Section 2.2

吴梦轩

P48 Q33

已知，当 x 趋于负无穷时，$F(x)$ 趋于 0。
当 x 趋于正无穷时，$F(x)$ 趋于 1：

$$\begin{aligned}\lim_{x\to\infty} F(x) = &\lim_{x\to\infty} 1 - e^{-\alpha x^\beta} \\ =&1 - e^{-\infty} \\ =&1 - 0 \\ =&1\end{aligned}$$

由于 $x \geqslant 0, \alpha > 0, \beta > 0$，可知 $-\alpha x^\beta \leqslant 0$，故 $0 < e^{-\alpha x^\beta} \leqslant 1$，故 $0 \leqslant F(x) \leqslant 1$。
$F(x)$ 是单调不减函数：

$$F'(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta} \geqslant 0$$

$F(x)$ 是右连续的：

$$\lim_{x\to x_0^+} F(x) = \lim_{x\to x_0^+} 1 - e^{-\alpha x^\beta} = 1 - e^{-\alpha x_0^\beta} = F(x_0)$$

综上，$F(x)$ 是分布函数。
$F(x)$ 的密度函数为：

$$f(x) = \frac{dF(x)}{dx} = \alpha\beta x^{\beta-1} e^{-\alpha x^\beta}$$

P48 Q40

a.

$$\begin{aligned}\int_{-\infty}^{\infty} f(x)\ dx =& \int_0^1 cx^2\ dx\\ =&\frac{c}{3}x^3\Big|_0^1\\ =&\frac{c}{3}\\ =&1\end{aligned}$$

可得 $c=3$。

b.

当 $0 \leqslant x \leqslant 1$ 时，$F(x)$ 的值为：

$$\begin{aligned}F(x) =& \int_{-\infty}^{x} f(t)\ dt\\ =& \int_0^x 3t^2\ dt\\ =&t^3\Big|_0^x\\ =&x^3\end{aligned}$$

可得累计分布函数为：

$$F(x)=\begin{cases}0 & x<0\\ x^3 & 0\leqslant x\leqslant 1\\ 1 & x>1\end{cases}$$

c.

$$\begin{aligned}P\{0.1\leqslant x\leqslant 0.5\} =&F(0.5)-F(0.1)\\ =&0.5^3-0.1^3\\ =&0.124\end{aligned}$$

P48 Q45

a.

$$\begin{aligned}P\{x<10\} =& F(10)\\ =&1-e^{-0.1\times 10}\\ =&1-e^{-1}\end{aligned}$$

b.

$$\begin{aligned}P\{5<x<15\} =& F(15)-F(5)\\ =&e^{-0.1\times 5}-e^{-0.1\times 15}\\ =&e^{-0.5}-e^{-1.5}\end{aligned}$$

c.

$$\begin{aligned}P\{x>t\} =& 1-F(t)\\ =&e^{-0.1t}\\ =&0.01\end{aligned}$$

可得 $t=\ln 0.01\times -10\approx 46.05$。

P49 Q52

a.

$$\begin{aligned}P\{X>72\} =& 1-P\{X\leqslant 72\}\\ =&1-\Phi\left(\frac{72-70}{3}\right)\\ =&1-\Phi\left(\frac{2}{3}\right)\\ \approx&1-0.7475\\ =&0.2525\end{aligned}$$

b.

$$
\begin{aligned}
X \sim & N(70 \text{ inch}, 9 \text{ inch}^2) \\
X \sim & N(177.8 \text{ cm}, 58.06 \text{ cm}^2) \\
X \sim & N(1.778 \text{ m}, 0.005806 \text{ m}^2)
\end{aligned}
$$

P49 Q53

(a)

$$
\begin{aligned}
P\{X > 10\} =& 1 - P\{X \leqslant 10\} \\
=& 1 - \Phi\left(\frac{10-5}{10}\right) \\
=& 1 - \Phi\left(\frac{1}{2}\right) \\
\approx& 1 - 0.6915 \\
=& 0.3085
\end{aligned}
$$

