



CS215 DISCRETE MATH

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Cryptography

- History of almost 4000 years (from 1900 B.C.)

Cryptography = kryptos + graphos

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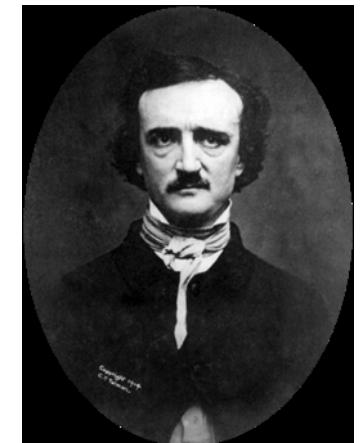
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The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



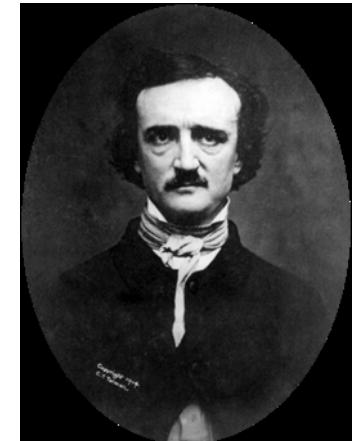
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(secret) (writing)

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“Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve.” – 1941



Cryptography

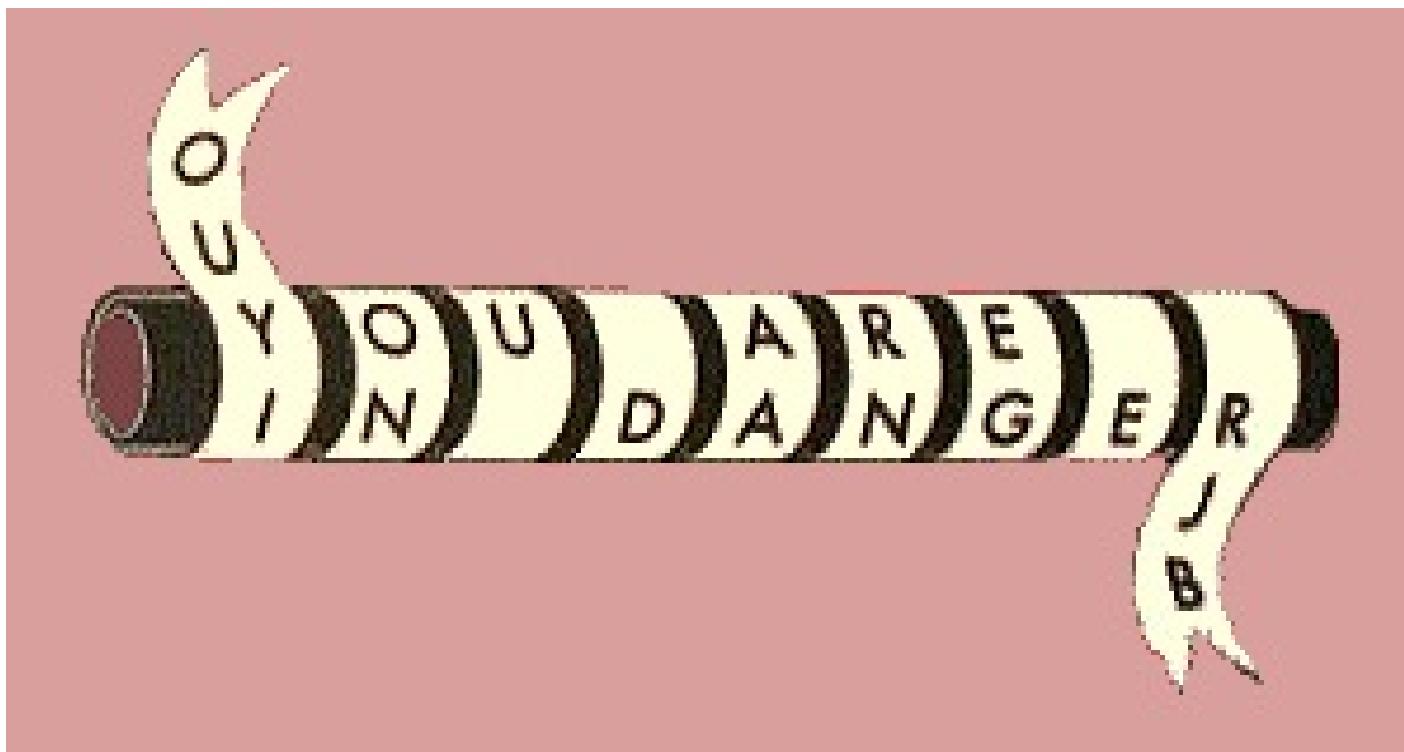
- One-sentence definition:

“Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*.” – Ronald L. Rivest



Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



Some Examples

- The Greeks also invented a cipher which changed **letters** to **numbers**. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Some Examples

- Caesar Cipher (after the name of JULIUS CAESAR)



VENI, VIDI, VICI

YHQL YLGL YLFL

Some Examples

- Morse Code: created by Samuel Morse in 1838

Morse Alphabet	
A	• —
B	— • • •
C	— • — •
D	— • •
E	•
F	• • — •
G	— — •
H	• • • •
I	• •
J	• — — —
K	— • —
L	• — • •
M	— —
N	— •
O	— — —
P	• — — — •
Q	— — • —
R	• — •
S	• • •
T	—
U	• • —
V	• • • —
W	• — —
X	— • • —
Y	— • — —
Z	— — • •
Full stop (.)	• — • — • —
Break signal or fresh line	— • • • —
Apostrophe (')	• — — — — •
Hyphen (-)	— • • • • —
Exclamation (!)	— — • • — —
Interrogation (?)	• • — — • •
Underline (_____)	• • — — • —
Parenthesis ()	— • — — • —
Inverted commas (" ")	• — • • — •

