CS215: Discrete Math (H)
2023 Fall Semester   Written Assignment # 2
Due: Oct. 30th, 2023, please submit at the beginning of class

Q.1 Suppose that A, B and C are three finite sets. For each of the following, determine whether or not it is true. Explain your answers.

(a) $(A \cap B \neq \emptyset) \rightarrow ((A - B) \subset A)$

(b) $(A - B = \emptyset) \rightarrow (A \cap B = B \cap A)$

(c) $(A \subseteq B) \rightarrow (|A \cup B| \geq 2|A|)$

Q.2 Let's formulate the "Barber's paradox" in the language of predicate logic. In English, the paradox may be stated as:

"The barber of the village Seville shaves those residents of Seville who do not shave themselves."

Assume that S is the set of all residents of Seville, which includes the barber. We have the following predicates over elements of the set S:

- $Shaves(x, y)$: true if x shaves y, false otherwise.
- $Barber(x)$: true if x is the barber of Seville (you may assume that Seville has just one barber), false otherwise.

Rewrite the statement of the paradox using only these two predicates, along with the notation of mathematical logic. Please also state the reason why the paradox occurs in the logical statement.

Q.3 In addition to union ($\cup$), intersection ($\cap$), difference ($-$), and power set ($\mathcal{P}(S)$), let's now add two more operations to our dealings with sets:

- Pairwise addition: $A \oplus B := \{a + b \mid a \in A, b \in B\}$ (This is also called the *Minkowski addition* of sets A and B.)
- Pairwise multiplication: $A \otimes B := \{a \times b \mid a \in A, b \in B\}$.

For example, if A is $\{1, 2\}$ and B is $\{10, 100\}$, then $A \oplus B = \{11, 12, 101, 102\}$ and $A \otimes B = \{10, 20, 100, 200\}$. Answer the following questions, and explain your answers.

(1) Briefly describe the following sets, where $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$:

(a) $\mathbb{N} \oplus \emptyset$

(b) $\mathbb{N} \oplus \mathbb{N}$

(c) $\mathbb{N}^+ \oplus \mathbb{N}^+$

(d) $\mathbb{N}^+ \otimes \mathbb{N}^+$

(2) If E denotes the set of all positive *even* numbers, how to represent the set of all positive multiples of 4 in terms of E and the set operations above? And, how to represent the set of all positive multiples of 8?

(3) Let $S := \{n^2 : n \in \mathbb{N}^+\}$. A Pythagorean triple consists of three positive integers x, y and z such that $x^2 + y^2 = z^2$. Construct the set of all possible z^2 such that z is the last element of a Pythegorean triple using only the set S and the set operations we have so far.

Q.4 Let A, B and C be sets. Prove the following using set identities.

(1) $(B - A) \cup (C - A) = (B \cup C) - A$

(2) $(A \cap B) \cap \overline{(B \cap C)} \cap (A \cap C) = \emptyset$

Q.5 Give an example of two uncountable sets A and B such that the intersection $A \cap B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.6 The *symmetric difference* of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. Give an example of two uncountable sets A and B such that the intersection $A \oplus B$ is

(a) finite,

(b) countably infinite,

(c) uncountable.

Q.7 For each of the following mappings, indicate what type of function they are (if any). Use the following options to describe them, and explain your answers.
i. Not a function.
ii. A function which is neither one-to-one nor onto.
iii. A function which is onto but not one-to-one.
iv. A function which is one-to-one but not onto.
v. A function which is both one-to-one and onto.

(a) The mapping f from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $f(x) = |2x|$.

(b) The mapping f from $\{1, 3\}$ to $\{2, 4\}$ defined by $f(x) = 2x$.

(c) The mapping f from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = 8 - 2x$.

(d) The mapping f from $\mathbb{R}$ to $\mathbb{Z}$ defined by $f(x) = \lfloor x + 1 \rfloor$.

(e) The mapping f from $\mathbb{R}^+$ to $\mathbb{R}^+$ defined by $f(x) = x - 1$.

