CS215 DISCRETE MATH

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Compound Propositions

- A *proposition* is a declarative statement that is either true or false. More complex propositions can be built from elementary statements using logical connectives.

- Logical connectives:
  - ◇ $\neg p$ (*Negation*)
  - ◇ $p \wedge q$ (*Conjunction*)
  - ◇ $p \vee q$ (*Disjunction*)
  - ◇ $p \oplus q$ (*Exclusive or*)
  - ◇ $p \rightarrow q$ (*Implication*)
  - ◇ $p \leftrightarrow q$ (*Biconditional*)

Negation

- Let p be a proposition. The statment "It is not the case that p." is called the *negation of p*, denoted by $\neg p$, and read as "not p".

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Examples:

  ◇ p – SUSTech is located in Shenzhen. (T)

  ◇ $\neg p$ – SUSTech is not located in Shenzhen.
      It is not the case that SUSTech is located in Shenzhen.

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  - ◇ $\neg p$ − SUSTech is not located in Shenzhen.
    It is not the case that SUSTech is located in Shenzhen.

  - ◇ $5 + 2 \neq 8$

  - ◇ 10 is not a prime number.

  - ◇ It is not the case that classes begin at 8:00am.

Negation

- A *truth table* displays the relationships between truth values (T or F) of different propositions.

p	$\neg p$
T	F
F	T

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Rows: contains all possible values of elementary propositions.

Conjunction (and)

- Let p and q be propositions. The *conjunction* of p and q, denoted by $p \wedge q$, is true when both p and q are true and is false otherwise.

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  ◇ q − $5 + 2 = 8$

  ◇ $p \wedge q$ − SUSTech is located in Shenzhen and $5 + 2 = 8$

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Examples:

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- ◇ q − $5 + 2 = 8$

- ◇ $p \wedge q$ − SUSTech is located in Shenzhen and $5 + 2 = 8$

- ◇ There are infinitely many twin prime numbers and $2 + 2 = 3$

- ◇ 2 is a prime number and 9 is a prime power.

- Let p and q be propositions. The *disjunction* of p and q, denoted by $p \lor q$, is false when both p and q are false and is true otherwise.

Disjunction (or)

- Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is false when both p and q are false and is true otherwise.

  Examples:

  ◇ p − SUSTech is located in Shenzhen.
  ◇ q − $5 + 2 = 8$

  ◇ $p \vee q$ − SUSTech is located in Shenzhen or $5 + 2 = 8$

Disjunction (or)

- Let p and q be propositions. The *disjunction* of p and q, denoted by $p \lor q$, is false when both p and q are false and is true otherwise.

Examples:

- ⋄ p − SUSTech is located in Shenzhen.
- ⋄ q − $5 + 2 = 8$

- ⋄ $p \lor q$ − SUSTech is located in Shenzhen or $5 + 2 = 8$

- ⋄ There are infinitely many twin prime numbers or $2 + 2 = 3$

- ⋄ 2 is a prime number or 9 is a prime power.

- **Conjunction** and **disjunction**

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Rows: all possible values of elementary propositions (2^n).

- Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is true when exactly one of p and q is true and is false otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- Let p and q be propositions. The *conditional statment* (a.k.a. *implication*) $p \rightarrow q$, is the proposition "if p, then q", is false when p is true and q is false, and true otherwise. In $p \rightarrow q$, p is called the *hypothesis* and q is called the *conclusion*.

Conditional Statement (implication)

- Let *p* and *q* be propositions. The *conditional statment* (a.k.a. *implication*) $p \rightarrow q$, is the proposition "if *p*, then *q*", is false when *p* is true and *q* is false, and true otherwise. In $p \rightarrow q$, *p* is called the *hypothesis* and *q* is called the *conclusion*.

p	*q*	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication

- $p \rightarrow q$ is read in a variety of equivalent ways:

  - ◇ if p then q
  - ◇ p implies q
  - ◇ p is sufficient for q
  - ◇ q is necessary for p
  - ◇ q follows from p
  - ◇ q unless $\neg p$
  - ◇ p only if q

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    ◇ q follows from p

    ◇ q unless $\neg p$

    ◇ p only if q

    Example:

    ◇ If you get 100 on the final, then you will get an A.

$$p \qquad\qquad\qquad q$$

Implication

- The *converse* of $p \to q$ is $q \to p$.
  The *contrapositive* of $p \to q$ is $\neg q \to \neg p$.
  The *inverse* of $p \to q$ is $\neg p \to \neg q$.

