Documentation for the hiding computations in Planetarium

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A and B are the extremities of the segment; C is the centre of the sphere; R is the radius of the sphere; K is the location of the camera.

For simplicity we will do most of our analysis in the plane KAB. We will use $(\overrightarrow{\mathrm{KA}}, \overrightarrow{\mathrm{KB}})$ as a basis of that plane. Let H be the orthogonal projection of C on KAB. Define α, β to be its coordinates in KAB:

$$\overrightarrow{\mathrm{KH}} = \alpha\overrightarrow{\mathrm{KA}} + \beta\overrightarrow{\mathrm{KB}}.$$

Note that $\overrightarrow{\mathrm{KH}} = \overrightarrow{\mathrm{KC}} + \overrightarrow{\mathrm{CH}}$. By its definition, $\overrightarrow{\mathrm{CH}}$ is orthogonal to both $\overrightarrow{\mathrm{KA}}$ and $\overrightarrow{\mathrm{KB}}$:

$$\begin{aligned}\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{CH}} &= 0\\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{CH}} &= 0,\end{aligned}$$

or:

$$\begin{aligned}\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}\\ \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}.\end{aligned}$$

Expanding $\overrightarrow{\mathrm{KH}}$ gives a linear system of two equations with two unknowns:

$$\begin{aligned}\alpha\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}} + \beta\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}\\ \alpha\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}} + \alpha\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}} &= \overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}.\end{aligned}$$

The determinant of this system is:

$$D = (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})^2,$$

which is non-zero if and only if $\mathrm{A} \neq \mathrm{B}$. The solutions are thus

$$\begin{aligned}\alpha &= \frac{(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}})}{D}\\ \beta &= \frac{(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KC}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KC}})}{D}.\end{aligned}$$

Now that $\overrightarrow{\mathrm{KH}}$ is determined we can compute $\overrightarrow{\mathrm{CH}} = \overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{KC}}$ and $\overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}}$. If $\overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}} \geq R^2$, the sphere is either tangent to the plane KAB or doesn't intersect it. Thus, there is no hiding.

Let's look at a figure in the plane KAB when the sphere intersects that plane, considering first the situation where the intersection circle is between the camera and the segment. r is the radius of the circle, it is such that $r^2 = R^2 - \overrightarrow{\mathrm{CH}} \cdot \overrightarrow{\mathrm{CH}}$. P is a point where a line going through K is tangent to the circle. Q is the point where KP intersects AB; it may be between K and P or behind P.

We need to find P. It is defined by two equations:

$$\begin{aligned}\overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{PH}} &= r^2\\ \overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{KP}} &= 0.\end{aligned}$$

The second equation may be rewritten:

$$\begin{aligned}\overrightarrow{\mathrm{PH}} \cdot (\overrightarrow{\mathrm{KH}} + \overrightarrow{\mathrm{HP}}) &= 0\\ \overrightarrow{\mathrm{PH}} \cdot \overrightarrow{\mathrm{KH}} &= r^2.\end{aligned}$$

Let γ and δ be the coordinates of $\overrightarrow{\mathrm{PH}}$ in KAB:

$$\overrightarrow{\mathrm{PH}} = \gamma\overrightarrow{\mathrm{KA}} + \delta\overrightarrow{\mathrm{KB}}.$$

The second equation is linear:

$$\gamma\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}} + \delta\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}} = r^2.$$

Thus:

$$\gamma = \frac{r^2 - \delta\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}}{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}}.$$

Note that $\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}$ and $\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}$ cannot both be 0 unless K is on AB.

The first equation is quadratic:

$$(\gamma\overrightarrow{\mathrm{KA}} + \delta\overrightarrow{\mathrm{KB}})^2 = r^2.$$

Pluging the value of γ above we get:

$$\begin{aligned}\delta^2((\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})^2 + 2(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}}) + \overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}}(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}})^2) +& \\ 2\delta r^2((\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KB}})(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}})(\overrightarrow{\mathrm{KB}} \cdot \overrightarrow{\mathrm{KH}})) +& \\ r^2(r^2(\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KA}}) - (\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{KH}})^2) = 0.&\end{aligned}$$

This equation always has two solutions because the sphere intersects KAB. Having determined $\overrightarrow{\mathrm{PH}}$, we can find Q. Q is on the line AB, thus:

$$\overrightarrow{\mathrm{AQ}} = \lambda\overrightarrow{\mathrm{AB}},$$

which can be written

$$\overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{KA}} = \lambda\overrightarrow{\mathrm{AB}}.$$

Noting that $\overrightarrow{\mathrm{KQ}}$ is orthogonal to $\overrightarrow{\mathrm{PH}}$ we have

$$-\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{PH}} = \lambda\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{PH}}, \text{ or, } \lambda = -\frac{\overrightarrow{\mathrm{KA}} \cdot \overrightarrow{\mathrm{PH}}}{\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{PH}}}.$$

It is possible though that $pointQ$ would be between K and P, in which case the segment would not intersect the cone. To determine if this is the case we write:

$$\overrightarrow{\mathrm{KP}} = \overrightarrow{\mathrm{KH}} - \overrightarrow{\mathrm{PH}} = (\alpha - \gamma)\overrightarrow{\mathrm{KA}} + (\beta - \delta)\overrightarrow{\mathrm{KB}}.$$

The line AB is the set of points of the form $\eta\overrightarrow{\mathrm{KA}} + (1 - \eta)\overrightarrow{\mathrm{KB}}$ so P is in front of AB if and only if

$$\alpha - \gamma + \beta - \delta < 1.$$

To complete the analysis we need to compute the intersection of the sphere (not the cone) with the line AB. Q is on the sphere, thus

$$\overrightarrow{\mathrm{CQ}} \cdot \overrightarrow{\mathrm{CQ}} = R^2.$$

It is also on the line AB thus

$$\overrightarrow{\mathrm{KQ}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{AB}}.$$

We have:

$$\overrightarrow{\mathrm{CQ}} = \overrightarrow{\mathrm{KQ}} - \overrightarrow{\mathrm{QC}} = \overrightarrow{\mathrm{KA}} + \mu\overrightarrow{\mathrm{KC}} = \overrightarrow{\mathrm{CA}} + \mu\overrightarrow{\mathrm{AB}},$$

and therefore

$$R^2 = (\overrightarrow{\mathrm{CA}} + \mu\overrightarrow{\mathrm{AB}})^2,$$

meaning that μ is a solution of

$$\mu\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{AB}} + 2\mu\overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{CA}} \cdot \overrightarrow{\mathrm{CA}} - R^2 = 0.$$

Depending on the location of the segment with respect to the sphere, there can be 0, 1, or 2 intersections.

If we take the union of the values of λ and μ and order them, it is straightforward to find the visible segments. Remember that $0 < \lambda, \mu < 1$ for points that are in the segment AB.