# CS215 DISCRETE MATH

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# Binary Relation

- **Definition**: Let $A$ and $B$ be two sets. A *binary relation from $A$ to $B$* is a subset of a Cartesian product $A \times B$.

  Let $R \subseteq A \times B$ denote $R$ is a set of ordered pairs of the form $(a, b)$ where $a \in A$ and $b \in B$.

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- **Definition**: A relation on the set $A$ is a relation from $A$ to itself.

- **Theorem** The number of binary relations on a set $A$, where $|A| = n$ is $2^{n^2}$.

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A relation $R$ is reflexive if and only if MR has 1 in every position on its main diagonal.

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  No. $(1,1) \notin R$

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  Yes. $(1, 1), (2, 2), (3, 3), (4, 4) \notin R_{\neq}$

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  No. $(1, 2) \in R_{div}$ but $(2, 1) \notin R$

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  Yes. If $(a,b) \in R_{\neq}$ then $(b,a) \in R_{\neq}$.

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A relation $R$ is symmetric if and only if MR is symmetric.

# Properties of Relations

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A relation $R$ is antisymmetric if and only if $m_{ij} = 1$ implies $m_{ji} = 0$ for $i \neq j$.

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  Set operations: union, intersection, difference, etc.

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- **Example**: Let $A = \{1, 2, 3\}$, $B = \{u, v\}$, and $R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$, $R_2 = \{(1, v), (3, u), (3, v)\}$

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  What is $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $R_2 - R_1$?

  We may also combine relations by matrix operations.

# Composite of Relations

- **Definition**: Let $R$ be a relation from a set $A$ to a set $B$ and $S$ be a relation from $B$ to $C$. The *composite of $R$ and $S$* is the relation consisting of the ordered pairs $(a, c)$ where $a \in A$ and $c \in C$ and for which there is a $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

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  $$S \circ R = \{(1, b), (3, a), (3, b)\}$$

# Implementation of Composite

- **Example**: Let $A = \{1, 2\}$, $B = \{1, 2, 3\}$, and $C = \{a, b\}$

  $R = \{(1, 2), (1, 3), (2, 1)\}$ is a relation from $A$ to $B$

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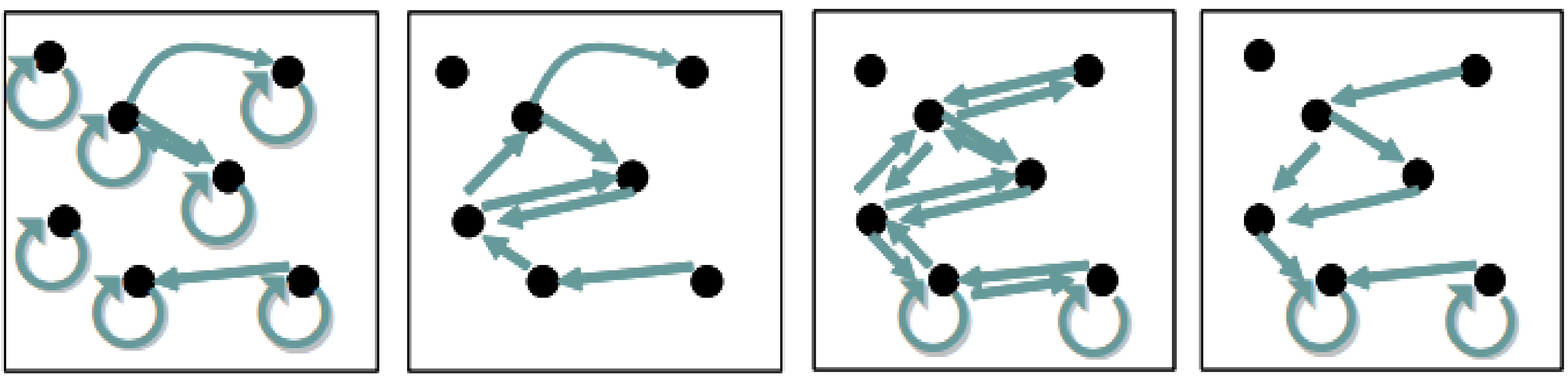

