南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

08 Computational Intractability

CS216 Algorithm Design and Analysis (H)

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Algorithm Design Patterns and Antipatterns

- **Algorithm design patterns:**
  - Greedy
  - Divide and conquer
  - Dynamic programming
  - Duality (e.g., network flow)
  - Reductions
  - Randomization
- **Algorithm design antipatterns:**
  - **NP**-completeness — Polynomial-time algorithm unlikely.
  - **PSPACE**-completeness — Polynomial-time certification algorithm unlikely.
  - Undecidability — No algorithm possible.

1. Polynomial-Time Reductions

Classify Problems by Computational Requirements

- **Q.** Which problems will we be able to solve in practice?
- **A working definition.** Those with polynomial-time (poly-time) algorithms.

von Neumann (1953) | Nash (1955) | Gödel (1956) | Cobham (1964) | Edmonds (1965) | Rabin (1966)

Turing machine, word RAM, uniform circuits, …

- **Theory.** Definition is broad and robust.
- **Practice.** Polynomial-time algorithms scale to huge problems.

constants tend to be small, e.g., $3n^2$

Classify Problems by Computational Requirements

- **Q.** Which problems will we be able to solve in practice?
- **A working definition.** Those with polynomial-time (poly-time) algorithms.

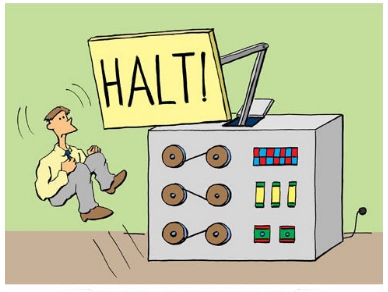
Yes	Probably No
Shortest Path	Longest Path
Matching	3D Matching
Min Cut	Max Cut
2-SAT	3-SAT
Planar 4-Color	Planar 3-Color
Bipartite Vertex Cover	Vertex Cover
Primality Testing	Factoring
Linear Programming	Integer Linear Programming

Classify Problems

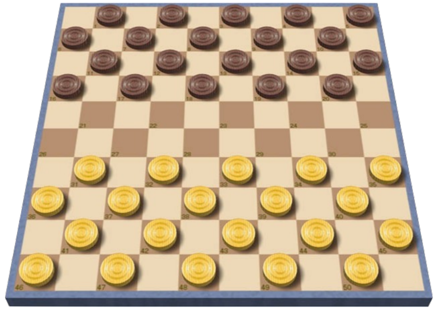
- **Goal.** Classify problems according to those that can be solved in polynomial time and those that cannot.
- **Provably requires exponential time:**
  - Given a constant-size program, does it halt in at most *k* steps?
  - Given a board position in an *n*-by-*n* generalization of checkers, can black guarantee a win?

input size = c + log k

forced capture rule



Alan designed the perfect computer



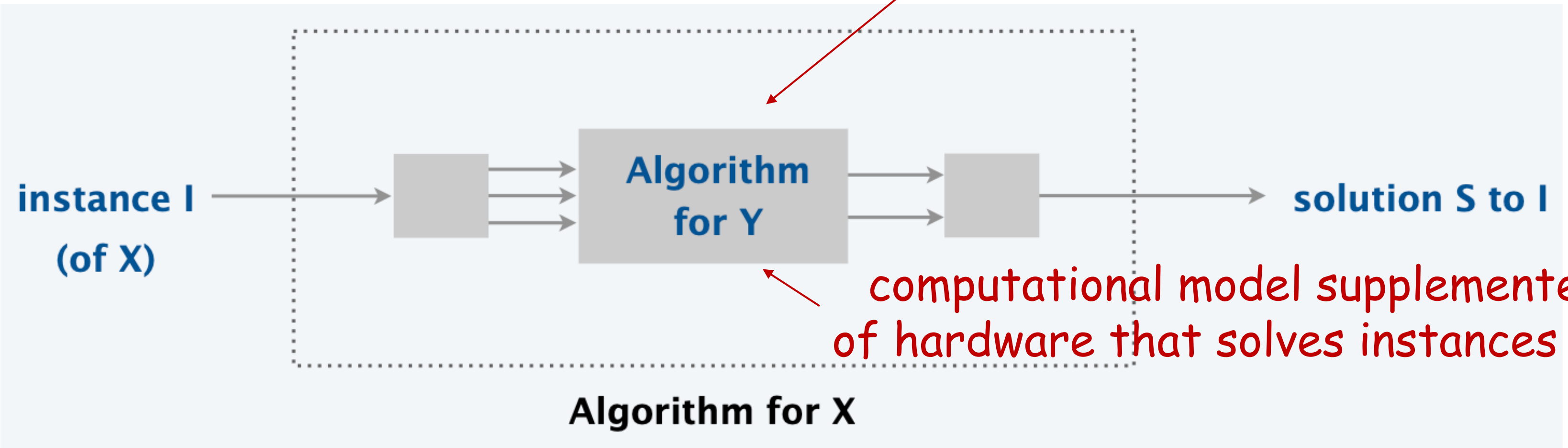
- **Frustrating news.** Huge number of fundamental problems have defied classification for decades.

Polynomial-Time Reductions

- **Q.** Suppose we could solve problem *Y* in polynomial time. What else could we solve in polynomial time?
- **Reduction.** Problem *X* polynomial-time (Cook) reduces to problem *Y* if arbitrary instances of problem *X* can be solved using:
  - Polynomial number of standard computational steps, plus
  - Polynomial number of calls to oracle that solves problem *Y*.

don't confuse with "reduce from"

can be viewed as a magic black box



computational model supplemented by special piece of hardware that solves instances of Y in a single step

Polynomial-Time Reductions

- **Q.** Suppose we could solve problem *Y* in polynomial time. What else could we solve in polynomial time?
- **Reduction.** Problem *X* polynomial-time (Cook) reduces to problem *Y* if arbitrary instances of problem *X* can be solved using:
  - Polynomial number of standard computational steps, plus
  - Polynomial number of calls to oracle that solves problem *Y*.

don't confuse with "reduce from"

can be viewed as a magic black box

Karp reduction allows only one oracle call

- **Notation.** $X \leq_P Y$
- **Note.** We pay for time to write down instances of *Y* sent to black box $\Rightarrow$ instances of *Y* must be of polynomial size.

Polynomial-Time Reductions

- **Design algorithms.** If $X \leq_P Y$ and Y can be solved in polynomial time, then X can also be solved in polynomial time.

- **Establish intractability.** If $X \leq_P Y$ and X cannot be solved in polynomial time, then Y cannot be solved in polynomial time.

up to cost of reduction

- **Establish equivalence.** If both $X \leq_P Y$ and $Y \leq_P X$, we use notation $X \equiv_P Y$. In this case, X can be solved in polynomial time if and only if Y can be.

- **Bottom line.** Reductions classify problems according to relative difficulty.

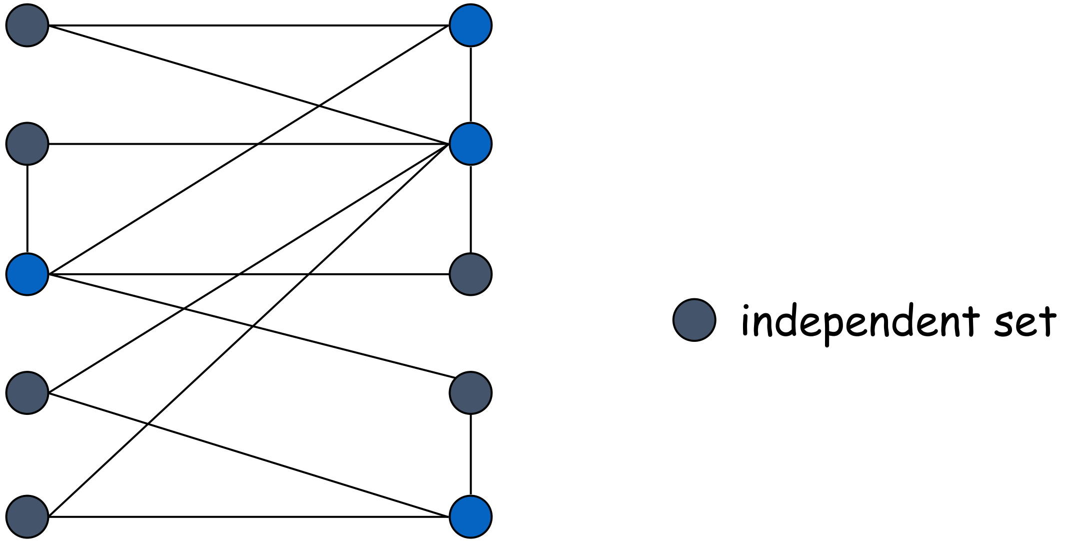
2. Reduction By Simple Equivalence

Basic reduction strategies:

- Reduction by simple equivalence
- Reduction from special case to general case
- Reduction via “gadgets”

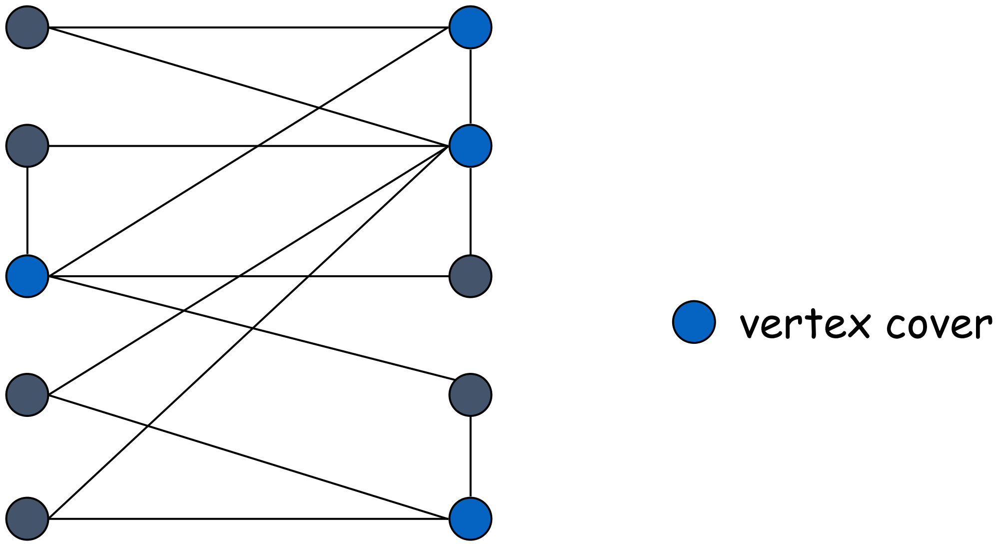
Independent Set

- **INDEPENDENT-SET.** Given a graph $G = (V, E)$ and an integer k, is there a subset S of k (or more) vertices such that no two in S are adjacent?
  - An independent set S has no edge connecting vertices in S.
- **Example.** Is there an independent set of size ≥ 6 ? Yes.
- **Example.** Is there an independent set of size ≥ 7 ? No.



