CS215 DISCRETE MATH

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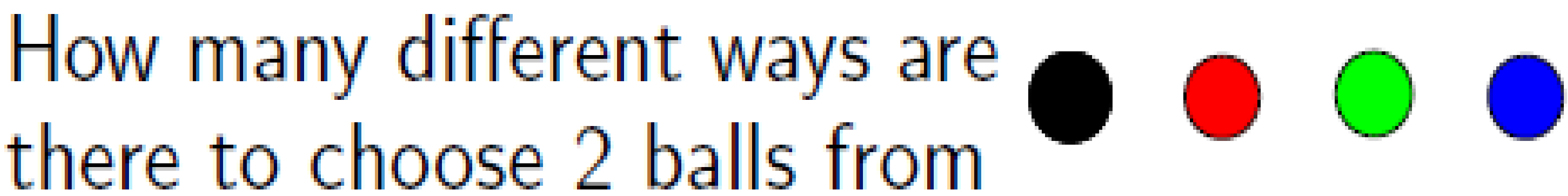

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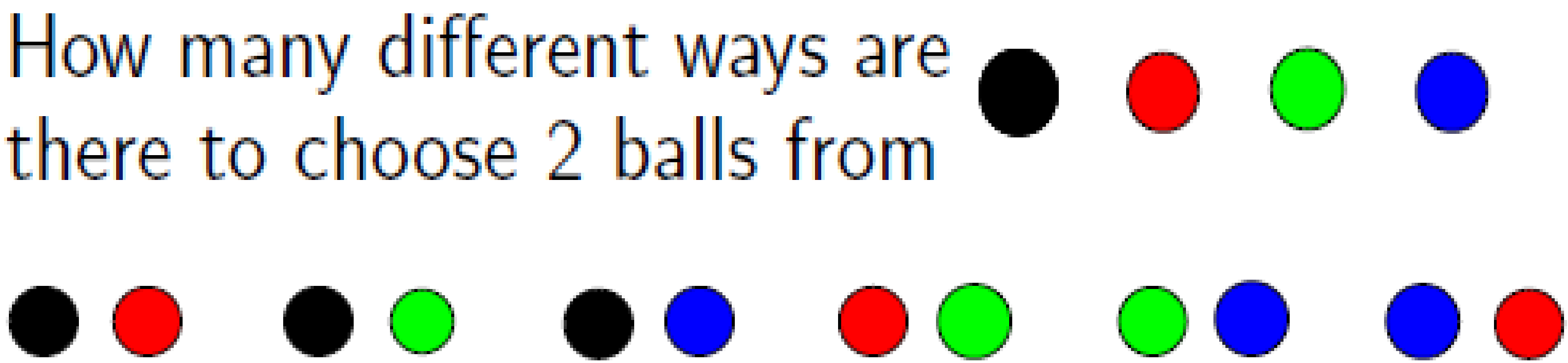

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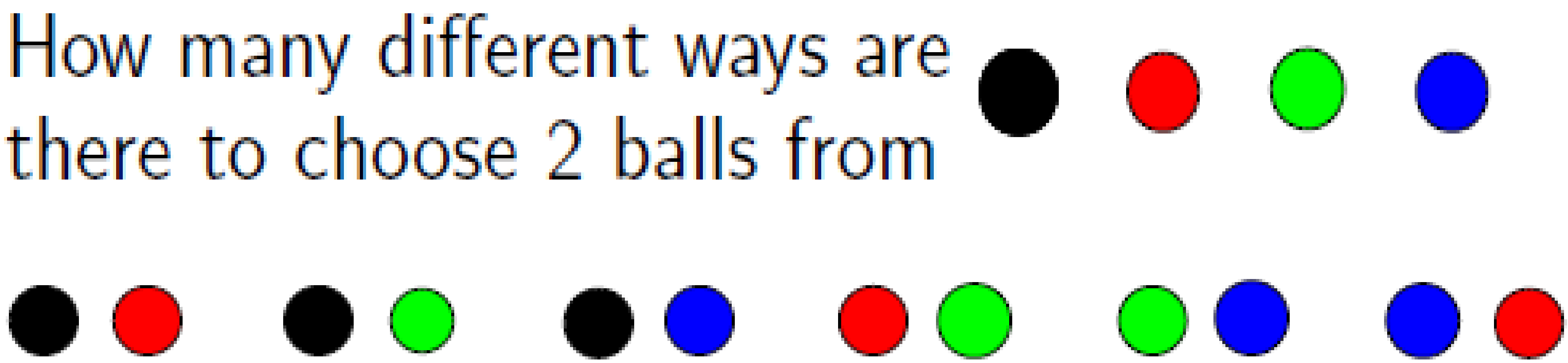

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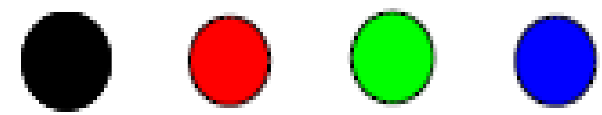
How many different ways are there to choose 2 balls from



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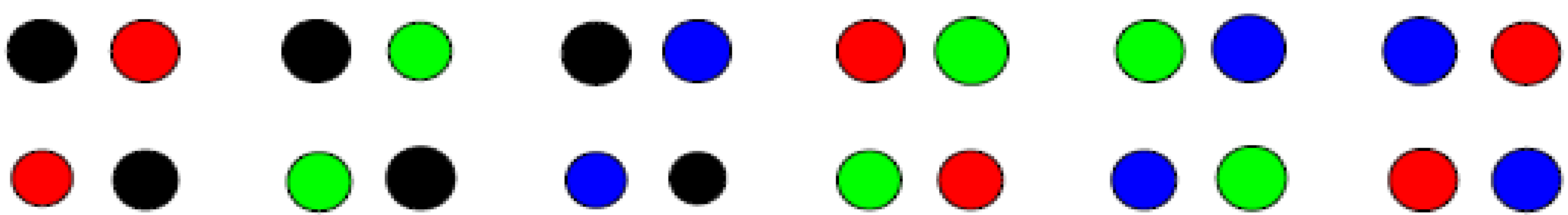

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  – simplify the solution by decomposing the problem

Basic Counting Rules

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In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

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Example

In an auditorium, the seats are labeled by a letter and numbers in between 1 to 50 (e.g., $A23$). What is the total number of seats?

We may either list all or use the product rule.

$$26 \times 50 = 1300$$

The Product Rule

- **Product Rule**: If a count of elements can be broken down into a sequence of dependent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is

$$n = n_1 \cdot n_2 \cdot \cdots \cdot n_k$$

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How many one-to-one functions are there from a set with m elements to a set with n elements?

How many onto functions?

The Product Rule

- The following loop is a part of program computing the product of two matrices.

```
for i = 1 to r
    for j = 1 to m
        S = 0
        for k = 1 to n
            S = S + A[i,k] * B[k,j]
        C[i,j] = S
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How many multiplications (in terms of r, m, n) does this program carry out in total among all iterations of line 5?

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Example

You need to travel from city A to B. You may either fly, take a train, or a bus. There are 12 different flights, 5 different trains and 10 buses. How many options do you have to get from A to B?

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We may use the sum rule.

$$12 + 5 + 10$$

The Sum Rule

- **Sum Rule**: If a count of elements can be broken down into a set of independent counts where the first count yields n_1 elements, the second n_2 elements, and kth count n_k elements, then the total number of elements is
$$n = n_1 + n_2 + \cdots + n_k$$

The Sum Rule

- The following loop is from selection sort.

```
(1) for i = 1 to n-1
(2)    for j = i+1 to n
(3)       if (A[i] > A[j])
(4)          exchange A[i] and A[j]
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How many comparisons (in terms of n) does this program carry out in total among all iterations of line 3?

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Example

Each password is 6 to 8 characters long, where each character is an lowercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

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$$P = P_6 + P_7 + P_8$$

Tree Diagrams

- A *tree* is a structure that consists of a root, branches and leaves.

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  Example

  What is the number of bit strings of length 4 that do not have two consecutive 1's?