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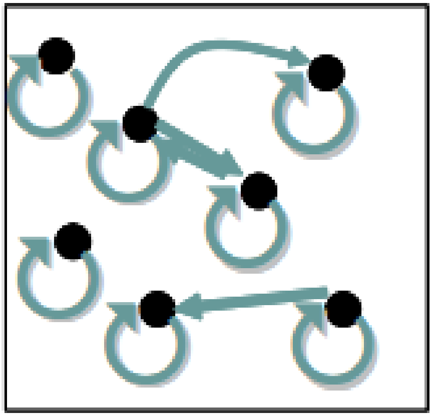

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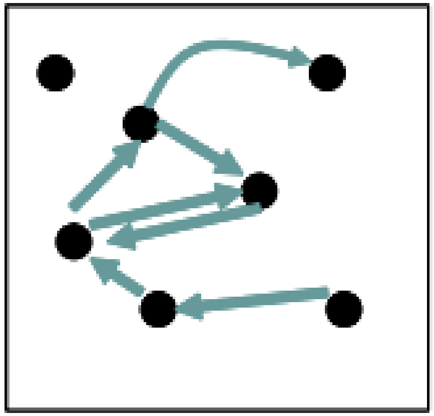

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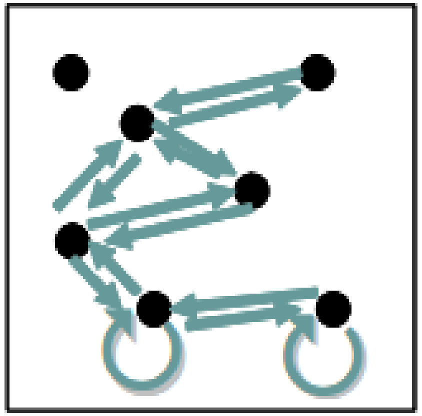

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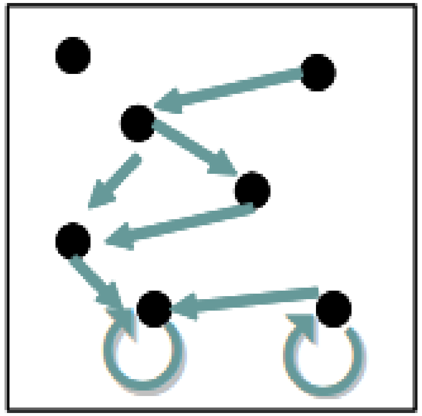

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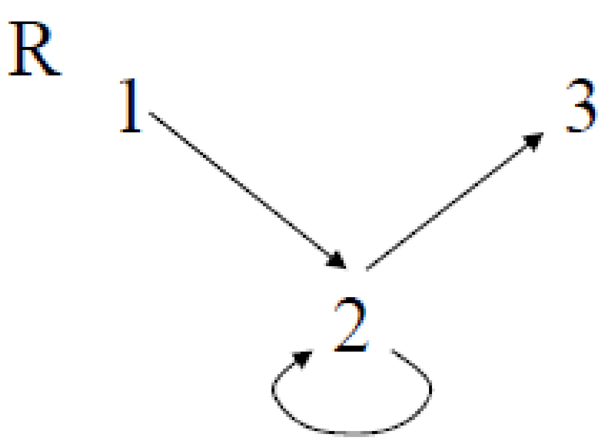

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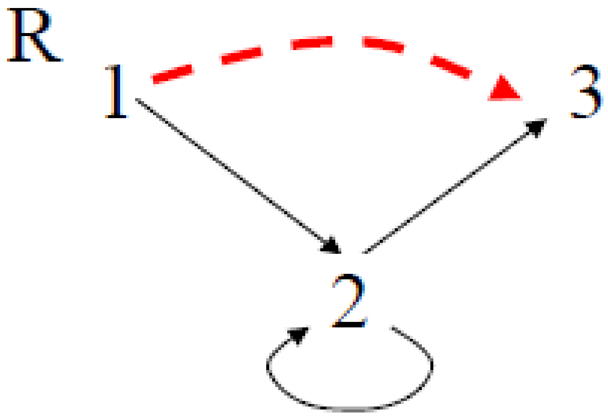

