CS215: Discrete Math (H)
2023 Fall Semester Written Assignment # 6
Due: Jan. 3rd, 2024, please submit at the beginning of class

Q.1 Let G be a simple graph. Show that the relation R on the set of vertices of G such that rRv if and only if there is an edge associated to $\{u, v\}$ is a symmetric, irreflexive relation on G.
Solution:

If uRv, then there is an edge associated with $\{u, v\}$. But $\{u, v\} = \{v, u\}$, so this edge is associated with $\{v, u\}$ and therefore vRu. Thus, by definition, R is a symmetric relation. A simple graph does not allow loops; therefore uRu never holds, and so by definition R is irreflexive.

□

Q.2 Let G be a *simple* graph with n vertices. Show that if the minimum degree of any vertex of G is greater than or equal to $(n-1)/2$, then G must be connected.
Solution: We prove this by contradiction. Suppose that the minimum degree is $(n-1)/2$ and G is not connected. Then G has at least two connected components. In each of the components, the minimum vertex degree is still $(n-1)/2$, and this means that each connected component must have at least $(n-1)/2+1$ vertices. Since there are at least two components, this means that the graph has at least $2(\frac{n-1}{2}+1) = n+1$ vertices, which is a contradiction.

□

Q.3 Let $n \geq 5$ be an integer. Consider the graph G_n whose vertices are the sets $\{a, b\}$, where $a, b \in \{1, \ldots, n\}$ and $a \neq b$, and whose adjacency rule is *disjointness*, that is, $\{a, b\}$ is adjacent to $\{a', b'\}$ whenever $\{a, b\} \cap \{a', b'\} = \emptyset$.

(a) Draw G_5.

(b) Find the degree of each vertex in G_n.

Solution:

(a) omitted.

(b) The degree of each vertex is $\binom{n-2}{2}$.

$\square$

Q.4 Let $G = (V, E)$ be a graph on n vertices. Construct a new graph, $G' = (V', E')$ as follows: the vertices of G' are the edges of G (i.e., $V' = E$), and two distinct edges in G are adjacent in G' if they share an endpoint.

(a) Draw G' for $G = K_4$.

(b) Find a formula for the number of edges of G' in terms of the vertex degrees of G.

Solution:

(a) omitted.

(b) The degree $\deg(v)$ of a vertex v in the graph G means that the number of $\deg(v)$ edges share the same vertex v. This will generate $\binom{\deg(v)}{2}$ edges in G'. In all the number of edges in G' is

$$\sum_{v \in V} \binom{\deg(v)}{2}.$$

$\square$

Q.5 Let $G = (V, E)$ be an undirected graph and let $A \subseteq V$ and $B \subseteq V$. Show that

(1) $N(A \cup B) = N(A) \cup N(B)$.

(2) $N(A \cap B) \subseteq N(A) \cap N(B)$, and give an example where $N(A \cap B) \neq N(A) \cap N(B)$.

Solution:

(1) If $x \in N(A \cup B)$, then x is adjacent to some vertex $v \in A \cup B$. W.l.o.g., suppose that $v \in A$. Then $x \in N(A)$ and therefore also in $N(A) \cup N(B)$. Conversely, if $x \in N(A) \cup N(B)$, then w.l.o.g. suppose that $x \in N(A)$. Thus, x is adjacent to some vertex $x \in A \subseteq A \cup B$, so $x \in N(A \cup B)$.

(2) If $x \in N(A \cap B)$, then x is adjacent to some vertex $v \in A \cap B$. Since both $v \in A$ and $v \in B$, it follows that $x \in N(A)$ and $x \in N(B)$, whence $x \in N(A) \cap N(B)$. For the counterexample, let $G = (\{u, v, w\}, \{\{u, v\}, \{v, w\}\})$, $A = \{u\}$, and $B = \{w\}$.

□

Q.6 Given a graph $G = (V, E)$, an edge $e \in E$ is said to be a *bridge* if the graph $G' = (V, E \setminus \{e\})$ has more connected components than G. Let G be a bipartite k-regular graph (the degree of every vertex is k) for $k \geq 2$. Prove that G has no bridge.

