Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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NP-complete Problems

- Class **NP** vs Class **P**
  - **P**: decision problems solvable in polynomial time
  - **NP**: decision problems with certificates verifiable in polynomial time (polynomial time verification)

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- Approximation Algorithm
  Natural idea: settle for *non-optimal* solutions for these “hard” problems, if we can find such close-to-the-optimal solutions reasonably fast.

Satisfiability Problem

- Satisfiability (*SAT*) – one of the most important **NP** problems
- **Definition** A *Boolean formula* is a logical formula consisting of
  - Boolean variables (0 = false, 1 = true),
  - logical operations
    - $\neg x$: Negation
    - $x \vee y$: Disjunction
    - $x \wedge y$: Conjunction

  With the truth table defined by:

x	y	$\neg x$	$x \vee y$	$x \wedge y$
0	0	1	0	0
0	1		1	0
1	0	0	1	0
1	1		1	1

Satisfiable

- **Definition** For a fixed k, Boolean formulas in the following form are called k-conjunctive normal form (k-CNF):
$$f_1 \wedge f_2 \wedge \cdots \wedge f_n$$
where each f_i is of the form $f_i = y_{i,1} \vee y_{i,2} \vee \cdots \vee y_{i,k}$, and each $y_{i,j}$ is a variable or the negation of a variable.

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- **2SAT**
Instance: A 2-CNF formula f
Problem: To decide whether f is *satisfiable*
Example a 2-CNF formula
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Theorem **2SAT** $\in$ Class **P**

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    a yes-input of L_1 maps to a yes-input of L_2, and a no-input of L_1 maps to a no-input of L_2

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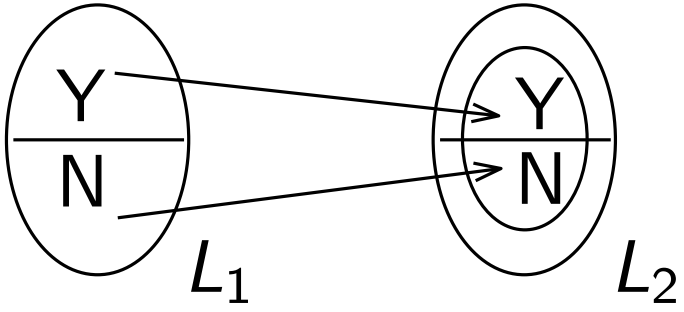

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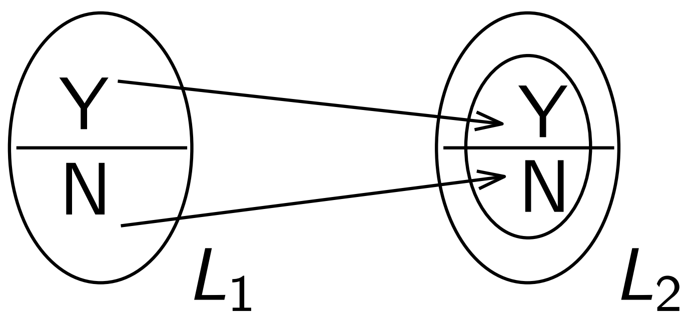

If such an f exists, we say that L_1 is *polynomial-time reducible* to L_2, and write $L_1 \leq_P L_2$.

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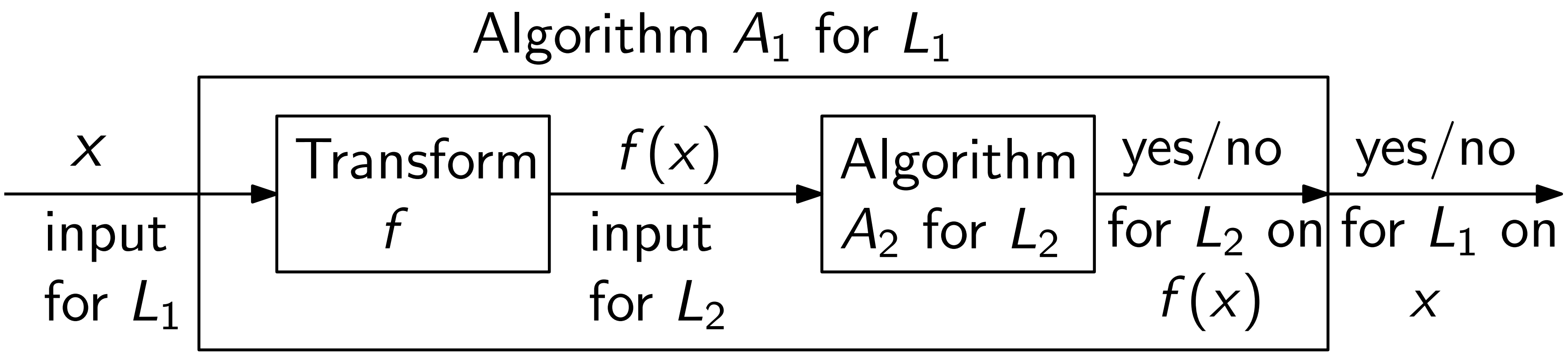

