Discrete Mathematics(H)
Southern University of Science and Technology
Mengxuan Wu
12212006

Assignment 2

Mengxuan Wu

Q.1

(a)

True.

Proof.
Assume there exists two sets A_1 and A_2 that $A_1 = A - B$ and $A_2 = A \cap B$. Therefore, we have $A_2 \neq \emptyset$, $A_1 \cap A_2 = \emptyset$, and $A_1 \cup A_2 = A$.

Since $A_2 \neq \emptyset$, there exists an element $x \in A_2$. Since $A_2 = A \cap B$, we have $x \in A$. Also, since $A_1 \cap A_2 = \emptyset$, we have $x \notin A_1$. Therefore, we can find an element $x \in A$ that $x \notin A_1$.

Hence, we can infer that A_1 is a true subset of A. Equivalently, we can say that $(A - B) \subset A$. □

(b)

True.

Proof.
According to the commutative law, the statement on the right side of the implication $A \cap B = B \cap A$ is a tautology. Therefore, the whole statement is always true. □

(c)

False.

Disproof.
For the sets A and B that $A = B = \{1, 2\}$, we have $A \subseteq B$. However, $|A \cup B| = 2 \ngeq 2|A| = 4$. □

Q.2

The "Barber's paradox" can be stated as:

$$\exists x(Barber(x) \rightarrow \forall y(\neg Shaves(y, y) \leftrightarrow Shaves(x, y)))$$

When we let y be the barber himself, we have:

$$\exists x(Barber(x) \rightarrow \neg Shaves(x,x) \leftrightarrow Shaves(x,x))$$

Since the statement on the right side of the implication is a contradiction, the whole statement can only be true when the statement on the left side of the implication is false, which means there does not exist a barber.

Q.3

(1)

(a)

Since there is no element in an empty set, we cannot find any $a+b$ pair. Therefore, the result of pairwise addition is still an empty set.

$$\mathbb{N} \oplus \emptyset = \emptyset$$

(b)

$$\mathbb{N} \oplus \mathbb{N} = \{0, 1, 2, \cdots\} = \mathbb{N}$$

(c)

$$\mathbb{N}^+ \oplus \mathbb{N}^+ = \{2, 3, 4, \cdots\} = \mathbb{N} \backslash \{0, 1\}$$

(d)

$$\mathbb{N}^+ \otimes \mathbb{N}^+ = \{1, 2, 3, \cdots\} = \mathbb{N}^+$$

(2)

$$\{x | x \text{ is positive multiple of } 4\} = E \otimes E$$
$$\{x | x \text{ is positive multiple of } 8\} = E \otimes E \otimes E$$

(3)

$$\{z^2\} = (S \oplus S) \cap S$$

Q.4

(1)

Proof.

$$
\begin{aligned}
(B-A)\cup(C-A) =& (B\cap\bar{A})\cup(C\cap\bar{A}) \\
=& (B\cup C)\cap\bar{A} \\
=& (B\cup C)-A
\end{aligned}
$$

□

(2)

Proof.

$$
\begin{aligned}
(A\cap B)\cap\overline{(B\cap C)}\cap(A\cap C) =& A\cap(B\cap C)\cap\overline{(B\cap C)} \\
=& A\cap\emptyset \\
=& \emptyset
\end{aligned}
$$

□

Q.5

(a)

$$
\begin{aligned}
A =& \{x \mid 0\le x\le 1 \text{ and } x\in\mathbb{R}\} \\
B =& \{x \mid 1\le x\le 2 \text{ and } x\in\mathbb{R}\}
\end{aligned}
$$

Therefore, $A\cap B=\{1\}$, which is finite.

(b)

$$
\begin{aligned}
A =& \{x \mid 0\le x\le 1 \text{ and } x\in\mathbb{R}\}\cup\mathbb{Z} \\
B =& \{x \mid 2\le x\le 3 \text{ and } x\in\mathbb{R}\}\cup\mathbb{Z}
\end{aligned}
$$

Therefore, $A\cap B=\mathbb{Z}$, which is countably infinite.

(c)

$$
A=B=\mathbb{R}
$$

Therefore, $A\cap B=\mathbb{R}$, which is uncountable.

Q.6

We can find that

$$A \oplus B = (A \cup B) - (A \cap B)$$

(a)

$$\begin{aligned} A =& \{x \mid 0 < x \leq 1 \text{ and } x \in \mathbb{R}\} \\ B =& \{x \mid 0 \leq x < 1 \text{ and } x \in \mathbb{R}\} \end{aligned}$$

Therefore, $A \oplus B = \{0, 1\}$, which is finite.