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  Examples:

  - If you get 100 on the final, then you will get an A. ($p \to q$)

  - If you get an A, then you get 100 on the final. ($q \to p$)
  - If you don't get an A, then you don't get 100 on the final. ($\neg q \to \neg p$)
  - If you don't get 100 on the final, then you don't get an A. ($\neg p \to \neg q$)

Implication

- The *converse* of $p \rightarrow q$ is $q \rightarrow p$.
  The *contrapositive* of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.
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  Examples:

  ◇ If you get 100 on the final, then you will get an A. ($p \rightarrow q$)

  ◇ If you get an A, then you get 100 on the final. ($q \rightarrow p$)
  ◇ If you don't get an A, then you don't get 100 on the final. ($\neg q \rightarrow \neg p$)
  ◇ If you don't get 100 on the final, then you don't get an A. ($\neg p \rightarrow \neg q$)

  $\neg q \rightarrow \neg p$ is *equivalent* to $p \rightarrow q$

- Let p and q be propositions. The *biconditional statment* (a.k.a. *bi-implications*) $p \leftrightarrow q$, is the proposition "p if and only if q", is true when p and q have the same truth values, and false otherwise.

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  - $\diamond$ p is necessary and sufficient for q
  - $\diamond$ if p then q, and conversely
  - $\diamond$ p iff q

Biconditional

- Let p and q be propositions. The *biconditional statment* (a.k.a. *bi-implications*) $p \leftrightarrow q$, is the proposition "p if and only if q", is true when p and q have the same truth values, and false otherwise.

  - ◇ p is necessary and sufficient for q
  - ◇ if p then q, and conversely
  - ◇ p iff q

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- Construct a truth table for $p \lor q \to \neg r$

Constructing the Truth Table

■ Construct a truth table for $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Computer Representation of True and False

- A *bit* is sufficient to represent two possible values: 0 (false) or 1 (true)

- A varaiable that takes on values 0 and 1 is called a *Boolean variable*.

- A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

Computer Representation of True and False

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- *bitwise operations*: replace "T" and "F" with 1 and 0.

$$
\begin{array}{r}
1011\ 0011 \\
\vee\ \underline{0110\ 1010} \\
1111\ 1011
\end{array}
$$

George Boole

- British mathematician (b. 1815, d. 1864)

  ◇ The inventor of Boolean algebra

  ◇ Truth tables are an example of B.A.

  ◇ Although Booles's work was not originally perceived as particularly interesting, even by other mathematicians, he is now seen as one of the founders of the field of Computer Science.
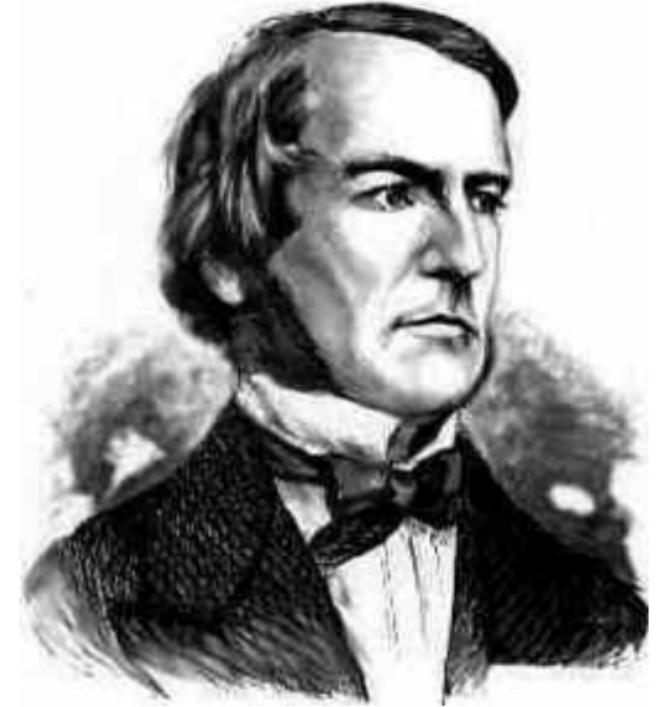
- **Translation of English sentences**

  ◇ use atomic (elementary) propositions

- **Inference and reasoning**

  ◇ new true propositions are inferred from existing ones

  ◇ used in *Artificial Intelligence*:

  – rule based (expert) systems, automatic theorem provers, ...

- **Design of logic circuit**

- If you are older than 13 or you are with your parents then you can watch this movie.