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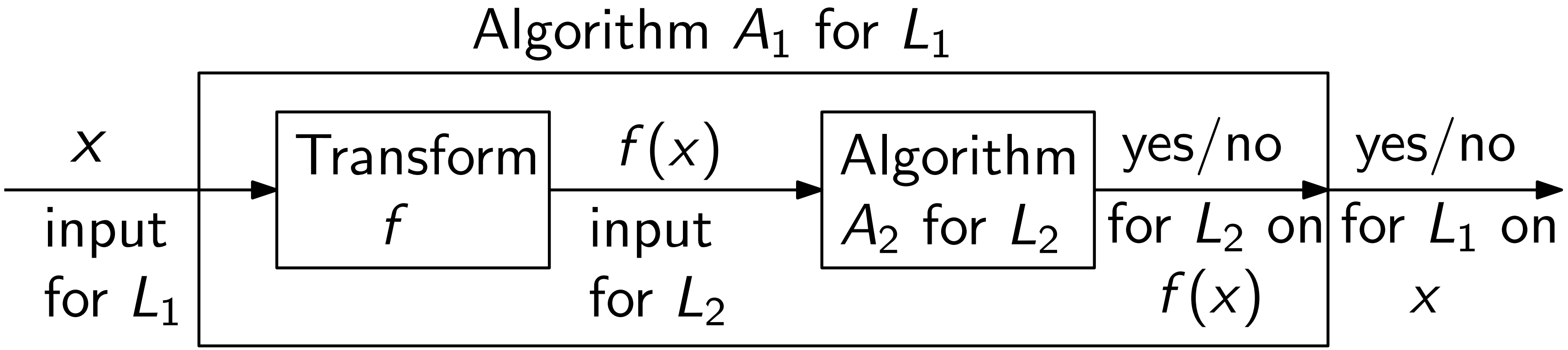

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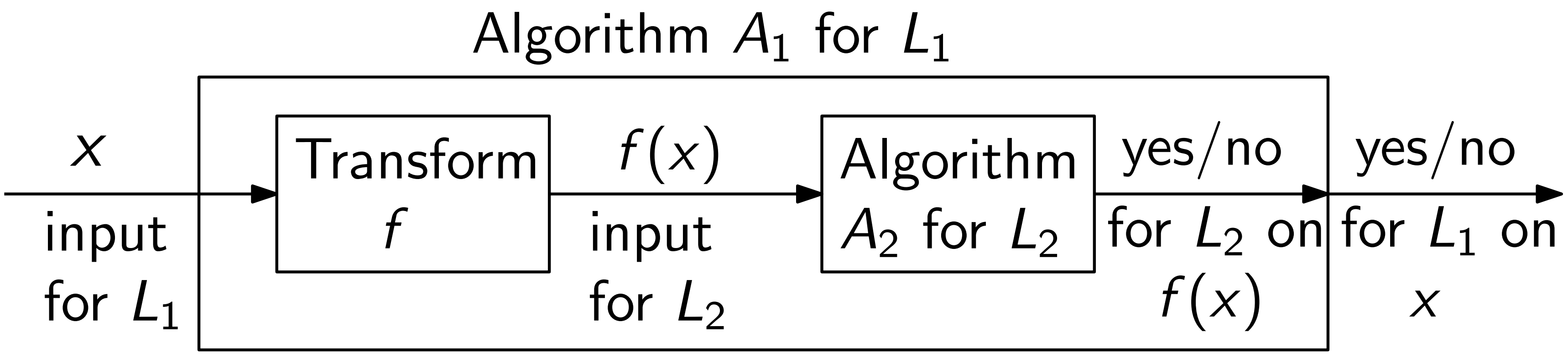

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Theorem If $L_1 \le_P L_2$ and $L_2 \in P$, then $L_1 \in P$

Lemma If $L_1 \le_P L_2$ and $L_2 \le_P L_3$, then $L_1 \le_P L_3$.

The Class NP-Complete (NPC)

- The Class *NPC* consists of all decision problems L s.t.
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  Proof. Let L'' be any problem in NP. Since $L' \in NPC$, by definition we have $L'' \leq_P L'$. Since $L' \leq_P L$, then by transitivity, we have $L'' \leq_P L$.

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  We will prove:
  3SAT $\leq_P$ DCLIQUE
  DCLIQUE $\leq_P$ DVC

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- **Definition** A *clique* in an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ of vertices s.t. each pair $u, v \in V'$ is connected by an edge $(u, v) \in E$. In other words, a clique is a complete subgraph of G.

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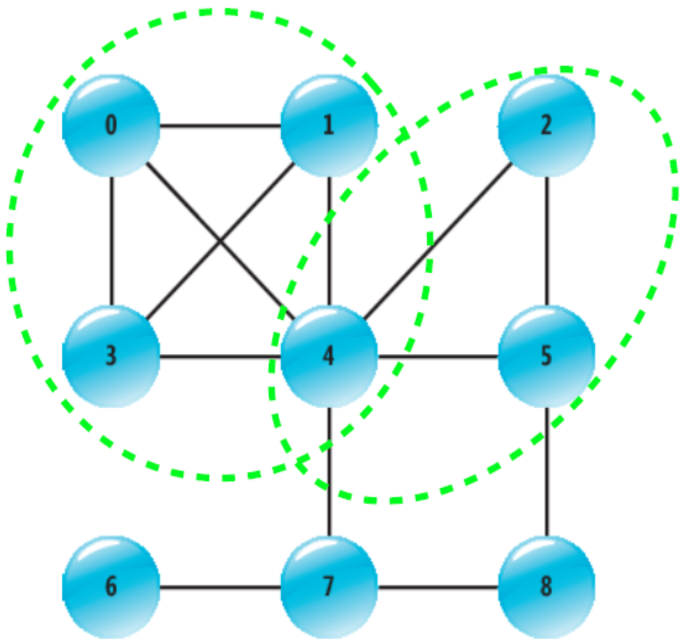

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  We will define a polynomial transformation f from 3SAT to DCLIQUE $f : \phi \mapsto (G, k)$ that builds a graph G and integer k s.t. ϕ is a Yes-input to 3SAT if and only if (G, k) is a Yes-input to DCLIQUE.