Some Examples

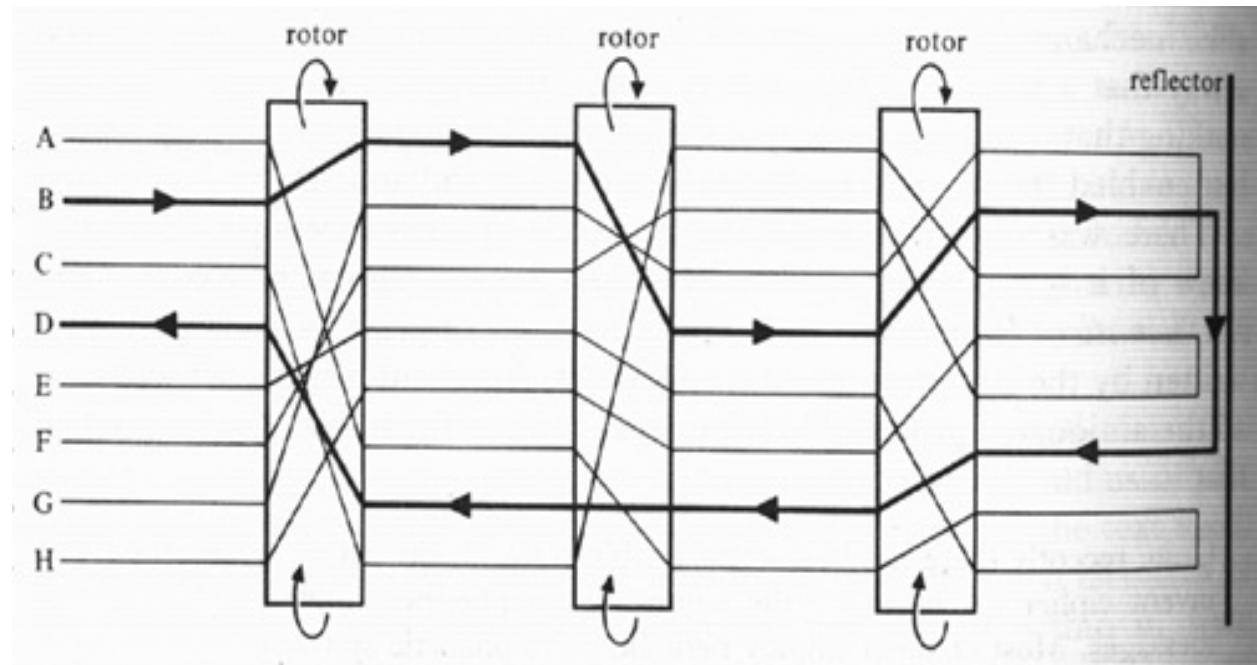
- Crytograms from the Chinese gold bars



<http://www.iacr.org/misc/china/china.html>

Some Examples

- Enigma, Germany coding machine in *World War II*.



Some Examples

- Sigaba, used by U.S. during *World War II*.



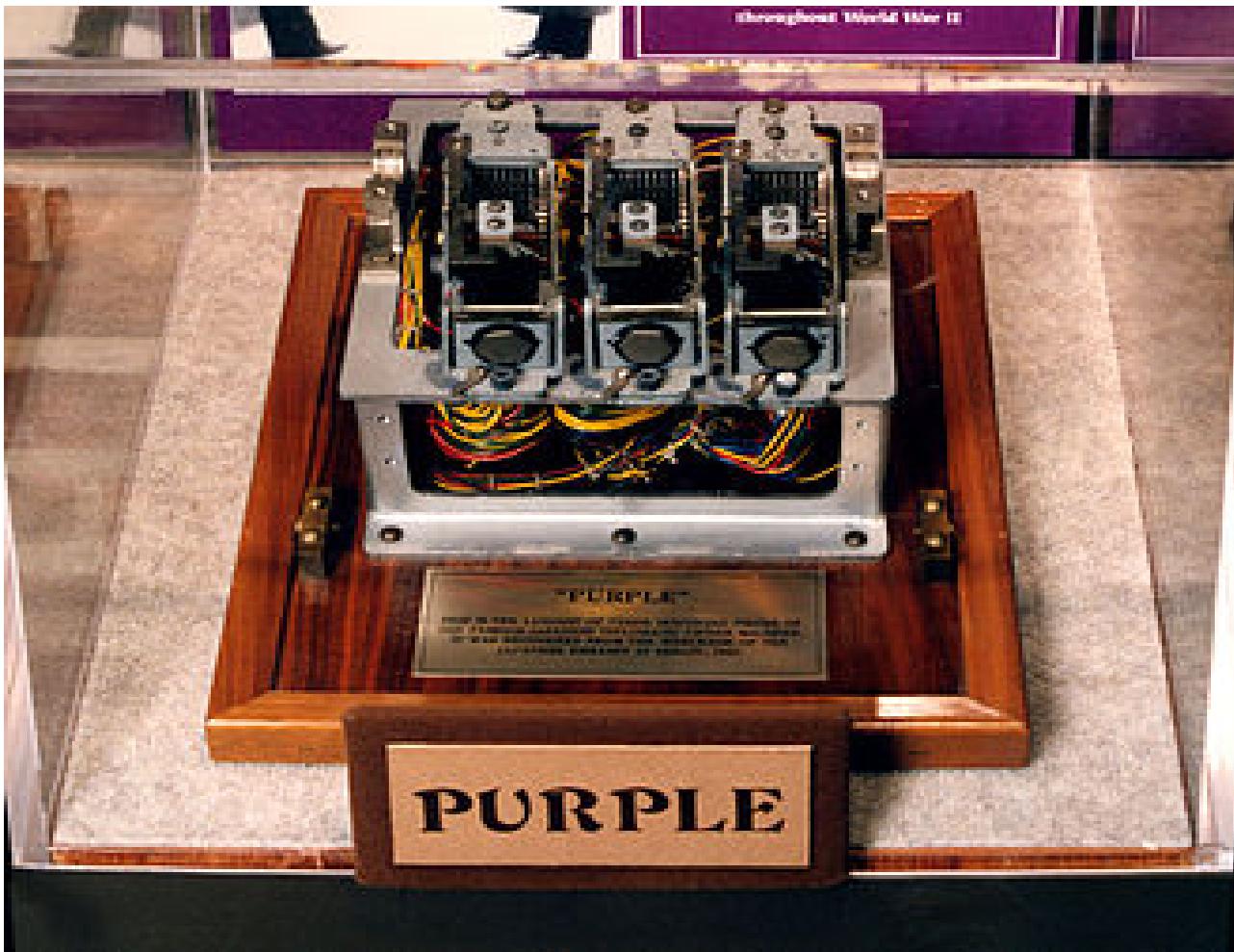
Some Examples

- Japanese “Enigma” Rotor Cipher Machine



Some Examples

- Japanese Purple Machine (97-shiki obun inji-ki)



People Working in Breaking Codes



Alan Turing
(1912-1954)



Claude E. Shannon
(1916-2001)

Cryptography History

- History (until 1970's)

“*Symmetric*” cryptography

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“*Symmetric*” cryptography



They need agree in advance on the *secret key k*.

