南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

07 Network Flow

CS216 Algorithm Design and Analysis (H)

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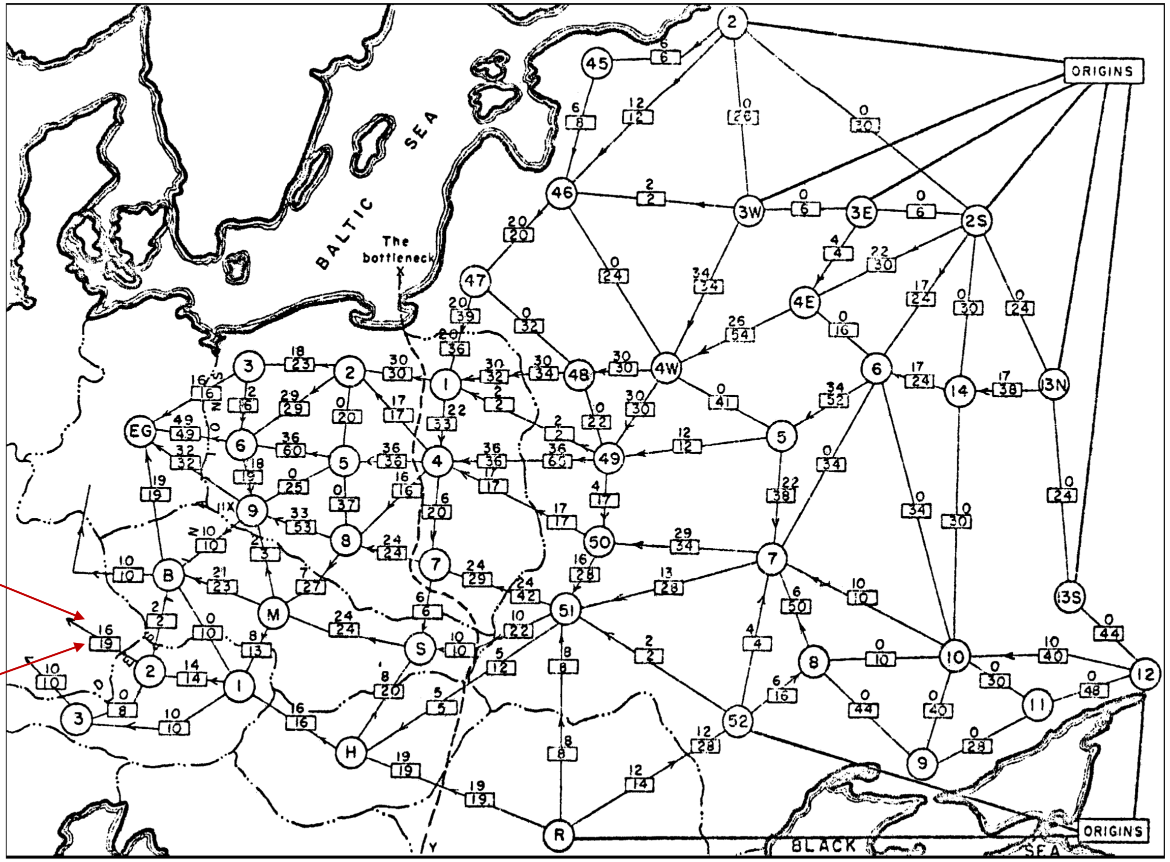
Maximum Flow Application (Tolstoǐ 1930s)

- **Soviet Union goal.** Maximize flow of supplies to Eastern Europe.

flow

capacity



rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

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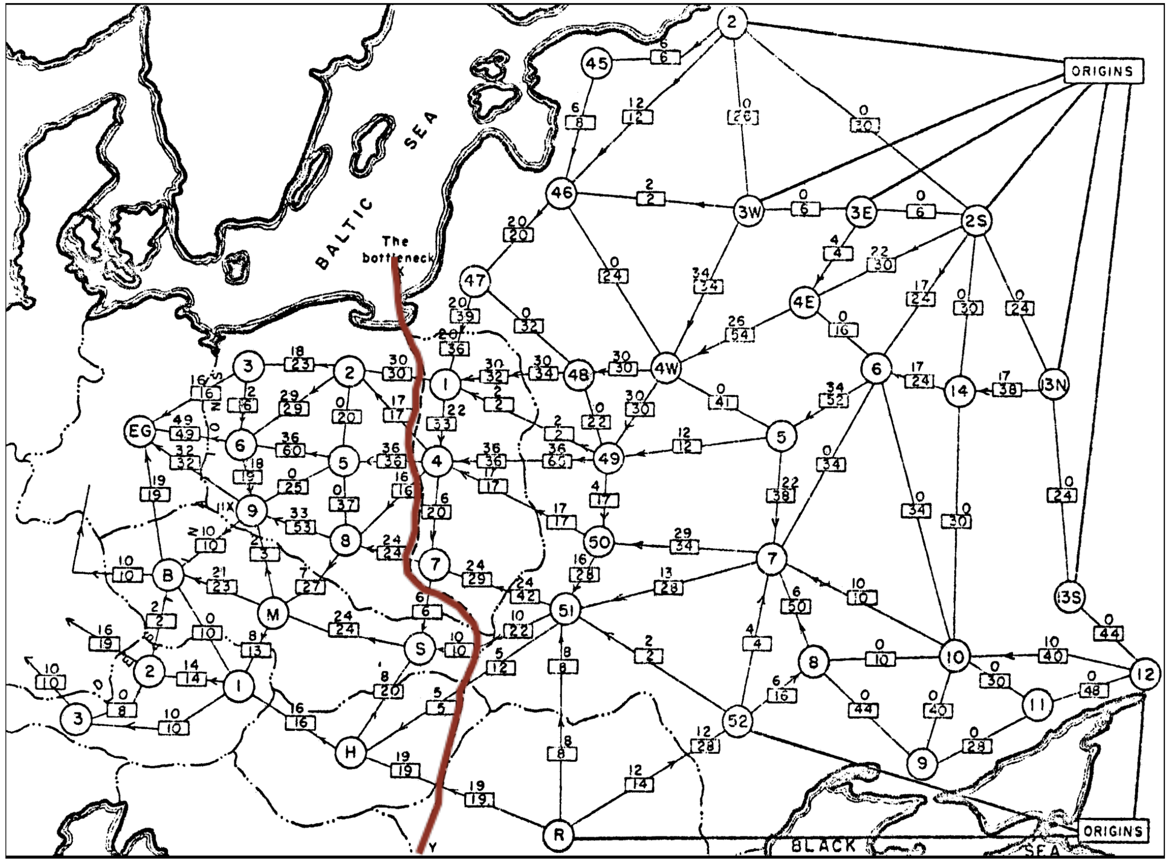
Minimum Cut Application (RAND 1950s)

- **"Free world" goal.** Cut supplies (if Cold War turns into real war).



rail network connecting Soviet Union with Eastern European countries
(map declassified by Pentagon in 1999)

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Maximum Flow and Minimum Cut

- **Max-flow and min-cut problems.**
  - Beautiful mathematical duality.
  - Cornerstone problems in combinatorial optimization.

- **They are widely applicable models.**
  - Data mining, open-pit mining, bipartite matching, network reliability, baseball elimination, image segmentation, network connectivity, Markov random fields, distributed computing, security of statistical data, egalitarian stable matching, network intrusion detection, multi-camera scene reconstruction, sensor placement for homeland security, etc.

we will learn some of the applications later

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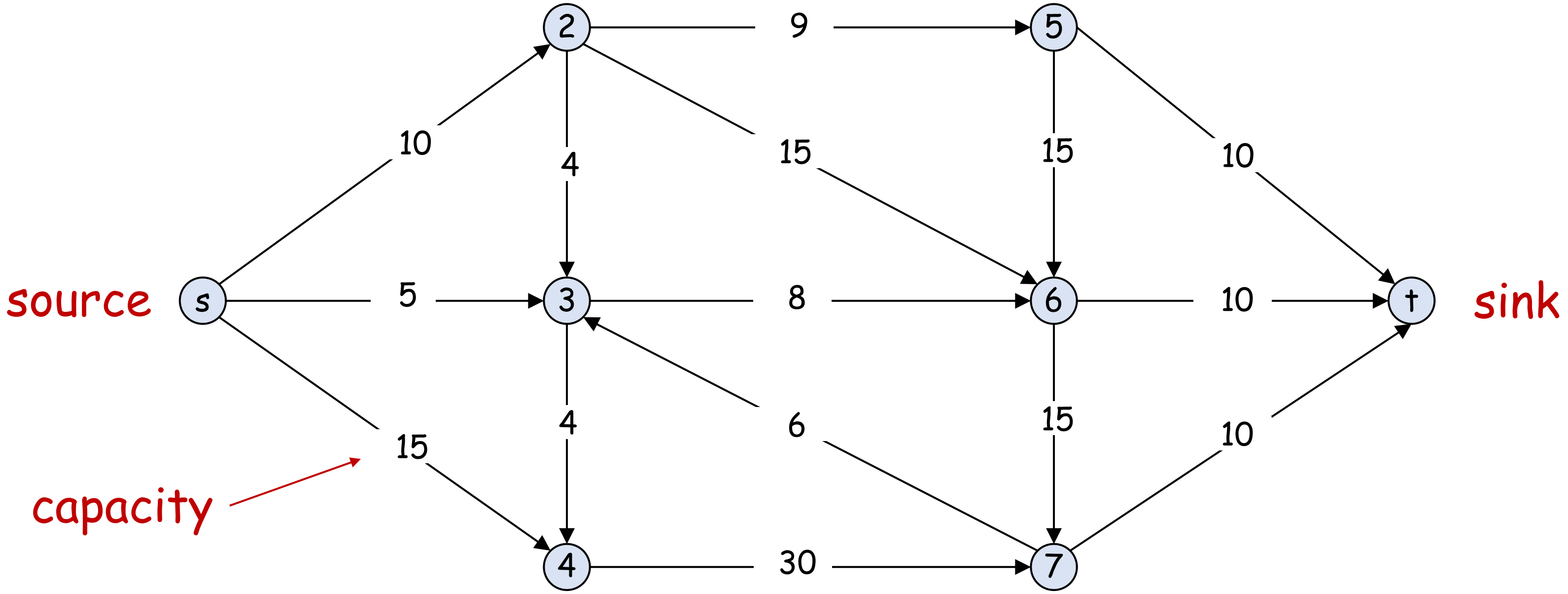
1. Max Flow and Min Cut

Flow Network

- **Def.** A flow network is a tuple $G = (V, E, s, t, c)$.
  - Directed graph $G = (V, E)$ with source s and sink t, and no parallel edges.
  - Capacity $c(e) \geq 0$ for each edge e.

assume all nodes are reachable from s

- **Intuition.** Material flowing through a transportation network, originating from source and being sent to sink.



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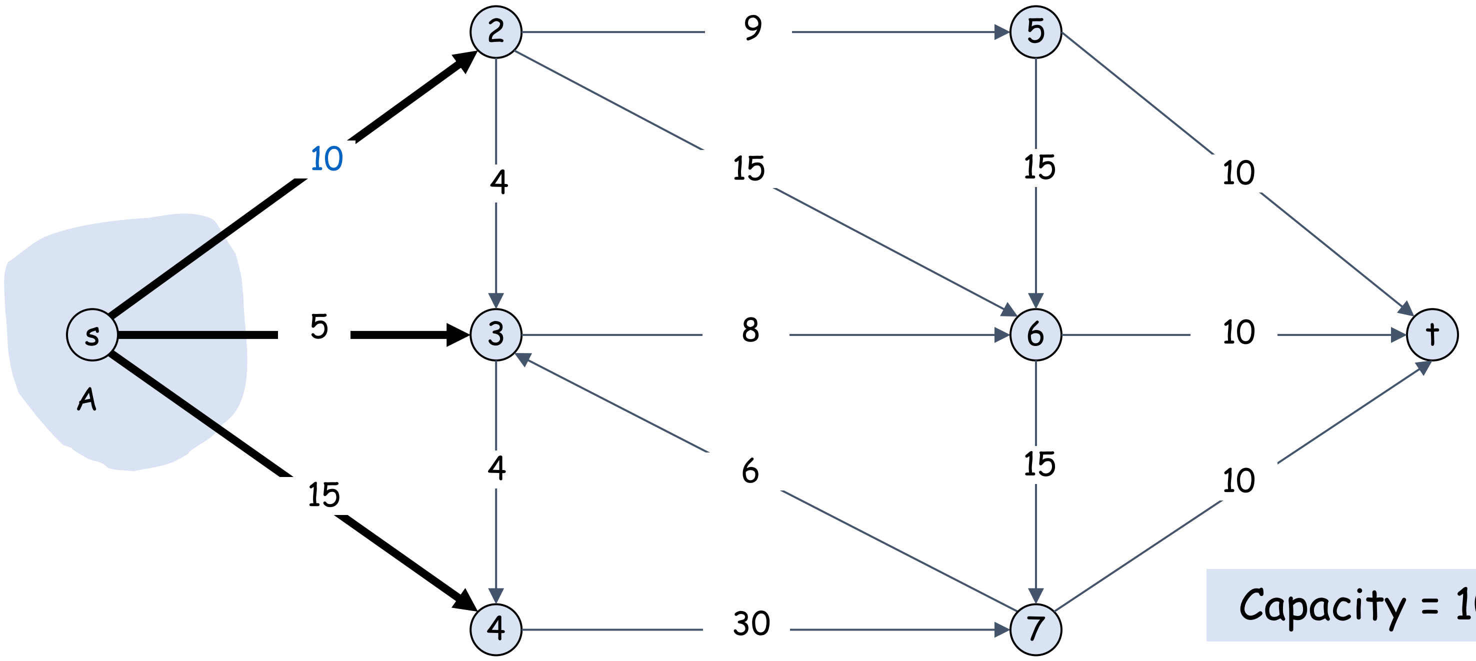
Minimum-Cut Problem

- **Def.** An *st*-cut (or cut) is a partition *(A, B)* of *V* with $s \in A$ and $t \in B$.
- **Def.** The capacity of a cut *(A, B)* is $c(A, B) = \sum_{e \text{ out of } A} c_e$

- **Example 1:**



Capacity = 10 + 5 + 15 = 30

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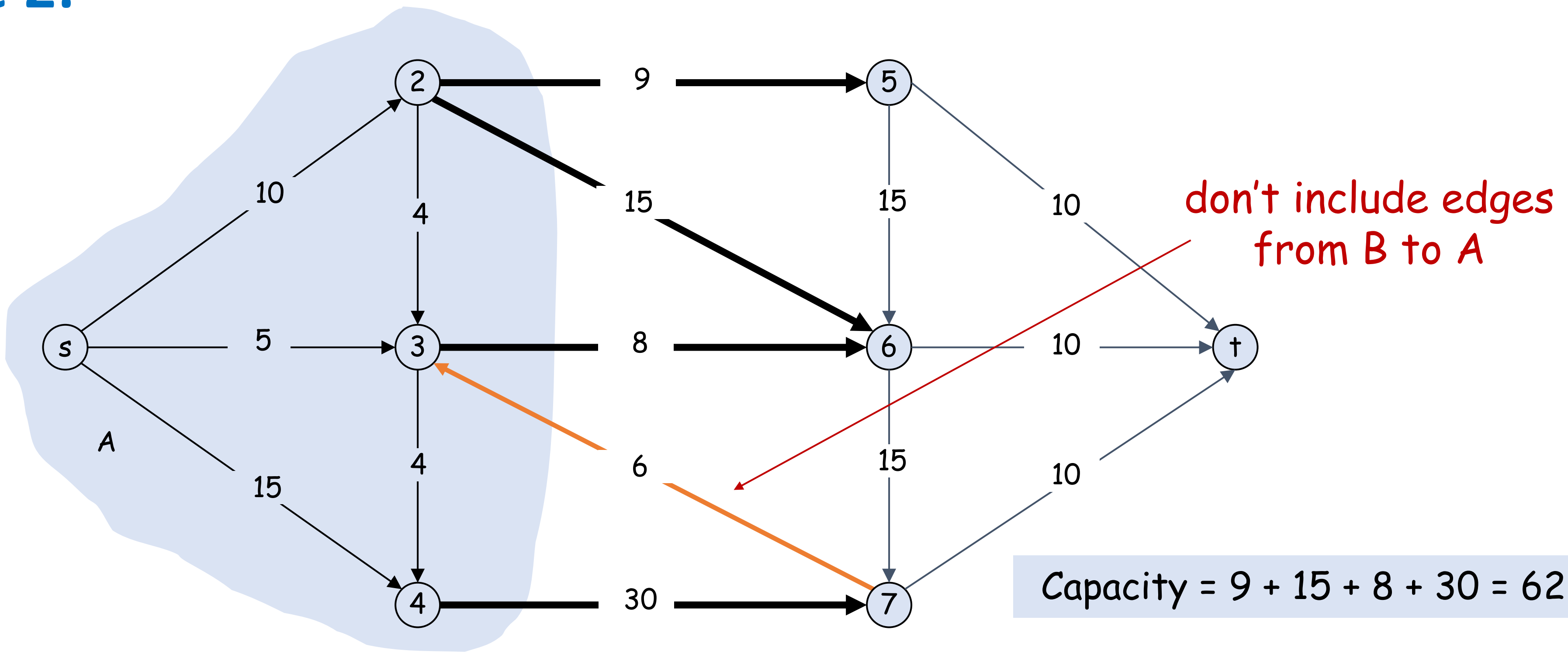
Minimum-Cut Problem

- **Def.** An *st*-cut (or cut) is a partition *(A, B)* of *V* with *s* ∈ *A* and *t* ∈ *B*.
- **Def.** The capacity of a cut *(A, B)* is $c(A, B) = \sum_{e \text{ out of } A} c_e$

- **Example 2:**



Capacity = 9 + 15 + 8 + 30 = 62

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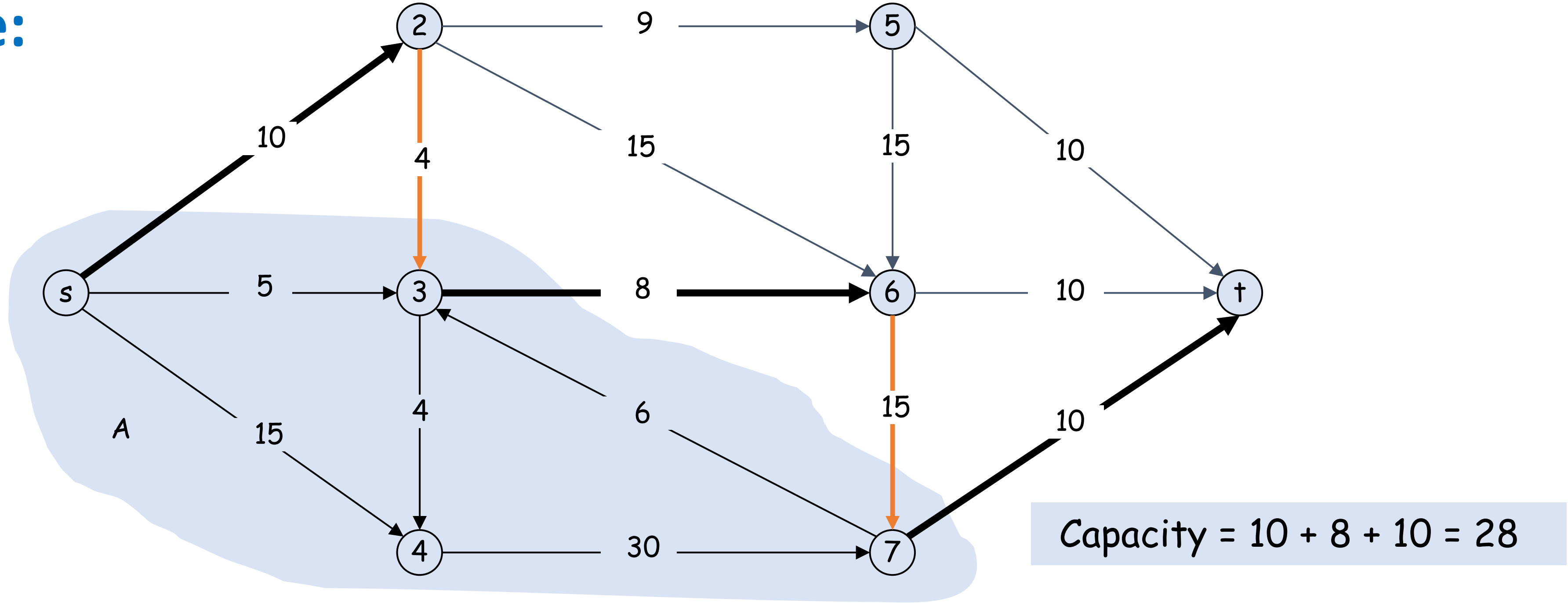
Minimum-Cut Problem

- **Def.** An *st*-cut (or cut) is a partition *(A, B)* of *V* with *s* ∈ *A* and *t* ∈ *B*.
- **Def.** The capacity of a cut *(A, B)* is $c(A, B) = \sum_{e \text{ out of } A} c_e$
- **Min-cut problem.** Find a cut of minimum capacity.

- **Example:**



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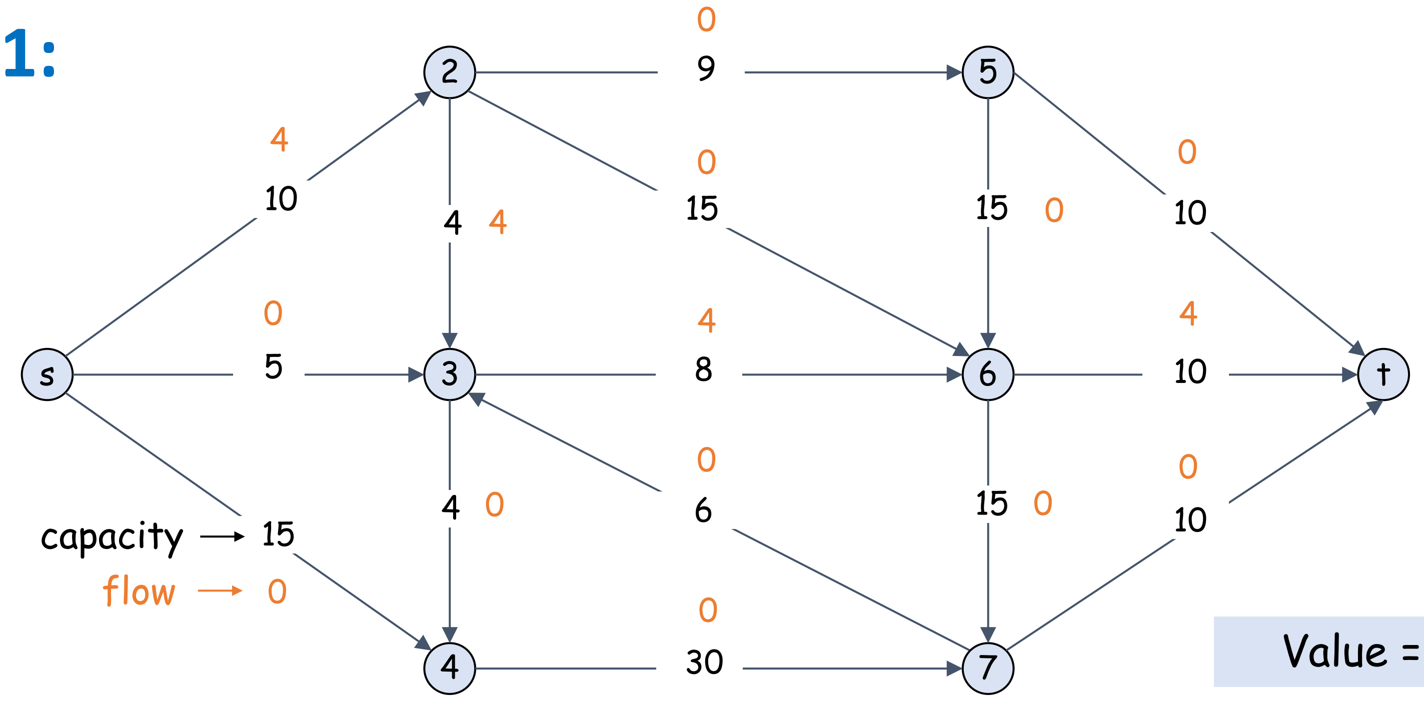
Maximum-Flow Problem

- **Def.** An *st*-flow (or flow) f is a function that satisfies
  - [Capacity] For each $e \in E: 0 \le f(e) \le c_e$
  - [Flow conservation] For each $v \in V - \{s,t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

simple notation: $f^{in}(v) = f^{out}(v)$

- **Def.** The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e) = f^{out}(s)$.

