Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Linear Congruences

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  Systems of linear congruences have been studied since ancient times.

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  About 1500 years ago, the Chinese mathematician Sun-Tsu asked: "There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?"

Modular Inverse

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  When does an inverse of a modulo m exist?

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\begin{aligned}
&4620 = 45 \cdot 101 + 75\\
&101 = 1 \cdot 75 + 26\\
&75 = 2 \cdot 26 + 23\\
&26 = 1 \cdot 23 + 3\\
&23 = 7 \cdot 3 + 2\\
&3 = 1 \cdot 2 + 1\\
&2 = 2 \cdot 1
\end{aligned}
$$

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$$75 = 2\cdot26 + 23$$
$$26 = 1\cdot23 + 3$$
$$23 = 7\cdot3 + 2$$
$$3 = 1\cdot2 + 1$$
$$2 = 2\cdot1$$

$$1 = 3 - 1\cdot2$$
$$1 = 3 - 1\cdot(23 - 7\cdot3) = -1\cdot23 + 8\cdot3$$
$$1 = -1\cdot23 + 8\cdot(26 - 1\cdot23) = 8\cdot26 - 9\cdot23$$
$$1 = 8\cdot26 - 9\cdot(75 - 2\cdot26) = 26\cdot26 - 9\cdot75$$
$$1 = 26\cdot(101 - 1\cdot75) - 9\cdot75$$
$$= 26\cdot101 - 35\cdot75$$
$$1 = 26\cdot101 - 35\cdot(4620 - 45\cdot101)$$
$$= -35\cdot4620 + 1601\cdot101$$

Using Inverses to Solve Congruences

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  Solution: We found that -2 is an inverse of 3 modulo 7. Multiply both sides of the congruence by -2, we have $x \equiv -8 \equiv 6 \pmod{7}$.

Number of Solutions to Congruences *

- **Theorem*** Let $d = \gcd(a, m)$ and $m' = m/d$. The congruence $ax \equiv b \pmod{m}$ has solutions if and only if $d|b$. If $d|b$, then there are exactly d solutions. If x_0 is a solution, then the other solutions are given by $x_0 + m', x_0 + 2m', \ldots, x_0 + (d-1)m'$.

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Proof.
1) "only if": If x_0 is a solution, then $ax_0 - b = km$. Thus, $ax_0 - km = b$. Since d divides $ax_0 - km$, we must have $d|b$.

2) "if": Suppose that $d|b$. Let $b = kd$. There exist integers s, t such that $d = as + mt$. Multiply both sides by k. Then $b = ask + mtk$. Let $x_0 = sk$. Then $ax_0 \equiv b \pmod{m}$.

3) "$\# = d$": $ax_0 \equiv b \pmod{m}$ $ax_1 \equiv b \pmod{m}$ imply that $m|a(x_1 - x_0)$ and $m'|a'(x_1 - x_0)$. This implies further that $x_1 = x_0 + km'$, where $k = 0, 1, \ldots, d-1$.

The Chinese Remainder Theorem

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x &\equiv 2 \pmod{3} \\
x &\equiv 3 \pmod{5} \\
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- **Theorem** (*The Chinese Remainder Theorem*) Let $m_1, m_2, \ldots, m_n$ be pairwise relatively prime positive integers greater than 1 and $a_1, a_2, \ldots, a_n$ arbitrary integers. Then the system

$$\begin{aligned} x &\equiv a_1 \pmod{m_1} \\ x &\equiv a_2 \pmod{m_2} \\ &\cdots \\ x &\equiv a_n \pmod{m_n} \end{aligned}$$

has a unique solution modulo $m = m_1 m_2 \cdots m_n$.

The Chinese Remainder Theorem

- **Proof** Let $M_k = m/m_k$ for $k = 1, 2, \ldots, n$ and $m = m_1 m_2 \cdots m_n$. Since $\gcd(m_k, M_k) = 1$, there is an integer y_k, an inverse of M_k modulo m_k such that $M_k y_k \equiv 1 \pmod{m_k}$. Let

$$x = a_1 M_1 y_1 + a_2 M_2 y_2 + \cdots a_n M_n y_n.$$

It is checked that x is a solution to the n congruences.

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Let $m = 3 \cdot 5 \cdot 7 = 105$, $M_1 = m/3 = 35$, $M_2 = m/5 = 21$, $M_3 = m/7 = 15$.

$$\begin{aligned} &35 \cdot 2 \equiv 1 \pmod{3} \\ &21 \equiv 1 \pmod{5} \\ &15 \equiv 1 \pmod{7} \end{aligned}$$

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$$x = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 \equiv 233 \equiv 23 \pmod{105}$$

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七子团圆正月半，除百零五便得知。
-- 程大位 《算法统要》（1593年）

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Fermat's Little Theorem

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$\mathcal{Q}$: How to prove Fermat's little theorem?

$$\{1, 2, \ldots, p-1\} = \{x, 2x, \ldots, x(p-1) \pmod p\}$$

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  Theorem * There is a primitive root modulo n if and only if $n = 2, 4, p^e$ or $2p^e$, where p is an odd prime.

  $\mathcal{Q}$: proof? The number of primitive roots? $*$

Number Theory and Cryptography

- Division, Primes
- Congruence
- Greatest Common Divisor (GCD)
- Euler's Theorem / Fermart's Little Theorem

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  $$a = dq + r$$

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- Greatest Common Divisor (GCD)

  Find the GCD of 286 and 503.

$$
\begin{array}{ll}
\gcd(503, 286) & 503 = 1 \cdot 286 + 217 \\
= \gcd(286, 217) & 286 = 1 \cdot 217 + 69 \\
= \gcd(217, 69) & 217 = 3 \cdot 69 + 10 \\
= \gcd(69, 10) & 69 = 6 \cdot 10 + 9 \\
= \gcd(10, 9) & 10 = 1 \cdot 9 + 1 \\
= 1 & 9 = 9 \cdot 1
\end{array}
$$

$$
\begin{array}{rcl}
1 & = & 10 - 1 \cdot 9 \\
1 & = & 7 \cdot 10 - 1 \cdot 69 \\
1 & = & 7 \cdot 217 - 22 \cdot 69 \\
1 & = & 29 \cdot 217 - 22 \cdot 286 \\
1 & = & 29 \cdot 503 - 51 \cdot 286
\end{array}
$$

- E

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  (extended) Euclidean algorithm
  find the modular inverse
  solve linear congruence $ax \equiv b \pmod{m}$ $(\gcd(a, m) = 1)$

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Number Theory Summary

- Division, Primes

  $a = dq + r \quad q = a \; div \; d \quad r = a \; mod \; d$

- Congruence

  $a \equiv b \pmod{m}$ if m divides $a - b$

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  (extended) Euclidean algorithm
  find the modular inverse
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- Euler's Theorem / Fermart's Little Theorem

  $x^{\phi(n)} \equiv 1 \bmod n$ if $\gcd(x, n) = 1$
  $x^{p-1} \equiv 1 \bmod p$ if $x \not\equiv 0 \bmod p$

Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - Pseudorandom number generators
  - Hash functions
  - Cryptography

Pseudorandom Number Generators

- *Linear congruential method*

  We choose four numbers:

  - the modulus m
  - multiplier a
  - increment c
  - seed x_0

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  We choose four numbers:

  - the modulus m
  - multiplier a
  - increment c
  - seed x_0

  We generate a sequence of numbers $x_1, x_2, \ldots, x_n, \ldots$ with $0 \leq x_i < m$ by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$

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Example:

- Assume : m=9,a=7,c=4, $x_0 = 3$

- x_1= 7*3+4 mod 9=25 mod 9 =7
- x_2 = 53 mod 9 = 8
- x_3 = 60 mod 9 = 6
- x_4= 46 mod 9 =1
- x_5 = 11 mod 9 =2
- x_6 = 18 mod 9 =0
- ....