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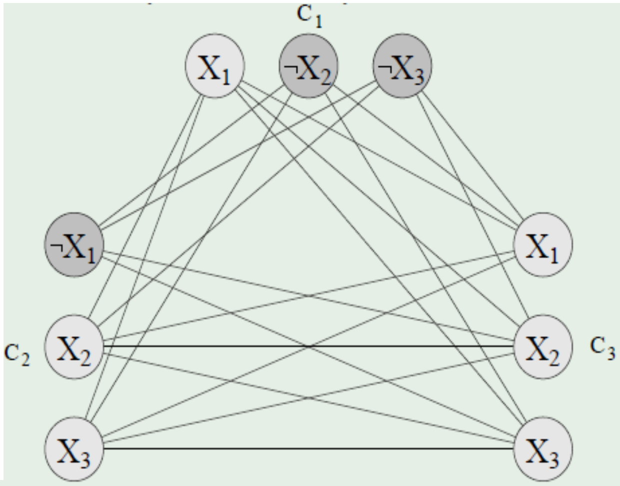

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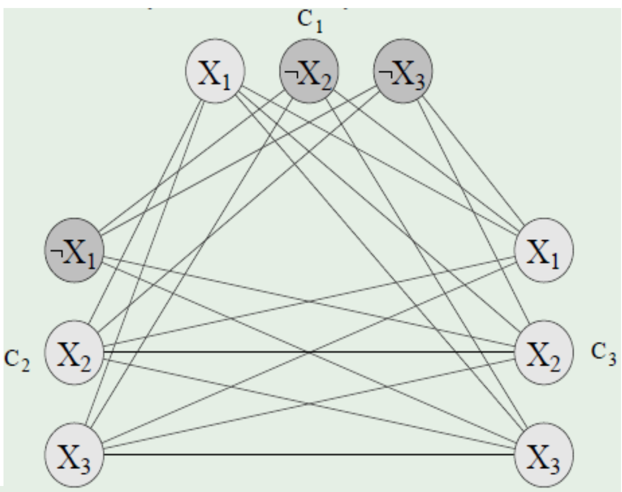

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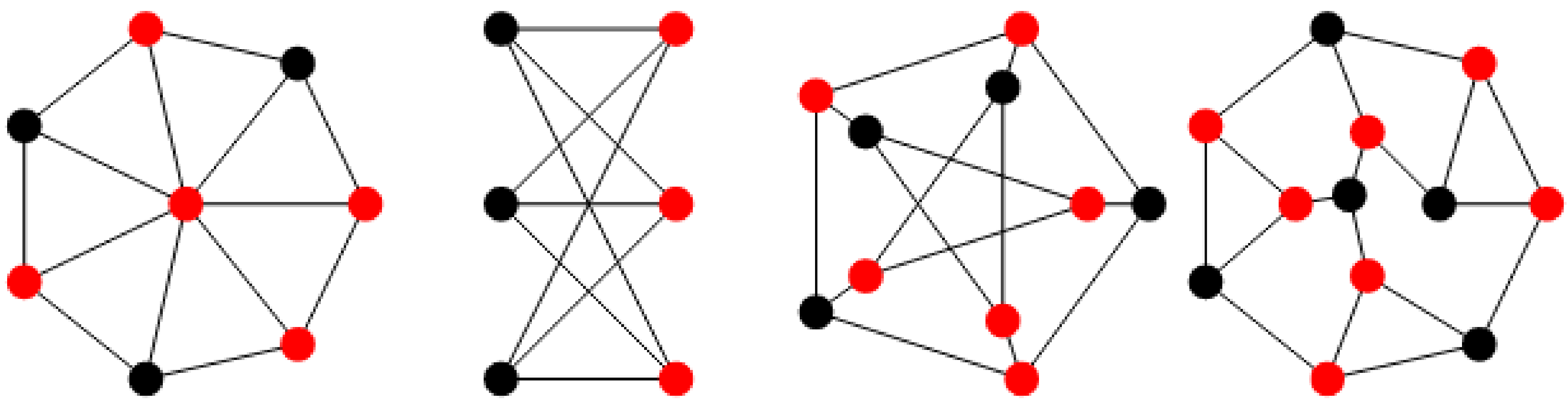

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Definition The *complement* of a graph $G = (V, E)$ is defined by $\overline{G} = (V, \overline{E})$ where

$$\overline{E} = \{(u, v) | u, v \in V, \ u \neq v, \ (u, v) \notin E\}.$$

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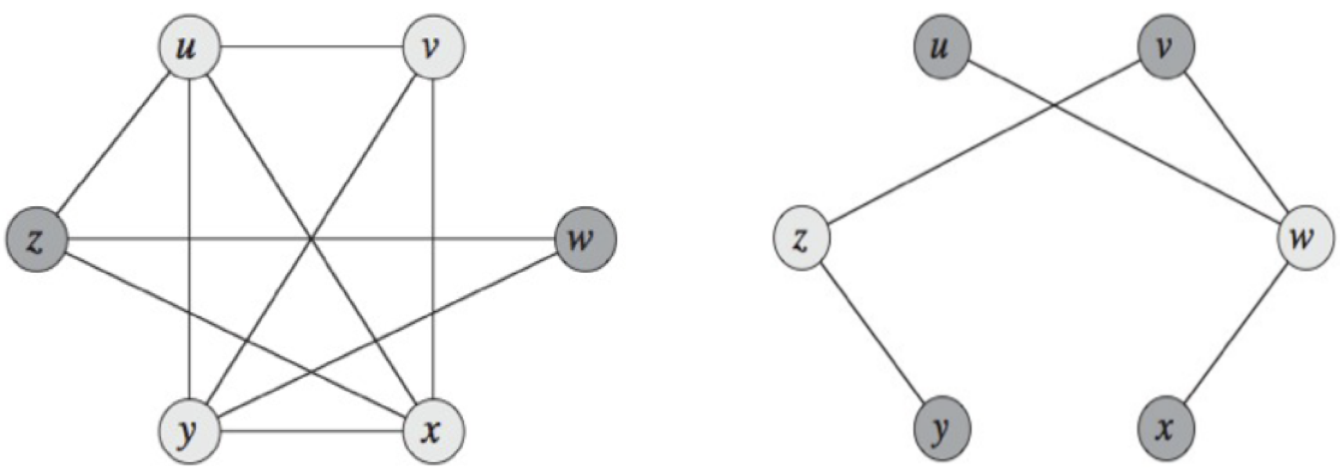