- **Example 1:**



Value = 4 + 0 + 0 = 4

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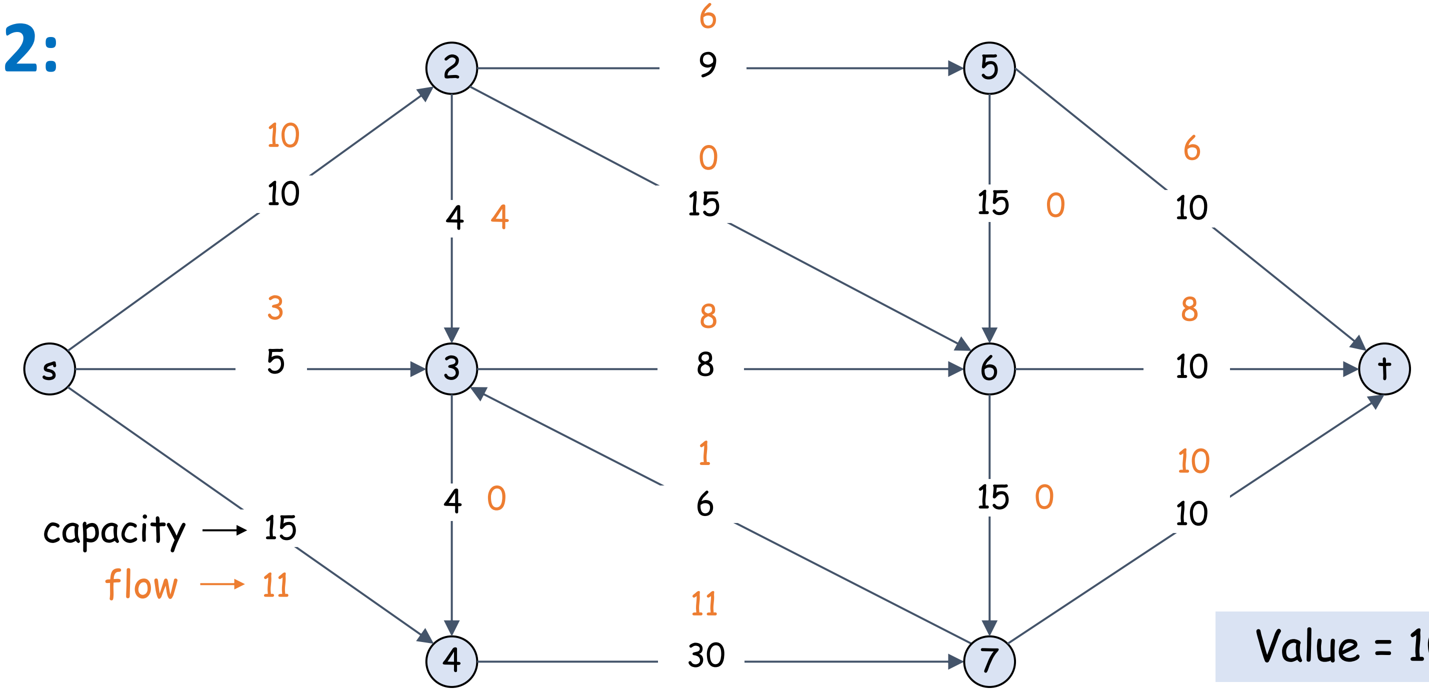
Maximum-Flow Problem

- **Def.** An *st*-flow (or flow) f is a function that satisfies
  - [Capacity] For each $e \in E$: $0 \le f(e) \le c_e$
  - [Flow conservation] For each $v \in V - \{s,t\}$: $\sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

simple notation:
$f^{in}(v) = f^{out}(v)$

- **Def.** The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e) = f^{out}(s)$.

- **Example 2:**



Value = 10 + 3 + 11 = 24

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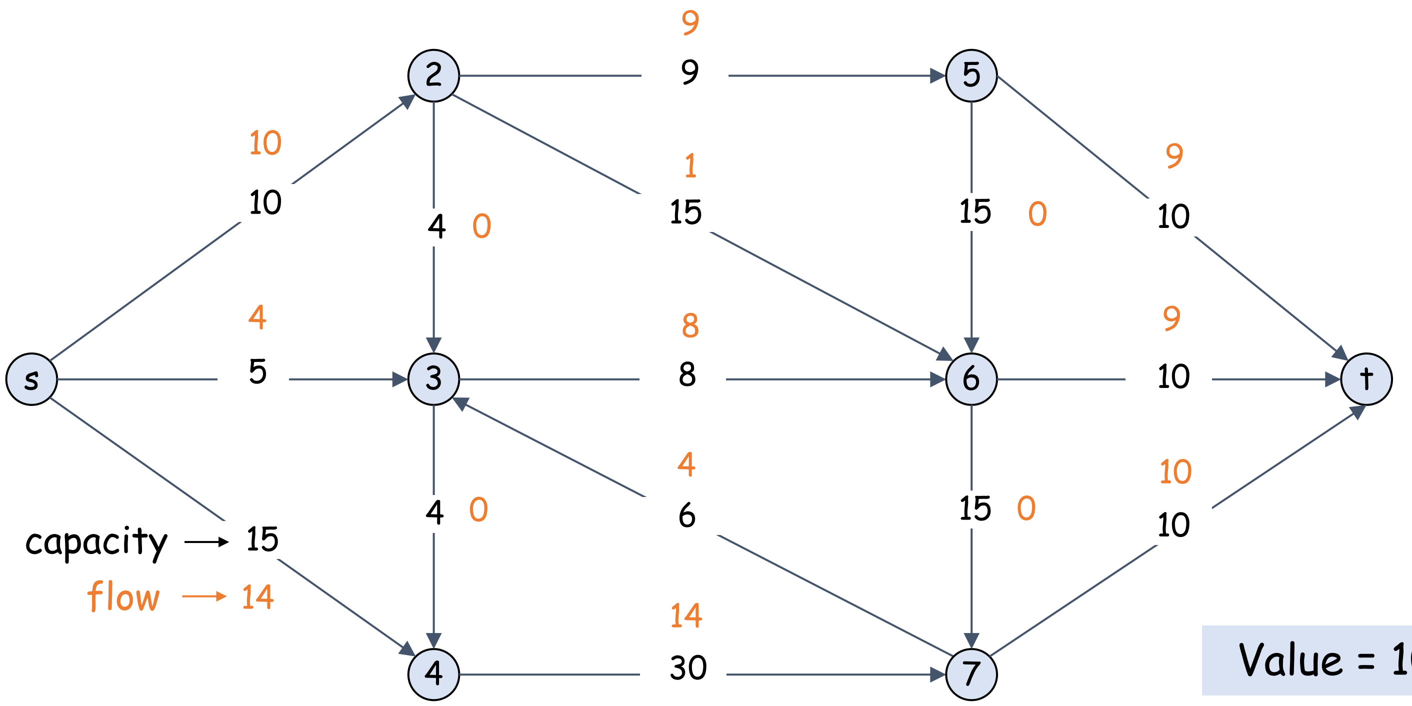
Maximum-Flow Problem

- **Def.** An *st*-flow (or flow) f is a function that satisfies
- **Def.** The value of a flow f is $v(f) = \sum_{e \text{ out of } s} f(e)$
- **Max-flow problem.** Find a flow of maximum value.

- **Example:**



Value = 10 + 4 + 14 = 28

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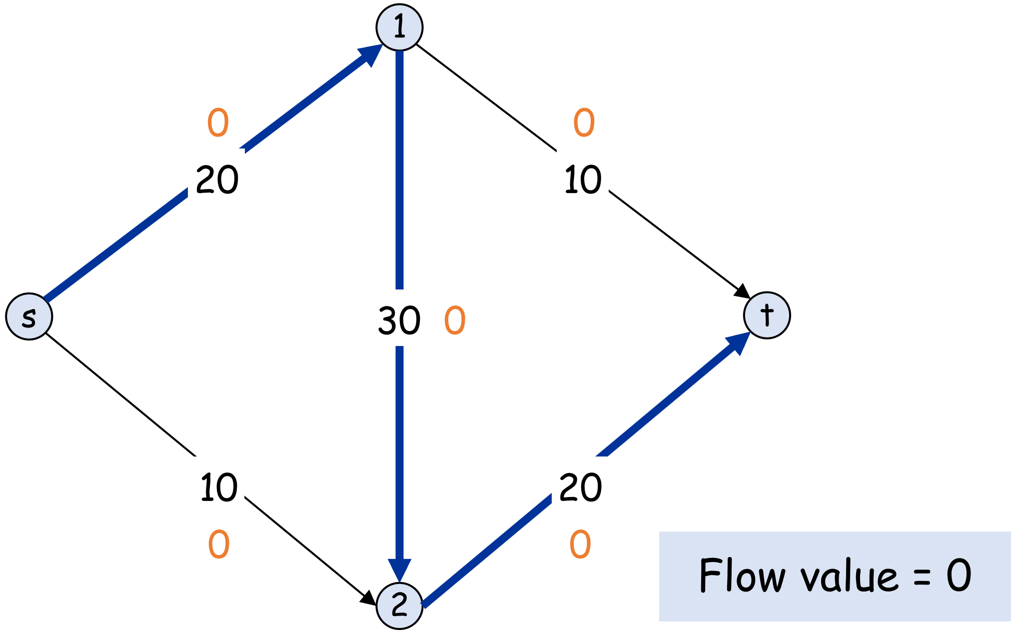
2. Ford-Fulkerson Algorithm

Toward a Max-Flow Algorithm

- **Greedy algorithm:**
  - Start with *f(e) = 0* for all edges *e* ∈ *E*.
  - Find an *s-t* path *P* where each edge has *f(e) < c(e)*.
  - Augment flow along path *P*.
  - Repeat until you get stuck.



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Toward a Max-Flow Algorithm

- **Greedy algorithm:**
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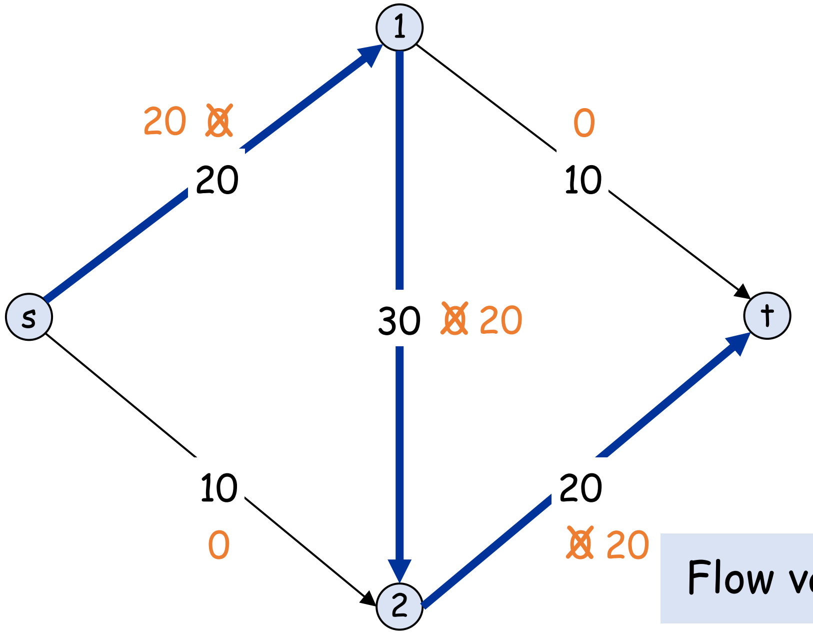

Flow value = 20

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Toward a Max-Flow Algorithm

- **Greedy algorithm:**
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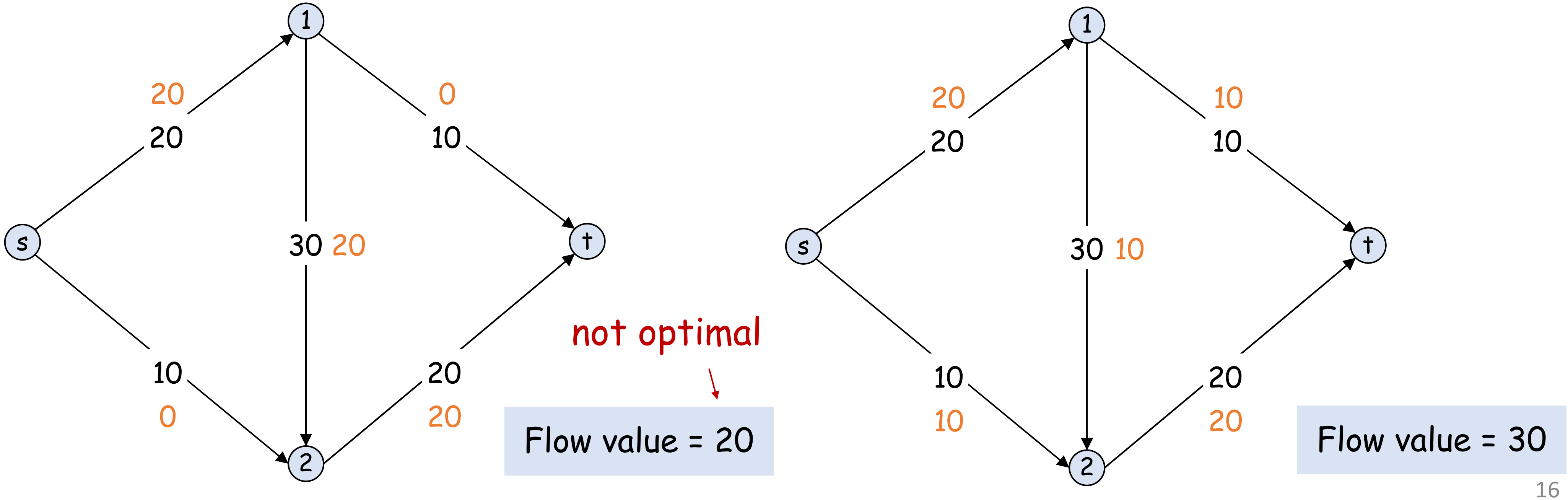

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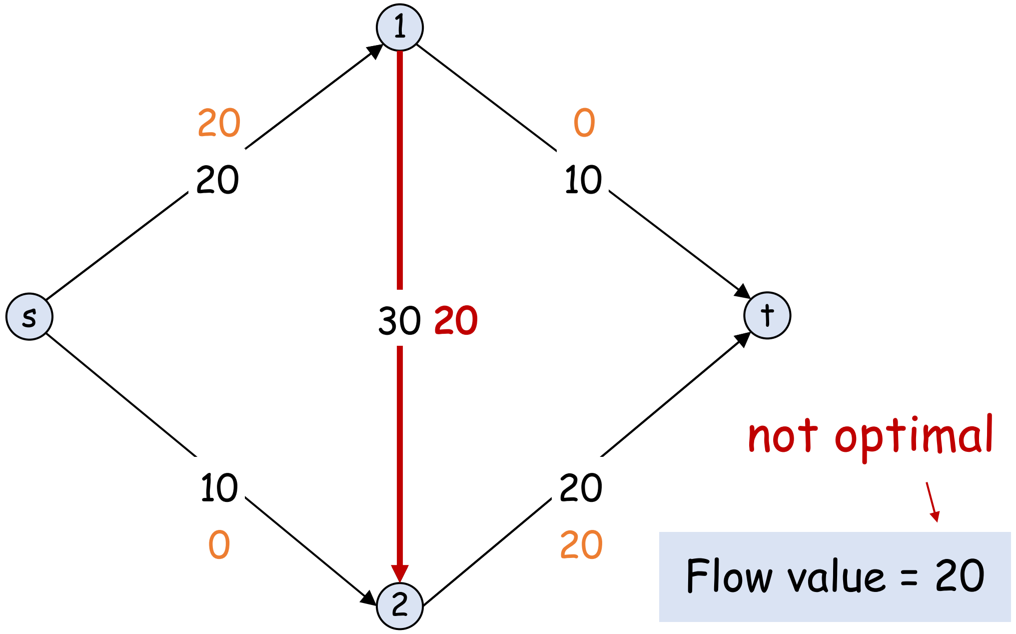
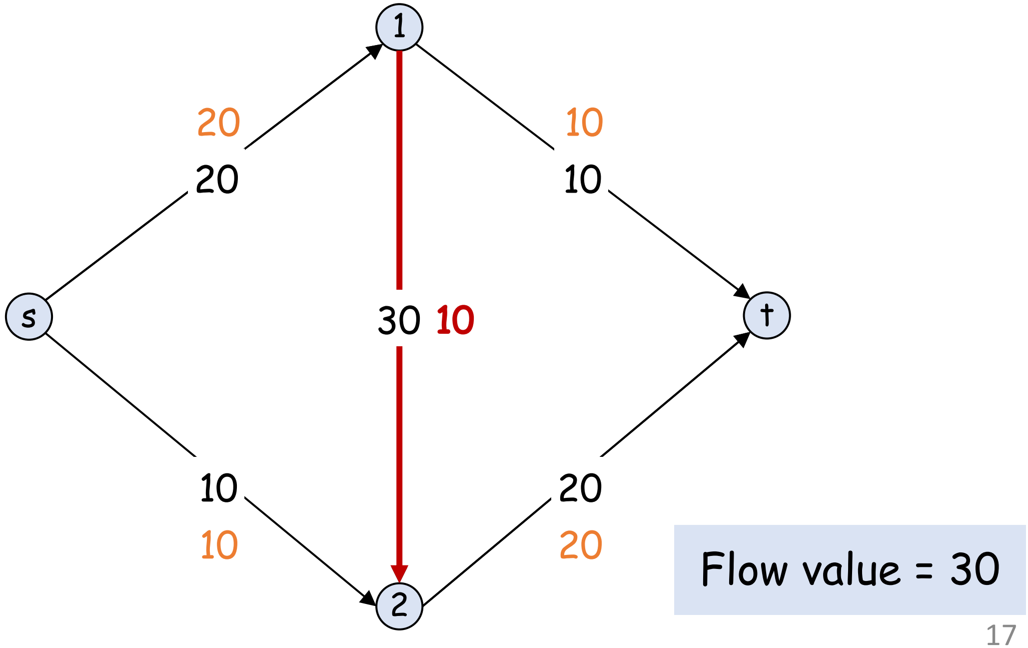
Toward a Max-Flow Algorithm

- **Q.** Why does the greedy algorithm fail?
- **A.** Once flow on an edge is increased, it never decreases.

- **Bottom line.** Need some mechanism to "undo" a bad decision.





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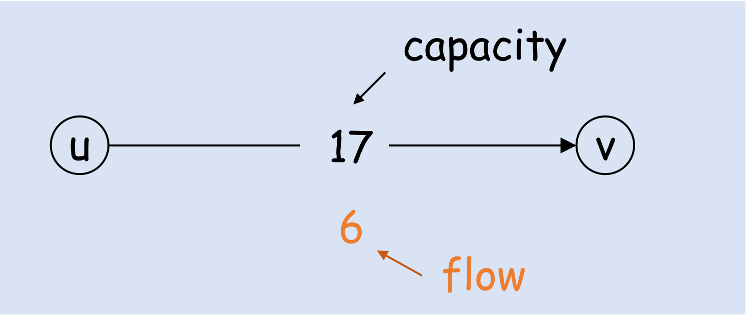
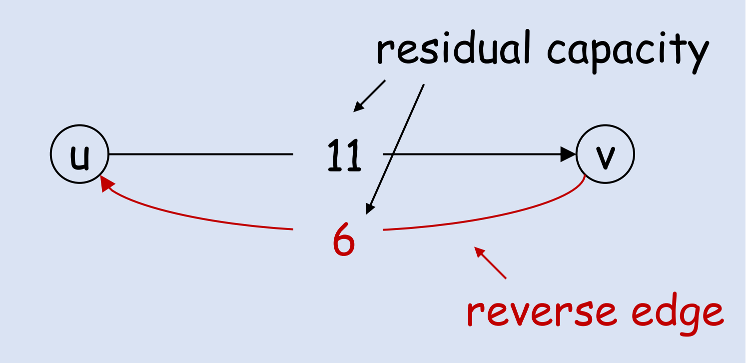
Residual Network

- **Original edge:** $e = (u, v) \in E$
  - flow $f(e)$, capacity $c(e)$
- **Reverse edge:** $e^R = (v, u)$
  - used to "undo" flow
- **Residual capacity:** c_f
  - original edge: $c_f(e) = c(e) - f(e)$
  - reverse edge: $c_f(e^R) = f(e)$
- **Residual network:** $G_f = (V, E_f, s, t, c_f)$
  - $E_f = \{e: f(e) < c(e)\} \cup \{e^R: f(e) > 0\}$: edges with positive residual capacity
- **Key property.** f' is a flow in G_f if and only if $f + f'$ is a flow in G.

flow on a reverse edge negates flow on corresponding original forward edge





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Augmenting Path

- **Def.** An augmenting path P is a simple s-t path in the residual network G_f.
- **Def.** The bottleneck capacity of an augmenting path P in G_f, denoted by bottleneck(G_f, P), is the minimum residual capacity of any edge in P.
- **Key property.** Consider a flow network $G = (V, E, s, t, c)$ with flow f and augmenting path P in G_f. After calling the following Augment(f, c, P), the returned f' is a flow in G and $v(f') = v(f)$ + bottleneck(G_f, P).

```
Augment(f, c, P) {
   δ = bottleneck(Gf, P)
   foreach e ∈ P {
      if (e ∈ E) f(e) = f(e) + δ
      else       f(eR) = f(eR) - δ
   }
   return f
}
```

e is a reverse edge
so e^R is a forward edge

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Ford-Fulkerson Algorithm

- **Ford-Fulkerson (FF) algorithm.**
  - Start with $f(e) = 0$ for each edge $e \in E$.
  - Find an s-t path P in the residual network G_f .
  - Augment flow along path P.
  - Repeat until you get stuck.

```
Ford-Fulkerson(G, s, t, c) {
   foreach e ∈ E: f(e) = 0
   Gf = residual network of G with respect to flow f

   while (there exists an augmenting path P) {
      f = Augment(f, c, P)
      update Gf
   }
   return f
}
```


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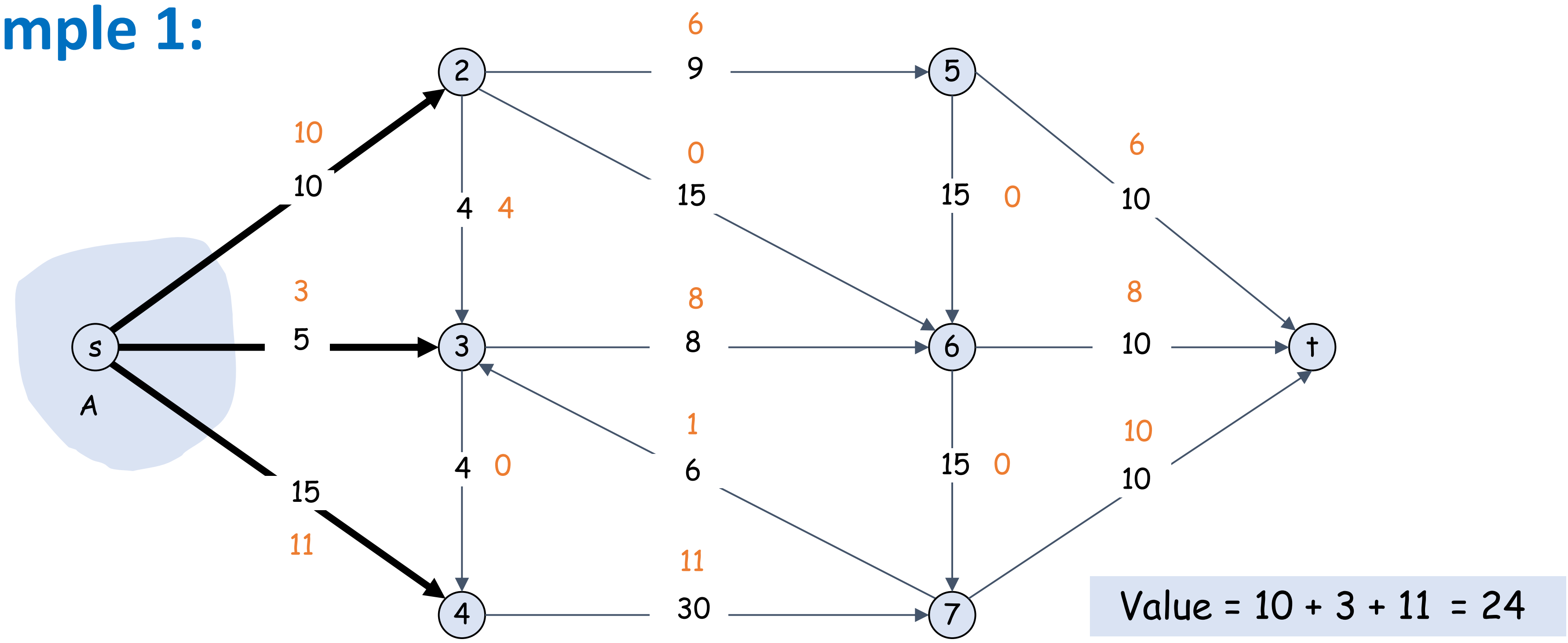
3. Max-Flow Min-Cut Theorem

Relationship between Flows and Cuts

- **Flow value lemma.** Let *f* be any flow, and let *(A, B)* be any cut. Then, the value of the flow *f* equals the net flow across the cut *(A, B)*.

