



# CS215 DISCRETE MATH

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# Permutation Group

- Let  $s_n = \langle 1, 2, \dots, n \rangle$  denote a *sequence* of integers 1 through  $n$ . Denote by  $P_n$  the set of all *permutations* of the sequence  $s_n$ .

For example,  $s_3 = \langle 1, 2, 3 \rangle$

$$P_3 = \{ \langle 1, 2, 3 \rangle, \langle 1, 3, 2 \rangle, \langle 2, 1, 3 \rangle, \langle 2, 3, 1 \rangle, \langle 3, 1, 2 \rangle, \langle 3, 2, 1 \rangle \}$$

- Define a binary operation  $\circ$  on the elements of  $P_n$ :  
for  $\rho, \pi \in P_n$ ,  $\pi \circ \rho$  denotes a *re-permutation* of the elements of  $\rho$  according to the elements of  $\pi$ .



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 $P_3 = \{ \langle p_1, p_2, p_3 \rangle \mid p_1, p_2, p_3 \in s_3 \text{ with } p_1 \neq p_2 \neq p_3 \}$ .



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- $\pi = \langle 3, 2, 1 \rangle$ ,  $\rho = \langle 1, 3, 2 \rangle$ , what is  $\pi \circ \rho$ ?



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- $\pi = \langle 3, 2, 1 \rangle$ ,  $\rho = \langle 1, 3, 2 \rangle$ , what is  $\pi \circ \rho$ ?  
 $\pi \circ \rho = \langle 2, 3, 1 \rangle \in P_3$
- We can verify the other three properties.  
$$\rho_1 \circ (\rho_2 \circ \rho_3) = (\rho_1 \circ \rho_2) \circ \rho_3$$
$$\langle 1, 2, 3 \rangle \circ \rho = \rho \circ \langle 1, 2, 3 \rangle = \rho$$

For each  $\rho \in P_3$ , there exists another unique  $\pi \in P_3$  such that

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For each  $\rho \in P_3$ , there exists another unique  $\pi \in P_3$  such that

$$\rho \circ \pi = \pi \circ \rho = \langle 1, 2, 3 \rangle$$

$(P_n, \circ)$  is called a *permutation group*.



# Representations of Integers

- We may use *decimal* (*base 10*) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.





# Representations of Integers

- We may use *decimal* (*base 10*) or *binary* or *octal* or *hexadecimal* or other notations to represent integers.
- Let  $b > 1$  be an integer. Then if  $n$  is a positive integer, it can be expressed **uniquely in the form**  
$$n = a_k b^k + a_{k-1} b^{k-1} + \dots + a_1 b + a_0,$$
 where  $k$  is nonnegative,  $a_i$ 's are nonnegative integers less than  $b$ . The representation of  $n$  is called *the base- $b$  expansion of  $n$*  and is denoted by  $(a_k a_{k-1} \dots a_1 a_0)_b$ .



# Algorithm: Constructing Base- $b$ Expansions

```
procedure base  $b$  expansion( $n, b$ : positive integers with  $b > 1$ )  
   $q := n$   
   $k := 0$   
  while ( $q \neq 0$ )  
     $a_k := q \bmod b$   
     $q := q \operatorname{div} b$   
     $k := k + 1$   
  return( $a_{k-1}, \dots, a_1, a_0$ ) { ( $a_{k-1} \dots a_1 a_0$ ) $_b$  is base  $b$  expansion of  $n$  }
```

# Example

- $(12345)_{10} = (30071)_8$

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$$12345 = 8 \cdot 1543 + 1$$

$$1543 = 8 \cdot 192 + 7$$

$$192 = 8 \cdot 24 + 0$$

$$24 = 8 \cdot 3 + 0$$

$$3 = 8 \cdot 0 + 3$$

# Binary Addition of Integers

$$a = (a_{n-1}a_{n-2} \dots a_1a_0), \quad b = (b_{n-1}b_{n-2} \dots b_1b_0)$$

```
procedure add(a, b: positive integers)
{the binary expansions of a and b are  $(a_{n-1}, a_{n-2}, \dots, a_0)_2$  and  $(b_{n-1}, b_{n-2}, \dots, b_0)_2$ , respectively}
c := 0
for j := 0 to n - 1
    d :=  $\lfloor (a_j + b_j + c) / 2 \rfloor$ 
    sj :=  $a_j + b_j + c - 2d$ 
    c := d
sn := c
return(s0, s1, ..., sn) {the binary expansion of the sum is  $(s_n, s_{n-1}, \dots, s_0)_2$ }
```

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$O(n)$  bit additions

# Algorithm: Binary Multiplication of Integers

$$a = (a_{n-1}a_{n-2} \dots a_1a_0)_2, \quad b = (b_{n-1}b_{n-2} \dots b_1b_0)_2$$

$$\begin{aligned} ab &= a(b_02^0 + b_12^1 + \dots + b_{n-1}2^{n-1}) \\ &= a(b_02^0) + a(b_12^1) + \dots + a(b_{n-1}2^{n-1}) \end{aligned}$$

```
procedure multiply(a, b: positive integers)
{the binary expansions of a and b are  $(a_{n-1}, a_{n-2}, \dots, a_0)_2$  and  $(b_{n-1}, b_{n-2}, \dots, b_0)_2$ , respectively}
for j := 0 to n - 1
    if  $b_j = 1$  then  $c_j = a$  shifted j places
    else  $c_j := 0$ 
{ $c_0, c_1, \dots, c_{n-1}$  are the partial products}
p := 0
for j := 0 to n - 1
    p := p +  $c_j$ 
return p {p is the value of ab}
```

