Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Graph Concepts

- $G = (V, E)$, *simple* graph, *multigraph*, *pseudograph*
- *Undirected*, *directed* graph
- Special graphs

  K_n, C_n, W_n, Q_n, $K_{m,n}$

  Hall's Marriage Theorem on *bipartite* graphs

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  Hall's Marriage Theorem on *bipartite* graphs
- Representation of graphs

  adjacency list, *adjacency matrix*, *incidence matrix*

Isomorphism of Graphs

- **Definition** The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function is called an *isomorphism*.

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- Useful graph invariants include the number of vertices, number of edges, degree sequence, the existence of a simple circuit of length k, etc.

Counting Paths between Vertices

- **Theorem** Let G be a graph with adjacency matrix $\mathbf{A}$ with respect to the ordering $v_1, v_2, \ldots, v_n$ of vertices. The number of different paths of length r from v_i to v_j, where $r > 0$ is positive, equals the (i, j)-th entry of $\mathbf{A}^r$.

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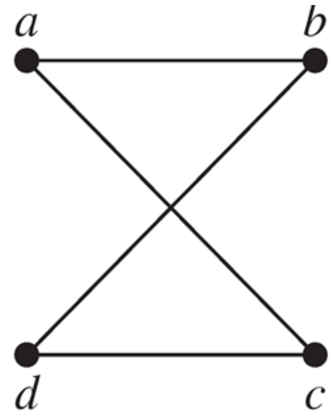
Proof (by **induction**)

$\mathbf{A}^{r+1} = \mathbf{A}^r\mathbf{A}$, the (i, j)-th entry of $\mathbf{A}^{r+1}$ equals $b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{in}a_{nj}$, where b_{ik} is the (i, k)-th entry of $\mathbf{A}^r$.

Counting Paths between Vertices

- **Example** How many paths of length 4 are there from a to d in the graph G?



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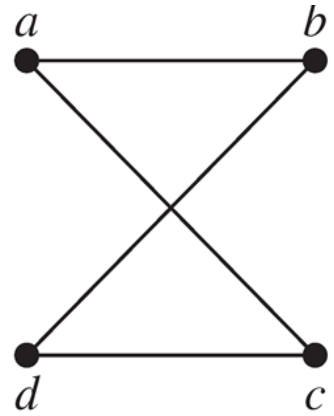

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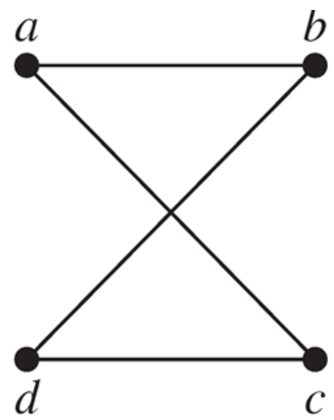

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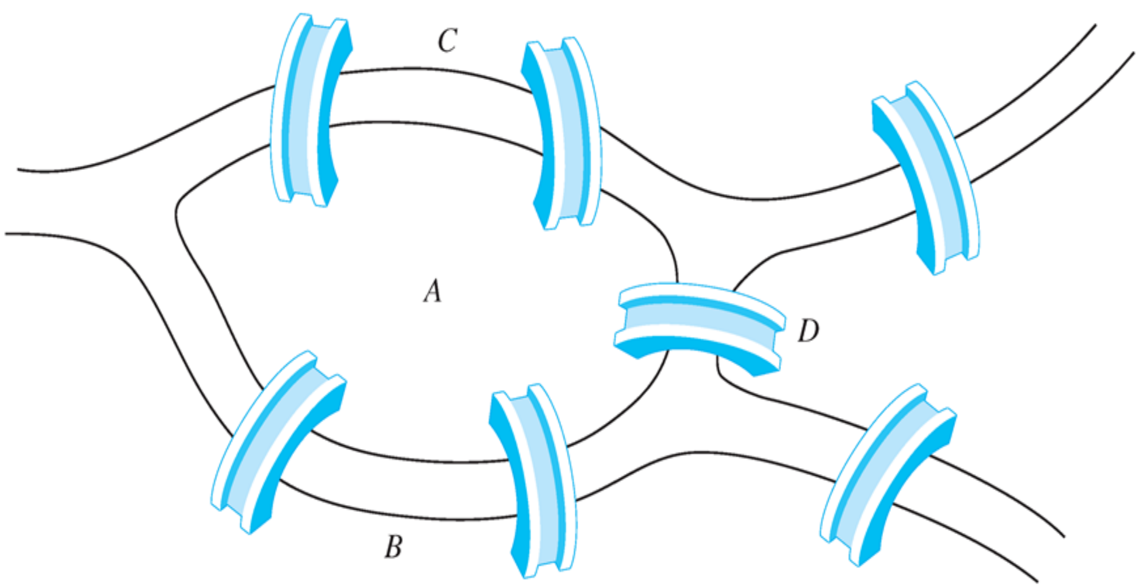
- **Königsberg seven-bridge problem**

  People wondered whether it was possible to start at some location in the town, travel across all the bridges once without crossing any bridge twice, and return to the starting point.



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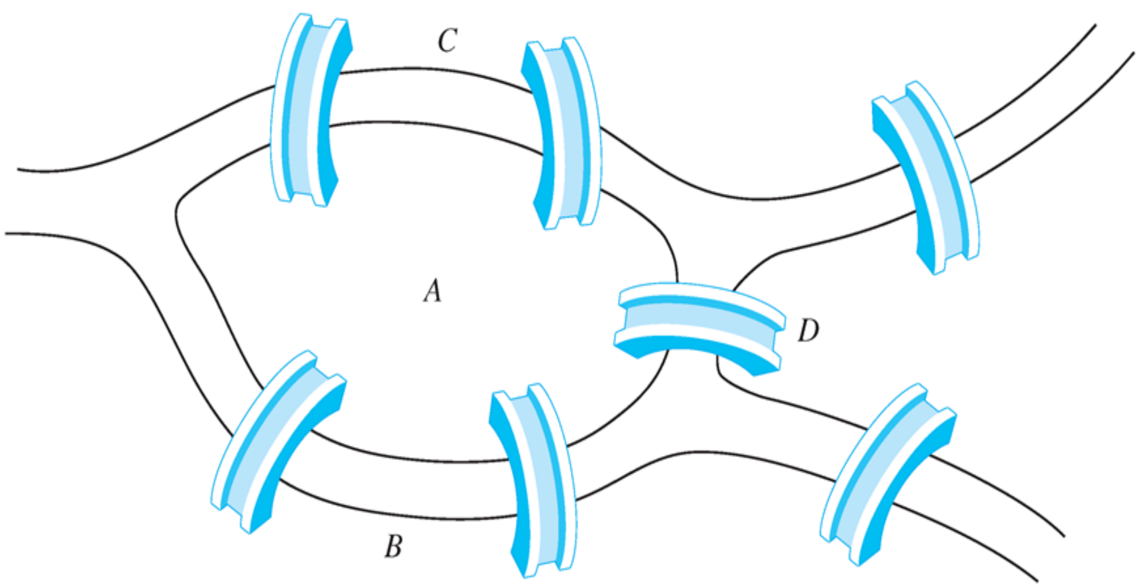

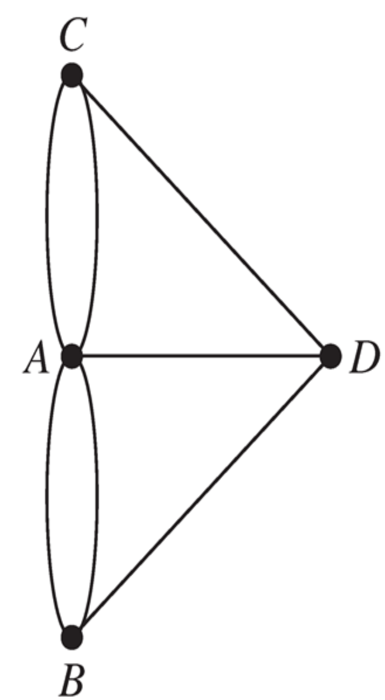

Euler Paths and Circuits

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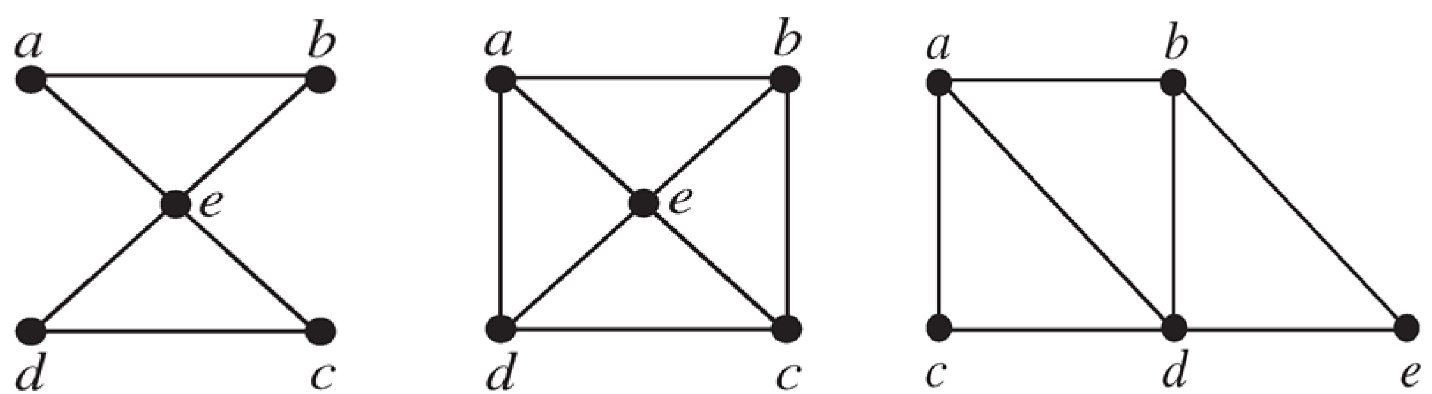

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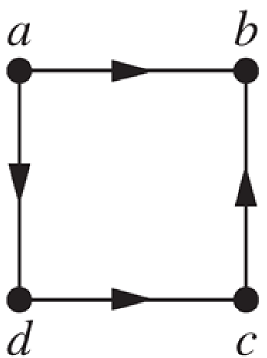

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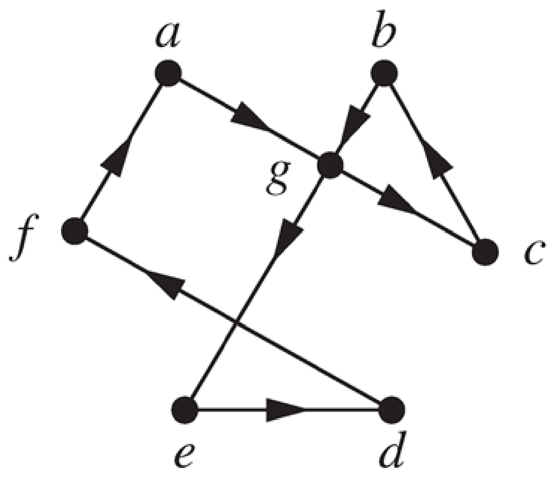

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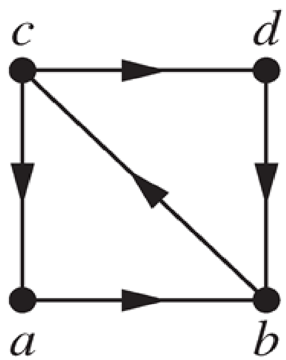

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- ◇ The initial vertex and the final vertex of an Euler path have odd degree.