Solution: We will prove the result by contradiction. Assume that G has a bridge $e = \{u, v\}$. Let's start with a couple of easy observations. Firstly, note that a bridge affects only the connected component it belongs to. Every connected component of a bipartite k-regular graph is itself bipartite k-regular, so we can assume, without loss of generality, that G is a connected bipartite k-regular graph. Secondly, removal of an edge can split a connected graph into at most two connected component – to see why, observe that if we restore the edge, the graph should be connected, but three or more disjoint components cannot be linked by a single edge.

Now assume that G has classes A and B, where $u \in A$ and $v \in B$. Removal of e splits G into disjoint components G_1 and G_2. Let A' be the set of vertices of A in G_1 and A'' be those in G_2 – both these sets are non-empty. Similarly let B', B'' be the vertices of B in G_1 and G_2, respectively. Observe that the bridge e must be the only edge linking G_1 and G_2, and assume without loss of generality that $u \in A'$ and $v \in B''$.

Now look at G_1, which is a bipartite graph with classes A' and B'. Since e is the only edge linking G_1 and G_2, every other edge of G incident on A' or B' is retained in G_1. So every vertex in A' and B' still has degree k in G_1, except u which has degree $k - 1$. Let $a := |A'|$ and $b := |B'|$. Since no edge links two vertices in B' (bipartite property), the number of edges in G_1 is simply kb (every edge is incident to some vertex in B', so we can add up the degrees of the vertices in B'). Similarly, adding up the degrees in A' instead, the number of edges is $k(a-1) + k - 1$. Equating the two formulae, we have $k(a - 1) + k - 1 = kb$, which implies that $k(a - b) = 1$. But this implies that $k = 1$, contradicting the given condition that $k \geq 2$. Hence, the bridge cannot exist.

□

Q.7 In an n-player *round-robin tournament*, every pair of distinct players compete in a single game. Assume that every game has a winner – there are no ties. The results of such a tournament can then be represented with a *tournament directed graph* where the vertices correspond to players and there is an edge $x \to y$ iff x beats y in their game.

(a) Explain whey a tournament directed graph cannot have cycles of length 1 or 2.

(b) Is the "beats" relation for a tournament graph always/sometimes/never: antisymmetric? reflexive? irreflexive? transitive?

(c) Show that a tournament graph represents a total ordering iff there are no cycles of length 3.

Solution:

(a) There are no self-loops in a tournament graph since no player plays with himself, so no length 1 cycles. Also, it cannot be that x beats y and y beats x for $x \neq y$, since every pair competes exactly once and there are no ties. This means there are no length 2 cycles.

(b) No self-loops implies the relation is irreflexive. It is also antisymmetric since for every pair of distinct players, exactly one game is played and results in a win for one of the players. Some tournament graphs represent transitive relations and others don't.

(c) As observed in (b), the "beats" relation whose graphp is a tournament is antisymmetric and irreflexive. Since every pair of players is comparable, the relation is a total ordering iff it is transitive. "Beats" is transitive iff for any players x, y and z, $x \to y$ and $y \to z$ implies that $x \to z$, and consequently that there is no edge $z \to x$. Therefore, "beats" is transitive iff there are no cycles of length 3.

□

Q.8 Let G be a connected graph, with the vertex set V. The *distance* between two vertices u and v, denoted by $dist(u, v)$, is defined as the *minimal* length of a path from u to v. Show that $dist(u, v)$ is a metric, i.e., the following properties hold for any $u, v, w \in V$:

(i) $dist(u,v) \geq 0$ and $dist(u,v) = 0$ if and only if $u = v$.

(ii) $dist(u,v) = dist(v,u)$.

(iii) $dist(u,v) \leq dist(u,w) + dist(w,v)$.

Solution:

(i) By definition, the $dist(u,v)$ is the minimal length of a path from u to v, and the length if the number of edges in the path. Thus, $dist(u,v)$ cannot be negative. Furthermore, $dist(u,v) = 0$ if and only if there is a path of length 0 from u to v, which means that $u = v$.

(ii) Suppose that P is path from u to v of the minimal length. We reverse all the edges in the path P, and will get a path P' from v to u. Note that P' must be the minimal path from v to u. Otherwise, we reverse P' and will get a shorter path from u to v, which is a contradiction. Thus, $dist(u,v) = dist(v,u)$.

