Discrete Mathematics(H)
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Assignment 5

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Q.1

(1)

R_1 is irreflexive, symmetric.

Because for any string a, a and a always have letters in common, thus $(a, a) \notin R_1$. Thus, R_1 is irreflexive, and it can't be reflexive.

Because for any strings a, b, if $(a, b) \in R_1$, then a and b have no letter in common. Then b and a also have no letter in common, thus $(b, a) \in R_1$. However, (a, b) and (b, a) is both in R_1 does not imply $a = b$, for two different strings can have no letter in common. Thus, R_1 is symmetric, and it can't be antisymmetric.

R_1 is not transitive. For strings $a =$ "A", $b =$ "B", $c =$ "A", $(a, b) \in R_1$ and $(b, c) \in R_1$, but $(a, c) \notin R_1$.

(2)

R_2 is irreflexive, symmetric.

Because for any string a, a and a always have the same length, thus $(a, a) \notin R_2$. Thus, R_2 is irreflexive, and it can't be reflexive.

Because for any strings a, b, if $(a, b) \in R_2$, then a and b don't have the same length. Then b and a also don't have the same length, thus $(b, a) \in R_2$. However, (a, b) and (b, a) is both in R_2 does not imply $a = b$, for two different strings can have different lengths. Thus, R_2 is symmetric, and it can't be antisymmetric.

R_2 is not transitive. For strings $a =$ "A", $b =$ "BB", $c =$ "C", $(a, b) \in R_2$ and $(b, c) \in R_2$, but $(a, c) \notin R_2$.

(3)

R_3 is irreflexive, antisymmetric and transitive.

Because for any string a, a can't be longer than itself, thus $(a, a) \notin R_3$. Thus, R_3 is irreflexive, and it can't be reflexive.

Because for any strings a, b, if $(a, b) \in R_3$, then a is longer than b. Then b can't be longer than a, thus $(b, a) \notin R_3$. Thus, R_3 can't be symmetric.

For any strings a, b, c, if $(a, b) \in R_3$ and $(b, a) \in R_3$, then a is longer than b and b is longer than a. However, this is a contradiction. Then, $(a, b) \in R_3$ and $(b, a) \in R_3$ imply $a = b$ is a tautology. Thus, R_3 is antisymmetric.

For any strings a, b, c, if $(a, b) \in R_3$ and $(b, c) \in R_3$, then a is longer than b and b is longer than c. Then, a is longer than c, thus $(a, c) \in R_3$. Thus, R_3 is transitive.

Q.2

(1)

R is reflexive. For any $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Q}$. So $(a, a) \in R$.

(2)

R is symmetric. For any $a, b \in \mathbb{R}$, if $(a, b) \in R$, then $a - b \in \mathbb{Q}$. So $b - a = -(a - b) \in \mathbb{Q}$. So $(b, a) \in R$.

(3)

R is not antisymmetric. For $a = 1$ and $b = 0$, $a - b = 1 \in \mathbb{Q}$ and $b - a = -1 \in \mathbb{Q}$. So $(a, b) \in R$ and $(b, a) \in R$. But $a \neq b$.

(4)

R is transitive. For any $a, b, c \in \mathbb{R}$, if $(a, b) \in R$ and $(b, c) \in R$, then $a - b \in \mathbb{Q}$ and $b - c \in \mathbb{Q}$. Let $\frac{m_1}{n_1} = a - b$ and $\frac{m_2}{n_2} = b - c$, where $m_1, n_1, m_2, n_2 \in \mathbb{Z}$ and $n_1, n_2 \neq 0$. Then $a - c = (a - b) + (b - c) = \frac{m_1}{n_1} + \frac{m_2}{n_2} = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2} \in \mathbb{Q}$. So $(a, c) \in R$.

Q.3

(a)

The number of symmetric relations is $2^{\frac{n^2+n}{2}}$.

(b)

The number of antisymmetric relations is $3^{\frac{n^2-n}{2}} 2^n$.

(c)

The number of irreflexive relations is 2^{n^2-n}.

(d)

The number of reflexive and symmetric relations is $2^{\frac{n^2-n}{2}}$.

(e)

The number of not reflexive nor irreflexive relations is $(2^n - 2)2^{n^2-n}$.

(f)

The number of reflexive and antisymmetric relations is $3^{\frac{n^2-n}{2}}$.

(g)

The number of symmetric, antisymmetric and transitive relations is 2^n.

Q.4

No. R^2 is not irreflexive.

