

# CS215 DISCRETE MATH

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Note that the case of k = n is special;

An *n*-element permutation of a set N of size |N| = n is what we earlier simply called a permutation.



• How many three-element permutations of  $\{1, 2, \ldots, n\}$  are there?



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*n* choices for first number



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Ex: When n = 4, there are 4 \times 3 \times 2 = 24
3 -element permutations of \{1, 2, 3, 4\}
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L = \{123, 124, 132, 134, 142, 143, 213, 214, 231, 234, 241, 243, 312, 314, 321, 324, 341, 342, 412, 413, 421, 423, 431, 432\}.
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Note: This type of "dictionary" ordering of tuples (assuming that we treat numbers the same as letters) is called a *lexicographic ordering* and is used quite often.



■ **Theorem** If N is a positive integer and k is an integer with  $1 \le k \le n$ , then there are

$$P(n,k) = n(n-1)(n-2)\cdots(n-k+1)$$

k-element permutations with n distinct elements.



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$$P(n,3) = 3! \cdot C(n,3)$$



■ **Theorem** For integers n and k with  $0 \le k \le n$ , the number of k-element subsets of an n-element set is

$$\binom{n}{k} = C(n, k) = \frac{P(n, k)}{k!} = \frac{n!}{k!(n-k)!}.$$

This is the number of k-combinations of a set with n elements.



$${}^{\bullet}\binom{n}{k} = \frac{n!}{k!(n-k)!}$$
 is the number of k-element subsets of an n-element set.

$$\binom{n}{0} = 1$$
 only one set of size 0.

$$\binom{n}{n} = 1$$
 only one set of size  $n$ .

 $\binom{n}{k} = \binom{n}{n-k}$  Obvious from equation. Can you think of a simple bijection that explains this?



 $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ 



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Use Sum Rule

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Use Sum Rule

Let 
$$P = \text{set of all subsets of } \{1,2,\ldots,n\}$$
  
 $S_i = \text{set of all } i\text{-subsets of } \{1,2,\ldots,n\}$ 

$$\Rightarrow |P| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i}$$



Let  $L = L_1 L_2 \dots L_n$  be a list of size n from  $\{0, 1\}$ If  $\mathcal{L} = \text{set of all such lists} \Rightarrow |\mathcal{L}| = 2^n$ There is a *bijection* between  $\mathcal{L}$  and P so  $|P| = 2^n$  and we are done.

Let L = L<sub>1</sub>L<sub>2</sub>...L<sub>n</sub> be a list of size n from {0,1}
If L = set of all such lists ⇒ |L| = 2<sup>n</sup>
There is a bijection between L and P so |P| = 2<sup>n</sup> and we are done.
Define the following function f: L → P
If L ∈ L then f(L) is the set S ⊆ {1,2,...,n} defined by

 $i \in S \Leftrightarrow L_i = 1$ 

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Ex: n = 5  $f(10101) = \{1, 3, 5\}, \ f(11101) = \{1, 2, 3, 5\}, \ f(00000) = \emptyset$ 9 - 4

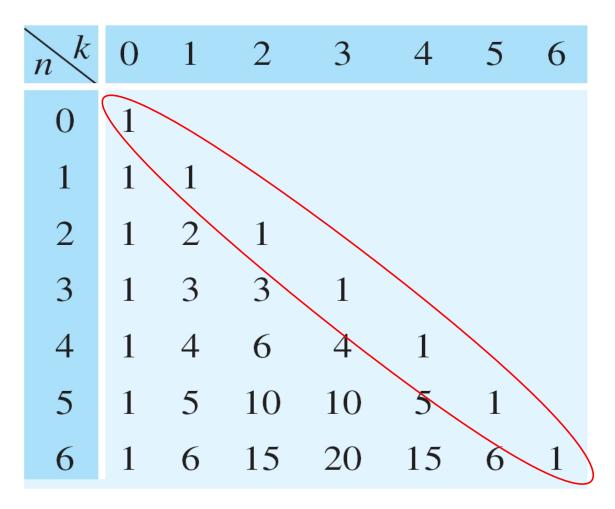
$n^{k}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1