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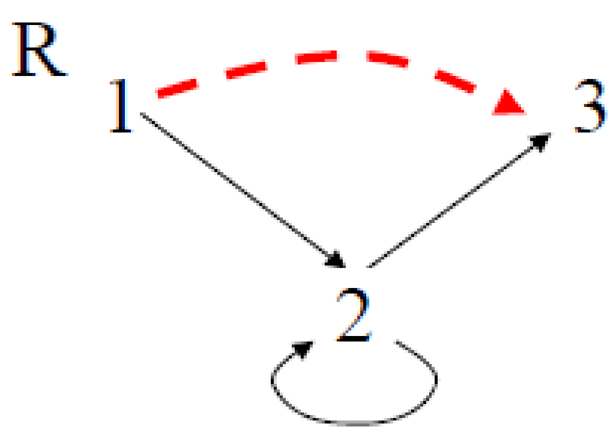


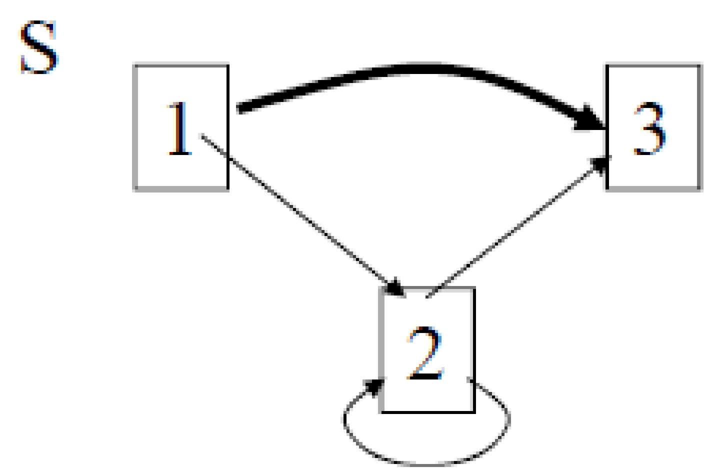

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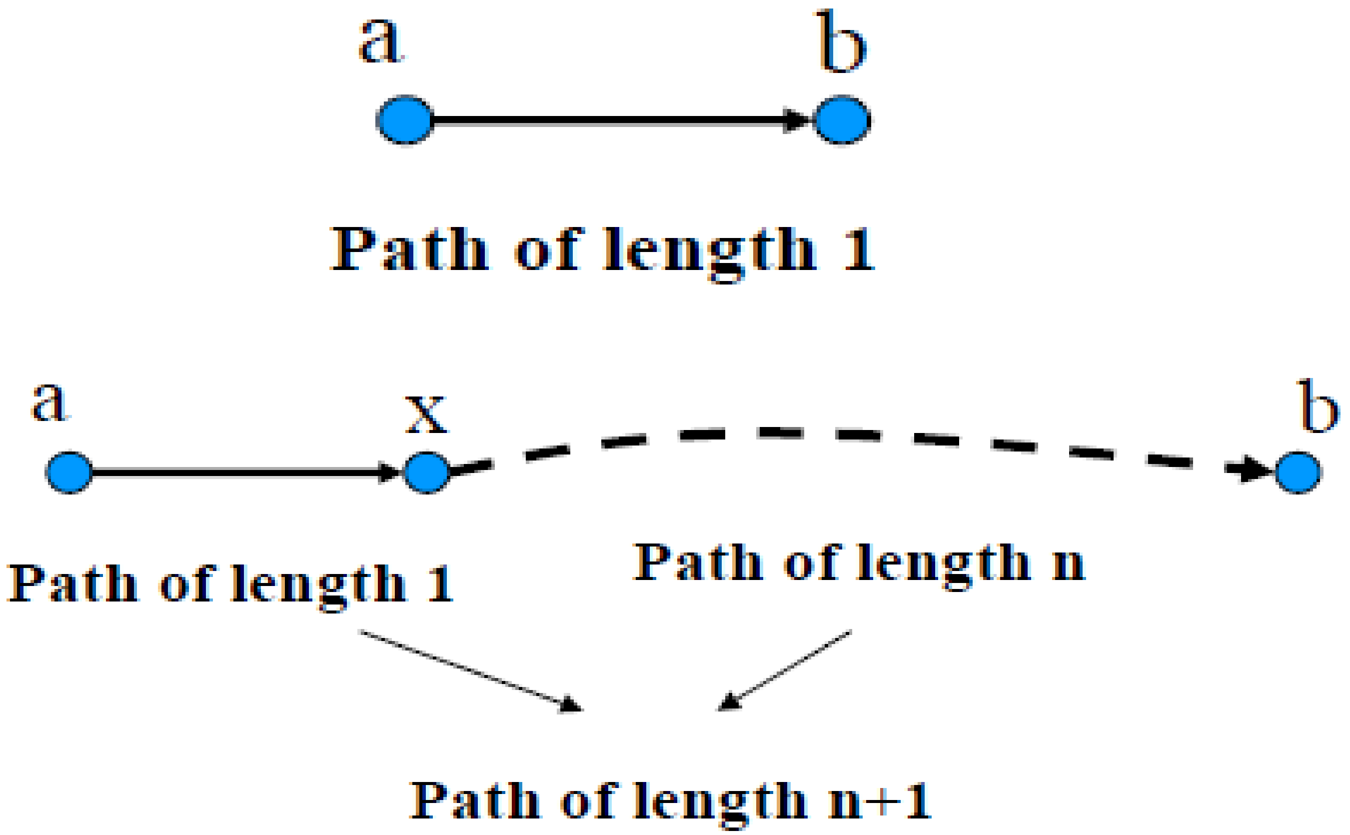

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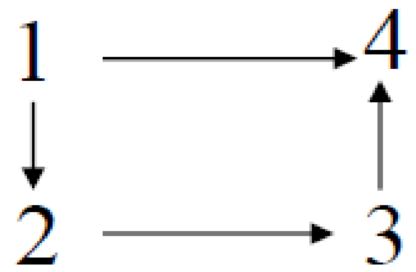