Translation

- If you are older than 13 or you are with your parents then you can watch this movie.

  If (you are older than 13) **or** (you are with your parents) **then** (you can watch this movie).

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  Atomic (**elementary**) propositions:
    - p − you are older than 13
    - q − you are with your parents
    - r − you can watch this movie

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  Atomic (**elementary**) propositions:

  p − you are older than 13

  q − you are with your parents

  r − you can watch this movie

  Translation: $p \lor q \rightarrow r$

- **If** (you are older than 13) **or** (you are with your parents) **then** (you can watch this movie).

  Translation: $p \lor q \rightarrow r$

Inference

- **If** (you are older than 13) **or** (you are with your parents) **then** (you can watch this movie).

  Translation: $p \vee q \rightarrow r$

  Given that p is true.

  With the help of the logic, we can infer the following statement:
  - You can watch this movie.

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  ◇ builds programs that act intelligently

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 - ◇ support inferences where new facts are inferred from existing ones following the semantics of logic

Inference

- *Artificial intelligence*

  ◇ builds programs that act intelligently

  ◇ programs often rely on symbolic manipulations

- *Expert systems*

  ◇ encode knowledge about the world in logic

  ◇ support inferences where new facts are inferred from existing ones following the semantics of logic

- *Theorem provers*

  ◇ encode existing knowledge (e.g., math) using logic
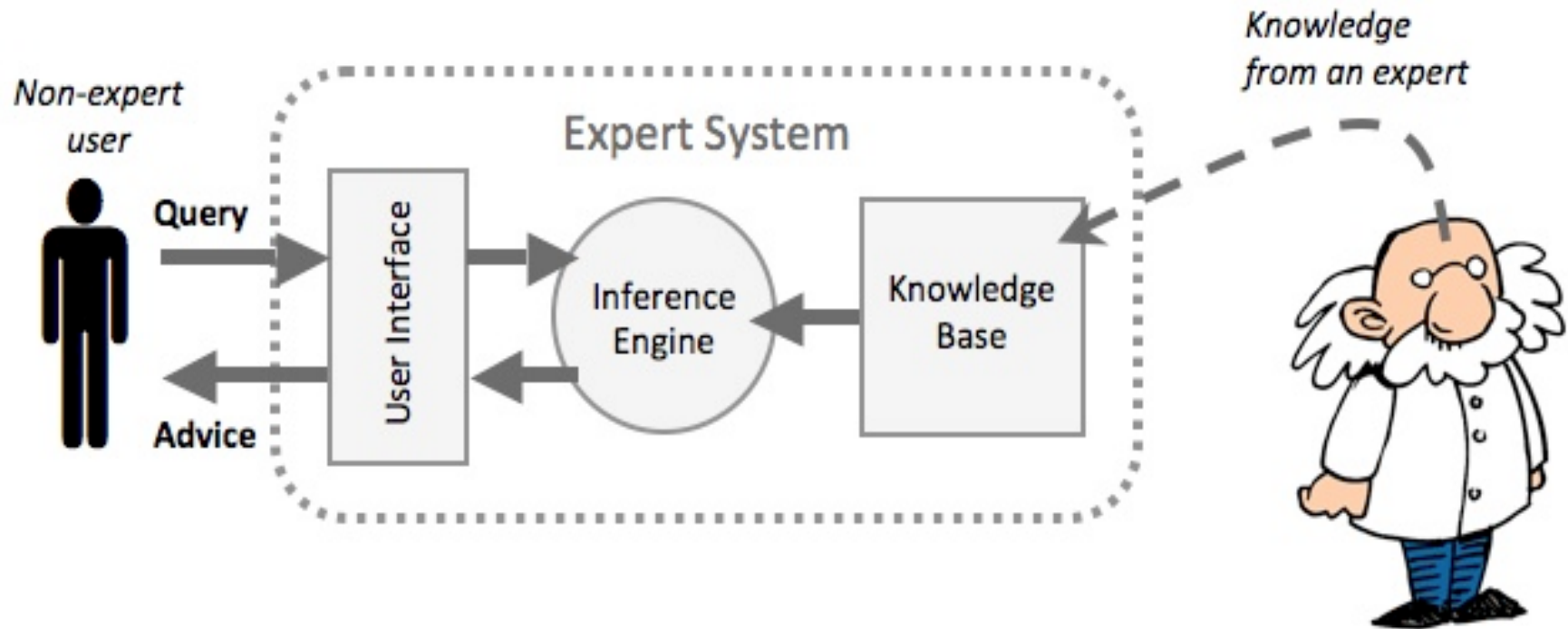
