Probability and Statistics
Southern University of Science and Technology
吴梦轩
12212006

Section 3.3

吴梦轩

P76 Q5

假设所有平行线竖直放置，令变量 X 表示针的左端距离其右侧最近的平行线的距离，令变量 Y 表示以左端为顶点时针与竖直方向的夹角。可知 X 服从均值为 $\frac{1}{D}$ 的均匀分布，且 $0 \leqslant X \leqslant D$。$Y$ 服从均值为 $\frac{1}{\pi}$ 的均匀分布，且 $0 \leqslant Y \leqslant \pi$。$X$ 与 Y 相互独立，故 $f_{X,Y}(x,y) = f_X(x)f_Y(y) = \frac{1}{\pi D}$。因此针与平行线相交的概率为：

$$
\begin{aligned}
P\{L\sin Y \geqslant X\} &= \iint\limits_{L\sin y \leqslant x} f_{X,Y}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\pi}\int_0^{L\sin y}\frac{1}{\pi D}\mathrm{d}x\mathrm{d}y \\
&= \frac{L}{\pi D}\int_0^{\pi}\sin y\mathrm{d}y \\
&= \frac{2L}{\pi D}
\end{aligned}
$$

由于 L，D 为已知常数，故可用 $\frac{2L}{\pi D}$ 估计 π。

P76 Q6

设 X 为该点的横坐标，Y 为该点的纵坐标。由于该点在椭圆 $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 内随机选择，故其概率函数为 $f_{X,Y}(x,y)=\frac{1}{\pi ab}$。其边际密度为：

$$
\begin{aligned}
f_X(x) &= \int_{-\sqrt{b^2(1-\frac{x^2}{a^2})}}^{\sqrt{b^2(1-\frac{x^2}{a^2})}}\frac{1}{\pi ab}\mathrm{d}y \\
&= \frac{2}{\pi a}\sqrt{1-\frac{x^2}{a^2}} \\
f_Y(y) &= \int_{-\sqrt{a^2(1-\frac{y^2}{b^2})}}^{\sqrt{a^2(1-\frac{y^2}{b^2})}}\frac{1}{\pi ab}\mathrm{d}x \\
&= \frac{2}{\pi b}\sqrt{1-\frac{y^2}{b^2}}
\end{aligned}
$$

考虑 X，Y 的取值范围，有：

$$f_X(x) = \begin{cases} \frac{2}{\pi a}\sqrt{1-\frac{x^2}{a^2}} & -a \leqslant x \leqslant a \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{2}{\pi b}\sqrt{1-\frac{y^2}{b^2}} & -b \leqslant y \leqslant b \\ 0 & \text{otherwise} \end{cases}$$

P76 Q7

已知 $F(x,y) = (1-e^{-\alpha x})(1-e^{-\beta y})$ 且 $x \geqslant 0$，$y \geqslant 0$，$\alpha > 0$，$\beta > 0$。则其联合密度为：

$$\begin{aligned} f(x,y) &= \frac{\partial^2 F(x,y)}{\partial x \partial y} \\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial x}(1-e^{-\alpha x})(1-e^{-\beta y}) \\ &= \frac{\partial}{\partial y}\alpha e^{-\alpha x}(1-e^{-\beta y}) \\ &= \alpha\beta e^{-\alpha x}e^{-\beta y} \\ &= \alpha\beta e^{-\alpha x-\beta y} \end{aligned}$$

其边际密度为：

$$\begin{aligned} f_X(x) &= \int_0^\infty \alpha\beta e^{-\alpha x-\beta y}\mathrm{d}y \\ &= \alpha e^{-\alpha x} \\ f_Y(y) &= \int_0^\infty \alpha\beta e^{-\alpha x-\beta y}\mathrm{d}x \\ &= \beta e^{-\beta y} \end{aligned}$$

考虑 X，Y 的取值范围，有：

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \beta e^{-\beta y} & y \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

P76 Q8

a.

