

CS215: Discrete Math (H)
2023 Fall Semester Written Assignment # 4
Due: Dec. 11th, 2023, please submit at the beginning of class

Q.1 Prove by induction that, for any integer $n \geq 2$,

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

Solution:

We begin by showing the **base case** $n = 2$ is satisfied. In fact, we have $\left(1 - \frac{1}{2^2}\right) = \frac{3}{4} = \frac{2+1}{2 \cdot 2}$. Suppose that the property holds for $n = k$, then we show it must hold for $n = k + 1$. We have

$$\begin{aligned} \prod_{i=2}^{k+1} \left(1 - \frac{1}{i^2}\right) &= \left(1 - \frac{1}{(k+1)^2}\right) \cdot \prod_{i=2}^k \left(1 - \frac{1}{i^2}\right) \\ &= \left(1 - \frac{1}{(k+1)^2}\right) \cdot \frac{k+1}{2k} \quad \text{by i.h} \\ &= \frac{(k+1)^2 - 1}{(k+1)^2} \cdot \frac{k+1}{2k} \\ &= \frac{(k+1)^2 - 1}{2k(k+1)} \\ &= \frac{k^2 + 2k}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)}. \end{aligned}$$

By mathematical induction, we have proved the conclusion.

□

Q.2 Prove that if A_1, A_2, \dots, A_n and B are sets, then

$$\begin{aligned} (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \\ = (A_1 \cap A_2 \cap \cdots \cap A_n) - B. \end{aligned}$$

Solution:

If $n = 1$, there is nothing to prove, and then $n = 2$, this says that $(A_1 \cap \bar{B}) \cap (A_2 \cap \bar{B}) = (A_1 \cap A_2) \cap \bar{B}$, which is the distributive law. For the inductive step, assume that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) = (A_1 \cap A_2 \cap \cdots \cap A_n) - B;$$

we must show that

$$(A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) = (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B.$$

We have

$$\begin{aligned} & (A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B) \cap (A_{n+1} - B) \\ &= ((A_1 - B) \cap (A_2 - B) \cap \cdots \cap (A_n - B)) \cap (A_{n+1} - B) \\ &= ((A_1 \cap A_2 \cap \cdots \cap A_n) - B) \cap (A_{n+1} - B) \\ &= (A_1 \cap A_2 \cap \cdots \cap A_n \cap A_{n+1}) - B. \end{aligned}$$

The third line follows from the inductive hypothesis, and the fourth line follows from the $n = 2$ case.

□

Q.3 Prove that if $h > -1$, then $1 + nh \leq (1 + h)^n$ for all nonnegative integers n . This is called **Bernoulli's inequality**.

Solution:

Let $P(n)$ be “ $1 + nh \leq (1 + h)^n$, $h > -1$.”

Basic step: $P(0)$ is true because $1 + 0 \cdot h = 1 \leq 1 = (1 + h)^0$.

Inductive step: Assume that $1 + kh \leq (1 + h)^k$. Then because $(1 + h) > 0$, $(1 + h)^{k+1} = (1 + h)(1 + h)^k \geq (1 + h)(1 + kh) = 1 + (k + 1)h + kh^2 \geq 1 + (k + 1)h$.

Inductive conclusion: By mathematical induction, we have $P(n)$ is true for all nonnegative integers n .

□

Q.4 Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

- (a) Show statements $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true, completing the basis step of the proof.
- (b) What is the inductive hypothesis of the proof?
- (c) What do you need to prove in the inductive step?
- (d) Complete the inductive step for $k \geq 21$.
- (e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Solution:

- (a) $P(18)$ is true, because we can form 18 cents of postage with one 4-cent stamp and two 7-cent stamps. $P(19)$ is true, because we can form 19 cents of postage with three 4-cent stamps and one 7-cent stamp. $P(20)$ is true, because we can form 20 cents of postage with five 4-cent stamps. $P(21)$ is true, because we can form 20 cents of postage with three 7-cent stamps.
- (b) The inductive hypothesis is the statement that using just 4-cent and 7-cent stamps we can form j cents postage for all j with $18 \leq j \leq k$, where we assume that $k \geq 21$.
- (c) In the inductive step we must show, assuming the inductive hypothesis, that we can form $k+1$ cents postage using just 4-cent and 7-cent stamps.
- (d) We want to form $k+1$ cents of postage. Since $k \geq 21$, we know that $P(k-3)$ is true, that is, we can form $k-3$ cents of postage. Put one more 4-cent stamp on the envelope, and we have formed $k+1$ cents of postage, as desired.
- (e) We have completed both the basis step and the inductive step, so by the principle of strong induction, the statement is true for every integer n greater than or equal to 18.