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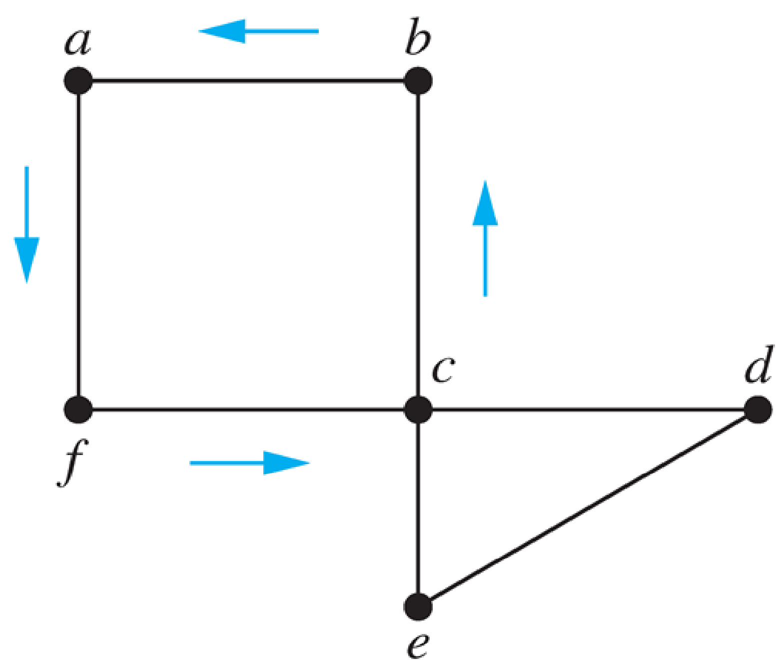

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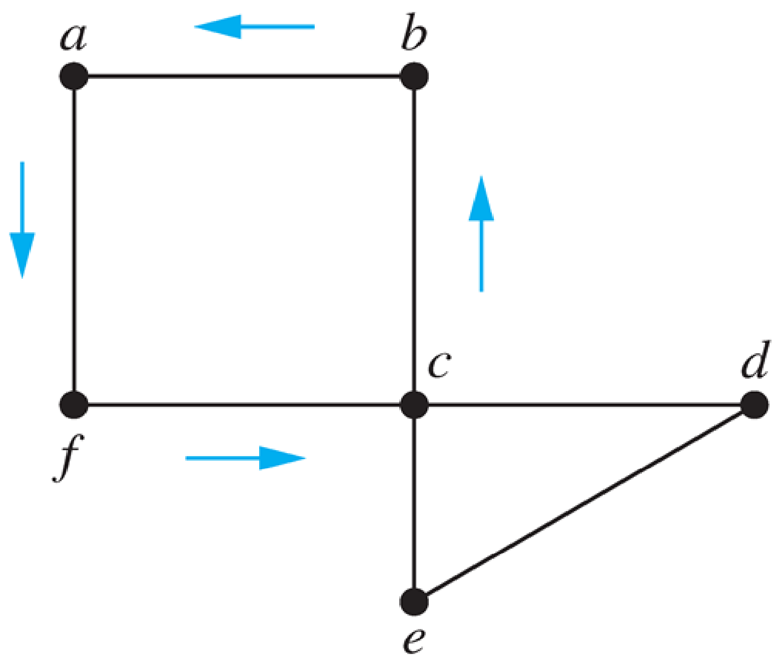

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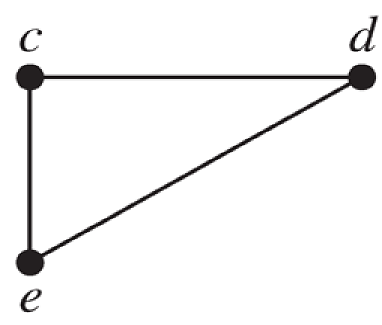

H

Algorithm for Constructing an Euler Circuit

procedure *Euler*(G: connected multigraph with all vertices of even degree)
circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges
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H := G with the edges of this circuit removed
while H has edges
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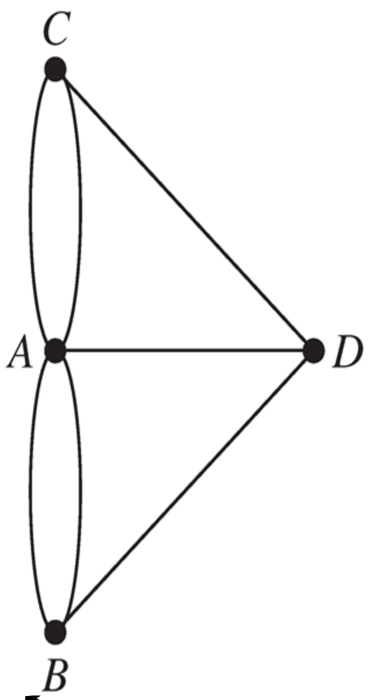

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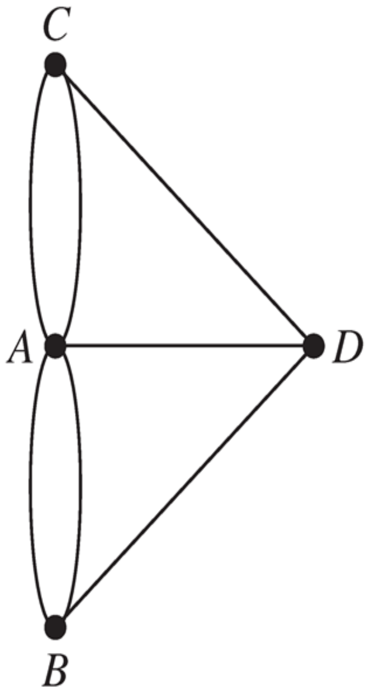

No Euler circuit

- **Example**

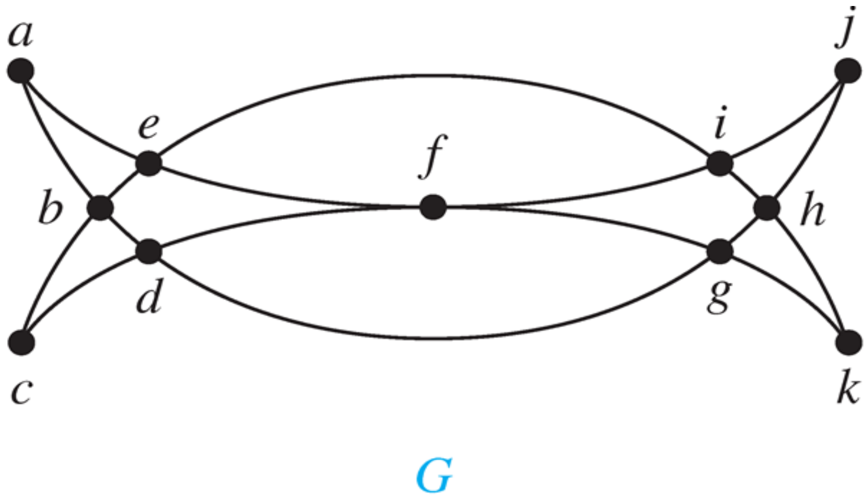

FIGURE 6 **Mohammed's Scimitars.**

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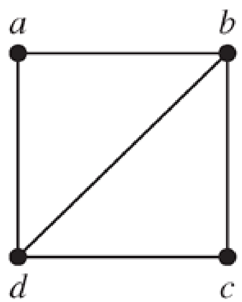

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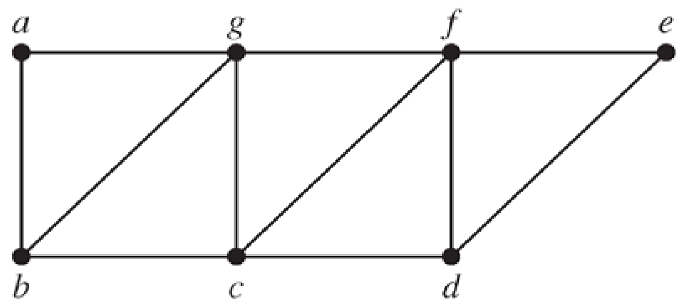

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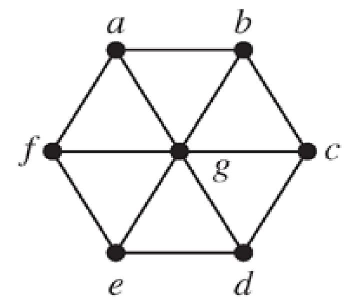

G_3

Applications of Euler Paths and Circuits

- Finding a path or circuit that traverses each
  - ◇ street in a neighborhood
  - ◇ road in a transportation network
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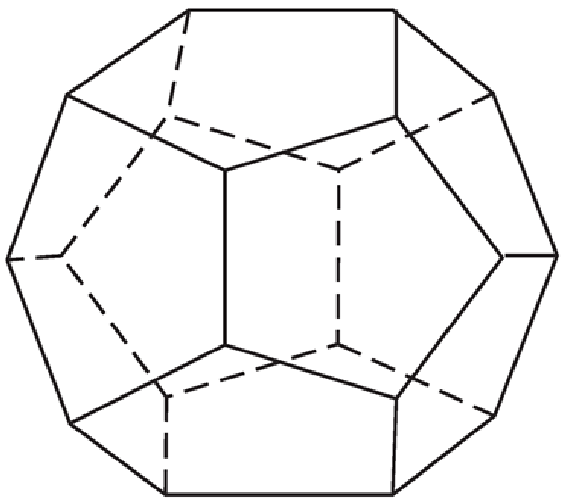

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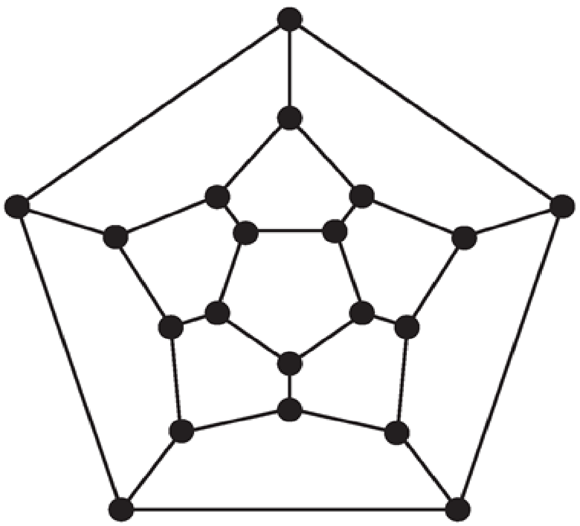