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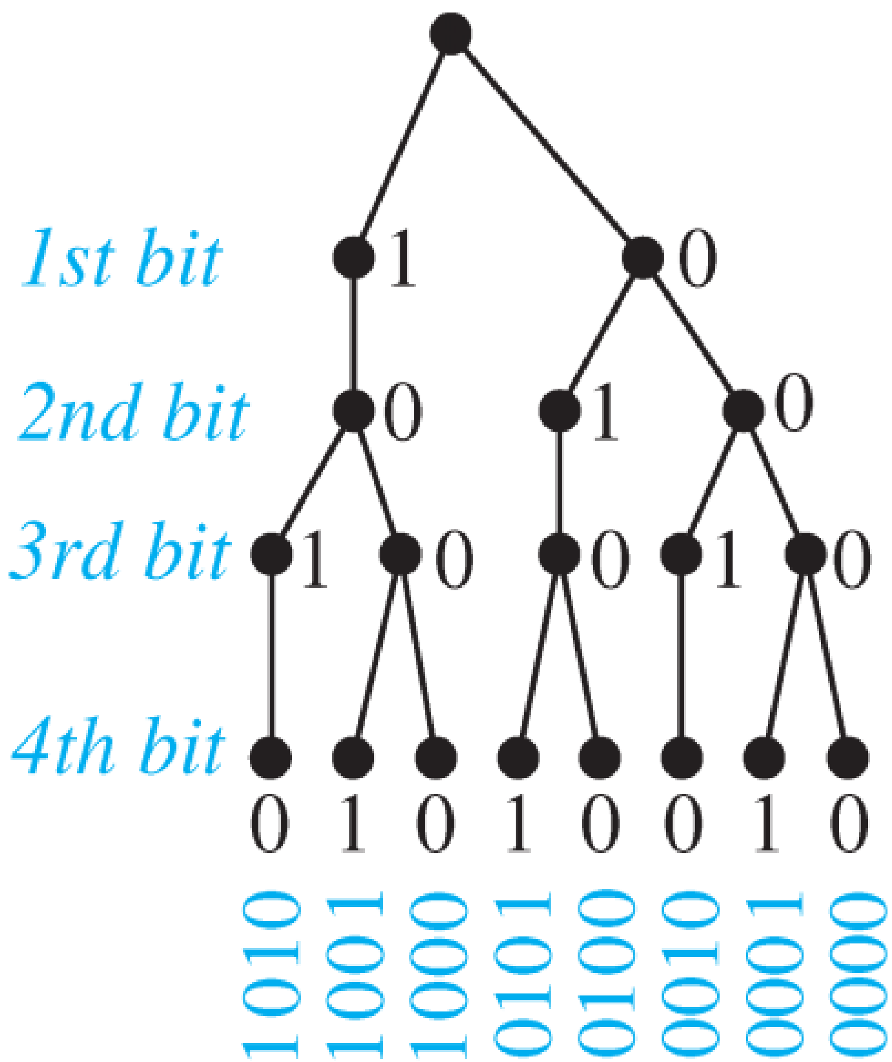

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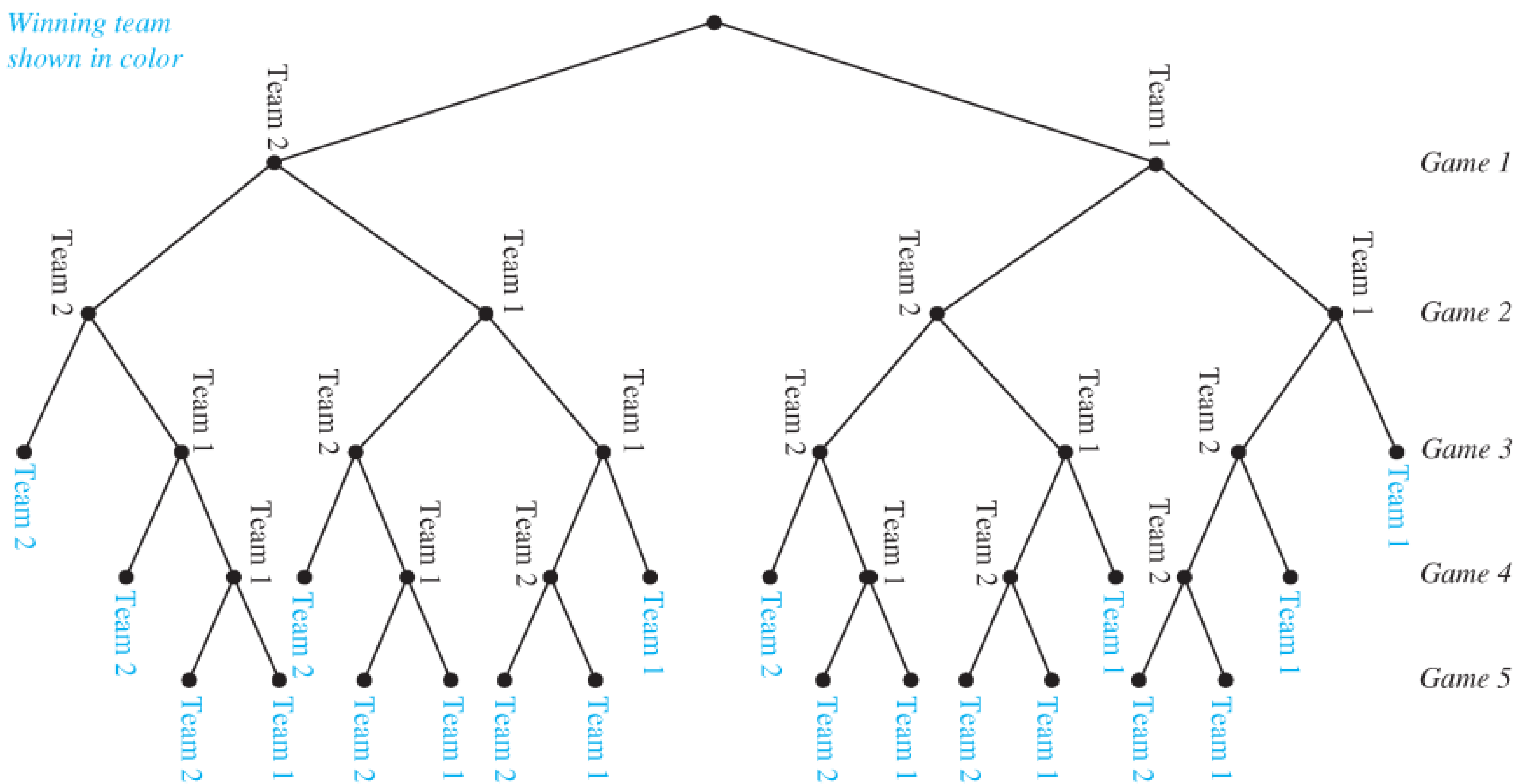

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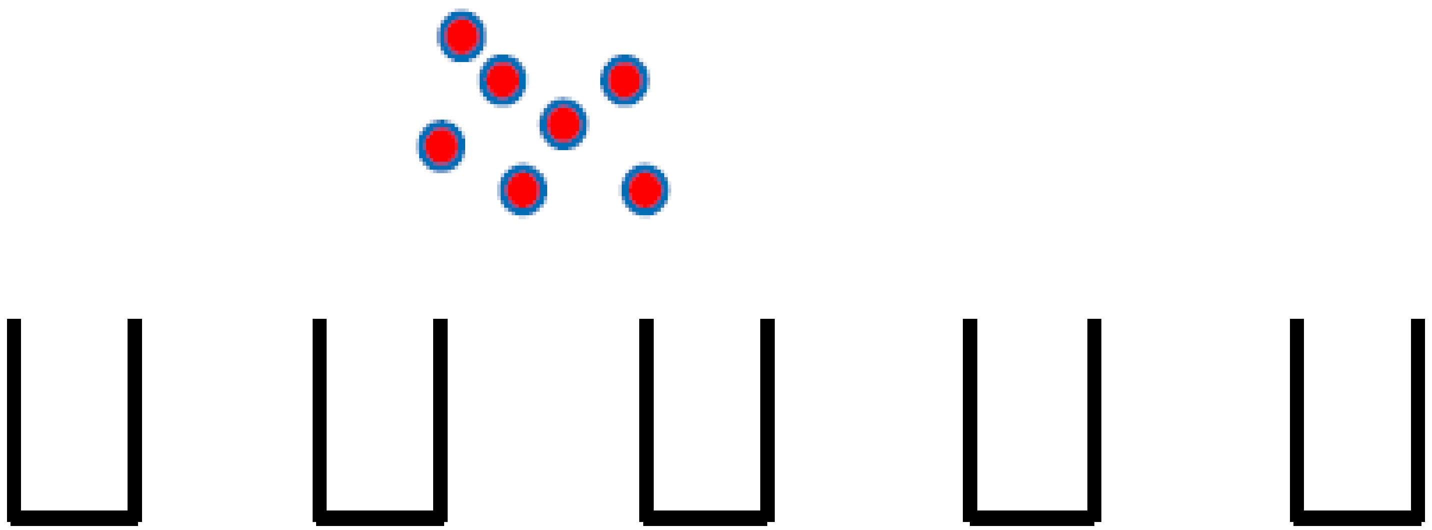