$n^{k}$	0	1	2	3	4	5	6
0	$\sqrt{1}$		1 3 6 10 15				
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
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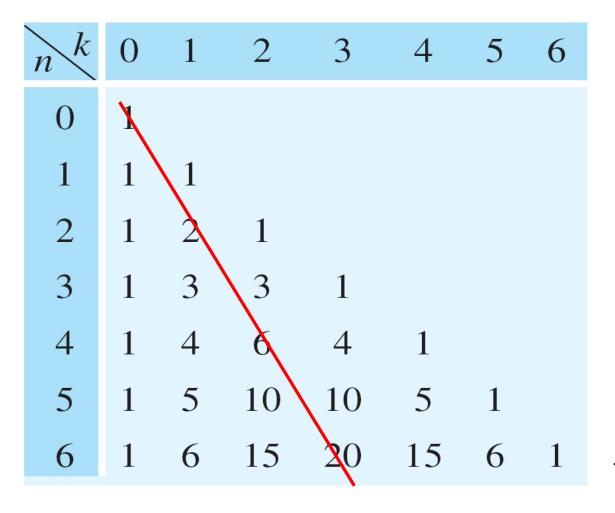
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Each row increases at first then decreases.



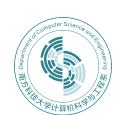


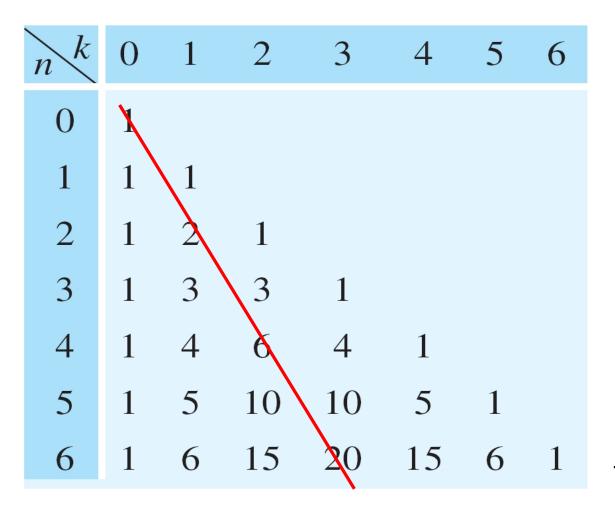
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Second half of each row is the reverse of the first half. Sum of items on n-th row is  $2^n$ 



## Pascal's Triangle

#### Take the table

$n^{k}$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1



### Pascal's Triangle

#### Take the table

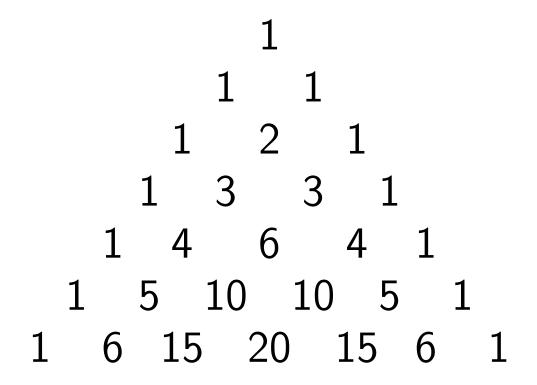
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5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

and shift each row slightly so that middle element is in middle



### Pascal's Triangle



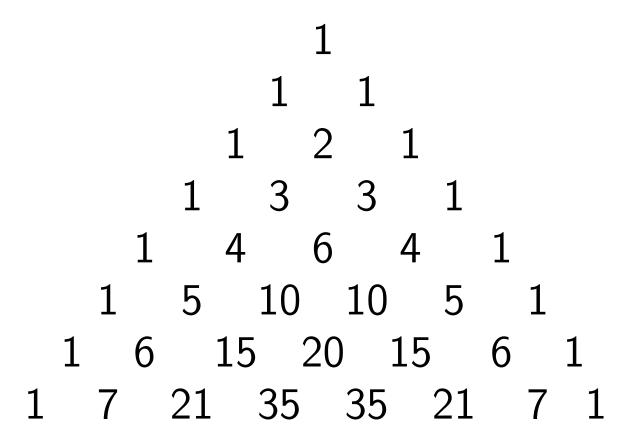


What is the next row in the table?



```
10 10
      15 20 15
1 7 21 35 35 21
```





#### **Pascal identity**

Each (non-1) entry in Pascal's Triangle is the sum of the two entries directly above it (to left and to right).