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p := 0
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    p := p +  $c_j$ 
return p {p is the value of ab}
```

$O(n^2)$  shifts and  $O(n^2)$  bit additions





# Algorithm: Computing div and mod

```
procedure division algorithm ( $a$ : integer,  $d$ : positive integer)
   $q := 0$ 
   $r := |a|$ 
  while  $r \geq d$ 
     $r := r - d$ 
     $q := q + 1$ 
  if  $a < 0$  and  $r > 0$  then
     $r := d - r$ 
     $q := -(q+1)$ 
  return ( $q, r$ ) { $q = a \text{ div } d$  is the quotient,  $r = a \text{ mod } d$  is the remainder }
```

# Algorithm: Computing div and mod

```
procedure division algorithm (a: integer, d: positive integer)
  q := 0
  r := |a|
  while r ≥ d
    r := r - d
    q := q + 1
  if a < 0 and r > 0 then
    r := d - r
    q := -(q+1)
  return (q, r) {q = a div d is the quotient, r = a mod d is the
  remainder }
```

$O(q \log a)$  bit operations. But there exist more efficient algorithms with complexity  $O(n^2)$ , where  $n = \max(\log a, \log d)$

# Algorithm: Computing div and mod (cont)

```
■ procedure division2 ( $a, d \in \mathbb{N}, d \geq 1$ )  
  if  $a < d$   
    return  $(q, r) = (0, a)$   
   $(q, r) = \text{division2}(\lfloor a/2 \rfloor, d)$   
   $q = 2q, r = 2r$   
  if  $a$  is odd  
     $r = r + 1$   
  if  $r \geq d$   
     $r = r - d$   
     $q = q + 1$   
  return  $(q, r)$ 
```

# Algorithm: Computing div and mod (cont)

```
■ procedure division2 ( $a, d \in \mathbb{N}, d \geq 1$ )  
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     $r = r + 1$   
  if  $r \geq d$   
     $r = r - d$   
     $q = q + 1$   
  return  $(q, r)$ 
```

$O(\log q \log a)$  bit operations.

# Algorithm: Binary Modular Exponentiation

$$b^n = b^{a_{k-1} \cdot 2^{k-1} + \dots + a_1 \cdot 2 + a_0} = b^{a_{k-1} \cdot 2^{k-1}} \dots b^{a_1 \cdot 2} \cdot b^{a_0}$$

Successively finds  $b \bmod m$ ,  $b^2 \bmod m$ ,  $b^4 \bmod m$ ,  $\dots$ ,  $b^{2^{k-1}} \bmod m$ , and multiplies together the terms  $b^{2^j} \bmod m$  where  $a_j = 1$ .

```
procedure modular_exponentiation( $b$ : integer,  $n = (a_{k-1}a_{k-2}\dots a_1a_0)_2$ ,  $m$ : positive integers)
   $x := 1$ 
   $power := b \bmod m$ 
  for  $i := 0$  to  $k - 1$ 
    if  $a_i = 1$  then  $x := (x \cdot power) \bmod m$ 
     $power := (power \cdot power) \bmod m$ 
  return  $x$  { $x$  equals  $b^n \bmod m$ }
```

# Algorithm: Binary Modular Exponentiation

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Successively finds  $b \bmod m$ ,  $b^2 \bmod m$ ,  $b^4 \bmod m$ ,  $\dots$ ,  $b^{2^{k-1}} \bmod m$ , and multiplies together the terms  $b^{2^j} \bmod m$  where  $a_j = 1$ .