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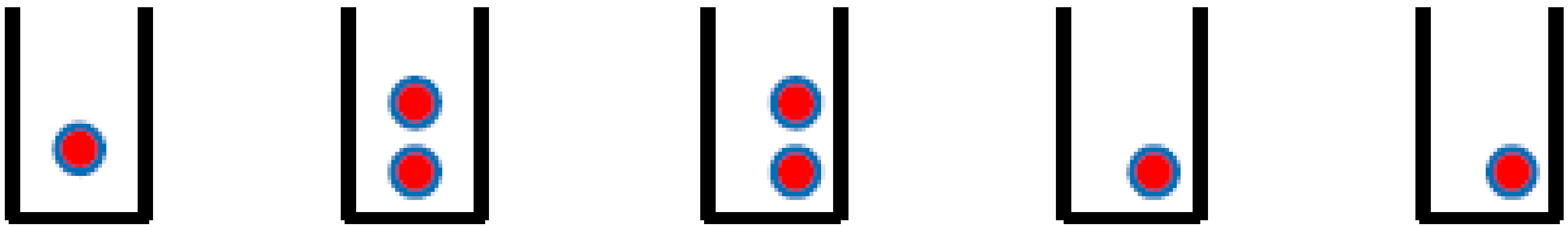

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  Example

  Assume that there are 367 students. Are there any two people who have the same birthday?

  There are 5 bins and 12 objects. Then there must be a bin with at least 3 objects. Why?

Generalized Pigeonhole Principle

- If N objects are placed into k bins, then there is at least one bin containing at least $\lceil N/k \rceil$ objects.

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Example

Assume there are 100 students. How many of them were born in the same month?

Bijections and Permutations

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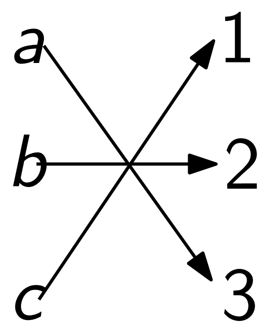

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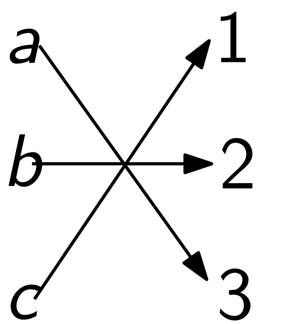

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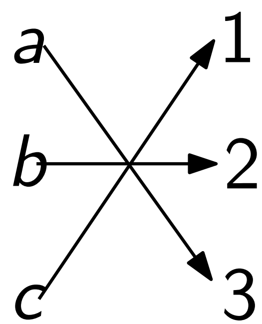

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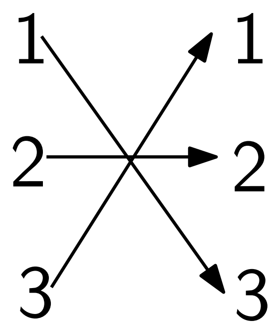

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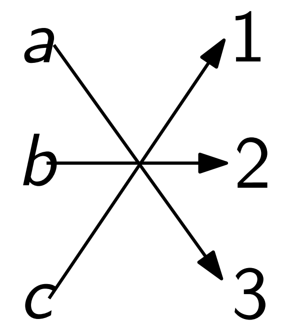

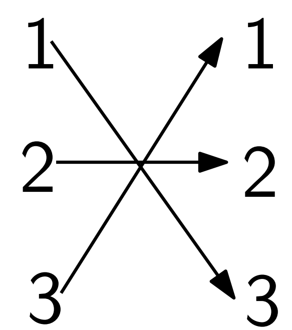

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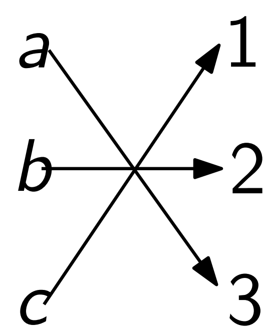
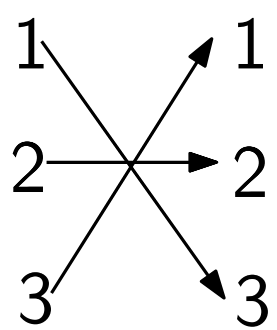
  Thus,
  the left and right sides must have the same size.





The Bijection Principle

- The following loop is a part of program to determine the number of triangles formed by n points in the plane.

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(1) trianglecount = 0
(2)    for i = 1 to n
(3)       for j = i+1 to n
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Among all iterations of line 5, what is the total number of times this line checks three points to see if they are collinear?

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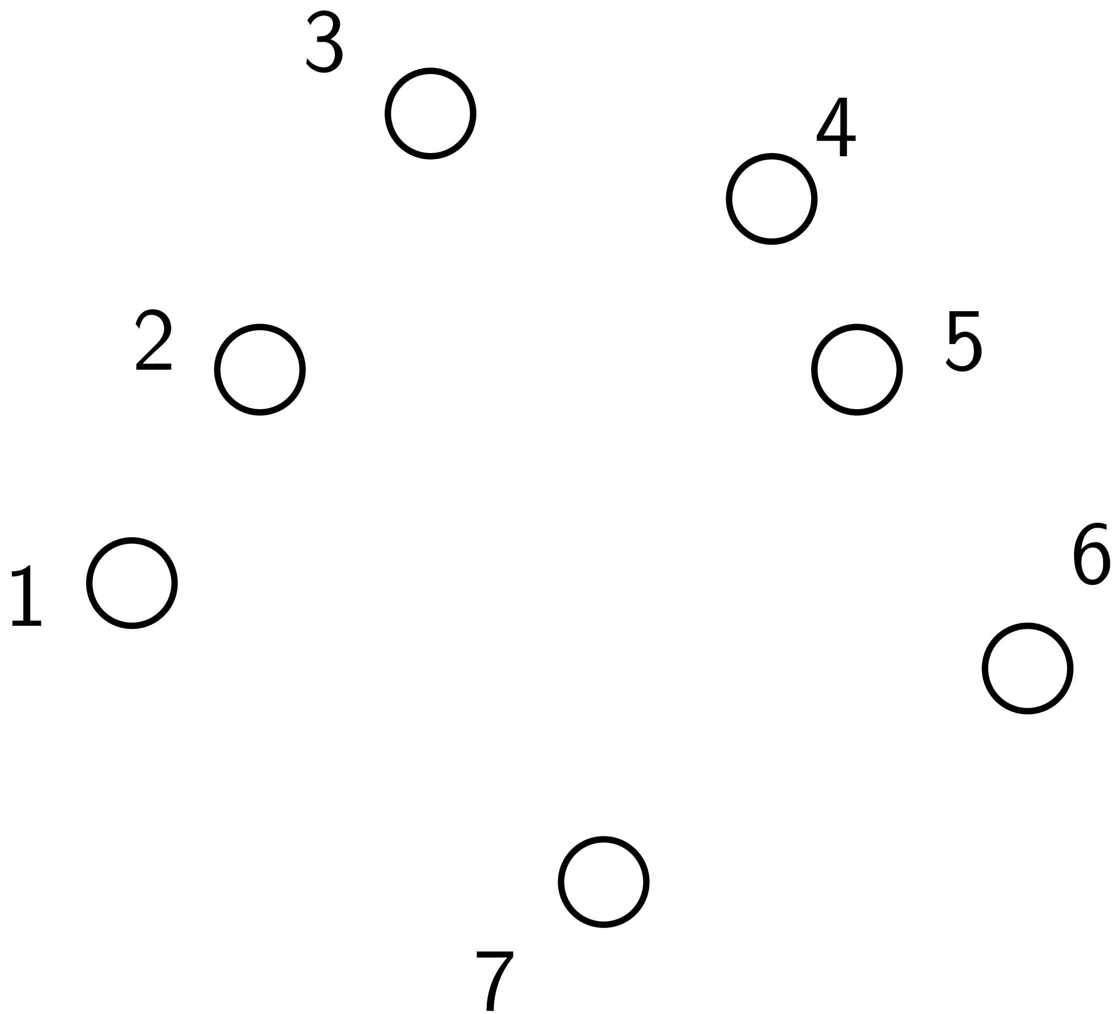