(b)

$$\begin{aligned} A =& \{x \mid 0 \leq x \leq 1 \text{ and } x \in \mathbb{R}\} \cup \mathbb{Z} \\ B =& \{x \mid 0 \leq x \leq 1 \text{ and } x \in \mathbb{R}\} \cup \mathbb{Z}^+ \end{aligned}$$

Therefore, $A \oplus B = \{0, -1, -2, \cdots\}$, which is countably infinite.

(c)

$$\begin{aligned} A =& \{x \mid x \geq 0 \text{ and } x \in \mathbb{R}\} \\ B =& \{x \mid x < 0 \text{ and } x \in \mathbb{R}\} \end{aligned}$$

Therefore, $A \oplus B = \mathbb{R}$, which is uncountable.

Q.7

(a)

Type ii.

The function is not a one-to-one function, since $f(-1) = f(1) = 2$. The function is also not an onto function, since there is no $x \in \mathbb{Z}$ that $f(x) = -1$.

(b)

Type i.

It is not a function since $f(3)$ is not defined.

(c)

Type v.

The function is a one-to-one function, since $f(x) = f(y) \leftrightarrow 8 - 2x = 8 - 2y \leftrightarrow x = y$. The function is also an onto function, since for any $y \in \mathbb{R}$, we can find an $x = \frac{8-y}{2} \in \mathbb{R}$ that $f(x) = 8 - 2x = 8 - 2 \cdot \frac{8-y}{2} = y$.

(d)

Type iii.

The function is not a one-to-one function, since $f(1) = f(1.5) = 2$. The function is an onto function, since for any $y \in \mathbb{Z}$, we can find an $x = y - 1 \in \mathbb{R}$ that $f(x) = \lfloor x + 1 \rfloor = \lfloor y \rfloor = y$.

(e)

Type i.

It is not a function since $f(0.5)$ is not defined.

(f)

Type iv.

The function is a one-to-one function, since $f(x) = f(y) \leftrightarrow x + 1 = y + 1 \leftrightarrow x = y$. The function is not an onto function, since there is no $x \in \mathbb{Z}^+$ that $f(x) = 1$.

Q.8

Proof by contradiction.

It's obvious that the *identity function* 1_A is a one-to-one and onto function.

Assume f is not a one-to-one function, then there exists $x_1, x_2 \in A$ that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Therefore, we have $g(f(x_1)) = g(f(x_2))$, which means $g \circ f$ is not a one-to-one function. This contradicts the fact that 1_A is a one-to-one function.

Assume g is not an onto function, then there exists $y \in A$ that for any x, $g(x) \neq y$. Therefore, we have $g(f(x)) \neq y$, which means $g \circ f$ is not an onto function. This contradicts the fact that 1_A is an onto function. $\square$

Q.9

(a)

False.

Disproof.

f does not must be a one-to-one function.

For example, let $f(x)$ be defined as $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1$. Then f is not a one-to-one function.

We assume $g(x)$ is defined as $1 \mapsto 1$, which is a one-to-one function, and $A = \{1\}$, $B = \{1, 2, 3\}$, $C = \{1\}$. Then $f \circ g$ is defined as $1 \mapsto 1$, which is also a one-to-one function. $\square$

(b)

True.

Proof by contradiction.

Assume g is not a one-to-one function, then there exists $x_1, x_2 \in A$ that $x_1 \neq x_2$ and $g(x_1) = g(x_2)$.

Therefore, we have $f(g(x_1)) = f(g(x_2))$, which means $f \circ g$ is not a one-to-one function. This contradicts the fact that $f \circ g$ is a one-to-one function. □

(c)

True.

Proof by contradiction.

This proof is the same as the proof in (b). □

(d)

True.

Proof by contradiction.

Assume f is not an onto function, then there exists $y \in C$ that for any $x \in B$, $f(x) \neq y$. Therefore, for any $x \in A$, we have $f(g(x)) \neq y$, which means $f \circ g$ is not an onto function. □

(e)

False.

Disproof.

g does not have to be an onto function.

For example, let $g(x)$ be defined as $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1$ with $A = \{1, 2, 3\}$, $B = \{1, 2\}$. Then g is not an onto function.

We assume $f(x)$ is defined as $1 \mapsto 1, 2 \mapsto 1$ and $C = \{1\}$. Then $f \circ g$ is defined as $1 \mapsto 1, 2 \mapsto 1, 3 \mapsto 1$ with $A = \{1, 2, 3\}$, $C = \{1\}$, which is an onto function. □

Q.10

Proof by cases.