□

Q.5 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Solution: The strong induction principle clearly implies ordinary induction, for if one has shown that $P(k) \rightarrow P(k+1)$, then it automatically follows that $[P(1) \wedge \cdots \wedge P(k)] \rightarrow P(k+1)$; in other words, strong induction can always be invoked whenever ordinary induction is used.

Conversely, suppose that $P(n)$ is a statement that one can prove using strong induction. Let $Q(n)$ be $P(1) \wedge \cdots \wedge P(n)$. Clearly $\forall n P(n)$ is logically equivalent to $\forall n Q(n)$. We show how $\forall n Q(n)$ can be proved using ordinary induction. First, $Q(1)$ is true because $Q(1) = P(1)$ and $P(1)$ is true by the basis step for the proof of $\forall n P(n)$ by strong induction. Now suppose that $Q(k)$ is true, i.e., $P(1) \wedge \cdots \wedge P(k)$ is true. By the proof of $\forall n P(n)$ by strong induction, it follows that $P(k+1)$ is true. But $Q(k) \wedge P(k+1)$ is just $Q(k+1)$. Thus, we have proved $\forall n Q(n)$ by ordinary induction.

□

Q.6 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

- (a) Find $f(16)$
- (b) Find a big- O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Solution:

- (a) $f(16) = 2f(4) + 4 = 2(2f(2) + 2) + 4 = 2(2 \cdot 1 + 2) + 4 = 12$.
- (b) Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + m$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of g we have $g(m) = 2g(m/2) + m$. Theorem 2 (with $a = 2, b = 2, c = 1$, and $d = 1$ now tells us that $g(m)$ is $O(m \log m)$. Since $m = \log n$, this means that our function is $O(\log n \cdot \log \log n)$.

□

Q.7 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

- (a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.
- (b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.
- (c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Solution:

- (a) By repeated substitution, we get

$$\begin{aligned} S(n) &= 9S(n/2) + n^2 \\ &= 9 \left[9S\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2 \right] + n^2 \\ &= 9^2 S\left(\frac{n}{2^2}\right) + \left(\frac{9}{4}\right) n^2 + n^2 \\ &= 9^2 \left[9S\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2 \right] + \left(\frac{9}{4}\right) n^2 + n^2 \\ &= 9^3 S\left(\frac{n}{2^3}\right) + \left(\frac{9}{4}\right)^2 n^2 + \left(\frac{9}{4}\right) n^2 + n^2 \\ &= \dots \\ &= 9^{\log_2 n} S(1) + n^2 \sum_{i=0}^{\log_2 n - 1} \left(\frac{9}{4}\right)^i \\ &= bn^{\log_2 9} + \frac{4}{5} n^{\log_2 9} - \frac{4}{5} n^2 \\ &= \left(b + \frac{4}{5}\right) n^{\log_2 9} - \frac{4}{5} n^2, \end{aligned}$$

where we are using the fact that

$$\left(\frac{9}{4}\right)^{\log_2 n} = \frac{9^{\log_2 n}}{n^2} = \frac{n^{\log_2 9}}{n^2}.$$

(b) Similar to (a), we get

$$\begin{aligned} T(n) &= aT\left(\frac{n}{4}\right) + n^2 \\ &= a \left[aT\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^2 \right] + n^2 \\ &= a^2 T\left(\frac{n}{4^2}\right) + \left(\frac{a}{16}\right) n^2 + n^2 \\ &= a^2 \left[aT\left(\frac{n}{4^3}\right) + \left(\frac{n}{4^2}\right)^2 \right] + \left(\frac{a}{16}\right) n^2 + n^2 \\ &= a^3 T\left(\frac{n}{4^3}\right) + \left(\frac{a}{16}\right)^2 n^2 + \left(\frac{a}{16}\right) n^2 + n^2 \\ &= \dots \\ &= a^{\log_4 n} T(1) + n^2 \sum_{i=0}^{\log_4 n - 1} \left(\frac{a}{16}\right)^i \\ &= cn^{\log_4 a} + \frac{16}{a-16} n^{\log_4 a} - \frac{16}{a-16} n^2 \\ &= \left(c + \frac{16}{a-16}\right) n^{\log_4 a} - \frac{16}{a-16} n^2, \end{aligned}$$

where we are using the fact that

$$\left(\frac{a}{16}\right)^{\log_4 n} = \frac{a^{\log_4 n}}{n^2} = \frac{n^{\log_4 a}}{n^2}.$$

(c) For $T(n) = O(S(n))$, we should have

$$\begin{aligned} n^{\log_4 a} &\leq n^{\log_2 9} \\ \log_4 a &\leq \log_2 9 \\ a &\leq 9^2 = 81. \end{aligned}$$

So the range of values is $16 < a \leq 81$.

