Department of
Computer Science and Engineering
计算机科学与工程系

CS215 DISCRETE MATH

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Properties of Relations

- **Reflexive Relation**: A relation R on a set A is called *reflexive* if $(a, a) \in R$ for every element $a \in A$.

  Irreflexive Relation: A relation R on a set A is called *irreflexive* if $(a, a) \notin R$ for every element $a \in A$.

  Symmetric Relation: A relation R on a set A is called *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

  Antisymmetric Relation: A relation R on a set A is called *antisymmetric* if $(b, a) \in R$ and $(a, b) \in R$ implies $a = b$ for all $a, b \in A$.

  Transitive Relation: A relation R on a set A is called *transitive* if $(a, b) \in R$ and $(b, c) \in R$ implies $(a, c) \in R$ for all $a, b, c \in A$.

Connectivity

- **Lemma**: Let A be a set with n elements, and R a relation on A. If there is a path from a to b with $a \neq b$, then there exists a path of length $\leq n-1$.

$$R^* = \bigcup_{k=1}^{n} R^k$$

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Theorem: The transitive closure of a relation R equals the connectivity relation R^*.

Recall Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

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  R has the following pairs:

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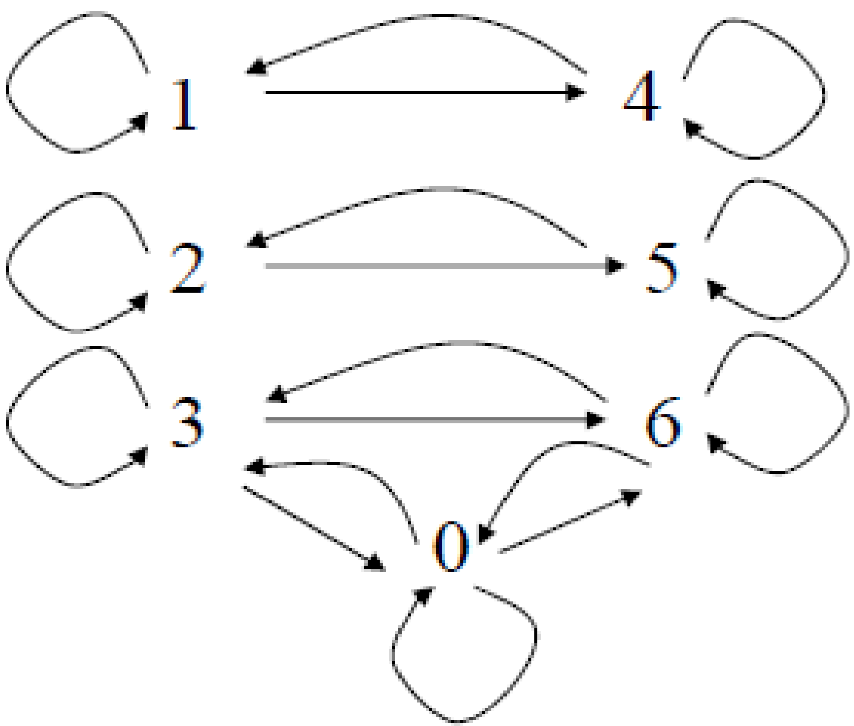

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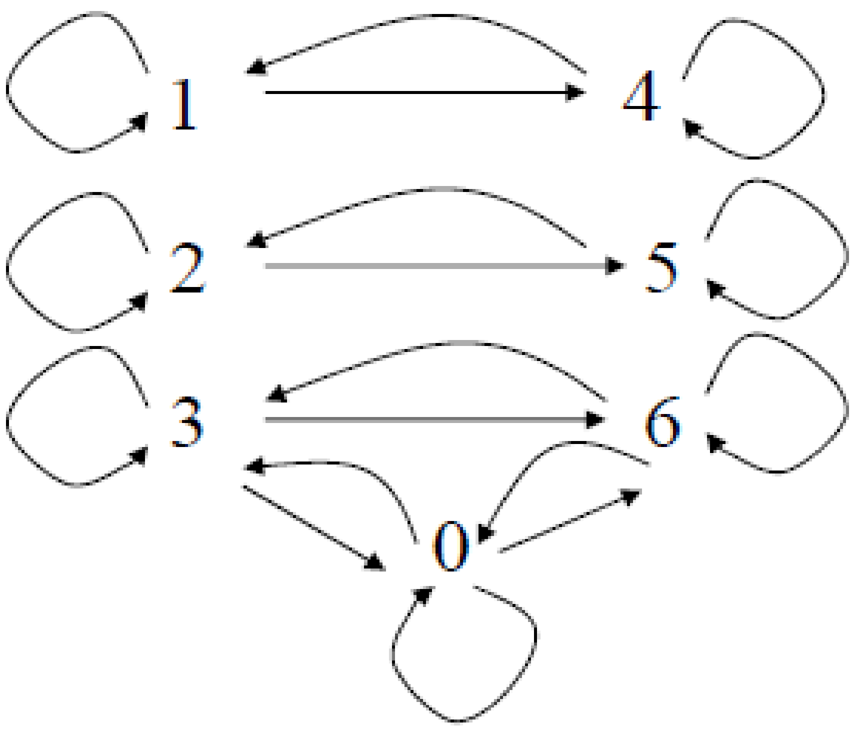

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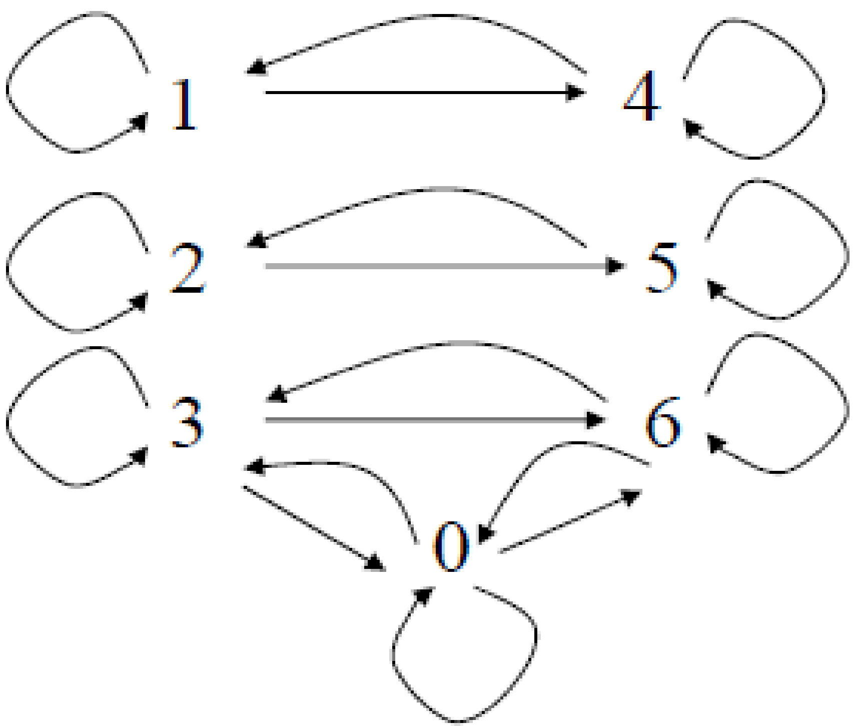

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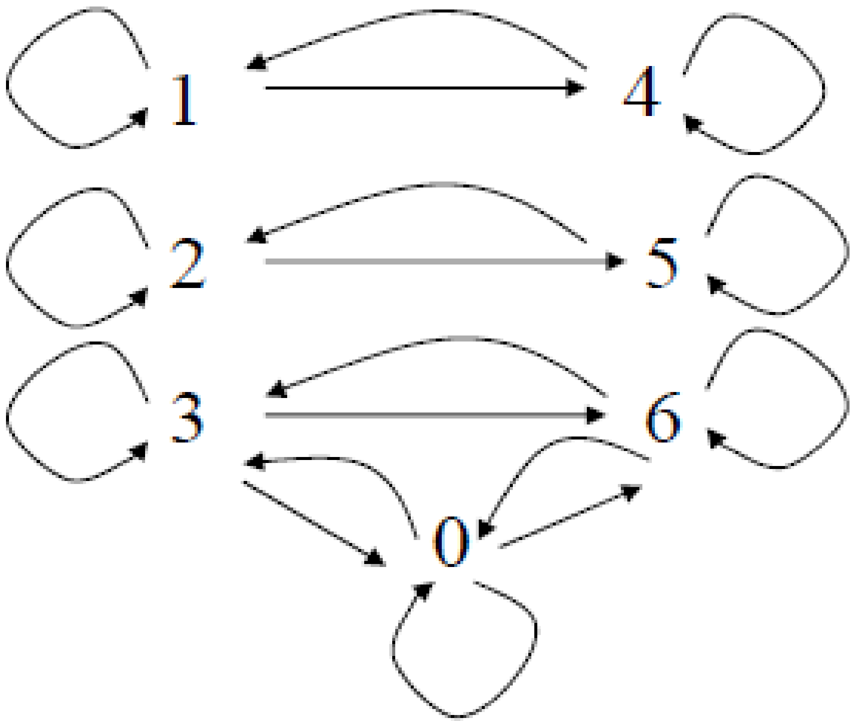

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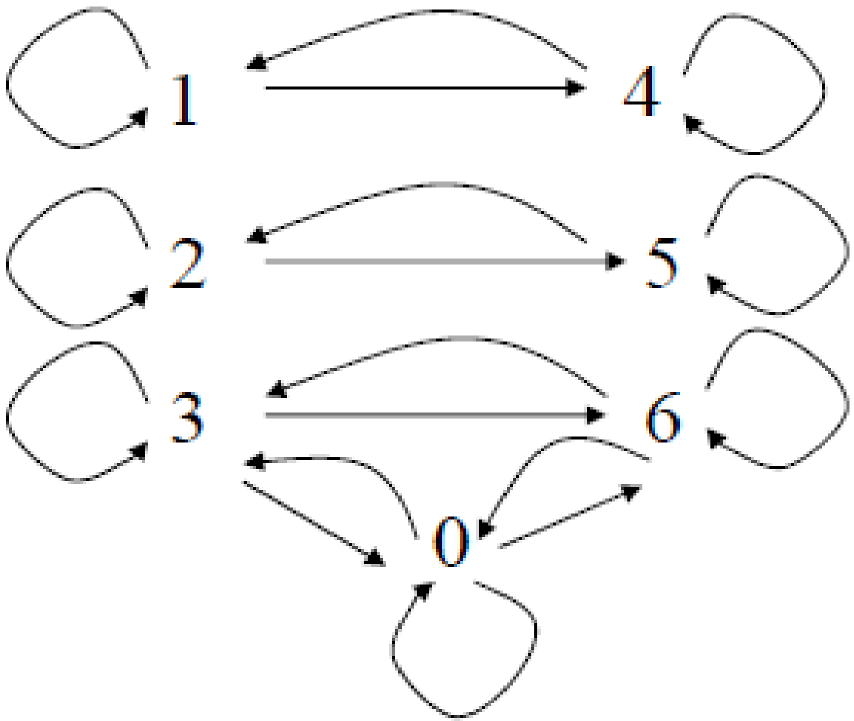

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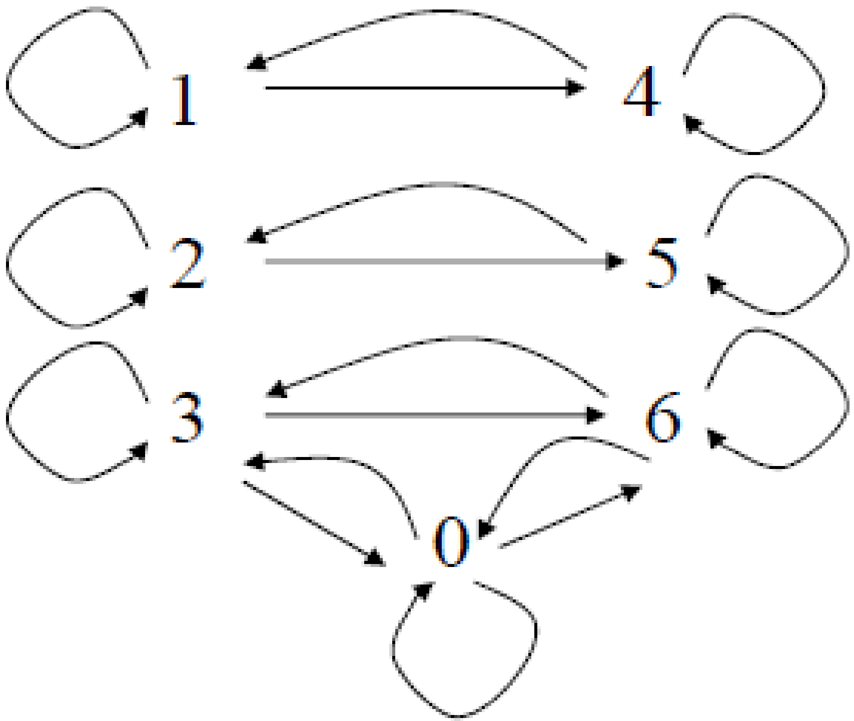

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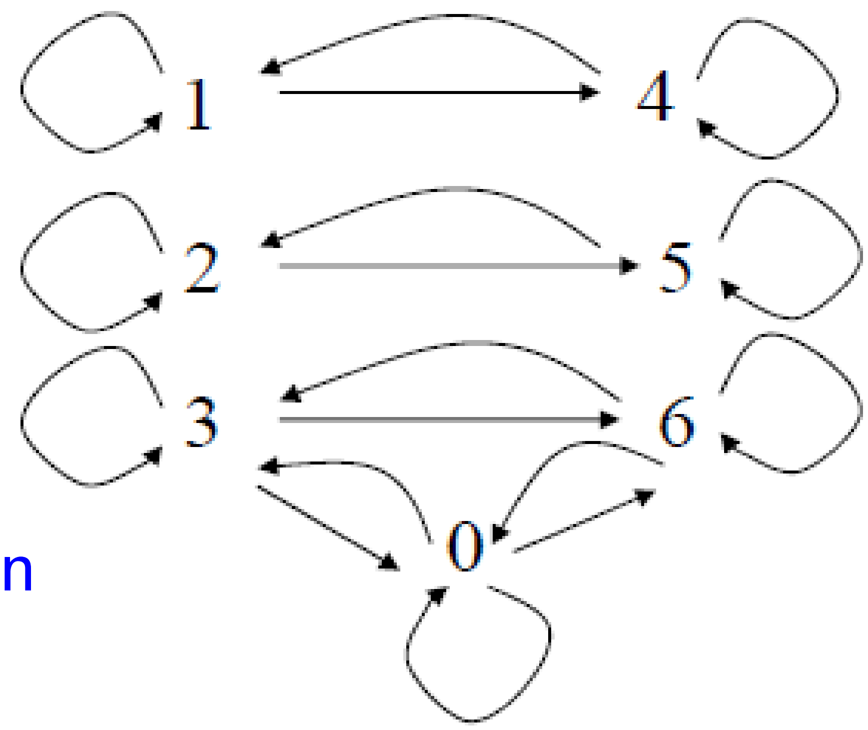

R is an equivalence relation

Examples of Equivalence Relations

- **Examples**

  "Strings a and b have the same length."

  "Integers a and b have the same absolute value."

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

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  "The relation $\geq$ between real numbers."

  "has a common factor greater than 1 between natural numbers."

Equivalence Class

- **Definition** Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a, denoted by $[a]_R$. When only one relation is considered, we use the notation $[a]$.

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$$[0] = [3] = [6] = \{0, 3, 6\}$$
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  "Strings a and b have the same length."

  $[a]$ = the set of all strings of the same length as a

  "Integers a and b have the same absolute value."

  $[a]$ = the set $\{a, -a\}$

  "Real numbers a and b have the same fractional part (i.e., $a - b \in \mathbf{Z}$)."