Q: How can they do this?

Q: What if Bob could send Alice a “special key” useful only for **encryption** but no help for **decryption**?

Caesar Cipher

- Key: $k = 0, 1, \dots, 25$

Encryption: encode i as $(i + k) \bmod 26$

Decryption: decode j as $(j - k) \bmod 26$

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plaintext: SEND REINFORCEMENT

Key: 2

ciphertext: UGPF TGKPHQTEGOGPV

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Problem: only 26 possibilities for keys!

Kerchoff's Principle (1883): System should be secure even if algorithms are known, as long as key is secret.

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C		Z
V	R	E		D

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However, substitution cipher is still **insecure!**

Key observation: can recover plaintext using *statistics* on *letter frequencies*.

Substitution Cipher

■ Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXXLEYVEOIEWHRXEXIPFE
MVEWHKVSTYLYZIXLIKIIXPPIJVSZEYPERRGERIMWQL
MGLMXQERIWGPSRIHMXQEREKI

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I – *most common letter*

LI – *most common pair*

XLI – *most common triple*

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Y = g

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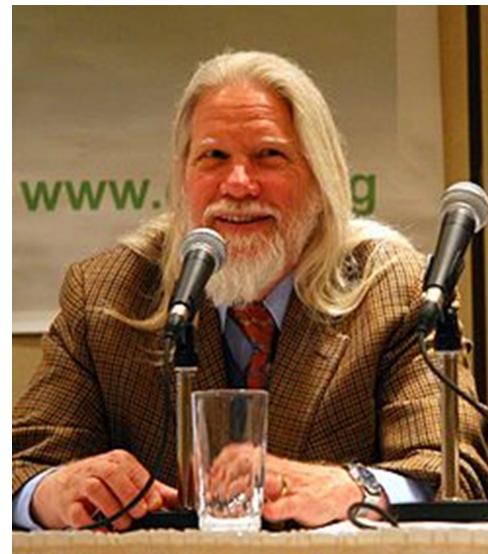
Y = g

HereUpOnLeGrandAroseWithAGraveAndStatelyAirAndBroug
MeTheBeetleFromAGlassCaselnWhichItWasEnclosedIt-
WasABe

Cryptography History

- History (from 1976)
 - ◊ W. Diffie, M. Hellman, “New direction in cryptography”, *IEEE Transactions on Information Theory*, vol. 22, pp. 644-654, 1976.

“We stand today on the brink of a revolution in cryptography.”



Bailey W. Diffie

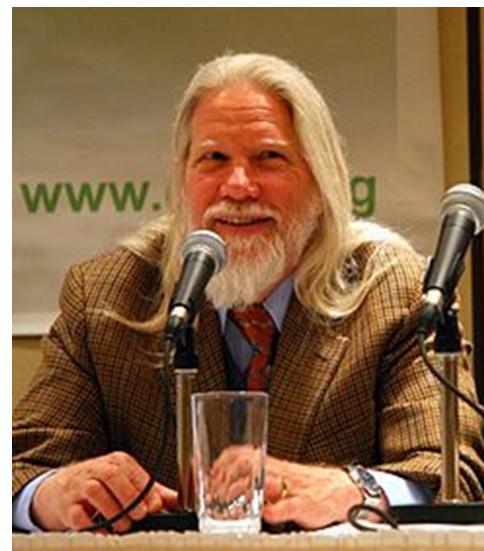
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2015 Turing Award

Bailey W. Diffie

Martin E. Hellman

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to modern cryptography . Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography," ^[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today. ^[40]
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Public Key Cryptography

- Alice wants to send a message to Bob



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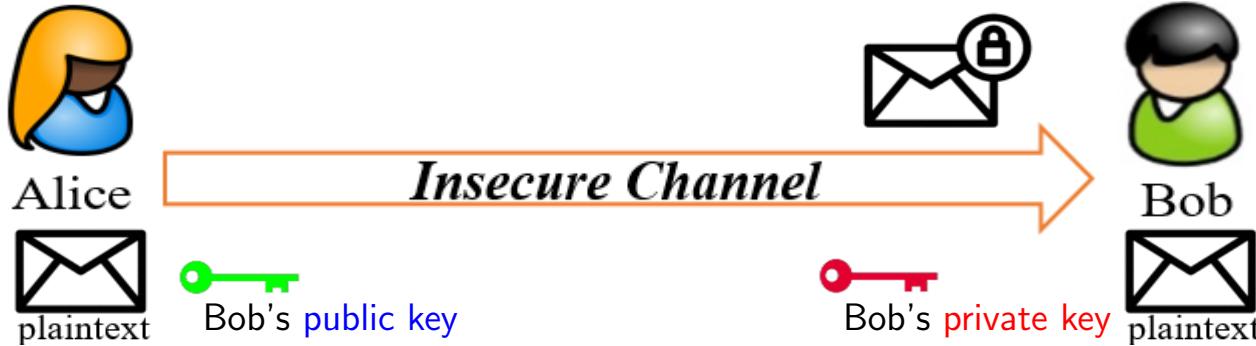
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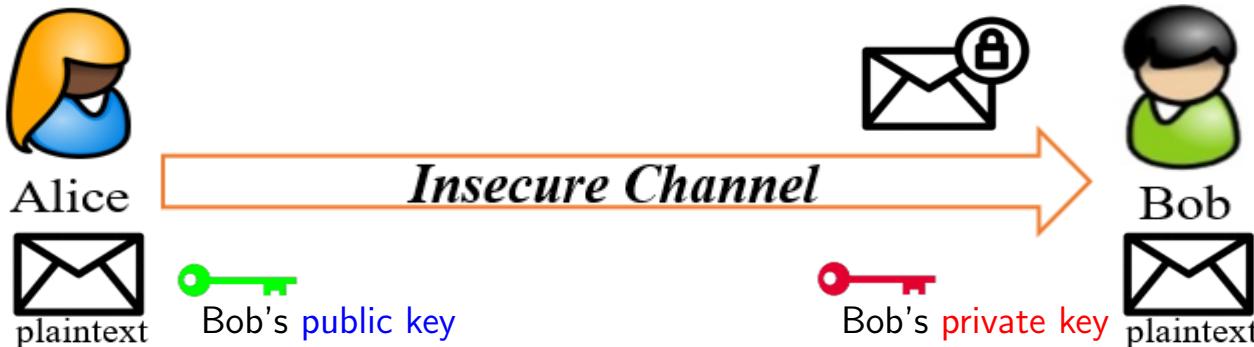
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Ronald L. Rivest