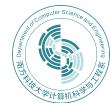
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We will use a combinatorial proof.



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$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Therefore, each term (left and right) represents the number of subsets of a particular size chosen from an appropriately sized set.



$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$



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Number of k-subsets of an n-element set.



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Try to use sum principle to explain relationship among these three terms.

Example: 
$$n = 5$$
,  $k = 2$ 

$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$



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Set  $S_1$  of 2-subsets of S

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$



$$\binom{5}{2} = \binom{4}{1} + \binom{4}{2}.$$

Consider  $S = \{A, B, C, D, E\}$ .

Set  $S_1$  of 2-subsets of S can be partitioned into 2 disjoint parts.

 $S_2$  the 2-subsets that contain E and

 $S_3$ , the set of 2-subsets that do not contain E.

$$S_1 = \{\{A, B\}, \{A, C\}, \{A, D\}, \{A, E\}, \{B, C\}, \{B, D\}, \{B, E\}, \{C, D\}, \{C, E\}, \{D, E\}\}\}.$$



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If n and k are integers satisfying 0 < k < n, then

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**Proof:** Apply sum rule.

Let  $S_1$  be set of all k-element subsets.

To apply sum rule, partition  $S_1$  into  $S_2$  and  $S_3$ .

Let  $S_2$  be set of k-element subsets that contain  $x_n$ .

Let  $S_3$  be set of k-element subsets that don't contain  $x_n$ .



### Blaise Pascal

Born 1623; Died 1662

French Mathematician

A Founder of Probability Theory

Inventor of one of the first mechanical calculating machines

Pascal Programming Language named for him





$$(x+y) = \binom{1}{0}x + \binom{1}{1}y$$



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$$(x+y)^2 = x^2 + 2xy + y^2 = {2 \choose 0}x^2 + {2 \choose 1}x^1y^1 + {2 \choose 2}y^2$$



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$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$= {3 \choose 0}x^3 + {3 \choose 1}x^2y + {3 \choose 2}xy^2 + {3 \choose 3}y^3$$



Number of k-element subsets of an n-element set is called a binomial coefficient because of its role in the algebraic expansion of a binomial  $(x + y)^n$ .



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**The Binomial Theorem** For any integer  $n \geq 0$ ,

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}$$



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$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$



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$$(x+y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

**Proof**?



# Application of the Binomial Theorem

We may use the Binomial Theorem to prove

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$



Suppose we have k labels of one kind, e.g., red and n-k labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?



Suppose we have k labels of one kind, e.g., red and n-k labels of another, e.g., blue. In how many different ways can we apply these labels to n objects?

Show that if we have  $k_1$  labels of one kind, e.g., red,  $k_2$  labels of a second kind, e.g., blue, and  $k_3 = n - k_1 - k_2$  labels of a third kind, then there are  $\frac{n!}{k_1!k_2!k_3!}$  ways to apply these labels to n objects



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What is the coefficient of  $x^{k_1}y^{k_2}z^{k_3}$  in  $(x+y+z)^n$ ?



There are  $\binom{n}{k_1}$  ways to choose the red items There are then  $\binom{n-k_1}{k_2}$  ways to choose the blue items from the remaining  $n-k_1$ . The remaining  $k_3$  items get labelled a third color.



#### Labelling and Trinomial Coefficients

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Using the *product rule* the total number of labellings is

$$\binom{n}{k_1} \binom{n-k_1}{k_2} = \frac{n!}{k_1!(n-k_1)!} \frac{(n-k_1)!}{(k_2)!(n-k_1-k_2)!}$$

$$= \frac{n!}{k_1!k_2!(n-k_1-k_2)!} = \frac{n!}{k_1!k_2!k_3!}$$



## Labelling and Trinomial Coefficients

• When  $k_1 + k_2 + k_3 = n$ , we call

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This will be very similar to the analysis of hashing *n* keys into a table of size 365.