□

Q.8 Consider three subsets A, B, C of a set S .

- (1) Write a formula of $|\overline{A} \cap \overline{B} \cap \overline{C}|$ using the inclusion-exclusion principle.
- (2) Use the formula in (1) to count the number of integers from 1 to 1000 (inclusive) which are not multiples of 10, 4 or 15.

Solution:

(1)

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| \\ = |S| - (|A| + |B| + |C|) + (|A \cap B| + |A \cap C| + |B \cap C|) - |A \cap B \cap C|. \end{aligned}$$

- (2) 667. Let A be the set of integers from 1 to 1000 that are multiples of 10, B be the set of integers from 1 to 1000 that are multiples of 4, and C be the set of integers from 1 to 1000 that are multiples of 15, respectively. We aim to find the number of integers in $\overline{A} \cap \overline{B} \cap \overline{C}$.

We first see that

$$\begin{aligned} |A| &= \lfloor \frac{1000}{10} \rfloor = 100, \\ |B| &= \lfloor \frac{1000}{4} \rfloor = 250, \\ |C| &= \lfloor \frac{1000}{15} \rfloor = 66. \end{aligned}$$

Integers in the set $A \cap B$ are divisible by both 10 and 4. But an integer is divisible by both 10 and 4 if and only if it is divisible by $\text{lcm}(10, 4) = 20$. Since $\text{lcm}(10, 4) = 20$, $\text{lcm}(10, 15) = 30$, and $\text{lcm}(4, 15) = 60$, we have

$$|A \cap B| = \lfloor \frac{1000}{20} \rfloor = 50, |A \cap C| = \lfloor \frac{1000}{30} \rfloor = 33, |B \cap C| = \lfloor \frac{1000}{60} \rfloor = 16.$$

Because $\text{lcm}(10, 4, 15) = 60$, we conclude that

$$|A \cap B \cap C| = \lfloor \frac{1000}{60} \rfloor = 16.$$

Therefore, by the inclusion-exclusion principle, the number of integers from 1 to 1000 that are not multiples of 10, 4 or 15 is

$$\begin{aligned} |\overline{A} \cap \overline{B} \cap \overline{C}| \\ = 1000 - (100 + 250 + 66) + (50 + 33 + 16) - 16 \\ = 667. \end{aligned}$$

Q.9 Suppose that $n \geq 1$ is an integer.

- (a) How many functions are there from the set $\{1, 2, \dots, n\}$ to the set $\{1, 2, 3\}$?
- (b) How many of the functions in part (a) are one-to-one functions?
- (c) How many of the functions in part (a) are onto functions?

Solution:

- (a) There are 3^n functions.
- (b) If $n \leq 3$, there are $P(3, n)$ one-to-one functions. Hence, there are 3 when $n = 1$, 6 when $n = 2$, and 6 when $n = 3$. If $n > 3$, then there are 0 injective functions; there cannot be a one-to-one function from A to B if $|A| > |B|$.
- (c) By the Inclusion-Exclusion Principle, we have

$$\begin{aligned} \# &= \#\{f : f(A) \subseteq \{1, 2, 3\}\} - \#\{f : f(A) \subseteq \{1, 2\}\} - \#\{f : f(A) \subseteq \{1, 3\}\} \\ &\quad - \#\{f : f(A) \subseteq \{2, 3\}\} + \#\{f : f(A) \subseteq \{1\}\} + \#\{f : f(A) \subseteq \{2\}\} \\ &\quad + \#\{f : f(A) \subseteq \{3\}\} \\ &= 3^n - 2^n - 2^n - 2^n + 1 + 1 + 1 \\ &= 3^n - 3 \cdot 2^n + 3. \end{aligned}$$

□

Q.10 Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Solution:

