Thèse de doctorat

NNT: 2023UPASM001

Generating family homologies of Legendrian submanifolds and moduli spaces of gradient staircases

Homologies des familles génératrices de sous-variétés legendriennes et espaces de modules d'escaliers de gradient

Thèse de doctorat de l'université Paris-Saclay

École doctorale n°574, Mathématiques Hadamard (EDMH)
Spécialité de doctorat : Mathématiques fondamentales
Graduate School : Mathématiques, Référent : Faculté des sciences d'Orsay

Thèse préparée au **Laboratoire de Mathématiques d'Orsay** (Université Paris-Saclay, CNRS), sous la direction de **Frédéric BOURGEOIS**, Professeur

Thèse soutenue à Paris-Saclay, le 25 janvier 2023, par

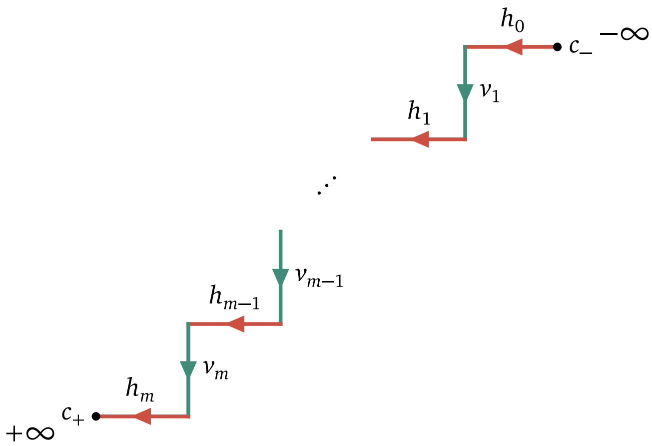
Cyril FALCON

Composition du jury

Claude VITERBO Professeur, Université Paris-Saclay	Président
Alexandru OANCEA Professeur, Sorbonne Université	Rapporteur & Examinateur
Lisa TRAYNOR Professeure, Bryn Mawr College	Rapporteure & Examinatrice
Baptiste CHANTRAINE Maître de conférences, HDR, Nantes Université	Examinateur
Urs FRAUENFELDER Professeur, Universität Augsburg	Examinateur

Generating family homologies of Legendrian submanifolds and moduli spaces of gradient staircases

Cyril Falcon



$$\mathscr{M}\left(c_-, c_+; \delta, \left(s^{-1} g_B\right) \oplus g_F\right) \xrightarrow[s \longrightarrow 0]{\text{Floer–Gromov}} \overline{\mathscr{M}}^{\mathrm{st}}\left(c_-, c_+; \delta, g_B \oplus g_F\right)$$

« Pour que tout soit consommé, pour que je me sente moins seul, il me restait à souhaiter qu'il y ait beaucoup de spectateurs le jour de mon exécution et qu'ils m'accueillent avec des cris de haine. »

Albert Camus, *L'Étranger*. (1942)

Acknowledgments/Remerciements

Longtemps, j'ai imaginé le moment où ces remerciements noirciraient les premières pages de mon mémoire. Tant l'issue de cette thèse était incertaine, je les ai souvent assez égoïstement perçus comme le symbole de ma libération. Aujourd'hui, alors que ma délivrance n'a jamais été aussi proche, ils me semblent retrouver leur première signification.
Sans le soutien et la bienveillance de ceux que j'écris, j'aurais été condamné à éternellement errer dans les limbes de mon doctorat.

Vient en tout premier lieu Frédéric Bourgeois, mon directeur de thèse, qui n'a jamais cessé de croire en l'aboutissement de mon doctorat et de s'investir dans ma réussite sans compter son temps, alors que ma propre confiance s'écaillait pourtant toujours un peu plus. Merci, Frédéric, pour ton soutien sans faille.
J'admire ta capacité à encadrer tes doctorants avec autant de dévouement, d'enthousiasme et de clairvoyance sur des sujets aussi passionnants et ambitieux que les nôtres. Je salue aussi la mesure de tes directives qui n'ont rien enlevé à mon autonomie, me laissant toujours pleinement explorer les idées que nous esquissions ensemble. Je crois pouvoir affirmer qu'à tes côtés, j'ai appris à faire de la recherche en mathématiques de la plus gratifiante des manières.

I am also honoured to defend my thesis in front of inspiring researchers such as Baptiste Chantraine, Urs Frauenfelder, Alexandru Oancea, Lisa Traynor, and Claude Viterbo. I hope that their influence reads in my thesis.
I am even more indebted to Alexandru and Lisa for their early and genuine interest in my work, carefully reviewing my dissertation, and always asking enthusiastic and enlightening questions. Merci, Alex, de m'avoir à plusieurs reprises encouragé à donner des exposés. Thank you, Lisa, for our multiple calls and your elegant and pioneering work on generating families.
Merci, Claude, de me faire l'honneur de présider ce jury de soutenance, ainsi que d'avoir porté à ma connaissance les travaux de Neil Fenichel sur les systèmes dynamiques lents-rapides qui m'ont permis de mieux situer mes propres recherches.

Je sais gré à tous les membres du personnel administratif et informatique du laboratoire. Stéphane Nonnenmacher et Benjamin Schraen pour leur gestion toujours efficace et flexible de notre école doctorale, ainsi que Mathilde Rousseau pour son assistance technique infaillible.

Je suis également reconnaissant envers mes collègues symplecticiens ou non, d'ici ou d'ailleurs, pour avoir parfois prêté une oreille attentive à mes interrogations et mes doutes, et pour avoir souvent su les lever. Je pense plus précisément à Agnès Gadbled, Hugo Federico et Antoine Fermé.

Merci d'avoir si souvent mis à l'épreuve mes premières esquisses de démonstrations.

Je remercie mes camarades orcéens avec qui j'ai partagé les errances inhérentes au doctorat et qui ont rendu les miennes moins éprouvantes. Je témoigne tout particulièrement mon amitié à Elio et Hugo dont la présence s'est étendue au-delà de notre bureau.

Depuis des années avant mon doctorat, mes parents et ma sœur m'épaulent inconditionnellement. Merci pour votre patience pendant mes longues années d'études.

Ces remerciements, cette thèse et encore bien d'autres choses, qui ne peuvent pas être énumérées, ne sauraient vraiment être complets sans Charlotte. Pudiquement, merci.

Virelade (France),
Le 18 décembre 2022.

Cyril Falcon

université PARIS-SACLAY FACULTÉ DES SCIENCES D'ORSAY

Mathématiques Orsay

FMJH

Contents

Introduction (en français)

Cette thèse est consacrée à l'étude des aspects de rigidité de contact des isotopies legendriennes parmi les sous-variétés legendriennes des variétés de contact. Toutes ces questions de rigidité sont aujourd'hui classiquement abordées en travaillant avec l'une ou plusieurs des trois techniques avancées suivantes :

- les courbes pseudo-holomorphes [EES05a, EES05b, EES07, BC14] ;
- les familles génératrices [Tra01, ST13, SS16] ; ou
- les faisceaux constructibles [GKS12, STZ17].

Depuis quelques années, l'étude de la correspondance entre ces trois différentes approches a été un terrain fertile pour la recherche en topologie de contact [FR11, HR13, RS18, NRS⁺20, RS21]. Les recherches contenues ici concernent seulement les invariants par familles génératrices des sous-variétés legendriennes, mais elles devraient également offrir des directions prometteuses pour mieux comprendre la correspondance entre les familles génératrices et les augmentations de l'algèbre de Tchekanov-Eliashberg [HR13].

Plus concrètement, ce travail réalise la première étape nécessaire pour démontrer que l'homologie des familles génératrices des sous-variétés legendriennes peut être en pratique calculée à partir de certains espaces de modules de trajectoires brisées. Son résultat principal est un théorème de compacité (Théorème A) établissant que les *escaliers de gradient* proviennent d'une perturbation singulière du flot de gradient qui est utilisé pour définir l'homologie pour les familles génératrices. Il est en effet attendu que ce *procédé de dégénérescence de Henry et Rutherford* sépare les différentes contributions qui interviennent dans le calcul de l'application de bord du complexe de chaînes des familles génératrices (Conjecture A). D'un côté, les contributions de la sous-variété legendrienne, et de l'autre, celles qui sont à imputer aux bifurcations dans le complexe des points critiques des familles génératrices.

Cette introduction est désormais consacrée à la description précise du procédé de dégénérescence de Henry et Rutherford et des espaces de modules d'escaliers de gradient (voir la Section 1.3), puis énonce rigoureusement les principaux résultats de ce mémoire de thèse (voir la Section 2). Il est toutefois plus commode de commencer par quelques rappels élémentaires concernant les sous-variétés legendriennes et leurs familles génératrices.

Un peu de contexte et de motivation

Une *variété de contact* est une paire (M^{2n+1}, ξ), où M est variété de dimension impaire et ξ est un champ maximalement non-intégrable d'hyperplans tangents à M, appelé *structure de contact*. Si α est une équation locale pour ξ, c'est-à-dire que ξ est localement $\ker(\alpha)$, alors $\alpha \wedge (\mathrm{d}\,\alpha)^n$

est une forme volume qui ne s'annule jamais, ou de manière équivalente, $\mathrm{d}\,\alpha_{|\xi}$ est une forme bilinéaire alternée et non-dégénérée (de sorte que M ne puisse pas être de dimension paire). Alors, le théorème de Frobenius implique non seulement que les structures de contact ne sont pas des feuilletages (puisque $\alpha \wedge \mathrm{d}\,\alpha \neq 0$), mais elles en sont aussi éloignées autant que possible. En effet, si une sous-variété de M qui est partout tangente à ξ, alors elle est dimension au plus n, et une telle variété intégrale de ξ est dite être une *sous-variété legendrienne* dès qu'elle est de dimension maximale n pour cette propriété.

Contrairement aux variétés riemanniennes, les variétés de contact n'ont pas d'invariants locaux comme la courbure, seules leurs propriétés topologiques globales sont pertinentes dans leur étude. En effet, toutes les variétés de contact sont localement modelées sur la variété $\mathbf{R}^{2n+1}$ munie des coordonnées $(x, y, z) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, et de la structure de contact $\xi_{\mathbf{R}^n}$ donnée par

$$\xi_{\mathbf{R}^n} = \ker\left(\alpha_{\mathbf{R}^n} = \mathrm{d}\,z - \sum_{k=1}^{n} y_k\,\mathrm{d}\,x_k\right),$$

[Gei08, Théorème 2.5.1] et cette construction peut être généralisée aux fibrés de premiers jets. Soient B une variété, son fibré de premiers jets $J^1B = T^*B \times \mathbf{R}$ a des coordoonées locales (q, p, z), où q sont des coordonnées locales sur B, p est duale à q et z est une coordonnée globale sur $\mathbf{R}$. Alors, J^1B est muni d'une structure de contact canonique donnée par $\xi_B = \ker(\alpha_B = \mathrm{d}\,z - p\,\mathrm{d}\,q)$ et pour laquelle la section nulle de J^1B est naturellement une sous-variété legendrienne.

Comme les sous-variétés legendriennes ont une grande codimension, il peut être attendu qu'elles soient des sous-variétés extrêmement flexibles, au sens du h-principe de Gromov [Gro86, EM02]. Grossièrement, les obstructions à l'existence d'isotopies lisses entre sous-variétés legendriennes devraient simplement se réduire à des contraintes homotopiques. Par exemple, toute sous-variété fermée de dimension n dans une variété de contact de dimension $2n+1$ est C^0-proche d'une sous-variété legendrienne, dès que son fibré tangent satisfait une certaine condition homotopique [Gei08, Proposition 6.3.6]. Par contre, les isotopies parmi les sous-variétés legendriennes sont elles nettement plus rigides, puisque par exemple, un théorème de la communauté affirme que toute classe d'isotopie lisse d'une sous-variété legendrienne se scinde en une infinité de classes d'isotopie legendrienne.

À cause de ces phénomènes de rigidité de contact, il est délicat de comprendre les obstructions à l'existence d'isotopies parmi les sous-variétés legendriennes, notamment car la frontière entre flexibilité la rigidité des sous-variétés legendriennes est intriquée et encore largement méconnue. En particulier, puisque distinguer des objets munis d'une structure algébrique est bien plus facile, ce problème requiert de construire de nouveaux invariants pour les sous-variétés legendriennes qui raffinent ceux qui proviennent déjà de constructions classiques en topologie algébrique.

Certaines de ces constructions sont maintenant décrites, et plus des détails sur la géométrie et la topologie de contact mettant l'accent sur leurs aspects mathématiques et physique peuvent être respectivement trouvés dans [Gei08] and [Arn89, Kho13].

1. La rigidité de contact des isotopies legendriennes

Comme mentionné ci-dessus, afin de mieux comprendre les obstructions à l'existence d'isotopies parmi les sous-variétés legendriennes, des données provenant de la topologie de contact doivent être prises en compte. Cette question est maintenant étudiée en vue de progresser vers l'approche par les familles génératrices à la rigidité de contact et les contributions plus précises de cette thèse.

1.1. Les invariants classiques et leurs limites.

Afin de développer de nouvelles techniques pour aborder cette question, il est raisonnable de d'abord essayer d'adapter les invariants algébriques qui existent déjà afin qu'ils puissent encoder des données topologiques provenant de la structure de contact définie dans un voisinage de la sous-variété legendrienne.

Soit $(M, \xi = \ker \alpha)$ une variété de contact *coorientée* avec l'orientation induite par $\alpha \wedge (\mathrm{d}\,\alpha)^n \neq 0$, et soit Λ une sous-variété legendrienne connexe, fermée et orientée dans (M, ξ), dans ce contexte, il est possible de définir deux invariants pour les isotopies parmi les sous-variétés legendriennes. Informellement, ces invariants sont définis pour respectivement capturer le comportement de Λ dans les directions tangentes et normales à ξ, grossièrement:

- la *classe de rotation* est une classe d'homotopie qui encode la rotation de $T\Lambda$ dans ξ ; et
- le *nombre de Thurston–Bennequin* compte les vrilles faites par ξ en parcourant Λ.

Le reste de cette sous-section est consacrée à leur définition plus précise.

1.1.1. *La classe de rotation.*

Soit $J : \xi \to \xi$ une *structure presque complexe* sur ξ qui est *compatible* avec la forme bilinéaire alternée et non dégénérée $\mathrm{d}\,\alpha_{|\xi}$ [MS95, Proposition 2.63], c'est-à-dire que les conditions suivantes sont satisfaites :

- J est un endomorphisme de fibrés vectoriels sur ξ tel que $J^2 = -\mathrm{id}_\xi$;
- pour tout $X \in \xi \setminus \{\mathbf{0}_\xi\}$, $\mathrm{d}\,\alpha(X, JX) > 0$; et
- pour tous $X \in \xi$ et $Y \in \xi$, $\mathrm{d}\,\alpha(JX, JY) = \mathrm{d}\,\alpha(X, Y)$.

Dès lors, J induit une métrique riemannienne $g_J = \mathrm{d}\,\alpha(\cdot, J\cdot)$ sur ξ pour laquelle $J(T\Lambda) = T\Lambda^\perp$, de sorte que Λ fournisse aussi un isomorphisme de fibrés vectoriels complexes $T\Lambda \otimes_{\mathbf{R}} \mathbf{C} \to \xi_{|\Lambda}$. Alors, la *classe de rotation* de Λ, noté $r(\Lambda)$, est la classe d'homotopie de $T\Lambda \otimes_{\mathbf{R}} \mathbf{C} \to \xi_{|\Lambda}$ parmi les isomorphismes de fibrés vectoriels complexes, et elle indépendante de J, car le sous-ensemble des structures presque complexes compatibles sur $(\xi, \mathrm{d}\,\alpha_{|\xi})$ est contractile [MS95, Proposition 4.1].

La classe de rotation est un invariant complet pour les homotopies régulières parmi les sous-variétés legendriennes immergées [EM02, Théorème 16.1.3], mais elle est souvent inextricable. Elle peut cependant être utilisée pour démontrer le théorème sur la rigidité de contact des classes d'isotopies des sous-variétés legendriennes.

1.1.2. *Le nombre de Thurston–Bennequin.*

Supposons pour simplifier que l'homologie relative de Λ dans M est nulle et soit alors $L \subset M$ une surface de Seifert immergée et orientée de Λ telle que l'orientation de Λ soit aussi induite par L. Soit R_ξ une section qui ne s'annule pas de TM/ξ et soit Λ_ξ un décalage de Λ par R_ξ tel que $L \pitchfork \Lambda_\xi$, alors $L \cap \Lambda_\xi$ est une sous-variété compacte de dimension 0 de M, car $\dim(\Lambda) + \dim(L) = \dim(M)$. Alors, le *nombre de Thurston-Bennequin* de Λ, noté $\mathrm{tb}(\Lambda)$, est le *nombre d'intersection algébrique* entre L et Λ_ξ, ou de manière équivalente le *nombre d'enlacement* de Λ avec Λ_ξ, définis par:

$$\mathrm{tb}(\Lambda) = \#(L, \Lambda_\xi) = \mathrm{enl}(\Lambda, \Lambda_\xi) = \sum_{x \in L \cap \Lambda_\xi} \varepsilon_x,$$

où $\varepsilon_x \in \{-1, 1\}$ et $\varepsilon_x = 1$ si, et seulement si, les orientations de $T_x L \oplus T_x \Lambda_\xi$ et $T_x M$ coïncident. En particulier, il est immédiat de vérifier que le nombre de Thurston–Bennequin de Λ ne dépend pas des choix faits pour R_ξ, L et de l'orientation de Λ.

Quand $(M, \xi) = (\mathbf{R}^3, \xi_{\mathbf{R}})$, (r, tb) est un invariant complet pour les isotopies legendriennes

- des nœuds legendriens topologiquement triviaux [EF09, Theorem 1.5] ;
- des nœuds legendriens toriques [EH01] ; et
- des nœuds legendriens en huit [EH01].

Ces résultats ne se généralisent aux nœuds legendriens avec des types de nœuds plus compliqués. Par exemple, il existe au moins deux nœuds legendriens de $(\mathbf{R}^3, \xi_{\mathbf{R}})$ qui sont lissement isotopes à des images miroirs de 5_2, mais qui ne sont pas legendriennement isotopes dans $(\mathbf{R}^3, \xi_{\mathbf{R}})$, bien qu'ils aient mêmes classe de rotation et nombre de Thurston–Bennequin [Che02, Théorème 1.1]. Cependant, il n'existe qu'un nombre fini de classes d'isotopie legendriennes ayant même type de nœud et mêmes classes de rotation et nombre de Thurston–Bennequin [CGH09, Théorème 10]. Donc, les nœuds legendriens de $(\mathbf{R}^3, \xi_{\mathbf{R}})$ sont quasiment classés par le type de nœud, la classe de rotation et le nombre de Thurston-Bennequin.

Cette approche n'est plus viable en grande dimension, puisque lorsque $n \geqslant 2$, il existe notamment une infinité de plongement legendrien de la sphère $\mathbf{S}^n$ de dimension n dans $(\mathbf{R}^{2n+1}, \xi_{\mathbf{R}^n})$ qui ont la même classe de rotation et le même nombre de Thurston–Bennequin, mais qui ne sont pourtant pas legendriennement isotopes dans $(\mathbf{R}^{2n+1}, \xi_{\mathbf{R}^n})$ [EES05b, Théorème 1.1].

1.2. Homologie pour les familles génératrices des sous-variétés legendriennes.

Beaucoup plus de sophistications sont nécessaires pour dépasser les simples invariants classiques des sous-variétés legendriennes. Dans cette thèse, toutes les variétés de contacts seront des fibrés de premiers jets, ce qui permettra de travailler avec des familles génératrices [Vit92, Cha95].

La section nulle du fibré de premiers jets $(J^1 B, \xi_B)$ est une sous-variété legendrienne qui se projette difféomorphiquement sur la base. Réciproquement, quand la projection d'une sous-variété sur la base est un difféomorphisme, alors elle provient d'une application lisse $f : B \to \mathbf{R}$ comme suit :

$$\{(b, T_b f, f(b)); b \in B\},$$

et il s'agit forcément d'une sous-variété legendrienne, legendriennement isotope à la section nulle. Cependant, une sous-variété legendrienne qui est obtenue par isotopie legendrienne de la section nulle peut avoir des singularités, de sorte qu'elle ne puisse plus être obtenue comme ci-dessus. Néanmoins, elle est encore obtenue à partir d'une application lisse $f : B \times \mathbf{R}^N \to \mathbf{R}$ comme suit :

$$\left\{\left(b, \frac{\partial f}{\partial b}(b,\eta), f(b,\eta)\right); \frac{\partial f}{\partial \eta}(b,\eta) = \mathbf{0}\right\},$$

[Che96, Théorème 2.5], et qui sera désormais appelée une *famille génératrice* (de fonctions). En s'inspirant de constructions en théorie de Morse, les familles génératrices permettent de définir de nouveaux invariants pour les isotopies parmi les sous-variétés legendriennes.

Avant de construire ces invariants par familles génératrices pour les sous-variétés legendriennes, il est d'abord préférable de rappeler l'approche moderne à l'homologie de Morse [Wit82, Flo89]. Non seulement cela fournira un contexte utile pour comprendre le cadre plus subtil de l'homologie pour les familles génératrices, mais cela permettra aussi ensuite de mieux décrire et motiver le procédé de dégénérescence de Henry et Rutherford qui est au cœur de cette thèse.

Soit M une variété fermée et soit $f : M \to \mathbf{R}$ une fonction de Morse, définissons $\mathrm{CM}_\bullet(f)$ comme étant l'espace vectoriel librement engendré sur $\mathbf{F}_2$ par les points critiques de f et qui est équipé de la graduation μ induite par l'indice de Morse des points critiques. Ensuite, pour faire de $\mathrm{CM}_\bullet(f)$ un complexe de chaînes, commençons d'abord par choisir une métrique riemannienne g sur M, puis pour tous les points critiques c_- et c_+ de f, définissons $\widehat{\mathscr{M}}(c_-, c_+; f, g)$ comme étant l'espace de modules des trajectoires de gradient de f pour g qui sont paramétrées et qui joignent c_- à c_+. Plus précisément, $\widehat{\mathscr{M}}(c_-, c_+; f, g)$ est l'ensemble des applications lisses $\gamma : \mathbf{R} \to M$ telles que

$$\begin{cases} \forall t \in \mathbf{R}, \partial_t \gamma(t) = -\nabla_g f(\gamma(t)), \\ \gamma(t) \xrightarrow[t \to -\infty]{} c_-, \\ \gamma(t) \xrightarrow[t \to +\infty]{} c_+. \end{cases}$$

Comme le groupe $(\mathbf{R}, +)$ agit librement par décalage temporel sur $\widehat{\mathscr{M}}(c_-, c_+; f, g)$, quand $c_- \neq c_+$, il est naturel de définir l'ensemble des orbites de cette action

$$\mathscr{M}(c_-, c_+; f, g) = \widehat{\mathscr{M}}(c_-, c_+; f, g)/\mathbf{R},$$

et de l'identifier avec l'espace de modules des trajectoires de gradient de f pour g qui sont ne sont pas paramétrisées et qui joignent c_- à c_+. Par ailleurs, la métrique riemannienne g peut être perturbée pour que le flot du gradient de f pour g soit un système dynamique de Morse-Smale, et que $\mathscr{M}(c_-, c_+; f, g)$ soit alors une variété de dimension $\mu(c_-) - \mu(c_+) - 1$ [Sch93, Théorème 1]. De plus, lorsque $\mathscr{M}(c_-, c_+; f, g)$ est de dimension 0, elle est compacte [AD10, Corollaire 3.2.4] et il est possible de construire une application linéaire $\partial_{f,g} : \mathrm{CM}_\bullet(f) \to \mathrm{CM}_{\bullet-1}(f)$ en la définissant sur les générateurs par la formule suivante

$$\partial_{f,g} c_- = \sum_{c_+} \#_{\mathbf{F}_2} \mathscr{M}(c_+, c_+; f, g) c_+,$$

où c_+ est un point critique d'indice de Morse $\mu(c_-) - 1$ et $\#_{\mathbf{F}_2}$ est un compte modulo 2 d'éléments. Afin d'avoir un complexe de chaînes, il suffit de démontrer que $\partial_{f,g}$ est bien un opérateur de bord, et pour cela il suffit de vérifier que le carré de $\partial_{f,g}$ s'annule bien sur les générateurs de $\mathrm{CM}_\bullet(f)$. Après un calcul direct, il s'agit de démontrer que

$$\sum_p \#_{\mathbf{F}_2}\mathscr{M}(q,p;f,g)\#_{\mathbf{F}_2}\mathscr{M}(p,r;f,g)=0,$$

où q, r et p sont des points critiques de f avec des indices de Morse décroissants et consécutifs. Comme pour dans la plupart des problèmes d'espaces de modules dans les théories homologiques, cette démonstration se fait en deux temps:

- **Compacité.** Comprendre les points d'adhérence des suites d'élements de $\mathscr{M}(q,p)$ afin de construire une compactification $\overline{\mathscr{M}(q,p)}$ de $\mathscr{M}(q,p)$.
- **Recollement.** Démontrer que les bords de l'espace compact $\overline{\mathscr{M}(q,p)}$ sont exactement des produits fibrés des espaces de modules utilisés pour définir l'application de bord.

Dans le cadre de l'homologie de Morse, le théorème de compacité [AD10, Théorème 3.2.2], ainsi que le théorème de recollement [AD10, Proposition 3.2.7] démontrent que les points sur les bords des compactifications des espaces de modules de dimension un sont en correspondance bijective avec des trajectoires de gradient brisées et non-paramétrisées :

$$\partial\overline{\mathscr{M}(q,p;f,g)}=\bigcup_p \mathscr{M}(q,p;f,g)\times\mathscr{M}(p,r;f,g).$$

En conclusion, la somme ci-dessus est bien nulle comme souhaité, puisqu'il y a un nombre pair de points sur le bord d'une variété compacte de dimension un, et l'*homologie de Morse* $\mathrm{HM}_\bullet(f)$ est alors définie comme l'homologie du complexe de chaînes $(\mathrm{CM}_\bullet(f),\partial_{f,g})$, c'est-à-dire

$$\mathrm{HM}_\bullet(M)=\ker\left(\partial_{f,g}\colon\ \mathrm{CM}_\bullet(f)\to\mathrm{CM}_{\bullet-1}(f)\right)/\,\mathrm{im}\left(\partial_{f,g}\colon\ \mathrm{CM}_{\bullet+1}(f)\to\mathrm{CM}_\bullet(f)\right).$$

L'homologie de Morse ne dépend finalement pas des choix de f et de g [AD10, Théorème 3.4.2], et il s'agit d'un invariant d'homotopie des variétés fermées [AD10, Théorème 4.6.2].

En mécanique classique, le principe fondamental de la dynamique assure qu'un objet qui est soumis à un champ de force doit suivre un chemin dans l'espace des phases qui minimise l'énergie. Dans une variété de contact coorientée (M,α), cette philosophie s'incarte dans l'étude de la théorie de Morse de la *fonctionnelle d'action* qui est défini sur les lacets par :

$$\mathscr{A}_\alpha\colon\gamma\mapsto\int_\gamma\alpha.$$

Sous des hypothèses techniques additionnelles, la théorie de Morse en dimension infinie de la fonctionnelle d'action, qui est aussi appelée *théorie symplectique des champs*, fournit des invariants intéressants des variétés de contact [EGH00]. En particulier, quand les chemins utilisés pour définir la fonctionnelle d'action débutent et terminent sur une même sous-variété legendrienne, alors la théorie symplectique des champs qui en résulte permet de construire des invariants par isotopies legendriennes [EES05a, EES07]. Bien que les invariants qui proviennent de la théorie symplectique des champs sont inconditionnellement puissant, leurs constructions reposent sur une analyse poussé et nécessitent de comprendre le comportement des solutions d'une équation aux dérivées partielles non linéaire.

En mécanique classique, les familles génératrices apparaissent très naturellement comme étant des lagrangiens de systèmes physiques et en topologie de contact, ils fournissent de manière similaire des modèles de dimension finie pour la fonctionnelle d'action. Plus concrètement, si une sous-variété legendrienne de (J^1B, ξ_B) a une famille génératrice $f : B \times \mathbf{R}^N \to \mathbf{R}$, sa *fonction différence* $\delta : B \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ est définie par

$$\delta(b, \eta_1, \eta_2) = f(b, \eta_1) - f(b, \eta_2),$$

et elle mesure la longueur entre des paires de points verticalement alignés dans la legendrienne. Choississons une métrique riemannienne g générique sur $B \times \mathbf{R}^N \times \mathbf{R}^N$, alors sous une hypothèse convenable et raisonnable sur le comportement à l'infini de la famille génératrice f, les espaces de modules de Morse $\mathscr{M}(c_-, c_+; \delta, g)$ qui sont associés à la fonction différence δ se comportent suffisamment bien pour reproduire la construction de l'homologie de Morse des variétés fermées. Cette construction aboutit à la définition de l'*homologie des familles génératrices* $\mathrm{HFG}_\bullet(f)$ de f, qui est indépendante du choix de g et permet de construire des invariants intéressants pour les isotopies legendriennes [Tra01]. En particulier, la caractéristique d'Euler–Poincaré de l'homologie des familles génératrices permet génériquement de calculer le nombre de Thurston–Bennequin des legendriennes dans $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ [EES05b, Propositions 3.2 et 3.3] et [ST13, Proposition 3.2]. En d'autres termes, l'homologie des familles génératrices réalise une *catégorification* du nombre de Thurston–Bennequin.

Dans le cadre des variétés fermées, l'homologie de Morse est un invariant d'homotopie effectif, puisqu'elle se calcule à partir d'espaces de modules d'objets géométriques qui sont facilement indentifiables sur la variétés, sans en connaître beaucoup d'informations. Cela n'est cependant plus le cas pour l'homologie des familles génératrices. Non seulement les familles génératrices des sous-variétés legendriennes sont souvent décrites qu'en des termes trop qualitatifs pour permettre des calculs en homologie, mais les espaces de modules utilisés ne sont aussi plus géométriques. En effet, le flot du gradient de la fonction différence n'a désormais plus lieu sur la sous-variété legendrienne en elle-même, mais plutôt sur un fibré vectoriel auxiliaire dans lequel elle se plonge non-canoniquement (par l'intermédiaire de la famille génératrice).

1.3. Le procédé de dégénérescence adiabatique de Henry et Rutherford.

Puisque l'homologie pour les familles génératrices est indépendante de la métrique riemannienne, en faire un choix soigneux peut aider à calculer en pratique l'homologie des familles génératrices. Idéalement, les lignes de gradient de la fonction différence devrait être tracée sur la legendrienne. Intuitivement, cet objectif est atteint en accélérant graduellement le flot de gradient dans la fibre, mais pour des raisons qui seront claires plus tard, il est plus commode de ralentir progressivement sa composante dans la base, cela revient à réaliser la *dégénérescence de Henry et Rutherford*. Rigoureusement, pour $s \in]0, 1]$, définissons d'abord

$$g_s = \left(s^{-1} g_B\right) \oplus g_F,$$

où g_B et g_F sont respectivement des métriques riemanniennes sur B et $\mathbf{R}^{2N}$, puis prenons $s \to 0$ dans l'équation différentielle qui caractérise le flot de gradient pour g_s de la fonction différence. Cependant, les lignes de flot qui résultent de ce flot de gradient singulièrement perturbé ne sont pas toujours tracées sur la sous-variété legendrienne elle-même, puisqu'un calcul (**Lemme 3.1**) montre que les trajectoires du flot de gradient pour g_s

- sont tangentes aux fibres loin du lieu fibrement critique de la fonction différence ; et
- s'annulent le long du lieu fibrement critique de la fonction différence, mais si elles sont préalablement redimensionnées par un facteur d'ordre s^{-1}, alors elles sont tangentes au lieu fibrement critique de la fonction différence.

Ainsi, le procédé de dégénérescence de Henry–Rutherford semble donc « converger » vers des trajectoires brisées obtenues par concaténation de deux types de trajectoires de gradient alternant entre *fragments verticaux* (dans le premier cas) et *fragments horizontaux* (dans le second cas). Quand ces trajectoires de gradient brisées n'ont qu'un nombre fini de fragments élementaires, elles sont appelées *escaliers de gradients* et l'ensemble de tous les escaliers de gradients qui sont non-paramétrisés et joignent c_- à c_+ est noté $\mathscr{M}^{\mathrm{esc}}(c_-, c_+; \delta, g_1)$ (**Définition 3.2**).

La démonstration de la version standard de l'équivalence de cobordisme obtenue par compacité et recollement montre que si $s_0 \in]0,1]$ est suffisamment petit, $\mathscr{M}(c_-, c_+; \delta, g_s)$ et $\mathscr{M}(c_-, c_+; \delta, g_{s_0})$ sont en correspondance bijective, dès qu'elles sont des variétés finies [**AD10**, Proposition 3.4.3]. Quand $\mathscr{M}(c_-, c_+; \delta, g_0)$ est remplacé par $\mathscr{M}^{\mathrm{esc}}(c_-, c_+; \delta, g_1)$, il est également conjecturé que cette correspondance bijective s'étende à $s = 0$, il s'agit de l'énoncé de la

Conjecture A (inspiré de [**HR13**, Conjecture 6.3], **Conjecture 3.1**)**.** *Si $s_0 \in]0,1]$ est assez petit, alors il existe une correspondance bijective ensembliste entre $\mathscr{M}(c_-, c_+; \delta, g_{s_0})$ et $\mathscr{M}^{\mathrm{esc}}(c_-, c_+; \delta, g_1)$, dès qu'il s'agit de variétés compactes de dimension* 0.

Cependant, comme $(g_s)_s$ n'est pas une homotopie de métriques riemmanniennes (puisque g_0 n'est même pas définie correctement), la **Conjecture A** ne peut pas être directement déduite du résultat usuel de compacité-recollement en théorie de Morse et qui est mentionné juste au-dessus.

Supposons que la **Conjecture A** soit vraie, le procédé de dégénérescence de Henry–Rutherford ne peut alors pas « converger » vers des trajectoires de gradient qui sont entièrement contenues dans la sous-variété legendrienne, mais cet objectif était de toute façon naïf, puisque cela signifierait que l'homologie pour les familles génératrices ne dépend que de la sous-variété legendrienne. Cependant, la **Conjecture A** permet tout de même de calculer en pratique l'homologie des familles génératrices, puisque

- les fragments verticaux sont liés aux glissement d'anses des familles génératrices ; et
- les fragments horizontaux sont entièrement déterminé par le front legendrien.

Travailler avec des espaces de modules d'escaliers de gradient permet de séparer les contributions qui interviennent dans le calcul des bords dans le complexe de chaînes des familles génératrices. D'une part, les fragments verticaux tiennent compte des bifurcations dans les points fibrement critiques de la famille génératrice et d'autre part, les fragments horizontaux de la géométrie de la sous-variété legendrienne.

La **Conjecture A** a d'abord été introduite comme fondement géométrique à l'algèbre différentielle graduée combinatoire qui est construite à partir de familles génératrices [**HR13**, Théorème 5.4].

En particulier, la Conjecture A en conjonction avec les [HR13, Théorèmes 5.4 and 5.5] permettent de retrouver le [FR11, Théorème 5.1] qui fait la construction d'une augmentation de l'algèbre de Tchekanov-Eliashberg d'un nœud legendrien à partir d'une famille génératrice linéaire à l'infini. Plus généralement, il est attendu que la Conjecture A joue un rôle important pour attaquer en dimensions quelconques, l'équivalence complète entre familles génératrices et augmentations de l'algèbre de Tchekanov-Eliashberg.

Il est important de mentionner que des espaces de modules de trajectoires brisées et des résultats similaires à la Conjecture A existent déjà dans la littérature. Par exemple, dans le contexte

- de homologie de Morse–Bott [Fra04, Appendix A], [BH13] ;
- de homologie de contact [Bou02] ; et
- de l'homologie symplectique [BO09a, BO09b].

Les démonstratiosn sont toutes basées sur des techniques de compacité-recollement, mais elles ont toutes leurs propres spécificités et difficultés analytiques, et la Conjecture A n'y fait pas exception.

2. Principaux résultats de la thèse

Le contexte est désormais bien mis en place pour énoncer les principaux résultats de ce mémoire. En particulier, cette section explique dans quelle mesure cette thèse réalise les premiers pas vers une démonstration complète de la Conjecture A.

2.1. Analyse sur les espaces de modules des chaînes d'escaliers de gradient.

Comme nous l'avons mentionné juste au-dessus, les correspondances entre espaces de modules, qui sont similaires à la Conjecture A, sont généralement démontrés en exploitant des techniques par compacité et recollement, et ce mémoire s'occupe précisément de la partie compacité pour le procédé de dégénérescence de Henry et Rutherford.

Afin de donner des énoncés plus précis, fixons quelques notations pour le reste de l'introduction. Soit Λ une sous-variété legendrienne fermée et connexe de (J^1B, ξ_B), soit $\delta\colon B \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ une fonction différence d'une famille génératrice linéaire à l'infini de Λ et soit

$$g_s = \left(s^{-1} g_B\right) \oplus g_N \oplus g_N,$$

où $s \in]0,1]$, g_B est une métrique riemannienne sur B et g_N est une métrique riemannienne sur $\mathbf{R}^N$. Dès lors, cette thèse étudie le comportement limite du flot de gradient de δ pour g_s quand $s \to 0$. Avant de pouvoir énoncer le théorème principal de cette thèse, il faut encore se rappeler qu'en théorie de Morse standard pour toute suite $\gamma_k \in \mathcal{M}(c_-, c_+; \delta, g_1)$, il existe une sous-suite (toujours noté de la même manière), un multiplet $(\gamma_1, \dots, \gamma_m)$, appelé *trajectoire brisée de gradient*, avec

- pour tout $i \in \{1, \dots, m\}$, $\gamma_i \in \mathcal{M}(c_i, c_{i+1}; \delta, g_1)$; et
- $c_1 = c_-$ et $c_{m+1} = c_+$;

et pour tout $i \in \{1, \dots, m\}$, une suite $(\tau_k^i)_{k \in \mathbf{N}}$ de décalages telle que $\gamma_k(\cdot + \tau_k^i) \underset{k}{\longrightarrow} \gamma_i$ en topologie C^1_{loc}. De même, pour étudier les accumulations des suites $\gamma_k \in \mathcal{M}(c_-, c_+; \delta, g_{s_k})$, où $s_k \in]0,1]$ et $s_k \underset{k}{\longrightarrow} 0$, il est pertinent de d'abord introduire les *chaînes d'escaliers de gradient* (Définition 3.3) qui sont des multiplets $\underline{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_\mathbf{m})$ avec

- pour tout $i \in \{1, \ldots, m\}$, $\mathbf{e_i} \in \mathscr{M}^{\mathrm{esc}}(c_i, c_{i+1}; \delta, g_1)$; et
- $c_1 = c_-$ et $c_{m+1} = c_+$;

et l'ensemble de toutes les chaînes d'escaliers non-paramétrées de c_- à c_+ est $\overline{\mathscr{M}^{\mathrm{esc}}}(c_-, c_+; \delta, g_1)$.
Avec tous ces rappels, l'énoncé de compacité associé au procédé de dégénérescence de Henry et Rutherford qui est présenté dans cette thèse se formule désormais ainsi

Théorème A (**Théorème 4.1**). *Supposons que $\Lambda \to B$ n'a que des singularités de type pli de Whitney. Alors Λ peut être perturbée de sorte que pour tout $\gamma_k \in \mathscr{M}(c_-, c_+; \delta, g_{s_k})$, avec $s_k \in]0, 1]$ et $s_k \to 0$, il existe une sous-suite (noté de la même manière) $(\gamma_k)_{k \in \mathbf{N}}$ et $\underline{\mathbf{e}} \in \overline{\mathscr{M}^{\mathrm{esc}}}(c_-, c_+; \delta, g_{s_k})$ tels que $\gamma_k \xrightarrow[k]{} \underline{\mathbf{e}}$ dans la topologie de Floer–Gromov.*

La convergence pour la topologie de Floer–Gromov est expliquée dans les **Définitions 3.4** et **3.5**, mais il convient d'y penser comme étant une convergence vers des trajectoires de gradient brisées, avec tout de même un facteur d'échelle temporel de s_k^{-1} dans le cas des fragments horizontaux, sans quoi les trajectoires récupérées seraient constantes (voir la **Section 1.3** ci-dessus).

Le **Théorème A** est probablement vrai sans restriction sur les singularités de la caustique de Λ, mais cela permet d'éviter des arguments combinatoires élaborés, surtout pour la **Proposition 4.7**. Quoi qu'il en soit, cette hypothèse revient à une question homotopique [**AG18**, Théorème 1.11] et elle satisfaite par de très nombreux exemples significatifs de legendriennes [**Ekh07**, **BST15**]. Cependant, il est incontournable de travailler avec des sous-variété legendriennes génériques, sans quoi le **Théorème A** peut être mis en échec pour des raisons véritablement profondes et dont nous allons désormais discuter.

2.2. Une difficulté technique majeure : un phénomène de bouillonnement infini.

Quand un groupe non-compact agit librement sur l'ensemble des solutions d'une équation aux dérivées partielles, alors les espaces de modules de solutions non-paramétrées qui lui sont associés sont généralement compactifiés par l'ajout d'objets brisés.
Pour mieux illustrer ce phénomène commun, considérons deux exemples clefs :

- En théorie de Morse, $(\mathbf{R}, +)$ agit librement par décalage temporel sur le flot du gradient, et les espaces de modules des trajectoires de gradient non-paramétrées sont compactifiés par l'ajout de trajectoires de gradient brisées [**AD10**, Théorème 3.2.2].
- En théorie de Gromov–Witten, $\mathrm{PSL}(2, \mathbf{R})$ agit librement par homographie sur l'équation de Cauchy–Riemann, et les espaces de modules des courbes pseudo-holomorphes qui sont non-paramétrées sont compactifiés par des courbes pseudo-holomorphes brisées [**MS04**, Théorème 4.4.3].

Pour démontrer de tels résultats de compacité, il faut d'abord s'assurer que le comportement analytique des solutions au bord des espaces de modules qu'elles forment n'est pas trop sauvage, ce qui signifie typiquement que leurs limites ne peuvent avoir qu'un nombre fini de fragments. Cette étape est généralement automatique, elle s'accomplit seulement en démontrant que chaque fragment non-constant consomme une quantité prédéterminée d'une « énergie » disponible en quantité totale finie.
Une fois encore, cette stratégie est bien comprise dans les deux exemples ci-dessus :

- En théorie de Morse, la valeur de la fonction de Morse décroît strictement le long des trajectoires de gradient non-constantes.
- En théorie de Gromov–Witten, l'aire symplectique des courbes pseudo-holomorphes qui sont non-constantes est uniformément minorée [MS04, Lemme 4.5.2].

Dans le cadre du Théorème A, même si la quantité totale de longueur verticale disponible est finie, rien n'empêche les fragments verticaux de devenir arbitrairement courts au voisinage des plis. Pour cette raison, les arguments énergétiques standards précédents ne suffisent pas à garantir que le procédé de degénéréscence de Henry et Rutherford ne récupère que des chaînes d'escaliers de gradient ayant un nombre fini de fragments élémentaires.

Cependant, une analyse bien plus élaborée montre que lorsque le procédé de dégénérescence de Henry et Rutherford récupère un escalier de gradient avec une infinité de fragments élémentaires, alors ses fragments horizontaux ont nécessairement des « tangences arbitrairement profondes » avec l'ensemble des points critiques de la projection caustique de la sous-variété legendrienne. Cela motive la définition des sous-variétés legendriennes *génériques de gradient* (Definition 1.11), puisqu'un champ de vecteur ne peut pas être tangent à une sous-variété générique à un ordre supérieur à sa dimension.
Avec toutes ces observations, nous pouvons formuler le

Théorème B (Théorème 4.3)**.** *Sur une sous-variété legendrienne générique de gradient, le procédé de degénérescence de Henry–Rutherford peut seulement « converger » vers des chaînes d'escaliers de gradient avec un nombre fini de fragments élémentaires.*

La démonstration du Théorème B explique pourquoi il est préférable de ralentir graduellement le flot du gradient de la fonction différence dans la base, plutôt que de l'accélérer dans la fibre, comme il était pourtant originellement suggéré de procéder dans [HR13].

Le Théorème B est intéressant, car les legendriennes génériques de gradient sont nombreuses. Plus précisément, la généricité de gradient est (comme son nom l'indique) une propriété générique parmi les sous-variétés legendriennes, il s'agit de l'énoncé du

Théorème C (Théorème 1.2)**.** *Le sous-ensemble formé des sous-variétés legendriennes génériques de gradient de type topologique fixé est ouvert et dense dans l'ensemble des plongements legendriens, muni de la topologie induite par la topologie C^∞ sur les applications lisses.*

Le Théorème A est parfaitement adapté pour faire des calculs en pratique, puisque :

- vérifier qu'une sous-variété legendrienne est générique de gradient ; et
- perturber une sous-variété legendrienne pour qu'elle soit générique de gradient ;

sont des procédures entièrement explicites.

Organisation du mémoire

Cette thèse s'organise autour de quatre chapitres mutuellement dépendants qui aboutissent à la démonstration du Théorème A, et elle se termine avec un cinquième chapitre indépendant qui contient des exemples illustrant la pertinence de la Conjecture A. Un grand soin a été apporté pour s'assurer que ce mémoire soit autosuffisant. En particulier, plusieurs résultats standards ont été adaptés afin de mieux convenir à l'utilisation qui allait être faite dans la thèse et plusieurs lacunes mineures dans des articles de recherche ont également été comblées.

- Chapitre 1 – **Conditions de transversalité pour les sous-variétés legendriennes**

Ce chapitre est consacré aux sous-variétés legendriennes génériques de gradient (Définition 1.11) et fournit, en particulier, la démonstration du Théorème C (Théorème 1.2) dans la Section 3. Il contient des rappels élémentaires de topologie de contact dans la Section 1, d'abord dans des variétés de contact abstraites (Section 1.1), puis dans des fibrés de premiers jets (Section 1.2). Des rappels supplémentaires sur l'étude des sous-variétés legendriennes sont également donnés dans la Section 2 et à la fin de la Section 3 qui contient les définitions respectives de la généricité de front (Définition 1.8) et de la généricité des cordes (Définition 1.12).

- Chapitre 2 – **Homologies pour les familles génératrices des sous-variétés legendriennes**

Ce chapitre est consacré à l'étude des sous-variétés legendriennes à l'aide des familles génératrices. La Section 1 débute avec un panoramique de la théorie des familles génératrices des sous-variétés legendriennes, avant de rappeller leur propriété de relèvement des homotopies (Théorème 2.1). La Section 1.2 établit un dictionnaire entre les propriétés génériques du Chapitre 1 pour les sous-variétés legendriennes et certaines propriétés de transversalité des familles génératrices. Ensuite, en vue de commencer à aborder la démonstration du Théorème A, la Section 2 étudie les familles génératrices du point de vue de la théorie de Morse (Proposition 2.4), puis deux versions simple/mixte de l'homologie des familles génératrices sont construites (Définition 2.6). Finalement, la Section 3 fait l'inventaire des résultats connus concernant la structure algébrique de l'homologie pour les familles génératrices dans sa version simple (Théorèmes 2.3 et 2.5). Puis, plusieurs énoncés similaires sont également étendus à sa version mixte (Théorème 2.4), ou sont simplement discutés et conjecturés (Conjectures 2.1 et 2.2).

- Chapitre 3 – **Le procédé de dégénérescence de Henry et Rutherford**

Ce chapitre contient tout le contenu nécessaire pour formuler la Conjecture A (Conjecture 3.1) et commencer à démontrer le Théorème A. Dans la Section 1, le procédé de dégénérescence de Henry et Rutherford est présenté comme étant un système dynamique lent-rapide pour lequel le sous-ensemble maximalement invariant n'est pas normalement hyperbolique en codimension un. Ensuite, la Section 2 est consacré aux escaliers et chaînes d'escaliers (Définitions 3.4 et 3.5), avec la Section 2.1 qui étudie la théorie de Morse des fragments verticaux (Proposition 3.1). Dans la Section 3, la convergence dans la topologie de Floer–Gromov (Définitions 3.4 et 3.5) est étudiée et plusieurs propriétés importantes pour décrire le comportement limite du procédé de dégenérescence de Henry et Rutherford sont également démontrées (Proposition 3.2).

- **Chapitre 4 – Compacité de Floer et Gromov pour les chaînes d'escaliers de gradient**

Ce chapitre est consacré à la démonstration du **Théorème A**, résultat principal de cette thèse. Ainsi, la **Section 1** décrit des procédures récursives permettant de récupérer l'un après l'autre des fragments élémentaires de chaînes d'escaliers de gradient dans la limite de Henry et Rutherford. La **Section 1.1** explique d'abord comment récupérer les fragments verticaux (**Proposition 4.1**), puis la **Section 1.2** s'occupe du cas plus compliqué des fragments horizontaux (**Théorème 4.2**). Ensuite, dans la **Section 2**, le **Théorème B** (**Théorème 4.3**) est démontré afin d'assurer que sur une sous-variété legendrienne générique de gradient, les procédures récursives de la **Section 1** s'arrêtent après un nombre fini d'étapes. Finalement, dans la **Section 3**, tous ces résultats sont mis ensemble pour démontrer le **Théorème A** (**Théorème 4.1**).

- **Chapitre 5 – Exemples**

Ce chapitre explique comment les espaces de modules d'escaliers de gradient peuvent être utilisés pour réaliser en dimension quelconque des calculs homologiques avec des familles génératrices. Ces exemples permettent de justifier l'importance de la version mixte de l'homologie pour les familles génératrices et sont le point de départ essentiel pour aborder les conjectures soulevées dans le **Chapitre 2**.

- **Perspectives de recherche**

Ce mémoire est enfin clôturé par la description de plusieurs directions de recherche futures qui suivent les perspectives ouvertes par le **Théorème A**.

Introduction (in English)

This thesis is concerned with the investigation of contact rigidity aspects of isotopies through Legendrian submanifolds in contact manifolds. Nowadays, these rigidity questions are classically tackled by working with one or several of the three following advanced techniques:

- pseudo-holomorphic curves [EES05a, EES05b, EES07, BC14];
- generating families [Tra01, ST13, SS16]; or
- constructible sheaves [GKS12, STZ17].

In recent years, the study of the correspondence between these three different approaches has been a fruitful research framework for contact topology [FR11, HR13, RS18, NRS+20, RS21]. The research featured here is about generating family invariants of Legendrian submanifolds, but it should offer promising directions to better understand the correspondence between generating families and augmentations of the Chekanov-Eliashberg algebra [HR13].

More concretely, this work is the first step towards showing that the generating family homology of Legendrian submanifolds can be computed from convenient moduli spaces of broken trajectories. Its main result is a compactness theorem (**Theorem A**), showing that *gradient staircases* arise from a singular perturbation of the genuine gradient flow used to define generating family homology. This *Henry–Rutherford limiting process* is expected to split apart the distinct inputs involved in the computation of the boundary operator of the generating family chain complex (**Conjecture A**). On one side, the contributions coming from the Legendrian submanifold, and on the other side, the ones due to bifurcations in the complex of critical points of the generating families.

The rest of the introduction is devoted to the more precise exposition of the Henry–Rutherford limiting process and the moduli spaces of gradient staircases (see **Subsection 1.3**), and then states the main results of this dissertation (see **Section 2**). It is, however, more appropriate to first go through some recollections from Legendrian submanifolds and generating families.