$$v(f) = f^{out}(A) - f^{in}(A)$$

- **Example 1:**



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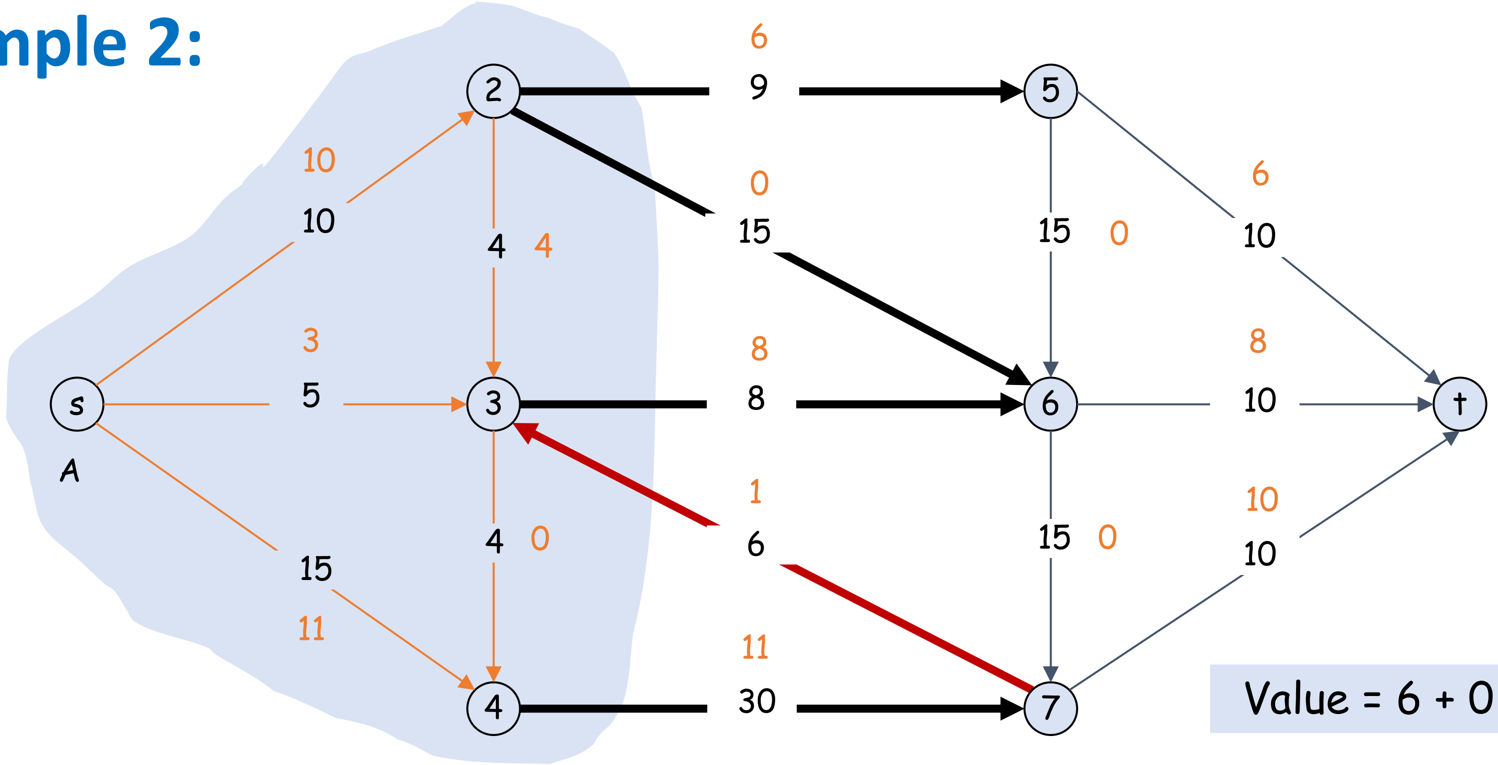
$$v(f) = f^{out}(A) - f^{in}(A)$$

- **Example 2:**



Value = 6 + 0 + 8 - 1 + 11 = 24

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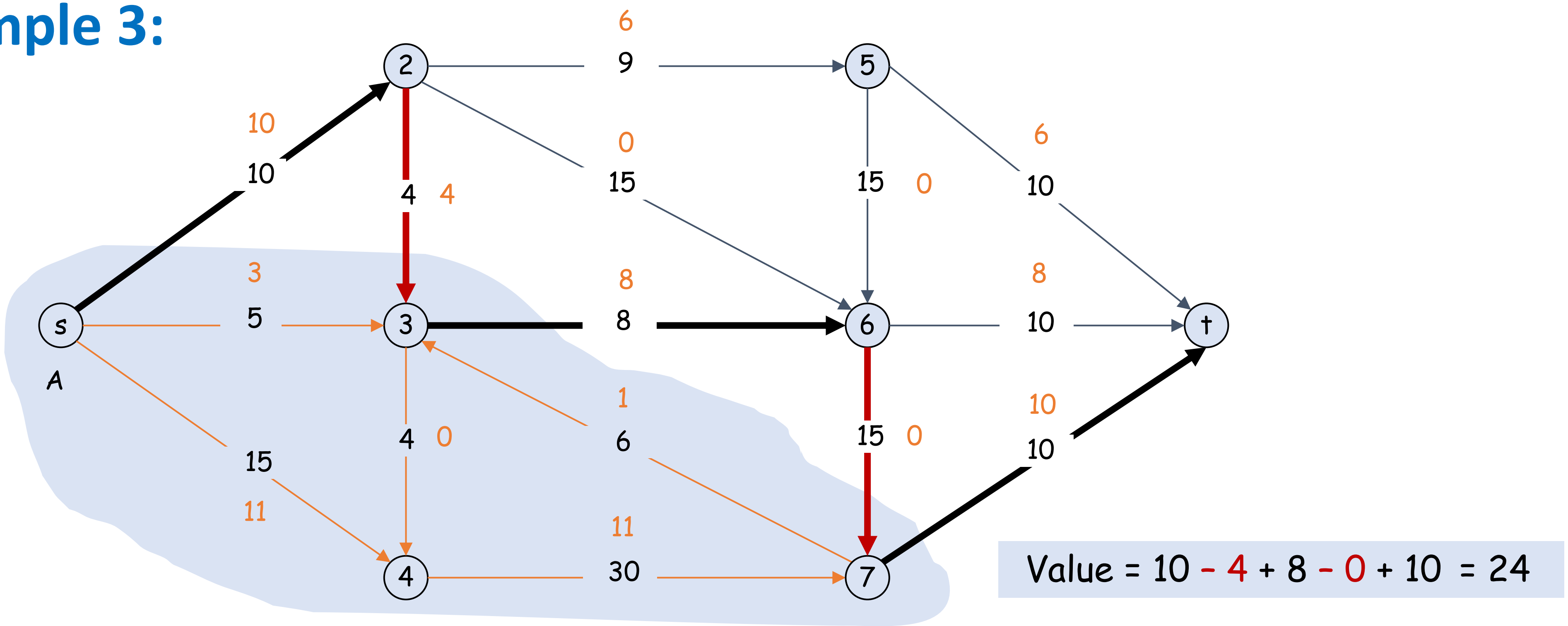
Relationship between Flows and Cuts

- **Flow value lemma.** Let *f* be any flow, and let *(A, B)* be any cut. Then, the value of the flow *f* equals the net flow across the cut *(A, B)*.

$$v(f) = f^{out}(A) - f^{in}(A)$$

- **Example 3:**



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Relationship between Flows and Cuts

- **Flow value lemma.** Let *f* be any flow, and let *(A, B)* be any cut. Then, the value of the flow *f* equals the net flow across the cut *(A, B)*.

$$v(f) = f^{out}(A) - f^{in}(A) \qquad v(f) = f^{in}(B) - f^{out}(B)$$

- **Pf.**

$$v(f) = \sum_{e \text{ out of } s} f(e) = f^{out}(s) = f^{out}(s) - f^{in}(s)$$

$f^{in}(s) = 0$

$$= \sum_{v \in A} (f^{out}(v) - f^{in}(v))$$

by flow conservation, $f^{out}(v) - f^{in}(v) = 0$ for all v in A except for v = s

$$= \sum_{e \text{ out of } A} f(e) - \sum_{e \text{ into } A} f(e) = f^{out}(A) - f^{in}(A)$$

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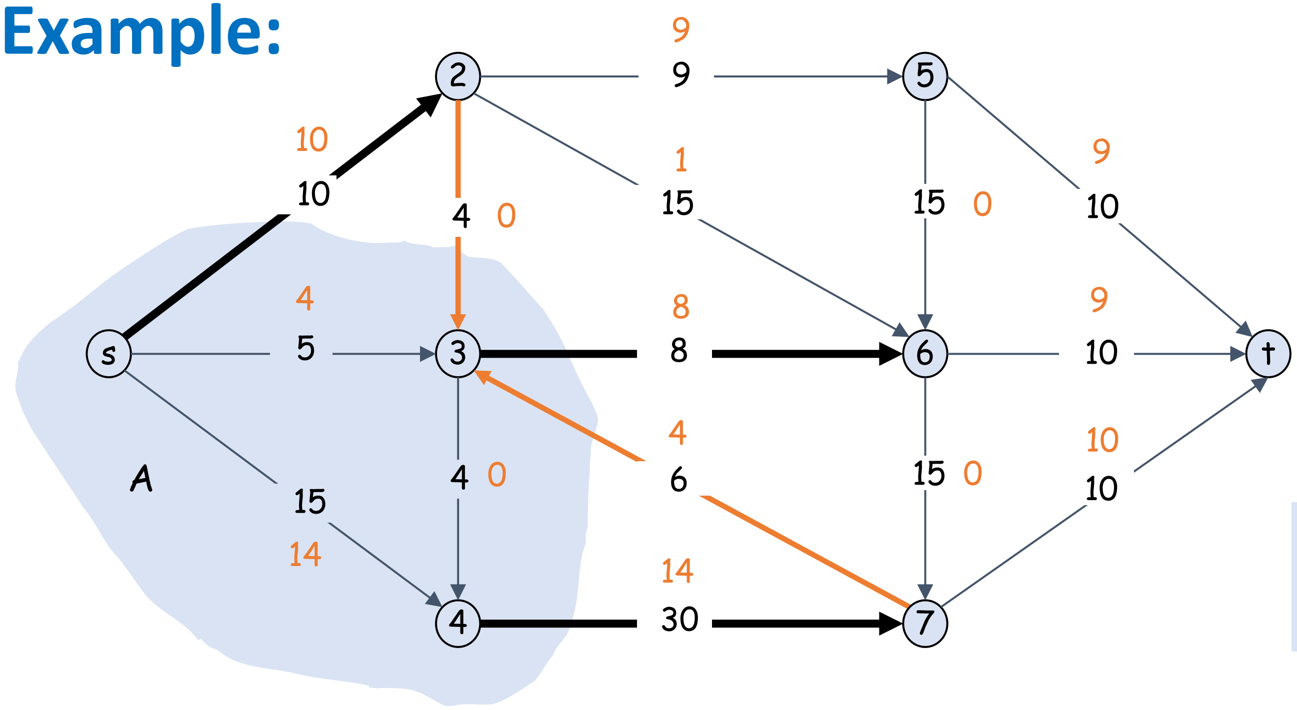
Relationship between Flows and Cuts

- **Weak duality.** Let *f* be any flow, and let *(A, B)* be any cut. Then the value of the flow *f* is at most the capacity of the cut: *v(f) ≤ c(A, B)*.
  - Recall that $c(A,B) = \sum_{e \text{ out of } A} c_e$.

- **Example:**



Flow value = 28
Cut capacity = 48 ⇒ Flow value ≤ 48

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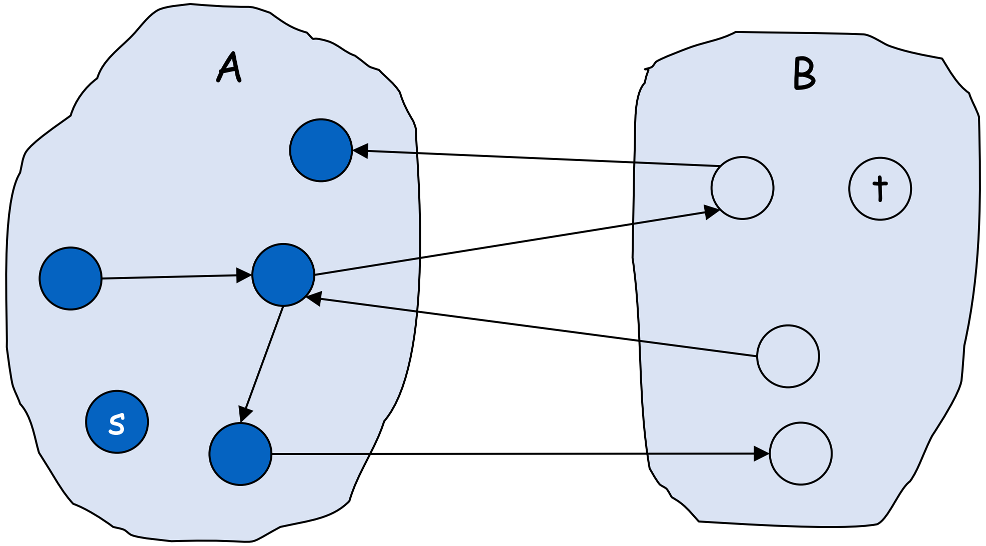
Relationship between Flows and Cuts

- **Weak duality.** Let f be any flow, and let (A, B) be any cut. Then the value of the flow f is at most the capacity of the cut: $v(f) \le c(A, B)$.
  - Recall that $c(A,B) = \sum_{e \text{ out of } A} c_e$.

- **Pf.** $$v(f) = f^{out}(A) - f^{in}(A)$$

flow value lemma

$$\le f^{out}(A)$$
$$= \sum_{e \text{ out of } A} f(e)$$
$$\le \sum_{e \text{ out of } A} c(e)$$
$$= c(A,B)$$



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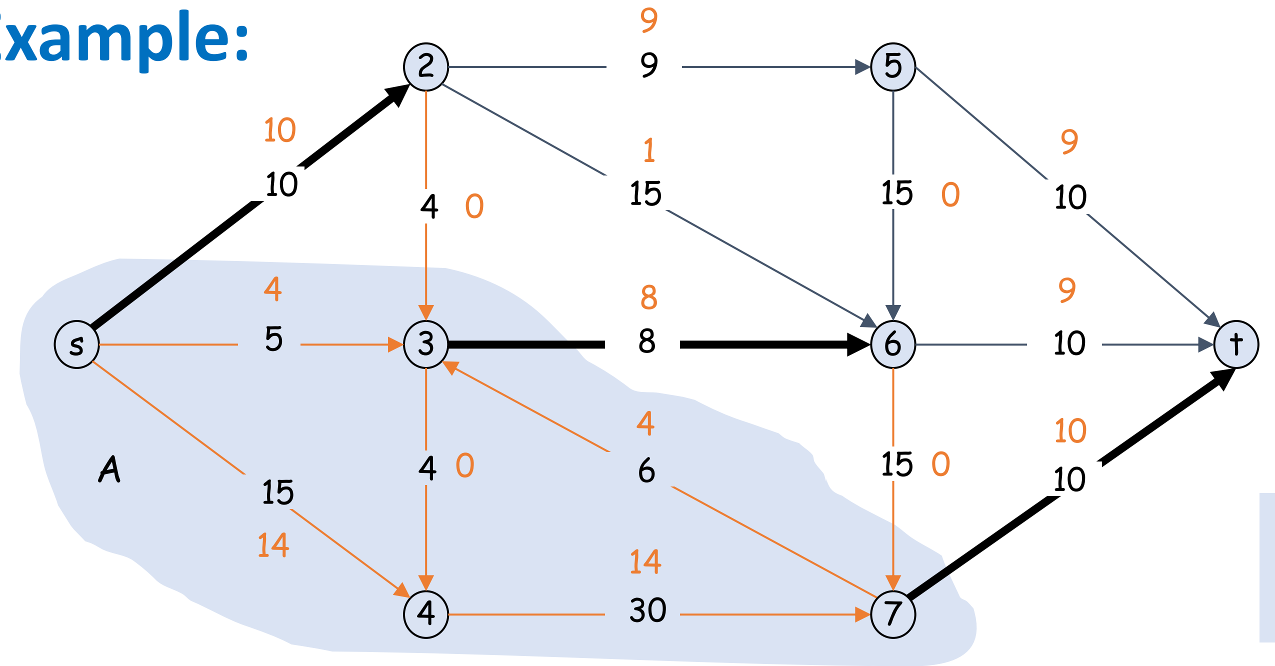
Certificate of Optimality

- **Corollary.** Let *f* be any flow, and let *(A, B)* be any cut. If $v(f) = c(A, B)$, then *f* is a max flow and *(A, B)* is a min cut.
- **Pf.** Every flow value is upper bounded by every cut capacity!
- **Example:**



Flow value = 28
Cut capacity = 28 $\Rightarrow$ Flow value $\leq$ 28

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Max-Flow Min-Cut Theorem

- **Max-flow min-cut theorem.** [Ford-Fulkerson 1956] Value of a max flow is equal to capacity of a min cut.
- **Augmenting path theorem.** Flow f is a max flow iff no augmenting paths.

- **Pf.** We prove both by showing the following are equivalent:
  - (i) There exists a cut (A, B) such that $v(f) = c(A, B)$.
  - (ii) f is a max flow.
  - (iii) There is no augmenting path with respect to f.

(i) $\Rightarrow$ (ii): This was the previous **Corollary**.

(ii) $\Rightarrow$ (iii): We prove the contrapositive.

- ➢ Let f be a flow. If there exists an augmenting path, then we can improve flow f by sending flow along this path. Then, f is not a max flow.

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Max-Flow Min-Cut Theorem

- **Pf continued.**
  - (i) There exists a cut *(A, B)* such that *v(f) = c(A, B)*.
  - (iii) There is no augmenting path with respect to *f*.

(iii) ⇒ (i):

- Let *f* be a flow with no augmenting paths.
- Let *A* be the set of nodes reachable from *s* in residual network G_f.
- By definition of $A \Rightarrow s \in A$. No augmenting path for $f \Rightarrow t \notin A$.

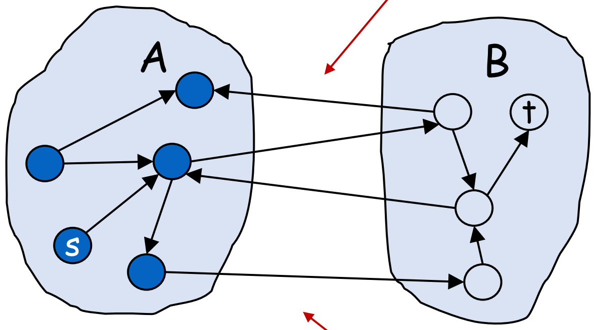
given a max flow f (so no augmenting path)
can find a min cut (A, B) in O(m) time

$$v(f) = f^{out}(A) - f^{in}(A)$$

flow value lemma

$$= \sum_{e \text{ out of } A} c(e) - 0$$

$$= c(A, B)$$



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4. Capacity-Scaling Algorithm

Ford-Fulkerson Algorithm: Analysis

- **Assumption.** Every edge capacity c_e is an integer between 1 and C.
- **Integrality invariant.** Throughout FF, every edge flow $f(e)$ and residual capacity $c_f(e)$ are integers.

- **Theorem.** FF terminates after at most $v(f^*) \le nC$ augmenting paths, where f^* is a max flow. (Assume flow network has no parallel edges.)
- **Pf.** Consider cut where $A = \{ s \}$ and note that each augmentation increases the value of the flow by at least 1. ▪
- **Corollary.** The running time of Ford-Fulkerson is $O(mnC)$.
- **Pf.** Can use either BFS or DFS to find an augmenting path in $O(m)$ time. ▪

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Ford-Fulkerson Algorithm: Analysis

- **Assumption.** Every edge capacity c_e is an integer between *1* and *C*.
- **Integrality invariant.** Throughout FF, every edge flow *f(e)* and residual capacity $c_f(e)$ are integers.

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- **Corollary.** The running time of Ford-Fulkerson is $O(mnC)$.

- **Integrality theorem.** There exists an integral max flow f^*.
- **Pf.** Since FF always terminates if capacities are integral, theorem follows from **integrality invariant** (and **augmenting path theorem**). ▪

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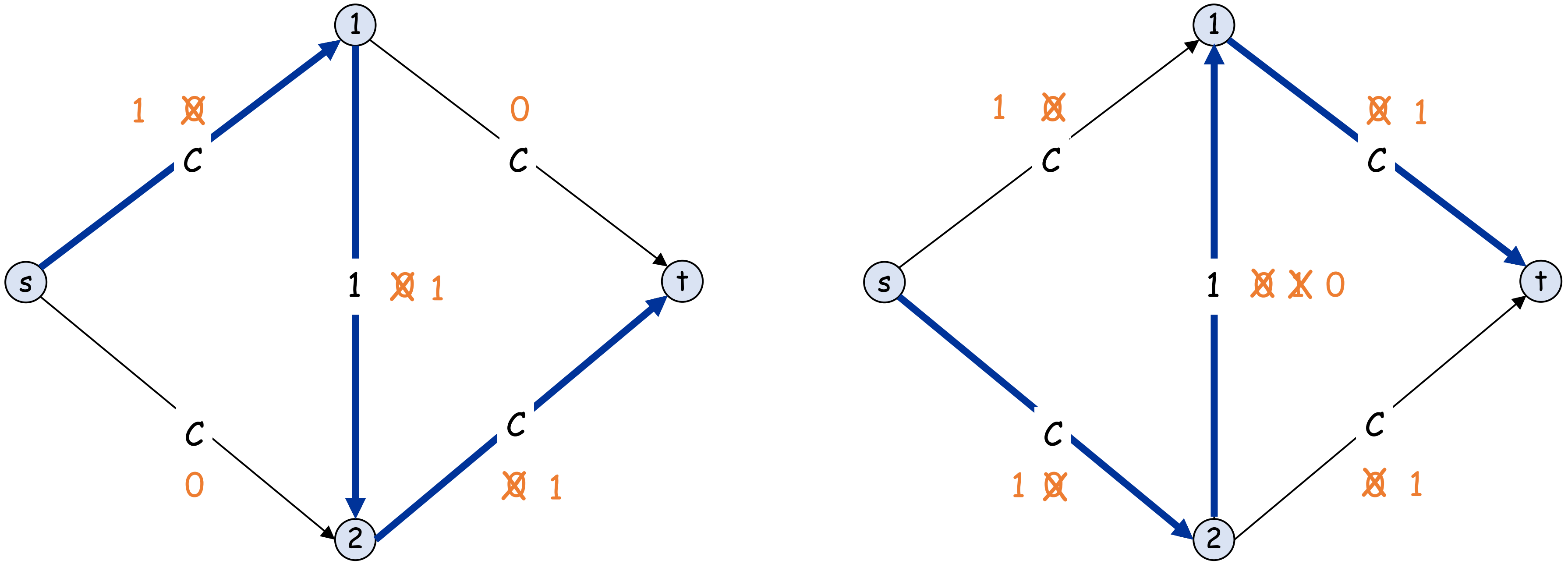
Ford-Fulkerson: Exponential Example

- **Q.** Is Ford-Fulkerson algorithm polynomial in input size? ← m, n, $\log_2 C$
- **A.** No. If max capacity is C, then algorithm can take $2C$ iterations.