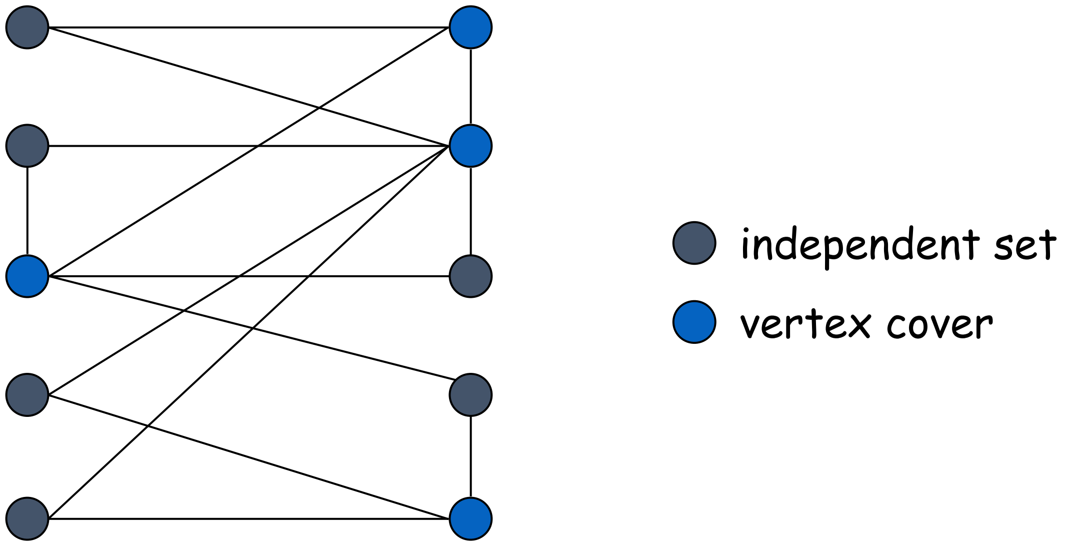
Vertex Cover

- **VERTEX-COVER.** Given a graph $G = (V, E)$ and an integer k, is there a subset S of k (or fewer) vertices such that each edge is incident to at least one vertex in S ?
- **Example.** Is there a vertex cover of size ≤ 4 ? Yes.
- **Example.** Is there a vertex cover of size ≤ 3 ? No.



Vertex Cover and Independent Set

- **Theorem.** INDEPENDENT-SET $\equiv_P$ VERTEX-COVER
- **Pf.** We show S is an independent set if and only if $V - S$ is a vertex cover.



Vertex Cover and Independent Set

- **Theorem.** INDEPENDENT-SET $\equiv_P$ VERTEX-COVER
- **Pf.** We show S is an independent set if and only if $V - S$ is a vertex cover.

  "only if" $\Rightarrow$: Let S be any independent set.

  ✓Consider an arbitrary edge $(u, v) \in E$.

  ✓S independent $\Rightarrow u \notin S$ or $v \notin S \Rightarrow u \in V - S$ or $v \in V - S$.

  ✓Thus, $V - S$ covers (u, v).

  "if" $\Leftarrow$: Let $V - S$ be any vertex cover.

  ✓Consider an arbitrary edge $(u, v) \in E$.

  ✓$V - S$ is a vertex cover $\Rightarrow u \in V - S$ or $v \in V - S \Rightarrow u \notin S$ or $v \notin S$.

  ✓Thus, S is an independent set. ▪

3. Reduction from Special Case to General Case

Basic reduction strategies:

- Reduction by simple equivalence
- Reduction from special case to general case
- Reduction via “gadgets”

Set Cover

- **SET-COVER.** Given a set U of elements, a collection of subsets of U, and an integer k, are there $\leq k$ of these subsets whose union is equal to U?
- **Sample application:**
  - Consider m available pieces of software.
  - Set U consists of n capabilities that we would like our system to have.
  - The i-th piece of software provides the subset $S_i \subseteq U$ of capabilities.
  - Goal: achieve all n capabilities using fewest pieces of software.
- **Example:**

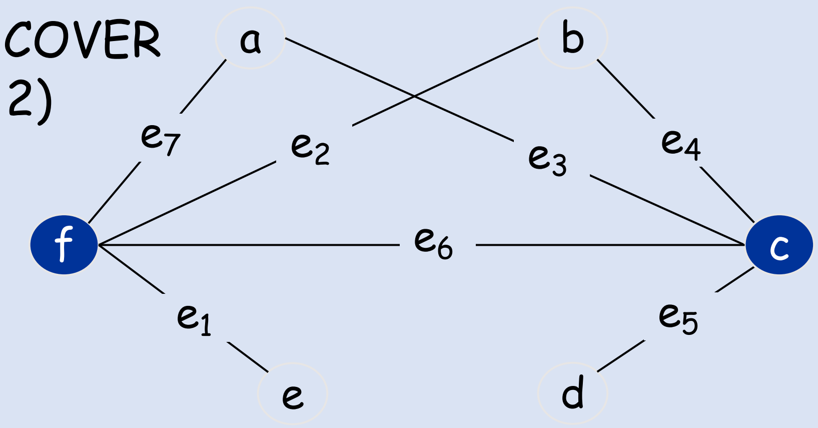
$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$ $k = 2$

$S_1 = \{3, 7\}$	$S_4 = \{2, 4\}$
$S_2 = \{3, 4, 5, 6\}$	$S_5 = \{5\}$
$S_3 = \{1\}$	$S_6 = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

- **Theorem.** VERTEX-COVER $\leq_P$ SET-COVER
- **Pf.** Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) such that it has set cover of size k if and only if G has a vertex cover of size k.
- **Construction.** $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$, $S = \{S_v : v \in V\}$.
- **Example:**

VERTEX COVER
(k = 2)



SET COVER (k = 2)

$U = \{ 1, 2, 3, 4, 5, 6, 7 \}$

$S_a = \{3, 7\}$ $\quad$ $S_b = \{2, 4\}$

$S_c = \{3, 4, 5, 6\}$ $\quad$ $S_d = \{5\}$

$S_e = \{1\}$ $\quad$ $S_f = \{1, 2, 6, 7\}$

Vertex Cover Reduces to Set Cover

- **Theorem.** VERTEX-COVER $\leq_P$ SET-COVER
- **Pf.** Given a VERTEX-COVER instance $G = (V, E)$ and k, we construct a SET-COVER instance (U, S, k) such that it has set cover of size k if and only if G has a vertex cover of size k.
- **Construction.** $U = E$, $S_v = \{e \in E : e \text{ incident to } v\}$, $S = \{S_v : v \in V\}$.
- **Lemma.** $G = (V, E)$ contains a vertex cover of size k if and only if (U, S, k) contains a set cover of size k.
- **Pf.** Let $X \subseteq V$ be a vertex cover of size k; let $Y \subseteq S$ be a set cover of size k.
  - "only if" $\Rightarrow$: $Y = \{ S_v : v \in X \}$ is a set cover of size k.
  - "if" $\Leftarrow$: $X = \{ v : S_v \in Y \}$ is a vertex cover of size k. ▪

PEARSON Addison Wesley

4. Reductions via “Gadgets”

Basic reduction strategies:

- Reduction by simple equivalence
- Reduction from special case to general case
- Reduction via “gadgets”

Satisfiability

- **Q.** Given a propositional formula Φ, is there a truth assignment to its variables such that $\Phi = 1$, i.e., is there a satisfying truth assignment?
- **Example:**

a	b	c	yes $(a \wedge b) \vee c$	no $(a \wedge \neg a) \vee (c \wedge \neg c)$
1	1	1	1	0
0	1	1	1	0
0	0	1	1	0
0	1	0	0	0
1	0	1	1	0
1	0	0	0	0
1	0	1	1	0
0	0	0	0	0

- **Key application.** Electronic design automation (EDA).