(iii) By definition, $dist(u,v) = \#$ of edges in the path P, where P is the path from u to v with the minimum length. Suppose that P_1 and P_2 are the paths of minimal length from u to w, and from w to v, respectively. Then $u\tilde{w}\tilde{v}$ is a new path P' from u to v. By the minimality of P, we must have $dist(u,v) \leq dist(u,w) + dist(w,v)$.

$\square$

Q.9 Use paths either to show that these graphs are not isomorphic or to find an isomorphism between these graphs.
Solution: The graph G has a simple closed path containing exactly the vertices of degree 3, namely $u_1u_2u_6u_5u_1$. The graph H has no simple closed path containing exactly the vertices of degree 3. Therefore the two graphs are not isomorphic.

$\square$

Q.10 Show that isomorphism of simple graphs is an equivalence relation.
Solution:

G is isomorphic to itself by the identity function, so isomorphism is reflexive. Suppose that G is isomorphic to H. Then there exists a one-to-one

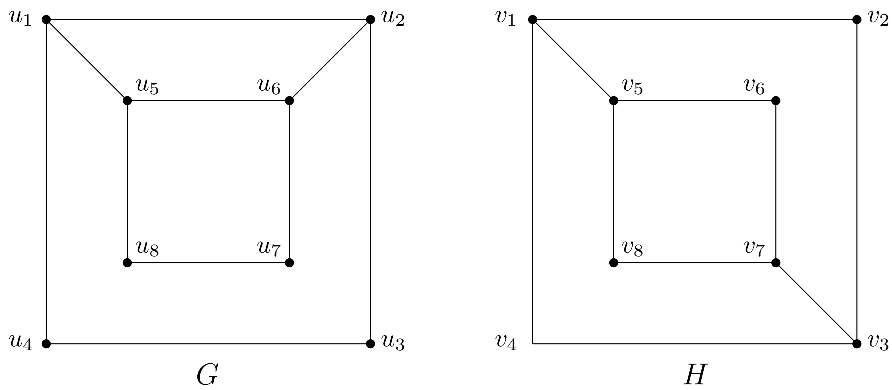

Figure 1: Q.9

correspondence f from G to H that preserves adjacency and nonadjacency. It follows that f^{-1} is a one-to-one correspondence from H to G that preserves adjacency and nonadjacency. Hence, isomorphism is symmetric. If G is isomorphic to H and H is isomorphic to K, then there are one-to-one correspondences f and g from G to H and from H to K that preserve adjacency and nonadjacency. It follows that $g \circ f$ is a one-to-one correspondence from G to K that preserves adjacency and nonadjacency. Hence, isomorphism is transitive.

$\square$

Q.11 Suppose that G_1 and H_1 are isomorphic and that G_1 and H_2 are isomorphic. Prove or disprove that $G_1 \cup G_2$ and $H_1 \cup H_2$ are isomorphic.
Solution: The isomorphism need not hold. For the simplest counterexample, let G_1, G_2 and H_1 each be the graph consisting of the single vertex v, and let H_2 be the graph consisting of the single vertex w. Then of course G_1 and H_1 are isomorphic, as are G_2 and H_2. But $G_1 \cup G_2$ is a graph with one vertex, and $H_1 \cup H_2$ is a graph with two vertices.

$\square$

Q.12 Given a graph G, its *line graph* $L(G)$ is defined as follows: every edge of G corresponds to a unique vertex of $L(G)$; any two vertices of $L(G)$ are adjacent if and only if their corresponding edges of G share a common endpoint.

Prove that if G is regular (all vertices have the same degree) and connected, then $L(G)$ has an Euler circuit.
Solution: If the degree of regular graph G is d, then every edge of G has $2(d-1)$ neighbours in $L(G)$. Since this is even, $L(G)$ has an Euler circuit.

$\square$

Q.13 Suppose that a connected planar simple graph with e edges and v vertices contains no simple circuits of length 4 or less. Show that $e \leq (5/3)v - (10/3)$ if $v \geq 4$.
Solution:

As in the argument in the proof of Corollary 1, we have $2e \geq 5r$ and $r = e - v + 2$. Thus $e - v + 2 \leq 2e/5$, which implies that $e \leq (5/3)v - (10/3)$.