(i)

$$\begin{aligned}
P\{X > Y\} &= \int_0^1 \int_0^x f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^x \frac{6}{7}(x+y)^2\mathrm{d}y\mathrm{d}x \\
&= \frac{6}{7}\int_0^1 \left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^x \mathrm{d}x \\
&= \frac{6}{7}\int_0^1 \left(x^3 + x^3 + \frac{x^3}{3}\right)\mathrm{d}x \\
&= \frac{6}{7}\cdot\frac{7}{3}\int_0^1 x^3\mathrm{d}x \\
&= 2\cdot\frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

(ii)

$$\begin{aligned}
P\{X + Y < 1\} &= P\{Y < 1 - X\} \\
&= \int_0^1 \int_0^{1-x} f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \int_0^{1-x} \frac{6}{7}(x+y)^2\mathrm{d}y\mathrm{d}x \\
&= \frac{6}{7}\int_0^1 \left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^{1-x} \mathrm{d}x \\
&= \frac{6}{7}\int_0^1 \frac{1}{3} - \frac{x^3}{3}\mathrm{d}x \\
&= \frac{6}{7}\cdot\frac{1}{3}\int_0^1 1 - x^3\mathrm{d}x \\
&= \frac{6}{7}\cdot\frac{1}{3}\cdot\left(x - \frac{x^4}{4}\right)\Big|_0^1 \\
&= \frac{6}{7}\cdot\frac{1}{3}\cdot\frac{3}{4} \\
&= \frac{3}{14}
\end{aligned}$$

(iii)

$$
\begin{aligned}
P\{X \leqslant \frac{1}{2}\} &= \int_0^{\frac{1}{2}} \int_0^1 f_{X,Y}(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_0^{\frac{1}{2}} \int_0^1 \frac{6}{7}(x+y)^2\mathrm{d}y\mathrm{d}x \\
&= \frac{6}{7}\int_0^{\frac{1}{2}} \left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^1 \mathrm{d}x \\
&= \frac{6}{7}\int_0^{\frac{1}{2}} \left(x^2 + x + \frac{1}{3}\right)\mathrm{d}x \\
&= \frac{6}{7}\cdot\frac{1}{3}\int_0^{\frac{1}{2}} 3x^2 + 3x + 1\mathrm{d}x \\
&= \frac{6}{7}\cdot\frac{1}{3}\cdot\left(x^3 + \frac{3}{2}x^2 + x\right)\Big|_0^{\frac{1}{2}} \\
&= \frac{6}{7}\cdot\frac{1}{3} \\
&= \frac{2}{7}
\end{aligned}
$$

b.

X，Y 的边际密度为：

$$
\begin{aligned}
f_X(x) &= \int_0^1 \frac{6}{7}(x+y)^2\mathrm{d}y \\
&= \frac{6}{7}\left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^1 \\
&= \frac{6}{7}\left(x^2 + x + \frac{1}{3}\right) \\
&= \frac{6x^2+6x+2}{7} \\
f_Y(y) &= \int_0^1 \frac{6}{7}(x+y)^2\mathrm{d}x \\
&= \frac{6}{7}\left(x^2y + xy^2 + \frac{y^3}{3}\right)\Big|_0^1 \\
&= \frac{6}{7}\left(y^2 + y + \frac{1}{3}\right) \\
&= \frac{6y^2+6y+2}{7}
\end{aligned}
$$

考虑 X，Y 的取值范围，有：

$$
f_X(x) = \begin{cases} \frac{6x^2+6x+2}{7} & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$f_Y(y) = \begin{cases} \frac{6y^2+6y+2}{7} & 0 \leqslant y \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

c.

