# An Introduction to Runge-Kutta Integrators

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In this post I shall assume understanding of the concepts described in chapter 8 (Motion) as well as chapter 11 (Vectors) of Richard Feynmann's *Lectures on Physics*.

## 1 Motivation

We want to be able to predict the position $\boldsymbol{s}(t)$ as a function of time of a satellite around a fixed planet of mass $M$. In order to do this, we recall that the velocity is given by

$$\boldsymbol{v} = \frac{\mathrm{d}\,\boldsymbol{s}}{\mathrm{d}\,t}$$

and the acceleration by

$$\boldsymbol{a} = \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}\,t} = \frac{\mathrm{d}^2\,\boldsymbol{s}}{\mathrm{d}\,t^2}.$$

We assume the mass of the satellite is constant and that the planet sits at the origin of our reference frame. Newton's law of universal gravitation tells us that the magnitude (the length) of the acceleration vector will be

$$a = \frac{GM}{s^2},$$

where $s$ is the length of $\boldsymbol{s}$. and that the acceleration will be directed towards the planet, so that

$$\boldsymbol{a} = -\frac{GM}{s^2}\frac{\boldsymbol{s}}{s}.$$

We don't really care that much about the specifics, but we see that this is a function of $\boldsymbol{s}$. We'll write it $\boldsymbol{a}(\boldsymbol{s})$. Putting it all together we could rewrite this as

$$\frac{\mathrm{d}^2\,\boldsymbol{s}}{\mathrm{d}\,t^2} = \boldsymbol{a}(\boldsymbol{s})$$

and go ahead and solve this kind of problem, but we don't like having a second derivative. Instead we see that we can write it as

$$\begin{cases} \frac{\mathrm{d}\,\boldsymbol{s}}{\mathrm{d}\,t} = \boldsymbol{v} \\ \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}\,t} = \boldsymbol{a}(\boldsymbol{s}) \end{cases}.$$

Let us define a vector $\boldsymbol{y}$ with 6 entries instead of 3,

$$\boldsymbol{y} = (\boldsymbol{s}, \boldsymbol{v}) = (s_x, s_y, s_z, v_x, v_y, v_z).$$

Similarly, define a function $\boldsymbol{f}$ as follows:

$$\boldsymbol{f}(\boldsymbol{y}) = (\boldsymbol{v}, \boldsymbol{a}(\boldsymbol{s})).$$

Our problem becomes

$$\frac{\mathrm{d}\,\boldsymbol{y}}{\mathrm{d}\,t} = \left(\frac{\mathrm{d}\,\boldsymbol{s}}{\mathrm{d}\,t}, \frac{\mathrm{d}\,\boldsymbol{v}}{\mathrm{d}\,t}\right) = (\boldsymbol{v}, \boldsymbol{a}(\boldsymbol{s})) = \boldsymbol{f}(\boldsymbol{y}).$$

So we have gotten rid of that second derivative.

## 2 Ordinary differential equations

We are interested computing solutions to equations of the form

$$\frac{\mathrm{d}\,\boldsymbol{y}}{\mathrm{d}\,t} = \boldsymbol{f}(\boldsymbol{y}, t).$$

These are called *ordinary differential equations* (ODES). The function $\boldsymbol{f}$ is called the *right-hand side* (RHS).

Recall that if the right-hand side didn't depend on $\boldsymbol{y}$, the answer would be the integral,

$$\frac{\mathrm{d}\,\boldsymbol{y}}{\mathrm{d}\,t} = \boldsymbol{f}(t) \implies \boldsymbol{y} = \int \boldsymbol{f}(t)\,\mathrm{d}\,t.$$

Here we will restrict ourselves to the case where the right-hand side doesn't depend on $t$ (but depends on $\boldsymbol{y}$), as was the case in the previous section. The equation becomes

$$\frac{\mathrm{d}\,\boldsymbol{y}}{\mathrm{d}\,t} = \boldsymbol{f}(\boldsymbol{y}).$$

We call such a right-hand side *autonomous*.