$\square$

Q.14 The **distance** between two distinct vertices v_1 and v_2 of a connected simple graph is the length (number of edges) of the shortest path between v_1 and v_2. The **radius** of a graph is the *minimum* over all vertices v of the maximum distance from v to another vertex. The **diameter** of a graph is the maximum distance between two distinct vertices. Find the radius and diameter of

(1) K_6

(2) $K_{4,5}$

(3) Q_3

(4) C_6

Solution:

(1) K_6: The diameter is clearly 1, since the maximum distance between two vertices is 1; the radius is also 1, with any vertex serving as the center.

(2) $K_{4,5}$: The diameter is clearly 2, since vertices in the same part are not adjacent, but no pair of vertices are at a distance greater than 2. Similarly, the radius is 2 with any vertex serving as the center.

(3) Q_3: Vertices at diagonally opposite corners of the cube are a distance of 3 from each other, and this is the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3.

(4) C_6: Vertices at opposite corners of the hexagon are a distance 3 from each other, and this it the worst case, so the diameter is 3. By symmetry we can take any vertex as the center, so it is clear that the radius is also 3. (Note that despite the appearances in this exercise, it is not always the case that the radius equals the diameter; for example, $K_{1,n}$ has radius 1 and diameter 2.)

$\square$

Q.15 Let n be a positive integer. Construct a **connected** graph with $2n$ vertices, such that there are *exactly* **two** vertices of degree i for each $i = 1, 2, \ldots, n$. (You can sketch some pictures, but your graph has to be described by a concise adjacency rule. Remember to prove that your graph is connected.)

Solution:

We draw a bipartite graph in the following way:

The vertex set contains two sets of vertices, V_1 and V_2, with each containing n vertices. For the ith vertex in V_1, it is connected to the first i vertices in the set V_2. In this way, the ith vertex in V_1 has degree i, and the ith vertex in V_2 has degree $n-i+1$, since the previous $i-1$ vertices in $V-1$ are not connected to the ith vertex in V_2 by the adjacency rule. The constructed graph is connected in an obvious way: $(1, 1, 2, 2, 3, 3, 4, 4, \ldots, n, n)$, where the first i denotes the ith vertex in V_1 and the second i denotes the ith vertex in V_2 (see the following figure).

$\square$

Q.16 An n-cube is a cube in n dimensions, denoted by Q_n. The 1-cube, 2-cube, 3-cube are a line segment, a square, a normal cube, respectively, as shown below. In general, you can construct the $(n+1)$-cube Q_{n+1} from the n-cube Q_n by making two copies of Q_n, prefacing the labels on the vertices with a 0 in one copy of Q_n and with a 1 in the other copy of Q_n, and adding

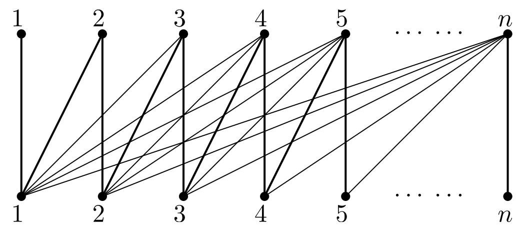

Figure 2: Q.15

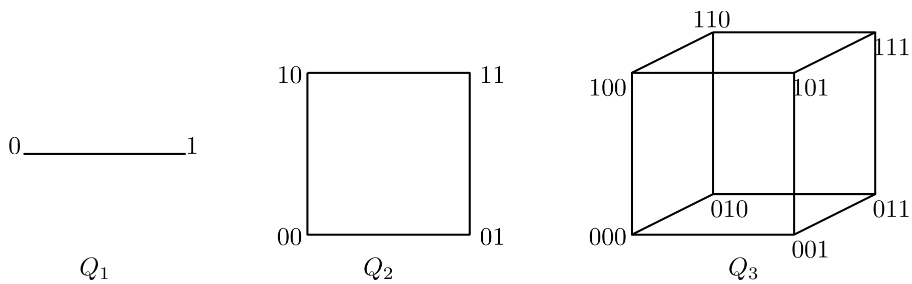

Figure 3: Q.16

edges connecting two vertices that have labels differing only in the first bit. Answer the following questions, and explain your answers.

