

# Hiding Computations in Projection

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Notation: A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; K is the location of the camera.

We start our analysis by eliminating a case that would cause anomalies in the computations below. If

$$\overrightarrow{KC} \cdot \overrightarrow{KC} < R^2$$

then the camera is inside the sphere and the segment is hidden irrespective of its position.

Next, consider the plane that contains K and is orthogonal to  $\overrightarrow{KC}$ ; it separates the entire space into two half-spaces. If the segment  $\overline{AB}$  is entirely within the half-space that does not contain C then the segment is not hidden. This is the case if the following inequalities are both true

$$\overrightarrow{KA} \cdot \overrightarrow{KC} < 0$$

$$\overrightarrow{KB} \cdot \overrightarrow{KC} < 0.$$

For simplicity we will do the rest of our analysis in the plane KAB. We will use  $(\overrightarrow{KA}, \overrightarrow{KB})$  as a basis of that plane. Let H be the orthogonal projection of C on KAB. Define  $\alpha, \beta$  to be its coordinates in KAB

$$\overrightarrow{KH} = \alpha \overrightarrow{KA} + \beta \overrightarrow{KB}.$$

Note that  $\overrightarrow{KH} = \overrightarrow{KC} + \overrightarrow{CH}$ . By its definition,  $\overrightarrow{CH}$  is orthogonal to both  $\overrightarrow{KA}$  and  $\overrightarrow{KB}$

$$\overrightarrow{KA} \cdot \overrightarrow{CH} = 0$$

$$\overrightarrow{KB} \cdot \overrightarrow{CH} = 0,$$

or

$$\overrightarrow{KA} \cdot \overrightarrow{KH} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$

$$\overrightarrow{KB} \cdot \overrightarrow{KH} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

Expanding  $\overrightarrow{KH}$  gives a linear system of two equations with two unknowns

$$\alpha \overrightarrow{KA} \cdot \overrightarrow{KA} + \beta \overrightarrow{KA} \cdot \overrightarrow{KB} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$

$$\alpha \overrightarrow{KA} \cdot \overrightarrow{KB} + \beta \overrightarrow{KB} \cdot \overrightarrow{KB} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

The determinant of this system is

$$D = (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2,$$

which is non-zero if and only if A  $\neq$  B. The solutions are thus

$$\alpha = \frac{(\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KB} \cdot \overrightarrow{KC})}{D}$$
$$\beta = \frac{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC})}{D}.$$

Now that  $\overrightarrow{KH}$  is determined we can compute  $\overrightarrow{CH} = \overrightarrow{KH} - \overrightarrow{KC}$ . If  $\overrightarrow{CH} \cdot \overrightarrow{CH} \geq R^2$ , the sphere is either tangent to the plane KAB or doesn't intersect it. Thus, there is no hiding.

Let's look at a figure in the plane KAB when the sphere intersects that plane. Let  $r$  be the radius of the intersection circle; it is such that  $r^2 = R^2 - \overrightarrow{CH} \cdot \overrightarrow{CH}$ .

If the circle doesn't intersect the wedge KAB then the segment AB is entirely visible. To find if this is the case, first observe that, if  $\theta$  is the angle between  $\overrightarrow{KA}$  and  $\overrightarrow{KB}$ , we have

$$\overrightarrow{KA} \cdot \overrightarrow{KB} = |\overrightarrow{KA}| |\overrightarrow{KB}| \cos \theta$$

Assume that H is outside the wedge KAB and in a position where the circle is tangent to KB at point N. Let M be the point where H projects on  $\overrightarrow{KB}$  parallel to  $\overrightarrow{KA}$ . We have

$$\overrightarrow{HM} = \alpha \overrightarrow{KA}$$

and

$$\overrightarrow{HM} \cdot \overrightarrow{HM} = \frac{r^2}{\sin^2 \theta} = \frac{r^2}{1 - \cos^2 \theta}$$

Eliminating  $\overrightarrow{HM}$  we obtain the following conditions on  $\alpha$  for H to be in the position indicated above

$$\alpha < 0$$

$$\alpha^2 > r^2 \frac{\overrightarrow{KB} \cdot \overrightarrow{KB}}{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2}.$$

We have similar conditions on  $\beta$  for the circle to be outside the wedge KAB and tangent to  $\overrightarrow{KA}$ .

