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Section 3.6

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P79 Q43

由于 U_1 与 U_2 相互独立，所以 $Z = U_1 + U_2$ 的密度函数为:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{U_1}(u_1) f_{U_2}(z - u_1) \mathrm{d}u_1$$

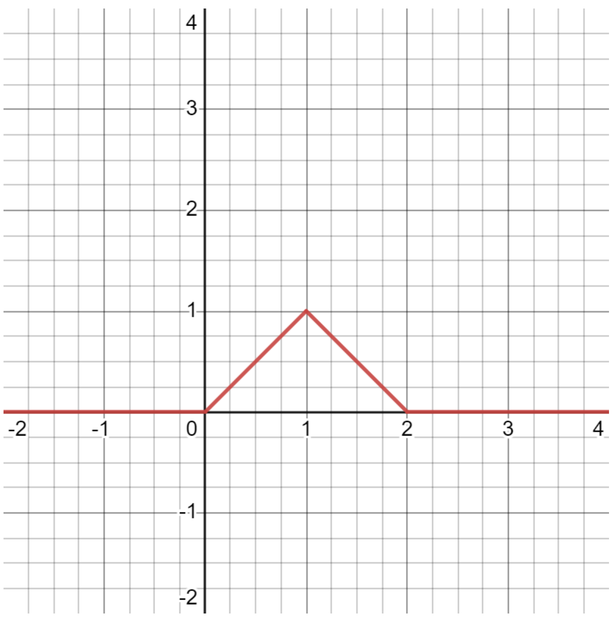
已知 U_1 与 U_2 的密度函数为:

$$f_{U_1}(u_1) = f_{U_2}(u_2) = \begin{cases} 1 & 0 \leqslant u_1, u_2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

所以 $S = U_1 + U_2$ 的密度函数为:

$$f_Z(z) = \begin{cases} \int_0^z f_{U_1}(u_1) f_{U_2}(z - u_1) \mathrm{d}u_1 = z & 0 \leqslant z \leqslant 1 \\ \int_{z-1}^1 f_{U_1}(u_1) f_{U_2}(z - u_1) \mathrm{d}u_1 = 2 - z & 1 < z \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

其图像如下:



P79 Q44

令 $Z = X + Y$，则有:

$$
\begin{aligned}
&P\{Z=0\} = P\{X=0, Y=0\} = \frac{1}{9} \\
&P\{Z=1\} = P\{X=0, Y=1\} + P\{X=1, Y=0\} = \frac{2}{9} \\
&P\{Z=2\} = P\{X=1, Y=1\} + P\{X=0, Y=2\} + P\{X=2, Y=0\} = \frac{1}{3} \\
&P\{Z=3\} = P\{X=1, Y=2\} + P\{X=2, Y=1\} = \frac{2}{9} \\
&P\{Z=4\} = P\{X=2, Y=2\} = \frac{1}{9}
\end{aligned}
$$

所以 $X + Y$ 的频率函数为:

$$
f_Z(z) = \begin{cases} \frac{1}{9} & z = 0 \\ \frac{2}{9} & z = 1 \\ \frac{1}{3} & z = 2 \\ \frac{2}{9} & z = 3 \\ \frac{1}{9} & z = 4 \\ 0 & \text{otherwise} \end{cases}
$$

P79 Q51

令 $z = xy$，则有:

$$
\begin{aligned}
F_Z(z) &= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) |J| \mathrm{d}x \mathrm{d}z \\
&= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|} \mathrm{d}x \mathrm{d}z \\
f_Z(z) &= \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|} \mathrm{d}x
\end{aligned}
$$

此时，将变量 x 改名为 y，有:

$$
f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|} \mathrm{d}y
$$

P79 Q52

假设两个变量为 $[0, 1]$ 上的均匀分布，则有:

$$
f_X(x) = f_Y(y) = \begin{cases} 1 & 0 \leqslant x, y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}
$$

当 $Z \geqslant 1$ 时，$Z = \frac{X}{Y}$ 的分布函数为：

$$\begin{aligned}
f_Z(z) &= P\{Z \leqslant z\} \\
&= P\left\{\frac{X}{Y} \leqslant z\right\} \\
&= P\left\{X \leqslant zY\right\} \\
&= \int_0^{\frac{1}{z}} \int_0^{zy} f_X(x) f_Y(y) \mathrm{d}x \mathrm{d}y + \int_{\frac{1}{z}}^1 \int_0^1 f_X(x) f_Y(y) \mathrm{d}x \mathrm{d}y \\
&= \int_0^{\frac{1}{z}} \int_0^{zy} \mathrm{d}x \mathrm{d}y + \int_{\frac{1}{z}}^1 \int_0^1 \mathrm{d}x \mathrm{d}y \\
&= \frac{1}{2z} + 1 - \frac{1}{z} \\
&= 1 - \frac{1}{2z}
\end{aligned}$$