```
procedure modular_exponentiation(b: integer,  $n = (a_{k-1}a_{k-2}\dots a_1a_0)_2$ , m: positive integers)
  x := 1
  power := b mod m
  for i := 0 to k - 1
    if  $a_i = 1$  then x := (x · power) mod m
    power := (power · power) mod m
  return x {x equals  $b^n \bmod m$ }
```

$O((\log m)^2 \log n)$  bit operations



# Primes

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# Primes

- A positive integer  $p$  that is greater than 1 and is **divisible only by 1 and by itself** is called a ***prime***.
- A positive integer  $p$  that is greater than 1 and is **not a prime** is called a ***composite***.
- **Fundamental Theorem of Arithmetic** Every integer greater than 1 can be written **uniquely as a prime or as the product of two or more primes** where the prime factors are written in order of nondecreasing size.



# Primes and Composites

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# Primes and Composites

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**Approach 1:** test if **each number**  $x < n$  divides  $n$ .

**Approach 2:** test if each **prime** number  $x < n$  divides  $n$ .

**Approach 3:** test if each **prime** number  $x \leq \sqrt{n}$  divides  $n$ .

# Primes and Composites

- If  $n$  is composite, then  $n$  has a prime divisor less than or equal to  $\sqrt{n}$ .



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- If  $n$  is composite, then  $n$  has a prime divisor less than or equal to  $\sqrt{n}$ .

## Proof.

◇ if  $n$  is composite, then it has a positive integer factor  $a$  such that  $1 < a < n$  by definition. This means that  $n = ab$ , where  $b$  is an integer greater than 1.

◇ assume that  $a > \sqrt{n}$  and  $b > \sqrt{n}$ . Then  $ab > n$ , contradiction. So either  $a \leq \sqrt{n}$  or  $b \leq \sqrt{n}$ .

◇ Thus,  $n$  has a divisor less than  $\sqrt{n}$ .

◇ By the Fundamental Theorem of Arithmetic, this divisor is either prime, or is a product of primes. In either case,  $n$  has a prime divisor less than  $\sqrt{n}$ .



# Primes

- There are infinitely many primes.

**Proof** (by contradiction)



# Greatest Common Divisor (GCD)

- Let  $a$  and  $b$  be integers, not both 0. The largest integer  $d$  such that  $d|a$  and  $d|b$  is called the *greatest common divisor* of  $a$  and  $b$ , denoted by  $\gcd(a, b)$ .



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The integers  $a$  and  $b$  are *relatively prime* if their greatest common divisor is 1.

A systematic way to find the gcd is **factorization**. Let

$a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$  and  $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$ . Then

$$\gcd(a, b) = p_1^{\min(a_1, b_1)} p_2^{\min(a_2, b_2)} \cdots p_n^{\min(a_n, b_n)}$$



# Least Common Multiple (LCM)

- Let  $a$  and  $b$  be integers. The *least common multiple* of  $a$  and  $b$  is the smallest positive integer that is divisible by both  $a$  and  $b$ , denoted by  $\text{lcm}(a, b)$ .



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We can also use **factorization** to find the lcm. Let  $a = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$  and  $b = p_1^{b_1} p_2^{b_2} \cdots p_n^{b_n}$ . Then

$$\text{lcm}(a, b) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_n^{\max(a_n, b_n)}$$


# Euclidean Algorithm

- Factorization can be **cumbersome** and **time consuming** since we need to find all factors of the two integers.



# Euclidean Algorithm

- Factorization can be **cumbersome** and **time consuming** since we need to find all factors of the two integers.
- Luckily, we have an efficient algorithm, called **Euclidean algorithm**. This algorithm has been known since ancient times and named after the ancient Greek mathematician Euclid.



# Euclidean Algorithm

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$$\text{Step 1: } 287 = 91 \cdot 3 + 14$$

$$\text{Step 2: } 91 = 14 \cdot 6 + 7$$



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- For two integers 287 and 91, we want to find  $\gcd(287, 91)$ .

Step 1:  $287 = 91 \cdot 3 + 14$

Step 2:  $91 = 14 \cdot 6 + 7$

Step 3:  $14 = 7 \cdot 2 + 0$



# Euclidean Algorithm

- For two integers 287 and 91, we want to find  $\gcd(287, 91)$ .

$$\text{Step 1: } 287 = 91 \cdot 3 + 14$$

$$\text{Step 2: } 91 = 14 \cdot 6 + 7$$

$$\text{Step 3: } 14 = 7 \cdot 2 + 0$$

$$\gcd(287, 91) = \gcd(91, 14) = \gcd(14, 7) = 7$$



# Euclidean Algorithm

- The Euclidean algorithm in pseudocode

## ALGORITHM 1 The Euclidean Algorithm.

**procedure**  $\text{gcd}(a, b$ : positive integers)

$x := a$

$y := b$

**while**  $y \neq 0$

$r := x \bmod y$

$x := y$

$y := r$

**return**  $x$  {gcd( $a, b$ ) is  $x$ }

# Euclidean Algorithm

- The Euclidean algorithm in pseudocode

## ALGORITHM 1 The Euclidean Algorithm.