Some context and motivation

A *contact manifold* is a pair (M^{2n+1}, ξ), where M is a smooth odd-dimensional manifold and ξ is a maximally non-integrable smooth tangent hyperplane field of M, called a *contact structure*. If α is a local one-form equation for ξ, meaning that ξ is locally equal to $\ker(\alpha)$, then $\alpha \wedge (\mathrm{d}\,\alpha)^n$ is a nowhere vanishing top form, or equivalently that $\mathrm{d}\,\alpha_{|\xi}$ is a non-degenerate skew-symmetric bilinear form (so that M cannot be even-dimensional). Not only Frobenius theorem implies that contact structures are not foliations (as $\alpha \wedge \mathrm{d}\,\alpha \neq 0$), they are as far away as possible from them. Indeed, if a smooth submanifold of M is everywhere tangent to ξ, then it is at most n-dimensional, and such a smooth integral submanifold of ξ is a *Legendrian submanifold* whenever it has maximal dimension n for this property.

Unlike Riemannian manifolds, contact manifolds cannot have any local invariant, like curvature, or to put it slightly differently, only their global topological properties are relevant to their study. Indeed, all contact manifolds are locally modelled on $\mathbf{R}^{2n+1}$ with coordinates $(x, y, z) \in \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, and contact structure $\xi_{\mathbf{R}^n}$ given by

$$\xi_{\mathbf{R}^n} = \ker\left(\alpha_{\mathbf{R}^n} = \mathrm{d}\, z - \sum_{k=1}^{n} y_k \,\mathrm{d}\, x_k\right),$$

[Gei08, Theorem 2.5.1] and this construction can also be generalised to first order jet bundles. Let B be a smooth manifold, its first order jet bundle $J^1B = T^*B \times \mathbf{R}$ has local coordinates (q, p, z), where q are local coordinates on B, p are dual coordinates of q and z is a global coordinate on $\mathbf{R}$. In this setting, J^1B can be endowed with a contact structure $\xi_B = \ker(\alpha_B = \mathrm{d}\, z - p \,\mathrm{d}\, q)$ for which the zero-section of J^1B naturally defines a Legendrian submanifold.

Since Legendrian submanifolds have large codimension in contact manifolds, it can be expected that they are highly flexible submanifolds, in the Gromov's h-principle sense [Gro86, EM02]. Roughly speaking, the obstructions to the existence of isotopies through smooth submanifolds between Legendrian submanifolds are expected to amount to constraints of homotopical nature. Arguing in this direction, any closed smooth n-dimensional submanifold of a $(2n+1)$-dimensional contact manifold is C^0-close to a Legendrian submanifold, provided some homotopical condition on its tangent bundle is met [Gei08, Proposition 6.3.6]. In contrast, isotopies through Legendrian submanifolds exhibit way more rigidity. For example, a folklore theorem asserts that the smooth isotopy class of any Legendrian submanifold splits into infinitely many Legendrian isotopy classes.

Due to the aforementioned contact rigidity phenomena, grasping obstructions to the existence of isotopies through Legendrian submanifolds is tricky, since for example, the interplay between flexibility and rigidity of Legendrian submanifolds is intricate and still not yet well understood. In particular, since it is easier to distinguish objects with an algebraic structure, making progress on this question requires to construct invariants for Legendrian submanifolds refining the ones already coming from classical constructions in algebraic topology.

Some of these constructions are now discussed, but in the meantime, more details about contact geometry and topology, respectively emphasising the mathematical and physical viewpoints, can be found in [Gei08] and [Arn89, Kho13].

1. Contact rigidity of Legendrian isotopies

As it was just mentioned, in order to better understand obstructions to the existence of isotopies through Legendrian submanifolds, inputs from contact topology need to be taken into account. This matter is now discussed, building towards the generating family approach to contact rigidity of Legendrian submanifolds and the more precise contributions of this thesis.

1.1. Classical invariants and their limitations.

Before developing new techniques to tackle this question, it is first reasonable to adapt already existing algebraic invariants so that they can encode topological data coming from the existing contact structure defined in a neighborhood of a Legendrian submanifold.

Let $(M, \xi = \ker \alpha)$ be a *cooriented* contact manifold with the orientation induced by $\alpha \wedge (\mathrm{d}\,\alpha)^n \neq 0$, and let Λ be a connected, closed and oriented Legendrian submanifold of (M, ξ), in this setting, it is then possible to introduce two invariants under isotopies through Legendrian isotopies of Λ. Informally, these invariants are defined to respectively capture the behaviour of Λ in the tangent and normal directions of ξ, roughly speaking:

- the *rotation class* of Λ is a homotopy class characterising how $T\Lambda$ spins in ξ; and
- the *Thurston–Bennequin number* of Λ counts how many twists ξ makes along Λ.

The rest of this subsection is now devoted to their more rigorous definitions.

1.1.1. *The rotation class.*

Let $J : \xi \to \xi$ be an *almost complex structure* on ξ which is *compatible* with the non-degenerate skew-symmetric bilinear form $\mathrm{d}\,\alpha_{|\xi}$ [MS95, Proposition 2.63], meaning that the following three conditions are satisfied:

- J is a bundle endomorphism of ξ such that $J^2 = -\mathrm{id}_\xi$;
- for all $X \in \xi \setminus \{\mathbf{0}_\xi\}$, $\mathrm{d}\,\alpha(X, JX) > 0$; and
- for all $X \in \xi$ and $Y \in \xi$, $\mathrm{d}\,\alpha(JX, JY) = \mathrm{d}\,\alpha(X, Y)$.

In particular, J induces a Riemannian bundle metric $g_J = \mathrm{d}\,\alpha(\cdot, J\cdot)$ on ξ for which $J(T\Lambda) = T\Lambda^\perp$ (since Λ is Legendrian), so that Λ also gives a complex vector bundles isomorphism $T\Lambda \otimes_{\mathbf{R}} \mathbf{C} \to \xi_{|\Lambda}$. Then, the *rotation class*, denoted by $r(\Lambda)$, is the homotopy class of $T\Lambda \otimes_{\mathbf{R}} \mathbf{C} \to \xi_{|\Lambda}$ among complex vector bundles isomorphisms, and it is independent of J, since the subset of compatible almost complex structures on $(\xi, \mathrm{d}\,\alpha_{|\xi})$ is contractible [MS95, Proposition 4.1].

The rotation class is a complete invariant for regular homotopy through immersed Legendrian submanifolds [EM02, Theorem 16.1.3], but it is often intractable. It can nonetheless be used to prove the folklore theorem on the contact rigidity of isotopy classes of Legendrian submanifolds.

1.1.2. *The Thurston–Bennequin number.*

Assume, for the sake of simplicity, that Λ is null-homologous in M and let L be a immersed and oriented Seifert surface of Λ such that the orientation of Λ agrees with this that is induced by L. Let R_ξ be a nowhere vanishing section of TM/ξ and let Λ_ξ be a shift of Λ by R_ξ such that $L \pitchfork \Lambda_\xi$, then $L \cap \Lambda_\xi$ is a compact 0-dimensional submanifold of M, since $\dim(\Lambda) + \dim(L) = \dim(M)$. Then, the *Thurston-Bennequin number* of Λ, denoted by $\mathrm{tb}(\Lambda)$, is the *algebraic intersection number* between L and Λ_ξ, or equivalently the *linking number* of Λ and Λ_ξ, which are defined as:

$$\mathrm{tb}(\Lambda) = \#(L, \Lambda_\xi) = \mathrm{lk}(\Lambda, \Lambda_\xi) = \sum_{x \in L \cap \Lambda_\xi} \varepsilon_x,$$

where $\varepsilon_x \in \{-1, 1\}$ and $\varepsilon_x = 1$ if, and only if, the orientations of $T_x L \oplus T_x \Lambda_\xi$ and $T_x M$ are equal. In particular, it is straightforward to check that the Thurston–Bennequin number of Λ does not dependent on the choices of R_ξ, L and the orientation of Λ.

When $(M, \xi) = (\mathbf{R}^3, \xi_{\mathbf{R}})$, the pair (r, tb) is a complete invariant for Legendrian isotopies of:

- topologically trivial Legendrian knots [EF09, Theorem 1.5];
- Legendrian torus knots [EH01]; and
- Legendrian figure-eight knots [EH01].

These results cannot be extended to Legendrian knots with more complicated smooth knot types. For example, there exist at least two Legendrian knots of $(\mathbf{R}^3, \xi_{\mathbf{R}})$ which are smoothly isotopic to mirror images of 5_2 (3-twist knots), but not isotopic through Legendrian knots of $(\mathbf{R}^3, \xi_{\mathbf{R}})$, despite having the same rotation class and Thurston–Bennequin number [Che02, Theorem 1.1]. However, within any smooth knot type, there are only finitely many isotopy classes of Legendrian knots having the same rotation class and Thurston–Bennequin number [CGH09, Théorème 10]. Therefore, Legendrian knots of $(\mathbf{R}^3, \xi_{\mathbf{R}})$ are almost completely classified by the smooth knot type, the rotation class and the Thurston–Bennequin number.

However, this approach is no longer viable in higher dimension, since when $n \geqslant 2$, there exist, for example, infinitely many Legendrian embeddings of the n-dimensional sphere $\mathbf{S}^n$ into $(\mathbf{R}^{2n+1}, \xi_{\mathbf{R}^n})$ with the same rotation class and Thurston–Bennequin number, but which are not isotopic through Legendrian submanifolds of $(\mathbf{R}^{2n+1}, \xi_{\mathbf{R}^n})$ [EES05b, Theorem 1.1].

1.2. Generating family homology of Legendrian submanifolds.

Going beyond classical invariants of Legendrian submanifolds requires some more sophistication. In this thesis, the contact manifolds considered are always first order jet bundles of manifolds, which allows working with generating families [Vit92, Cha95].

The zero-section of a first order jet bundle $(J^1 B, \xi_B)$ is a Legendrian submanifold whose projection onto the base is a diffeomorphism. Conversely, when the projection of a submanifold onto the base is a diffeomorphism, it is obtained from a smooth map $f : B \to \mathbf{R}$ as follows:

$$\{(b, T_b f, f(b)); b \in B\},$$

and it is automatically a Legendrian submanifold which is Legendrian isotopic to the zero-section. However, a Legendrian submanifold which is obtained from the zero-section by an isotopy through Legendrian submanifolds can have singularities, so that it can no longer be obtained as above. Nonetheless, it is still given by a smooth map $f : B \times \mathbf{R}^N \to \mathbf{R}$ by:

$$\left\{ \left(b, \frac{\partial f}{\partial b}(b, \eta), f(b, \eta) \right); \frac{\partial f}{\partial \eta}(b, \eta) = \mathbf{0} \right\},$$

[Che96, Theorem 2.5] and from now on it will be referred to as a *generating family* (of functions). Mimicking constructions borrowed from Morse theory, generating families allow the definition of new invariants for isotopies through Legendrian submanifolds.

Before constructing invariants of Legendrian submanifolds from generating families, it is better to start with some basic recollections from the modern approach to Morse homology [Wit82, Flo89]. It will not only provide useful background to understand the more subtle setting of generating family homology, but also helps to describe and motivate the Henry–Rutherford limiting process which is at the heart of this thesis.

Let M be a closed manifold and let $f : M \to \mathbf{R}$ be a Morse function, then let us define $\mathrm{MC}_\bullet(f)$ as the free vector space generated over $\mathbf{F}_2$ by the critical points of f and endowed with the grading μ induced by the Morse index of critical points. In order to upgrade $\mathrm{MC}_\bullet(f)$ to a chain complex, pick a Riemannian metric g on M and for all critical points c_- and c_+, let us define $\widehat{\mathscr{M}}(c_-, c_+; f, g)$ to be the moduli space of parametrized g-gradient flow lines of f that are going from c_- to c_+. More explicitly, the set $\widehat{\mathscr{M}}(c_-, c_+; f, g)$ is made of the smooth maps $\gamma\colon \mathbf{R} \to M$ such that

$$\begin{cases} \forall t \in \mathbf{R}, \partial_t \gamma(t) = -\nabla_g f(\gamma(t)), \\ \gamma(t) \xrightarrow[t\to-\infty]{} c_-, \\ \gamma(t) \xrightarrow[t\to+\infty]{} c_+. \end{cases}$$

Since the group $(\mathbf{R}, +)$ acts freely by time-translation on the set $\widehat{\mathscr{M}}(c_-, c_+; f, g)$, as soon as $c_- \neq c_+$, it is then only natural to define the orbit space of this action

$$\mathscr{M}(c_-, c_+; f, g) = \widehat{\mathscr{M}}(c_-, c_+; f, g)/\mathbf{R},$$

and identify it with the moduli space of unparameterized g-gradient flow lines of f from c_- to c_+. Besides, g can be smoothly perturbed so that the g-gradient flow of f is a Morse-Smale system, and the set $\mathscr{M}(c_-, c_+; f, g)$ is then a manifold of dimension $\mu(c_-) - \mu(c_+) - 1$ [Sch93, Theorem 1]. Moreover, when $\mathscr{M}(c_-, c_+; f, g)$ is zero-dimensional, it is also compact [AD10, Corollaire 3.2.4], so that it allows to define a linear map $\partial_{f,g}\colon \mathrm{MC}_\bullet(f) \to \mathrm{MC}_{\bullet-1}(f)$, given on the generators by

$$\partial_{f,g} c_- = \sum_{c_+} \#_{\mathbf{F}_2} \mathscr{M}(c_+, c_+; f, g) c_+,$$

where c_+ is a critical point of Morse index $\mu(c_-) - 1$ and $\#_{\mathbf{F}_2}$ is the modulo 2 count of elements. In order to have a chain complex, it only remains to show that $\partial_{f,g}$ is a boundary operator and for that purpose, it suffices to prove that $\partial_{f,g}$ indeed squares to zero on the generators of $\mathrm{MC}_\bullet(f)$. After a straightforward computation, it amounts showing that

$$\sum_p \#_{\mathbf{F}_2} \mathscr{M}(q, p; f, g) \#_{\mathbf{F}_2} \mathscr{M}(p, r; f, g) = 0,$$

where q, r and p are arbitrary critical points with strictly decreasing and consecutive Morse index. As it is customary with moduli problems in homology theories, the proof is carried in two steps:

- **Compactness.** Understand the limit points of the sequences of elements in $\mathscr{M}(q, p)$ in order to construct a compactification $\overline{\mathscr{M}(q, p)}$ of $\mathscr{M}(q, p)$.
- **Gluing.** Prove that the boundary components of the compact space $\overline{\mathscr{M}(q, p)}$ are exactly fibred products of the moduli spaces used to define the boundary operator.

In the Morse setting, the compactness theorem [AD10, Théorème 3.2.2] and the gluing theorem [AD10, Proposition 3.2.7] show that the boundary points of the compactified one-dimensional moduli spaces are in bijective correspondence with broken unparameterized gradient flow lines:

$$\partial\overline{\mathcal{M}(q,p;f,g)} = \bigcup_{p} \mathcal{M}(q,p;f,g) \times \mathcal{M}(p,r;f,g).$$

In the end, the above sum is indeed zero, as required, since there is a finite even number of points on the boundary of a compact one-dimensional manifold, and the *Morse homology* $\mathrm{MH}_\bullet(M)$ is then defined as the homology of the chain complex $(\mathrm{MC}_\bullet(f), \partial_{f,g})$, namely

$$\mathrm{MH}_\bullet(M) = \ker\left(\partial_{f,g} \colon \mathrm{MC}_\bullet(f) \to \mathrm{MC}_{\bullet-1}(f)\right) / \operatorname{im}\left(\partial_{f,g} \colon \mathrm{MC}_{\bullet+1}(f) \to \mathrm{MC}_\bullet(f)\right).$$

At last, Morse homology is independent of the choices for f and g [AD10, Théorème 3.4.2] and is an homotopy invariant for closed manifolds [AD10, Théorème 4.6.2].

In classical mechanics, the fundamental principle of dynamics states that an object in a force field follows an energy minimising path of the phase space. In a cooriented contact manifold (M, α), this philosophy incarnates in the study of the Morse theoretical properties of its *action functional*, which is defined on paths by

$$\mathcal{A}_\alpha \colon \gamma \mapsto \int_\gamma \alpha.$$

Under some technical assumptions, the infinite-dimensional Morse theory of the action functional, also called *symplectic field theory*, results in interesting invariants for contact manifolds [EGH00]. In particular, when the paths considered in the definition of the action functional are required to start and end on the same Legendrian submanifold, the corresponding symplectic field theory allows to construct invariants for isotopies through Legendrian submanifolds [EES05a, EES07]. Even though these invariants arising from symplectic field theory are without a doubt powerful, their constructions rely on quite involved analysis and require to understand the behaviour of the solutions of a non-linear partial differential equation.

In classical mechanics, generating families naturally appear as Lagrangians of physical systems and in contact topology, they similarly provide finite dimensional models of the action functional. More concretely, if a Legendrian submanifold in (J^1B, ξ_B) has a generating family $f : B \times \mathbf{R}^N \to \mathbf{R}$, its *difference function* $\delta : B \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is defined by

$$\delta(b, \eta_1, \eta_2) = f(b, \eta_1) - f(b, \eta_2),$$

and measures the length between vertically aligned pair of points in the Legendrian submanifold. Then, letting g be a generic Riemannian metric on $B \times \mathbf{R}^N \times \mathbf{R}^N$, under a suitable mild assumption on the behaviour at infinity of the generating family f, the Morse moduli spaces $\mathcal{M}(c_-, c_+; \delta, g)$ associated to the difference function δ are sufficiently well-behaved to mimick the construction of the Morse homology for closed smooth manifolds. This construction yields the definition of the *generating family homology* $\mathrm{GFH}_\bullet(f)$ of f, which is still independent of the choice for g, and allows to construct interesting invariants for isotopies through Legendrian submanifolds [Tra01]. In particular, the Euler–Poincaré characteristic of the generating family homology in $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ generically recovers the Thurston-Bennequin number [EES05b, Propositions 3.2 and 3.3] and [ST13, Proposition 3.2]. In other words, the generating family homology is a *categorification* of the Thurston-Bennequin number.

In the setting of closed manifolds, Morse homology is an effective homotopy invariant, since it is computed from moduli spaces of geometric objects that are easily identified on the manifold, without knowing much of it. However, this is no longer true for generating family homology. Not only the generating families of Legendrian submanifolds are often only understood in too qualitative terms to allow any homological computation, but the moduli spaces involved are also no longer constituted of geometric objects. Indeed, the gradient flow of the difference function does not occur on the Legendrian submanifold itself, but rather on an auxiliary vector bundle in which it non-canonically embeds (by the mean of the generating family).

1.3. The adiabatic limiting process of Henry and Rutherford.

Since the generating family homology is independent of the Riemmanian metric, wisely picking it could help with the effective computation of the generating family homology, and ideally the gradient flow lines of the difference function should be drawn on the Legendrian submanifold. Intuitively, this goal is achieved by progressively speeding up the fibrewise gradient flow, but for reasons that will be clear later, it is more convenient to rather slow down the base component, this amounts to perform what will be from now on called the *Henry-Rutherford limiting process*. Rigorously, for $s \in (0,1]$, let us first define

$$g_s = \left(s^{-1} g_B\right) \oplus g_F,$$

where g_B is a Riemannian metric on the base B, g_F is a Riemannian metric on the fibre F, and then let $s \to 0$ in the differential equation for the g_s-gradient flow of the difference function. However, the resulting flow lines of this singularly perturbed gradient flow are not always drawn on the Legendrian submanifold itself, since a computation (**Lemma 3.1**) shows that when $s \to 0$, the gradient flow lines of the difference function with respect to g_s

- are tangent to fibres away from the fibrewise critical set of the difference function; and
- vanish along the fibrewise critical set of the difference function, but if rescaled by s^{-1}, they are then tangent to the fibrewise critical set of the difference function.

Therefore, the Henry–Rutherford limiting process seems to "converge" towards broken trajectories obtained by concatenating two alternating types of elementary gradient flow lines, first *vertical fragments* (in the first case above), and then *horizontal fragments* (in the second case above). When these broken gradient flow lines only have a finite number of elementary fragments, they are called *gradient staircases* and the set of all unparameterized gradient staircases from c_- to c_+ is denoted by $\mathcal{M}^{\mathrm{st}}(c_-, c_+; \delta, g_1)$ (**Definition 3.2**).

Recall from the proof of the standard cobordism-equivalence from compactness and gluing that if $s_0 \in (0,1]$ is small enough, then for all $s \in (0, s_0]$, $\mathcal{M}(c_-, c_+; \delta, g_s)$ and $\mathcal{M}(c_-, c_+; \delta, g_{s_0})$ are in one-to-one correspondence, whenever they are finite manifolds [**AD10**, Proposition 3.4.3]. Replacing $\mathcal{M}(c_-, c_+; \delta, g_0)$ by $\mathcal{M}^{\mathrm{st}}(c_-, c_+; \delta, g_1)$, it is conjectured that the previous one-to-one correspondence extends to $s = 0$, this is the statement of

Conjecture A (based on [**HR13**, Conjecture 6.3], **Conjecture 3.1**). *If $s_0 \in (0,1]$ is small enough, then there exists a one-to-one set correspondence between $\mathcal{M}(c_-, c_+; \delta, g_{s_0})$ and $\mathcal{M}^{\mathrm{st}}(c_-, c_+; \delta, g_1)$, as soon as they are zero-dimensional compact manifolds.*

Since $(g_s)_s$ is not an homotopy of Riemannian metrics (in particular, g_0 is not even well-defined), Conjecture A cannot be directly deduced from the usual compactness-gluing from Morse theory mentioned just above.

Provided Conjecture A is true, the Henry–Rutherford limiting process does not "converge" towards gradient flow lines entirely drawn on the Legendrian submanifold, but this goal was naive, since it would imply that generating family homology only depends on the Legendrian submanifold. However, Conjecture A still helps with the effective computation of generating family homology, since

- vertical fragments are related to handleslides of the generating family; and
- horizontal fragments are entirely determined by the Legendrian front.

In a manner of speaking, working with moduli spaces of gradient staircases allows splitting apart the different types of contributions to the differential of the generating family chain complex. Vertical fragments account for bifurcations of the fibrewise critical points of the generating family and horizontal fragments for the geometry of the Legendrian submanifold.

Conjecture A was first introduced as a geometrical grounding for a combinatorial differential graded algebra for Legendrian knots constructed from generating families [HR13, Theorem 5.4]. In particular, Conjecture A with [HR13, Theorems 5.4 and 5.5] recover [FR11, Theorem 5.1] which constructs an augmentation of the Chekanov-Eliashberg algebra of a Legendrian knot from a linear-at-infinity generating family. More generally, Conjecture A is expected to play a crucial role to tackle in arbitrary dimensions, the complete equivalence between generating families and augmentations of the Chekanov-Eliashberg.

It is worth mentioning that moduli spaces of broken trajectories and results like Conjecture A have already appeared before in the litterature. For example, in the context of

- Morse–Bott homology [Fra04, Appendix A], [BH13];
- contact homology [Bou02]; and
- symplectic homology [BO09a, BO09b].

The proofs are all based on compactness-gluing techniques, but they also all have their own specificities and analytical difficulties, and Conjecture A is no exception.

2. Main statements of the dissertation

The context has now been correctly set up to state the main results featured in this dissertation. In particular, this section explains to what extent this thesis constitutes the first steps towards a complete proof of Conjecture A.

2.1. Analysis on the moduli spaces of gradient staircases chains.

As it was mentioned before, correspondence results between moduli spaces, like Conjecture A, are generally proved using compactness-gluing techniques, and this dissertation precisely deals with the compactness part for the Henry–Rutherford limiting process.

In order to give more precise statements, let us fix some notations for the rest of this introduction. Let Λ be a closed and connected Legendrian submanifold of (J^1B, ξ_B), let $\delta : B \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ be the difference function of some linear-at-infinity generating family of Λ, and let us define

$$g_s = \left(s^{-1} g_B\right) \oplus g_N \oplus g_N,$$

for all $s \in (0,1]$, where g_B is a Riemannian metric on B and g_N is a Riemannian metric on $\mathbf{R}^N$. Then, this thesis is concerned with the limiting behaviour of the g_s-gradient flow of δ as $s \to 0$. Before being able to state the main theorem of this thesis, recall from standard Morse theory that for all sequences $\gamma_k \in \mathscr{M}(c_-, c_+; \delta, g_1)$, there exist a subsequence (still denoted in the same way) and a tuple $(\gamma_1, \ldots, \gamma_m)$, called a *broken gradient flow line*, where

- for all $i \in \{1, \ldots, m\}$, $\gamma_i \in \mathscr{M}(c_i, c_{i+1}; \delta, g_1)$; and
- $c_1 = c_-$ and $c_{m+1} = c_+$;

and for all $i \in \{1, \ldots, m\}$, a sequence $(\tau_k^i)_{k \in \mathbf{N}}$ of timeshifts such that $\gamma_k(\cdot + \tau_k^i) \underset{k}{\longrightarrow} \gamma_i$ in C^1_{loc}-topology. Similarly, to study the behaviour of sequences $\gamma_k \in \mathscr{M}(c_-, c_+; \delta, g_{s_k})$, where $s_k \in (0,1]$ and $s_k \underset{k}{\longrightarrow} 0$, it is relevant to define *gradient staircases chains* (Definition 3.3) as tuples $\underline{\mathbf{e}} = (\mathbf{e_1}, \ldots, \mathbf{e_m})$, where

- for all $i \in \{1, \ldots, m\}$, $\mathbf{e_i} \in \mathscr{M}^{\mathrm{st}}(c_i, c_{i+1}; \delta, g_1)$; and
- $c_1 = c_-$ and $c_{m+1} = c_+$;

and the set of all unparameterized staircases chains from c_- to c_+ is denoted by $\overline{\mathscr{M}^{\mathrm{st}}}(c_-, c_+; \delta, g_1)$. With these reminders at hand, the compactness result for the Henry–Rutherford limiting sequence that is featured in this thesis is now given by

Theorem A (Theorem 4.1)**.** *Assume that the projection $\Lambda \to B$ has only Whitney pleat singularities. Then, Λ can be perturbed in a way that for all $\gamma_k \in \mathscr{M}(c_-, c_+; \delta, g_{s_k})$, where $s_k \in (0,1]$ and $s_k \underset{k}{\longrightarrow} 0$, there exist a subsequence (still denoted in the same way) and $\underline{\mathbf{e}} \in \overline{\mathscr{M}^{\mathrm{st}}}(c_-, c_+; \delta, g_1)$ such that $\gamma_k \underset{k}{\longrightarrow} \underline{\mathbf{e}}$ in the Floer–Gromov topology.*

The meaning for convergence in the Floer–Gromov topology is given in Definitions 3.4 and 3.5. As for now, it is enough to think about it as a convergence towards broken gradient flow lines, but with an additional time-rescaling by ${s_k}^{-1}$ in the case of horizontal fragments to avoid recovering constant flow lines (see Section 1.3 above).

Restricting the type of caustic singularities of Λ is most likely unnecessary for Theorem A to hold. However, it avoids involved combinatorial arguments, especially in the proof of Proposition 4.7. In any case, this mild assumption only amounts to a homotopical question [AG18, Theorem 1.11], and it is also satisfied by many meaningful examples of Legendrian submanifolds [Ekh07, BST15]. However, it is mandatory to work with generic Legendrian submanifolds, otherwise Theorem A can fail to be true for deep reasons that will now be discussed.

2.2. A major technical difficulty: an infinite bubbling-like phenomenon.

When a noncompact group acts freely on solutions of a partial differential equation, the associated moduli spaces of unparametrized solutions are expected to be compactified with broken objects. To better illustrate this common phenomenon, consider the following two key examples:

- In standard Morse theory, $\mathbf{R}$ acts freely by time-translation on the gradient flow, and the moduli spaces of unparametrized gradient trajectory are compactified by broken gradient flow lines [AD10, Théorème 3.2.2].
- In Gromov–Witten theory, $\mathrm{PSL}(2,\mathbf{R})$ acts freely by Möbius transformations on the Cauchy-Riemann equation, and the moduli spaces of unparametrized pseudo-holomorphic curves are compactified by broken pseudo-holomorphic curves [MS04, Theorem 4.4.3].

Proving such compactness results first requires showing that the borderline analytical behaviour of solutions is not too wild, typically meaning that their limits have a finite number of fragments. This step is generally straightforward and accomplished showing that each nonconstant fragment consumes a predetermined amount of some kind of "energy", available in finite total amount. Once more, this strategy is well understood in the above two examples:

- In Morse theory, the value of the Morse function strictly decreases along nonconstant gradient flow lines.
- In Gromov-Witten theory, the symplectic area of a nonconstant pseudo-holomorphic curves is uniformly bounded from below [MS04, Lemma 4.5.2].

In the setting of Theorem A, even though the total amount of vertical length available is finite, nothing prevents vertical fragments to become arbitrarily short in neighbourhoods of cusp-edges. For this reason, the previous standard energetical arguments are not enough to ensure that the Henry–Rutherford limiting process only recovers gradient staircases chains with a finite number of elementary fragments.

However, a much more involved analysis shows that when the Henry–Rutherford limiting process recovers a gradient staircases chain with infinitely many elementary fragments, then its horizontal fragments have arbitrarily "deep tangency" with the singular locus of the Legendrian submanifold. Then, the definition of *gradient generic* Legendrian submanifolds (Definition 1.11) is only natural since the order of tangency of a vector field with a generic manifold is bounded by its dimension. With all these observations at hand goes the statement of

Theorem B (Theorem 4.3)**.** *On a gradient generic Legendrian submanifold, the Henry–Rutherford limiting process can can only "converge" towards gradient staircases chains with a finite number of elementary fragments.*

Going through the proof of Theorem B explains why it was more convenient to slow down the base component of the difference function gradient flow rather than speeding it up in the fibre, as it was originally suggested in [HR13].

Theorem B is interesting because gradient generic Legendrian submanifolds are highly numerous. More rigorously speaking, gradient genericity is (as the name suggests) a generic property among all Legendrian submanifolds, this is the statement of

Theorem C (Theorem 1.2)**.** *The subset of all gradient generic Legendrian submanifolds of fixed smooth type is open and dense in the set of all Legendrian embeddings, endowed with the topology induced by the C^∞-topology on smooth maps.*

Theorem A is completely computational-friendly, since

- checking whether a Legendrian submanifold is gradient generic or not; and
- perturbing a Legendrian submanifold to make it gradient generic;

are entirely effective and explicit procedures.

Structure of the dissertation

This thesis is organised in four interdependent chapters culminating to the proof of Theorem A, and ends with a fifth independent chapter of examples to illustrate the relevance of Conjecture A. Great care has been taken to ensure that this dissertation is fully self-contained and in particular, proofs of several standard results have been adapted to better suit their intended use in the thesis, and multiple minor gaps in research articles have also been filled.

- Chapter 1 – **Transversality conditions for Legendrian submanifolds**

This first chapter is devoted to gradient genericity of Legendrian submanifolds (Definition 1.11) and in particular, the proof of Theorem C (Theorem 1.2) is thoroughly given in Section 3. It starts with reminders of elementary contact topology in Section 1, first for general abstract contact manifolds in Section 1.1, then in the setting of first-order jet bundles in Section 1.2. Some more recollections from the study of Legendrian submanifolds are found in Section 2 and at the end of Section 3, respectively with the definitions of front genericity (Definition 1.8) and chord genericity (Definition 1.12).

- Chapter 2 – **Generating family homologies of Legendrian submanifolds**

This second chapter is devoted to the study of Legendrian submanifolds from generating families. Section 1 starts with an overview of generating family theory for Legendrian submanifolds and the homotopy lifting property of generating families (Theorem 2.1) is recalled in Section 1.1. Then, Section 1.2 establishes a complete dictionary between transversal properties of generic Legendrian submanifolds from Chapter 1 and transversal properties of their generating families. This chapter is lastly concerned with generating family homologies of Legendrian submanifolds. To address Theorem A, Section 2 studies Morse theory of generating families (Proposition 2.4), and then simple/mixed versions of generating family homology are constructed (Definition 2.6). Finally, Section 3 starts by reviewing all the results that are already known about the algebraic structure of the generating family homology in its usual simple version (Theorem 2.3 and 2.5). Then, similar statements are extended to the mixed version (Theorem 2.4) or simply discussed and conjectured (Conjectures 2.1 and 2.2).

- Chapter 3 – **The Henry–Rutherford limiting process**

This third chapter provides all the necessary background to state Conjecture A (Conjecture 3.1) and start working towards a complete proof of Theorem A. In Section 1, the Henry–Rutherford limiting process is described in terms of some fast-slow dynamical system for which the maximal flow invariant subset fails to be normally hyperbolic for a whole codimension one subset of points. Then, **Section 2** is concerned with staircases and staircases chains (**Definition 3.4** and **3.5**), with **Section 2.1** studying several Morse theoretical aspects of vertical fragments (**Proposition 3.1**). Finally, **Section 3** reviews convergence in the Floer-Gromov topology (**Definitions 3.4** and **3.5**) and some of its relevant properties to describe the limiting behaviour of the Henry–Rutherford limiting process are also proved (**Proposition 3.2**).

- **Chapter 4** – **Floer-Gromov compactness for gradient staircases chains**

This fourth chapter is devoted to **Theorem A** (**Theorem 4.1**), the main result of this dissertation. **Section 1** describes recursive procedures to recover elementary fragments of gradient staircases chains from the Henry–Rutherford limiting process. **Section 1.1** explains how to recover vertical fragments (**Proposition 4.1**) and **Section 1.2** deals with horizontal fragments (**Theorem 4.2**). Then, **Section 2** is concerned with the proof of **Theorem B** (**Theorem 4.3**), stating that on a gradient generic Legendrian submanifold, these procedures stop after a finite number of steps. Finally, **Section 3** gathers all these statements to prove **Theorem A** (**Theorem 4.1**).

- **Chapter 5** – **Examples**

This fifth chapter explains how moduli spaces of gradient staircases can be used to carry explicit homological computations with generating families in all dimension. These examples emphasise the relevance of the mixed version of the generating family homology and are the main starting point to tackle the conjectures from **Chapter 2**.

- **Research prospects**

This dissertation is concluded with the description of several future research directions following the perspectives opened by **Theorem A**.

1.

Transversality conditions for Legendrian submanifolds

This chapter provides useful background on Legendrian submanifolds of first-order jet bundles and in particular, several classical generic properties of Legendrian submanifolds are recalled. However, these standard transversality conditions do not prevent the Henry-Rutherford limiting process from recovering gradient staircases with infinitely many breaking points and fragments. Then, the main goal of this chapter is to define new transversality conditions (**Definition 1.11**) playing a crucial role in the proof of **Theorem B**, and to establish **Theorem C**, stating that these new properties are generically satisfied by Legendrian submanifolds (**Theorem 1.2**).

1. Contact manifolds and their Legendrian submanifolds

This section reviews basic notions from contact topology and provides the relevant background to formulate the transversality conditions that are needed for the proof of **Theorem B**.

1.1. General setting.

First, these recollections from contact topology are carried in general contact manifolds.

Definition 1.1. A one-form α on a $(2n+1)$-dimensional smooth manifold M is a *contact form* whenever $\alpha \wedge (\mathrm{d}\,\alpha)^n$ is a nowhere vanishing top form on M and its associated *Reeb vector field* R_α is uniquely defined by $\mathrm{d}\,\alpha(R_\alpha, \cdot) = 0$ and $\alpha(R_\alpha) = 1$.

Remark. A differential one-form α is a contact form whenever $\mathrm{d}\,\alpha$ is non-degenerate on $\ker(\alpha)$, then the rank of $\xi = \ker(\alpha)$ is even and $\mathrm{d}\,\alpha$ has a one-dimensional kernel which is transverse to ξ. In particular, contact forms only exist on odd-dimensional manifolds and the Reeb vector field of a contact form is always well-defined.

Definition 1.2. A pair (M, ξ) is a *contact manifold* whenever M is an odd-dimensional smooth manifold and ξ is a *maximally non-integrable* hyperplane distribution of the tangent bundle TM, for all open sets U of M, there exists a contact form α_U on U such that $\xi_{|U} = \ker(\alpha_U)$.

Definition 1.3. A submanifold L of a contact manifold (M, ξ) is *isotropic* whenever $TL \subset \xi_{|L}$. If M has dimension $2n+1$, then L is *Legendrian* whenever it is isotropic and has dimension n.

Remark. If α is a global contact form on M, a submanifold L of M is isotropic whenever $\alpha_{|L} \equiv 0$. In particular, $\mathrm{d}\,\alpha_{|L} \equiv 0$ and since $\mathrm{d}\,\alpha_{|\xi}$ is non-degenerate, then $\dim(L)$ is less than $\mathrm{rk}(\xi) - \dim(L)$. Therefore, Legendrian submanifolds are isotropic submanifolds of maximal dimension.

Definition 1.4. Let (M, ξ) be a contact manifold and let Λ be a Legendrian submanifold of (M, ξ). Assume that α is a globally defining contact form for ξ, a path $\gamma\colon [0, T] \to M$ is a *Reeb chord* of Λ whenever it is a smooth integral curve of R_α whose both *ends* $\gamma(0)$ and $\gamma(T)$ are elements of Λ.

The *action functional* $\mathscr{A}_\alpha$ in (M, α) is defined on smooth paths $\gamma\colon [0, T] \to M$ by

$$\mathscr{A}_\alpha(\gamma) = \int_\gamma \alpha = \int_0^T \gamma^* \alpha.$$

A Reeb chord $\gamma\colon [0, T] \to M$ of Λ is *non-degenerate* when $(\phi_T)_* T_{\gamma(0)}\Lambda$ is transverse to $T_{\gamma(T)}\Lambda$ in ξ, where ϕ_T is the flow at time T of the Reeb vector field of α.

1.2. First-order jet bundle setting.

This section now focuses on the contact topology of first-order jet bundles.

Definition 1.5. The *Liouville form* λ_B of T^*B is the differential one-form on T^*B defined by

$$\forall \zeta \in T T^*B, \lambda_B(\zeta) = \zeta \circ T\pi_B,$$

where $\pi_B\colon T^*B \to B$ is the canonical cotangent bundle projection.

Remark. Under the canonical identification between the sections of π_B and the one-forms of B, the Liouville form is also called the *tautological one-form*, since it is the unique one-form of T^*B such that for all sections σ of π_B, the equality $\sigma^* \lambda_B = \sigma$ holds.

Example 1.1. If q are local coordinates on B and p are dual coordinates to q, then

$$\lambda_B = \sum_{k=1}^n p_k \,\mathrm{d}\, q_k,$$

The first-order jet bundle of B is $J^1B = T^*B \times \mathbf{R}$.

Definition 1.6. The *standard* or *canonical contact structure* ξ_B on J^1B is defined by

$$\xi_B = \ker(\alpha_B = \mathrm{d}\, z - \lambda_B),$$

where z is the coordinate along $\mathbf{R}$ in $J^1B = T^*B \times \mathbf{R}$.

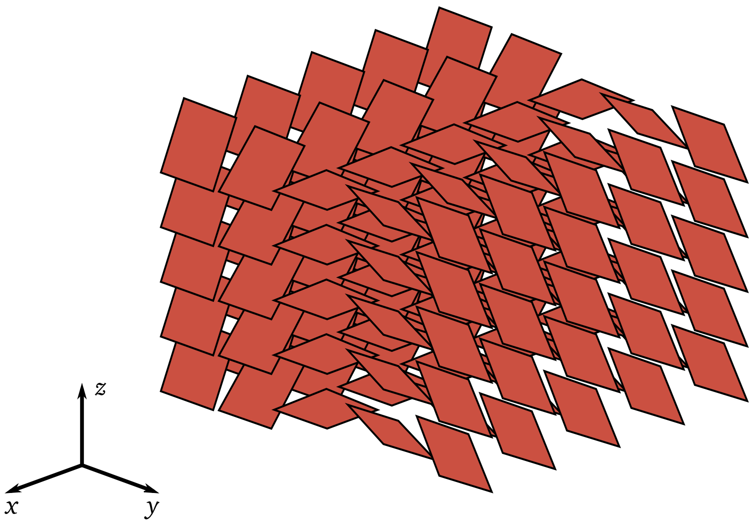


Figure 1. The canonical contact structure $\xi_{\mathbf{R}} = \ker(\mathrm{d}\, z - y\, \mathrm{d}\, x)$ on $J^1\mathbf{R} = {\mathbf{R}^3}_{(x,y,z)}$.

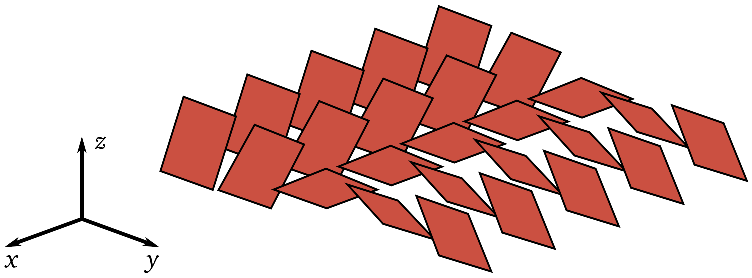


Figure 2. A horizontal layer ($z = z_0$) of $\xi_{\mathbf{R}}$ as represented in Figure 1.

Remark. The Reeb vector field of α_B is the vertical vector field given by ∂_z.

Remark. The tautological property of the Liouville form implies that

$$\{(b, T_b f(b), f(b)); b \in B\}$$

is automatically a Legendrian submanifold of (J^1B, ξ_B), where $f : B \to \mathbf{R}$ is a smooth function. In other words, graphs of *holonomic sections* are natural Legendrian submanifolds of (J^1B, ξ_B).

To conclude, let us now emphasise the relevance of first-order jet bundles in contact topology with the standard Weinstein's neighbourhood theorem.

Theorem 1.1 ([Gei08, Corollary 2.5.9]). *Let (M, ξ) be a contact manifold and let Λ be a closed Legendrian submanifold of (M, ξ), then there exists a smooth diffeomorphism $\varphi : U \to V$ between open neighbourhoods of Λ in M and 0_Λ in $J^1\Lambda$ such that $\varphi^*\xi_{\Lambda|V} = \xi_{|U}$ and $\varphi(\Lambda) = 0_\Lambda$.*

Remark. Not only all contact manifold locally look alike [Gei08, Theorem 2.5.1], Theorem 1.1 further states that neighbourhoods of Legendrian submanifolds are locally modelled on their first-order jet bundles.

2. Thom-Boardman hierarchy for singularities of generic Legendrian caustics

First, the classical definition of front genericity (Definition 1.8) is recalled, since it is the starting point to introduce the new crucial transversality conditions needed for the proof of Theorem B. Then, the mild assumption appearing in Theorem A to restrict the singularities of the Legendrian caustics is precisely defined (Definition 1.9).

Let M be a closed and connected n-dimensional manifold and let us define $\mathrm{Leg}(M, B)$ to be the space of Legendrian embeddings of M into (J^1B, ξ_B) endowed with the subspace topology induced by the Whitney C^∞-topology on the set of smooth maps from M to J^1B.

Definition 1.7. The *caustic projection* is defined as the canonical bundle projection $\Pi_B : J^1B \to B$, and the front projection is defined as the canonical bundle projection $\Pi_{\mathrm{fr}} : J^1B \to J^0B = B \times \mathbf{R}$. Let $\Lambda \in \mathrm{Leg}(M, B)$ be a Legendrian embedding, then its *caustic* is the set of critical values of $\Pi_{B|\Lambda}$ and its *front* is the image of Λ under Π_{fr}.

Remark. Even though Legendrian submanifold are always smooth, their caustics and fronts are in general singular submanifolds.

For all $\Lambda \in \mathrm{Leg}(M,B)$ and for all integers $k \in \{0,\dots,n\}$, let us define Λ_k to be the subset of Λ consisting of elements $\lambda \in \Lambda$ at which the differential of $\Pi_{B|\Lambda}$ has a k-dimensional kernel, so that

$$\Lambda = \Lambda_0 \cup \Lambda_1 \cup \cdots \cup \Lambda_{n-1} \cup \Lambda_n,$$

and also define $\Lambda^{\prec}$ to be the union of Λ_k for all $k \in \{1,\dots,n\}$.

Definition 1.8. A Legendrian embedding $\Lambda \in \mathrm{Leg}(M,B)$ is *front generic* whenever the following two conditions:

(1) for all $k \in \{0,\dots,n\}$, Λ_k is a transversally cut out codimension $k(k+1)/2$ submanifold;
(2) for all integers k and ℓ in $\{0,\dots,n\}$, $\Pi_{B|\Lambda_k}$ and $\Pi_{B|\Lambda_\ell}$ are transverse maps;

are simultaneously met.

Remark. If a Legendrian embedding is front generic, then its caustic projection is stratified by transversally intersecting smooth immersed submanifolds whose self-intersections are transverse. Indeed, by the lower semi-continuity of the rank, for all integers $k \in \{0,\dots,n\}$, the closure of Λ_k is the finite union of the lower-dimensional strata Λ_j for $j \in \{k,\dots,n\}$.

Remark. If k or ℓ is 0, then $\Pi_{B|\Lambda_k}$ and $\Pi_{B|\Lambda_\ell}$ are automatically transverse, provided Λ_0 is open.

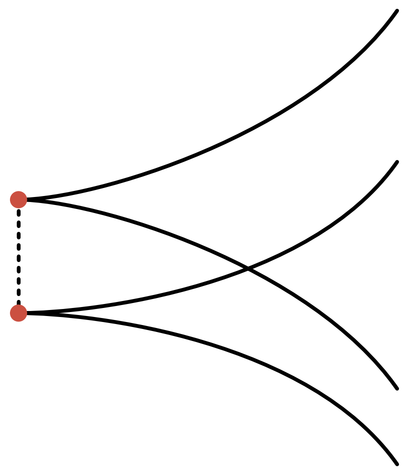


Figure 3. A piece of a Legendrian knot front projection where front genericity fails.

Remark. A front generic Legendrian submanifold of (J^1B,ξ_B) is uniquely determined by its front. Indeed, if $(q,p,z) \in T^*B \times \mathbf{R}$ is a local parametrization for a Legendrian submanifold Λ, then

$$\forall (q,p,z) \notin \Lambda^{\prec}, p_i = \frac{\partial z}{\partial q_i},$$

and when Λ is front generic, $\Lambda \setminus \Lambda^{\prec}$ is an open and dense subset of Λ, so that the above formula continuously extends to the whole Legendrian submanifold.

Following [AGZV85, Section 2.1–2.3], in the case of smooth maps, and [AG18, Section 1.3], in the specific case of Legendrian fibrations, Thom's transversality theorem ensures that front genericity is a generic property in $\mathrm{Leg}(M,B)$.

Proposition 1.1. *The subset of front generic elements of* $\mathrm{Leg}(M,B)$ *is open and dense.*

The stratification by the rank of the differential does not suffice to fully classify the critical points of smooth maps and in particular, front genericity is not enough to completely understand the singularities of Legendrian caustics. Indeed, if $\Lambda \in \mathrm{Leg}(M, B)$ is front generic, then the map $\Pi_{B|\Lambda_0}$ is a submersion, but for all integers $k \in \{1, \dots, n\}$, the map $\Pi_{B|\Lambda_k}$ need not to have constant rank. This issue is in general solved by considering the Thom-Boardman hierarchy Σ^I of critical points, whose construction has been initiated in [Tho56] and developed more thoroughly in [Boa67]. In the case of Legendrian caustics, the first step toward the Thom-Boardman hierarchy consists in defining for all $k \in \{0, \dots, n\}$ and all $\ell \in \{0, \dots, k\}$, $\Lambda_{k,\ell}$ as the subset of $\lambda \in \Lambda_k$ at which the differential of $\Pi_{B|\Lambda_k}$ has a ℓ-dimensional kernel, so that

$$\Lambda_k = \Lambda_{k,0} \cup \Lambda_{k,1} \cup \dots \cup \Lambda_{k,k-1} \cup \Lambda_{k,k},$$

see [AGZV85, Section 2.4], in the case of smooth maps, and [AG18, Section 1.3], in the specific case of Legendrian fibrations, for more details.

In this thesis, the same mild assumption as the one found in [Ekh07] is made on the Legendrian caustics considered to avoid having to resort to the complete Thom-Boardman hierarchy, as well as making other tedious combinatorial discussions (like induction on the number of strata).

Definition 1.9. The subset $\mathrm{Leg}_{1,0}(M, B)$ consists of $\Lambda \in \mathrm{Leg}(M, B)$ which are *at most* $\Sigma^{1,0}$*-singular*, meaning that Λ is front generic and $\Lambda^{\prec} = \Lambda_{1,0}$.

Remark. A Legendrian embedding is at most $\Sigma^{1,0}$-singular, if and only if, all the singularities of its caustic are of the simplest type, they are Whitney pleats.

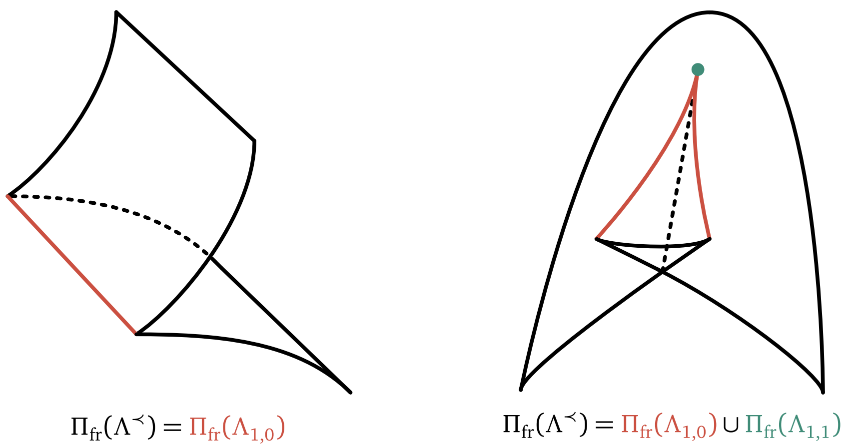


Figure 4. Front projections of a pleat (on the left) and a swallowtail (on the right).

Remark. In the one-dimensional case $n = 1$, using Proposition 1.1 shows that any Legendrian embedding is isotopic through Legendrian embeddings to an at most $\Sigma^{1,0}$-singular one.

Remark. The simplification of singularities of Legendrian embeddings satisfies a full h-principle, in the sense of [EM02], [AG18, Theorem 1.1]. In particular, roughly speaking, it implies that a Legendrian embedding is isotopic through Legendrian embeddings to an at most $\Sigma^{1,0}$-singular one whenever some obstructions of homotopical nature vanish.

3. Gradient genericity for Legendrian submanifolds

This section first carefully defines gradient generic Legendrian submanifolds (**Definition 1.11**), then shows **Theorem C** (**Theorem 1.2**) stating that the subset of all gradient generic Legendrian submanifolds is open and dense in the subset of all at most $\Sigma^{1,0}$-singular Legendrian submanifolds. For this thesis, gradient genericity is essential to the proof of **Theorem A**, since it is designed to prevent the Henry-Rutherford limiting process from recovering gradient staircases chains with infinitely many breaking points and elementary fragments (**Theorem B**).

The definition of gradient genericity amounts to the existence of

- a gradient flow coming from vertical length in the front projection (**Lemma 1.2**); and
- associated iterated tangency loci with the front projection singularities.

In order to define *preliminary transversality conditions*, similar constructions have already been sketched in [**Ekh07**, Section 3.1.1], but not in a global and computational-friendly manner.

Definition 1.10. The *smooth sheets* of Λ are the connected components of $\Lambda \setminus \Lambda^{\prec}$ and its *strands* are the front projection of its smooth sheets.