Satisfiability: SAT and 3-SAT

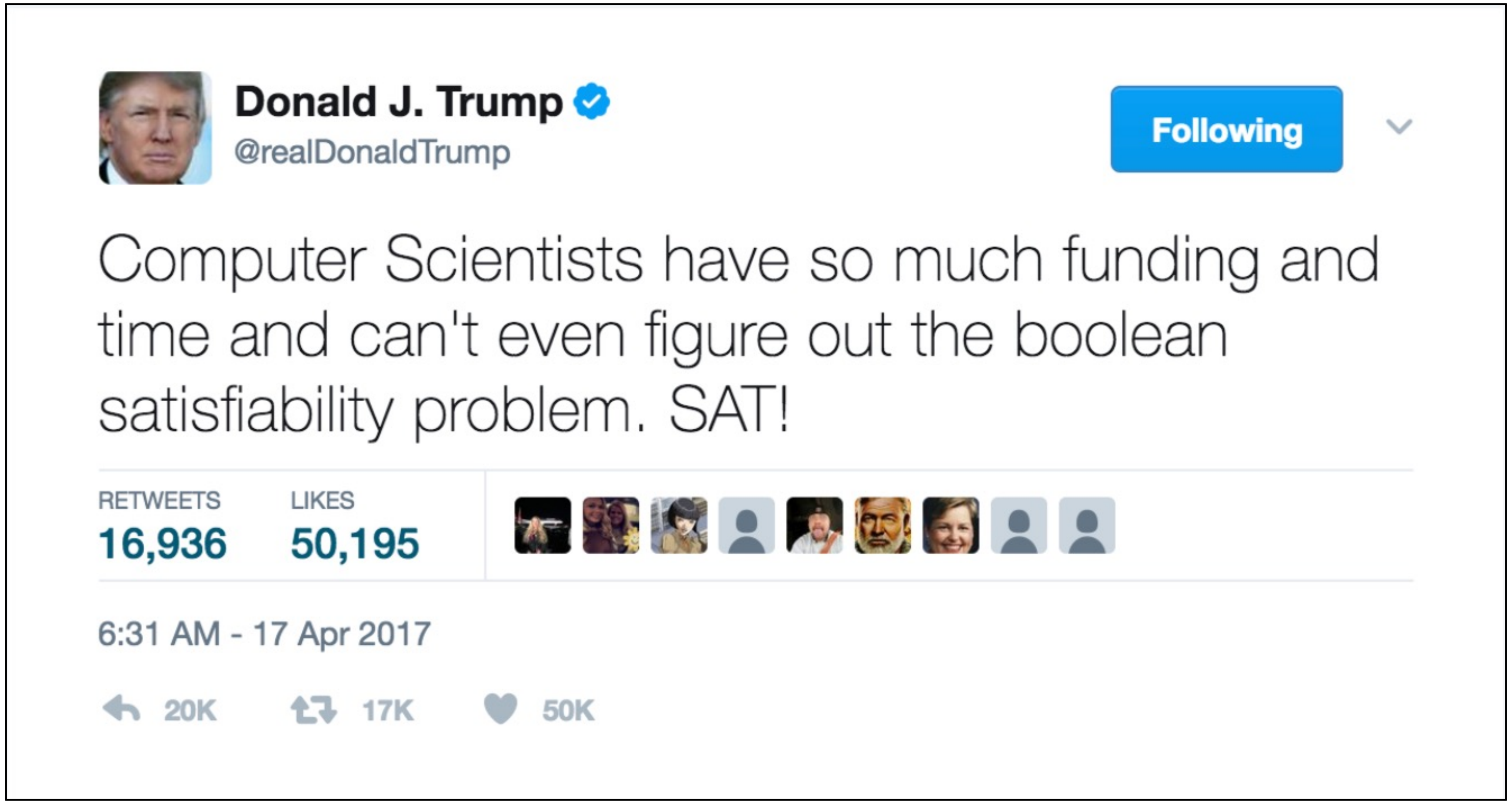
- **Literal.** A Boolean variable or its negation. x_i or $\overline{x_i}$
- **Clause.** A disjunction of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$
- **Conjunctive normal form (CNF).** A propositional formula Φ that is the conjunction of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$
- **SAT.** Given a CNF formula Φ, does it have a satisfying truth assignment?
- **3-SAT.** SAT where each clause contains exactly *3* literals (and each literal corresponds to a different variable).
- **Example:**

$$\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$$

yes assignment: x_1 = true, x_2 = true, x_3 = false, x_4 = false

Satisfiability is Hard

- **Hypothesis.** There does not exist a polynomial-time algorithm for 3-SAT.
- **Remark.** This hypothesis is equivalent to **P ≠ NP** conjecture.



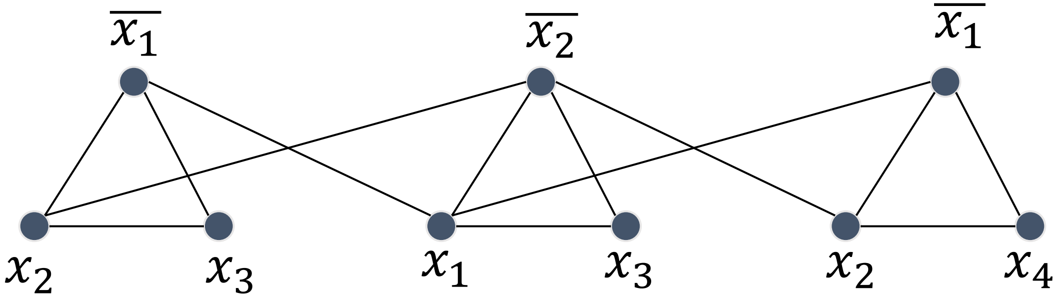
3-Satisfiability Reduces to Independent Set

- **Theorem.** 3-SAT $\leq_P$ INDEPENDENT-SET
- **Pf idea.** Given 3-SAT instance Φ, construct an INDEPENDENT-SET instance (G, k) that has an independent set of size k ($k = |\Phi|$) iff Φ is satisfiable.

number of clauses

- **Construction:**
  - *G* contains *3* vertices for each clause, one for each literal.
  - Connect *3* literals in a clause in a triangle.
  - Connect literal to each of its negations.
- **Example:**

G

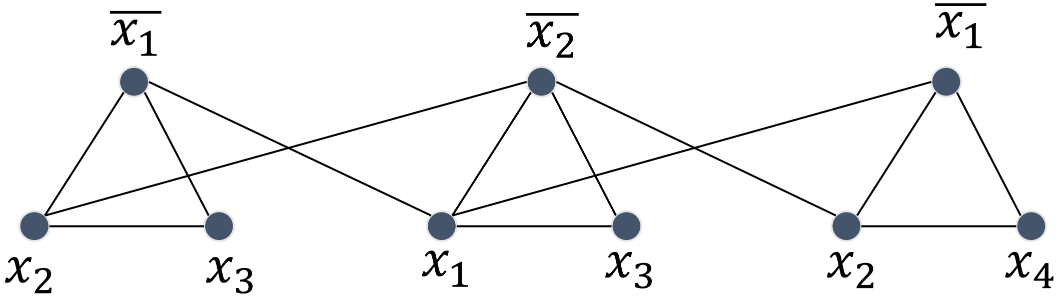


k = 3 $\quad \Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

3-Satisfiability Reduces to Independent Set

- **Lemma.** Φ is satisfiable iff G contains independent set of size $k = |\Phi|$.
- **Pf.** "only if" $\Rightarrow$:
  - Consider any satisfying assignment for Φ.
  - Choose one true literal from each clause/triangle:
    - ✓no two literals chosen in one triangle
    - ✓complementary literals not both chosen
  - This is an independent set of size $k = |\Phi|$.

G

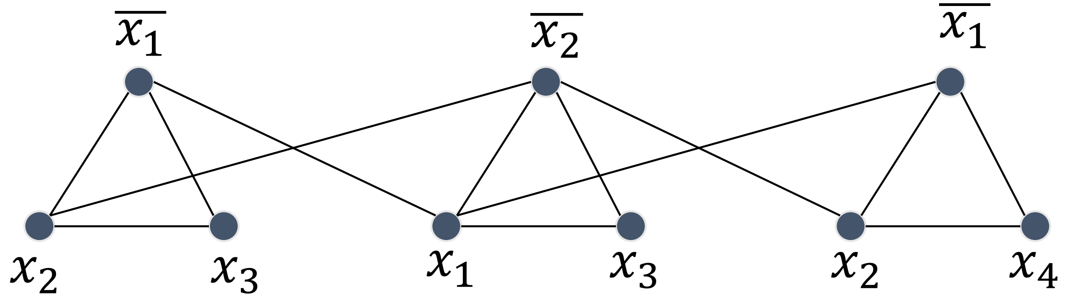


k = 3 $\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4)$

3-Satisfiability Reduces to Independent Set

- **Lemma.** Φ is satisfiable iff *G* contains independent set of size $k = |\Phi|$.
- **Pf.** "if" $\Leftarrow$:
  - Let *S* be an independent set of size $k = |\Phi|$.
  - At most one vertex in each triangle can be contained in an independent set.
  - *S* must contain exactly one vertex in each triangle.
  - Set these literals to true (and remaining literals consistently).
  - All clauses in Φ are satisfied. ▪

G



k = 3 $\quad \Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$

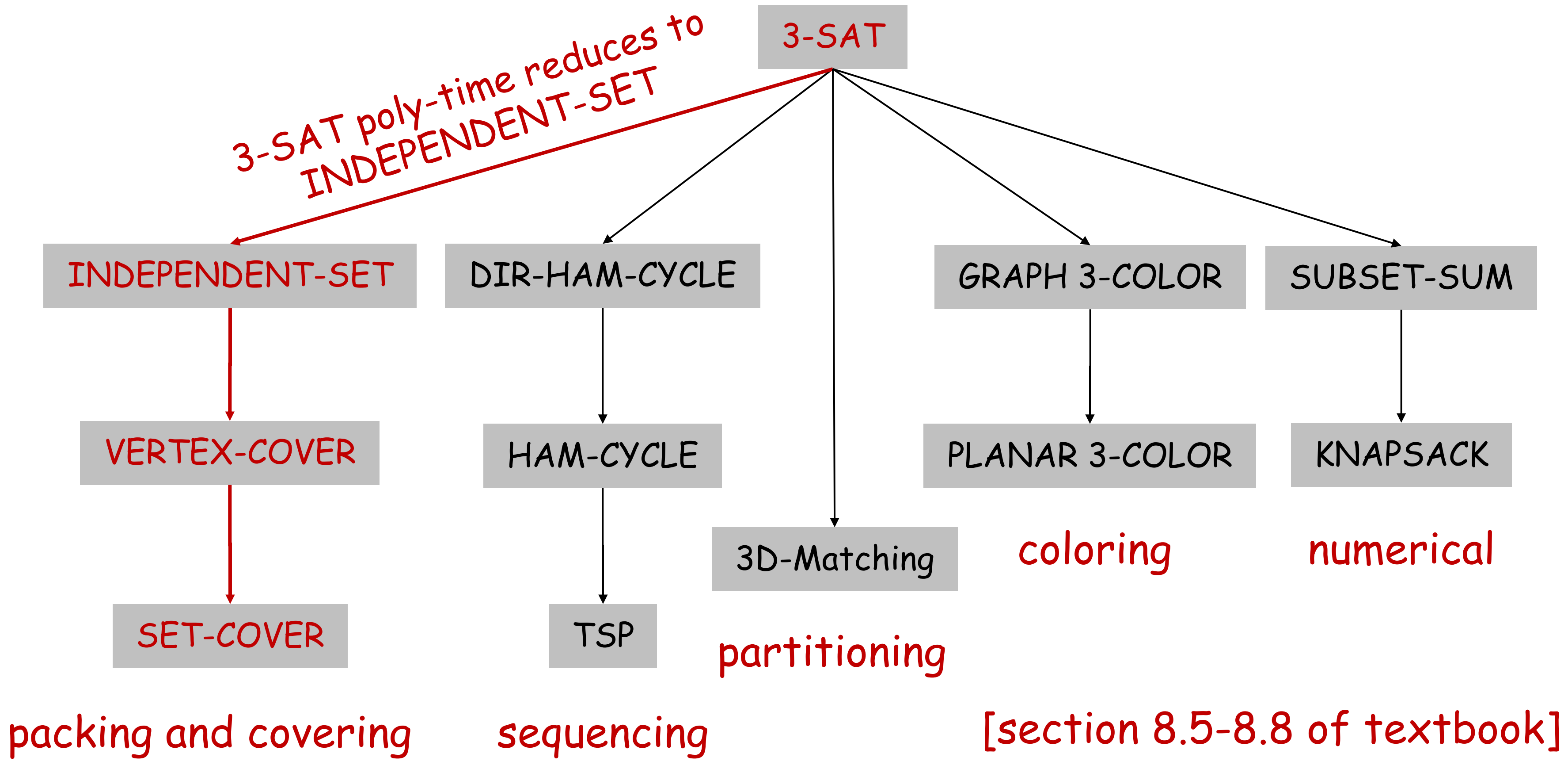
Reductions: Quick Summary

- **Basic reduction strategies:**
  - Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER
  - Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER
  - Encoding with "gadgets": 3-SAT $\leq_P$ INDEPENDENT-SET