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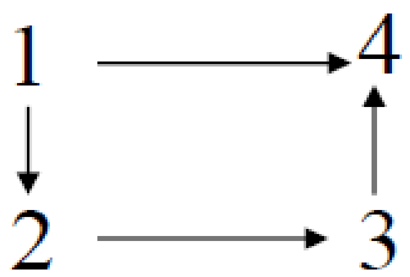

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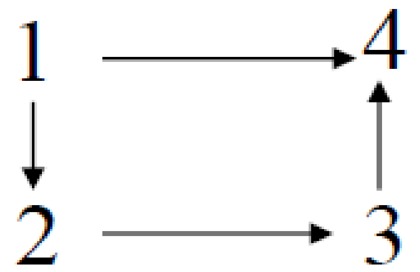


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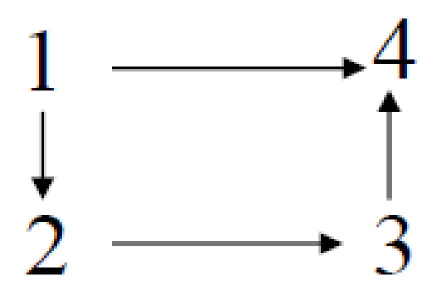


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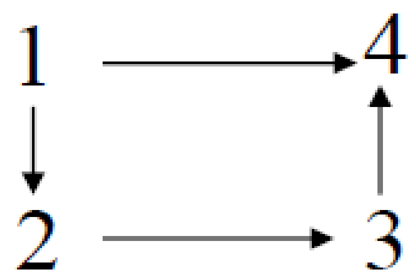