(f) The mapping f from $\mathbb{Z}^+$ to $\mathbb{Z}^+$ defined by $f(x) = x + 1$.

Q.8 For each set A, the *identity function* $1_A : A \to A$ is defined by $1_A(x) = x$ for all x in A. Let $f : A \to B$ and $g : B \to A$ be the functions such that $g \circ f = 1_A$. Show that f is one-to-one and g is onto.

Q.9 Suppose that two functions $g : A \to B$ and $f : B \to C$ and $f \circ g$ denotes the *composition* function.

(a) If $f \circ g$ is one-to-one and g is one-to-one, must f be one-to-one? Explain your answer.

(b) If $f \circ g$ is one-to-one and f is one-to-one, must g be one-to-one? Explain your answer.

(c) If $f \circ g$ is one-to-one, must g be one-to-one? Explain your answer.

(d) If $f \circ g$ is onto, must f be onto? Explain your answer.

(e) If $f \circ g$ is onto, must g be onto? Explain your answer.

Q.10 Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Q.11 Derive the formula for $\sum_{k=1}^{n} k^2$.

Q.12 Derive the formula for $\sum_{k=1}^{n} k^3$.

Q.13 Find a formula for $\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor$, when m is a positive integer.

Q.14 Show that a subset of a countable set is also countable.

Q.15 Assume that $|S|$ denotes the cardinality of the set S. Show that if $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Q.16 Show that if A, B and C are sets such that $|A| \leq |B|$ and $|B| \leq |C|$, then $|A| \leq |C|$.

Q.17 If A is an uncountable set and B is a countable set, must $A - B$ be uncountable?

Q.18 By the Schröder-Bernstein theorem, prove that $(0, 1)$ and $[0, 1]$ have the same cardinality.

Q.19 Show that if $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_{n-1}$, and a_n are real numbers and $a_n \neq 0$, then $f(x)$ is $\Theta(x^n)$.

Q.20 Prove that $n \log n = \Theta(\log n!)$ for all positive integers n.

Q.21

(1) Show that $(\sqrt{2})^{\log n} = O(\sqrt{n})$, where the base of the logarithm is 2.

(2) Arrange the functions

$$n^n,\ (\log n)^2,\ n^{1.0001},\ (1.0001)^n,\ 2^{\sqrt{\log_2 n}},\ n(\log n)^{1001}$$

in a list such that each function is big-O of the next function.

Q.22 Compare the following pairs of functions in terms of order of growth. In each of the following, determine if $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$, $f(n) = \Theta(g(n))$. There is **no need** to explain your answers.

	$f(n)$	$g(n)$	
(1)	$(\log_2 n)^a$	n^b	here $a, b > 0$
(2)	$2^{n \log_2 n}$	$10n!$	
(3)	$\sqrt{n}$	$(\log_2 n)^5$	
(4)	$\frac{n^2}{\log_2 n}$	$(n \log_2 n)^4$	
(5)	$\log_2 n$	$\log_2(66n)$	
(6)	$1000(\log_2 n)^{0.9999}$	$(\log_2 n)^{1.001}$	
(7)	n^2	$n(\log_2 n)^{15}$	

Q.23 Suppose that $T_1(n) = O(f(n))$ and $T_2(n) = O(f(n))$. Determine whether each of the following is true or false. Justify your answers.

(1) $T_1(n) + T_2(n) = O(f(n))$

(2) $\frac{T_1(n)}{T_2(n)} = O(1)$

(3) $T_1(n) = O(T_2(n))$

Q.24 Aliens from another world come to the Earth and tell us that the *3SAT* problem is *solvable* in $O(n^8)$ time. Which of the following statements follow as a consequence?

A. All NP-Complete problems are solvable in polynomial time.

B. All NP-Complete problems are solvable in $O(n^8)$ time.

C. All problems in NP, even those that are not NP-Complete, are solvable in polynomial time.

D. No NP-Complete problem can be solved *faster* than in $O(n^8)$ in the worst case.

E. P = NP.