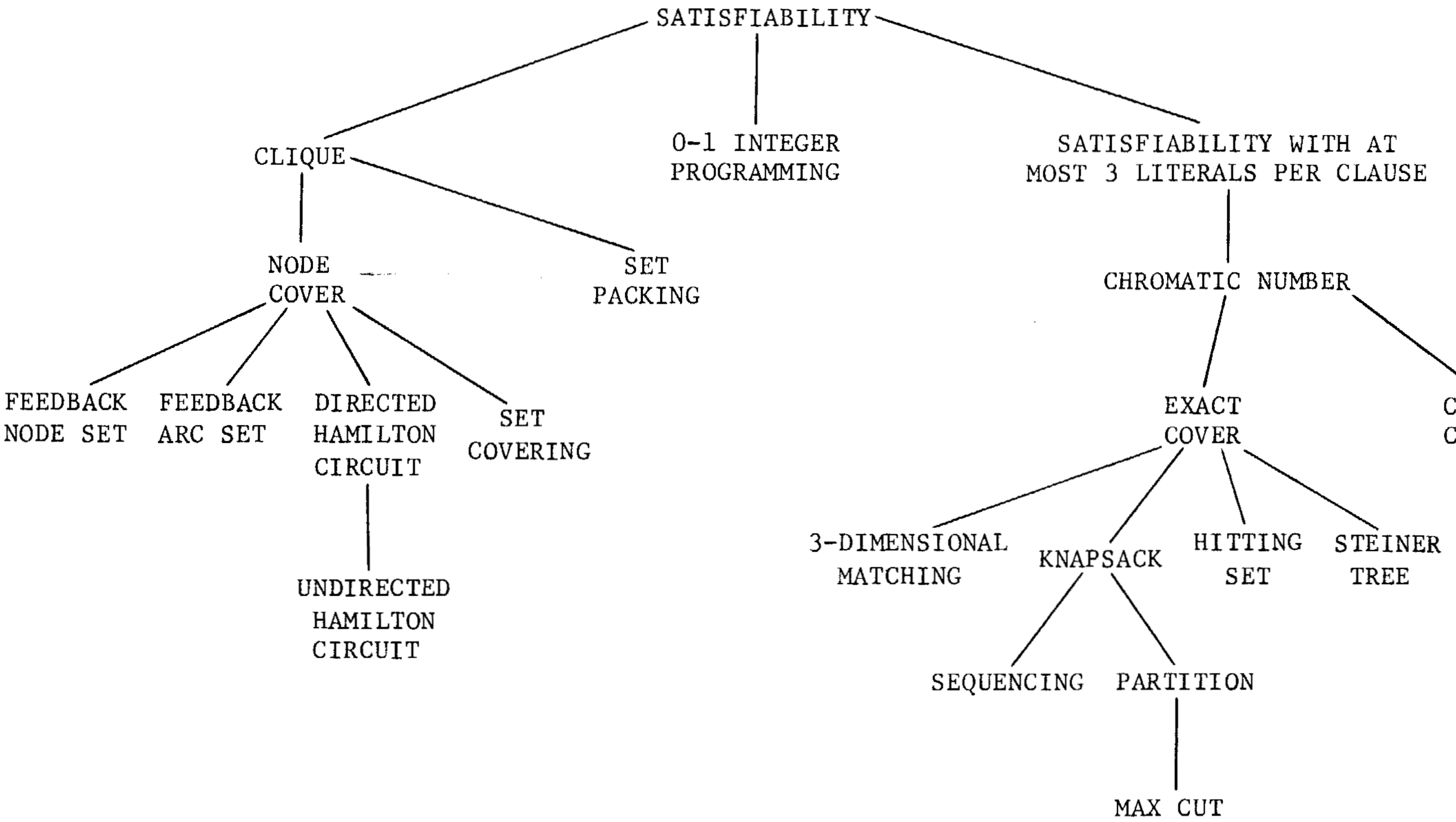
- **Transitivity.** If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
- **Pf idea.** Compose the two algorithms.

- **Example.** 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER

Polynomial-Time Reductions in Textbook



Karp's Poly-Time Reductions from SAT



Richard M. Karp
1985 Turing Award

Exercise: Three Types of Problems

- **Decision problem.** Does there exist a vertex cover of size $\leq k$?
- **Search problem.** Find a vertex cover of size $\leq k$.
- **Optimization problem.** Find a vertex cover of minimum size.

- **Note.** The above are all variants of the vertex cover problem.

- **Q.** Can you prove that the above *3* problems polynomial-time reduce to each other?

Decision Problems vs. Search Problems

- **VERTEX-COVER.** Does there exist a vertex cover of size $\leq k$?
- **FIND-VERTEX-COVER.** Find a vertex cover of size $\leq k$.

- **Theorem.** VERTEX-COVER $\equiv_P$ FIND-VERTEX-COVER
- **Pf.** $\leq_P$: Decision problem is a special case of search problem.

  $\geq_P$: To find a vertex cover of size $\leq k$:
  - ➢ Determine if there exists a vertex cover of size $\leq k$.
  - ➢ Find a vertex v such that $G - \{v\}$ has a vertex cover of size $\leq k - 1$. (Any vertex in any vertex cover of size $\leq k$ will have this property.)
  - ➢ Include v in the vertex cover.
  - ➢ Recursively find a vertex cover of size $\leq k - 1$ in $G - \{v\}$. ▪

run VERTEX-COVER algorithm n = |V| times

delete v and all incident edges

Search Problems vs. Optimization Problems

- **FIND-VERTEX-COVER.** Find a vertex cover of size $\leq k$.
- **FIND-MIN-VERTEX-COVER.** Find a vertex cover of minimum size.

- **Theorem.** FIND-VERTEX-COVER $\equiv_P$ FIND-MIN-VERTEX-COVER
- **Pf.** $\leq_P$: Search problem is a special case of optimization problem.

  $\geq_P$: To find a vertex cover of minimum size:
  - ➢ Binary search (or linear search) for size k^* of minimum vertex cover.
  - ➢ Solve search problem for given k^* to find the minimum vertex cover. ▪

Announcement

- **Assignment 6 has been released and the deadline is 2pm, June 5.**

5. **P** vs. **NP**

P

- **Decision problem:**
  - Problem X is a set of strings.
  - Instance s is one string.
  - Algorithm A solves problem X: $A(s) = yes$ if and only if $s \in X$.
- **Def.** Algorithm A runs in polynomial time if for every string s: $A(s)$ terminates in $\leq p(|s|)$ "steps", for some polynomial function $p(\cdot)$.

  length of s

- **P.** Set of decision problems for which there exists a poly-time algorithm.

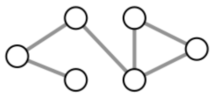
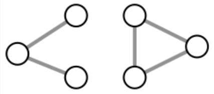
  on a deterministic Turing machine

- **Example:**

problem PRIMES:	$\{ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots \}$
instance s:	592335744548702854681
algorithm:	Agrawal–Kayal–Saxena (2002)

Some Problems in **P**

- **P.** Decision problems for which there exists a poly-time algorithm.

problem	description	poly-time algorithm	yes	no
MULTIPLE	Is x a multiple of y ?	grade-school division	`51, 17`	`51, 16`
REL-PRIME	Are x and y relatively prime ?	Euclid's algorithm	`34, 39`	`34, 51`
PRIMES	Is x prime ?	Agrawal–Kayal–Saxena	`53`	`51`
EDIT-DISTANCE	Is the edit distance between x and y less than 5 ?	Needleman–Wunsch	`niether` `neither`	`acgggt` `ttttta`
L-SOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss–Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
U-CONN	Is an undirected graph G connected?	depth-first search		

NP

- **Certification intuition:**
  - Certifier views things from "managerial" viewpoint.
  - Certifier doesn't determine whether $s \in X$ on its own; rather, it checks a proposed proof t that shows $s \in X$.
- **Def.** Algorithm $C(s, t)$ is a certifier for problem X if for every string s: $s \in X$ if and only if there exists a string t such that $C(s, t) = yes$.
- **NP.** Set of decision problems for which there exists a poly-time certifier.
  - $C(s, t)$ is a polynomial-time algorithm.
  - Certificate t is of polynomial size: $|t| \leq p(|s|)$ for some polynomial $p(\cdot)$.

    also called "witness" or "proof"
- **Remark.** **NP** stands for nondeterministic polynomial time.

Certifiers and Certificates: Composites

- **COMPOSITES.** Given an integer s, is s a composite number?