```
procedure gcd(a, b: positive integers)
  x := a
  y := b
  while y ≠ 0
    r := x mod y
    x := y
    y := r
  return x{gcd(a, b) is x}
```

The number of **divisions** required to find  $\text{gcd}(a, b)$  is  $O(\log b)$ , where  $a \geq b$ . (this will be proved later.)



# Correctness of Euclidean Algorithm

- **Lemma** Let  $a = bq + r$ , where  $a, b, q$  and  $r$  are integers. Then  $\gcd(a, b) = \gcd(b, r)$ .



# Correctness of Euclidean Algorithm

- **Lemma** Let  $a = bq + r$ , where  $a, b, q$  and  $r$  are integers. Then  $\gcd(a, b) = \gcd(b, r)$ .

## Proof.

- ◇ suppose that  $d|a$  and  $d|b$ . Then  $d$  also divides  $a - bq = r$ . Hence, any common divisor of  $a$  and  $b$  must also be any common divisor of  $b$  and  $r$ .
- ◇ suppose that  $d|b$  and  $d|r$ . Then  $d$  also divides  $bq + r = a$ . Hence, any common divisor of  $b$  and  $r$  must also be a common divisor of  $a$  and  $b$ .
- ◇ Therefore,  $\gcd(a, b) = \gcd(b, r)$ .





# Correctness of Euclidean Algorithm

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$$r_0 = r_1 q_1 + r_2 \quad 0 \leq r_2 < r_1,$$

$$r_1 = r_2 q_2 + r_3 \quad 0 \leq r_3 < r_2,$$

.

.

.

$$r_{n-2} = r_{n-1} q_{n-1} + r_n \quad 0 \leq r_n < r_{n-1},$$

$$r_{n-1} = r_n q_n.$$

# Correctness of Euclidean Algorithm

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$$r_{n-2} = r_{n-1} q_{n-1} + r_n \quad 0 \leq r_n < r_{n-1},$$

$$r_{n-1} = r_n q_n.$$

$$\gcd(a, b) = \gcd(r_0, r_1) = \cdots = \gcd(r_{n-1}, r_n) = \gcd(r_n, 0) = r_n$$



# GCD as Linear Combinations

- **Bezout's Theorem** If  $a$  and  $b$  are positive integers, then there exist integers  $s$  and  $t$  such that  $\gcd(a, b) = sa + tb$ . This is called *Bezout's identity*.



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We may use *extended Euclidean algorithm* to find Bezout's identity.

**Example:** Express 1 as the linear combination of 503 and 286.

$$503 = 1 \cdot 286 + 217$$

$$286 = 1 \cdot 217 + 69$$

$$217 = 3 \cdot 69 + 10$$

$$69 = 6 \cdot 10 + 9$$

$$10 = 1 \cdot 9 + 1$$



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$$503 = 1 \cdot 286 + 217$$

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$$217 = 3 \cdot 69 + 10$$

$$69 = 6 \cdot 10 + 9$$