Adi Shamir



Leonard M. Adleman

R. Rivest, A. Shamir, L. Adleman, “A method for obtaining digital signatures and public-key cryptosystems”,
Communications of the ACM, vol. 21-2, pages 120-126, 1978.

RSA Public Key Cryptosystem

■ Rivest-Shamir-Adleman

2002 **Turing Award**

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Pick two **large** primes, p and q . Let $n = pq$, then $\phi(n) = (p - 1)(q - 1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
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RSA Public Key Cryptosystem

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Theorem (Correctness) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

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Q : How to prove this?

RSA Public Key Cryptosystem: Example

Parameters:	p	q	n	$\phi(n)$	e	d
	5	11	55	40	7	23

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	5	11	55	40	7	23

Public key: (7, 55)

Private key: 23

Encryption: $M = 28, C = M^7 \text{ mod } 55 = 52$

Decryption: $M = C^{23} \text{ mod } 55 = 28$

RSA Public Key Cryptosystem: Parameters

Parameters: p q n $\phi(n)$ e d

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CS 208 – Algorithm Design and Analysis

The Security of the RSA

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Q : Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p - 1, q - 1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

- SSL/TLS protocol

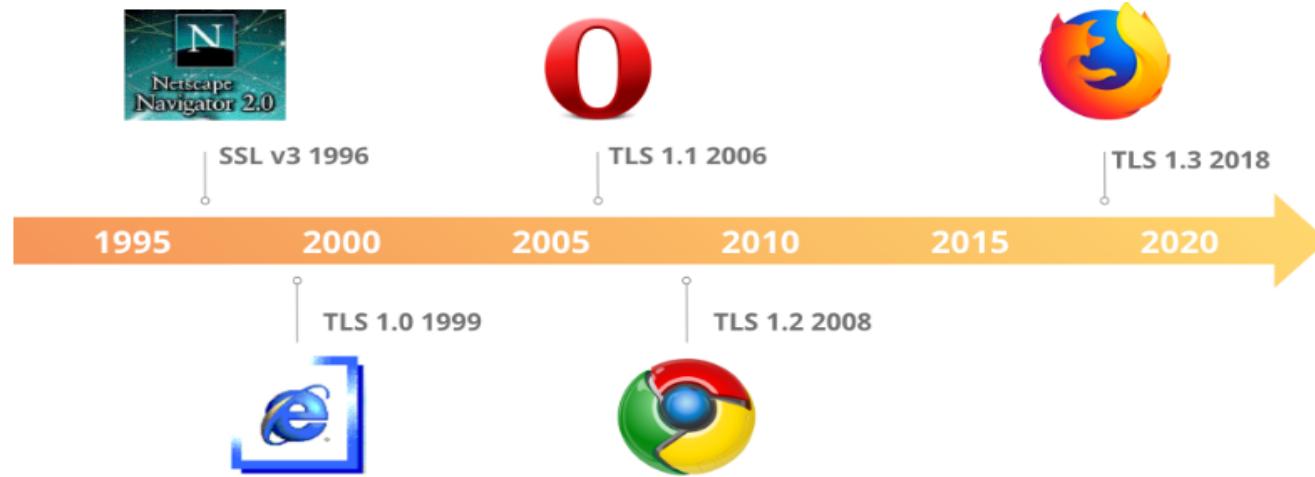
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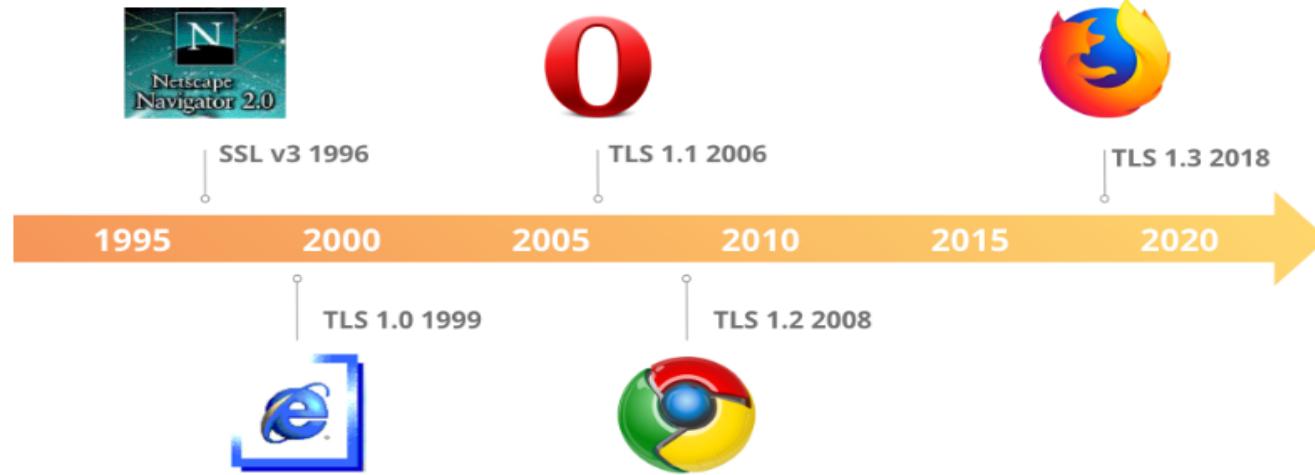
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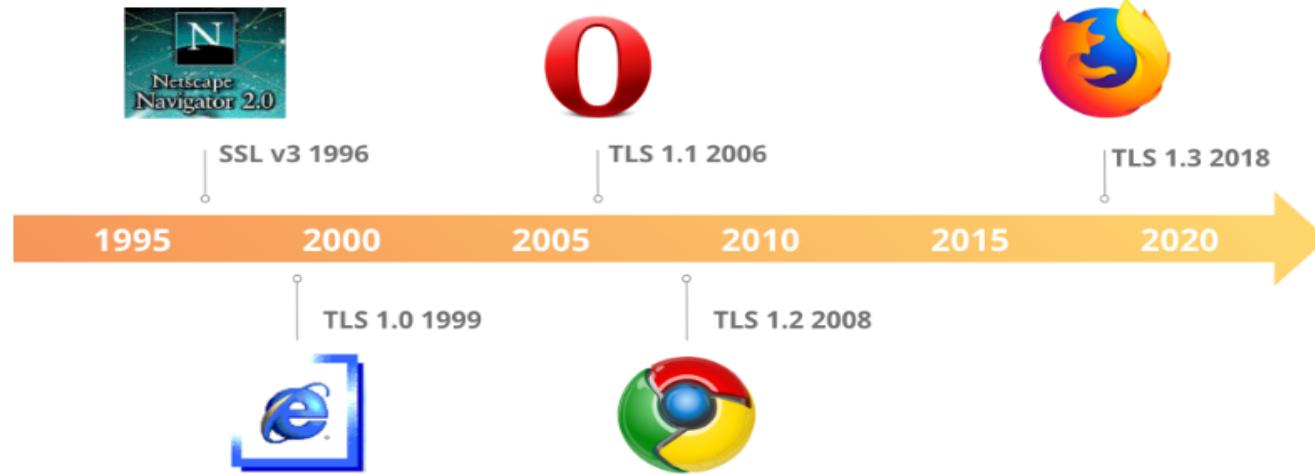


Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
ECDH-RSA	No	No	Yes	Yes	Yes	No
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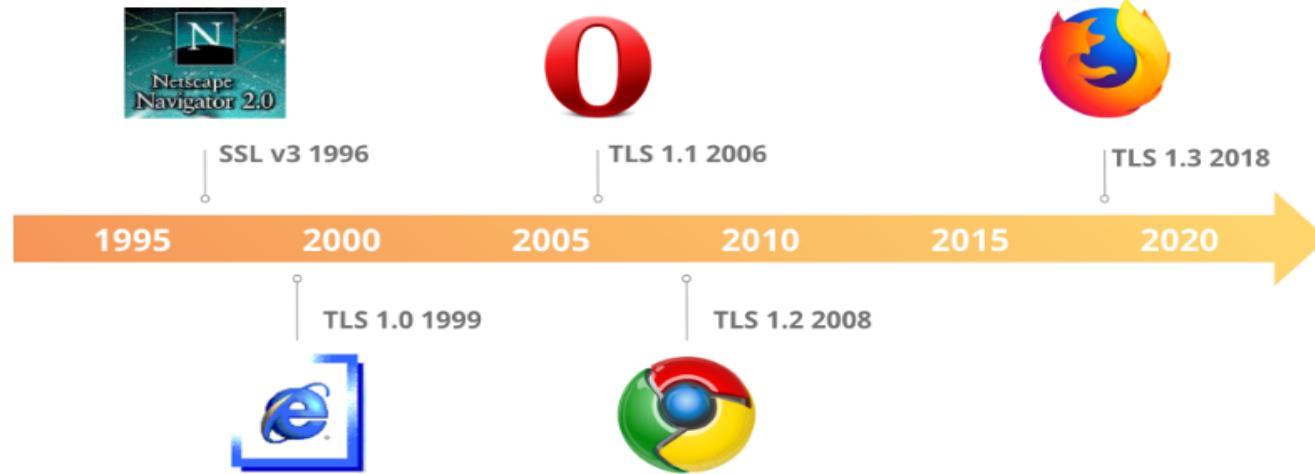
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CS 305 – Computer Networks

Applications of RSA

SSL/TLS protocol



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CS 305 – Computer Networks

CS 403 – Cryptography and Network Security

Using RSA for Digital Signature

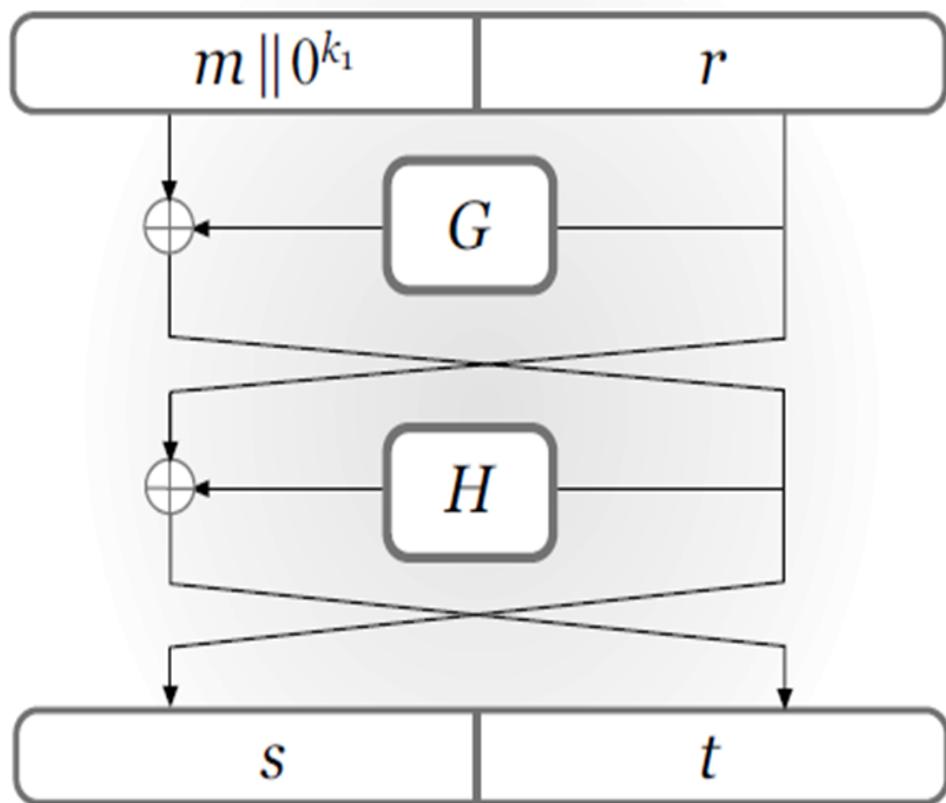
$$S = M^d \bmod n \text{ (RSA signature)}$$

$$M = S^e \bmod n \text{ (RSA verification)}$$

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



The Discrete Logarithm

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Discrete Logarithm Problem:

Given n , b and y , find x .