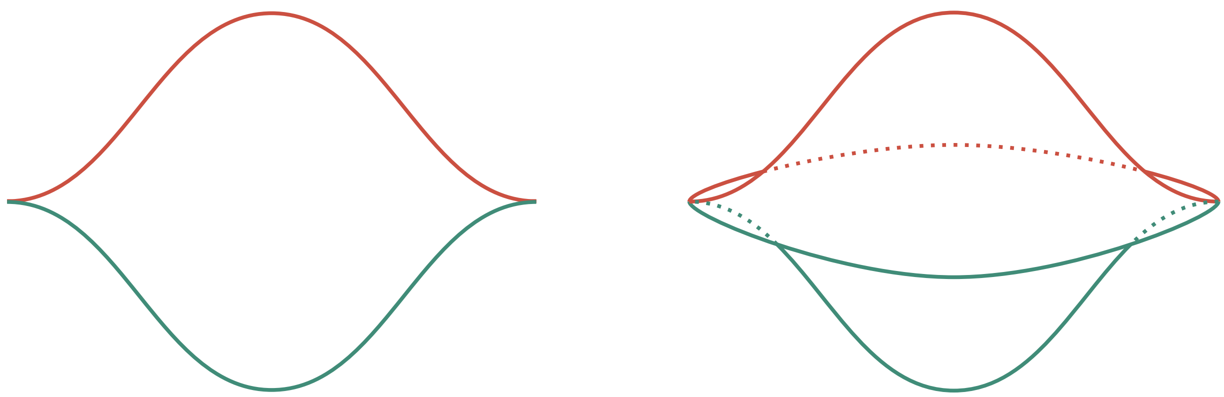


Figure 5. Strands of a Legendrian knot (on the left) and surface (on the right).

At most $\Sigma^{1,0}$-singular Legendrian submanifolds only have a finite number of smooth sheets.

Lemma 1.1. *Let $\Lambda \in \mathrm{Leg}_{1,0}(M,B)$, then Λ has a finite number of smooth sheets and they are open.*

Proof. According to **Definition 1.9**, $\Lambda^{\prec}$ is a codimension 1 submanifold of Λ, and in particular, each connected component of $\Lambda^{\prec}$ can give rise to at most two connected components of $\Lambda \setminus \Lambda^{\prec}$. Therefore, the proof boils down showing that $\Lambda^{\prec}$ has a finite number of connected components. From the remark following **Definition 1.8** and **Definition 1.9**, notice that $\Lambda^{\prec}$ is a compact subset, then using tubular neighbourhoods, conclude that the connected components of $\Lambda^{\prec}$ are isolated. In conclusion, $\Lambda^{\prec}$ only have a finite number of connected components and they are closed in Λ, thus yielding the result. ∎

Smooth sheets of Legendrian submanifolds are graphs of real-multivalued smooth maps.

Lemma 1.2. *Let $\Lambda \in \mathrm{Leg}_{1,0}(M,B)$, then for every smooth sheet S of Λ, there exists a unique $f_S : \overline{S} \to \mathbf{R}$ such that $\overline{S}$ is locally everywhere the graph of the first-extension of $f_S \circ \Pi_B{}^{-1}$.*

Proof. Start by noticing that Definitions 1.8 and 1.10 ensure that for $\lambda \in S$, $T_\lambda \Pi_{B|S}$ is invertible. Therefore, applying the inverse function theorem provides an open neighbourhood $U_{S,\lambda}$ of λ in S and an open neighbourhood $V_{S,\lambda}$ of $\Pi_B(\lambda)$ in B such that $\Pi_{B|S}\colon U_{S,\lambda} \to V_{S,\lambda}$ is a diffeomorphism. Hence, there exists a unique $u_{S,\lambda}\colon V_{S,\lambda} \to \mathbf{R}$ such that $u_{S,\lambda}$ is smooth $\Pi_{\mathrm{fr}}(U_{S,\lambda})$ is the graph of $u_{S,\lambda}$ and from the third remark after Definition 1.8, $U_{S,\lambda}$ is the graph of the first-jet extension of $u_{S,\lambda}$. In particular, $u_{S,\lambda}$ and all its partial derivatives of any order are uniformly bounded from above, whence $u_{S,\lambda}$ uniquely extends to a smooth map $u^*_{S,\lambda}\colon \overline{V_{S,\lambda}} \to \mathbf{R}$ such that $\overline{U_{S,\lambda}}$ is the graph of $j^1 u^*_{S,\lambda}$. To conclude, patch these locally defined smooth maps $\{u^*_{S,\lambda}\}_{\lambda\in S}$ together to recover

$$f_S(\lambda) = u^*_{S,\lambda}(\Pi_B(\lambda)),$$

which is the desired globally defined smooth map, thus yielding the result. ■

Remark. Even though front projections of Legendrian submanifolds have no vertical tangencies, as soon as $\dim(B) \geqslant 2$, it is possible that some of their strands cross the vertical axis several times, thus preventing Π_B from being injective on the associated smooth sheets, see Figure 7.

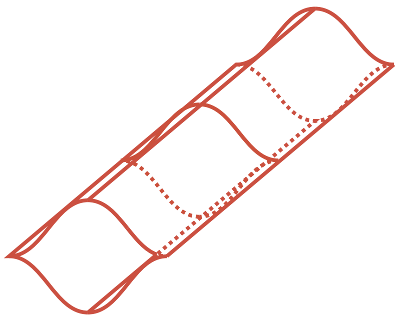


Figure 6. In front projection, tubes for connected sums are coiled as parking ramps.

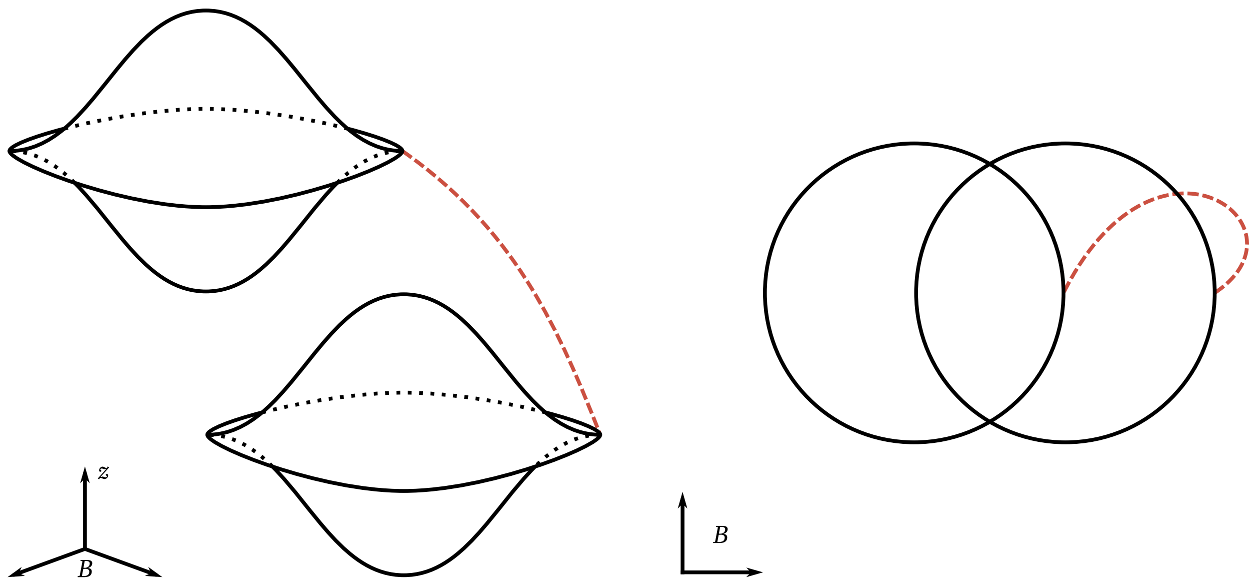


Figure 7. Connected sum of two vertically overlapping "flying saucers".

This discussions explains why the functions constructed in Lemma 1.2 can only be defined on the closure of the smooth sheets and not directly on their caustic projections.

From now on, let $\Lambda \in \mathrm{Leg}_{1,0}(M,B)$, then let us define $L = \Lambda \times_B \Lambda$ and $L^{\prec} = (\Lambda^{\prec} \times_B \Lambda) \cup (\Lambda \times_B \Lambda^{\prec})$. According to **Lemma 1.1**, Λ has a finite number of smooth sheets and let denote them by $S_1, \ldots, S_r$, then let also $f_1, \ldots, f_r$ be the smooth maps constructed in **Lemma 1.2** from $S_1, \ldots, S_r$, respectively. For all $\beta = \{k, \ell\} \subset \{1, \ldots, r\}$ with $k \neq \ell$, let also define

$$\Lambda_\beta = \partial S_k \cap \partial S_\ell = \overline{S_k} \cap \overline{S_\ell}.$$

Without requiring any assumption on the singularities of Λ, Λ_β is a subset of $\Lambda^{\prec}$, but if Λ is assumed to be at most $\Sigma^{1,0}$-singular, then Λ_β is open in $\Lambda^{\prec}$, as a union of connected components. Indeed, since $\Lambda^{\prec}$ is a codimension one submanifold, for all $\lambda \in \Lambda^{\prec}$, there exists a sufficiently small chart (U, φ) of Λ around λ such that $\varphi(U) \setminus \varphi(\Lambda^{\prec} \cap U)$ has exactly two connected components, and there exist $k \in \{1, \ldots, r\}$ and $\ell \in \{1, \ldots, r\}$ such that $k \neq \ell$ and for all $\lambda' \in U$, $\lambda' \in \partial S_k \cap \partial S_\ell$. Therefore, $\Lambda^{\prec}$ is the union of Λ_β, for all $\beta \subset \{1, \ldots, r\}$ such that $|\beta| = 2$.

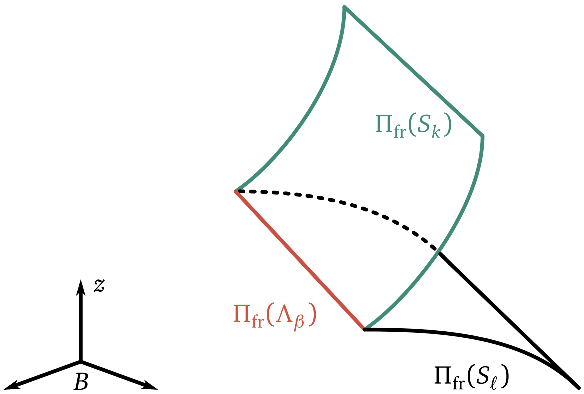


Figure 8. A connected component of Λ_β with $\beta = \{k, \ell\}$.

Remark. Notice that Λ_β is independent of the order of elements of β, since $\partial S_k \cap \partial S_\ell = \partial S_\ell \cap \partial S_k$. Moreover, notice that both definitions of Λ_β agree, since **Lemma 1.1** shows that whenever $k \neq \ell$, then S_k and S_ℓ are disjoint open sets, so that

$$\begin{aligned}
\overline{S_k} \cap \overline{S_\ell} &= (S_k \cup \partial S_k) \cap (S_\ell \cup \partial S_\ell), \\
&= (S_k \cap S_\ell) \cup (S_k \cap \partial S_\ell) \cup (\partial S_k \cap S_\ell) \cup (\partial S_k \cap \partial S_\ell), \\
&= \varnothing \cup \varnothing \cup \varnothing \cup (\partial S_k \cap \partial S_\ell), \\
&= \partial S_k \cap \partial S_\ell.
\end{aligned}$$

In conclusion, Λ_β is a well-defined subset of $\Lambda^{\prec}$.

Let Δ_β be the diagonal of the product $\Lambda_\beta \times \Lambda_\beta$, then for all $i \in \{1, \ldots, r\}$, let us define

$$\begin{aligned}
L_0^{i,\beta} &= \overline{S_i} \times_B \Lambda_\beta, && \text{if } i \notin \beta, \\
&= \left(\overline{S_i} \times_B \Lambda_\beta\right) \setminus \Delta_\beta, && \text{otherwise,}
\end{aligned}$$

which is, by construction, a subset of $L^{\prec}$.

Remark. In the definition of $L_0^{i,\beta}$, it is not possible to require that $i \notin \beta$, since when $\dim(B) \geqslant 2$, closure of strands can cross the vertical axis several times, as shown in **Figure 7**.

For all $i \in \{1,\dots,r\}$, all $\beta \subset \{1,\dots,r\}$ and all $\alpha \subset \{1,\dots,r\}$ such that $|\beta| = 2$, $|\alpha| = 2$ and $i \in \alpha$, then let us define

$$L_0^{\alpha,\beta} \begin{array}{ll} = \Lambda_\alpha \times_B \Lambda_\beta, & \text{if } \alpha \neq \beta, \\ = \left(\Lambda_\alpha \times_B \Lambda_\beta\right) \setminus \Delta_\beta, & \text{otherwise.} \end{array},$$

which is, by construction, a subset of $L_0^{i,\beta}$.

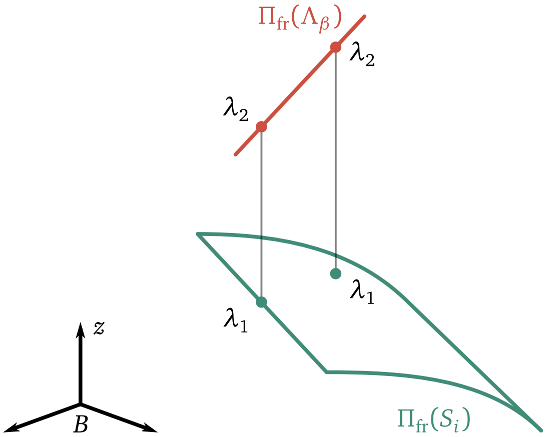


Figure 9. Schematic description of points $(\lambda_1, \lambda_2) \in L_0^{i,\beta}$.

The geometrical relevance of $L_0^{\alpha,\beta}$ is given by the following

Proposition 1.2. *Let $i \in \{1,\dots,r\}$ and $\beta \subset \{1,\dots,r\}$ such that $|\beta| = 2$, then $L_0^{i,\beta}$ is a manifold with boundary of dimension* $\dim(B) - 1$ *and whose boundary is given by*

$$\partial L_0^{i,\beta} = \bigcup_{\substack{\alpha \subset \{1,\dots,r\} \\ |\alpha|=2, \alpha \ni i}} L_0^{\alpha,\beta}.$$

Proof. Discuss whether the closure of the three smooth sheets share a cusp-edge or not:

- **Case 1.** Assume that $i \notin \beta$ and notice that as topological subspaces of $L^{\prec}$, then

$$\operatorname{Int} L_0^{i,\beta} = S_i \times_B \Lambda_\beta, \quad L_0^{i,\beta} \setminus \operatorname{Int} L_0^{i,\beta} = \partial S_i \times_B \Lambda_\beta,$$

  since **Lemma 1.1** shows that S_i is open in Λ and it was observed that Λ_β is open in $\Lambda^{\prec}$. Besides, since $\Lambda^{\prec}$ is the union of Λ_β for all β, ∂S_i is the union of $\partial S_i \cap \partial S_j$ for all $j \neq i$, and the boundary of $L_0^{i,\beta}$ can therefore be written as

$$L_0^{i,\beta} \setminus \operatorname{Int} L_0^{i,\beta} = \bigcup_{\substack{\alpha \subset \{1,\dots,r\} \\ |\alpha|=2, \alpha \ni i}} L_0^{\alpha,\beta}.$$

  Let $\alpha \subset \{1,\dots,r\}$ such that $|\alpha| = 2$ and $i \in \alpha$, since Λ_α and Λ_β are open subsets of $\Lambda^{\prec}$, using **Definition 1.8**, $\Pi_{B_{|\Lambda_\alpha}}$ and $\Pi_{B_{|\Lambda_\beta}}$ are transverse and $L_0^{\alpha,\beta}$ is a $(\dim(B)-2)$-manifold, but since smooth sheets only meet pairwise on their boundaries, their union is disjoint. In conclusion, the topological boundary of $L_0^{i,\beta}$ is a manifold of dimension $\dim(B)$–2, and for similar reasons, the topological interior of $L_0^{i,\beta}$ is a manifold of dimension $\dim(B) - 1$, thus yielding the result.
- **Case 2.** Assume that $i \in \beta$, then since Δ_β is closed in $\overline{S_i} \times_B \Lambda_\beta$, $L_0^{i,\beta}$ is open in $\overline{S_i} \times_B \Lambda_\beta$ and thus **Case 1** yields the result.

Together, the above two cases yield the result. ∎

Remark. Let $f : X \to Z$ and $g : Y \to Z$ be smooth maps between manifolds with boundaries. Recall that the *fibre product* of f and g, symply denoted by $X \times_Z Y$, is defined by

$$X \times_Z Y = \{(x,y) \in X \times Y ; f(x) = g(y)\},$$

and if f and g are transverse, then $X \times_Z Y$ is a manifold of dimension $\dim(X)+\dim(Y)-\dim(Z)$. Indeed, let Δ_Z be the diagonal of $Z \times Z$, then notice that by construction $X \times_Z Y = (f,g)^{-1}(\Delta_Z)$. Besides, if (f,g) is transverse to Δ_Z if, and only if, f and g are transverse [Hir76, Exercise 3.2.14], but according to [Hir76, Theorem 1.3.3], whenever (f,g) is transverse to Δ_Z, then $(f,g)^{-1}(\Delta_Z)$ is a submanifold of codimension $\operatorname{codim}(\Delta_Z) = \dim(Z)$.

To define gradient generic Legendrian submanifolds, let g be a Riemannian metric on Λ, and then let us define a gradient flow in B, induced by the vertical length between pairs of vertically aligned points belonging to the closure of smooth sheets of Λ.
Then, for all $i \in \{1,\dots,r\}$ and all $j \in \{1,\dots,r\}$, let us define $X_{i,j} : \overline{S_i} \times_B \overline{S_j} \to TB$ by

$$X_{i,j}(\lambda_1,\lambda_2) = T_{\lambda_1}\Pi_B(\nabla_g f_i(\lambda_1)) - T_{\lambda_2}\Pi_B(\nabla_g f_j(\lambda_2)),$$

which is a well-defined smooth map, since by construction of fibre products, $\Pi_B(\lambda_1) = \Pi_B(\lambda_2)$, and according to Proposition 1.2, this map is entirely determined by Λ (and g).

In order to proceed further towards the definition of gradient generic Legendrian submanifolds, define iterated tangency loci of integral curves of all the vector fields $X_{i,j}$ with the singular locus of the Legendrian caustic.
Then, for all $i \in \{1,\dots,r\}$ and all $\beta = \{k,\ell\} \subset \{1,\dots,r\}$ such that $|\beta| = 2$, let us define

$$L_1^{i,\beta} = \left\{(\lambda_1,\lambda_2) \in L_0^{i,\beta} ; X_{i,k}(\lambda_1,\lambda_2) \in T_{\lambda_2}\Pi_B\left(T_{\lambda_2}\Lambda_\beta\right)\right\},$$

which is the set of points at which $X_{i,k}$ is tangent to the caustic projection of the cusp-edge Λ_β.

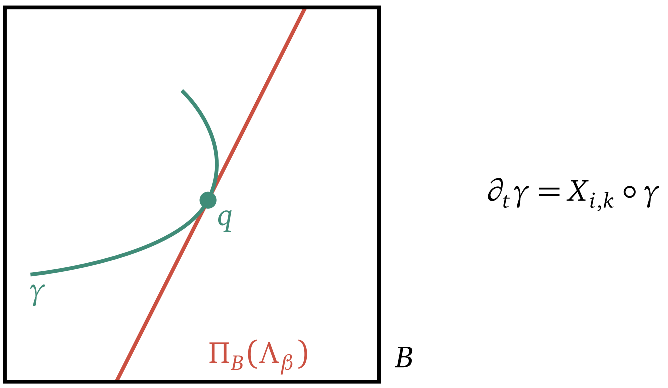


Figure 10. Schematic description of points $(\lambda_1,\lambda_2) \in L_1^{i,\beta}$, $\Pi_B(\lambda_1) = q = \Pi_B(\lambda_2)$.

Remark. Notice that the definition of $L_1^{i,\beta}$ is independent of the choice of an element in $\beta = \{k,\ell\}$, since Proposition 1.2 shows that the functions $f_k : \overline{S_k} \to \mathbf{R}$ and $f_\ell : \overline{S_i} \to \mathbf{R}$ agree along their common cusp-edge Λ_β, and thus $X_{i,k}$ and $X_{i,\ell}$ agree on $\overline{S_i} \times_B \Lambda_\beta$.

Define $\Pi_B^{[2]}\colon J^1(B)\times J^1(B)\to B$ as the canonical projection onto the base of the second factor, then according to **Proposition 1.2**, $L_1^{i,\beta}$ can be written, in a more iteratively-suitable way, as

$$L_1^{i,\beta}=\left\{(\lambda_1,\lambda_2)\in L_0^{i,\beta}; X_{i,k}(\lambda_1,\lambda_2)\in T_{(\lambda_1,\lambda_2)}\Pi_B^{[2]}\left(T_{(\lambda_1,\lambda_2)}L_0^{i,\beta}\right)\right\}.$$

Then, for all nonnegative integers m, building on the above equality, let us recursively define $L_m^{i,\beta}$. In particular, provided that $L_m^{i,\beta}$ is well-defined and is a manifold, let us define

$$L_{m+1}^{i,\beta}=\left\{(\lambda_1,\lambda_2)\in L_m^{i,\beta}; X_{i,k}(\lambda_1,\lambda_2)\in T_{(\lambda_1,\lambda_2)}\Pi_B^{[2]}\left(T_{(\lambda_1,\lambda_2)}L_m^{i,\beta}\right)\right\},$$

which is the set of points at which $X_{i,k}$ is tangent at least to order $m+1$ to the caustic projection of the cusp-edge Λ_β.

For all $i\in\{1,\dots,r\}$, all $\beta\subset\{1,\dots,r\}$ and all $\alpha\subset\{1,\dots,r\}$ such that $|\beta|=2$, $|\alpha|=2$ and $i\in\alpha$, for all nonnegative integers m, mimicking the construction of $L_m^{i,\beta}$, let us recursively define $L_m^{\alpha,\beta}$. In particular, provided that $L_m^{\alpha,\beta}$ is well-defined and is a manifold, let us define

$$L_{m+1}^{\alpha,\beta}=\left\{(\lambda_1,\lambda_2)\in L_m^{\alpha,\beta}; X_{i,k}(\lambda_1,\lambda_2)\in T_{(\lambda_1,\lambda_2)}\Pi_B^{[2]}\left(T_{(\lambda_1,\lambda_2)}L_m^{\alpha,\beta}\right)\right\},$$

which is, by construction, a subset of $L_{m+1}^{i,\beta}$.

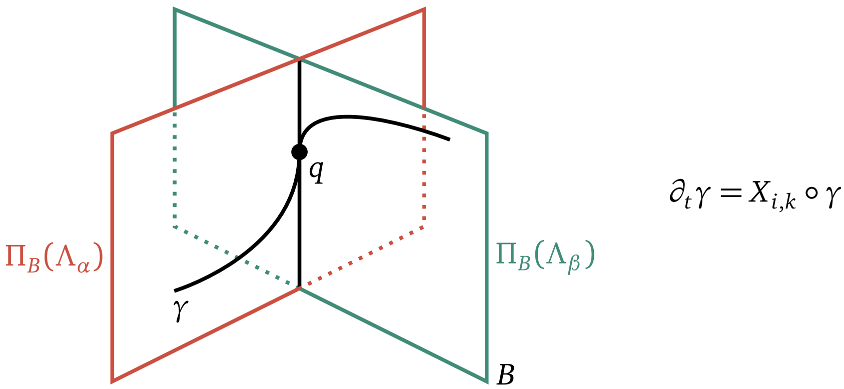


Figure 11. Schematic description of points $(\lambda_1,\lambda_2)\in L_1^{\alpha,\beta}$, $\Pi_B(\lambda_1)=q=\Pi_B(\lambda_2)$.

Remark. Using the front genericity of Λ, **Definition 1.8** shows that the caustic projection of Λ_β is an immersed submanifold of B with self-transverse intersections, but generally not embedded, and at all its double points, there is an ambiguity on the component, $X_{i,k}$ should be tangent to. Solving this difficulty requires to lift the tangency condition to Λ, where everything is embedded, and selecting components of $\Pi_B(\Lambda_\beta)$ with the projection $\Pi_B^{[2]}$ at step $m=1$.

Definition 1.11. The Legendrian embedding $\Lambda\in\mathrm{Leg}_{1,0}(M,B)$ is *gradient generic* whenever for all nonnegative integers m, all $i\in\{1,\dots,r\}$ and all $\beta\subset\{1,\dots,r\}$ with two elements, $L_m^{i,\beta}$ is a manifold with boundary of dimension $\dim(B)-m-1$ and whose boundary is given by

$$\partial L_m^{i,\beta}=\bigcup_{\substack{\alpha\subset\{1,\dots,r\}\\|\alpha|=2,\alpha\ni i}} L_m^{\alpha,\beta}.$$

Remark. It is a straightforward consequence that on gradient generic Legendrian submanifolds, trajectories of the gradient flow, induced in the caustic by the vertical length between pairs of smooth sheets, are tangent at most to order $\dim(B)-1$ to the singular locus.

Gradient genericity is a generic property among at most $\Sigma^{1,0}$-singular Legendrian submanifolds.

Theorem 1.2. *The subset of gradient generic elements of* $\mathrm{Leg}_{1,0}(M,B)$ *is open and dense.*

Proof. Openness is trivial, since requiring finitely many sets to be manifolds is an open condition. Showing denseness requires more work, for an at most $\Sigma^{1,0}$-singular Legendrian submanifold Λ, it amounts to prove recursively on the nonnegative integer m that there exists a perturbation of Λ such that for $i \in \{1,\dots,r\}$ and $\beta \subset \{1,\dots,r\}$, $L_m^{i,\beta}$ is a $(\dim(B)-m-1)$-manifold with boundary.

- **Base step.** The base step $m=0$ has already been dealt with in **Proposition 1.2**.
- **Inductive step.** Let m be a nonnegative integer, and assume that for all nonnegative integers $m' \leqslant m$, all $i \in \{1,\dots,r\}$ and all $\beta \subset \{1,\dots,r\}$ such that $|\beta| = 2$, $L_{m'}^{i,\beta}$ is a manifold with boundary of dimension $\dim(B)-m'-1$ and whose boundary is given by
$$\partial L_{m'}^{i,\beta} = \bigcup_{\substack{\alpha\subset\{1,\dots,r\}\\ |\alpha|=2,\alpha\ni i}} L_{m'}^{\alpha,\beta}.$$
Then, let us show that Λ can be perturbed so that
  - for $m' \in \{0,\dots,m\}$, $i \in \{1,\dots,r\}$ and $\beta \subset \{1,\dots,r\}$, $L_{m'}^{i,\beta}$ is left unperturbed;
  - for $i \in \{1,\dots,r\}$ and $\beta \subset \{1,\dots,r\}$, $L_m^{i,\beta}$ is a codimension 1 submanifold of $L_m^{i,\beta}$;

  and this second property is itself achieved by carrying the following steps:
  - **Step 1.** Perturb the topological boundary components of $L_{m+1}^{i,\beta}$ to be manifolds.
  - **Step 2.** Show that $L_{m+1}^{i,\beta}$ is a manifold in an open neighbourhood of its boundary.
  - **Step 3.** Conclude by extending the perturbation constructed in **Step 1** away from an open neighbourhood of the topological boundary of $L_{m+1}^{i,\beta}$.

  For that purpose, it is more convenient to perturb the functions provided by **Lemma 1.2**, rather than the Legendrian submanifold itself.

  Step 1. It is no surprise that this step relies on the appropriate use of the Sard's theorem. Let $\alpha \subset \{1,\dots,r\}$ such that $|\alpha|=2$ and $i \in \alpha$, then let us give a local description for $L_{m+1}^{\alpha,\beta}$. Since Λ is at most $\Sigma^{1,0}$-singular, using the induction hypothesis, $\Pi_B^{[2]}{}_{|L_m^{\alpha,\beta}}$ is an immersion. In particular, for all $\boldsymbol{\lambda} \in L_m^{\alpha,\beta}$, there must exist an open neighbourhood $U_{\boldsymbol{\lambda}}^{\alpha,\beta}$ of $\boldsymbol{\lambda}$ in $L_m^{\alpha,\beta}$, and a section $\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}$ of $\Pi_B^{[2]*}TB_{|U_{\boldsymbol{\lambda}}^{\alpha,\beta}}$ such that there are vector bundle direct sum splittings given by
$$TB_{|\Pi_B^{[2]}(U_{\boldsymbol{\lambda}}^{\alpha,\beta})} = T\Pi_B^{[2]}\left(TU_{\boldsymbol{\lambda}}^{\alpha,\beta}\right) \oplus \mathbf{R}\nu_{\boldsymbol{\lambda}}^{\alpha,\beta} \quad \text{, if } m=0,$$
$$T\Pi_B^{[2]}\left(TL_{m-1}^{\alpha,\beta}\right)_{|\Pi_B^{[2]}(U_{\boldsymbol{\lambda}}^{\alpha,\beta})} = T\Pi_B^{[2]}\left(TU_{\boldsymbol{\lambda}}^{\alpha,\beta}\right) \oplus \mathbf{R}\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}, \text{ otherwise.}$$
since the induction hypothesis shows that $L_m^{\alpha,\beta}$ is a codimension one submanifold of $L_{m-1}^{\alpha,\beta}$. Associated to the above decompositions, there exist a bundle projection $\pi_{\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}}$ onto $\mathbf{R}\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}$, and a smooth map $\mu_{\boldsymbol{\lambda}}^{\alpha,\beta} : U_{\boldsymbol{\lambda}}^{\alpha,\beta} \to \mathbf{R}$ such that if $i \in \alpha$ and $k \in \beta$, then
$$\pi_{\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}}(X_{i,k}(\boldsymbol{\lambda})) = \mu_{\boldsymbol{\lambda}}^{\alpha,\beta}(\boldsymbol{\lambda})\nu_{\boldsymbol{\lambda}}^{\alpha,\beta}.$$
In particular, by construction, $L_{m+1}^{\alpha,\beta} \cap U_{\boldsymbol{\lambda}}^{\alpha,\beta}$ is the reverse image of 0 by $\mu_{\boldsymbol{\lambda}}^{\alpha,\beta} : U_{\boldsymbol{\lambda}}^{\alpha,\beta} \to \mathbf{R}$ and the proof boils down showing that Λ can be perturbed in a way that 0 is a regular

value for all $\mu_{\boldsymbol{\lambda}}^{\alpha,\beta} : U_{\boldsymbol{\lambda}}^{\alpha,\beta} \to \mathbf{R}$.

Let pr_1 and pr_2 be the first and second projection of $\Lambda^{\prec} \times_B \Lambda^{\prec}$ onto its factors, respectively, since Λ is at most $\Sigma^{1,0}$-singular, $\Pi_{B|\Lambda^{\prec}}$ is self-transverse, and pr_1 and pr_2 are immersions. Thus, pr_1 is a local embedding and there exists a refinement of $\{U_{\boldsymbol{\lambda}}^{\alpha,\beta}\}_{\boldsymbol{\lambda} \in L_m^{\alpha,\beta}}$ such that

(H1) For all $\boldsymbol{\lambda} \in L_m^{\alpha,\beta}$, pr_1 is an embedding over all the compact subsets of $U_{\boldsymbol{\lambda}}^{\alpha,\beta}$.

Since $L_m^{\alpha,\beta}$ and Δ_β are disjoint, argue by contradiction with the continuity of pr_1 and pr_2 to show to that there exists an even finer refinement of $\{U_{\boldsymbol{\lambda}}^{\alpha,\beta}\}_{\boldsymbol{\lambda} \in L_m^{\alpha,\beta}}$ such that

(H2) $$\forall \boldsymbol{\lambda} \in L_m^{\alpha,\beta}, \mathrm{pr}_1\left(U_{\boldsymbol{\lambda}}^{\alpha,\beta}\right) \cap \mathrm{pr}_2\left(U_{\boldsymbol{\lambda}}^{\alpha,\beta}\right) = \varnothing.$$

Then, by compactness of $L_m^{\alpha,\beta}$, extract a finite subcover $\{U_N^{\alpha,\beta}\}_{N \in \{1,\dots,N_{\alpha,\beta}\}}$ from $\{U_{\boldsymbol{\lambda}}^{\alpha,\beta}\}_{\boldsymbol{\lambda} \in L_m^{\alpha,\beta}}$. Let now define the dimension of the deformation space of parameters d by

$$d = \sum_{\substack{\alpha \subset \{1,\dots,r\} \\ |\alpha|=2, \alpha \ni i}} \sum_{\substack{\beta \subset \{1,\dots,r\} \\ |\beta|=2}} N_{\alpha,\beta}.$$

then for all $N \in \{1, \dots, N_{\alpha,\beta}\}$, using the induction hypothesis and the assumption **(H1)**, there exists a compactly supported smooth map $\varepsilon_N^{\alpha,\beta} : \Lambda \to \mathbf{R}$ such that

(H3) The smooth map $\varepsilon_N^{\alpha,\beta}$ has compact support in $\mathrm{pr}_1\left(U_N^{\alpha,\beta}\right)$;

(H4) If $m \geqslant 1$, then the vector field $\mathrm{pr}_1{}^*\nabla_g \varepsilon_N^{\alpha,\beta}$ is tangent to $L_{m-1}^{\alpha,\beta}$;

(H5) The vector field $\mathrm{pr}_1{}^*\nabla_g \varepsilon_N^{\alpha,\beta}$ is transverse to $L_m^{\alpha,\beta}$.

since pr_1 is a local embedding on $U_N^{\alpha,\beta}$ and $L_m^{\alpha,\beta}$ is a codimension 1 submanifold of $L_{m-1}^{\alpha,\beta}$. First, for all $i \in \{1, \dots, r\}$, let us define a linear deformation $F_i : \overline{S_i} \times \mathbf{R}^d \to \mathbf{R}$ of f_i by

$$F_i(\lambda, \mathbf{t}) = f_i(\lambda) + \sum_{\substack{\alpha \subset \{1,\dots,r\} \\ |\alpha|=2, \alpha \ni i}} \sum_{\substack{\beta \subset \{1,\dots,r\} \\ |\beta|=2}} \sum_{N=1}^{N_{\alpha,\beta}} t_N^{\alpha,\beta} \varepsilon_N^{\alpha,\beta}(\lambda),$$

and then for $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, r\}$, let us define $\widetilde{X}_{i,j} : \overline{S_i} \times_B \overline{S_j} \times \mathbf{R}^d \to TB$ by

$$X_{i,j}(\lambda_1, \lambda_2, \mathbf{t}) = T_{\lambda_1}\Pi_B\left(\nabla_g F_i(\cdot, \mathbf{t})(\lambda_1)\right) - T_{\lambda_2}\Pi_B\left(\nabla_g F_k(\cdot, \mathbf{t})(\lambda_2)\right).$$

Besides, for all $\alpha \subset \{1, \dots, r\}$ and all $\beta \subset \{1, \dots, r\}$ such that $|\alpha| = 2$, $i \in \alpha$ and $|\beta| = 2$, introduce a thickened version $\widetilde{L}_{m+1}^{\alpha,\beta}$ of $L_{m+1}^{\alpha,\beta}$ defined as the subset of $(\boldsymbol{\lambda}, \mathbf{t}) \in L_{m+1}^{\alpha,\beta} \times \mathbf{R}^d$ such that

$$X_{i,j}(\boldsymbol{\lambda}, \mathbf{t}) \in T_{\boldsymbol{\lambda}}\Pi_B^{[2]}\left(T_{\boldsymbol{\lambda}} L_m^{\alpha,\beta}\right),$$

where $i \in \{1, \dots, r\}$ and $k \in \{1, \dots, r\}$ are indifferently chosen in α and β, respectively. Associated to the vector bundle decompositions introduced at the beginning of **Step 1**, and for all $N \in \{1, \dots, N_{\alpha,\beta}\}$, there must exist a smooth map $\widetilde{\mu}_N^{\alpha,\beta} : U_N^{\alpha,\beta} \to \mathbf{R}$ such that

$$\pi_{\nu_N^{\alpha,\beta}}\left(\widetilde{X}_{i,k}(\boldsymbol{\lambda}, \mathbf{t})\right) = \widetilde{\mu}_N^{\alpha,\beta}(\boldsymbol{\lambda}, \mathbf{t})\, \nu_{\boldsymbol{\lambda}}^{\alpha,\beta}.$$

Then, by construction, $\widetilde{L}_{m+1}^{\alpha,\beta} \cap (U_N^{\alpha,\beta} \times \mathbf{R}^d)$ is the reverse image of 0 by $\widetilde{\mu}_N^{\alpha,\beta} : U_N^{\alpha,\beta} \times \mathbf{R}^d \to \mathbf{R}$, but now, everything has been arranged for 0 to automatically be a regular value of $\widetilde{\mu}_N^{\alpha,\beta}$.

Indeed, from assumptions (H2) and (H3), for all $(\lambda_1,\lambda_2)\in U_N^{\alpha,\beta}$ and all $t\in\mathbf{R}^d$, then

$$\begin{aligned}\partial_{t_N^\alpha}\widetilde{\mu}_N^{\alpha,\beta}(\lambda_1,\lambda_2,\mathbf{t}) &= \partial_{t_N^{\alpha,\beta}}\left(\mu_N^{\alpha,\beta}(\lambda_1)+t_N^{\alpha,\beta}\pi_{\nu_N^{\alpha,\beta}}\left(T_{\lambda_1}\Pi_B\left(\nabla_g\varepsilon_N^{\alpha,\beta}(\lambda_1)\right)\right)\right),\\ &= \pi_N^{\alpha,\beta}\left(T_{\lambda_1}\Pi_B\left(\nabla_g\varepsilon_N^{\alpha,\beta}(\lambda_1)\right)\right),\\ &= \pi_N^{\alpha,\beta}\left(T_{\lambda_1,\lambda_2}\Pi_B^{[2]}\left(\mathrm{pr}_1{}^*\nabla_g\varepsilon_N^{\alpha,\beta}(\lambda_1,\lambda_2)\right)\right),\\ &\neq 0,\end{aligned}$$

which is nonzero from assumption (H5), since Λ is assumed to be at most $\Sigma^{1,0}$-singular, so that $\Pi_{B|\Lambda}$ is an immersion and vectors of $T\Lambda$ are determined by their caustic projection. Therefore, $\widetilde{L}_{m+1}^{\alpha,\beta}$ is a manifold, since $\{U_N^{\alpha,\beta}\times\mathbf{R}^d\}_{N\in\{1,\dots,N_{\alpha,\beta}\}}$ is an open covering of $L_m^{\alpha,\beta}$, and it remains to use the Sard's theorem to pick a suitable deformation parameter in $\mathbf{R}^d$. For that purpose, let us define $\widetilde{\pi}_{\mathbf{R}^d}^{\alpha,\beta}:\widetilde{L}_{m+1}^{\alpha,\beta}\to\mathbf{R}^d$ to be the projection onto the second factor, and there exists $\mathbf{t}^*\in\mathbf{R}^d$, arbitrarily close to $\mathbf{0}$, such that $\mathbf{t}^*$ is a regular value of $\widetilde{\pi}_{\mathbf{R}^d}^{\alpha,\beta}$. To conclude, define a perturbation of Λ by replacing f_i by $f_i^*=F_i(\cdot,\mathbf{t}^*)$, for $i\in\{1,\dots,N\}$. Indeed, for all $\alpha\subset\{1,\dots,r\}$ and all $\beta\subset\{1,\dots,r\}$ such that $|\alpha|=2$, $i\in\alpha$ and $|\beta|=2$, notice that

- According to (H4), for all nonnegative integers $m'\leqslant m$, $L_{m'}^{\alpha,\beta}$ is left unchanged.
- Since t^* is a regular value of $\pi_{\mathbf{R}^d}^{\alpha,\beta}$ and $L_{m+1}^{\alpha,\beta}\times\{\mathbf{t}^*\}=(\pi_{\mathbf{R}^d}^{\alpha,\beta})^{-1}(\mathbf{t}^*)$, $L_{m+1}^{\alpha,\beta}$ is a manifold.

Moreover, according to Proposition 1.1, if $\mathbf{t}^*\in\mathbf{R}^d$ is sufficiently close to $\mathbf{0}$, then the above procedure constructs a Legendrian submanifold that is still at most $\Sigma^{1,0}$-singular, thus almost yielding Step 1.

One subtlety has not yet been addressed, as it is still needed to show that

$$\forall i\in\{1,\dots,r\},\forall j\in\{1,\dots,r\},\forall t\in\mathbf{R}^d,\forall\lambda\in\Lambda_{\{i,j\}},F_i(\lambda,\mathbf{t})=F_j(\lambda,\mathbf{t}),$$

otherwise, the deformation will tear apart the smooth sheets of Λ along their cusp-edges. For that purpose, let us pick $i\in\{1,\dots,r\}$ and $j\in\{1,\dots,r\}$ such that $i\neq j$ and $\Lambda_{\{i,j\}}\neq\varnothing$, and let $\alpha\subset\{1,\dots,r\}$ and $\beta\subset\{1,\dots,r\}$ such that $|\alpha|=2$, $i\in\alpha$ and $|\beta|=2$, then

$$\forall N\in\{1,\dots,N_{\alpha,\beta}\},\forall\lambda\in\Lambda_{\{i,j\}},\lambda\in\mathrm{pr}_1\left(U_N^{\alpha,\beta}\right)\implies\alpha=\{i,j\},\tag{1}$$

since $\mathrm{pr}_1(U_N^{\alpha,\beta})\subset\Lambda_\alpha$ and the smooth sheets of Λ only meet pairwise on their boundaries. Therefore, for all $\lambda\in\Lambda_{\{i,j\}}$ and all $t\in\mathbf{R}^d$, then

$$F_i(\lambda,\mathbf{t})=f_i(\lambda)+\sum_{\substack{\alpha\subset\{1,\dots,r\}\\|\alpha|=2,\alpha\ni i}}\sum_{\substack{\beta\subset\{1,\dots,r\}\\|\beta|=2}}\sum_{N=1}^{N_{\alpha,\beta}}t_N^{\alpha,\beta}\varepsilon_N^{\alpha,\beta}(\lambda),$$

[using (H3) and (1)]
$$=f_i(\lambda)+\sum_{\substack{\beta\subset\{1,\dots,r\}\\|\beta|=2}}\sum_{N=1}^{N}t_N^{\{i,j\},\beta}\varepsilon_N^{\{i,j\},\beta}(\lambda),$$

[using Lemma 1.2]
$$=f_j(\lambda)+\sum_{\substack{\beta\subset\{1,\dots,r\}\\|\beta|=2}}\sum_{N=1}^{N}t_N^{\{j,i\},\beta}\varepsilon_N^{\{j,i\},\beta}(\lambda),$$

[using (H3) and (1)]
$$= f_j(\lambda) + \sum_{\substack{\alpha\subset\{1,\dots,r\}\\ |\alpha|=2,\alpha\ni j}} \sum_{\substack{\beta\subset\{1,\dots,r\}\\ |\beta|=2}} \sum_{N=1}^{N_{\alpha,\beta}} t_N^{\alpha,\beta}\varepsilon_N^{\alpha,\beta}(\lambda),$$
$$= F_j(\lambda,\mathbf{t}),$$

thus yielding Step 1.

Step 2. Mimicking the proof of Proposition 1.2, the topological boundary of $L_{m+1}^{i,\beta}$ is

$$\partial L_{m+1}^{i,\beta} = \bigcup_{\substack{\alpha\subset\{1,\dots,r\}\\ |\alpha|=1,\alpha\ni i}} L_{m+1}^{\alpha,\beta},$$

but Step 1 provides a perturbation defined on the whole Λ such that $L_{m+1}^{\alpha,\beta}$ are manifolds, and their union is disjoint, since smooth sheets only meet pairwise on their boundaries. Thus, the topological boundary of $L_{m+1}^{i,\beta}$ is manifold and by openness of being a manifold, there must exist an open neighbourhood of $\partial L_{m+1}^{i,\beta}$ in L such that $L_{m+1}^{i,\alpha}\cap V$ is manifold, thus yielding Step 2.

Step 3. Mimimicking the proof of Step 1 shows that the perturbation can be extended away from the open neighbourhood provided in Step 2 in a way that $L_{m+1}^{i,\beta}$ is a manifold. However, this time the construction is even easier, since it is no more required to take care of the cusp-edges shared by the smooth sheets of Λ, thus yielding Step 3.

The above three steps yields the inductive step.

Mathematical induction thus yields the result. ■

Remark. Let $f: X\to Z$ and $g: Y\to Z$ be transverse maps between manifolds with boundaries. Assume that g is an immersion, then the canonical projection $p: X\times_Z Y\to X$ is also an immersion. Indeed, using the remark following Proposition 1.2, $X\times_Z Y$ is a manifold such that

$$T(X\times_Z Y) = \{(v_X, v_Y)\in TX\times TY; Tf(v_X) = Tg(v_Y)\}.$$

In particular, if $Tp(v_X, v_Y) = \mathbf{0}_{TX}$, then $v_X = Tp(v_X, v_Y) = \mathbf{0}_{TX}$, so that $Tg(v_Y) = Tf(v_X) = \mathbf{0}_{TZ}$, but since g is an immersion, $v_Y = \mathbf{0}_{TY}$ and in conclusion, $(v_X, v_Y) = \mathbf{0}_{TX\times_Z Y}$.

At last, further generic properties of Legendrian embeddings will also be of use in this thesis.

Definition 1.12. A Legendrian embedding $\Lambda\in \mathrm{Leg}(M,B)$ is *chord generic* whenever the following four conditions

(1) the Reeb chords of Λ are non-degenerate;
(2) the Reeb chords of Λ have distinct action values;
(3) the Reeb chords of Λ have disjoint images under Π_B;
(4) the ends of the Reeb chords of Λ stay away from an open neighbourhood of $\Lambda^{\prec}$ in Λ;

are simultaneously met.

It is a folklore result that chord genericity is a generic property among Legendrian submanifolds, see, for example, [EES05a].

Proposition 1.3. *The subset of chord generic elements of* $\mathrm{Leg}(M,B)$ *is open and dense.*

Remark. Gradient and chord genericity are independent and compatible properties of Legendrian embeddings, so that **Theorem 1.2** and **Proposition 1.3** show that the subset of gradient and chord generic Legendrian embeddings is open and dense in $\mathrm{Leg}_{1,0}(M,B)$.

2.

Generating family homologies of Legendrian submanifolds

This chapter is an overview of the theory of generating families for Legendrian submanifolds. First, their relevance in the study of contact rigidity phenomena of Legendrian submanifolds is emphasised, and the homotopy lifting property of generating families (**Theorem 2.1**) is recalled. Then, in view of the proof of **Theorem B**, the transversality conditions defined for Legendrian submanifolds in **Chapter 1** are now rephrased using strictly generating families (**Section 1.2**). Afterwards, while the construction of a Morse-like homology theory for Legendrian submanifolds from generating families is reviewed (**Section 2**), a useful technical result (**Proposition 2.4**) for crucial estimates in the proof of **Theorem A** is established. At last, several results concerning the algebraic structure of the generating family homology are discussed and conjectured (**Section 3**) to motivate the study of the *Henry-Rutherford limiting process* in the next chapter.

1. Generating family theory overview

It was observed in **Section 1.2** that holonomic sections of first-order jet bundles tautologically are Legendrian submanifolds which are isotopic to the zero-section through Legendrian submanifolds. Conversely, not all Legendrian submanifolds which are Legendrian isotopic to the zero-section are holonomic sections, but they are nonetheless always obtained by reduction of holonomic sections of stabilised first-order jet bundles. This observation is the starting point of the generating family theory for Legendrian submanifolds.

Definition 2.1. A smooth function $f : B \times \mathbf{R}^N \to \mathbf{R}$ is a *generating family* whenever its *fibrewise critical set* $\Sigma_f = \partial_\eta f^{-1}(\mathbf{0})$ is a transversely cut out submanifold of $B \times \mathbf{R}^N$, meaning that $\mathbf{0}$ is a regular value of the *fibrewise derivative* $\partial_\eta f : B \times \mathbf{R}^N \to \mathbf{R}^N$ of f.

Remark. The generating family condition explicitly reads as follows, for all $(b, \eta) \in B \times \mathbf{R}^N$ such that $\partial_\eta f(b, \eta) = \mathbf{0}$, the $N \times (n + N)$ matrix $\operatorname{Jac}_{(b,\eta)} \partial_\eta f = \begin{bmatrix} \partial^2_{b\eta} f(b, \eta) & \partial^2_{\eta\eta} f(b, \eta) \end{bmatrix}$ has full rank, namely it has rank N.

Remark. Sard's theorem directly ensures that being a generating family is a generic property in the set of smooth maps from $B \times \mathbf{R}^N$ to $\mathbf{R}$ endowed with the C^∞-topology.

Definition 2.2. A Legendrian submanifold Λ of (J^1B, ξ_B) is *generated* by a function $f : B \times \mathbf{R}^N \to \mathbf{R}$ whenever f is a generating family and the map $j_f : B \times \mathbf{R}^N \to J^1B$ defined in local coordinates by

$$j_f(b, \eta) = (b, \partial_b f(b, \eta), f(b, \eta)),$$

restricts to a diffeomorphism between Σ_f and Λ.

Remark. Let $C = T^*B \times \{0_{T^*\mathbf{R}^N}\}$ and let $\Gamma_{\mathrm{d}f}$ be the graph of $\mathrm{d}f$, then Λ is generated by f if, and only if, it is obtained by lifting the symplectic reduction $(\Gamma_{\mathrm{d}f} \cap C)/C^\perp$ to J^1B, by the means of f. Indeed, for their canonical symplectic structures, C is a coisotropic submanifold of $T^*(B \times \mathbf{R}^N)$

and the symplectic reduction $T^*(B\times\mathbf{R}^N)/C^\perp$ is seen to be canonically symplectomorphic to T^*B. Moreover, the exact Lagrangian submanifold $\Gamma_{\mathrm{d}f}$ of $T^*(B\times\mathbf{R}^N)$ is transverse to C exactly when f is a generating family.

Remark. The front projection of Λ is the *Cerf's diagram* of the family of functions $(f_{|\{b\}\times\mathbf{R}^N})_{b\in B}$, namely a point $(b,z)\in B\times\mathbf{R}$ belongs to the front projection of Λ if, and only if, z is a critical value of the function $f_{|\{b\}\times\mathbf{R}^N}$.

Remark. Every Legendrian germ is generated by a generating family [AGZV85, Section 2.20], but nonetheless not all Legendrian submanifolds are globally generated by a generating family. For example, if there exists a Darboux chart (U,φ) of Λ such that the front projection of $\varphi(\Lambda\cap U)$ has a zig-zag disjoint from the rest of the front, then Λ has no generating family [AGI21, Proof of Proposition 2.8].

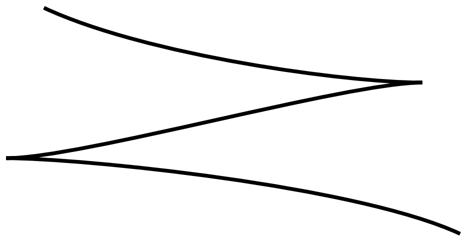


Figure 12. A zig-zag that is disjoint from the rest of the front of a Legendrian knot. Higher-dimensional zig-zags are obtained by trivial products with $\mathbf{R}^{n-1}$, $n\geqslant 2$.

Example 2.1. For all integers $n\geqslant 1$, let us define $f^n_\prec\colon \mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ by $f^n_\prec(b_1,\dots,b_n,\eta)=\frac{\eta^3}{3}-b_1\eta$, then the fibrewise critical of f is a transversely cut out nth-dimensional parabolic cylinder in $\mathbf{R}^{n+1}$ which is given by

$$\Sigma_{f^n_\prec}=\left\{(b_1,\dots,b_n,\eta)\in\mathbf{R}^n\times\mathbf{R};\,\eta^2=b_1\right\}.$$

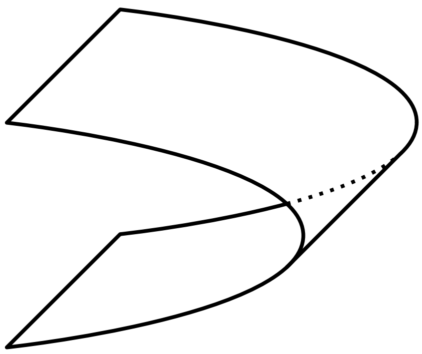


Figure 13. A 2-dimensional parabolic cylinder in $\mathbf{R}^3$.

