Probability and Statistics
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Section 3.5

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P75 Q1

a.

变量 X 与 Y 的边际频率函数为:

X	1	2	3	4
$f_X(x)$	0.19	0.32	0.31	0.18

X

Y	1	2	3	4
$f_Y(y)$	0.19	0.32	0.31	0.18

Y

b.

k	1	2	3	4
$P\{X=k\|Y=1\}$	$\frac{10}{19}$	$\frac{5}{19}$	$\frac{2}{19}$	$\frac{2}{19}$

$P\{X=k|Y=1\}$

k	1	2	3	4
$P\{Y=k\|X=1\}$	$\frac{10}{19}$	$\frac{5}{19}$	$\frac{2}{19}$	$\frac{2}{19}$

$P\{Y=k|X=1\}$

P76 Q9

a.

$$\begin{aligned}
\int_{-1}^{1}\int_{0}^{1-x^2} f(x,y)\mathrm{d}y\mathrm{d}x &= \int_{-1}^{1}\int_{0}^{1-x^2} c\ \mathrm{d}y\mathrm{d}x \\
&= c\int_{-1}^{1}(1-x^2)\mathrm{d}x \\
&= c\left[x-\frac{x^3}{3}\right]_{-1}^{1} \\
&= \frac{4c}{3} \\
&= 1
\end{aligned}$$

所以 $c = \frac{3}{4}$。因此，边际密度函数为：

$$\begin{aligned} f_X(x) &= \int_0^{1-x^2} f(x,y)\mathrm{d}y \\ &= \frac{3}{4}\int_0^{1-x^2} \mathrm{d}y \\ &= \frac{3}{4}(1-x^2) \\ &= \frac{3-3x^2}{4} \\ f_Y(y) &= \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\mathrm{d}x \\ &= \frac{3}{4}\int_{-\sqrt{1-y}}^{\sqrt{1-y}} \mathrm{d}x \\ &= \frac{3}{2}\sqrt{1-y} \end{aligned}$$

$$f_X(x) = \begin{cases} \frac{3-3x^2}{4}, & -1 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3}{2}\sqrt{1-y}, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

b.

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\ &= \frac{\frac{3}{4}}{\frac{3}{2}\sqrt{1-y}} \\ &= \frac{1}{2\sqrt{1-y}} \\ f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\ &= \frac{\frac{3}{4}}{\frac{3-3x^2}{4}} \\ &= \frac{1}{1-x^2} \end{aligned}$$

P76 Q10

a.

边际密度为：

$$
\begin{aligned}
f_X(x) &= \int_0^{\infty} x e^{-x(y+1)} \mathrm{d}y \\
&= x e^{-x} \int_0^{\infty} e^{-xy} \mathrm{d}y \\
&= x e^{-x} \left[-\frac{1}{x} e^{-xy} \right]_0^{\infty} \\
&= e^{-x} \\
f_Y(y) &= \int_0^{\infty} x e^{-x(y+1)} \mathrm{d}x \\
&= \left[-\frac{xy+x+1}{(y+1)^2} e^{-x(y+1)} \right]_0^{\infty} \\
&= \frac{1}{(y+1)^2}
\end{aligned}
$$

$$
f_X(x) = \begin{cases} e^{-x}, & xgeq0 \\ 0, & \text{otherwise} \end{cases}
$$

$$
f_Y(y) = \begin{cases} \frac{1}{(y+1)^2}, & y \geqslant 0 \\ 0, & \text{otherwise} \end{cases}
$$

明显地，X 和 Y 不独立，因为 $f(x,y) = xe^{-x(y+1)} \neq f_X(x) f_Y(y) = \frac{e^{-x}}{(y+1)^2}$。

b.

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{x e^{-x(y+1)}}{\frac{1}{(y+1)^2}} \\
&= x(y+1)^2 e^{-x(y+1)} \\
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{x e^{-x(y+1)}}{e^{-x}} \\
&= x e^{-xy}
\end{aligned}
$$

c.