  $[a]$ = the set $\{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

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- **Theorem** Let R be an equivalence relation on a set A. The following statements are equivalent:

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  $(iii) \rightarrow (i)$: there exists a c s.t. $c \in [a]$ and $c \in [b]$

Partition of a Set S

- **Definition** Let S be a set. A collection of nonempty subsets of S $A_1, A_2, \ldots, A_k$ is called *a partition of S* if:

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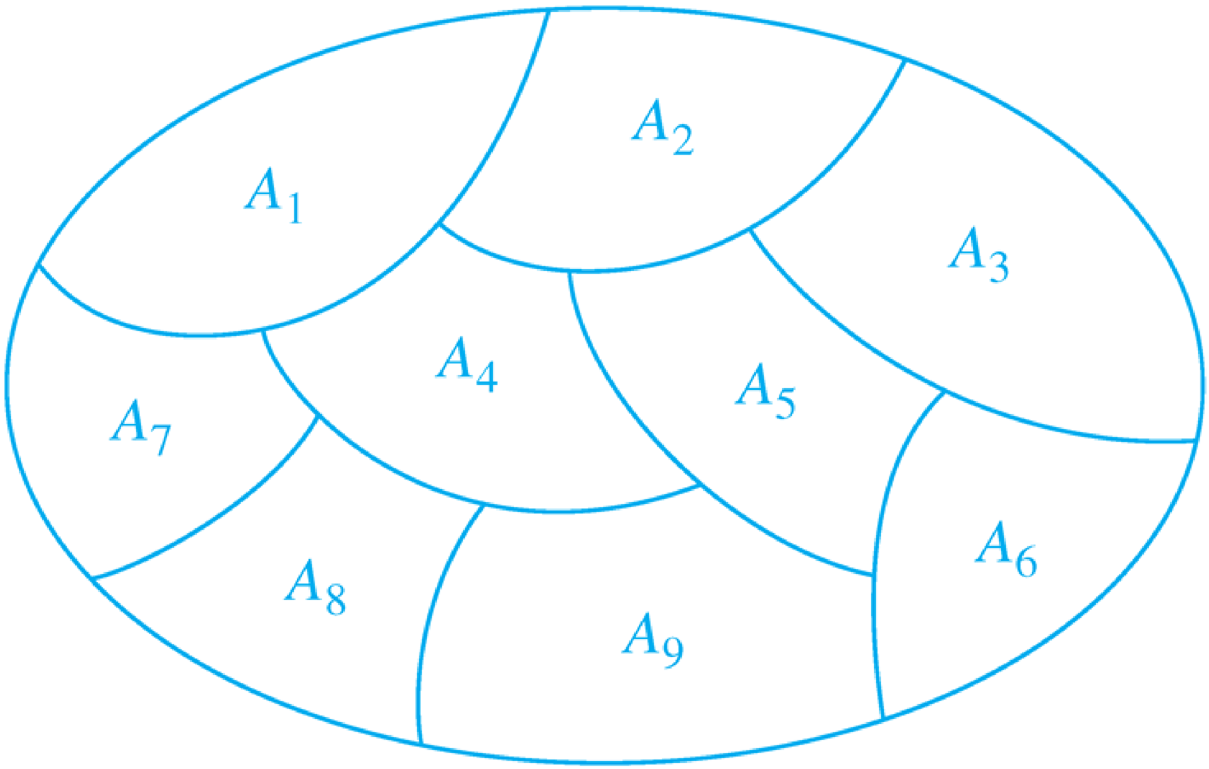

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Is A_1, A_2, A_3 a partition of S?

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Theorem Let $\{A_1, A_2, \ldots, A_i, \ldots\}$ be a partition of S. Then there is an equivalence relation R on S, that has the sets A_i as its equivalence classes.

Partial Ordering

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  $2, 4$ are comparable, $3, 5$ are incomparable.

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S is a chain.

Lexicographic Ordering

- **Definition** Given two posets $(A_1, \preccurlyeq_1)$ and $(A_2, \preccurlyeq_2)$, the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2), i.e., $(a_1, a_2) \preccurlyeq (b_1, b_2)$, either if $a_1 \prec_1 b_1$ or if $a_1 = b_1$ then $a_2 \preccurlyeq_2 b_2$.

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  - $discreet \prec discrete$
  - $discreet \prec discreetness$

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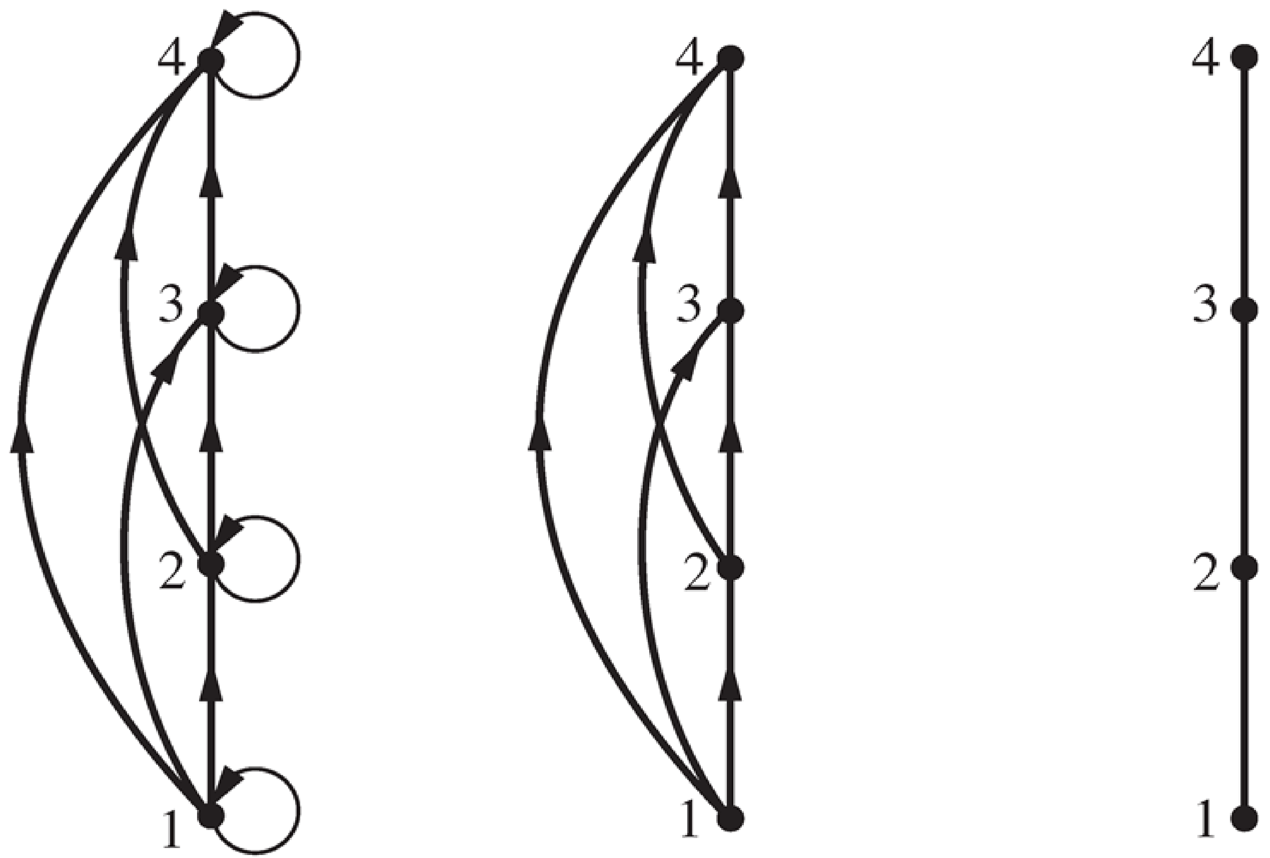
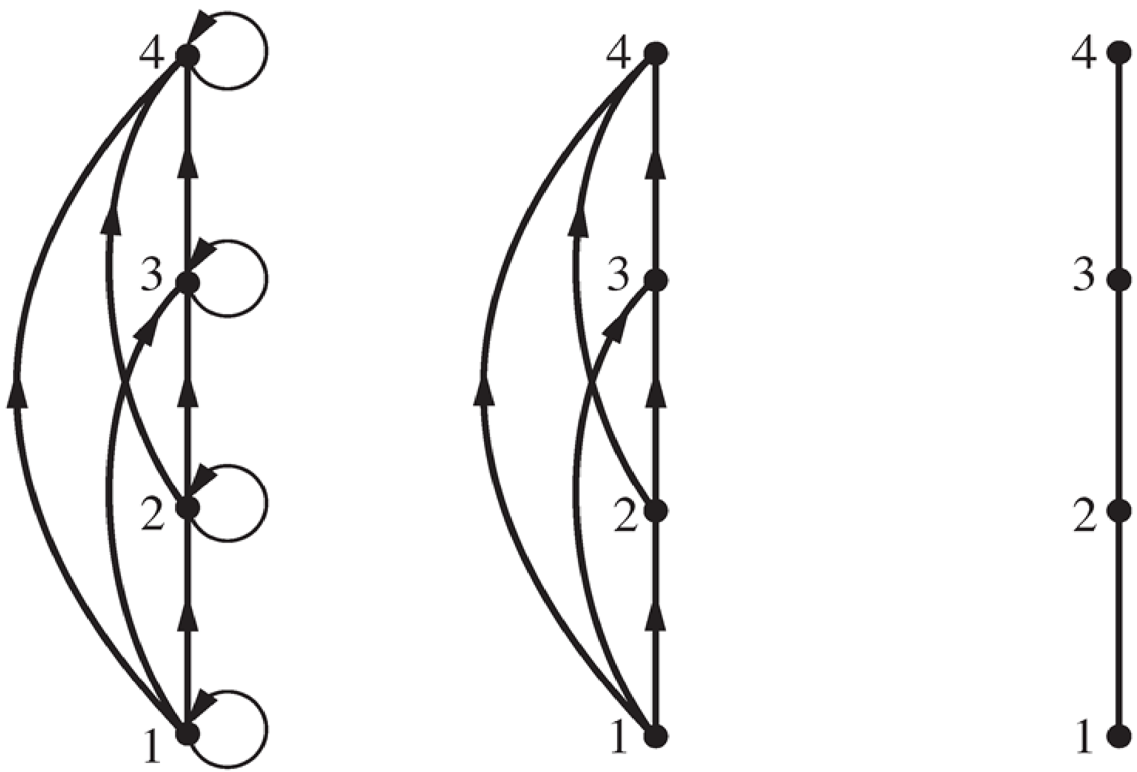


Hasse Diagram

- (a) A partial ordering. The loops are due to the reflexive property

  (b) The edges that must be present due to the transitive property are deleted

  (c) The Hasse diagram for the partial ordering (a)



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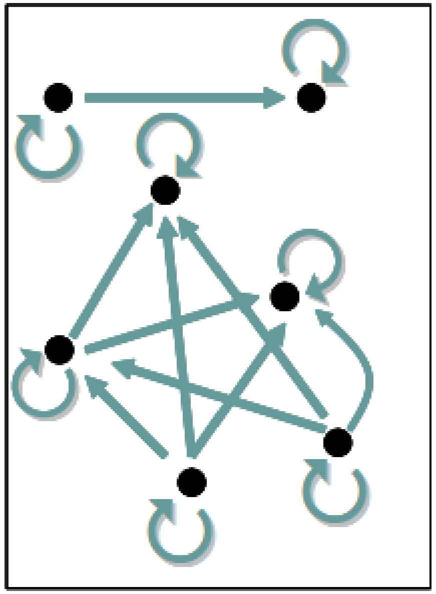
  ◇ Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.

Hasse Diagram Example



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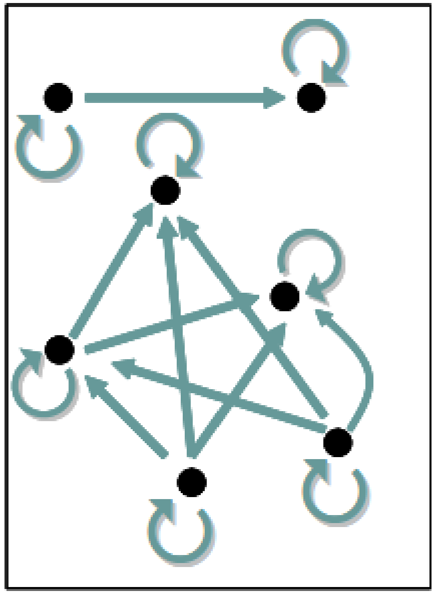

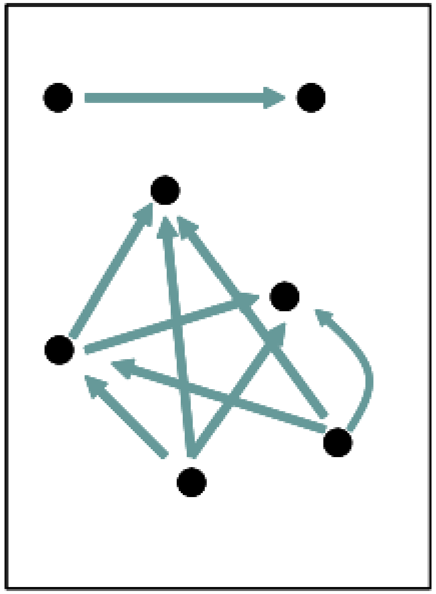

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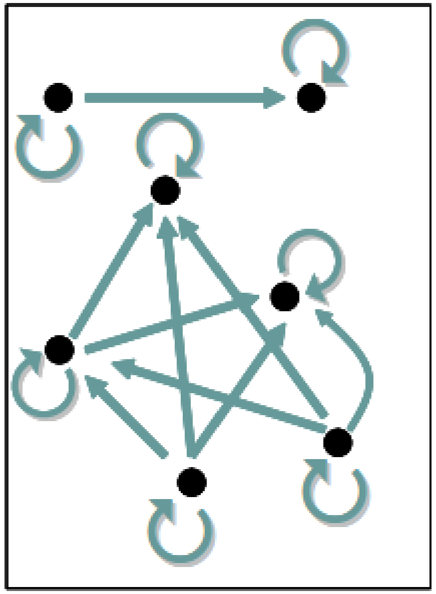

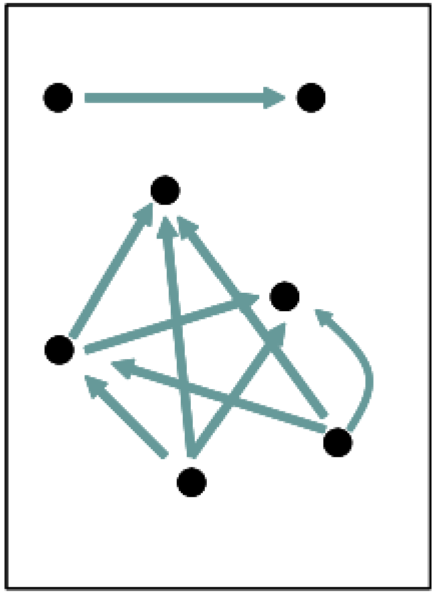

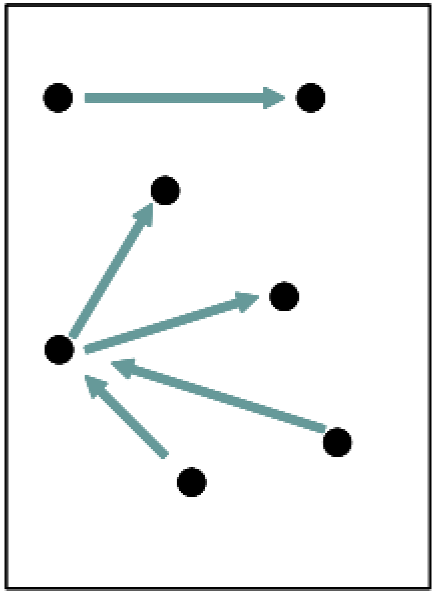

Maximal and Minimal Elements

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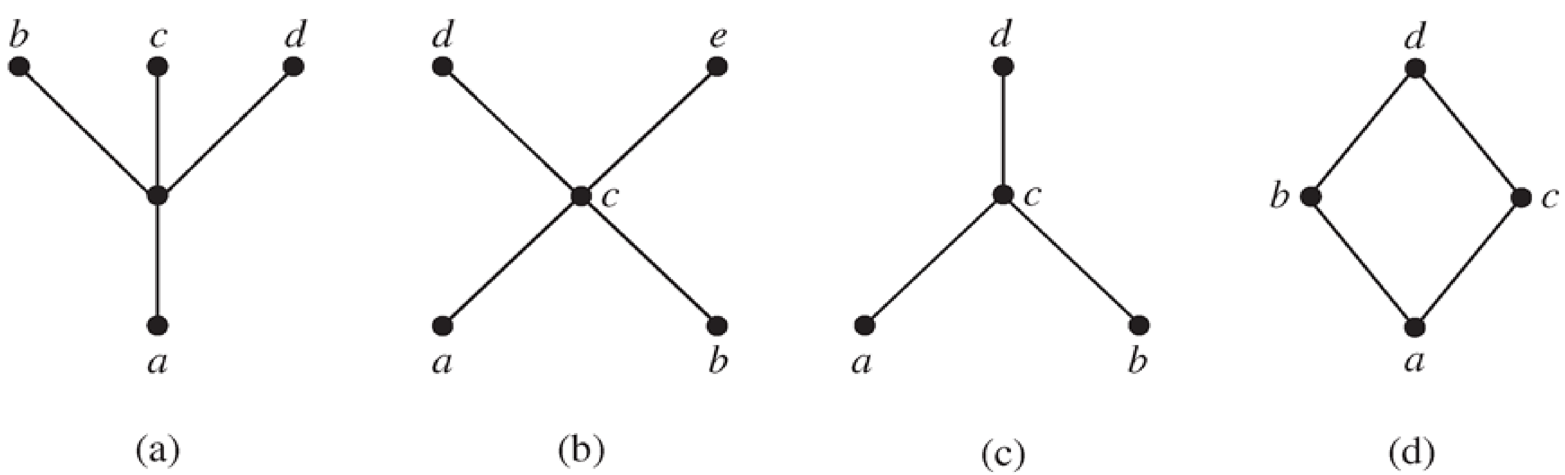

Maximal and Minimal Elements

- **Definition** Let A be a subset of a poset $(S, \preccurlyeq)$.
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Example Find the greatest lower bound and the least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(\mathbf{Z}^+, |)$.