Since $\gcd(2, 121) = 1$, it suffices to prove that $2 \mid \binom{240}{120}$ and $121 \mid \binom{240}{120}$. We prove these two divisibilities in general, i.e.,

$$2 \mid \binom{2n}{n}, \text{ and } (n+1) \mid \binom{2n}{n}.$$

Since

$$\begin{aligned}
 \binom{2n}{n} &= \frac{(2n)!}{n!n!} \\
 &= \frac{2n \cdot (2n-1)!}{n!n!} \\
 &= \frac{2 \cdot (2n-1)!}{(n-1)!n!} \\
 &= 2 \cdot \binom{2n-1}{n},
 \end{aligned}$$

we have 2 divides $\binom{2n}{n}$. Since

$$\begin{aligned}
 \binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!n!} - \frac{(2n)!}{(n+1)!(n-1)!} \\
 &= \frac{(2n)!}{(n+1)!n!} \\
 &= \frac{1}{n+1} \binom{2n}{n},
 \end{aligned}$$

which is an integer, we have $n+1$ divides $\binom{2n}{n}$. This completes the proof.

□

Q.11 Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection?

Solution:

This is equivalent to count the number of solutions $x_1 + x_2 + \cdots + x_{25} = 12$, where all x_i 's are nonnegative. The number of solutions is $\binom{36}{12}$.

□

Q.12 Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$

with five variables.

- (1) Count the number of integer solutions, with $x_1 \geq 3$, $x_2 \geq 0$, $x_3 \geq -2$, $x_4 \geq 0$, and $x_5 \geq 0$.
- (2) Count the number of integer solutions, with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$.

Solution:

- (1) Let $y_1 = x_1 - 3$, $y_2 = x_2$, $y_3 = x_3 + 2$, $y_4 = x_4$, and $y_5 = x_5$. Then the number of integer solutions is equal to the number of integer solutions of the equation

$$y_1 + y_2 + y_3 + y_4 + y_5 = 9,$$

with all y_i 's ≥ 0 . Then the number is $\binom{13}{4} = 715$.

- (2) We first count the number of integer solutions with $x_1 \geq 6$ and $x_2, x_3, x_4, x_5 \geq 0$. Let $y_1 = x_1 - 6$ and $y_i = x_i$ for $2 \leq i \leq 5$, we have the number of integer solutions is $\binom{8}{4}$. Thus, the number of integer solutions with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$ is equal to the number of integer solutions with $x_1, x_2, x_3, x_4, x_5 \geq 0$ minus $\binom{8}{4}$. This leads to

$$\binom{14}{4} - \binom{8}{4} = 931.$$

Q.13 16 points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Solution: The area of the rectangle is 15 square units. By the pigeonhole principle, two of these points must lie in the same 1 by 1 square. Therefore, they are no further apart than the diagonal of the square $\sqrt{2}$.

□

Q.14 Prove the hockeystick identity

$$\sum_{k=0}^r \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

- (a) using a combinatorial argument
- (b) using Pascal's identity.

Solution:

- (a) $\binom{n+r+1}{r}$ counts the number of ways to choose a sequence of r 0s and $n+1$ 1s by choosing the positions of the 0s. Alternatively, suppose that the $(j+1)$ st term is the last term equal to 1, so that $n \leq j \leq n+r$. Once we have determined where the last 1 is, we decide where the 0s are to be placed in the j spaces before the last 1. There are n 1s and $j-n$ 0s in this range. By the sum rule it follows that there are $\sum_{j=n}^{n+r} \binom{j}{j-n} = \sum_{k=0}^r \binom{n+k}{k}$ ways to this.
- (b) Let $P(r)$ be the statement to be proved. The basis step is the equation $\binom{n}{0} = \binom{n+1}{0}$, which is just $1 = 1$. Assume that $P(r)$ is true. Then

$$\begin{aligned}
 & \sum_{k=0}^{r+1} \binom{n+k}{k} \\
 &= \sum_{k=0}^r \binom{n+k}{k} + \binom{n+r+1}{r+1} \\
 &= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \\
 &= \binom{n+r+2}{r+1},
 \end{aligned}$$

using the inductive hypothesis and Pascal's identity.

□

Q.15 For $0 \leq k \leq n$, show that

$$\sum_{r=k}^n \binom{n}{r} \binom{r}{k} = \binom{n}{k} 2^{n-k}.$$

Your proof may be either combinatorial or algebraic.