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  The reduction takes polynomial time

  A *clique* of size k in G $\Rightarrow$ a *vertex cover* of size $|V| - k$ in $\overline{G}$
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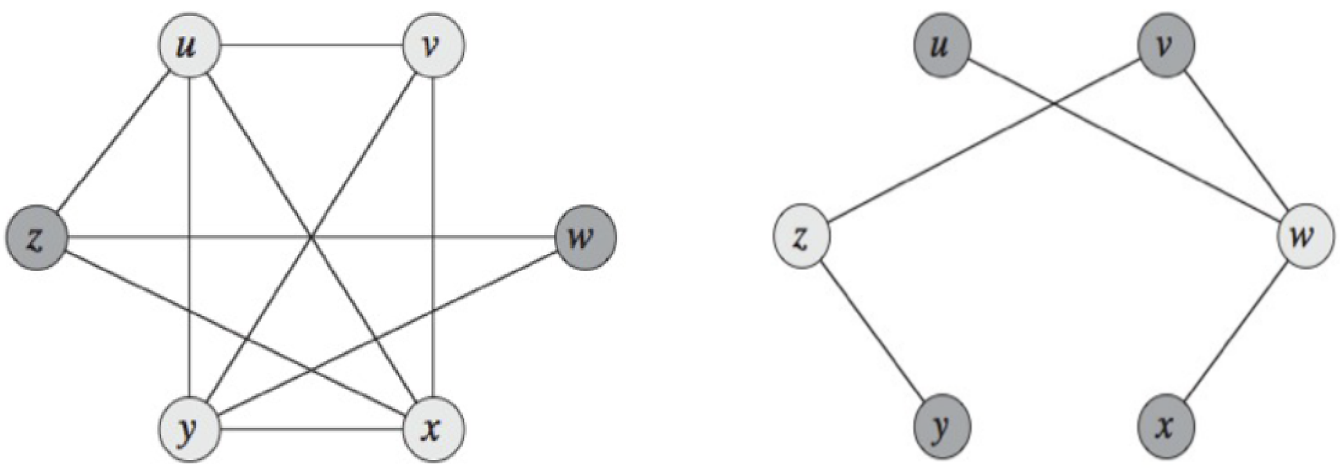

Approximation Algorithm Example: VC

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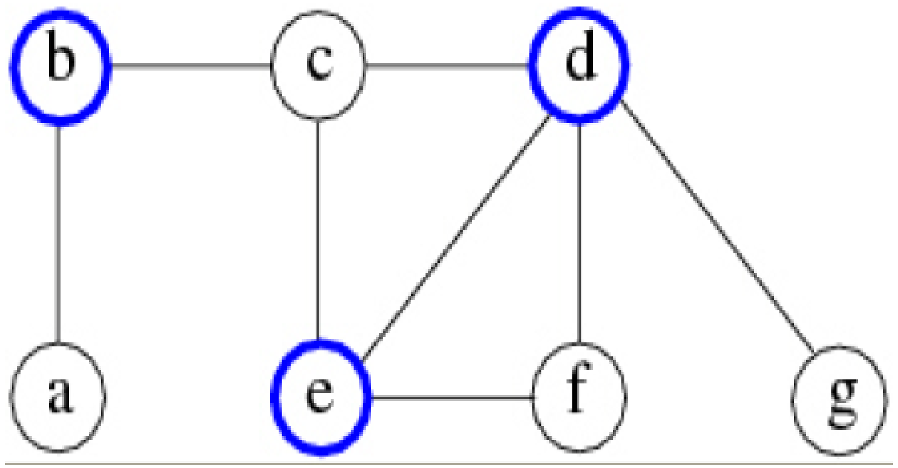

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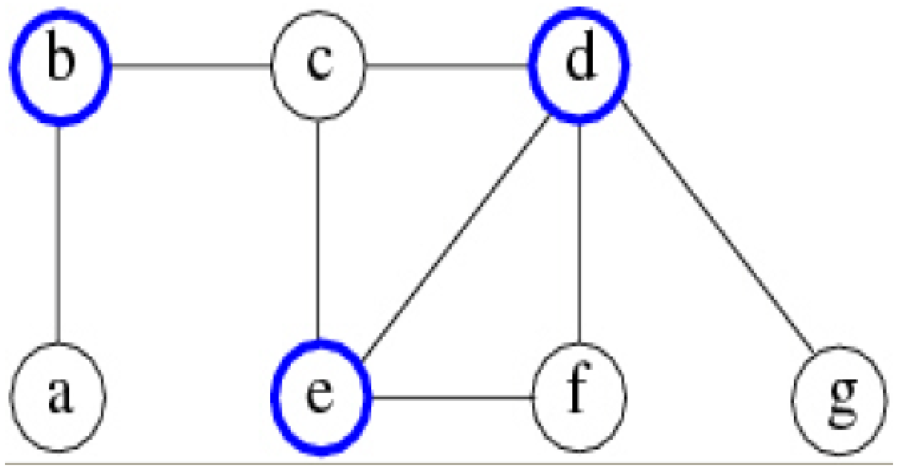