(1) How many edges does an n-cube Q_n have?

(2) For what n, the n-cube Q_n has an Euler circuit?

(3) Is an n-cube Q_n bipartite or not?

(4) For what n, the n-cube Q_n is planar?

(5) For what n, the n-cube Q_n has an Hamilton circuit?

Solution:

(1) Fix any vertex v of Q_n. All its neighbors differ from v in exactly one position. There are n positions possible to differ at. Hence, every vertex has degree n. Then by the Handshaking Theorem, we have $2e = \sum_{v \in V} \deg(v) = n \cdot 2^n$. Thus, the number of edges is $n \cdot 2^{n-1}$.

(2) Q_n has an Euler circuit if and only if all its degrees are even. Since for each vertex of Q_n, its degree is n, we have that Q_n has an Euler circuit if and only if n is even.

(3) Q_n is bipartite. Let V_1 be the set of vertices of Q_n with an even number of 0's, and V_2 be the set of vertices of Q_n with an odd number of 0's. Clearly, every vertex must either have an odd or and even number of 0's. Hence, the disjoint union of V_1 and V_2 constitutes the vertex set of Q_n. For two vertices $x, y \in V_1$, there is an edge between x and y if and only if x and y differ in exactly one position. But this would imply that if one of them has an even number of 0's while the other has an odd number of 0's. So these two vertices cannot be both from V_1. This is a contradiction. Similarly we can prove that it is not possible to have an edge with both vertices from V_2.

(4) Q_n is planar only for $n \leq 3$. By (4), we know that Q_n is bipartite, and thereby does not have a circuit of length 3. Then by the necessary condition in Corollary 3, we have $e \leq 2v - 4$. For Q_4, there are $v = 16$ vertices and $e = 4 \cdot 2^{4-1} = 32$. It is easily seen that $32 > 2 \cdot 16 - 4$. Thus, Q_4 cannot be a planar graph. Obviously Q_4 is a minor of Q_n for $n > 4$. Therefore, Q_n is only planar for $n = 1, 2, 3$.

(5) For all n, Q_n has a Hamilton circuit. We prove this by induction. If $n = 1$, we simply need to visit each vertex of a two-vertex graph with an edge connecting them.

Assume that it is true for $n = k$. To build a $(k+1)$-cube, we take two copies of the k-cube and connect the corresponding edges. Take that Hamilton circuit on one cube and reverse it on the other. Then choose an edge on one that is part of the circuit and the corresponding edge on the other and delete them from the circuit. Finally, add to the path connections from the corresponding endpoints on the cubes which will produce a circuit on the $(k+1)$-cube.

Q.17 Consider the two graphs G and H. Answer the following three questions, and explain your answers.

(1) Which of the two graphs is/are *bipartite*?

(2) Are the two graphs *isomorphic* to each other?

(3) Which of the two graphs has/have an *Euler circuit*?

Solution:

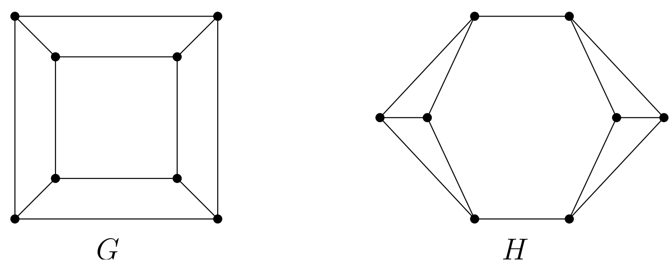

Figure 4: Q.17

(1) The graph G is bipartite, and the graph H is not. It is checked below that the graph G can be 2-colored, which H cannot.

(2) The two graphs are not isomorphic, since there exists a circuit of length 3 in H, which does not exist in G.

(3) Since both of the two graphs have only degree-3 vertices, neither of these two graphs has an Euler circuit.

□

Q.18 There are 17 students who communicates with each other discussing problems in discrete math. They are only 3 possible problems, and each pair of students discuss one of these three 3 problems. Prove that there are at least 3 students who are all pairwise discussing the same problem.