Well-Ordered Set

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p. 620, Theorem 1

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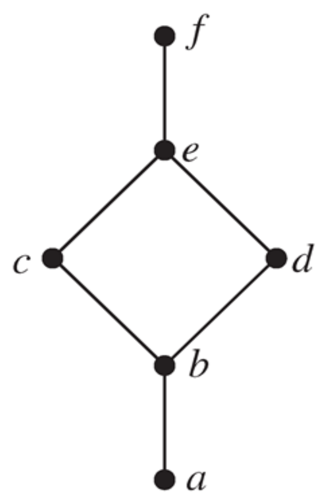

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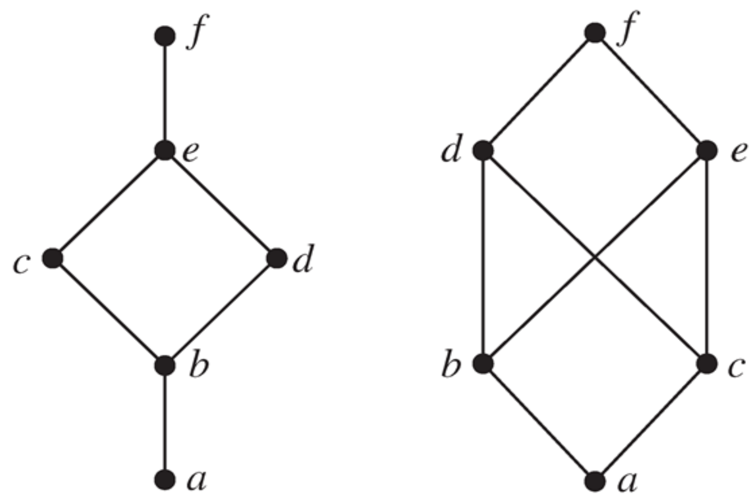

Example Determine whether the posets $(\{1, 2, 3, 4, 5\}, |)$ and $(\{1, 2, 4, 8, 16\}, |)$ are lattices.

Topological Sorting

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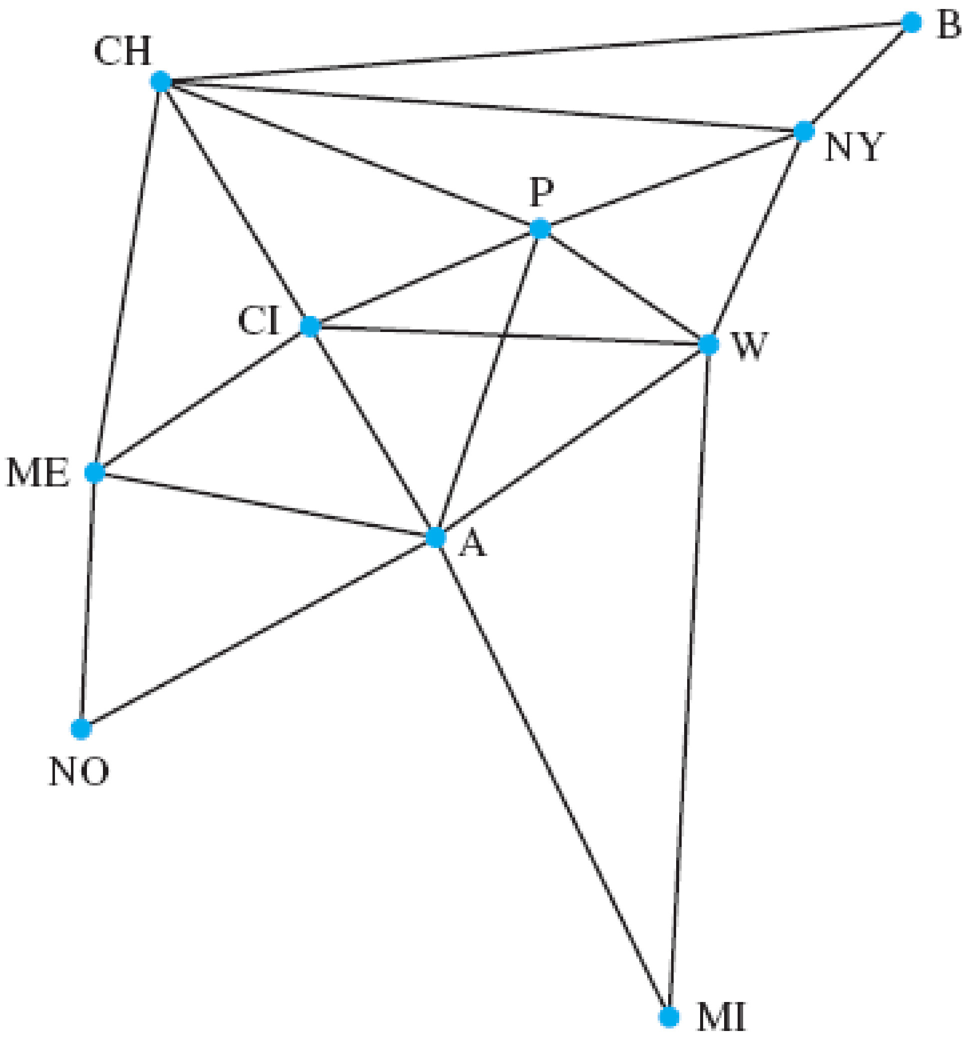
  Topological sorting: Given a partial ordering R, find a total ordering $\preccurlyeq$ such that $a \preccurlyeq b$ whenever $a \; R \; b$. $\preccurlyeq$ is said *compatible with R*.

Topological Sorting for Finite Posets

procedure topological_sort (S: finite poset)

$k := 1;$
while $S \neq \emptyset$
  $a_k :=$ a minimal element of S
  $S := S \setminus \{a_k\}$
  $k := k + 1$
end while
// $\{a_1, a_2, \ldots, a_n\}$ is a compatible total ordering of S

Example



What is the minimum number of links to send a message from B to NO?

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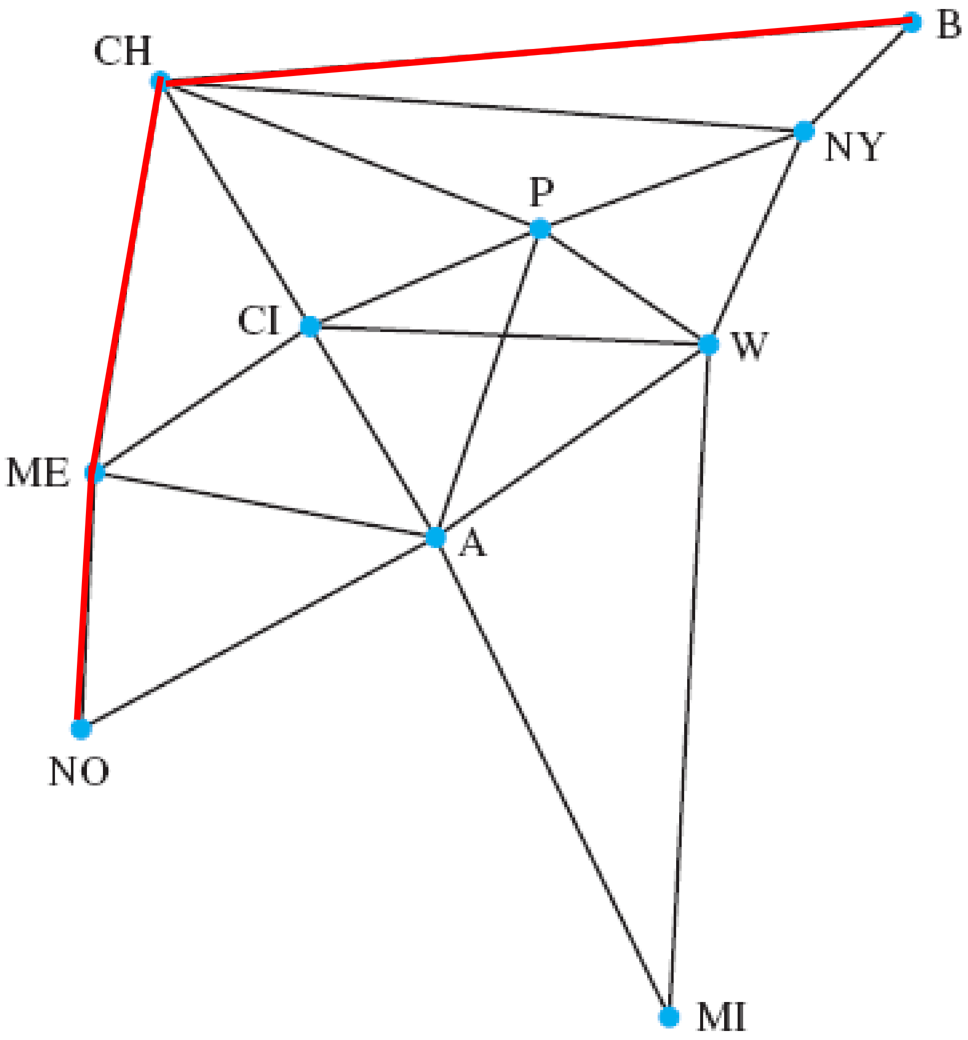

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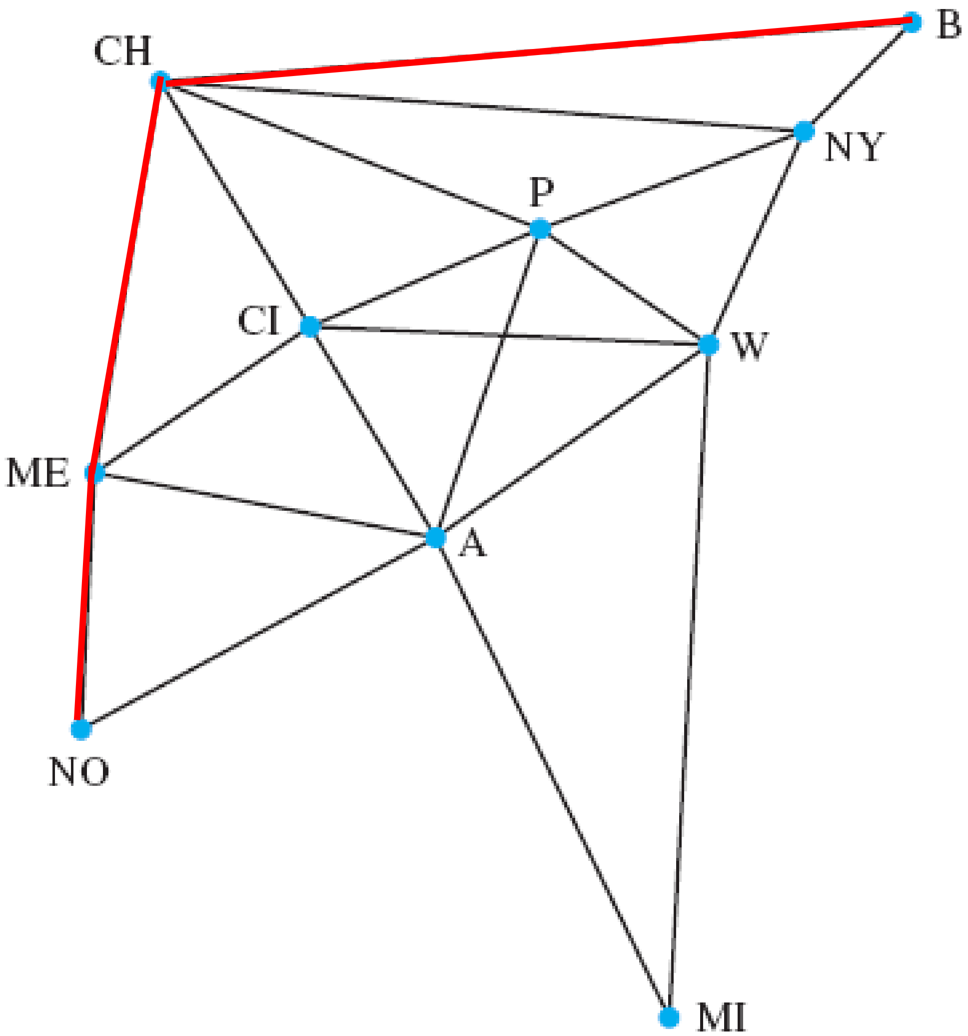

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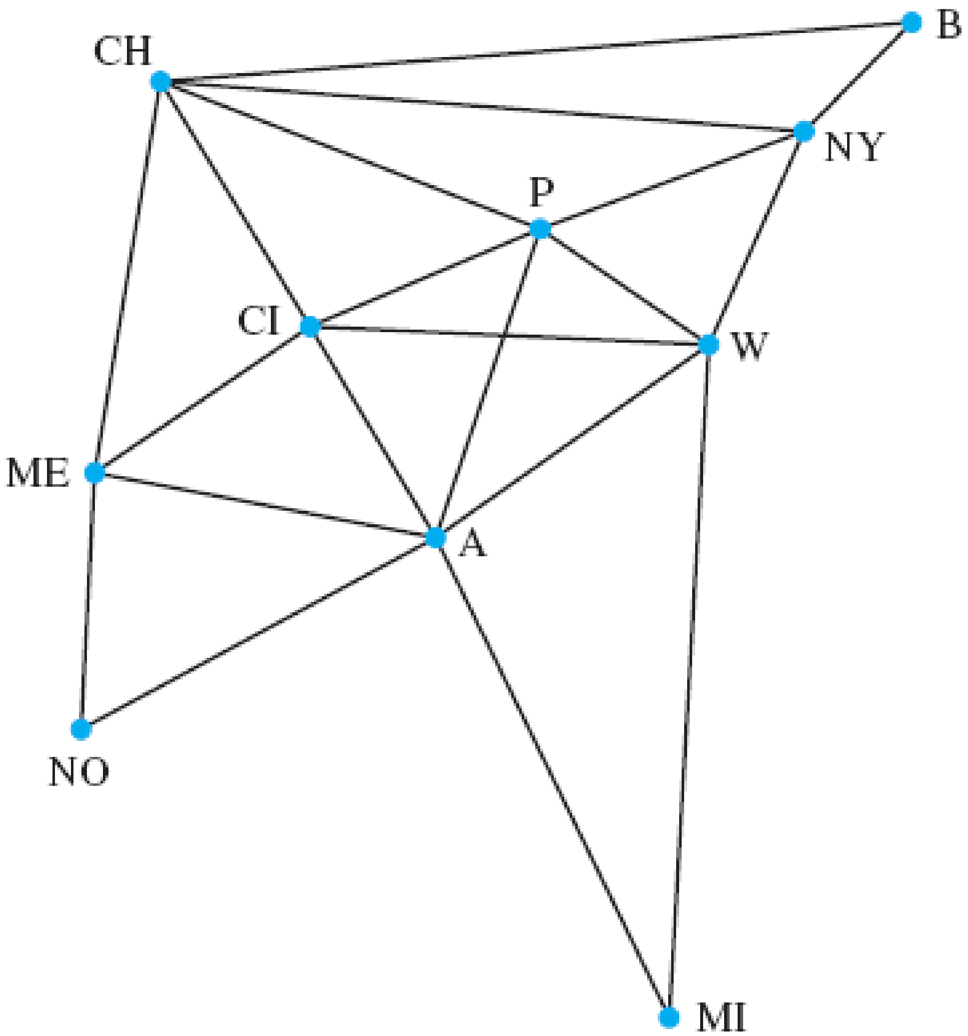

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Which city/cities has/have the most communication links emanating from it/them?