- **Certificate.** A non-trivial factor t of s.
  - Note that such a certificate exists if and only if s is composite. Also, $|t| \leq |s|$.
- **Certifier.** Grade-school division.
- **Example:**

instance s:	437669	
certificate t:	541 ⟵	$437{,}669 = 541 \times 809$

- **Conclusion.** COMPOSITES $\in$ **NP**

Certifiers and Certificates: Satisfiability

- **SAT.** Given a CNF formula Φ, is there a satisfying truth assignment?
- **3-SAT.** SAT where each clause contains *exactly 3* literals.

- **Certificate.** An assignment of truth values to the Boolean variables.
- **Certifier.** Check that each clause in Φ has at least one true literal.
- **Example:**

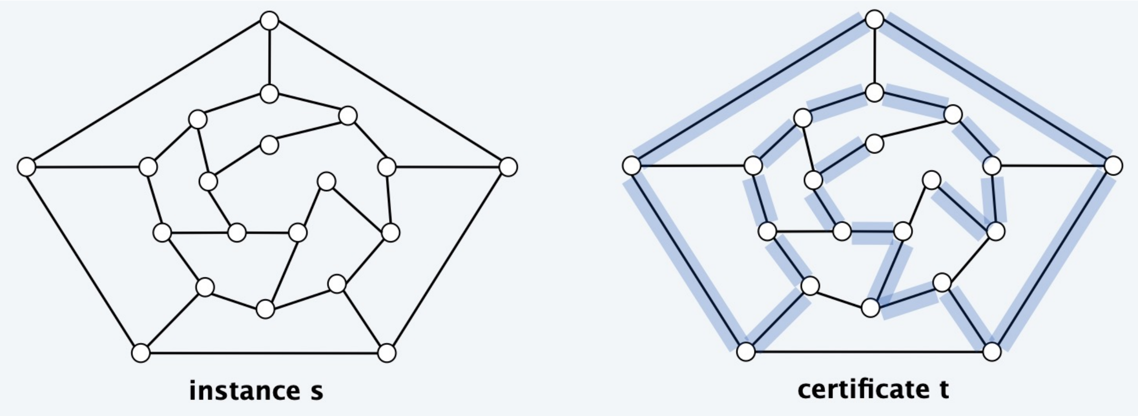
instance s	$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$
certificate t	$x_1 = \mathit{true},\ x_2 = \mathit{true},\ x_3 = \mathit{false},\ x_4 = \mathit{false}$

- **Conclusions.** SAT $\in$ **NP**, 3-SAT $\in$ **NP**.

Certifiers and Certificates: Hamiltonian Path

- **HAMILTONIAN-PATH.** Given an undirected graph $G = (V, E)$, does there exist a simple path that visits every node?

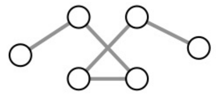
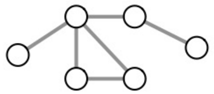
- **Certificate.** A permutation π of the n nodes.
- **Certifier.** Check that π contains each node in V exactly once, and that G contains an edge between each pair of adjacent nodes in π.
- **Example:**



- **Conclusion.**
  HAMILTONIAN-PATH $\in$ **NP**

Some Problems in **NP**

- **NP.** Decision problems for which there exists a poly-time certifier.

problem	description	poly-time algorithm	yes	no
L-SOLVE	Is there a vector x that satisfies $Ax = b$?	Gauss–Edmonds elimination	$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 4 & -2 \\ 0 & 3 & 15 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 36 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$
COMPOSITES	Is x composite ?	Agrawal–Kayal–Saxena	51	53
FACTOR	Does x have a nontrivial factor less than y ?	???	(56159, 50)	(55687, 50)
SAT	Given a CNF formula, does it have a satisfying truth assignment?	???	$\neg x_1 \lor x_2 \lor \neg x_3$ $x_1 \lor \neg x_2 \lor x_3$ $\neg x_1 \lor \neg x_2 \lor x_3$	$\neg x_2$ $x_1 \lor x_2$ $\neg x_1 \lor x_2$
HAMILTON-PATH	Is there a simple path between u and v that visits every node?	???		

Significance of **NP**

- **NP.** Decision problems for which there exists a poly-time certifier.

> " NP captures vast domains of computational, scientific, and mathematical endeavors, and seems to roughly delimit what mathematicians and scientists have been aspiring to compute feasibly. " — Christos Papadimitriou

> " In an ideal world it would be renamed P vs VP. " — Clyde Kruskal

verifiable in polynomial time

P ⊆ NP

- **P.** Decision problems for which there is a poly-time algorithm.
- **NP.** Decision problems for which there is a poly-time certifier.

- **Theorem.** **P ⊆ NP**
- **Pf.** Consider any problem $X \in$ **P**:
  - By definition, there exists a poly-time algorithm $A(s)$ that solves X.
  - Certificate: $t = \varepsilon$ (empty string), certifier $C(s, t) = A(s)$. ▪

NP ⊆ EXP

- **NP.** Decision problems for which there is a poly-time certifier.
- **EXP.** Decision problems for which there is an exponential-time algorithm.

- **Theorem.** **NP ⊆ EXP**
- **Pf.** Consider any problem $X \in$ **NP**:
  - By definition, there exists a poly-time certifier $C(s, t)$ for X, where certificate t satisfies $|t| \le p(|s|)$ for some polynomial $p(\cdot)$.
  - To solve instance s, run $C(s, t)$ on all strings t with $|t| \le p(|s|)$.
  - Return *yes* iff $C(s, t)$ returns *yes* for any of these potential certificates. ▪

- **Fact.** **P ≠ EXP** ⇒ either **P ≠ NP**, or **NP ≠ EXP**, or both.

The Main Question: **P** vs **NP**

- **Q.** How to solve an instance of 3-SAT with *n* variables?
- **A.** Exhaustive search: try all 2^n truth assignments.

- **Q.** Can we do anything substantially more clever?
- **Conjecture.** $\underbrace{\text{No poly-time algorithm}}_{\text{"intractable"}}$ for 3-SAT.

The Main Question: **P** vs **NP**

- **Does P = NP?** [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the certification problem?
  - One of Millennium Problems by Clay Mathematics Institute: $ 1 million prize

break RSA cryptosystem and potentially crash economy

- **If yes...** Efficient algorithms for 3-SAT, TSP, VERTEX-COVER, FACTOR...
- **If no...** No efficient algorithms possible for 3-SAT, TSP, VERTEX-COVER...
- **Consensus opinion.** Probably no.

Possible Outcomes: **P ≠ NP**

" I conjecture that there is no good algorithm for the traveling salesman problem. My reasons are the same as for any mathematical conjecture: (i) It is a legitimate mathematical possibility and (ii) I do not know." — Jack Edmonds (1966)

" In my view, there is no way to even make intelligent guesses about the answer to any of these questions. If I had to bet now, I would bet that P is not equal to NP. I estimate the half-life of this problem at 25–50 more years, but I wouldn't bet on it being solved before 2100. " — Bob Tarjan (2002)

" We seem to be missing even the most basic understanding of the nature of its difficulty... All approaches tried so far probably (in some cases, provably) have failed. In this sense P = NP is different from many other major mathematical problems on which a gradual progress was being constantly done (sometimes for centuries) whereupon they yielded, either completely or partially. " — Alexander Razborov (2002)

Possible Outcomes: **P = NP**

" I think that in this respect I am on the loony fringe of the mathematical community: I think (not too strongly!) that P = NP and this will be proved within twenty years. Some years ago, Charles Read and I worked on it quite bit, and we even had a celebratory dinner in a good restaurant before we found an absolutely fatal mistake. " — Béla Bollobás (2002)

" In my opinion this shouldn't really be a hard problem; it's just that we came late to this theory, and haven't yet developed any techniques for proving computations to be hard. Eventually, it will just be a footnote in the books. " — John Conway

Other Possible Outcomes

- **P = NP**, but only $\Omega(n^{100})$ algorithm for 3-SAT.
- **P ≠ NP**, but with $O(n^{\log^* n})$ algorithm for 3-SAT.
- **P = NP** is independent (of ZFC axiomatic set theory).

can neither prove nor disprove

> " It will be solved by either 2048 or 4096. I am currently somewhat pessimistic. The outcome will be the truly worst case scenario: namely that someone will prove P = NP because there are only finitely many obstructions to the opposite hypothesis; hence there exists a polynomial time solution to SAT but we will never know its complexity! "
>
> — Donald Knuth

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6. **NP**-Complete

Two Types of Reductions

- **Def.** Problem *X* polynomial (Cook) reduces to problem *Y* if arbitrary instances of problem *X* can be solved using:
  - Polynomial number of standard computational steps, plus
  - Polynomial number of calls to oracle that solves problem *Y*.
- **Def.** Problem *X* polynomial (Karp) transforms to problem *Y* if given any instance *x* of *X*, we can construct an instance *y* of *Y* such that *x* is a *yes* instance of *X* iff *y* is a *yes* instance of *Y*. we require |y| to be of size polynomial in |x|
- **Note.** Polynomial transformation is polynomial reduction with just one call to oracle for *Y*, exactly at the end of the algorithm for *X*. Almost all previous reductions were of this form.
- **Open question.** Are these two concepts the same with respect to **NP**?