$C = 2^{\log_2 C}$



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Choosing Good Augmenting Paths

- **Caveat.** If capacities can be irrational, FF may not terminate or converge!

capacities are rational in practice
but FF could run in exponential time

- **Use care when selecting augmenting paths:**
  - Some choices lead to exponential algorithms.
  - Clever choices lead to polynomial algorithms.
- **Goal.** Choose augmenting paths so that:
  - Can find augmenting paths efficiently.
  - Few iterations.
- **Choose augmenting paths with:**
  - Max bottleneck capacity ("fattest"). ← difficult, how to find?
  - Sufficiently large bottleneck capacity. ← coming next
  - Fewest edges. [Edmonds-Karp 1972, Dinitz 1970] ← later sections

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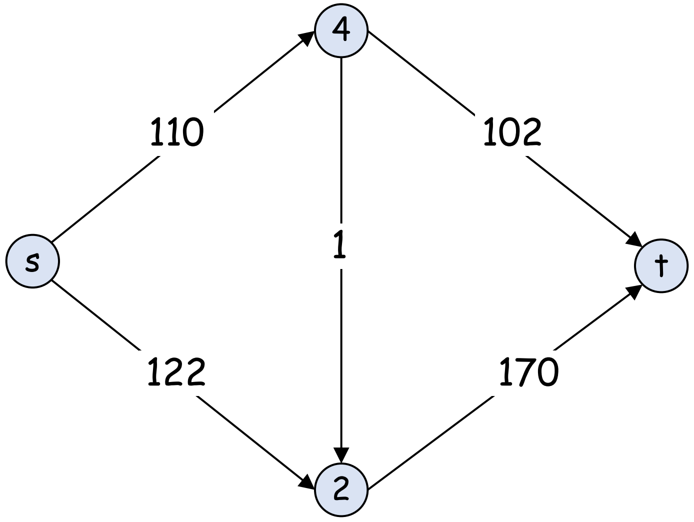
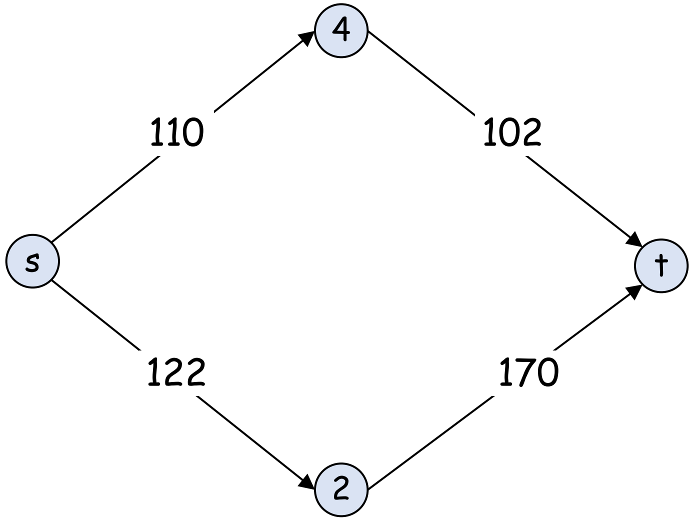
Capacity-Scaling Algorithm

- **Overview.** Choosing augmenting paths with "large" bottleneck capacity.
  - Maintain scaling parameter Δ.
  - Let $G_f(\Delta)$ be the subnetwork of the residual network containing only those edges with capacity $\geq \Delta$.
  - Any augmenting path in $G_f(\Delta)$ has bottleneck capacity $\geq \Delta$.

not necessarily largest



G_f



G_f (100)

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Capacity-Scaling Algorithm

- **Greedy algorithm.** Halve Δ when no augmenting path can be found.

```
Capacity-Scaling(G, s, t, c) {
   foreach e ∈ E: f(e) = 0
   Δ = largest power of 2 that is ≤ C
   Gf = residual network with respect to flow f

   while (Δ ≥ 1) {
      Gf(Δ) = Δ-residual network of G with respect to flow f
      while (there exists an augmenting path P in Gf(Δ)) {
         f = Augment(f, c, P)
         update Gf(Δ)
      }
      Δ = Δ / 2
   }
   return f
}
```


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Capacity-Scaling Algorithm: Correctness

- **Assumption.** Every edge capacity c_e is an integer between 1 and C.
- **Integrality invariant.** Throughout capacity-scaling algorithm, every edge flow $f(e)$ and residual capacity $c_f(e)$ are integers.

- **Theorem.** If capacity-scaling algorithm terminates, then f is a max flow.
- **Pf.** (direct proof)
  - **Integrality invariant** $\Rightarrow G_f(1) = G_f$.
  - Upon termination of the $\Delta = 1$ phase, there are no augmenting paths.
  - **Augmenting path theorem** $\Rightarrow$ the resulting flow f is a max flow. ▪

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Capacity-Scaling Algorithm: Running Time

- **Lemma 1.** The outer while loop repeats $1 + \lfloor \log_2 C \rfloor$ times.
- **Pf.** Initially $C/2 < \Delta \leq C$; Δ decreases by a factor of 2 each iteration. ▪

- **Lemma 2.** Let f be the flow at the end of a Δ-scaling phase. Then the value of a maximum flow $\leq v(f) + m\Delta$. (Proof shown later.)
- **Pf idea.** Adding edges with capacity $< \Delta$ can only increase flow by $\leq m\Delta$.

- **Lemma 3.** There are $\leq 2m$ augmentations per scaling phase.
- **Pf.** Let f be the flow at the end of the previous scaling phase $\Delta' = 2\Delta$.
  - **Lemma 2** $\Rightarrow$ maximum flow value $\leq v(f) + m\Delta' = v(f) + m(2\Delta)$.
  - Each augmentation in a Δ-phase increases $v(f)$ by at least Δ. ▪

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Capacity-Scaling Algorithm: Running Time

- **Lemma 1.** The outer while loop repeats $1 + \lfloor \log_2 C \rfloor$ times.
- **Lemma 3.** There are $\leq 2m$ augmentations per scaling phase.

- **Theorem.** The capacity-scaling algorithm takes $O(m^2 \log C)$ time.
- **Pf. Lemma 1 + Lemma 3** $\Rightarrow$ $O(m \log C)$ augmentations. Finding an augmenting path takes $O(m)$ time. ▪

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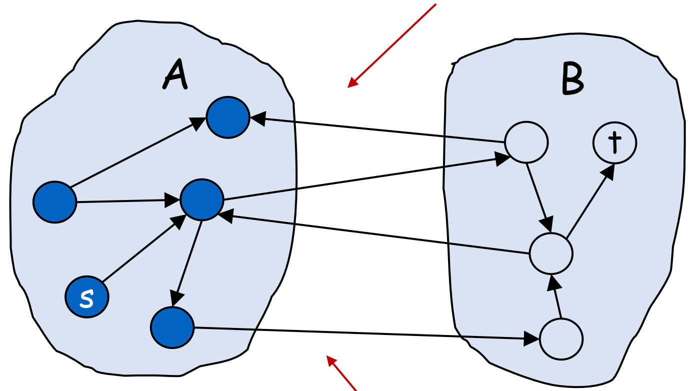
Capacity-Scaling Algorithm: Running Time

- **Lemma 2.** Let f be the flow at the end of a Δ-scaling phase. Then the value of a maximum flow $\leq v(f) + m\Delta$.
- **Pf.** (similar to the proof of **max-flow min-cut theorem**)
  - We show that there exists a cut (A, B) such that $c(A, B) \leq v(f) + m\Delta$.
  - Choose A to be the set of nodes reachable from s in $G_f(\Delta)$.
  - By definition of $A \Rightarrow s \in A$. No augmenting path for $f \Rightarrow t \notin A$.

$$
\begin{aligned}
v(f) &= f^{out}(A) - f^{in}(A) \\
&\geq \sum_{e \text{ out of } A} (c(e) - \Delta) - \sum_{e \text{ in to } A} \Delta \\
&\geq \sum_{e \text{ out of } A} c(e) - \sum_{e \text{ out of } A} \Delta - \sum_{e \text{ in to } A} \Delta \\
&\geq c(A, B) - m\Delta
\end{aligned}
$$

flow value lemma



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5. Edmonds-Karp Algorithm

Shortest Augmenting Path (Edmonds-Karp)

- **Q.** How to choose next augmenting path in Ford-Fulkerson?
- **A.** Pick a path that uses the fewest edges. ← can find via BFS
- **Edmonds-Karp (EK) algorithm:**

```
Edmonds-Karp(G, s, t, c) {
   foreach e ∈ E: f(e) = 0
   Gf = residual network of G with respect to flow f

   while (there exists an augmenting path P in Gf) {
      P = Breath-First-Search(Gf)
      f = Augment(f, c, P)          core procedure
      update Gf
   }
   return f
}
```


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Edmonds-Karp Algorithm: Analysis Overview

- **Lemma 1.** The length of a shortest augmenting path never decreases. (Proof shown later.)
- **Lemma 2.** After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases. (Proof shown later.)

- **Theorem.** Edmonds-Karp algorithm takes $O(m^2 n)$ time.
- **Pf.** (direct proof)
  - $O(m)$ time to find a shortest augmenting path via BFS.
  - There are $\leq mn$ augmentations.
    - ✓Augmenting paths are simple $\Rightarrow$ at most $n - 1$ different lengths
    - ✓**Lemma 1** + **Lemma 2** $\Rightarrow$ at most m augmenting paths for each length ▪

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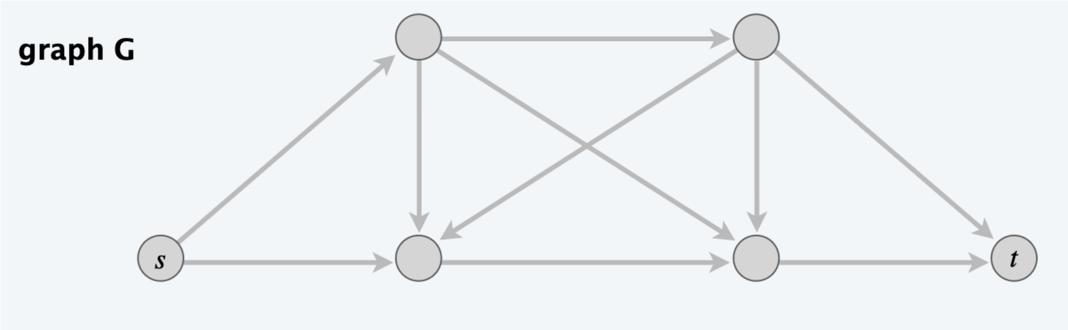
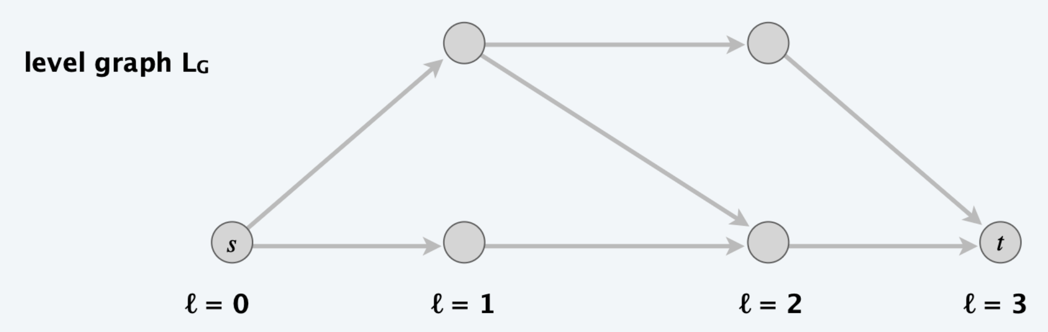
Edmonds-Karp Algorithm: Analysis

- **Def.** Given a directed graph $G = (V, E)$ with source s, its level graph is defined by:
  - $\ell(v)$ = number of edges in shortest s-v path.
  - $L_G = (V, E_G)$ is the subgraph of G that contains only those edges $(v, w) \in E$ such that $\ell(w) = \ell(v) + 1$.





- **Key property.** P is a shortest s-v path in G iff P is an s-v path in L_G.

all possible shortest s-v paths are captured in L_G

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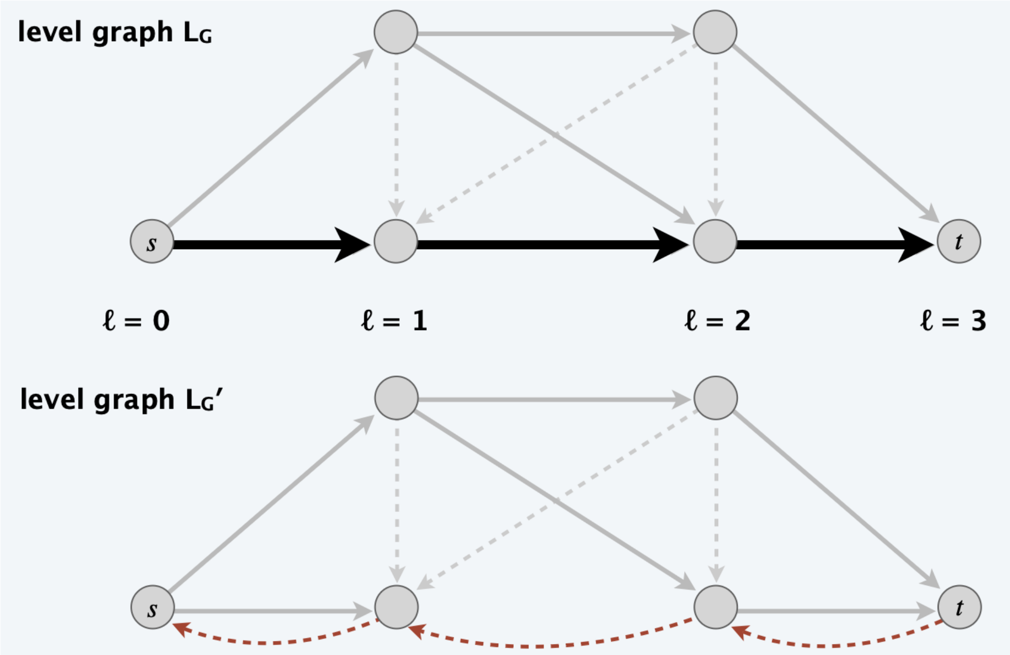
Edmonds-Karp Algorithm: Analysis

- **Pf of Lemma 1:** (Length of a shortest augmenting path never decreases.)
  - Let f and f' be flows before and after a shortest-path augmentation.
  - Let L_G and $L_{G'}$ be level graphs of G_f and $G_{f'}$. Only reverse edges added to $G_{f'}$.
  - Any s-t path that uses a reverse edge is longer than previous length. ▪



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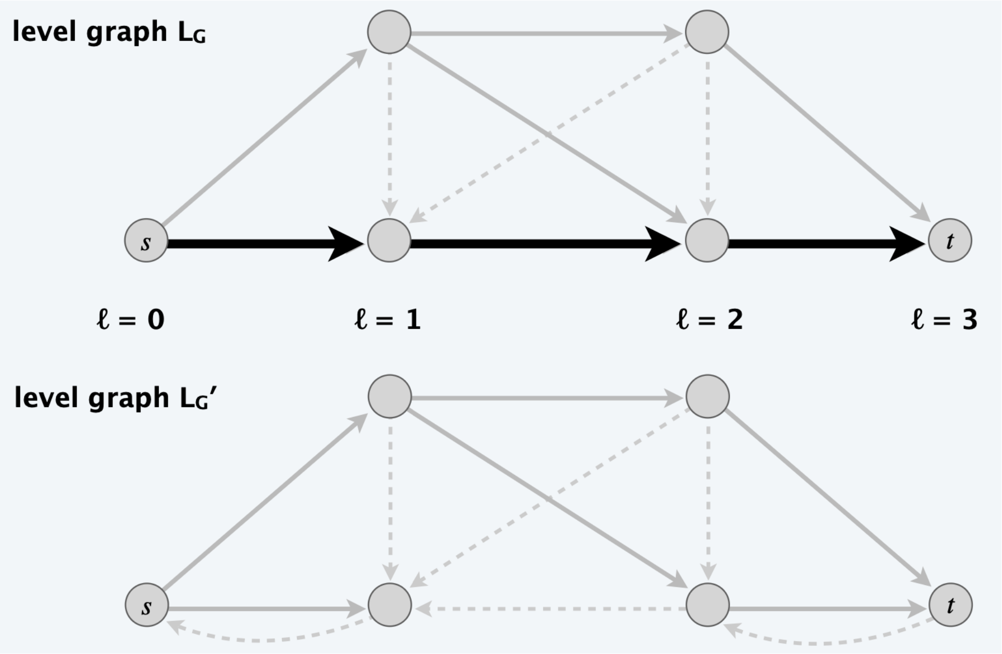
Edmonds-Karp Algorithm: Analysis

- **Pf of Lemma 2:** (After at most *m* shortest-path augmentations, the length of a shortest augmenting path strictly increases.)
  - At least one (bottleneck) edge is deleted from L_G per augmentation.
  - No new edge added to L_G until it does not have any *s*-*t* path, then shortest path length in the new level graph strictly increases. ▪



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Edmonds-Karp Algorithm: Summary

- **Lemma 1.** The length of a shortest augmenting path never decreases.
- **Lemma 2.** After at most m shortest-path augmentations, the length of a shortest augmenting path strictly increases.

- **Theorem.** Edmonds-Karp algorithm takes $O(m^2 n)$ time.

- **Fact.** $\Theta(mn)$ augmentations are necessary for some flow networks.
- **Solution.** Try to decrease time per augmentation instead.
  - Simple idea $\Rightarrow O(mn^2)$ [Dinitz 1970] ← next section
  - Dynamic trees $\Rightarrow O(mn \log n)$ [Sleator-Tarjan 1982]

invented by Dinitz in response to a class exercise by Adel'son-Vel'skiĭ

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6. Dinitz’s Algorithm

Dinitz’s Algorithm

- **Two types of augmentations:**
  - Normal: length of shortest path does not change.
  - Special: length of shortest path strictly increases.
- **Phase of normal augmentations:**
  - Construct level graph L_G.
  - Start at s, advance along an edge in L_G until reach t or get stuck.
  - If reach t, augment flow; update L_G ; and restart from s.
  - If get stuck, delete node from L_G and retreat to previous node.

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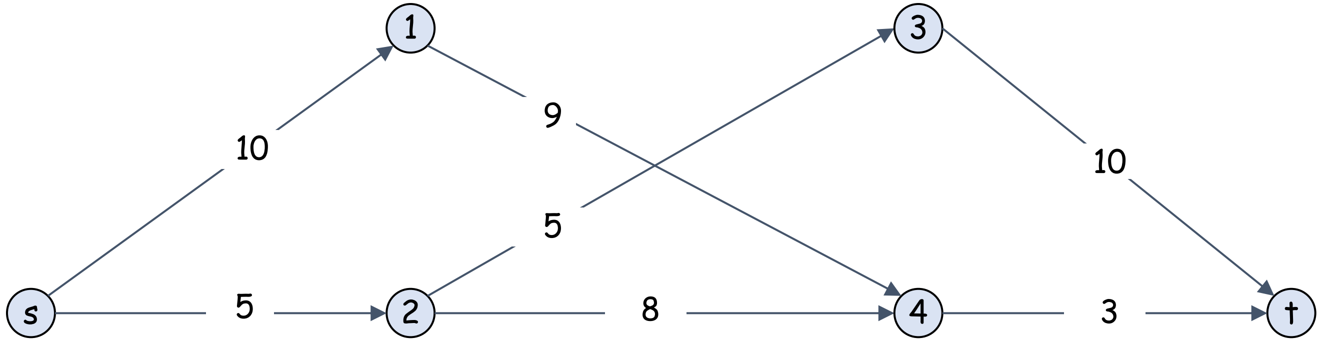

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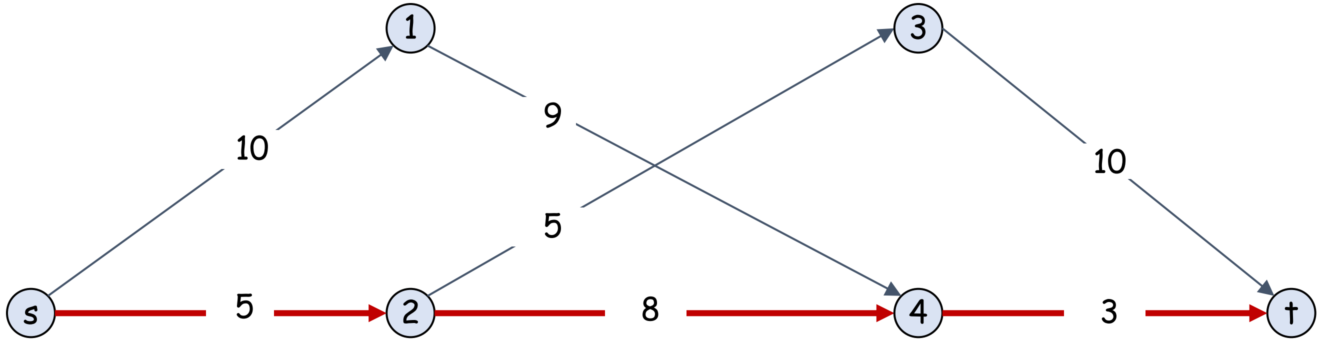

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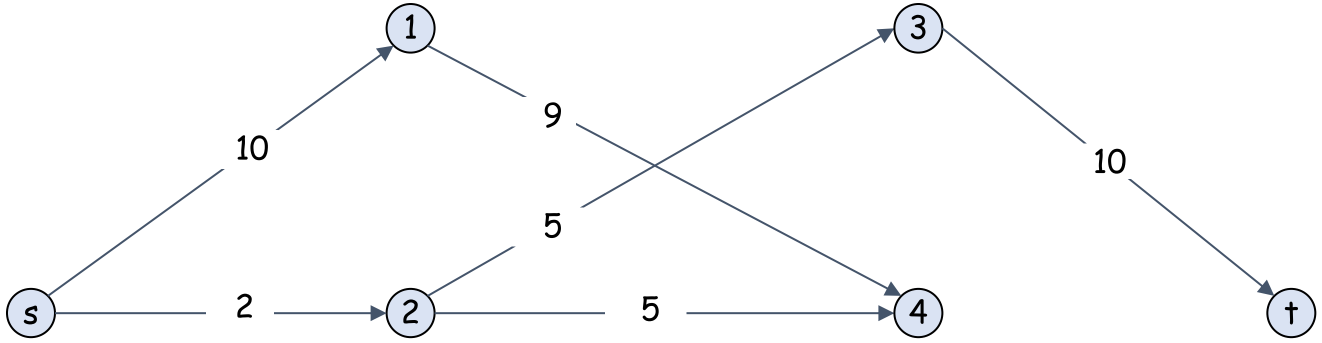

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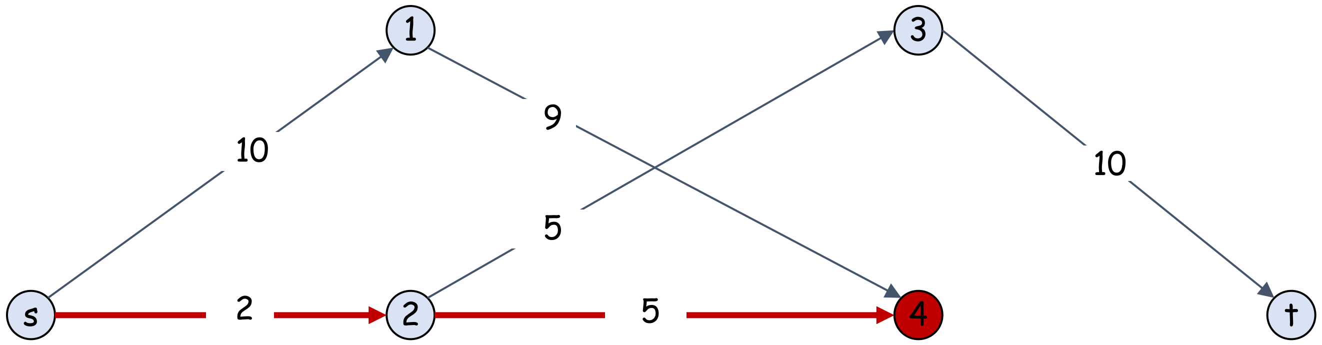

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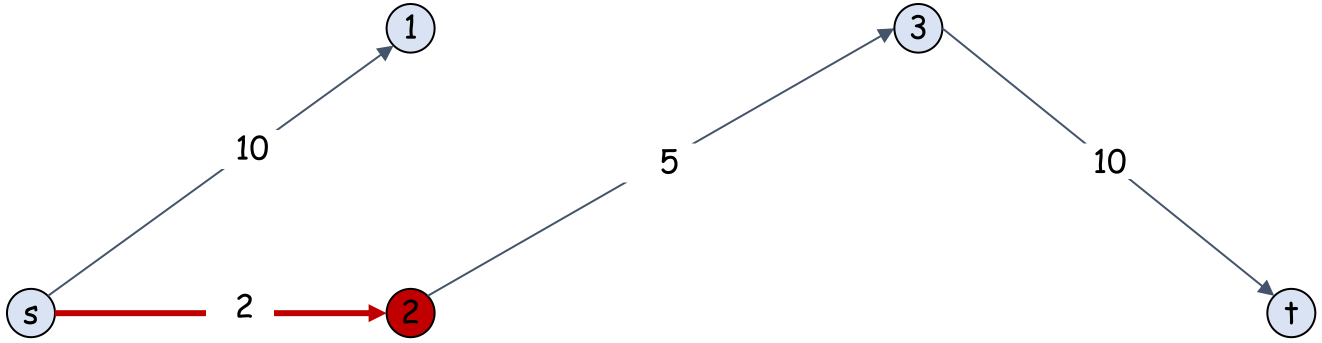