Furthermore, the bifurcations of $(f^n_{\prec|\{b\}\times\mathbf{R}}\colon\mathbf{R}\to\mathbf{R})_{b\in\mathbf{R}^2}$ can be described as follows:

- If $b_1<0$, then $f^n_{\prec|\{b\}\times\mathbf{R}}$ is strictly increasing.
- If $b_1=0$, then $f^n_{\prec|\{b\}\times\mathbf{R}}$ has a unique critical point and it is degenerate.
- If $b_1>0$, then $f^n_{\prec|\{b\}\times\mathbf{R}}$ has two critical points and they are non-degenerate.

Therefore, $f^n_\prec$ is a generating family for the Legendrian submanifold of $(J^1\mathbf{R}^n,\xi_{\mathbf{R}^n})$ whose front projection is a standard nth-dimensional Whitney pleat in $\mathbf{R}^{n+1}$.

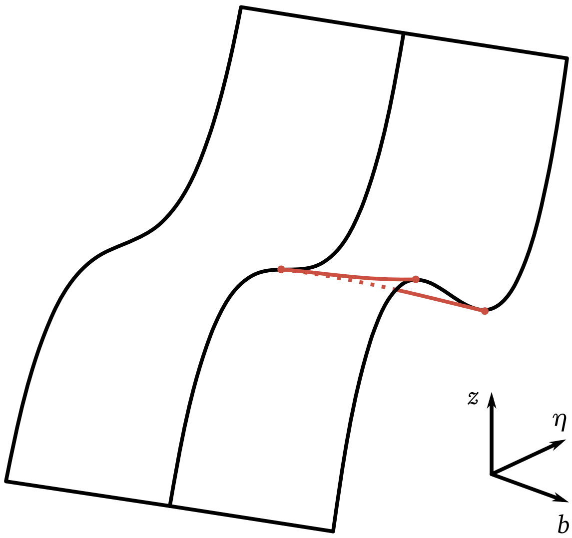


Figure 14. Graph of the generating family $f^1_{\prec}$ with fibrewise critical values in red.

Example 2.2. For all integers $n \geqslant 1$, let us consider the Legendrian submanifold Λ^n_0 of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ which is recursively defined as follows:

- **Base case.** If $n = 1$, then Λ^1_0 is the standard Legendrian unknot of $(J^1\mathbf{R}, \xi_{\mathbf{R}})$ with maximal Thurston-Bennequin number.
- **Inductive step.** If $n \geqslant 2$, then Λ^n_0 is the Legendrian submanifold whose front projection is obtained by spinning the front projection of Λ^{n-1}_0 around the z-axis.

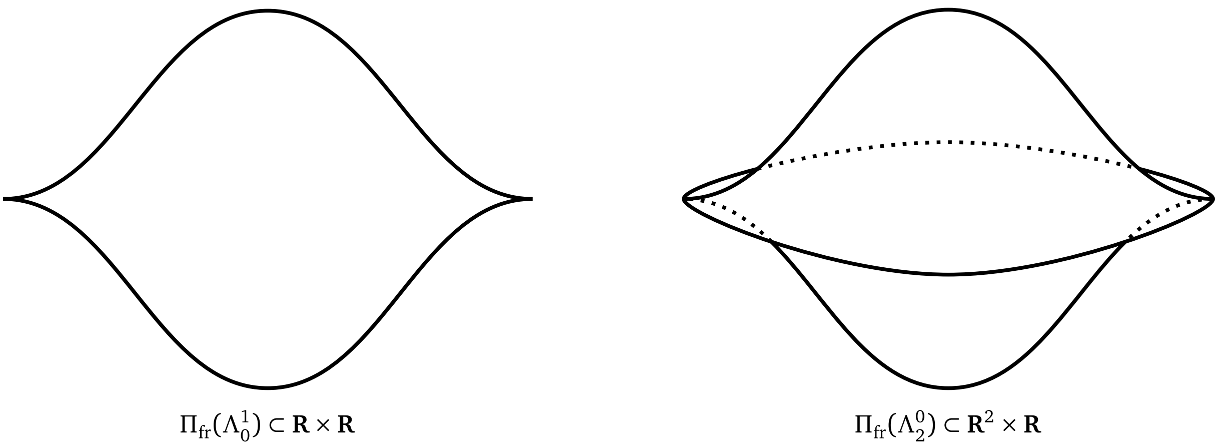


Figure 15. Front projections of Λ^1_0 (on the left) and Λ^2_0 (on the right).

Let $\|\cdot\|$ be the usual 2-norm on $\mathbf{R}^n$ and let us define $f^n_0 : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ by $f^n_0(b, \eta) = \eta^3 + 3(\|b\|^2 - 1)\eta$, then the fibrewise critical set of f^n_0 is a transversely cut out standard n-sphere of $\mathbf{R}^{n+1}$ given by

$$\Sigma_{f^n_0} = \left\{(b, \eta) \in \mathbf{R}^n \times \mathbf{R}; \|b\|^2 + \eta^2 = 1\right\}.$$

Furthermore, the bifurcations of $({f^n_0}_{|\{b\}\times\mathbf{R}} : \mathbf{R} \to \mathbf{R})_{b\in\mathbf{R}^n}$ can be described as follows:

- If $\|b\| > 1$, then ${f^n_0}_{|\{b\}\times\mathbf{R}^n}$ is strictly increasing.
- If $\|b\| = 1$, then ${f^n_0}_{|\{b\}\times\mathbf{R}^n}$ has a unique critical point and it is degenerate.
- If $\|b\| < 1$, then ${f^n_0}_{|\{b\}\times\mathbf{R}^n}$ has two critical points and they are non-degenerate.

In conclusion, f^n_0 is a generating family of Λ^n_0.

1.1. Homotopy lifting property for generating families.

Only the values taken by a generating family onto its fibrewise critical set are relevant to describe a Legendrian submanifold, but the rest of the generating family has no geometrical significance. In particular, if a Legendrian submanifold has a generating family, it can be naturally modified to directly produce infinitely many other, but all these generating families really should be identified and treated as one.

Definition 2.3. Two generating families defined over vector bundles with the same base are *equivalent* whenever they can be made equal by applying to both of them a finite number of the following two types of moves:

- **Stabilisation move.** Let $f : B \times \mathbf{R}^{N_1} \to \mathbf{R}$ be a function, then given any non-degenerate quadratic form Q on $\mathbf{R}^{N_2}$, let us define $f \oplus Q : B \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2} \to \mathbf{R}$ by
$$(f \oplus Q)(b, \eta_1, \eta_2) = f(b, \eta_1) + Q(\eta_2).$$
- **Fibre-preserving diffeomorphism move.** Let $f : B \times \mathbf{R}^N \to \mathbf{R}$ be a function, then given any diffeomorphism $\Phi = \mathrm{id}_B \times \varphi$ of $B \times \mathbf{R}^N$ over B, let us define $f_\Phi : B \times \mathbf{R}^N \to \mathbf{R}$ by
$$f_\Phi(b, \eta) = f(\Phi(b, \eta)) = f(b, \varphi(b, \eta)).$$

Remark. If f is a generating family, then so are $f \oplus Q$ and f_Φ for the same Legendrian immersion. Indeed, notice first that $\Sigma_{f \oplus Q} = \Sigma_f \times \{\mathbf{0}_{\mathbf{R}^{N_2}}\}$ and that the chain rule also shows that $\Sigma_{f_\Phi} = \Phi^{-1}(\Sigma_f)$. Moreover, if $(b, \eta) \in \Sigma_f$, then a straightforward computation implies that
$$\mathrm{Jac}_{(b,\eta,\mathbf{0})}\,\partial_\eta(f \oplus Q) = \begin{bmatrix} \partial^2_{b\eta} f(b,\eta) & \partial^2_{\eta\eta} f(b,\eta) & \mathbf{0}_{n,N_2} \\ \mathbf{0}_{N_2,n} & \mathbf{0}_{N_2,N_1} & \mathrm{Hess}_\mathbf{0}\, Q \end{bmatrix} = \begin{bmatrix} \mathrm{Jac}_{(b,\eta)}\,\partial_\eta f & \mathbf{0}_{n,N_2} \\ \mathbf{0}_{N_2,n+N_1} & \mathrm{Hess}_\mathbf{0}\, Q \end{bmatrix},$$
which is a $(N_1 + N_2) \times (n + N_1 + N_2)$ matrix of rank $\mathrm{rk}(\mathrm{Jac}_{(b,\eta)}\,\partial_\eta f) + \mathrm{rk}(\mathrm{Hess}_\mathbf{0}\, Q)$, namely $N_1 + N_2$. Besides, if $(b, \eta) \in \Sigma_f$, then using the chain rule shows that $\mathrm{Jac}_{\Phi(b,\eta)}\,\partial_\eta f_\Phi$ is a $N \times (n + N)$ matrix of rank $\mathrm{rk}(\mathrm{Jac}_{(b,\eta)}\,\partial_\eta f)$, namely N.

Unlike real valued functions defined over the base, generating families and their equivalence relation are well-behaved with respect to isotopies of closed Legendrian submanifolds.

Theorem 2.1 ([Cha84], [LS85], [Sik86] and [Che96, Theorem 2.5]). *Let $(\Lambda_t)_{t\in[0,1]}$ be an isotopy of closed Legendrian submanifolds of (J^1B, ξ_B), then for all generating families f of Λ_0, there exists a homotopy $(f_t)_{t\in[0,1]}$ such that f_t is a generating family of Λ_t and f_0 and f are equivalent.*

1.2. Properties of generating families of generic Legendrian submanifolds.

Some of the properties that were discussed in Chapter 1 for Legendrian submanifolds are now translated within the language of generating families.

Let $f : B \times \mathbf{R}^N \to \mathbf{R}$ be a generating family, then for all $k \in \{0, \ldots, N\}$, let us define $\Sigma_k(f)$ to be the subset of Σ_f consisting of elements $(b, \eta) \in \Sigma_f$ at which the differential of the canonical projection $\pi_{f,B} : \Sigma_f \to B$ has a k-dimensional kernel, so that
$$\Sigma_f = \Sigma_0(f) \cup \Sigma_1(f) \cup \cdots \cup \Sigma_{N-1}(f) \cup \Sigma_N(f),$$
and also define $\Sigma_f^{\prec}$ to be the union of $\Sigma_k(f)$ for all $k \in \{1, \ldots, N\}$.

Proposition 2.1. *Let $f : B \times \mathbf{R}^N \to \mathbf{R}$ be a generating family, then for all $k \in \{0, \dots, n\}$, the equality*

$$\Sigma_k(f) = \left\{(b,\eta) \in \Sigma_f \,;\, \dim\ker\left(\mathrm{Hess}_\eta f_b\right) = k\right\},$$

holds true, where for all $b \in B$, f_b stands for $f_{|\{b\}\times\mathbf{R}^N}$.

Proof. Let $(b,\eta) \in \Sigma_f$, then using the transversality assumption from Definition 2.1, it holds

$$\begin{aligned} T_{(b,\eta)}\Sigma_f &= \ker T_{(b,\eta)}\partial_\eta f, \\ &= \ker\begin{bmatrix} \partial^2_{b\eta} f(b,\eta) & \partial^2_{\eta\eta} f(b,\eta) \end{bmatrix}, \\ &= \ker\begin{bmatrix} \partial^2_{b\eta} f(b,\eta) & \mathrm{Hess}_\eta f_b \end{bmatrix}. \end{aligned}$$

Let $v = v_B \oplus v_{\mathbf{R}^N} \in T_{(b,\eta)}\Sigma_f$, then $\partial^2_{b,\eta} f(b,\eta) v_B + \mathrm{Hess}_\eta f_b\, v_{\mathbf{R}^N} = \mathbf{0} \in \mathbf{R}^N$ and $T_{(b,\eta)}\pi_{f,B}(v) = v_B$. Therefore, it shows that $v \in \ker T_{(b,\eta)}\pi_{f,B}$ if, and only if, $v_B = \mathbf{0} \in T_b B$ and $v_{\mathbf{R}^N} \in \ker \mathrm{Hess}_\eta f_b$, thus yielding the result. ■

Remark. The computations carried in the proof of Proposition 2.1 show that if $f : B \times \mathbf{R}^N \to \mathbf{R}$ is a generating family, then $f_{|\{b\}\times\mathbf{R}^N}$ is a Morse function if, and only if, b is a regular value of $\pi_{f,B}$, thus for b in a codimension 0 submanifold of B, according to Sard's theorem.

Let $k \in \{0, \dots, N\}$, if $\Sigma_k(f)$ is a submanifold, then for all $\ell \in \{0, \dots, k\}$, let us define $\Sigma_{k,\ell}(f)$ to be the subset of $\Sigma_k(f)$ consisting of elements $(b,\eta) \in \Sigma_k(f)$ at which the differential of $\pi_{f,B_{|\Sigma_k(f)}}$ has a ℓ-dimensional kernel, so that

$$\Sigma_k(f) = \Sigma_{k,0}(f) \cup \Sigma_{k,1}(f) \cup \dots \cup \Sigma_{k,k-1}(f) \cup \Sigma_{k,k}(f).$$

Proposition 2.2. *Let Λ be a front generic Legendrian submanifold of (J^1B, ξ_B) and let $f : B \times \mathbf{R}^N \to \mathbf{R}$ be a generating family of Λ, then the following two conditions are satisfied:*

(1) For all $k \in \{0, \dots, N\}$, $\Sigma_k(f)$ is a transversally cut out codimension $k(k+1)/2$ submanifold.
(2) For all integers k and ℓ in $\{0, \dots, N\}$, $\pi_{f,B_{|\Sigma_k(f)}}$ and $\pi_{f,B_{|\Sigma_\ell(f)}}$ are transverse maps.

Moreover, if Λ is at most $\Sigma^{1,0}$-singular, then $\Sigma_f{}^{\prec} = \Sigma_{1,0}(f)$.

Proof. It suffices to notice that $\pi_{f,B} : \Sigma_f \to B$ fits in the following commutative diagram

$$\begin{array}{ccc} \Sigma_f & \xrightarrow{j^1 f_{|\Sigma_f}} & \Lambda \\ {\scriptstyle \pi_{f,B}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{B|\Lambda}} \\ B & \xrightarrow[\mathrm{id}_B]{} & B \end{array}.$$

Therefore, since Definition 2.2 shows that the above two horizontal maps are diffeomorphisms, then Definition 1.8 directly yields the result. ■

Example 2.3. Let $f^n_{\prec} : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be the generating family from Example 2.1, then

$$\begin{aligned} \Sigma_{f^n_\prec} &= \left\{(b,\eta) \in \mathbf{R}^n \times \mathbf{R};\, \eta^2 = b_1\right\}, \\ \Sigma_0(f^n_\prec) &= \Sigma(f^n_\prec) \setminus \{b_1 = 0\}, \\ \Sigma_{f^n_\prec}{}^{\prec} &= \Sigma_{1,0}(f^n_\prec) = \Sigma(f^n_\prec) \cap \{b_1 = 0\}, \end{aligned}$$

and the corresponding Legendrian submanifold is indeed at most $\Sigma^{1,0}$-singular.

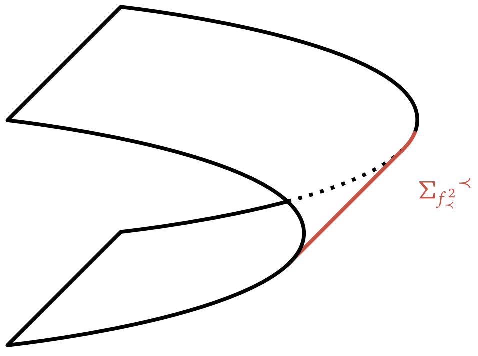


Figure 16. The stratification of the fibrewise critical set of $f^2_{\prec}$.

Example 2.4. Let $f^n_0 \colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ be the generating family from Example 2.1, then

$$\Sigma_{f^n_0} = \left\{(b,\eta) \in \mathbf{R}^n \times \mathbf{R};\, \|b\|^2 + \eta^2 = 1\right\},$$
$$\Sigma_0(f^n_0) = \Sigma(f^n_0) \setminus \{b_n = 0\},$$
$$\Sigma_{f^n_0}{}^{\prec} = \Sigma_{1,0}(f^n_0) = \Sigma(f^n_0) \cap \{b_n = 0\},$$

and the corresponding Legendrian submanifold is indeed at most $\Sigma^{1,0}$-singular.

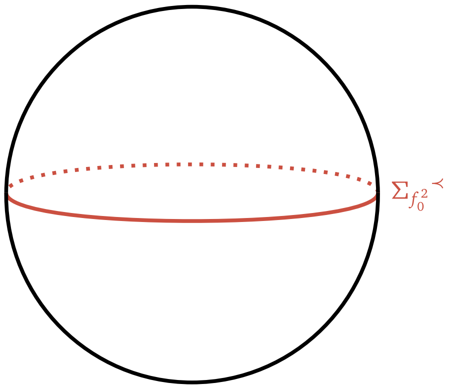


Figure 17. The stratification of the fibrewise critical set of $f^2_{\prec}$.

Remark. As it is clear from the proofs of Propositions 2.1 and 2.2, the singularities of the caustic of Λ correspond to the points $(b,\eta) \in \Sigma_f$ such that $T_{(b,\eta)}\Sigma_f \subset \{\mathbf{0}\} \times \mathbf{R}^N$.

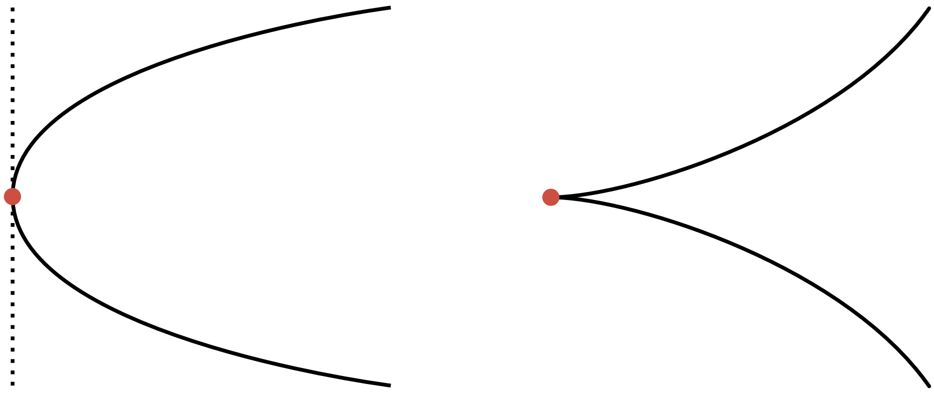


Figure 18. Vertical tangencies and singularities of Legendrian caustics.

Corollary 2.1. *Let $f : B\times\mathbf{R}^N \to \mathbf{R}$ be a generating family of a Legendrian submanifold $\Lambda \subset (J^1B, \xi_B)$. If Λ is at most $\Sigma^{1,0}$-singular, then for all $b \in B$, the critical points of $f_{|\{b\}\times\mathbf{R}^N}$ are isolated and thus in finite number when Λ is closed.*

Proof. The critical set of $f_{|\{b\}\times\mathbf{R}^N}$ is equal to ${\pi_{f,B}}^{-1}(b)$, but using Proposition 2.2, then

- the union of $\Sigma_0(f)$ and $\Sigma_1(f)$ is equal to Σ_f;
- the set $\Sigma_0(f)$ is a codimension 0 submanifold of Σ_f and $\pi_{f,B_{|\Sigma_0(f)}}$ has rank n; and
- the set $\Sigma_1(f)$ is a codimension 1 submanifold of Σ_f and $\pi_{f,B_{|\Sigma_1(f)}}$ has rank $n-1$.

Therefore, the critical set of $f_{|\{b\}\times\mathbf{R}^N}$, which can also be written as

$$
{\pi_{f,B}}^{-1}(b) = {\pi_{f,B_{|\Sigma_0(f)}}}^{-1}(b) \cup {\pi_{f,B_{|\Sigma_1(f)}}}^{-1}(b),
$$

is, by the constant rank level set theorem, the union of two 0-dimensional submanifolds of Σ_f, but since Λ is compact, so is Σ_f, thus yielding the result. ∎

Remark. Despite the remark made after the proof of Proposition 2.2, Corollary 2.1 is false for arbitrary generating families, as it is seen considering $f : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ defined by

$$
f(b_1,\dots,b_N,\eta_1,\dots,\eta_N) = \sum_{i=1}^{N} b_i\eta_i.
$$

It is a generating family, since $T\partial_\eta f = \begin{bmatrix} I_N & 0_N \end{bmatrix}$ has rank N, but $f_{|\{\mathbf{0}\}\times\mathbf{R}^N}$ has non-isolated critical points, since its critical set is the whole $\mathbf{R}^N$, while for all $b \neq \mathbf{0}$, $f_{|\{b\}\times\mathbf{R}^N}$ has no critical points. However, the (non-compact) Legendrian submanifold

$$
\Lambda = \{\mathbf{0}\} \times \mathbf{R}^N \times \{0\} \subset (J^1\mathbf{R}^N, \xi_{\mathbf{R}^N}),
$$

generated by f is not even front generic, since $\Sigma_0(\Lambda)$ is empty.

2. Morse theory of the difference function

In this section, the construction of a Morse-like homological theory for Legendrian submanifolds from generating families is recalled and several technical results are established as preliminaries for the proof of Theorem A.

Definition 2.4 ([Tra01, Definition 3.5])**.** Let $f_1 : B\times\mathbf{R}^{N_1} \to \mathbf{R}$ and $f_2 : B\times\mathbf{R}^{N_2} \to \mathbf{R}$ be functions, then the *difference function* of f_1 with f_2 is the function $\delta : B\times\mathbf{R}^{N_1}\times\mathbf{R}^{N_2} \to \mathbf{R}$ defined as

$$
\delta(b,\eta_1,\eta_2) = f_1(b,\eta_1) - f_2(b,\eta_2).
$$

The next proposition shows that critical points of difference functions are relevant to the study of the contact rigidity phenomena of Legendrian submanifolds.

Proposition 2.3 ([FR11, Lemma 6.1], [ST13, Proposition 3.2])**.** *Let $\Lambda \subset (J^1B, \xi_B)$ be a Legendrian submanifold, and let $f_1 : B\times\mathbf{R}^N \to \mathbf{R}$ and $f_2 : B\times\mathbf{R}^N \to \mathbf{R}$ be two generating families of Λ, then if δ is the difference function of f_1 with f_2, the critical points of δ splits in two types:*

- *There exist one-to-one correspondences between the Reeb chords of Λ, and the negatively and positively valued critical points of δ, respectively.*
- *If $\varepsilon > 0$ is small enough, then in the region $\{-\varepsilon < \delta < \varepsilon\}$, δ is a Morse-Bott function with a unique critical submanifold Σ, Σ is diffeomorphic to Λ and has Morse-Bott index N. Furthermore, if $f_1 = f_2$, then Σ is contained in the level set $\{\delta = 0\}$.*

Moreover, if $B = \mathbf{R}^n$ and Λ is chord generic, then for all Reeb chords γ of Λ, the non-zero valued critical points of δ associated to γ are non-degenerate and have Morse index $\mathrm{CZ}(\gamma) + N$, where CZ *is the Conley-Zehnder index defined in* [EES05b, Subsection 2.3].

There is however no hope at all to extract any meaningful homological invariant for Legendrian submanifolds from general generating families, since Morse theory is known to be ill-behaved on non-compact manifolds, mainly for the following two reasons:

- critical points can escape to infinity; and
- gradient flow lines can wander to infinity.

In particular, Morse homology can depend on the choice of the Morse-Smale pair.

Definition 2.5. A function $f : B \times \mathbf{R}^N \to \mathbf{R}$ is *linear-at-infinity* whenever there exist a compact subset K of $B \times \mathbf{R}^N$ and a nonzero linear form u of $\mathbf{R}^N$ such that for all $(b, \eta) \notin K$, $f(b, \eta) = u(\eta)$. The maximal compact subset on which this equality holds is referred to as the *support* of f.

Remark. In view of Theorem 2.1 and the stabilisation move, it seems more natural to consider generating families that agree outside a compact subset with a non-degenerate quadratic form, but such generating families can generate compact Legendrian submanifolds only if B is compact. It is no longer an issue when working with linear-at-infinity generating families, but it is needed to show that after a suitable fibre-preserving diffeomorphism, the stabilisation of a linear-at-infinity generating family can also be made linear-at-infinity [ST13, Lemma 3.8].

Lemma 2.1. *Let $g = g_B \oplus g_1 \oplus g_2$ be a product Riemannian metric on $B \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$, then*

$$\begin{aligned}\nabla_g \delta(b, \eta_1, \eta_2) &= \nabla_{g_B} \delta(b, \eta_1, \eta_2) \oplus \nabla_{g_1} \delta(b, \eta_1, \eta_2) \oplus \nabla_{g_2} \delta(b, \eta_1, \eta_2),\\ &= \nabla_{g_B} \delta(b, \eta_1, \eta_2) \oplus \nabla_{g_1} f_1(b, \eta_1) \oplus -\nabla_{g_2} f_2(b, \eta_2),\end{aligned}$$

where $T(B \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}) \simeq TB \oplus T\mathbf{R}^{N_1} \oplus T\mathbf{R}^{N_2}$ through the canonical bundle isomorphism.

Proof. Let $v = v_B \oplus v_1 \oplus v_2 \in T(B \times \mathbf{R}^N \times \mathbf{R}^N)$, then it follows from the definition of g that

$$\begin{aligned} g\left(\nabla_{g_B} \delta \oplus \nabla_{g_1} \delta \oplus \nabla_{g_2} \delta, v\right) &= g_B\left(\nabla_{g_B} \delta, v_B\right) + g_1\left(\nabla_{g_1} \delta, v_1\right) + g_2\left(\nabla_{g_2} \delta, v_2\right),\\ \text{[by definition of } \nabla] \quad &= \partial_B \delta(v_B) + \partial_{\eta_1} \delta(v_1) + \partial_{\eta_2} \delta(v_2),\\ [\text{since } T = \partial_b \oplus \partial_{\eta_1} \oplus \partial_{\eta_2}] \quad &= T\delta(v),\\ \text{[by definition of } \nabla] \quad &= g\left(\nabla_g \delta, v\right),\end{aligned}$$

thus the equality $\nabla_g \delta = \nabla_{g_B} \delta \oplus \nabla_{g_1} \delta \oplus \nabla_{g_2} \delta$ holds, since g is non-degenerate and v is arbitrary. Let $(b, \eta_1, \eta_2) \in B \times \mathbf{R}^{N_1} \times \mathbf{R}^{N_2}$ and let $v_1 \in T_{\eta_1} \mathbf{R}^{N_1}$, then it follows from the definition of ∇_{g_1} that

$$\begin{aligned} g_1\left(\nabla_{g_1} \delta(b, \eta_1, \eta_2), v_1\right) &= \partial_{\eta_1} \delta(b, \eta_1, \eta_2)(v_1),\\ &= \partial_{\eta_1} f_1(b, \eta_1)(v_1),\\ &= g_1\left(\nabla_{g_1} f_1(b, \eta_1), v_1\right),\end{aligned}$$

thus showing that $\nabla_{g_1} \delta(b, \eta_1, \eta_2) = \nabla_{g_1} f_1(b, \eta_1)$, since g_1 is non-degenerate and v_1 is arbitrary. Finally, proceeding in the exact same way, but now using that $\partial_{\eta_2} \delta(b, \eta_1, \eta_2) = -\partial_{\eta_2} f_2(b, \eta_2)$, shows that $\nabla_{g_2} \delta(b, \eta_1, \eta_2) = -\nabla_{g_2} f_2(b, \eta_2)$, thus yielding the result. ■

Given any Riemannian metric, the gradient flow of the difference function of two linear-at-infinity generating families is always globally defined, since

- in every compact subset, it has a uniform existence-time; and
- there exists a compact subset outside of which the flow is linear.

Thus, for all positively valued critical points c_- and c_+ of δ and all Riemannian metric g on $B\times\mathbf{R}^{2N}$, it is possible to define a Morse moduli space $\widehat{\mathscr{M}}(c_-,c_+;\delta,g)$ of parametrized trajectories as

$$\widehat{\mathscr{M}}(c_-,c_+;\delta,g) = \left\{\gamma\colon \mathbf{R}\to B\times\mathbf{R}^{2N};\, \partial_t\gamma = -\nabla_g\delta\circ\gamma \text{ and } \lim_{\pm\infty}\gamma = c_\pm\right\}.$$

Then, since the additive group of real numbers acts freely by time-translations on $\widehat{\mathscr{M}}(c_-,c_+;\delta,g)$, a Morse moduli space $\mathscr{M}(c_-,c_+;\delta,g)$ of unparametrized trajectories should also be defined as

$$\mathscr{M}(c_-,c_+;\delta,g) = \widehat{\mathscr{M}}(c_-,c_+;\delta,g)/\mathbf{R}.$$

It is possible to mimick the above definition and define Morse moduli spaces for $\delta_{|\{b\}\times\mathbf{R}^{2N}}$, $b\in B$. At last, notice that if Λ chord generic, using **Definition 1.12 (1)** and **Proposition 2.3** shows that the positively valued critical points of δ are all non-degenerate and for an open dense subset of Riemannian metrics g, the sets $\mathscr{M}(c_-,c_+;\delta,g)$ are manifolds of dimension $\mu(c_-)-\mu(c_+)-1$, where μ is the Morse index associated to δ.

Proposition 2.4. *If f_1 and f_2 are linear-at-infinity, there exists a compact subset K_δ of $B\times\mathbf{R}^N\times\mathbf{R}^N$ such that all the critical points of δ are located in K_δ. Moreover, if $g = g_B\oplus g_1\oplus g_2$ is any product Riemannian metric on $B\times\mathbf{R}^{N_1}\times\mathbf{R}^{N_2}$, then any $\gamma\in\widehat{\mathscr{M}}(c_-,c_+;\delta,g)$ takes its values in K_δ.*

Proof. Let $C = C_B\times C_1\times C_2$ be a compact subset of $B\times\mathbf{R}^N\times\mathbf{R}^N$ such that $C_B\times C_1$ and $C_B\times C_2$ contains the compact subsets provided by **Definition 2.5**, when applied to the linear-at-infinity functions f_1 and f_2, respectively. Then, having also fixed global trivializations of $T\mathbf{R}^{N_1}$ and $T\mathbf{R}^{N_2}$ and using **Lemma 2.1**, there exist non-zero vectors $u_1\in\mathbf{R}^{N_1}$ and $u_2\in\mathbf{R}^{N_2}$ such that

$$\forall b\in {C_B}^{c}, \forall\eta_1\in\mathbf{R}^{N_1}, \forall\eta_2\in\mathbf{R}^{N_2}, \nabla_g\delta(b,\eta_1,\eta_2) = (0,-u_1,u_2), \tag{2}$$

$$\begin{aligned}&\forall b\in B, \forall\eta_1\in {C_1}^{c}, \forall\eta_2\in\mathbf{R}^{N_2}, \nabla_{g_1}\delta(b,\eta_1,\eta_2) = -u_1,\\ &\forall b\in B, \forall\eta_1\in\mathbf{R}^{N_1}, \forall\eta_2\in {C_2}^{c}, \nabla_{g_2}\delta(b,\eta_2,\eta_2) = u_2.\end{aligned} \tag{3}$$

In particular, $\nabla_g\delta$ does not vanish outside of C, so that the critical points of δ are located in C. Let also $K_\delta = K_B\times K_1\times K_2$ be a compact subset of $B\times\mathbf{R}^{N_1}\times\mathbf{R}^{N_2}$ containing C in its interior and such that C_B, C_1 and C_2 are contained in K_B, K_1 and K_2, respectively. At last, replacing K_1 and K_2 by their convex hulls, which are still compact subsets, assume that K_1 and K_2 are convex subsets. From now on, proceed to show that if γ is a negative g-gradient flow line of δ leaving K, then it is either trapped in the past or in the future outside of it. Write γ in components as $\gamma = (\gamma_B,\gamma_1,\gamma_2)$, so that using **Lemma 2.1**, the differential equation satisfied by γ reads in components as:

$$\begin{cases}\partial_t\gamma_B(t) = -\nabla_{g_B}\delta(\gamma(t)),\\ \partial_t\gamma_1(t) = -\nabla_{g_1}\delta(\gamma(t)),\,,\\ \partial_t\gamma_2(t) = -\nabla_{g_2}\delta(\gamma(t)),\end{cases} \tag{4}$$

and now discuss on the component of γ leaving K_δ.

- **Case 1.** Assume there exists $t_0 \in \mathbf{R}$ such that $b_0 = \gamma_B(t_0) \notin K_B$, then let us introduce
$$t_- = \inf\{t \leqslant t_0 \mid \gamma_B(t) = b_0\} \in \mathbf{R} \cup \{-\infty\},$$
$$t_+ = \sup\{t \geqslant t_0 \mid \gamma_B(t) = b_0\} \in \mathbf{R} \cup \{+\infty\},$$
and argue, for the sake of contradiction, that t_- or t_+ is finite. Assume that t_- is finite, then $\gamma_B(t_-) = b_0 \in {C_B}^{\mathrm{c}}$ and there exists $\varepsilon > 0$ such that $\gamma_B((t_- - \varepsilon, t_- + \varepsilon)) \subset {C_B}^{\mathrm{c}}$. Therefore, it follows from equations (2) and (4) that
$$\forall t \in (t_\pm - \varepsilon, t_\pm + \varepsilon), \partial_t \gamma_B(t) = -\nabla_{g_M} \delta(\gamma(t)) = 0,$$
thus γ_B is constant, equal to b_0, on $(t_\pm - \varepsilon, t_\pm + \varepsilon)$, contradicting the minimality of t_-. A similar argument shows that t_+ is also infinite, which reads that $\gamma_B \equiv b_0 \notin K_B$ on $\mathbf{R}$, thus showing that for all $t \in \mathbf{R}$, $\gamma(t) \notin K_\delta$.
- **Case 2.** Assume there exists $t_0 \in \mathbf{R}$ such that $\eta_0 = \gamma_1(t_0) \notin K_1$, then let us define
$$D_- = \eta_0 + (-\infty, 0]u_1 \subset \mathbf{R}^{N_1},$$
$$D_+ = \eta_0 + [0, +\infty)u_1 \subset \mathbf{R}^{N_1},$$
and argue, for the sake of contradiction, that $D_- \cap K_1$ and $D_+ \cap K_1$ are non-empty, then the convexity of K_1 implies that $\eta_0 \in K$, hence a contradiction. Now arguing exactly as in the previous case with the following times
$$t_- = \inf\{t \leqslant t_0 \mid \gamma_1(t) = \eta_0 + (t - t_0)u_1\} \in \mathbf{R} \cup \{-\infty\},$$
$$t_+ = \sup\{t \geqslant t_0 \mid \gamma_1(t) = \eta_0 + (t - t_0)u_1\} \in \mathbf{R} \cup \{+\infty\},$$
it follows from equations (3) and (4) that if $D_\pm \cap K_1 = \varnothing$, then $t_\pm = \pm\infty$, thus
$$D_- \cap K_1 = \varnothing \Rightarrow \forall t \in (-\infty, t_0], \gamma_1(t) \in D_- \subset {K_1}^{\mathrm{c}}$$
$$D_+ \cap K_1 = \varnothing \Rightarrow \forall t \in [t_0, +\infty), \gamma_1(t) \in D_+ \subset {K_1}^{\mathrm{c}}.$$
This concludes that either for all $t \leqslant t_0$, $\gamma(t) \notin K_\delta$ or for all $t \geqslant t_0$, $\gamma(t) \notin K_\delta$.
- **Case 3.** Assume there exists $t_0 \in \mathbf{R}$ such that $\gamma_2(t_0) \notin K_2$, then proceeding exactly as in the previous case shows that either for all $t \leqslant t_0$, $\gamma(t) \notin K_\delta$ or for all $t \geqslant t_0$, $\gamma(t) \notin K_\delta$.

If $\gamma \in \widehat{\mathscr{M}}(q, p; \delta, g)$, then γ has range in K_δ; otherwise, it follows from the previous discussion that either $c_- = \lim_{-\infty} \gamma \in \overline{{K_\delta}^{\mathrm{c}}} = {\mathring{K}_\delta}^{\mathrm{c}}$ or $c_+ = \lim_{+\infty} \gamma \in {\mathring{K}_\delta}^{\mathrm{c}}$, but since c_- and c_+ are critical points of δ, they are in C, hence contradicting that $C \subset \mathring{K}_\delta$, thus yielding the result. ∎

Remark. The standard uniqueness result for solutions of ordinary differential equations cannot be applied directly in the proof of Proposition 2.4, since the vector field giving the derivative of any component of γ does not only depend on this component, but also on all the others.

Definition 2.6 ([Tra01, Definition 3.9]). Let $f_1 \colon B \times \mathbf{R}^N \to \mathbf{R}$ and $f_2 \colon B \times \mathbf{R}^N \to \mathbf{R}$ be two linear-at-infinity generating families, and let $\delta \colon B \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ be the difference function of f_1 and f_2. Assume that $\omega > \varepsilon > 0$ are such that all the positive critical values of δ are contained in (ε, ω), then the *generating family homology* of (f_1, f_2), denoted by $\mathrm{GFH}_\bullet(f_1, f_2)$, is defined to be
$$\mathrm{GFH}_\bullet(f_1, f_2) = H_{\bullet+N+1}(\{\delta < \omega\}, \{\delta < \varepsilon\}; \mathbf{F}_2).$$

Moreover, the *Poincaré polynomial* of (f_1, f_2) is the Laurent polynomial $\Gamma_{f_1,f_2}(t)$ defined by

$$\Gamma_{f_1,f_2}(t) = \sum_{k\in\mathbf{Z}} \dim(\mathrm{GFH}_k(f_1, f_2))t^k.$$

When $f_1 = f_2$, $\mathrm{GFH}_\bullet(f_1, f_2)$ and $\Gamma_{f_1,f_2}(t)$ are simply denoted by $\mathrm{GFH}_\bullet(f)$ and $\Gamma_f(t)$, respectively.

Remark. Using a one-parametric version of the standard structural result for sublevel sets of Morse functions [ST13, Lemma 2.4], the isomorphism classes of $\mathrm{GFH}_\bullet(f_1, f_2)$ are independent of the choices of ε and ω [ST13, Corollary 3.10].

Proposition 2.5 ([ST13, Lemma 3.6])**.** *Let f_1 and f_2 be two linear-at-infinity generating families. If f_1 and f_2 are in the same equivalence class, then* $\mathrm{GFH}_\bullet(f_1)$ *and* $\mathrm{GFH}_\bullet(f_2)$ *are isomorphic.*

Remark. The invariance of the isomorphism class of $\mathrm{GFH}_\bullet$ with respect to the stabilisation move is guaranteed by the choice of the grading shift.

Remark. Let Λ be a Legendrian submanifold and f be a linear-at-infinity generating family of Λ, then the isomorphism class of $\mathrm{GFH}_\bullet(f)$ is not necessarily a Legendrian isotopy invariant of Λ itself, but rather an invariant of (Λ, f) for Legendrian isotopies and equivalences of generating families. Indeed, using Theorem 2.1 and Proposition 2.5, the collection of all the isomorphism classes of generating family homologies of all linear-at-infinity generating families of Λ is a Legendrian isotopy invariant of Λ.

Theorem 2.2. *If f_1 and f_2 are linear-at-infinity generating families in the same equivalence class, then there exist integers $N_1 \in \mathbf{Z}$ and $N_2 \in \mathbf{Z}$ such that for all linear-at-infinity generating families f, there exist isomorphisms of graded $\mathbf{F}_2$-vector spaces such that*

$$\mathrm{GFH}_\bullet(f_1, f) \simeq \mathrm{GFH}_{\bullet+N_1}(f_2, f),$$
$$\mathrm{GFH}_\bullet(f, f_1) \simeq \mathrm{GFH}_{\bullet+N_2}(f, f_2),$$

provided f_1, f_2 and f are defined over the same vector bundle.

Proof. Let define $\delta_{f_1,f}$, $\delta_{f_2,f}$, δ_{f,f_1} and δ_{f,f_2} to be difference functions associated to f_1, f_2 and f, then it is straightforward to check that $\delta_{f_1,f}$ and δ_{f,f_1} are equivalent to $\delta_{f_2,f}$ and δ_{f,f_2}, respectively. Thus, [ST13, Lemma 2.2] directly yields the result, as it did for Proposition 2.5. ■

Remark. Grading shifts in Theorem 2.2 account for the asymmetry of mixed difference functions, so that equivalence moves of generating families can now be applied independently to f_1 and f_2. Indeed, if f_2 is a Nth dimensional index k stabilisation of f_1, then $N_1 = k$ and $N_2 = N - k$, and similarly, if f_1 is a Nth dimensional index k stabilisation of f_2, then $N_1 = -k$ and $N_2 = k - N$. However, if f_1 and f_2 only differ by a fibre-preserving diffeomorphism, then $N_1 = 0$ and $N_2 = 0$. Thus, these observations compromise the construction of a Legendrian isotopy invariant of Λ itself from the mixed version of the generating family homology.

Remark. Theorem 2.2 has a pseudo-holomorphic curves analogue given in [BC14, Theorem 1.4] and for which no grading shifts appears, meaning that N_1 and N_2 always vanish.

3. Geography of the generating family homology

First, the long exact sequence for generating family homology (Theorem 2.3) is generalised to the mixed version of generating family homology (Theorem 2.4). In particular, it leads to conjecture that the mixed version of generating family homology is a complete invariant for equivalence of generating families (Conjecture 2.1). However, the duality long exact sequence is not enough to carry any concrete computation of the mixed version of generating family homology and only the geography of the usual version of generating family homology can be recalled (Theorem 2.5).

Theorem 2.3 ([BST15, Theorem 6.1]). *Let Λ^n be a closed Legendrian submanifold of (J^1B, ξ_B) and let f be a linear-at-infinity generating family of Λ, then there exists a long exact sequence*

$$\text{(LES-GFH)} \qquad \cdots \longrightarrow \mathrm{GFH}_k(f) \xrightarrow{\tau_k} H_k(\Lambda; \mathbf{F}_2) \xrightarrow{\sigma_k} \mathrm{GFH}^{n-k}(f) \xrightarrow{\rho_k} \mathrm{GFH}_{k-1}(f) \longrightarrow \cdots.$$

Moreover, the following further properties are satisfied:

(1) If $D_\bullet \colon H^{n-\bullet}(\Lambda; \mathbf{F}_2) \to H_\bullet(\Lambda; \mathbf{F}_2)$ is the Poincaré duality, then τ_{n-k} is adjoint to $\sigma_k \circ D_k$.

(2) If Λ is chord generic, then $\tau_n \colon \mathrm{GFH}_n(f) \to H_n(\Lambda; \mathbf{F}_2)$ is surjective.

Sketch of a proof. Pick $\omega > 0$ sufficiently large and $\varepsilon \in (0, \omega)$ sufficiently small so that the real interval (ε, ω) contains all the positive critical values of the difference function δ of f with itself. Then, using the isomorphisms induced by the diffeomorphism exchanging the two $\mathbf{R}^N$ factors in the domain of δ [ST13, Lemma 7.1], the homology long exact sequence of $(\delta^\omega, \delta^\varepsilon, \delta^{-\varepsilon})$ and the cohomology long exact sequence of $(\delta^\varepsilon, \delta^{-\varepsilon}, \delta^{-\omega})$ fit into the following commutative diagram:

$$\begin{tikzcd}
{} \arrow[r, "\tau_k"] & H_{k+N}(\delta^\varepsilon, \delta^{-\varepsilon}; \mathbf{F}_2) \arrow[r, "s_k"] \arrow[d, "\alpha_k", "\simeq"'] & H_{k+N}(\delta^\omega, \delta^{-\varepsilon}; \mathbf{F}_2) \arrow[r, "r_k"] \arrow[d, "\beta_k", "\simeq"'] & H_{k+N}(\delta^\omega, \delta^{\varepsilon}; \mathbf{F}_2) \arrow[r] \arrow[d, "\gamma_k", "\simeq"'] & {} \\
{} \arrow[r] & H^{n-k+N}(\delta^\varepsilon, \delta^{-\varepsilon}; \mathbf{F}_2) \arrow[r, "s'_{n-k}"] & H^{n-k+N}(\delta^\varepsilon, \delta^{-\omega}; \mathbf{F}_2) \arrow[r, "r'_{n-k}"] & H^{n-k+N}(\delta^{-\varepsilon}, \delta^{-\omega}; \mathbf{F}_2) \arrow[r] & {}
\end{tikzcd},$$

where for all $a \in \mathbf{R}$, δ^a is a shorthand notation for $\{\delta < a\}$, then it now only remains to identify several terms in the above diagram.

- Let $D^u(\Lambda)$ be the unit negative normal bundle of Λ, then according to Proposition 2.3 and the structural result for sublevel sets of Morse-Bott functions [Nic11, Theorem 2.44], δ^ε is homotopic to the space obtained from $\delta^{-\varepsilon}$ by attaching $D^u(\Lambda)$ along its boundary. Therefore, since $D^u(\Lambda)$ is a $\mathbf{D}^N$-bundle over Λ, Thom's isomorphism yields
$$H_{k+N}(\delta^\varepsilon, \delta^{-\varepsilon}; \mathbf{F}_2) \simeq H_{k+N}(D^u(\Lambda), \partial D^u(\Lambda); \mathbf{F}_2) \simeq H_k(\Lambda; \mathbf{F}_2).$$
- Since f is linear-at-infinity, there exists a smooth homotopy $(\delta_s)_{s \in [0,1]}$ such that $\delta_0 = \delta$, δ_1 has no critical points and for all $s \in [0,1]$, $-\omega$ and ω are both regular values of δ_s. Then, using a one-parametric version of the standard structural result for sublevel sets of Morse functions [ST13, Lemma 2.4], $(\delta^\omega, \delta^{-\omega})$ is homotopy equivalent to $({\delta_1}^\omega, {\delta_1}^{-\omega})$. Moreover, the structural result for sublevel sets of Morse functions [Nic11, Theorem 2.6] shows that ${\delta_1}^\omega$ is a deformation retract of ${\delta_1}^{-\omega}$, so that the pair $({\delta_1}^\omega, {\delta_1}^{-\omega})$ is acyclic. Therefore, $(\delta^\omega, \delta^{-\omega})$ is acyclic [ST13, Lemma 3.12] and the connecting homomorphism
$$\partial_{n-k} \colon H^{n-k+N}(\delta^\varepsilon, \delta^{-\omega}; \mathbf{F}_2) \to H^{n-k+N+1}(\delta^\omega, \delta^\varepsilon; \mathbf{F}_2),$$
of the cohomology long exact sequence of $(\delta^\omega, \delta^\varepsilon, \delta^{-\omega})$ is an isomorphism.

- Using Definition 2.6, $H_{k+N}(\delta^\omega, \delta^\varepsilon) = \mathrm{GFH}_{k-1}(f)$.

At last, the desired long exact sequence is obtained by defining σ_k and ρ_k as $\sigma_k = \partial_{n-k} \circ \beta_k \circ s_k$ and $\rho_k = r_k \circ \beta_k{}^{-1} \circ \partial_{n-k}{}^{-1}$, respectively.

Let us now establish the further two properties claimed.

(1) Using the homomorphism induced by the inclusion $i\colon \delta^{-\omega} \hookrightarrow \delta^{-\varepsilon}$, which also appears in the cohomology long exact sequence of $(\delta^\varepsilon, \delta^{-\varepsilon}, \delta^{-\omega})$, the cohomology long exact sequences of $(\delta^\omega, \delta^\varepsilon, \delta^{-\varepsilon})$ and $(\delta^\omega, \delta^\varepsilon, \delta^{-\omega})$ fit into the following commutative diagram:

$$\begin{tikzcd} {} \arrow[r] & H^{n-k+N}(\delta^\varepsilon, \delta^{-\varepsilon}; \mathbf{F}_2) \arrow[r, "d_{n-k}=\tau_{n-k}{}^*"] \arrow[d, "i_*=s'_{n-k}"] & H^{n-k+N+1}(\delta^\omega, \delta^\varepsilon; \mathbf{F}_2) \arrow[r] \arrow[d, "\mathrm{id}"] & {} \\ {} \arrow[r] & H^{n-k+N}(\delta^\varepsilon, \delta^{-\omega}; \mathbf{F}_2) \arrow[r, "\partial_{n-k}"] & H^{n-k+N+1}(\delta^\omega, \delta^\varepsilon; \mathbf{F}_2) \arrow[r] & {} \end{tikzcd},$$

where d_{n-k} is adjoint to τ_{n-k}, since the cohomology long exact sequence of $(\delta^\omega, \delta^\varepsilon, \delta^{-\varepsilon})$ is adjoint to its homology long exact sequence, whenever field coefficients are used. Therefore, using the two commutative diagrams built in this proof shows that

$$\begin{aligned} \tau_{n-k}{}^* &= \partial_{n-k} \circ s'_{n-k}, \\ &= \partial_{n-k} \circ \left(\beta_k \circ s_k \circ \alpha_k{}^{-1}\right), \\ &= (\partial_{n-k} \circ \beta_k \circ s_k) \circ \alpha_k{}^{-1}, \\ &= \sigma_k \circ \alpha_k{}^{-1}. \end{aligned}$$

Moreover, recalling that the Poincaré duality isomorphism is given by flipping the sign of the Morse function used to compute the homology [AD10, Proposition 4.3.1] shows that $D_k = \alpha_k{}^{-1}$ and yields $\tau_{n-k}{}^* = \sigma_k \circ D_k$.

(2) According to Theorem 2.3 (1), τ_n is surjective if, and only if, σ_0 is injective, but using the long exact sequence (LES-GFH) shows that σ_0 is injective if, and only if, τ_0 is zero. Let Δ be the diagonal of $B \times \mathbf{R}^N \times \mathbf{R}^N$ and let g be a Riemannian metric of $B \times \mathbf{R}^N \times \mathbf{R}^N$ such that for all positively valued critical points c of δ, $W^u(c; \delta, g) \pitchfork (B \times \Delta)$ holds, then let us define $\mathcal{M}(c, B \times \Delta; \delta, g)$ to be the set of $\gamma \in W^u(c; \delta, g)$ such that $\gamma(0) \in B \times \Delta$. If c has Morse index $N+1$, then $\mathcal{M}(c, B \times \Delta; \delta, g)$ is a one-dimensional manifold which is compactified, as customary in Morse theory, by adding broken negative g-gradient flow lines $\widetilde{\gamma} = (\gamma_1, \gamma_2)$ of δ such that γ_1 starts from c and $\gamma_2(0) \in B \times \Delta$ [AB95, Lemma 3.3]. However, discussing on the breaking point of $\widetilde{\gamma}$ and using Proposition 2.3, shows that
   - either γ_1 ends on a positively valued critical point q of δ and $\gamma_2 \in \mathcal{M}(q, B \times \Delta)$; or
   - γ_1 ends on Σ and γ_2 is constant, since Σ and $B \times \Delta$ are both contained in $\{\delta = 0\}$;

   therefore, on the chains level, τ_0 is homotopic to 0, thus yielding the result.

■

Theorem 2.3 directly implies a quantitative version of the Arnol'd's chord conjecture [Arn86], predicting that the minimal number of Reeb chords is constrained by the singular homology of the Legendrian submanifold, provided it is closed and generic.

Corollary 2.2 ([ST13, Corollary 1.10])**.** *Let Λ be a closed Legendrian submanifold of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$. If Λ is chord generic and has a linear-at-infinity generating family, then*

$$\forall k \in \{0,\dots,n\}, r_{k+1}(\Lambda) + r_{n-k+1}(\Lambda) \geqslant \dim H_k(\Lambda;\mathbf{F}_2),$$

where $r_\ell(\Lambda)$ is the number of Reeb chords of Λ of Conley-Zenhder index ℓ.