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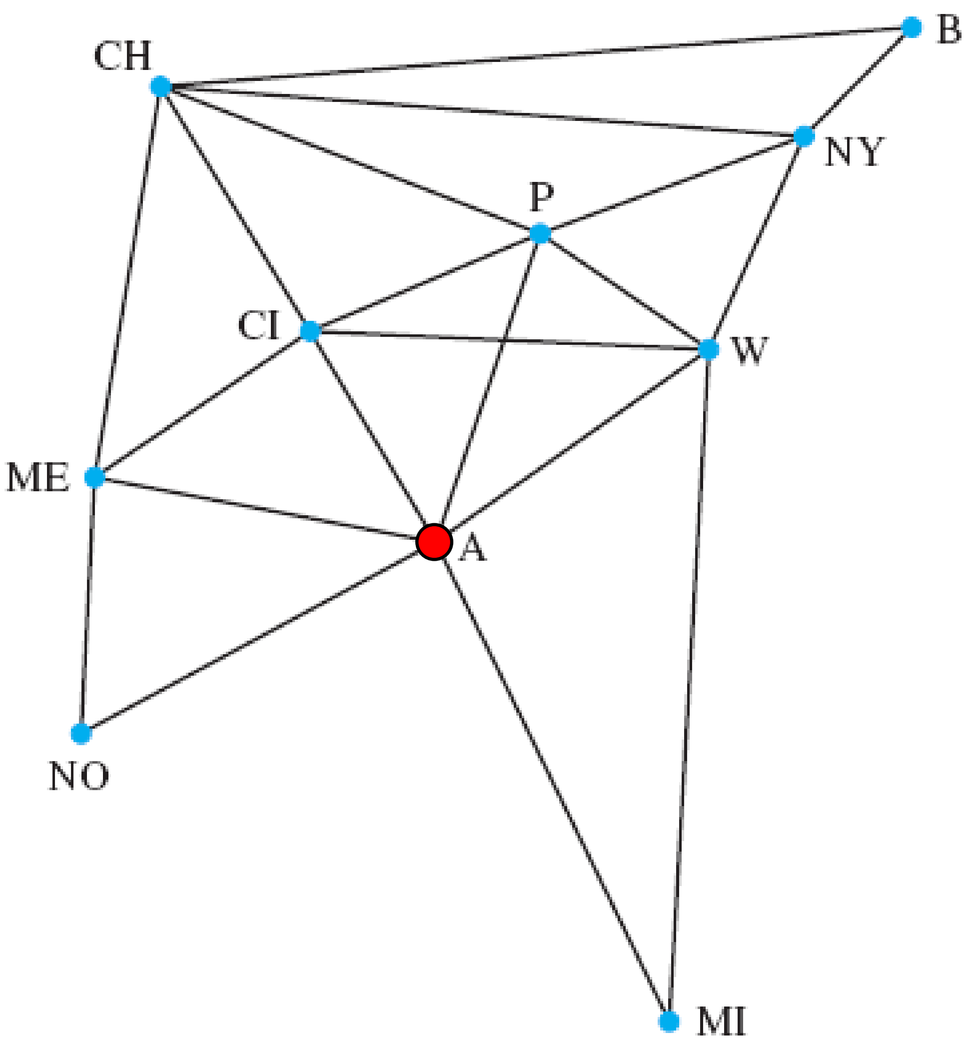

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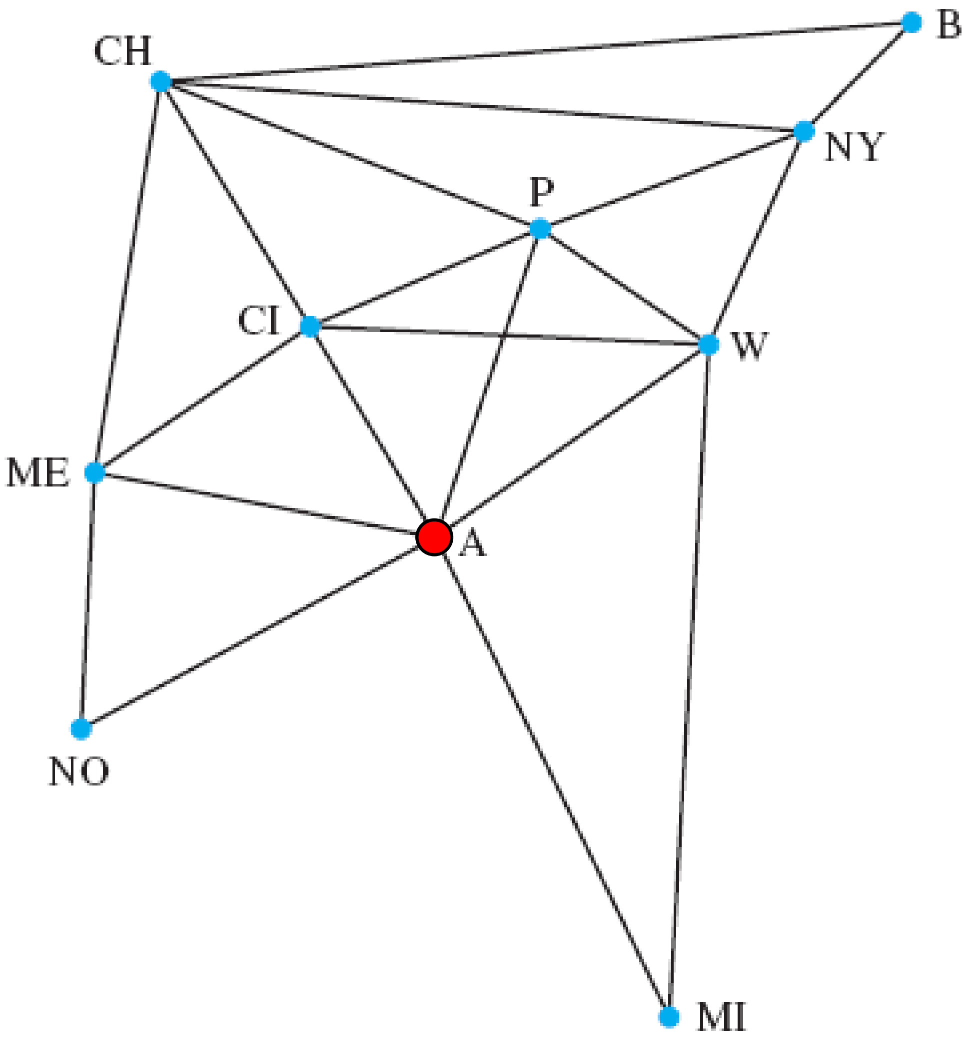

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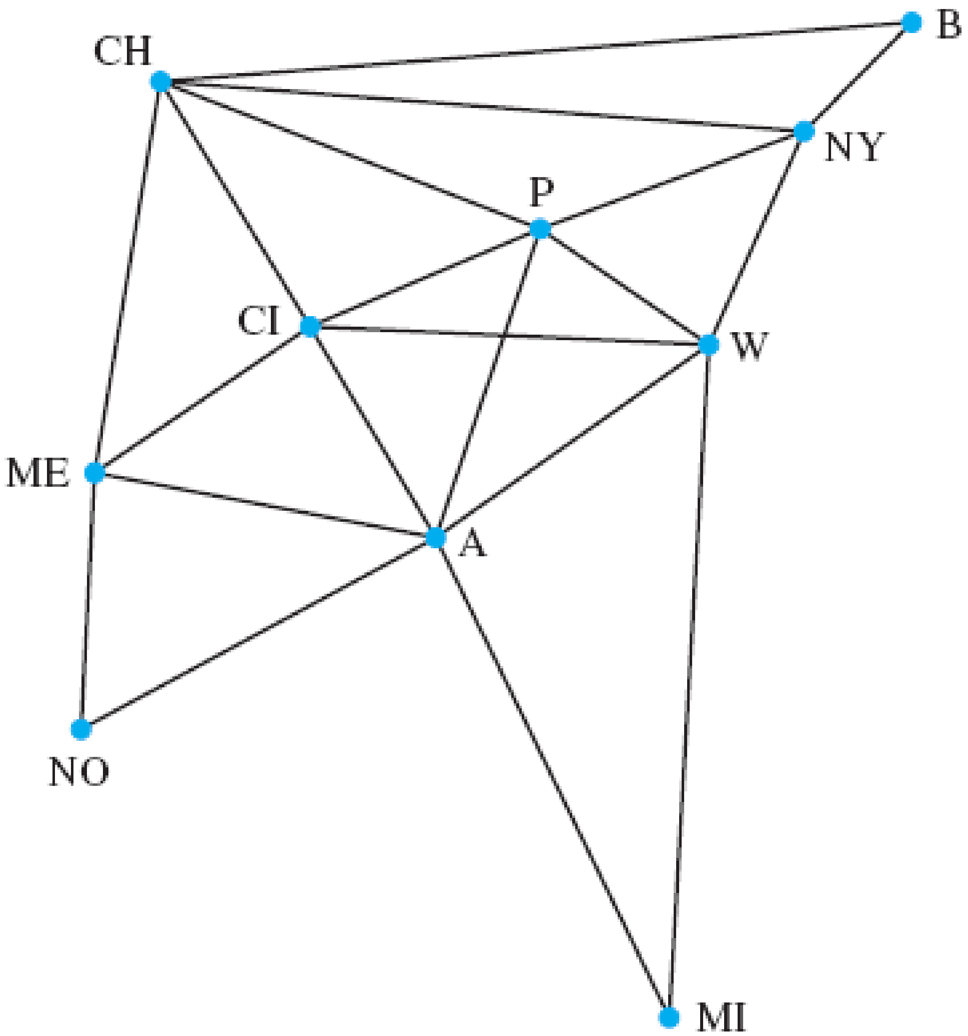

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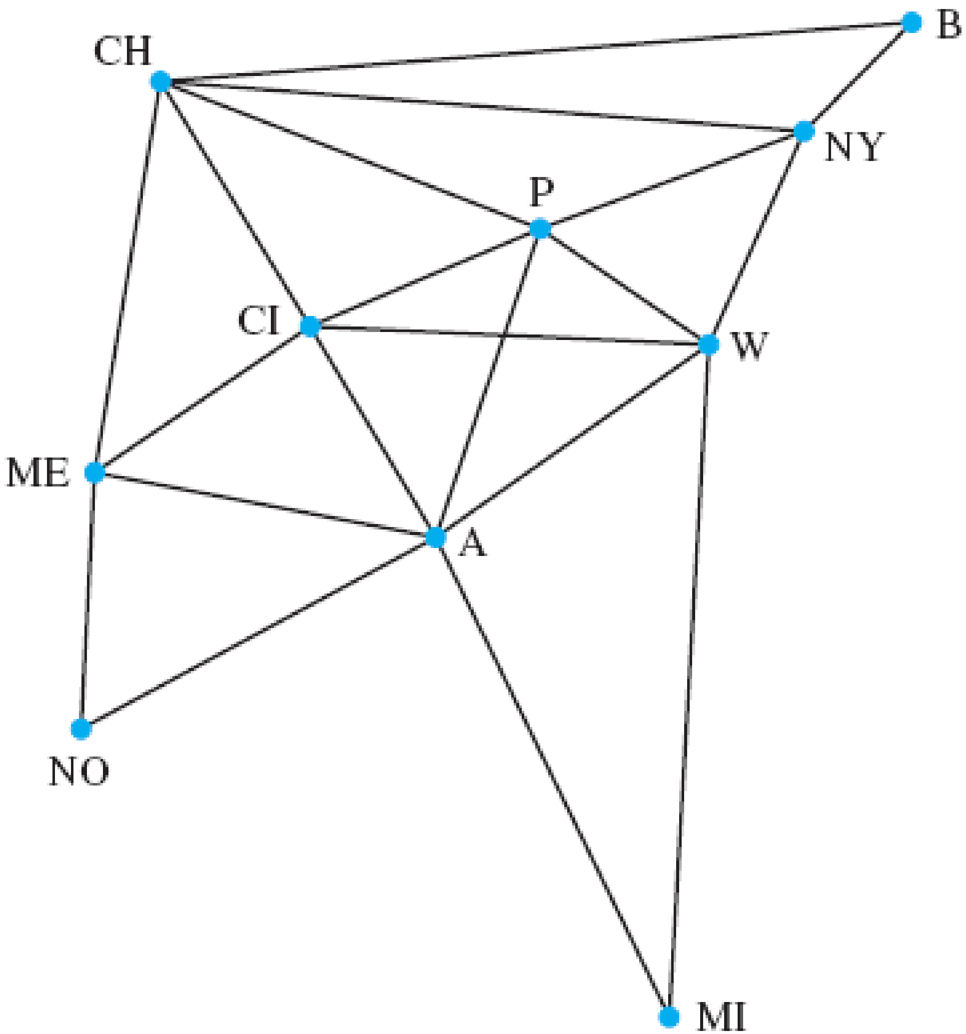

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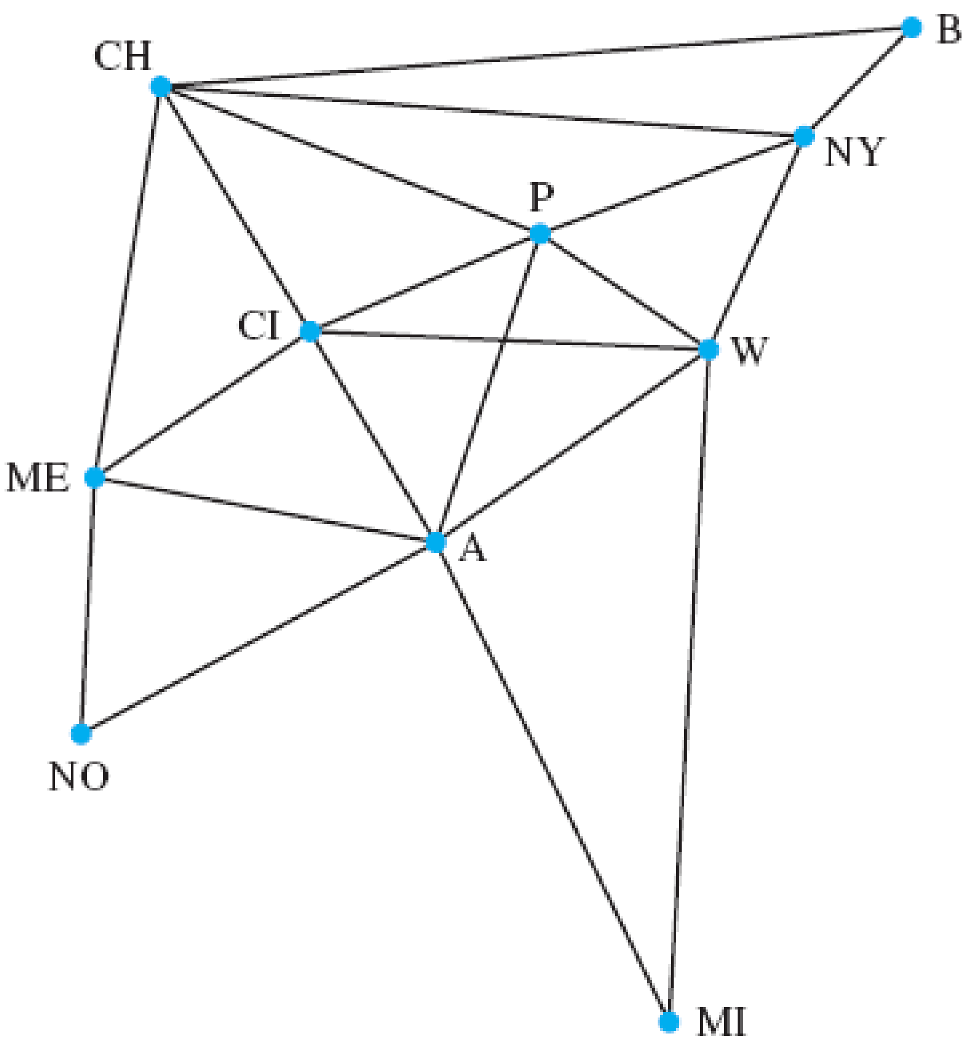