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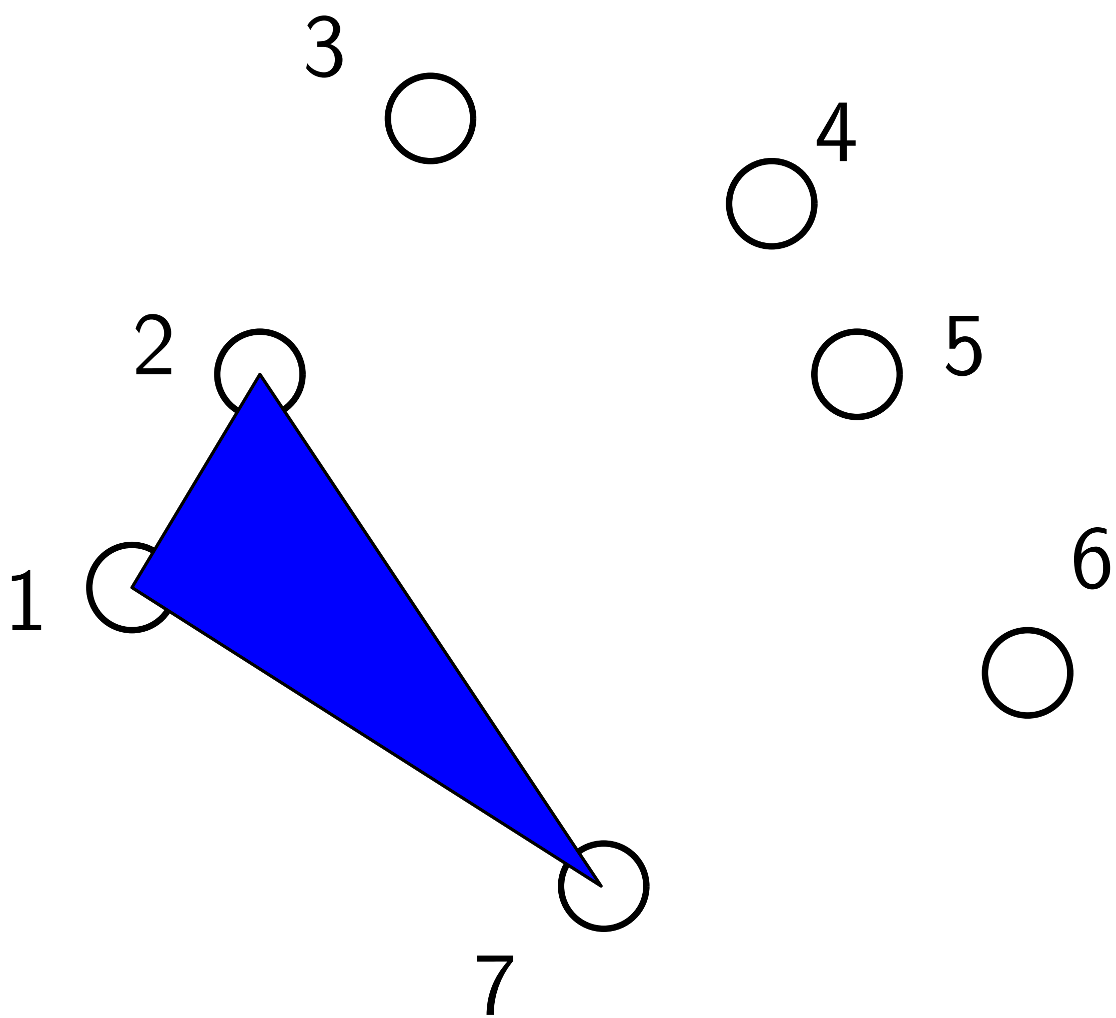

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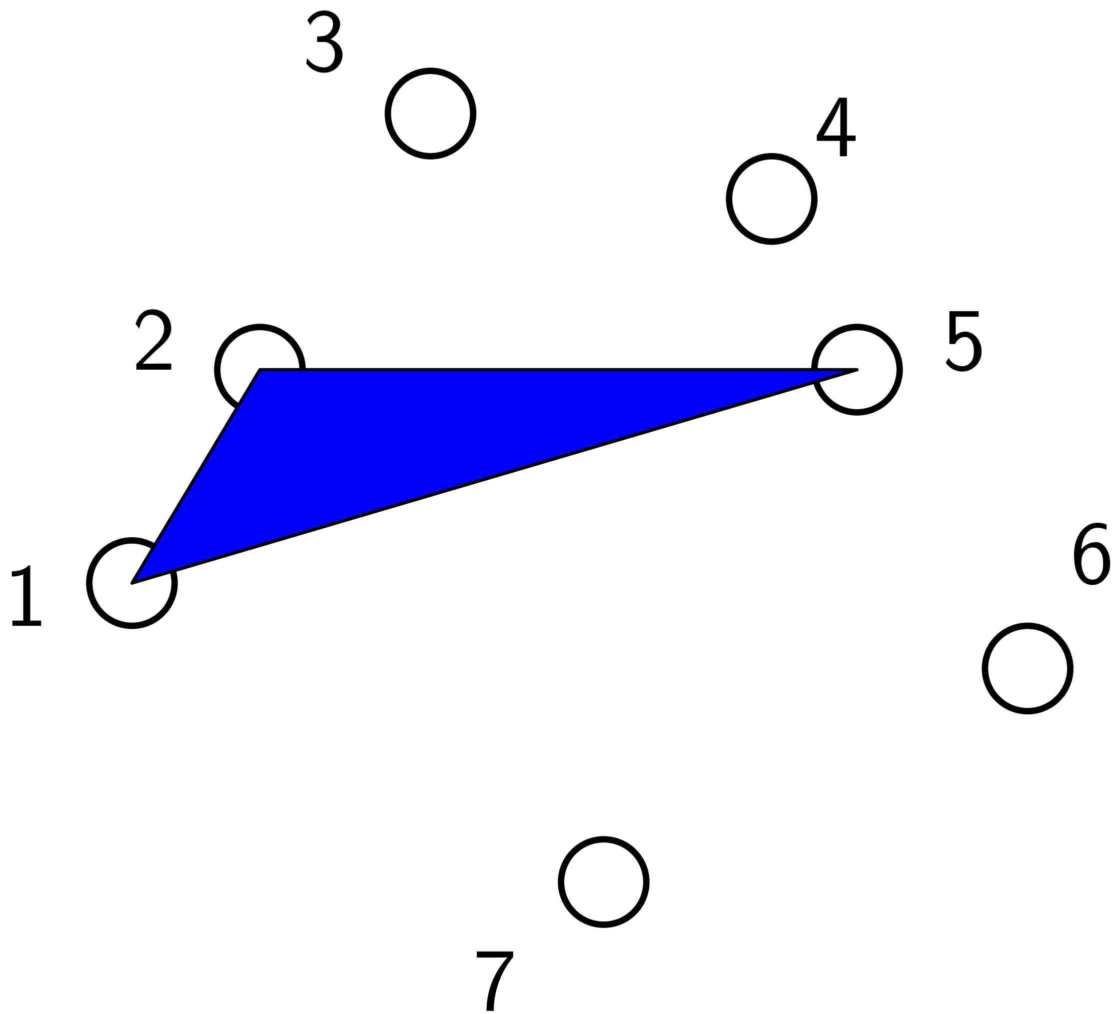