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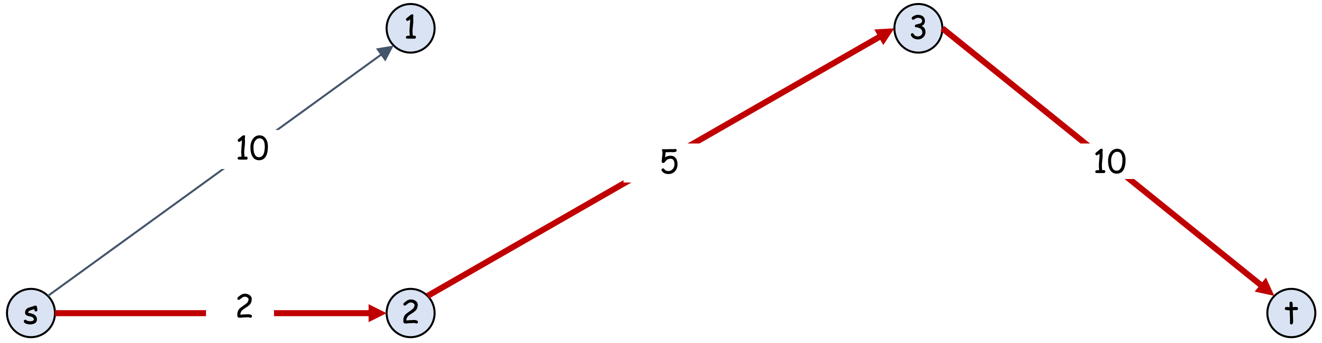

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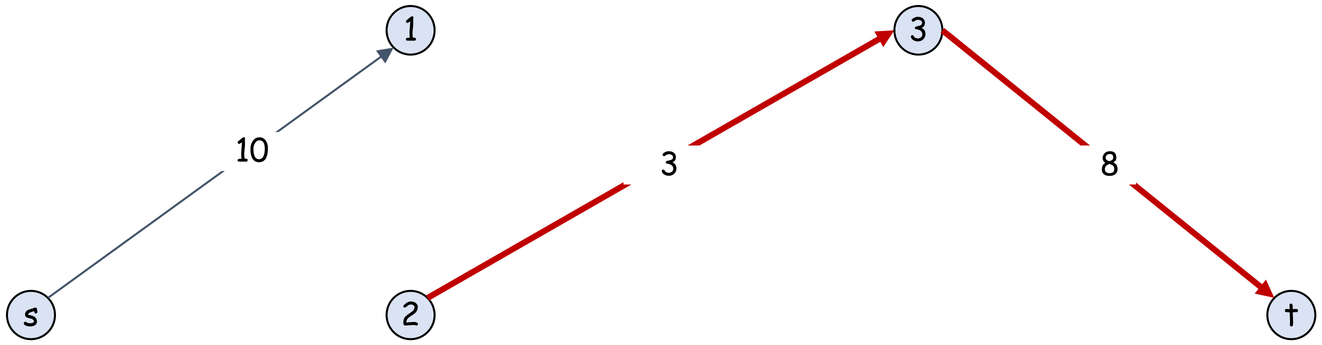

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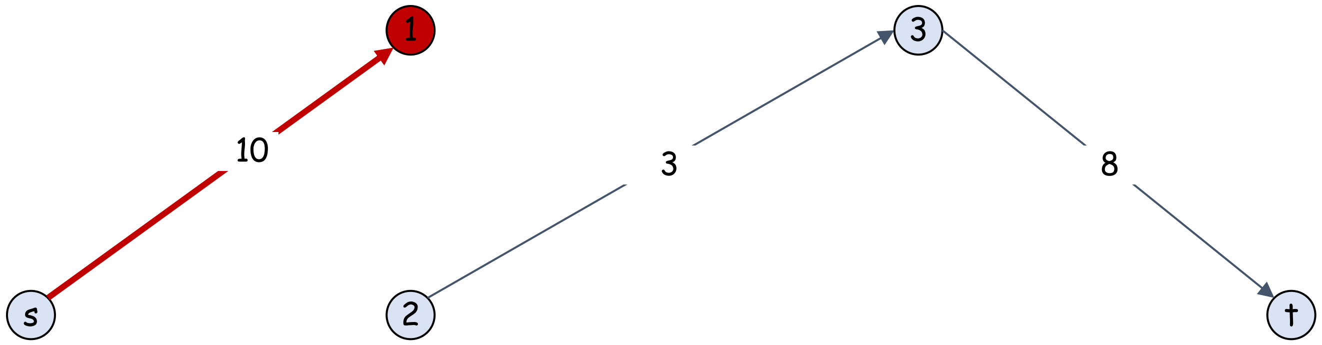

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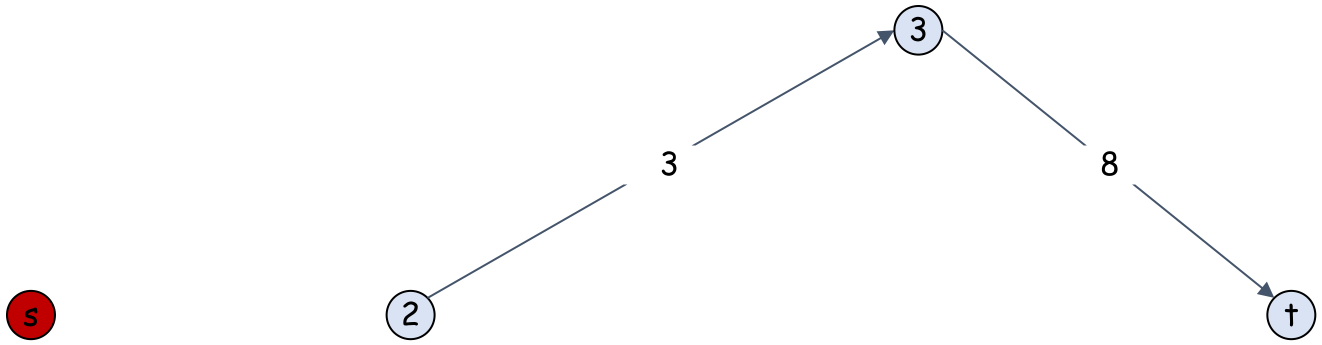

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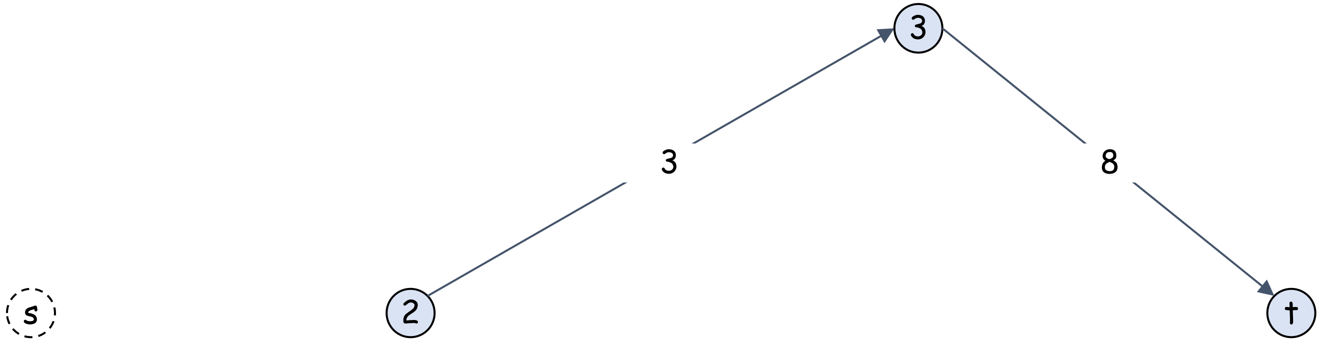

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Dinitz's Algorithm

- **Two types of augmentations:**
  - Normal: length of shortest path does not change.
  - Special: length of shortest path strictly increases.
- **Dinitz's algorithm per normal phase:** (as refined by Even and Itai)

```
Dinitz-Normal-Phase(Gf, s, t) {
   LG = level graph of Gf
   P = empty path
   Advance(s)
}

Retreat(v) {
   if (v = s) return
   else
      delete v and incident edges from LG
      remove last edge (u, v) from P
      Advance(u)
}

Advance(v) {
   if (v = t)
      f = Augment(f, c, P)
      remove bottleneck edges from LG
      P = empty path
      Advance(s)

   if (there exists (v, w) ∈ LG)
      add edge (v, w) to P
      Advance(w)

   Retreat(v)
}
```

optimization: can instead advance from the nearest advancable node

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Dinitz's Algorithm

- **Two types of augmentations:**
  - Normal: length of shortest path does not change.
  - Special: length of shortest path strictly increases.
- **Dinitz's algorithm:**

```
Dinitz(G, s, t, c) {
   foreach e ∈ E: f(e) = 0
   Gf = residual network of G with respect to flow f

   while (there exists an augmenting path P in Gf) {
      Dinitz-Normal-Phase(Gf, s, t)
   }
   return f
}
```


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Dinitz's Algorithm: Analysis

- **Lemma.** A phase can be implemented to run in $O(mn)$ time.
- **Pf.** (direct proof)
  - Level graph initialization happens once per phase. ← O(m) per phase using BFS
  - At most m augmentations per phase. ← O(mn) per phase
    (because an augmentation deletes at least one edge from L_G)
  - At most n retreats per phase. ← O(m + n) per phase
    (because a retreat deletes one node and all incident edges from L_G)
  - At most mn advances per phase. ← O(mn) per phase
    (because at most n advances before retreat or augmentation) ▪
- **Theorem.** [Dinitz 1970] Dinitz's algorithm runs in $O(mn^2)$ time.
- **Pf.** There are at most $n - 1$ phases and each phase runs in $O(mn)$ time. ▪

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Augmenting-Path Algorithms: Summary

year	method	# augmentations	running time
1955	augmenting path	$n\, C$	$O(m\, n\, C)$
1972	fattest path	$m \log (mC)$	$O(m^2 \log n \log (mC))$
1972	capacity scaling	$m \log C$	$O(m^2 \log C)$
1985	improved capacity scaling	$m \log C$	$O(m\, n \log C)$
1970	shortest augmenting path	$m\, n$	$O(m^2\, n)$
1970	level graph	$m\, n$	$O(m\, n^2)$
1983	dynamic trees	$m\, n$	$O(m\, n \log n)$

augmenting-path algorithms with m edges, n nodes, and integer capacities between 1 and C

fat paths

shortest paths

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Max-Flow Algorithms: Theory Highlights

year	method	worst case	discovered by
1951	**simplex**	$O(m\, n^2\, C)$	Dantzig
1955	**augmenting paths**	$O(m\, n\, C)$	Ford–Fulkerson
1970	**shortest augmenting paths**	$O(m\, n^2)$	Edmonds–Karp, Dinitz
1974	**blocking flows**	$O(n^3)$	Karzanov
1983	**dynamic trees**	$O(m\, n \log n)$	Sleator–Tarjan
1985	**improved capacity scaling**	$O(m\, n \log C)$	Gabow
1988	**push-relabel**	$O(m\, n \log (n^2 / m))$	Goldberg–Tarjan
1998	**binary blocking flows**	$O(m^{3/2} \log (n^2 / m) \log C)$	Goldberg–Rao
2013	**compact networks**	$O(m\, n)$	Orlin
2014	**interior-point methods**	$\tilde{O}(m\, n^{1/2} \log C)$	Lee–Sidford
2016	**electrical flows**	$\tilde{O}(m^{10/7}\, C^{1/7})$	Mądry
20xx		???	

← best in practice

max-flow algorithms with m edges, n nodes, and integer capacities between 1 and C

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Max-Flow Algorithms: Practice

- **Caveat.** Worst-case running time is generally not useful for predicting or comparing max-flow algorithm performance in practice.

- **Best in practice.** Push-relabel algorithm [Goldberg-Tarjan 1988] with gap relabeling: $O(m^{3/2})$ in practice. [section 7.4 of textbook]
  - Increases flow one edge at a time instead of one augmenting path at a time.

- **Computer vision.** Different algorithms work better for some dense problems that arise in applications to computer vision.

- **Implementation.** MATLAB, Google OR-Tools, etc.

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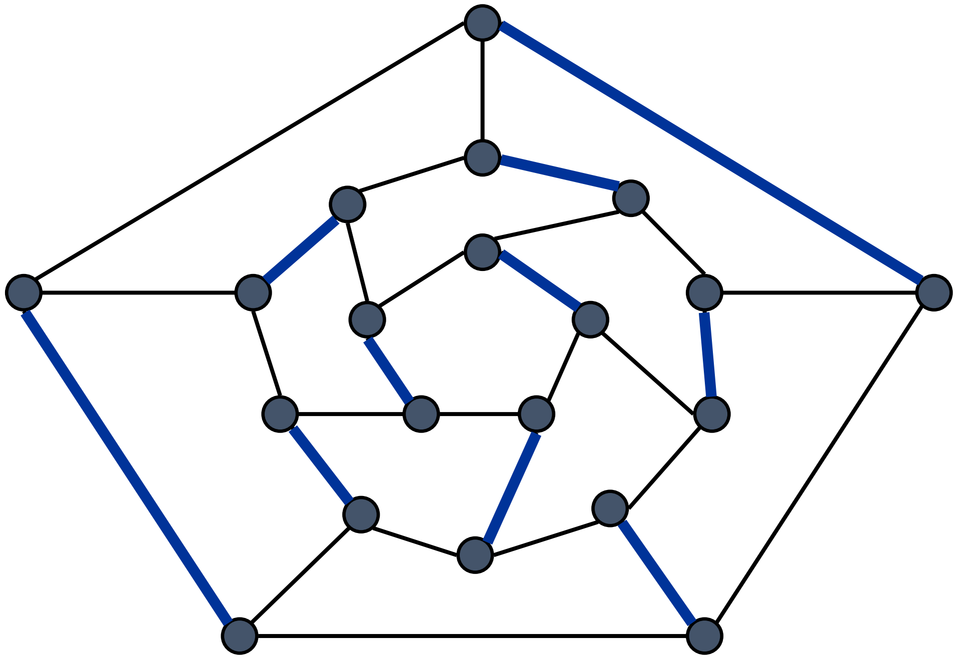
Announcement

- **The Bonus Assignment has been released and the deadline is June 7.**
  - ➢ This bonus assignment is optional and the captured material is out of the scope of this course.
  - ➢ The grade of this assignment can be used to replace the lowest grade of your other 6 mandatory assignments.

7. Bipartite Matching

Matching

- **Def.** Given an undirected graph $G = (V, E)$, the subset of edges $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- **Max matching.** Find max-cardinality matching.



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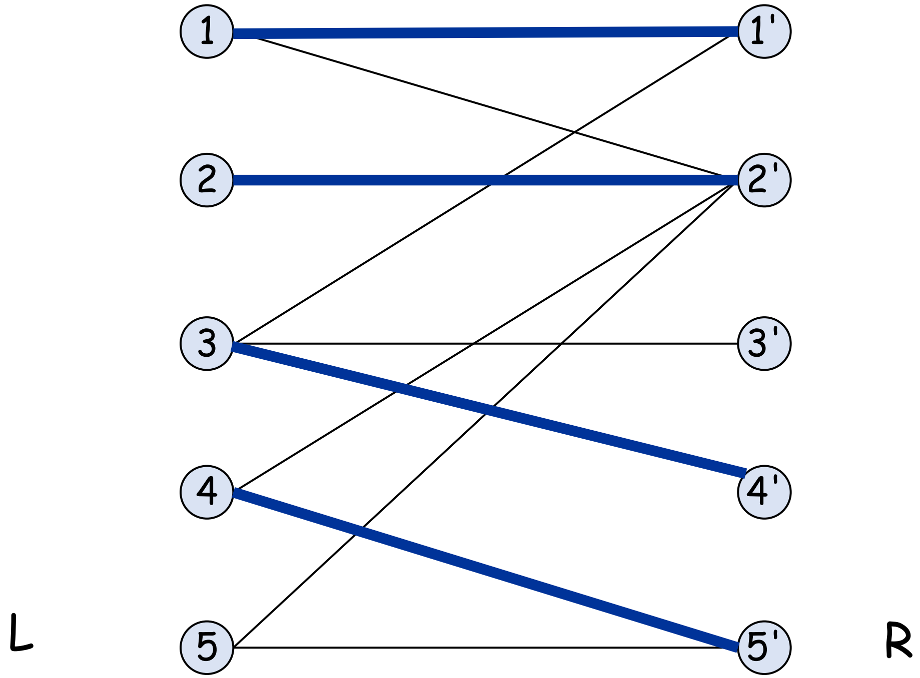
Bipartite Matching

- **Def.** Given an undirected bipartite graph $G = (L \cup R, E)$, the subset of edges $M \subseteq E$ is a matching if each node appears in at most one edge in M.
- **Max bipartite matching.** Find max-cardinality matching in bipartite graph.



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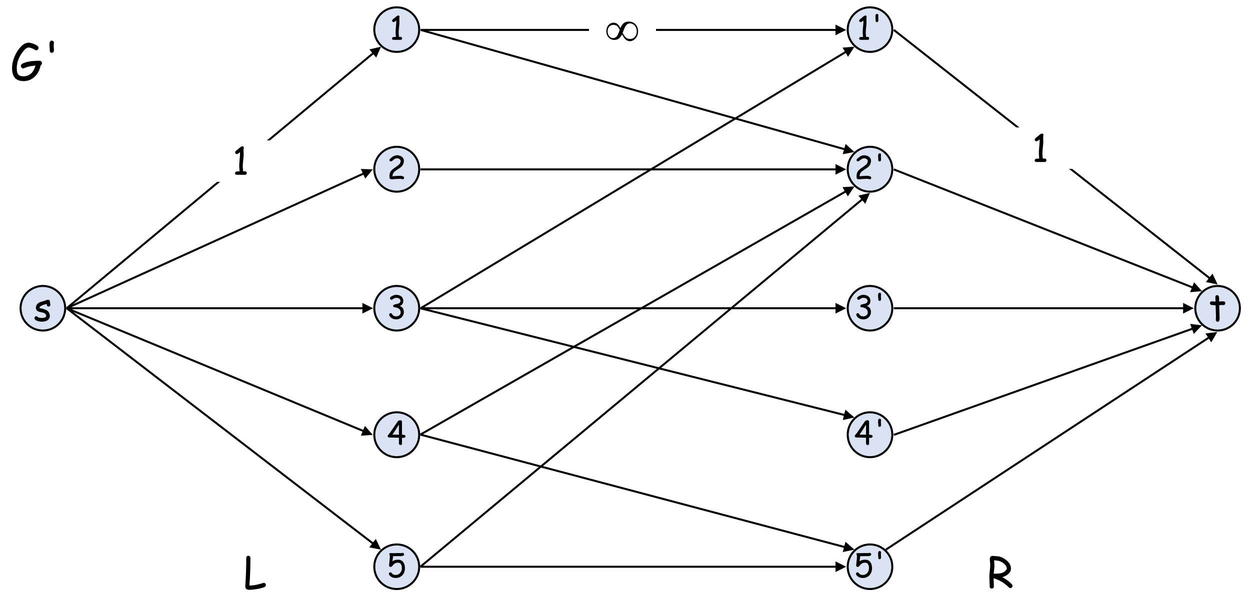
Bipartite Matching: Max-Flow Formulation

- **Max-flow formulation:**
  - Create a directed graph $G' = (L \cup R \cup \{s, t\}, E')$.
  - Direct all edges from L to R, and assign infinite (or unit) capacity.
  - Add source s, and unit-capacity edges from s to each node in L.
  - Add sink t, and unit-capacity edges from each node in R to t.



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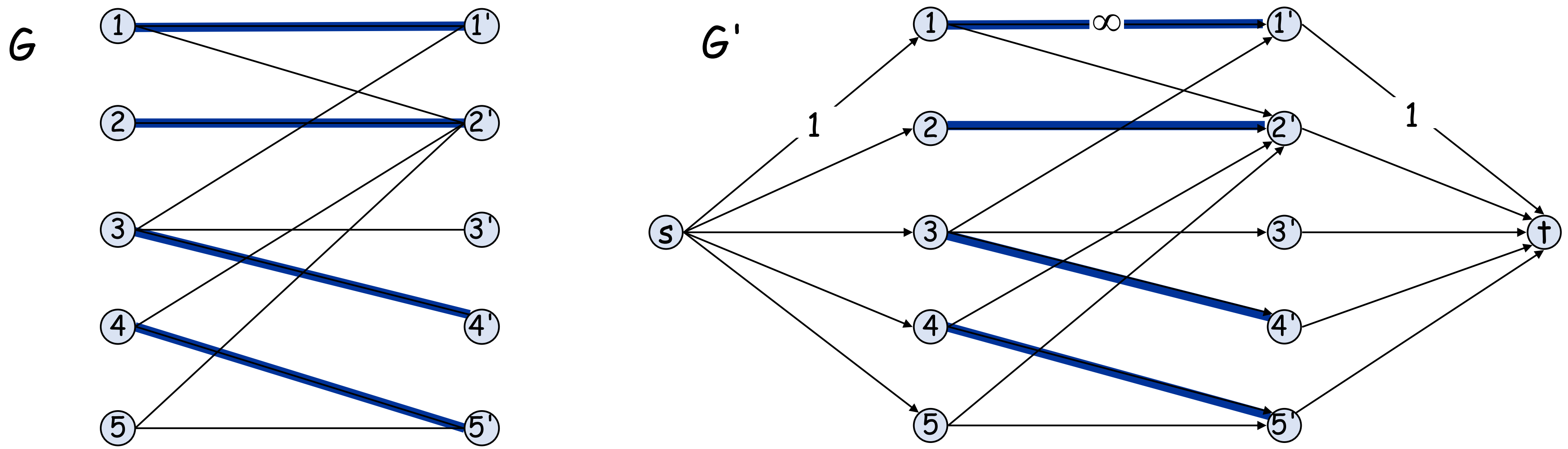
Max-Flow Formulation: Correctness

- **Theorem.** There exists a 1-1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.
- **Pf.** ⇒: Let M be a matching in G of cardinality k.
  - Consider flow f that sends *1* unit flow on each of the k corresponding paths.
  - f is an integral flow of value k. ▪



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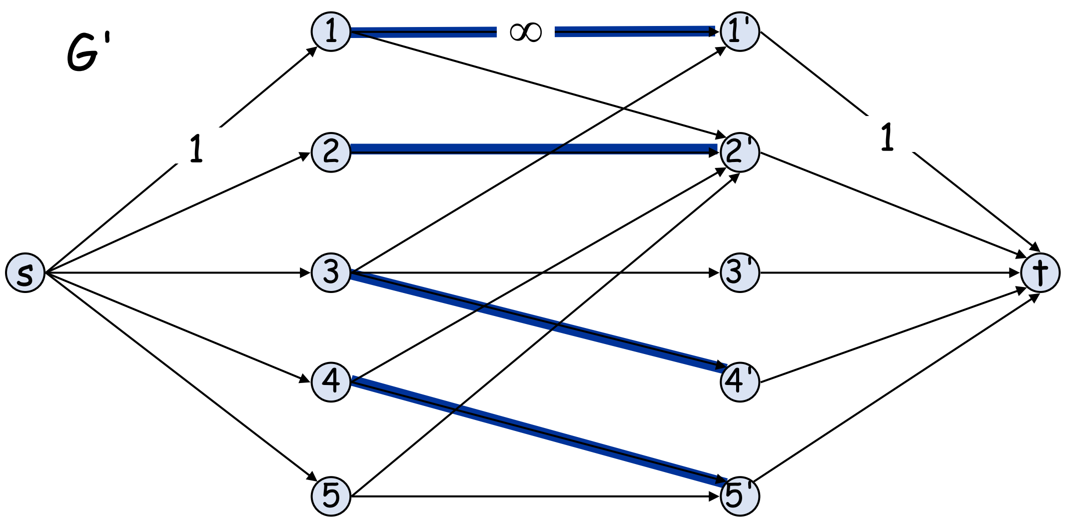
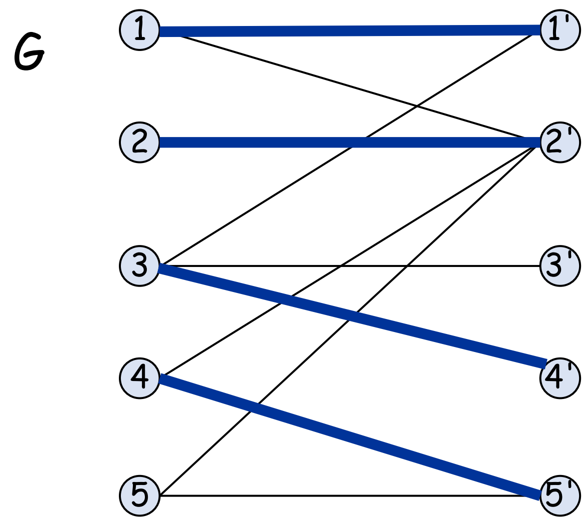
Max-Flow Formulation: Correctness

- **Theorem.** There exists a 1-1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.
- **Pf.** $\Leftarrow$: Let f be an integral flow in G' of value k.
  - Consider M = set of edges from L to R with $f(e) = 1$:
    - ✓Each node in L and R participates in at most one edge in M.
    - ✓$|M| = k$: apply **flow value lemma** to cut $(L \cup \{s\}, R \cup \{t\})$. ▪





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Max-Flow Formulation: Correctness

- **Theorem.** There exists a 1-1 correspondence between matchings of cardinality k in G and integral flows of value k in G'.

- **Corollary.** Can find max bipartite matching via max-flow formulation.
- **Pf.** (direct proof)
  - **Integrality theorem** $\Rightarrow$ there exists a max flow f^* in G' that is integral.
  - **Theorem** $\Rightarrow$ f^* corresponds to max-cardinality matching. ▪

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Perfect Matchings in Bipartite Graphs

- **Def.** Given a graph $G = (V, E)$, a subset of edges $M \subseteq E$ is a perfect matching if each node appears in exactly one edge in M.