$$10 = 1 \cdot 9 + 1$$

$$1 = 10 - 1 \cdot 9$$

$$= 7 \cdot 10 - 1 \cdot 69$$

$$= 7 \cdot 217 - 22 \cdot 69$$

$$= 29 \cdot 217 - 22 \cdot 286$$

$$= 29 \cdot 503 - 51 \cdot 286$$





# Corollaries of Bezout's Theorem

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**Proof.** by induction. Will be given later.



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**Proof.** (by contradiction) Suppose that the positive integer  $n$  can be written as a product of primes in two distinct ways:

$$n = p_1 p_2 \cdots p_s \text{ and } n = q_1 q_2 \cdots q_t$$

Remove all common primes from the factorizations to get

$$p_{i_1} p_{i_2} \cdots p_{i_u} = q_{j_1} q_{j_2} \cdots q_{j_v}$$

It then follows that  $p_{i_1}$  divides  $q_{j_k}$  for some  $k$ , **contradicting** the assumption that  $p_{i_1}$  and  $q_{j_k}$  are distinct primes.



# Dividing Congruences by an Integer

- **Theorem** Let  $m$  be a positive integer and let  $a, b, c$  be integers. If  $ac \equiv bc \pmod{m}$  and  $\gcd(c, m) = 1$ , then  $a \equiv b \pmod{m}$ .



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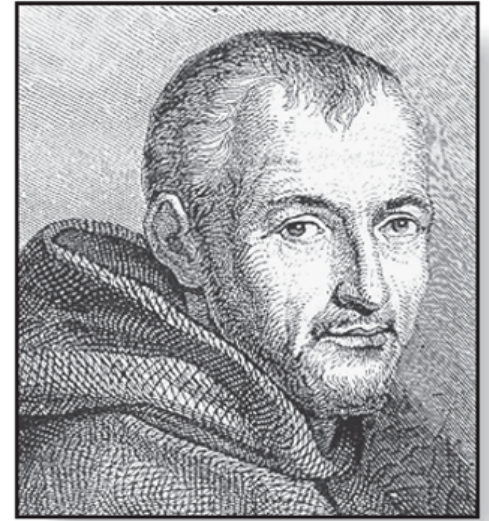
**Proof.** Since  $ac \equiv bc \pmod{m}$ , we have  $m \mid ac - bc = c(a - b)$ . Because  $\gcd(c, m) = 1$ , it follows that  $m \mid a - b$ .





# Mersenne Primes

- Prime numbers of the form  $2^p - 1$ , where  $p$  is a prime.



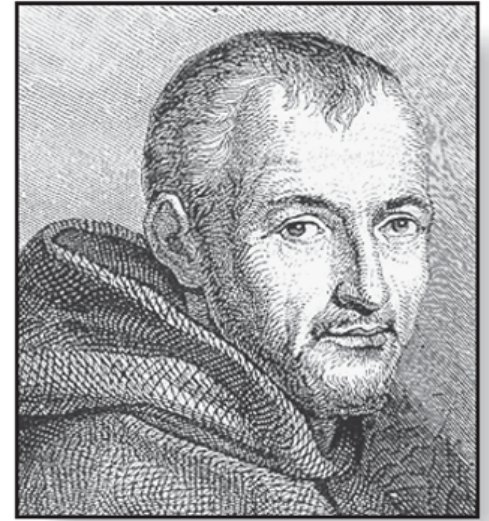
Marin Mersenne

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◇  $2^2 - 1 = 3$ ,  $2^3 - 1 = 7$ ,  $2^5 - 1 = 31$ ,  
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◇ The largest known prime numbers are Mersenne primes.



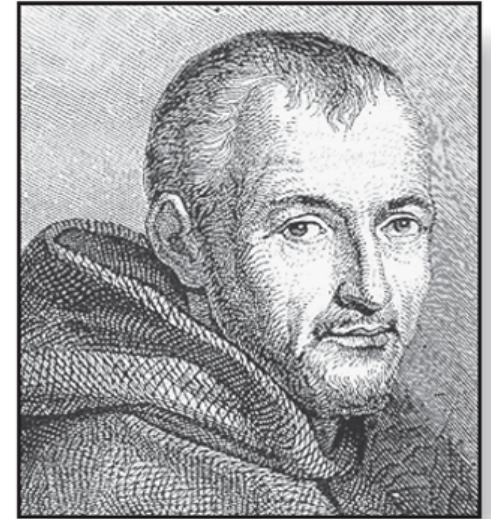
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## Largest Known Prime, 49th Known Mersenne Prime Found!

**January 7, 2016** — GIMPS celebrated its 20th anniversary with the discovery of the largest known prime number,  $2^{74,207,281}-1$ .

## 50th Known Mersenne Prime Found!

**January 3, 2018** — Persistence pays off. Jonathan Pace, a GIMPS volunteer for over 14 years, discovered the 50th known Mersenne prime,  $2^{77,232,917}-1$  on December 26, 2017. The prime number is calculated by multiplying together 77,232,917 twos, and then subtracting one. It weighs in at 23,249,425 digits, becoming the largest prime number known to mankind. It bests the previous record prime, also discovered by GIMPS, by 910,807 digits.