$1 - 2 - 7$: yes

$1 - 2 - 5$: yes

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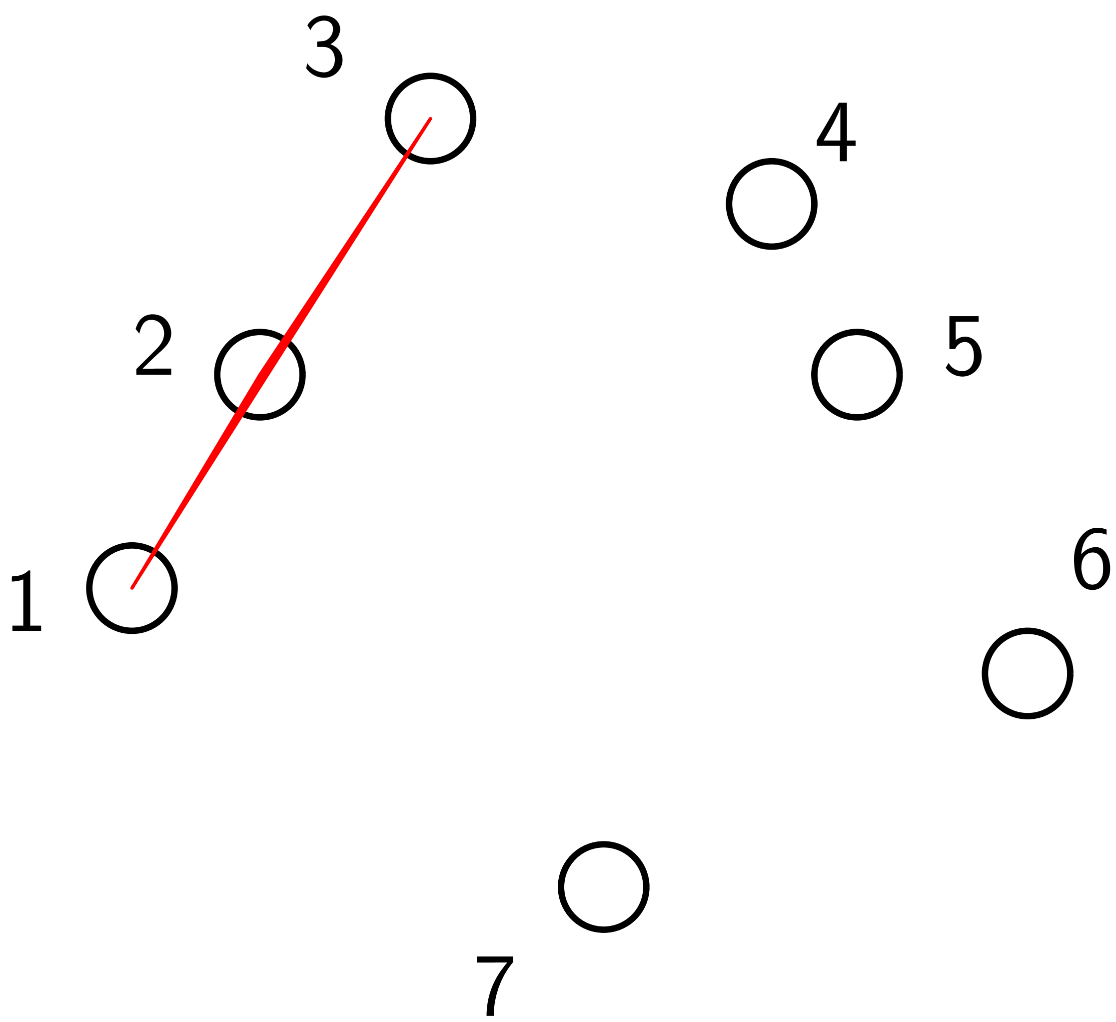

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$1 - 2 - 5$: yes

$1 - 2 - 3$: no

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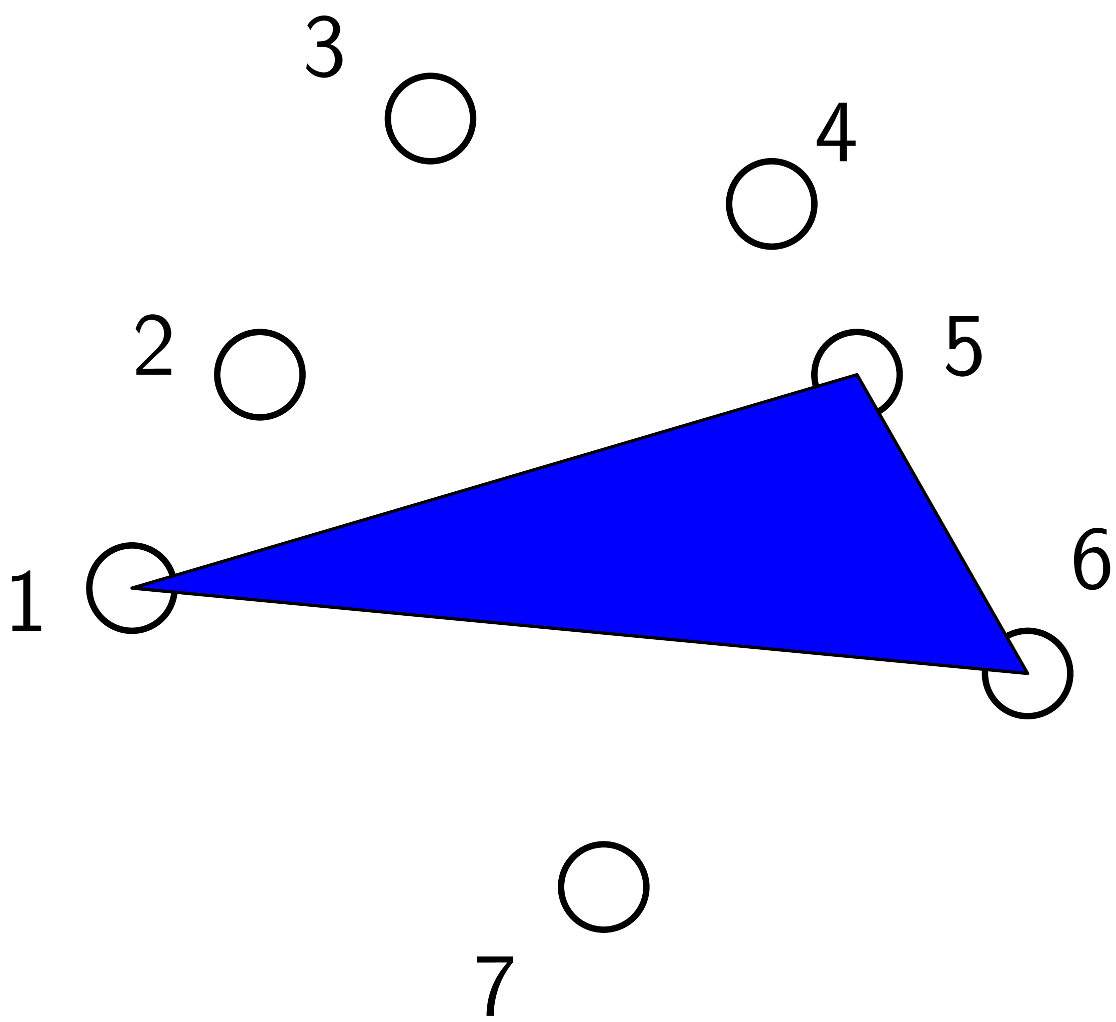

