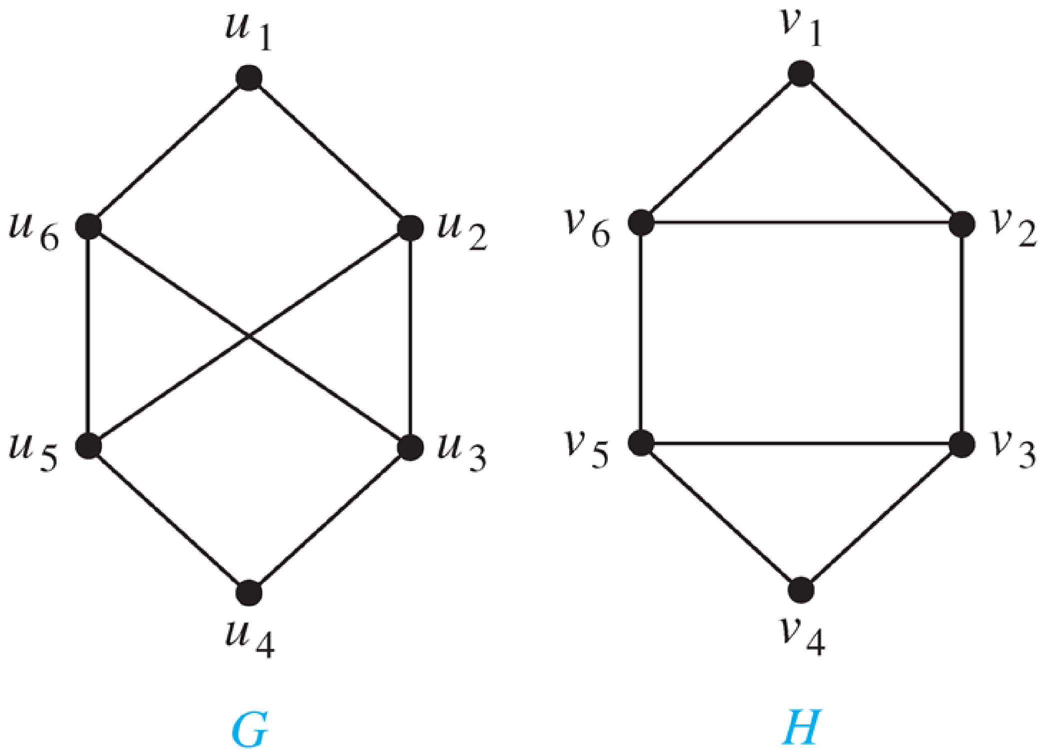

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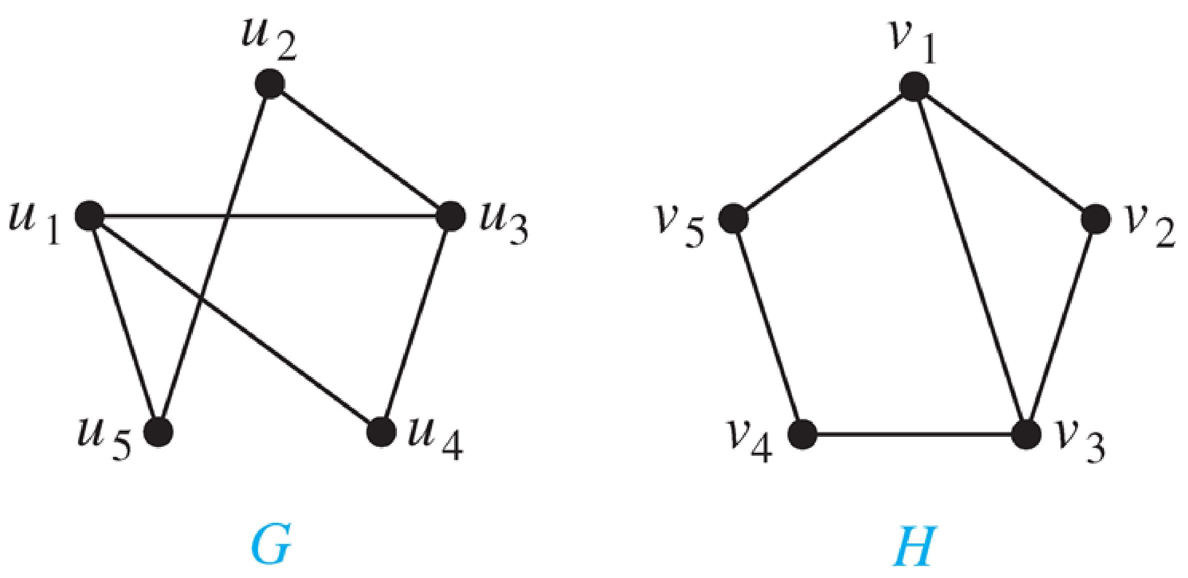

Counting Paths between Vertices

- **Theorem** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where $r > 0$ is positive, equals the (i, j)-th entry of $\mathbf{A}^r$.

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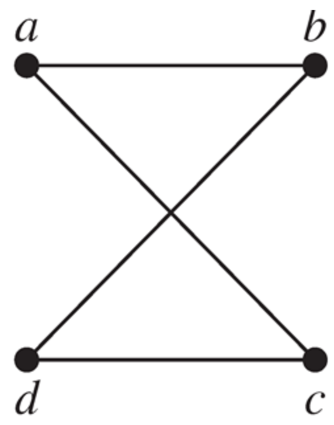
Proof (by **induction**)

$\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)-th entry of $\mathbf{A}^{r+1}$ equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}$, where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$.

Counting Paths between Vertices

- **Example** How many paths of length 4 are there from a to d in the graph G?



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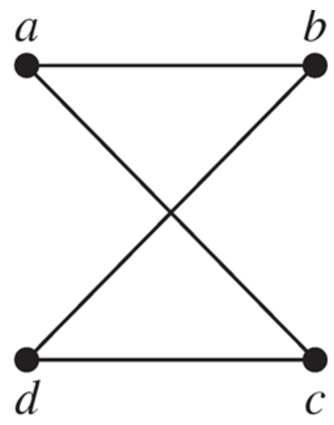

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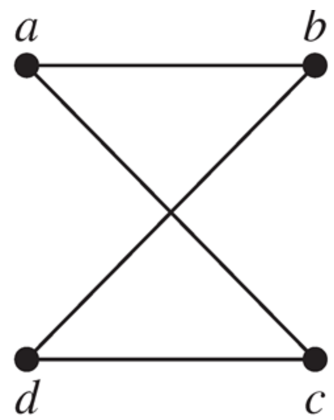

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}$$

Next Lecture

- Graph theory II ...