  ◇ show that some hypothesis is true
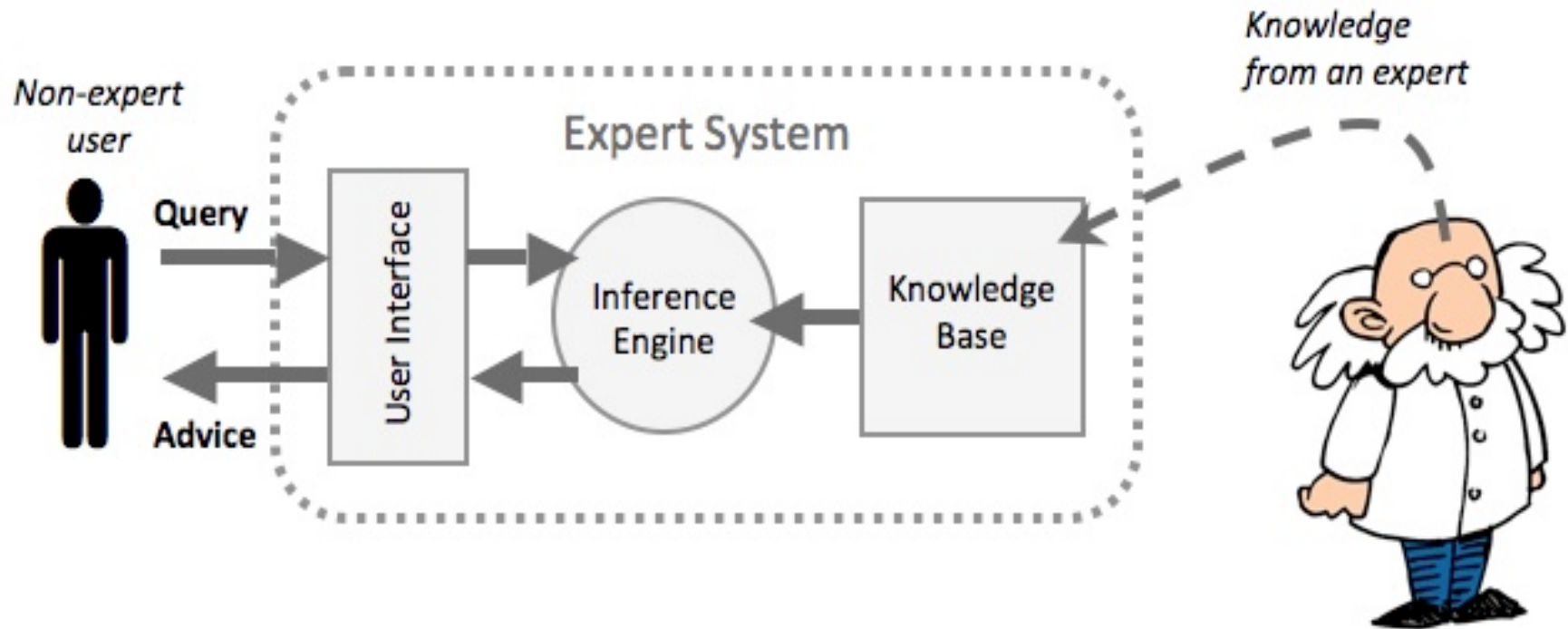
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Expert System

Computer-aided proofs of Arrow's and other impossibility theorems ☆

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A R T I C L E   I N F O

A B S T R A C T

Arrow's impossibility theorem is one of the landmark results in social choice theory. Over the years since the theorem was proved in 1950, quite a few alternative proofs have been put forward. In this paper, we propose yet another alternative proof of the theorem. The basic idea is to use induction to reduce the theorem to the base case with 3 alternatives and 2 agents and then use computers to verify the base case. This turns out to be an effective approach for proving other impossibility theorems such as Muller–Satterthwaite and Sen's theorems as well. Motivated by the insights of the proof, we discover a new theorem with the help of computer programs. We believe this new proof opens an exciting prospect of using computers to discover similar impossibility or even possibility results.

Other Applications

Tautology and Contradiction

- A compound proposition that is always true for all possible truth values is called a *tautology*.

- A compound proposition that is always false for all possible truth values is called a *contradiction*.

- A compound proposition that is neither a tautology nor a contradiction is called a *contingency*.

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P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
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- Two propositions are *equivalent* if they always have the same truth value.