当 $0 < Z < 1$ 时，$Z = \frac{X}{Y}$ 的分布函数为：

$$\begin{aligned}
f_Z(z) &= \int_0^1 \int_0^{zy} f_X(x) f_Y(y) \mathrm{d}x \mathrm{d}y \\
&= \int_0^1 \int_0^{zy} \mathrm{d}x \mathrm{d}y \\
&= \frac{z}{2}
\end{aligned}$$

综上可得：

$$f_Z(z) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}z} \frac{z}{2} = \frac{1}{2} & 0 < z < 1 \\ \frac{\mathrm{d}}{\mathrm{d}z} \left(1 - \frac{1}{2z}\right) = \frac{1}{2z^2} & z \geqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

P80 Q57

由题可知：

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$
\begin{aligned}
f_{Y_1,Y_2}(y_1,y_2) &= \frac{1}{2\pi}e^{-\frac{1}{2}(2y_1^2-2y_1y_2+y_2^2)} \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \\
&= \frac{1}{2\pi}|J|e^{-\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}(J^{-1})^T\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}J^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \\
&= \frac{1}{2\pi}e^{-\frac{1}{2}x_1^2-\frac{1}{2}x_2^2}
\end{aligned}
$$

因此可知:

$$
(J^{-1})^T\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}J^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

可知当 $a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = -1$ 时，上述方程组成立。

因此 $x_1 = y_1, x_2 = y_2 - y_1$。

补充 1

$$
\begin{aligned}
f_{U,V}(u,v) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\
&= \frac{1}{4\pi}e^{-\frac{1}{4}(u^2+v^2)}
\end{aligned}
$$

其边缘密度函数为:

$$
\begin{aligned}
f_U(u) &= \int_{-\infty}^{+\infty} f_{U,V}(u,v)\mathrm{d}v \\
&= \int_{-\infty}^{+\infty} \frac{1}{4\pi}e^{-\frac{1}{4}(u^2+v^2)}\mathrm{d}v \\
&= \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}u^2}\int_{-\infty}^{+\infty}\frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}v^2}\mathrm{d}v \\
&= \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}u^2}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2}\sqrt{2\pi}}e^{-\frac{1}{2}\frac{v^2}{\sqrt{2}^2}}\mathrm{d}v \\
&= \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}u^2}
\end{aligned}
$$

同理可知 $f_V(v) = \frac{1}{2\sqrt{\pi}}e^{-\frac{1}{4}v^2}$

易知 U 与 V 相互独立，因为 $f_{U,V}(u,v) = f_U(u)f_V(v)$。

补充 2

(1)

边缘密度函数为:

$$\begin{aligned} f_X(x) &= \int_0^{2x} f_{X,Y}(x,y)\mathrm{d}y \\ &= \int_0^{2x} 1\mathrm{d}y \\ &= 2x \\ f_Y(y) &= \int_{\frac{y}{2}}^1 f_{X,Y}(x,y)\mathrm{d}x \\ &= \int_{\frac{y}{2}}^1 1\mathrm{d}x \\ &= 1-\frac{y}{2} \end{aligned}$$

$$f_X(x) = \begin{cases} 2x & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1-\frac{y}{2} & 0 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

(2)

令 $u = 2x - y$，则有:

$$\begin{aligned} F_Z(z) &= \int_0^z \int_{\frac{u}{2}}^1 f_{X,Y}(x,y) \left|\frac{\partial(x,y)}{\partial(x,u)}\right| \mathrm{d}x\mathrm{d}u \\ &= \int_0^z \int_{\frac{u}{2}}^1 \mathrm{d}x\mathrm{d}u \\ &= \int_0^z \left(1-\frac{u}{2}\right)\mathrm{d}u \\ &= z - \frac{z^2}{4}\ (0 \leqslant z \leqslant 2) \end{aligned}$$

所以 Z 的密度函数为:

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z}\left(z-\frac{z^2}{4}\right) = 1-\frac{z}{2}\ (0 \leqslant z \leqslant 2)$$

(3)

$$
\begin{aligned}
P\{Y<\frac{1}{2}|X<\frac{1}{2}\} &= \frac{P\{Y<\frac{1}{2},X<\frac{1}{2}\}}{P\{X<\frac{1}{2}\}} \\
&= \frac{\int_0^{\frac{1}{4}}\int_0^{2x} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x+\int_{\frac{1}{4}}^{\frac{1}{2}}\int_0^{\frac{1}{2}} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x}{\int_0^{\frac{1}{2}} f_X(x)\mathrm{d}x} \\
&= \frac{\frac{3}{16}}{\frac{1}{4}} \\
&= \frac{3}{4}
\end{aligned}
$$