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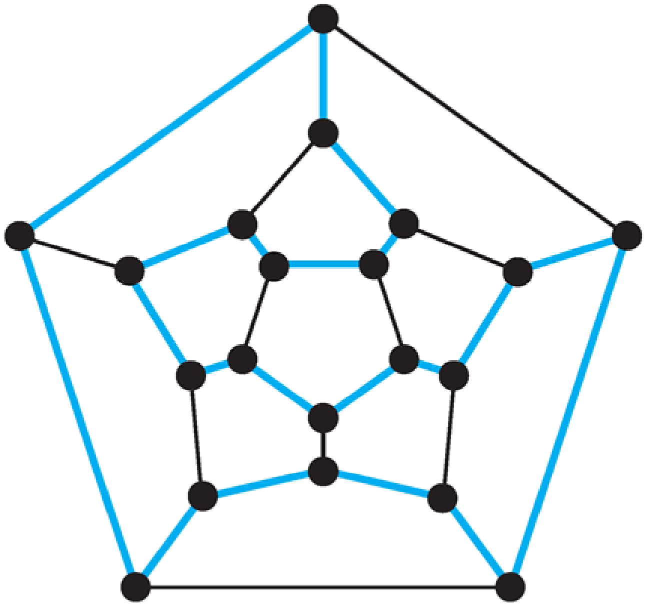

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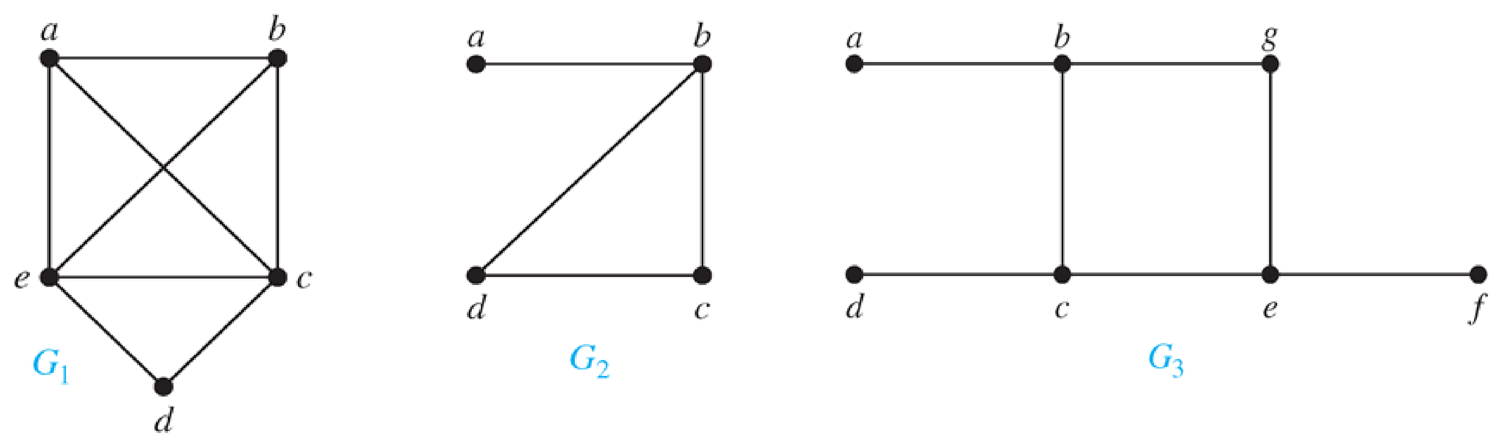

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  the decision version of the TSP $\in$ NPC

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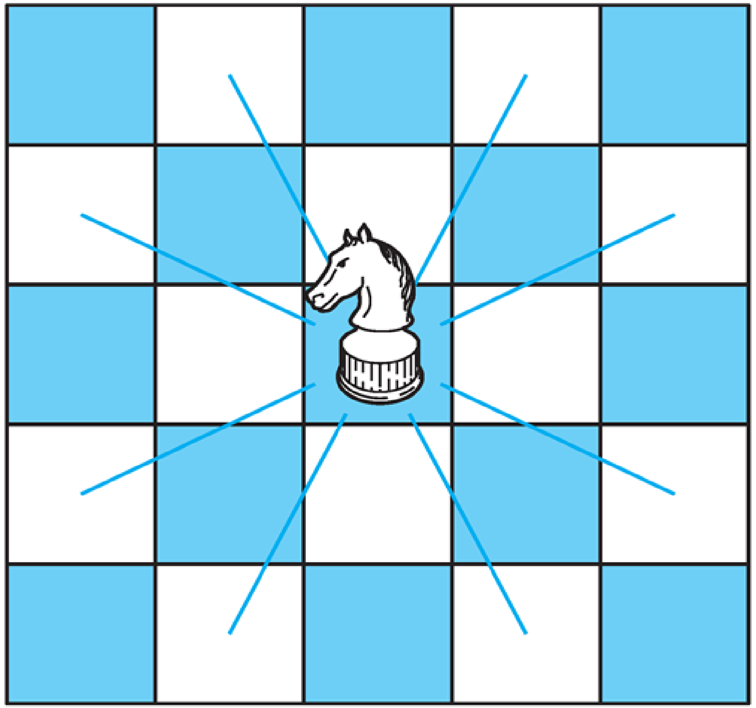

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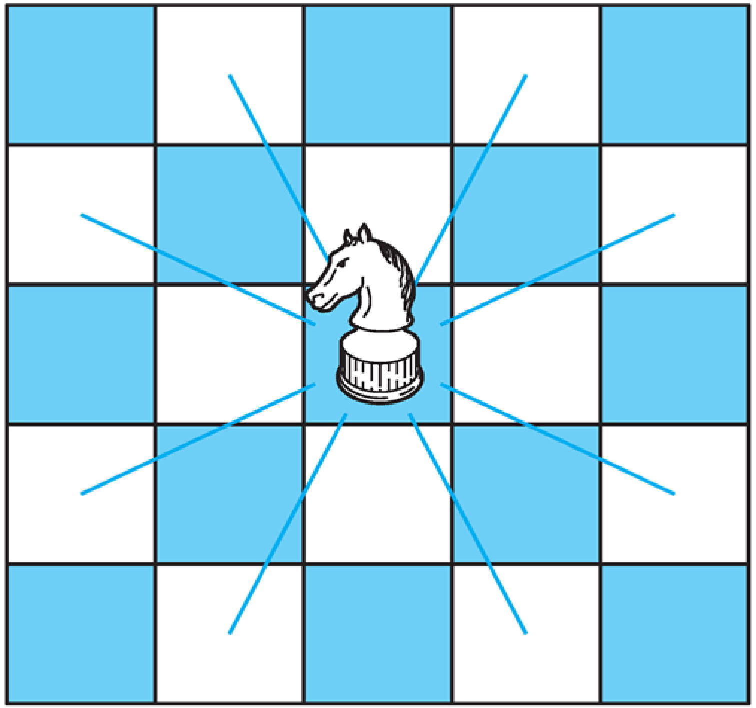

What about in 6×6 chessboard?

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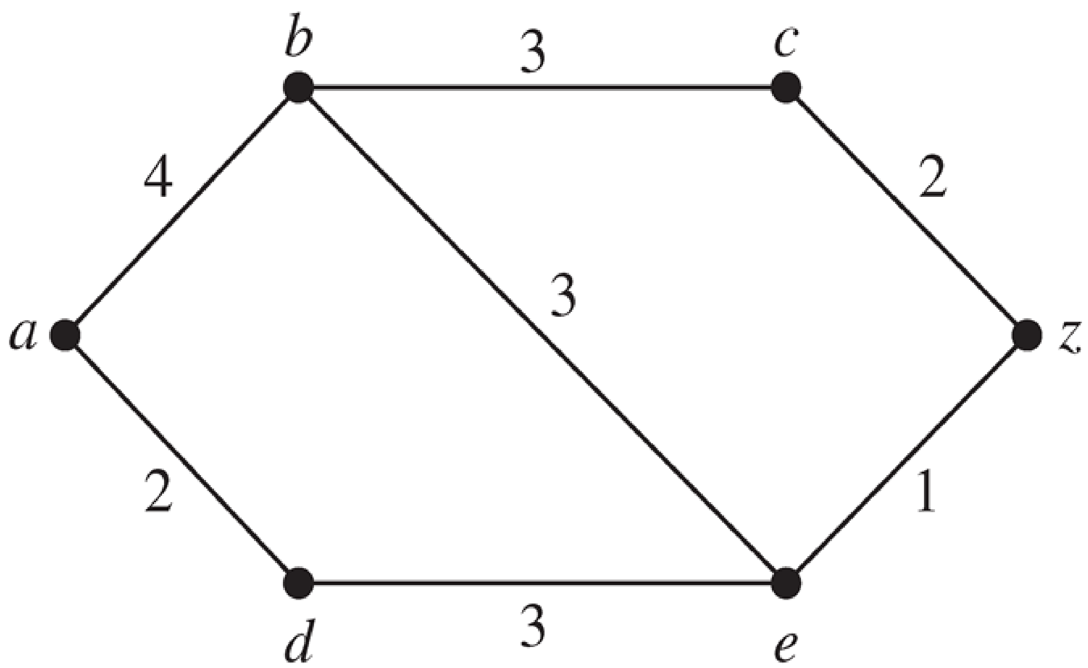

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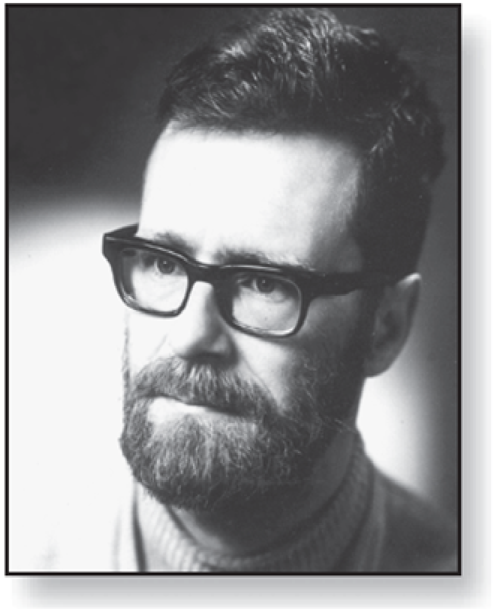

Edsger Wybe Dijkstra

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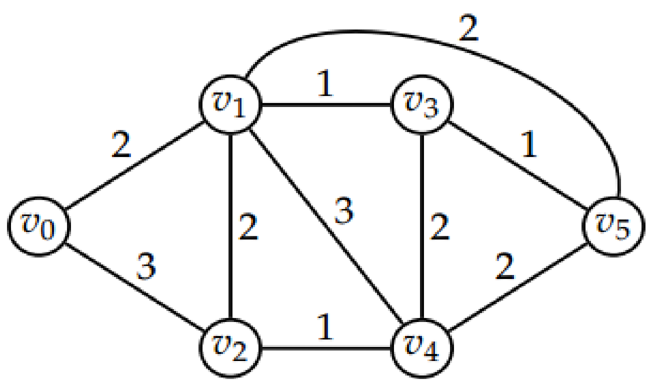