Suppose R is an irreflexive relation on A, where $A = \{0, 1\}$ and $R = \{(0, 1), (1, 0)\}$. Then $R^2 = \{(0, 0), (1, 1)\}$. So R^2 is not irreflexive.

Q.5

(1)

Since R_1 and R_2 are reflexive, then for any $a \in A$, $(a, a) \in R_1$ and $(a, a) \in R_2$. Therefore, $(a, a) \notin R_1 \oplus R_2$. So $R_1 \oplus R_2$ is irreflexive.

(2)

Yes. $R_1 \cap R_2$ is reflexive.

Since R_1 and R_2 are reflexive, then for any $a \in A$, $(a, a) \in R_1$ and $(a, a) \in R_2$. Therefore, $(a, a) \in R_1 \cap R_2$. So $R_1 \cap R_2$ is reflexive.

(3)

Yes. $R_1 \cup R_2$ is reflexive.

Since R_1 and R_2 are reflexive, then for any $a \in A$, $(a, a) \in R_1$ and $(a, a) \in R_2$. Therefore, $(a, a) \in R_1 \cup R_2$. So $R_1 \cup R_2$ is reflexive.

Q.6

(1)

R is an equivalence relation.

Reflexive:

For any ordered pair (a, b) where $a, b \in \mathbb{Z}^+$, $ab = ba$ is always true, thus $((a, b), (a, b)) \in R$. Hence, R is reflexive.

Symmetric:

For any ordered pair (a, b), (c, d) where $a, b, c, d \in \mathbb{Z}^+$, if $((a, b), (c, d)) \in R$, then $ad = bc$. Therefore, $cb = da$, and $((c, d), (a, b)) \in R$. Hence, R is symmetric.

Transitive:

For any ordered pair (a, b), (c, d), (e, f) where $a, b, c, d, e, f \in \mathbb{Z}^+$, if $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, then $ad = bc$ and $cf = de$. Therefore, $af = \frac{adcf}{dc} = \frac{bcde}{dc} = be$, thus $((a, b), (e, f)) \in R$. Hence, R is transitive.

(2)

Let $(a,b) = (1,2)$, then $2a = b$. Hence, the equivalence class of $(1,2)$ is $\{(a,b)|2a = b \text{ and } a,b \in \mathbb{Z}^+\}$.

(3)

Each equivalence class is a set of ordered pairs with the same value of $\frac{a}{b}$.

Q.7

(1)

Proof.

R is an equivalence relation.

Reflexive:

For any tuple (a,b,c) where $a,b,c \in \mathbb{R}$, $(a,b,c) = 1 \cdot (a,b,c)$ and $1 \in \mathbb{R}\backslash\{0\}$. Hence, R is reflexive.

Symmetric:

For any tuple (a,b,c), (d,e,f) where $a,b,c,d,e,f \in \mathbb{R}$, if $(a,b,c) = k(d,e,f)$ where $k \in \mathbb{R}\backslash\{0\}$. Then $(d,e,f) = \frac{1}{k}(a,b,c)$, where $\frac{1}{k} \in \mathbb{R}\backslash\{0\}$. Hence, R is symmetric.

Transitive:

For any tuple (a,b,c), (d,e,f), (g,h,i) where $a,b,c,d,e,f,g,h,i \in \mathbb{R}$, if $(a,b,c) = k_1(d,e,f)$ and $(d,e,f) = k_2(g,h,i)$ where $k_1,k_2 \in \mathbb{R}\backslash\{0\}$. Then $(a,b,c) = k_1k_2(g,h,i)$, where $k_1k_2 \in \mathbb{R}\backslash\{0\}$. Thus, $(a,b,c)R(g,h,i)$. Hence, R is transitive. □

(2)

$$[(1,1,1)]_R = \{(1,1,1),(2,2,2),(3,3,3),(4,4,4),\dots\}$$

$$[(1,0,3)]_R = \{(1,0,3),(2,0,6),(3,0,9),(4,0,12),\dots\}$$

(3)

No. Not all equivalence classes have the same cardinality.

Disproof.

The equivalence class $[(1,1,1)]_R$ has infinite elements, while the equivalence class $[(0,0,0)]_R$ has only one element. □

Q.8

Proof.

T is an equivalence relation.

Reflexive:

Since R and S are both reflexive, then for any $a \in A$, $(a,a) \in R$ and $(a,a) \in S$. Therefore, $(a,a) \in R \cap S$. Hence, T is reflexive.