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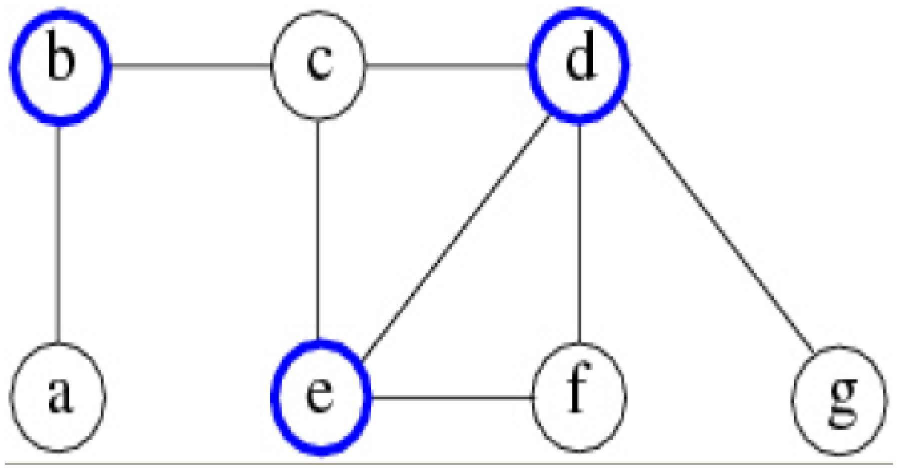
  It is very unlikely to give an exact polynomial time algorithm (Why?)



An Approximation Algorithm for VC

Approx-Vertex-Cover(G=(V, E))

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E'= E;
while E' is not empty do do
    let (u, v) be any edge in E'          (*);
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But, how good is C?

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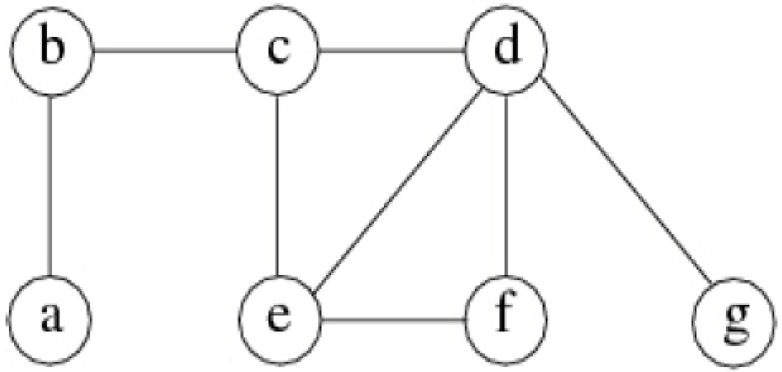

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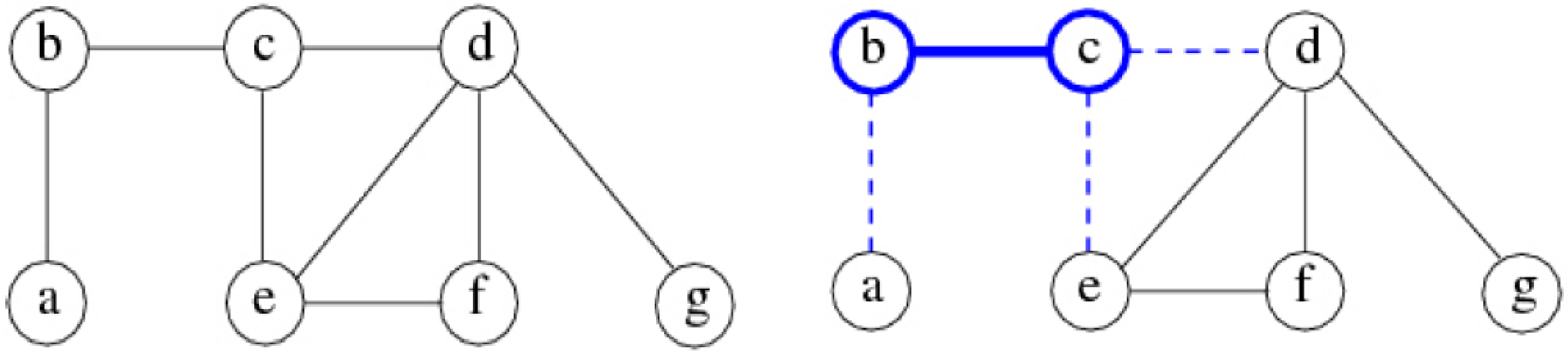