Case 1: $c \leq x < c + \frac{1}{3}$ for some $c \in \mathbb{Z}$.

$$
\begin{aligned}
LHS =& \lfloor 3x \rfloor \\
=& 3c \\
RHS =& \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor \\
=& c + c + c \\
=& 3c
\end{aligned}
$$

Case 2: $c + \frac{1}{3} \leq x < c + \frac{2}{3}$ for some $c \in \mathbb{Z}$.

$$\begin{aligned} LHS =& \lfloor 3x \rfloor \\ =& 3c + 1 \\ RHS =& \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor \\ =& c + c + (c + 1) \\ =& 3c + 1 \end{aligned}$$

Case 3: $c + \frac{2}{3} \leq x < c + 1$ for some $c \in \mathbb{Z}$.

$$\begin{aligned} LHS =& \lfloor 3x \rfloor \\ =& 3c + 2 \\ RHS =& \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor \\ =& c + (c + 1) + (c + 1) \\ =& 3c + 2 \end{aligned}$$

$\square$

Q.11

$$\begin{aligned} \sum_{k=1}^{n} [k^3 - (k-1)^3] =& n^3 - (n-1)^3 + (n-1)^3 - (n-2)^3 + \cdots + 2^3 - 1^3 + 1^3 - 0^3 \\ =& n^3 - 0^3 \\ =& n^3 \\ \sum_{k=1}^{n} [k^3 - (k-1)^3] =& \sum_{k=1}^{n} [3k^2 - 3k + 1] \\ =& 3 \sum_{k=1}^{n} k^2 - 3 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \\ =& 3 \sum_{k=1}^{n} k^2 - 3 \cdot \frac{n(n+1)}{2} + n \end{aligned}$$

Therefore, we have

$$\begin{aligned} n^3 =& 3 \sum_{k=1}^{n} k^2 - 3 \cdot \frac{n(n+1)}{2} + n \\ \sum_{k=1}^{n} k^2 =& \frac{n(n+1)}{2} - \frac{n}{3} + \frac{n^3}{3} \\ =& \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

Q.12

$$
\begin{aligned}
\sum_{k=1}^{n}[k^4-(k-1)^4] =& n^4-(n-1)^4+(n-1)^4-(n-2)^4+\cdots+2^4-1^4+1^4-0^4 \\
=& n^4-0^4 \\
=& n^4 \\
\sum_{k=1}^{n}[k^4-(k-1)^4] =& \sum_{k=1}^{n}[4k^3-6k^2+4k-1] \\
=& 4\sum_{k=1}^{n}k^3-6\sum_{k=1}^{n}k^2+4\sum_{k=1}^{n}k-\sum_{k=1}^{n}1 \\
=& 4\sum_{k=1}^{n}k^3-6\cdot\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}-n
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
n^4 =& 4\sum_{k=1}^{n}k^3-6\cdot\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}-n \\
\sum_{k=1}^{n}k^3 =& \frac{n^4+n(n+1)(2n+1)-2n(n+1)+n}{4} \\
=& \frac{n^2(n+1)^2}{4}
\end{aligned}
$$

Q.13

$$
\begin{aligned}
\sum_{k=0}^{m}\lfloor\sqrt{k}\rfloor =& \lfloor\sqrt{0}\rfloor+\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\cdots+\lfloor\sqrt{m}\rfloor \\
=& (1^2-0^2)\times 0+(2^2-1^2)\times 1+(3^2-2^2)\times 2+\cdots
\end{aligned}
$$

Let $n = \lfloor \sqrt{m} \rfloor$, we have

$$\begin{aligned}
\sum_{k=0}^{m} \lfloor \sqrt{k} \rfloor =& (1^2 - 0^2) \times 0 + (2^2 - 1^2) \times 1 + \cdots + (n^2 - (n-1)^2) \cdot (n-1) \\
& + (m - n^2 + 1) \cdot n \\
=& \sum_{k=0}^{n-1} [(k+1)^2 - k^2] \cdot k + (m - n^2 + 1) \cdot n \\
=& \sum_{k=0}^{n-1} [2k^2 + k] + (m - n^2 + 1) \cdot n \\
=& \frac{2n(n-1)(2n-1)}{6} + \frac{n(n-1)}{2} + (m - n^2 + 1) \cdot n \\
=& \frac{n(n-1)(4n+1)}{6} + (m - n^2 + 1) \cdot n \\
=& -\frac{n(n-1)(2n+5)}{6} + mn \\
=& -\frac{\lfloor \sqrt{m} \rfloor (\lfloor \sqrt{m} \rfloor - 1)(2\lfloor \sqrt{m} \rfloor + 5)}{6} + m\lfloor \sqrt{m} \rfloor
\end{aligned}$$

Q.14

Proof.