Symmetric:

For any $a, b \in A$, if $(a, b) \in T$, then $(a, b) \in R$ and $(a, b) \in S$. Because R and S are both symmetric, then $(b, a) \in R$ and $(b, a) \in S$. Therefore, $(b, a) \in R \cap S$. Hence, T is symmetric.

Transitive:

For any $a, b, c \in A$, if $(a, b) \in T$ and $(b, c) \in T$, then $(a, b), (b, c) \in R$ and $(a, b), (b, c) \in S$. Because R and S are both transitive, then $(a, c) \in R$ and $(a, c) \in S$. Thus, $(a, c) \in R \cap S$. Hence, T is transitive. $\square$

Q.9

(a)

Yes. $(\mathbb{R}, =)$ is a partially ordered set and $=$ is a partial order.

Reflexive:

For any $a \in \mathbb{R}$, $a = a$. Therefore, $(a, a) \in R$. Hence, $=$ is reflexive.

Antisymmetric:

For any $a, b \in \mathbb{R}$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ and $b = a$. Therefore, (a, b) and (b, a) imply $a = b$. Hence, $=$ is antisymmetric.

Transitive:

For any $a, b, c \in \mathbb{R}$, if $(a, b) \in R$ and $(b, c) \in R$, then $a = b$ and $b = c$. Therefore, $a = c$, thus $(a, c) \in R$. Hence, $=$ is transitive.

(b)

No. $(\mathbb{R}, <)$ is not a partially ordered set.

This is because $<$ is not reflexive. For any $a \in \mathbb{R}$, $a < a$ is false.

(c)

Yes. $(\mathbb{R}, \leq)$ is a partially ordered set and $\leq$ is a partial order.

Reflexive:

For any $a \in \mathbb{R}$, $a \leq a$. Therefore, $(a, a) \in R$. Hence, $\leq$ is reflexive.

Antisymmetric:

For any $a, b \in \mathbb{R}$, if $(a, b) \in R$ and $(b, a) \in R$, then $a \leq b$ and $b \leq a$. Therefore, $a = b$. Hence, $\leq$ is antisymmetric.

Transitive:

For any $a, b, c \in \mathbb{R}$, if $(a, b) \in R$ and $(b, c) \in R$, then $a \leq b$ and $b \leq c$. Therefore, $a \leq c$, thus $(a, c) \in R$. Hence, $\leq$ is transitive.

(d)

No. $(\mathbb{R}, \neq)$ is not a partially ordered set.

This is because $\neq$ is not reflexive. For any $a \in \mathbb{R}$, $a \neq a$ is false.

Q.10

(a)

Proof.

$\preceq$ is a partial order.

Reflexive:

For any function $f : \mathbb{R} \to \mathbb{R}$, $f(x) \le f(x)$ is true for any $x \in \mathbb{R}$. Hence, $\preceq$ is reflexive.

Antisymmetric:

For any functions $f, g : \mathbb{R} \to \mathbb{R}$, if $f \preceq g$ and $g \preceq f$, then $f(x) \le g(x)$ and $g(x) \le f(x)$ for any $x \in \mathbb{R}$. Therefore, $f(x) = g(x)$ for any $x \in \mathbb{R}$. Hence, $\preceq$ is antisymmetric.

Transitive:

For any functions $f, g, h : \mathbb{R} \to \mathbb{R}$, if $f \preceq g$ and $g \preceq h$, then $f(x) \le g(x)$ and $g(x) \le h(x)$ for any $x \in \mathbb{R}$. Therefore, $f(x) \le h(x)$ for any $x \in \mathbb{R}$, thus $f \preceq h$. Hence, $\preceq$ is transitive. □

(b)

Disproof.

$\preceq$ is not a total order.

Let $f(x) = x$ and $g(x) = -x$ for any $x \in \mathbb{R}$. Then $f \preceq g$ and $g \preceq f$ are both false. □

Q.11

(a)

Yes. $\preceq$ is reflexive.

For any positive integer a, let m_a be the sum of distinct prime factors of a. Then obviously $m_a \le m_a$, thus $a \preceq a$. Hence, $\preceq$ is reflexive.

(b)

No. $\preceq$ is not antisymmetric.

For integer 2 and 4, the sum of prime factors are both 2. Then, $(2, 4), (4, 2) \in R$. However, $2 \ne 4$. Hence, $\preceq$ is not antisymmetric.

(c)

Yes. $\preceq$ is transitive.

For any positive integers a, b, c, if $a \preceq b$ and $b \preceq c$, then $m_a \le m_b$ and $m_b \le m_c$. Therefore, $m_a \le m_c$, thus $a \preceq c$. Hence, $\preceq$ is transitive.