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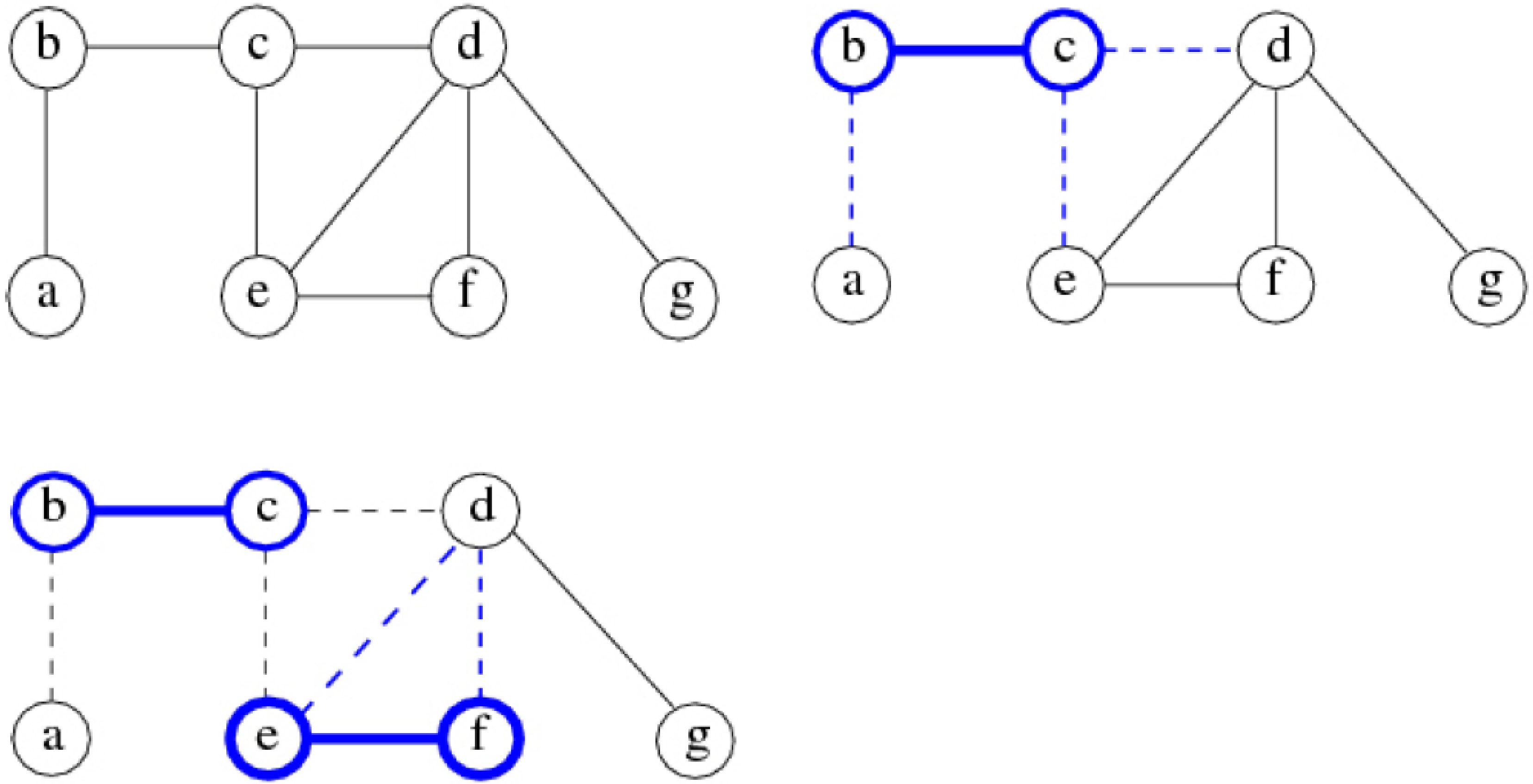

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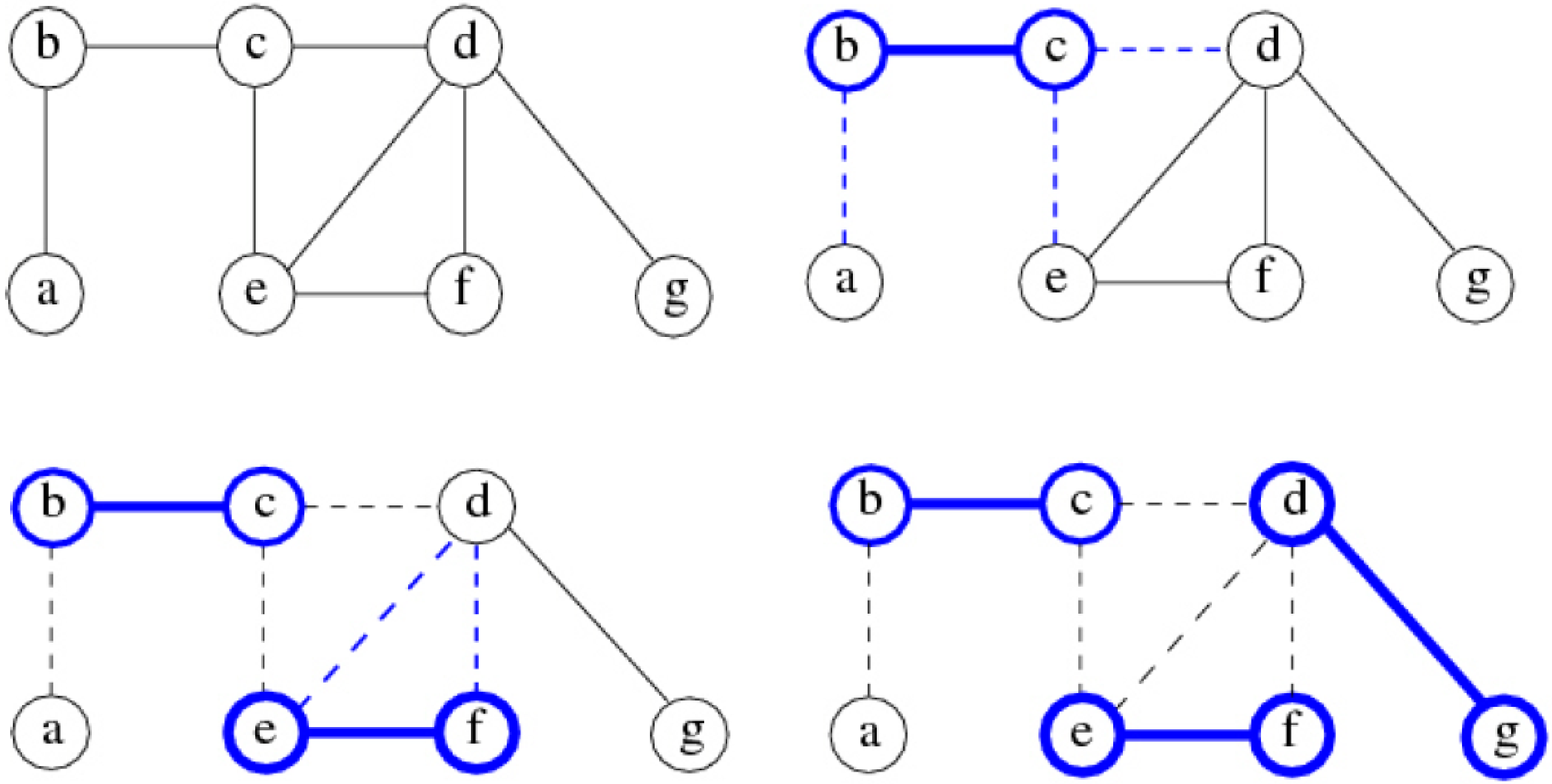