This is very hard!

El Gamal Encryption

- **Setup** Let p be a prime, and g be a generator of \mathbb{Z}_p . The **private key** x is an integer with $1 < x < p - 2$. Let $y = g^x \bmod p$. The **public key** for *El Gamal encryption* is (p, g, y) .

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El Gamal Encryption: Pick a **random** integer k from \mathbb{Z}_{p-1} ,

$$\begin{aligned} a &= g^k \bmod p \\ b &= My^k \bmod p \end{aligned}$$

The ciphertext C consists of the pair (a, b) .

El Gamal Decryption:

$$M = b(a^x)^{-1} \bmod p$$

Using El Gamal for Digital Signature

$$\begin{aligned} a &= g^k \pmod{p} \\ b &= k^{-1}(M - xa) \pmod{p-1} \end{aligned}$$

(El Gamal **signature**)

$$y^a a^b \equiv g^M \pmod{p}$$

(El Gamal **verification**)

Using El Gamal for Digital Signature

$a = g^k \pmod{p}$
 $b = k^{-1}(M - xa) \pmod{p-1}$
(El Gamal **signature**)

$y^a a^b \equiv g^M \pmod{p}$
(El Gamal **verification**)

Q : How to verify it?

An Example

Choose $p = 2579$, $g = 2$, and $x = 765$. Hence
 $y = 2^{765} \bmod 2579 = 949$.

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- **(Public key)** $k_e = (p, g, y) = (2579, 2, 949)$
- **(Private key)** $k_d = x = 765$

Encryption: Let $M = 1299$ and choose a random $k = 853$,

$$\begin{aligned}(a, b) &= (g^k \bmod p, My^k \bmod p) \\ &= (2^{853} \bmod 2579, 1299 \cdot 949^{853} \bmod 2579) \\ &= (435, 2396).\end{aligned}$$

Decryption:

$$M = b(a^x)^{-1} \bmod p = 2396 \times (435^{765})^{-1} \bmod 2579 = 1299.$$

Security of the El Gamal Cryptosystem

Question 1: Is it feasible to derive x from (p, g, y) ?

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It is equivalent to solving the DLP. It is **believed** that there is **NO** polynomial-time algorithm. p should be large enough, typically 160 bits.

Security of the El Gamal Cryptosystem

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Attack 1: Use $M = by^{-k}$. However, k is **randomly** picked.

Attack 2: Use $M = b(a^x)^{-1} \bmod p$, but x is **secret**.