Hash Functions

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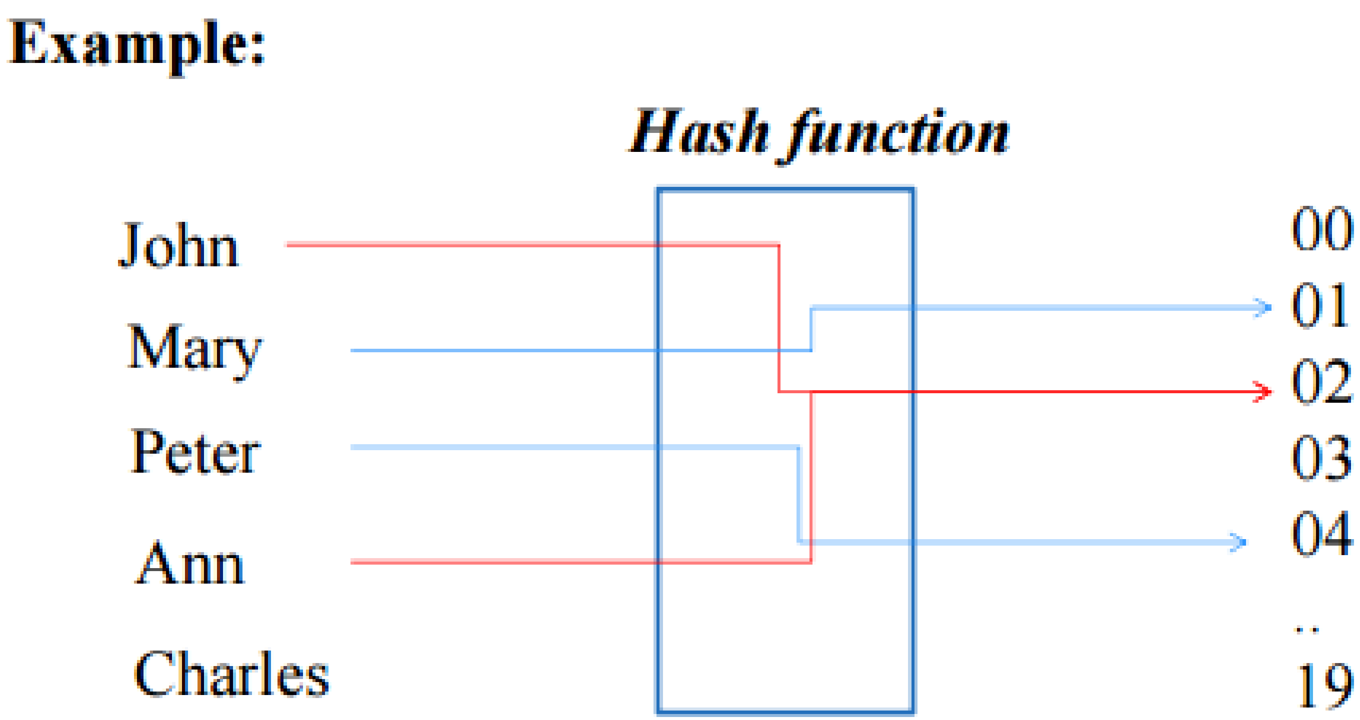

Hash Functions

- **Problem**: Given a large collection of records, how can we store and find a record quickly?

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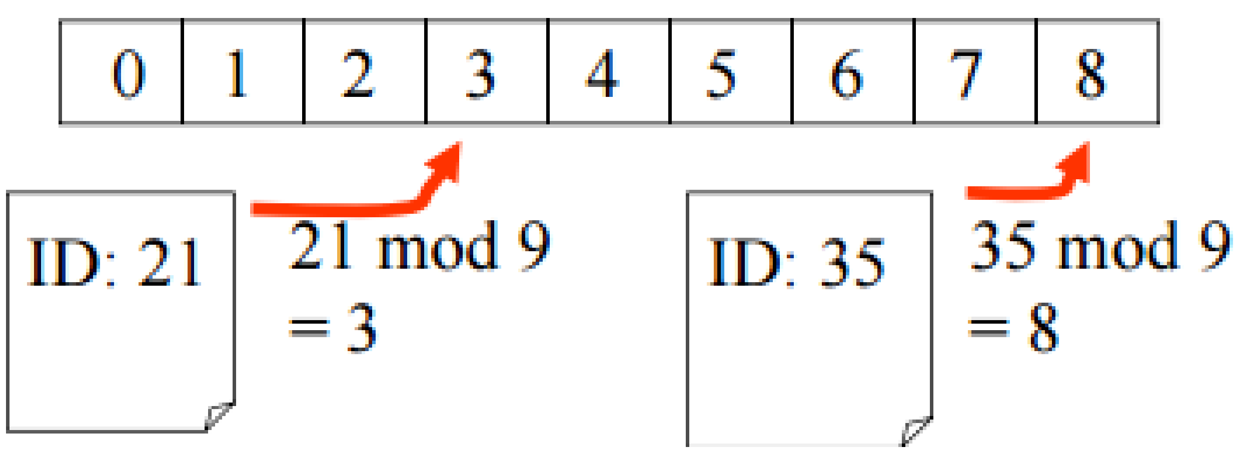
  Solution: Use a hash function, calculate the location of the record based on the record's ID.

Example: A common hash function is

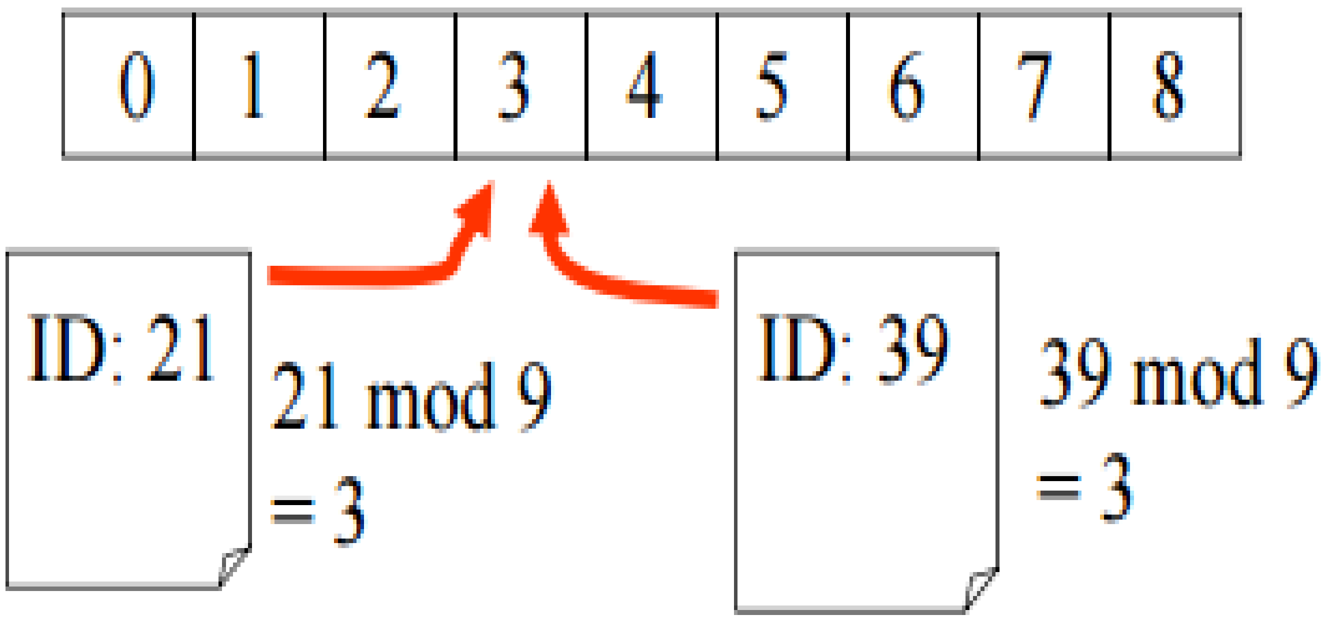
- $h(k) = k \textbf{ mod } n$,

where n is the number of available storage locations.