- **Q.** When does a bipartite graph have a perfect matching?

- **Structure of bipartite graphs with perfect matchings:**
  - Clearly we must have $|L| = |R|$.
  - What other conditions are necessary?
  - What other conditions are sufficient?

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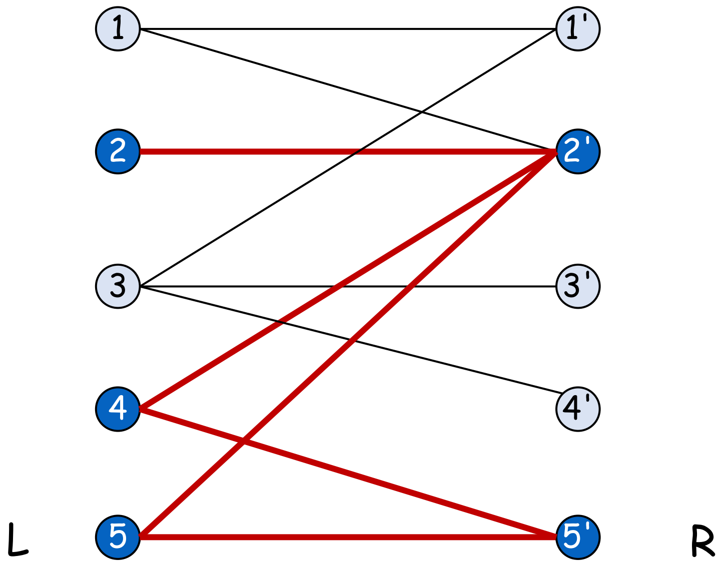
Perfect Matchings in Bipartite Graphs

- **Notation.** Let S be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in S.
- **Claim.** If a bipartite graph $G = (L \cup R, E)$ has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.
- **Pf.** Each node in S has to be matched to a different node in $N(S)$.
- **Example:**



No perfect matching exists:
S = { 2, 4, 5 }
N(S) = { 2', 5' }.

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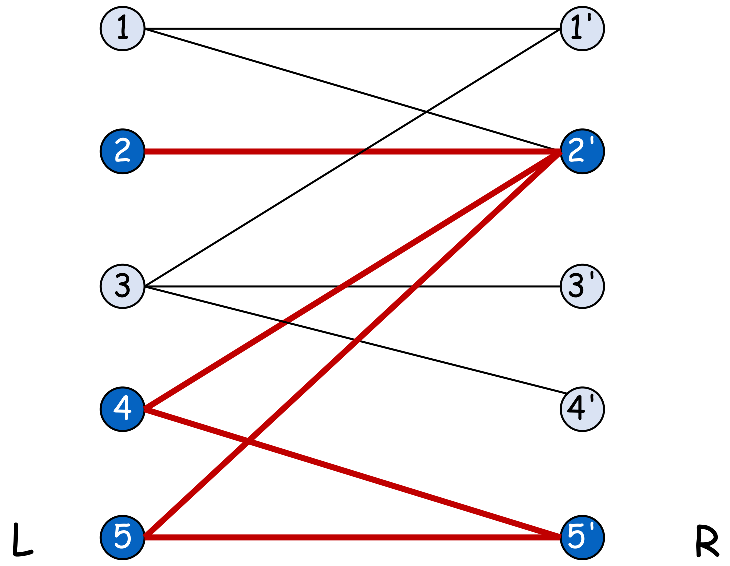
Hall's Marriage Theorem

- **Theorem.** [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. G has a perfect matching if and only if $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.
- **Pf.** "only if" $\Rightarrow$: This was the previous **Claim**.



No perfect matching exists:
$S = \{ 2, 4, 5 \}$
$N(S) = \{ 2', 5' \}$.

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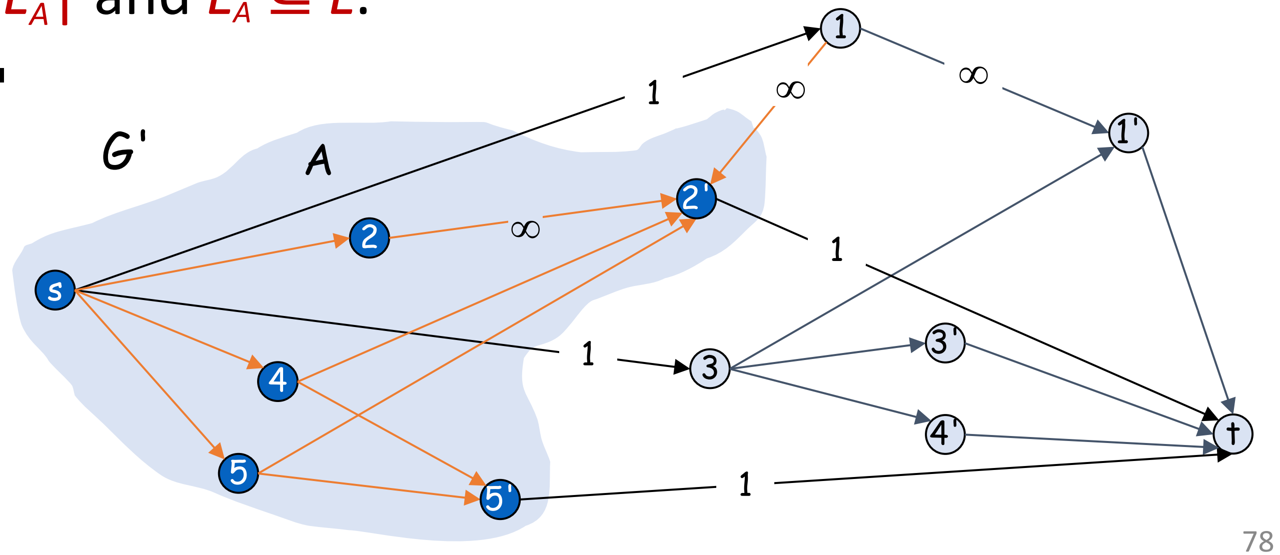
Hall's Marriage Theorem

- **Pf.** "if" $\Leftarrow$: Suppose G does not have a perfect matching. (contrapositive)
  - Formulate as a max flow problem and let (A, B) be a min cut in G'.
  - **Max-flow min-cut theorem** $\Rightarrow c(A, B) < |L|$
  - Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$.
  - Min cut cannot use ∞ edges $\Rightarrow N(L_A) \subseteq R \cap A = R_A$
  - $c(A, B) = |L_B| + |R_A| < |L| = |L_A| + |L_B| \Rightarrow |R_A| < |L_A|$
  - Together, $|N(L_A)| \leq |R_A| < |L_A|$ and $L_A \subseteq L$.
  - The contrapositive is true. ▪

$L_A = \{2, 4, 5\}$
$L_B = \{1, 3\}$
$R_A = \{2', 5'\}$
$N(L_A) = \{2', 5'\}$



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Max Matching: Closing Remarks

- **Algorithms for bipartite matching:**
  - Edmonds-Karp: $O(m^2 n)$
  - Capacity-scaling: $O(m^2 \log C) = O(m^2)$
  - Ford-Fulkerson: $O(m\ v(f^*)) = O(mn)$
  - Dinitz: $O(mn^{1/2})$ [Even-Tarjan 1975]

  learned max-flow algorithms

  - Fast matrix multiplication: $O(n^{2.378})$ [Mucha-Sankowsi 2003]
- **Non-bipartite matching:**
  - Structure of non-bipartite (undirected) graphs is more complicated.
  - But well-understood. [Tutte-Berge formula, Edmonds-Galai]
  - Blossom algorithm: $O(n^4)$ [Edmonds 1965]
  - Best known: $O(mn^{1/2})$ [Micali-Vazirani 1980, Vazirani 1994]

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Historical Significance (Jack Edmonds 1965)

2. Digression. An explanation is due on the use of the words "efficient algorithm." First, what I present is a conceptual description of an algorithm and not a particular formalized algorithm or "code."

For practical purposes computational details are vital. However, my purpose is only to show as attractively as I can that there is an efficient algorithm. According to the dictionary, "efficient" means "adequate in operation or performance." This is roughly the meaning I want—in the sense that it is conceivable for maximum matching to have no efficient algorithm. Perhaps a better word is "good."

I am claiming, as a mathematical result, the existence of a *good* algorithm for finding a maximum cardinality matching in a graph.

There is an obvious finite algorithm, but that algorithm increases in difficulty exponentially with the size of the graph. It is by no means obvious whether *or not* there exists an algorithm whose difficulty increases only algebraically with the size of the graph.

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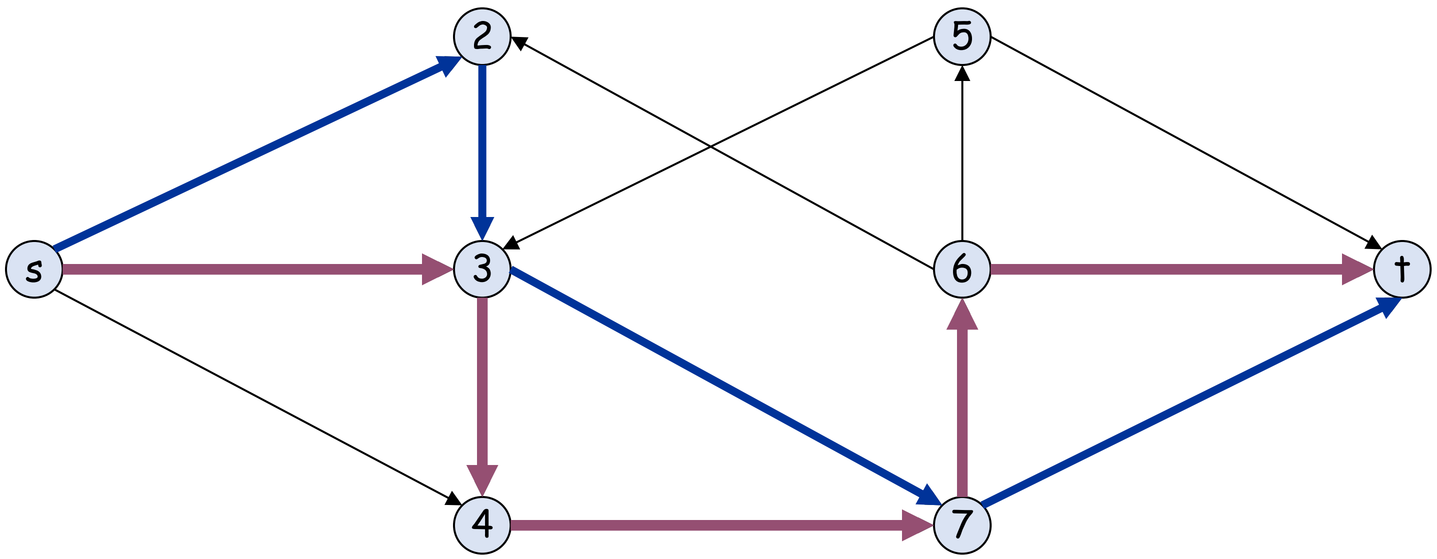
Announcement

- **Assignment 5 has been released and the deadline is May 29.**

8. Disjoint Paths

Edge-Disjoint Paths

- **Def.** Two paths are edge-disjoint if they have no edge in common.
- **Edge-disjoint paths problem.** Given a directed graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Example.** Communication networks.



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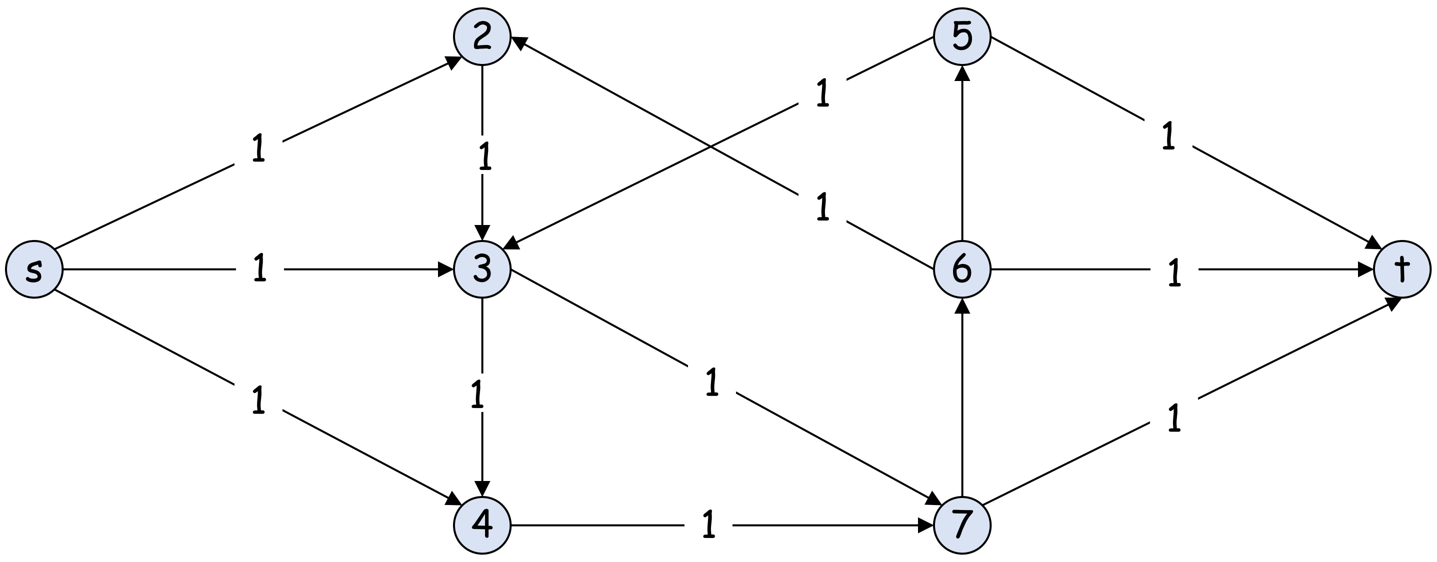
Edge-Disjoint Paths: Max-Flow Formulation

- **Def.** Two paths are edge-disjoint if they have no edge in common.
- **Edge-disjoint paths problem.** Given a directed graph $G = (V, E)$ and two nodes s and t, find the max number of edge-disjoint s-t paths.
- **Max-flow formulation.** Assign unit capacity to every edge.



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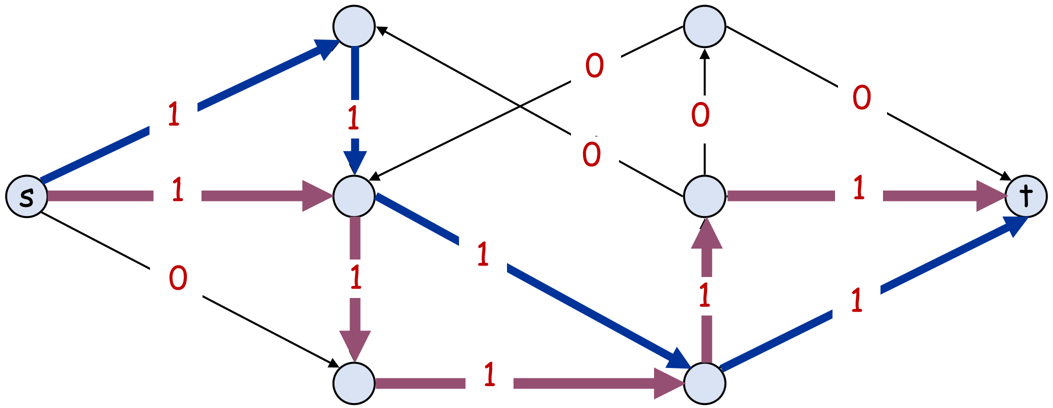
Max-Flow Formulation: Correctness

- **Theorem.** There exists a 1-1 correspondence between k edge-disjoint s-t paths in G and integral flows of value k in G'.
- **Pf.** $\Rightarrow$:
  - Let $P_1, \ldots, P_k$ be k edge-disjoint paths in G.
  - Set $f(e) = 1$ if edge e participates in some path P_i; else set $f(e) = 0$.
  - Since paths are edge-disjoint, f is a flow of value k. ▪



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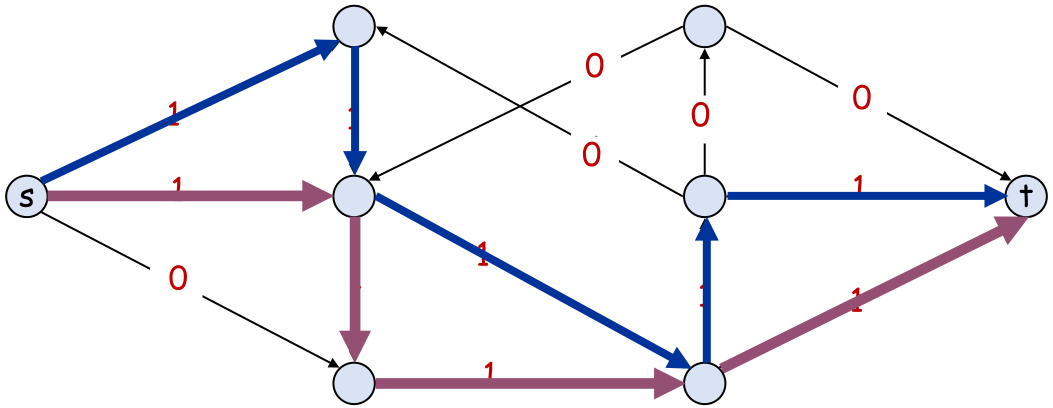
Max-Flow Formulation: Correctness

- **Theorem.** There exists a 1-1 correspondence between k edge-disjoint s-t paths in G and integral flows of value k in G'.
- **Pf.** $\Leftarrow$:
  - Let f be an integral flow in G' of value k.
  - Consider edge (s, u) with $f(s, u) = 1$:
    - ✓ flow conservation $\Rightarrow$ there exists an edge (u, v) with $f(u, v) = 1$
    - ✓ continue until reach t, always choosing a new edge
  - Produces k edge-disjoint paths (not necessarily simple) . ▪



can eliminate cycles and get simple paths if desired

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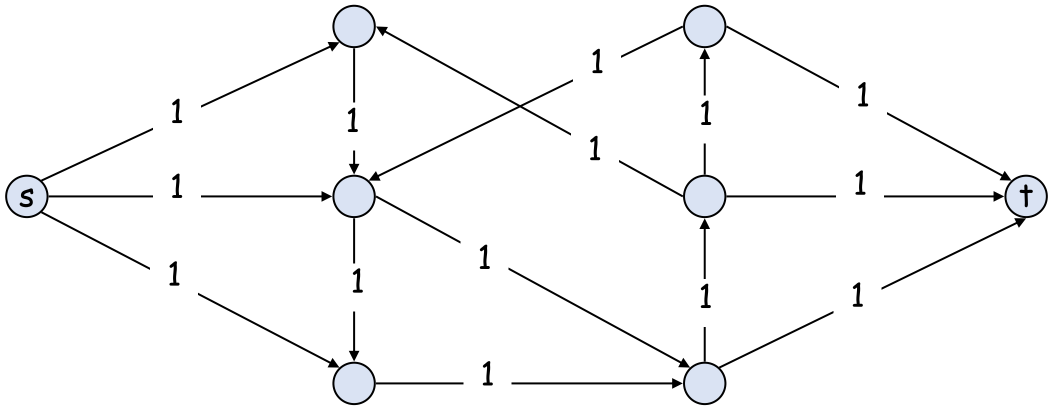
Max-Flow Formulation: Correctness

- **Max-flow formulation.** Assign unit capacity to every edge.
- **Theorem.** There exists a 1-1 correspondence between k edge-disjoint s-t paths in G and integral flows of value k in G'.
- **Corollary.** Can solve edge-disjoint paths via max-flow formulation.
- **Pf.** (direct proof)
  - **Integrality theorem** $\Rightarrow$ there exists a max flow f^* in G' that is integral.
  - **Theorem** $\Rightarrow$ f^* corresponds to max number of edge-disjoint s-t paths in G. ▪



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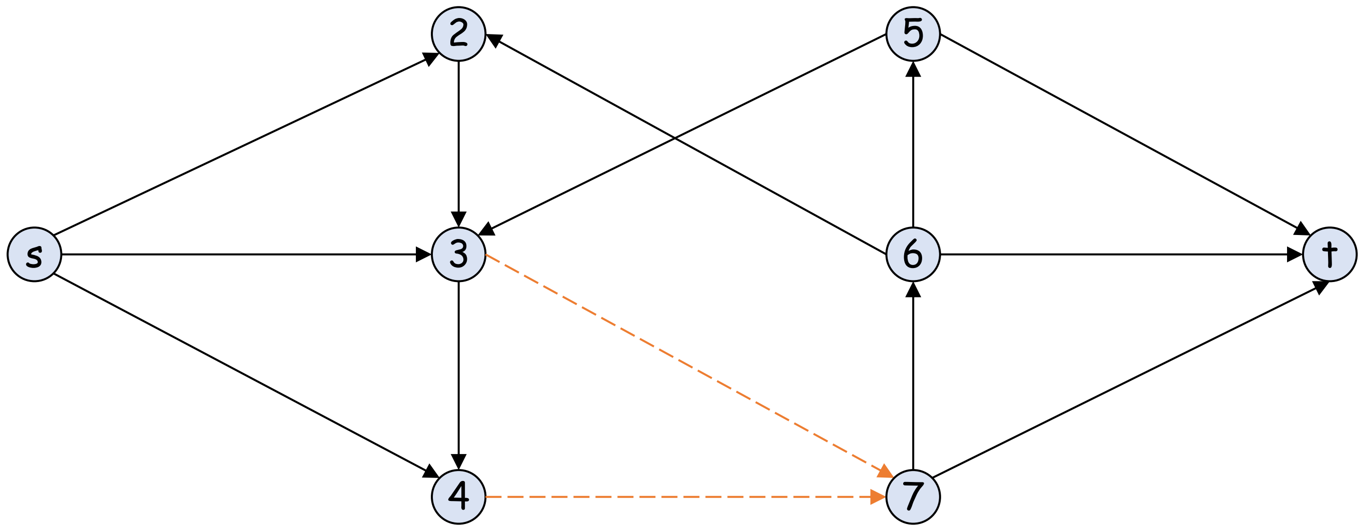
Network Connectivity

- **Def.** A set of edges $F \subseteq E$ disconnects t from s if every s-t path uses at least one edge in F.

- **Network connectivity.** Given a directed graph $G = (V, E)$ and two nodes s and t, find min number of edges whose removal disconnects t from s.