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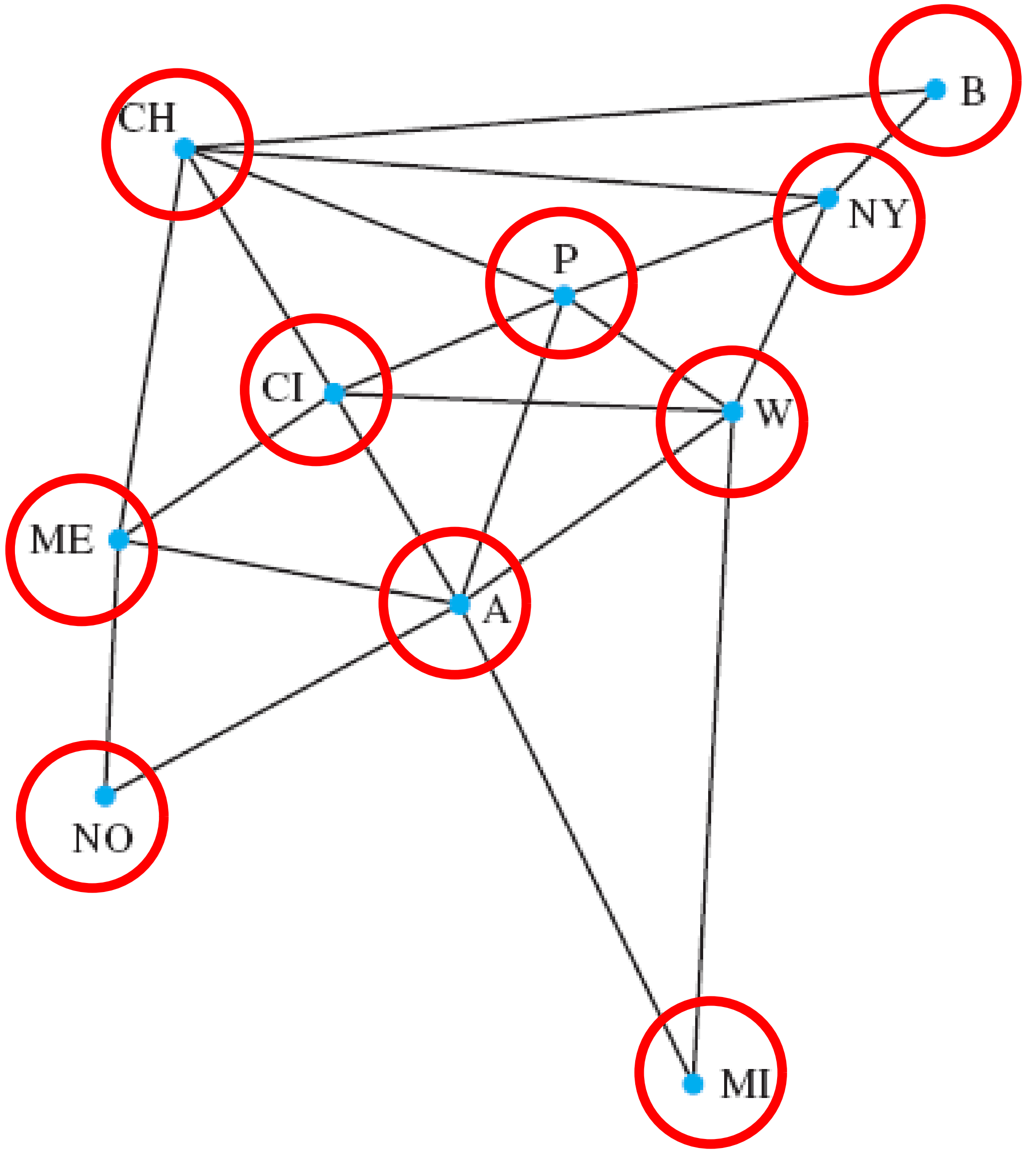

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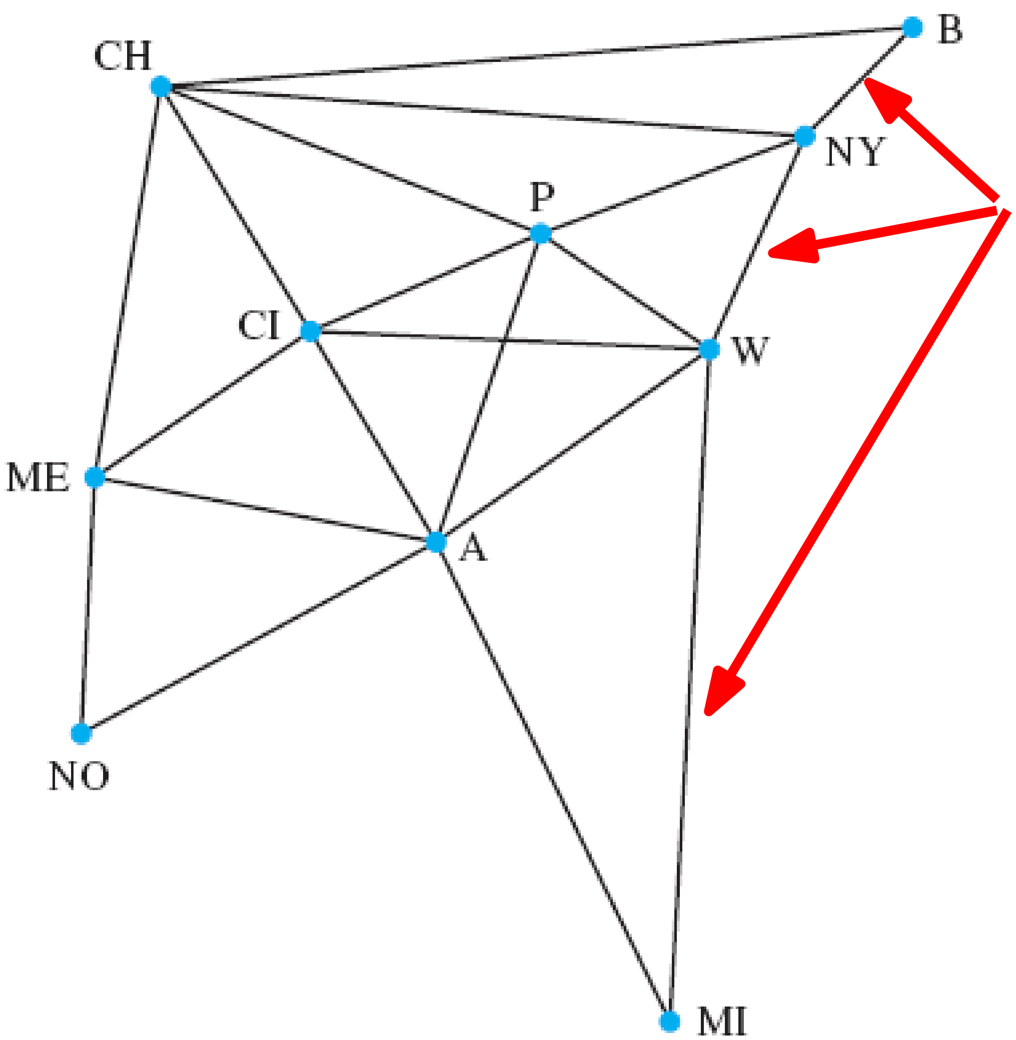

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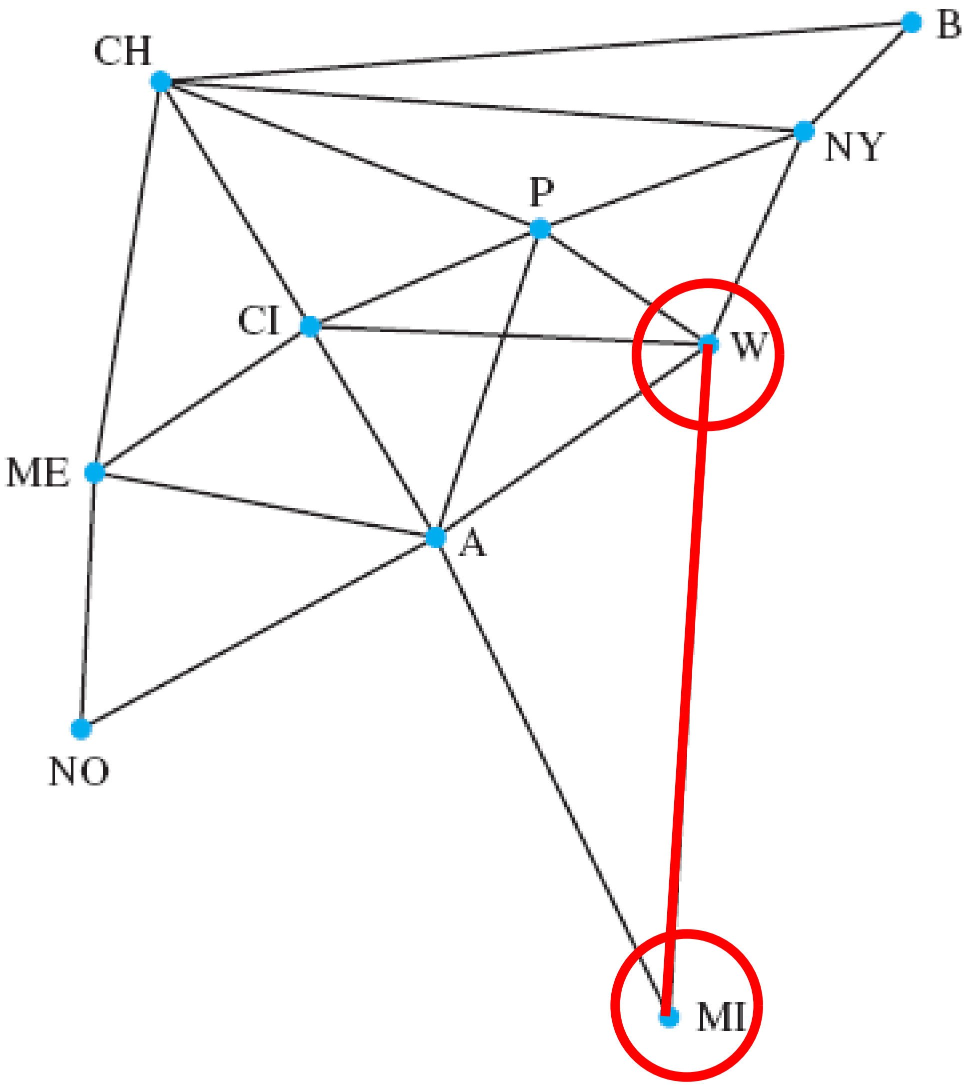

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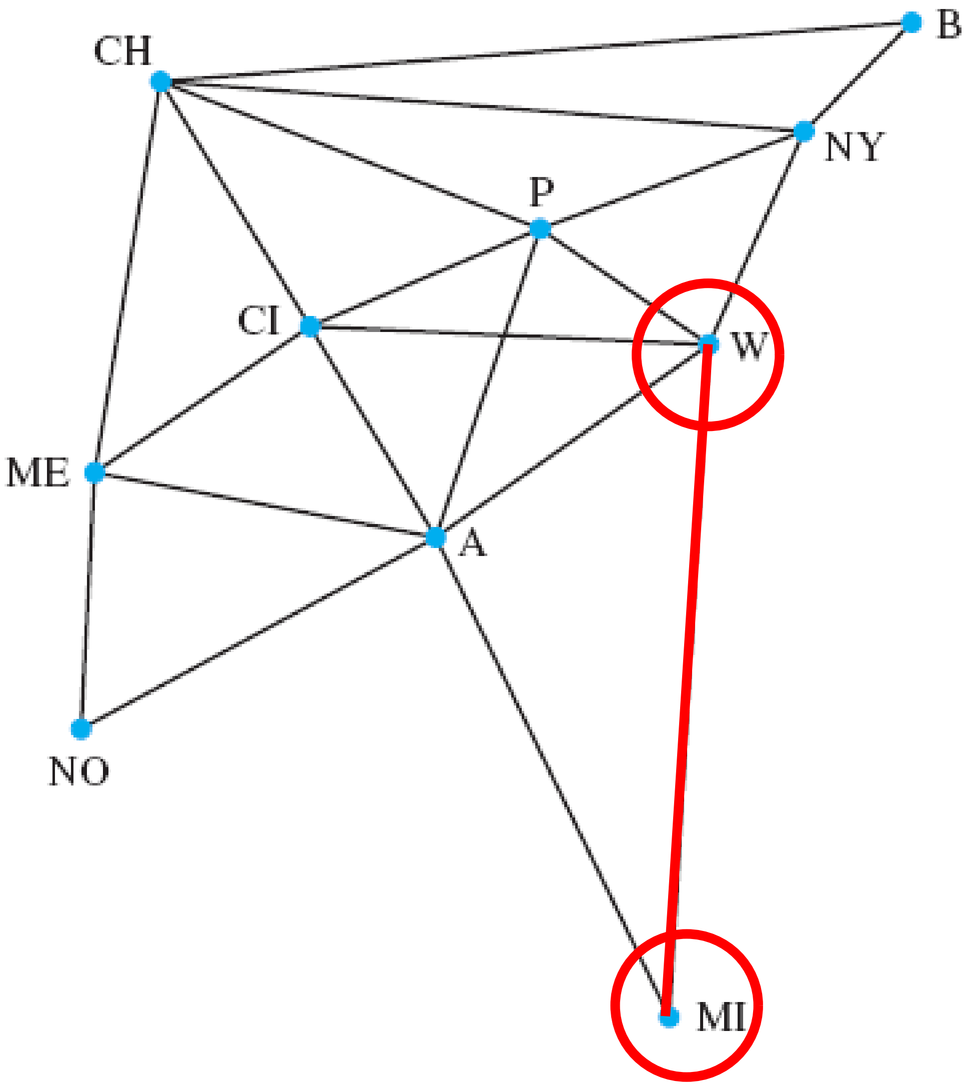

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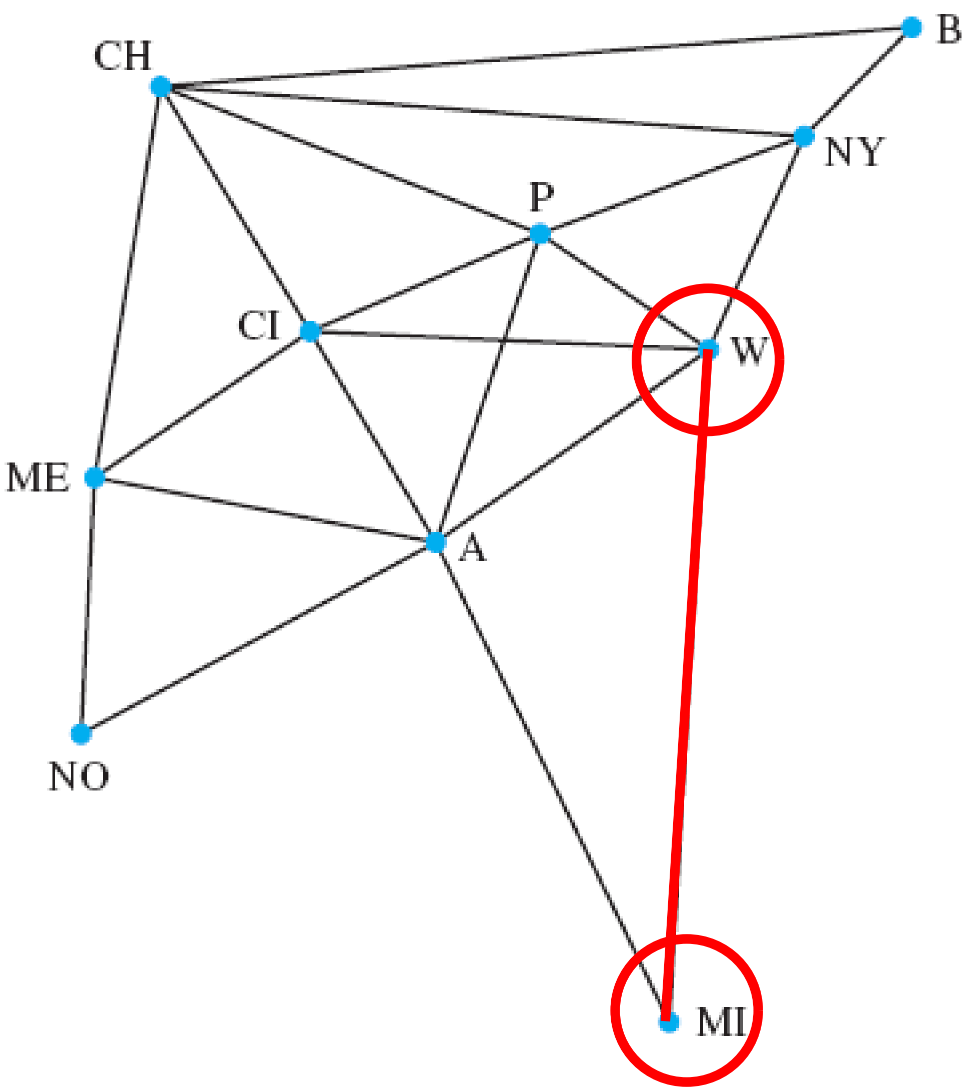

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When a vertex is an endpoint of an edge, we say that the edge and the vertex are incident to each other

Definition of a Graph

- **Definition**. A *graph* $G = (V, E)$ consists of a nonempty set V of *vertices* (or *nodes*) and a set E of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to be *incident to* (or *connect* its endpoints.