The next step is to find out the extension of the wedge formed by the circle when seen from K. Let P be a point where a line going through K is tangent to the circle, and let Q be the point where KP intersects AB; it may be between K and P or behind P.

We need to find P. It is defined by two equations

$$\overrightarrow{PH} \cdot \overrightarrow{PH} = r^2$$

$$\overrightarrow{PH} \cdot \overrightarrow{KP} = 0.$$

The second equation may be rewritten

$$\overrightarrow{PH} \cdot (\overrightarrow{KH} + \overrightarrow{HP}) = 0$$

from which we obtain

$$\overrightarrow{PH} \cdot \overrightarrow{KH} = r^2.$$

Let  $\gamma$  and  $\delta$  be the coordinates of  $\overrightarrow{PH}$  in KAB

$$\overrightarrow{PH} = \gamma \overrightarrow{KA} + \delta \overrightarrow{KB}.$$

The second equation is linear

$$\gamma \overrightarrow{KA} \cdot \overrightarrow{KH} + \delta \overrightarrow{KB} \cdot \overrightarrow{KH} = r^2,$$

thus

$$\gamma = \frac{r^2 - \delta \overrightarrow{KB} \cdot \overrightarrow{KH}}{\overrightarrow{KA} \cdot \overrightarrow{KH}}.$$

Note that  $\overrightarrow{KA} \cdot \overrightarrow{KH}$  and  $\overrightarrow{KB} \cdot \overrightarrow{KH}$  cannot both be 0 unless K is on AB. The first equation is quadratic

$$(\gamma \overrightarrow{KA} + \delta \overrightarrow{KB})^2 = r^2.$$

Plugging the value of  $\gamma$  above we get

$$\begin{aligned} \delta^2 ((\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})^2 + 2(\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})(\overrightarrow{KB} \cdot \overrightarrow{KH}) + \overrightarrow{KA} \cdot \overrightarrow{KA}(\overrightarrow{KB} \cdot \overrightarrow{KH})^2) + \\ 2\delta r^2 ((\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH}) - (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KH})) + \\ r^2 (r^2 (\overrightarrow{KA} \cdot \overrightarrow{KA}) - (\overrightarrow{KA} \cdot \overrightarrow{KH})^2) = 0. \end{aligned}$$

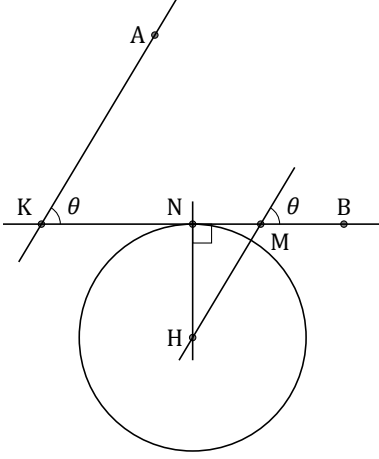


Figure 1. Circle outside of the wedge KAB.

This equation always has two solutions because the sphere intersects KAB. Having determined  $\overrightarrow{PH}$ , we can find Q. Q is on the line AB, thus

$$\overrightarrow{AQ} = \lambda \overrightarrow{AB},$$

which can be written

$$\overrightarrow{KQ} - \overrightarrow{KA} = \lambda \overrightarrow{AB}.$$

Noting that  $\overrightarrow{KQ}$  is orthogonal to  $\overrightarrow{PH}$  we have

$$-\overrightarrow{KA} \cdot \overrightarrow{PH} = \lambda \overrightarrow{AB} \cdot \overrightarrow{PH}, \text{ or, } \lambda = -\frac{\overrightarrow{KA} \cdot \overrightarrow{PH}}{\overrightarrow{AB} \cdot \overrightarrow{PH}}.$$

Having determined the values of  $\lambda$  we need to find out where Q is located with respect to the segment  $\overline{AB}$ . Let S be the intersection of AB with the line orthogonal to KH at K. We locate S on AB as follows

$$\overrightarrow{KS} = \overrightarrow{KA} + \sigma \overrightarrow{AB}.$$

Noting that  $\overrightarrow{KS} \cdot \overrightarrow{KH} = 0$  we obtain

$$\sigma = -\frac{\overrightarrow{KA} \cdot \overrightarrow{KH}}{\overrightarrow{AB} \cdot \overrightarrow{KH}}$$