(b)

$$
\begin{aligned}
P\{-20 < X < 15\} =& P\{X < 15\} - P\{X < -20\} \\
=& \Phi\left(\frac{15-5}{10}\right) - \Phi\left(\frac{-20-5}{10}\right) \\
=& \Phi(1) - \Phi\left(\frac{-5}{2}\right) \\
=& \Phi(1) + \Phi\left(\frac{5}{2}\right) - 1 \\
\approx& 0.8413 + 0.9938 - 1 \\
=& 0.8351
\end{aligned}
$$

(c)

$$\begin{aligned} P\{X > x\} =& 1 - P\{X \leqslant x\} \\ =& 1 - \Phi\left(\frac{x-5}{10}\right) \\ =& 0.05 \end{aligned}$$

可得 $\Phi\left(\frac{x-5}{10}\right) = 0.95$，故 $\frac{x-5}{10} = 1.645$，$x = 21.45$。

补充 1

(1)

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)\ dx =& \int_{-\infty}^{\infty} Ae^{-|x|}\ dx \\ =& 2A\int_{0}^{\infty} Ae^{-x}\ dx \\ =& 2A\left(-e^{-x}\right)\Big|_{0}^{\infty} \\ =& 2A(-0+1) \\ =& 2A \\ =& 1 \end{aligned}$$

可得 $A = \frac{1}{2}$。

(2)

$$\begin{aligned} P\{0 < X < 1\} =& \int_{0}^{1} f(x)\ dx \\ =& \int_{0}^{1} \frac{1}{2}e^{-|x|}\ dx \\ =& \frac{1}{2}\int_{0}^{1} e^{-x}\ dx \\ =& \frac{1}{2}\left(-e^{-x}\right)\Big|_{0}^{1} \\ =& \frac{1}{2}(1-e^{-1}) \\ =& \frac{1-e^{-1}}{2} \end{aligned}$$

(3)

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} f(t)\ dt \\
&= \int_{-\infty}^{x} \frac{1}{2} e^{-|t|}\ dt
\end{aligned}
$$

当 $x < 0$ 时，有：

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{1}{2} e^{-(-t)}\ dt \\
&= \frac{1}{2} \int_{-\infty}^{x} e^{t}\ dt \\
&= \frac{1}{2} \left(e^{t}\right) \Big|_{-\infty}^{x} \\
&= \frac{1}{2}(e^{x} - e^{-\infty}) \\
&= \frac{1}{2} e^{x}
\end{aligned}
$$

当 $x \geqslant 0$ 时，有：

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{1}{2} e^{-t}\ dt \\
&= \frac{1}{2} \int_{-\infty}^{0} e^{-t}\ dt + \frac{1}{2} \int_{0}^{x} e^{-t}\ dt \\
&= \frac{1}{2} + \frac{1}{2} \left(-e^{-t}\right) \Big|_{0}^{x} \\
&= \frac{1}{2} + \frac{1}{2}(-e^{-x} + e^{-0}) \\
&= 1 - \frac{1}{2} e^{-x}
\end{aligned}
$$

综上，累计分布函数为：

$$
F(x) = \begin{cases} \frac{1}{2} e^{x} & x < 0 \\ 1 - \frac{1}{2} e^{-x} & x \geqslant 0 \end{cases}
$$

补充 2

每一次未等到服务而离开的概率为：

$$
\begin{aligned}
P\{X > 10\} &= 1 - F(10) \\
&= 1 - (1 - e^{-0.2 \times 10}) \\
&= e^{-2}
\end{aligned}
$$

每月次数 Y 服从二项分布，即 $Y \sim b(5, e^{-2})$。则可得 Y 的分布律为：

y	$P\{Y=y\}$
0	$(1-e^{-2})^5$
1	$5e^{-2}(1-e^{-2})^4$
2	$10e^{-4}(1-e^{-2})^3$
3	$10e^{-6}(1-e^{-2})^2$
4	$5e^{-8}(1-e^{-2})$
5	e^{-10}

故 $P\{Y \geqslant 1\}$ 为:

$$\begin{aligned} P\{Y \geqslant 1\} =& 1 - P\{Y=0\} \\ =& 1-(1-e^{-2})^5 \\ \approx& 1-0.4833 \\ =& 0.5167 \end{aligned}$$