we abuse notation $\leq_p$ and blur distinction

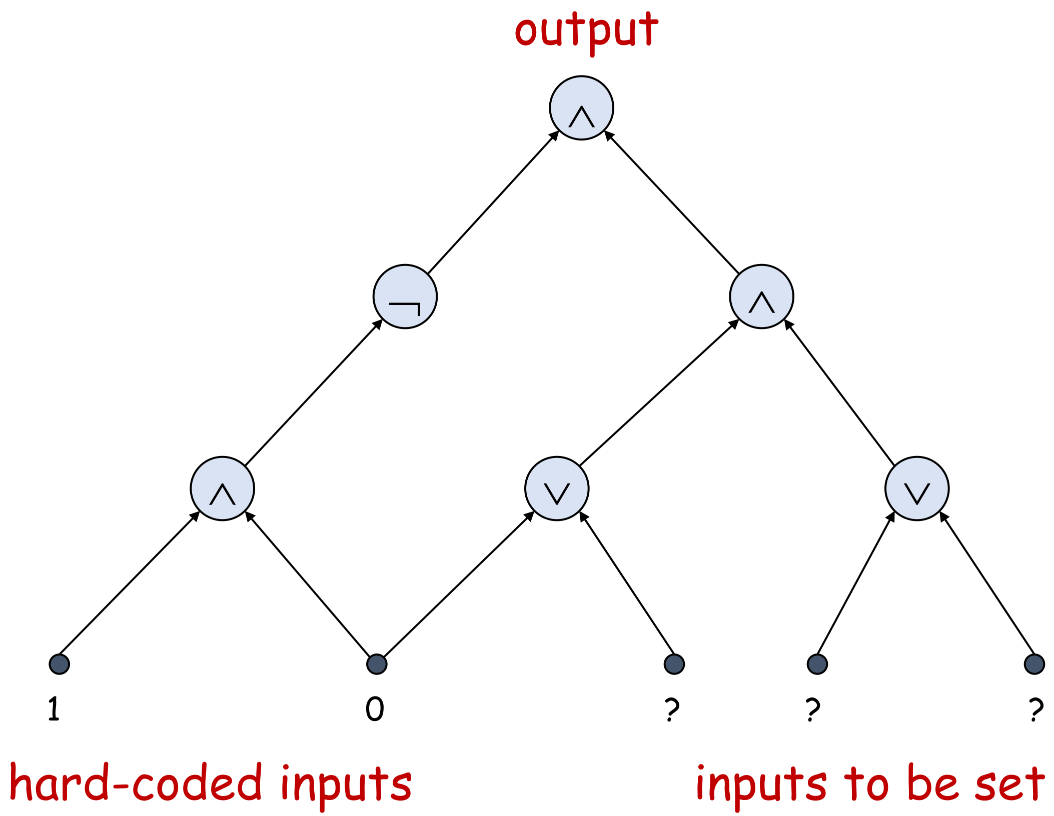
NP-Complete and **NP**-hard

- **NP-complete.** Set of problems Y in **NP** with the property that for every problem X in **NP**, $X \leq_p Y$.
- **NP-hard.** [Bell Labs, Steve Cook, Ron Rivest, Sartaj Sahni]
  Set of problems such that every problem in **NP** poly-time reduces to it.
- **Theorem.** Suppose Y is an **NP**-complete problem. Then $Y \in$ **P** iff **P** = **NP**.
- **Pf.** ("if" + "only if")
  - "if" $\Leftarrow$: If **P** = **NP** then $Y \in$ **P** because $Y \in$ **NP**.
  - "only if" $\Rightarrow$: Suppose $Y \in$ **P**. Consider any problem $X \in$ **NP**. Since $X \leq_p Y$, we have $X \in$ **P**. This implies **NP** $\subseteq$ **P**. We already know **P** $\subseteq$ **NP**. Thus, **P** = **NP**. ▪

- **Fundamental question.** Are there any "natural" **NP**-complete problems?

Circuit Satisfiability

- **CIRCUIT-SAT.** Given a combinational circuit built out of AND, OR, and NOT gates, is there a way to set the circuit inputs so that the output is *1*?



yes inputs: 1 0 1

The "First" **NP**-Complete Problem

- **Theorem.** [Cook 1971, Levin 1973] CIRCUIT-SAT is **NP**-complete.
- **Pf idea.** (direct proof)

reflects basic distinction between algorithms and circuits

  - Any deterministic algorithm that takes a fixed number of bits as input and produces a yes/no answer can be represented by such a circuit. Moreover, if algorithm runs in poly-time, then circuit is of poly-size.
  - Consider any problem $X \in$ **NP**. It has a poly-time certifier $C(s, t)$. To determine whether $s \in X$, one needs to know if there exists a certificate t of length $p(|s|)$ such that $C(s, t) = yes$.
  - View $C(s, t)$ as an algorithm on $|s| + p(|s|)$ bits (input s, certificate t) and convert it into a poly-size circuit K.

    ✓first $|s|$ bits are hard-coded with s

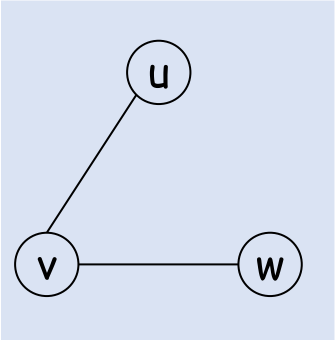
    ✓remaining $p(|s|)$ bits represent bits of t

sketchy part of proof: fixing number of bits

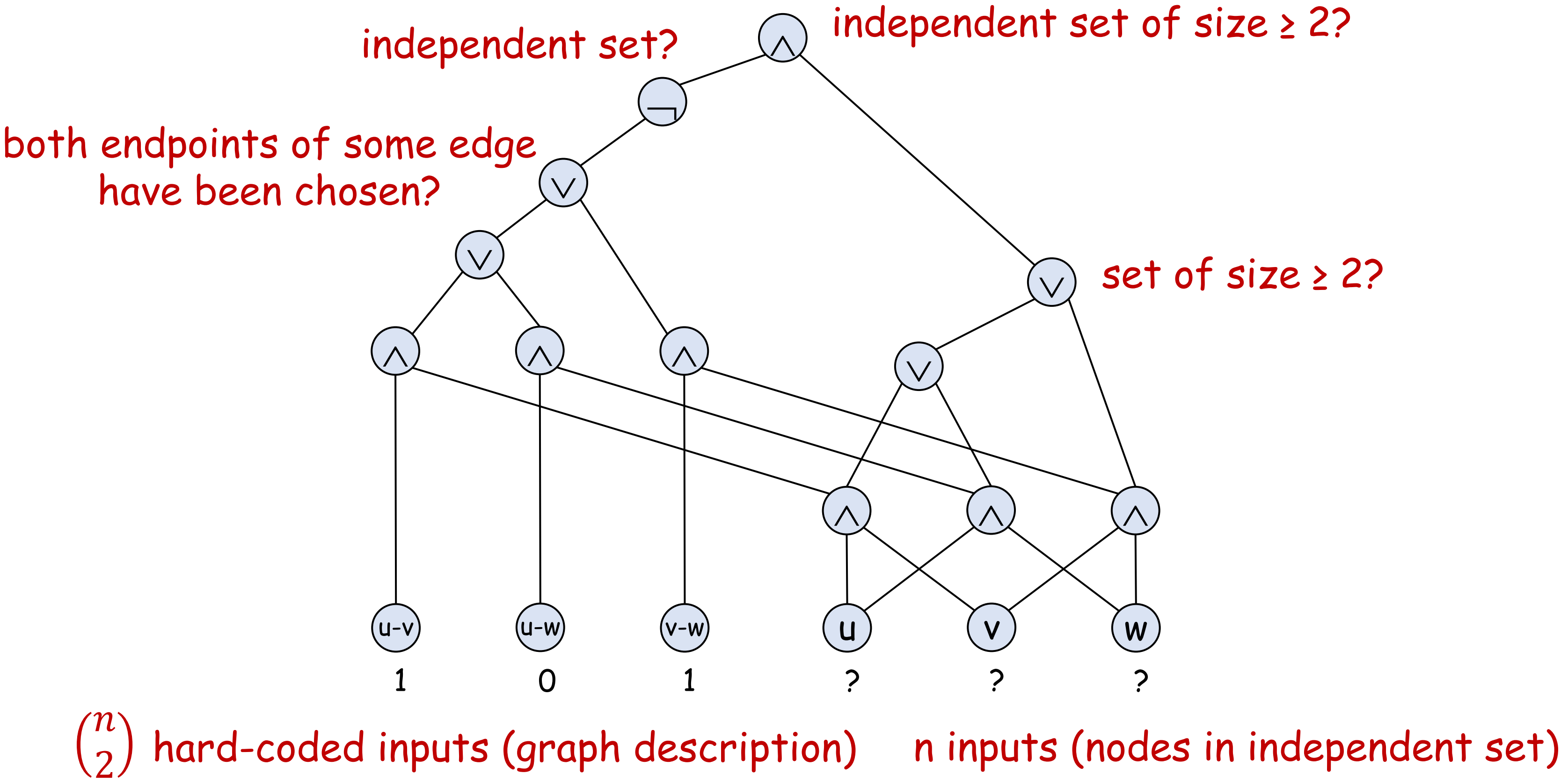
  - Circuit K is satisfiable iff $C(s, t) = yes$.

CIRCUIT-SAT Representation Example

- **Example.** Construction below creates a circuit *K* whose inputs can be set such that *K* outputs true iff graph *G* has an independent set of size ≥ *2*.



G = (V, E), n = 3



Establishing **NP**-Completeness

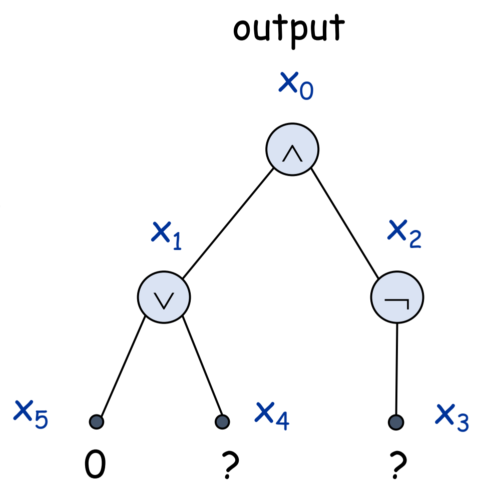
- **Remark.** Once we establish first "natural" **NP**-complete problem, others fall like dominoes.
- **Recipe.** To prove that $Y \in$ **NP**-complete:
  - Step 1. Show that Y is in **NP**.
  - Step 2. Choose an **NP**-complete problem X.
  - Step 3. Prove that $X \leq_p Y$.