Diffie-Hellman Key Exchange Protocol

User A

Generate random
 $X_A < p$
calculate
 $Y_A = \alpha^{X_A} \text{ mod } p$

Calculate
 $k = (Y_B)^{X_A} \text{ mod } p$

Y_A
→
←
 Y_B

User B

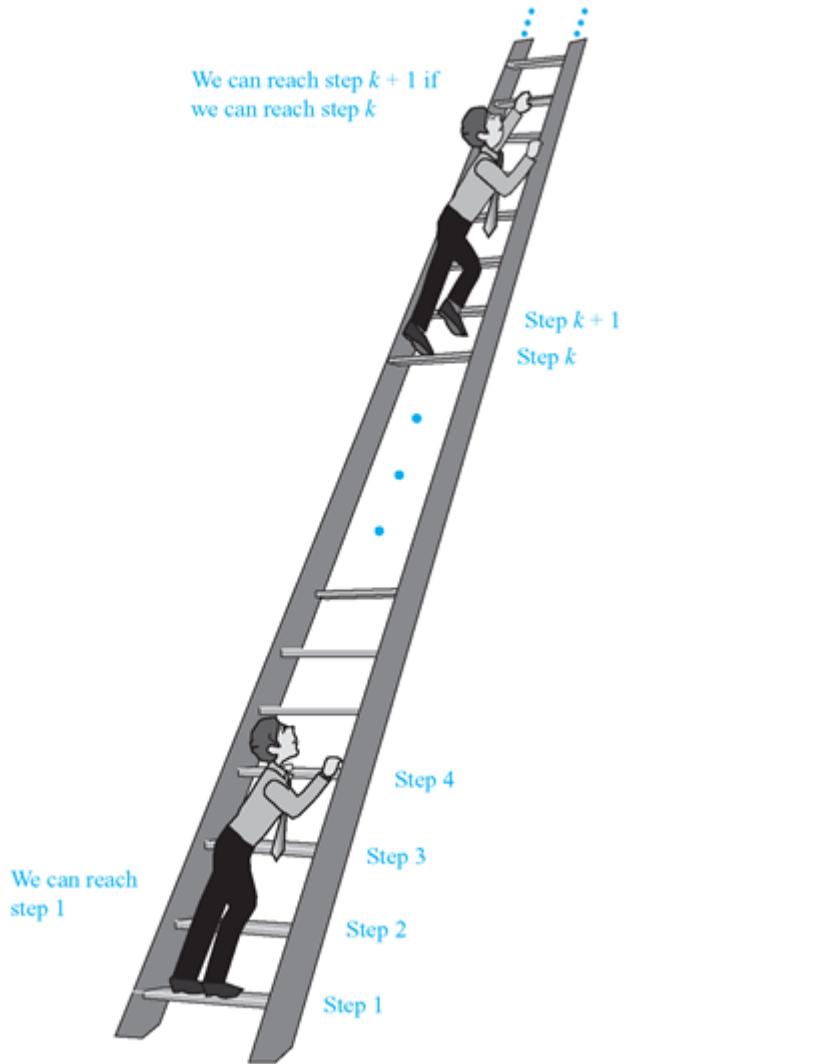
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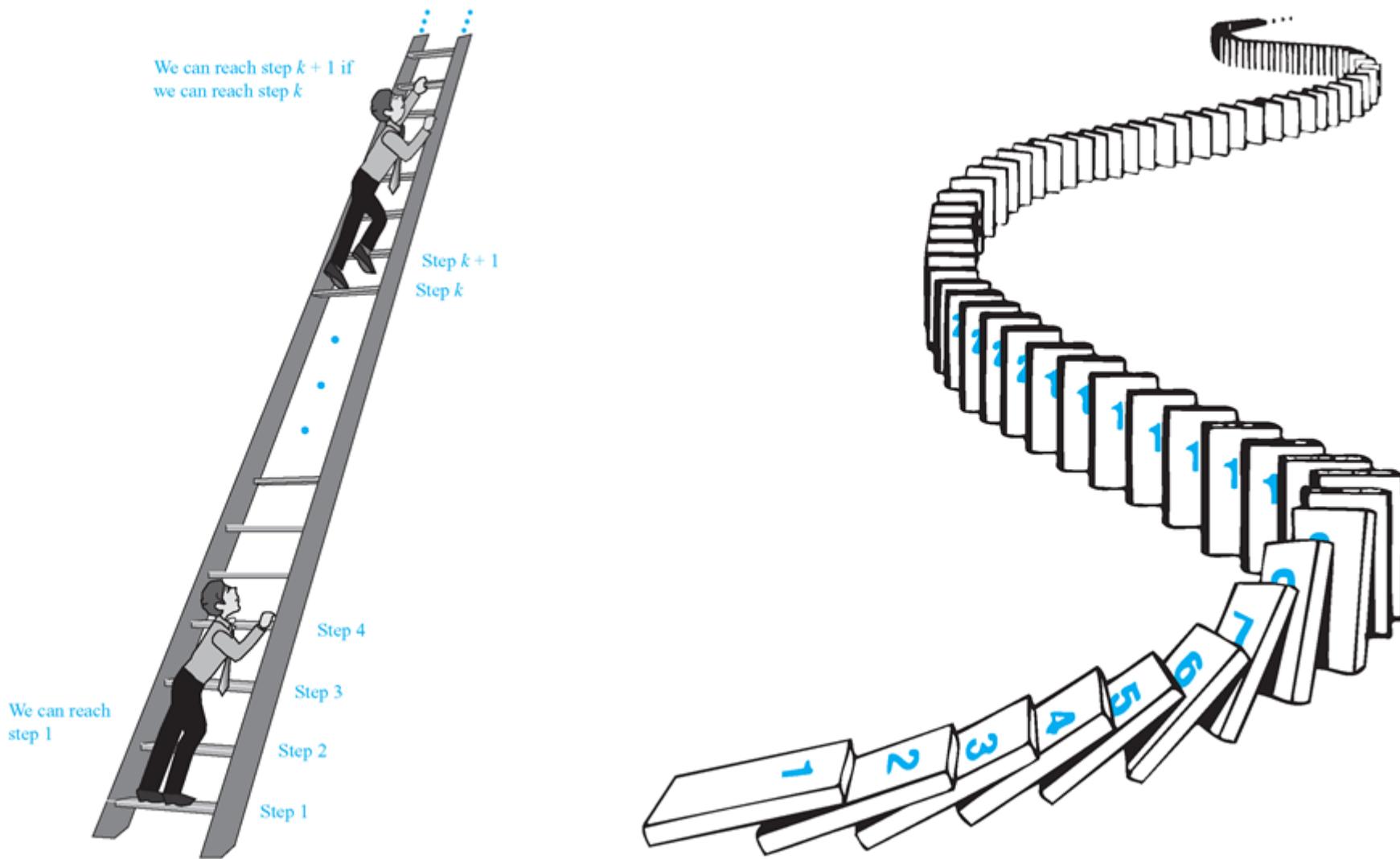
Cryptography Wonders

- *Digital Signatures.* Electronically sign documents
- Zero-knowledge Proofs.* Alice proves to Bob that she earns $< \$50k$ without Bob learning her income.
- Privacy-preserving data mining.* Bob holds DB. Alice gets answer to one query, without Bob knowing what she asked.
- Playing poker over the net.* Alice, Bob, Carol and David can play Poker over the net without trusting each other or any central server. (*E-Voting*)
- Electronic Auctions.* Can run auctions s.t. no one (even not seller) learns anything other than winning party and bid.
- Fully Homomorphic Encryption.* Encrypt $E(m)$ in a way that allows to compute $E(f(m))$.

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- We conclude by distinguishing between the *weak principle* of mathematical induction and the *strong principle* of mathematical induction.

The *strong principle* can actually be derived from the *weak principle*.