Solution: Note that

$$\begin{aligned}
\binom{n}{r} \binom{r}{k} &= \frac{n!}{r! \cdot (n-r)!} \frac{r!}{k!(r-k)!} \\
&= \frac{n!}{k!(r-k)!(n-r)!} \\
&= \frac{n!}{k!(n-k)!} \frac{(n-k)!}{(r-k)!(n-r)!} \\
&= \binom{n}{k} \binom{n-k}{r-k}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\sum_{r=k}^n \binom{n}{r} \binom{r}{k} &= \sum_{r=k}^n \binom{n}{k} \binom{n-k}{r-k} \\
&= \binom{n}{k} \sum_{r=k}^n \binom{n-k}{r-k} \\
&= \binom{n}{k} \sum_{s=0}^{n-k} \binom{n-k}{s} \quad \text{let } s = r - k \\
&= \binom{n}{k} 2^{n-k}.
\end{aligned}$$

For a combinatorial proof, observe that the right-hand side is equal to the number of ways of doing the following with a set X of size n :

- Pick a subset A of X consisting of k elements. This can be done in $\binom{n}{k}$ ways.
- Pick any subset B of $X \setminus A$. This can be done in 2^{n-k} ways.

An equivalent way of forming the sets A and B is to do the following:

- For any r such that $k \leq r \leq n$, pick a subset C of X consisting of r elements. For a given r , this can be done in $\binom{n}{r}$ ways.
- Pick a subset A of C consisting of k elements, and let $B = C \setminus A$. This can be done in $\binom{r}{k}$ ways.

Summing over all possible r , we obtain the desired result.

Q.16

Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

Solution: The CE is

$$r^3 - 2r^2 - r + 2 = (r + 1)(r - 1)(r - 2).$$

The roots are $r = -1$, $r = 1$ and $r = 2$. Hence, the solutions to this recurrence are of the form

$$a_n = \alpha_1(-1)^n + \alpha_2 1^n + \alpha_3 2^n.$$

To find the constants α_1, α_2 and α_3 , we use the initial conditions. Plugging in $n = 0, n = 1$, and $n = 2$, we have

$$a_0 = 1 = \alpha_1 + \alpha_2 + \alpha_3 a_1 = 0 = -\alpha_1 + \alpha_2 + 2\alpha_3 a_2 = 7 = \alpha_1 + \alpha_2 + 4\alpha_3.$$

We then have $\alpha_1 = 3/2$, $\alpha_2 = -5/2$, and $\alpha_3 = 2$. Hence,

$$a_n = 3/2 \cdot (-1)^n - 5/2 \cdot 1^n + 2 \cdot 2^n = 2^{n+1} + (-1)^n \cdot 3/2 - 5/2.$$

□

Q.17

- (a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.
- (b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Solution:

- (a) The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = p_2 n^2 + p_1 n + p_0$. (Note that $s = 1$ here, and

1 is not a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $p_2n^2 + p_1n + p_0 = 2(p_2(n-1)^2 + p_1(n-1) + p_0) + 2n^2$. We rewrite this by grouping terms with equal powers of n , obtaining $(-p_2 - 2)n^2 + (4p_2 - p_1)n + (-2p_2 + 2p_1 - p_0) = 0$. In order for this equation to be true for all n , we must have $p_2 = -2$, $4p_2 = p_1$, and $-2p_2 + 2p_1 - p_0 = 0$. This tells us that $p_1 = -8$ and $p_0 = -12$. Therefore the particular solution we seek is $a_n^{(p)} = -2n^2 - 8n - 12$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n - 2n^2 - 8n - 12$.

- (b) We plug the initial condition into our solution from part (a) to obtain $4 = a_1 = 2\alpha - 2 - 8 - 12$. This tells us that $\alpha = 13$. So the solution is $a_n = 13 \cdot 2^n - 2n^2 - 8n - 12$.

□

Q.18 Denote by a_n the number of *ternary* strings (with elements 0, 1, 2) of length n that contain either 00 or 11.

- (1) Find a recurrence relation for a_n with initial conditions.
- (2) Find a closed-form expression for a_n .

Solution:

- (1) $a_n = 2a_{n-1} + a_{n-2} + 2 \cdot 3^{n-2}$.