- **Theorem.** If $X \in$ **NP**-complete, $Y \in$ **NP**, and $X \leq_P Y$, then $Y \in$ **NP**-complete.
- **Pf.** Consider any problem $W \in$ **NP**. Then $\underbrace{W \leq_P X}_{\text{by definition of NP-complete}} \underbrace{\leq_P Y}_{\text{by assumption}}$.
  - By transitivity, $W \leq_P Y$.
  - Hence, $Y \in$ **NP**-complete. ▪

3-SAT is **NP**-Complete

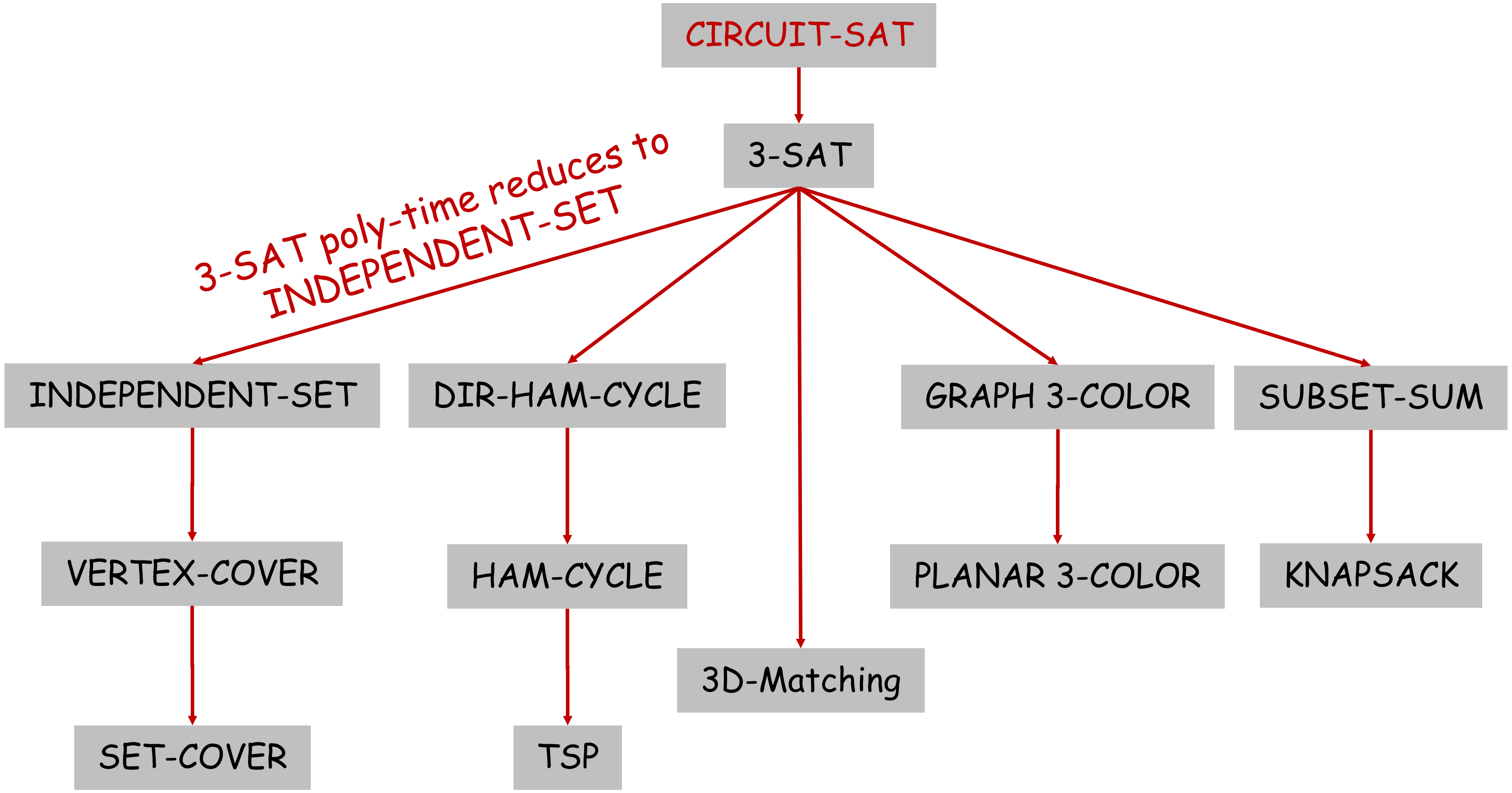
- **Theorem.** 3-SAT is **NP**-complete.
- **Pf.** Suffices to show that CIRCUIT-SAT $\leq_P$ 3-SAT since 3-SAT $\in$ **NP**.
  - Let K be any circuit. Create a 3-SAT variable x_i for each circuit element i.
  - Make circuit compute correct values at each node:
    - ✓ $x_2 = \neg\, x_3 \;\Rightarrow$ add 2 clauses: $x_2 \vee x_3\,,\;\; \overline{x_2} \vee \overline{x_3}$
    - ✓ $x_1 = x_4 \vee x_5 \;\Rightarrow$ add 3 clauses: $x_1 \vee \overline{x_4}\,,\;\; x_1 \vee \overline{x_5}\,,\;\; \overline{x_1} \vee x_4 \vee x_5$
    - ✓ $x_0 = x_1 \wedge x_2 \;\Rightarrow$ add 3 clauses: $\overline{x_0} \vee x_1\,,\;\; \overline{x_0} \vee x_2\,,\;\; x_0 \vee \overline{x_1} \vee \overline{x_2}$
  - Output value and hard-coded input values:
    - ✓ $x_5 = 0 \;\Rightarrow$ add 1 clause: $\overline{x_5}$
    - ✓ $x_0 = 1 \;\Rightarrow$ add 1 clause: x_0
  - Final step: turn clauses of length < 3 into clauses of length exactly 3.
    - ✓ Add extra variables z_1, z_2 such that they do not affect satisfiability. ▪

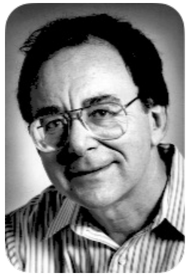
think how?



Implications of Karp

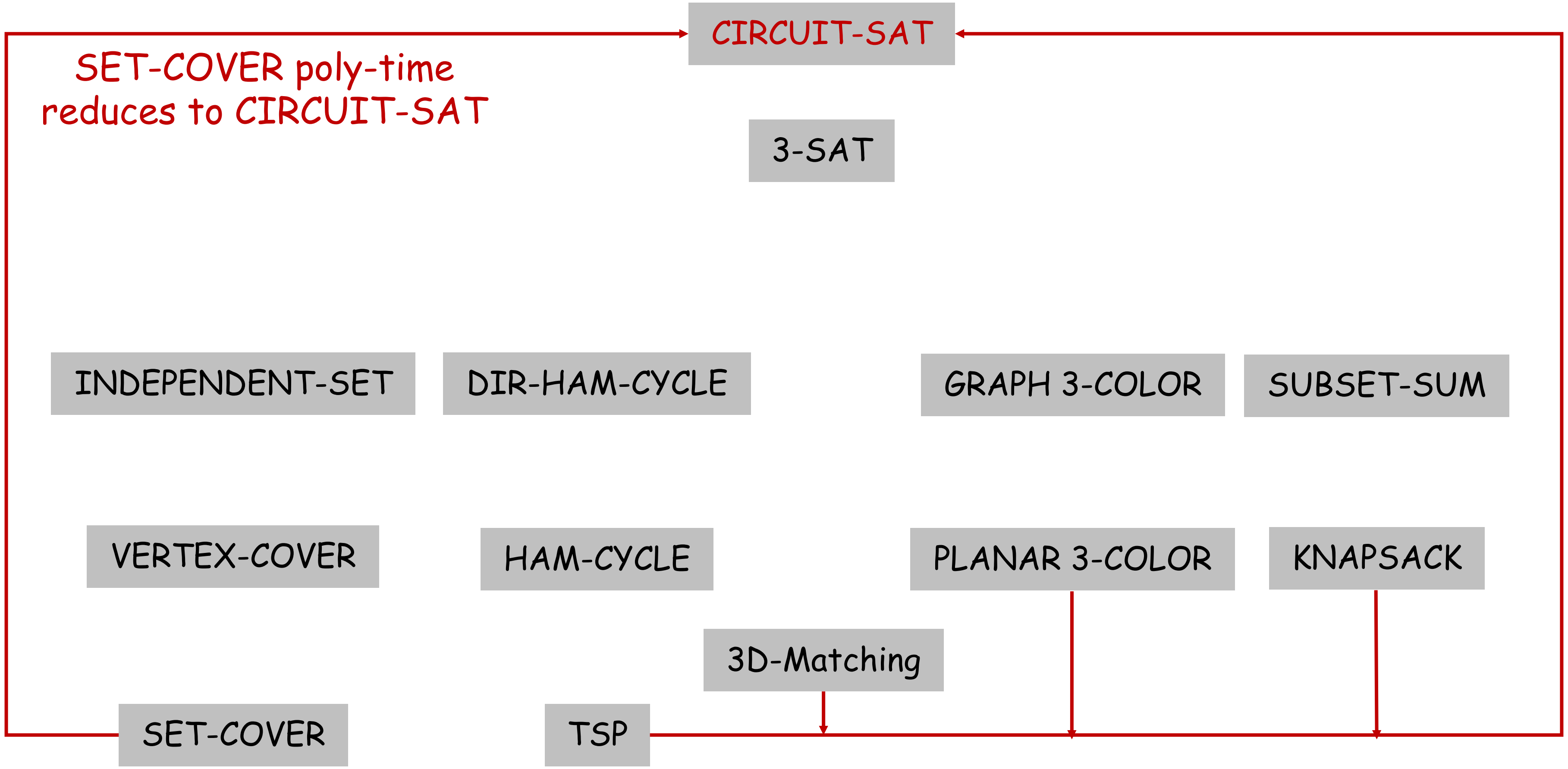
- CIRCUIT-SAT polynomial-time reduces to all problems below (and more).