Solution: We use vertices to denote the 17 students and edges to denote the communication among these students. In addition, we use 3 different colors to color the edges to denote the 3 problems they are discussing. For one fixed student A, A communicates with the other 16 students. By the Pigeonhole Principle, at least 6 edges are of the same color, w.l.o.g., we assume that the edges AB, AC, AD, AE, AF, AG are all of color red.

If among the six students B, C, D, E, F, G there is one edge, e.g., BC whose color is also red, then all the three edges of the triangle ABC are red.

If among the six students B, C, D, E, F, G there is no red edge, we consider the edges BC, BD, BE, BF, BG. There are only two colors for these five edges, so at least there are three of these five edges of the same color. W.l.o.g., assume that the three edges BC, BD, BE are of the same color, yellow. We consider the triangle CDE. If there is one yellow edge of the triangle CDE, say, CD is yellow, then the triangle BCD is a triangle with three edges all yellow. If the triangle CDE does not have yellow edge, which means all edges of CDE are blue, we again have a triangle with three edges of the same color.

$\square$

Q.19 Which complete bipartite graphs $K_{m,n}$, where m and n are positive integers, are trees?

Solution:

If both m and n are at least 2, then clearly there is a simple circuit of length 4 in $K_{m,n}$. On the other hand, $K_{m,1}$ is cleraly a tree (as is $K_{1,n}$). Thus we conclude that $K_{m,n}$ is a tree if and only if $m = 1$ or $n = 1$.

$\square$

Q.20

What is the value of each of these postfix expressions?

(a) $9\ 3\ /\ 5\ +\ 7\ 2\ -\ *$

(b) $3\ 2\ *\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ -$

Solution:

We exhibit the answers by showing with parentheses the operation that is applied next, working from left to right (it always involves the first occurrence of an operator symbol).

(a) $(9\ 3\ /)\ 5 + 7\ 2\ -\ * = (3\ 5\ +)\ 7\ 2\ -\ * = 8(\ 7\ 2\ -) * = (8\ 5\ *) = 40$

(b) $(3\ 2\ *)\ 2\ \uparrow\ 5\ 3\ -\ 8\ 4\ /\ *\ - = (6\ 2\ \uparrow)\ 5\ 3\ -\ 8\ 4\ /\ *\ - = 36\ (5\ 3\ -)\ 8\ 4\ /\ *\ - = 36\ 2(8\ 4\ /)\ *\ - = 36(2\ 2\ *)\ - = (36\ 4\ -) = 32$

$\square$

Q.21

How many different spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) $K_{2,2}$ d) C_5

Solution:

a) 3 b) 16 c) 4 d) 5

□

Q.22

How many nonisomorphic spanning trees does each of these simple graphs have?

a) K_3 b) K_4 c) K_5

Solution:

a) 1 b) 2 c) 3

□

Q.23

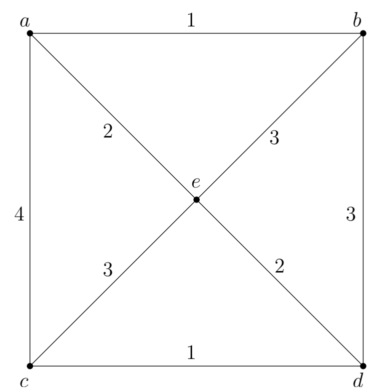

Figure 5: Q.23

(1) Use Prim's algorithm to find a minimum spanning tree for the given weighted graph.

(2) Use Kruskal's algorithm to find a minimum spanning tree for the same weighted graph.

Solution:

(1) We start with the minimum weight edge $\{a, b\}$. The least weight edge incident to the tree constructed so far is edge $\{a, e\}$, with weight 2, so we add it to the tree. Next we add edge $\{d, e\}$, and then edge $\{c, d\}$. This completes the tree, whose total weight is 6.

(2) With Kruskal's algorithm, we add at each step the shortest edge and will not complete a simple circuit. Thus we pick edge $\{a, b\}$ first, and then edge $\{c, d\}$ (alphabetical order breaks ties), followed by $\{a, e\}$ and $\{d, e\}$. The total weight is 6.

$\square$