Q.12

(1)

Proof.

R is a partial ordering.

Reflexive:

For a tuple (a, b, c) where $a, b, c \in \mathbb{N}$, $2^a3^b5^c \leq 2^a3^b5^c$ is always true. Hence, R is reflexive.

Antisymmetric:

For tuples (a, b, c), (d, e, f) where $a, b, c, d, e, f \in \mathbb{N}$, if $2^a3^b5^c \leq 2^d3^e5^f$ and $2^d3^e5^f \leq 2^a3^b5^c$, then $2^a3^b5^c = 2^d3^e5^f$. By the fundamental theorem of arithmetic, $a = d$, $b = e$ and $c = f$. Therefore, $(a, b, c) = (d, e, f)$. Hence, R is antisymmetric.

Transitive:

For tuples (a, b, c), (d, e, f), (g, h, i) where $a, b, c, d, e, f, g, h, i \in \mathbb{N}$, and $(a, b, c)R(d, e, f)$ and $(d, e, f)R(g, h, i)$, then $2^a3^b5^c \leq 2^d3^e5^f$ and $2^d3^e5^f \leq 2^g3^h5^i$. Therefore, $2^a3^b5^c \leq 2^g3^h5^i$, thus $(a, b, c)R(g, h, i)$. Hence, R is transitive. $\square$

(2)

$(1, 1, 1)$ and $(2, 2, 2)$ are comparable elements.

There is no incomparable elements.

(3)

To find the least upper bound for $(5, 0, 1)$ and $(1, 1, 2)$, we need to find the smallest number $2^a3^b5^c$ that is greater than or equal to both $2^53^05^1 = 160$ and $2^13^15^2 = 150$. Hence, the least upper bound is $(5, 0, 1)$.

Similarly, we can find the greatest lower bound for $(5, 0, 1)$ and $(1, 1, 2)$, which is $(1, 1, 2)$.

(4)

The minimal element is $(0, 0, 0)$.

There is no maximal element.

Q.13

$(\mathbb{Z} \times \mathbb{Z}, \preceq)$ is a partially ordered set.

Reflexive:

For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b) = (a, b)$ is always true. Therefore, $(a, b) \preceq (a, b)$. Hence, $\preceq$ is reflexive.

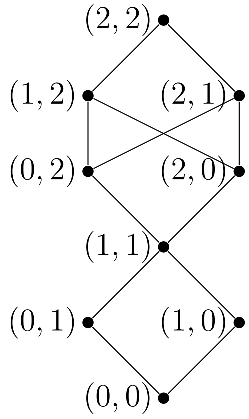
Antisymmetric:

For any $(a, b), (c, d) \in \mathbb{Z}\times\mathbb{Z}$, suppose $(a, b) \preceq (c, d)$ and $(c, d) \preceq (a, b)$. If $(a, b) \neq (c, d)$, then the two relationships above would be equivalent to $a^2 + b^2 < c^2 + d^2$ and $c^2 + d^2 < a^2 + b^2$, which is a contradiction. Therefore, $(a, b) = (c, d)$. Hence, $\preceq$ is antisymmetric.

Transitive:

For any $(a, b), (c, d), (e, f) \in \mathbb{Z} \times \mathbb{Z}$, suppose $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$. If $(a, b) \preceq (c, d)$ is equivalent to $a^2+b^2 < c^2+d^2$, then $a^2+b^2 < c^2+d^2 \leq e^2+f^2 \to a^2+b^2 < e^2 + f^2$. Therefore, $(a, b) \preceq (e, f)$. If $(c, d) \preceq (e, f)$ is equivalent to $c^2 + d^2 < e^2 + f^2$, we can get the same result as shown above. If $(a, b) \preceq (c, d)$ is equivalent to $(a, b) = (c, d)$ and $(c, d) \preceq (e, f)$ is equivalent to $(c, d) = (e, f)$, then $(a, b) = (e, f)$. Therefore, $(a, b) \preceq (e, f)$. Hence, $\preceq$ is transitive.

Hasse diagram:



Q.14

(a)

The maximal elements are l and m.

(b)

The minimal elements are a, b, and c.

(c)

No. There is no greatest element.

(d)

No. There is no least element.

(e)

The upper bounds of $\{a, b, c\}$ are k, l, and m.

(f)

The least upper bound of $\{a, b, c\}$ is k.

(g)

The lower bound does not exist.

(h)

The greatest lower bound does not exist.