Let A be a countable set and B be a subset of A, denoted by $B \subseteq A$.

Case 1: B is finite:

Then B is obviously countable.

Case 2: B is infinite:

Since B is a subset of A, the mapping $f(x) = x$ from B to A is a one-to-one function. Hence, $|B| \leq |A| = |\mathbb{N}^+|$.

Let $A = \{a_1, a_2, a_3, \cdots\}$. For every $a_n \in B$, assume a_n is the mth element in B, then we can define a mapping $g(m) = a_n$ from $\mathbb{N}^+$ to B, which is also a one-to-one function. Hence, $|B| \geq |\mathbb{N}^+|$.

Therefore, we have $|B| = |\mathbb{N}^+|$, which means B is countable. $\square$

Q.15

Proof.

If the sets are finite, then it is obviously true.

If the sets are infinite, since $|A| = |B|$ and $|B| = |C|$, we can find a one-to-one function f_1 from A to B and a one-to-one function f_2 from B to C. Therefore, the mapping $f_2 \circ f_1$ from A to C is also a one-to-one function.

Also, we can find a one-to-one function g_1 from B to A and a one-to-one function g_2 from C to B. Therefore, the mapping $g_1 \circ g_2$ from C to A is also a one-to-one function.

Hence, we have $|A| = |C|$. $\square$

Q.16

Proof.

If the sets are finite, then it is obviously true.

If the sets are infinite, since $|A| \leq |B|$ and $|B| \leq |C|$,we can find a one-to-one function f_1 from A to B and a one-to-one function f_2 from B to C. Therefore, the mapping $f_2 \circ f_1$ from A to C is also a one-to-one function.

Hence, we have $|A| \leq |C|$. $\square$

Q.17

True.

Proof by contradiction.

Let us assume $A - B$ is a countable set. Let $A - B = \{a_1, a_2, a_3, \cdots\}$ and $B = \{b_1, b_2, b_3, \cdots\}$, we can find such mapping f:

$$\begin{aligned} f(a_n) =& n \\ f(b_n) =& -n \end{aligned}$$

Since $(A - B) \cup B = A$, f is a mapping from A to $\mathbb{Z}\backslash\{0\}$. We can infer that $|A| \leq |\mathbb{Z}|$. Therefore, A is a countable set, which contradicts the fact that A is an uncountable set. $\square$

Q.18

Proof.

Let $A = \{x \mid x \in [0, 1] \text{ and } x \in \mathbb{R}\}$ and $B = \{x \mid x \in (0, 1) \text{ and } x \in \mathbb{R}\}$.

Mapping $f(x) = x$ from B to A is a one-to-one function. Therefore, $|B| \leq |A|$.

Mapping $g(x) = \frac{x}{2} + \frac{1}{4}$ from A to B is a one-to-one function. Therefore, $|A| \leq |B|$.

Using the Schröder-Bernstein theorem, we have $|A| = |B|$. Equivalently, we can say that $[0, 1]$ and $(0, 1)$ have the same cardinality. $\square$

Q.19

Proof.

First, we show that $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = O(x^n)$.

$$\begin{aligned} |f(x)| =& |a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0| \\ \leq& |a_n| x^n + |a_{n-1}| x^{n-1} + \cdots + |a_1| x + |a_0| \\ \leq& (|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_0|) x^n \\ =& c \cdot x^n \end{aligned}$$

We can conclude that:

$$|f(x)| \leq c \cdot x^n$$

Then, we show that $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = \Omega(x^n)$.

$$\begin{aligned}|f(x)| =& |a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0| \\ =& |a_n + a_{n-1}x^{-1} + \cdots + a_1 x^{1-n} + a_0 x^{-n}| \cdot x^n \\ \geq& \big||a_n| - |a_{n-1}x^{-1} + \cdots + a_1 x^{1-n} + a_0 x^{-n}|\big| \cdot x^n\end{aligned}$$

$|a_{n-1}x^{-1} + \cdots + a_1 x^{1-n} + a_0 x^{-n}|$ approaches 0 as x approaches infinity. Since $a_n \neq 0$, we can find a c that $0 < c < |a_n|$ and $|a_n| - |a_{n-1}x^{-1} + \cdots + a_1 x^{1-n} + a_0 x^{-n}| \geq c$ when x is large enough. Therefore, we have:

$$|f(x)| \geq c \cdot x^n$$

Therefore, we have $f(x) = \Theta(x^n)$. □

Q.20

Proof.