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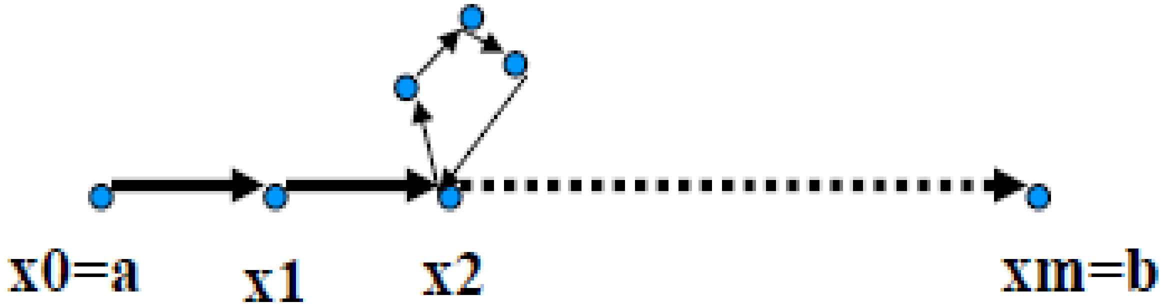

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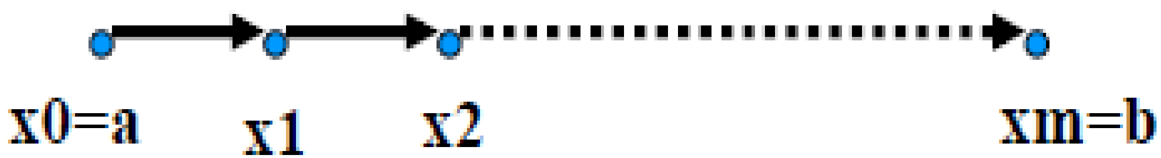


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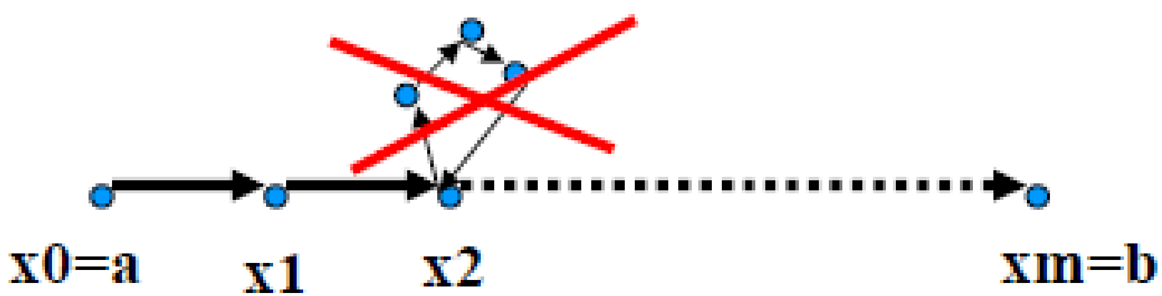

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**Recall** Finding a transitive closure corresponds to finding all pairs of elements that are connected with a directed path

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1. If $(a, b) \in R^*$ and $(b, c) \in R^*$, then there are paths from $a$ to $b$ and from $b$ to $c$ in $R$. Thus, there is a path from $a$ to $c$ in $R$. This means that $(a, c) \in R^*$.

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We have $S^* \subseteq S$. Thus, $R^* \subseteq S^* \subseteq S$

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**Example**

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$\mathbf{M}_{R^*} = ?$

# Simple Transitive Closure Algorithm

- **Lemma**: Let $A$ be a set with $n$ elements, and $R$ a relation on $A$. If there is a path from $a$ to $b$ with $a \neq b$, then there exists a path of length $\leq n - 1$.