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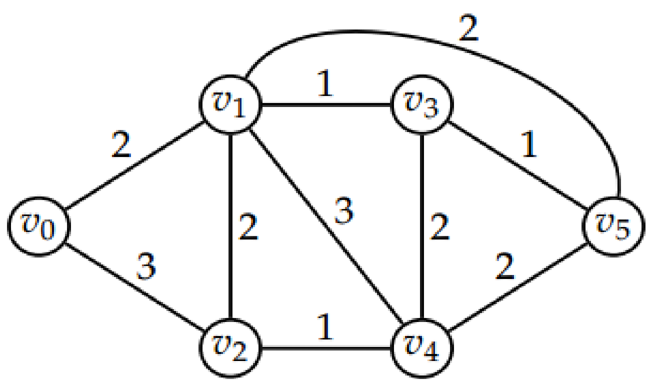

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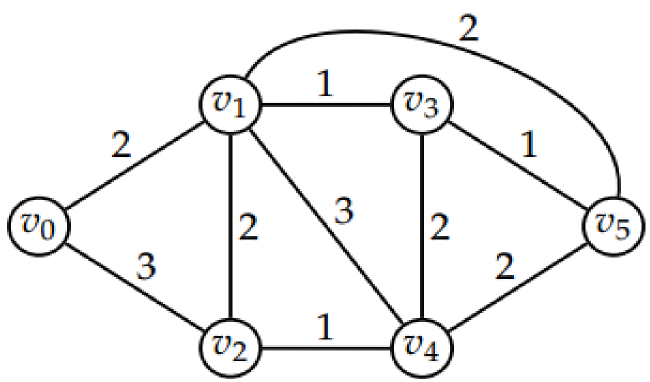

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0	∞	∞	∞	∞	∞

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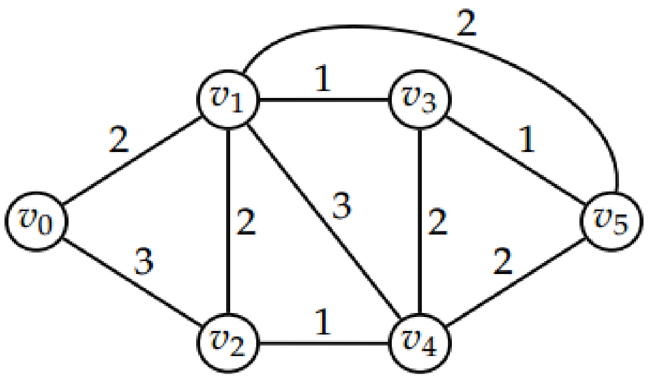

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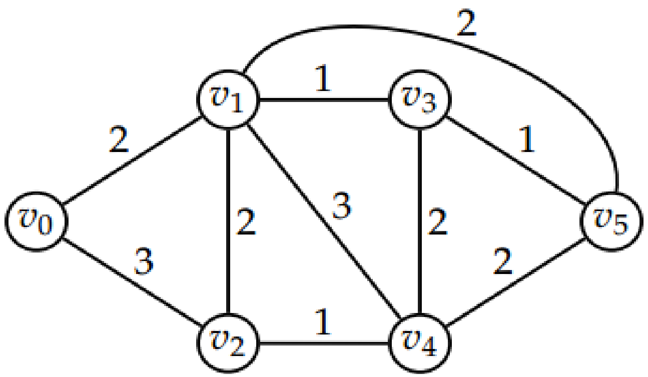

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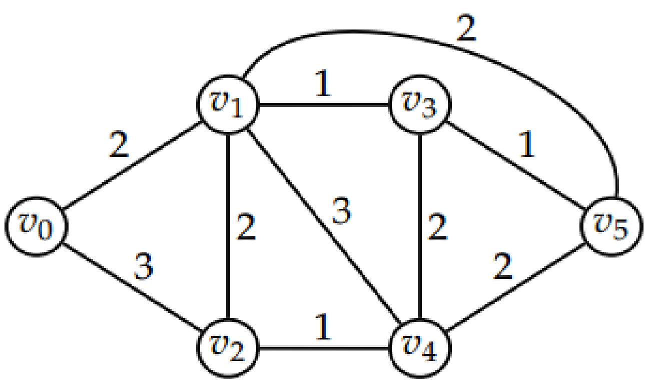

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$d(v_3) = 2 + 1 = 3$, $d(v_4) = 2 + 3 = 5$,

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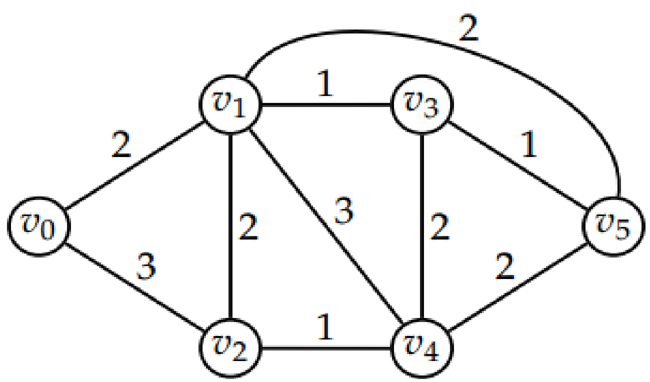

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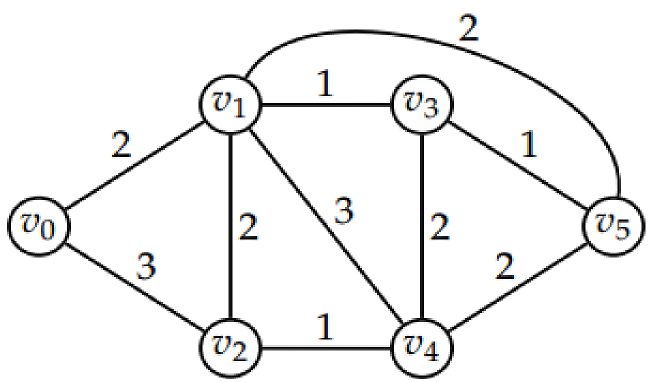

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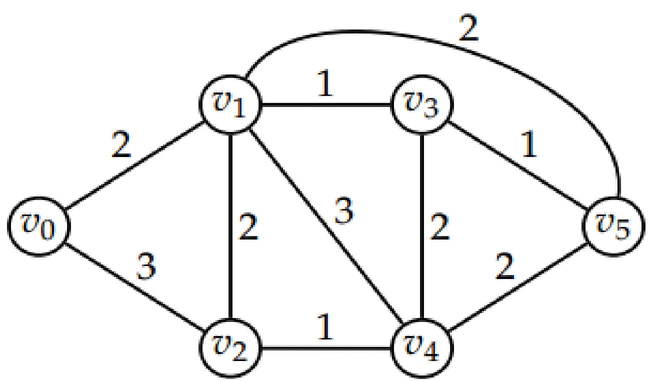

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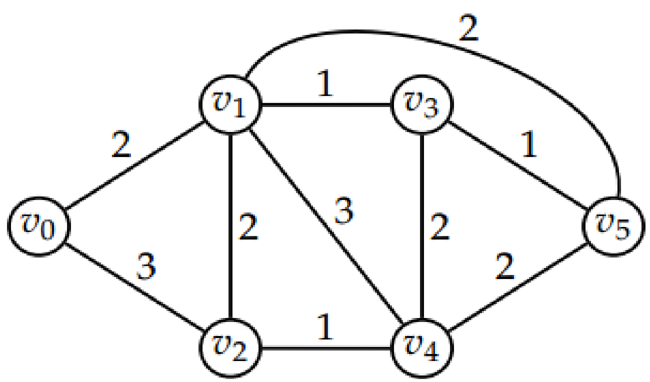

v_0	v_1	v_2	v_3	v_4	v_5
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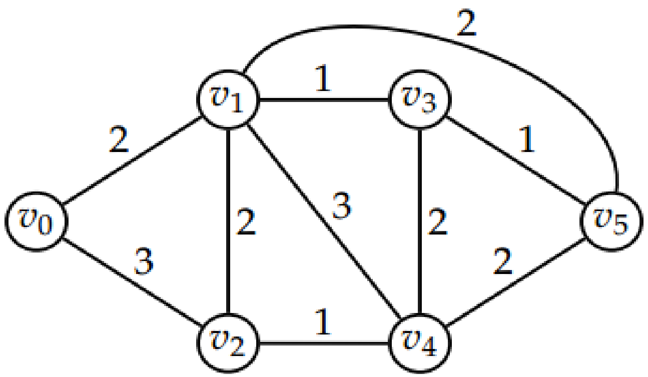

v_0	v_1	v_2	v_3	v_4	v_5
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$i = 4$

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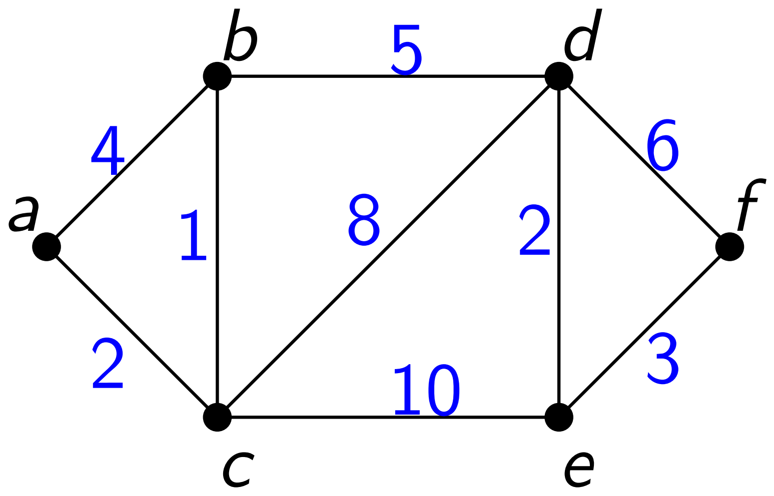
  Complexity

  read the Textbook p.712 – p.714

Another Example



The Single-Source Shortest Paths (SSSP) Problem

- Dijkstra's algorithm

  $O(v^2)$ using an *array* [Dijkstra 1956]