Hash Functions

- Two records mapped to the same location

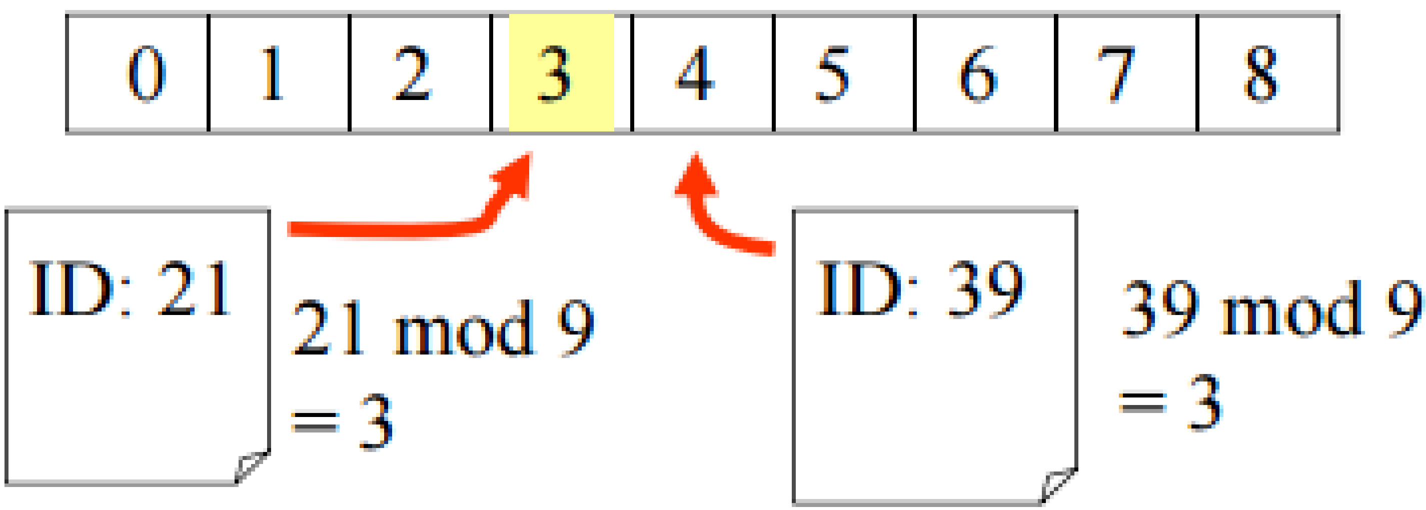


Hash Functions

- **Solution 1**: move to the next available location

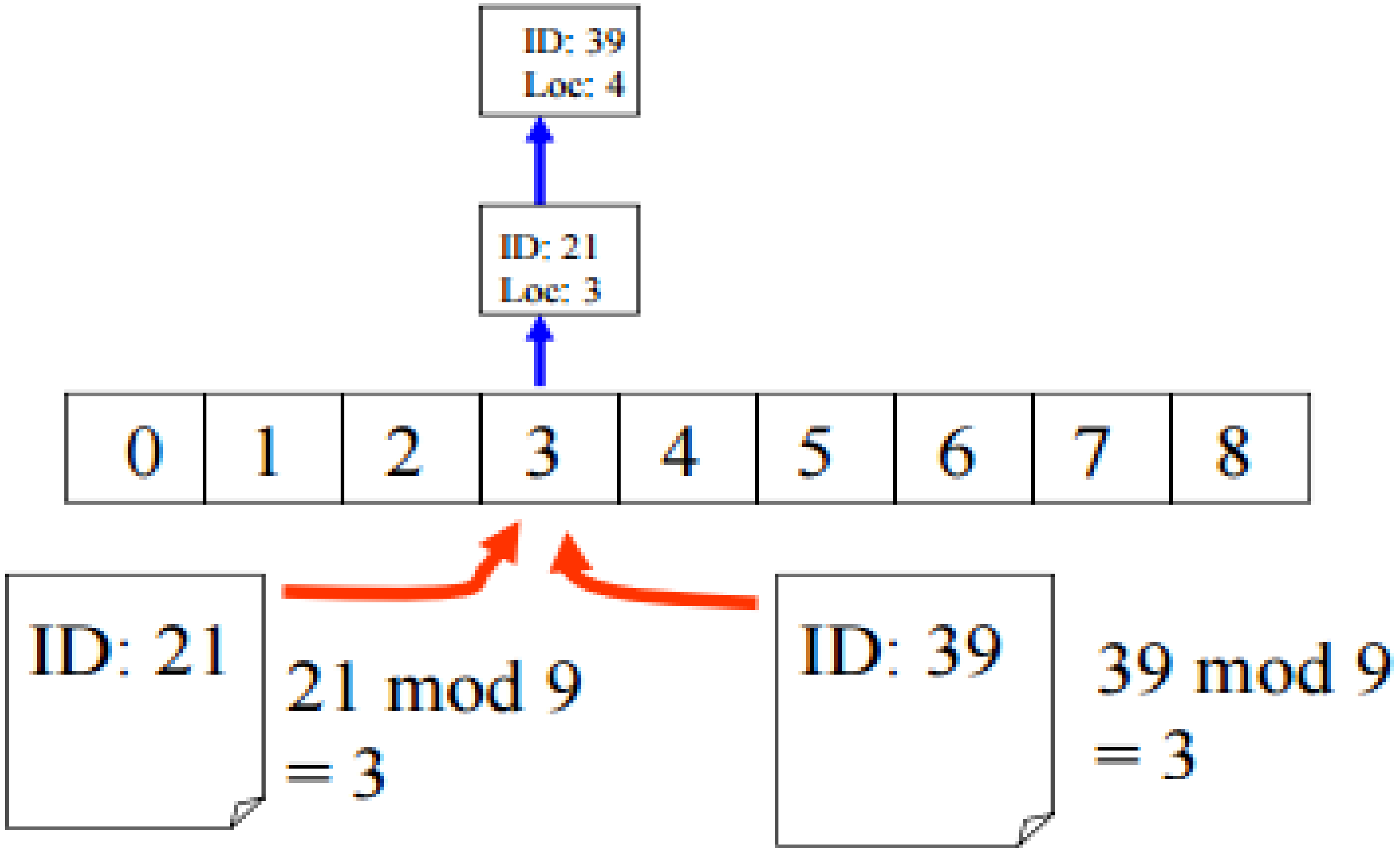
try

$$h_0(k) = k \textbf{ mod } n$$
$$h_1(k) = (k+1) \textbf{ mod } n$$
$$\ldots$$
$$h_m(k) = (k+m) \textbf{ mod } n$$



Hash Functions

- **Solution 2**: remember the exact location in a secondary structure that is searched sequentially



Applications of Number Theory in Cryptography

- Introduction
- Symmetric cryptography
- Asymmetric cryptography
- RSA Cryptosystem
- DLP and El Gamal cryptography
- Diffie-Hellman key exchange protocol
- Crytocurrency, e.g., bitcoin

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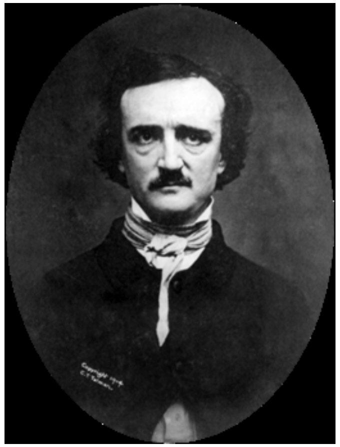
The term was first used in *The Gold-Bug*, by Edgar Allan Poe (1809 - 1849).