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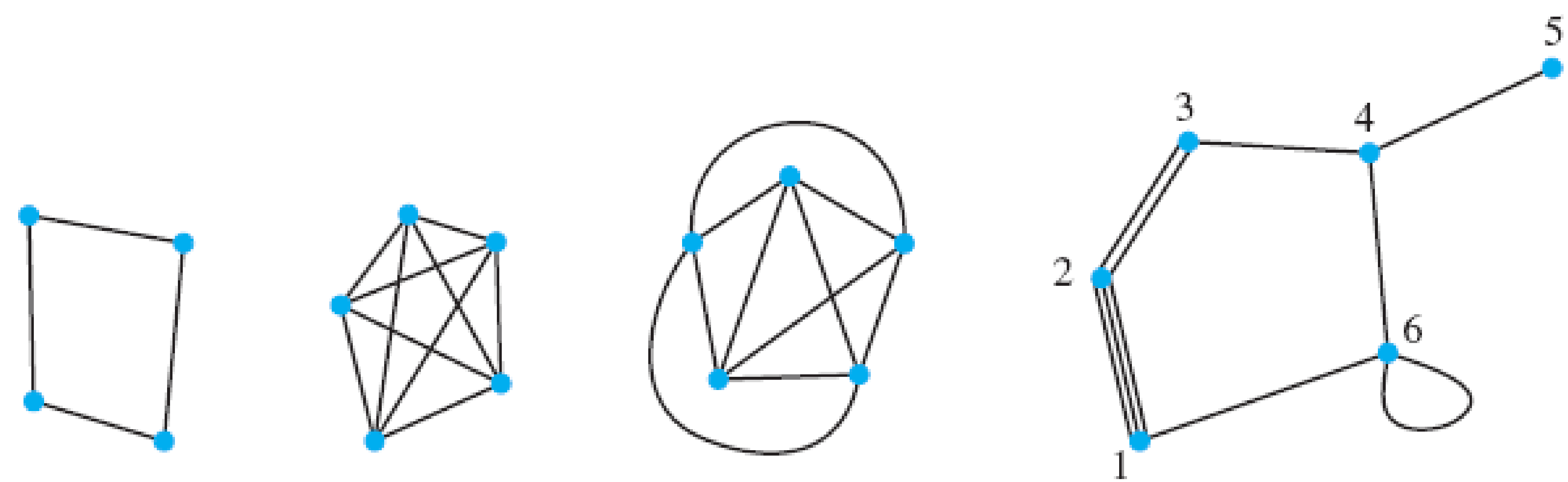

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  Complete graph K_n

  A graph with n vertices that has an edge between each pair of vertices

Graphs

- Graphs and graph theory can be used to model:
  - Computer networks
  - Social networks
  - Communication networks
  - Information networks
  - Software design
  - Transportation networks
  - Biological networks

Graph Models

- Computer Networks

  Vertices: computers
  Edges: connections

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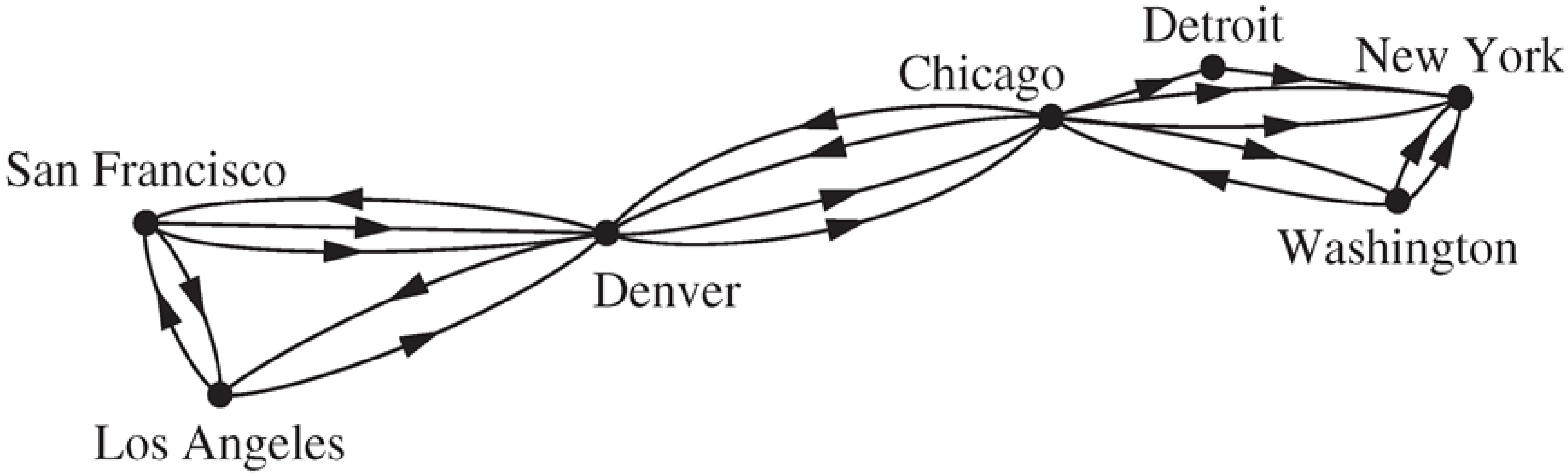

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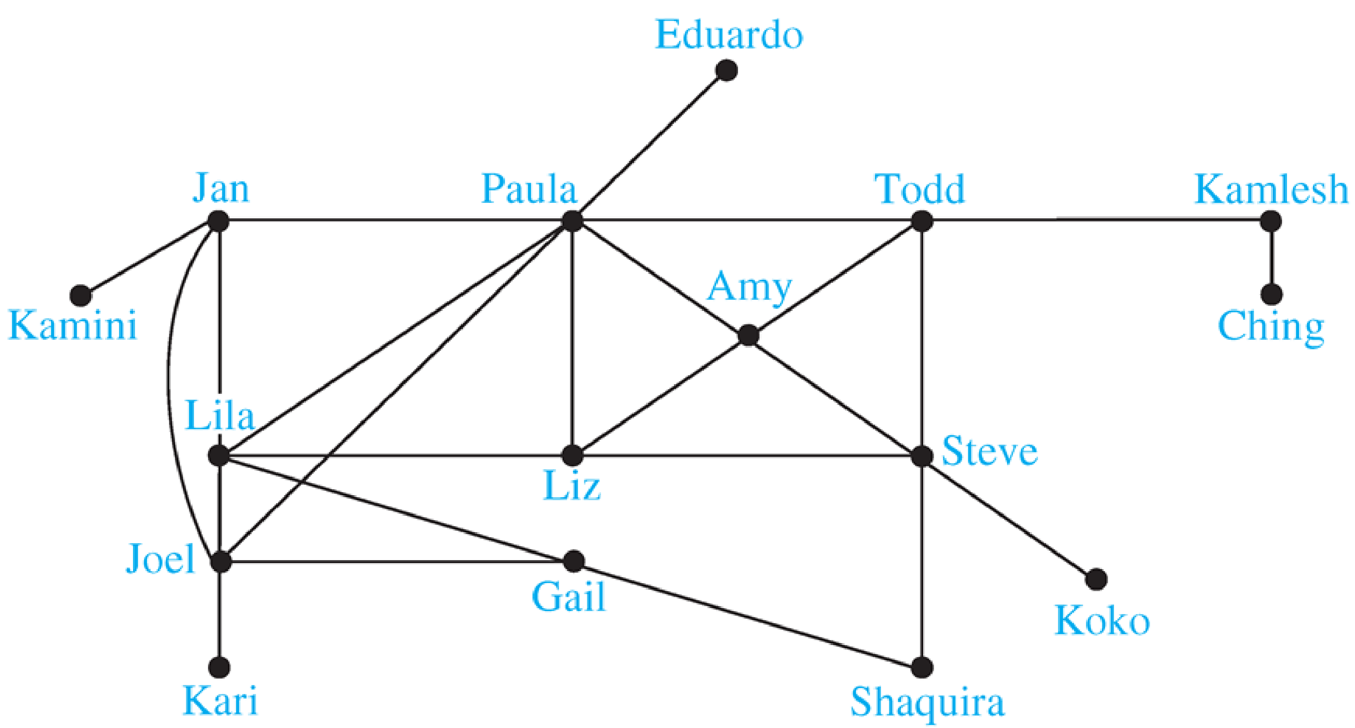

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  Example

  the Hollywood graph

  the Erdős number

The Erdős Number

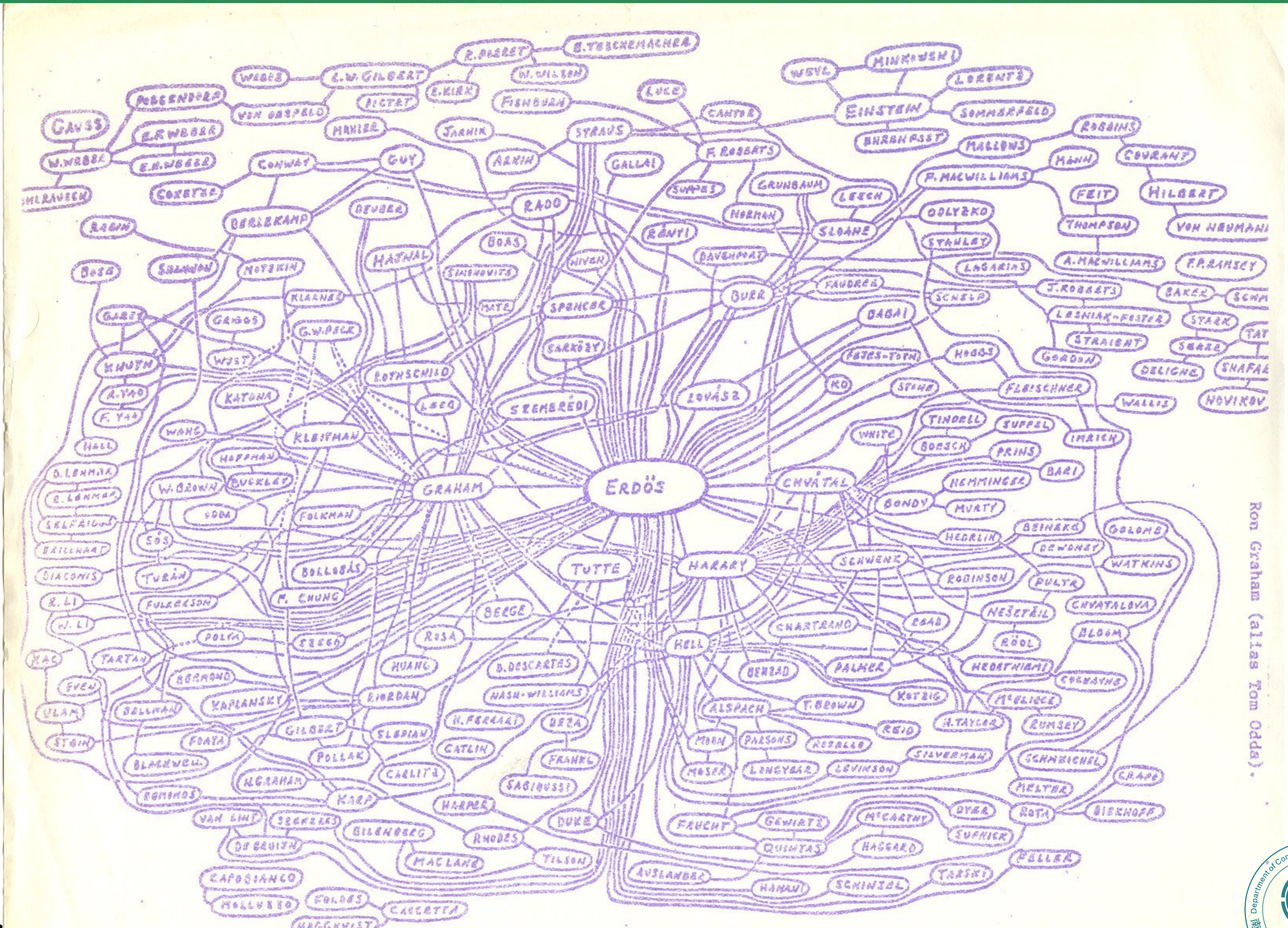
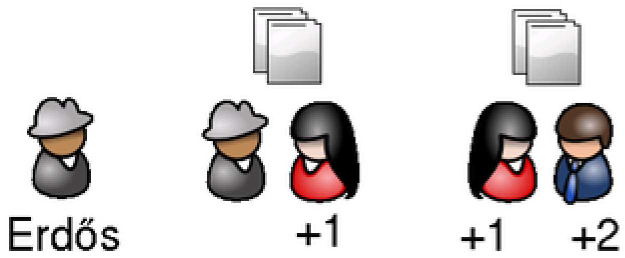


Figure 1
To appear in Topics in Graph Theory (F. Harary, ed.) New York Academy of Sciences (1979).

The Erdős Number



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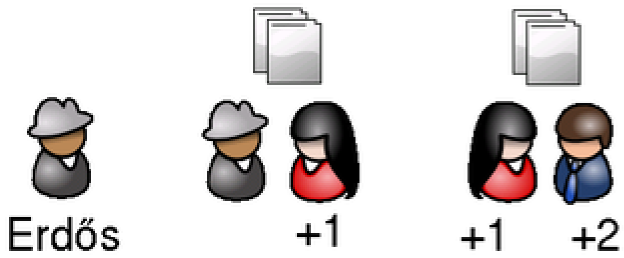

Erdös number	0	---	1 person
Erdös number	1	---	504 people
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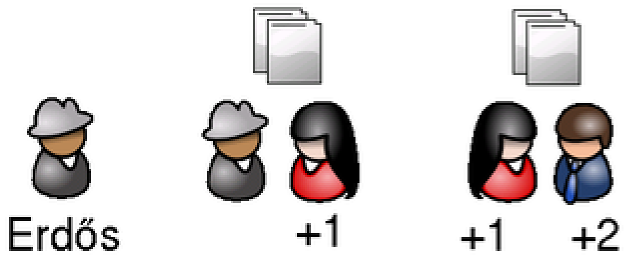


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Statistics on Mathematical Collaboration, 1903-2016

	#Laureates	#Erdős	%Erdős	Min	Max	Average	Median
Fields Medal	56	56	100.0%	2	6	3.36	3
Nobel Economics	76	47	61.84%	2	8	4.11	4
Nobel Chemistry	172	42	24.42%	3	10	5.48	5
Nobel Medicine	210	58	27.62%	3	12	5.50	5
Nobel Physics	200	159	79.50%	2	12	5.63	5

Undirected Graphs

- **Definition** Two vertices u, v in an undirected graph G are called *adjacent* (or *neighbors*) in G if there is an edge e between u and v. Such an edge e is called *incident* with the vertices u and v and e is said to connect u and v.

