

An Introduction Hamiltonian Mechanics

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In this post I shall assume understanding of the concepts described in chapter 4 (Conservation of Energy), chapter 8 (Motion) as well as sections 11-4 and 11-5 (Vectors and Vector algebra) of chapter 11 of Richard Feynmann's *Lectures on Physics*.

It is the continuation of my *Introduction to Runge-Kutta Integrators*, but it does not rely on the concepts described in that post.

1 Motivation

We have previously seen how to compute the evolution of physical systems while keeping the buildup of error in check. However, the error will still build up over time. We would like to ensure that fundamental properties of the physical system are preserved. For instance, we'd like a low strongly-bound orbit not to turn into an escape trajectory (or a reentry) over time: we need conservation of energy.

In order to make an integrator that conserves energy, it is helpful to look at physics from a viewpoint where the conservation of energy is the fundamental hypothesis, rather than a consequence of the application of some forces.

2 Gravitation from a Newtonian viewpoint

Recall the formulation by forces of the gravitational N -body problem: each body i , located at position \mathbf{q}_i exerts a force \mathbf{F}_{ij} on every other body j , located at \mathbf{q}_j whose magnitude is

$$F_{ij} = \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^2}.$$

The force on j is directed toward i , so

$$\mathbf{F}_{ij} = F_{ij} \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|} = \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^2} \frac{\mathbf{q}_i - \mathbf{q}_j}{|\mathbf{q}_i - \mathbf{q}_j|} = \frac{Gm_i m_j}{|\mathbf{q}_i - \mathbf{q}_j|^3} (\mathbf{q}_i - \mathbf{q}_j).$$

Adding up all the forces exerted on j by the bodies $1, \dots, j-1, j+1, \dots, N$, we get the total force \mathbf{F}_j on j ,

$$\begin{aligned} \mathbf{F}_j &= \mathbf{F}_{1j} + \mathbf{F}_{2j} + \dots + \mathbf{F}_{j-1j} + \mathbf{F}_{j+1j} + \mathbf{F}_{Nj} \\ &= \sum_{i \in \{1, N\} \setminus \{j\}} \mathbf{F}_{ij} \end{aligned}$$

This force then changes the velocity of j according to Newton's second law,

$$\frac{d\mathbf{v}_j}{dt} = \frac{\mathbf{F}_j}{m_j}.$$

What is the energy of that N -body system? Part of it is the energy due to the motion of the bodies, the so-called *kinetic*¹ energy T ,

$$T = \sum_{j=1}^N \frac{1}{2} m_j v_j^2.$$

¹From ancient Greek *κινεῖν*, to move.