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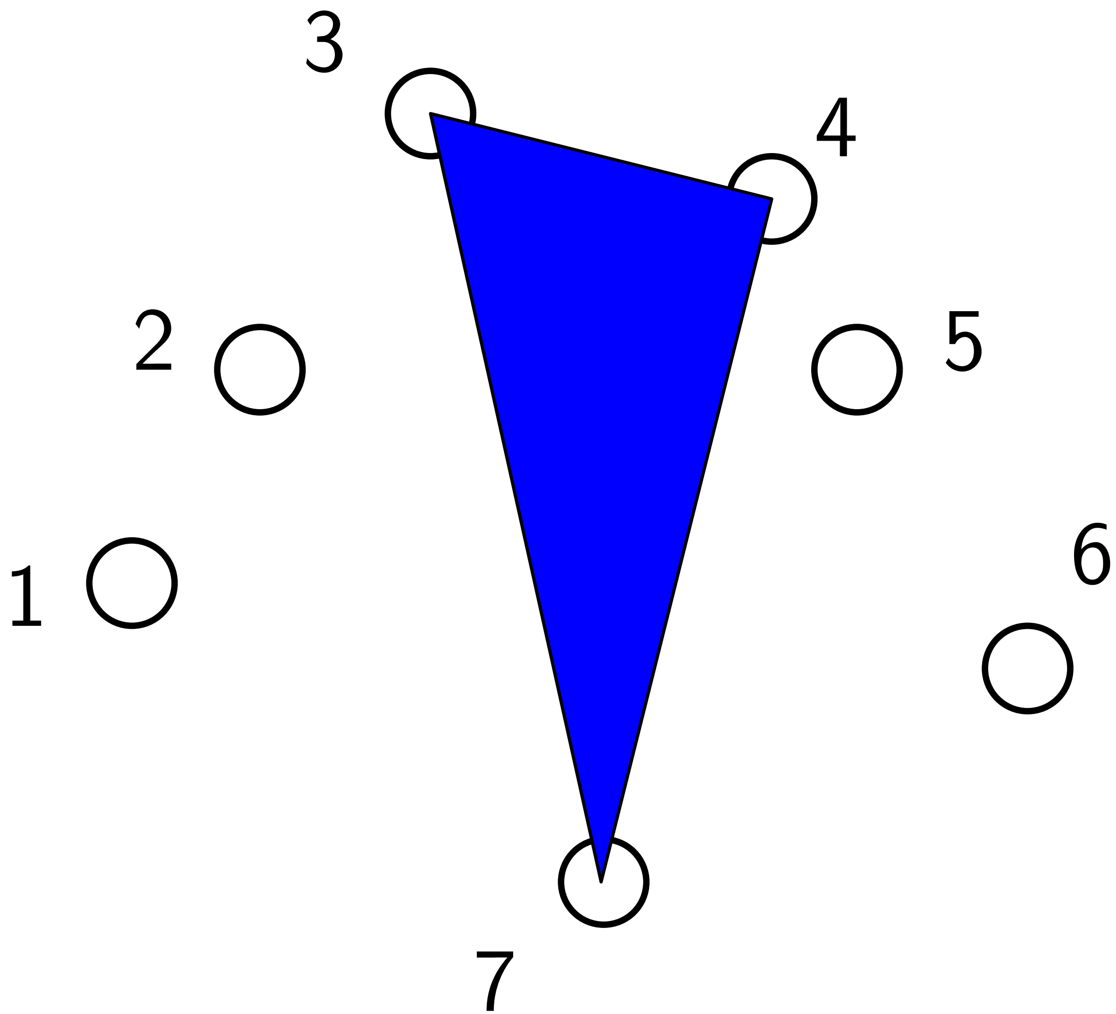

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$3 - 4 - 7$: yes

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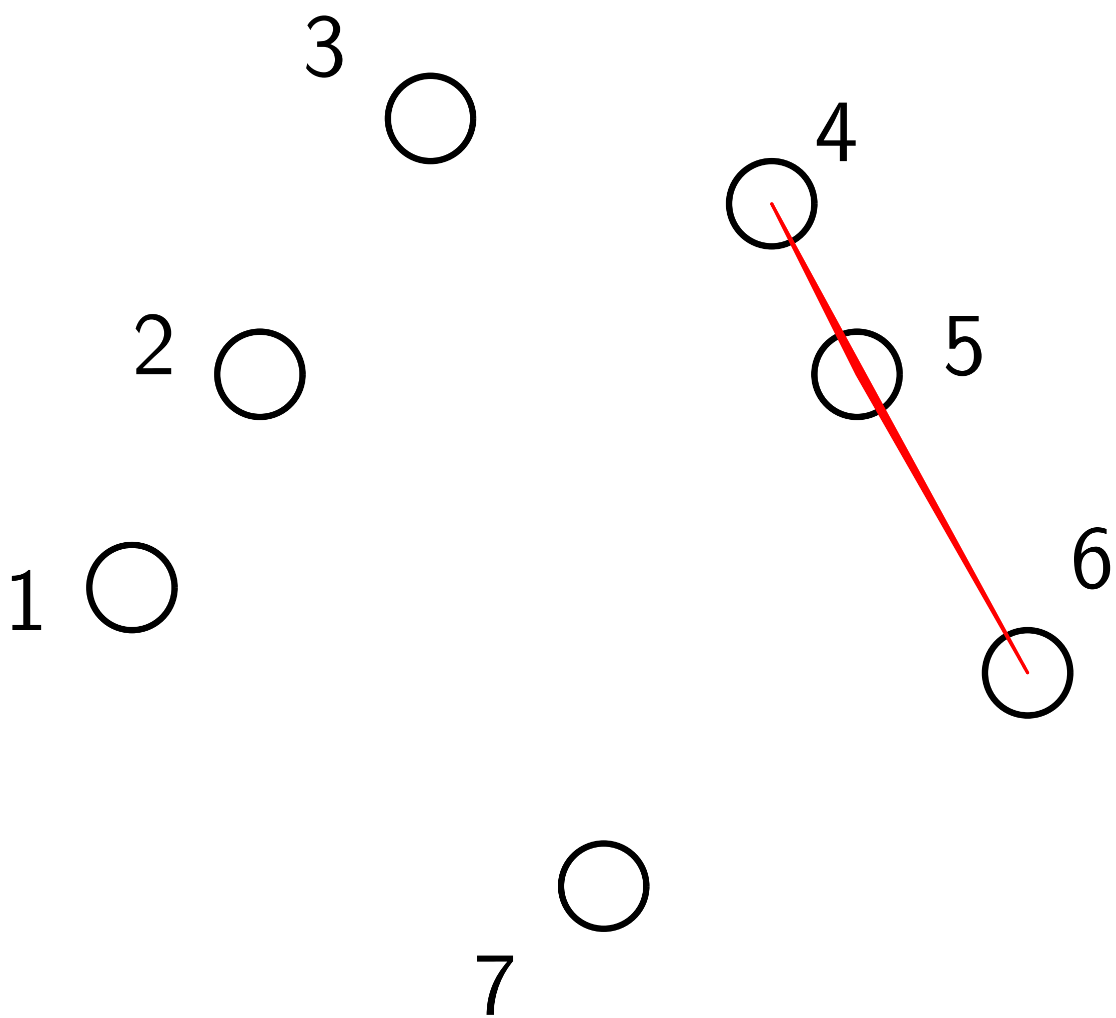

