

P79 Q44

令 $Z = X + Y$, 则有:

$$P\{Z = 0\} = P\{X = 0, Y = 0\} = \frac{1}{9}$$

$$P\{Z = 1\} = P\{X = 0, Y = 1\} + P\{X = 1, Y = 0\} = \frac{2}{9}$$

$$P\{Z = 2\} = P\{X = 1, Y = 1\} + P\{X = 0, Y = 2\} + P\{X = 2, Y = 0\} = \frac{1}{3}$$

$$P\{Z = 3\} = P\{X = 1, Y = 2\} + P\{X = 2, Y = 1\} = \frac{2}{9}$$

$$P\{Z = 4\} = P\{X = 2, Y = 2\} = \frac{1}{9}$$

所以 $X + Y$ 的频率函数为:

$$f_Z(z) = \begin{cases} \frac{1}{9} & z = 0 \\ \frac{2}{9} & z = 1 \\ \frac{1}{3} & z = 2 \\ \frac{2}{9} & z = 3 \\ \frac{1}{9} & z = 4 \\ 0 & \text{otherwise} \end{cases}$$

P79 Q51

令 $z = xy$, 则有:

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^z \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) |J| dx dz \\ &= \int_{-\infty}^z \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|} dx dz \\ f_Z(z) &= \int_{-\infty}^{\infty} f\left(x, \frac{z}{x}\right) \frac{1}{|x|} dx \end{aligned}$$

此时, 将变量 x 改名为 y , 有:

$$f_Z(z) = \int_{-\infty}^{\infty} f\left(y, \frac{z}{y}\right) \frac{1}{|y|} dy$$

P79 Q52

假设两个变量为 $[0, 1]$ 上的均匀分布, 则有:

$$f_X(x) = f_Y(y) = \begin{cases} 1 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

当 $Z \geq 1$ 时, $Z = \frac{X}{Y}$ 的分布函数为:

$$\begin{aligned}
 f_Z(z) &= P\{Z \leq z\} \\
 &= P\left\{\frac{X}{Y} \leq z\right\} \\
 &= P\{X \leq zY\} \\
 &= \int_0^{\frac{1}{z}} \int_0^{zy} f_X(x)f_Y(y)dx dy + \int_{\frac{1}{z}}^1 \int_0^1 f_X(x)f_Y(y)dx dy \\
 &= \int_0^{\frac{1}{z}} \int_0^{zy} dx dy + \int_{\frac{1}{z}}^1 \int_0^1 dx dy \\
 &= \frac{1}{2z} + 1 - \frac{1}{z} \\
 &= 1 - \frac{1}{2z}
 \end{aligned}$$

当 $0 < Z < 1$ 时, $Z = \frac{X}{Y}$ 的分布函数为:

$$\begin{aligned}
 f_Z(z) &= \int_0^1 \int_0^{zy} f_X(x)f_Y(y)dx dy \\
 &= \int_0^1 \int_0^{zy} dx dy \\
 &= \frac{z}{2}
 \end{aligned}$$

综上可得:

$$f_Z(z) = \begin{cases} \frac{d}{dz} \frac{z}{2} = \frac{1}{2} & 0 < z < 1 \\ \frac{d}{dz} \left(1 - \frac{1}{2z}\right) = \frac{1}{2z^2} & z \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

P80 Q57

由题可知:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\begin{aligned}
f_{Y_1, Y_2}(y_1, y_2) &= \frac{1}{2\pi} e^{-\frac{1}{2}(2y_1^2 - 2y_1y_2 + y_2^2)} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2} \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}} \\
&= \frac{1}{2\pi} |J| e^{-\frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} (J^{-1})^T \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} J^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \\
&= \frac{1}{2\pi} e^{-\frac{1}{2}x_1^2 - \frac{1}{2}x_2^2}
\end{aligned}$$

因此可知：

$$(J^{-1})^T \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} J^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

可知当 $a_{11} = a_{22} = 1, a_{12} = 0, a_{21} = -1$ 时，上述方程组成立。

因此 $x_1 = y_1, x_2 = y_2 - y_1$ 。

补充 1

$$\begin{aligned}
f_{U,V}(u, v) &= f_{X,Y}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \\
&= \frac{1}{4\pi} e^{-\frac{1}{4}(u^2 + v^2)}
\end{aligned}$$

其边缘密度函数为：

$$\begin{aligned}
f_U(u) &= \int_{-\infty}^{+\infty} f_{U,V}(u, v) dv \\
&= \int_{-\infty}^{+\infty} \frac{1}{4\pi} e^{-\frac{1}{4}(u^2 + v^2)} dv \\
&= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2} \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}v^2} dv \\
&= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sqrt{2\pi}}} e^{-\frac{1}{2} \frac{v^2}{\sqrt{2^2}}} dv \\
&= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}u^2}
\end{aligned}$$

同理可知 $f_V(v) = \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}v^2}$

易知 U 与 V 相互独立，因为 $f_{U,V}(u, v) = f_U(u)f_V(v)$ 。

补充 2

(1)

边缘密度函数为：

$$\begin{aligned} f_X(x) &= \int_0^{2x} f_{X,Y}(x,y) dy \\ &= \int_0^{2x} 1 dy \\ &= 2x \end{aligned}$$

$$\begin{aligned} f_Y(y) &= \int_{\frac{y}{2}}^1 f_{X,Y}(x,y) dx \\ &= \int_{\frac{y}{2}}^1 1 dx \\ &= 1 - \frac{y}{2} \end{aligned}$$

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 1 - \frac{y}{2} & 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

(2)

令 $u = 2x - y$, 则有：

$$\begin{aligned} F_Z(z) &= \int_0^z \int_{\frac{u}{2}}^1 f_{X,Y}(x,y) \left| \frac{\partial(x,y)}{\partial(x,u)} \right| dx du \\ &= \int_0^z \int_{\frac{u}{2}}^1 dx du \\ &= \int_0^z \left(1 - \frac{u}{2} \right) du \\ &= z - \frac{z^2}{4} \quad (0 \leq z \leq 2) \end{aligned}$$

所以 Z 的密度函数为：

$$f_Z(z) = \frac{d}{dz} \left(z - \frac{z^2}{4} \right) = 1 - \frac{z}{2} \quad (0 \leq z \leq 2)$$

(3)

$$\begin{aligned}P\{Y < \frac{1}{2} | X < \frac{1}{2}\} &= \frac{P\{Y < \frac{1}{2}, X < \frac{1}{2}\}}{P\{X < \frac{1}{2}\}} \\&= \frac{\int_0^{\frac{1}{4}} \int_0^{2x} f_{X,Y}(x,y) dy dx + \int_{\frac{1}{4}}^{\frac{1}{2}} \int_0^{\frac{1}{2}} f_{X,Y}(x,y) dy dx}{\int_0^{\frac{1}{2}} f_X(x) dx} \\&= \frac{\frac{3}{16}}{\frac{1}{4}} \\&= \frac{3}{4}\end{aligned}$$