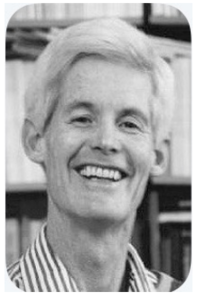




Implications of Cook-Levin

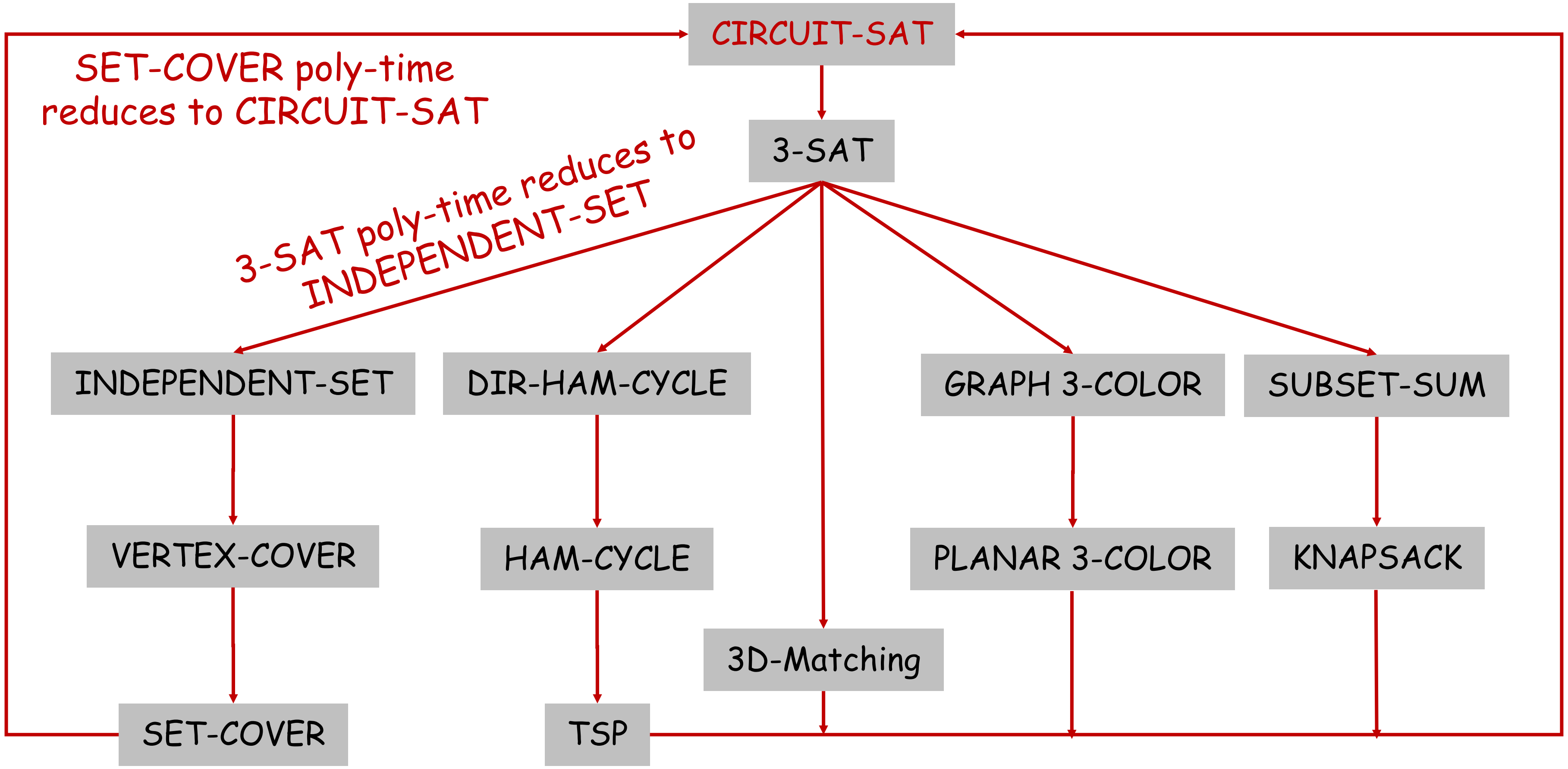
- All problems below (and more) polynomial-time reduces to CIRCUIT-SAT.

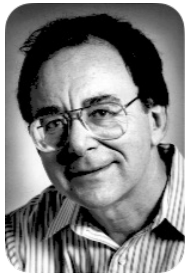


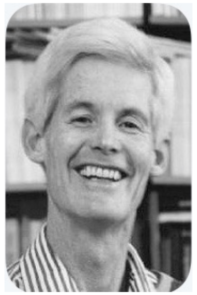


Implications of Karp and Cook-Levin

- **Observation.** All of the following problems are **NP**-complete, i.e., they are manifestations of the same really hard problem.







Some **NP**-Complete Problems

- **Basic genres of NP-complete problems and paradigmatic examples:**
  - Packing/covering problems: SET-COVER, VERTEX-COVER, INDEPENDENT-SET.
  - Constraint satisfaction problems: CIRCUIT-SAT, SAT, 3-SAT.
  - Sequencing problems: HAMILTONIAN-PATH, HAMILTONIAN-CYCLE, TSP.
  - Partitioning problems: 3D-MATCHING, 3-COLOR.
  - Numerical problems: SUBSET-SUM, KNAPSACK.
- **Practice.** Most **NP** problems are known to be either in **P** or **NP**-complete.
- **Notable exceptions.** FACTOR, DISCRETE-LOG, GRAPH-ISOMORPHISM, …

- **Theorem.** [Ladner 1975] Unless **P = NP**, there exist problems in **NP** that are neither in **P** nor **NP**-complete.

More Hard Computational Problems

Aerospace engineering. Optimal mesh partitioning for finite elements.

Biology. Phylogeny reconstruction.

Chemical engineering. Heat exchanger network synthesis.

Chemistry. Protein folding.

Civil engineering. Equilibrium of urban traffic flow.

Economics. Computation of arbitrage in financial markets with friction.

Electrical engineering. VLSI layout.

Environmental engineering. Optimal placement of contaminant sensors.

Financial engineering. Minimum risk portfolio of given return.

Game theory. Nash equilibrium that maximizes social welfare.

Mathematics. Given integer $a_1, \ldots, a_n$, compute $\int_0^{2\pi} \cos(a_1\theta) \times \cos(a_2\theta) \times \cdots \times \cos(a_n\theta)\, d\theta$

Mechanical engineering. Structure of turbulence in sheared flows.

Medicine. Reconstructing 3d shape from biplane angiocardiogram.

Operations research. Traveling salesperson problem.

Physics. Partition function of 3d Ising model.

Politics. Shapley–Shubik voting power.

Recreation. Versions of Sudoku, Checkers, Minesweeper, Tetris, Rubik's Cube.

Statistics. Optimal experimental design.

Extent and Impact of **NP**-Completeness

- **Extent of NP-completeness:** [Papadimitriou 1995]
  - Prime intellectual export of CS to other disciplines.
  - 6,000 citations per year (more than “compiler”, “OS”, “database”).
  - Broad applicability and classification power.

- **NP-completeness can guide scientific inquiry:**
  - 1926: Ising introduces simple model for phase transitions.
  - 1944: Onsager finds closed-form solution to 2D-ISING in tour de force.
  - 19xx: Feynman and other top minds seek solution to 3D-ISING.
  - 2000: Istrail proves 3D-ISING $\in$ **NP**-complete.

a holy grail of statistical mechanics

search for closed formula appears doomed

Exploiting Intractability

- **Q.** Is FACTOR $\in$ **P**?
- **A.** Unknown.

- **Challenge.** Factor this number:

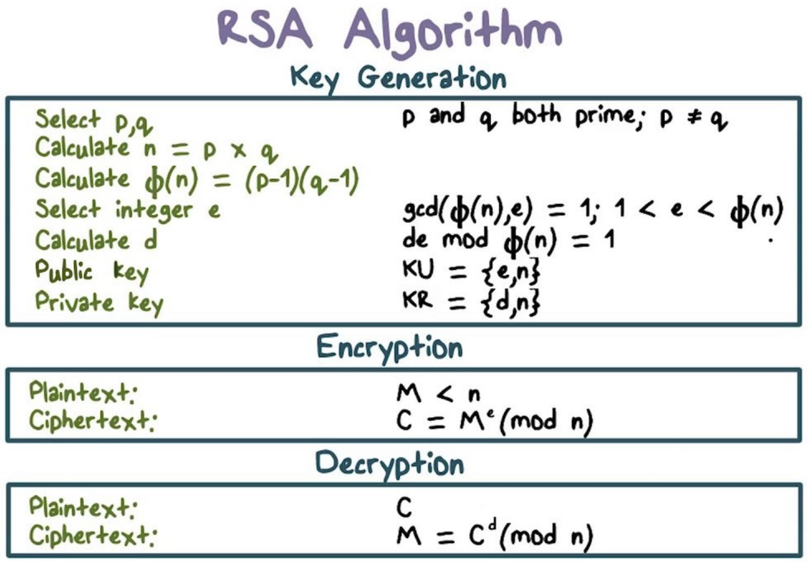
74037563479561712828046796097429573142593188889231289
08493623263897276503402826627689199641962511784399589
43305021275853701189680982867331732731089309005525051
16877063299072396380786710086096962537934650563796359

RSA-704
($30,000 prize if you can factor)

Exploiting Intractability

- **Modern cryptography:**
  - ➢ E.g., securely browsing on the Internet, digitally sign an e-document, etc.
  - ➢ Enables freedom of privacy, speech, press, political association.
- **RSA.** Based on dichotomy between complexity of two problems.
  - ➢ To use: generate two random *n*-bit primes and multiply.
  - ➢ To break: suffices to factor a 2*n*-bit integer.

Sold for $2.1 billion



CS 6035 @ Georgia Tech

Factoring on a Quantum Computer

- **Theorem.** [Shor 1994] Can factor an *n*-bit integer in $O(n^3)$ steps on a quantum computer.
  - *15 = 3 x 5* factored in 2001; *21 = 3 x 7* factored in 2012.

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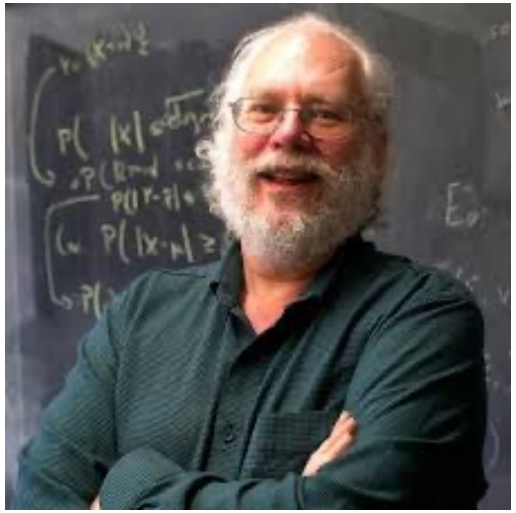
Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*

Peter W. Shor†



- **Fundamental question.** Does **P = BQP**?

quantum analog of P
(bounded error quantum polynomial time)