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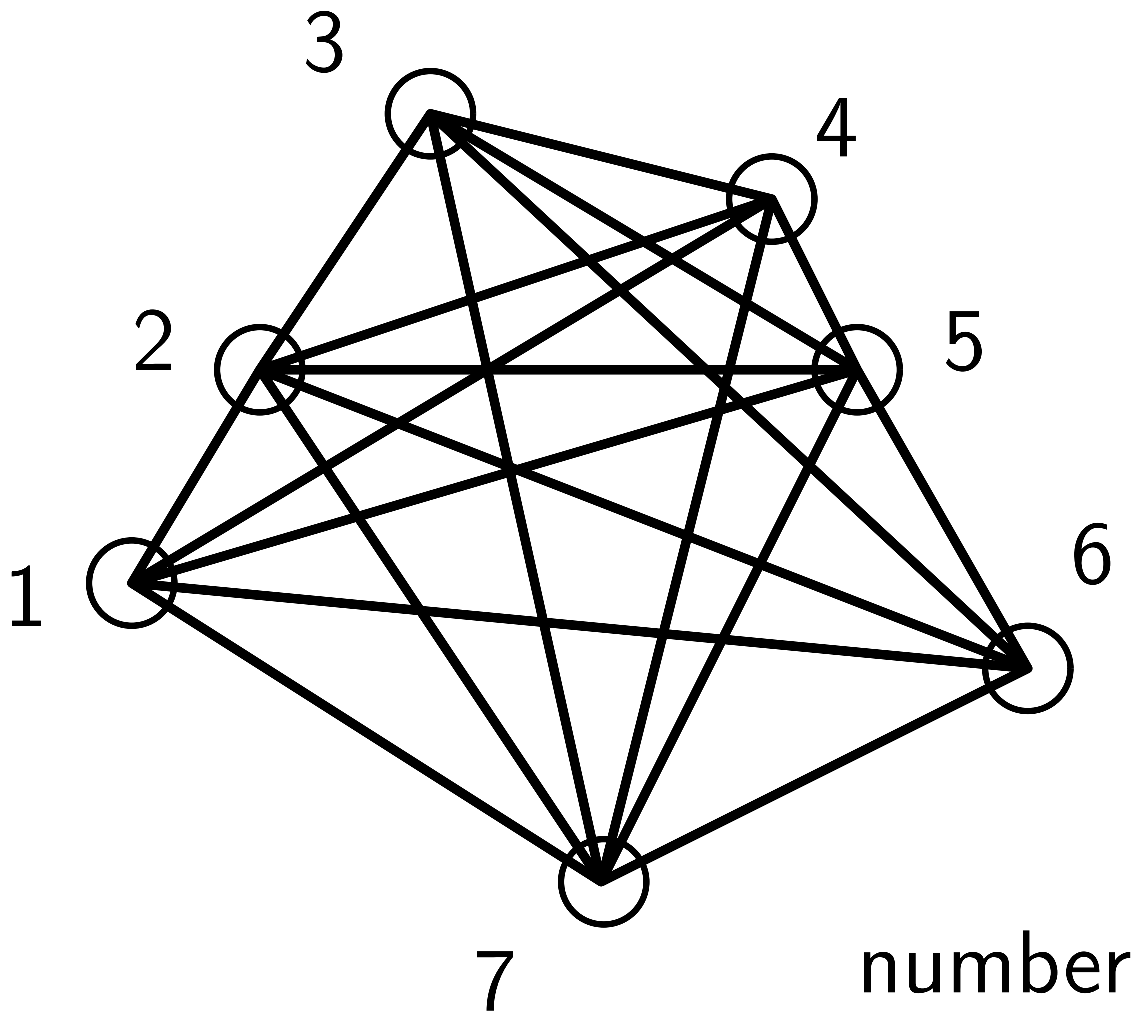

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$1 - 5 - 6$: yes

$3 - 4 - 7$: yes

$4 - 5 - 6$: no

number of triangles: 33

Counting Triangles

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triangleCount = 0
  for i = 1 to n
    for j = i+1 to n
      for k = j+1 to n
        if points i, j, k are not collinear
          triangleCount = triangleCount + 1
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A loop

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Thus each triple i, j, k with $i < j < k$ is examined exactly once.

For example, if $n = 4$, then triples (i, j, k) used by algorithm are (1,2,3), (1,2,4), (1,3,4), and (2,3,4).

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- Want to compute the number of
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  Claim: Number of increasing triples is exactly the same as number of 3-element subsets from $\{1, 2, \ldots, n\}$

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f is a bijection because

f is one-to-one
if $(i, j, k) \neq (i', j', k') \Rightarrow f(\,(i, j, k)\,) \neq f(\,(i', j', k'))$

f is onto
if γ is a 3-element subset then it can be written as $\gamma = \{i, j, k\}$
where $i < j < k$ so $f(\,(i, j, k)\,) = \gamma$.

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  We actually already saw that $|X| = |Y| = \binom{n}{2}$

The Bijection Principle

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  Currently, we started with the problem of counting the # of increasing triples and changed it to the problem of counting the # of 3-element sets from $\{1, 2, \ldots, n\}$

Inclusion-Exclusion Principle

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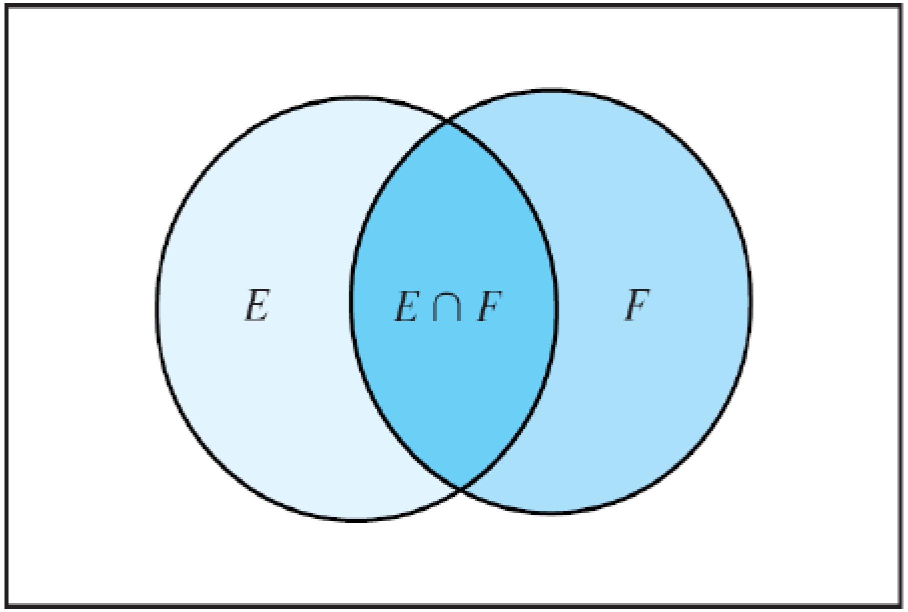

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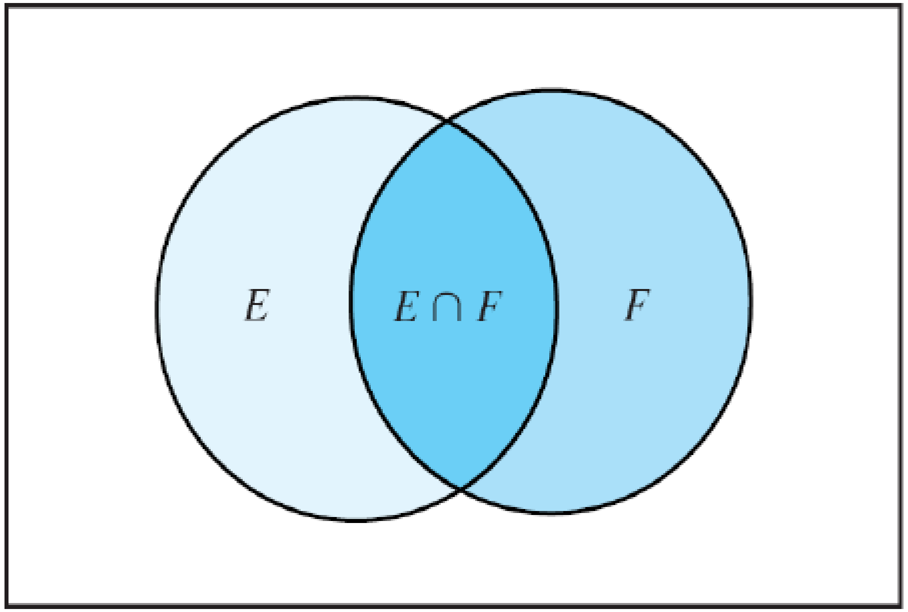

