

CS215 DISCRETE MATH

Dr. QI WANG

Department of Computer Science and Engineering

Office: Room413, CoE South Tower

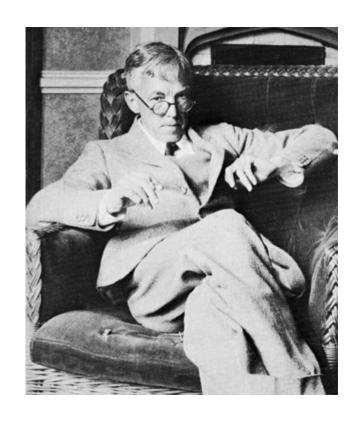
Email: wangqi@sustech.edu.cn

Application of Number Theory

G. H. Hardy (1877 - 1947)

In his 1940 autobiography *A Mathematician's Apology*, Hardy wrote

"The great modern achievements of applied mathematics have been in relativity and quantum mechanics, and these subjects are, at present, almost as 'useless' as the theory of numbers."



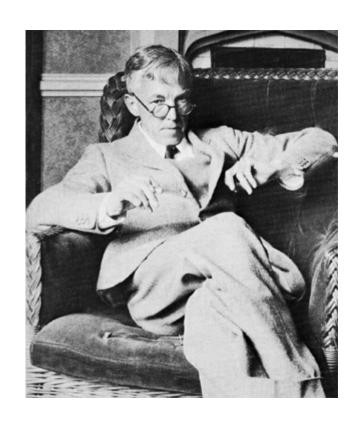


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If he could see the world now, Hardy would be spinning in his grave.



Number Theory

Number theory is a branch of mathematics that explores integers and their properties, is the basis of cryptography, coding theory, computer security, e-commerce, etc.



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At one point, the largest employer of mathematicians in the United States, and probably the world, was the National Security Agency (NSA). The NSA is the largest spy agency in the US (bigger than CIA, Central Intelligence Agency), and has the responsibility for code design and breaking.



Division

If a and b are integers with $a \neq 0$, we say that a divides b if there is an integer k such that b = ak, or equivalently b/a is an integer. In this case, we say that a is a factor or divisor of b, and b is a multiple of a. (We use the notations $a \mid b$, $a \nmid b$)



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Example

- ♦ 4 | 24
- ♦ 3 ∤ 7



■ All integers divisible by d > 0 can be enumerated as:

$$\dots, -kd, \dots, -2d, -d, 0, d, 2d, \dots, kd, \dots$$



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Question: Let n and d be two positive integers. How many positive integers not exceeding n are divisible by d?

Answer: Count the number of integers such that $0 < kd \le n$. Therefore, there are $\lfloor n/d \rfloor$ such positive integers.



Properties

Let a, b, c be integers. Then the following hold:

- (i) if a|b and a|c, then a|(b+c)
- (ii) if a|b then a|bc for all integers c
- iii) if a|b and b|c, then a|c



Properties

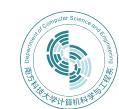
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Proof.



Corollary If a, b, c are integers, where $a \neq 0$, such that a|b and a|c, then a|(mb + nc) whenever m and n are integers.



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Proof. By part (ii) and part (i) of Properties.



The Division Algorithm

If a is an integer and d a positive integer, then there are unique integers q and r, with $0 \le r < d$, such that a = dq + r. In this case, d is called the divisor, a is called the dividend, q is called the quotient, and r is called the remainder.



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In this case, we use the notations $q = a \, div \, d$ and $r = a \, mod \, d$.



Congruence Relation

If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides a - b, denoted by $a \equiv b \pmod{m}$. This is called congruence and m is its modulus.



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Example



More on Congruences

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that a = b + km.



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Proof.

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"only if" part

"if" part
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(mod m) and mod m Notations

- $\blacksquare a \equiv b \pmod{m}$ and $a \mod m = b$ are different.
 - $\diamond a \equiv b \pmod{m}$ is a relation on the set of integers
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Let a and b be integers, and let m be a positive integer. Then $a \equiv b \mod m$ if and only if $a \mod m = b \mod m$



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Congruences of Sums and Products

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$



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Algebraic Manipulation of Congruences