Let b_n be the number of n -ternary strings that begin with 0 and contain neither 00 nor 11. Let c_n and d_n be the same except starting with 1 or 2, respectively. Let $t_n = b_n + c_n + d_n$ be the total number of n -ternary strings that contain neither 00 nor 11. We want $a_n = 3^n - t_n$.

To find a recurrence relation for b_n , observe that an n -ternary string which begins with 0 and contains neither 00 nor 11 is

- 0, followed by an $(n-1)$ -digit string which begins with 1 and contains neither 00 nor 11; or
- 0, followed by an $(n-1)$ -digit string which begins with 2 and contains neither 00 nor 11.

Therefore, we have $b_n = c_{n-1} + d_{n-1}$. Similarly, we have

$$c_n = b_{n-1} + d_{n-1}, d_n = b_{n-1} + c_{n-1} + d_{n-1} = t_{n-1}.$$

Adding all of these gives

$$t_n = 2t_{n-1} + d_{n-1} = 2t_{n-1} + t_{n-2}.$$

Writing this in terms of a_n , we have

$$3^n - a_n = 2(3^{n-1} - a_{n-1}) + (3^{n-2} - a_{n-2}),$$

which simplifies to

$$a_n = 2a_{n-1} + a_{n-2} + 2 \cdot 3^{n-2}.$$

The two initial conditions are $a_0 = a_1 = 0$.

- (2) The characteristic equation of the homogeneous part is $x^2 - 2x - 1 = 0$, and its two distinct roots are $1 + \sqrt{2}$ and $1 - \sqrt{2}$. Thus, the solutions to the original recurrence relation of a_n is

$$a_n = c_1(1 + \sqrt{2})^n + c_2(1 - \sqrt{2})^n + p(n).$$

We try $p(n) = c \cdot 3^n$, and we then have

$$c \cdot 3^n = 2 \cdot c \cdot 3^{n-1} + c \cdot 3^{n-2} + 2 \cdot 3^{n-2},$$

which gives

$$(2c - 2)3^{n-2} = 0$$

for all n . Therefore, we have $c = 1$, and $p(n) = 3^n$. Thus, the closed-form solution is

$$a_n = c_1(1 + \sqrt{2})^n + c_2(1 - \sqrt{2})^n + 3^n.$$

By the two initial conditions $a_0 = a_1 = 0$, we have

$$\begin{aligned} c_1 + c_2 + 1 &= 0(n=0) \\ (1 + \sqrt{2})c_1 + (1 - \sqrt{2})c_2 + 3 &= 0(n=1) \end{aligned}$$

which has solution

$$c_1 = \frac{-1 - \sqrt{2}}{2}, \quad c_2 = \frac{\sqrt{2} - 1}{2}.$$

Hence the closed-form solution is

$$a_n = \frac{-1 - \sqrt{2}}{2}(1 + \sqrt{2})^n + \frac{\sqrt{2} - 1}{2}(1 - \sqrt{2})^n + 3^n.$$

Q.19 Let $S_n = \{1, 2, \dots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n . Note that $a_0 = 0$ and $a_1 = 1$.

Solution: We may split S_n into 3 cases :

Case (1): item 1 is not in the subset. We must now choose a non-empty subset of $\{2, \dots, n\}$. There are a_{n-1} ways to do this.

Case (2): item 1 is in the subset, and there are more elements. We must now choose a non-empty subset of $\{3, \dots, n\}$. There are a_{n-2} ways to do this.

Case (3): item 1 is in the subset, and no other elements are. There is 1 way to do this.

Thus, we have the recurrence relation as: $a_n = a_{n-1} + a_{n-2} + 1$.

□

Q.20 Use generating functions to prove Vandermonde's identity:

$$C(m+n, r) = \sum_{k=0}^r C(m, r-k)C(n, k),$$

whenever m, n , and r are nonnegative integers with r not exceeding either m or n . [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

Solution: Applying the binomial theorem to the equality $(1+x)^{m+n} = (1+x)^m(1+x)^n$, shows that $\sum_{r=0}^{m+n} C(m+n, r)x^r = \sum_{r=0}^m C(m, r)x^r \cdot \sum_{r=0}^n C(n, r)x^r = \sum_{r=0}^{m+n} [\sum_{k=0}^r C(m, r-k)C(n, k)]x^r$. Comparing coefficients gives the desired identity.

□

Q.21 Generating functions are very useful, for example, provide an approach to solving linear recurrence relations. Read pp. 537-548 of the textbook. [You do not need to write anything for this problem on your submitted assignment paper.]