Cryptography

- History of almost 4000 years (from 1900 B.C.)

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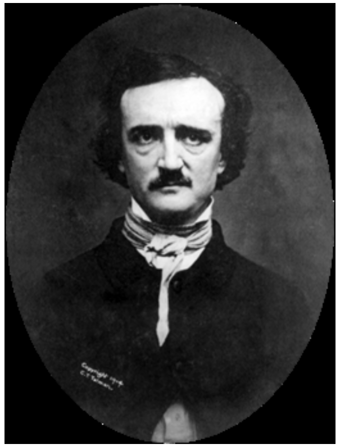


“Human ingenuity cannot concoct a cipher which human ingenuity cannot resolve.” – 1941

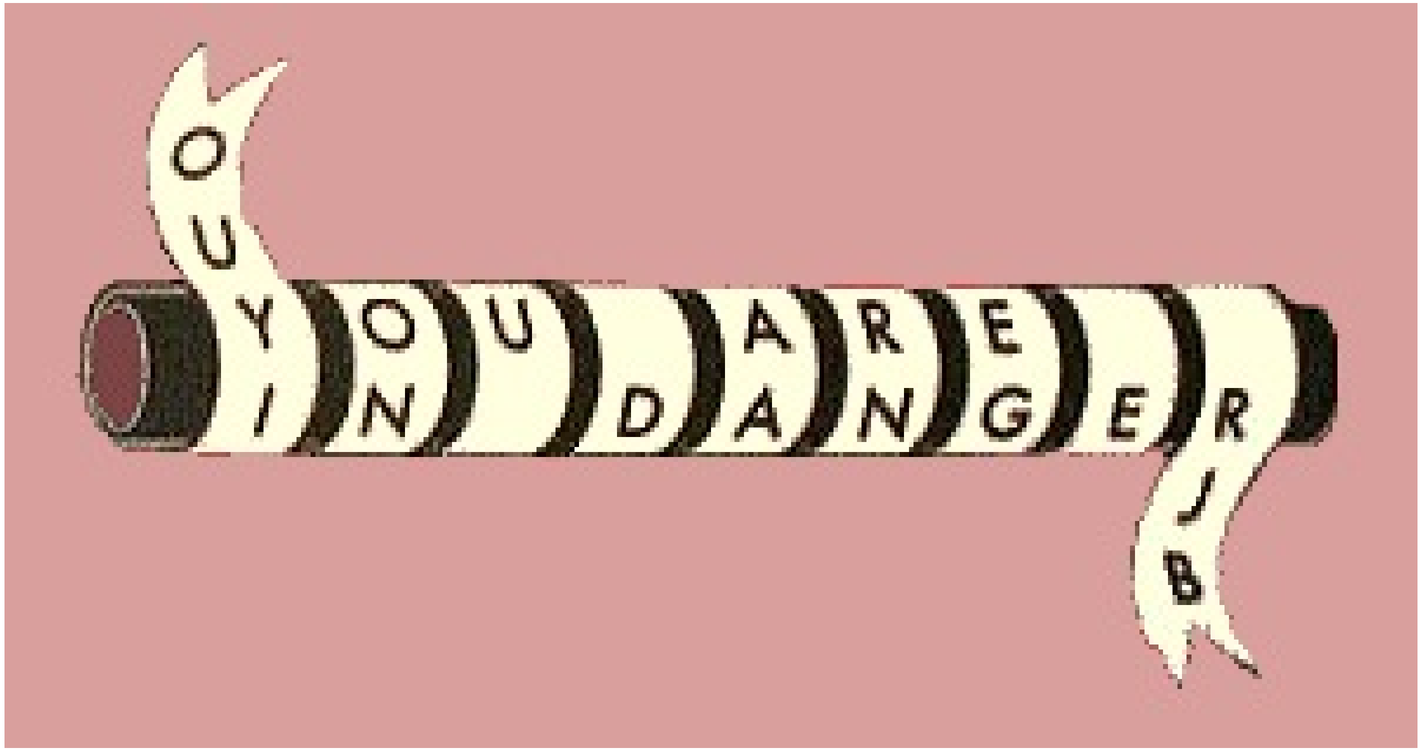
Cryptography

- One-sentence definition:

  "Cryptography is the practice and study of techniques for secure communication in the presence of third parties called *adversaries*." – Ronald L. Rivest

Some Examples

- In 405 BC, the Greek general LYSANDER OF SPARTA was sent a coded message written on the inside of a servant's belt.



Some Examples

- The Greeks also invented a cipher which changed letters to numbers. A form of this code was still being used during *World War I*.

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I/J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

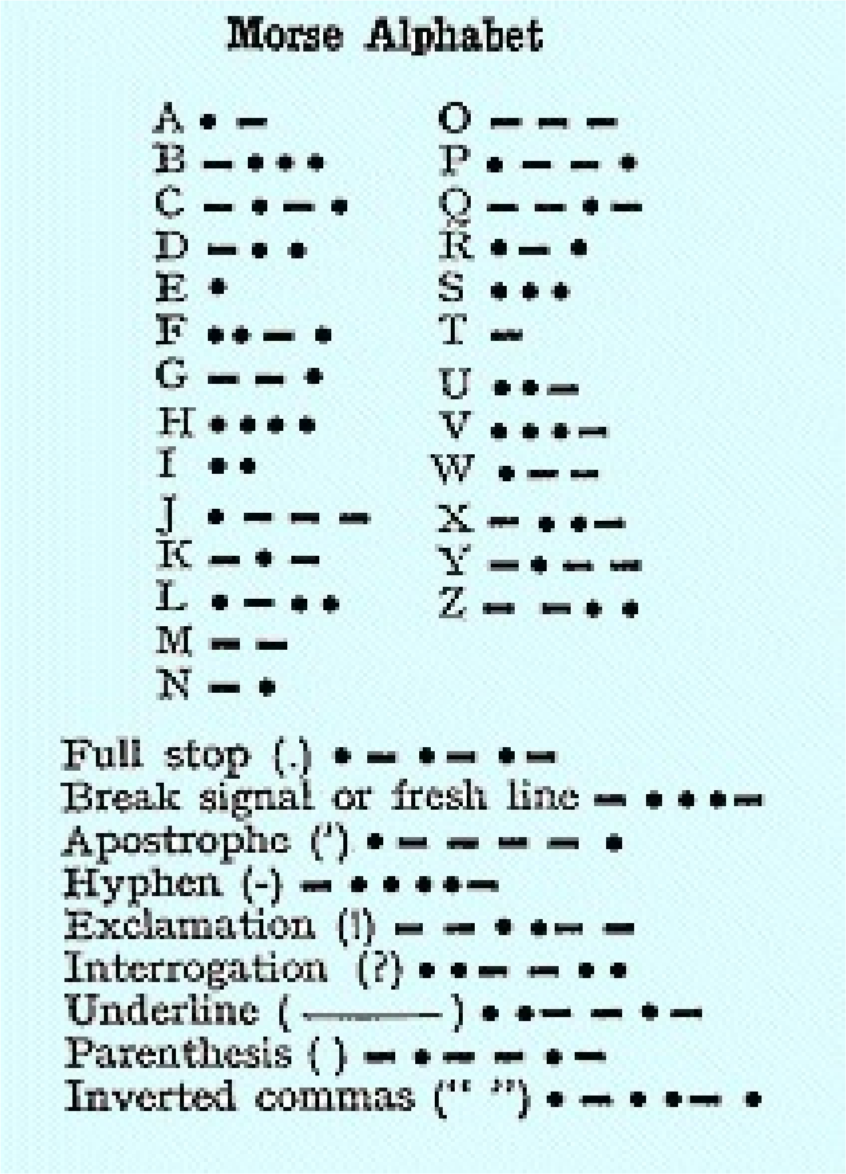
- Caesar Cipher (after the name of JULIUS CAESAR)