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- New result

Negative-Weight Single-Source Shortest Paths in Near-linear Time

Aaron Bernstein* Danupon Nanongkai† Christian Wulff-Nilsen‡

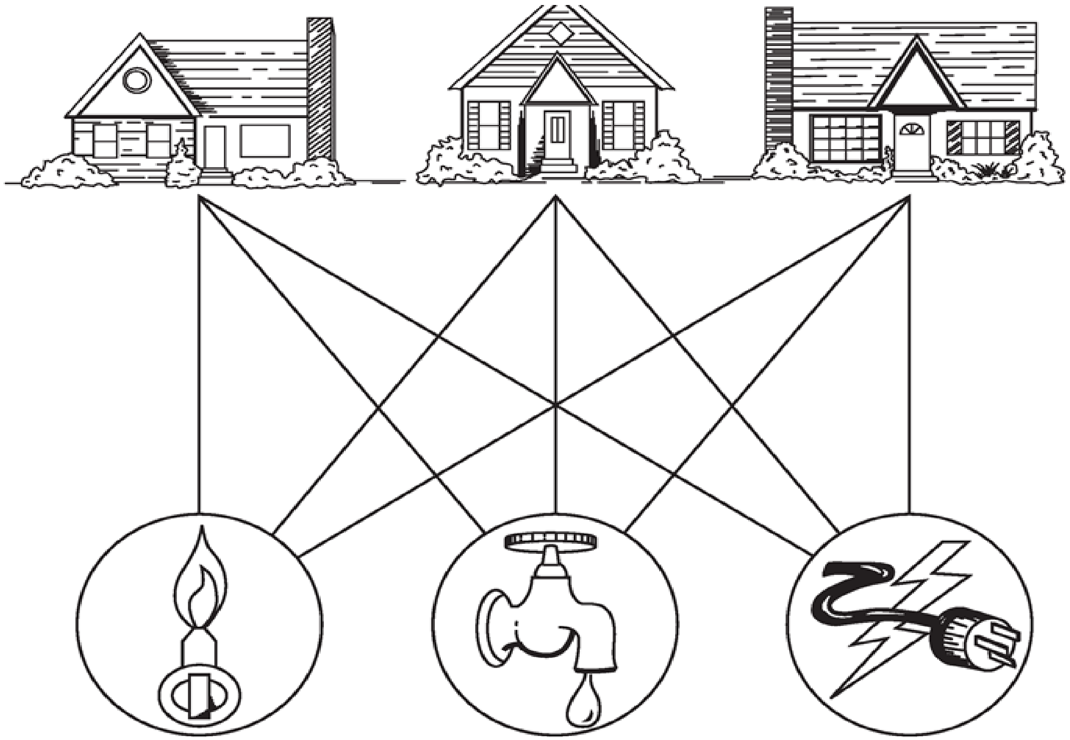
Abstract

We present a randomized algorithm that computes single-source shortest paths (SSSP) in $O(m \log^8(n) \log W)$ time when edge weights are integral and can be negative.[1] This essentially resolves the classic negative-weight SSSP problem. The previous bounds are $\tilde{O}((m+n^{1.5}) \log W)$ [BLNPSSSW FOCS'20] and $m^{4/3+o(1)} \log W$ [AMV FOCS'20]. Near-linear time algorithms were known previously only for the special case of planar directed graphs [Fakcharoenphol and Rao FOCS'01].

In contrast to all recent developments that rely on sophisticated continuous optimization methods and dynamic algorithms, our algorithm is simple: it requires only a simple graph decomposition and elementary combinatorial tools. In fact, ours is the first combinatorial algorithm for negative-weight SSSP to break through the classic $\tilde{O}(m\sqrt{n} \log W)$ bound from over three decades ago [Gabow and Tarjan SICOMP'89].

Planar Graphs

- Join three houses to each of three seperate utilities.



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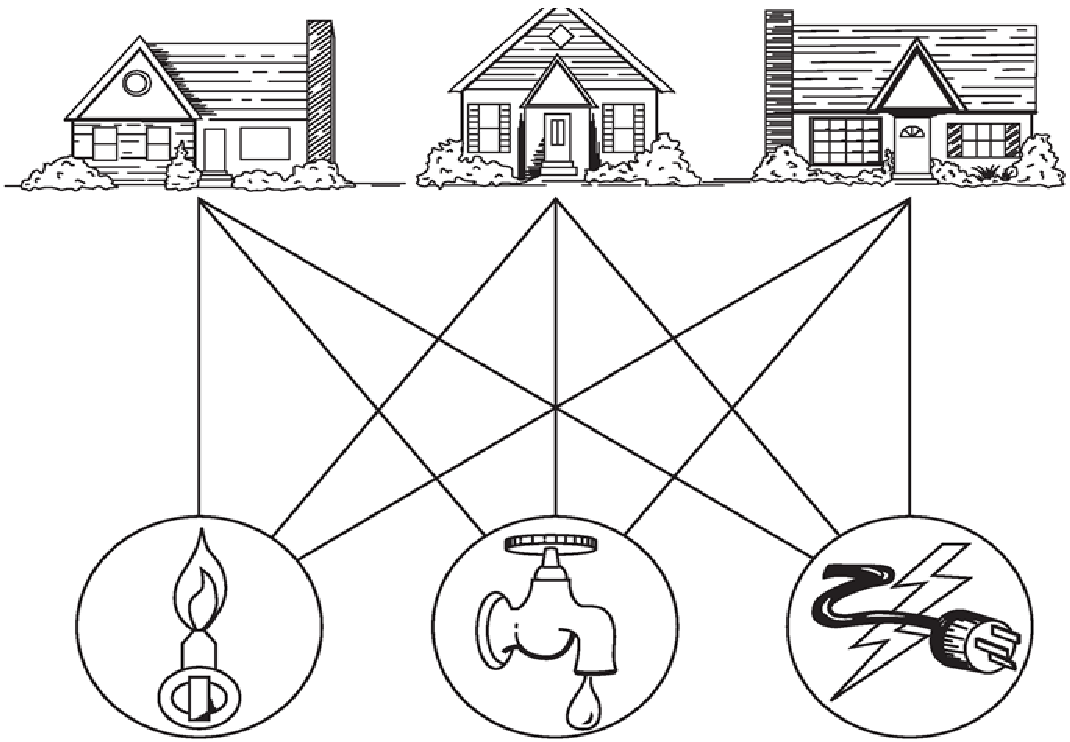

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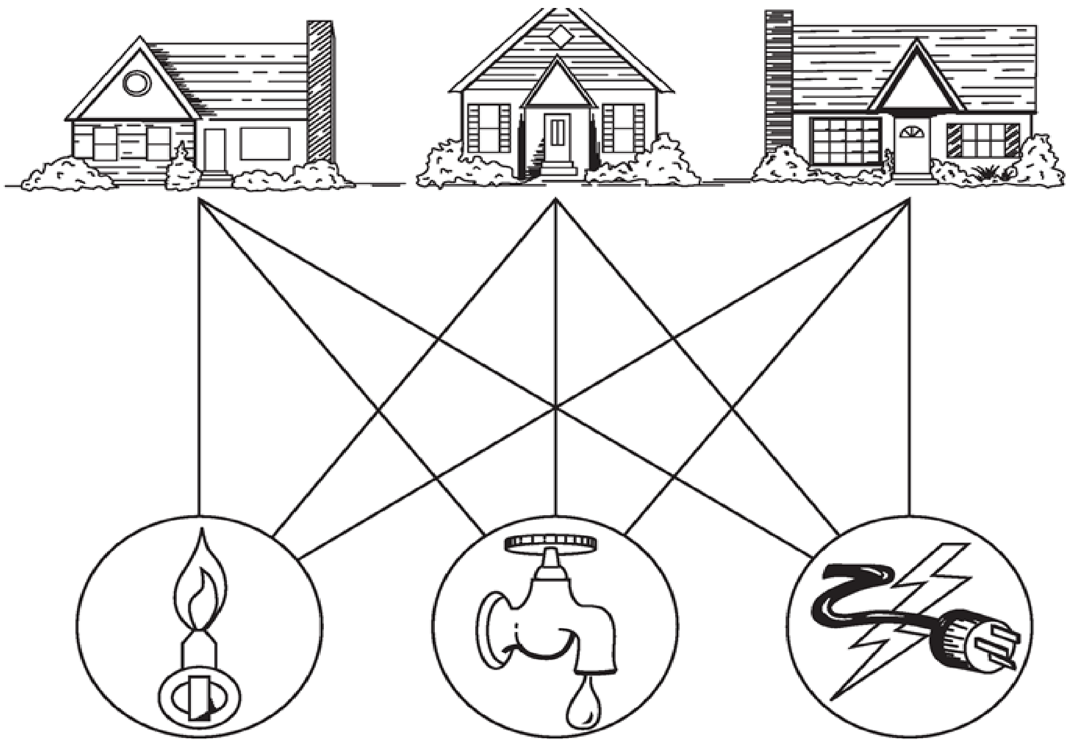

Can this graph be drawn in the plane s.t. no two of its edges cross? $K_{3,3}$

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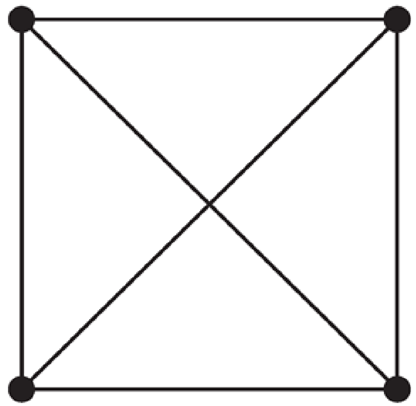

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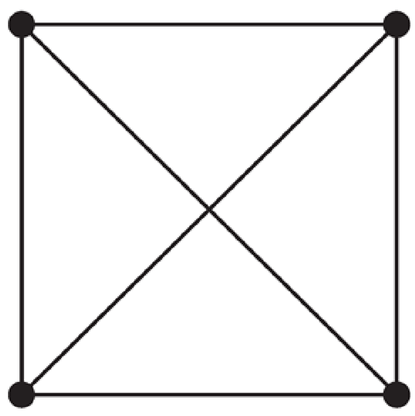

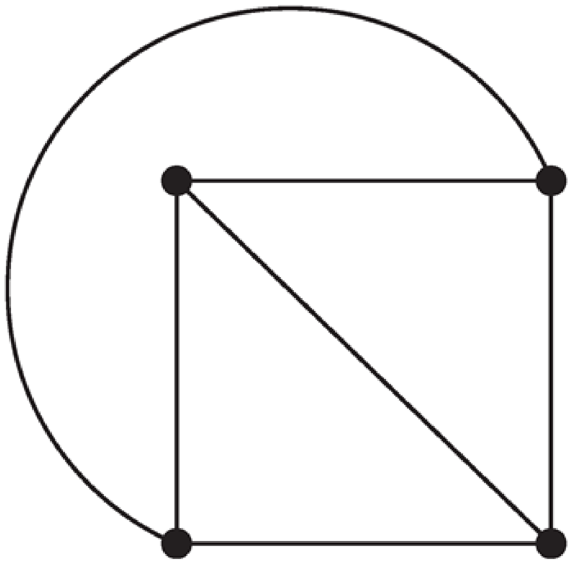

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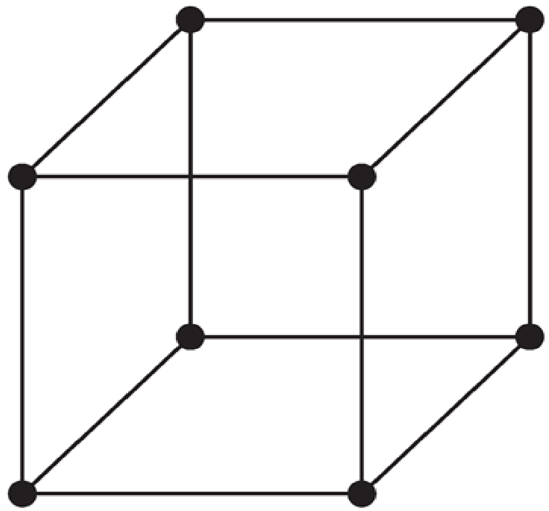