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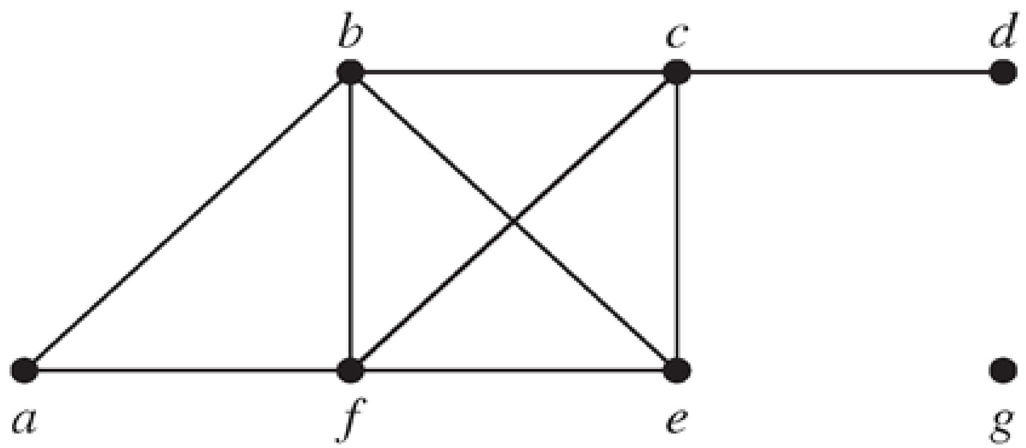
  Definition The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes two to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.

Undirected Graphs

- **Example**: What are the degrees and neighborhoods of the vertices in the graph G?



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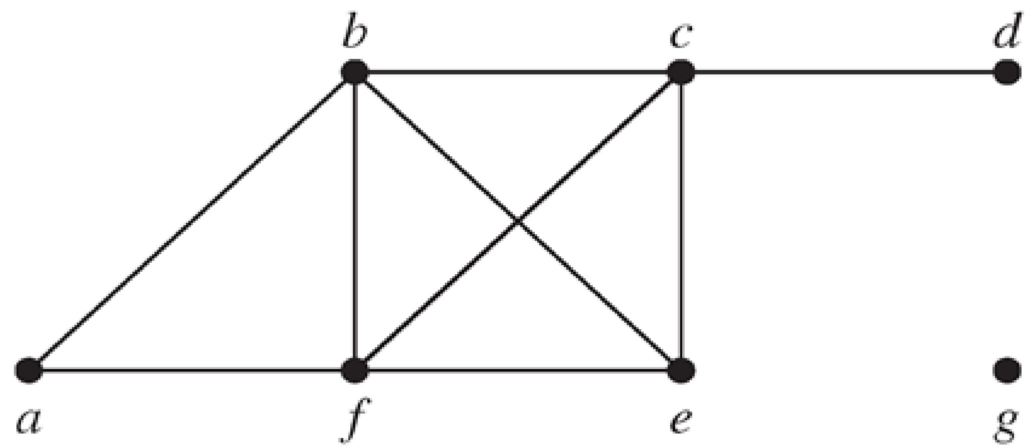

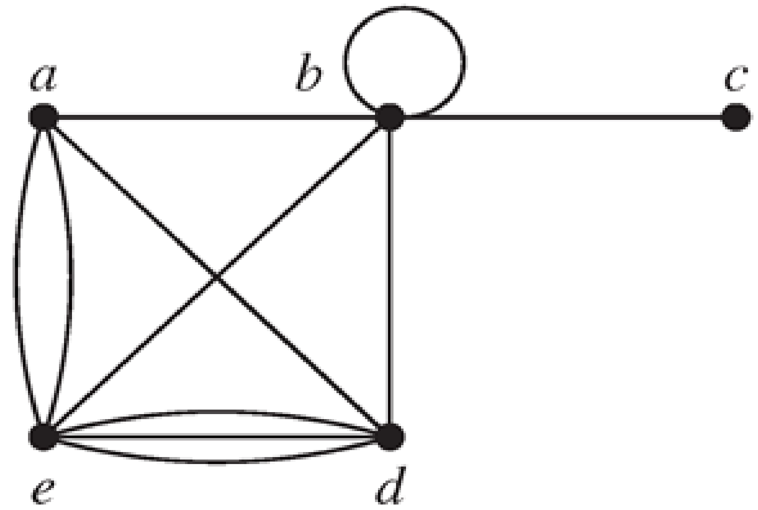

Undirected Graphs

- **Theorem 1** (Handshaking Theorem) If $G = (V, E)$ is an undirected graph with m edges, then

$$2m = \sum_{v \in V} \deg(v)$$

Proof

Undirected Graphs

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Directed Graphs

- **Definition** An *directed graph* $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of directed edges. Each edge is an ordered pair of vertices. The directed edge (u, v) is said to start at u and end at v.

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  Definition Let (u, v) be an edge in G. Then u is the *initial vertex* of the edge and is *adjacent to v* and v is the *terminal vertex* of this edge and is *adjacent from u*. The initial and terminal vertices of a loop are the same.

Directed Graphs

- **Definition** The *in-degree* of a vertex v, denoted by $\deg^-(v)$, is the number of edges which terminate at v. The *out-degree* of v, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of the vertex.

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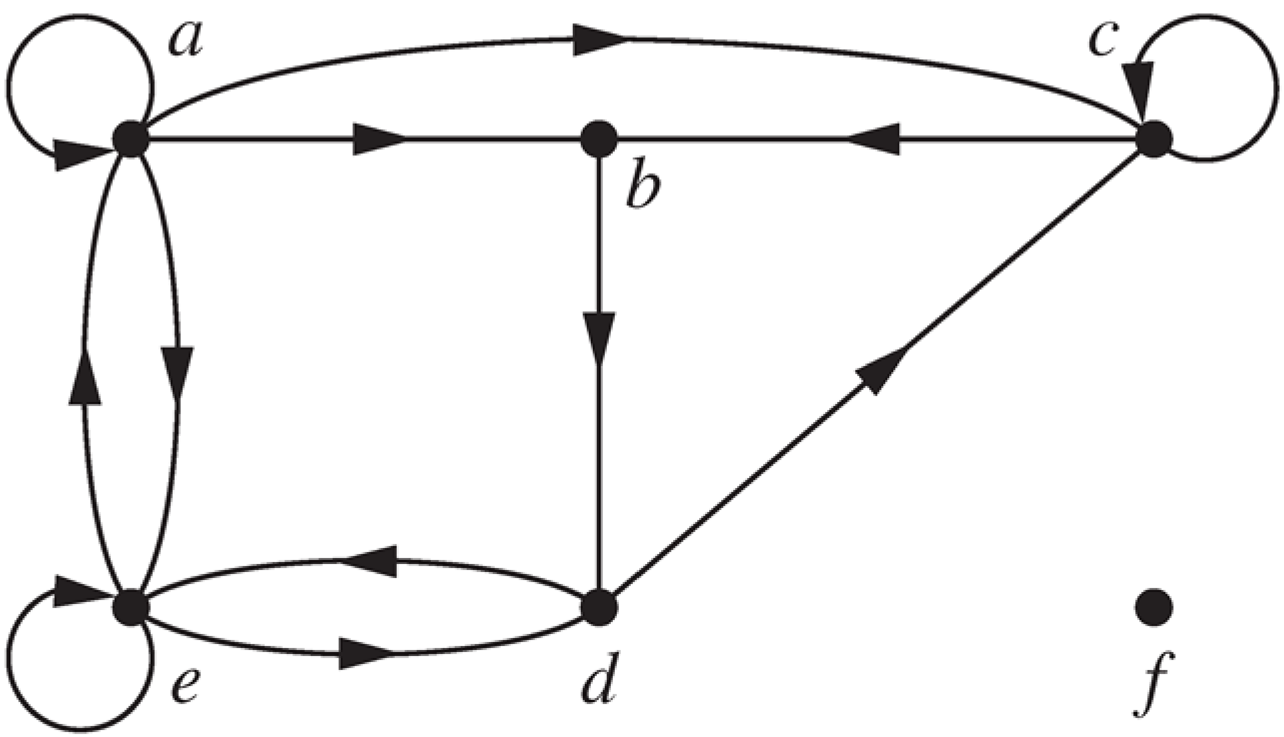

- **Theorem 3** Let $G = (V, E)$ be a graph with directed edges. Then

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v)$$

Proof

Complete Graphs

- A *complete graph* on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.

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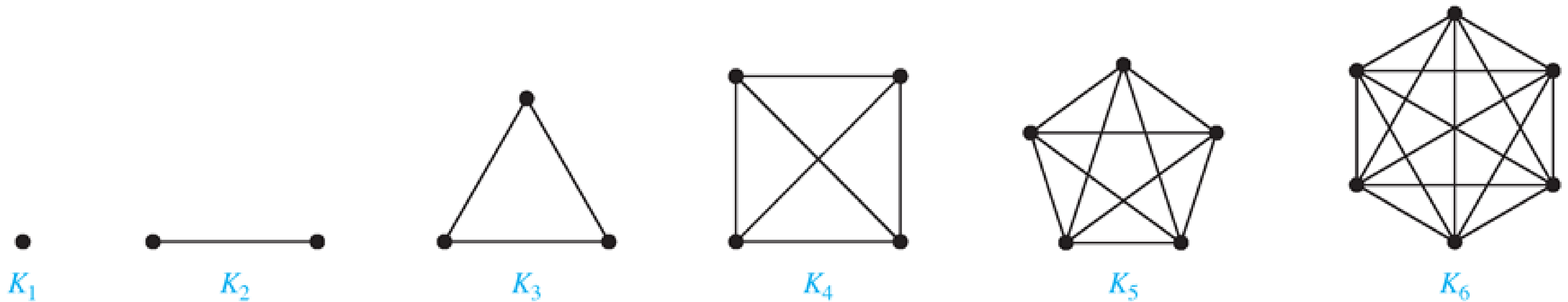

Cycles

- A *cycle* C_n for $n \geq 3$ consists of n vertices $v_1, v_2, \ldots, v_n$, and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.

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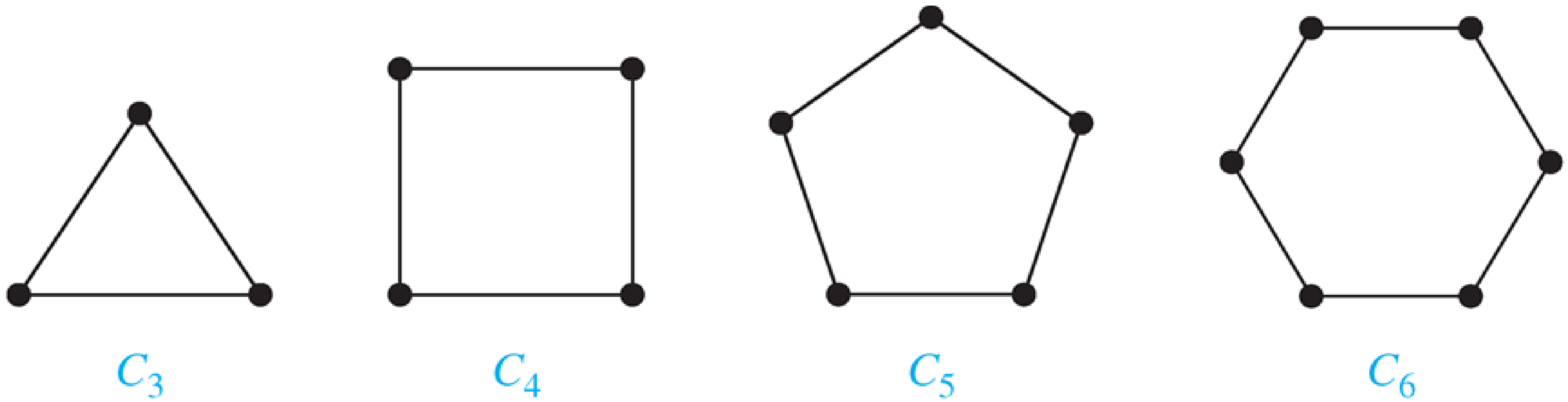

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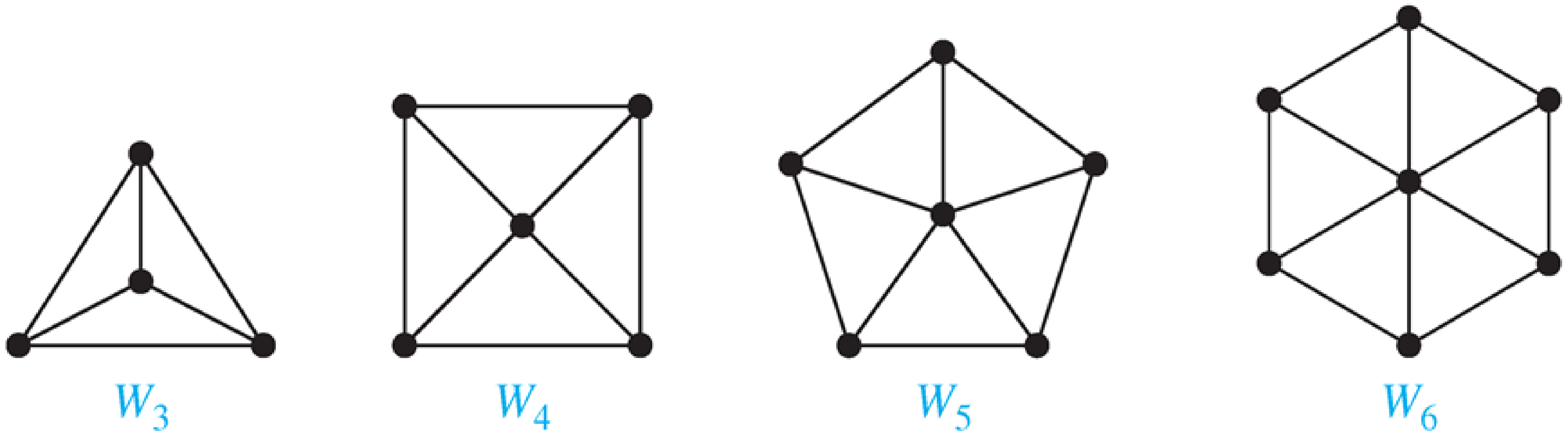

N-dimensional Hypercube

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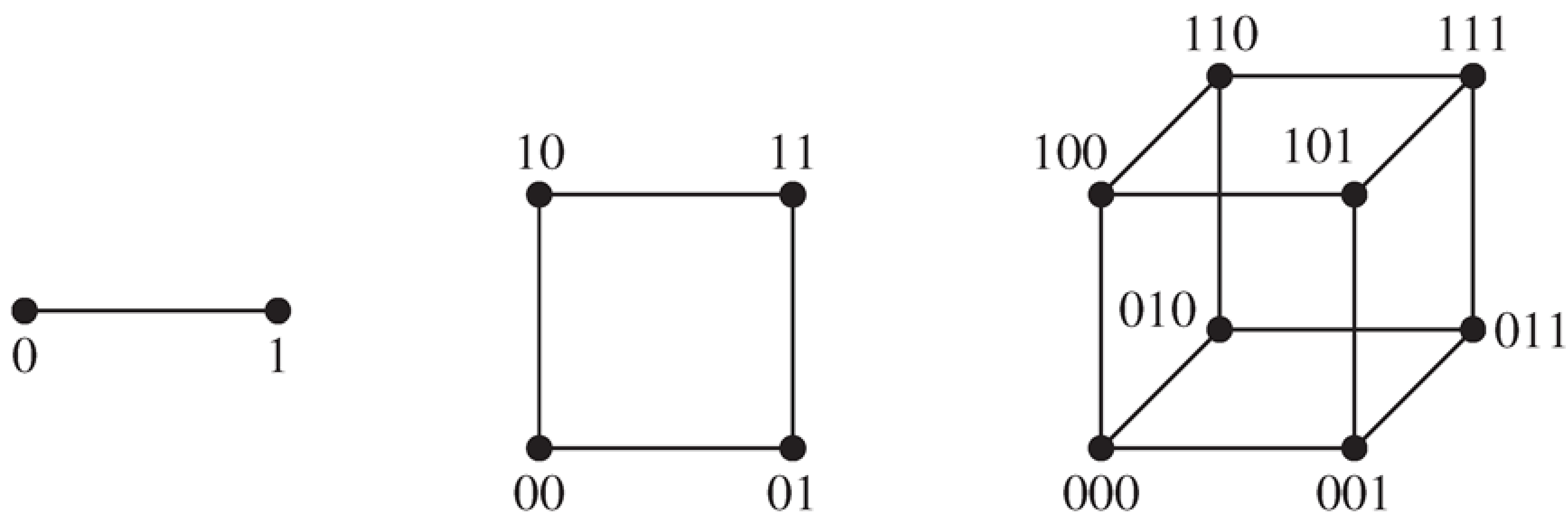

