CS215: Discrete Math (H)
2023 Fall Semester   Written Assignment # 5
Due: Dec. 20, 2023, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b)|a \text{ and } b \text{ have no letters in common}\}$

(2) $R_2 = \{(a, b)|a \text{ and } b \text{ are not the same length}\}$

(3) $R_3 = \{(a, b)|a \text{ is longer than } b\}$

Solution:

(1) Irreflexive, symmetric

(2) Irreflexive, symmetric

(3) Irreflexive, antisymmetric, transitive

$\square$

Q.2 Define a relation R on $\mathbb{R}$, the set of real numbers, as follows: For all x and y in $\mathbb{R}$, $(x, y) \in R$ if and only if $x - y$ is rational. Answer the followings, and explain your answers.

(1) Is R reflexive?

(2) Is R symmetric?

(3) Is R antisymmetric?

(4) Is R transitive?

Solution:

(1) Yes. Note that for all x we have $x - x = 0$, which is rational.

(2) Yes. Suppose that $(x, y) \in R$. Then $x - y = \frac{m}{n}$ for two integers m and n. Hence $y - x = \frac{-m}{n}$, which is again rational.

(3) No. Let $x = \sqrt{2}$ and $y = \sqrt{2} + 2$. Then we have $(x, y) \in R$ and $(y, x) \in R$, but $x \neq y$.

(4) Yes. Let $(x, y) \in R$ and $(y, z) \in R$. Then by definition both $x - y$ and $y - z$ are rational. Consequently, their sum $(x - y) + (y - z) = x - z$ is also rational. By definition, we have $(x, z) \in R$.

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Solution:

(a) $2^{n(n+1)/2}$

(b) $2^n 3^{n(n-1)/2}$

(c) $2^{n(n-1)}$

(d) $2^{n(n-1)/2}$

(e) $2^{n^2} - 2 \cdot 2^{n(n-1)}$

(f) $3^{n(n-1)/2}$

(g) 2^n

□

Q.4 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Solution: R^2 might not be irreflexive. For example, $R = \{(1,2),(2,1)\}$.

$\square$

Q.5 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Solution:

(1) Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \oplus R_2$ for all $a \in A$. Thus, $R_1 \oplus R_2$ is irreflexive.

(2) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cap R_2$

(3) Yes. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \cup R_2$

$\square$

Q.6 Let R be the relation on the set of ordered pairs of positive integers such that $((a,b),(c,d)) \in R$ if and only if $ad = bc$.

(a) Show that R is an equivalence relation.

(b) What is the equivalence class of $(1,2)$ with respect to the equivalence relation R?

(c) Give an interpretation of the equivalence classes for the equivalence relation R.

Solution:

(a) For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

(b) The equivalence classes of $(1,2)$ is the set of all pairs (a,b) such that the fraction a/b equals $1/2$.

(c) The equivalence classes are the positive rational numbers.

$\square$

Q.7 For the relation R on the set $X = \{(a,b,c) : \ a,b,c \in \mathbb{R}\}$ with $(a_1,b_1,c_1)R(a_2,b_2,c_2)$ if and only if $(a_1,b_1,c_1) = k(a_2,b_2,c_2)$ for some $k \in \mathbb{R} \setminus \{0\}$.

(1) Prove that this is an *equivalence* relation.

(2) Write at least three elements of the equivalence classes $[(1,1,1)]$ and $[(1,0,3)]$.

(3) Do all the equivalence classes in this relation have the same cardinality?

Solution:

(1) Reflexive: Consider $(a,b,c) \in X$. Note that $(a,b,c) = 1(a,b,c)$. Thus, the relation R is reflexive.

Symmetric: Consider $(a_1,b_1,c_1),(a_2,b_2,c_2) \in X$ such that $(a_1,b_1,c_1)R(a_2,b_2,c_2)$. By definition of the relation

$$\begin{aligned} (a_1,b_1,c_1) &= k(a_2,b_2,c_2) \\ \frac{1}{k}(a_1,b_1,c_1) &= (a_2,b_2,c_2). \end{aligned}$$

Since $1/k \in \mathbb{R}$, $(a_2,b_2,c_2)R(a_1,b_1,c_1)$. Thus, the relation is symmetric.