Equivalent Propositions

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    Examples:

```
(1) if ((i+j ≤ p+q) && (i ≤ p) &&
        ((j > q) || (List1[i] ≤ List2[j])))
(2)    List3[k] = List1[i]
(3)    i = i+1
(4) else
(5)    List3[k] = List2[j]
(6)    j = j+1
(7) k = k+1
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Consider the two pieces of codes taken from two different versions of *Mergesort*. Do they do the same thing?

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- Let's rewrite using
  $s \sim (i + j \leq p + q)$, $t \sim (i \leq p)$, $u \sim (j > q)$
  $v \sim (List1[i] \leq List2[j])$

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(1) $s \wedge t \wedge (u \vee v)$ (1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

Equivalent Propositions

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  (1) $s \wedge t \wedge (u \vee v)$ (1') $(s \wedge t \wedge u) \vee (s \wedge t \wedge v)$

  Now set $w \sim (s \wedge t)$

  (1) $w \wedge (u \vee v)$ (1') $(w \wedge u) \vee (w \wedge v)$

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(1) $w \wedge (u \vee v)$

w	u	v	$u \vee v$	$w \wedge (u \vee v)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(1') $(w \wedge u) \vee (w \wedge v)$

w	u	v	$w \wedge u$	$w \wedge v$	$(w \wedge u) \vee (w \wedge v)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Distributive Laws

- $(w \wedge (u \vee v))$ is *equivalent* to $(w \wedge u) \vee (w \wedge v)$

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- Equivalent statements are important for logical reasoning since they can be substituted and can help us to:
  - make a logical argument
  - infer new propositions

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  Example $p \rightarrow q \equiv \neg q \rightarrow \neg p$

De Morgan's Laws

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$

  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

De Morgan's Laws

- $\neg(p \lor q) \equiv \neg p \land \neg q$

  $\neg(p \land q) \equiv \neg p \lor \neg q$

p	q	$\neg p$	$\neg q$	$(p \lor q)$	$\neg(p \lor q)$	$\neg p \land \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Important Logical Equivalences

- **_Identity laws_**

  - $\diamond\ p \wedge T\ \equiv\ p$
  - $\diamond\ p \vee F\ \equiv\ p$

- **_Domination laws_**

  - $\diamond\ p \vee T\ \equiv\ T$
  - $\diamond\ p \wedge F\ \equiv\ F$

- **_Idempotent laws_**

  - $\diamond\ p \vee p\ \equiv\ p$
  - $\diamond\ p \wedge p\ \equiv\ p$

Important Logical Equivalences

- *Double negation laws*
  - ◇ $\neg(\neg p) \equiv p$

- *Commutative laws*
  - ◇ $p \vee q \equiv q \vee p$
  - ◇ $p \wedge q \equiv q \wedge p$

- *Associative laws*
  - ◇ $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - ◇ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Important Logical Equivalences

- *Distributive laws*

  - $\diamond\ p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$
  - $\diamond\ p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

- *De Morgan's laws*

  - $\diamond\ \neg(p \lor q) \equiv \neg p \land \neg q$
  - $\diamond\ \neg(p \land q) \equiv \neg p \lor \neg q$

- *Others*

  - $\diamond\ p \lor (p \land q) \equiv p$
  - $\diamond\ p \land (p \lor q) \equiv p$ *Absorption laws*

  - $\diamond\ p \lor \neg p \equiv T$
  - $\diamond\ p \land \neg p \equiv F$ *Negation laws*

  - $\diamond\ p \to q \equiv \neg p \lor q$ *Useful law*

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Using Logical Equivalences

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- **Example**: Show that $(p \land q) \to p$ is a tautology.

$$
\begin{aligned}
\textbf{Proof}: (p \land q) \to p &\equiv \neg(p \land q) \lor p && \text{Useful} \\
&\equiv (\neg p \lor \neg q) \lor p && \text{De Morgan's} \\
&\equiv (\neg q \lor \neg p) \lor p && \text{Commutative} \\
&\equiv \neg q \lor (\neg p \lor p) && \text{Associative} \\
&\equiv \neg q \lor T && \text{Negation} \\
&\equiv T && \text{Domination}
\end{aligned}
$$

Using Logical Equivalences

- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

- **Example**: Show that $(p \wedge q) \to p$ is a tautology.

  Proof (alternatively):

p	q	p ∧ q	(p ∧ q)→p
T	T	T	**T**
T	F	F	**T**
F	T	F	**T**
F	F	F	**T**

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- Equivalences can be used in proofs. A proposition or its part can be transformed using equivalences.

- **Example**: Show that $p \rightarrow q \equiv \neg q \rightarrow \neg p$.

  Proof:
  $$\begin{aligned}
  \neg q \rightarrow \neg p &\equiv \neg(\neg q) \vee (\neg p) & \text{Useful}\\
  &\equiv q \vee (\neg p) & \text{Double negation}\\
  &\equiv (\neg p) \vee q & \text{Communtative}\\
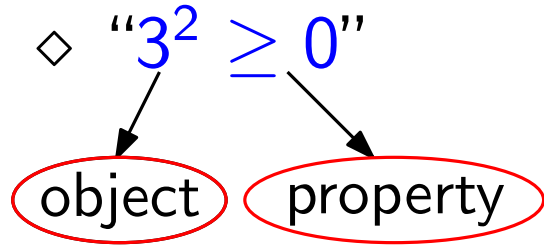 &\equiv p \rightarrow q & \text{Useful}
  \end{aligned}$$

Limitations of Propositional Logic

- Propositional logic: the world is described in terms of elementary propositions and their logical combinations. Elementary statements typically refer to objects, their properties and relations.
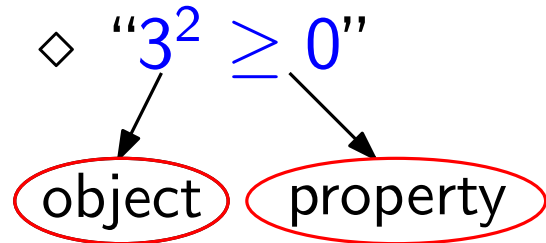
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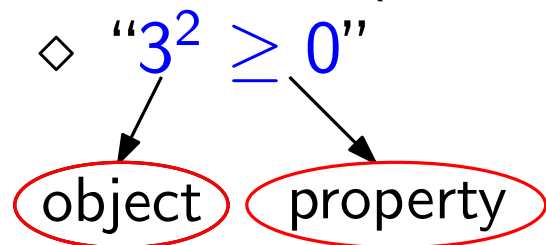
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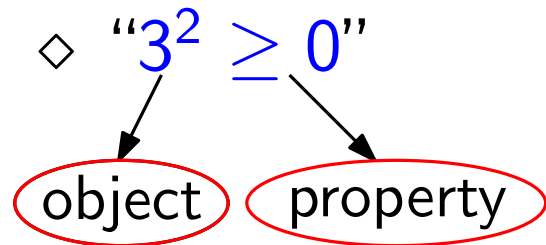
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Solution: make statements with variables
- If x is an integer, then $x^2 \geq 0$.
- x is an integer $\rightarrow x^2 \geq 0$.

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  - ◇ "Some of the integers are prime."

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Solutions: make statements with *quantifiers*
  – universal quantifier: the property is satisfied by all members of the group
  – existential quantifier: at least one member of the group satisfies the property

- Remedies the limitations of propositional logic:
  - ◇ explicitly models objects and their properties
  - ◇ allows to make statements with variables and quantify them

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- Basic building blocks of the predicate logic:
    - ◇ *Constant* – models a specific object
      Examples: "1", "SUSTech", ...

    - ◇ *Variable* – represents object of specific type
      Examples: x, y, ... (universe can be people, numbers ...)

    - ◇ *Predicate* – represents properties or relations among objects
      Examples: Red(car23), student(x), married(A, B)...

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  Example: Assume $\text{Prime}(x)$ where the universe of discourse are integers

  - Prime(2) ... T
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    – Prime(6) ... F
    – ...

  Is $\text{Prime}(x)$ a proposition?

    No, but after the substitution it becomes one.

- A *predicate* is a statement $P(x_1, x_2, \ldots, x_n)$ that contains n variables $x_1, x_2, \ldots, x_n$ and becomes a proposition when specific values are substituted for the variables x_i.

- The *universe* (*domain*) D of the *predicate variables* $(x_1, x_2, \ldots, x_n)$ is the set of all values that may be substituted in place of the variables.

- The *truth set* of $P(x_1, x_2, \ldots, x_n)$ is the set of all values of the predicate variables $(x_1, x_2, \ldots, x_n)$ such that the proposition $P(x_1, x_2, \ldots, x_n)$ is true.

- **Example 1**: (Predicates with one variable)
  Let $P(x)$ be the predicate "$x^2 > x$" with universe of the real numbers.

  - ◇ What are the truth values of $P(2)$ and $P(1)$?
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  - ◇ $P(2) = \text{T}$, $P(1) = \text{F}$
  - ◇ $x > 1$ or $x < 0$

- **Example 2**: (Predicates with two variables)
  Let $Q(x, y)$ be the predicate "$x = y + 3$" with universe of the real numbers.

  - ◇ What are the truth values of $Q(1, 2)$ and $Q(3, 0)$?
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  - ◇ $Q(1, 2) = F$, $Q(3, 0) = T$
  - ◇ $(a, a - 3)$ for all real numbers a

- Compound statements are obtained via logical connectives.

Compound Statements in Predicate Logic

■ Compound statements are obtained via logical connectives.

