Probability and Statistics

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Section 4.2

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P118 Q49

a.

$$
\begin{aligned}
E(Z) =& E(\alpha X + (1-\alpha)Y) \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\alpha x + (1-\alpha)y]f(x,y)\mathrm{d}x\mathrm{d}y \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[\alpha x + (1-\alpha)y]f(x)f(y)\mathrm{d}x\mathrm{d}y \\
=& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha x f(x)f(y)\mathrm{d}x\mathrm{d}y + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(1-\alpha)y f(x)f(y)\mathrm{d}x\mathrm{d}y \\
=& \alpha\int_{-\infty}^{\infty}x f(x)\mathrm{d}x + (1-\alpha)\int_{-\infty}^{\infty}y f(y)\mathrm{d}y \\
=& \alpha E(X) + (1-\alpha)E(Y) \\
=& \alpha\mu + (1-\alpha)\mu \\
=& \mu
\end{aligned}
$$

b.

$$
\begin{aligned}
Var(Z) =& Var[\alpha X + (1-\alpha)Y] \\
=& Var(\alpha X) + Var[(1-\alpha)Y] \\
=& \alpha^2 Var(X) + (1-\alpha)^2 Var(Y) \\
=& \alpha^2\sigma_X^2 + (1-\alpha)^2\sigma_Y^2 \\
=& (\sigma_X^2 + \sigma_Y^2)\alpha^2 - 2\sigma_Y^2\alpha + \sigma_Y^2
\end{aligned}
$$

可知，当 $\alpha = \frac{\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}$ 时，$Var(Z)$ 取得最小值，最小值为 $\frac{\sigma_X^2\sigma_Y^2}{\sigma_X^2+\sigma_Y^2}$。

c.

由上式可知，$\alpha = \frac{1}{2}$ 时，$Var(Z)$ 为

$$Var(Z) = \frac{\sigma_X^2 + \sigma_Y^2}{4}$$

若要使使用平均值优于单独使用 X 或 Y，则需要

$$\frac{\sigma_X^2 + \sigma_Y^2}{4} < \sigma_X^2 \quad \text{且} \quad \frac{\sigma_X^2 + \sigma_Y^2}{4} < \sigma_Y^2$$

即

$$\sigma_Y < \sqrt{3}\sigma_X$$
$$\sigma_X < \sqrt{3}\sigma_Y$$

P118 Q50

$$\begin{aligned} E(\overline{X}) =& E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ =& \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\ =& \frac{1}{n}\sum_{i=1}^{n} \mu \\ =& \mu \end{aligned}$$

$$\begin{aligned} Var(\overline{X}) =& Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ =& \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i) \\ =& \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\ =& \frac{1}{n}\sigma^2 \end{aligned}$$

P119 Q55

$$
\begin{aligned}
E(T) =& E\left(\sum_{k=1}^{n} kX_k\right) \\
=& \sum_{k=1}^{n} kE(X_k) \\
=& \sum_{k=1}^{n} k\mu \\
=& \mu \sum_{k=1}^{n} k \\
=& \mu \frac{n(n+1)}{2}
\end{aligned}
$$

$$
\begin{aligned}
Var(T) =& Var\left(\sum_{k=1}^{n} kX_k\right) \\
=& \sum_{k=1}^{n} k^2 Var(X_k) \\
=& \sum_{k=1}^{n} k^2 \sigma^2 \\
=& \sigma^2 \sum_{k=1}^{n} k^2 \\
=& \sigma^2 \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

补充 1

$$
\begin{aligned}
E(Z) =& E(5X - 2Y + 15) \\
=& 5E(X) - 2E(Y) + 15 \\
=& 5 \times 3 - 2 + 15 \\
=& 28
\end{aligned}
$$

$$
\begin{aligned}
Var(Z) =& Var(5X - 2Y + 15) \\
=& 5^2 Var(X) + (-2)^2 Var(Y) \\
=& 5^2 \times 4 + (-2)^2 \times 9 \\
=& 136
\end{aligned}
$$

补充 2

$$
\begin{aligned}
E(Z) =& E(2X_1 - X_2 + 3X_3 - \frac{1}{2}X_4) \\
=& 2E(X_1) - E(X_2) + 3E(X_3) - \frac{1}{2}E(X_4) \\
=& 2 \times 2 - 4 + 3 \times 6 - \frac{1}{2} \times 8 \\
=& 14
\end{aligned}
$$

$$
\begin{aligned}
Var(Z) =& Var(2X_1 - X_2 + 3X_3 - \frac{1}{2}X_4) \\
=& 2^2 Var(X_1) + (-1)^2 Var(X_2) + 3^2 Var(X_3) + (-\frac{1}{2})^2 Var(X_4) \\
=& 2^2 \times 4 + (-1)^2 \times 3 + 3^2 \times 2 + (-\frac{1}{2})^2 \times 1 \\
=& \frac{149}{4}
\end{aligned}
$$