- If $a \equiv b \mod m$, then
 - $c \cdot a \equiv c \cdot b \pmod{m}$?
 - $c + a \equiv c + b \pmod{m}$?
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```
14 \equiv 8 \pmod{6} but 7 \not\equiv 4 \pmod{6}
```



Computing the mod Function

Corollary Let m be a positive integer and let a and b be integers. Then

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(a+b) \mod m = ((a \mod m) + (b \mod m)) \mod m

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Example

$$\diamond$$
 7 +₁₁ 9 =?

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Closure: if $a, b \in \mathbf{Z}_m$, then $a +_m b$, $a \cdot_m b \in \mathbf{Z}_m$



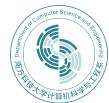
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- **Associativity**: if $a, b, c \in \mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$



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- **Commutativity**: if $a, b \in \mathbf{Z}_m$, then $a +_m b = b +_m a$
- **Distributivity**: if $a, b, c \in \mathbf{Z}_m$, then $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$



Group

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Example:

$$(\mathbb{Z},+)$$
, $(\mathbb{Q},+)$, $(\mathbb{R},+)$, $(\mathbb{M}_{n\times n},+)$? (\mathbb{Z}^*,\times) , (\mathbb{Q}^*,\times) , (\mathbb{R}^*,\times) , $(\mathbb{M}_{n\times n}^*,\cdot)$?



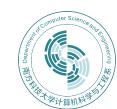
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■ Define a binary operation \circ on the elements of P_n : for $\rho, \pi \in P_n$, $\pi \circ \rho$ denotes a *re-permutation* of the elements of ρ according to the elements of π .



• Consider $s_3 = <1, 2, 3>$, and $P_3 = \{< p_1, p_2, p_3> | p_1, p_2, p_3 \in s_3 \text{ with } p_1 \neq p_2 \neq p_3\}.$



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- $\pi = <3, 2, 1>$, ho = <1, 3, 2>, what is $\pi \circ \rho$? $\pi \circ \rho = <2, 3, 1> \in P_3$



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- We can verify the other three properties.

$$\rho_1 \circ (\rho_2 \circ \rho_3) = (\rho_1 \circ \rho_2) \circ \rho_3$$
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For each $\rho \in P_3$, there exists another unique $\pi \in P_3$ such that $\rho \circ \pi = \pi \circ \rho = <1,2,3>$



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 (P_n, \circ) is called a *permutation group*.



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If the group operation is referred to as addition (*multiplication*), then the group also allows for *subtraction* (*division*).

$$a - b = a + (-b)$$
$$a/b = a \cdot b^{-1}$$



Ring

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Associativity: $(a \times b) \times c = a \times (b \times c)$

Distributivity: $a \times (b + c) = a \times b + a \times c$ $(a + b) \times c = a \times c + b \times c$



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Example:

$$(\mathbb{Z},+, imes)$$
, $(\mathbb{Q},+, imes)$, $(\mathbb{R},+, imes)$, $(\mathbb{M}_{n imes n},+,\cdot)$?



Commutative Ring, Integral Domain

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```
Identity element for multiplication: a1 = 1a = a

Nonzero product for any two nonzero elements: if ab = 0, then either a or b must be 0.
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$$(\mathbb{Z},+,\times)$$
, $(\mathbb{Q},+,\times)$, $(\mathbb{R},+,\times)$? $(\mathbb{Z}_m,+,\times)$, $(\mathbb{M}_{n\times n},+,\cdot)$?



Field

A *field*, denoted by $(F, +, \times)$, is an *integral domain* whose elements satisfy the following additional property.

Inverse for multiplication: For every $a \in F$, there exists an element b, denoted by a^{-1} , such that ab = ba = 1.



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, $(\mathbb{Q},+, imes)$, $(\mathbb{R},+, imes)$? $(\mathbb{Z}_p,+, imes)$?



Representations of Integers

We may use decimal (base 10) or binary or octal or hexadecimal or other notations to represent integers.



Representations of Integers

- We may use *decimal* (*base* 10) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.
- Let b > 1 be an integer. Then if n is a positive integer, it can be expressed uniquely in the form $n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0$, where k is nonnegative, a_i 's are nonnegative integers less than b. The representation of n is called the base-b expansion of n and is denoted by $(a_k a_{k-1} \dots a_1 a_0)_b$.



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Example

$$(101011111)_2 = 2^8 + 2^6 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 = 351$$

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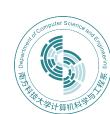
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Example



$$n = a_k b^k + a_{k-1} b^{k-1} + a_{k-2} b^{k-2} + \dots + a_2 b^2 + a_1 b + a_0$$

$$= b(a_k b^{k-1} + a_{k-1} b^{k-2} + a_{k-2} b^{k-3} + \dots + a_2 b + a_1) + a_0$$

$$= b(b(a_k b^{k-2} + a_{k-1} b^{k-3} + a_{k-2} b^{k-4} + \dots + a_2) + a_1) + a_0$$

$$= \dots$$



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$$= \dots$$

To construct the base-b expansion of an integer n,

- Divide n by b to obtain $n = bq_0 + a_0$, with $0 \le a_0 < b$
- The remainder a_0 is the rightmost digit in the base-b expansion of n. Then divide q_0 by b to get $q_0 = bq_1 + a_1$ with $0 \le a_1 < b$
- a₁ is the second digit from the right. Continue by successively dividing the quotients by b until the quotient is 0



Algorithm: Constructing Base-b Expansions