Proof. Let $\delta\colon \mathbf{R}^n \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ be the difference function of some linear-at-infinity generating family of Λ and for all $\ell \in \{-N-1,\dots,n-1\}$, let m_ℓ be the number of positively valued critical points of δ of Morse index $\ell+N+1$. Then, using the the rank nullity theorem shows that

$$\begin{aligned}
& & \dim H_k(\Lambda;\mathbf{F}_2) &= \dim\ker(\sigma_k) + \mathrm{rk}(\sigma_k),\\
&[\text{using (LES-GFH)}] & &= \mathrm{rk}(\tau_k) + \mathrm{rk}(\sigma_k),\\
&[\text{by definition of } \tau_k \text{ and } \sigma_k] & &\leqslant \dim \mathrm{GFH}_k(f) + \dim \mathrm{GFH}^{n-k}(f),\\
&[\text{since fields coefficients are used}] & &= \dim \mathrm{GFH}_k(f) + \dim \mathrm{GFH}_{n-k}(f),\\
&[\text{by definition of GFH}] & &\leqslant m_k + m_{n-k},\\
&[\text{using Proposition 2.3}] & &= r_{k+1} + r_{n-k+1},
\end{aligned}$$

thus yielding the result. ■

Remark. Forgetting the grading computations, the same proof shows that if a closed and chord generic Legendrian submanifold of (J^1B, ξ_B) has a linear-at-infinity generating family, then its number of Reeb chords is bounded from below by half the sum of its $\mathbf{F}_2$-Betti numbers.

The long exact sequence (LES-GFH) from Theorem 2.3 translates straightforwardly to the mixed version of generating family homology.

Theorem 2.4. *Let Λ^n be a closed Legendrian submanifold of (J^1B, ξ_B), and also f_1 and f_2 be two linear-at-infinity generating families of Λ. If f_1 and f_2 are both defined over the same vector bundle, then there exists a long exact sequence*

$$\text{(LES-mGFH)} \qquad \cdots \longrightarrow \mathrm{GFH}_k(f_1,f_2) \xrightarrow{\tau_k} H_k(\Lambda;\mathbf{F}_2) \xrightarrow{\sigma_k} \mathrm{GFH}^{n-k}(f_2,f_1) \xrightarrow{\rho_k} \mathrm{GFH}_{k-1}(f_1,f_2) \longrightarrow \cdots.$$

and if $D_\bullet\colon H^{n-\bullet}(\Lambda;\mathbf{F}_2) \to H_\bullet(\Lambda;\mathbf{F}_2)$ is the Poincaré duality, then τ_{n-k} is adjoint to $\sigma_k \circ D_k$.

Proof. It suffices to mimick the relevant parts of the proof of Theorem 2.3, now noticing that if

$$\delta_{f_1,f_2}(b,\eta_1,\eta_2) = f_1(b,\eta_1) - f_2(b,\eta_2),$$
$$\delta_{f_2,f_1}(b,\eta_1,\eta_2) = f_2(b,\eta_1) - f_1(b,\eta_2),$$

then exchanging the two fibre factors in the domain of δ_{f_1,f_2} leads to the following equality

$$\delta_{f_1,f_2}(b,\eta_2,\eta_1) = -\delta_{f_2,f_1}(b,\eta_1,\eta_2),$$

so that the desired long exact sequence features $\mathrm{GFH}^\bullet(f_2,f_1)$ instead of $\mathrm{GFH}^\bullet(f_1,f_2)$. ■

Remark. However, the proof of Theorem 2.3 (2) does not straightforwardly extend to the mixed version of generating family homology, since δ is not necessarily identically zero on the diagonal. In particular, if f_1 and f_2 are not equivalent, then $\tau_n\colon \mathrm{GFH}_n(f_1,f_2) \to H_n(\Lambda;\mathbf{F}_2)$ from the long exact sequence (LES-mGFH) does not need to be surjective.

If f_1 and f_2 are equivalent linear-at-infinity generating families, then Theorems 2.2 and 2.3 (2) ensure that $\tau_n\colon \mathrm{GFH}_n(f_1, f_2) \to H_n(\Lambda; \mathbf{F}_2)$ from the long exact sequence (LES-mGFH) is surjective, and the converse is conjectured to be true.

Conjecture 2.1. *Let Λ^n be a closed, connected and chord generic Legendrian submanifold of (J^1B, ξ_B), and let f_1 and f_2 be two linear-at-infinity generating families of Λ, then f_1 and f_2 are in the same equivalence class if, and only if, the map $\tau_n\colon \mathrm{GFH}_n(f_1, f_2) \to H_n(\Lambda; \mathbf{F}_2)$ in (LES-mGFH) is surjective. In particular, the mixed version of generating family homology is a complete invariant for equivalence of generating families.*

Remark. The long exact sequence (LES-mGFH) has a pseudo-holomorphic curves analogue which was established in [BC14, Theorem 1.5], and for which a version of the Conjecture 2.1 is shown to hold in [BG19, Theorem 1.1].

Theorem 2.5 ([BST15, Theorems 1.1 and 1.2])**.** *The following assertions hold true.*

(1) *Let Λ^n be a closed and connected Legendrian submanifold of (J^1B, ξ_B) and let also f be a linear-at-infinity generating family of Λ. Then, there exist nonnegative integer coefficients polynomials q and p such that*

$$\Gamma_f(t) = (q_0 + q_1 t + \ldots + q_n t^n) + p(t) + t^{n-1} p(t^{-1}), \tag{GFH-P}$$

where for all $k \in \{0, \ldots, n\}$, $q_k + q_{n-k} = \dim H_k(\Lambda; \mathbf{F}_2)$, $q_n \neq 0$ and $p(t) = \sum_{k \geqslant \lfloor (n-1)/2 \rfloor} p_k t^k$.

A Laurent polynomial satisfying all conditions (GFH-P) above is GFH-admissible.

(2) *Let $P(t)$ be a* GFH*-admissible Laurent polynomial such that $n = \deg P \geqslant 2$ and $P(0) = 0$, then there exist a closed and connected Legendrian submanifold Λ of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ and a linear-at-infinity generating family f of Λ such that $P(t) = \Gamma_f(t)$.*

Sketch of a proof. The two statements are proved independently.

(1) For all $k \in \mathbf{Z}$, let us define the following nonnegative integer coefficients:

$$q_k = \dim \mathrm{GFH}_k(f) - \dim \ker(\tau_k),$$
$$p_k = \dim \ker(\tau_k),$$

then using the exactness of the sequence (LES-GFH) and Theorem 2.3 (1) shows that $q_k + q_{n-k} = \dim H_k(\Lambda; \mathbf{F}_2)$ and $p_{n-1-k} = p_k$, while Theorem 2.3 (2) implies that $q_n \neq 0$, thus yielding the result.

(2) The proof relies on the explicit construction of two types of elementary examples of linear-at-infinity generating families of Legendrian submanifolds of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ and, more importantly, on the computation of their Poincaré polynomials.
   - **Manifold class building blocks.** ([BST15, Corollary 6.7]) Let $a \in \{1, \ldots, n-1\}$, then there exists a linear-at-infinity generating family of a connected Legendrian submanifold of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ whose Poincaré polynomial is $t^n + t^a$.
   - **Duality classes building blocks.** ([BST15, Lemma 6.10]) Let $a \in \mathbf{N}$, then there exists a linear-at-infinity generating family of a Legendrian sphere of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ whose Poincaré polynomial is $t^n + t^a + t^{n-1-a}$.

It now suffices to take the connected sum, as defined in [BST15, Corollary 6.4], of the relevant building blocks constructed above.

■

Remark. The Legendrian submanifolds needed for the proof of Theorem 2.5 (2) are constructed using spinning and surgery operations on the usual Legendrian unknot and Hopf link of $(J^1\mathbf{R}, \xi_{\mathbf{R}})$ and all the resulting Legendrian submanifolds of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ are at most $\Sigma^{1,0}$-singular.

Conjecture 2.2. *The following assertions hold true.*

(1) Let Λ^n be a closed and connected Legendrian submanifold of (J^1B, ξ_B), and let f_1 and f_2 be two non-equivalent linear-at-infinity generating families of Λ. Then, there exist nonnegative integer coefficients polynomial q and Laurent polynomial p such that

$$\Gamma_{f_1,f_2}(t) = q(t) + p(t), \tag{mGFH-P}$$

where q has degree at most $n-1$ and $q(0) = 1$, and p further satisfies

$$\begin{cases} p(-1) \equiv 0 \bmod 2, & \text{if } n = 1, \\ p(-1) \leqslant \dfrac{1-q(-1)}{2}, & \text{if } n = 2, \end{cases} .$$

A Laurent polynomial satisfying all conditions (mGFH-P) above is mGFH-admissible.

(2) Let $P(t)$ be a mGFH-admissible Laurent polynomial, then there exist an integer $n \geqslant 1$, a closed and connected Legendrian submanifold Λ of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$, and two non-equivalent linear-at-infinity generating families f_1 and f_2 of Λ such that $P(t) = \Gamma_{f_1,f_2}(t)$.

Remark. The pseudo-holomorphic curves analogues of Conjecture 2.2 (1) and (2) have already been answered in [BG19, Proposition 4.2] and [BG19, Theorem 1.3], respectively.

3.

The Henry-Rutherford limiting process

In this chapter, the Henry-Rutherford limiting process is now introduced as a singular perturbation of the gradient flow used in **Chapter 2** to construct the generating family homology (**Section 1**). In particular, in order to explain how **Conjecture A** arises from the Henry-Rutherford process, gradient staircases are precisely defined, and their analytical properties are studied (**Section 2**), and at last, a topology is defined to describe convergence for this limiting process (**Section 3**). All along this chapter, several useful estimates for the proof of **Theorem A** are derived.

1. Geometrising the boundary operators of the generating family chain complexes

Despite being similar to the Morse homology of smooth manifolds, homologies for generating families of Legendrian submanifolds cannot be efficiently computed from geometrical data only. Indeed, the boundary operators of the corresponding chain complexes are not easily understood in term of the Legendrian submanifolds themselves. Even if **Theorem 2.5** can sometimes be used to overcome this difficulty in the non-mixed version of generating family homology, the long exact sequence **(LES-mGFH)** is of no use for practical computations.

In any case, the differentials of the generating family chain complexes can be geometrised taking advantage of the invariance of Morse homology with respect to the choice of a Riemannian metric. More concretely, if $\delta : B \times F \to \mathbf{R}$ is a difference function, g_B and g_F are respectively Riemannian metrics on B and F, then according to **Lemma 3.1**, for all parameters $s \in (0,1]$, the gradient flow of δ with respect to $g = g_B \oplus g_F$ can be singularly perturbed in the following fashion

$$\begin{cases} \partial_t b_s(t) = -s\nabla_{g_B}\delta(b_s(t), \eta_s(t)), \\ \partial_t \eta_s(t) = -\nabla_{g_F}\delta(b_s(t), \eta_s(t)), \end{cases} \tag{5}$$

where b_s and η_s respectively stands for the coordinates of the solutions of **(5)** along B and F. Besides, writting this system in the *slow time* scale $\tau = st$ leads to

$$\begin{cases} \partial_\tau b_s(\tau) = -\nabla_{g_B}\delta(b_s(\tau), \eta_s(\tau)), \\ s\partial_\tau \eta_s(\tau) = -\nabla_{g_F}\delta(b_s(\tau), \eta_s(\tau)). \end{cases} \tag{6}$$

And notice that as long as $s \neq 0$, both systems **(5)** and **(6)** are equivalent, but

- in the limit $s \to 0$, system **(5)** converges to the *vertical system* given by

$$\begin{cases} \partial_t b_s(t) = \mathbf{0}, \\ \partial_t \eta_s(t) = -\nabla_{g_F}\delta(b_s(t), \eta_s(t)), \end{cases} \tag{7}$$

  whose solutions are drawn in a fibre and are determined by the generating families,

- whereas when $s \to 0$, system (6) converges to the *horizontal system* given by

$$\begin{cases} \partial_\tau b_s(t) = -\nabla_{g_B}\delta(b_s(\tau), \eta_s(\tau)), \\ \qquad \mathbf{0} = -\nabla_{g_F}\delta(b_s(\tau), \eta_s(\tau)), \end{cases} \tag{8}$$

whose solutions are drawn on $\{\nabla_{g_F}\delta = \mathbf{0}\}$ and are determined by the Legendrian front.

Intuitively and roughly speaking, if when $s \to 0$, the solutions of (5) and (6) correspond to the solutions of (7) and (8), respectively, then this degenerating process is expected to split apart the distinct inputs involved to compute the differentials of the generating family chain complexes. On one side, the contributions coming from the Legendrian submanifold, and on the other side, the ones due to bifurcations in the complex of critical points of the generating families.

Even if the limiting behaviour of solutions of such *fast-slow systems* has been extensively studied, the techniques developped in [Fen71, Fen74, Fen77, Fen79] fail to apply to systems (5) and (6). Since singularities of the Legendrian caustic correspond to points where normal hyperbolicity of the maximal flow invariant set $\{\nabla_{g_F}\delta = \mathbf{0}\}$ is lost.

The rest of this section is now devoted to the first technical preliminaries needed to analytically describe and understand this degeneration process.

Lemma 3.1. *Let $g = g_B \oplus g_N \oplus g_N$ be a product Riemannian metric on $B \times \mathbf{R}^N \times \mathbf{R}^N$ and $g_1 = g_2 = g_N$. Let also $s \in (0,1]$ and let us define $g_s = (s^{-1}g_B) \oplus g_N \oplus g_N$, then*

$$\begin{aligned} \nabla_{g_s}\delta(b,\eta_1,\eta_2) &= s\nabla_{g_B}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_1}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_2}\delta(b,\eta_1,\eta_2), \\ &= s\nabla_{g_B}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_N} f_1(b,\eta_1) \oplus -\nabla_{g_N} f_2(b,\eta_2), \end{aligned}$$

where $T(B \times \mathbf{R}^N \times \mathbf{R}^N) \simeq TB \oplus T\mathbf{R}^N \oplus T\mathbf{R}^N$ through the canonical bundle isomorphism.

Proof. Since g_s is a product Riemannian metric, Lemma 2.1 applies and shows that

$$\begin{aligned} \nabla_{g_s}\delta(b,\eta_1,\eta_2) &= \nabla_{s^{-1}g_B}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_1}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_2}\delta(b,\eta_1,\eta_2), \\ &= \nabla_{s^{-1}g_B}\delta(b,\eta_1,\eta_2) \oplus \nabla_{g_N} f_1(b,\eta_1) \oplus -\nabla_{g_N} f_2(b,\eta_2). \end{aligned}$$

Besides, let $v_B \in TB$, then the definitions of ∇_{g_B} and $\nabla_{s^{-1}g_B}$ implies that

$$\begin{aligned} s^{-1}g_B\left(s\nabla_{g_B}\delta, v_B\right) &= g_B\left(\nabla_{g_B}\delta, v_B\right), \\ &= \partial_b\delta(v_B), \\ &= s^{-1}g_B\left(\nabla_{s^{-1}g_B}\delta, v_B\right), \end{aligned}$$

therefore showing that $\nabla_{s^{-1}g_B}\delta = s\nabla_{g_B}\delta$ holds, since $s^{-1}g_B$ is non-degenerate and v_B is arbitrary, thus yielding the result. ■

Lemma 3.2. *Let $\widehat{g}$ be the* Sasaki bundle metric *induced by g on $T(B \times \mathbf{R}^N \times \mathbf{R}^N)$, then*

$$\begin{aligned} \left\|\nabla_{g_s}\delta\right\|_g &\leqslant \left\|\nabla_g\delta\right\|_g, \\ \left\|T\nabla_{g_s}\delta\right\|_{\widehat{g}} &\leqslant \left\|T\nabla_g\delta\right\|_{\widehat{g}}, \end{aligned}$$

hold true pointwise on $B \times \mathbf{R}^N \times \mathbf{R}^N$ and $T(B \times \mathbf{R}^N \times \mathbf{R}^N)$, respectively.

Proof. Using **Lemma 3.1**, first for $s=1$, then for arbitrary s, shows that

$$\begin{aligned}
\left\|\nabla_{g_s}\delta\right\|_g^2 &= g\left(\nabla_{g_s}\delta,\nabla_{g_s}\delta\right) = g\left(s\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta, s\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta\right),\\
\text{[by definition of } g\text{]} \quad &= s^2 g_B\left(\nabla_{g_B}\delta,\nabla_{g_B}\delta\right)+g_1\left(\nabla_{g_1}\delta,\nabla_{g_1}\delta\right)+g_2\left(\nabla_{g_2}\delta,\nabla_{g_2}\delta\right),\\
\text{[since } s\leqslant 1\text{]} \quad &\leqslant g_B\left(\nabla_{g_B}\delta,\nabla_{g_B}\delta\right)+g_1\left(\nabla_{g_1}\delta,\nabla_{g_1}\delta\right)+g_2\left(\nabla_{g_2}\delta,\nabla_{g_2}\delta\right),\\
\text{[by definition of } g\text{]} \quad &= g\left(\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta,\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta\right),\\
&= g\left(\nabla_g\delta,\nabla_g\delta\right),\\
&= \left\|\nabla_g\delta\right\|_g^2.
\end{aligned}$$

Taking the square root on both sides of the above inequality thus yields the first desired inequality. At last, proceeding in the exact same way for $\|T\nabla_{g_s}\delta\|_{\widehat{g}}^2$, but now using that

$$T\nabla_{g_s}\delta = s\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta,$$

since by **Lemma 3.1**, $\nabla_{g_s}\delta = s\nabla_{g_B}\delta\oplus\nabla_{g_1}\delta\oplus\nabla_{g_2}\delta$, and that $\widehat{g}$ is a product Riemannian metric, shows the last desired inequality, thus yielding the result. ∎

Definition 3.1. A sequence $(\gamma_k)_{k\in\mathbf{N}}$ is a *Henry-Rutherford sequence*, sometimes called *HR-sequence* for short, whenever there exist positively valued critical points c_- and c_+ of δ, and a sequence of real numbers $(s_k)_{k\in\mathbf{N}}$ in $(0,1]$ such that $s_k \underset{k}{\longrightarrow} 0$ and for all $k\in\mathbf{N}$, $\gamma_k\in\widehat{\mathscr{M}}(c_-,c_+;g_{s_k},\delta)$.

Remark. When HR-sequences are mentionned, the following data has implicitely been fixed

- a difference function δ of linear-at-infinity generating families; and
- a product Riemannian metric $g = g_B\oplus g_N\oplus g_N$ on the domain of the difference function;

then Λ will always be endowed with the Sasaki bundle Riemannian metric induced by g_B on J^1B.

Lemma 3.3. *Let $(\gamma_k)_{k\in\mathbf{N}}$ is a Henry-Rutherford sequence, then for all $k\in\mathbf{N}$, $\delta\circ\gamma_k$ is decreasing.*

Proof. Let $k\in\mathbf{N}$, then for all $t\in\mathbf{R}$, the chain rule shows that

$$\begin{aligned}
\partial_t(\delta\circ\gamma_k)(t) &= T_{\gamma_k(t)}\delta(\partial_t\gamma_k(t)),\\
\text{[by definition of } \nabla\text{]} \quad &= g_{s_k}\left(\nabla_{g_{s_k}}\delta(\gamma_k(t)),\partial_t\gamma_k(t)\right),\\
\text{[since } \partial_t\gamma_k = -\nabla_g\delta\circ\gamma_k\text{]} \quad &= g_{s_k}\left(\nabla_{g_{s_k}}\delta(\gamma_k(t)),-\nabla_{g_{s_k}}\delta(\gamma_k(t))\right),\\
&= -\left\|\nabla_{g_{s_k}}\delta(\gamma_k(t))\right\|_{g_{s_k}}^2,
\end{aligned}$$

thus the derivative of $\delta\circ\gamma_k$ is everywhere negative, thus yielding the result. ∎

2. Gradient staircases and gradient staircases chains

The main goal of this section is to give a precise definition of gradient staircases (**Definition 3.2**) in order to more precisely state **Conjecture A** (**Conjecture 3.1**) and start working towards its resolution with the first preliminaries to the proof of **Theorem A** (**Section 2.2**).

2.1. Definitions of vertical and horizontal fragments of gradient staircases.

Gradient staircases (**Definition 3.2**) and gradient staircases chains (**Definition 3.3**) are defined, and at last, **Conjecture A** is stated (**Conjecture 3.1**).

Let us define two subsets $S_\delta = \Sigma_{f_1} \times_B \Sigma_{f_2} \subset B \times \mathbf{R}^{2N}$ and $S_\delta{}^{\prec} = (\Sigma_{f_1}{}^{\prec} \times_B \Sigma_{f_2}) \cup (\Sigma_{f_1} \times_B \Sigma_{f_2}{}^{\prec}) \subset S_\delta$. Then, using Lemma 3.1 and Proposition 2.2, notice that

- S_δ is the set of $(b, \eta) \in B \times \mathbf{R}^{2N}$ such that η is a critical point of $\delta_{|\{b\}\times\mathbf{R}^{2N}}$; and
- $S_\delta{}^{\prec}$ is the set of $(b, \eta) \in B \times \mathbf{R}^{2N}$ such that η is a degenerate critical point of $\delta_{|\{b\}\times\mathbf{R}^{2N}}$.

Moreover, since Λ is at most $\Sigma^{1,0}$-singular, using the proof of Proposition 2.2 shows that

- $S_\delta{}^{\prec}$ is a closed subset of S_δ, as ensured by the remark following Definition 1.8; and
- $S_\delta \setminus S_\delta{}^{\prec}$ and $S_\delta{}^{\prec}$ are submanifolds of dimensions $\dim(B)$ and $\dim(B) - 1$, respectively, since it was observed in the remark following the proof of Proposition 1.2 that fibre products of transverse maps are smooth.

Definition 3.2 (based on [HR13, Definition 6.2])**.** Let c_- and c_+ be two positively valued critical points of δ, the set $\widehat{\mathscr{M}}^{\mathrm{st}}(c_-, c_+; g, \delta)$ of *parametrized gradient staircases from c_- to c_+* consists of tuples $\mathbf{e} = (h_0, v_1, h_1, \dots, v_{m-1}, h_{m-1}, v_m, h_m)$ such that $m \in \mathbf{N}$ and $m \geqslant 1$, and furthermore

- **Horizontal fragments (steps).** For all $i \in \{0, \dots, m\}$, $h_i \colon I_i \to S_\delta$ is a continuous map, where $I_0 = (-\infty, 0]$, $I_m = [0, +\infty)$ and for all $i \in \{1, \dots, m-1\}$, $I_i = [0, t_i]$ with $t_i > 0$. Moreover, the following differential equation
$$\partial_t h_i = -\nabla_g \left(\delta_{|S_\delta \setminus S_\delta{}^{\prec}}\right) \circ h_i,$$
is satisfied wherever it makes sense.
- **Vertical fragments (risers).** For all $j \in \{1, \dots, m\}$, there exists $b_j \in B$ such that v_j is non-constant and $v_j \in \mathscr{M}(v_j{}^-, v_j{}^+; \delta_{b_j}, g)$, where δ_{b_j} stands for $\delta_{|\{b_j\}\times\mathbf{R}^N\times\mathbf{R}^N}$.
- **Junctions of the fragments.** For all $i \in \{1, \dots, m-1\}$, $h_i(0) = v_i{}^+$ and $h_i(t_i) = v_{i+1}{}^-$, moreover $h_0(0) = v_1{}^-$ and $v_m{}^+ = h_m(0)$.
- **Gradient staircase endpoints.** The equalities $c_- = \lim_{-\infty} h_0$ and $c_+ = \lim_{+\infty} h_m$ hold.

Moreover, the set $\mathscr{M}^{\mathrm{st}}(c_-, c_+; g, \delta)$ of *gradient staircases from c_- to c_+* consists of equivalence classes $[\mathbf{e}] = (h_0, [v_1], h_1, \dots, [v_{m-1}], h_{m-1}, [v_m], h_m)$ such that $\mathbf{e} \in \widehat{\mathscr{M}}^{\mathrm{st}}(c_-, c_+; g, \delta)$.

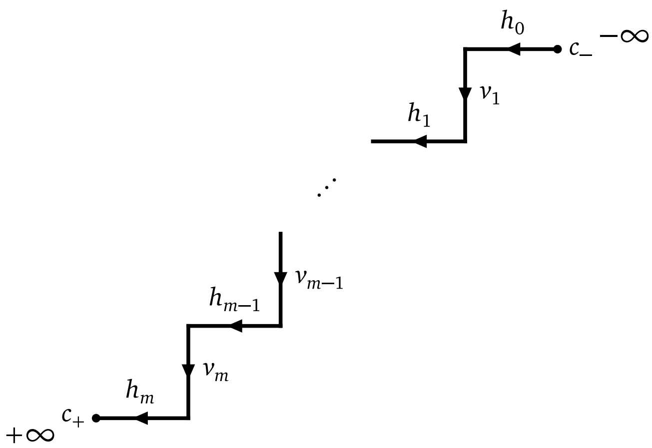


Figure 19. Schematic representation of a gradient staircase from c_- to c_+.

Remark. Even though the differential equation for horizontal fragments of gradient staircases is defined by the intermediate of the difference function, it is uniquely determined by the front. Indeed, since the canonical projection $\pi_B \colon B \times \mathbf{R}^{2N} \to B$ restricts to an immersion over $S_\delta \setminus S_\delta{}^{\prec}$, in a neighbourhood of every point, $S_\delta \setminus S_\delta{}^{\prec}$ is a graph over B, so that $\delta_{|S_\delta \setminus S_\delta{}^{\prec}}$ is a function over B.

Remark. On an a gradient generic Legendrian submanifold, the horizontal fragments of gradient staircases are uniquely determined by the front, since they are required to be continuous and their defining differential equations make sense everywhere apart from isolated points.

Remark. Despite what is claimed in [HR13, Subsection 6.1], their whole discussion does not apply to difference functions of generating families of arbitrary Legendrian submanifolds, since their fibrewise critical sets are not transversally cut out submanifolds, except when the caustic of the Legendrian submanifolds considered are nonsingular (which is a strong assumption).

Conjecture 3.1 ([HR13, Conjecture 6.3]). *If Λ is* generic, *then for all positively valued critical points c_- and c_+ of δ such that $\mu(c_-) = \mu(c_+) - 1$, there exists $s_0 \in (0,1]$ such that for all $s \in (0, s_0]$, there exists a bijective correspondence between $\mathscr{M}(c_-, c_+; \delta, g_s)$ and $\mathscr{M}^{\mathrm{st}}(c_-, c_+; g, \delta)$.*

Morse moduli spaces are compactified by broken gradient trajectories, [AD10, Théorème 3.2.2]. Gradient staircases chains play the same role for moduli spaces of gradient staircases.

Definition 3.3. Let c_- and c_+ be positively valued critical points of δ, the set $\widehat{\overline{\mathscr{M}}}^{\mathrm{st}}(c_-, c_+; g, \delta)$ of *parametrized gradient staircases chains from c_- to c_+* consists of tuples $\underline{\mathbf{e}} = (\mathbf{e}_1, \ldots, \mathbf{e}_m)$, such that m is a positive integer, for all $i \in \{1, \ldots, m\}$, $\mathbf{e}_i \in \widehat{\mathscr{M}}^{\mathrm{st}}(c_i, c_{i+1}; g, \delta)$, $c_1 = c_-$ and $c_{m+1} = c_+$. Moreover, as in Definition 3.2, the set $\overline{\mathscr{M}}^{\mathrm{st}}(c_-, c_+; g, \delta)$ of *gradient staircases chains from c_- to c_+* consists of equivalence classes $[\underline{\mathbf{e}}] = ([\mathbf{e}_1], \ldots, [\mathbf{e}_m])$ such that $\underline{\mathbf{e}} \in \widehat{\overline{\mathscr{M}}}^{\mathrm{st}}(c_-, c_+; g, \delta)$.

2.2. Analysis of vertical fragments.

In this section, several standard results from Morse theory are extended to the fibrewise gradient flow of generating families, and in particular, vertical fragments with relatively compact range are shown to join critical points (Proposition 3.1).

Lemma 3.4. *Let v be a vertical fragment, then for all $t \in \mathbf{R}$, the following equalities hold*

$$\partial_t \delta(v(t)) = -\left\|\nabla_g \delta(v(t))\right\|_g^2 = -\|\partial_t v(t)\|_g^2,$$

Moreover, $\delta \circ v$ is decreasing and it is strictly decreasing if, and only if, v is non-constant.

Proof. For all $t \in \mathbf{R}$, the chain rule shows that

$$\begin{aligned}
& \partial_t(\delta \circ v)(t) = T_{v(t)}\delta(\partial_t v(t)), \\
\text{[by definition of } \nabla] \quad & = g\left(\nabla_g \delta(v(t)), \partial_t v(t)\right), \\
\text{[since } \partial_t v = -\nabla_g \delta \circ v] \quad & = g\left(\nabla_g \delta(v(t)), -\nabla_g \delta(v(t))\right), \\
& = -\left\|\nabla_g \delta(v(t))\right\|_g^2, \\
\text{[since } \partial_t v = -\nabla_g \delta \circ v] \quad & = -\|\partial_t v(t)\|_g^2.
\end{aligned}$$

In particular, the derivative of $\delta \circ v$ is everywhere negative, thus showing that it is decreasing. Moreover, $\delta \circ v$ is constant if, and only if, its derivative is everywhere vanishing, namely

$$\forall t \in \mathbf{R}, \nabla_g \delta(v(t)) = \mathbf{0},$$

and then by the uniqueness result for solutions of ordinary differential equations, v is constant, thus showing the negation of the claimed equivalence and yielding the result. ■

Lemma 3.5. *Let v be a vertical fragment and let us define its* energy $E(v) \in [0, +\infty]$ *as*

$$E(v) = \int_{\mathbf{R}} \|\partial_t v(t)\|_g{}^2 \,\mathrm{dt}.$$

If v has relatively compact range, then $E(v)$ is finite and $E(v) = \sup_{t\in\mathbf{R}} \delta(v(t)) - \inf_{t\in\mathbf{R}} \delta(v(t))$.

Proof. Let $a \in [0, +\infty)$, then Lemma 3.4 implies that

$$\int_{-a}^{a} \|\partial_t v(t)\|_g{}^2 \,\mathrm{dt} = \int_{a}^{-a} \partial_t \delta(v(t)) \,\mathrm{dt} = \delta(v(-a)) - \delta(v(a)). \tag{9}$$

However, $\delta \circ v$ is decreasing (by Lemma 3.4) and bounded (since v has relatively compact range), thus its limits at $-\infty$ and $+\infty$ both exist and are finite, and they furthermore satisfy:

$$\lim_{t\to-\infty} \delta(v(t)) = \sup_{t\in\mathbf{R}} \delta(v(t)),$$
$$\lim_{t\to+\infty} \delta(v(t)) = \inf_{t\in\mathbf{R}} \delta(v(t)).$$

Therefore, taking the limit as $a \to +\infty$ in (9) shows that

$$\begin{aligned}\lim_{a\to-\infty} \int_{a}^{0} \|\partial_t v(t)\|_g{}^2 \,\mathrm{dt} &= \lim_{t\to-\infty} \delta(v(t)) - \lim_{t\to+\infty} \delta(v(t)),\\ &= \sup_{t\in\mathbf{R}} \delta(v(t)) - \inf_{t\in\mathbf{R}} \delta(v(t)),\end{aligned}$$

thus yielding the result. ■

Lemma 3.6. *Let K be a compact subset of $B \times \mathbf{R}^N \times \mathbf{R}^N$, then there exists a constant $M > 0$ such that any vertical fragment v having range in K satisfies $\|v\|_{C^2,g} \leqslant M$.*

Proof. Since v has range in K, for all $t \in \mathbf{R}$, it holds that

$$\|\partial_t v(t)\|_g = \left\|\nabla_g \delta(v(t))\right\|_g \leqslant \sup_K \left\|\nabla_g \delta\right\| = M_1.$$

Moreover, the chain rule and the above inequality show that for all $t \in \mathbf{R}$, it holds that

$$\left\|\partial_{tt}^2 v(t)\right\|_{\widehat{g}} = \left\|T_{v(t)} \nabla_g \delta(\partial_t v(t))\right\|_{\widehat{g}} \leqslant M_1 \cdot \sup_K \left\|T \nabla_g \delta\right\|_{\widehat{g}} = M_2,$$

hence, if d_g is the Riemannian distance associated to g, letting $M = \max(\mathrm{diam}_{d_g}(K), M_1, M_2)$, which depends only on K, δ and g, yields the result. ■

Remark. Lemmata 3.4, 3.5 and 3.6 hold for gradient flow lines of arbitrary smooth functions.

Proposition 3.1. *Let v be a vertical fragment. If Λ is at most $\Sigma^{1,0}$-singular and v has relatively compact range, then the limits of v at $-\infty$ and $+\infty$ both exist, and are points of S_δ.*

Proof. Using Definition 3.2, there exists $b \in B$ such that v is a g-gradient flow line of $-\delta_{|\{b\}\times\mathbf{R}^N\times\mathbf{R}^N}$, let us write $\delta_b = \delta_{|\{b\}\times\mathbf{R}^N\times\mathbf{R}^N}$ and focus on the negative end of v, then the proof is carried through the following steps:

- **Step 1.** The critical points of δ_b are the $(\eta_1, \eta_2) \in \mathbf{R}^N \times \mathbf{R}^N$ such that $(b, \eta_1, \eta_2) \in S_\delta$. Moreover, the number of critical points of δ_b is finite.
- **Step 2.** It holds that $\nabla_g \delta(v(t)) \xrightarrow[t\to-\infty]{} 0$.
- **Step 3.** The limit of v at $-\infty$ exists and is a critical point of δ_b.

Step 1. Let us write $f_{1_b} = f_{1|\{b\}\times\mathbf{R}^N}$ and $f_{2_b} = f_{2|\{b\}\times\mathbf{R}^N}$, it follows from Lemma 3.1 that

$$\forall(\eta_1,\eta_2)\in\mathbf{R}^N\times\mathbf{R}^N, \nabla_g\delta_b(\eta_1,\eta_2) = \nabla_{g_N}f_{1_b}(\eta_1)\oplus -\nabla_{g_N}f_{2_b}(\eta_2),$$

then (η_1,η_2) is a critical point of δ_b if, and only if, $(b,\eta_1,\eta_2)\in S_\delta$, thus yielding the first claim. Moreover, Corollary 2.1 ensures that the number of critical points of f_{1_b} and f_{2_b} are both finite, therefore the number of critical points of δ_b is also finite, thus yielding Step 1.

Step 2. Argue, for the sake of contradiction, that $\partial_t v(t) \underset{t\to-\infty}{\not\longrightarrow} 0$, then there exists $\varepsilon > 0$ and a sequence $(t_k)_{k\in\mathbf{N}}$ of real numbers such that $t_k \underset{k}{\to} -\infty$ and for all integers $k \geqslant 0$, $\|\partial_t v(t_k)\|_g \geqslant \varepsilon$. Besides, using Lemma 3.6 and the mean value inequality, there exists $M > 0$ such that

$$\forall t_1\in\mathbf{R}, \forall t_2\in\mathbf{R}, \|\partial_t v(t_1)-\partial_t v(t_2)\|_g \leqslant M|t_1-t_2|,$$

thus, defining $I_k = \left[t_k - \frac{\varepsilon}{2M}, t_k + \frac{\varepsilon}{2M}\right]$, for $k\in\mathbf{N}$, it then follows from the triangle inequality that

$$\forall t\in I_k, \|\partial_t v(t)\|_g \geqslant \|\partial_t v(t_k)\|_g - \|\partial_t v(t_k)-\partial v(t)\|_g \geqslant \varepsilon - M|t_k-t| \geqslant \frac{\varepsilon}{2}.$$

However, since $t_k \underset{k}{\to} -\infty$, there exists a subsequence (still denoted in the same way) such that for all $k\in\mathbf{N}$, $t_{k+1} < t_k - \frac{\varepsilon}{2M}$, thus the intervals $\{I_k\}_{k\in\mathbf{N}}$ are pairwise disjoints, implying that

$$E(v) \geqslant \int_{(-\infty,0]} \|\partial_t v(t)\|_g{}^2\,\mathrm{d}t \geqslant \sum_{k=0}^{+\infty}\int_{I_k}\|\partial_t v(t)\|_g{}^2\,\mathrm{d}t \geqslant \sum_{k=0}^{+\infty}\left(\frac{\varepsilon}{2}\right)^2\frac{\varepsilon}{M} = +\infty$$

hence contradicting Lemma 3.5, thus $\partial_t v \underset{t\to-\infty}{\longrightarrow} 0$ and Lemma 3.4 yields Step 2.

Step 3. According to Step 1, let us define $\sigma_1\in S_\delta,\ldots,\sigma_\ell\in S_\delta$ to be the distinct critical points of δ_b, then for all $k\in\{1,\ldots,\ell\}$, let U_k be any connected open neighbourhood of σ_k in $\{b\}\times\mathbf{R}^N\times\mathbf{R}^N$. Moreover, since $\sigma_1,\ldots,\sigma_\ell$ are pairwise distincts, assume, without loss of generality, that $U_1,\ldots,U_\ell$ are pairwise disjoints, and let us define U to be their union. It then follows from Step 2 that there exists $T<0$ such that for all $t<T$, $v(t)\in U$. Hence, by continuity of v, there exists $k\in\{1,\ldots,\ell\}$ such that for all $t<T$, $v(t)\in U_k$, but since U_k was arbitrary, it yields $\lim\limits_{-\infty} v = \sigma_k$ and Step 3.

A similar argument applied to the positive end of v shows that $\lim\limits_{+\infty} v$ exists and is a point of S_δ, thus yielding the result. ■

Remark. The proof of Proposition 3.1 more generally show that the bounded gradient flow lines of any smooth function, having isolated critical points, converge to critical points.

Remark. If δ_b was a Morse function, the proof of Proposition 3.1 would be rather standard and straightforward (see, for example, [AD10, Proposition 2.1.6]). However, as it was remarked after the proof of Proposition 2.1 that δ_b fails to be a Morse function when b is in the complement a codimension 0 submanifold of B.

3. Floer-Gromov convergence for Henry–Rutherford sequences

This section defines and studies a good Floer-Gromov-like topology (Definitions 3.4 and 3.5) well-behaved to analytically describe how the Henry-Rutherford limiting process straighten the gradient flow lines of difference functions onto gradient staircases chains (Theorem A).

Definition 3.4. A Henry-Rutherford sequence $\gamma_k \in \widehat{\mathcal{M}}(c_-, c_+; \delta, g_{s_k})$ *converges in the Floer-Gromov topology* towards a parametrized gradient staircase $\mathbf{e} = (h_0, v_1, h_1, \dots, v_m, h_m) \in \widehat{\mathcal{M}}^{\mathrm{st}}(c_-, c_+; \delta, g)$ whenever there exist sequences of real numbers $(\tau_k^{h_0})_{k\in\mathbf{N}}, \dots, (\tau_k^{h_m})_{k\in\mathbf{N}}$ and $(\tau_k^{v_1})_{k\in\mathbf{N}}, \dots, (\tau_k^{v_m})_{k\in\mathbf{N}}$ such that

- for all $i \in \{0, \dots, m\}$, $\gamma_k(s_k{}^{-1}(\cdot + \tau_k^{h_i})) \underset{k}{\longrightarrow} h_i$ in the C^1_{loc}-topology; and
- for all $j \in \{1, \dots, m\}$, $\gamma_k(\cdot + \tau_k^{v_j}) \underset{k}{\longrightarrow} v_j$ in the C^1_{loc}-topology.

Furthermore, an unparameterized Henry-Rutherford sequence $[\widetilde{\gamma}_k] \in \mathcal{M}(c_-, c_+; \delta, g_{s_k})$ *converges in the Floer-Gromov topology* towards an unparameterized gradient staircase $[\widetilde{\mathbf{e}}] \in \mathcal{M}^{\mathrm{st}}(c_-, c_+; \delta, g)$ whenever there exist representatives $\gamma_k \in \widehat{\mathcal{M}}(c_-, c_+; \delta, g_{s_k})$ of $[\widetilde{\gamma}_k]$ and $\mathbf{e} \in \widehat{\mathcal{M}}^{\mathrm{st}}(c_-, c_+; \delta, g)$ of $[\widetilde{\mathbf{e}}]$ such that $\gamma_k \underset{k}{\longrightarrow} \mathbf{e}$ in the Floer-Gromov topology.

Remark. Since sequences of real numbers act by time-translation on Henry–Rutherford sequences, the Floer-Gromov convergence must account for Henry–Rutherford sequences drifting to infinity. In particular, time-shifts are used to centre Henry–Rutherford sequences around a fixed point, and scaling allows to recover nonconstant horizontal fragments.

Remark. Definition 3.4 directly extends to Henry-Rutherford sequences $(\gamma_k : I_k \to B \times \mathbf{R}^N \times \mathbf{R}^N)_{k\in\mathbf{N}}$ only defined over a sequence of real intervals $(I_k)_{k\in\mathbf{N}}$ such that $|I_k| \underset{k}{\longrightarrow} +\infty$.

Remark. For large enough nonnegative integers k, the sequence of time-shifts in Definition 3.4 are intertwined as follows:

$$\tau_k^{h_0} \leqslant \tau_k^{v_1} \leqslant \tau_k^{h_1} \leqslant \cdots \leqslant \tau_k^{v_{m-1}} \leqslant \tau_k^{h_{m-1}} \leqslant \tau_k^{v_m} \leqslant \tau_k^{h_m},$$

as it is seen from the junctions of fragments conditions in Definition 3.2, using Lemma 3.3.

In particular, Floer-Gromov convergence can be extended to gradient staircases chains.

Definition 3.5. A Henry-Rutherford sequence $(\gamma_k)_{k\in\mathbf{N}}$ *converges in the Floer-Gromov topology* to a parametrized gradient staircases chain $\underline{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_m)$ whenever there exist sequences of real intervals $(I_k^1)_{k\in\mathbf{N}}, \dots, (I_k^m)_{k\in\mathbf{N}}$ such that

- for all $k \in \mathbf{N}$, $\{I_k^1, \dots, I_k^m\}$ is a partition of $\mathbf{R}$; and
- for all $i \in \{1, \dots, m\}$, $\gamma_{k|I_k^i} \underset{k}{\longrightarrow} \mathbf{e}_i$ in the Floer-Gromov topology.

Moreover, an unparameterized Henry-Rutherford sequence $[\widetilde{\gamma}_k] \in \mathcal{M}(c_-, c_+; \delta, g_{s_k})$ *converges in the Floer-Gromov topology* to an unparametrized gradient staircases chain $[\underline{\widetilde{\mathbf{e}}}] \in \overline{\mathcal{M}}^{\mathrm{st}}(c_-, c_+; \delta, g)$ whenever there exist representatives $\gamma_k \in \widehat{\mathcal{M}}(c_-, c_+; \delta, g_{s_k})$ of $[\widetilde{\gamma}_k]$ and $\underline{\mathbf{e}} \in \widehat{\overline{\mathcal{M}}}^{\mathrm{st}}(c_-, c_+; \delta, g)$ of $[\underline{\widetilde{\mathbf{e}}}]$ such that $\gamma_k \underset{k}{\longrightarrow} \underline{\mathbf{e}}$ in the Floer-Gromov topology.

Remark. The first condition in Definition 3.5 ensures that no other gradient staircases can be extracted from the Henry-Rutherford sequence considered.

Proposition 3.2. *Let v and w be non-constant vertical fragments. Assume that v and w are recovered from $(\gamma_k)_{k\in\mathbf{N}}$ using sequences of time-shifts $(\tau_k^v)_{k\in\mathbf{N}}$ and $(\tau_k^w)_{k\in\mathbf{N}}$, respectively, then v and w are equivalent if, and only if, $(|\tau_k^w - \tau_k^v|)_{k\in\mathbf{N}}$ is a bounded sequence.*

Proof. For all $k \in \mathbf{N}$, let us define $T_k = \tau_k^w - \tau_k^v$, so that $\tau_k^w = T_k + \tau_k^v$.

- **Necessary condition.** Assume that v and w are equivalent, then there exists $T \in \mathbf{R}$ such that $w = v(\cdot + T)$ and assume, for the sake of contradiction, that $(|T_k|)_{k\in\mathbf{N}}$ is unbounded. Besides, for all $k \in \mathbf{N}$, there exists $\varepsilon_k \in \{-1,1\}$ such that $|T_k| = \varepsilon_k T_k$, then since $\{-1,1\}$ is finite, using the pigeonhole principle, there exist $\varepsilon \in \{-1,1\}$ and a subsequence (still denoted in the same way) such that
$$\forall k \in \mathbf{N}, |T_k| = \varepsilon T_k.$$
If v and w are exchanged, the signs of $(\varepsilon_k)_{k\in\mathbf{N}}$ are flipped, thus it can be assumed, without loss of generality, that $\varepsilon = 1$, so that $(T_k)_{k\in\mathbf{N}}$ is a nonnegative unbounded sequence and there exists a subsequence (denoted in the same way) such that for all $k \in \mathbf{N}$, $T_k \geqslant T+1$. Therefore, using Lemma 3.3 shows that
$$\delta\left(\gamma_k\left(-T+\tau_k^w\right)\right) = \delta\left(\gamma_k\left(-T+T_k+\tau_k^v\right)\right) \leqslant \delta\left(\gamma_k\left(1+\tau_k^v\right)\right),$$
and taking the limit for k of the above inequality yields, $\delta(w(-T)) = \delta(v(0)) \leqslant \delta(v(1))$. However, according to Lemma 3.4, $\delta \circ v$ is strictly decreasing, hence a contradiction.
- **Sufficient condition.** Assume that $(|T_k|)_{k\in\mathbf{N}}$ is bounded, then there exists $T \in \mathbf{R}$ and a subsequence such that $T_k \xrightarrow[k]{} T$ and proceed to show that for all $t \in \mathbf{R}$, $v(t) = w(t+T)$. Let k be an integer, then making several uses of the triangle inequality, notice that

$$\begin{aligned}
\|w(t)-v(t+T)\|_g &\leqslant \left\|\gamma_k\left(t+\tau_k^w\right)-w(t)\right\|_g + \left\|\gamma_k\left(t+\tau_k^w\right)-v(t+T)\right\|_g, \\
&= \left\|\gamma_k\left(t+\tau_k^w\right)-w(t)\right\|_g + \left\|\gamma_k\left(t+T_k+\tau_k^v\right)-v(t+T)\right\|_g, \\
&\leqslant \left\|\gamma_k\left(t+\tau_k^w\right)-w(t)\right\|_g + \left\|\gamma_k\left((t+T)+\tau_k^v\right)-v(t+T)\right\|_g, \\
&\quad + \left\|\gamma_k\left(t+T_k+\tau_k^v\right)-\gamma_k\left(t+T+\tau_k^v\right)\right\|_g.
\end{aligned} \tag{10}$$

Besides, since γ_k is negative g_{s_k}-gradient trajectory of δ, the mean value inequality yields

$$\begin{aligned}
\left\|\gamma_k\left(t+T_k+\tau_k^v\right)-\gamma_k\left(t+T+\tau_k^v\right)\right\|_g &\leqslant \sup_{\mathbf{R}} \left\|\nabla_{g_{s_k}}\delta\circ\gamma_k\right\|_g \cdot |T-T_k|, \\
\text{[by Proposition 2.4]} \qquad &\leqslant \sup_{K_\delta} \left\|\nabla_{g_{s_k}}\delta\right\|_g \cdot |T-T_k|, \\
\text{[by Lemma 3.2]} \qquad &\leqslant \sup_{K_\delta} \left\|\nabla_g\delta\right\|_g \cdot |T_k-T| \xrightarrow[k]{} 0.
\end{aligned}$$

Therefore, taking the limit for k on both sides of (10) shows that v and w are equivalent.

Both sides of the claimed equivalence have thus been proved. ■

4.

Floer-Gromov compactness for gradient staircases chains

The goal of this chapter is to provide a complete proof of Theorem A, whose rigorous and precise statement now reads as follows:

Theorem 4.1. *Let Λ be a closed Legendrian submanifold of (J^1B, ξ_B) with at most $\Sigma^{1,0}$-singularities. If Λ is gradient and chord generic, then for all parametrized Henry–Rutherford sequences $(\gamma_k)_{k\in\mathbf{N}}$, there exist a parametrized gradient staircases chain $\underline{\mathbf{e}}$ and a subsequence (denoted in the same way) such that $\gamma_k \underset{k}{\longrightarrow} \underline{\mathbf{e}}$ in the Floer-Gromov topology.*

Remark. Gradient genericity is meant for the Riemannian metric induced by a metric on the base, see the remark following Definition 3.1, for more details on this construction.

Each section of this chapter corresponds to a step in the proof of Theorem 4.1:

- **Step 1.** First, elementary fragments of gradient staircases are recursively recovered. Applying Arzelà-Ascoli theorem easily recovers vertical fragments (Proposition 4.1), while horizontal fragments need a more careful analysis (Theorem 4.2).
- **Step 2.** The recursive procedure recovering one after the other the elementary fragments of gradient staircases is then shown to end after a finite number of steps (Theorem 4.3). It is not straightforward and heavily relies on gradient genericity defined in Chapter 1 to prevent elementary fragments of gradient staircases from becoming arbitrarily short and accumulating.
- **Step 3.** At last, it is shown that after the recursive procedure has ended, it has recovered a complete gradient staircases chain without missing elementary fragments (Section 3). In other words, the elementary fragments recovered have continuous junctions.

This chapter is the culmination of the thesis and materials from all previous chapters are needed. In particular, the assumptions in the statement of Theorem 4.1 were explained in Chapter 1, while some of the technicalities needed for the proofs were dealt with in Chapter 3.

1. Recovering elementary fragments of a gradient staircases chain from a HR-sequence

This section explains how elementary fragments of gradient staircases chains are recovered from Henry–Rutherford-sequences in the Floer-Gromov topology sense. It first deals with the easier case of vertical fragments (Section 1.1) and then with horizontal fragments (Section 1.2).

1.1. Recovering one vertical fragment at a time.

The convergence in the Floer-Gromov topology of Henry–Rutherford sequences towards vertical fragments is straightforward from Ascoli-Arzelà theorem, since Proposition 2.4 and Lemma 3.2 directly provides uniform C^2-bounds allowing to take $s \to 0$ in the ordinary differential equation associated to the g_s-gradient flow.

Proposition 4.1. *Let U be an open neighbourhood of S_δ in $B \times \mathbf{R}^N \times \mathbf{R}^N$. Assume that for all $k \in \mathbf{N}$, there exists $\tau_k \in \mathbf{R}$ such that $\gamma_k(\tau_k) \notin U$, then there exist a non-constant vertical fragment v and a subsequence (still denoted in the same way) such that $\gamma_k(\cdot + \tau_k) \underset{k}{\longrightarrow} v$ in the C^1_{loc}-topology.*

Proof. For $k \in \mathbf{N}$, let us define $c_k = \gamma_k(\cdot + \tau_k)$ and let us write c_k in components as $c_k = (c_{k,B}, c_{k,F})$, then c_k is a negative g_{s_k}-gradient flow line of δ, since γ_k is and it is an autonomous vector field. Besides, using **Lemma 3.1**, the differential equation satisfied by c_k reads in components as:

$$\begin{cases} \partial_t c_{k,B}(t) = -s_k \nabla_{g_B} \delta\left(c_k(t)\right), \\ \partial_t c_{k,F}(t) = -\nabla_{g_F} \delta\left(c_k(t)\right), \end{cases} \tag{11}$$

Then, **Proposition 2.4** and **Lemma 3.2** show that $(c_k)_{k\in\mathbf{N}}$ is a bounded sequence in C^2-topology, and the mean value inequality shows that $(c_k)_{k\in\mathbf{N}}$ and $(\partial_t c_k)_{k\in\mathbf{N}}$ are bounded and equicontinuous, so that Arzelà-Ascoli theorem applies and provides a limit point v for $(c_k)_{k\in\mathbf{N}}$ in the C^1_{loc}-topology. Therefore, if v is written in components as $v = (v_B, v_F)$, taking the limit for k in the system of differential equations **(11)** satisfied by c_k and recalling that $s_k \underset{k}{\longrightarrow} 0$, then yields

$$\begin{cases} \partial_t v_B(t) = 0, \\ \partial_t v_F(t) = -\nabla_{g_F} \delta(v(t)). \end{cases}$$

The above equations shows there exists $b \in B$ such that v is a g-gradient flow line of $-\delta_{|\{b\}\times\mathbf{R}^N\times\mathbf{R}^N}$, therefore v is a vertical fragment and to conclude, it only remains to show that v is non-constant. For this purpose, start by noticing that **Proposition 2.4** ensures that v has relatively compact range, so that **Proposition 3.1** applies and shows that the ends of v are both elements of S_δ. Besides, argue that, by construction of $(\tau_k)_{k\in\mathbf{N}}$, for all $k \in \mathbf{N}$, it holds that $c_k(0) \in U^c$, then taking the limit for k yields $v(0) \in \overline{U^c} = U^c \subset {S_\delta}^c$ and the result. ■

Remark. Whenever the conclusion of **Proposition 4.1** holds, v is said to be *recovered* from $(\gamma_k)_{k\in\mathbf{N}}$, and this definition is compatible with **Definition 3.5**.