## 51st Known Mersenne Prime Found!

**December 21, 2018** — The Great Internet Mersenne Prime Search (GIMPS) has discovered the largest known prime number,  $2^{82,589,933}-1$ , having 24,862,048 digits. A computer volunteered by Patrick Laroche from Ocala, Florida made the find on December 7, 2018. The new prime number, also known as M82589933, is calculated by multiplying together 82,589,933 twos and then subtracting one. It is more than one and a half million digits larger than the previous record prime number.

27 - 4 <http://www.mersenne.org/>

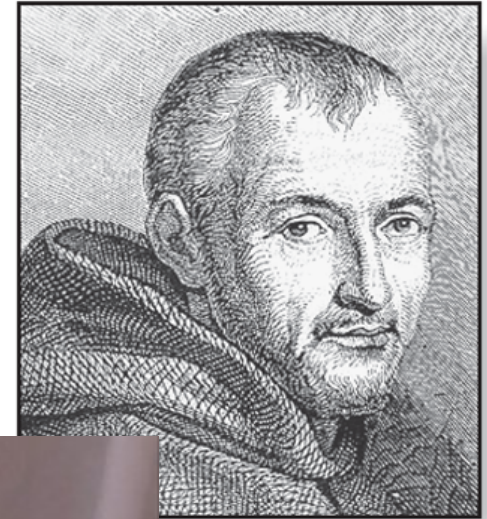




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**Prime Found!**

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- *Twin-prime Conjecture*: There are infinitely many twin primes.





# Linear Congruences

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Systems of linear congruences have been studied since ancient times.

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About 1500 years ago, the Chinese mathematician Sun-Tsu asked: “There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?”



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When does an inverse of  $a$  modulo  $m$  exist?



# Inverse of $a$ modulo $m$

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How to prove the uniqueness of the inverse?



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**Example.** Find an inverse of 101 modulo 4620.

$$4620 = 45 \cdot 101 + 75$$

$$101 = 1 \cdot 75 + 26$$

$$75 = 2 \cdot 26 + 23$$

$$26 = 1 \cdot 23 + 3$$

$$23 = 7 \cdot 3 + 2$$

$$3 = 1 \cdot 2 + 1$$

$$2 = 2 \cdot 1$$



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$$2 = 2 \cdot 1$$

$$1 = 3 - 1 \cdot 2$$

$$1 = 3 - 1 \cdot (23 - 7 \cdot 3) = -1 \cdot 23 + 8 \cdot 3$$

$$1 = -1 \cdot 23 + 8 \cdot (26 - 1 \cdot 23) = 8 \cdot 26 - 9 \cdot 23$$

$$1 = 8 \cdot 26 - 9 \cdot (75 - 2 \cdot 26) = 26 \cdot 26 - 9 \cdot 75$$

$$1 = 26 \cdot (101 - 1 \cdot 75) - 9 \cdot 75$$

$$= 26 \cdot 101 - 35 \cdot 75$$

$$1 = 26 \cdot 101 - 35 \cdot (4620 - 45 \cdot 101)$$

$$= -35 \cdot 4620 + 1601 \cdot 101$$



