CS215: Discrete Math (H)
2023 Fall Semester Written Assignment # 6
Due: Jan. 3rd, 2024, please submit at the beginning of class

Q.1 Let G be a simple graph. Show that the relation R on the set of vertices of G such that rRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.

Q.2 Let G be a *simple* graph with n vertices. Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.

Q.3 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Q.4 Let $G = (V, E)$ be a graph on n vertices. Construct a new graph, $G' = (V', E')$ as follows: the vertices of G' are the edges of G (i.e., $V' = E$), and two distinct edges in G are adjacent in G' if they share an endpoint.

(a) Draw G' for $G = K_4$.

(b) Find a formula for the number of edges of G' in terms of the vertex degrees of G.

Q.5 Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Q.6 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph (the degree of every vertex is k) for $k \geq 2$. Prove that G has no bridge.

Q.7 In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner – there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ iff x beats y in their game.

(a) Explain whey a tournament directed graph cannot have cycles of length 1 or 2.

(b) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) Show that a tournament graph represents a total ordering iff there are no cycles of length 3.

Q.8 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u, v) \geq 0$ and $dist(u, v) = 0$ if and only if $u = v$.

(ii) $dist(u, v) = dist(v, u)$.

(iii) $dist(u, v) \leq dist(u, w) + dist(w, v)$.

Q.9 Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.

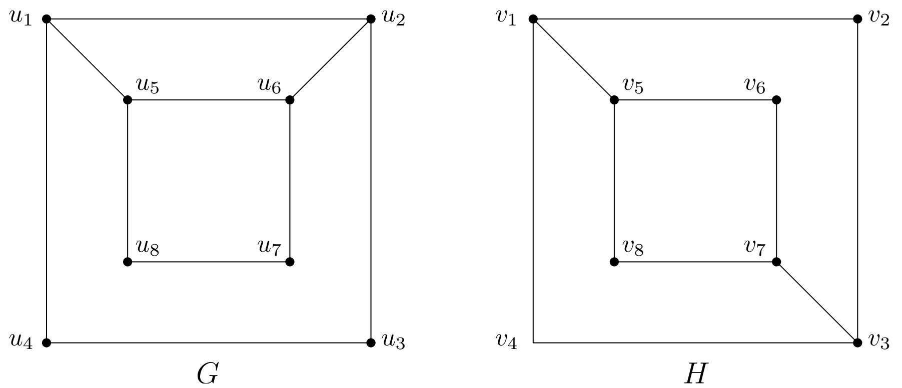

Figure 1: Q.9

Q.10 Show that isomorphism of simple graphs is an equivalence relation.

Q.11 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.

Q.12 Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint. Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.

Q.13 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.

Q.14 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Q.15 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Q.16 An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding edges connecting two vertices that have labels differing only in the first bit. Answer the following questions, and explain your answers.

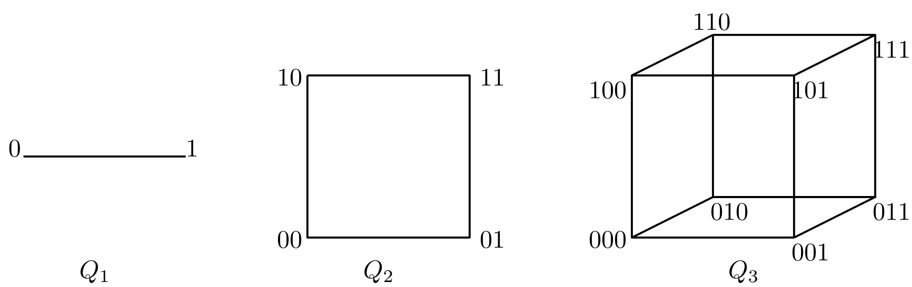

Figure 2: Q.16

(1) How many edges does an n-cube Q_n have?

(2) For what n, the n-cube Q_n has an Euler circuit?

(3) Is an n-cube Q_n bipartite or not?

(4) For what n, the n-cube Q_n is planar?

(5) For what n, the n-cube Q_n has an Hamilton circuit?

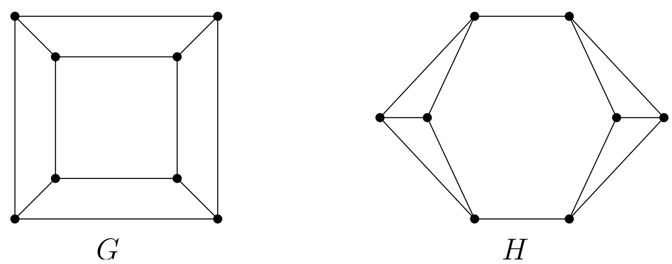

Figure 3: Q.17

Q.17 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Q.18 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Q.19 Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Q.20

What is the value of each of these postfix expressions?

(a) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

(b) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Q.21

How many different spanning trees does each of these simple graphs have?
a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Q.22

How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

Q.23

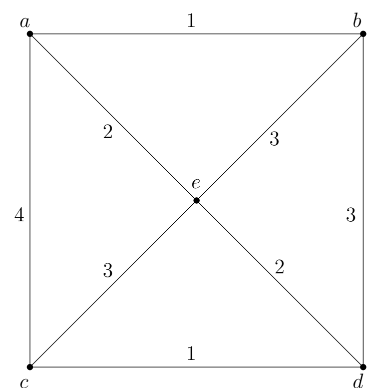

Figure 4: Q.23

(1) Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

(2) Use Kruskal's algorithm to find a minimum spanning tree for the same weighted graph.