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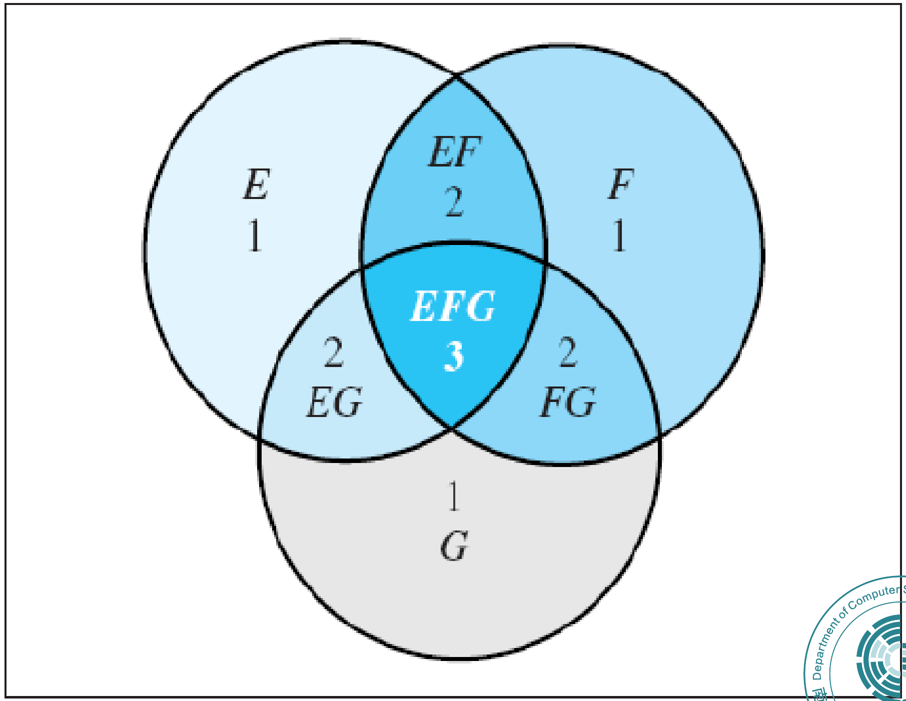

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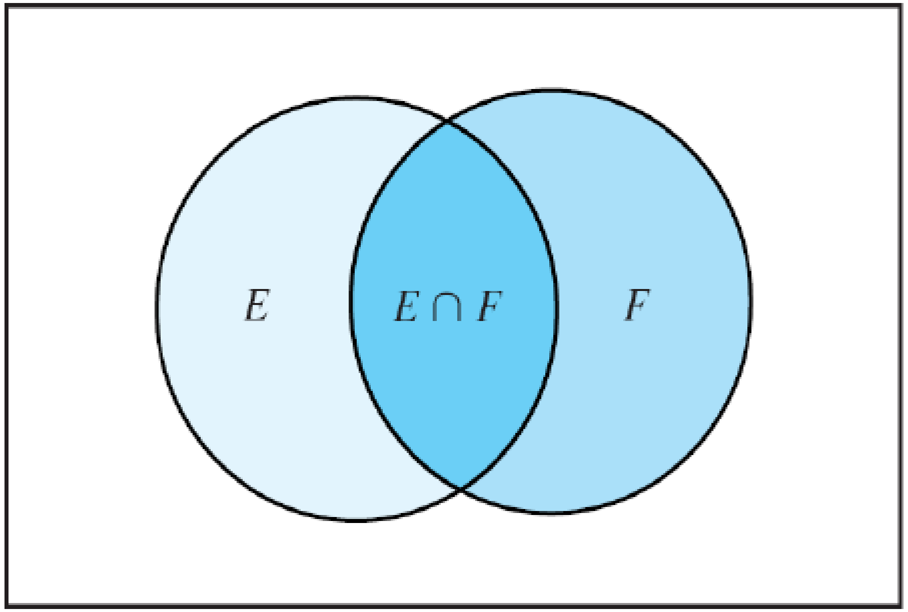
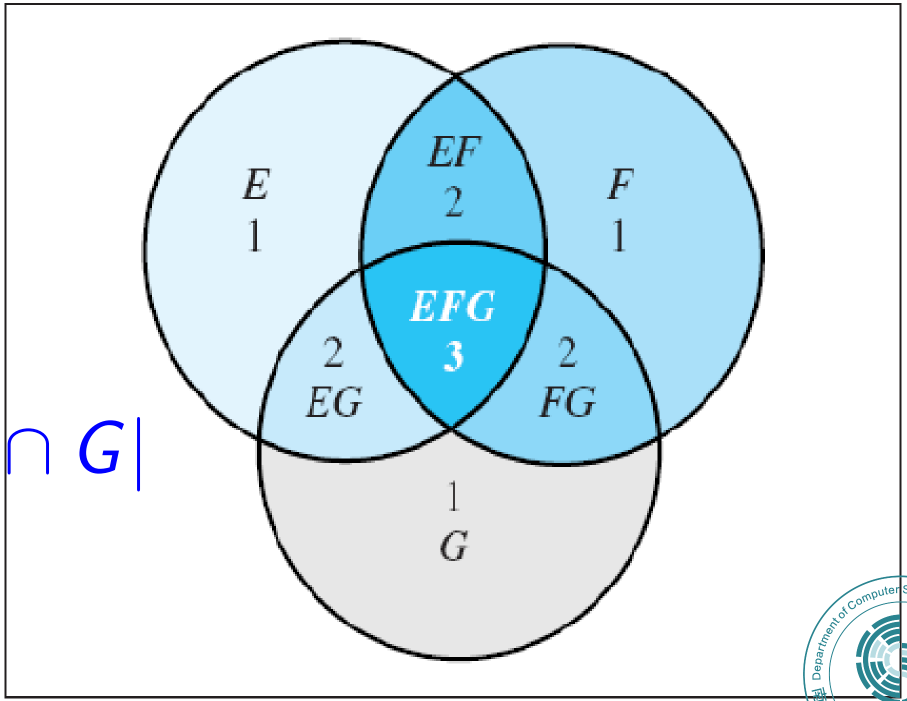


Three sets

$$\begin{aligned} & |E \cup F \cup G| \\ = & |E| + |F| + |G| \\ & -|E \cap F| - |E \cap G| - |F \cap G| \\ & +|E \cap F \cap G| \end{aligned}$$



Inclusion-Exclusion Principle

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Inductive Hypothesis

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  For the third term, by distributive law,

$$\left|\left(\cup_{i=1}^{n-1} E_i\right) \cap E_n\right| = \left|\cup_{i=1}^{n-1} (E_i \cap E_n)\right| = \left|\cup_{i=1}^{n-1} G_i\right|$$

  where $G_i = E_i \cap E_n$.

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Note that (why?)

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Some discussion:
first summation sums $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do not contain n
$|E_n|$ and second summation together sum $(-1)^{k+1}|E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_k}|$ over all lists $i_1, i_2, \ldots, i_k$ that do contain n

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$= \sum_{k=1}^{n}(-1)^{k+1} \binom{n}{k}(n-k)^m$

Quiz #1

- 1. For the following logic statements, using "Yes" or "No" to answer whether they are *tautology/tautologies* or not.

  (1) $\neg p \to (p \to q)$

  (2) $(p \vee q) \wedge (\neg p \vee r) \to (q \vee r)$
- 2. For the following pairs of logic statements, using "Yes" or "No" to answer whether the two are *logically equivalent* or not.

  (1) p and $(p \wedge \neg q) \vee (p \wedge q)$

  (2) $\exists x\ (P(x) \to Q(x))$ and $\forall x\ P(x) \to \exists x\ Q(x)$
- 3. Let $f : A \to B$ and $g : B \to C$ be two functions. Use "Yes" or "No" to answer the following.

  (1) If the composition $g \circ f$ is a *bijection*, then (a) must f be one-to-one? (b) must g be onto?

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  $$\forall n \in \mathbb{N}(n^3 + 6n + 4 \text{ is odd} \Rightarrow n \text{ is even})$$

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Next Lecture

- counting II ...