CS215: Discrete Math (H)
2023 Fall Semester   Written Assignment # 4
Due: Dec. 11th, 2023, please submit at the beginning of class

Q.1 Prove by induction that, for any integer $n \geq 2$,

$$\left(1-\frac{1}{2^2}\right)\cdot\left(1-\frac{1}{3^2}\right)\cdot\left(1-\frac{1}{4^2}\right)\cdot\cdots\cdot\left(1-\frac{1}{n^2}\right)=\frac{n+1}{2n}.$$

Q.2 Prove that if $A_1, A_2, \ldots, A_n$ and B are sets, then

$$\begin{aligned}&(A_1 - B)\cap(A_2 - B)\cap\cdots\cap(A_n - B)\\ =\ &(A_1\cap A_2\cap\cdots\cap A_n) - B.\end{aligned}$$

Q.3 Prove that if $h > -1$, then $1 + nh \leq (1 + h)^n$ for all nonnegative integers n. This is called **Bernoulli's inequality**.

Q.4 Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The parts of this exercise outline a strong induction proof that $P(n)$ is true for $n \geq 18$.

(a) Show statements $P(18)$, $P(19)$, $P(20)$ and $P(21)$ are true, completing the basis step of the proof.

(b) What is the inductive hypothesis of the proof?

(c) What do you need to prove in the inductive step?

(d) Complete the inductive step for $k \geq 21$.

(e) Explain why these steps show that this statement is true whenever $n \geq 18$.

Q.5 Show that the principle of mathematical induction and strong induction are equivalent; that is, each can be shown to be valid from the other.

Q.6 Suppose that the function f satisfies the recurrence relation $f(n) = 2f(\sqrt{n}) + \log n$ whenever n is a perfect square greater than 1 and $f(2) = 1$.

(a) Find $f(16)$

(b) Find a big-O estimate for $f(n)$. [Hint: make the substitution $m = \log n$.]

Q.7 The running time of an algorithm A is described by the following recurrence relation:

$$S(n) = \begin{cases} b & n = 1 \\ 9S(n/2) + n^2 & n > 1 \end{cases}$$

where b is a positive constant and n is a power of 2. The running time of a competing algorithm B is described by the following recurrence relation:

$$T(n) = \begin{cases} c & n = 1 \\ aT(n/4) + n^2 & n > 1 \end{cases}$$

where a and c are positive constants and n is a power of 4. For the rest of this problem, you may assume that n is always a power of 4. You should also assume that $a > 16$. (Hint: you may use the equation $a^{\log_2 n} = n^{\log_2 a}$)

(a) Find a solution for $S(n)$. Your solution should be in *closed form* (in terms of b if necessary) and should *not* use summation.

(b) Find a solution for $T(n)$. Your solution should be in *closed form* (in terms of a and c if necessary) and should *not* use summation.

(c) For what range of values of $a > 16$ is Algorithm B at least as efficient as Algorithm A asymptotically ($T(n) = O(S(n))$)?

Q.8 Consider three subsets A, B, C of a set S.

(1) Write a formula of $|\overline{A} \cap \overline{B} \cap \overline{C}|$ using the inclusion-exclusion principle.

(2) Use the formula in (1) to count the number of integers from 1 to 1000 (inclusive) which are not multiples of 10, 4 or 15.

Q.9 Suppose that $n \geq 1$ is an integer.

(a) How many functions are there from the set $\{1, 2, \ldots, n\}$ to the set $\{1, 2, 3\}$?

(b) How many of the functions in part (a) are one-to-one functions?

(c) How many of the functions in part (a) are onto functions?

Q.10 Prove that the binomial coefficient

$$\binom{240}{120}$$

is divisible by $242 = 2 \cdot 121$.

Q.11 Alice is going to choose a selection of 12 chocolates. There are 25 different brands of them and she can have as many as she wants of each brand (but can only choose 12 pieces). How many ways can she make this selection?

Q.12 Consider the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 10.$$

with five variables.

(1) Count the number of integer solutions, with $x_1 \geq 3$, $x_2 \geq 0$, $x_3 \geq -2$, $x_4 \geq 0$, and $x_5 \geq 0$.

(2) Count the number of integer solutions, with $0 \leq x_1 \leq 5$ and $x_2, x_3, x_4, x_5 \geq 0$.

Q.13 16 points are chosen inside a 5×3 rectangle. Prove that two of these points lie within $\sqrt{2}$ of each other.

Q.14 Prove the hockeystick identity

$$\sum_{k=0}^{r} \binom{n+k}{k} = \binom{n+r+1}{r}$$

whenever n and r are positive integers,

(a) using a combinatorial argument

(b) using Pascal's identity.

Q.15 For $0 \leq k \leq n$, show that

$$\sum_{r=k}^{n} \binom{n}{r}\binom{r}{k} = \binom{n}{k} 2^{n-k}.$$

Your proof may be either combinatorial or algebraic.

Q.16

Solve the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with initial conditions $a_0 = 1$, $a_1 = 0$, and $a_2 = 7$.

Q.17

(a) Find all solutions of the recurrence relation $a_n = 2a_{n-1} + 2n^2$.

(b) Find the solution of the recurrence relation in part (a) with initial condition $a_1 = 4$.

Q.18 Denote by a_n the number of *ternary* strings (with elements $0, 1, 2$) of length n that contain either 00 or 11.

(1) Find a recurrence relation for a_n with initial conditions.

(2) Find a closed-form expression for a_n.

Q.19 Let $S_n = \{1, 2, \ldots, n\}$ and let a_n denote the number of *non-empty* subsets of S_n that contain **no** two consecutive integers. Find a recurrence relation for a_n. Note that $a_0 = 0$ and $a_1 = 1$.

Q.20 Use generating functions to prove Vandermonde's identity:

$$C(m+n, r) = \sum_{k=0}^{r} C(m, r-k)C(n, k),$$

whenever m, n, and r are nonnegative integers with r not exceeding either m or n. [Hint: Look at the coefficient of x^r in both sides of $(1+x)^{m+n} = (1+x)^m(1+x)^n$.]

Q.21 Generating functions are very useful, for example, provide an approach to solving linear recurrence relations. Read pp. 537-548 of the textbook. [You do not need to write anything for this problem on your submitted assignment paper.]