CS215: Discrete Math (H)
2023 Fall Semester Written Assignment # 5
Due: Dec. 20, 2023, please submit at the beginning of class

Q.1 Let S be the set of all strings of English letters. Determine whether these relations are *reflexive, irreflexive, symmetric, antisymmetric*, and/or *transitive*.

(1) $R_1 = \{(a, b)|a \text{ and } b \text{ have no letters in common}\}$

(2) $R_2 = \{(a, b)|a \text{ and } b \text{ are not the same length}\}$

(3) $R_3 = \{(a, b)|a \text{ is longer than } b\}$

Q.2 Define a relation R on $\mathbb{R}$, the set of real numbers, as follows: For all x and y in $\mathbb{R}$, $(x, y) \in R$ if and only if $x - y$ is rational. Answer the followings, and explain your answers.

(1) Is R reflexive?

(2) Is R symmetric?

(3) Is R antisymmetric?

(4) Is R transitive?

Q.3 How many relations are there on a set with n elements that are

(a) symmetric?

(b) antisymmetric?

(c) irreflexive?

(d) both reflexive and symmetric?

(e) neither reflexive nor irreflexive?

(f) both reflexive and antisymmetric?

(g) symmetric, antisymmetric and transitive?

Q.4 Suppose that the relation R is irreflexive. Is the relation R^2 necessarily irreflexive?

Q.5 Suppose that R_1 and R_2 are both *reflexive* relations on a set A.

(1) Show that $R_1 \oplus R_2$ is *irreflexive*.

(2) Is $R_1 \cap R_2$ also *reflexive*? Explain your answer.

(3) Is $R_1 \cup R_2$ also *reflexive*? Explain your answer.

Q.6 Let R be the relation on the set of ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$.

(a) Show that R is an equivalence relation.

(b) What is the equivalence class of $(1, 2)$ with respect to the equivalence relation R?

(c) Give an interpretation of the equivalence classes for the equivalence relation R.

Q.7 For the relation R on the set $X = \{(a, b, c) : a, b, c \in \mathbb{R}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $(a_1, b_1, c_1) = k(a_2, b_2, c_2)$ for some $k \in \mathbb{R} \setminus \{0\}$.

(1) Prove that this is an *equivalence* relation.

(2) Write at least three elements of the equivalence classes $[(1, 1, 1)]$ and $[(1, 0, 3)]$.

(3) Do all the equivalence classes in this relation have the same cardinality?

Q.8 Let A be a set, let R and S be relations on the set A. Let T be another relation on the set A defined by $(x, y) \in T$ if and only if $(x, y) \in R$ and $(x, y) \in S$. Prove or disprove: If R and S are both *equivalence relations*, then T is also an equivalence relation.

Q.9 Which of these are posets?

(a) $(\mathbf{R}, =)$

(b) $(\mathbf{R}, <)$

(c) $(\mathbf{R}, \leq)$

(d) $(\mathbf{R}, \neq)$

Q.10 Given functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, f is **dominated** by g if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Write $f \preceq g$ if f is dominated by g.

(a) Prove that $\preceq$ is a partial ordering.

(b) Prove or disprove: $\preceq$ is a total ordering.

Q.11 For two positive integers, we write $m \preceq n$ if the sum of the (distinct) prime factors of the first is less than or equal to the product of the (distinct) prime factors of the second. For instance $75 \preceq 14$, because $3 + 5 \leq 2 \cdot 7$.

(a) Is this relation reflexive? Explain.

(b) Is this relation antisymmetric? Explain.

(c) Is this relation transitive? Explain.

Q.12 The relation R on the set $X = \{(a, b, c) : \ a, b, c \in \mathbb{N}\}$ with $(a_1, b_1, c_1)R(a_2, b_2, c_2)$ if and only if $2^{a_1}3^{b_1}5^{c_1} \leq 2^{a_2}3^{b_2}5^{c_2}$.

(1) Prove that R is a partial ordering.

(2) Write two comparable and two incomparable elements if they exist.

(3) Find the least upper bound and the greatest lower bound of the two elements $(5, 0, 1)$ and $(1, 1, 2)$.

(4) List a minimal and a maximal element if they exist.

Q.13 Define the relation $\preceq$ on $\mathbb{Z} \times \mathbb{Z}$ according to

$$(a, b) \preceq (c, d) \Leftrightarrow (a, b) = (c, d) \text{ or } a^2 + b^2 < c^2 + d^2.$$

Show that $(\mathbb{Z}\times\mathbb{Z}, \preceq)$ is a poset; Construct the Hasse diagram for the subposet $(B, \preceq)$, where $B = \{0, 1, 2\} \times \{0, 1, 2\}$.

Q.14 Answer these questions for the partial order represented by this Hasse diagram.

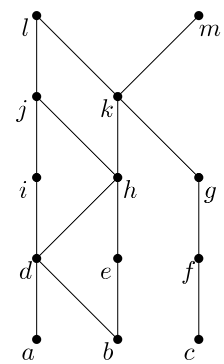

Figure 1: Q.14

(a) Find the maximal elements.

(b) Find the minimal elements.

(c) Is there a greatest element?

(d) Is there a least element?

(e) Find all upper bounds of $\{a, b, c\}$.

(f) Find the least upper bound of $\{a, b, c\}$, if it exists.

(g) Find all lower bounds of $\{f, g, h\}$.

(h) Find the greatest lower bound of $\{f, g, h\}$, if it exists.