Transitive: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_3, b_3, c_3)$. By definition of the relation, we have

$$\begin{aligned}
(a_1, b_1, c_1) &= j(a_2, b_2, c_2) \\
(a_2, b_2, c_2) &= k(a_3, b_3, c_3) \\
(a_1, b_1, c_1) &= kj(a_3, b_3, c_3)
\end{aligned}$$

Since $jk \in \mathbb{R}$, we have $(a_1, b_1, c_1)R(a_3, b_3, c_3)$ and the relation is transitive. To sum up, the relation is an equivalence relation.

(2) We have

$$[(1, 1, 1)] = \{(1, 1, 1), (-1, -1, -1), (2, 2, 2), \ldots\}.$$

$$[(1, 0, 3)] = \{(1, 0, 3), (-1, 0, -3), (2, 0, 6), \ldots\}.$$

(3) No. Note that $[(0, 0, 0)] = \{(0, 0, 0)\}$. All the others are infinite.

Q.8 Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Solution:
We need to show that T is reflexive, symmetric, and transitive.
Reflexive: For any x, we have $(x, x) \in R$ and $(x, x) \in S$, then $(x, x) \in T$.
Symmetric: Suppose that $(x, y) \in T$. This means $(x, y) \in R$ and $(x, y) \in S$. Since R and S are both symmetric, we have $(y, x) \in R$ and $(y, x) \in S$. Then $(y, x) \in T$.
Transitive: Suppose that $(x, y) \in T$ and $(y, z) \in T$. Then $(x, y) \in R$ and $(y, x) \in R$ imply that $(x, z) \in R$. Similarly, we have $(x, z) \in S$. This will imply that $(x, z) \in T$.

Q.9 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Solution:

(a) Yes. (It is the smallest partial order: reflexivity ensures that very partial order contains at least all pairs (a, b).)

(b) No. It is not reflexive.

(c) Yes.

(d) No. The relations is not reflexive, not antisymmetric, not transitive.

$\square$

Q.10 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Solution:

(a) **Reflexive** For all $x \in \mathbb{R}$, $f(x) \leq f(x)$, so $f \preceq f$.

Antisymmetric Let $f \preceq g$ and $g \preceq f$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq f(x)$ and thus $f(x) = g(x)$. Since this holds for all x, we have $f = g$.

Transitive Let $f \preceq g \preceq h$. Then for all $x \in \mathbb{R}$, $f(x) \leq g(x) \leq h(x)$, giving $f(x) \leq h(x)$. So, $f \preceq h$.

(b) It is not a total ordering. Let $f(x) = x$ and $g(x) = -x$. Then $f(1) = 1 \not\leq -1 = g(1)$ and $g(-1) = 1 \not\leq -1 = f(-1)$. So it is not the case that for all x, $f(x) \leq g(x)$, and it is not the case that for all x, $g(x) \leq f(x)$. That is, these two functions are incomparable.

$\square$

Q.11 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Solution:

(a) Yes, because the product of positive integers greater than or equal to 2 is less than their sum.

(b) No, because $33 \preceq 26$ and $26 \preceq 33$, but $26 \neq 33$.

(c) No, because $33 \preceq 35$ and $35 \preceq 13$, but we do not have $33 \preceq 13$.

□

Q.12 The relation R on the set $X = \{(a, b, c) : a, b, c \in \mathbb{N}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $2^{a_1}3^{b_1}5^{c_1} \leq 2^{a_2}3^{b_2}5^{c_2}$.

(1) Prove that R is a partial ordering.

(2) Write two comparable and two incomparable elements if they exist.

(3) Find the least upper bound and the greatest lower bound of the two elements $(5, 0, 1)$ and $(1, 1, 2)$.

(4) List a minimal and a maximal element if they exist.

Solution:

(1) Reflexive: Consider $(a, b, c) \in X$. Note that $2^a3^b5^c \leq 2^a3^b5^c$ by definition of $\leq$ (equals). Thus, the relation is reflexive.

