Section 3.7

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P81 Q70

令 $X = X_{(1)}, Y = X_{(n)}$，则有：

$$
\begin{aligned}
P\{x \leqslant X_{(1)} \leqslant x + \mathrm{d}x, y \leqslant X_{(n)} \leqslant y + \mathrm{d}y\} &= n(n-1) \cdot [F(y) - F(x)]^{n-2} \cdot f(x)\mathrm{d}x \cdot f(y)\mathrm{d}y \\
&= f(x, y)\mathrm{d}y\mathrm{d}x
\end{aligned}
$$

因此可知 U, V 的联合密度函数为：

$$f(x, y) = n(n-1)f(x)f(y)[F(y) - F(x)]^{n-2} \ (x \leqslant y)$$

其联合累计分布函数为：

$$
\begin{aligned}
F(x, y) &= P\{X \leqslant x, Y \leqslant y\} \\
&= \int_{-\infty}^{x} \int_{u}^{y} f(u, v)\mathrm{d}v\mathrm{d}u \\
&= \int_{-\infty}^{x} \int_{u}^{y} n(n-1)f(u)f(v)[F(v) - F(u)]^{n-2}\mathrm{d}v\mathrm{d}u \\
&= \int_{-\infty}^{x} nf(u) \int_{u}^{y} (n-1)f(v)[F(v) - F(u)]^{n-2}\mathrm{d}v\mathrm{d}u \\
&= \int_{-\infty}^{x} nf(u) \left[[F(v) - F(u)]^{n-1}\right]_{u}^{y} \mathrm{d}u \\
&= \int_{-\infty}^{x} nf(u)[F(y) - F(u)]^{n-1}\mathrm{d}u \\
&= [-[F(y) - F(u)]^{n}]_{-\infty}^{x} \\
&= F(y)^{n} - [F(y) - F(x)]^{n}
\end{aligned}
$$

补充 1

可知 $\{x, y, z\}$ 共有 $3 = 6$ 种等可能的排列，分别为：

$$\{1, 2, 2\}, \{1, 3, 3\}, \{2, 3, 3\}, \{2, 1, 2\}, \{3, 1, 3\}, \{3, 2, 3\}$$

因此可知 (X, Y) 的联合频率函数及边缘频率函数为：

$X \backslash Y$	1	2	3	$f_X(x)$
1	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{1}{3}$
3	$\frac{1}{6}$	$\frac{1}{6}$	0	$\frac{1}{3}$
$f_Y(y)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

(X, Z) 的联合频率函数及边缘频率函数为：

$X \backslash Z$	2	3	$f_X(x)$
1	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
2	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$
3	0	$\frac{1}{3}$	$\frac{1}{3}$
$f_Z(z)$	$\frac{1}{3}$	$\frac{2}{3}$	1

补充 2

$$\begin{aligned}
F_{\min}(z) &= P\{Z \leqslant z\} \\
&= P\{\min(X, Y) \leqslant z\} \\
&= 1 - P\{\min(X, Y) > z\} \\
&= 1 - P\{X > z, Y > z\} \\
&= 1 - P\{X > z\}P\{Y > z\} \\
&= 1 - (1 - F_X(z))(1 - F_Y(z)) \\
&= 1 - (1 - \Phi(z))(1 - \Phi(z)) \\
&= 2\Phi(z) - \Phi^2(z)
\end{aligned}$$