Example:

◇ Prime(2) ∧ Prime(3)
 – Translation: "Both 2 and 3 are primes." (T)
◇ City(Shenzhen) ∨ River(Shenzhen)
 – Translation: "Shenzhen is a city or a river." (T)
◇ CS-major(x) → Student(x)
 – Translation: "If x is CS-major then x is a student." (not a proposition)

- The statement $P(x)$ is not a proposition since there are many objects that it can be applied to.

Predicates

- The statement $P(x)$ is not a proposition since there are many objects that it can be applied to.

- But the difference is:

    ◇ predicate logic allows us to explicitly manipulate and substitute for the objects

    ◇ predicate logic permits quantified sentences where variables are substituted for statements about the group of objects

- Two types of quantified statements:

  ◇ *Universal*

    Example: "All CS-major graduates have to pass CS201".
    (This is true for all CS-major graduates.)

  ◇ *Existential*

    Example: "Some CS-major students graduate with honor."
    (This is true for some students.)

- The *universal quantification* of $P(x)$ is the proposition: "$P(x)$ is true for all values of x in the universe of discourse." denoted by $\forall x \, P(x)$, and is expressed as for every x, $P(x)$.

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Universal Quantifier

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  ◇ What is the truth value of $\forall x\, P(x)$?

  ◇ Assume that the universe is all real numbers
  ◇ $\forall x\, P(x)$ is true.

  ◇ Is $P(x)$ a proposition?   No. Many possible substitutions.
  ◇ Is $\forall x\, P(x)$ a proposition?
    Yes. True if for all x from the universe $P(x)$ is ture.

- The *existential quantification* of $P(x)$ is the proposition: "There exists an element in the universe of discourse such that $P(x)$ is true ." denoted by $\exists x\, P(x)$, and is expressed as there is an x such that $P(x)$.

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  ◇ Assume that the universe is all real numbers
  ◇ $\exists x\ P(x)$ is true.

  ◇ Is $P(x)$ a proposition?   No. Many possible substitutions.
  ◇ Is $\exists x\ P(x)$ a proposition?
    Yes. True if there is even an x s.t. $P(x)$ is ture (e.g. 10).

- **Example**:

  ◇ $Q(x) -$ "$x = x + 2$ where x is a real number".

  ◇ What is the truth value of $\exists x\ Q(x)$?

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  ◇ $\exists x\ Q(x)$ is false.

- **Example**:

  - $Q(x) -$ "$x = x + 2$ where x is a real number".
  - What is the truth value of $\exists x \; Q(x)$?

  - $\exists x \; Q(x)$ is false.

  - $C(x) -$ CS-major(x) $\land$ Honor-student(x).
  - What is the truth value of $\exists x \; C(x)$?
  - Translation: "There is a person who is a CS-major student and who also graduated with honor." (T)

- When $\forall x\ P(x)$ and $\exists x\ P(x)$ are true and false?

Statement	When true?	When false?
$\forall$x P(x)	P(x) true for all x	There is an x where P(x) is false.
$\exists$x P(x)	There is some x for which P(x) is true.	P(x) is false for all x.

Summary of Quantified Statements

- When $\forall x\; P(x)$ and $\exists x\; P(x)$ are true and false?

Statement	When true?	When false?
$\forall x\; P(x)$	$P(x)$ true for all x	There is an x where $P(x)$ is false.
$\exists x\; P(x)$	There is some x for which $P(x)$ is true.	$P(x)$ is false for all x.

- Suppose that the elements in the universe can be enumerated as $x_1, x_2, \ldots, x_n$ then:
  - ◇ $\forall x\; P(x)$ is true whenever $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$ is true
  - ◇ $\exists x\; P(x)$ is true whenever $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$ is true.

- The truth values of $\exists x\ P(x)$ and $\forall x\ P(x)$ depend on both the propositional function $P(x)$ and the universe.

- The truth values of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and the universe.

Example: $P(x) - $ "$x < 2$"

◇ universe: the positive integers
$\exists x\, P(x) - $ T, $\forall x\, P(x) - $ F

◇ universe: the negative integers