First, we show that $n \log n = O(\log n!)$.

$$\begin{aligned}2\log n! =& 2(\log 1 + \log 2 + \cdots + \log n) \\ =& (\log 1 + \log n) + [\log 2 + \log(n-1)] + \cdots + (\log n + \log 1) \\ =& \sum_{k=1}^{n} \log[k \cdot (n+1-k)] \\ =& \sum_{k=\frac{n-1}{2}}^{\frac{1-n}{2}} \log[(\frac{n+1}{2} - k) \cdot (\frac{n+1}{2} + k)] \\ =& \sum_{k=\frac{n-1}{2}}^{\frac{1-n}{2}} \log[(\frac{n+1}{2})^2 - k^2]\end{aligned}$$

It's obvious that the term $(\frac{n+1}{2})^2 - k^2$ reach its maximum when $k = 0$, and reach its minimum when $k = \frac{n-1}{2}$ or $k = \frac{1-n}{2}$. Therefore, we have:

$$\begin{aligned}2\log n! =& \sum_{k=\frac{n-1}{2}}^{\frac{1-n}{2}} \log[(\frac{n+1}{2})^2 - k^2] \\ \geq& \sum_{k=\frac{n-1}{2}}^{\frac{1-n}{2}} \log[(\frac{n+1}{2})^2 - (\frac{n-1}{2})^2] \\ =& \sum_{k=\frac{n-1}{2}}^{\frac{1-n}{2}} \log n \\ =& n\log n\end{aligned}$$

We can conclude that:

$$n \log n \leq 2 \log n!$$

Then, we show that $n \log n = \Omega(\log n!)$.

$$\begin{aligned} n^n &\geq n! \\ \log n^n &\geq \log n! \\ n \log n &\geq \log n! \end{aligned}$$

Since $\log n! \leq n \log n \leq 2 \log n!$, we have $n \log n = \Theta(\log n!)$. □

Q.21

(1)

Proof.

$$\begin{aligned} (\sqrt{2})^{\log_2 n} =& 2^{\frac{1}{2}\log_2 n} \\ =& 2^{\log_2 \sqrt{n}} \\ =& \sqrt{n} \\ =& O(\sqrt{n}) \end{aligned}$$

□

(2)

$$(\log n)^2,\ 2^{\sqrt{\log_2 n}},\ n(\log n)^{1001},\ n^{1.0001},\ (1.0001)^n,\ n^n$$

Q.22

(1)

$$f(n) = O(g(n))$$

(2)

$$f(n) = \Omega(g(n))$$

(3)

$$f(n) = \Omega(g(n))$$

(4)

$$f(n) = O(g(n))$$

(5)

$$f(n) = \Theta(g(n))$$

(6)

$$f(n) = O(g(n))$$

(7)

$$f(n) = \Omega(g(n))$$

Q.23

(1)

True.

Proof.
Assume $T_1(n) \leq c_1 \cdot f(n)$ for $n \geq n_1$ and $T_2(n) \leq c_2 \cdot f(n)$ for $n \geq n_2$. Then we have $T_1(n) + T_2(n) \leq (c_1 + c_2) \cdot f(n)$ for $n \geq \max(n_1, n_2)$. Therefore, we have $T_1(n) + T_2(n) = O(f(n))$. □

(2)

False.

Disproof.
Let $f(n) = n^2$, $T_1(n) = n^2$ and $T_2(n) = n$. Then we have $\frac{T_1(n)}{T_2(n)} = n$, which is not in $O(1)$. □

(3)

False.

Disproof.
Let $f(n) = n^2$, $T_1(n) = n^2$ and $T_2(n) = n$. Then $T_1(n)$ is not in $O(T_2(n))$. □

Q.24

A, C, and E.

Proof.

Let the *3SAT* problem be L_0.

Since the *3SAT* problem is NP-complete, we can infer that for every L in *NP*, there exists a polynomial-time reduction f from L to L_0. Equivalently, we have $L \leq_p L_0$ for every L in *NP*. Since L_0 is solvable in $O(n^8)$, we can infer that every L in *NP* is solvable in polynomial time, which means $NP = P$. Hence, A, C, and E are true.

Since the transformation function in f is in polynomial time but not necessarily in $O(n^8)$ time, we can't say that every L in *NP* is solvable in $O(n^8)$ time. Hence, B is false.

By limiting the input of the *3SAT* problem, we can create another NP-complete problem L_1 that is solvable in $O(n^7)$ time. It's obvious the transformation function is in $O(1)$ time. Therefore, we can infer that L_1 is solvable in $O(n^7)$ time, which is faster than $O(n^8)$ time. Hence, D is false. □