Documentation for the hiding computations in Planetarium

Pascal Leroy (pleroy)

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A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; K is the location of the camera.

We start our analysis by eliminating a case that would cause anomalies in the computations below. If:

$$\overrightarrow{\mathrm{KC}} \cdot \overrightarrow{\mathrm{KC}} < R^2$$

then the camera is inside the sphere and the segment is hidden irrespective of its position.

Next, consider the plane that contains K and is orthogonal to $\overrightarrow{\mathrm{KC}}$; it separates the entire space into two half-spaces. If the segment $\overline{\mathrm{AB}}$ is entirely within the half-space that does not contain C then the segment is not hidden. This is the case if the following inequalities are both true:

$$\begin{aligned} \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}} &< 0 \\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}} &< 0. \end{aligned}$$

For simplicity we will do the rest of our analysis in the plane KAB. We will use $(\overrightarrow{\mathrm{KA}}, \overrightarrow{\mathrm{KB}})$ as a basis of that plane. Let H be the orthogonal projection of C on KAB. Define α, β to be its coordinates in KAB:

$$\overrightarrow{\mathrm{KH}} = \alpha \overrightarrow{\mathrm{KA}} + \beta \overrightarrow{\mathrm{KB}}.$$

Note that $\overrightarrow{\mathrm{KH}} = \overrightarrow{\mathrm{KC}} + \overrightarrow{\mathrm{CH}}$. By its definition, $\overrightarrow{\mathrm{CH}}$ is orthogonal to both $\overrightarrow{\mathrm{KA}}$ and $\overrightarrow{\mathrm{KB}}$:

$$\begin{aligned} \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{CH}} &= 0 \\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{CH}} &= 0, \end{aligned}$$

or:

$$\begin{aligned} \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}} \\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}. \end{aligned}$$

Expanding $\overrightarrow{\mathrm{KH}}$ gives a linear system of two equations with two unknowns:

$$\begin{aligned} \alpha \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}} + \beta \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}} \\ \alpha \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} + \alpha \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}. \end{aligned}$$

The determinant of this system is:

$$D = (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})^2,$$

which is non-zero if and only if $\mathrm{A} \neq \mathrm{B}$. The solutions are thus

$$\begin{aligned} \alpha &= \frac{(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}})}{D} \\ \beta &= \frac{(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}})}{D}. \end{aligned}$$

Now that $\overrightarrow{\mathrm{KH}}$ is determined we can compute $\overrightarrow{\mathrm{CH}} = \overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{KC}}$ and $\overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}}$. If $\overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}} \geq R^2$, the sphere is either tangent to the plane KAB or doesn't intersect it. Thus, there is no hiding.

Let's look at a figure in the plane KAB when the sphere intersects that plane. Let r be the radius of the intersection circle; it is such that $r^2 = R^2 - \overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}}$.

If the circle doesn't intersect the wedge KAB then the segment $\overrightarrow{\mathrm{AB}}$. To find if this is the case, first observe that, if θ is the angle between $\overrightarrow{\mathrm{KA}}$ and $\overrightarrow{\mathrm{KB}}$, we have:

$$\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} = KAKBcos\theta$$

Assume that H is outside the wedge KAB and in a position where the circle is tangent to $\overrightarrow{\mathrm{KB}}$ at point N. Let M be the point where H projects on $\overrightarrow{\mathrm{KB}}$ parallel to $\overrightarrow{\mathrm{KA}}$. We have:

$$\overrightarrow{\mathrm{HM}} = \alpha\overrightarrow{\mathrm{KA}}$$

and:

$$\overrightarrow{\mathrm{HM}} \cdot \overrightarrow{\mathrm{HM}} = \frac{r^2}{sin^2\theta} \qquad = \frac{r^2}{1 - cos^2\theta}$$

Eliminating $\overrightarrow{\mathrm{HM}}$ we obtain the following conditions on α for H to be in the position indicated above:

$$\alpha < 0$$
$$\alpha^2 > r^2 \frac{\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}}}{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}}\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}} - \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}}^2}.$$

We have similar conditions on β for the circle to be outside the wedge KAB and tangent to $\overrightarrow{\mathrm{KA}}$.

The next step is to find out the extension of the wedge formed by the circle when seen from K. Let P be a point where a line going through K is tangent to the circle, and let Q be the point where KP intersects AB; it may be between K and P or behind P.