条件密度 $f_{X|Y}(x|y)$ 为:

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{\frac{6}{7}(x+y)^2}{\frac{6y^2+6y+2}{7}} \\ &= \frac{3(x+y)^2}{3y^2+3y+1} \end{aligned}$$

条件密度 $f_{Y|X}(y|x)$ 为:

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{X,Y}(x,y)}{f_X(x)} \\ &= \frac{\frac{6}{7}(x+y)^2}{\frac{6x^2+6x+2}{7}} \\ &= \frac{3(x+y)^2}{3x^2+3x+1} \end{aligned}$$

补充 1

$$\begin{aligned} \lim_{x\to\infty, y\to\infty} F(x,y) &= \lim_{x\to\infty, y\to\infty} k(1-e^{-x})(1-e^{-y}) \\ &= k \cdot 1 \cdot 1 \\ &= k \end{aligned}$$

故 $k=1$, (X,Y) 的联合密度为:

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\partial^2 F(x,y)}{\partial x \partial y} \\ &= \frac{\partial^2 (1-e^{-x})(1-e^{-y})}{\partial x \partial y} \\ &= \frac{\partial}{\partial y}\frac{\partial}{\partial x}(1-e^{-x})(1-e^{-y}) \\ &= \frac{\partial}{\partial y} e^{-x}(1-e^{-y}) \\ &= e^{-x}e^{-y} \\ &= e^{-(x+y)} \end{aligned}$$

$$f_{X,Y}(x,y)=\begin{cases} e^{-(x+y)} & x\geqslant 0, y\geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

故边缘密度函数为:

$$\begin{aligned} f_X(x) &= \int_0^{\infty} e^{-(x+y)}\mathrm{d}y \\ &= e^{-x} \\ f_Y(y) &= \int_0^{\infty} e^{-(x+y)}\mathrm{d}x \\ &= e^{-y} \end{aligned}$$

$$f_X(x)=\begin{cases} e^{-x} & x\geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y)=\begin{cases} e^{-y} & y\geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$

则 $P\{1<X<3, 1<Y<2\}$ 为:

$$\begin{aligned} P\{1<X<3,1<Y<2\} &= \int_1^3\int_1^2 e^{-(x+y)}\mathrm{d}y\mathrm{d}x \\ &= \int_1^3 e^{-x}\int_1^2 e^{-y}\mathrm{d}y\mathrm{d}x \\ &= \int_1^3 e^{-x}\left(-e^{-y}\right)\Big|_1^2\mathrm{d}x \\ &= \int_1^3 e^{-x}\left(-e^{-2}+e^{-1}\right)\mathrm{d}x \\ &= \left(-e^{-2}+e^{-1}\right)\int_1^3 e^{-x}\mathrm{d}x \\ &= \left(-e^{-2}+e^{-1}\right)\left(-e^{-x}\right)\Big|_1^3 \\ &= \left(-e^{-2}+e^{-1}\right)\left(-e^{-3}+e^{-1}\right) \\ &= e^{-5}-e^{-4}-e^{-3}+e^{-2} \end{aligned}$$

补充 2

(1)

其边缘密度函数为：

$$
\begin{aligned}
f_X(x) &= \int_0^1 x + y \mathrm{d}y \\
&= x + \frac{1}{2} \\
f_Y(y) &= \int_0^1 x + y \mathrm{d}x \\
&= y + \frac{1}{2}
\end{aligned}
$$

$$
f_X(x) = \begin{cases} x + \frac{1}{2} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}
$$

$$
f_Y(y) = \begin{cases} y + \frac{1}{2} & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}
$$

(2)

$$
\begin{aligned}
P\{X > Y\} &= \int_0^1 \int_0^x f_{X,Y}(x, y) \mathrm{d}y \mathrm{d}x \\
&= \int_0^1 \int_0^x (x + y) \mathrm{d}y \mathrm{d}x \\
&= \int_0^1 \left(xy + \frac{y^2}{2} \right) \Big|_0^x \mathrm{d}x \\
&= \frac{3}{2} \int_0^1 x^2 \mathrm{d}x \\
&= \frac{3}{2} \cdot \frac{1}{3} \\
&= \frac{1}{2}
\end{aligned}
$$

(3)

$$\begin{aligned}
P\{X < 0.5\} &= \int_0^{0.5} f_X(x)\mathrm{d}x \\
&= \int_0^{0.5} x + \frac{1}{2}\mathrm{d}x \\
&= \left(\frac{x^2}{2} + \frac{x}{2}\right)\Big|_0^{0.5} \\
&= \frac{1}{8} + \frac{1}{4} \\
&= \frac{3}{8}
\end{aligned}$$