VENI, VIDI, VICI

YHQL YLGL YLFL

Some Examples

- Morse Code: created by Samuel Morse in 1838

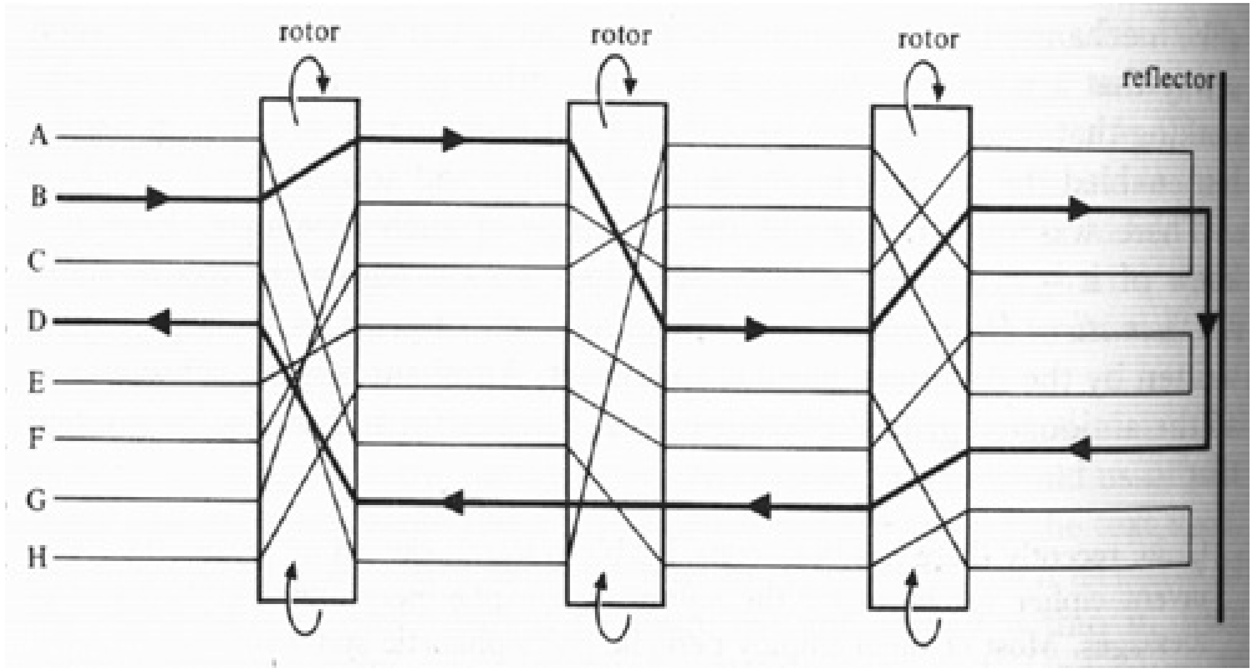


- Crytograms from the Chinese gold bars

http://www.iacr.org/misc/china/china.html

Some Examples

- Enigma, Germany coding machine in *World War II.*

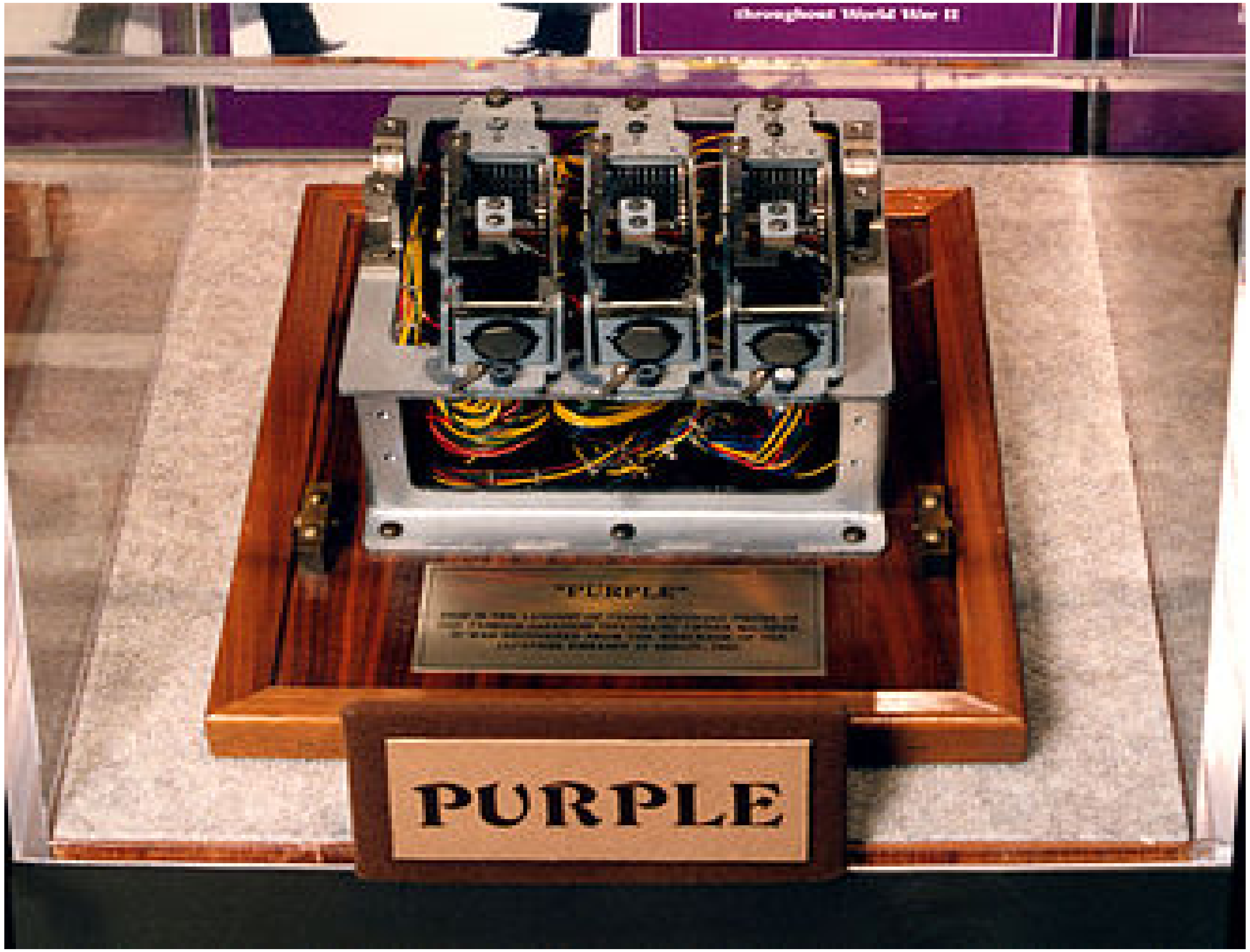


- Sigaba, used by U.S. during *World War II.*

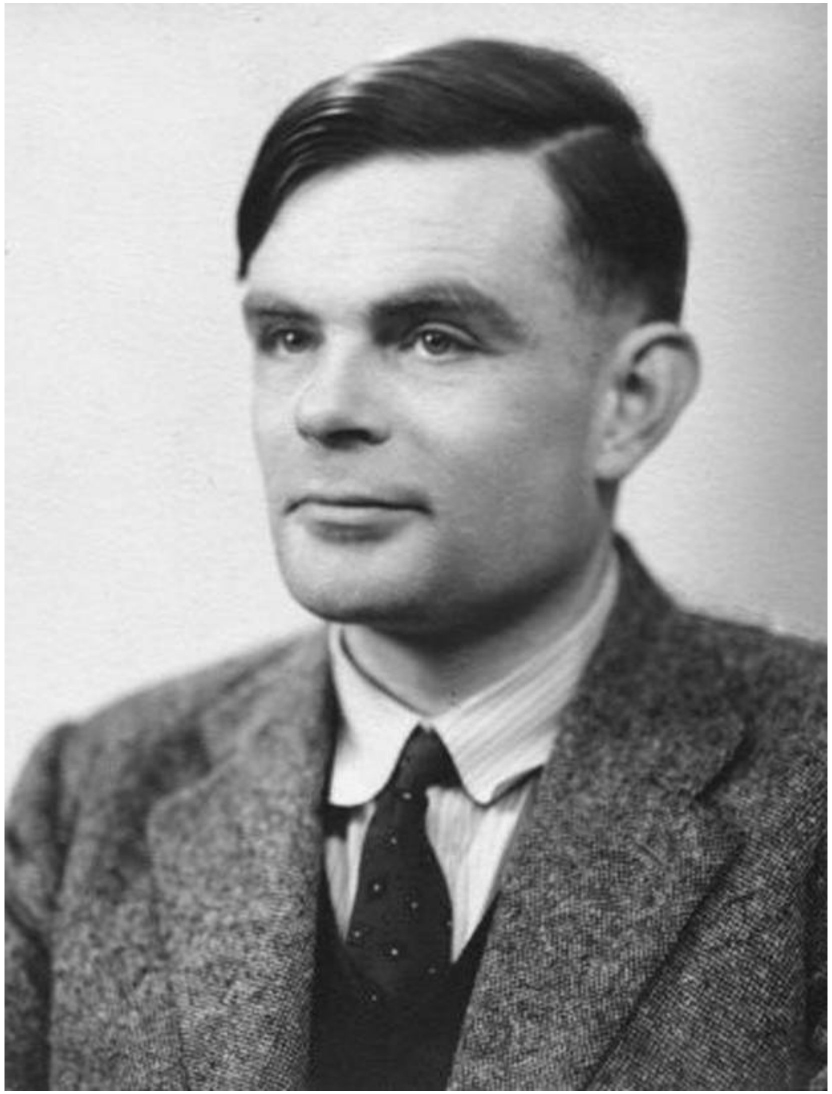
- Japanese “Enigma” Rotor Cipher Machine

Some Examples

- Japanese Purple Machine (97-shiki obun inji-ki)



People Working in Breaking Codes



Alan Turing
(1912-1954)

Claude E. Shannon
(1916-2001)

- History (until 1970's)

  "*Symmetric*" cryptography

Cryptography History

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$\mathcal{Q}$: How can they do this?

$\mathcal{Q}$: What if Bob could send Alice a "special key" useful only for encryption but no help for decryption?

Caesar Cipher

- Key: $k = 0, 1, \ldots, 25$

  Encryption: encode i as $(i + k) \bmod 26$

  Decryption: decode j as $(j - k) \bmod 26$

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  ciphertext: UGPF TGKPHQTEGOGPV

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  Kerchoff's Principle (1883): System should be secure even if algorithms are known, as long as key is secret.

Substitution Cipher

- Key: table mapping each letter to another letter

A	B	C		Z
V	R	E		D

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However, substitution cipher is still insecure!

Key observation: can recover plaintext using *statistics* on *letter frequencies.*