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**Solution:** We found that  $-2$  is an inverse of  $3$  modulo  $7$ . Multiply both sides of the congruence by  $-2$ , we have  $x \equiv -8 \equiv 6 \pmod{7}$ .





# Number of Solutions to Congruences \*

- **Theorem\*** Let  $d = \gcd(a, m)$  and  $m' = m/d$ . The congruence  $ax \equiv b \pmod{m}$  has solutions if and only if  $d|b$ . If  $d|b$ , then there are exactly  $d$  solutions. If  $x_0$  is a solution, then the other solutions are given by  $x_0 + m', x_0 + 2m', \dots, x_0 + (d - 1)m'$ .

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## Proof.

- 1) “only if”: If  $x_0$  is a solution, then  $ax_0 - b = km$ . Thus,  $ax_0 - km = b$ . Since  $d$  divides  $ax_0 - km$ , we must have  $d|b$ .
- 2) “if”: Suppose that  $d|b$ . Let  $b = kd$ . There exist integers  $s, t$  such that  $d = as + mt$ . Multiply both sides by  $k$ . Then  $b = ask + mtk$ . Let  $x_0 = sk$ . Then  $ax_0 \equiv b \pmod{m}$ .
- 3) “ $\# = d$ ”:  $ax_0 \equiv b \pmod{m}$   $ax_1 \equiv b \pmod{m}$  imply that  $m|a(x_1 - x_0)$  and  $m'|a'(x_1 - x_0)$ . This implies further that  $x_1 = x_0 + km'$ , where  $k = 0, 1, \dots, d-1$ .

# The Chinese Remainder Theorem

- About 1500 years ago, the Chinese mathematician Sun-Tsu asked:

“There are certain things whose number is unknown. When divided by 3, the remainder is 2; when divided by 5, the remainder is 3; when divided by 7, the remainder is 2. What will be the number of things?”

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# The Chinese Remainder Theorem

- **Theorem** (*The Chinese Remainder Theorem*) Let  $m_1, m_2, \dots, m_n$  be pairwise relatively prime positive integers greater than 1 and  $a_1, a_2, \dots, a_n$  arbitrary integers. Then the system

$$x \equiv a_1 \pmod{m_1}$$

$$x \equiv a_2 \pmod{m_2}$$

...

$$x \equiv a_n \pmod{m_n}$$

has a unique solution modulo  $m = m_1 m_2 \cdots m_n$ .

# The Chinese Remainder Theorem

- **Proof** Let  $M_k = m/m_k$  for  $k = 1, 2, \dots, n$  and  $m = m_1 m_2 \cdots m_n$ . Since  $\gcd(m_k, M_k) = 1$ , there is an integer  $y_k$ , an inverse of  $M_k$  modulo  $m_k$  such that  $M_k y_k \equiv 1 \pmod{m_k}$ . Let