```
procedure base b expansion(n, b): positive integers with b > 1)
q := n
k := 0
while (q \neq 0)
a_k := q \mod b
q := q \operatorname{div} b
k := k + 1
return(a_{k-1}, ..., a_1, a_0) \{(a_{k-1} ... a_1 a_0)_b \text{ is base } b \text{ expansion of } n\}
```



Example

 \blacksquare (12345)₁₀ = (30071)₈



Example

 \blacksquare (12345)₁₀ = (30071)₈

$$12345 = 8 \cdot 1543 + 1$$

$$1543 = 8 \cdot 192 + 7$$

$$192 = 8 \cdot 24 + 0$$

$$24 = 8 \cdot 3 + 0$$

$$3 = 8 \cdot 0 + 3$$



Binary Addition of Integers

$$a = (a_{n-1}a_{n-2} \dots a_1a_0), b = (b_{n-1}b_{n-2} \dots b_1b_0)$$

```
procedure add(a, b): positive integers)
{the binary expansions of a and b are (a_{n-1}, a_{n-2}, ..., a_0)_2 and (b_{n-1}, b_{n-2}, ..., b_0)_2, respectively}
c := 0

for j := 0 to n - 1
d := \lfloor (a_j + b_j + c)/2 \rfloor
s_j := a_j + b_j + c - 2d
c := d
s_n := c
return(s_0, s_1, ..., s_n){the binary expansion of the sum is (s_n, s_{n-1}, ..., s_0)_2}
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O(n) bit additions



Algorithm: Binary Multiplication of Integers

```
a = (a_{n-1}a_{n-2} \dots a_1 a_0)_2, \ b = (b_{n-1}b_{n-2} \dots b_1 b_0)_2
ab = a(b_0 2^0 + b_1 2^1 + \dots + b_{n-1} 2^{n-1})
= a(b_0 2^0) + a(b_1 2^1) + \dots + a(b_{n-1} 2^{n-1})
```

```
procedure multiply(a, b: positive integers) {the binary expansions of a and b are (a_{n-1}, a_{n-2}, ..., a_0)_2 and (b_{n-1}, b_{n-2}, ..., b_0)_2, respectively} for j := 0 to n-1

if b_j = 1 then c_j = a shifted j places

else c_j := 0
{c_0, c_1, ..., c_{n-1} are the partial products}

p := 0
for j := 0 to n-1

p := p + c_j

return p {p is the value of ab}
```



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 $O(n^2)$ shifts and $O(n^2)$ bit additions 30 - 2



Algorithm: Computing div and mod

```
procedure division algorithm (a: integer, d: positive integer)
q := 0
r := |a|
while r \ge d
    r := r - d
    q := q + 1
if a < 0 and r > o then
     r := d - r
     q := -(q+1)
return (q, r) {q = a \operatorname{div} d is the quotient, r = a \operatorname{mod} d is the
remainder }
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 $O(q \log a)$ bit operations. But there exist more efficient algorithms with complextiy $O(n^2)$, where $n = \max(\log a, \log d)$

Algorithm: Computing div and mod (cont)

procedure division2 (a, $d \in \mathbb{N}$, $d \ge 1$) if a < d**return** (q, r) = (0, a)(q, r) = division2(|a/2|, d)q = 2q, r = 2rif a is odd r = r + 1if r > dr = r - dq = q + 1return (q, r)



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 $O(\log q \log a)$ bit operations.



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b^n = b^{a_{k-1} \cdot 2^{k-1} + \dots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \cdots b^{a_1 \cdot 2} \cdot b^{a_0}
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Successively finds $b \mod m$, $b^2 \mod m$, $b^4 \mod m$, ..., $b^{2^{k-1}} \mod m$, and multiplies together the terms b^{2^j} where $a_j = 1$.

```
procedure modular exponentiation(b: integer, n = (a<sub>k-1</sub>a<sub>k-2</sub>...a<sub>1</sub>a<sub>0</sub>)<sub>2</sub>, m: positive integers)
x := 1
power := b mod m
for i := 0 to k - 1
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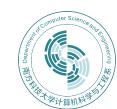
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 $O((\log m)^2 \log n)$ bit operations



Next Lecture

number theory, cryptography ...