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The optimal vertex cover C^* must cover every edge in M, so $|C^*| \ge |M|$. But notice that the algorithm returns a vertex set of size $2|M|$. Therefore, we have

$$|C| = 2|M| \le 2|C^*|.$$

Field

- A *field* is a set $\mathbb{F}$ equipped with two operations, *addition* $(+)$ and *multiplication* $(\cdot)$, and two special elements $0, 1$, s.t.:
  - $(\mathbb{F}, +)$ is an *abelian group* with identity element 0
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  - For all $a \in \mathbb{F}$, $0 \cdot a = a \cdot 0 = 0$
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  - The properties can be verified

  Every $a \in \mathbb{F}_p^*$ has a *multiplicative inverse*: since $a \in \mathbb{F}_p^*$ and p is a prime, we have $\gcd(a, p) = 1$, and by extended Euclidean algorithm, there exist x, y s.t. $ax + py = 1$, and then $x = a^{-1} \bmod p$.

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- Any finite field $\mathbb{F}$ is a *finite dimensional vector space* over $\mathbb{F}_p$, with $n = \dim_{\mathbb{F}_p}(\mathbb{F})$, $|\mathbb{F}| = p^n$, i.e., the cardinality of $\mathbb{F}$ must be a prime power.

Finite Fields

- **Uniqueness** of finite fields:
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- *Binary field* – *characteristic-2* finite fields $\mathbb{F}_{2^m}$
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- An *irreducible polynomial* $f(x)$ of degree m is chosen:
  $f(x)$ cannot be factered as a product of binary polynomials each of degree less than m
  - *Addition*: usual
  - *Multiplication*: modulo $f(x)$

Elements of Finite Fields

- An *irreducible polynomial* $f(x)$ of degree m
  - $f(x) = x^4 + 1$ over $\mathbb{F}_2$
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  - *Addition*: $(z^3 + z^2 + 1) + (z^2 + z + 1) = z^3 + z$
  - *Subtraction*: $(z^3 + z^2 + 1) - (z^2 + z + 1) = z^3 + z$
  - *Multiplication*: $(z^3 + z^2 + 1) \cdot (z^2 + z + 1) = z^5 + z + 1 = z^2 + 1$
  - *Inversion*: $(z^3 + z^2 + 1)^{-1} = z^2$
    since $(z^3 + z^2 + 1) \cdot z^2 = z^5 + z^4 + z^2 = 1 \bmod z^4 + z + 1$

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  If $\psi : z \mapsto c$ is an *ismorphism* between K_1 and K_2, then $f_1(c) \equiv 0 \pmod{f_2}$ for some $c \in K_2$. The choices for c are $z^2 + z$, $z^2 + z + 1$, $z^3 + z^2$, and $z^3 + z^2 + 1$.

Extension Fields and Subfields

- Let p be a prime and $m \geq 2$. Let $\mathbb{F}_p[z]$ denote the set of all polynomials in the variable z with coefficients from $\mathbb{F}_p$. Let $f(z)$ be an *irreducible polynomial of degree m in $\mathbb{F}_p[z]$*.

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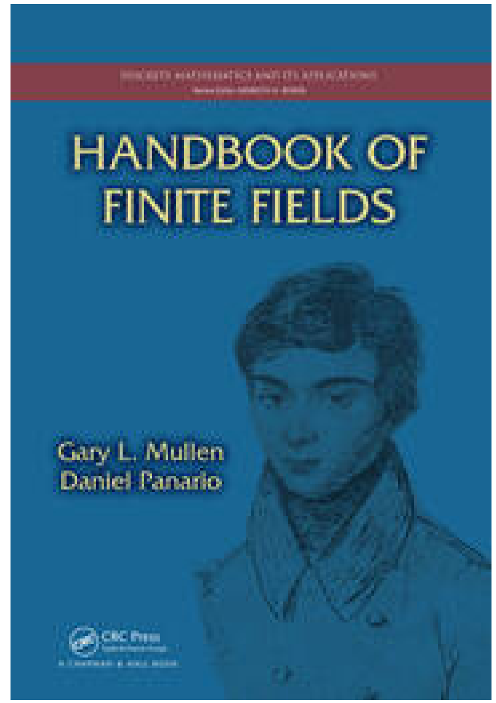
- A finite field $\mathbb{F}_{p^m}$ has precisely one subfield of order p^ℓ for each positive divisor ℓ of m.
  The elements of this subfield are the elements $a \in \mathbb{F}_{p^m}$ satisfying $a^{p^\ell} = a$; Conversely, every subfield of $\mathbb{F}_{p^m}$ has order p^ℓ for some positive divisor ℓ of m.

Applications of Finite Fields



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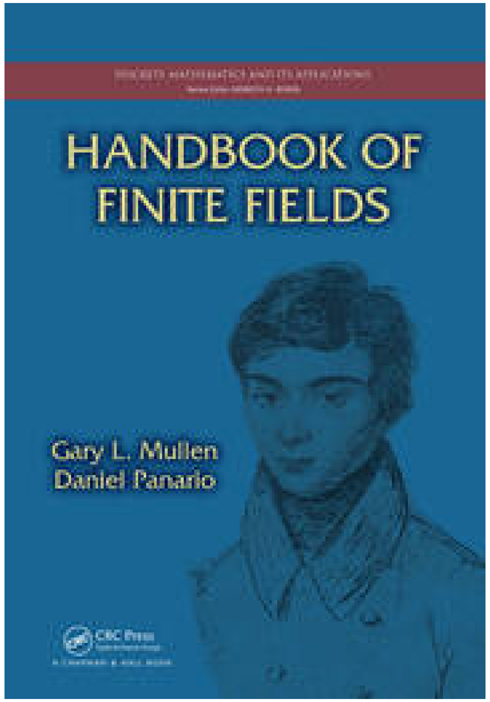

coding theory, cryptography, combinatorics, data storage systems, simulation, communications, signal design, ...

