1. a) Write down the formal definition of $\Theta(g(n))$. [10%]

   b) Simplify the following functions by expressing them in Θ-notation. Give a formal justification for each case, referring to the definition of Θ-notation.

      (i) $n \cdot (3 + \log n)$

      (ii) $4n^2 + n - 100$

   [30%]

   c) For each of the following statements, decide whether the statement is true or false. Explain your answers.

   (i) $o(n) = O(n)$ [15%]

   (ii) $\Omega(n^2) = \Theta(n^2)$ [15%]

   d) The following algorithm counts the number of zeros within an array $A[1 \ldots n]$ of length $n \geq 1$.

COUNT-ZEROS(A)

```
x = 0
for i = 1 to A.length do
    if A[i] = 0 then
        x = x + 1
return x
```

Prove the correctness of COUNT-ZEROS by stating an appropriate loop invariant and showing the three properties: initialisation, maintenance, and termination.

[30%]

2. a) Copy the following table to your answer booklet and fill in asymptotic statements that best describe the running time of the given algorithms across inputs of n elements, using appropriate symbols Θ, O, and/or Ω.

Algorithm	running time
INSERTIONSORT	
SELECTIONSORT	
MERGESORT	
QUICKSORT	
BUBBLESORT	

[25%]

b)

(i) Define the term *max-heap property*, referring to an array $A[1\ldots n]$. [10%]

(ii) Does the following array represent a max-heap? Justify your answer.

42	33	15	20	24	18	4	5

[10%]

c) Recall that QUICKSORT uses the last element of the input as pivot element. Write down the contents of the following array $A[1\ldots n]$ after the execution of PARTITION$(A,1,8)$.

4	3	8	2	7	5	1	6

[20%]

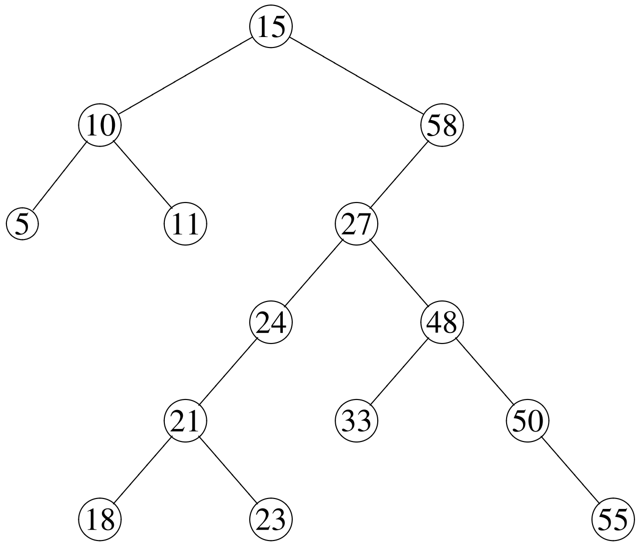
d) Consider an array $A[1\ldots n]$ of n integers in the range 0 to k. Give two algorithms PRE-PROCESS(A,n,k) and COUNT-LESS-OR-EQUAL-ELEMENTS(a) in pseudocode (or Java syntax) such that PREPROCESS preprocesses the input A in time $O(n+k)$. After pre-processing, COUNT-LESS-OR-EQUAL-ELEMENTS(a) must be able to return the number of elements in A which are less or equal to a ($\leq a$) in time $O(1)$, for arbitrary inputs $0 \leq a \leq k$. Explain why your algorithms meet the stated running time bounds.

[35%]

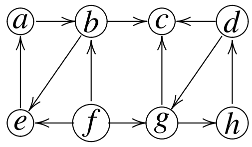
3. a) Prove by induction that every nonempty binary tree satisfies $|V| = |E| + 1$.

   b) Insert the numbers 12, 5, 9, 18, 15, 2, 17, 19 and 13 in that order into a binary search tree, which is initially empty.

   c) Delete the nodes labelled with 15, 58, 55, 48, 18, 10, 5 and 24 in that order from the following binary search tree. Show the resulting binary search tree.

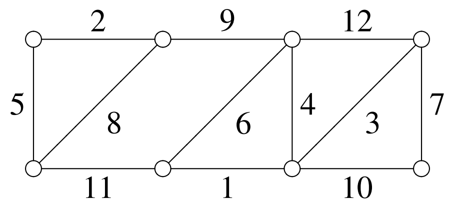


4. a) Perform a depth-first search on the directed graph below, visiting nodes in alphabetical order. Write down the timestamps of each node.



b) Write down the strongly connected component graph of the graph from (a).

c) With Kruskal's algorithm, compute the minimal spanning tree of the following weighted graph.



d) Prove that every directed graph, which can be topologically sorted, is acyclic.

END OF QUESTION PAPER