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It is checked that  $x$  is a solution to the  $n$  congruences.



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How to prove the **uniqueness** of the solution modulo  $m$ ?



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Let  $m = 3 \cdot 5 \cdot 7 = 105$ ,  $M_1 = m/3 = 35$ ,  $M_2 = m/5 = 21$ ,  
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-- 程大位 《算法统要》 (1593年)

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# Modular Arithmetic in CS

- Modular arithmetic and congruencies are used in CS:
  - ◇ Pseudorandom number generators
  - ◇ Hash functions
  - ◇ Cryptography



# Pseudorandom Number Generators

## ■ *Linear congruential method*

We choose four numbers:

- ◇ the modulus  $m$
- ◇ multiplier  $a$
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We generate a sequence of numbers  $x_1, x_2, \dots, x_n, \dots$  with  $0 \leq x_i < m$  by using the congruence

$$x_{n+1} = (ax_n + c) \pmod{m}$$



# Pseudorandom Number Generators

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### **Example:**

- Assume :  $m=9, a=7, c=4, x_0 = 3$
- $x_1 = 7*3+4 \pmod{9} = 25 \pmod{9} = 7$
- $x_2 = 53 \pmod{9} = 8$
- $x_3 = 60 \pmod{9} = 6$
- $x_4 = 46 \pmod{9} = 1$
- $x_5 = 11 \pmod{9} = 2$
- $x_6 = 18 \pmod{9} = 0$
- ....



# Hash Functions

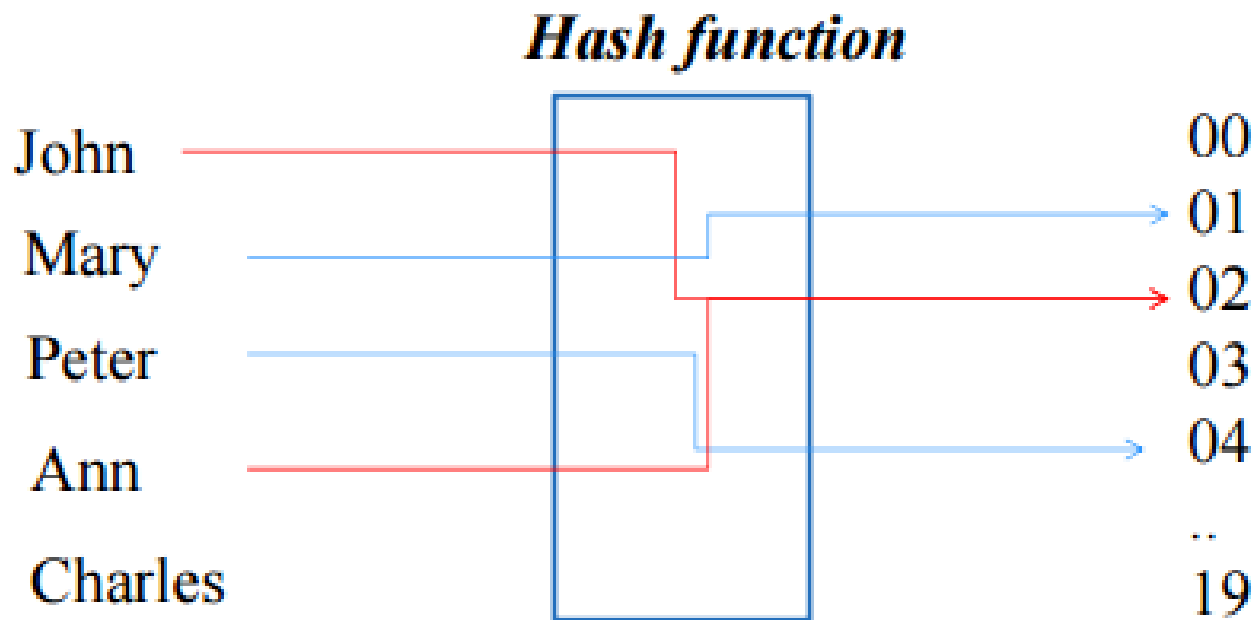
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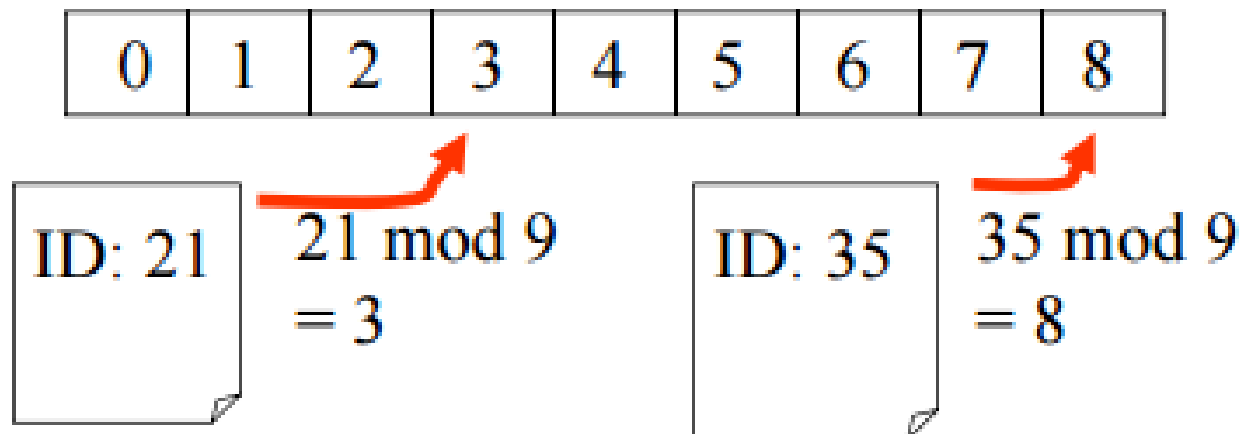
- **Problem:** Given a large collection of records, how can we store and find a record quickly?

**Solution:** Use a hash function, calculate the location of the record based on the record's ID.

**Example:** A common hash function is

- $h(k) = k \bmod n$ ,

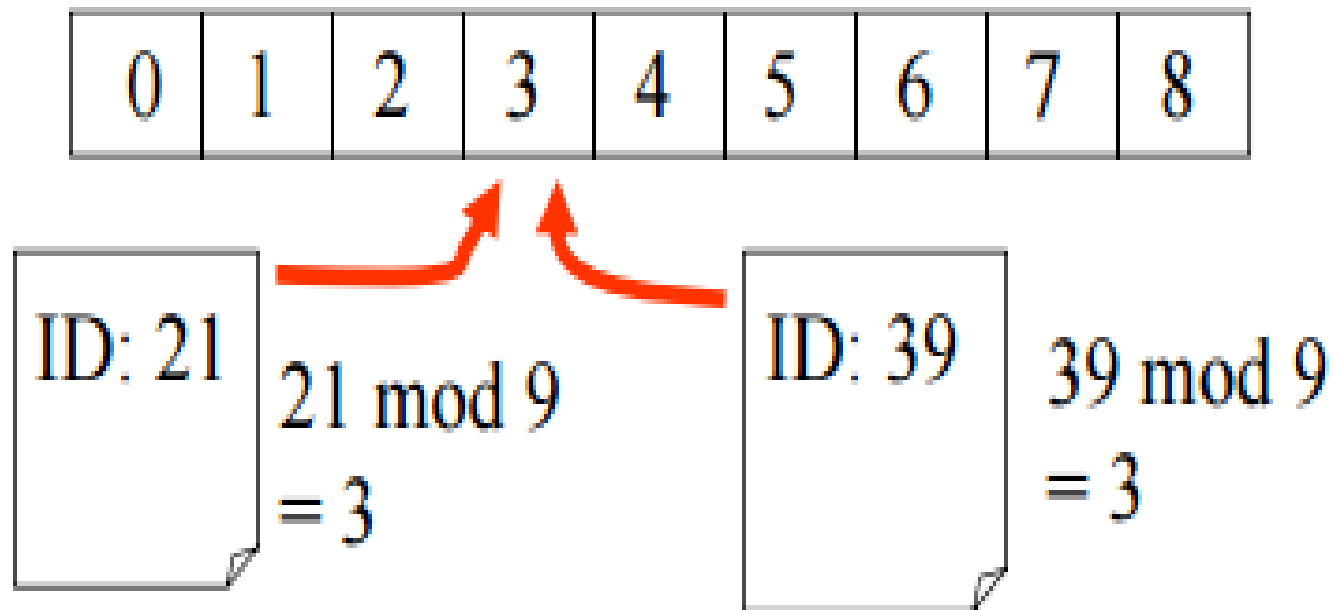
where  $n$  is the number of available storage locations.





# Hash Functions

- Two records mapped to the same location



# Hash Functions

- **Solution 1:** move to the next available location

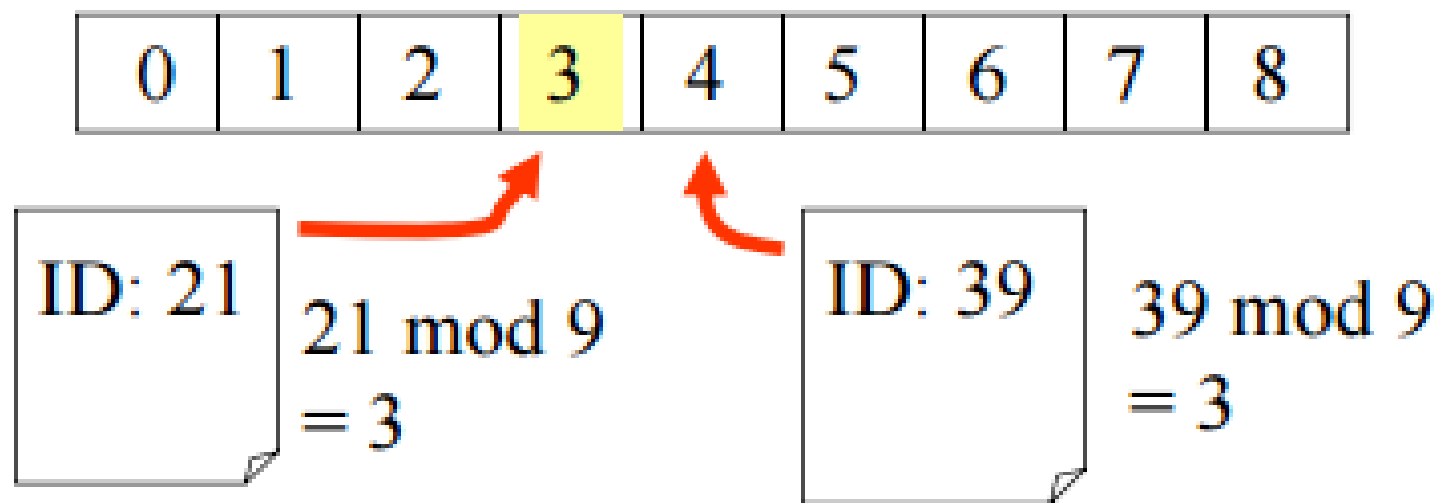
try

$$h_0(k) = k \bmod n$$

$$h_1(k) = (k+1) \bmod n$$

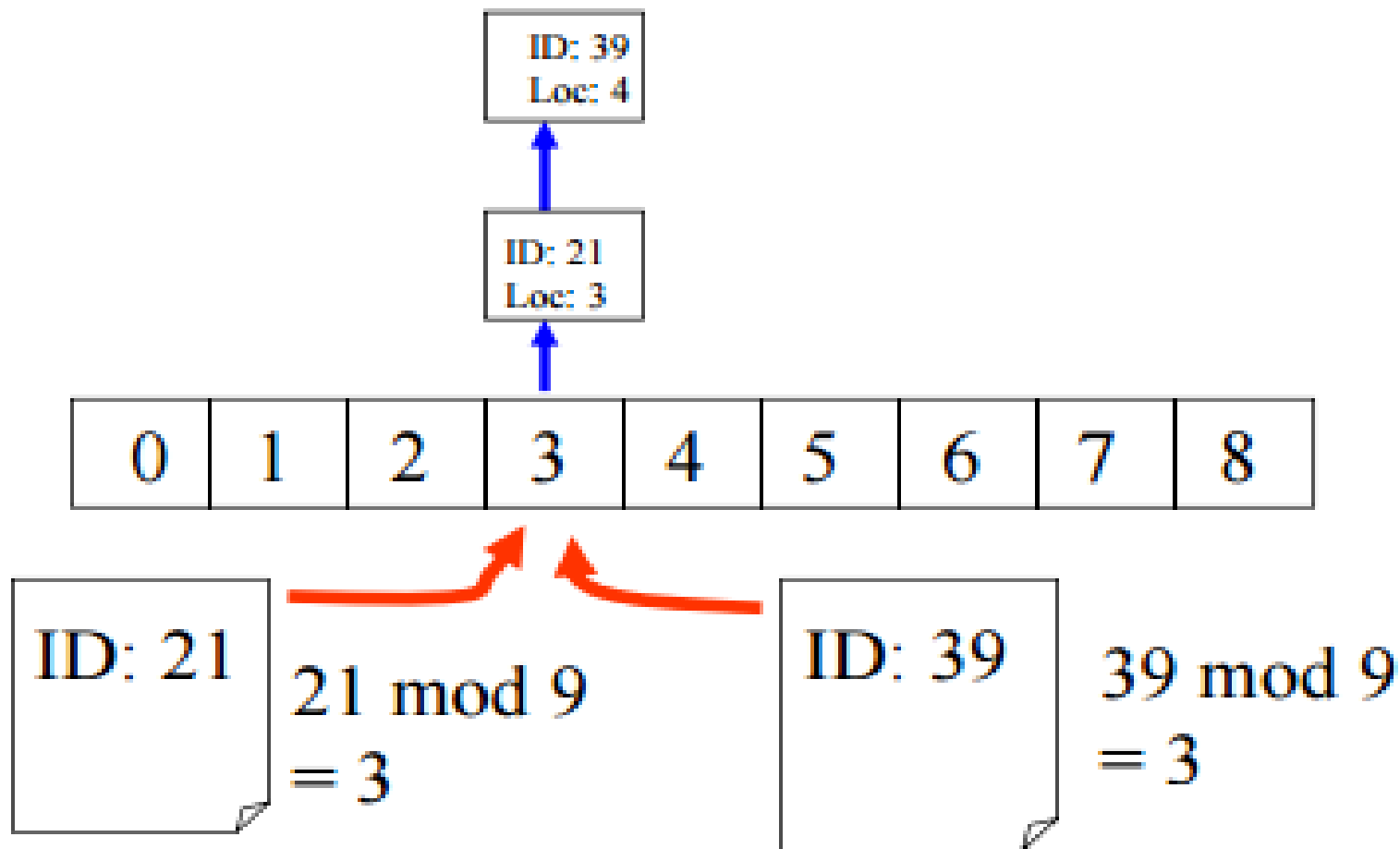
...

$$h_m(k) = (k+m) \bmod n$$



# Hash Functions

- **Solution 2:** remember the exact location in a secondary structure that is searched sequentially



# Next Lecture

- cryptography ...

