Documentation for the hiding computations in Planetarium

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Notation: A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; K is the location of the camera.

We start our analysis by eliminating a case that would cause anomalies in the computations below. If

$$\overrightarrow{\mathrm{KC}} \cdot \overrightarrow{\mathrm{KC}} < R^2$$

then the camera is inside the sphere and the segment is hidden irrespective of its position.

Next, consider the plane that contains K and is orthogonal to $\overrightarrow{\mathrm{KC}}$; it separates the entire space into two half-spaces. If the segment $\overline{\mathrm{AB}}$ is entirely within the half-space that does not contain C then the segment is not hidden. This is the case if the following inequalities are both true

$$\begin{aligned}\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}} &< 0\\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}} &< 0.\end{aligned}$$

For simplicity we will do the rest of our analysis in the plane KAB. We will use $(\overrightarrow{\mathrm{KA}}, \overrightarrow{\mathrm{KB}})$ as a basis of that plane. Let H be the orthogonal projection of C on KAB. Define α, β to be its coordinates in KAB

$$\overrightarrow{\mathrm{KH}} = \alpha\overrightarrow{\mathrm{KA}} + \beta\overrightarrow{\mathrm{KB}}.$$

Note that $\overrightarrow{\mathrm{KH}} = \overrightarrow{\mathrm{KC}} + \overrightarrow{\mathrm{CH}}$. By its definition, $\overrightarrow{\mathrm{CH}}$ is orthogonal to both $\overrightarrow{\mathrm{KA}}$ and $\overrightarrow{\mathrm{KB}}$

$$\begin{aligned}\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{CH}} &= 0\\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{CH}} &= 0,\end{aligned}$$

or

$$\begin{aligned}\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}\\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}.\end{aligned}$$

Expanding $\overrightarrow{\mathrm{KH}}$ gives a linear system of two equations with two unknowns

$$\begin{aligned}\alpha\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}} + \beta\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}\\ \alpha\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} + \alpha\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}.\end{aligned}$$

The determinant of this system is

$$D = (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})^2,$$

which is non-zero if and only if $\mathrm{A} \neq \mathrm{B}$. The solutions are thus

$$\begin{aligned}\alpha &= \frac{(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}})}{D}\\ \beta &= \frac{(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}})}{D}.\end{aligned}$$

Now that $\overrightarrow{KH}$ is determined we can compute $\overrightarrow{CH} = \overrightarrow{KH} - \overrightarrow{KC}$. If $\overrightarrow{CH} \cdot \overrightarrow{CH} \geq R^2$, the sphere is either tangent to the plane KAB or doesn't intersect it. Thus, there is no hiding.

Let's look at a figure in the plane KAB when the sphere intersects that plane. Let r be the radius of the intersection circle; it is such that $r^2 = R^2 - \overrightarrow{CH} \cdot \overrightarrow{CH}$.

If the circle doesn't intersect the wedge KAB then the segment AB is entirely visible. To find if this is the case, first observe that, if θ is the angle between $\overrightarrow{KA}$ and $\overrightarrow{KB}$, we have

$$\overrightarrow{KA} \cdot \overrightarrow{KB} = |\overrightarrow{KA}||\overrightarrow{KB}| \cos\theta$$

Assume that H is outside the wedge KAB and in a position where the circle is tangent to KB at point N. Let M be the point where H projects on $\overrightarrow{KB}$ parallel to $\overrightarrow{KA}$. We have

$$\overrightarrow{HM} = \alpha\overrightarrow{KA}$$

and

$$\overrightarrow{HM} \cdot \overrightarrow{HM} = \frac{r^2}{\sin^2\theta} = \frac{r^2}{1 - \cos^2\theta}$$

Eliminating $\overrightarrow{HM}$ we obtain the following conditions on α for H to be in the position indicated above

$$\alpha < 0$$

$$\alpha^2 > r^2 \frac{\overrightarrow{KB} \cdot \overrightarrow{KB}}{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2}.$$

We have similar conditions on β for the circle to be outside the wedge KAB and tangent to $\overrightarrow{KA}$.

