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Section 1.6

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P24 Q68

该证明为假。反例如下：

若令 $\Omega = \{1,2,3,4\}$，$A = \{1,2\}$，$B = \{2,3\}$，$C = \{3,4\}$，则此时 $P(A) = P(B) = P(C) = \frac{1}{2}$，$P(AB) = P(BC) = \frac{1}{4}$。

由于 $P(AB) = P(A)P(B)$ 且 $P(BC) = P(B)P(C)$，所以 A 与 B 独立，B 与 C 独立。但此时 $P(AC) = 0 \neq P(A)P(C)$，所以 A 与 C 不独立。

P24 Q71

已知 A, B 和 C 相互独立，所以 $P(ABC) = P(A)P(B)P(C)$，$P(AB) = P(A)P(B)$，$P(AC) = P(A)P(C)$，$P(BC) = P(B)P(C)$。

由于：

$$
\begin{aligned}
P((A \cap B) \cap C) &= P(ABC) \\
&= P(A)P(B)P(C) \\
&= P(A \cap B)P(C)
\end{aligned}
$$

所以 $A \cap B$ 与 C 独立。

由于：

$$
\begin{aligned}
P((A \cup B) \cap C) &= P((AC) \cup (BC)) \\
&= P(AC) + P(BC) - P(ABC) \\
&= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C) \\
&= (P(A) + P(B) - P(AB))P(C) \\
&= P(A \cup B)P(C)
\end{aligned}
$$

所以 $A \cup B$ 与 C 独立。

P24 Q74

记 $A=\{$整个系统正常工作$\}$，$A_i=\{$第i个单元正常工作$\}$。则有：

$$\begin{aligned}
P(A)=&P(A_1A_2\cup A_3\cup A_4A_5)\\
=&P(A_1A_2)+P(A_3)+P(A_4A_5)\\
&-P(A_1A_2A_3)-P(A_1A_2A_4A_5)-P(A_3A_4A_5)\\
&+P(A_1A_2A_3A_4A_5)\\
=&P(A_1)P(A_2)+P(A_3)+P(A_4)P(A_5)\\
&-P(A_1)P(A_2)P(A_3)-P(A_1)P(A_2)P(A_4)P(A_5)-P(A_3)P(A_4)P(A_5)\\
&+P(A_1)P(A_2)P(A_3)P(A_4)P(A_5)\\
=&(1-p)+2(1-p)^2-2(1-p)^3-(1-p)^4+(1-p)^5\\
=&-p^5+4p^4-4p^3+1
\end{aligned}$$

P24 Q77

记 $A=\{$命中靶心至少一次$\}$，$A_i=\{$第i次命中靶心$\}$。则有：

$$\begin{aligned}
P(A)=&P(\bigcup_{i=1}^{n}A_i)\\
=&1-P(\bigcap_{i=1}^{n}\bar{A}_i)\\
=&1-\prod_{i=1}^{n}P(\bar{A}_i)\\
=&1-0.95^n
\end{aligned}$$

为使 $P(A)\geqslant 0.5$，则有：

$$\begin{aligned}
1-0.95^n&\geqslant 0.5\\
0.95^n&\leqslant 0.5\\
n\geqslant \log_{0.95}0.5&\approx 13.5
\end{aligned}$$

所以至少需要扔 14 次。

P25 Q79

a.

记 $A_i=\{$第i个人给出 A 基因$\}$，易知对于携带者有 $P(A_i)=0.5$，则有：

$$\begin{aligned}P(\text{后代为 AA}) =& P(A_1A_2)\\ =& P(A_1)P(A_2)\\ =& 0.5^2\\ =& 0.25\\ P(\text{后代为 Aa}) =& P(A_1\bar{A}_2\cup\bar{A}_1A_2)\\ =& P(A_1\bar{A}_2)+P(\bar{A}_1A_2)\\ =& P(A_1)P(\bar{A}_2)+P(\bar{A}_1)P(A_2)\\ =& 2\times0.5\times0.5\\ =& 0.5\\ P(\text{后代为 aa}) =& P(\bar{A}_1\bar{A}_2)\\ =& P(\bar{A}_1)P(\bar{A}_2)\\ =& 0.5^2\\ =& 0.25\end{aligned}$$

b.

$$\begin{aligned}P(\text{后代为 Aa}|\text{后代为 Aa 或 aa}) =& \frac{P(\text{后代为 Aa})}{P(\text{后代为 Aa})+P(\text{后代为 aa})}\\ =& \frac{0.5}{0.5+0.25}\\ =& \frac{2}{3}\end{aligned}$$

c.