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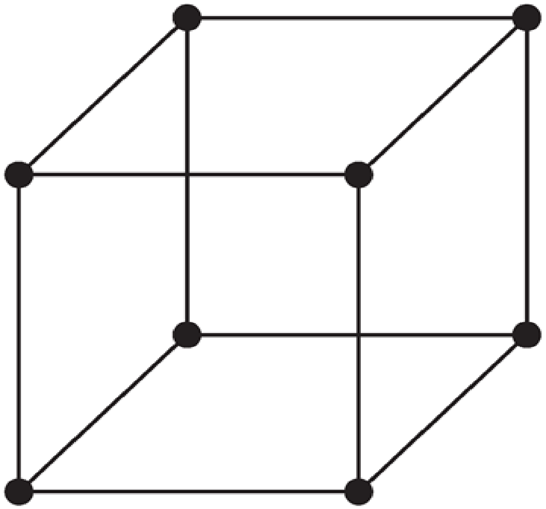

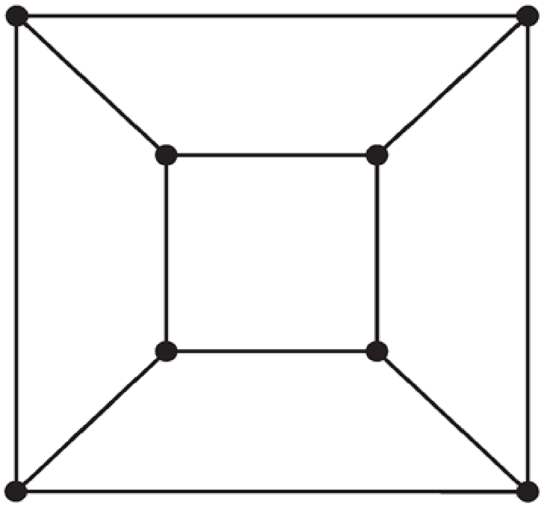

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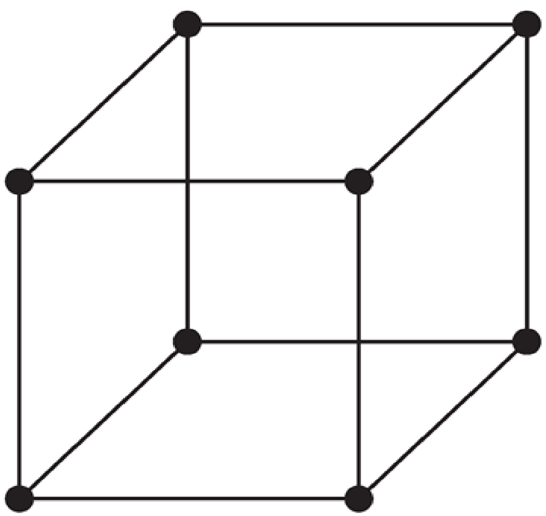

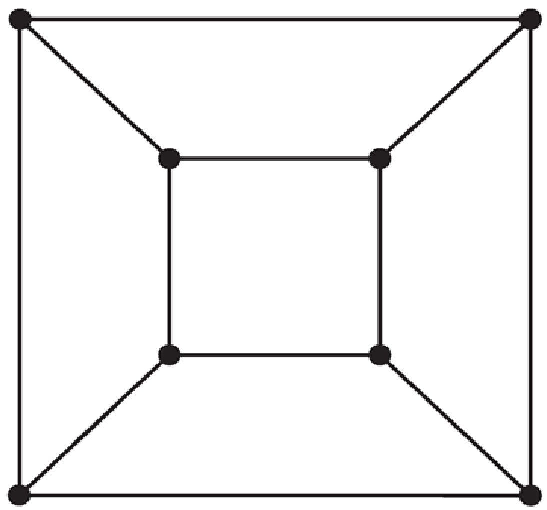

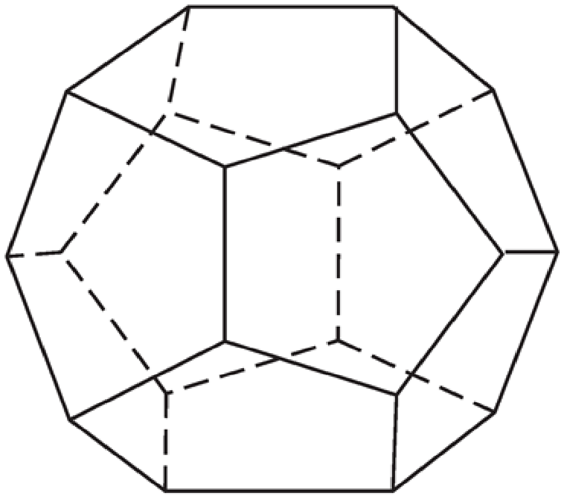

(a)

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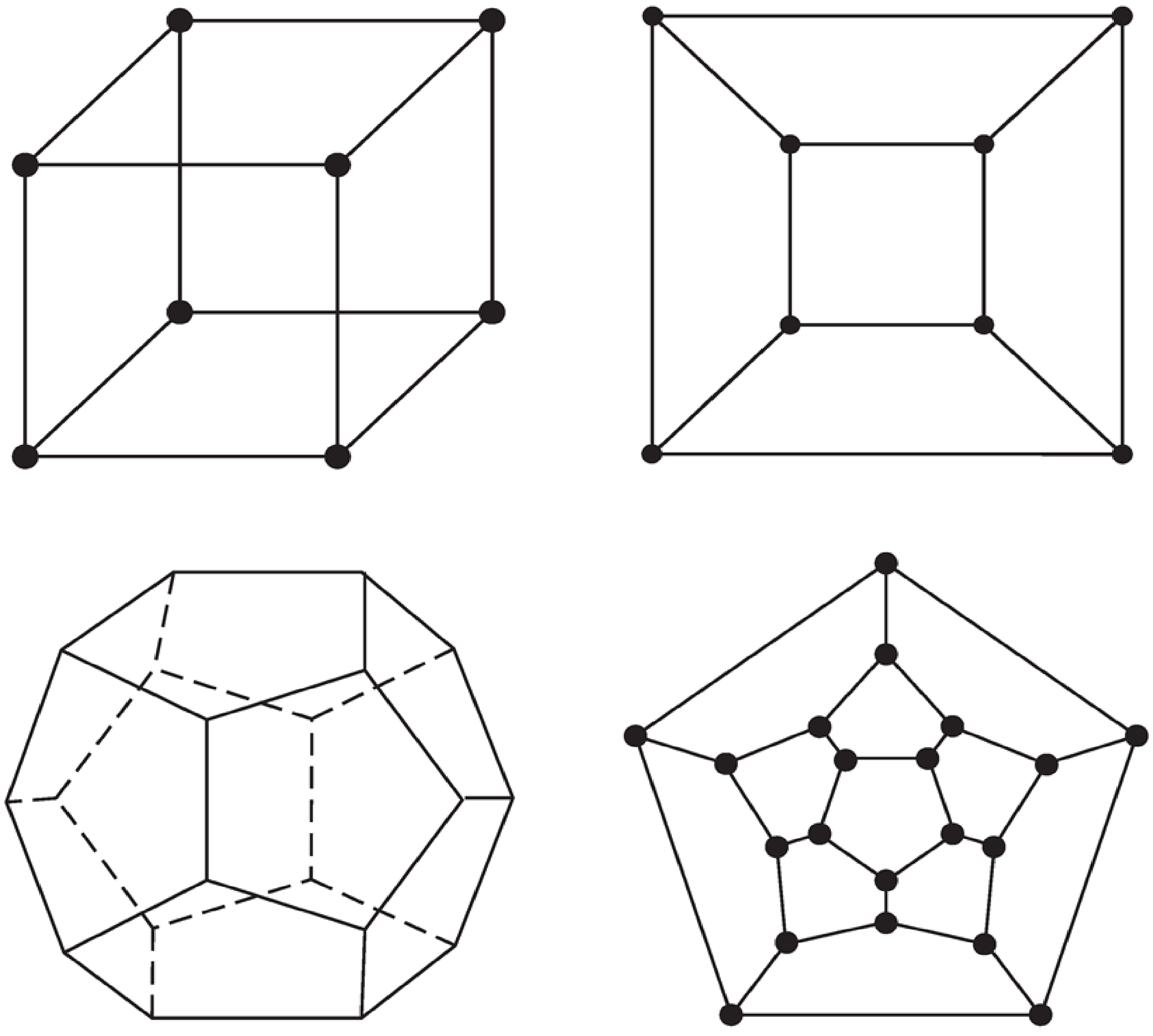

(a) (b)

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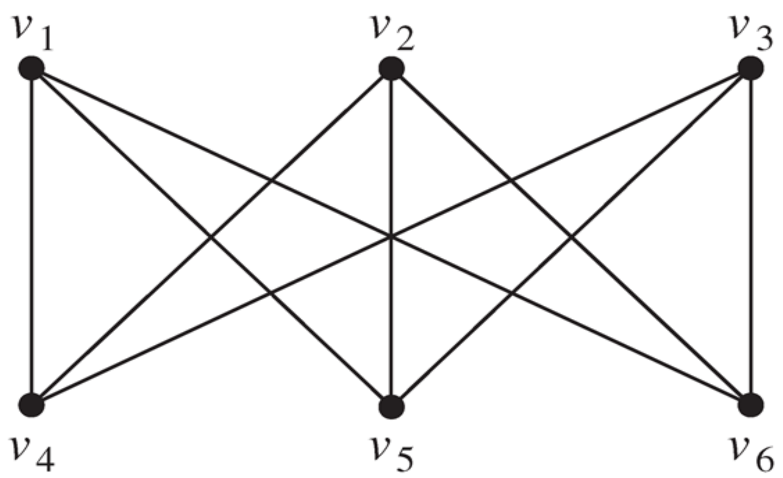

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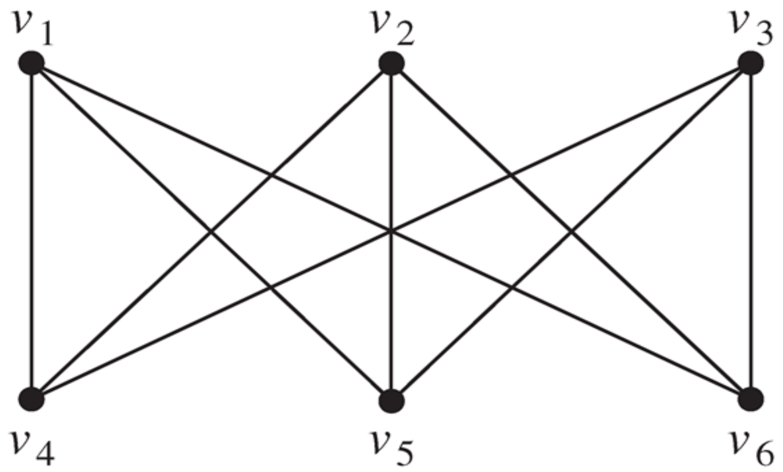