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$$|S| = 365^n$$

 $B_n$  – "there are n students in a room and none of them share a birthday."

$$\#B_n = 365 \times 364 \times \cdots \times (365 - (n-1))$$



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$$|S| = 365^n$$

 $B_n$  – "there are n students in a room and none of them share a birthday."

$$\#B_n = 365 \times 364 \times \cdots \times (365 - (n-1))$$

$$\#A_n + \#B_n = 365^n$$



n	$A_n$	$B_n$	n	$A_{n}$	$B_n$
1	0.00000000	1.00000000	16	0.28360400	0.71639599
2	0.00273972	0.99726027	17	0.31500766	0.68499233
3	0.00820416	0.99179583	18	0.34691141	0.65308858
4	0.01635591	0.98364408	19	0.37911852	0.62088147
5	0.02713557	0.97286442	20	0.41143838	0.58856161
6	0.04046248	0.95953751	21	0.44368833	0.55631166
7	0.05623570	0.94376429	22	0.47569530	0.52430469
8	0.07433529	0.92566470	23	0.50729723	0.49270276
9	0.09462383	0.90537616	24	0.53834425	0.46165574
10	0.11694817	0.88305182	25	0.56869970	0.43130029
11	0.14114137	0.85885862	26	0.59824082	0.40175917
12	0.16702478	0.83297521	27	0.62685928	0.37314071
13	0.19441027	0.80558972	28	0.65446147	0.34553852
14	0.22310251	0.77689748	29	0.68096853	0.31903146
15	0.25290131	0.74709868	30	0.70631624	0.29368375



Event A: at least two people in the room have the same birthday
Event B: no two people in the room have the same birthday

$$Pr[A] = 1 - Pr[B]$$

$$\Pr[B] = \left(1 - \frac{1}{365}\right) \cdot \left(1 - \frac{2}{365}\right) \cdot \dots \cdot \left(1 - \frac{n-1}{365}\right)$$
$$= \prod_{i=1}^{n-1} \left(1 - \frac{i}{365}\right).$$

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$$p(n; H) := 1 - \prod_{i=1}^{n-1} (1 - \frac{i}{H})$$



Since  $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ , for  $|x| \ll 1$ ,  $e^x \approx 1 + x$ 



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Recall that 
$$p(n; H) := 1 - \prod_{i=1}^{n-1} (1 - \frac{i}{H})$$

This probability can be approximated as

$$p(n; H) \approx 1 - e^{-n(n-1)/2H} \approx 1 - e^{-n^2/2H}$$
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Let n(p; H) be the smallest number of values we have to choose, such that the probability for finding a collision is at least p. By inverting the expression above, we have

$$n(p; H) \approx \sqrt{2H \ln \frac{1}{1-p}}.$$



The Euclidean algorithm in pseudocode

#### ALGORITHM 1 The Euclidean Algorithm.

```
procedure gcd(a, b): positive integers)
x := a
y := b
while y \neq 0
r := x \mod y
x := y
y := r
return x\{\gcd(a, b) \text{ is } x\}
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The number of divisions required to find gcd(a, b) is  $O(\log b)$ , where  $a \ge b$ . (this will be proved later.)



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#### Why?



Key steps in the Euclidean algorithm

```
egin{array}{lll} r_0 &= r_1 q_1 + r_2 & 0 \leq r_2 < r_1, \\ r_1 &= r_2 q_2 + r_3 & 0 \leq r_3 < r_2, \\ & \cdot & \\ & \cdot & \\ & \cdot & \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 \leq r_n < r_{n-1}, \\ r_{n-1} &= r_n q_n \ . \end{array}
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Key steps in the Euclidean algorithm

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r_0 = r_1q_1 + r_2 0 \le r_2 < r_1, r_1 = r_2q_2 + r_3 0 \le r_3 < r_2, 0 \le r_3 < r_3, 0 \le r_3 < r_2, 0 \le r_3 < r_3, 0 \le r_3 < r_3
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#### **Observation:**

$$r_{i+2} = r_i \mod r_{i+1}$$

Key steps in the Euclidean algorithm

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■ **Definition** A *linear homogeneous relation of degree k* with constant coefficients is a recurrence relation of the form

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By induction, such a recurrence relation is uniquely determined by this recurrence relation, and k initial conditions  $a_0, a_1, \ldots, a_{k-1}$ .