已知该无病后代为 Aa 的概率为 $\frac{2}{3}$，为 aa 的概率为 $\frac{1}{3}$。由全概率公式可知：

$$\begin{aligned}P(A_1) &= \frac{1}{2}\times\frac{2}{3}+0\times\frac{1}{3}\\ &= \frac{1}{3}\end{aligned}$$

其配偶为 Aa 的概率为 p，为 aa 的概率为 $1-p$。由全概率公式可知：

$$\begin{aligned}P(A_2) &= \frac{1}{2}\cdot p+0\cdot(1-p)\\ &= \frac{p}{2}\end{aligned}$$

故有：

$$
\begin{aligned}
P(\text{后代为 AA}) =& P(A_1A_2) \\
=& P(A_1)P(A_2) \\
=& \frac{1}{3}\cdot\frac{p}{2} \\
=& \frac{p}{6} \\
P(\text{后代为 Aa}) =& P(A_1\bar{A}_2 \cup \bar{A}_1A_2) \\
=& P(A_1\bar{A}_2) + P(\bar{A}_1A_2) \\
=& P(A_1)P(\bar{A}_2) + P(\bar{A}_1)P(A_2) \\
=& \frac{1}{3}\cdot(1-\frac{p}{2}) + \frac{2}{3}\cdot\frac{p}{2} \\
=& \frac{1}{3} + \frac{p}{6} \\
P(\text{后代为 aa}) =& P(\bar{A}_1\bar{A}_2) \\
=& P(\bar{A}_1)P(\bar{A}_2) \\
=& \frac{2}{3}\cdot(1-\frac{p}{2}) \\
=& \frac{2}{3} - \frac{p}{3}
\end{aligned}
$$

d.

由贝叶斯公式可知：

$$
\begin{aligned}
P(\text{父亲是携带者}|\text{后代不患病}) =& \frac{P(\text{后代不患病}|\text{父亲是携带者})P(\text{父亲是携带者})}{P(\text{后代不患病})} \\
=& \frac{(1-p+\frac{3p}{4})\cdot\frac{2}{3}}{(1-p+\frac{3p}{4})\cdot\frac{2}{3}+1\cdot\frac{1}{3}} \\
=& \frac{4-p}{6-p}
\end{aligned}
$$

补充 1

已知事件 A 与事件 B 独立，且有 $P(\bar{A}\bar{B}) = \frac{1}{9}$，$P(A\bar{B}) = P(\bar{A}B)$。则有：

$$
\begin{aligned}
P(A\bar{B}) &= P(\bar{A}B) \\
P(A)P(\bar{B}) &= P(\bar{A})P(B) \\
P(A) - P(A)P(B) &= P(B) - P(A)P(B) \\
P(A) &= P(B)
\end{aligned}
$$

由于 $P(A) = P(B)$，所以 $P(\bar{A}) = P(\bar{B})$。
又由于 $P(\bar{A}\bar{B}) = \frac{1}{9}$，所以 $P(A) = P(B) = \frac{2}{3}$。

补充 2

$$
\begin{aligned}
P(A \cup B \cup C) =& P(A) + P(B) + P(C) \\
& - P(AB) - P(AC) - P(BC) \\
& + P(ABC) \\
=& 3P(A) - 3P(A)^2 + 0 \\
=& \frac{9}{16}
\end{aligned}
$$

解方程可得 $P(A) = \frac{3}{4}$或$\frac{1}{4}$。然而，若 $P(A) = P(B) = P(C) = \frac{3}{4}$，则 $P(ABC) \geqslant \frac{1}{4}$，与题设矛盾。

所以 $P(A) = P(B) = P(C) = \frac{1}{4}$。

练习

1.

正确的是 3

$$
\begin{aligned}
P(A|B) =& \frac{P(AB)}{P(B)} \\
=& \frac{0}{P(B)} \\
=& 0
\end{aligned}
$$

2.

正确的是 1，2，4

$$
\begin{aligned}
P(B|A) =& \frac{P(AB)}{P(A)} \\
=& \frac{P(A)P(B)}{P(A)} \\
=& P(B) \\
>& 0
\end{aligned}
$$

$$
\begin{aligned}
P(A|B) =& \frac{P(AB)}{P(B)} \\
=& \frac{P(A)P(B)}{P(B)} \\
=& P(A)
\end{aligned}
$$

由独立的定义可知 4 正确。