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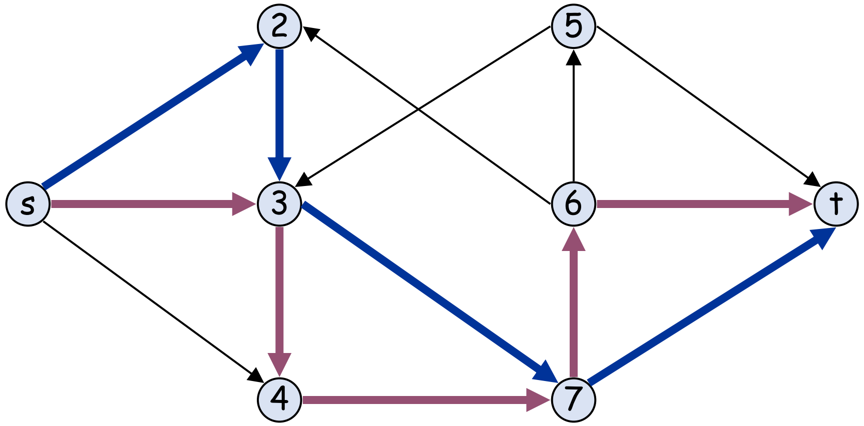
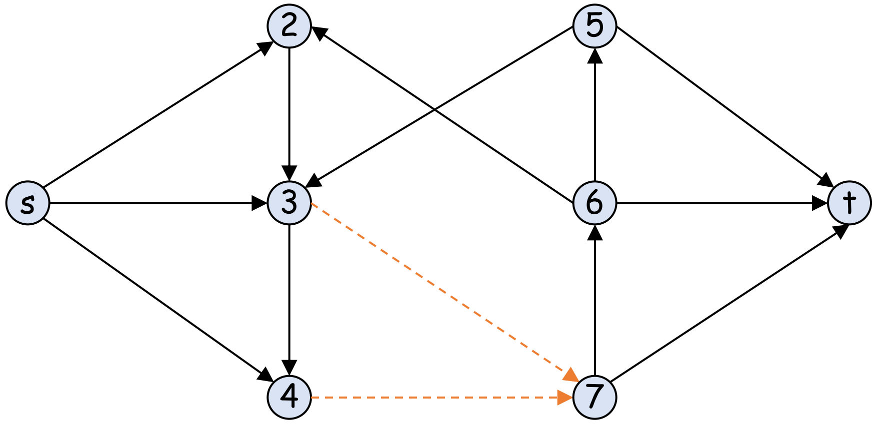
Menger's Theorem

- **Theorem.** [Menger 1927] The max number of edge-disjoint s-t paths equals the min number of edges whose removal disconnects t from s.
- **Pf.** $\leq$:
  - Suppose the removal of $F \subseteq E$ disconnects t from s, and $|F| = k$.
  - Every s-t path uses at least one edge in F.
  - Hence, the number of edge-disjoint paths is $\leq k$. ▪





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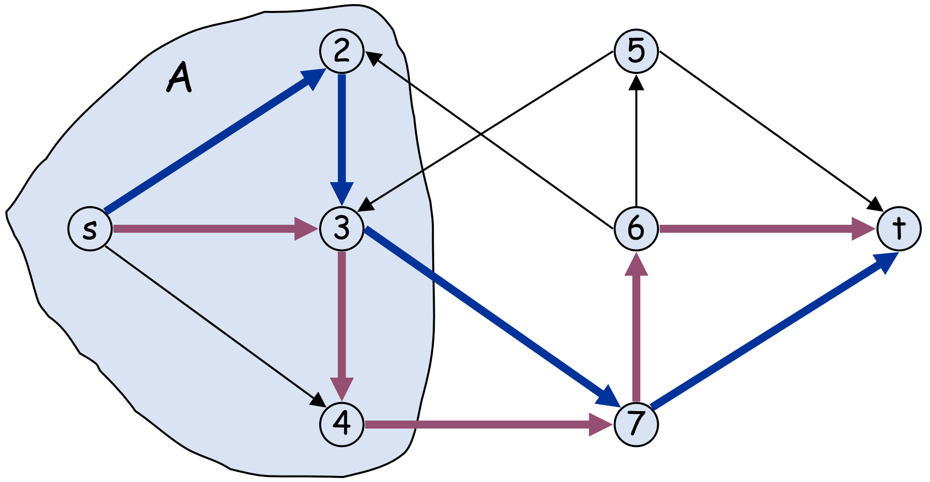
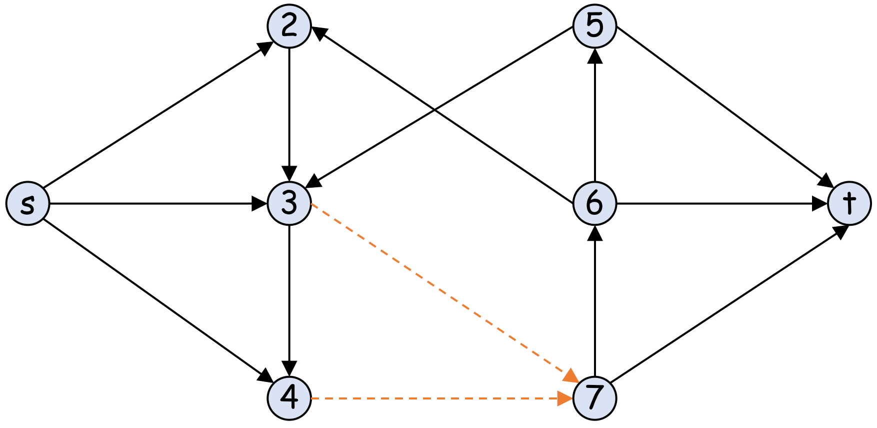
Menger's Theorem

- **Theorem.** [Menger 1927] The max number of edge-disjoint s-t paths equals the min number of edges whose removal disconnects t from s.
- **Pf.** $\geq$:
  - ➢ Suppose max number of edge-disjoint s-t paths is k. Then, value of max flow = k.
  - ➢ **Max-flow min-cut theorem** $\Rightarrow$ there exists a cut (A, B) of capacity k.
  - ➢ Let F be set of edges going from A to B.
  - ➢ $|F| = k$ and removing F disconnects t from s. ▪





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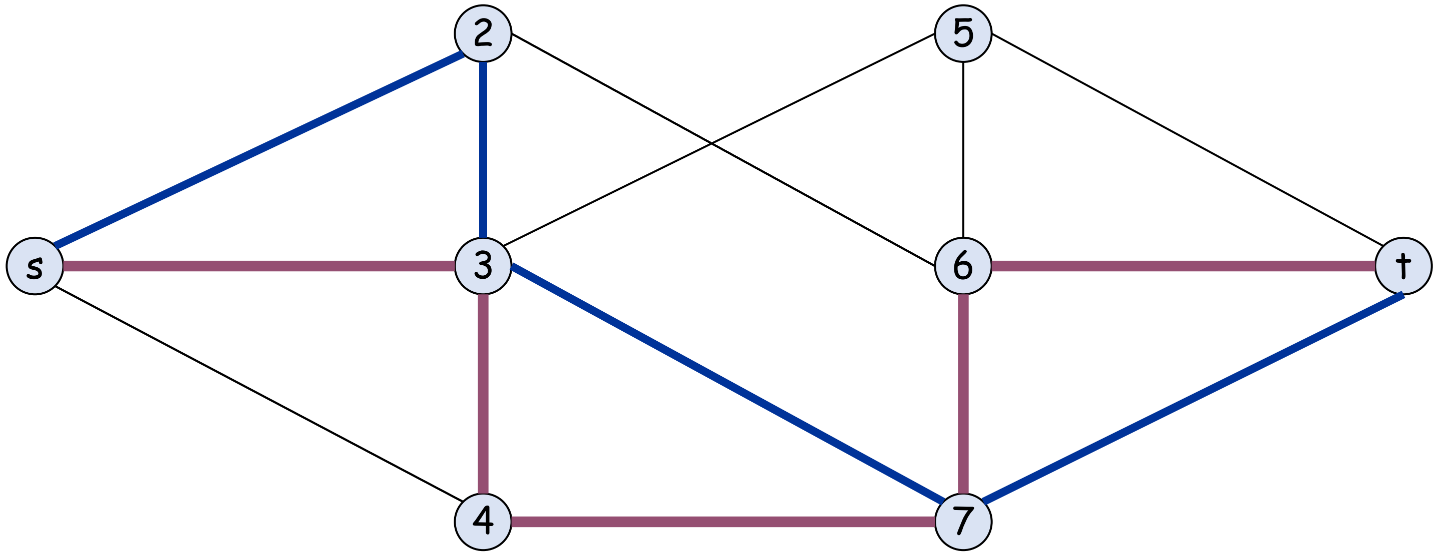
Edge-Disjoint Paths in Undirected Graphs

- **Edge-disjoint paths problem in undirected graphs.** Given an undirected graph $G = (V, E)$ and two nodes s and t, find max number of edge-disjoint s-t paths.
- **Example.** 2 edge-disjoint paths.



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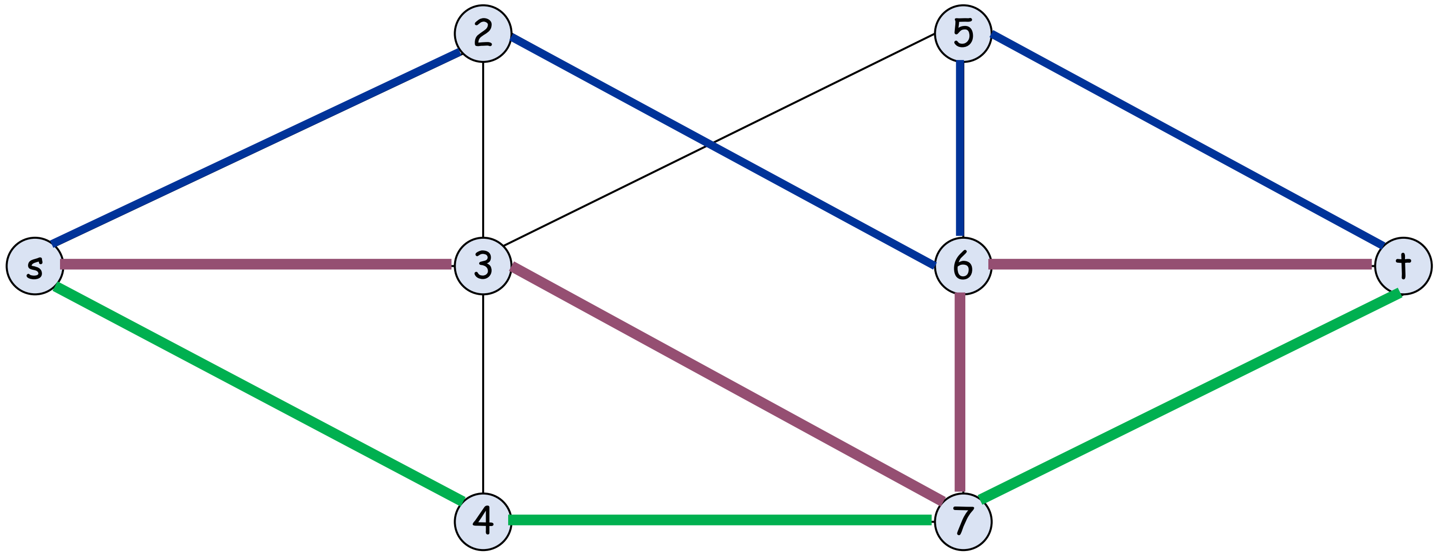
Edge-Disjoint Paths in Undirected Graphs

- **Edge-disjoint paths problem in undirected graphs.** Given an undirected graph $G = (V, E)$ and two nodes s and t, find max number of edge-disjoint s-t paths.
- **Example.** *3* edge-disjoint paths (max number).



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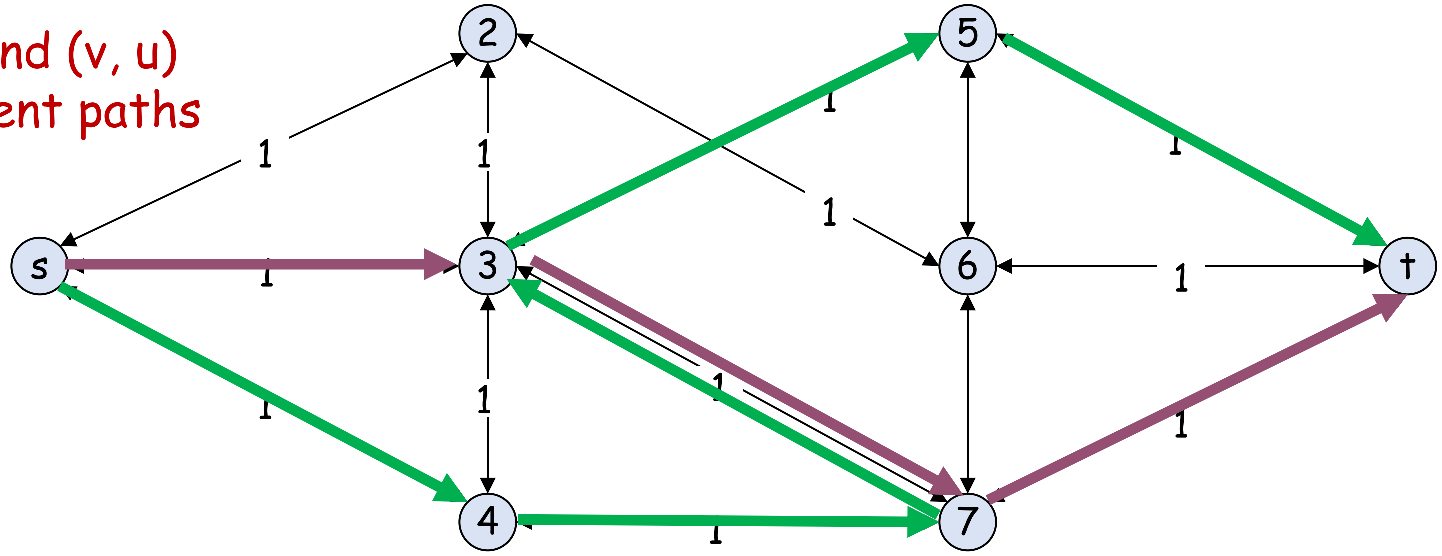
Edge-Disjoint Paths: Max-Flow Formulation

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.
- **Caveat.** Two paths P_1 and P_2 may be edge-disjoint in the directed graph but not edge-disjoint in the undirected graph.

edges (u, v) and (v, u) used in different paths



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Max-Flow Formulation: Correctness

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.

- **Lemma.** In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e': $f(e) = 0$ or $f(e') = 0$ or both. Moreover, **integrality theorem** still holds.
- **Pf.** (by induction on the number of such edge pairs)
  - Suppose $f(e) > 0$ and $f(e') > 0$ for a pair of antiparallel edges e and e'.
  - Set $f'(e) = f(e) - \delta$ and $f'(e') = f(e') - \delta$, where $\delta = \min\{ f(e), f(e') \}$.
  - f' is still a flow of the same value but has one fewer such pair. ▪

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Max-Flow Formulation: Correctness

- **Max-flow formulation.** Replace each edge with two antiparallel edges and assign unit capacity to every edge.

- **Lemma.** In any flow network, there exists a maximum flow f in which for each pair of antiparallel edges e and e': $f(e) = 0$ or $f(e') = 0$ or both. Moreover, **integrality theorem** still holds.

- **Theorem.** Max number of edge-disjoint s-t paths = value of max flow.
- **Pf.** Similar to the proof in directed graphs. Use the above **Lemma**.

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More Menger Theorems

- **Theorem.** Given an undirected graph and two nodes *s* and *t*, the max number of edge-disjoint *s*-*t* paths equals the min number of edges whose removal disconnects *s* and *t*.

- **Theorem.** Given an undirected graph and two nonadjacent nodes *s* and *t*, the max number of internally node-disjoint *s*-*t* paths equals the min number of internal nodes whose removal disconnects *s* and *t*.

- **Theorem.** Given a directed graph with two nonadjacent nodes *s* and *t*, the max number of internally node-disjoint *s*-*t* paths equals the min number of internal nodes whose removal disconnects *t* from *s*.

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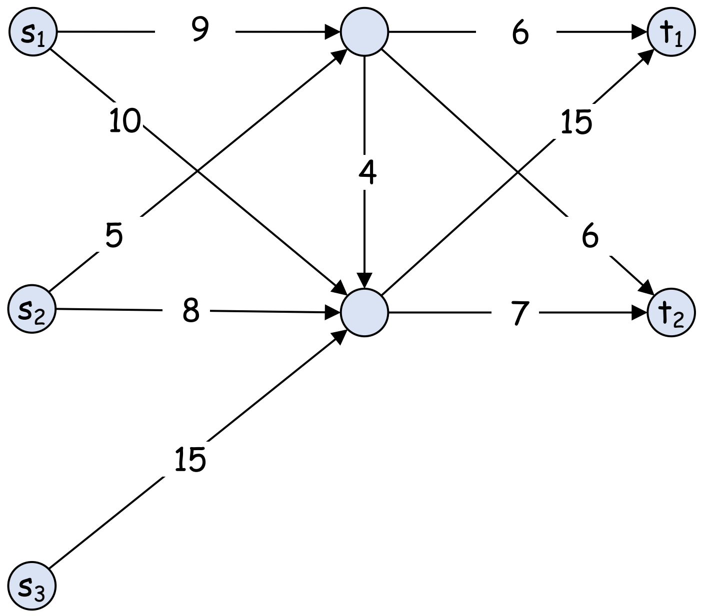
9. Extensions to Max Flow

Multiple Sources and Sinks

- **Def.** Given a directed graph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and multiple source nodes and multiple sink nodes, find a max flow that can be sent from the source nodes to the sink nodes.

flow network G



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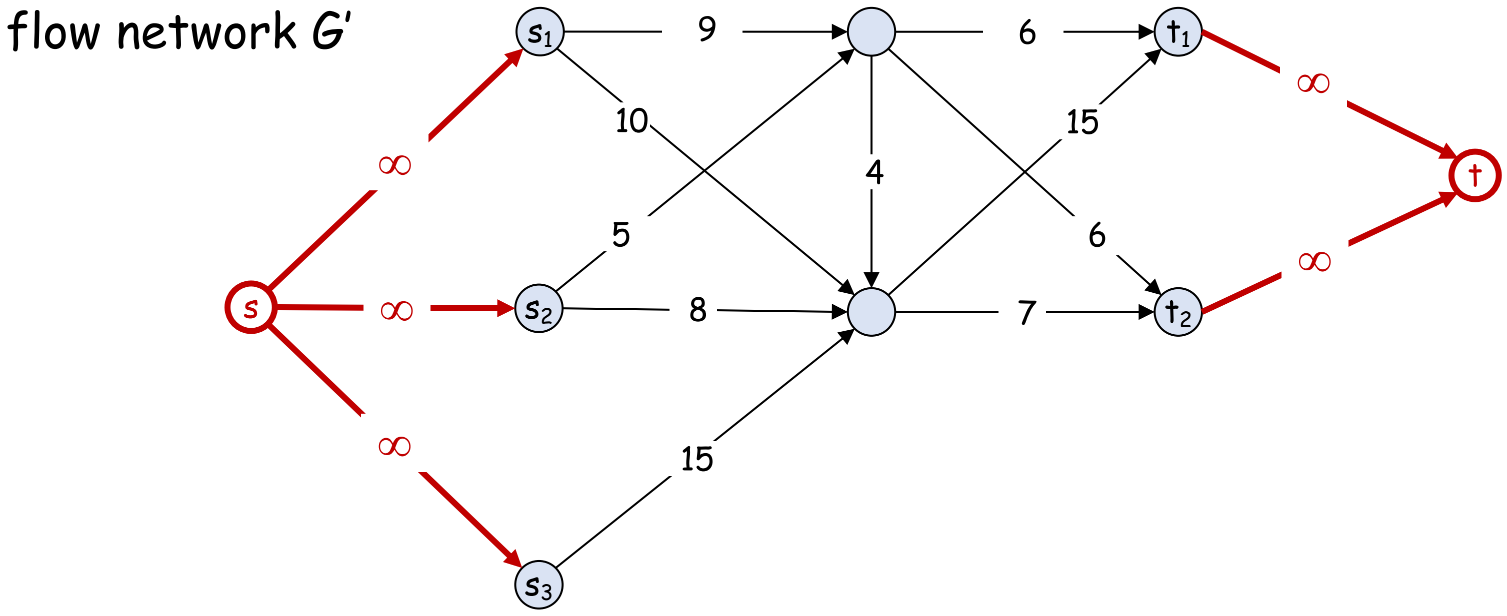
Multiple Sources and Sinks

- **Def.** Given a directed graph $G = (V, E)$ with edge capacities $c(e) \geq 0$ and multiple source nodes and multiple sink nodes, find a max flow that can be sent from the source nodes to the sink nodes.
- **Claim.** There exists a 1-1 correspondence between flows in G and G'.



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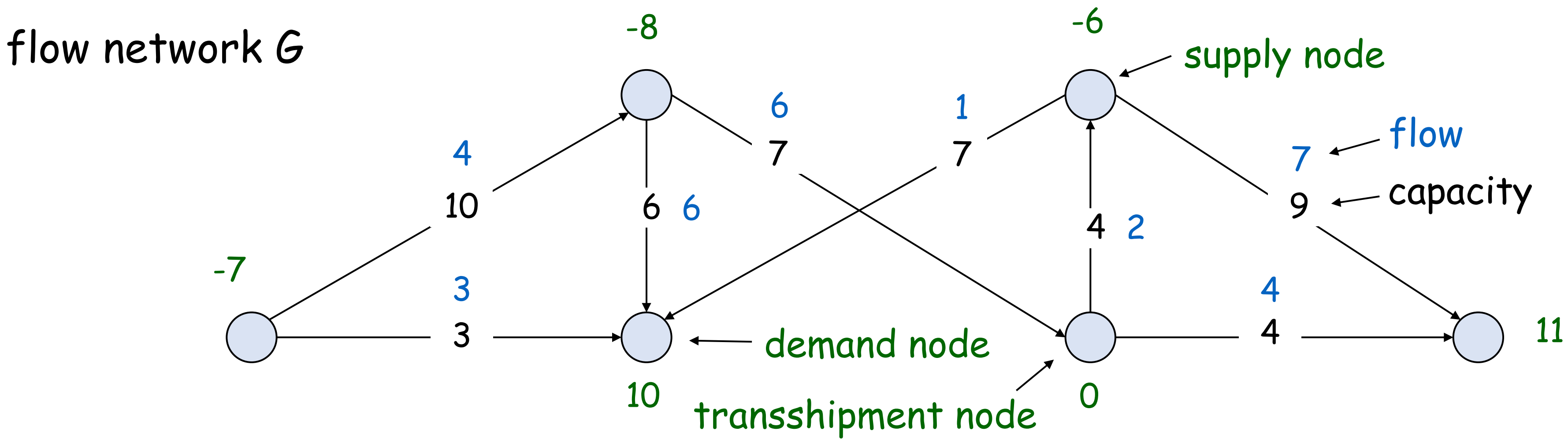
Circulation with Supplies and Demands

- **Def.** Given a directed graph *G = (V, E)* with edge capacities *c(e) ≥ 0* and node demands *d(v)*, a circulation is a function *f(e)* that satisfies:
  - [Capacity] For each $e \in E$: $0 \le f(e) \le c(e)$
  - [Flow conservation] For each $v \in V$: $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$
- **Circulation problem.** Given *(V, E, c, d)*, find a circulation.

simple notation:
$f^{in}(v) - f^{out}(v) = d(v)$



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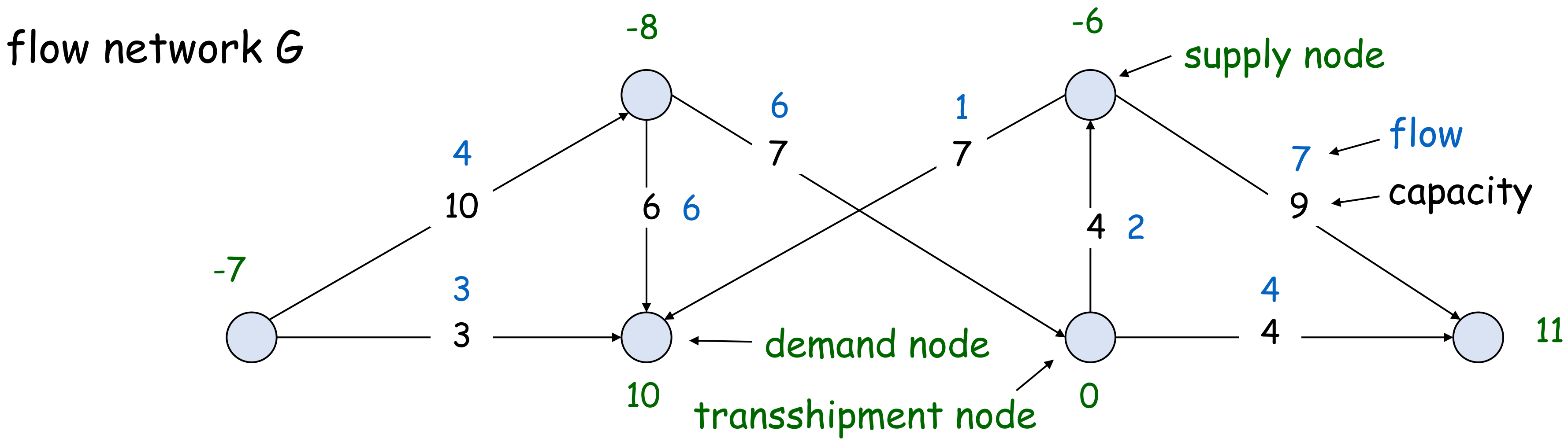
Circulation with Supplies and Demands

- **Circulation problem.** Given *(V, E, c, d)*, find a circulation.
- **Observation.** *G* has a circulation $\Rightarrow \sum_{v:d(v)>0} d(v) = \sum_{v:d(v)<0} -d(v)$

  total demand = total supply
- **Pf.** (equivalent to show $\sum_v d(v) = 0$)
  - Sum flow conservation over all $v \in V$: $\sum_v f^{in}(v) - \sum_v f^{out}(v) = \sum_v d(v)$
  - Each *f(e)* appears once in $\sum_v f^{in}(v)$ and once in $\sum_v f^{out}(v)$ so cancel out.
  - Therefore, $\sum_v d(v) = 0$.