Substitution Cipher

Table 1: Relative frequencies of the letters of the English language

Letter	Relative Frequency (%)	Letter	Relative Frequency (%)
a	8.167	n	6.749
b	1.492	o	7.507
c	2.782	p	1.929
d	4.253	q	0.095
e	12.702	r	5.987
f	2.228	s	6.327
g	2.015	t	9.056
h	6.094	u	2.758
i	6.966	v	0.978
j	0.153	w	2.360
k	0.772	x	0.150
l	4.025	y	1.974
m	2.406	z	0.074

Substitution Cipher

- Table 2: Number of Diagraphs Expected in 2,000 Letters of English Text

th	-	50	at	-	25	st	-	20
er	-	40	en	-	25	io	-	18
on	-	39	es	-	25	le	-	18
an	-	38	of	-	25	is	-	17
re	-	36	or	-	25	ou	-	17
he	-	33	nt	-	24	ar	-	16
in	-	31	ea	-	22	as	-	16
ed	-	30	ti	-	22	de	-	16
ne	-	30	to	-	22	rt	-	16
ha	-	26	it	-	20	ve	-	16

Table 3: The 15 Most Common Trigraphs in the English Language

1	-	the	6	-	tio	11	-	edt
2	-	and	7	-	for	12	-	tis
3	-	tha	8	-	nde	13	-	oft
4	-	ent	9	-	has	14	-	sth
5	-	ion	10	-	nce	15	-	men

Substitution Cipher

- LIVITCSWPIYVEWHEVSRIQMXLEYVEOIEWHRXEXIPFE MVEWHKVSTYLXZIXLIKIIXPIJVSZEYPERRGERIMWQL MGLMXQERIWGPSRIHMXQEREKI

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  I – *most common letter*
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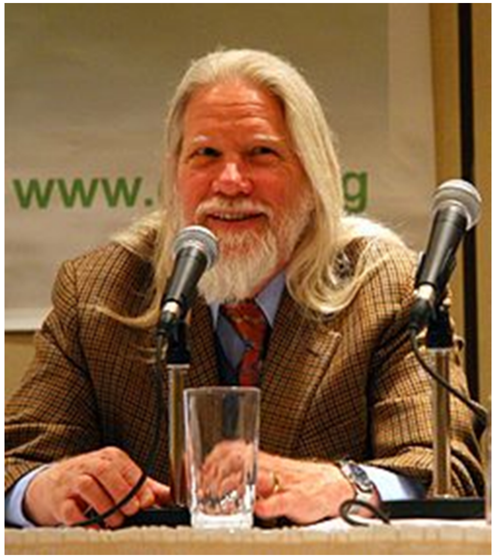
I = e
L = h
X = t

V = r
E = a
Y = g

HereUpOnLeGrandAroseWithAGraveAndStatelyAirAndBroug MeTheBeetleFromAGlassCaseInWhichItWasEnclosedIt-WasABe

- History (from 1976)
  - ◇ W. Diffie, M. Hellman, "New direction in cryptography", *IEEE Transactions on Information Theory, vol. 22, pp. 644-654, 1976.*

"We stand today on the brink of a revolution in cryptography."