1.2. Recovering one horizontal fragment at a time.

Unlike what was done in **Proposition 4.1** for vertical fragments, horizontal fragments cannot be recovered by a direct application of the Ascoli-Arzelà theorem to a Henry–Rutherford sequence. In particular, since Henry–Rutherford sequences need to be speed-up to converge to nonconstant horizontal fragments (**Definition 3.4** and **3.5**), working with horizontal fragments is trickier. Unfortunately, this speed-up parameter also messes up the uniform C^2-bounds required to apply the Ascoli–Arzelà theorem and recover horizontal fragments from Henry–Rutherford sequences. Indeed, if $\gamma_s \in \widehat{\mathscr{M}}(c_-, c_+; \delta, g_s)$, then using the chain rule and **Lemma 3.1** shows that

$$\partial_t \gamma_s(s^{-1}t) = -\nabla_{g_B} \delta(\gamma_s(s^{-1}t)) \oplus s^{-1} \nabla_{g_F} \delta(\gamma_s(s^{-1}t)),$$

which does not converge in the limit $s \to 0$, unless $\gamma_s(s^{-1}t)$ converges sufficiently quickly onto S_δ.

This difficulty is overcome by conducting a careful geometrical analysis of the difference function gradient flow resulting in upper bounds (**Propositions 4.2**, **4.3**, **4.4** and **4.5**), allowing to prove uniform convergence on an interval from pointwise convergence on its boundary (**Theorem 4.2**). Similar techniques were previously applied to the Floer equation to get [**BO09b**, Proposition A.3].

Moreover, **Propositions 4.2** and **4.4** are also expected to play a crucial role in the proof of the Floer–Gromov gluing result needed for **Conjecture 3.1** to hold, since [**BO09b**, Proposition A.3] was at the heart of [**BO09b**, Proposition 4.22] and [**BO09b**, Theorem 3.7].

The rest of this section now conducts a careful quantitative analysis of the negative gradient flow dynamics of the difference function when sufficiently near to its maximal flow invariant subset. In particular, it is shown that

- fibre directions are uniformly contracted (**Propositions 4.2** and **4.3**); and
- base directions are controlled by the fibre ones (**Propositions 4.4** and **4.5**);

and in both cases, the analysis and the resulting estimates are slightly different whether or not the open neighbourhood considered intersects the singular part of the maximal flow invariant subset; e.g., not only nonsingular fibre directions are uniformly contracted, but with an exponential rate. From these estimates, horizontal fragments are finally recovered (**Theorem 4.2**).

First, let us start working on the nonsingular fibre directions of the maximal flow invariant subset. For that purpose, let $\pi_B : B \times \mathbf{R}^{2N} \to B$ be the canonical projection, then by construction of $S_\delta{}^{\prec}$, every point $\sigma \in S_\delta \backslash S_\delta{}^{\prec}$ has an open neighbourhood V^σ in $S_\delta \backslash S_\delta{}^{\prec}$ such that $\pi_{B_{|V^\sigma}}$ is an embedding. Indeed, π_B restricts to an immersion on $S_\delta \backslash S_\delta{}^{\prec}$ and is therefore locally everywhere an embedding. Besides, since $S_\delta \setminus S_\delta{}^{\prec}$ is a submanifold of $B \times \mathbf{R}^{2N}$, it is locally closed in $B \times \mathbf{R}^{2N}$, and in particular, there exists an open neighbourhood U^σ of σ in $B \times \mathbf{R}^{2N}$ such that $U^\sigma \cap (S_\delta \setminus S_\delta{}^{\prec})$ is closed in U^σ. Moreover, by shrinking U^σ, it can be assumed that $U^\sigma = U_B^\sigma \times U_F^\sigma$ and $U^\sigma \cap (S_\delta \setminus S_\delta{}^{\prec}) \subset V^\sigma$. Therefore, by construction of U^σ, $\pi_B : B \times \mathbf{R}^{2N} \to B$ restricts to an embedding on $U^\sigma \cap (S_\delta \setminus S_\delta{}^{\prec})$, and in conclusion, there exists a unique smooth map $\eta^\sigma : U_B^\sigma \to U_F^\sigma$ such that

$$U^\sigma \cap (S_\delta \setminus S_\delta{}^{\prec}) = \left\{(b, \eta^\sigma(b)), b \in U_B^\sigma\right\} = \Gamma_{\eta^\sigma}.$$

Then, let also $K^\sigma = K_B^\sigma \times K_F^\sigma$ be a compact subset of U^σ.

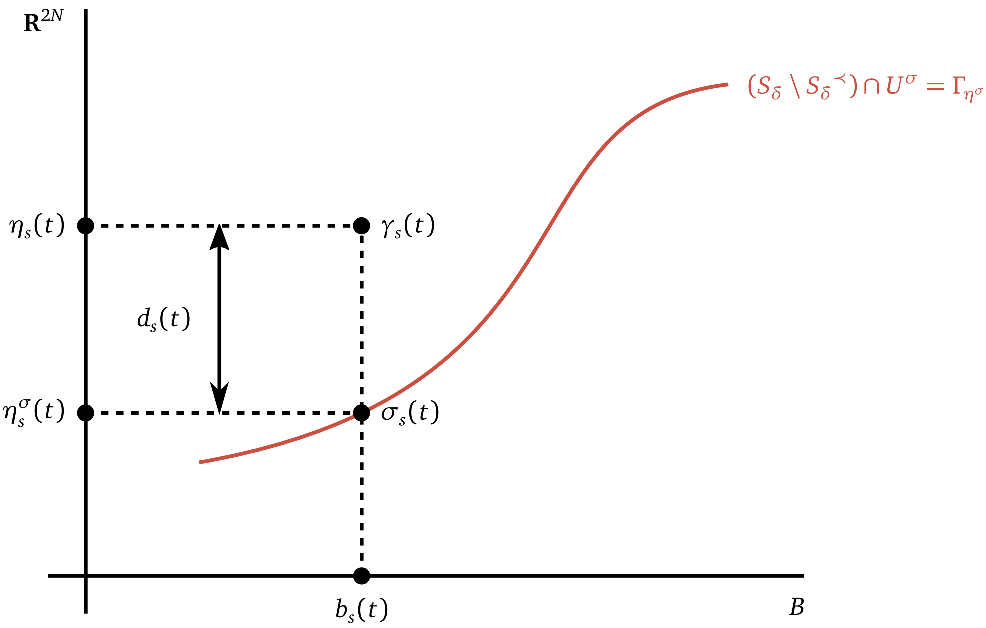


Figure 20. Visualising the quantities introduced in the proof of **Proposition 4.2**.

Proposition 4.2. *There exist uniform constants $d_0 > 0$, $C > 0$ and $\rho > 0$, depending only on δ, such that for all $s \in (0,1]$, for all $\gamma_s \in \widehat{\mathcal{M}}(c_-, c_+; \delta, g_s)$ and all interval $[t_s^-, t_s^+] \subset \mathbf{R}$ such that*

- *the range of $\gamma_s = (b_s, \eta_s)$ restricted to $[t_s^-, t_s^+]$ is contained in K^σ; and*
- *$\|\eta_s - \eta_s^\sigma\|_{g_F} \leqslant d_0$ on $[t_s^-, t_s^+]$, where $\gamma_s = (b_s, \eta_s)$ and $\eta_s^\sigma = \eta^\sigma \circ b_s$;*

then for all $t \in [t_s^-, t_s^+]$, $\Delta\eta_s(t) = \eta_s(t) - \eta_s^\sigma(t)$ and its first-order derivative at t are bounded by

$$C\left(\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right)\frac{\cosh(\rho(t-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2)} + s\right),$$

for the norm that are associated to g_F on $\mathbf{R}^{2N}$ and $T\mathbf{R}^{2N}$, respectively.

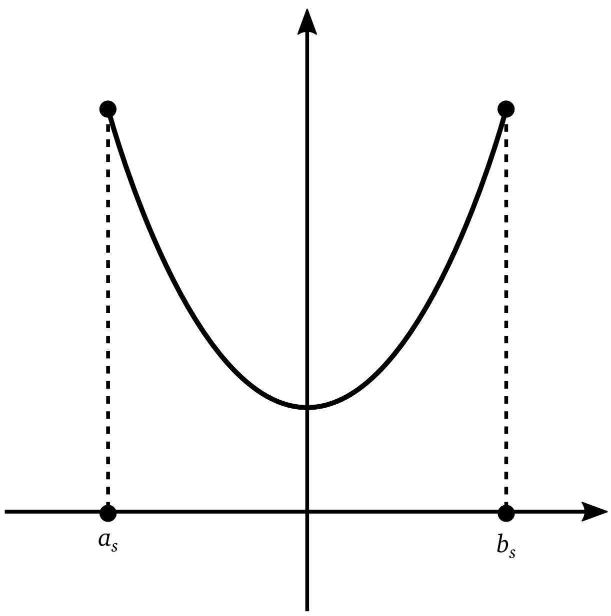


Figure 21. Shape of the upper bound derived in Proposition 4.2.

Proof. Let $d_s = \|\Delta\eta_s\|_{g_F} : [t_s^-, t_s^+] \to \mathbf{R}$ and $t \in (t_s^-, t_s^+)$, then applying twice the chain rule yields

$$d_s''(t) = \frac{g_F(\Delta\eta_s(t), \partial_{tt}^2\Delta\eta_s(t)) + \|\partial_t\Delta\eta_s(t)\|_{g_F}{}^2}{\|\Delta\eta_s(t)\|_{g_F}} - \frac{g_F(\Delta\eta_s(t), \partial_t\Delta\eta_s(t))^2}{\|\Delta\eta_s(t)\|_{g_F}{}^3}. \tag{12}$$

Besides, the Cauchy-Schwarz inequality shows that

$$-\frac{g_F(\Delta\eta_s(t), \partial_t\Delta\eta_s(t))^2}{\|\Delta\eta_s(t)\|_{g_F}{}^3} \geqslant -\frac{\|\partial_t\Delta\eta_s(t)\|_{g_F}{}^2}{\|\Delta\eta_s(t)\|_{g_F}},$$

and plugging the above inequality in equation (12) yields

$$\begin{aligned} d_s''(t) &\geqslant \frac{g_F(\Delta\eta_s(t), \partial_{tt}^2\Delta\eta_s(t)}{\|\Delta\eta_s(t)\|_{g_F}}, \\ &= \frac{g_F(\Delta\eta_s(t), \partial_{tt}^2\eta_s(t)) - g_F(\Delta\eta_s(t), \partial_{tt}^2\eta_s^\sigma(t))}{\|\Delta\eta_s(t)\|_{g_F}}, \end{aligned} \tag{13}$$

where the equality follows from the definition of $\Delta\eta_s$, the linearity of ∂_{tt}^2 and the bilinearity of g_F. Recall from Lemma 3.1 that the differential equation satisfied by γ_s reads in components as:

$$\begin{cases} \partial_t b_s(t) = -s\nabla_{g_B}\delta(\gamma_s(t)), \\ \partial_t\eta_s(t) = -\nabla_{g_F}\delta(\gamma_s(t)), \end{cases} \tag{14}$$

and use these equations to separately derive upper bounds for both terms on the numerator of the right-hand side of inequality (13).

- First, using the chain rule and taking the derivative on both sides of (14) yield

$$\begin{aligned} \partial^2_{tt}\eta_s(t) &= -T_{\gamma_s(t)}\nabla_{g_F}\delta(\partial_t\gamma_s(t)),\\ &= -T_{\gamma_s(t)}\nabla_{g_F}\delta(\partial_t b_s(t)\oplus\partial_t\eta_s(t)),\\ \text{[since } T=\partial_b\oplus\partial_\eta\text{]} \qquad &= -\partial_b\nabla_{g_F}\delta(\gamma_s(t))\cdot\partial_t b_s(t) - \partial_\eta\nabla_{g_F}\delta(\gamma_s(t))\cdot\partial_t\eta_s(t),\\ &= -\partial_b\nabla_{g_F}\delta(\gamma_s(t))\cdot\partial_t b_s(t) - \mathrm{Hess}_{\eta_s(t)}\,\delta_{b_s(t)}\cdot\partial_t\eta_s(t), \end{aligned}$$

where in the last equality, $\delta_{b_s(t)}$ is used as a shorthand notation standing for $\delta_{|\{b_s(t)\}\times\mathbf{R}^{2N}}$. Therefore, plugging equation (14) in the equality above shows that

$$\partial^2_{tt}\eta_s(t) = s\partial_b\nabla_{g_F}\delta(\gamma_s(t))\cdot\nabla_{g_B}\delta(\gamma_s(t)) + \mathrm{Hess}_{\eta_s(t)}\,\delta_{b_s(t)}\cdot\nabla_{g_F}\delta(\gamma_s(t)), \tag{15}$$

and the first term of (13) splits in two components that will now be separately dealt with. For the first term, start using that δ is C^2 and that γ_s has range in K^σ to show there exists a constant $c_1>0$, depending only on δ, such that $\|\partial_b\nabla_{g_F}\delta(\gamma_s(t))\cdot\nabla_{g_B}\delta(\gamma_s(t))\|_{g_F}\leqslant c_1$, then use that $\|\Delta\eta_s(t)\|_{g_F}=d_s(t)\leqslant d_0$ and the Cauchy-Schwarz inequality to get

$$g_F\left(\Delta\eta_s(t), s\partial_b\nabla_{g_F}\delta(\gamma_s(t))\cdot\nabla_{g_B}\delta(\gamma_s(t))\right)\geqslant -c_1 s d_s(t). \tag{16}$$

The second of this term is dealt with applying Taylor's formulas to $X_{b_s(t)}=(\nabla_{g_F}\delta)_{|\{b_s(t)\}\times\mathbf{R}^{2N}}$. For that purpose, start using that γ_s has range in K^σ and the compactness of S_δ (since the generating families considered are linear-at-infinity) to show that for all $\tau\in[0,1]$, $(1-\tau)\eta_s(t)+\tau\eta^\sigma_s(t)$ stays in a uniform compact subset of $B\times\mathbf{R}^{2N}$, depending only on δ. Thus, since δ is C^3, there exist $c_2>0$ and $c_3>0$, depending only on δ, such that

$$\sup_{\tau\in[0,1]}\left\|\mathrm{Hess}_{(1-\tau)\eta_s(t)+\tau\eta^\sigma_s(t)}\,\delta_{b_s(t)}\right\|_{g_F}\leqslant c_2,$$
$$\sup_{\tau\in[0,1]}\left\|T_{(1-\tau)\eta_s(t)+\tau\eta^\sigma_s(t)}\,\mathrm{Hess}\,\delta_{b_s(t)}\right\|_{\widehat{g_F}}\leqslant c_3,$$

where $\widehat{g_F}$ is the *Sasaki bundle metric* induced by the Riemannian metric g_F on $T\mathbf{R}^{2N}$. Recalling from Lemma 3.1 that $X_{b_s(t)}(\eta^\sigma_s(t))=\mathbf{0}$, since $\nabla_{g_F}\delta$ vanishes exactly on S_δ, applying Taylor's formula to $X_{b_s(t)}$ between $\eta_s(t)$ and $\eta^\sigma_s(t)$ shows that

$$\|X_{b_s(t)}(\eta_s(t))\|_{g_F}\leqslant c_2\|\Delta\eta_s(t)\|_{g_F}, \tag{17}$$

then letting $A_s(t)=\mathrm{Hess}_{\eta^\sigma_s(t)}\,\delta_{b_s(t)}$ and going one degree higher in the expansion gives

$$\begin{cases} X_{b_s(t)}(\eta_s(t)) = A_s(t)\Delta\eta_s(t)+R_s(t),\\ \|A_s(t)\|_{g_F}\leqslant c_2,\\ \|R_s(t)\|_{g_F}\leqslant \dfrac{c_3}{2}\|\Delta\eta_s(t)\|_{g_F}{}^2, \end{cases}, \tag{18}$$

Besides, since γ_s takes its values in K^σ, $(b_s(t),\eta^\sigma_s(t))$ stays in a compact subset of $S_\delta\setminus S_\delta{}^{\prec}$ depending only on δ, and using Proposition 2.1, $A_s(t)=\mathrm{Hess}_{\eta^\sigma_s(t)}\,\delta_{b_s(t)}$ is invertible. Thus, since δ is C^2, there exists $r>0$, depending only on δ, such that $r<\|A_s(t)^{-1}\|_{g_F}{}^{-1}$. Hence, since $1/r$ is, by construction, an upper bound for the spectral radius of $A_s(t)^{-1}$, in particular (bounding from above $\|A_s(t)^{-1}\cdot A_s(t)\Delta\eta_s(t)\|_{g_F}$), it then follows that

(19) $$\|A_s(t)\Delta\eta_s(t)\|_{g_F}\geqslant r\|\Delta\eta_s(t)\|_{g_F}.$$

Moreover, let us introduce $D_s(t)=\operatorname{Hess}_{\eta_s(t)}\delta_{b_s(t)}-A_s(t)=\operatorname{Hess}_{\eta_s(t)}\delta_{b_s(t)}-\operatorname{Hess}_{\eta_s^\sigma(t)}\delta_{b_s(t)}$, and then use mean value inequality to get

(20) $$\|D_s(t)\|_{g_F}\leqslant c_3\|\Delta\eta_s(t)\|_{g_F}.$$

With all these preliminaries at hand, start noticing that $\operatorname{Hess}_{\eta_s(t)}\delta_{b_s(t)}=A_s(t)+D_s(t)$, and then proceed as follows to derive the desired lower bound:

$$\begin{aligned}
g_F(\Delta\eta_s(t),\operatorname{Hess}_{\eta_s(t)}\delta_{b_s(t)}X_{b_s(t)}(\eta_s(t))) = \quad & g_F(\Delta\eta_s(t),A_s(t)X_{b_s(t)}(\eta_s(t)))\\
&+g_F(\Delta\eta_s(t),D_s(t)X_{b_s(t)}(\eta_s(t))),\\
\text{[using (17)]}\qquad = \quad & g_F(\Delta\eta_s(t),A_s(t)^2\Delta\eta_s(t))+g_F(\Delta\eta_s(t),A_s(t)R_s(t))\\
&+g_F(\Delta\eta_s(t),D_s(t)X_{b_s(t)}(\eta_s(t))),\\
\text{[since } \delta \text{ is } C^2, A_s(t) \text{ is symmetric]}\qquad = \quad & \|A_s(t)\Delta\eta_s(t)\|_{g_F}{}^2+g_F(\Delta\eta_s(t),A_s(t)R_s(t))\\
&+g_F(\Delta\eta_s(t),D_s(t)X_{b_s(t)}(\eta_s(t))),\\
\text{[using (19) and}\qquad \geqslant \quad & r^2\|\Delta\eta_s(t)\|_{g_F}{}^2-\|A_s(t)R_s(t)\|_{g_F}\|\Delta\eta_s(t)\|_{g_F}\\
\text{the Cauchy-Schwarz inequality]}\qquad & -\|D_s(t)X_{b_s(t)}(\eta_s(t))\|_{g_F}\|\Delta\eta_s(t)\|_{g_F},\\
\text{[using (17), (18) and (20)]}\qquad \geqslant \quad & r^2\|\Delta\eta_s(t)\|_{g_F}{}^2-2c_2c_3\|\Delta\eta_s(t)\|_{g_F}{}^3,\\
\text{[since } \|\Delta\eta_s(t)\|_{g_F}\leqslant d_0\text{]}\qquad \geqslant \quad & (r^2-2c_2c_3d_0)\|\Delta\eta_s(t)\|_{g_F}{}^2.
\end{aligned}$$

Therefore, provided $d_0<\frac{r^2}{2c_2c_3}$, let us define $\rho=\sqrt{r^2-2c_2c_3d_0}>0$ so that

(21) $$g_F(\Delta\eta_s(t),\operatorname{Hess}_{\eta_s(t)}\delta_{b_s(t)}\nabla_{g_F}\delta(\gamma_s(t)))\geqslant\rho^2\|\Delta\eta_s(t)\|_{g_F}{}^2=\rho^2d_s{}^2(t).$$

Finally, plugging equality (15), and using inequalities (16) and (21) yield

(22) $$g_F(\Delta\eta_s(t),\partial^2_{tt}\eta_s(t))\geqslant\rho^2d_s{}^2(t)-c_1sd_s(t),$$

ending the analysis for the first term of inequality (13).

- First, recall that $\eta_s^\sigma=\eta^\sigma\circ b_s$, then use the chain rule and equation (14) to show that

(23) $$\partial_t\eta_s^\sigma(t)=T_{b_s(t)}\eta^\sigma(\partial_tb_s(t))=-sT_{b_s(t)}\eta^\sigma(\nabla_{g_B}\delta(\gamma_s(t))).$$

In particular, notice from equation (23) that $s^{-1}\partial_t\eta_s^\sigma$ is a composition made from:
  - $T\eta^\sigma$ and $\nabla_{g_B}\delta$, which are continuously differentiable maps depending only on δ, since δ is C^2 and η^σ is entirely determined by S_δ; and
  - b_s and γ_s, which are continuously differentiable maps whose values and the values of their first-order derivatives stay in some compact subsets depending only on δ, as it is seen from equation (14), since δ is C^2, γ_s has range in K^σ and $s\leqslant 1$.

Therefore, applying the chain rule to $\partial_t\eta_s^\sigma$ shows there exists a uniform constant $c_4>0$, depending only on δ, such that $\|\partial^2_{tt}\eta_s^\sigma\|_{g_F}\leqslant c_4s$, then by the Cauchy-Schwarz inequality

(24) $$-g_F(\Delta\eta_s(t),\partial^2_{tt}\eta_s^\sigma(t))\geqslant-c_4sd_s(t),$$

ending the analysis for the second term of inequality (13).

Finally, letting $c = c_1 + c_4 > 0$, and using inequalities (22) and (24) in inequality (13) shows that

$$d_s'' \geqslant \rho^2 d_s(t) - cs. \tag{25}$$

Then, let us define $p_s : [t_s^-, t_s^+] \to \mathbf{R}$ by

$$p_s(t) = \max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right) \frac{\cosh(\rho(t - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2)} + \frac{cs}{\rho^2},$$

and let also $u_s = d_s - p_s : [t_s^-, t_s^+] \to \mathbf{R}$, so that by construction of p_s, $u_s(t_s^-) \leqslant 0$ and $u_s(t_s^+) \leqslant 0$. Let $t_M \in [t_s^-, t_s^+]$ such that $u_s \leqslant u_s(t_M)$ and assume, for the sake of contradiction, that $u_s(t_M) \geqslant 0$, then because u_s is negative on the boundary of $[t_s^-, t_s^+]$ and t_M is a maximum of u_s, $u_s''(t_M) \leqslant 0$. However, by construction of u_s, $u_s'' \geqslant 4\rho^2 u_s$ on (t_s^-, t_s^+) and $u_s''(t_M) \geqslant 0$, which is a contradiction. Therefore, it shows that $u_s \leqslant u_s(t_M) \leqslant 0$ holds on $[t_s^-, t_s^+]$, and equivalently for all $t \in [t_s^-, t_s^+]$, the following inequality holds true

$$d_s(t) \leqslant \max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right) \frac{\cosh(\rho(t - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2)} + \frac{cs}{\rho^2}, \tag{26}$$

yielding the desired upper bound for the zeroth-order derivative.

In order to derive the estimate for the first-order derivative, notice that since δ and η^σ are C^2, using that γ_s takes its values in K^σ and equation (23), there exists a uniform constant $c_5 > 0$, depending only on δ (recall that η^σ is entirely determined by S_δ), such that

$$\|\partial_t \eta_s^\sigma(t)\|_{g_F} \leqslant c_5 s. \tag{27}$$

Then, use the definition of $\Delta\eta_s$ and the triangle inequality to show that

$$\begin{aligned}
\|\partial_t \Delta\eta_s(t)\|_{g_F} &= \|\nabla_{g_F} \delta(\gamma_s(t))\|_{g_F} + \|\partial_t \eta_s^\sigma(t)\|_{g_F}, \\
\text{[using (14)]} \quad &= \|X_{b_s(t)}(\eta_s(t))\|_{g_F} + \|\partial_t \eta_s^\sigma(t)\|_{g_F}, \\
\text{[using (17) and (27)]} \quad &\leqslant c_2 \|\Delta\eta_s(t)\|_{g_F} + c_5 s, \\
\text{[using (26)]} \quad &\leqslant c_2 p_s(t) + c_5 s.
\end{aligned}$$

Therefore, for all $t \in [t_s^-, t_s^+]$, the following inequality holds true

$$\|\partial_t \Delta\eta_s(t)\|_{g_F} \leqslant c_2 \max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right) \frac{\cosh(\rho(t - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2)} + \left(\frac{c_2 c}{\rho^2} + c_5\right) s,$$

yielding the desired upper bound for the first-order derivative.

Thus, letting the uniform constant

$$C = \max\left(1, c_2, \frac{c}{\rho^2}, \frac{c_2 c}{\rho^2} + c_5\right) > 0,$$

depending only on δ, yields the result. ■

Remark. The hyperbolic cosine part of the estimate from Proposition 4.2 is not surprising as it was already found analysing similar cases in [Bou02, Section 3.2] and [BO09a, Proposition A.3]. It is however unusual to recover a linear term in the degenerating process parameter (s here), but examining the proof of Proposition 4.2, it seems that it is unavoidable as soon as the front projection of the Legendrian submanifold has a non-horizontal strand.

Remark. To enlighten the statement of Proposition 4.2 and the computations carried in its proof, let us first consider the toy model given by exponential decay towards critical points for gradient flow of Morse functions:

- (M, g) is a Riemannian manifold;
- $f : M \to \mathbf{R}$ is a Morse function; and
- $\gamma : R \to M$ is a gradient trajectory of f;

then for $|t| \geqslant T$ large enough, γ stays in a Morse coordinate chart around a critical point $p_\pm$ of f. In this chart, the gradient flow f is linear, thus showing there exist $C, \rho_\pm > 0$ such that

$$\forall |t| \geqslant T, \|\gamma(t) - p_\pm\| \leqslant Ce^{-\rho_\pm |t|},$$

where the decay rates $\rho_\pm$ are controlled by the smallest absolute value of eigenvalues of $\mathrm{Hess}_{p_\pm} f$. Moreover, exponential decay can also be obtained without resorting to Morse coordinate charts. First, linearise the gradient flow of f around $p_\pm$ to show that

$$\forall |t| \geqslant T, \partial_{tt}^2 \|\gamma(t) - p_\pm\|^2 \geqslant 4{\rho_\pm}^2 \|\gamma(t) - p_\pm\|^2,$$

and then mimick the end of the proof of Proposition 4.2, but now with semi-infinite intervals, and thus the recovered upper bounds are negative exponentials rather than hyperbolic cosines. Going back to Proposition 4.2, inequality (25) is derived generalizing from the above toy model and noticing that:

- δ behaves like a Morse function in a fibre above $S_\delta \setminus {S_\delta}^{\prec}$; and
- gradient flow lines of δ move at speed s in the base direction.

It is worth mentioning that the notorious result from [HWZ96], stating the exponential decay of pseudo-holomorphic curves towards degenerate Reeb orbits has been obtained applying a strategy similar to the one developed in the proof of Proposition 4.2.

Example 4.1. Let us consider an example to check that the estimate in Proposition 4.2 is sharp. For that purpose, let us first define $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by $f(b, \eta) = \frac{\eta^2}{2} - b\eta$, then f is a generating family for the Legendrian submanifold of $(J^1\mathbf{R}, \xi_{\mathbf{R}})$ whose front projection is the diagonal of $\mathbf{R} \times \mathbf{R}$. Moreover, the fast-slow system describing the Henry–Rutherford limiting process associated to f is given in components by:

$$\begin{cases} b_s(t) = s\eta_s(t), \\ \eta_s(t) = b_s(t) - \eta_s(t). \end{cases}$$

Then, since this system of differential equations is linear, let us compute its eigenvalues λ_s^h and λ_s^v, their associated eigenvectors e_s^h and e_s^v, and then let us also determine their asymptotic behaviour in the limit $s \to 0$:

$$\begin{aligned} \lambda_s^h &= \frac{-1 + \sqrt{1 + 4s}}{2} \underset{s \to 0}{\sim} s, & e_s^h &= \begin{pmatrix} \dfrac{1 + \sqrt{1 + 4s}}{2} \\ 1 \end{pmatrix} \underset{s \to 0}{\sim} \begin{pmatrix} 1 + s \\ s \end{pmatrix}, \\ \lambda_s^v &= \frac{1 + \sqrt{1 + 4s}}{2} \underset{s \to 0}{\sim} -(1 + s), & e_s^v &= \begin{pmatrix} \dfrac{1 - \sqrt{1 + 4s}}{2} \\ 1 \end{pmatrix} \underset{s \to 0}{\sim} \begin{pmatrix} s \\ 1 \end{pmatrix}, \end{aligned}.$$

From these computations, there exist uniform constants A and B, independent of s, such that

$$\Delta\eta_s(t) = \eta_s(t) - b_s(t) \underset{s\to 0}{\sim} A(1-s)e^{-(1+s)t} + Bse^{st},$$

where the first term is responsible for the hyperbolic cosine in the estimate from Proposition 4.2 and the second one for the linear term in s.

Let us now focus on the fibrewise dynamics near a singularity of the maximal invariant subset. In particular, it is shown that

- transversal directions to the singularity behave as nonsingular fibre directions; and
- the tangencial direction to the singularity is controlled by the transversal directions;

thus providing an analog of Proposition 4.2 for singular fibre directions (Proposition 4.3).

First, every point $\sigma^{\prec}$ of $S_\delta{}^{\prec}$ has an open neighbourhood $V^{\prec}$ in $S_\delta{}^{\prec}$ such that $\pi_{B_{|V^{\prec}}}$ is an embedding. Indeed, since Λ is at most $\Sigma^{1,0}$-singular, by Proposition 2.2, π_B restricts to an immersion on $S_\delta{}^{\prec}$. Besides, since $S_\delta{}^{\prec}$ is a submanifold of $B\times\mathbf{R}^{2N}$, it is locally closed in $B\times\mathbf{R}^{2N}$, and in particular, there must exist an open neighbourhood $U^{\prec}$ of $\sigma^{\prec}$ in $B\times\mathbf{R}^{2N}$ such that $U^{\prec}\cap S_\delta{}^{\prec}$ is closed in $U^{\prec}$. Moreover, by shrinking $U^{\prec}$, it can also be assumed that $U^{\prec} = U_B^{\prec}\times U_F^{\prec}$ and $U^{\prec}\cap S_\delta{}^{\prec}\subset V^{\prec}$. Therefore, by construction of $U^{\prec}$, $\pi_B\colon B\times\mathbf{R}^{2N}\to B$ restricts to an embedding on $U^{\prec}\cap S_\delta{}^{\prec}$, and in conclusion, there exists a unique smooth map $\eta^{\prec}\colon U_B^{\prec}\to U_F^{\prec}$ such that

$$S_\delta{}^{\prec}\cap U^{\prec} = \left\{(b,\eta^{\sigma}(b));\, b\in U_B^{\prec}\right\} = \Gamma_{\eta^{\prec}}.$$

Then, let also $K^{\prec} = K_B^{\prec}\times K_F^{\prec}$ be a compact subset of $U^{\prec}$.

For $(b,\eta)\in S_\delta{}^{\prec}$, let us define $E^{//}_{(b,\eta)} = \ker \operatorname{Hess}_\eta \delta_b$ and let $E^{\perp}_{(b,\eta)}$ be its orthogonal in $\mathbf{R}^{2N}$ for g_F. Since δ is C^2 and Λ is at most $\Sigma^{1,0}$-singular, using Proposition 2.2 shows that

- $E^{//}$ and $E^{\perp}$ are vector bundles over $S_\delta{}^{\prec}$;
- $E^{//}$ and $E^{\perp}$ have orthogonal supplementary fibres in $\mathbf{R}^{2N}$ for g_F; and
- $E^{//}$ has rank one and $E^{\perp}$ has rank $2N-1$;

In particular, any section ν can be uniquely written as $\nu = \nu^{//}+\nu^{\perp}$, where $\nu^{//}\in E^{//}$ and $\nu^{\perp}\in E^{\perp}$. Finally, notice that by construction, for all $(b,\eta)\in S_\delta{}^{\prec}$, the following properties hold true:

- $\operatorname{Hess}_\eta \delta_b\colon \mathbf{R}^{2N}\to\mathbf{R}^{2N}$ stabilises $E^{//}_{(b,\eta)}$ and $E^{\perp}_{(b,\eta)}$; and
- $\operatorname{Hess}_\eta \delta_b\colon \mathbf{R}^{2N}\to\mathbf{R}^{2N}$ restricts to a linear invertible map between $E^{\perp}_{(b,\eta)}$ and itself.

With this preliminaries at hand, everything is now settled to state and prove Proposition 4.3.

Proposition 4.3. *There exist uniform constants $d_0>0$, $C>0$ and $\rho>0$, depending only on δ, such that for all $s\in(0,1]$, for all $\gamma_s\in\widehat{\mathcal{M}}(c_-,c_+;\delta,g_s)$ and all interval $[t_s^-,t_s^+]\subset\mathbf{R}$ such that*

- *the range of $\gamma_s=(b_s,\eta_s)$ restricted to $[t_s^-,t_s^+]$ is contained in $K^{\prec}$; and*
- *$\|\eta_s-\eta_s^{\prec}\|_{g_F}\leqslant d_0$ on $[t_s^-,t_s^+]$, where $\gamma_s=(b_s,\eta_s)$ and $\eta_s^{\prec}=\eta^{\prec}\circ b_s$;*

then for all $t\in[t_s^-,t_s^+]$, $\Delta\eta_s(t)=\eta_s(t)-\eta_s^{\prec}(t)$ and its first-order derivative at t are bounded by

$$C\left(\max\left(\left\|\Delta\eta_s{}^{\perp}(t_s^-)\right\|_{g_F},\left\|\Delta\eta_s{}^{\perp}(t_s^+)\right\|_{g_F}\right)+\max\left(\left\|\Delta\eta_s{}^{//}(t_s^-)\right\|_{g_F},\left\|\Delta\eta_s{}^{//}(t_s^+)\right\|_{g_F}\right)+s^{1/2}\right),$$

for the norms associated to g_F on $\mathbf{R}^{2N}$ and $T\mathbf{R}^{2N}$, respectively.

Proof. Let $t \in [t_s^-, t_s^+]$, then by assumption, $\gamma_s(t) \in S_\delta{}^{\prec}$, so that by construction, $\mathrm{Hess}_{\eta_s(t)} \delta_{b_s(t)}$ restricts to an invertible linear map between $E_{\gamma_s(t)}{}^{\perp}$ and itself, and moreover $\Delta\eta_s{}^{\perp}(t) \in E_{\gamma_s(t)}{}^{\perp}$. Therefore, mimicking the proof of Proposition 4.2, there exist uniform constants $d_0 > 0$, $c_1 > 0$ and $\rho > 0$, depending only on δ, such that if $\|\Delta\eta_s{}^{\perp}\|_{g_F}$ is assumed to be less than d_0 on $[t_s^-, t_s^+]$, then for all $t \in [t_s^-, t_s^+]$, $\Delta\eta_s{}^{\perp}(t)$ and its first-order derivative at t are bounded by

$$c_1\left(\max\left(\left\|\Delta\eta_s{}^{\perp}(t_s^-)\right\|_{g_F}, \left\|\Delta\eta_s{}^{\perp}(t_s^+)\right\|_{g_F}\right)\frac{\cosh(\rho(t-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2)} + s\right), \tag{28}$$

Let us define $\sigma_s : [t_s^-, t_s^+] \to S_\delta{}^{\prec} \subset B \times \mathbf{R}^{2N}$ by $\sigma_s(t) = (b_s(t), \eta_s^{\prec}(t))$, where $\gamma_s(t) = (b_s(t), \eta_s(t))$. Moreover, using Lemma 3.1, the differential equation γ_s satifies writes in components as follows:

$$\begin{cases} \partial_t b_s(t) = -s\nabla_{g_B}\delta(\gamma_s(t)), \\ \partial_t \eta_s(t) = -\nabla_{g_F}\delta(\gamma_s(t)), \end{cases}, \tag{29}$$

and it now remains to understand the behaviour of $\Delta\eta_s{}^{/\!/}$ from equation (29) and inequality (28). For that purpose, recall that a fibre bundle defined over a contractible base is globally trivialisable, but since $E^{/\!/}$ and $E^{\perp}$ are vector bundles of rank 1 and $2N-1$, respectively, and σ_s is contractible, there exist global trivialisations of $\sigma_s{}^*E^{/\!/}$ and $\sigma_s{}^*E^{\perp}$, and it is possible to define

- a unitary vector $e_1 = e_{/\!/}$ for the norm associated to g_F spanning $\sigma_s{}^*E^{/\!/}$; and
- a orthonormal family of vectors $(e_2, \dots, e_{2N})$ for g_F spanning $\sigma_s{}^*E^{\perp}$.

Besides, since $E^{/\!/}$ and $E^{\perp}$ have orthogonal fibres, $(e_1, \dots, e_{2N})$ is an orthonormal basis of $(\mathbf{R}^{2N}, g_F)$. Therefore, in these global trivialisations, the orthogonal projections onto $\sigma_s{}^*E^{/\!/}$, and onto $\sigma_s{}^*E^{\perp}$, are respectively given by

$$\begin{aligned} v^{/\!/} &= g_F(v, e_1)e_1 = g_F(v, e_{/\!/})e_{/\!/}, \\ v^{\perp} &= \sum_{k=2}^{2N} g_F(v, e_k)e_k. \end{aligned}$$

In particular, there exist smooth coordinates $x_s^1 = y_s : [t_s^-, t_s^+] \to \mathbf{R}$ and $x_s^2, \dots, x_s^{2N} : [t_s^-, t_s^+] \to \mathbf{R}$ such that for all $t \in [t_s^-, t_s^+]$, the following decompositions hold true

$$\begin{aligned} \Delta\eta_s{}^{/\!/}(t) &= x_s^1(t)e_1 = y_s(t)e_{/\!/}, \\ \Delta\eta_s{}^{\perp}(t) &= \sum_{k=2}^{2N} x_s^k(t)e_k. \end{aligned}$$

From now on, let $t \in (t_s^-, t_s^+)$, then using the definition of $\Delta\eta_s$ and the linearity of ∂_t shows that

$$\begin{aligned} \partial_t\Delta\eta_s(t) &= \partial_t\eta_s(t) - \partial_t\eta_s(t), \\ \text{[using the chain rule]} \quad &= \partial_t\eta_s(t) - T_{b_s(t)}\eta^{\prec}(\partial_t b_s(t)), \\ \text{[using equation (29)]} \quad &= -\nabla_{g_B}\delta(\gamma_s(t)) + sT_{b_s(t)}\eta^{\prec}(\nabla_{g_B}\delta(\gamma_s(t)). \end{aligned}$$

Therefore, since $\partial_t\Delta\eta_s{}^{/\!/}(t) = y_s{}'(t)e_{/\!/}$, orthogonally projecting the above equation onto $E^{/\!/}_{\gamma_s(t)}$, and then identifying the resulting components along $e_{/\!/}$ yields

$$y_s'(t) = -g_F(\nabla_{g_F}\delta(\gamma_s(t)), e_{/\!/}) + sg_F(T_{b_s(t)}\eta^{\prec}(\nabla_{g_B}\delta(\gamma_s(t))), e_{/\!/}), \tag{30}$$

and it remains to estimate separately both terms on the right-hand side of this equation.

- To deal with the first of these terms, define $\delta_{b_s(t)} = \delta_{|\{b_s(t)\}\times\mathbf{R}^{2N}}$ and $X_{b_s(t)} = -\nabla_{g_F}\delta_{b_s(t)}$, then observe that in the coordinates $(e_1,\dots,e_n)$, $X_{b_s(t)}$ is given by

$$X_{b_s(t)} = \left(\frac{\partial\delta_{b_s(t)}}{\partial e_{//}}, \frac{\partial\delta_{b_s(t)}}{\partial e_2}, \dots, \frac{\partial\delta_{b_s(t)}}{\partial e_{2N}}\right).$$

Recalling from **Lemma 3.1** that $X_{b_s(t)}(\eta_s^{\prec}(t)) = \mathbf{0}$, since $\nabla_{g_F}\delta$ vanishes exactly on S_δ, according to the quadratic Taylor's formula applied to $X_{b_s(t)}$ between $\eta_s(t)$ and $\eta_s^{\prec}(t)$, there exists $\tau\in[0,1]$ such that if $\eta_s^*(t) = (1-\tau)\eta_s(t)+\tau\eta_s^{\prec}(t)$ and

$$R_s(t) = -\frac{1}{6}\sum_{i=1}^{2N}\left(\sum_{j,k,\ell=1}^{2N}\frac{\partial^4\delta_{b_s(t)}}{\partial e_i\partial e_j\partial e_k\partial e_\ell}(\eta_s^*(t))x_s^j(t)x_s^k(t)x_s^\ell(t)\right)e_i,$$

then the following equality holds true

$$X_{b_s(t)}(\eta_s(t)) = \mathrm{Hess}_{\eta_s^{\prec}(t)}\,\delta_{b_s(t)}\cdot\Delta\eta_s(t) - \frac{1}{2}\sum_{i=1}^{2N}\left(\sum_{j,k=1}^{2N}\frac{\partial^3\delta_{b_s(t)}}{\partial e_i\partial e_j\partial e_k}(\eta_s^{\prec}(t))x_s^j(t)x_s^k(t)\right)e_i + R_s(t).$$

Besides, since by construction, $\mathrm{Hess}_{\eta_s^{\prec}(t)}\,\delta_{b_s(t)}\cdot\Delta\eta_s(t) = \mathrm{Hess}_{\eta_s^{\prec}(t)}\,\delta_{b_s(t)}\cdot\Delta\eta_s{}^{\perp}(t)\in E^{\perp}_{\gamma_s(t)}$, orthogonally projecting this equation onto $E^{//}_{\gamma_s(t)}$, only the terms with $i=1$ remain, and then identifying the resulting components along $e_{//}$ yields

$$\begin{aligned} g_F(X_{b_s(t)}(\eta_s(t)), e_{//}) = &-\frac{1}{2}\sum_{j_1,k_1=1}^{2N}\frac{\partial^3\delta_{b_s(t)}}{\partial e_{//}\partial e_{j_1}\partial e_{k_1}}(\eta_s^{\prec}(t))x_s^{j_1}(t)x_s^{k_1}(t)\\ &-\frac{1}{6}\sum_{j_2,k_2,\ell_2=1}^{2N}\frac{\partial^4\delta_{b_s(t)}}{\partial e_{//}\partial e_{j_2}\partial e_{k_2}\partial e_{\ell_2}}(\eta_s^*(t))x_s^{j_2}(t)x_s^{k_2}(t)x_s^{\ell_2}(t).\end{aligned}$$

The right-hand side of the above equation is a polynomial in $y_s(t)$ of degree at most 3, then since the partial derivatives of any order of $\delta_{b_s(t)}$ are symmetric (since δ is smooth), its coefficients can be extracted as follows:

- **At least quadratic coefficient.** Set $\#\{j_1,k_1=1\}=2\wedge\#\{j_2,k_2,\ell_2=1\}\geqslant 2$, then

$$a_s(t) = -\frac{1}{2}\frac{\partial^3\delta_{b_s(t)}}{\partial e_{//}{}^3}(\eta_s^{\prec}(t)) - \frac{1}{6}\frac{\partial^4\delta_{b_s(t)}}{\partial e_{//}{}^4}(\eta_s^*(t))y_s(t) - \frac{1}{2}\sum_{j=2}^{2N}\frac{\partial^4\delta_{b_s(t)}}{\partial e_{//}{}^3\partial e_j}(\eta_s^*(t))x_s^j(t).$$

- **Linear coefficient.** Set $\#\{j_1,k_1=1\}=1\wedge\#\{j_2,k_2,\ell_2=1\}=1$, then:

$$b_s(t) = -\sum_{j=2}^{2N}\frac{\partial^3\delta_{b_s(t)}}{\partial e_{//}{}^2\partial e_j}(\eta_s^{\prec}(t))x_s^j(t) - \frac{1}{2}\sum_{j,k=2}^{2N}\frac{\partial^4\delta_{b_s(t)}}{\partial e_{//}\partial e_j\partial e_k}(\eta_s^*(t))x_s^j(t)x_s^k(t).$$

- **Constant coefficient.** Set $\#\{j_1,k_1=1\}=0\wedge\#\{j_2,k_2,\ell_2=1\}=0$, then:

$$c_s(t) = -\frac{1}{2}\sum_{j,k=2}^{2N}\frac{\partial^3\delta_{b_s(t)}}{\partial e_j\partial e_k}(\eta_s^{\prec}(t))x_s^j(t)x_s^k(t) - \frac{1}{6}\sum_{j,k,\ell=2}^{2N}\frac{\partial^4\delta_{b_s(t)}}{\partial e_{//}\partial e_j\partial e_k\partial e_\ell}(\eta_s^*(t))x_s^j(t)x_s^k(t)x_s^\ell(t).$$

In conclusion, by construction of the smooth coefficients $a_s, b_s, c_s\colon (t_s^-, t_s^+)\to\mathbf{R}$ above, and by definition of $X_{b_s(t)}$, the following equality holds true

$$-g_F(\nabla_{g_F}\delta(\gamma_s(t)), e_{//}) = a_s(t)y_s(t)^2 + b_s(t)y_s(t) + c_s(t), \tag{31}$$

ending the analysis for the first term on the right-hand side of equality **(30)**.

- To deal with the second one of this term, start noticing that since δ and $\eta^{\prec}$ are both C^1, then using that γ_s has range in $K^{\prec}$ shows that there exists a uniform constant $C_3 > 0$, depending only on δ ($\eta^{\prec}$ is determined by S_δ), such that $\|T_{b_s(t)}\eta^{\prec}(\nabla_{g_B}\delta(\gamma_s(t)))\|_{g_F} \leqslant C_3$. Therefore, since $\|e_{//}\|_{g_F} = 1$, using the Cauchy-Schwarz inequality yields

$$\left|s g_F(T_{b_s(t)}\eta^{\prec}(\nabla_{g_B}\delta(\gamma_s(t))), e_{//})\right| \leqslant C_3 s, \tag{32}$$

ending the analysis for the second term on the right-hand side of equality (30).

In conclusion, first using equations (30) and (31) to derive an expression for $y_s{}'(t) - a_s(t)y_s{}^2(t)$, and in a second time applying the triangle inequality and inequality (32) on the resulting equality shows that

$$\left|y_s{}'(t) - a_s(t)y_s{}^2(t)\right| \leqslant |b_s(t)| \cdot |y_s(t)| + |c_s(t)| + C_3 s. \tag{33}$$

In order to estimate further the right-hand side of inequality (33), use that η_s has range in $K_F^{\prec}$ and the compactness of $S_\delta{}^{\prec}$ (the generating families considered are linear-at-infinity) to show that for all $\tau \in [0,1]$, $(1-\tau)\eta_s(t) + \tau\eta_s^{\prec}(t)$ stays in a uniform compact set, depending only on δ. Therefore, since δ is C^5, the mean value inequality provides uniform constants $c_2 > 0$ and $c_3 > 0$, depending only on δ, such that for all $i, j, k, \ell \in \{1, \dots, 2N\}$:

$$\sup_{\tau\in[0,1]} \left| \frac{\partial^3 \delta_{b_s(t)}}{\partial e_i \partial e_j \partial e_k}((1-\tau)\eta_s(t) + \tau\eta_s^{\prec}(t)) \right| \leqslant c_2,$$

$$\sup_{\tau\in[0,1]} \left| \frac{\partial^4 \delta_{b_s(t)}}{\partial e_i \partial e_j \partial e_k \partial e_\ell}((1-\tau)\eta_s(t) + \tau\eta_s^{\prec}(t)) \right| \leqslant c_3.$$

Besides, recall that

- from the Cauchy-Schwarz inequality, for all $k \in \{2, \dots, 2N\}$, $|x_s^k(t)| \leqslant \|\Delta\eta_s{}^{\perp}(t)\|_{g_F}$;
- from the triangle inequality, $\|\Delta\eta_s{}^{\perp}(t)\|_{g_F} \leqslant \|\Delta\eta_s(t)\|_{g_F}$; and
- $\|\Delta\eta_s(t)\|_{g_F}$ is assumed to be less than d_0;

thus defining uniform constants $C_1 = (2N-1)c_2 + \frac{(2N-1)^2 c_3 d_0}{2} > 0$ and $C_2 = \frac{(2N-1)^2 c_2}{2} + \frac{(2N-1)^3 c_3 d_0}{6} > 0$, and then applying the triangle inequality shows that

$$|b_s(t)| \leqslant (2N-1)c_2\|\Delta\eta_s{}^{\perp}(t)\|_{g_F} + \frac{(2N-1)^2 c_3}{2}\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2 \leqslant C_1\|\Delta\eta_s{}^{\perp}(t)\|_{g_F},$$

$$|c_s(t)| \leqslant \frac{(2N-1)^2 c_2}{2}\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2 + \frac{(2N-1)^3 c_3}{6}\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2 \leqslant C_2\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2.$$

In conclusion, plugging the above inequalities in inequality (33) yields

$$\left|y_s{}'(t) - a_s(t)y_s{}^2(t)\right| \leqslant C_1\|\Delta\eta_s{}^{\perp}(t)\|_{g_F} \cdot |y_s(t)| + C_2\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2 + C_3 s, \tag{34}$$

and then the desired zeroth-order estimate will be straighforwardly derived from inequality (34), but before, it remains to derives

- a nonzero uniform lower bound, depending only on δ, for $|a_s|$; and
- a uniform upper bound, depending only on δ, for $|a_s|$.

