Data Structure and Algorithm Analysis(H)
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Work Sheet 6

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Question 6.1

1.

Loop Invariant: At the end of each iteration, $A[j-1]$ is the smallest element in $A[j-1..A.\text{length}]$.

Initialization: Before the first iteration, we can assume a $j = A.\text{length}+1$, so the array $A[j-1..A.\text{length}]$ contains one element $A[A.\text{length}]$, which is the smallest element in the array.

Maintenance: Assume that $A[j]$ is the smallest element in $A[j..A.\text{length}]$ before iteration. Then, in the iteration, if $A[j-1] < A[j]$, then $A[j-1]$ is the smallest element in $A[j-1..A.\text{length}]$. Otherwise, $A[j]$ is the smallest element in $A[j-1..A.\text{length}]$, and we swap $A[j]$ and $A[j-1]$.

Termination: When the loop terminates, $j = i+1$, so $A[i]$ is the smallest element in $A[i..A.\text{length}]$.

2.

Loop Invariant: At the end of the i-th iteration, $A[1..i]$ contains the first i smallest elements in $A[1..A.\text{length}]$ in sorted order.

Initialization: After the first inner loop, $A[1]$ is the smallest element in $A[1..A.\text{length}]$, so $A[1..1]$ contains the first 1 smallest elements in $A[1..A.\text{length}]$ in sorted order.

Maintenance: Assume that $A[1..i-1]$ contains the first $i-1$ smallest elements in $A[1..A.\text{length}]$ in sorted order before the i-th iteration. Then, in the i-th iteration, we find the smallest element in $A[i..A.\text{length}]$, which is the i-th smallest element in $A[1..A.\text{length}]$. We put it in $A[i]$, then $A[1..i]$ contains the first i smallest elements in $A[1..A.\text{length}]$ in sorted order.

Termination: When the loop terminates, $i = A.\text{length}$, so $A[1..A.\text{length}]$ contains the first $A.\text{length}$ smallest elements in $A[1..A.\text{length}]$ in sorted order.

3.

BUBBLE-SORT(A)	Runtime (in one iteration)
1. **for** $i = 1$ **to** $A.\text{length}-1$ **do**	$\Theta(n-1)$
2. **for** $j = A.\text{length}$ **downto** $i+1$ **do**	$\Theta(n-i)$
3. **if** $A[j] < A[j-1]$ **then**	$\Theta(1)$
4. exchange $A[j]$ with $A[j-1]$	$\Theta(1)$

$$\begin{aligned}
T(n) =& \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (\Theta(1) + \Theta(1)) \\
=& \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Theta(1) \\
=& \frac{n(n-1)}{2} \Theta(1) \\
=& \Theta(n^2)
\end{aligned}$$

Hence, the asymptotic runtime of BUBBLE-SORT is $\Theta(n^2)$.

Question 6.2

First, we sort the array:

$A[4]$	$A[3]$	$A[7]$	$A[6]$	$A[9]$	$A[5]$	$A[2]$	$A[1]$	$A[8]$
z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	z_9
2	4	5	6	8	9	10	12	25

1.

The probability that $A[2] = 10 = z_7$ and $A[3] = 4 = z_2$ are compared is:

$$\begin{aligned}
\Pr(z_2 \text{ is compared to } z_7) =& \frac{2}{7-2+1} \\
=& \frac{1}{3}
\end{aligned}$$

2.

Likewise, we have the probability that z_8 is compared to z_9 is 1.

3.

Likewise, we have the probability that z_1 is compared to z_9 is $\frac{2}{9}$.

4.

Likewise, we have the probability that z_3 is compared to z_7 is $\frac{2}{5}$.

Question 6.3

Proof.

First, we proof this inequality $\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\frac{1}{k} \geq \sum_{i=1}^{\frac{n}{2}}\sum_{k=1}^{n-1}\frac{1}{k}$.

We can write the two sums by each term as follows:

$$\sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\frac{1}{k} = (n-1)\cdot\frac{1}{1} + (n-2)\cdot\frac{1}{2} + \cdots + \frac{n}{2}\cdot\frac{1}{\frac{n}{2}} + \cdots + 1\cdot\frac{1}{n-1}$$

$$\sum_{i=1}^{\frac{n}{2}}\sum_{k=1}^{n-1}\frac{1}{k} = \frac{n}{2}\cdot\frac{1}{1} + \frac{n}{2}\cdot\frac{1}{2} + \cdots + \frac{n}{2}\cdot\frac{1}{\frac{n}{2}} + \cdots + \frac{n}{2}\cdot\frac{1}{n-1}$$

And both sums have $\frac{n^2-n}{2}$ elements.

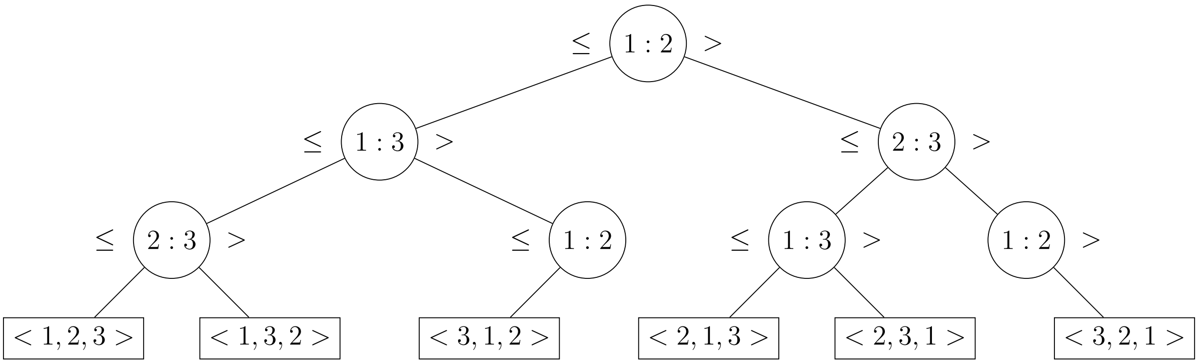
The inequality holds because we remove some elements that are bigger than $\frac{1}{\frac{n}{2}}$ from the first sum and add the same number of elements that are smaller than $\frac{1}{\frac{n}{2}}$ to the second sum.

Then, we have:

$$\begin{aligned}
E(X) =& \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2}{j-i+1} \\
=& \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\frac{2}{k+1} \\
\geq& \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\frac{2}{k+k} \\
=& \sum_{i=1}^{n-1}\sum_{k=1}^{n-i}\frac{1}{k} \\
\geq& \sum_{i=1}^{\frac{n}{2}}\sum_{k=1}^{n-1}\frac{1}{k} \\
=& \frac{n}{2}\sum_{k=1}^{n-1}\frac{1}{k} \\
\geq& \frac{n}{2}\ln(n-1) \\
=& \Theta(n\log n)
\end{aligned}$$

Therefore, $E(X) \geq \Theta(n\log n)$. Equally, we have $E(X) = \Omega(n\log n)$. □

Question 6.4



Note: The notation $i : j$ means to compare a_i and a_j, where a_n is the n-th element in the original array.

Question 6.5

The smallest possible depth of a leaf in a decision tree for a comparison sort is $n - 1$.

Proof.
For each comparison, we can concatenate at most one element to the sorted sequence. Hence, for a result of n elements, we need at least $n - 1$ comparisons and this cannot be optimized. By definition of decision tree, we know all leaves of comparison sort have depth at least $n - 1$. $\square$