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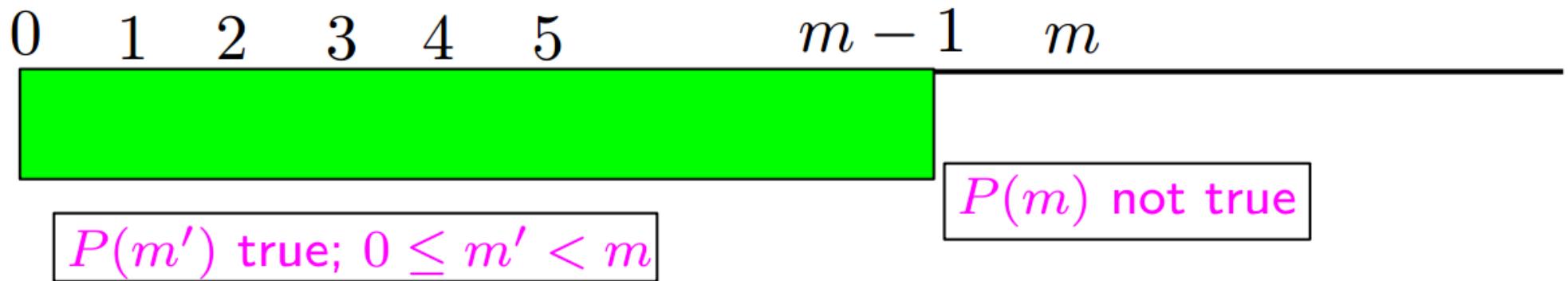
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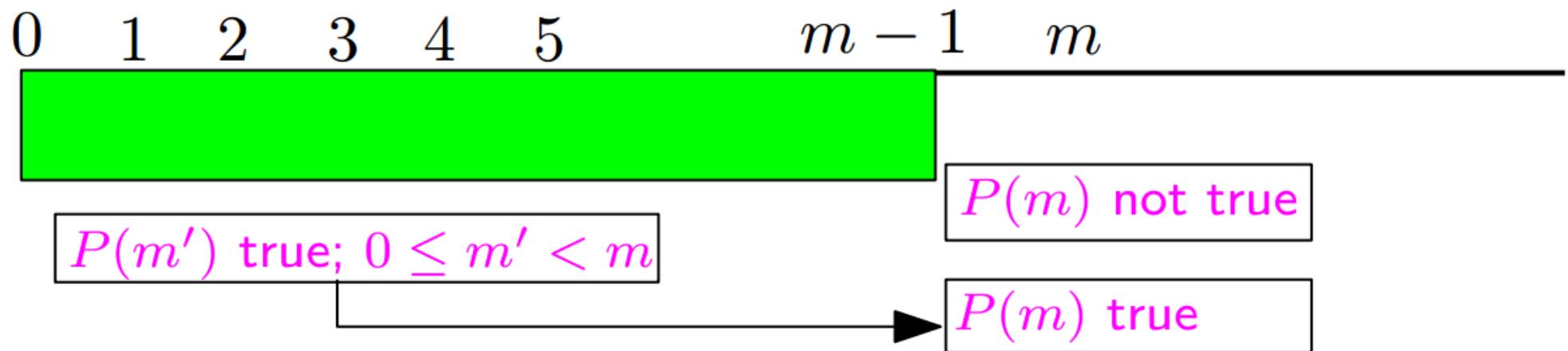


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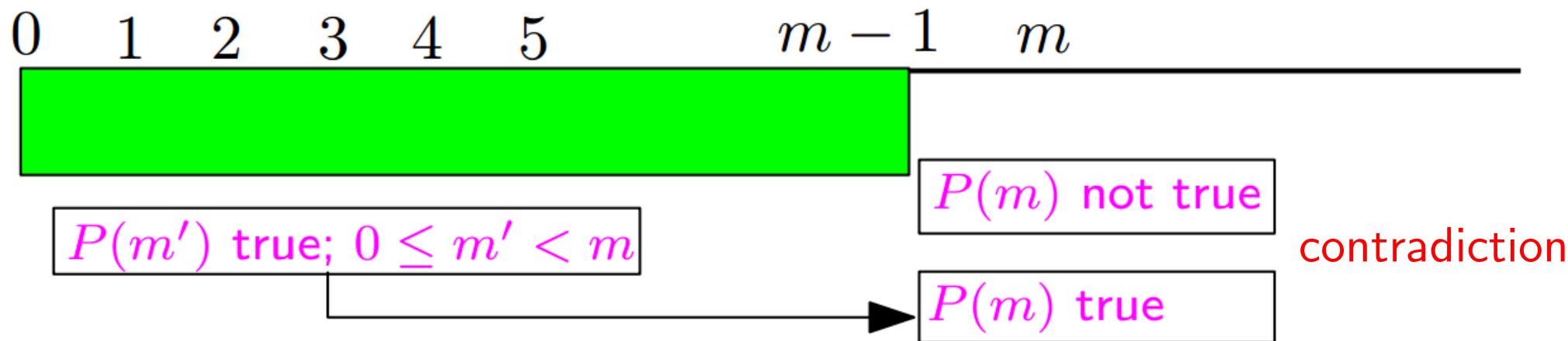
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- ◊ Therefore, (*) holds for all positive integers n .

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The **key step** was proving that

$$P(n - 1) \rightarrow P(n)$$

where $P(n)$ is the statement

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When a **for all** quantifier is false, there must be some n for which it is false. Let n be the smallest nonnegative integer for which $2^{n+1} \not\geq n^2 + 2$.

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Then setting $i = n - 1$ gives

$$2^{(n-1)+1} \geq (n-1)^2 + 2.$$

or

$$(*) \quad 2^n \geq n^2 - 2n + 1 + 2 = n^2 - 2n + 3$$

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46 - 5

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Since $P(n - 1) \rightarrow P(n)$, we see that

$P(0)$ implies $P(1)$, $P(1)$ implies $P(2)$, ...

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(a) – *Basic Step Inductive Hypothesis*

49 - 4 (b) – *Inductive Step Inductive Conclusion*

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By mathematical induction, $\forall n > 0, 2^{n+1} \geq n^2 + 2$.

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(ii) Suppose that $n > 2$ and that $2^n \geq (n - 1)^2 + 3$ (*)

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Hence, we've just prove that for $n > 2, P(n-1) \rightarrow P(n)$.

By mathematical induction, $\forall n > 2, 2^{n+1} \geq n^2 + 3$.

Proof by Induction

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Let $P(n) - 2^{n+1} \geq n^2 + 3$ Base Step

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 - ◊ Iterating gives us a proof of $P(n)$ for all n

Strong Induction

■ Principle (*The Strong Principle of Mathematical Induction*)

(a) If the statement $P(b)$ is true

(b) for all $n > b$, the statement

$P(b) \wedge P(b+1) \wedge \cdots \wedge P(n-1) \rightarrow P(n)$ is true.

then $P(n)$ is true for all integers $n \geq b$.

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 - ◊ Then, if n is not a prime power, it is a product of two smaller numbers, each of which is, by the **inductive hypothesis**, a power of a prime or a product of powers of primes.
 - ◊ Thus, by the **strong principle of mathematical induction**, every positive integer is a power of a prime or a product of powers of primes.

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3. We conclude on the basis of **the principle of mathematical induction** that $P(n)$ is true for all $n \geq b$.

Next Lecture

- recurrence ...