Applications

- ◇ IC design
- ◇ design of road networks

Euler's Formula

- **Theorem** (Euler's Formula) Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

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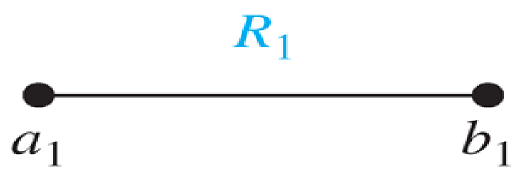

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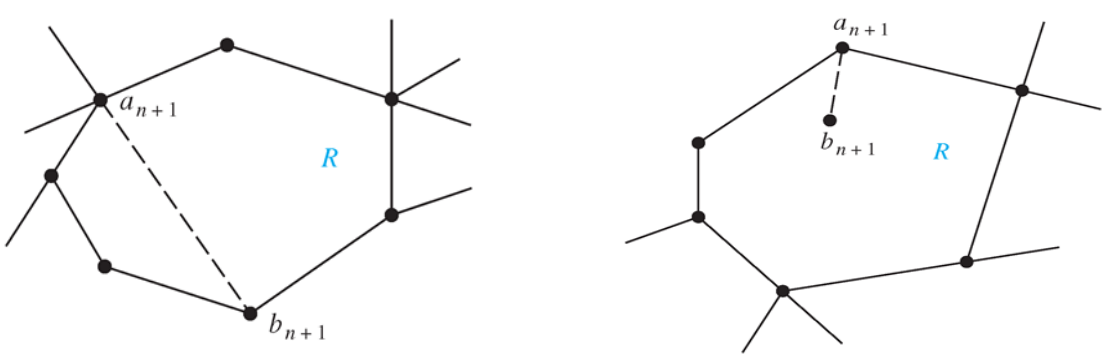

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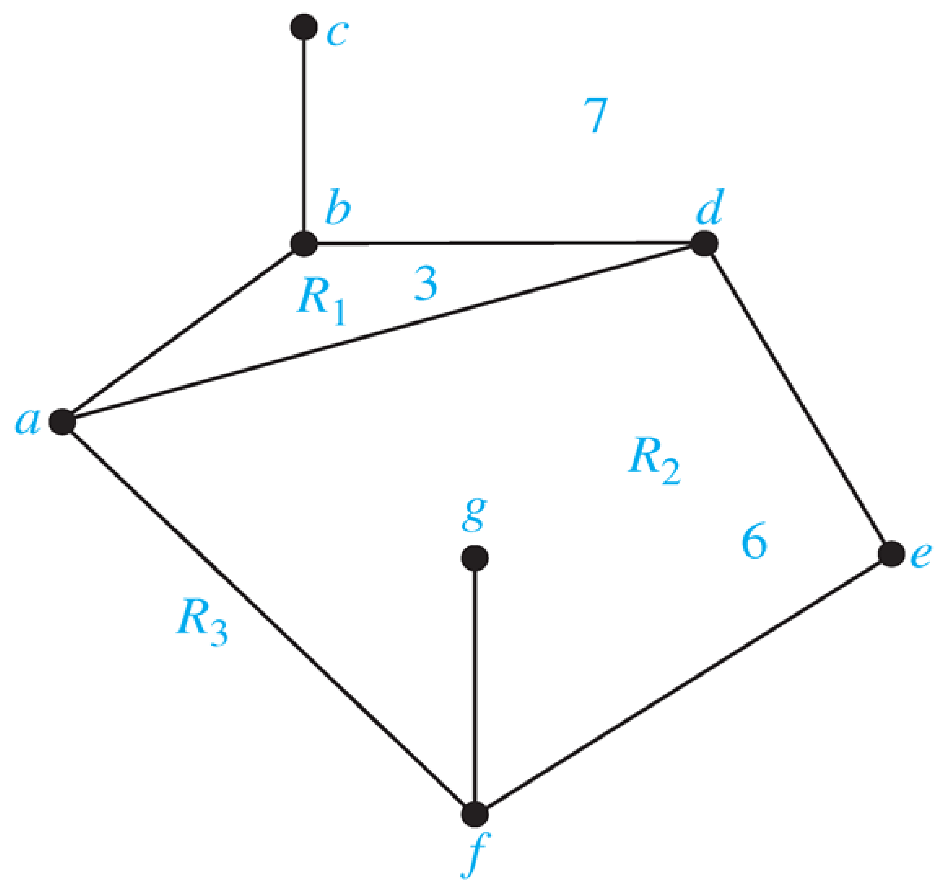

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  By Euler's formula, the proof is completed.

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  By Corollary 1 and the Handshaking Theorem.

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  Proof

  (By contradiction)

  By Corollary 1 and the Handshaking Theorem.

  Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

- **Corollary 2** If G is a connected planar simple graph, then G has a vertex of degree not exceeding 5.

  Proof

  (By contradiction)

  By Corollary 1 and the Handshaking Theorem.

  Corollary 3 In a connected planar simple graph has e edges and v vertices with $v \geq 3$ and no circuits of length three, then $e \leq 2v - 4$.

  Proof similar to that of Corollary 1.

Examples

- Show that K_5 is nonplanar.

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  Using Corollary 1

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  Show that $K_{3,3}$ is nonplanar.

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  Using Corollary 3

Examples

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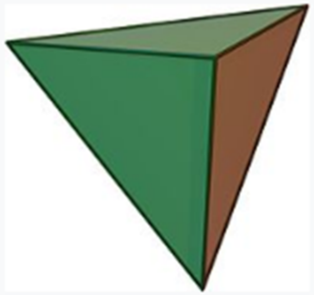
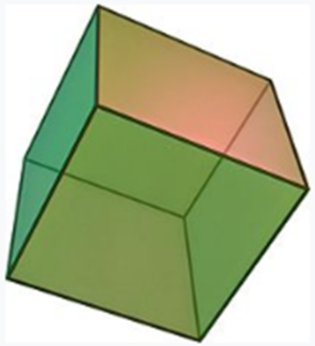
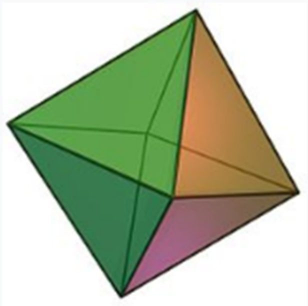
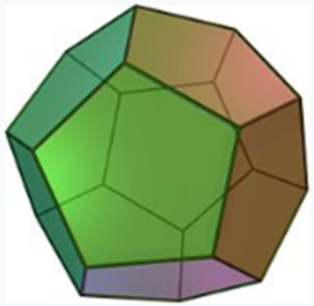
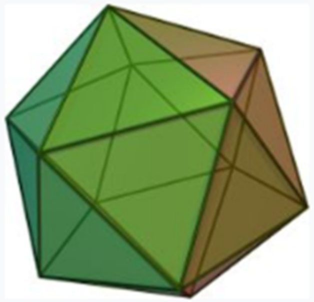
  Using Corollary 1

  Show that $K_{3,3}$ is nonplanar.

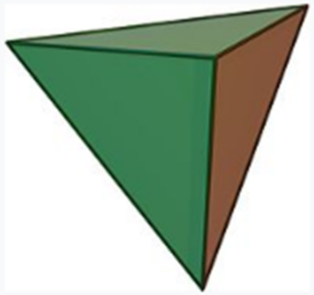
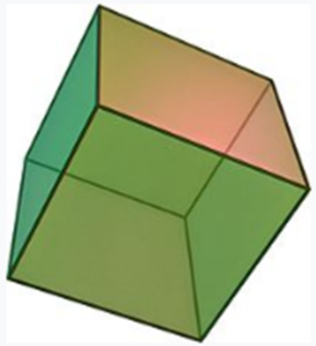
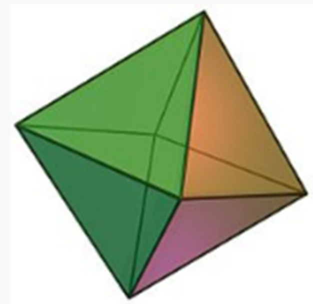
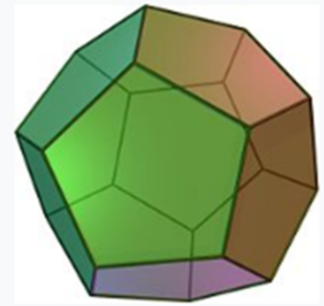
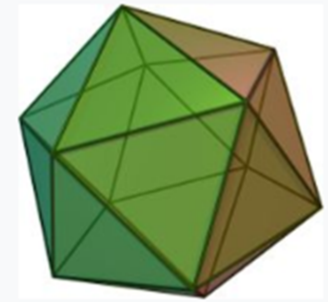
  Using Corollary 3

  Corollary 2 is used in the proof of Five Color Theorem.

Only 5 Platonic Solids

				
Tetrahedron {3, 3}	Cube {4, 3}	Octahedron {3, 4}	Dodecahedron {5, 3}	Icosahedron {3, 5}

Only 5 Platonic Solids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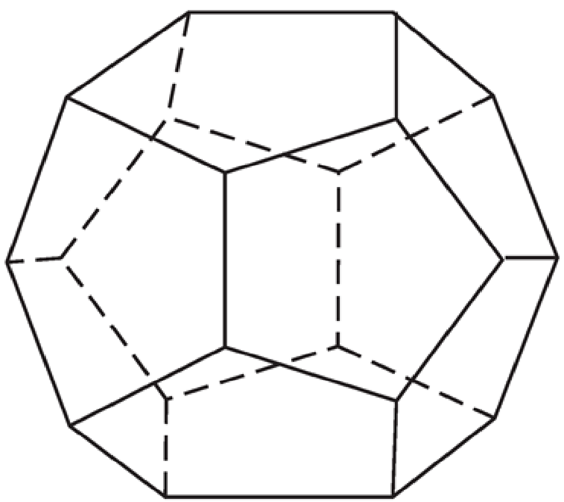
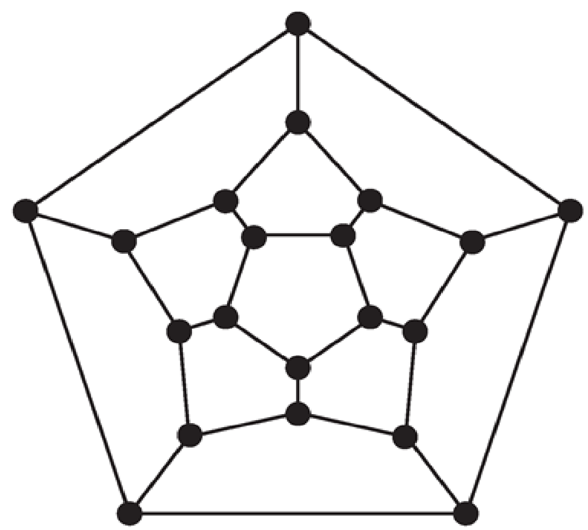