The next step is to find out the extension of the wedge formed by the circle when seen from K. Let P be a point where a line going through K is tangent to the circle, and let Q be the point where KP intersects AB; it may be between K and P or behind P.

We need to find P. It is defined by two equations

$$\overrightarrow{PH} \cdot \overrightarrow{PH} = r^2$$

$$\overrightarrow{PH} \cdot \overrightarrow{KP} = 0.$$

The second equation may be rewritten

$$\overrightarrow{PH} \cdot (\overrightarrow{KH} + \overrightarrow{HP}) = 0$$

from which we obtain

$$\overrightarrow{PH} \cdot \overrightarrow{KH} = r^2.$$

Let γ and δ be the coordinates of $\overrightarrow{PH}$ in KAB

$$\overrightarrow{PH} = \gamma\overrightarrow{KA} + \delta\overrightarrow{KB}.$$

The second equation is linear

$$\gamma\overrightarrow{KA} \cdot \overrightarrow{KH} + \delta\overrightarrow{KB} \cdot \overrightarrow{KH} = r^2,$$

thus

$$\gamma = \frac{r^2 - \delta\overrightarrow{KB} \cdot \overrightarrow{KH}}{\overrightarrow{KA} \cdot \overrightarrow{KH}}.$$

Note that $\overrightarrow{KA} \cdot \overrightarrow{KH}$ and $\overrightarrow{KB} \cdot \overrightarrow{KH}$ cannot both be 0 unless K is on AB. The first equation is quadratic

$$(\gamma\overrightarrow{KA} + \delta\overrightarrow{KB})^2 = r^2.$$

Pluging the value of γ above we get

$$\begin{aligned}\delta^2((\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})^2 + 2(\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})(\overrightarrow{KB} \cdot \overrightarrow{KH}) + \overrightarrow{KA} \cdot \overrightarrow{KA}(\overrightarrow{KB} \cdot \overrightarrow{KH})^2)+ \\ 2\delta r^2((\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH}) - (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KH}))+ \\ r^2(r^2(\overrightarrow{KA} \cdot \overrightarrow{KA}) - (\overrightarrow{KA} \cdot \overrightarrow{KH})^2) = 0.\end{aligned}$$

This equation always has two solutions because the sphere intersects KAB. Having determined $\overrightarrow{\mathrm{PH}}$, we can find Q. Q is on the line AB, thus

$$\overrightarrow{\mathrm{AQ}} = \lambda\overrightarrow{\mathrm{AB}},$$

which can be written

$$\overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KA}} = \lambda\overrightarrow{\mathrm{AB}}.$$

Noting that $\overrightarrow{\mathrm{KQ}}$ is orthogonal to $\overrightarrow{\mathrm{PH}}$ we have

$$-\overrightarrow{\mathrm{KA}}\cdot\overrightarrow{\mathrm{PH}} = \lambda\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{PH}}, \text{ or, } \lambda = -\frac{\overrightarrow{\mathrm{KA}}\cdot\overrightarrow{\mathrm{PH}}}{\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{PH}}}.$$

Having determined the values of λ we need to find out where Q is located with respect to the segment $\overrightarrow{\mathrm{AB}}$. Let S be the intersection of AB with the line orthogonal to KH at K. We locate S on AB as follows

$$\overrightarrow{\mathrm{KS}} = \overrightarrow{\mathrm{KA}} + \sigma\overrightarrow{\mathrm{AB}}.$$

Noting that $\overrightarrow{\mathrm{KS}}\cdot\overrightarrow{\mathrm{KH}} = 0$ we obtain

$$\sigma = -\frac{\overrightarrow{\mathrm{KA}}\cdot\overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{KH}}}$$

Now let T be the intersection of AB with the line orthogonal to KH at P. We locate T on AB similarly

$$\overrightarrow{\mathrm{KT}} = \overrightarrow{\mathrm{KA}} + \tau\overrightarrow{\mathrm{AB}}.$$