$\exists x\, P(x) - $ T, $\forall x\, P(x) - $ T

◇ universe: $\{\, 3,\, 4,\, 5\, \}$
$\exists x\, P(x) - $ F, $\forall x\, P(x) - $ F

- The quantifiers $\forall$ and $\exists$ have *higher precedence* than all the logical operators.

    ◇ $\forall x\ P(x) \vee Q(x)$ means $(\forall x\ P(x)) \vee Q(x)$ rather than $\forall x\ (P(x) \vee Q(x))$

- Sentence: All SUSTech students are smart.

  ◇ universe: SUSTech students

  translation: $\forall x \; Smart(x)$

- **Sentence:** All SUSTech students are smart.

  - ◇ universe: SUSTech students

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  - ◇ universe: all students

    translation: $\forall x \; (At(x, SUSTech) \rightarrow Smart(x))$

Translation with Quantifiers

- **Sentence:** All SUSTech students are smart.

    ◇ universe: SUSTech students
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    ◇ universe: all students
      translation: $\forall x\ (At(x, SUSTech) \rightarrow Smart(x))$

  $\mathcal{Q}$: What about this?
    $\forall x\ (At(x, SUSTech) \wedge Smart(x))$

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This means every student is at SUSTech and is smart!

- **Sentence:** All SUSTech students are smart.

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  $\forall x\ (At(x, SUSTech) \land Smart(x))$

  This means every student is at SUSTech and is smart!

  - ◇ universe: people

    translation: $\forall x\ (Student(x) \land At(x, SUSTech) \rightarrow Smart(x))$

- Sentence: Someone at SUSTech is smart.

  ◇ universe: all SUSTech affiliates

  translation: $\exists x\ Smart(x)$

■ Sentence: Someone at SUSTech is smart.

◇ universe: all SUSTech affiliates

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◇ universe: people

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- **Sentence:** Someone at SUSTech is smart.

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■ **Sentence:** Someone at SUSTech is smart.

◇ **universe:** all SUSTech affiliates

translation: $\exists x \; Smart(x)$

◇ **universe:** people

translation: $\exists x \; (At(x, SUSTech) \wedge Smart(x))$

$\mathcal{Q}$: What about this?
$\exists x \; (At(x, SUSTech) \rightarrow Smart(x))$

This is even **true** if there is anyone who is **not** at SUSTech!

- Sentence: Nothing is perfect.

  ◇ translation: $\neg \exists x \; Perfect(x)$

Negation of Quantifiers

- Sentence: Nothing is perfect.

  - ◇ translation: $\neg \exists x \; Perfect(x)$

  - ◇ translation: $\forall x \; \neg Perfect(x)$
    (Everything is imperfect.)

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- Sentence: Nothing is perfect.

  ◇ translation: $\neg \exists x\ Perfect(x)$

  ◇ translation: $\forall x\ \neg Perfect(x)$
    (Everything is imperfect.)

  Conclusion: $\neg \exists x\ P(x)$ is equivalent to $\forall x\ \neg P(x)$

- Sentence: Not all horses are white.

  ◇ translation: $\neg \forall x \ (Horse(x) \rightarrow White(x))$

■ Sentence: Not all horses are white.

◇ translation: $\neg\forall x\ (Horse(x) \rightarrow White(x))$

◇ translation: $\exists x\ (Horse(x) \wedge \neg White(x))$
(There is a horse that is not white.)

- Sentence: Not all horses are white.

  ◇ translation: $\neg \forall x \ (Horse(x) \to White(x))$

  ◇ translation: $\exists x \ (Horse(x) \wedge \neg White(x))$
  (There is a horse that is not white.)

  ◇ logically equivalent to
  $\exists x \ \neg (Horse(x) \to White(x))$

- Sentence: Not all horses are white.

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    (There is a horse that is not white.)

  ◇ logically equivalent to
    $\exists x \ \neg(Horse(x) \rightarrow White(x))$

  Conclusion: $\neg \forall x \ P(x)$ is equivalent to $\exists x \ \neg P(x)$

- a.k.a. De Morgan laws for quantifiers

Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg\, \exists x\; P(x)$	$\forall x\; \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\, \forall x\; P(x)$	$\exists x\; \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Next Lecture

- Predicate logic, proofs, sets …