Evolutionary Stable Strategies

Definition 1. *We call a two player game* (A, B) *symmetric if* $A = B^\top$.

Definition 2 (pure version)**.** *Let* A *be a payoff matrix for a symmetric game and let* $u(x, y) = x^\top Ay$. *A mixed strategy* x *is an* evolutionarily stable strategy (ESS) *if for every* **pure** *strategy* $y \neq x$:
(i) $u(x, x) \geqslant u(y, x)$, *and*
(ii) if $u(x, x) = u(y, x)$, *then* $u(x, y) > u(y, y)$.

Definition 3 (mixed version)**.** *Let* A *be a payoff matrix for a symmetric game and let* $u(x, y) = x^\top Ay$. *A mixed strategy* x *is an* evolutionarily stable strategy (ESS) *if for every* **mixed** *strategy* $y \neq x$:
(i) $u(x, x) \geqslant u(y, x)$, *and*
(ii) if $u(x, x) = u(y, x)$, *then* $u(x, y) > u(y, y)$.

Definition 4 (ESS invasion form)**.** *Let* A *be the payoff matrix of a symmetric game and* $u(p, q) = p^\top Aq$. *A mixed strategy* $x \in \Delta$ *is an* evolutionarily stable strategy (ESS) *if for all* $z \in \Delta$ *with* $z \neq x$ *there exists* $\bar{\varepsilon} > 0$ *such that for all* $\varepsilon \in (0, \bar{\varepsilon})$,

$$u(x, (1-\varepsilon)x + \varepsilon z) \;>\; u(z, (1-\varepsilon)x + \varepsilon z)\,,$$

i.e.

$$x^\top A((1-\varepsilon)x + \varepsilon z) \;>\; z^\top A((1-\varepsilon)x + \varepsilon z).$$

Exercise 1 (1p)**.** *Show that Definition 2 and Definition 3 are not equivalent.*

Exercise 2 (1p)**.** *Show that Definition 3 and Definition 4 are equivalent.*

Exercise 3 (1p)**.** *Find all ESS for the symmetric game with payoff matrix*

$$A = \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}.$$