For that purpose, recall that $\|e_{//}\|_{g_F} = 1$, so that the previous observations can be enhanced as

- $|y_s(t)| = \|\Delta\eta_s{}^{//}(t)\|_{g_F}$ and for all $k \in \{2, \dots, 2N\}$, $|x_s^k(t)| \leqslant \|\Delta\eta_s{}^{\perp}(t)\|_{g_F}$;
- $\|\Delta\eta_s{}^{//}(t)\|_{g_F} \leqslant \|\Delta\eta_s(t)\|_{g_F}$ and $\|\Delta\eta_s{}^{\perp}(t)\|_{g_F} \leqslant \|\Delta\eta_s(t)\|_{g_F}$; and
- $\|\Delta\eta_s(t)\|_{g_F}$ is assumed to be less than d_0;

In particular, defining the uniform constant $C_4 = \frac{c_2}{2} + \frac{(3N-1)c_3 d_0}{3} > 0$, which depends only on δ, and then using the triangle inequality shows that

$$|a_s(t)| \leqslant \frac{c_2}{2} + \frac{c_3}{6}|y_s(t)| + \frac{(2N-1)c_3}{2}\|\Delta\eta_s{}^{\perp}(t)\|_{g_F} \leqslant C_4.$$

Besides, also notice that for all $(b,\eta) \in S_\delta{}^{\prec}$, the partial derivative $\frac{\partial^3 \delta_b}{\partial e_{//}{}^3}(\eta)$ is necessarily nonzero, since using **Propositions 2.1** and **2.2** ensures that $\frac{\partial^2 \delta_b}{\partial e_{//}{}^2}(\eta) = 0$ is a transversally cut-out equation. Moreover, since δ is C^2 and $S_\delta{}^{\prec}$ is compact (since the generating families are linear-at-infinity), it is possible to define a uniform constant $\varepsilon > 0$, depending only on δ, by

$$\varepsilon = \frac{1}{4} \inf_{(b,\eta)\in S_\delta{}^{\prec}} \left| \frac{\partial^3 \delta_b}{\partial e_{//}{}^3}(\eta) \right| > 0.$$

Therefore, using the triangle inequality shows that

$$\begin{aligned}
\frac{1}{2}\left|\frac{\partial^3 \delta_{b_s(t)}}{\partial e_{//}{}^3}(\eta_s^{\prec}(t))\right| &= \left| a_s(t) + \frac{1}{6}\frac{\partial^4 \delta_{b_s(t)}}{\partial e_{//}{}^4}(\eta_s^{*}(t)) y_s(t) + \frac{1}{2}\sum_{j=2}^{2N} \frac{\partial^4 \delta_{b_s(t)}}{\partial e_{//}{}^3 \partial e_j}(\eta_s^{*}(t)) x_s^j(t) \right|, \\
&\leqslant |a_s(t)| + \frac{c_3}{2}|y_s(t)| + \frac{(2N-1)c_3}{6}\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}, \\
&\leqslant |a_s(t)| + \frac{(2N+1)c_3 d_0}{3},
\end{aligned}$$

and provided that $d_0 < \frac{3\varepsilon}{2(2N+1)c_3}$ (which is from now on assumed), then

$$|a_s(t)| \geqslant \frac{1}{2}\left|\frac{\partial^3 \delta_{b_s(t)}}{\partial e_{//}{}^3}(\eta_s^{\prec}(t))\right| - \frac{2(2N+1)c_3 d_0}{3} \geqslant \varepsilon.$$

It is now time to conclude by letting $t_M \in [t_s^-, t_s^+]$ such that for all $t \in [t_s^-, t_s^+]$, $|y_s(t)| \leqslant |y_s(t_M)|$, and then by discussing whether t_M belongs to the interior of the interval $[t_s^-, t_s^+]$ or not

- Assume that $t_M \in (t_s^-, t_s^+)$, then $y_s{}'(t_M) = 0$ and thus inequality (34) becomes
$$\left|a_s(t_M) y_s{}^2(t_M)\right| \leqslant C_1 \|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F} \cdot |y_s(t_M)| + C_2 \|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F}{}^2 + C_3 s,$$
and this inequality is plotted in **Figure 22** below.

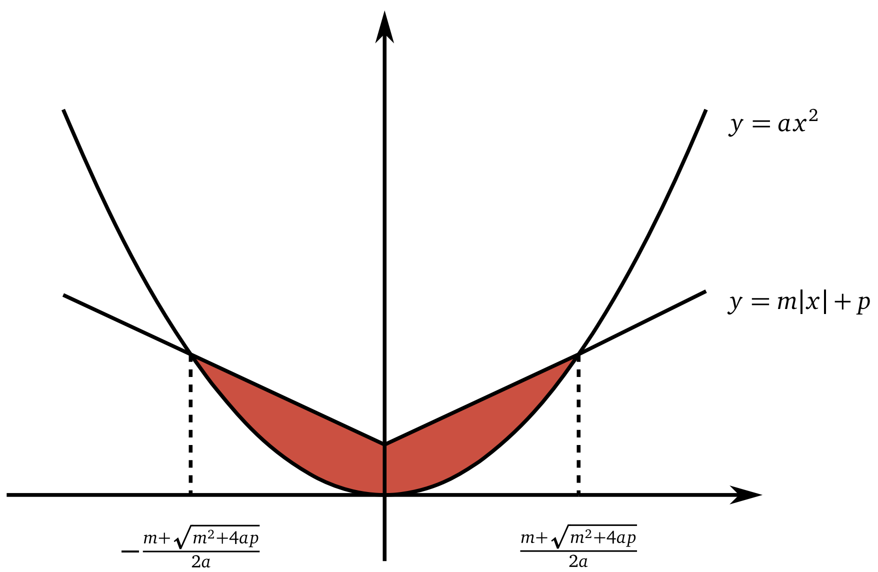


Figure 22. Solving the inequality $a|x|^2 \leqslant m|x| + p$, where $a > 0$, $m > 0$ and $p > 0$.

Then, discussing on the sign of $y_s(t_M)$, and solving with the quadratic formula, yields

$$|y_s(t_M)| \leqslant \frac{C_1\|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F} + \sqrt{C_1{}^2\|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F}{}^2 + 4a_s(t_M)(C_2\|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F}{}^2 + C_3 s)}}{2|a_s(t_M)|},$$
$$\leqslant \frac{(C_1+\sqrt{C_2C_4})}{\varepsilon}\|\Delta\eta_s{}^{\perp}(t_M)\|_{g_F} + \frac{\sqrt{C_3}}{2\varepsilon}s^{1/2},$$

where the last inequality follows from $(x+y)^{1/2} \leqslant x^{1/2}+y^{1/2}$, for all $x \geqslant 0$ and $y \geqslant 0$. In particular, plugging inequality (28) in the inequality above shows that

$$|y_s(t_M)| \leqslant \frac{c_1(C_1+\sqrt{C_2C_4})}{\varepsilon}\max(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}) + \left(c_1 + \frac{\sqrt{C_3}}{2\varepsilon}\right)s^{1/2},$$

since $s \in (0,1]$ and the quotient of hyperbolic cosines in inequality (28) is less than 1.

- Assume that $t_M \in \{t_s^-, t_s^+\}$, then $|y_s(t_M)| \leqslant \max(\|\Delta\eta_s{}^{//}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{//}(t_s^-))$.

Therefore, this discussion shows that for all $t \in [t_s^-, t_s^+]$, $\|\Delta\eta_s{}^{//}(t)\|_{g_F}$ is bounded by

$$\frac{c_1(C_1+\sqrt{C_2C_4})}{\varepsilon}\max(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}) + \max(\|\Delta\eta_s{}^{//}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{//}(t_s^-)) + \left(c_1 + \frac{\sqrt{C_3}}{2\varepsilon}\right)s^{1/2}. \tag{35}$$

To conclude, let us define uniform constants $C_5 > 0$ and $C_6 > 0$, depending only on δ, by

$$C_5 = c_1\left(1 + \frac{C_1+\sqrt{C_2C_4}}{\varepsilon}\right), \quad C_6 = 2c_1 + \frac{\sqrt{C_3}}{2\varepsilon},$$

therefore, using the triangle inequality on $\|\Delta\eta_s\|_{g_F}$, and then plugging inequalities (28) and (35), shows that for all $t \in [t_s^-, t_s^+]$, $\|\Delta\eta_s(t)\|_{g_F}$ (and in particular, $\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}$ and $\|\Delta\eta_s{}^{//}(t)\|_{g_F}$ also) is bounded by

$$C_5\max\left(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}\right) + \max\left(\|\Delta\eta_s{}^{//}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{//}(t_s^-)\|_{g_F}\right) + C_6 s^{1/2}, \tag{36}$$

since $s \in (0,1]$ and the quotient of hyperbolic cosines in inequality (28) is less than 1 on $[t_s^-, t_s^+]$. Yielding the desired estimate for the zeroth-order derivative.

For the first-order derivative estimate, use the triangle inequality and inequality (34) to show that

$$\left|y_s{}'(t)\right| \leqslant C_1\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}\cdot|y_s(t)| + C_2\|\Delta\eta_s{}^{\perp}(t)\|_{g_F}{}^2 + C_3 s + C_4\left|y_s{}^2(t)\right|,$$

then plugging $s \leqslant 1$, $\|\Delta\eta_s(t)\|_{g_F} \leqslant d_0$ and inequality (36) above yields

$$\|\partial_t\Delta\eta_s{}^{//}(t)\|_{g_F} \leqslant C_7\max(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}) + C_8\max(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}) + C_9 s^{1/2}, \tag{37}$$

where $C_7 = (C_1+C_2+C_4)C_5d_0 > 0$, $C_8 = (C_1+C_2+C_4)d_0 > 0$ and $C_9 = (C_1+C_2+C_4)C_6d_0 + C_3 > 0$. Hence, using the triangle inequality on $\|\partial_t\Delta\eta_s\|_{g_F}$, and then plugging inequalities (28) and (37), ensures that for all $t \in (t_s^-, t_s^+)$, $\|\partial_t\Delta\eta_s(t)\|_{g_F}$ is bounded by

$$(c_1+C_7)\max(\|\Delta\eta_s{}^{\perp}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{\perp}(t_s^+)\|_{g_F}) + C_8\max(\|\Delta\eta_s{}^{//}(t_s^-)\|_{g_F}, \|\Delta\eta_s{}^{//}(t_s^+)\|_{g_F}) + (c_1+C_9)s^{1/2},$$

since $s \in (0,1]$ and the quotient of hyperbolic cosines in inequality (28) is less than 1 on $[t_s^-, t_s^+]$. Yielding the desired estimate for the first-order derivative.

Conclude defining a uniform constant $C > 0$, depending only on δ, by

$$C = \max\left(1, C_5, C_6, c_1 + C_7, c_8, c_1 + C_9\right).$$

Despite being derived in some coordinate chart, the above estimates still yield **Proposition 4.3**, since using the chain rule and the compactness of $K^{\prec}$ ensures that working in coordinate chart only affects the exact value of C. ■

Applying a similar strategy to the one used to prove **Proposition 4.3**, but now substituting

- base directions to the tangencial fibre direction of the degeneracy; and
- fibre directions to transverse fibre directions of degeneracy;

shows that understanding the fibrewise dynamics of the negative gradient flow of the difference function is enough to tame the dynamics in the basis.

As earlier, let us first deals with the base dynamics away from the singularities.

Proposition 4.4. *Let us define $\ell\colon U_B^\sigma \to \mathbf{R}$ by $\ell(b) = \delta(b, \eta^\sigma(b))$ and let $\varphi_\bullet^\ell$ be the flow of $-\nabla_{g_B}\ell$. There exist uniform constants $C > 0$ and $a > 0$, depending only on δ, such that with the same notations and assumptions from* **Proposition 4.2**, *then the following statements hold true:*

(1) If the range of b_s does not intersect an open neighbourhood of the critical points of ℓ, then there exists $b_0 \in B$ such that $b_s(t) - \varphi_{st}^\ell(b_0)$ and its first-order derivative at t are bounded by

$$C\left(\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right)\frac{\cosh(\rho(t-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2))} + s\right),$$

for the norms associated to g_B on B and TB, respectively.

(2) If either $b_s(t_s^+)$ or $b_s(t_s^-)$ converges towards a critical point of ℓ in the limit $s \to 0$, then there exists $s_0 \in (0,1]$ such that for all $s \in (0, s_0]$, there also exists $b_0 \in B$ such that $b_s(t) - \varphi_{st}^\ell(b_0)$ and its first-order derivative at t are bounded by

$$C\left(\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right)\frac{\cosh(\rho(t-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2))} + (t_s^+-t_s^-)s^2\right)e^{as(t_s^+-t_s^-)},$$

for the norms associated to g_B on B and TB, respectively.

Proof. For all $b \in B$ and for all $v \in T_bB$, the definition of ∇_{g_B} and the chain rule show that

$$\begin{aligned}
&g_B\left(\nabla_{g_B}\ell(b), v\right) = T_b\ell(v) = T_{(b,\eta^\sigma(b))}\delta(v \oplus T_b\eta^\sigma(v)),\\
&[\text{by definition of } \nabla_g] \quad = g\left(\nabla_g\delta(b,\eta^\sigma(b)), v\oplus T_b\eta^\sigma(v)\right),\\
&[\text{using } \textbf{Lemma 3.1}] \quad = g\left(\nabla_{g_B}\delta(b,\eta^\sigma(b)) \oplus \nabla_{g_F}\delta(b,\eta^\sigma(b)), v\oplus T_b\eta^\sigma(v)\right),\\
&[\text{since } (b,\eta^\sigma(b)) \in S_\delta] \quad = g\left(\nabla_{g_B}\delta(b,\eta^\sigma(b)) \oplus \mathbf{0}_{\mathbf{R}^{2N}}, v\oplus T_b\eta^\sigma(v)\right),\\
&[\text{since } g = g_B \oplus g_F] \quad = g_B\left(\nabla_{g_B}\delta(b,\eta^\sigma(b)), v\right) + g_F(\mathbf{0}_{\mathbf{R}^{2N}}, T_b\eta^\sigma(v)),\\
&\qquad = g_B\left(\nabla_{g_B}\delta(b,\eta^\sigma(b)), v\right),
\end{aligned}$$

thus, the equality $\nabla_{g_B}\ell(b) = \nabla_{g_B}\delta(b,\eta^\sigma(b))$ holds, since g_B is non-degenerate and v is arbitrary. Besides, for all $t \in [t_s^-, t_s^+]$, let us define an error term $\varepsilon_s(t) \in T_{b_s(t)}B$ by

$$\begin{aligned}
\varepsilon_s(t) &= \nabla_{g_B}\ell(b_s(t)) - \nabla_{g_B}\delta(\gamma_s(t)),\\
&= \nabla_{g_B}\delta(b_s(t), \eta_s^\sigma(t)) - \nabla_{g_B}\delta(b_s(t), \eta_s(t)),\\
&= \nabla_{g_B}\delta(b_s(t), \eta_s^\sigma(t)) - \nabla_{g_B}\delta(b_s(t), \eta_s(t)).
\end{aligned}$$

Then, using that γ_s has range in K^σ and S_δ is compact (since f_1 and f_2 are linear-at-infinity) shows that for all $\tau \in [0,1]$, $(b_s(t), (1-\tau)\eta_s^\sigma(t) + \tau\eta_s(t))$ stays in a uniform compact subset of $B \times \mathbf{R}^{2N}$. Hence, since δ is C^2, the mean value inequality shows there exists a uniform constant $C_1 > 0$, depending only on δ, such that for all $t \in [t_s^-, t_s^+]$, the quantity $\|\varepsilon_s(t)\|_{g_B}$ is bounded by

$$\|\varepsilon_s(t)\|_{g_B} \leqslant C_1 \|\eta_s(t) - \eta_s^\sigma(t)\|_{g_F}. \tag{38}$$

Therefore, **Proposition 4.3** implies there exist uniform constants $d_0 > 0$, $C_2 > 0$ and $\rho > 0$, depending only on δ, such that if d_s is less than d_0 on $[t_s^-, t_s^+]$ (which is from now on assumed), then for all $t \in [t_s^-, t_s^+]$, the following inequality holds true:

$$\|\varepsilon_s(t)\|_{g_B} \leqslant C_1 C_2 \left(\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F}\right) \frac{\cosh(\rho(t-(t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2)} + s \right). \tag{39}$$

Let us write $\gamma_s \colon \mathbf{R} \to B \times \mathbf{R}^{2N}$ in components as $\gamma_s(t) = (b_s(t), \eta_s(t))$, then recall from **Lemma 3.1** that the differential equation it satifies translates in components as follows:

$$\begin{cases} \partial_t b_s(t) = -s\nabla_{g_B}\delta(\gamma_s(t)), \\ \partial_t \eta_s(t) = -\nabla_{g_F}\delta(\gamma_s(t)), \end{cases}. \tag{40}$$

Therefore, using differential equation (40) and the definition of ε_s shows that

$$\partial_t b_s(t) = -s\nabla_{g_B}\ell(b_s(t)) + s\varepsilon_s(t), \tag{41}$$

and the result will follow from inequality (39) by direct integration of equation (41), discussing whether or not b_s stays away from the critical set of ℓ, when s goes to zero.

(1) If the range of b_s does not intersect an open neighbourhood of the critical points of ℓ, then there exists a uniform constant $L > 0$, depending only on δ, such that $t_s^+ - t_s^- \leqslant L/s$. Indeed, since b_s has range in K_B^σ, there exists a uniform compact subset K_{reg} constituted of regular points of ℓ, depending only on δ, such that b_s has range in K_{reg}, then let us define

$$\varepsilon_{\mathrm{reg}} = \inf_{K_{\mathrm{reg}}} \|\nabla_{g_B}\ell\|_{g_B} > 0.$$

Besides, let us define $I_s =]st_s^-, st_s^+[$ and $u \colon I_s \to \mathbf{R}$ by $u(t) = -\ell(b_s(t/s))$, then for $t \in I_s$, introduce $\tau = t/s$ and use the chain rule to compute

$$\begin{aligned}
& u'(t) = -\frac{T_{b_s(\tau)}\ell(\partial_t b_s(\tau))}{s}, \\
&\text{[by definition of } \nabla_{g_B}] && = -\frac{g_B(\nabla_{g_B}\ell(b_s(\tau)), \partial_t b_s(\tau))}{s}, \\
&\text{[using equation (41)]} && = \|\nabla_{g_B}\ell(b_s(\tau))\|_{g_B}{}^2 - g_B(\nabla_{g_B}\ell(b_s(\tau)), \varepsilon_s(\tau)), \\
&\text{[using Cauchy-Schwarz inequality]} && \geqslant \|\nabla_{g_B}\ell(b_s(\tau))\|_{g_B}{}^2 - \|\nabla_{g_B}\ell(b_s(\tau))\|_{g_B}\|\varepsilon_s(\tau)\|_{g_B}, \\
&\text{[since } b_s(\tau) \in K_{\mathrm{reg}} \text{ and by definition of } \varepsilon_{\mathrm{reg}}] && \geqslant \varepsilon_{\mathrm{reg}}\left(\varepsilon_{\mathrm{reg}} - \|\varepsilon_s(\tau)\|_{g_B}\right), \\
&\text{[using inequality (38) and } \|\Delta\eta_s\|_{g_F} \leqslant d_0] && \geqslant \varepsilon_{\mathrm{reg}}\left(\varepsilon_{\mathrm{reg}} - C_1 d_0\right).
\end{aligned}$$

To proceed any further, let us define a uniform constant $\varepsilon = \frac{\varepsilon_{\mathrm{reg}}{}^2}{2} > 0$, depending only δ, and assume that $d_0 \leqslant \frac{\varepsilon_{\mathrm{reg}}}{2C_1}$, then the above inequalities show that for all $t \in I_s$, $u'(t) \geqslant \varepsilon$.

Hence, by the definition of u, integrating u' on $I_s = [st_s^-, st_s^+]$ shows that

$$\ell(b_s(t_s^-)) - \ell(b_s(t_s^+)) \geqslant \varepsilon(st_s^+ - st_s^-).$$

Let us define a uniform constant $L = (\sup_{K_{\text{reg}}} \ell - \inf_{K_{\text{reg}}} \ell)/\varepsilon + 1 > 0$, depending only on δ, then since b_s has range in K_{reg}, the previous inequality divided by εs yields $t_s^+ - t_s^- \leqslant L/s$. Moreover, using the straightening theorem for vector fields and the compactness of K_{reg}, there exist finitely many coordinate charts $\{(U_k, \varphi_k : U_k \to \mathbf{R}^n)\}_{1\leqslant k\leqslant r}$ of B covering K_{reg} and a fixed vector $X \in \mathbf{R}^n$ such that for all $k \in \{1,\dots,r\}$, $-\nabla_{g_B}\ell$ is equal to ${\varphi_k}^* X$ on U_k. Let $t \in (t_s^-, t_s^+)$ and $\tau_s = (t_s^+ - t_s^-)/2$, and assume, without loss of generality, that $\tau_s \leqslant t$, then since b_s is continuous and has range in K_{reg}, there exists a partition

$$\tau_s = t_s^1 < t_s^2 < \ldots < t_s^{m-1} < t_s^m = t,$$

such that for all $k \in \{1,\dots,m-1\}$, b_s restricted to $[t_s^k, t_s^{k+1}]$ has range in the chart U_k. Hence, in particular, for all $k \in \{1,\dots,m-1\}$, in the local coordinates of B given by φ_k, the differential equation (41) can be written as follows:

$$\forall \tau \in [t_s^k, t_s^{k+1}], \partial_t b_s(\tau) = sX + s\,\mathrm{Jac}_{b_s(\tau)}\,\varphi_k \cdot \varepsilon_s(\tau).$$

Let us define $b_0 = \varphi_1(b_s(\tau_s)) - s\tau_s X$, directly integrating the differential equation (41) on each subinterval of the partition $\tau_s = t_s^1 < t_s^2 < \ldots < t_s^{m-1} < t_s^m = t$ shows that

$$b_s(t) - \varphi_{st}^{\ell}(b_0) = b_s(t) - (sXt + b_0) = \int_{\tau_s}^{t_s^2} s\,\mathrm{Jac}_{b_s(\tau)}\,\varphi_1 \cdot \varepsilon_s(\tau)\,\mathrm{d}\tau + \sum_{k=2}^{m-1} \int_{t_s^k}^{t_s^{k+1}} s\,\mathrm{Jac}_{b_s(\tau)}\,\varphi_k \cdot \varepsilon_s(\tau)\,\mathrm{d}\tau, \tag{42}$$

provided that for all $k \in \{1,\dots,m-2\}$, the initial condition was carefully chosen for the resulting trajectories on $[t_s^k, t_s^{k+1}]$ and $[t_s^{k+1}, t_s^{k+2}]$ to agree on the intermediate time t_s^{k+1}. But since K_{reg} is compact, there exists a uniform constant $C_3 > 0$, depending only on δ, such that for all $k \in \{1,\dots,k\}$, the inequality $\|\mathrm{Jac}_{b_s(t)}\,\varphi_k \cdot \varepsilon_s(t)\|_{g_B} \leqslant C_3 \|\varepsilon_s(t)\|_{g_B}$ holds. In particular, using the monotonicity of the integral and inequality (39) show that each term on the right-hand side of equation (42) are bounded by

$$C_1C_2C_3\left(\max(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F})\left|\int_{\tau_s}^{t} \frac{\cosh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2)}\,\mathrm{d}\tau\right| + s\left|\int_{\tau_s}^{t} 1\,\mathrm{d}\tau\right|\right)s. \tag{43}$$

To keep notations short, let us define $M_s = \max(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F})$, then using the triangle inequality in equation (42) and inequality (43) yields

$$\begin{aligned}
\left\|b_s(t) - \varphi_{st}^{\ell}(b_0)\right\|_{g_B} &\leqslant mC_1C_2C_3\left(M_s\left|\int_{\tau_s}^{t} \frac{\cosh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))}\,\mathrm{d}\tau\right| + \left|\int_{\tau_s}^{t} 1\,\mathrm{d}\tau\right| s\right)s,\\
&= mC_1C_2C_3\left(M_s\left|\frac{\sinh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))}\right| s + |t - \tau_s| s^2\right),\\
[\text{since } t_s^- \leqslant t \leqslant t_s^+] &\leqslant mC_1C_2C_3\left(\frac{1}{\rho}M_s\left|\frac{\sinh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))}\right| s + \left|\frac{t_s^+ - t_s^-}{2}\right| s^2\right),\\
[\text{as } t_s^+ - t_s^- \leqslant L/s] &\leqslant mC_1C_2C_3\left(\frac{1}{\rho}M_s\left|\frac{\sinh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))}\right| s + \frac{L}{2}s\right),\\
[\text{as } |\sinh| \leqslant \cosh] &\leqslant mC_1C_2C_3\left(\frac{1}{\rho}M_s\frac{\cosh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))}s + \frac{L}{2}s\right),
\end{aligned}$$

$$[\text{since } s \leqslant 1] \qquad \leqslant mC_1C_2C_3\left(\frac{1}{\rho}M_s\frac{\cosh(\rho(\tau-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2))}+\frac{L}{2}s\right).$$

In conclusion, for all $t\in[t_s^-,t_s^+]$, $b_s(t)-\varphi_{st}^\ell(b_0)$ is bounded by

$$mC_1C_2C_3\left(\frac{1}{\rho}\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F},\|\Delta\eta_s(t_s^+)\|_{g_F}\right)\frac{\cosh(\rho(\tau-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2))}+\frac{L}{2}s\right), \tag{44}$$

yielding the desired estimate from Proposition 4.4 (1) for the zeroth-order derivative.

In order to control the first-order derivative from the zeroth-order derivative estimate, start noticing that since ℓ is C^2 and defined a compact subset, both depending only on δ, the mean value inequality provides a uniform constant $C_4>0$, depending only on δ, such that for all $t\in[t_s^-,t_s^+]$, the following inequality holds true

$$\left\|\nabla_{g_B}\ell(b_s(t))-\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)\right\|_{g_B}\leqslant C_4\left\|b_s(t)-\varphi_{st}^\ell(b_0)\right\|_{g_B}. \tag{45}$$

Then, derive the desired bootstrapping inequality, proceeding as follows

$$\begin{aligned}
& \left\|\partial_t b_s(t)-\partial_t\varphi_{st}^\ell(b_0)\right\|_{g_B} = \left\|\partial_t b_s(t)+s\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)\right\|_{g_B},\\
&[\text{using equality (41)}] \qquad = \left\|-s\nabla_{g_B}\ell(b_s(t))+s\varepsilon_s(t)+s\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)\right\|_{g_B},\\
& \qquad = s\left\|\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)-\nabla_{g_B}\ell(b_s(t))+\varepsilon_s(t)\right\|,\\
&[\text{since } s\leqslant 1] \qquad \leqslant \left\|\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)-\nabla_{g_B}\ell(b_s(t))+\varepsilon_s(t)\right\|_{g_B},\\
&[\text{by triangle inequality}] \qquad \leqslant \left\|\nabla_{g_B}\ell(b_s(t))-\nabla_{g_B}\ell\left(\varphi_{st}^\ell(b_0)\right)\right\|_{g_B}+\|\varepsilon_s(t)\|_{g_B},\\
&[\text{using inequality (45)}] \qquad \leqslant C_4\left\|b_s(t)-\varphi_{st}^\ell(b_0)\right\|_{g_B}+\|\varepsilon_s(t)\|_{g_B}.
\end{aligned}$$

To conclude, apply inequalities (39) and (44) in the bootstrapping inequality above to show that for all $t\in[t_s^-,t_s^+]$, $\partial_t b_s(t)-\partial_t\varphi_{st}^\ell(b_0)$ is bounded by

$$C_1C_2\left(\left(\frac{mC_2C_4}{\rho}+1\right)\max\left(\|\Delta\eta_s(t_s^-)\|_{g_F},\|\Delta\eta_s(t_s^+)\|_{g_F}\right)\frac{\cosh(\rho(\tau-(t_s^-+t_s^+)/2))}{\cosh(\rho(t_s^+-t_s^-)/2))}+\left(\frac{mLC_3C_4}{2}+1\right)s\right).$$

yielding the desired estimate from Proposition 4.4 (1) for the first-order derivative.

Thus, letting the uniform constant

$$C=C_1C_2\max\left(\frac{1}{\rho},\frac{L}{2},\frac{mC_3C_4}{\rho}+1,\frac{mLC_3C_4}{2}+1\right)>0,$$

depending only on δ, yields Proposition 4.4 (1).

(2) If either $b_s(t_s^+)$ or $b_s(t_s^-)$ converges towards some critical point p of ℓ in the limit $s\to 0$, then there exist a coordinate chart around p and a linear map A such that $-\nabla_{g_B}\ell(b)=Ab$. Indeed, this claim follows from the Morse lemma, noticing that since Λ is chord generic, then applying Definition 1.12 (1) and Proposition 2.3 shows that ℓ is a Morse function. Assume just for now that b_s takes its values in the domain of this Morse coordinate chart, then in this coordinate chart, the differential equation (41) translates as follows

$$\partial_t b_s(t)=sAb_s(t)+s\varepsilon_s(t).$$

Let $\tau_s = (t_s^+ - t_s^-)/2$ and $b_0 = e^{-s\tau_s A} b_s(\tau_s)$, then applying the Duhamel's formula yields

$$b_s(t) - \varphi_{st}^\ell(b_0) = b_s(t) - e^{stA} b_0 = \int_{\tau_s}^t s e^{s(t-\tau)A} \varepsilon_s(\tau)\,\mathrm{d}\tau. \tag{46}$$

There exists a uniform constant $a > 0$, depending only on δ, such that $\|Ab\|_{g_B} \leqslant 2a\|b\|_{g_B}$. In particular, let $t \in [t_s^-, t_s^+]$ and $\tau \in [\min(\tau_s, t), \max(\tau_s, t)]$, then using the power series expansion of the exponential function and the triangle inequality, it is well known that

$$\left\| e^{s(t-\tau)} \varepsilon_s(t) \right\|_{g_B} \leqslant e^{2as(t-\tau)} \|\varepsilon_s(t)\|_{g_B} \leqslant e^{as(t_s^+ - t_s^-)} \|\varepsilon_s(t)\|_{g_B},$$

where the last inequality follows from the exponential function being increasing on $\mathbf{R}$. Therefore, applying this last inequality in equality (46) yields

$$\left\| b_s(t) - \varphi_{st}^\ell(b_0) \right\|_{g_B} = \left| \int_{\tau_s}^t s \left\| e^{s(t-\tau)A} \varepsilon_s(\tau) \right\|_{g_B} \mathrm{d}\tau \right| \leqslant e^{as(t_s^+ - t_s^-)} \left| \int_{\tau_s}^t s \|\varepsilon_s(\tau)\|_{g_B} \,\mathrm{d}\tau \right|.$$

Besides, notice that this integral already appeared in the proof of **Proposition 4.4 (1)**, so that carrying the same computations shows that for all $t \in [t_s^-, t_s^+]$, $b_s(t_0) - \varphi_{st}^\ell(b_0)$ is bounded by

$$C_1 C_2 \left(\frac{1}{\rho} \max\left(\|\Delta\eta_s(t_s^-)\|_{g_F}, \|\Delta\eta_s(t_s^+)\|_{g_F} \right) \frac{\cosh(\rho(\tau - (t_s^- + t_s^+)/2))}{\cosh(\rho(t_s^+ - t_s^-)/2))} + \frac{1}{2}(t_s^+ - t_s^-) s^2 \right) e^{as(t_s^+ - t_s^-)}, \tag{47}$$

yielding the desired estimate from **Proposition 4.4 (2)** for the zeroth-order derivative. Moreover, **Proposition 4.4 (1)** shows that away from p, (47) can be improved to (44), and in particular, it can be assumed, without loss of generality, that b_s has range in the domain of a Morse coordinate chart around p.

In order to control the first-order derivative from the zeroth-order derivative estimate, start by observing that there exists $s_0 \in (0, 1]$ such that for all $s \in (0, s_0]$, $s(t_s^+ - t_s^-) \geqslant 1$. Indeed, let $T_s = s(t_s^+ - t_s^-)$ and assume, for the sake of contradiction, that $T_s \underset{s \to 0}{\not\longrightarrow} +\infty$, then there exist a sequence $(s_k)_{n \in \mathbf{N}}$ and $T > 0$ such that $s_k \in (0, 1]$, $s_k \underset{n}{\longrightarrow} 0$ and $T_{s_k} \underset{n}{\longrightarrow} T$. Assume, without loss of generality, that $b_{s_k}(t_{s_k}^+) \underset{n}{\longrightarrow} p$, then let us show that $b_{s_k}(t_{s_k}^-) \underset{n}{\longrightarrow} p$. For that purpose, use the triangle inequality twice to show that

$$\begin{aligned}
\left\| b_{s_k}(t_{s_k}^-) - p \right\|_{g_B} &\leqslant \left\| b_{s_k}(t_{s_k}^-) - \varphi_{s_k t_{s_k}^-}^\ell(b_0) \right\|_{g_B} + \left\| \varphi_{s_k t_{s_k}^-}^\ell(b_0) - p \right\|_{g_B}, \\
\text{[as } st_{s_k}^- = st_{s_k}^+ - T_{s_k}\text{]} \quad &= \left\| b_{s_k}(t_{s_k}^-) - \varphi_{s_k t_{s_k}^-}^\ell(b_0) \right\|_{g_B} + \left\| \varphi_{s_k t_{s_k}^+ - T_{s_k}}^\ell(b_0) - p \right\|_{g_B}, \\
&\leqslant \left\| b_{s_k}(t_{s_k}^-) - \varphi_{s_k t_{s_k}^-}^\ell(b_0) \right\|_{g_B} + \left\| \varphi_{s_k t_{s_k}^+ - T_{s_k}}^\ell(b_0) - \varphi_{s_k t_{s_k}^+ - T}^\ell(b_0) \right\|_{g_B} \\
&\quad + \left\| \varphi_{s_k t_{s_k}^+ - T}^\ell(b_0) - p \right\|_{g_B}, \\
&= \left\| b_{s_k}(t_{s_k}^-) - \varphi_{s_k t_{s_k}^-}^\ell(b_0) \right\|_{g_B} + \left\| \varphi_{s_k t_{s_k}^+ - T_{s_k}}^\ell(b_0) - \varphi_{s_k t_{s_k}^+ - T}^\ell(b_0) \right\|_{g_B} \\
\text{[as } \varphi_\bullet^\ell \text{ is a semigroup]} \quad &\quad + \left\| \varphi_{-T}^\ell\left(\varphi_{s_k t_{s_k}^+}^\ell(b_0) \right) - p \right\|_{g_B}.
\end{aligned}$$

Let us show that all the terms of the right-hand side of this inequality goes to 0 with n:

- Using inequality (47) shows that the first term converges towards 0 as $n \to +\infty$.

- Using the mean value inequality shows that the second term is bounded by
$$\left\|\varphi^{\ell}_{s_k t^+_{s_k} - T_{s_k}}(b_0) - \varphi^{\ell}_{s_k t^+_{s_k} - T}(b_0)\right\|_{g_B} \leqslant \underbrace{\sup_{K_B^\sigma} \|\nabla_{g_B}\ell\|_{g_B}}_{<+\infty} \cdot |T_{s_k} - T|,$$
but the right-hand side term of the above inequality converges to 0 when $n \to +\infty$, since ℓ is C^1 on the compact set K_B^σ and by construction $T_{s_k} \underset{n}{\longrightarrow} T$.
- First, let us see that $\varphi^{\ell}_{st^+_s}(b_0) \underset{n}{\longrightarrow} p$, using the triangle inequality to show that
$$\left\|\varphi^{\ell}_{st^+_s}(b_0) - p\right\|_{g_B} \leqslant \left\|\varphi^{\ell}_{st^+_s}(b_0) - b_s(t^+_s)\right\|_{g_B} + \left\|b_s(t^+_s) - p\right\|_{g_B},$$
and then concluding using inequality (47) and that by assumption, $b_{s_k}(t^+_{s_k}) \underset{n}{\longrightarrow} p$. Since $\varphi^{\ell}_{-T}(p) = p$ (as p is a critical point of ℓ), φ^{ℓ}_{-T} is continuous and $\varphi^{\ell}_{st^+_s}(b_0) \underset{n}{\longrightarrow} p$, the third term also converges towards 0 when $n \to +\infty$.

Therefore, $b_{s_k}(t^-_{s_k}) \underset{n}{\longrightarrow} p$, contradicting the starting assumption of Proposition 4.4 (2). In particular, there exists a constant $s_0 \in (0,1]$ such that for all $s \in (0, s_0]$, $s(t^+_s - t^-_s) \geqslant 1$. Assuming that $s \leqslant s_0$, then plugging $s(t^+_s - t^-_s) \geqslant 1$ and $e^{as(t^+_s - t^-_s)} \geqslant 1$ in inequality (39) shows that the quantity $\|\varepsilon_s(t)\|_{g_B}$ is also bounded by
$$C_1 C_2 \left(\max\left(\|\Delta\eta_s(t^-_s)\|_{g_F}, \|\Delta\eta_s(t^+_s)\|_{g_F}\right) \frac{\cosh(\rho(\tau - (t^-_s + t^+_s)/2))}{\cosh(\rho(t^+_s - t^-_s)/2))} + (t^+_s - t^-_s)s^2 \right) e^{as(t^+_s - t^-_s)}.$$
Hence, using the bootstrapping inequality derived in the proof of Proposition 4.4 (1) shows that for all $t \in [t^-_s, t^+_s]$, $\partial_t b_s(t) - \partial_t \varphi^{\ell}_{st}(b_0)$ is bounded by
$$C_1 C_2 \left(\left(\frac{C_4}{\rho} + 1\right) \max\left(\|\Delta\eta_s(t^-_s)\|_{g_F}, \|\Delta\eta_s(t^+_s)\|_{g_F}\right) \frac{\cosh(\rho(\tau - (t^-_s + t^+_s)/2))}{\cosh(\rho(t^+_s - t^-_s)/2))} + \left(\frac{C_4}{2} + 1\right)(t^+_s - t^-_s)s^2 \right) e^{as(t^+_s - t^-_s)}.$$
yielding the desired estimate from Proposition 4.4 (2) for the first-order derivative.

Thus, letting the uniform constant
$$C = C_1 C_2 \max\left(\frac{1}{\rho}, \frac{1}{2}, \frac{C_4}{\rho} + 1, \frac{C_4}{2} + 1\right) > 0,$$
depending only on δ, yields Proposition 4.4 (2).

Despite being derived in some coordinate charts, the above estimates still yield Proposition 4.4, since using the chain rule and the compactness of K_B^σ ensures that working in coordinate charts only affects the exact value of C. ∎

Let us now focus on the base dynamics near a singularity of the maximal flow invariant subset.

Proposition 4.5. *Let us define $\ell^{\prec} : U_B^{\prec} \to \mathbf{R}$ by $\ell^{\prec}(b) = \delta(b, \eta^{\prec}(b))$ and let $\varphi^{\ell^{\prec}}_{\bullet}$ be the flow of $-\nabla_{g_B}\ell^{\prec}$. There exists a uniform constant $C > 0$, depending only on δ such that with the same notations and assumptions from Proposition 4.3, there exists a constant $b_0 \in B$ such that for all $t \in [t^-_s, t^+_s]$, $b_s(t) - \varphi^{\ell^{\prec}}_{st}(b_0)$ and its first-order derivative at t are bounded by*
$$C\left(\max\left(\left\|\Delta\eta_s{}^{\perp}(t^-_s)\right\|_{g_F}, \left\|\Delta\eta_s{}^{\perp}(t^+_s)\right\|_{g_F}\right) + \max\left(\left\|\Delta\eta_s{}^{/\!/}(t^-_s)\right\|_{g_F}, \left\|\Delta\eta_s{}^{/\!/}(t^+_s)\right\|_{g_F}\right) + s^{1/2}\right),$$
for the norms associated to g_B on B and TB, respectively.

Proof. For all $t \in [t_s^-, t_s^+]$, let us define $\varepsilon_s(t) = \nabla_{g_B}\ell^{\prec}(b_s(t)) - \nabla_{g_B}\delta(\gamma_s(t)) \in T_{b_s(t)}B$ and mimick the proof of **Proposition 4.4** to show that there exist uniform constants $d_0 > 0$, $\rho > 0$ and $C_1 > 0$, depending only on δ, such that if $\|\eta_s^{\prec}(t) - \eta_s(t)\|_{g_F}$ is less than d_0 (which is from now on assumed), then for all $t \in [t_s^-, t_s^+]$, the quantity $\|\varepsilon_s(t)\|_{g_B}$ is bounded by

$$C_1\left(\max\left(\left\|\Delta\eta_s{}^{\perp}(t_s^-)\right\|_{g_F}, \left\|\Delta\eta_s{}^{\perp}(t_s^+)\right\|_{g_F}\right) + \max\left(\left\|\Delta\eta_s{}^{//}(t_s^-)\right\|_{g_F}, \left\|\Delta\eta_s{}^{//}(t_s^+)\right\|_{g_F}\right) + s\right).$$

Moreover, as in the proof of **Proposition 4.4**, the differential equation satisfied by γ_s implies that

$$\partial_t b_s(t) = -s\nabla_{g_B}\ell^{\prec}(b_s(t)) + s\varepsilon_s(t),$$

and the result will similarly follow by direct integration from the inequality satisfied by $\|\varepsilon_s(t)\|_{g_B}$, but this time without having to discuss whether b_s stays away from the critical points of $\ell^{\prec}$ or not. Indeed, since Λ is chord generic, **Definition 1.12 (4)** shows that $\ell^{\prec}$ has no critical points at all; otherwise, using **Propositions 2.2** and **2.3**, there would exist a Reeb chord with an end on $\Lambda^{\prec}$. In conclusion, repeating the proof of **Proposition 4.4 (1)** with the right estimate above for $\|\varepsilon_s\|_{g_B}$ thus yields the result. ∎

Using **Propositions 4.2**, **4.3**, **4.4** and **4.5**, horizontal fragments can now be recovered.

The following statement is used in situations where **Proposition 4.1** does not apply.

Theorem 4.2. *If Λ is gradient generic and $([t_k^-, t_k^+])_{k\in\mathbf{N}}$ is an increasing sequence of intervals of $\mathbf{R}$ such that for all nonnegative integers k and all $t \in [t_k^-, t_k^+]$, $(\gamma_k(t))_{k\in\mathbf{N}}$ converges to some point of S_δ, then there exist a horizontal fragment $h\colon I \to S_\delta$, a sequence of time-shifts $(\tau_k)_{k\in\mathbf{N}}$, with $\tau_k \in \mathbf{R}$, and a subsequence (still denoted in the same way) such that $\gamma_k(s_k{}^{-1}(\cdot + \tau_k)) \underset{k}{\longrightarrow} h$ in the C^1_{loc}-topology.*

Proof. All the technical work has already been done, the proof only amounts to:

- **Step 1.** Shift and rescale $[t_k^-, t_k^+]$ by $s_k{}^{-1}$ to recover the domain of a horizontal fragment. Discussing whether or not the Henry-Rutherford sequence gets near a critical point of δ and proceeding as in the proof of **Proposition 4.4** shows that there exists a sequence of times-shifts $(\tau_k)_{k\in\mathbf{N}}$ such that if $(I_k)_{k\in\mathbf{N}}$ is defined for all nonnegative integers k by
$$I_k = \left[s_k t_k^- - \tau_k, s_k t_k^+ - \tau_k\right],$$
then the sequence of intervals $(I_k)_{k\in\mathbf{N}}$ converges to a semi-infinity or compact interval I, either of the form $(-\infty, 0]$ or $[0, +\infty)$, or $[0, t]$ for some $t > 0$.
- **Step 2.** There exist a map $h\colon I \to S_\delta$ and a subsequence (still denoted in the same way) such that there is a pointwise convergence $\gamma_k(s_k{}^{-1}(\cdot + \tau_k)) \underset{k}{\longrightarrow} h$.
- **Step 3.** Since Λ is gradient generic, according to **Propositions 4.3** and **4.5** shows that the map h constructed in **Step 2** can only cross the singular locus $S_\delta{}^{\prec}$ at isolated points. For more details on **Step 3**, see the proof of **Theorem 4.3** in **Section 2.2**.
- **Step 4.** Since Λ is compact, according to **Step 3**, h does not cross the singular locus apart from a finite number of points, and applying **Propositions 4.2** and **4.4** shows that h is a horizontal fragment and the convergence in **Step 2** occurs in the C^1_{loc}-topology.

The above steps yields the result. ∎

Remark. Since the sequence of intervals $[t_k^-, t_k^+]$ from the statement of Theorem 4.2 is increasing, it is possible and easy to make sense of the pointwise convergence of $(\gamma_k\colon [t_k^-, t_k^+] \to B \times \mathbf{R}^{2N})_{k\in\mathbf{N}}$. Moreover, if $(t_k^+ - t_k^-)_{k\in\mathbf{N}}$ is bounded, then the horizontal fragment recovered is constant.

Remark. Whenever the conclusion of Theorem 4.2 holds, h is said to be *recovered* from $(\gamma_k)_{k\in\mathbf{N}}$, and this definition is compatible with Definition 3.5.

2. Finiteness of the number of vertical fragments recovered from a HR-sequence

Since vertical fragments can become arbitrarily short near singularities of the Legendrian front, they do not consume a positive predetermined amount of the total vertical length that is available. In particular, nothing seems to prevent the Henry-Rutherford limiting process from converging to a broken gradient trajectory with infinitely many pairwise non-equivalent vertical fragments. However, on a gradient generic Legendrian submanifold, the Henry–Rutherford limiting process must end in a finite number of steps, this is the statement of

Theorem 4.3. *Using the same notations and assumptions from Theorem 4.1, the number of vertical fragments that can be recovered from $(\gamma_k)_{k\in\mathbf{N}}$ by applying recursively Proposition 4.1 is finite.*

The proof of Theorem 4.3 is carried through the following steps:

- **Step 1.** First, argue, for the sake of contradiction, that the Henry–Rutherford limiting process recovers infinitely many pairwise non-equivalent vertical fragments and show they uniformly converge to a singularity of the Legendrian caustic (Proposition 4.6).
- **Step 2.** Then, in a short time interval for the domain of the Henry–Rutherford sequence, construct infinitely many horizontal fragments crossing the singular component of the accumulation point of the vertical fragments (Proposition 4.7).
- **Step 3.** At last, prove that these horizontal fragments have arbitrarily deep tangency with the singular locus of the Legendrian front, thus yielding a contradiction with the gradient genericity transversality conditions (Section 2.2).

This section is organised by the structure of the proof of Theorem 4.3.

2.1. Technical preliminaries and constructions.

The goal of this section is to construct the sequence of horizontal fragments (Proposition 4.7) contradicting the gradient genericity transversality conditions.

Assume, for the sake of contradiction, that there exists a sequence $(v_j)_{j\in\mathbf{N}}$ of pairwise non-equivalent vertical fragments that can be recovered from $(\gamma_k)_{k\in\mathbf{N}}$.

Lemma 4.1. *For all $j \in \mathbf{N}$, let us define $\ell_j = \delta({v_j}^-) - \delta({v_j}^+) \geqslant 0$, then $(\ell_j)_{j\in\mathbf{N}}$ is summable.*

Proof. Let $J \in \mathbf{N}$ such that $J \geqslant 1$ and let $I = \{1, \dots, J\}$, then for all $j \in I$, using Definition 3.5, there exist, a sequence of real numbers $\left(\tau_k^j\right)_{k\in\mathbf{N}}$ and a subsequence (still denoted in the same way) such that $\gamma_k\left(\cdot + \tau_k^j\right) \underset{k}{\longrightarrow} v_j$ in the C^1_{loc}-topology. Then, let $\varepsilon > 0$ and use the continuity of δ to show that for all $j \in I$, there exist $K_j \in \mathbf{N}$ and $T > 0$ such that

$$\forall k \in \mathbf{N}, k \geqslant K_j \Rightarrow \left|\delta\left(\gamma_k\left(\pm T + \tau_k^j\right)\right) - \delta\left({v_j}^{\pm}\right)\right| \leqslant \varepsilon. \tag{48}$$

Besides, for all $k \in \mathbf{N}$, there exist $\sigma_k \in \mathfrak{S}_I$ such that $(\tau_k^{\sigma_k(j)})_{j\in I}$ is decreasing, then using the pigeonhole principle, there exist $\sigma \in \mathfrak{S}_I$ a subsequence (still denoted in the same way) such that for all $k \in \mathbf{N}$, $\sigma_k = \sigma$, so that it can be assumed, without loss of generality, that

$$\forall j \in \{1,\dots,J-1\}, \forall k \in \mathbf{N}, \tau_k^j - \tau_k^{j+1} \geqslant 0.$$

Let $j \in I \setminus \{J-1\}$, then, using **Proposition 3.2**, the sequence $(\tau_k^j - \tau_k^{j+1})_{k\in\mathbf{N}}$ is also unbounded, so that there exists $K_j' \in \mathbf{N}$ such that for all integers $k \geqslant K_j'$, the inequality $\tau_k^j - \tau_k^{j+1} \geqslant 2T$ holds. Equivalently, for all integers $k \geqslant K_j'$, $-T + \tau_k^j \geqslant T + \tau_k^{j+1}$, so that **Lemma 3.3** shows that

$$\forall k \in \mathbf{N}, k \geqslant K_j' \Rightarrow \delta\left(\gamma_k\left(-T + \tau_k^j\right)\right) \leqslant \delta\left(\gamma_k\left(T + \tau_k^{j+1}\right)\right). \tag{49}$$

From now on, let $k \in \mathbf{N}$ such that $k \geqslant \max_{j\in I}(K_j, K_j')$, then inequalities **(48)** and **(49)** yields

$$\forall j \in \{1,\dots,J-1\}, -\delta(v_j{}^+) + \delta(v_{j+1}{}^-) \leqslant 2\varepsilon.$$

At last, using the inequalities above and a telescoping series like simplification shows that

$$\begin{aligned} \sum_{j=1}^{J} \ell_j &\leqslant \delta(v_1{}^-) - \delta(v_N{}^+) + 2(J-2)\varepsilon, \\ \text{[using Lemma 3.4]} \quad &\leqslant \delta(c_-) - \delta(c_+) + 2(J-2)\varepsilon, \\ \text{[since } \varepsilon > 0 \text{ is arbitrary]} \quad &\leqslant \delta(c_-) - \delta(c_+), \end{aligned}$$

then the partial sums of the series of $(\ell_j)_{j\in\mathbf{N}}$ are uniformly bounded from above, since J is arbitrary. Using **Lemma 3.4**, $(\ell_j)_{j\in\mathbf{N}}$ is a sequence of nonnegative real numbers, thus yielding the result. ∎

Remark. It is crucial that all the $(v_j)_{j\in\mathbf{N}}$ are recovered from a same Henry–Rutherford sequence, and the same will be true for all the energetical arguments used later.

Remark. The permutation of $\{1,\dots,J\}$ is given by the reverse order of the $\delta(v_j{}^-)$ on the real line.

Proposition 4.6. *There exist a singular point σ and a subsequence (still denoted in the same way) such that $v_j \underset{j}{\longrightarrow} \sigma$ in the C^0-topology.*

Proof. The proof is carried through the following steps:

- **Step 1.** There exist a point $\sigma \in S_\delta$ and a subsequence (still denoted in the same way) such that $v_j \underset{j}{\longrightarrow} \sigma$ in the C^0_{loc}-topology.
- **Step 2.** The point $\sigma \in S_\delta$ constructed in **Step 1** actually belongs to $S_\delta{}^{\prec}$.
- **Step 3.** The convergence in **Step 1** actually occurs in the C^0-topology.

Step 1. Since the generating families f_1 and f_2 are linear-at-infinity, S_δ is compact, and using **Proposition 3.1**, there exist points σ_- and σ_+ of S_δ, as well as a common subsequence (still denoted in the same way) such that $(v_j{}^-)_{j\in\mathbf{N}}$ and $(v_j{}^+)_{j\in\mathbf{N}}$ converge to σ_- and σ_+, respectively. Besides, using **Proposition 2.4** and **Lemma 3.6**, the sequence $(v_j)_{j\in\mathbf{N}}$ is bounded in C^2-topology, and the mean value inequality shows that $(v_j)_{j\in\mathbf{N}}$ and $(\partial_t v_j)_{j\in\mathbf{N}}$ are bounded and equicontinuous, then Arzelà-Ascoli theorem applies and show there exists a vertical fragment v joining σ_- to σ_+. However, **Lemma 4.1** and the continuity of δ show that $\delta(\sigma_-) = \delta(\sigma_+)$, then using **Lemma 3.4**, the vertical fragment v is constant, so that $\sigma_- = \sigma = \sigma_+$, thus yielding **Step 1**.

