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Section 6.3

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P136 Q3

$$\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}}=\frac{\overline{X}}{\frac{1}{\sqrt{16}}}=4\overline{X}\sim N(0,1)$$

因此有

$$\begin{aligned}P(|\overline{X}|<c)=&P(-c<\overline{X}<c)\\=&P(-4c<4\overline{X}<4c)\\=&\Phi(4c)-\Phi(-4c)\\=&2\Phi(4c)-1\\=&0.5\end{aligned}$$

即 $\Phi(4c)=0.75$，查表得 $c\approx 0.1686$。

P136 Q6

已知 $T\sim t_n$，则有

$$T=\frac{X}{\sqrt{\frac{Y}{n}}}\text{ 且}X\sim N(0,1),Y\sim\chi_n^2$$

因此有

$$T^2=\frac{X^2}{\frac{Y}{n}}=\frac{\frac{X^2}{1}}{\frac{Y}{n}}\text{ 且}X^2\sim\chi_1^2,Y\sim\chi_n^2$$

因此，$T^2\sim F_{1,n}$。

P136 Q8

若 X，Y 为 $\lambda = 1$ 的独立指数分布，令 $Z = \frac{X}{Y}$，则有

$$
\begin{aligned}
f_Z(z) &= \int_0^\infty f_X(zy) f_Y(y)|y|dy \\
&= \int_0^\infty e^{-zy}e^{-y}ydy \\
&= \int_0^\infty ye^{-(z+1)y}dy \\
&= \left[-\frac{y}{z+1}e^{-(z+1)y}\right]\Big|_0^\infty + \int_0^\infty \frac{1}{z+1}e^{-(z+1)y}dy \\
&= \frac{1}{z+1}\int_0^\infty e^{-(z+1)y}dy \\
&= \frac{1}{z+1}\left[-\frac{1}{z+1}e^{-(z+1)y}\right]\Big|_0^\infty \\
&= \frac{1}{(z+1)^2}
\end{aligned}
$$

即

$$
\begin{aligned}
f_Z(z) &= \frac{1}{(z+1)^2} \\
&= 4 \cdot \frac{1}{(2z+2)^2} \\
&= \frac{1}{1\cdot 1} \cdot 2^1 \cdot 2^1 \cdot \frac{z^{1-1}}{(2z+2)^2} \\
&= \frac{\Gamma(\frac{2+2}{2})}{\Gamma(\frac{2}{2})\cdot\Gamma(\frac{2}{2})} \cdot 2^{\frac{2}{2}} \cdot 2^{\frac{2}{2}} \cdot \frac{z^{\frac{2}{2}-1}}{(2z+2)^2}
\end{aligned}
$$

因此，$Z \sim F(2,2)$。

补充 1

$$
\frac{\overline{X}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\overline{X}-\mu}{\frac{4}{\sqrt{n}}} \sim N(0,1)
$$

因此有

$$
\begin{aligned}
P(|\overline{X}-\mu|<1) =& P(-1<\overline{X}-\mu<1) \\
=& P(-\frac{\sqrt{n}}{4}<\frac{\overline{X}-\mu}{\frac{4}{\sqrt{n}}}<\frac{\sqrt{n}}{4}) \\
=& \Phi(\frac{\sqrt{n}}{4})-\Phi(-\frac{\sqrt{n}}{4}) \\
=& 2\Phi(\frac{\sqrt{n}}{4})-1 \\
\geqslant& 0.95
\end{aligned}
$$

即 $\Phi(\frac{\sqrt{n}}{4})\geqslant 0.975$，查表得 $\frac{\sqrt{n}}{4}\geqslant 1.96$，解得 $n\geqslant 61.47$。

补充 2

$$
\frac{(\overline{X}-\overline{Y})-(\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}=\frac{\overline{X}-\overline{Y}}{20\sqrt{\frac{1}{36}+\frac{1}{49}}}\sim N(0,1)
$$

