Section 4.4

吴梦轩

P120 Q67

令底为 X，宽为 Y，则由条件概率公式有

$$P_{Y|X}(y|x) = \frac{1}{x}$$

由于 $X \sim U(0,1)$，易知 X 的条件期望为 $\frac{1}{2}$。Y 的条件期望为

$$\begin{aligned} E(Y|X=x) &= \int_0^x y \cdot \frac{1}{x} dy \\ &= \frac{1}{x} \cdot \frac{y^2}{2}\bigg|_0^x \\ &= \frac{x}{2} \end{aligned}$$

该长方形周长的期望为

$$\begin{aligned} E(2(X+Y)) &= E(E(2(X+Y)|X)) \\ &= E(2E(X|X) + 2E(Y|X)) \\ &= E\left(2 \cdot X + 2 \cdot \frac{X}{2}\right) \\ &= E(3X) \\ &= 3 \cdot \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

该长方形面积的期望为

$$\begin{aligned}
E(XY) &= E(E(XY|X)) \\
&= E(XE(Y|X)) \\
&= E\left(X \cdot \frac{X}{2}\right) \\
&= \frac{1}{2}E(X^2) \\
&= \frac{1}{2} \cdot \frac{1}{3} \\
&= \frac{1}{6}
\end{aligned}$$

P120 Q77

a.

$$\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= \int_0^\infty \int_x^\infty xye^{-y}dydx - \int_0^\infty \int_x^\infty xe^{-y}dydx \int_0^\infty \int_x^\infty ye^{-y}dydx \\
&= 3 - 2 \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{D(X)D(Y)}} \\
&= \frac{1}{\sqrt{(2-1^2)(6-2^2)}} \\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

b.

$$\begin{aligned}
f_X(x) &= \int_x^\infty e^{-y}dy \\
&= e^{-x} \ (x > 0) \\
f_Y(y) &= \int_0^y e^{-y}dx \\
&= ye^{-y} \ (y > 0)
\end{aligned}$$

可得条件概率为

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\
&= \frac{e^{-y}}{ye^{-y}} \\
&= \frac{1}{y}\ (0 < x < y) \\
f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\
&= \frac{e^{-y}}{e^{-x}} \\
&= e^{x-y}\ (0 < x < y)
\end{aligned}
$$

可得条件期望为

$$
\begin{aligned}
E(X|Y=y) &= \int_0^y x \cdot \frac{1}{y} dx \\
&= \frac{1}{y} \cdot \frac{x^2}{2}\bigg|_0^y \\
&= \frac{y}{2}\ (y > 0) \\
E(Y|X=x) &= \int_x^\infty y \cdot e^{x-y} dy \\
&= e^x \cdot \left(-ye^{-y}\right)\bigg|_x^\infty - e^x \cdot \int_x^\infty -e^{-y} dy \\
&= e^x \cdot xe^{-x} + e^x \cdot e^{-x} \\
&= x + 1\ (x > 0)
\end{aligned}
$$

c.

$$
\begin{aligned}
P\{E(X|Y) < z\} =& P\{\frac{Y}{2} < z\} \\
=& P\{Y < 2z\} \\
=& \int_0^{2z} ye^{-y} dy \\
=& [(-1-y)e^{-y}]\bigg|_0^{2z} \\
=& (-1-2z)e^{-2z} + 1\ (z > 0)
\end{aligned}
$$

$$
\begin{aligned}
P\{E(Y|X) < z\} =& P\{X + 1 < z\} \\
=& P\{X < z - 1\} \\
=& \int_0^{z-1} e^{-x} dx \\
=& (-e^{-x})\Big|_0^{z-1} \\
=& 1 - e^{1-z} \ (z > 1)
\end{aligned}
$$

补充 1

$$
\begin{aligned}
E(E(X|Y)) &= \int_{-\infty}^{\infty} E(X|Y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx \right) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x \frac{f_{X,Y}(x,y)}{f_Y(y)} dx \right) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X,Y}(x,y) dx \right) dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x f_{X,Y}(x,y) dy \right) dx \\
&= \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f_{X,Y}(x,y) dy \right) dx \\
&= \int_{-\infty}^{\infty} x f_X(x) dx \\
&= E(X)
\end{aligned}
$$