We have

$$\overrightarrow{\mathrm{PT}} = \overrightarrow{\mathrm{KA}} + \tau\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{KH}} + \overrightarrow{\mathrm{PH}}$$

Noting that $\overrightarrow{\mathrm{PT}}\cdot\overrightarrow{\mathrm{KH}} = 0$ we obtain

$$\tau = \frac{\overrightarrow{\mathrm{KH}}\cdot\overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{PH}}\cdot\overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{KA}}\cdot\overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{KH}}}$$

We can now determine if λ is an "interesting" intersection, i.e., one that intersects the cone behind the sphere when seen from the camera. First, assume that A and B are in the same order as S and T on AB. Then we have $\sigma \le \tau$ and the intersection is farther than T (as seen from the camera) if and only if $\tau < \lambda$. Conversely, if A and B are in the reverse order as S and T on AB we have $\tau \le \sigma$ and the intersection is farther than T if and only if $\lambda < \tau$

There is another special case to handle: if the line AB is in "hyperbolic" position, i.e. intersects both halves of the cone, then one value of λ is smaller than σ and one value is greater than σ. Exactly one of the values of λ will be retained by the preceding analysis, and we need to add an extra λ equal to an infinity with the sign of $\tau - \sigma$ to account for the fact that all the points farther than T are hidden by the cone.

To complete the analysis we need to compute the intersection Q of the sphere (not the cone) with the line AB. Q is on the sphere, thus

$$\overrightarrow{\mathrm{CQ}}\cdot\overrightarrow{\mathrm{CQ}} = R^2.$$

It is also on the line AB thus

$$\overrightarrow{\mathrm{KQ}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{AB}}.$$

We have

$$\overrightarrow{\mathrm{CQ}} = \overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{AB}} - \overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{CA}} + \mu\overrightarrow{\mathrm{AB}},$$

and therefore

$$R^2 = (\overrightarrow{\mathrm{CA}} + \mu\overrightarrow{\mathrm{AB}})^2,$$

meaning that μ is a solution of

$$\mu\overrightarrow{\mathrm{AB}}\cdot\overrightarrow{\mathrm{AB}} + 2\mu\overrightarrow{\mathrm{CA}}\cdot\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{CA}}\cdot\overrightarrow{\mathrm{CA}} - R^2 = 0.$$

Depending on the location of the segment with respect to the sphere, there can be 0, 1, or 2 intersections.

If we take the union of the values of λ and μ and order them, it is straightforward to find the visible segments. Remember that $0 < \lambda, \mu < 1$ for points that are in the segment AB.

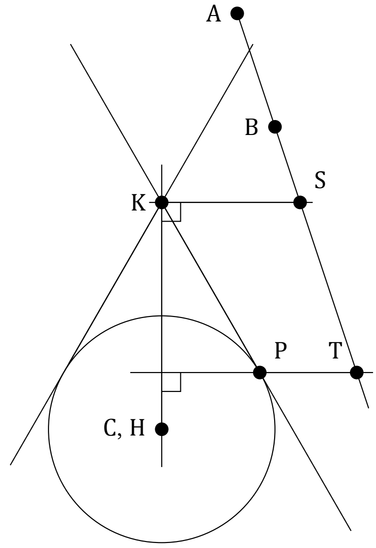

Figure 1. Definition of S and T.

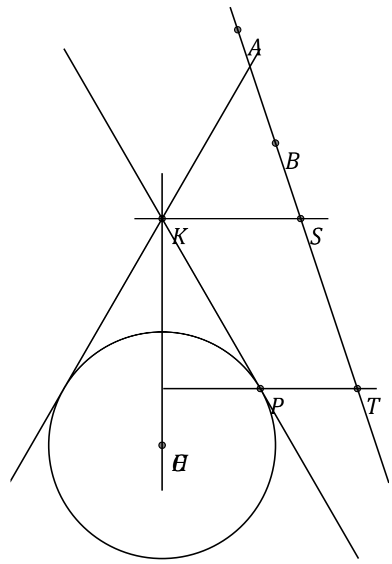

Figure 2. Definition2 of S and T.