

Documentation for the hiding computations in Planetarium

Pascal Leroy (pleroy)

2017-07-31

A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; K is the location of the camera.

For simplicity we will do most of our analysis in the plane KAB. We will use $(\overrightarrow{KA}, \overrightarrow{KB})$ as a basis of that plane. Let H be the orthogonal projection of C on KAB. Define α, β to be its coordinates in KAB:

$$\overrightarrow{KH} = \alpha \overrightarrow{KA} + \beta \overrightarrow{KB}.$$

Note that $\overrightarrow{KH} = \overrightarrow{KC} + \overrightarrow{CH}$. By its definition, \overrightarrow{CH} is orthogonal to both \overrightarrow{KA} and \overrightarrow{KB} :

$$\overrightarrow{KA} \cdot \overrightarrow{CH} = 0$$

$$\overrightarrow{KB} \cdot \overrightarrow{CH} = 0,$$

or:

$$\overrightarrow{KA} \cdot \overrightarrow{KH} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$

$$\overrightarrow{KB} \cdot \overrightarrow{KH} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

Expanding \overrightarrow{KH} gives a linear system of two equations with two unknowns:

$$\alpha \overrightarrow{KA} \cdot \overrightarrow{KA} + \beta \overrightarrow{KA} \cdot \overrightarrow{KB} = \overrightarrow{KA} \cdot \overrightarrow{KC}$$

$$\alpha \overrightarrow{KA} \cdot \overrightarrow{KB} + \beta \overrightarrow{KB} \cdot \overrightarrow{KB} = \overrightarrow{KB} \cdot \overrightarrow{KC}.$$

The determinant of this system is:

$$D = (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KB}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})^2,$$

which is non-zero if and only if A \neq B. The solutions are thus

$$\alpha = \frac{(\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KB} \cdot \overrightarrow{KC})}{D}$$
$$\beta = \frac{(\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KC}) - (\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KC})}{D}.$$

Now that \overrightarrow{KH} is determined we can compute $\overrightarrow{CH} = \overrightarrow{KH} - \overrightarrow{KC}$ and $\overrightarrow{CH} \cdot \overrightarrow{CH}$. If $\overrightarrow{CH} \cdot \overrightarrow{CH} \geq R^2$, the sphere is either tangent to the plane KAB or doesn't intersect it. Thus, there is no hiding.

Let's look at a figure in the plane KAB when the sphere intersects that plane, considering first the situation where the intersection circle is between the camera and the segment. r is the radius of the circle, it is such that $r^2 = R^2 - \overrightarrow{CH} \cdot \overrightarrow{CH}$. P is a point where a line going through K is tangent to the circle. Q is the point where KP intersects AB; it may be between K and P or behind P.

We need to find P. It is defined by two equations:

$$\overrightarrow{PH} \cdot \overrightarrow{PH} = r^2$$

$$\overrightarrow{PH} \cdot \overrightarrow{KP} = 0.$$

The second equation may be rewritten:

$$\overrightarrow{PH} \cdot (\overrightarrow{KH} + \overrightarrow{HP}) = 0$$

$$\overrightarrow{PH} \cdot \overrightarrow{KH} = r^2.$$

Let γ and δ be the coordinates of \overrightarrow{PH} in KAB:

$$\overrightarrow{PH} = \gamma \overrightarrow{KA} + \delta \overrightarrow{KB}.$$

The second equation is linear:

$$\gamma \overrightarrow{KA} \cdot \overrightarrow{KH} + \delta \overrightarrow{KB} \cdot \overrightarrow{KH} = r^2.$$

Thus:

$$\gamma = \frac{r^2 - \delta \overrightarrow{KB} \cdot \overrightarrow{KH}}{\overrightarrow{KA} \cdot \overrightarrow{KH}}.$$

Note that $\overrightarrow{KA} \cdot \overrightarrow{KH}$ and $\overrightarrow{KB} \cdot \overrightarrow{KH}$ cannot both be 0 unless K is on AB.

The first equation is quadratic:

$$(\gamma \overrightarrow{KA} + \delta \overrightarrow{KB})^2 = r^2.$$

Plugging the value of γ above we get:

$$\begin{aligned} \delta^2((\overrightarrow{KB} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})^2 + 2(\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH})(\overrightarrow{KB} \cdot \overrightarrow{KH}) + \overrightarrow{KA} \cdot \overrightarrow{KA}(\overrightarrow{KB} \cdot \overrightarrow{KH})^2) + \\ 2\delta r^2((\overrightarrow{KA} \cdot \overrightarrow{KB})(\overrightarrow{KA} \cdot \overrightarrow{KH}) - (\overrightarrow{KA} \cdot \overrightarrow{KA})(\overrightarrow{KB} \cdot \overrightarrow{KH})) + \\ r^2(r^2(\overrightarrow{KA} \cdot \overrightarrow{KA}) - (\overrightarrow{KA} \cdot \overrightarrow{KH})^2) = 0. \end{aligned}$$

This equation always has two solutions because the sphere intersects KAB. Having determined \overrightarrow{PH} , we can find Q. Q is on the line AB, thus:

$$\overrightarrow{AQ} = \lambda \overrightarrow{AB},$$

which can be written

$$\overrightarrow{KQ} - \overrightarrow{KA} = \lambda \overrightarrow{AB}.$$

Noting that \overrightarrow{KQ} is orthogonal to \overrightarrow{PH} we have

$$-\overrightarrow{KA} \cdot \overrightarrow{PH} = \lambda \overrightarrow{AB} \cdot \overrightarrow{PH}, \text{ or, } \lambda = -\frac{\overrightarrow{KA} \cdot \overrightarrow{PH}}{\overrightarrow{AB} \cdot \overrightarrow{PH}}.$$

It is possible though that Q would be between K and P, in which case the segment would not intersect the cone. To determine if this is the case we write:

$$\overrightarrow{KP} = \overrightarrow{KH} - \overrightarrow{PH} = (\alpha - \gamma)\overrightarrow{KA} + (\beta - \delta)\overrightarrow{KB}.$$

The line AB is the set of points of the form $\eta \overrightarrow{KA} + (1 - \eta)\overrightarrow{KB}$ so P is in front of AB if and only if

$$\alpha - \gamma + \beta - \delta < 1.$$

To complete the analysis we need to compute the intersection of the sphere (not the cone) with the line AB. Q is on the sphere, thus

$$\overrightarrow{CQ} \cdot \overrightarrow{CQ} = R^2.$$

It is also on the line AB thus

$$\overrightarrow{KQ} = \overrightarrow{KA} + \mu \overrightarrow{AB}.$$

We have:

$$\overrightarrow{CQ} = \overrightarrow{KQ} - \overrightarrow{KC} = \overrightarrow{KA} + \mu \overrightarrow{AB} - \overrightarrow{KC} = \overrightarrow{CA} + \mu \overrightarrow{AB},$$

and therefore

$$R^2 = (\overrightarrow{CA} + \mu \overrightarrow{AB})^2,$$

meaning that μ is a solution of

$$\mu \overrightarrow{AB} \cdot \overrightarrow{AB} + 2\mu \overrightarrow{CA} \cdot \overrightarrow{AB} + \overrightarrow{CA} \cdot \overrightarrow{CA} - R^2 = 0.$$

Depending on the location of the segment with respect to the sphere, there can be 0, 1, or 2 intersections.

If we take the union of the values of λ and μ and order them, it is straightforward to find the visible segments. Remember that $0 < \lambda, \mu < 1$ for points that are in the segment AB.