补充 2

(1)

$$
\begin{aligned}
\int_0^{\infty} \int_y^{\infty} k e^{-(x+y)} dx dy =& \frac{k}{2} \\
=& 1
\end{aligned}
$$

因此可得 $k = 2$。

$$\begin{aligned}
E(XY) &= \int_0^{\infty}\int_y^{\infty} xy2e^{-(x+y)}dxdy \\
&= \int_0^{\infty} ye^{-y}\int_y^{\infty} xe^{-x}dxdy \\
&= \int_0^{\infty} ye^{-y}(1+y)e^{-y}dy \\
&= \left[\frac{1}{2}(1+y)^2e^{-2y}\right]\Big|_0^{\infty} \\
&= 1 \\
E(X) &= \int_0^{\infty}\int_y^{\infty} x2e^{-(x+y)}dxdy \\
&= \frac{3}{2} \\
E(Y) &= \int_0^{\infty}\int_y^{\infty} y2e^{-(x+y)}dxdy \\
&= \frac{1}{2} \\
E(X^2) &= \int_0^{\infty}\int_y^{\infty} x^22e^{-(x+y)}dxdy \\
&= \frac{7}{2} \\
E(Y^2) &= \int_0^{\infty}\int_y^{\infty} y^22e^{-(x+y)}dxdy \\
&= \frac{1}{2}
\end{aligned}$$

由此可得

$$\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y) \\
&= 1 - \frac{3}{2}\cdot\frac{1}{2} \\
&= \frac{1}{4} \\
\rho(X,Y) &= \frac{Cov(X,Y)}{\sqrt{D(X)D(Y)}} \\
&= \frac{\frac{1}{4}}{\sqrt{(\frac{7}{2}-\frac{3}{2}^2)(\frac{1}{2}-\frac{1}{2}^2)}} \\
&= \frac{1}{\sqrt{5}}
\end{aligned}$$

(2)

$$
\begin{aligned}
f_X(x) = &\int_0^x 2e^{-(x+y)}dy \\
=&2e^{-x} - 2e^{-2x} \\
f_Y(y) = &\int_y^\infty 2e^{-(x+y)}dx \\
=&2e^{-2y}
\end{aligned}
$$

可得条件概率为

$$
\begin{aligned}
f_{X|Y}(x|y) =&\frac{f_{X,Y}(x,y)}{f_Y(y)} \\
=&\frac{2e^{-(x+y)}}{2e^{-2y}} \\
=&e^{y-x}\ (0 \leqslant y \leqslant x) \\
f_{Y|X}(y|x) =&\frac{f_{X,Y}(x,y)}{f_X(x)} \\
=&\frac{2e^{-(x+y)}}{2e^{-x} - 2e^{-2x}} \\
=&\frac{e^{-x-y}}{e^{-x} - e^{-2x}} \\
=&\frac{e^{-y}}{1 - e^{-x}}\ (0 \leqslant y \leqslant x)
\end{aligned}
$$

可得条件期望为

$$
\begin{aligned}
E(X|Y=y) = &\int_y^\infty xe^{y-x}dx \\
=&e^y \cdot \left(-xe^{-x}\right)\Big|_y^\infty - e^y \cdot \int_y^\infty -e^{-x}dx \\
=&e^y \cdot ye^{-y} + e^y \cdot e^{-y} \\
=&y + 1\ (y \geqslant 0) \\
E(Y|X=x) = &\int_0^x y\frac{e^{-y}}{1 - e^{-x}}dy \\
=&\frac{1}{1 - e^{-x}} \cdot \left(-ye^{-y}\right)\Big|_0^x + \frac{1}{1 - e^{-x}} \cdot \int_0^x e^{-y}dy \\
=&\frac{1}{1 - e^{-x}} \cdot \left(-xe^{-x} - e^{-x} + 1\right) \\
=&1 - \frac{x}{e^x - 1}\ (x \geqslant 0)
\end{aligned}
$$

(3)

$$\begin{aligned}
P\{E(X|Y) < z\} =& P\{Y + 1 < z\} \\
=& P\{Y < z - 1\} \\
=& \int_0^{z-1} 2e^{-2y} dy \\
=& 1 - e^{2-2z} \ (z \geqslant 1)
\end{aligned}$$