We need to find P. It is defined by two equations:

$$\overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{PH}} = r^2$$
$$\overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{KP}} = 0.$$

The second equation may be rewritten:

$$\overrightarrow{\mathrm{PH}} \cdot (\overrightarrow{\mathrm{KH}} + \overrightarrow{\mathrm{HP}}) = 0$$
$$\overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{KH}} = r^2.$$

Let γ and δ be the coordinates of $\overrightarrow{\mathrm{PH}}$ in KAB:

$$\overrightarrow{\mathrm{PH}} = \gamma\overrightarrow{\mathrm{KA}} + \delta\overrightarrow{\mathrm{KB}}.$$

The second equation is linear:

$$\gamma\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}} + \delta\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}} = r^2.$$

Thus:

$$\gamma = \frac{r^2 - \delta\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}}.$$

Note that $\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}$ and $\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}$ cannot both be 0 unless K is on AB.

The first equation is quadratic:

$$(\gamma\overrightarrow{\mathrm{KA}} + \delta\overrightarrow{\mathrm{KB}})^2 = r^2.$$

Pluging the value of γ above we get:

$$\begin{aligned}\delta^2((\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})^2 + 2(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}) + \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}}(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}})^2)+ \\ 2\delta r^2((\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}))+ \\ r^2(r^2(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})^2) = 0.\end{aligned}$$

This equation always has two solutions because the sphere intersects KAB. Having determined $\overrightarrow{\mathrm{PH}}$, we can find Q. Q is on the line AB, thus:

$$\overrightarrow{\mathrm{AQ}} = \lambda\overrightarrow{\mathrm{AB}},$$

which can be written

$$\overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KA}} = \lambda\overrightarrow{\mathrm{AB}}.$$

Noting that $\overrightarrow{\mathrm{KQ}}$ is orthogonal to $\overrightarrow{\mathrm{PH}}$ we have

$$-\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{PH}} = \lambda \overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{PH}}, \text{ or, } \lambda = -\frac{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{PH}}}{\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{PH}}}.$$

Having determined the values of λ we need to find out where Q is located with respect to the segment $\overrightarrow{\mathrm{AB}}$. Let S be the intersection of AB with the line orthogonal to KH at K. We locate S on AB as follows:

$$\overrightarrow{\mathrm{KS}} = \overrightarrow{\mathrm{KA}} + \sigma \overrightarrow{\mathrm{AB}}.$$

Noting that $\overrightarrow{\mathrm{KS}} \cdot \overrightarrow{\mathrm{KH}} = 0$ we obtain:

$$\sigma = -\frac{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{KH}}}$$

Now let T be the intersection of AB with the line orthogonal to KH at P. We locate T on AB similarly:

$$\overrightarrow{\mathrm{KT}} = \overrightarrow{\mathrm{KA}} + \tau \overrightarrow{\mathrm{AB}}.$$

We have:

$$\overrightarrow{\mathrm{PT}} = \overrightarrow{\mathrm{KA}} + \tau \overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{KH}} + \overrightarrow{\mathrm{PH}}$$

Noting that $\overrightarrow{\mathrm{PT}} \cdot \overrightarrow{\mathrm{KH}} = 0$ we obtain:

$$\tau = \frac{\overrightarrow{\mathrm{KH}} \cdot \overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{KH}}}$$

We can now determine if λ is an "interesting" intersection, i.e., one that intersects the cone behind the sphere when seen from the camera. First, assume that A and B are in the same order as S and T on AB. Then we have $\sigma <= \tau$ and the intersection is farther than T (as seen from the camera) if and only if $\tau < \lambda$. Conversely, if A and B are in the reverse order as S and T on AB we have $\tau <= \sigma$ and intersection is farther than T if and only if $\lambda < \tau$

There is another special case to handle: if the line AB is in "hyperbolic" position, i.e. intersects both halves of the cone, then one value of λ is smaller than σ and one value is greater than σ. Exactly one of the values of λ will be retained by the preceding analysis, and we need to add an extra λ equal to an infinity with the sign of $\tau - gs$ to account for the fact that all the points farther than T are hidden by the cone.

To complete the analysis we need to compute the intersection of the sphere (not the cone) with the line AB. Q is on the sphere, thus

$$\overrightarrow{\mathrm{CQ}} \cdot \overrightarrow{\mathrm{CQ}} = R^2.$$

It is also on the line AB thus

$$\overrightarrow{\mathrm{KQ}} = \overrightarrow{\mathrm{KA}} + \mu \overrightarrow{\mathrm{AB}}.$$

We have:

$$\overrightarrow{\mathrm{CQ}} = \overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{KA}} + \mu \overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{CA}} + \mu \overrightarrow{\mathrm{AB}},$$

and therefore

$$R^2 = (\overrightarrow{\mathrm{CA}} + \mu \overrightarrow{\mathrm{AB}})^2,$$

meaning that μ is a solution of

$$\mu \overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AB}} + 2\mu \overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{CA}} - R^2 = 0.$$

Depending on the location of the segment with respect to the sphere, there can be 0, 1, or 2 intersections.

If we take the union of the values of λ and μ and order them, it is straightforward to find the visible segments. Remember that $0 < \lambda, \mu < 1$ for points that are in the segment AB.