**procedure** transClosure ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
// computes $R^*$ with zero-one matrices
$A := B := \mathbf{M}_R$;
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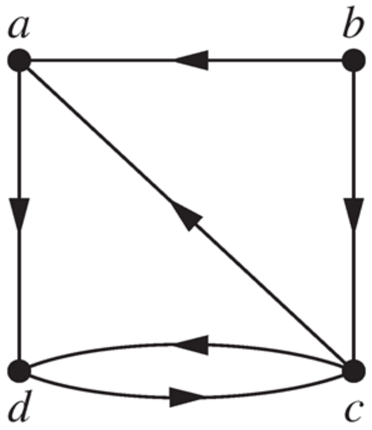


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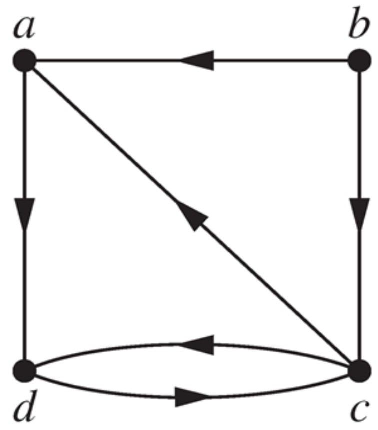


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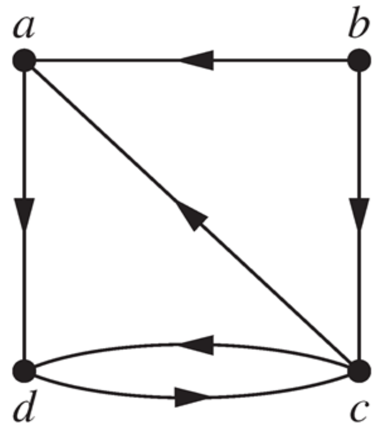


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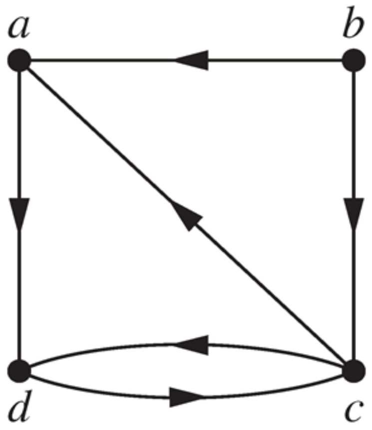


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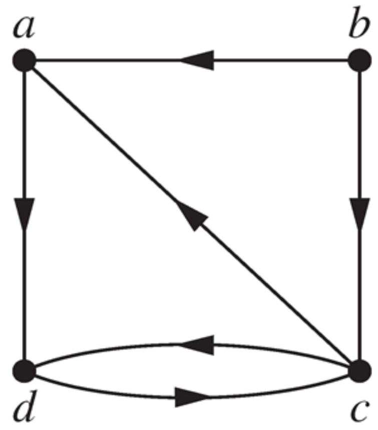