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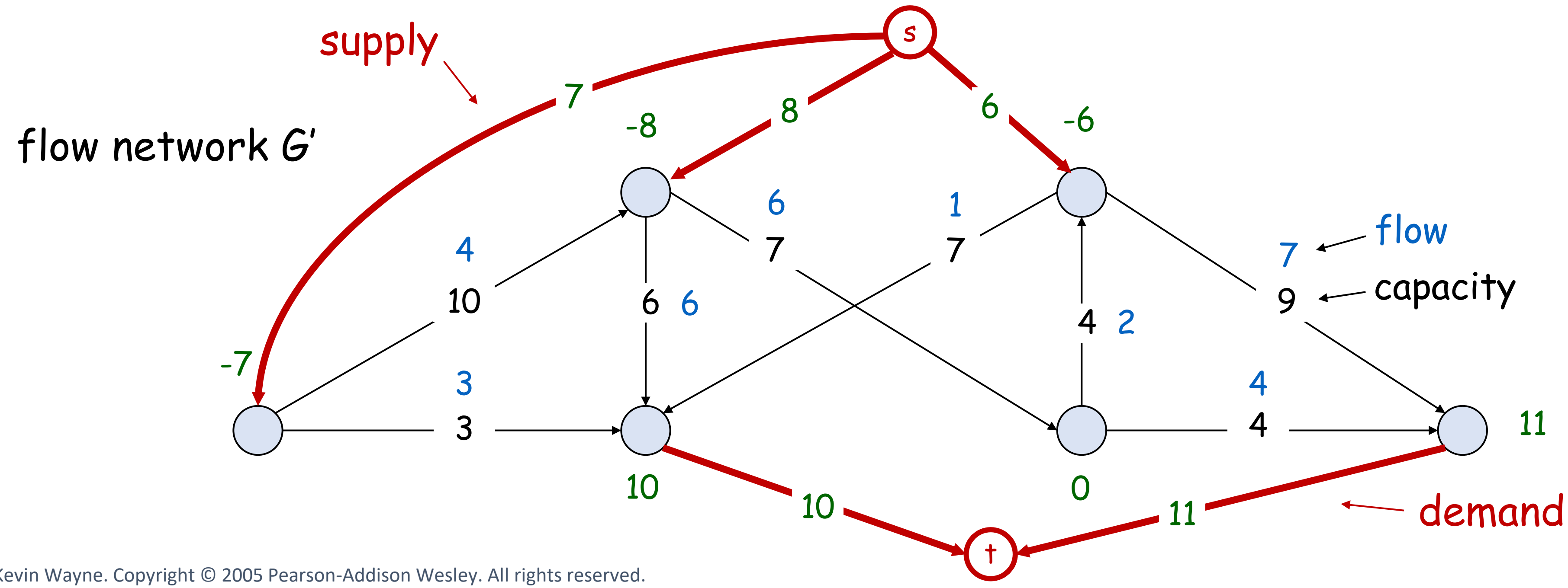
Circulation: Max-Flow Formulation

- **Max-flow formulation:**
  - Add new source *s* and sink *t*.
  - For each *v* with *d(v) < 0*, add edge *(s, v)* with capacity *−d(v)*.
  - For each *v* with *d(v) > 0*, add edge *(v, t)* with capacity *d(v)*.



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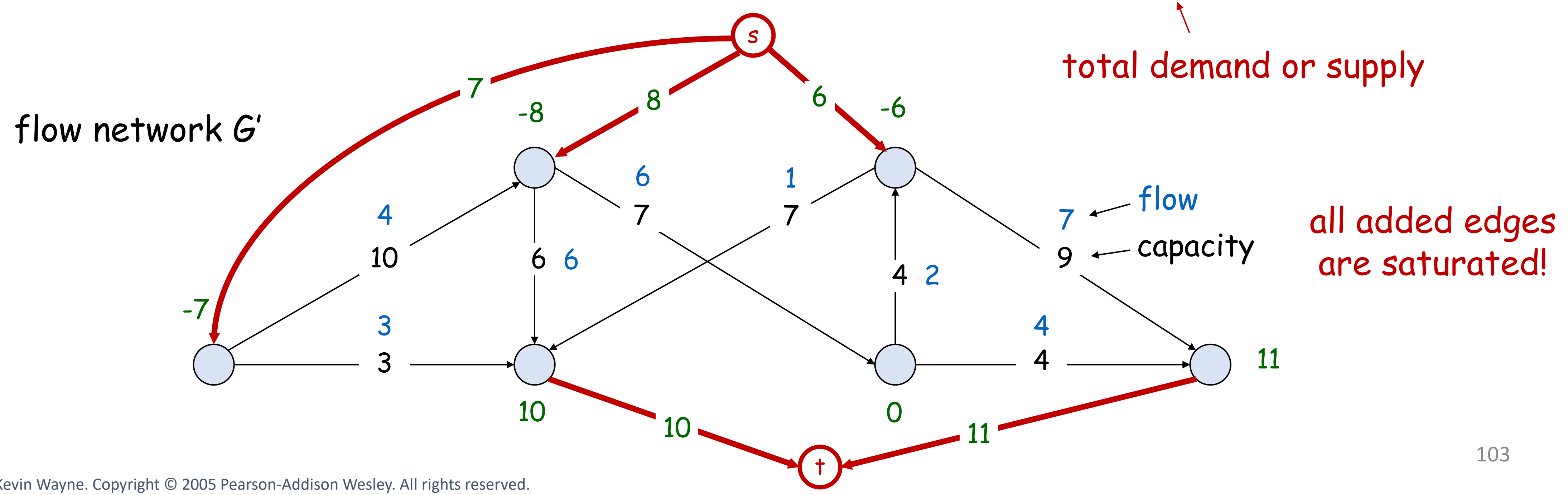
Circulation: Max-Flow Formulation

- **Max-flow formulation:**
  - Add new source *s* and sink *t*.
  - For each *v* with *d(v) < 0*, add edge *(s, v)* with capacity *−d(v)*.
  - For each *v* with *d(v) > 0*, add edge *(v, t)* with capacity *d(v)*.
- **Claim.** *G* has a circulation iff *G′* has max flow of value $D = \sum_{v:d(v)>0} d(v)$.



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Circulation with Supplies and Demands

- **Integrality theorem.** If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued.
- **Pf.** Follows from **integrality theorem** for max flow and previous **Claim**.

- **Theorem.** Given *(V, E, c, d)*, there does not exist a circulation iff there exists a node partition *(A, B)* such that $\sum_{v \in B} d(v) > c(A, B)$.
- **Pf idea.** Look at a min cut in *G'*.

exploit the relation between
cut in G' and node partition in G

demand by nodes in B exceeds
supply of nodes in B plus
max capacity of edges from A to B

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Circulation with Lower Bounds

- **Def.** Given a directed graph $G = (V, E)$ with edge capacities $c(e) \geq 0$, lower bounds $\ell(e) \geq 0$, and node demands $d(v)$, a circulation is a function $f(e)$ that satisfies:
  - [Capacity] For each $e \in E$: $\ell(e) \leq f(e) \leq c(e)$
  - [Flow conservation] For each $v \in V$: $\sum_{e \text{ into } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$

- **Circulation problem with lower bounds.** Given (V, E, ℓ, c, d), does there exist a feasible circulation?

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Circulation with Lower Bounds

- **Max-flow formulation.** Model lower bounds as circulation with demands.
  - Send $\ell(e)$ units of flow along edge e. ← this flow can be abstracted away from G'
  - Update demands of both endpoints.

- **Theorem.** There exists a circulation in G iff there exists a circulation in G'. Moreover, if all demands, capacities, and lower bounds in G are integers, then there exists a circulation in G that is integer-valued.
- **Pf idea.** $f(e)$ is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G'.

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10. Survey Design

Survey Design

- **Survey design:**
  - Design survey asking n_1 consumers about n_2 products. ← one question per consumer-product
  - Can survey consumer i about product j only if they own it.
  - Ask consumer i between c_i and c_i' questions.
  - Ask between p_j and p_j' consumers about product j.

- **Goal.** Design a survey that meets these specs, if possible.

- **Bipartite perfect matching.** Special case when $c_i = c_i' = p_j = p_j' = 1$.

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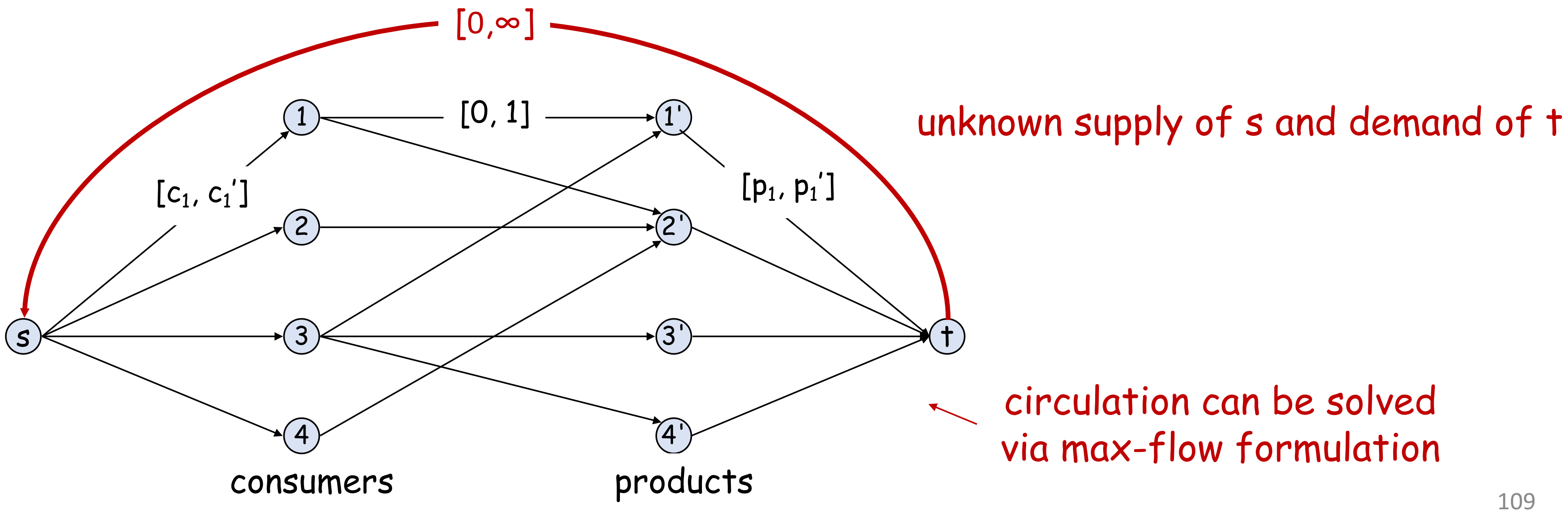
Survey Design: Circulation Formulation

- **Circulation formulation.** A circulation problem with lower bounds.
  - Add edge (i, j) if consumer i owns product j.
  - Add edge from s to consumer i. Add edge from product j to t.
  - Add edge from t to s with infinite capacity ∞. All demands = 0.
- **Claim.** Integer circulation $\Leftrightarrow$ feasible survey design.



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11. Airline Scheduling

Airline Scheduling

- **Airline scheduling:**
  - Complex computational problem faced by airline carriers.
  - Must produce schedules that are efficient in terms of equipment usage, crew allocation, and customer satisfaction.
  - One of largest consumers of high-powered algorithmic techniques.

even in presence of unpredictable events, e.g., weather and breakdowns

- **"Toy problem":**

one crew per flight

  - Manage flight crews by reusing them over multiple flights.
  - Input: set of k flights for a given day.
  - Flight i leaves origin o_i at time s_i and arrives at destination d_i at time f_i.
  - Minimize number of flight crews.

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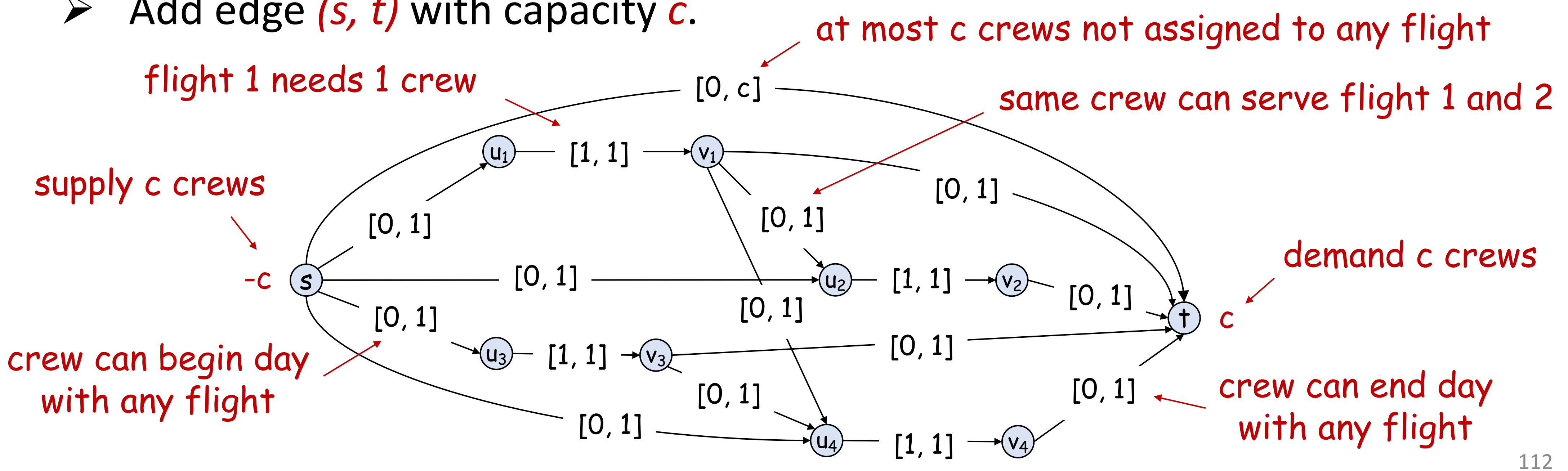
Airline Scheduling: Circulation Formulation

- **Circulation formulation.** (Check if c crews suffice.)
  - For flight i, add nodes u_i, v_i and edge (u_i, v_i) with lower bound and capacity 1.
  - Add source s with demand $-c$, and edges (s, u_i) with capacity 1.
  - Add sink t with demand c, and edges (v_i, t) with capacity 1.
  - If flight j reachable from i, add edge (v_i, u_j) with capacity 1.
  - Add edge (s, t) with capacity c.

u_i = start of flight i
v_i = end of flight i



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Airline Scheduling: Running Time

- **Theorem.** Airline scheduling problem can be solved in $O(k^3 \log k)$ time.
- **Pf.** (direct proof)
  - k = number of flights
  - $O(k)$ nodes, $O(k^2)$ edges.
  - c = number of crews (unknown)
  - At most k crews needed $\Rightarrow$ solve $\log_2 k$ circulation problems.

    binary search for min crew number c*
  - Any flow value is between 0 and $k \Rightarrow \leq k$ augmentations per circulation problem.
  - Overall time = $O(k^3 \log k)$.

- **Note.** Can solve in $O(k^3)$ time by formulating as minimum-flow problem.

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Airline Scheduling: Running Time

- **Remark.** We solved a toy version of a real problem.
- **Real-world problem models countless other factors:**
  - Flight crews can fly only a certain number of hours in a given time window.
  - Need optimal schedule over planning horizon, not just one day.
  - Flights don't always leave or arrive on schedule.
  - Simultaneously optimize both flight schedule and fare structure.
- **Take-away message:**
  - Our solution is a generally useful technique for efficient reuse of limited resources but trivializes real airline scheduling problem.
  - Flow techniques are useful for solving airline scheduling problems (and are widely used in practice).
  - Running an airline efficiently is a very difficult problem.

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12. Image Segmentation

Image Segmentation

- **Image segmentation.** Divide image into coherent regions.
  - Central problem in image processing.

- **Example.** Separate human and robot from background scene.

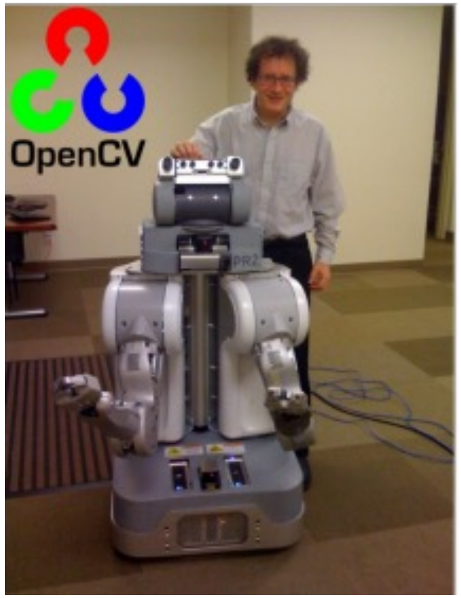

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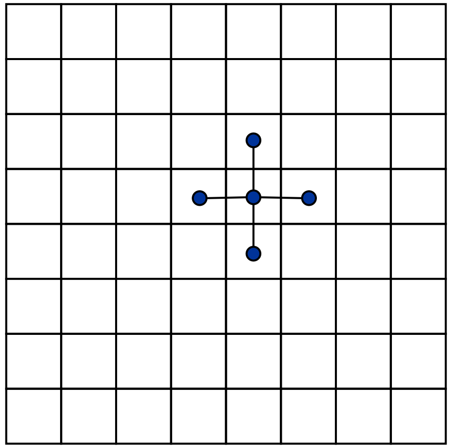
Image Segmentation

- **Foreground/background segmentation.** Label each pixel in picture as belonging to foreground or background.
  - V = set of pixels, E = pairs of neighboring pixels.
  - $a_i \geq 0$ is likelihood pixel i in foreground.
  - $b_i \geq 0$ is likelihood pixel i in background.
  - $p_{ij} \geq 0$ for each $(i, j) \in E$ is separation penalty for labeling one of i and j as foreground and the other as background.



- **Goals:**
  - Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground.
  - Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground.
  - Find partition (A, B) that maximizes $q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

foreground background

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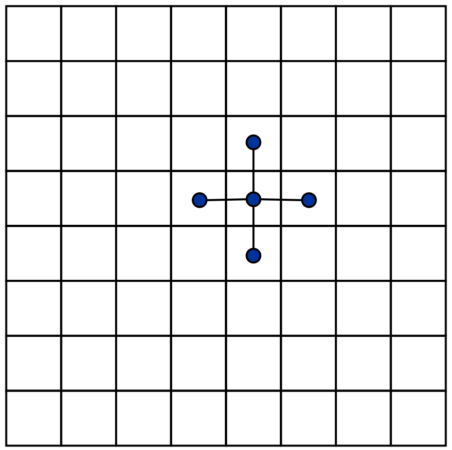
Image Segmentation

- **Formulate as min-cut problem:**
  - No source or sink.
  - Undirected graph.
  - Maximization. ⟵ how to convert max to min?
- **Turn into minimization problem:**
  - Maximizing $q(A,B) = \sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$



is equivalent to minimizing $\underbrace{\left(\sum_{i \in A \cup B} a_i + \sum_{j \in A \cup B} b_j \right)}_{\text{constant}} - q(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$

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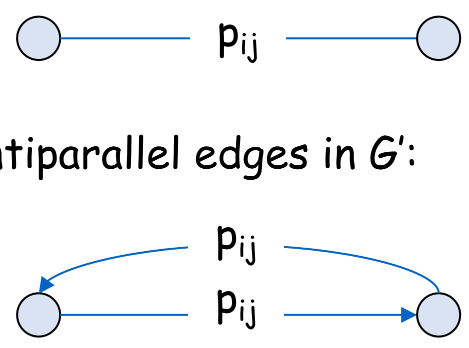
Image Segmentation

- **Formulate as min-cut problem $G' = (V', E')$:**
  - Include node for each pixel.
  - Use two anti-parallel edges instead of undirected edge.
  - Add source s to represent foreground.
  - Add sink t to represent background.

edge in G:

antiparallel edges in G':



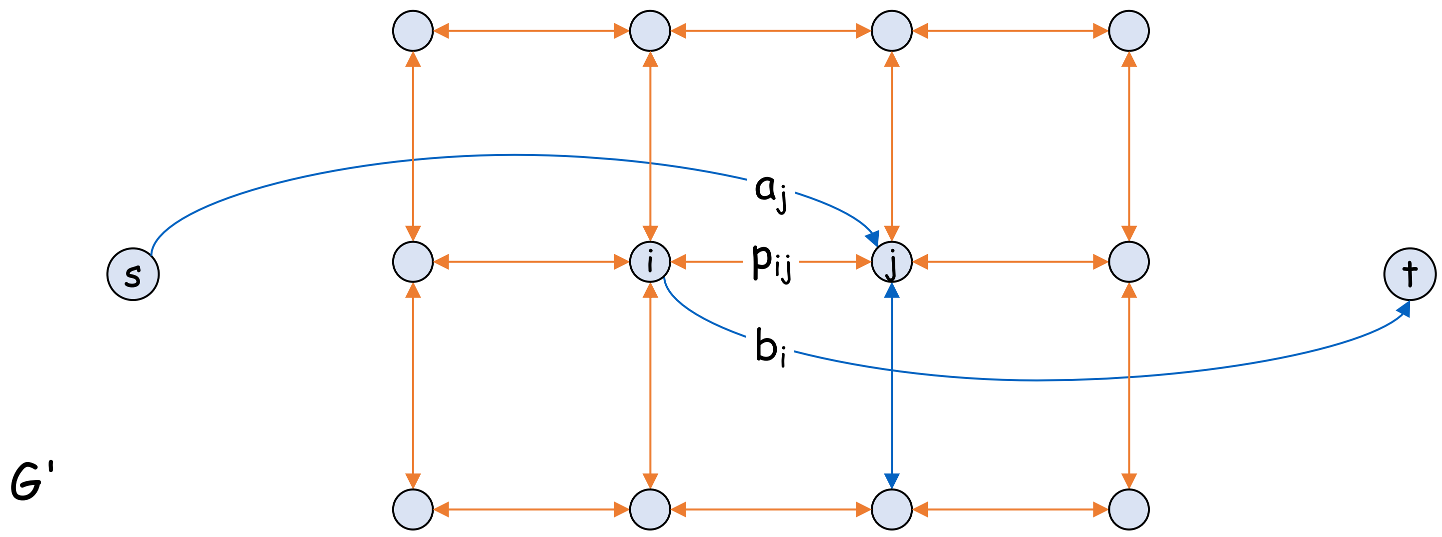


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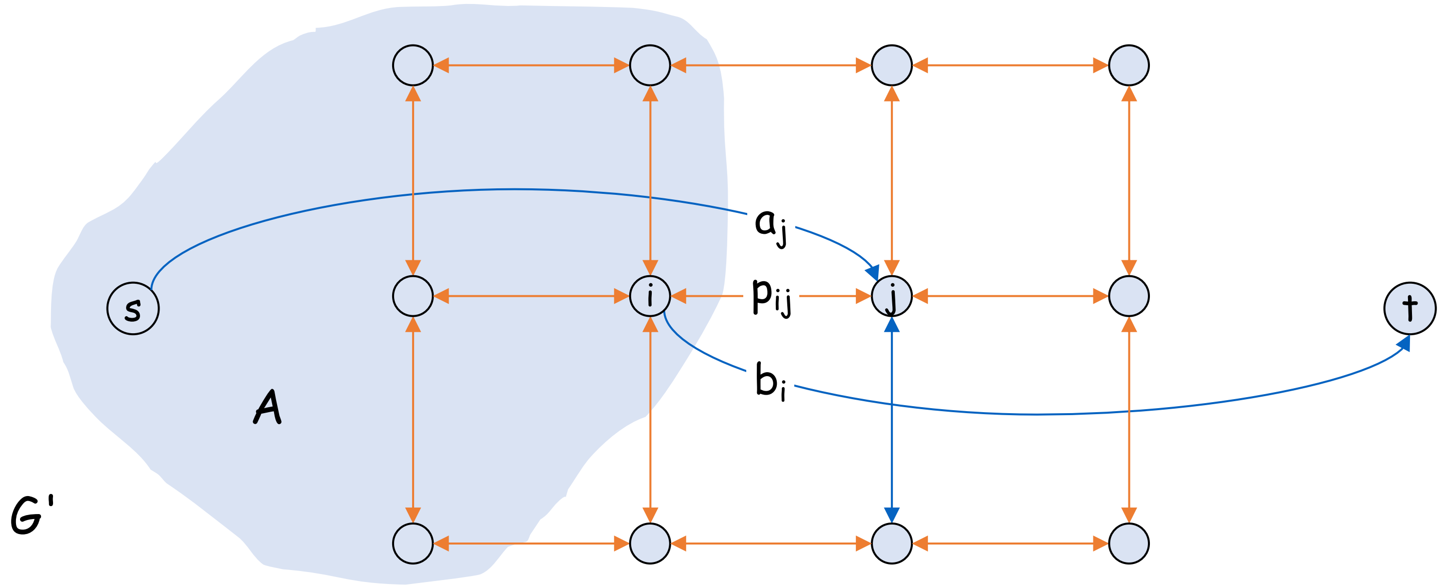
Image Segmentation

- **Min-cut formulation.** Consider a cut (A, B) in directed graph G':
  - A = foreground $\quad c(A,B) = \sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ i \in A, j \in B}} p_{ij}$ ← if i and j on different sides, p_{ij} counted exactly once
  - Precisely the quantity we want to minimize in original undirected graph.



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Network Flow: Closing Remarks

- **More applications:**
  - Project selection (maximum weight closure problem). [section 7.11 of textbook]
  - Baseball elimination. [section 7.12 of textbook]

- **Adding costs to max-flow problems:**
  - Min-cost max flow (a natural extension to max-flow problems). [lab assignment]
  - Min-cost perfect matching in bipartite graphs. [written assignment]