Calculations for the second-order zonal harmonic

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2016-10-26

Notations:

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, |\boldsymbol{j}| = 1, r = |\boldsymbol{r}|.$$

For oblateness along the z-axis, the potential is

$$\frac{J_2}{2r^5}(3z^2 - r^2).$$

For oblateness along $\boldsymbol{j}$,

$$U(\boldsymbol{r}) = \frac{J_2}{2r^5}(3(\boldsymbol{r}\cdot\boldsymbol{j})^2 - r^2).$$

Differentiating,

$$\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} = \frac{J_2}{2}\left(-\frac{5}{r^6}\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}}(3(\boldsymbol{r}\cdot\boldsymbol{j})^2 - r^2) + \frac{1}{r^5}\left(3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{j})^2 - 2r\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}}\right)\right).$$

Recall that $\frac{\mathrm{d}\,r}{\mathrm{d}\,\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$,

$$\begin{aligned}\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} &= \frac{J_2}{2}\left(-\frac{5\boldsymbol{r}}{r^7}(3(\boldsymbol{r}\cdot\boldsymbol{j})^2 - r^2) + \frac{1}{r^5}\left(3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{j})^2 - 2\boldsymbol{r}\right)\right)\\ &= \frac{J_2}{2}\left(-\frac{15\boldsymbol{r}}{r^7}(\boldsymbol{r}\cdot\boldsymbol{j})^2 + \frac{3\boldsymbol{r}}{r^5} + \frac{3}{r^5}3\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{j})^2\right).\end{aligned}$$

With $\frac{\mathrm{d}}{\mathrm{d}\,\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{j})^2 = 2(\boldsymbol{r}\cdot\boldsymbol{j})\frac{\mathrm{d}\,(\boldsymbol{r}\cdot\boldsymbol{j})}{\mathrm{d}\,\boldsymbol{r}} = 2(\boldsymbol{r}\cdot\boldsymbol{j})\boldsymbol{j}$,

$$\begin{aligned}\frac{\mathrm{d}\,U}{\mathrm{d}\,\boldsymbol{r}} &= \frac{J_2}{2}\left(-\frac{15\boldsymbol{r}}{r^7}(\boldsymbol{r}\cdot\boldsymbol{j})^2 + \frac{3\boldsymbol{r}}{r^5} + \frac{6\boldsymbol{j}}{r^5}(\boldsymbol{r}\cdot\boldsymbol{j})\right)\\ &= \frac{3J_2}{2r^5}\left(2\boldsymbol{j}(\boldsymbol{r}\cdot\boldsymbol{j}) + \boldsymbol{r}\left(1 - \frac{5(\boldsymbol{r}\cdot\boldsymbol{j})^2}{r^2}\right)\right).\end{aligned}$$

Note that this is invariant under $\boldsymbol{j} \mapsto -\boldsymbol{j}$ (but not under $\boldsymbol{r} \mapsto -\boldsymbol{r}$).

Implementation:

```
// If j is a unit vector along the axis of rotation, and r a vector from the
// center of |body| to some point in space, the acceleration computed here is:
//
//   -(J₂ / (μ ‖r‖⁵)) (3 j (r.j) + r (3 - 15 (r.j)² / ‖r‖²) / 2)
//
// Where ‖r‖ is the norm of r and r.j is the inner product.  It is the
// additional acceleration exerted by the oblateness of |body| on a point at
// position r.  J₂, J̃₂ and J̄₂ are normally positive and C̃₂₀ and C̄₂₀ negative
// because the planets are oblate, not prolate.  Note that this follows IERS
// Technical Note 36 and it differs from
// https://en.wikipedia.org/wiki/Geopotential_model which seems to want J̃₂ to be
// negative.
template<typename Frame>
FORCE_INLINE Vector<Quotient<Acceleration, GravitationalParameter>, Frame>
Order2ZonalAcceleration(OblateBody<Frame> const& body,
                        Displacement<Frame> const& r,
                        Exponentiation<Length, -2> const& one_over_r_squared,
                        Exponentiation<Length, -3> const& one_over_r_cubed) {
  Vector<double, Frame> const& axis = body.polar_axis();
  Length const r_axis_projection = InnerProduct(axis, r);
  auto const j2_over_r_fifth =
      body.j2_over_μ() * one_over_r_cubed * one_over_r_squared;
  Vector<Quotient<Acceleration,
                  GravitationalParameter>, Frame> const axis_effect =
      (-3 * j2_over_r_fifth * r_axis_projection) * axis;
  Vector<Quotient<Acceleration,
                  GravitationalParameter>, Frame> const radial_effect =
      (j2_over_r_fifth *
           (-1.5 +
            7.5 * r_axis_projection *
                  r_axis_projection * one_over_r_squared)) * r;
  return axis_effect + radial_effect;
}
```