$$\begin{aligned}
P\{X^2+Y^2 \leqslant \frac{1}{2}\} &= \iint\limits_{x^2+y^2\leqslant\frac{1}{2}} f(x,y)\mathrm{d}x\mathrm{d}y \\
&= \iint\limits_{x^2+y^2\leqslant\frac{1}{2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\mathrm{d}x\mathrm{d}y \\
&= \frac{3}{2\pi}\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-r^2}r\mathrm{d}r\mathrm{d}\theta \\
&= \frac{3}{2\pi}\int_0^{2\pi}\left[-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\right]_0^{\frac{1}{\sqrt{2}}}\mathrm{d}\theta \\
&= \frac{3}{2\pi}\int_0^{2\pi}\frac{1}{3}-\frac{1}{3}\cdot\frac{1}{2\sqrt{2}}\mathrm{d}\theta \\
&= \frac{2\sqrt{2}-1}{2\sqrt{2}}
\end{aligned}$$

d.

$$\begin{aligned}
f_X(x) &= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\mathrm{d}y \\
&= \frac{3}{2\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\mathrm{d}y \\
&= \frac{3}{4}(1-x^2)
\end{aligned}$$

同理可得 $f_Y(y) = \frac{3}{4}(1-y^2)$。

$$f_X(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{3}{4}(1-y^2), & -1 \leqslant y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

e.

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{4}(1-y^2)} \\
&= \frac{2\sqrt{1-x^2-y^2}}{\pi(1-y^2)} \\
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{\frac{3}{2\pi}\sqrt{1-x^2-y^2}}{\frac{3}{4}(1-x^2)} \\
&= \frac{2\sqrt{1-x^2-y^2}}{\pi(1-x^2)}
\end{aligned}
$$

补充 1

(1)

已知 $f_{Y|X}(y|x) = \frac{1}{x}$，则 X 与 Y 的联合分布函数为：

$$
\begin{aligned}
f_{X,Y}(x,y) &= f_{Y|X}(y|x)f_X(x) \\
&= \frac{1}{x} \cdot 1 \\
&= \frac{1}{x}
\end{aligned}
$$

$$
f_{X,Y}(x,y) = \begin{cases} \frac{1}{x}, & 0 < y < x < 1 \\ 0, & \text{otherwise} \end{cases}
$$

(2)

$$
\begin{aligned}
f_Y(y) &= \int_y^1 f(x,y)\mathrm{d}x \\
&= \int_y^1 \frac{1}{x}\mathrm{d}x \\
&= \ln x\bigg|_y^1 \\
&= -\ln y
\end{aligned}
$$

(3)

$$
\begin{aligned}
P\{X+Y>1\} &= P\{Y>1-X\} \\
&= \int_{\frac{1}{2}}^{1}\int_{1-x}^{x} f(x,y)\mathrm{d}y\mathrm{d}x \\
&= \int_{\frac{1}{2}}^{1}\int_{1-x}^{x} \frac{1}{x}\mathrm{d}y\mathrm{d}x \\
&= \int_{\frac{1}{2}}^{1} \frac{1}{x}\left[y\right]_{1-x}^{x}\mathrm{d}x \\
&= \int_{\frac{1}{2}}^{1} \frac{1}{x}(x-1+x)\mathrm{d}x \\
&= \int_{\frac{1}{2}}^{1} 2-\frac{1}{x}\mathrm{d}x \\
&= \left[2x-\ln x\right]_{\frac{1}{2}}^{1} \\
&= 1-\ln 2
\end{aligned}
$$

补充 2

(1)

边缘密度函数为：

$$
\begin{aligned}
f_X(x) &= \int_{x}^{\infty} e^{-y}\mathrm{d}y \\
&= -e^{-y}\Big|_{x}^{\infty} \\
&= e^{-x} \\
f_Y(y) &= \int_{0}^{y} e^{-y}\mathrm{d}x \\
&= ye^{-y}
\end{aligned}
$$

$$
f_X(x) = \begin{cases} e^{-x}, & x>0 \\ 0, & \text{otherwise} \end{cases}
$$

$$
f_Y(y) = \begin{cases} ye^{-y}, & y>0 \\ 0, & \text{otherwise} \end{cases}
$$

易知 X 和 Y 不独立，因为 $f(x,y)=e^{-y}\neq f_X(x)f_Y(y)=ye^{-x}e^{-y}$。

(2)

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f(x,y)}{f_Y(y)} \\
&= \frac{e^{-y}}{ye^{-y}} \\
&= \frac{1}{y} \\
f_{Y|X}(y|x) &= \frac{f(x,y)}{f_X(x)} \\
&= \frac{e^{-y}}{e^{-x}} \\
&= e^{x-y}
\end{aligned}
$$