Q_1 Q_2 Q_3

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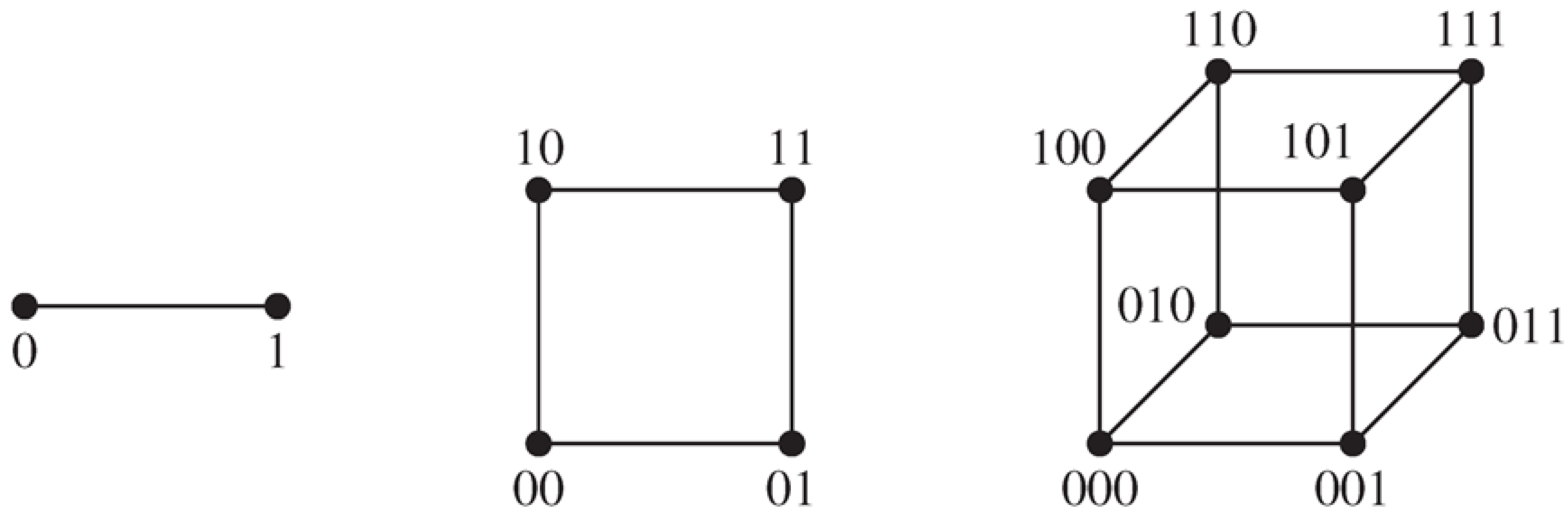

Q_1 Q_2 Q_3

How many vertices? How many edges?

Bipartite Graphs

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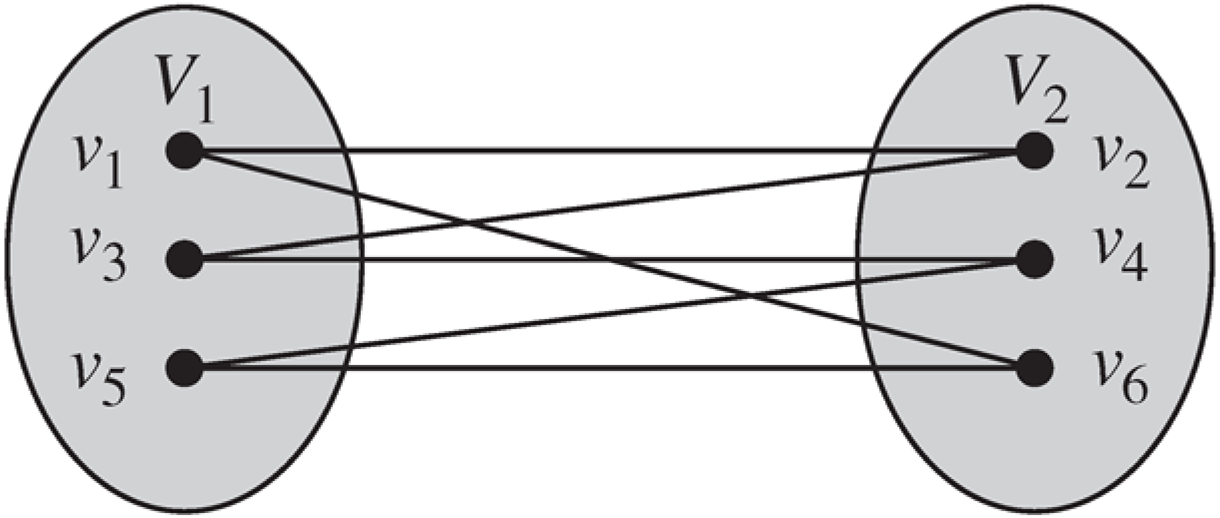
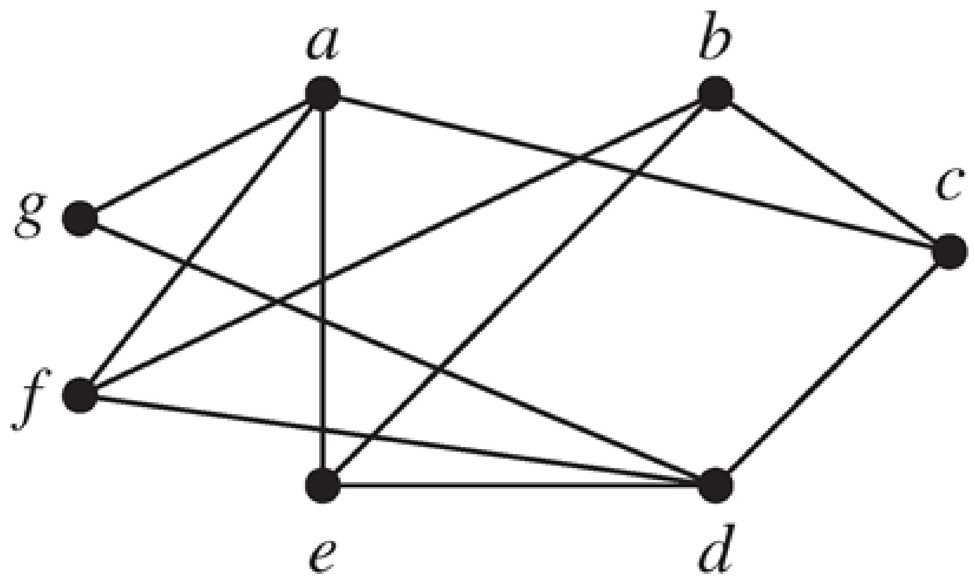


Bipartite Graphs



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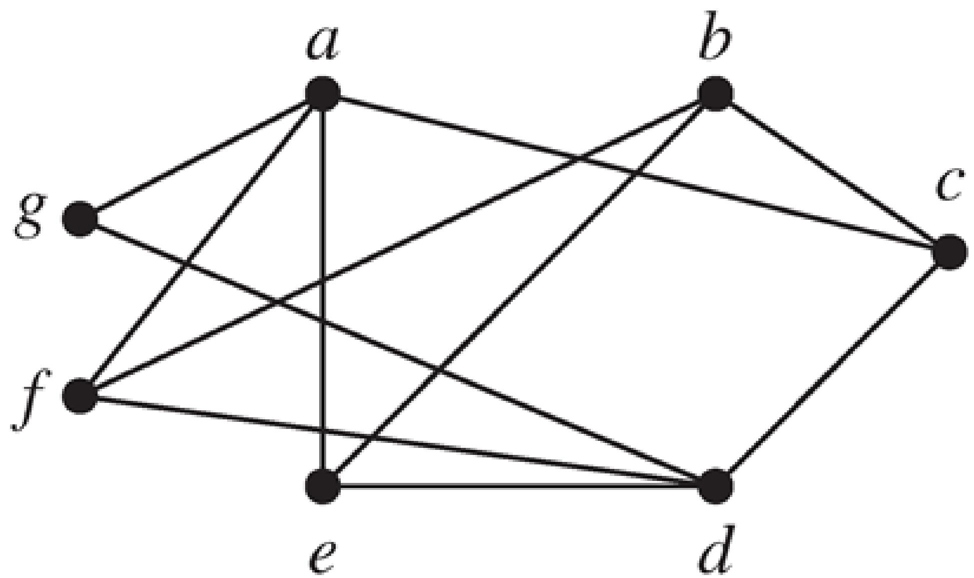
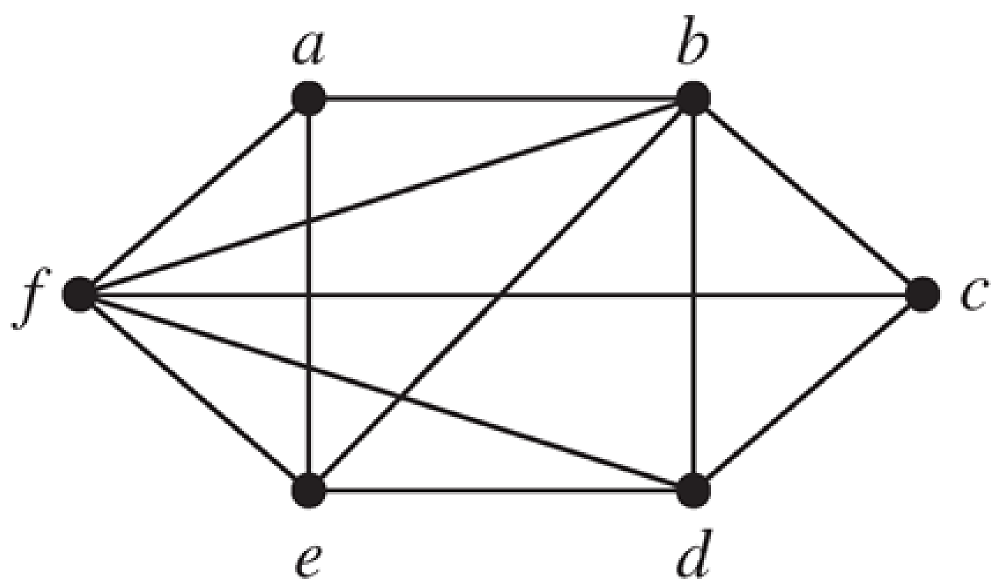
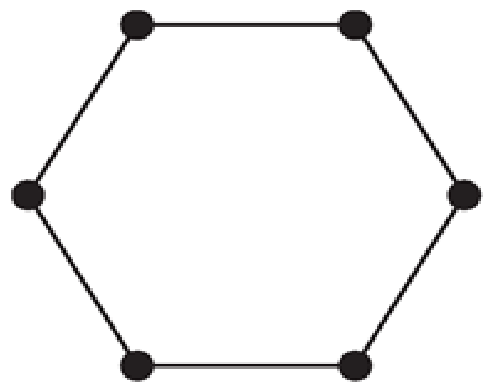




- **Example** Show that C_6 is bipartite.



C_6

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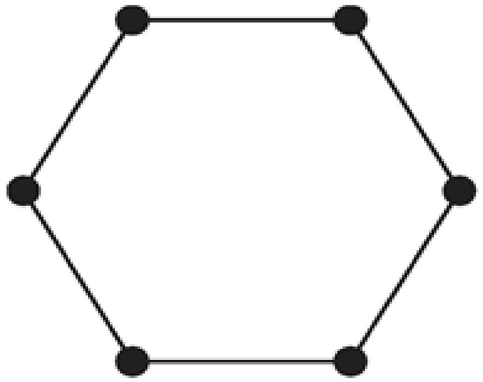
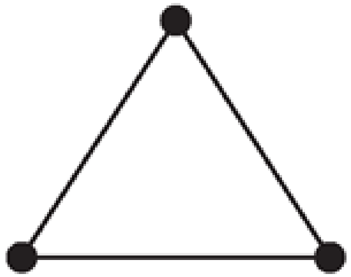


C_6

Example Show that C_3 is not bipartite.



C_3

Complete Bipartite Graphs

- **Definition** A *complete bipartite graph* $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

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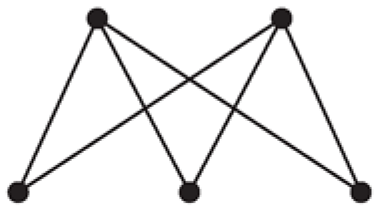
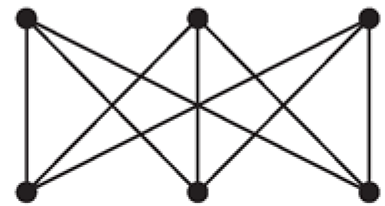


$K_{2,3}$



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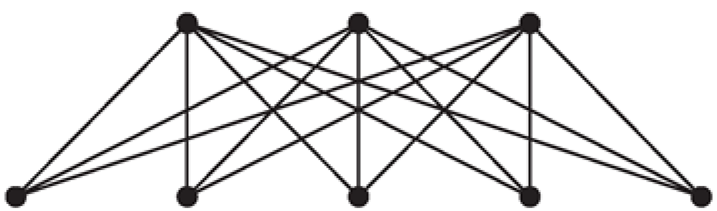
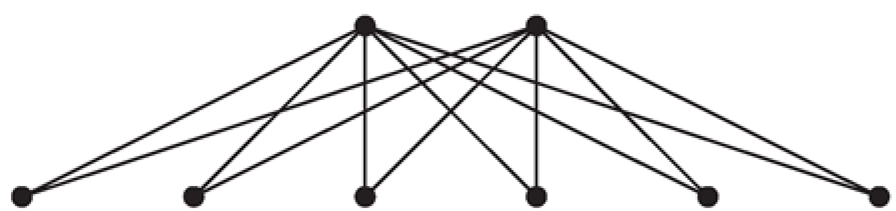


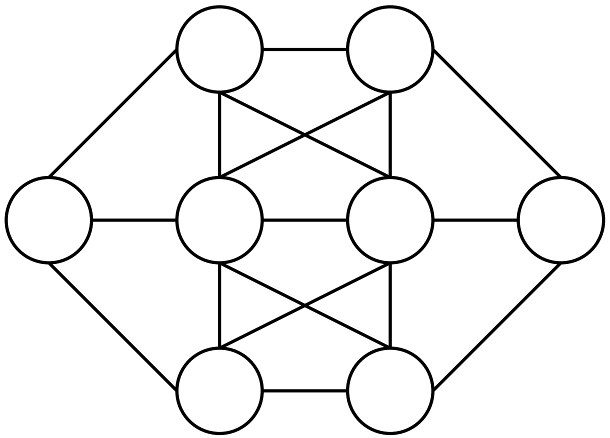
$K_{3,5}$



$K_{2,6}$

Puzzles using Graphs

- **The eight-circles problem** Place the letters A, B, C, D, E, F, G, H into the eight circles in the figure, in such a way that no letter is adjacent to a letter that is next to it in the alphabet.



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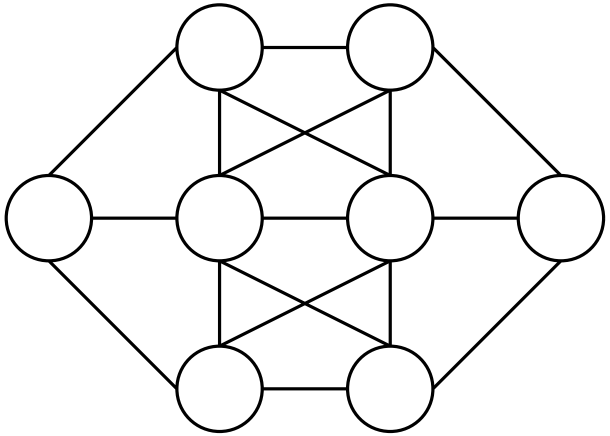

- **Six people at a party** Show that, in any gathering of six people, there are either three people who all know each other, or three people none of which knows either of the other two.

Next Lecture

- graph ...