因此有

$$
\begin{aligned}
P(|\overline{X}-\overline{Y}|\leqslant 10) =& P(-10\leqslant\overline{X}-\overline{Y}\leqslant 10) \\
=& P(-\frac{1}{2\sqrt{\frac{1}{36}+\frac{1}{49}}}\leqslant\frac{\overline{X}-\overline{Y}}{20\sqrt{\frac{1}{36}+\frac{1}{49}}}\leqslant\frac{1}{2\sqrt{\frac{1}{36}+\frac{1}{49}}}) \\
=& 2\Phi(\frac{1}{2\sqrt{\frac{1}{36}+\frac{1}{49}}})-1 \\
\approx& 0.9772
\end{aligned}
$$

补充 3

$$
\sum_{i=1}^{n}\frac{(X_i-\mu)^2}{\sigma^2}=\frac{\sum_{i=1}^{10}X_i^2}{0.3^2}\sim\chi^2_{10}
$$

因此有

$$
\begin{aligned}
P\{\sum_{i=1}^{10}X_i^2\leqslant c\} =& P\{\frac{\sum_{i=1}^{10}X_i^2}{0.3^2}\leqslant\frac{c}{0.3^2}\} \\
=& P\{\chi^2_{10}\leqslant\frac{c}{0.3^2}\} \\
=& 0.95
\end{aligned}
$$

查表得 $\frac{c}{0.3^2}\approx 18.307$，解得 $c\approx 1.64763$。

补充 4

(1)

可知 $X_1 - X_2 \sim N(0, 2\sigma^2)$，$X_1 + X_2 \sim N(0, 2\sigma^2)$，因此有

$$\frac{X_1 - X_2}{\sqrt{2}\sigma}^2 \sim \chi_1^2 \text{ 且} \frac{X_1 + X_2}{\sqrt{2}\sigma}^2 \sim \chi_1^2$$

两者相互独立，其非奇异线性组合也相互独立，因此有

$$\left(\frac{X_1 - X_2}{X_1 + X_2}\right)^2 = \frac{(\frac{X_1 - X_2}{\sqrt{2}\sigma})^2}{(\frac{X_1 + X_2}{\sqrt{2}\sigma})^2} \sim F_{1,1}$$

(2)

$$\frac{1}{\frac{(X_1+X_2)^2}{(X_1+X_2)^2+(X_1-X_2)^2}} = \frac{(X_1 + X_2)^2 + (X_1 - X_2)^2}{(X_1 + X_2)^2} = 1 + \frac{(X_1 - X_2)^2}{(X_1 + X_2)^2}$$

可将原式改为

$$P\{\frac{(X_1 + X_2)^2}{(X_1 + X_2)^2 + (X_1 - X_2)^2} > k\} = 0.10 \to P\{\frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} < \frac{1}{k} - 1\} = 0.10$$

查表可知 $F_{0.10}(1, 1) = \frac{1}{F_{0.90}(1,1)} \approx \frac{1}{39.9}$。因此有 $\frac{1}{k} - 1 \approx \frac{1}{39.9}$，解得 $k \approx 0.9755$。

补充 5

已知 $\overline{X_n} \sim N(\mu, \frac{\sigma^2}{n})$，$X_{n+1} \sim N(\mu, \sigma^2)$，则有

$$\begin{aligned} X_{n+1} - \overline{X_n} &\sim N(0, \frac{n+1}{n}\sigma^2) \\ \frac{X_{n+1} - \overline{X_n}}{\sigma\sqrt{\frac{n+1}{n}}} &\sim N(0, 1) \end{aligned}$$

又有

$$\frac{(n-1)S_n^2}{\sigma^2} \sim \chi_{n-1}^2$$

因此有

$$\frac{\frac{X_{n+1} - \overline{X_n}}{\sigma\sqrt{\frac{n+1}{n}}}}{\sqrt{\frac{(n-1)S_n^2}{\sigma^2} \cdot \frac{1}{n-1}}} = \frac{X_{n+1} - \overline{X_n}}{S_n\sqrt{\frac{n+1}{n}}} \sim t_{n-1}$$

因此 $c = \sqrt{\frac{n}{n+1}}$，$t_c \sim t_{n-1}$。