(a)



(b)

- **Definition** If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex w together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation is called an *elementary subdivision*. The graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *homomorphic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

Kuratowski's Theorem

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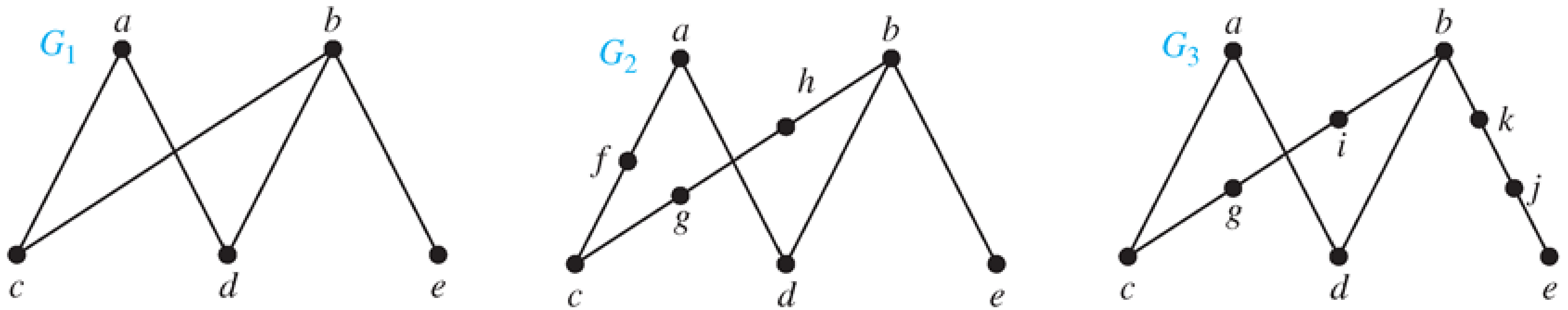

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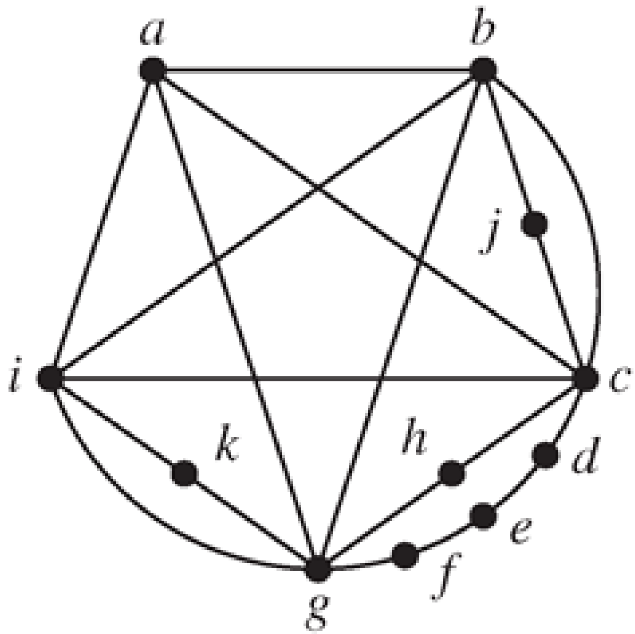
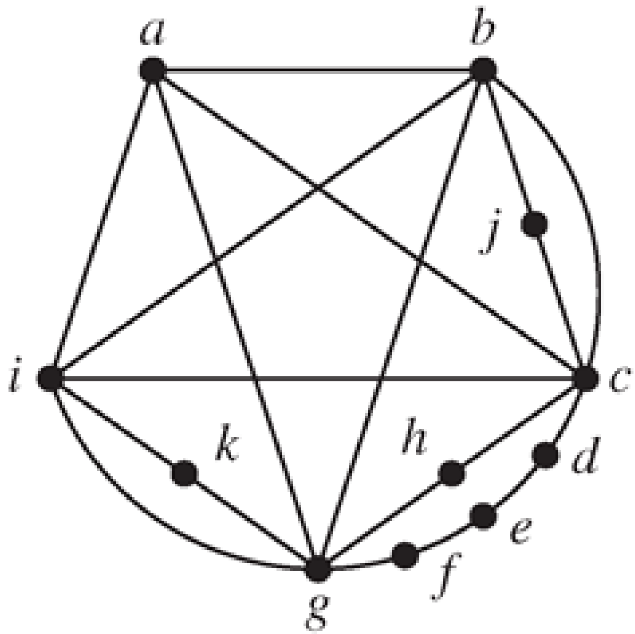
  Theorem A graph is nonplanar if and only if it contains a subgraph homomorphic to $K_{3,3}$ or K_5.

Examples



G

Examples



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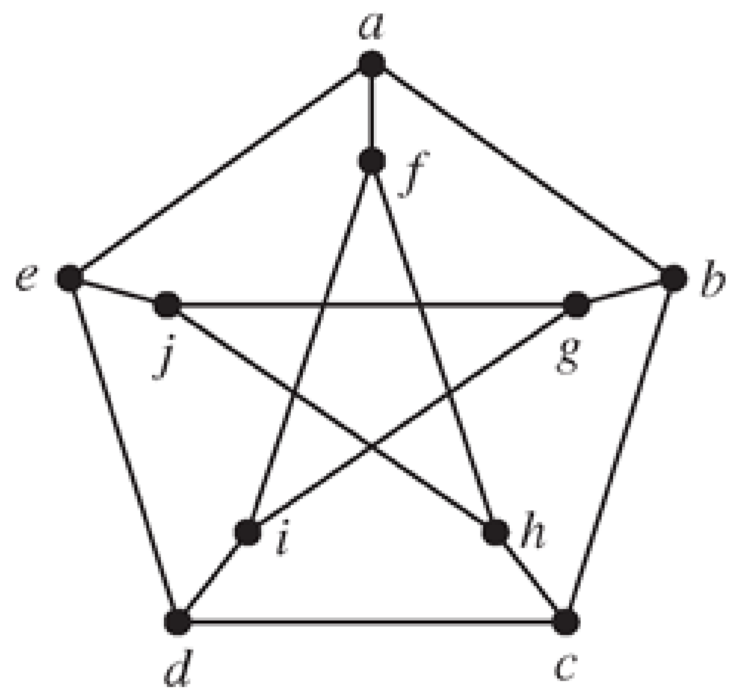
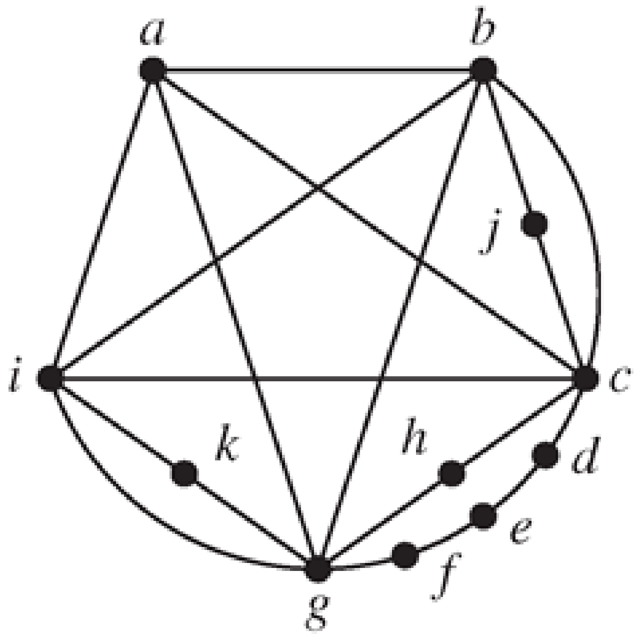


Examples



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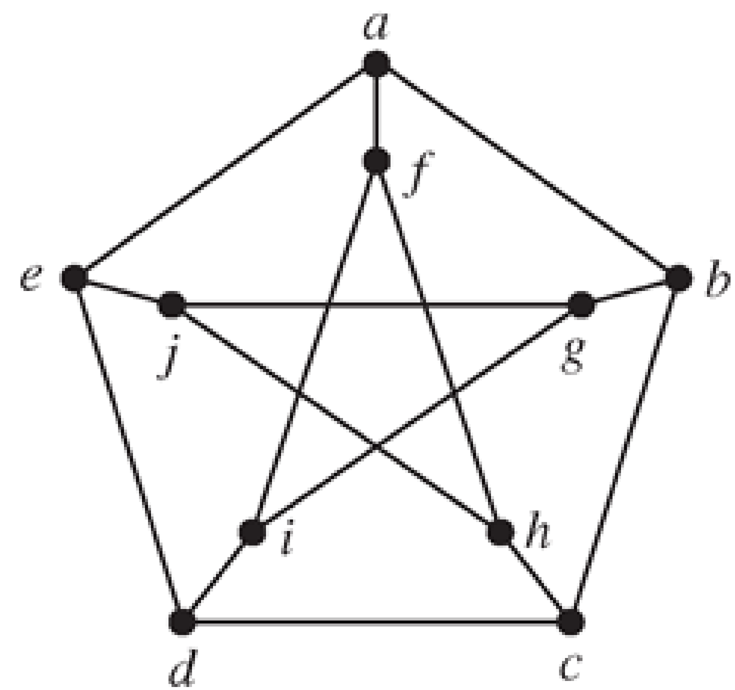

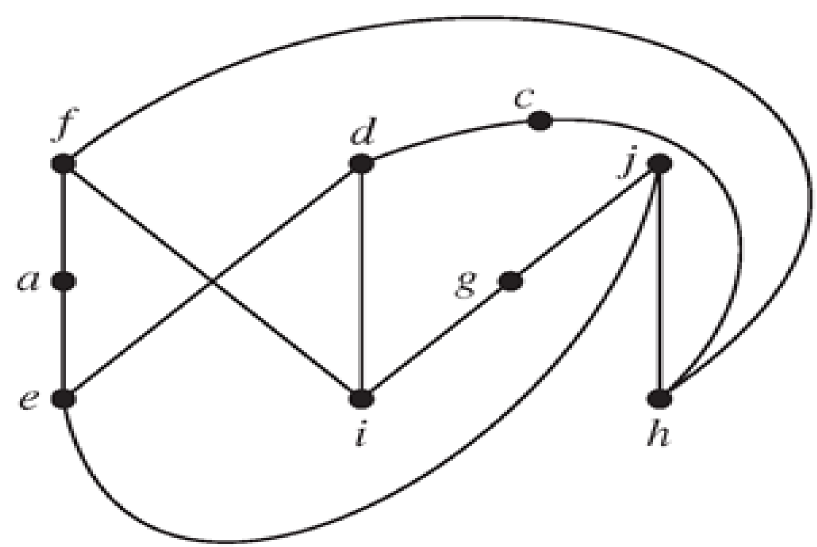

Next Lecture

- Graph theory III ...