Evolutionarily Stable Strategies and Replicator Dynamics

Evolutionarily Stable Strategies

Task 1. *Consider the following*

$$A = \begin{pmatrix} -6 & 2 & 1 \\ 2 & -11 & 3 \\ 1 & 3 & -22 \end{pmatrix}.$$

Find the ESS (mixed version) and provide the appropriate proof.

Definition 1 (Hawks–Dove–Retaliator). *For Hawk and Dove consider its extension, by adding an extra strategy* Retaliator *strategy R. Retaliator plays Dove unless the opponent escalates; against Hawk it retaliates (so the outcome is like H vs H), while against Dove (and against another Retaliator) it behaves like Dove. This yields the payoff matrix*

$$A = \begin{pmatrix} \frac{V-C}{2} & V & \frac{V-C}{2} \\ 0 & \frac{V}{2} & \frac{V}{2} \\ \frac{V-C}{2} & \frac{V}{2} & \frac{V}{2} \end{pmatrix}, \qquad \textit{(rows/cols ordered as } (H, D, R)\textit{)}.$$

Task 2. *Find all ESS for Hawks–Dove–Retaliator game. Consider two cases $V < C$ and $V \geqslant C$.*

Task 3 (*). *Can you propose and implement the algorithm that finds ESS (mixed version) for a given symmetric game? Prove its correctness.*

Replicator Dynamics

Definition 2. *For a symmetric game with payoff matrix $n \times n$ A, the replicator dynamics is the ODE system:*

$$\dot{x}_i = x_i((Ax)_i - x^\top Ax), \quad i = 1, 2, ...n \tag{1}$$

Task 4. *Show that if $\sum_i x_i(0) = 1$ then for all $t \geqslant 0$ we have $\sum_i x_i(t) = 1$.*

Task 5. *Show that if for some t_0 and i $x_i(t_0) = 0$ then for all $t \geqslant t_0$ we have $x_i(t) = 0$. Conclude that if $\forall i\ x_i(0) \geqslant 0$, then $\forall i, t\ x_i(t) \geqslant 0$*

Task 6. *What happens if $x(0) = x^*$, where x^* is the Nash Equilibrium.*

Task 7. *Investigate the replicator dynamics for Hawkss and Doves game for different initial conditions and different values of V and C. What is the limit $\lim_{t\to\infty} x(t)$? Illustrate your answer with appropriate graphs.*

Task 8. *Investigate the replicator dynamics for games from Task 1 and Task 2 for different parameters and initial conditions. Illustrate your answers with appropriate graph and formulate conclusions and hypothesis (Hint: use barycentric coordinates to draw appropriate 2D graphs).*

Task 9. *Check the behaviour of replicator dynamics for the games having pure version ESS but not having mixed version ESS.*

Task 10 (* if proven). *Is it true that if a symmetric game has ESS x^* (mixed version), then for every initial conditions that are sufficiently close to x^* we have $\lim_{t\to\infty} x(t) = x^*$?*

Task 11. *Consider classical Rock–Paper–Scissors (RPS) with payoff matrix*

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Find all NE and check if there is ESS. Analyse the replicator dynamics for different initial values.

Task 12. *Consider a modification of a classical Rock–Paper–Scissors (RPS), where the price for winning is v, with payoff matrix*

$$A = \begin{pmatrix} 0 & -1 & v \\ v & 0 & -1 \\ -1 & v & 0 \end{pmatrix}.$$

Find all NE and check if there is ESS if $v > 0$. Analyse the replicator dynamics for different initial values.

Task 13. *Consider the following*

$$A = \alpha \begin{pmatrix} -6 & 2 & 1 \\ 2 & -11 & 3 \\ 1 & 3 & -22 \end{pmatrix} + \beta \begin{pmatrix} 0 & -1 & v \\ v & 0 & -1 \\ -1 & v & 0 \end{pmatrix}.$$

Investigate the replicator dynamics model for different α, β and $v > 0$. If some patterns and examples are present some of them. If you can formulate some hypothesis/observations - do it.

Task 14. *Let us add a small mutation term $\mu > 0$ to the replicator dynamics:*

$$\dot{x}_i = x_i((Ax)_i - x^\top Ax) + \mu\left(\frac{1}{n} - x_i\right), \quad i = 1, \dots, n.$$

For RPS, simulate the replicator dynamics with small μ and different initial values. How would you interpret the results in biological terms? Can you slightly generalize the model?

Task 15. *Is it possible for a game that its replicator dynamics has cycles as well as attractors? Check the problem for different values of x^*, like all coordinates greater than zero (all pure strategies play) or similar. Propose some hypothesis.*