Now let T be the intersection of AB with the line orthogonal to KH at P. We locate T on AB similarly

$$\overrightarrow{KT} = \overrightarrow{KA} + \tau \overrightarrow{AB}.$$

We have

$$\overrightarrow{PT} = \overrightarrow{KA} + \tau \overrightarrow{AB} - \overrightarrow{KH} + \overrightarrow{PH}$$

Noting that  $\overrightarrow{PT} \cdot \overrightarrow{KH} = 0$  we obtain

$$\tau = \frac{\overrightarrow{KH} \cdot \overrightarrow{KH} - \overrightarrow{PH} \cdot \overrightarrow{KH} - \overrightarrow{KA} \cdot \overrightarrow{KH}}{\overrightarrow{AB} \cdot \overrightarrow{KH}}$$

We can now determine if  $\lambda$  is an "interesting" intersection, i.e., one that intersects the cone behind the sphere when seen from the camera. First, assume that A and B are in the same order as S and T on AB. Then we have  $\sigma \leq \tau$  and the intersection is farther than T (as seen from the camera) if and only if  $\tau < \lambda$ . Conversely, if A and B are in the reverse order as S and T on AB we have  $\tau \leq \sigma$  and the intersection is farther than T if and only if  $\lambda < \tau$ .

There is another special case to handle: if the line AB is in "hyperbolic" position, i.e. intersects both halves of the cone, then one value of  $\lambda$  is smaller than  $\sigma$  and one value is greater than  $\sigma$ . Exactly one of the values of  $\lambda$  will be retained by the preceding analysis, and we need to add an extra  $\lambda$  equal to an infinity with the sign of  $\tau - \sigma$  to account for the fact that all the points farther than T are hidden by the cone.

To complete the analysis we need to compute the intersection Q of the sphere (not the cone) with the line AB. Q is on the sphere, thus

$$\overrightarrow{CQ} \cdot \overrightarrow{CQ} = R^2.$$

It is also on the line AB thus

$$\overrightarrow{KQ} = \overrightarrow{KA} + \mu \overrightarrow{AB}.$$

We have

$$\overrightarrow{CQ} = \overrightarrow{KQ} - \overrightarrow{KC} = \overrightarrow{KA} + \mu \overrightarrow{AB} - \overrightarrow{KC} = \overrightarrow{CA} + \mu \overrightarrow{AB},$$

and therefore

$$R^2 = (\overrightarrow{CA} + \mu \overrightarrow{AB})^2,$$

meaning that  $\mu$  is a solution of

$$\mu \overrightarrow{AB} \cdot \overrightarrow{AB} + 2\mu \overrightarrow{CA} \cdot \overrightarrow{AB} + \overrightarrow{CA} \cdot \overrightarrow{CA} - R^2 = 0.$$

Depending on the location of the segment with respect to the sphere, there can be 0, 1, or 2 intersections.

If we take the union of the values of  $\lambda$  and  $\mu$  and order them, it is straightforward to find the visible segments. Remember that  $0 < \lambda, \mu < 1$  for points that are in the segment AB.

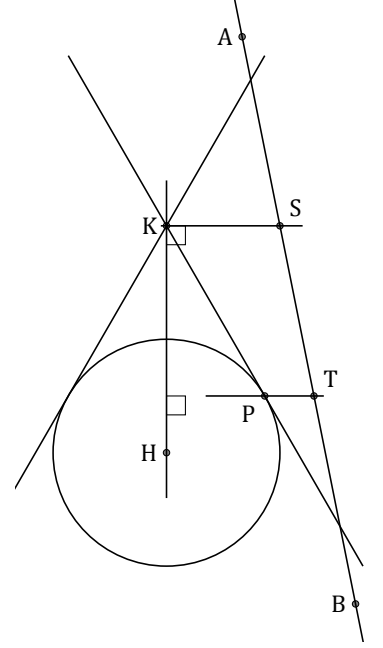


Figure 2. Definition of S and T.