Review

01. Propositional Logic
02. Predicate Logic
03. Mathematical Proofs
04. Sets
05. Functions
06. Complexity of Algorithms
07. Number Theory
    Groups, Rings and Fields
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- Quantified statements

  universal, existential, equivalence

Methods of Proving Theorems

- Basic methods to prove theorems:
  - *direct proof*
    - $p \to q$ is proved by showing that if p is true then q follows
  - *proof by contrapositive*
    - show the contrapositive $\neg q \to \neg p$
  - *proof by contradiction*
    - show that $(p \wedge \neg q)$ contradicts the assumptions
  - *proof by cases*
    - give proofs for all possible cases
  - *proof of equivalence*
    - $p \leftrightarrow q$ is replaced with $(p \to q) \wedge (q \to p)$

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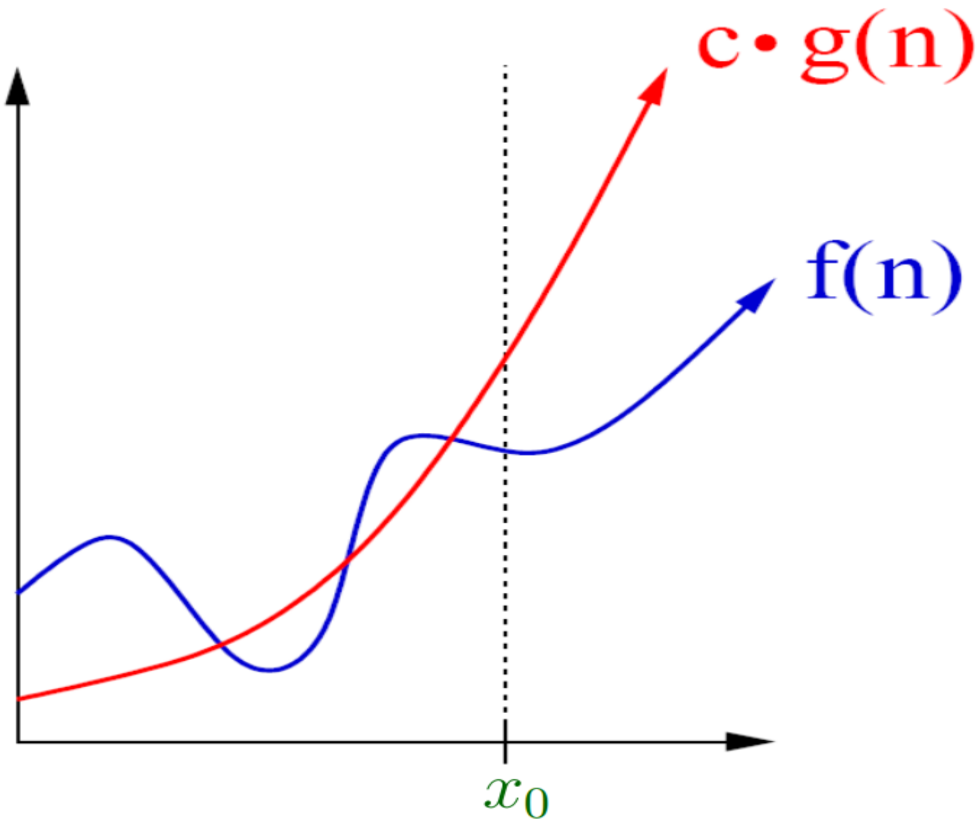
- counting the number of such functions?

Big-O Notation

- Let f and g be functions from the set of integers or the set of real numbers to the set of real numbers. We say that $f(n) = O(g(n))$ (reads: $f(n)$ is O of $g(n)$), if there exist some positive constants C and k such that $|f(n)| \leq C|g(n)|$, whenever $n > k$.



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$$
\begin{aligned}
x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
x &\equiv 2 \pmod{7}
\end{aligned}
$$

- Fermat's Little Theorem

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  Euler's Theorem

  Primitive roots, multiplicative order

Cryptography

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  Euler's Theorem

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  RSA cryptosystem

  DLP, Diffie-Hellman protocol

Mathematical Induction

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3. We conclude on the basis of the principle of mathematical induction that $P(n)$ is true for all $n \geq b$.

Recurrence

- Iterating a recurrence

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  bottom up or top down

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  bottom up or top down

  prove by induction, complexity, ...

Counting

- The sum rule and product rule

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  The Inclusion-Exclusion Principle

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  The Pigeonhole Principle

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  Theorem If N is a positive integer and k is an integer with $1 \leq k \leq n$, then there are
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  The Binomial Theorem, Trinomial

- **Definition** An *r-combination* with repetition allowed, or a *multiset of size r*, chosen from a set of n elements, is an unordered selection of elements with repetition allowed.

  Example Find # multisets of size 17 from the set $\{1, 2, 3\}$.

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- Combinatorial proof

Binary Relations

- Properties of relations

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  Definition A relation R on a set A is called a *partial ordering* if it is reflexive, antisymmetric, and transitive.

Graphs & Trees

- Basic concepts

- Basic concepts

  connected graph, simple graph, isomophism, chromatic number, planar graph, Euler circuit, Hamilton circuit, shortest path, bipartite graph, complete graph, special graphs (K_n, $K_{m,n}$, C_n, W_n, Q_n), m-ary tree, tree traversal, spanning tree ...

Good Luck!