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#### **Examples**

$$P_n = (1.11)P_{n-1}$$
 $f_n = f_{n-1} + f_{n-2}$ 
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 $f_n = f_{n-1} + f_{n-2}$  linear homogeneous recurrence relation of degree 2

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 linear homogeneous recurrence relation of degree 2

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 NOT linear

$$H_n = 2H_{n-1} + 1$$
 NOT homogeneous

$$B_n = nB_{n-1}$$
 coefficients are not constants



**Example** Consider the recurrence relation

$$a_n = 2a_{n-1} - a_{n-2}$$

Which of the following are solutions?

$$\diamond a_n = 3n$$
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$$\diamond a_n = 2^n$$
:

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**Fact**: Assume that the sequences  $a_n$  and  $a'_n$  both satisfy the recurrence, then  $b_n = a_n + a'_n$ ,  $d_n = \alpha a_n$  also satisfy the recurrence, where  $\alpha$  is a constant.



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So, try to find any solution of the form  $a_n = r^n$  that satisfies the recurrence.

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 $\diamond$  Bring  $a_n = r^n$  back to the recurrence relation:

i.e., 
$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$
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♦ The solutions to the *characteristic equation* can yield an explicit formula for the sequence.

$$(r^k - c_1 r^{k-1} - \cdots - c_k) = 0$$



### Recall: Problem IV

### **■** Fibonacci number

$$F_0 = 0$$
,  $F_1 = 1$ ,  $F_n = F_{n-1} + F_{n-2}$  for  $n \ge 2$ 



### Recall: Problem IV

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 $\diamond$  What is the closed-form expression of  $F_n$ ?



### Recall: Problem IV

### Fibonacci number

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 $\diamond$  What is the closed-form expression of  $F_n$ ?

Consider  $x^n = x^{n-1} + x^{n-2}$ , with  $x \neq 0$ . There are two different roots

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}$$

Then  $F_n$  can be the form of  $a\phi^n + b\psi^n$ . By  $F_0 = 0$  and  $F_1 = 1$ , we have a + b = 0 and  $\phi a + \psi b = 1$ , leading to  $a = \frac{1}{\sqrt{5}}$ , b = -a. Therefore,

$$F_n = \frac{\phi^n - \psi^n}{\sqrt{5}}$$



Consider an arbitrary linear homogeneous relation of degree 2 with constant coefficients:

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**Theorem** If this CE has 2 roots  $r_1 \neq r_2$ , then the sequence  $\{a_n\}$  is a solution of the recurrence relation if and only if  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$  for  $n \geq 0$  and constants  $\alpha_1, \alpha_2$ .



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**Proof?** 



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#### **Proof?**

See [Theorem 1 p. 515].



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Two roots are 2 and -1. So, assume that

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n.$$



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We get  $\alpha_1 = 3$  and  $\alpha_2 = -1$ . Thus,  $a_n = 3 \cdot 2^n - (-1)^n$ 



**Example 2**  $a_n = 7a_{n-1} - 10a_{n-2}$ , with  $a_0 = 2$ ,  $a_1 = 1$ 



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**Example** 
$$a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$$



### The Case of Degenerate Roots

**Theorem** If the CE  $r^2 - c_1 r - c_2 = 0$  has only 1 root  $r_0$ , then

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## The Case of Degenerate Roots in General

**Theorem** [Theorem 4, p.519] Suppose that there are t roots  $r_1, \ldots, r_t$  with multiplicities  $m_1, \ldots, m_t$ . Then

$$a_n = \sum_{i=1}^t \left( \sum_{j=0}^{m_i-1} \alpha_{i,j} n^j \right) r_i^n,$$

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#### **Example**

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$$
 with  $a_0 = 1$ ,  $a_1 = -2$ ,  $a_2 = -1$ 



■ **Definition** A *linear nonhomogeneous relation* with constant coefficients may contain some terms F(n) that depend only on n

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**Idea**: We already know how to find  $h_n$ . For many common f(n), a solution  $b_n$  to the non-homogeneous recurrence is similar to f(n). We then need find  $a_n = b_n + h_n$  to the non-homogeneous recurrence that satisfies both recurrence and initial conditions.

**Theorem** If  $a_n = p(n)$  is any particular solution to the linear nonhomogeneous relation with constant coefficients,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then all its solutions are of the form

$$a_n = p(n) + h(n),$$

where  $a_n = h(n)$  is any solution to the associated homogeneous recurrence relation

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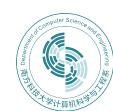
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We get 
$$c = -1$$
 and  $d = -3/2$ . Thus,  $p(n) = -n - 3/2$   
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#### Next Lecture

generating function, relation ...