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2015 **Turing Award**

Bailey W. Diffie

Martin E. Hellman

2015	Martin E. Hellman Whitfield Diffie	For fundamental contributions to **modern cryptography**. Diffie and Hellman's groundbreaking 1976 paper, "New Directions in Cryptography,"[39] introduced the ideas of public-key cryptography and digital signatures, which are the foundation for most regularly-used security protocols on the internet today.[40]

Public Key Cryptography

- Alice wants to send a message to Bob

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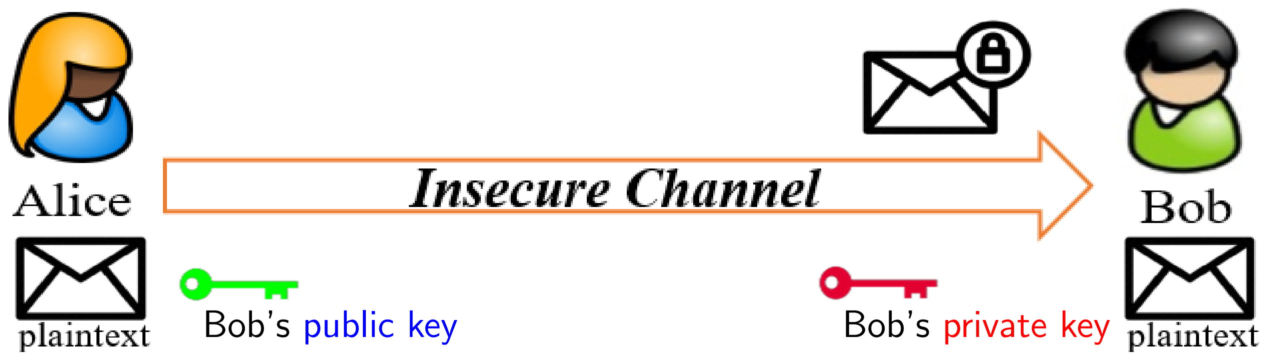

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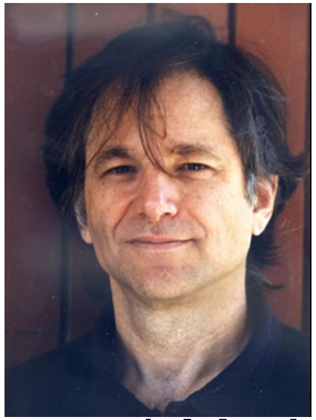
Leonard M. Adleman

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RSA Public Key Cryptosystem

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Pick two large primes, p and q. Let $n = pq$, then $\phi(n) = (p-1)(q-1)$. Encryption and decryption keys e and d are selected such that

- $\gcd(e, \phi(n)) = 1$
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Theorem (*Correctness*) : Let p and q be two odd primes, and define $n = pq$. Let e be relatively prime to $\phi(n)$ and let d be the multiplicative inverse of e modulo $\phi(n)$. For each integer x such that $0 \leq x < n$,

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RSA Public Key Cryptosystem

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$\mathcal{Q}$: How to prove this?

RSA Public Key Cryptosystem: Example

Parameters:	p	q	n	$\phi(n)$	e	d
	5	11	55	40	7	23

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Encryption: $M = 28, C = M^7 \bmod 55 = 52$

Decryption: $M = C^{23} \bmod 55 = 28$

RSA Public Key Cryptosystem: Parameters

Parameters: $p \quad q \quad n \quad \phi(n) \quad e \quad d$

Public key: (e, n)

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p, q, $\phi(n)$ must be kept secret!

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$\mathcal{Q}$: Consider the RSA system, where $n = pq$ is the modulus. Let (e, d) be a key pair for the RSA. Define

$$\lambda(n) = \text{lcm}(p-1, q-1)$$

and compute $d' = e^{-1} \bmod \lambda(n)$. Will decryption using d' instead of d still work?

Applications of RSA

- SSL/TLS protocol

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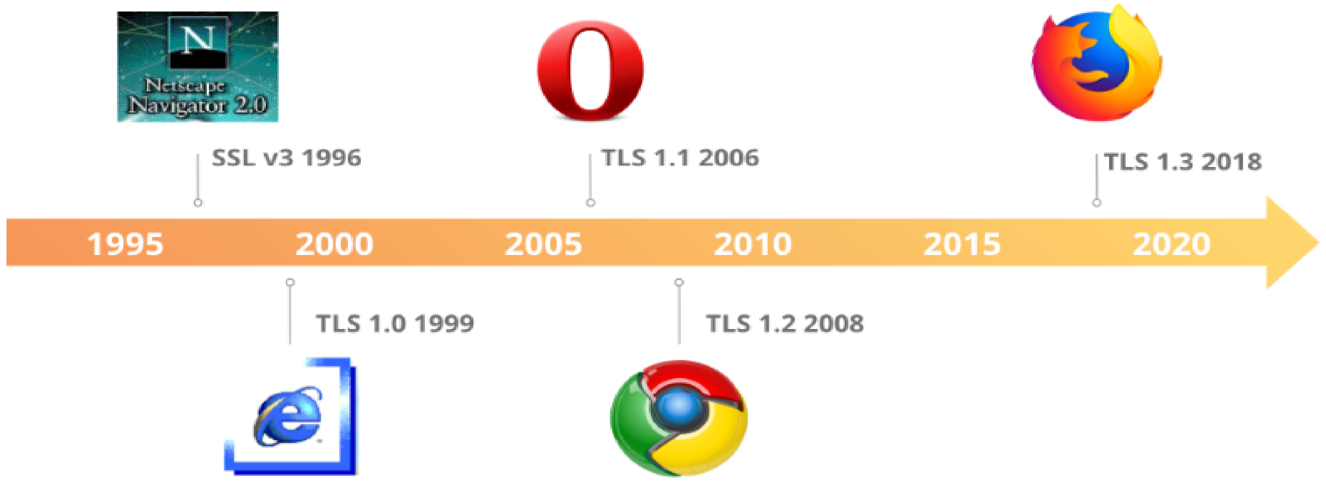

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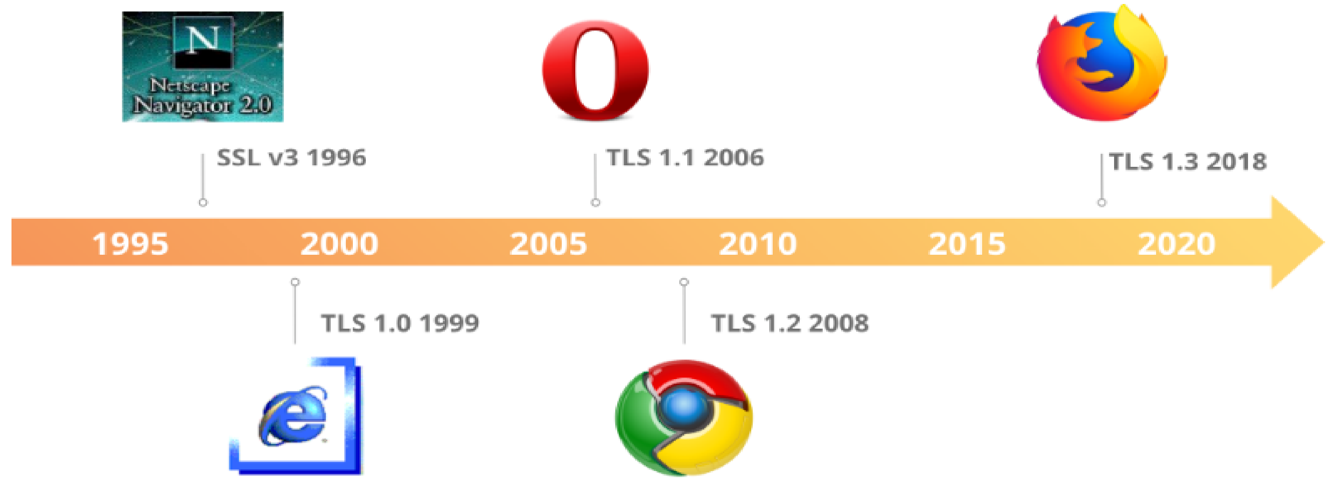