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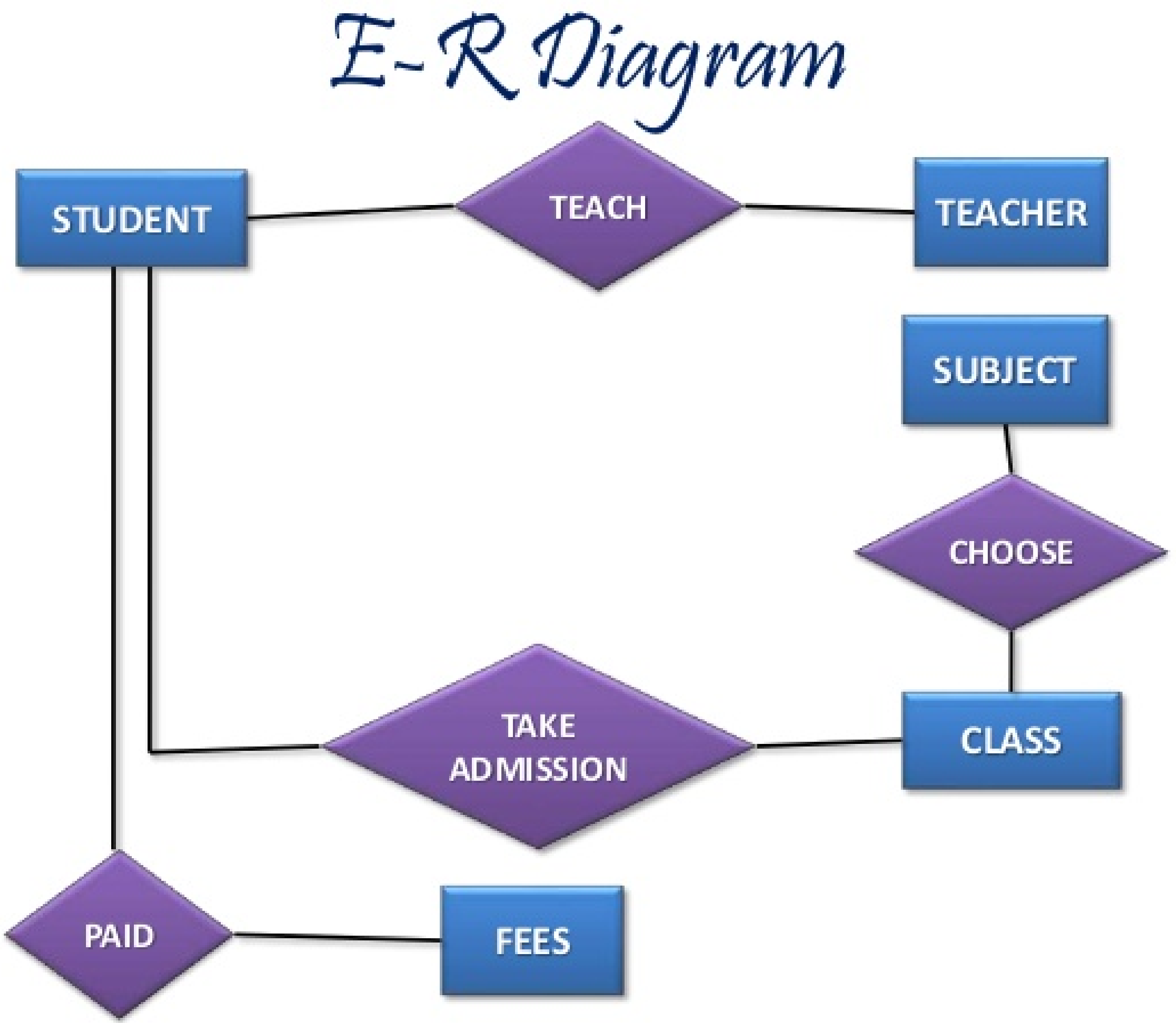

# Selection Operators

- Let $A$ be any *n-ary domain* $A = A_1 \times \cdots \times A_n$, and let $C : A \to \{T, F\}$ be any *condition* (predicate) on elements ($n$-tuples) of $A$.

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  – $\forall R \subseteq A$,

$$\begin{aligned} s_C(R) &= R \cap \{a \in A \mid s_C(a) = T\} \\ &= \{a \in R \mid s_C(a) = T\}. \end{aligned}$$

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- Then, $s_{\mathit{UpperLevel}}$ is the selection operator that takes any relation $R$ on $A$ (database of students) and produces a relation consisting of just the upper-level classes (juniors and seniors).

# Projection Operators

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  i.e., where $1 \leq i_k \leq n$ for all $1 \leq k \leq m$.

- Then the *projection operator* on $n$-tuples

  $$P_{\{i_k\}} : A \to A_{i_1} \times \cdots \times A_{i_m}$$

  is defined by

  $$P_{\{i_k\}}(a_1, \cdots, a_n) = (a_{i_1}, \cdots, a_{i_m})$$

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- This operator can be usefully applied to a whole relation $R \subseteq Cars$ (database of cars) to obtain a list of *model*/*color* combinations available.

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- $A, B, C$ can also be sequences of elements rather that single elements.

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- Suppose that $R_2$ is a room assignment table relating *Courses* to *Rooms* and *Times*.

- Then $J(R_1, R_2)$ is like your class schedule, listing $(professor, course, room, time)$.

# Next Lecture

- relation III ...