Antisymmetric: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_1, b_1, c_1)$. By definition of the relation, we have

$$\begin{aligned}
2^{a_1}3^{b_1}5^{c_1} &\leq 2^{a_2}3^{b_2}5^{c_2}, \\
2^{a_2}3^{b_2}5^{c_2} &\leq 2^{a_1}3^{b_1}5^{c_1}, \\
2^{a_1}3^{b_1}5^{c_1} &= 2^{a_2}3^{b_2}5^{c_2}, \\
a_1 &= a_2, \\
b_1 &= b_2, \\
c_1 &= c_2.
\end{aligned}$$

Transitive: Consider $(a_1, b_1, c_1), (a_2, b_2, c_2), (a_3, b_3, c_3) \in X$ such that $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ and $(a_2, b_2, c_2)R(a_3, b_3, c_3)$. By definition of the relation, we have

$$\begin{aligned}
2^{a_1}3^{b_1}5^{c_1} &\leq 2^{a_2}3^{b_2}5^{c_2}, \\
2^{a_2}3^{b_2}5^{c_2} &\leq 2^{a_3}3^{b_3}5^{c_3}, \\
2^{a_1}3^{b_1}5^{c_1} &\leq 2^{a_3}3^{b_3}5^{c_3}.
\end{aligned}$$

The latter is by transitivity of $\leq$. Thus, the relation is transitive.

(2) $(1, 2, 3)$ and $(4, 5, 6)$ are comparable. No pairs are incomparable. Every pair of integers has a lesser integer.

(3) Since $2^5 3^0 5^1 = 160$ and $2^1 3^1 5^2 = 150$. Thus, the least upper bound is $(5, 0, 1)$ and the greatest lower bound is $(1, 1, 2)$.

(4) The minimal element is $(0, 0, 0)$ because $2^0 3^0 5^0 = 1$ which is the s-mallest nonzero, nonnegative integer. There is no maximal element, because there is always a bigger integer.

Q.13 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a, b) \preceq (c, d) \Leftrightarrow (a, b) = (c, d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z} \times \mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0, 1, 2\} \times \{0, 1, 2\}$.
Solution: We now prove that $\preceq$ on the set $\mathbb{Z} \times \mathbb{Z}$ is a partial ordering. Obviously, $(a, b) \preceq (a, b)$, and we have $\preceq$ is reflexive; Suppose that $(a, b) \preceq$

(c, d) and $(c, d) \preceq (a, b)$, then the only possibility is that $(a, b) = (c, d)$. Then $\preceq$ is antisymmetric; Suppose that $(a, b) \preceq (c, d)$ and $(c, d) \preceq (e, f)$, then we have four possible cases: $(a, b) = (c, d)$ and $c^2 + d^2 < e^2 + f^2$; $(a, b) = (c, d)$ and $(c, d) = (e, f)$; $a^2 + b^2 < c^2 + d^2$ and $(c, d) = (e, f)$; $a^2 + b^2 < c^2 + d^2$ and $c^2 + d^2 < e^2 + f^2$. For each of the four cases above, we have $(a, b) \preceq (e, f)$ and thereby the relation $\preceq$ is transitive.

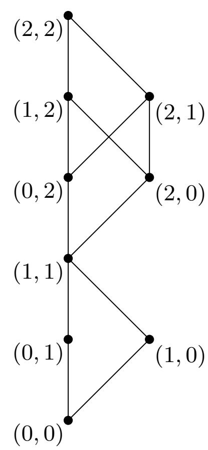

Figure 1: Q.13

$\square$

Q.14 Answer these questions for the partial order represented by this Hasse diagram.

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

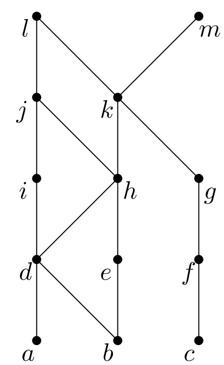

Figure 2: Q.14

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.

Solution:

(a) The maximal elements are the ones with no other elements above them, namely l and m.

(b) The minimal elements are the ones with no other elements below them, namely a, b and c.

(c) There is no greatest element, since neither l nor m is greater than the other.

(d) There is no least elements, since neither a nor b is less than the other.

(e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l and m are the elements fitting this description.

(f) Since k is less than both l and m, it is the least upper bound of a, b and c.

(g) No element is less than both f and h, so there are no lower bounds.

(h) Since there is no lower bound, there cannot be greatest lower bound.

$\square$