Key exchange/agreement and authentication

Algorithm	SSL 2.0	SSL 3.0	TLS 1.0	TLS 1.1	TLS 1.2	TLS 1.3
RSA	Yes	Yes	Yes	Yes	Yes	No
DH-RSA	No	Yes	Yes	Yes	Yes	No
DHE-RSA (forward secrecy)	No	Yes	Yes	Yes	Yes	Yes
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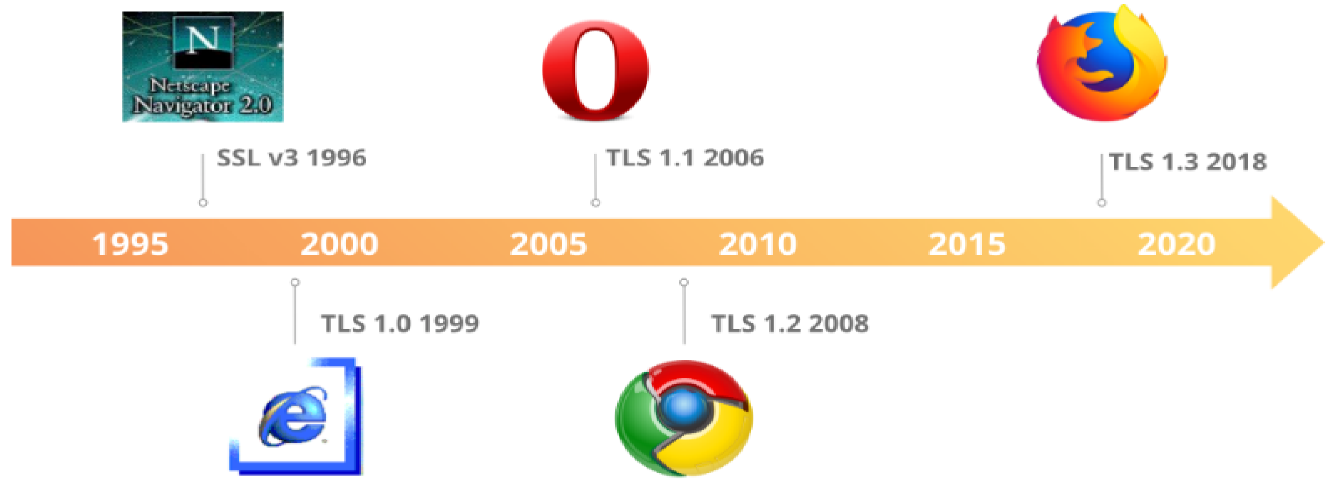

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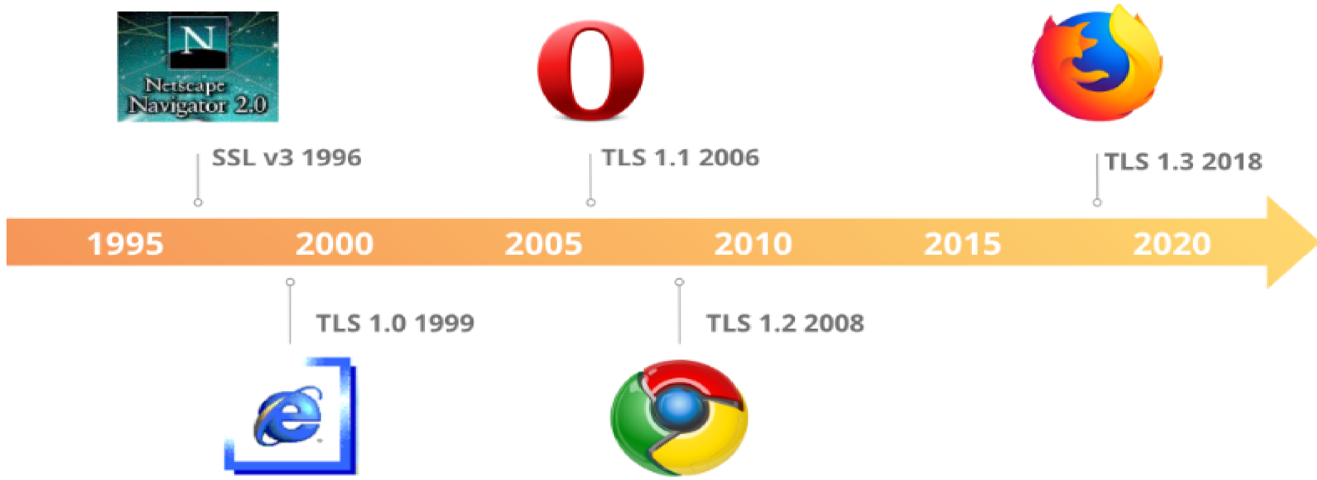

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Using RSA for Digital Signature

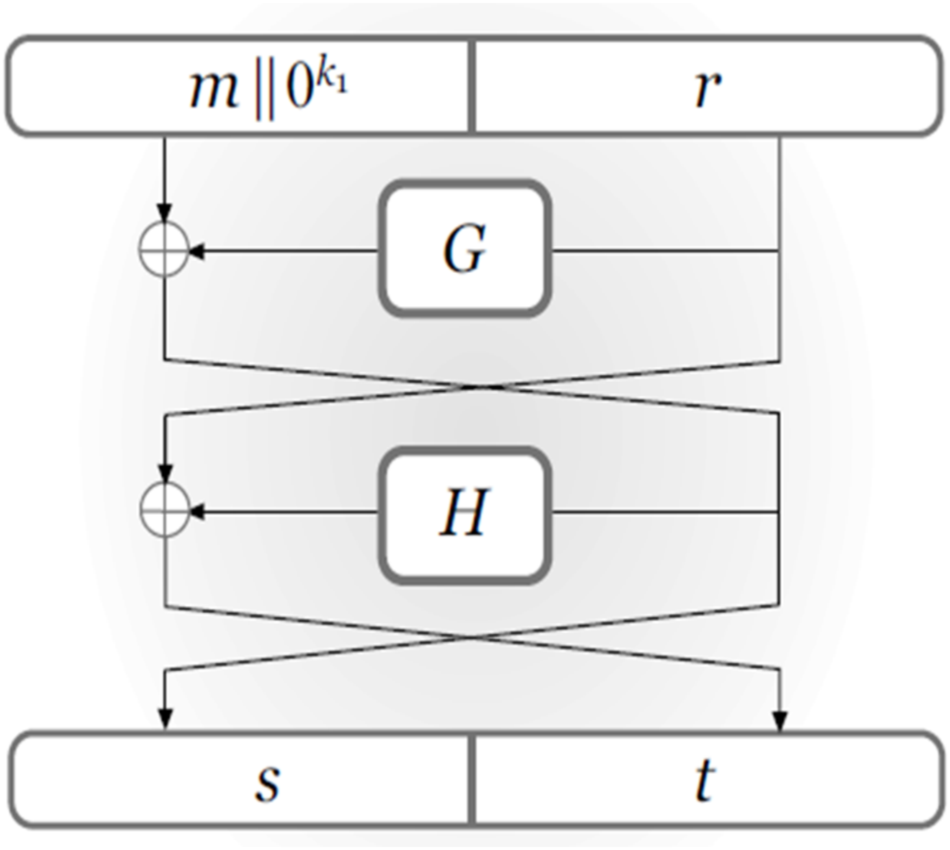
$S = M^d \bmod n$ (RSA **signature)**

$M = S^e \bmod n$ (RSA **verification)**

Why?

RSA-OAEP Standard

- RSA-OAEP (Optimal Asymmetric Encryption Padding) is *IND-CCA2 secure*.
- PKCS#1 V2, RFC2437 Standard



Next Lecture

- cryptography ...