Step 2. Let $j \in \mathbf{N}$ and write $v_j : \mathbf{R} \to B \times \mathbf{R}^N \times \mathbf{R}^N$ in components as $v_j(t) = (b_j(t), \eta_{1,j}(t), \eta_{2,j}(t))$, then, using Definition 3.2, its components satisfy the following conditions

- $b_j : \mathbf{R} \to B$ is constant;
- $\eta_{1,j} : \mathbf{R} \to \mathbf{R}^N$ is a g_N-gradient flow line of $-f_{1_{b_j}}$, where $f_{1_{b_j}}$ stands for $f_{1|\{b_j\}\times\mathbf{R}^N}$; and
- $\eta_{2,j} : \mathbf{R} \to \mathbf{R}^N$ is a g_N-gradient flow line of $f_{2_{b_j}}$, where $f_{2_{b_j}}$ stands for $f_{2|\{b_j\}\times\mathbf{R}^N}$.

Let us also write $v_j{}^-$ and $v_j{}^+$ in components as $v_j{}^- = (b_j, \eta_{1,j}{}^-, \eta_{2,j}{}^-)$ and $v_j{}^+ = (b_j, \eta_{2,j}{}^+, \eta_{2,j}{}^+)$, then, using Definition 3.2, the points $v_j{}^-$ and $v_j{}^+$ both belong to S_δ, which reads as

$$\partial_\eta f_1\left(b_j, \eta_{1,j}{}^-\right) = \mathbf{0}, \quad \partial_\eta f_2\left(b_j, \eta_{2,j}{}^-\right) = \mathbf{0},$$
$$\partial_\eta f_1\left(b_j, \eta_{1,j}{}^+\right) = \mathbf{0}, \quad \partial_\eta f_2\left(b_j, \eta_{2,j}{}^+\right) = \mathbf{0}.$$

Besides, using Definition 3.2, v_j is non-constant, then, since b_j is constant, there exists $i_j \in \{1,2\}$ such that $\eta_{i,j}$ is non-constant, then using the pigeonhole principle, there exist $i \in \{1,2\}$ and a subsequence (still denoted in the same way) such that for all $j \in \mathbf{N}$, $\eta_{i,j} : \mathbf{R} \to \mathbf{R}^N$ is non-constant. Therefore, since $\eta_{i,j}$ is a gradient flow line, $\eta_{i,j}{}^- \neq \eta_{i,j}{}^+$ and it is thus possible to define

$$\nu_j = \frac{\eta_{i,j}{}^+ - \eta_{i,j}{}^-}{\|\eta_{i,j}{}^+ - \eta_{i,j}{}^-\|_{g_N}} \in \mathbf{S}^{N-1}.$$

Moreover, since the unit sphere $\mathbf{S}^{N-1}$ of $\mathbf{R}^N$, with respect to the Riemannian metric g_N, is compact, there exist $\nu \in \mathbf{S}^{N-1}$ ($\nu \neq \mathbf{0}$) and a subsequence (still denoted in the same way) such that $\nu_j \underset{j}{\longrightarrow} \nu$. Let us write σ in components as $\sigma = (b, \eta_1, \eta_2)$, and use the triangle inequality to show that

$$\begin{aligned}\left\|\mathrm{Hess}_{\eta_i} f_{i_b}\, \nu\right\|_{g_N} \leqslant\ & \left\|\mathrm{Hess}_{\eta_i} f_{i_b}\, \nu - \mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu\right\|_{g_N} \\ &+ \left\|\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu - \mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu_j\right\|_{g_N} \\ &+ \left\|\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu_j\right\|_{g_N}.\end{aligned}$$

Let us show that each of the terms of the right-hand side of the above inequality goes to 0 with j:

- Recall that $\partial_\eta f_i$ is continuously differentiable and that $(b_j, \eta_{i,j}{}^-) \underset{j}{\longrightarrow} (b, \eta_i)$, since it was shown during Step 2 that $v_j{}^- \underset{j}{\longrightarrow} \sigma$, so that $\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu \underset{j}{\longrightarrow} \mathrm{Hess}_{\eta_i} f_{i_b}\, \nu$.
- Let us define $\mathbf{B}^N$ to be the closed unit ball of $\mathbf{R}^N$, with respect to the Riemannian metric g_N, and let $\widehat{g_N}$ be the *Sasaki bundle metric* associated to g_N, then for all $j \in \mathbf{N}$, applying the mean value inequality to $\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}} : \mathbf{R}^N \to \mathbf{R}^N$ between $\nu \in \mathbf{B}^N$ and $\nu_j \in \mathbf{B}^N$ shows that

$$\begin{aligned}&\left\|\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu - \mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu_j\right\|_{g_N} \leqslant \sup_{t\in[0,1]} \left\|T_{(1-t)\nu + t\nu_j} \mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\right\|_{\widehat{g_N}} \cdot \|\nu_j - \nu\|_{g_N}, \\ &[\text{since } \mathbf{B}^N \text{ is convex}] \qquad \leqslant \sup_{x\in\mathbf{B}^N} \left\|T_x \mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\right\|_{\widehat{g_N}} \cdot \|\nu_j - \nu\|_{g_N}, \\ &[\text{since } (b_j, \eta_{i,j}{}^-) \in \Sigma_{f_i}] \qquad \leqslant \underbrace{\sup_{(q,e)\in\Sigma_{f_i}} \sup_{x\in\mathbf{B}^N} \left\|T_x \mathrm{Hess}_e f_{i_q}\right\|_{\widehat{g_N}}}_{<+\infty} \cdot \|\nu_j - \nu\|_{g_N} \underset{j}{\longrightarrow} 0,\end{aligned}$$

  since $T\,\mathrm{Hess}\, f_i$ is continuous, Σ_{f_i} and $\mathbf{B}^N$ are both compact, and $\nu_j \underset{j}{\longrightarrow} 0$.
- Let $j \in \mathbf{N}$, applying Taylor's formula for $\partial_\eta f_{i_{b_j}} : \mathbf{R}^N \to \mathbf{R}^N$ between $\eta_{i,j}{}^-$ and $\eta_{i,j}{}^+$ and then dividing both sides by $\|\eta_{i,j}{}^+ - \eta_{i,j}{}^-\|_{g_N}$ directly yields $\mathrm{Hess}_{\eta_{i,j}{}^-} f_{i_{b_j}}\, \nu_j \underset{j}{\longrightarrow} \mathbf{0}$.

Therefore, $\nu \in \ker \operatorname{Hess}_{\eta_i} f_{i|\{b\}\times\mathbf{R}^N}$, so that **Proposition 2.1** shows that $(b,\eta_i) \in \Sigma_{f_i}{}^{\prec}$ and $\sigma \in S_\delta{}^{\prec}$. Geometrically, since ν is both tangent to the fibre and to S_δ at σ, σ must be a singularity of S_δ. Thus yielding **Step 2**.

Step 3. Let $t_0 \in \mathbf{R}$, then for all $j \in \mathbf{N}$ and $t \in \mathbf{R}$, the following inequalities hold

$$\begin{aligned}
&\text{[using the triangle inequality]} && d_g(v_j(t),\sigma) \leqslant d_g(v_j(t_0),\sigma) + d_g(v_j(t), v_j(t_0)),\\
&\text{[using the definition of } d_g\text{]} && \leqslant d_g(v_j(t_0),\sigma) + \left| \int_{t_0}^{t} \|\partial_t v_j(\tau)\|_g \,\mathrm{d}\tau \right|,\\
& && \leqslant d_g(v_j(t_0),\sigma) + \int_{\mathbf{R}} \|\partial_t v_j(\tau)\|_g \,\mathrm{d}\tau,\\
&\text{[using Lemma 3.4]} && \leqslant d_g(v_j(t_0),\sigma) + \delta(v_j{}^-) - \delta(v_j{}^+),
\end{aligned}$$

thus, since the last bound is uniform in t, **Step 1** and **Lemma 4.1** yield **Step 3**.

The above steps directly yield the result. ∎

Proposition 4.7. *There exists a sequence $(h_j)_{j\in\mathbf{N}}$ of horizontal fragments such that*

- *there exists $t_j > 0$ such that h_j is parametrized by $[0, t_j]$;*
- *the horizontal fragment h_j is recovered from $(\gamma_k)_{k\in\mathbf{N}}$; and*
- *there exists $t_j{}^* \in [0, t_j]$ such that $h_j(t_j{}^*) \in S_\delta{}^{\prec}$.*

Moreover, these data satisfy $t_j \underset{j}{\longrightarrow} 0$ and $h_j \underset{j}{\longrightarrow} \sigma$ in the C^0-topology.

Proof. All the technical work has already been done, the proof only amouts to:

- **Step 1.** Since Λ is chord generic, from **Definition 1.12 (4)** and **Proposition 2.3**, there exists an open neighbourhood U of σ in S_δ such that U contains no critical point of δ. Moreover, using **Proposition 4.6**, there exists a subsequence (denoted in the same way) such that for all nonnegative integers j, the vertical fragment v_j has range in U.
- **Step 2.** Using **Step 1**, for all nonnegative integers j, applying **Theorem 4.2** recovers a horizontal fragment $h_j\colon [0,t_j] \to S_\delta$ from $(\gamma_k)_{k\in\mathbf{N}}$ such that $h_j(0) = v_j{}^+$.
- **Step 3.** Applying **Proposition 4.1** for all nonnegative integers j shows that there exists a vertical fragment $w_j\colon \mathbf{R} \to B \times \mathbf{R}^{2N}$ recovered from $(\gamma_k)_{k\in\mathbf{N}}$ and such that $w_j{}^- = h_j(t_j)$. Then, mimicking the proof of **Lemma 4.1** shows that there exists a subsequence $(w_j)_{j\in\mathbf{N}}$ (denoted in the same way) such that for all nonnegative integers j, w_j has range in U.
- **Step 4.** Since Λ is assumed to be at most $\Sigma^{1,0}$-singular, using **Proposition 2.2** shows that it is possible shrink U so that $\pi_B(U) \setminus \pi_B(S_\delta{}^{\prec} \cap U)$ has two connected components, but according to **Step 3**, for all integers $j \geqslant 0$, the horizontal fragment h_j has range in U. Therefore, for connexity reasons, there must exist $t_j{}^* \in [0,t_j]$ such that $h_j(t_j{}^*) \in S_\delta{}^{\prec}$.
- **Step 5.** Since all the horizontal fragments constructed in **Step 2** are recovered from a same Henry–Rutherford sequence, mimicking the energetical argument from the proof of **Lemma 4.1** shows that $t_j \underset{j}{\longrightarrow} 0$.
- **Step 6.** Using **Proposition 4.6** and **Step 5** shows that $h_j \underset{j}{\longrightarrow} \sigma$ in the C^0-topology.

The above steps yields the result. ∎

Remark. It is crucial that all the $(h_j)_{j\in\mathbf{N}}$ are recovered from a same Henry–Rutherford sequence. Otherwise, Step 5 and Step 6 fails to be true and without $t_j \underset{j}{\longrightarrow} 0$ and $h_j \underset{j}{\longrightarrow} \sigma$ in the C^0-topology, it is impossible to contradict gradient genericity and end the proof of Theorem 4.3.

2.2. Proof of Theorem 4.3.

The proof of Theorem 4.3 now only amounts to show that the sequence of horizontal fragments constructed in Proposition 4.7 contradicts the gradient genericity of the Legendrian submanifold. A similar statement, but without a complete proof, is given in [Ekh07, Lemma 3.10].

Proof of Theorem 4.3. Constructions from Chapter 1 are transported to S_δ by Proposition 2.2. Besides, by exchanging the generating families in δ, it can be assumed, without loss of generality, that the the singular point σ, constructed in Proposition 4.6, belongs to $L_0^{i,\beta}$ for some i and β. With these preliminaries at hand, the proof now goes through the following steps:

- **Step 1.** For all $\theta > 0$, let us recursively defined $L_m^{i,\beta}(\theta)$ on m by
  - For $m = 0$, $L_0^{i,\beta}(\theta) = L_0^{i,\beta}$;
  - For $m \geqslant 1$, assume that $L_m^{i,\beta}(\theta)$ is constructed and let $L_{m+1}^{i,\beta}(\theta)$ be the set of points at which the angle between $X_{i,k}$ and $T\Pi_B^{[2]}(TL_m^{i,\beta})$ is less or equal to θ.

  By construction, for all nonnegative integers m, $L_m^{i,\beta}(\theta)$ is an open neighbourhood of $L_m^{i,\beta}$. Then, using Proposition 4.7, for all nonnegative integers j, $h_j({t_j}^*) \in L_0^{i,\beta}(\theta)$.
- **Step 2.** For all nonnegative integers m, there exists a subsequence (still denoted in the same way) of the sequence $(h_j)_{j\in\mathbf{N}}$ constructed in Proposition 4.7 such that

  $$h_j({t_j}^*), h_{j+1}({t_{j+1}}^*) \in L_m^{i,\beta}(\theta) \implies h_j({t_j}^*) \in L_{m+1}^{i,\beta}(\theta).$$

  Otherwise, when t_j is small enough, using Taylor's formula shows that $h_j(t_j) \notin L_m^{i,\beta}(\theta)$, but if $h_j(t_j)$ and $h_{j+1}({t_{j+1}}^*)$ are close enough, then $h_{j+1}({t_{j+1}}^*) \notin L_m^{i,\beta}(\theta)$, a contradiction. Indeed, intuitively picture a flow line leaving a submanifold with a certain angle, then the time taken by the flow line to come back to the submanifold is at least proportionnal to this angle.
- **Step 3.** Show by induction on m that for all integers $m \geqslant 0$, there exists an integer $J_m \geqslant 0$ such that for all integers $j \geqslant J_m$, $h_j({t_j}^*) \in L_m^{i,\beta}(\theta)$.
  - **Base step.** For $m = 0$, the proof is already contained in Step 1.
  - **Inductive step.** Let $m \geqslant 0$ and assume that there exists an integer $J_m \geqslant 0$ such that

    $$\forall j \in \mathbf{N}, j \geqslant J_m \implies h_j({t_j}^*) \in L_m^{i,\beta}(\theta),$$

    then applying Step 2 directly yields the inductive step.

Since Λ is gradient generic, applying Definition 1.11 shows that $L_m^{i,\beta}$ is empty whenever $m \geqslant n$, and thus, if $\theta > 0$ is small enough, then for all nonnegative integers $m \geqslant n$, $L_m^{i,\beta}(\theta)$ is also empty. However, applying Step 3 and Proposition 4.7 shows that $\sigma \in L_n^{i,\beta}(\theta)$, which is a contradiction, thus yielding a result.

To conclude this proof, notice that none of these steps relies on the statement of Theorem 4.2, which was indeed a crucial requirement. Indeed, showing that horizontal fragments cannot flow along the singular locus $S_\delta{}^{\prec}$ uses a toy model of this reasoning, where the sequence of horizontal fragments is obtained by chopping into small pieces the original horizontal fragment. ■

Remark. It is necessary to work with $L_m^{i,\beta}(\theta)$ rather than $L_m^{i,\beta}$, otherwise **Step 2** fails.

3. Proof of the Floer-Gromov compactness result

Proposition 4.1, **Theorem 4.2** and **Theorem 4.3** are now used to prove **Theorem 4.1**.

Proof of Theorem 4.1. All the technical work has already been done, the proof only amounts to:

- **Step 1.** Since Λ is gradient generic, **Theorem 4.3** applies and shows that only a finite number of non-equivalent and nonconstant vertical fragments can be recovered from recursive use of **Proposition 4.1**.
- **Step 2.** According to **Step 1**, **Proposition 4.1** can be recursively apply to recover all the possible vertical fragments $(v_1, \dots, v_m)$, ordered by decreasing value of δ.

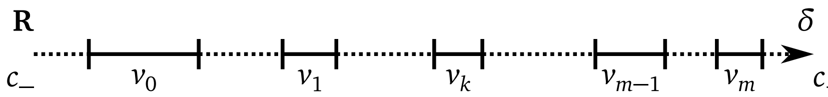


Figure 23. Schematic representation of the intervals provided by **Step 2**.

- **Step 3.** Apply **Theorem 4.2** in the complement of the union of intervals from **Step 2** to construct horizontal fragments $(h_0, \dots, h_m)$ such that

$$\forall k \in \{1, \dots, m-1\}, {h_k}^- = {v_k}^+, {h_k}^+ = {v_{k+1}}^-,$$

  and ${h_0}^- = c_-$, ${h_0}^+ = {v_1}^-$, and ${h_m}^- = {v_m}^+$, ${h_m}^+ = c_+$.
- **Step 4.** Construct a gradient staircases from the data provided by **Step 2** and **Step 3**. For that purpose, notice that the subset of $k \in \{1, \dots, m\}$ such that either h_k^- or h_k^+ is a critical point of δ uniquely define an ordered partition

$$\begin{cases} \mathbf{e}_1 = (h_{0,1}, v_{1,1}, h_{1,1}, \dots, v_{m_1,1}, h_{m_1,1}), \\ \qquad \dots \\ \mathbf{e}_k = (h_{0,k}, v_{1,k}, h_{1,k}, \dots, v_{m_k,k}, h_{m_k,k}), \\ \qquad \dots \\ \mathbf{e}_r = (h_{0,r}, v_{1,r}, h_{1,r}, \dots, v_{m_r,1}, h_{m_r,r}), \end{cases}$$

  of the ordered tuple $(h_0, v_1, h_1, \dots, v_m, h_m)$, with possible missing or constant $h_{0,k}$ or $h_{m_k,k}$. If so, then $h_{0,k}$ or $h_{m_k,k}$ is defined to be a constant trajectory on a semi-infinite interval, equal to the relevant critical point of δ.

Since Λ is chord generic, applying **Proposition 2.3**, and according to **Definition 1.12 (1)** and **(2)**, the set of critical values of δ is finite and injective, thus for all $k \in \{1, \dots, r-1\}$, ${h_{m_k,k}}^+ = {h_{0,k+1}}^-$. In conclusion, $\underline{\mathbf{e}} = (\mathbf{e}_1, \dots, \mathbf{e}_\mathbf{r})$ is a gradient staircases chain, thus yielding the result. ■

5.

Examples

This chapter explains how gradient staircases can be used to carry homological computations with linear-at-infinity generating families of generic Legendrian submanifolds of any dimension. In Section 1, generating families for higher-dimensional Legendrian Hopf links are constructed, and simple and mixed versions of their generating family homology are computed in Section 2. Following [BST15, Section 6.2] and [BG19, Section 4], these examples are the main and essential starting point needed to tackle Conjecture 2.2.

1. Constructions of generating families

Higher-dimensional analogues $\Lambda^n_{(2)}$ of the standard Legendrian Hopf link of $(J^1\mathbf{R}, \xi_{\mathbf{R}})$ are defined, and two different linear-at-infinity generating families $f^n_{//}$ and $f^n_{\#}$ of $\Lambda^n_{(2)}$ are constructed.

For all integers $n \geqslant 1$, the *standard Legendrian Hopf link* $\Lambda^n_{(2)}$ of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$ is defined via its front. More precisely, the front projection of $\Lambda^n_{(2)}$ consists in two overlapping vertically aligned copies of the front projection of the standard unknotted Legendrian sphere Λ^n_0, defined in Example 2.2.

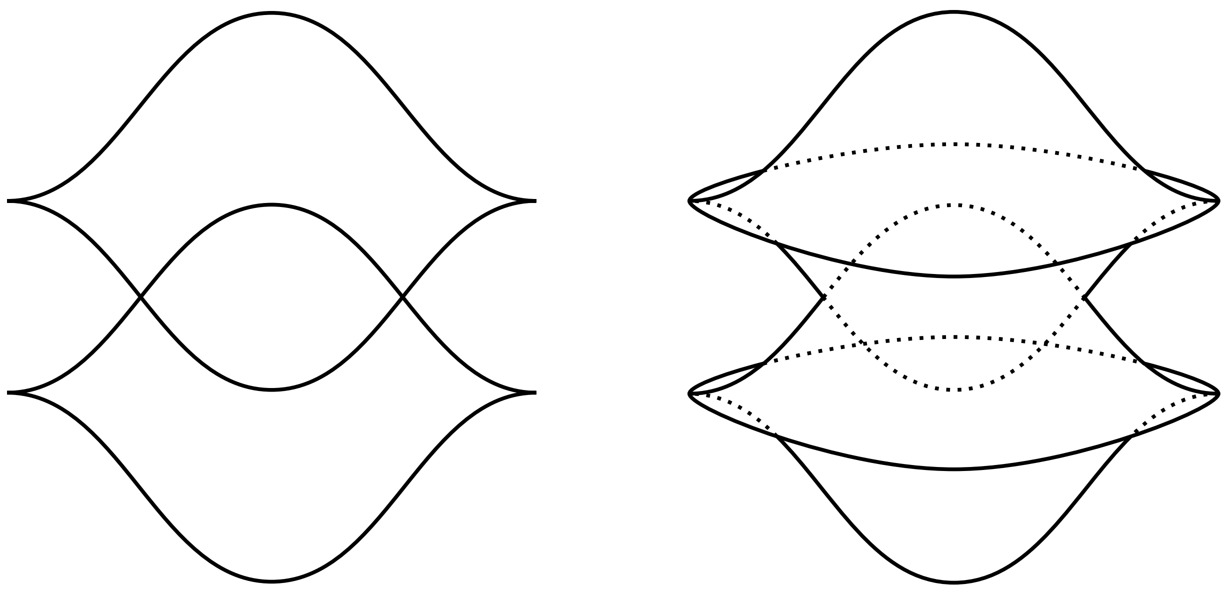


Figure 24. Front projections of $\Lambda^1_{(2)}$ (on the left) and $\Lambda^2_{(2)}$ (on the right).

Recall from Example 2.2 that Λ^n_0 has a generating family $f^n_0 : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by

$$f^n_0(b, \eta) = \eta^3 - 3(\|b\|^2 - 1)\eta,$$

then use suitable cut-off functions to turn f^n_0 into a linear-at-infinity generating family $\widetilde{f}^n_0$ of Λ^n_0. In this section, all the generating families constructed for $\Lambda^n_{(2)}$ will be modelled on $\widetilde{f}^n_0$.

- **Parallel copies and spinning of the standard Legendrian unknot: $f^n_{//}$.**

Let us first construct a generating family $f^n_{//}$ of $\Lambda^n_{(2)}$ by two taking parallel copies of $\widetilde{f}^n_0$.

- **Step 1.** Shrink $\widetilde{f}^n_0$ to define a linear-at-infinity generating family f^1_1 of $\frac{1}{2}\Lambda^1_0$.
- **Step 2.** Stabilise $\widetilde{f}^n_0$ by a negative one-dimensional quadratic form to produce a $+1$ shift of its fibrewise Morse indices and then use a fibre-preserving diffeomorphism to recover a linear-at-infinity generating family f^1_2 of Λ^1_0.
- **Step 3.** Stabilise f^1_1 by a positive one-dimensional quadratic form and then use a fibre-preserving diffeomorphism to recover a linear-at-infinity generating family $\widetilde{f}^1_1$ of $\frac{1}{2}\Lambda^1_0$, but now the fibre dimensions of $\widetilde{f}^1_1$ and f^1_2 are the same.
- **Step 4.** Shift slightly downward the values of $\widetilde{f}^1_1$ to define F^1_1.
- **Step 5.** Translate f^1_2 in the fibre to define F^1_2 such that $\operatorname{supp} F^1_1 \cap \operatorname{supp} F^1_2 = \varnothing$.
- **Step 6.** Define $f^1_{//}$ as the sum of F^1_1 and F^1_2 in the sense of [ST13, Definition 3.18].
- **Step 7.** Construct $f^n_{//} = F^n_1 + F^n_2$ by iteratively spinning $f^1_{//}$ [BST15, Proposition 3.2].

For schematic representations of some of these steps, see Figure 25.

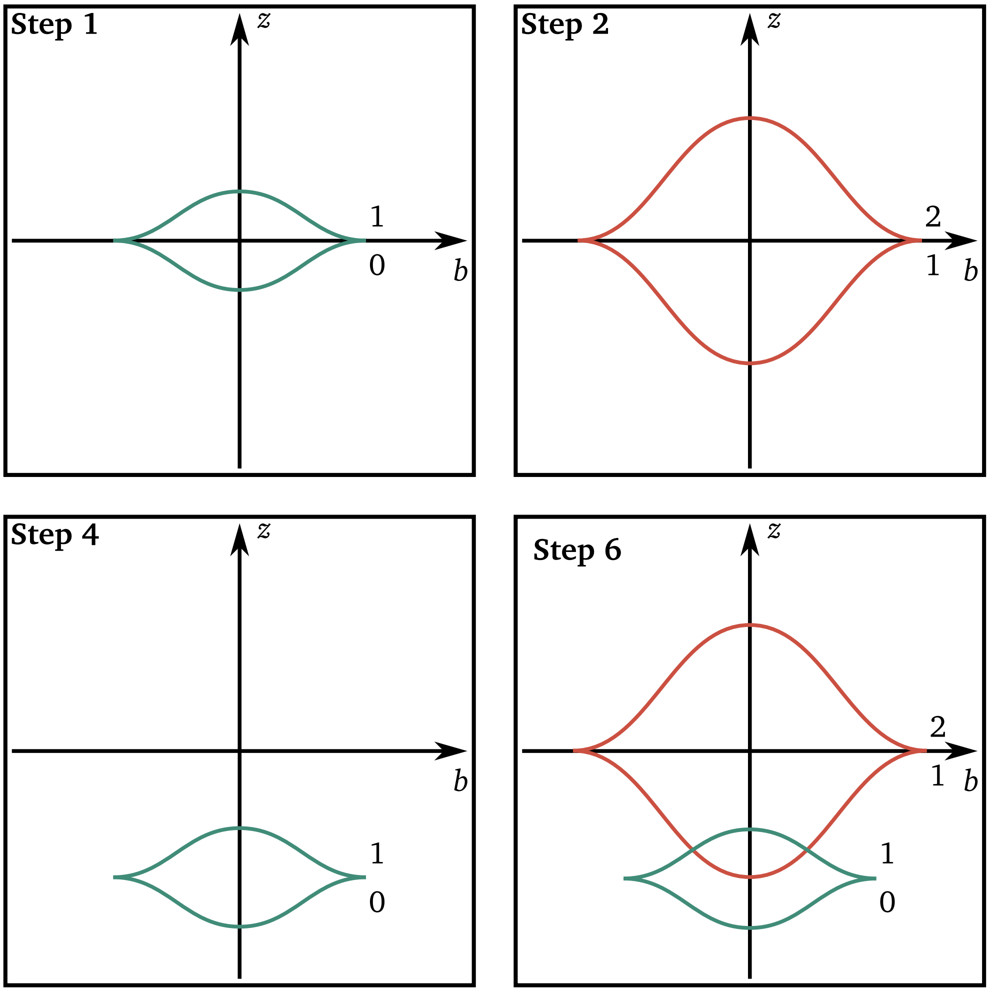


Figure 25. Construction of the generating family $f^n_{//}$.

Remark. Step 1 prevents $f^n_{//}$ from having a continuum of positively valued critical points.

Remark. Because of Step 5, the generating family $f^n_{//}$ has no handleslide, since it is obtained from generating families whose support are disjoint.

- **Surgery and spinning on the standard Legendrian unknot: $f^n_\#$.**

Let us now construct a generating family $f^n_\#$ of $\Lambda^n_{(2)}$ by isotopy and surgery on $\widetilde{f}^n_0$.

- **Step 1.** Perform a one-dimensional negative stabilisation of $\widetilde{f}^n_0$ to produce a $+1$ shift of its fibrewise Morse indices and then use a suitable fibred diffeomorphism to still recover a linear-at-infinity generating family of Λ^1_0.
- **Step 2.** Make two upside-down Reidemeister I moves on the lower smooth sheet of Λ^1_0 and then use [Swi92, Theorem B] and Theorem 2.1 to recover a new linear-at-infinity generating family of Λ^1_0.
- **Step 3.** Take a connected sum between the inner cusp points of the Reidemeister I moves and use [BST15, Theorem 4.2] to recover a linear-at-infinity generating family of $\Lambda^1_{(2)}$.
- **Step 4.** Use a one-parameter version of the usual Whitney trick [HW73, Lemma 1.2'] to simplify the pair of handleslides located in-between the crossings of the front of $\Lambda^1_{(2)}$. This construction provides a linear-at-infinity generating family $f^1_\#$ of $\Lambda^1_{(2)}$.
- **Step 5.** Construct $f^n_\#$ by iteratively spinning $f^1_\#$ [BST15, Proposition 3.2].

For schematic representations of some of these steps, see Figure 26.

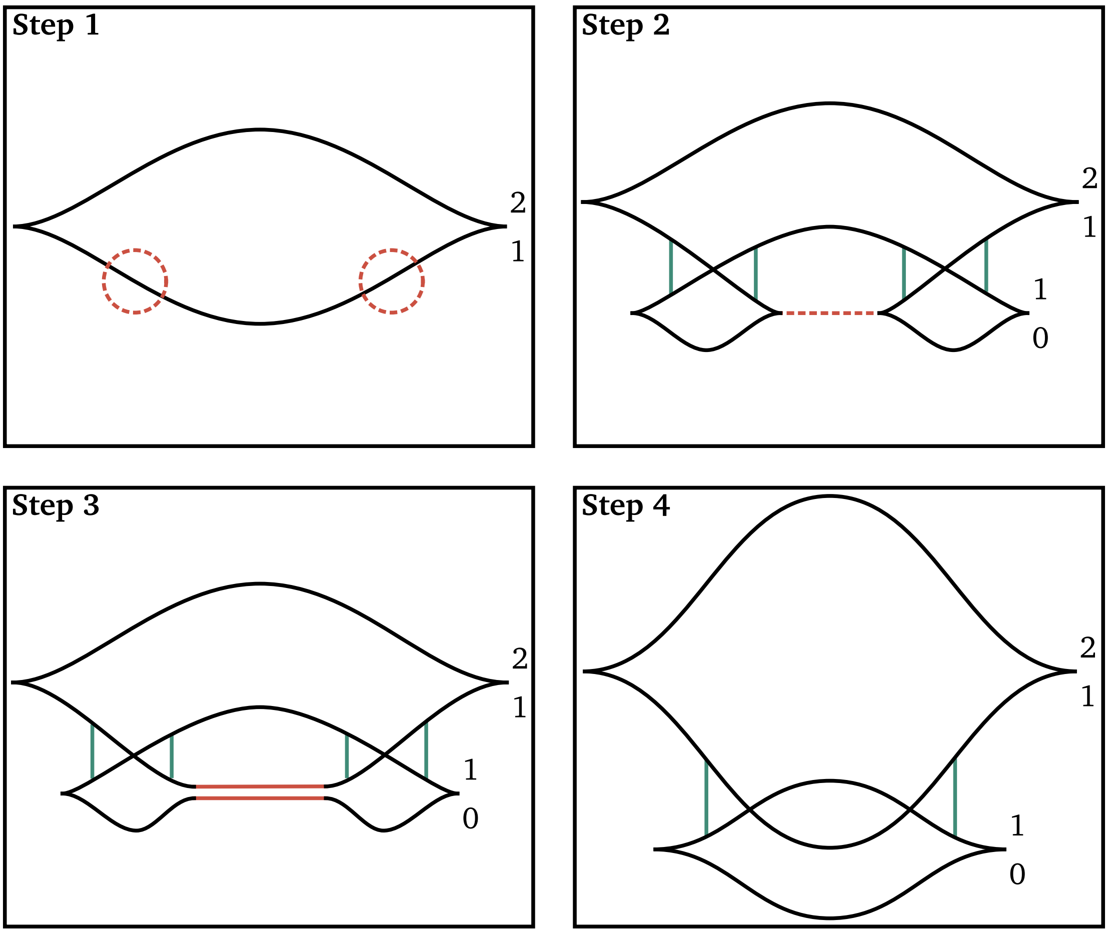


Figure 26. Construction of the generating family $f^n_\#$ (handleslides in green).

Remark. A specific construction provides one more linear-at-infinity generating family for $\Lambda^1_{(2)}$, which has a unique handleslide that is located in-between the crossings of its front projection.

2. Homological computations

To compute $\mathrm{GFH}_\bullet(f^n_{//})$, $\mathrm{GFH}_\bullet(f^n_{\#})$ and $\mathrm{GFH}_\bullet(f^n_{//}, f^n_{\#})$, Conjecture 3.1 is now assumed to hold true. According to [BST15, Section 6.2] and [BG19, Section 4], these computations constitute the main ingredients to settle Conjecture 2.2 and they also provide evidence towards Conjecture 2.1.

- **Generators and grading of the generating family chain complex.**

Let Λ be a chord generic Legendrian submanifold of (J^1B, ξ_B), let δ be the difference function of any pair of linear-at-infinity generating families of Λ and let $(C_\bullet, \partial)$ be the associated generating family chain complex.

According to Proposition 2.3, the grading and the generators of $(C_\bullet, \partial)$ only depend on Λ itself, and [EES05b, Lemma 3.4] shows that the grading can also be computed on the generators as

$$\mu(b, \eta_1, \eta_2) = \mathrm{ind}_{(\eta_1,\eta_2)} \delta_b + \mathrm{ind}_b \delta_{(\eta_1,\eta_2)} - 1,$$

where the first term is the difference of the fibrewise Morse indices of the generating families, and the second one is the Morse index of the vertical height function between sheets.

Observe that $\Lambda^n_{(2)}$ is not chord generic, since all its Reeb chords are located above a same point, but Proposition 1.3 shows there exists a Legendrian representative of $\Lambda^n_{(2)}$ that is chord generic. For the rest of this section, it is assumed that $\Lambda^n_{(2)}$ has been appropriately perturbed.

The number of critical points of δ is the number of pairs of smooth sheets of $\Lambda^n_{(2)}$, namely $\binom{4}{2} = 6$, and their gradings are given in the following table

Critical point	c_{12}	c_{11}	c_{22}	M_{12}	c_{21}	m_{12}
Grading	$n+1$	n	n	n	$n-1$	0

If $n \geqslant 2$, then using gradings: $\partial c_{12} \in \langle c_{11}, c_{22}, M_{12}\rangle$, $\partial c_{11}, \partial c_{22}, \partial M_{12} \in \langle c_{21}\rangle$ and $\partial m_{12} = \partial c_{21} = 0$, and these observations will be essential to compute $\mathrm{GFH}_\bullet(f^n_{//})$, $\mathrm{GFH}_\bullet(f^n_{\#})$ and $\mathrm{GFH}_\bullet(f^n_{//}, f^n_{\#})$.

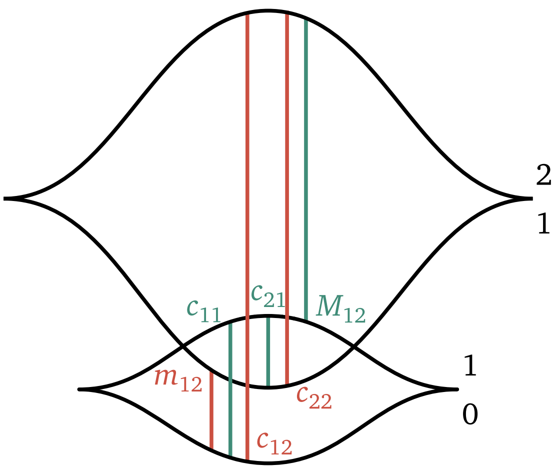


Figure 27. (Front of the) standard Legendrian Hopf link and its offset Reeb chords.

- **Simple generating family homology of $f^n_{//}$.**

Let us compute the differential of the generating family chain complex associated to $f^n_{//}$.

Proposition 5.1. *When $n \geqslant 2$, the chain complex differential $\partial^n_{//}$ associated to $f^n_{//}$ is given by*

$$\partial^n_{//} c_{12} = M_{12},$$

and the other generators (c_{11}, c_{22}, M_{12}, c_{21} and m_{12}) are sent to 0 by $\partial^n_{//}$.

Proof. Let us recall that $\partial^n_{//} c_{12} \in \langle c_{11}, c_{22}, M_{12} \rangle$, $\partial^n_{//} c_{11}, \partial^n_{//} c_{22}, \partial^n_{//} M_{12} \in \langle c_{21} \rangle$ and $\partial^n_{//} m_{12} = \partial^n_{//} c_{21} = 0$, and the proof amounts to describe all the possible gradient staircases between these generators. For this purpose, it suffices to examine the smooth sheets between which these generators lie, since gradient staircases decrease chord length and are determined by their vertical fragments. Therefore, since $f^n_{//}$ has no handleslides, vertical fragments can only be obtained from birth-death gradient trajectories of F^n_1 or F^n_2 [Cha18], and cannot jump between different components of $\Lambda^n_{(2)}$. In particular, the differential $\partial^n_{//}$ must preserve the indices decorating the generators.

- Let us first describe the gradient staircases from c_{12} to c_{11}, c_{22} and M_{12}.
  - It is not possible to reach c_{11} from c_{12}.
  - It is not possible to reach c_{22} from c_{12}.
  - There exists a unique gradient staircase from c_{12} to M_{12}. It starts with a semi-infinite horizontal fragment from c_{12} to the chord located above the base projection of M_{12}, followed by a vertical fragment made from the birth-death gradient trajectory of F^n_2, and it ends with a semi-infinite constant horizontal fragment at M_{12}.

  Therefore, $\partial^n_{//} c_{12} = M_{12}$.
- It is not possible to reach c_{21} from c_{11}, and thus $\partial^n_{//} c_{11} = 0$.
- It is not possible to reach c_{21} from c_{22}, and thus $\partial^n_{//} c_{22} = 0$.
- It is not possible to reach c_{21} from M_{12}, and thus $\partial^n_{//} M_{12} = 0$.

Conjecture 3.1 thus yields the result. ∎

In particular, Proposition 5.1 shows that the generating family homology of $f^n_{//}$ is equal to

$$\begin{aligned} \mathrm{GFH}_\bullet(f_{//}) &= \langle c_{11}, c_{22}, c_{21}, m_{12} \rangle, \\ &\simeq \mathbf{F}_2{}^2[n] \oplus \mathbf{F}_2[n-1] \oplus \mathbf{F}_2[0]. \end{aligned}$$

Moreover, the Poincaré's polynomial of $f^n_{//}$ is equal to $\Gamma_{f^n_{//}}(t) = 2t^n + t^{n-1} + 1$ (one dual pair).

- **Simple generating family homology of $f^n_{\#}$.**

Let us compute the differential of the generating family chain complex associated to $f^n_{\#}$.

Proposition 5.2. *When $n \geqslant 2$, the chain complex differential $\partial^n_{\#}$ associated to $f^n_{\#}$ is given by*

$$\begin{cases} \partial^n_{\#} c_{12} = c_{11} + c_{22} + M_{12}, \\ \partial^n_{\#} c_{11} = c_{21}, \\ \partial^n_{\#} c_{22} = c_{21}, \end{cases},$$

and the other generators (M_{12}, c_{21} and m_{12}) are sent to 0 by $\partial^n_{\#}$.

Proof. Let us recall that $\partial_\#^n c_{12} \in \langle c_{11}, c_{22}, M_{12}\rangle$, $\partial_\#^n c_{11}, \partial_\#^n c_{22}, \partial_\#^n M_{12} \in \langle c_{21}\rangle$ and $\partial_\#^n m_{12} = \partial_\#^n c_{21} = 0$, and the proof amounts to describe all the possible gradient staircases between these generators. For this purpose, it suffices to examine the smooth sheets between which these generators lie, since gradient staircases decrease chord length and are determined by their vertical fragments. Therefore, since $f_\#^n$ has handleslides, vertical fragments can be obtained

- from birth-death gradient trajectories of F_1^n or F_2^n [Cha18];
- from handleslides of $f_\#^n$; or
- by concatenating these two types of gradient trajectories.

In particular, vertical fragments can now jump between the two connected components of $\Lambda_{(2)}^n$ and the differential $\partial_\#^n$ no longer preserves the indices decorating the generators.

- Let us first describe the gradient staircases from c_{12} to c_{11}, c_{22} and M_{12}.
  - There exists a unique gradient staircase from c_{12} to c_{11}. It starts with a semi-infinite horizontal fragment from c_{21} to the chord located above the base projection of c_{11}, followed by a vertical fragment obtained by concatenating the birth-death gradient trajectory of F_1^n with an upward handleslide of $f_\#^n$, and it ends with a semi-infinite constant horizontal fragment at c_{11}.
  - There exists a unique gradient staircase from c_{12} to c_{22}. It starts with a semi-infinite horizontal fragment from c_{12} to the chord located above the base projection of c_{22}, followed by a vertical fragment obtained by concatenating the birth-death gradient trajectory of F_2^n with a downward handleslide of $f_\#^n$, and it ends with a semi-infinite constant horizontal fragment at c_{22}.
  - As in Proposition 5.1, there exists a unique gradient staircase from c_{12} to M_{12}.

  Therefore, $\partial_\#^n c_{12} = c_{11} + c_{22} + M_{12}$.
- There exists a unique gradient staircase between c_{11} and c_{21}. It starts with a semi-infinite horizontal fragment from c_{11} to the chord that is located above the base projection of c_{21}, followed by a vertical fragment obtained from the birth-death gradient trajectory of F_1^n and an upward handleslide of $f_\#^n$, and it ends with a semi-infinite constant horizontal fragment at c_{21}.
- There exists a unique gradient staircase between c_{22} and c_{21}. It starts with a semi-infinite horizontal fragment from c_{22} to the chord that is located above the base projection of c_{21}, followed by a vertical fragment obtained from the birth-death gradient trajectory of F_2^n and a downward handleslide of $f_\#^n$, and it ends with a semi-infinite constant horizontal fragment at c_{21}.
- It is not possible to reach c_{21} from M_{12}, and thus $\partial_\#^n M_{12} = 0$.

Conjecture 3.1 thus yields the result. ∎

In particular, Proposition 5.2 shows that the generating family homology of $f_\#^n$ is equal to

$$\begin{aligned} \mathrm{GFH}_\bullet(f_\#^n) &= \langle c_{11} + c_{22}, m_{12}\rangle, \\ &\simeq \mathbf{F}_2[n] \oplus \mathbf{F}_2[0]. \end{aligned}$$

Moreover, the Poincaré's polynomial of $f_\#^n$ is equal to $\Gamma_{f_\#^n}(t) = t^n + 1$ (no dual pair).

Remark. Proposition 2.5 implies that $f_{//}^n$ and $f_\#^n$ are nonequivalent, since $\mathrm{GFH}_\bullet(f_{//}^n) \not\simeq \mathrm{GFH}_\bullet(f_\#^n)$.

- **Mixed generating family homology of $(f^n_{//}, f^n_{\#})$.**

Let us compute the differential of the generating family chain complex of $(f^n_{//}, f^n_{\#})$.

Proposition 5.3. *When $n \geqslant 2$, the chain complex differential $\partial^n_{//,\#}$ associated to $(f^n_{//}, f^n_{\#})$ is given by*

$$\begin{cases} \partial^n_{//,\#} c_{12} = c_{22} + M_{12}, \\ \partial^n_{//,\#} c_{11} = c_{21}, \end{cases},$$

and the other generators (c_{22}, M_{12}, c_{21} and m_{12}) are sent to 0 by $\partial^n_{//,\#}$.

Proof. Since $f^n_{//}$ has no handleslide and $f^n_{\#}$ appears with a negative sign in the difference function, it suffices to mimick the proof Proposition 5.2 and remove all the gradient staircases containing at least one vertical fragment constructed by going downward an handleslide of $f^n_{\#}$. ∎

In particular, Proposition 5.3 shows that the generating family homology of $(f^n_{//}, f^n_{\#})$ is equal to

$$\begin{aligned} \mathrm{GFH}_\bullet(f^n_{//}, f^n_{\#}) &= \langle M_{12}, m_{12} \rangle, \\ &\simeq \mathbf{F}_2[n] \oplus \mathbf{F}_2[0]. \end{aligned}$$

Moreover, the Poincaré's polynomial of $(f^n_{//}, f^n_{\#})$ is equal to $\Gamma_{f^n_{//}, f^n_{\#}}(t) = t^n + 1$ (no dual pair).

Remark. Using the long exact sequence (LES-mGFH) shows that $\mathrm{GFH}_\bullet(f^n_{\#}, f^n_{//}) \simeq \mathrm{GFH}_\bullet(f^n_{//}, f^n_{\#})$.

Example 5.1. To conclude, let us investigate a connected example constructed from $f^n_{//}$ and $f^n_{\#}$. For all integers $n \geqslant 1$, let us define a connected Legendrian submanifold Λ^n_1 from $\Lambda^n_{(2)}$, as follows:

- **Step 1.** Start from $\Lambda^n_{(2)}$.
- **Step 2.** Perform a Legendrian isotopy to stretch to the right the lower copy of Λ^n_0 in $\Lambda^n_{(2)}$.
- **Step 3.** Make an upward Reidemeister I move on the upper sheet of this stretched Λ^n_0 to adjust the fibrewise Morse indices of the generating family.
- **Step 4.** Take a connected sum between the two facing cusp-edges.

For schematic representations of some of these steps, see Figure 28.

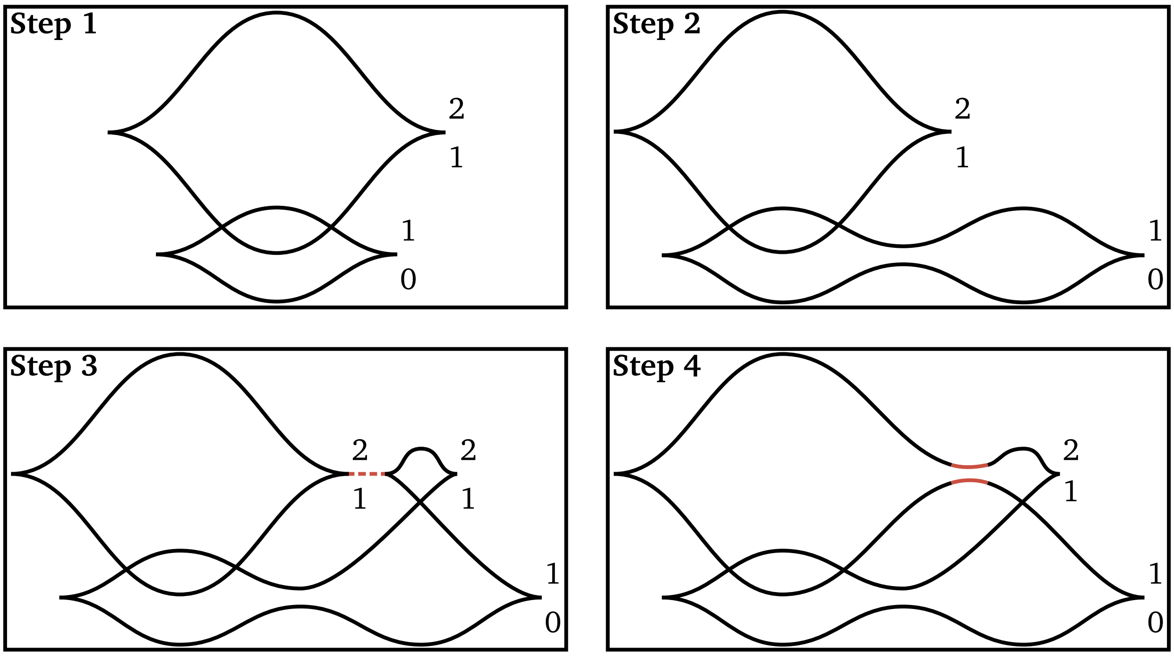


Figure 28. Construction of Λ^n_1 from $\Lambda^n_{(2)}$.

In particular, Λ_1^1 has the smooth knot type of a right-handed trefoil knot, see Figure 29.

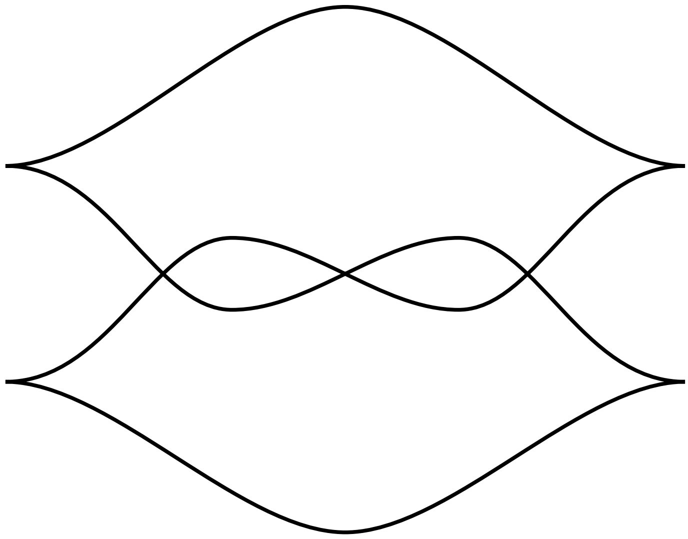


Figure 29. Front projection of a Legendrian right-handed trefoil knot.

Applying these steps to $f_{//}^n$ and $f_{\#}^n$ provides linear-at-infinity generating families $F_{//}^n$ and $F_{\#}^n$ of Λ_1^n. By mimicking the proof of [BG19, Proposition 3.5] for the long exact sequence (LES-mGFH) and identifying the manifold classes of $\mathrm{GFH}_\bullet(f_{//})$, $\mathrm{GFH}_\bullet(f_{\#})$ and $\mathrm{GFH}_\bullet(f_{//}, f_{\#})$ with gradient staircases, the Poincaré's polynomials of $F_{//}^n$, $F_{\#}^n$ and $(F_{//}^n, F_{\#}^n)$ are computed from these of $f_{//}^n$, $f_{\#}^n$ and $(f_{//}^n, f_{\#}^n)$. It is found that $\Gamma_{F_{//}^n}(t) = t^n + t^{n-1} + 1$, $\Gamma_{F_{\#}^n}(t) = t^n + t^{n-1} + 1$ and $\Gamma_{F_{//}^n, F_{\#}^n}(t) = 1$.

In particular, Proposition 2.5 does not allow to state whether $F_{//}^n$ and $F_{\#}^n$ are equivalent or not, but as as predicted by Conjecture 2.1, Theorem 2.2 shows that $F_{//}^n$ and $F_{\#}^n$ are not equivalent, since $\dim \mathrm{GFH}_\bullet(F_{//}^n, F_{\#}^n) \neq \dim \mathrm{GFH}_\bullet(F_{//}^n)$.

In conclusion, the simple version of generating family homology does not distinguish $F_{//}^n$ from $F_{\#}^n$, but the mixed version of generating family homology does.

Research prospects

This work opens several natural research directions, some of them have already been discussed. Not only this short chapter gathers them in one place, it also offers further open questions that could likely be settled by elaborating on the scientific content already provided in this dissertation. These research prospects are discussed by increasing level of accessibility, and for the first ones, guidelines to work towards their proofs are also developped.

1. Answering the Henry-Rutherford conjecture

The starting point of this thesis was the Henry-Rutherford conjecture, whose statement is:

Conjecture. *Let c_- and c_+ be positively valued critical points of δ such that $\mu(c_-) = \mu(c_+) - 1$ holds. There exists $s_0 \in (0,1]$ such that for all $s \in (0,s_0]$, $\mathscr{M}(c_-, c_+; \delta, g_s)$ and $\mathscr{M}^{\mathrm{st}}(c_-, c_+; g, \delta)$ are both finite and in one-to-one correspondence.*

It was adressed by a compactness and gluing strategy, but only the compactness part was tackled. Completely settling the Henry-Rutherford conjecture still requires to deal with the gluing result, whose statement reads as follows:

Conjecture. *Let c_- and c_+ be positively valued critical points of δ, and let also $e \in \mathscr{M}^{\mathrm{st}}(c_-, c_+; \delta, g)$. If Λ is generic, there exists $s_0 \in (0,1]$ such that for all $s \in (0,s_0]$, there exists $\gamma_s \in \mathscr{M}(c_-, c_+; \delta, g_s)$ such that*

$$\gamma_s \xrightarrow[s \to 0]{} e,$$

where the previous convergence occurs in the Floer-Gromov topology introduced in Definition 3.4. Moreoever, if $\dim \mathscr{M}^{\mathrm{st}}(c_-, c_+; \delta, g) = 0$, then the above one-parameter family is unique.

Showing this conjecture should be similar to the proof developped for [BO09a, Proposition 4.22], and should roughly go through the following steps:

- **Step 1.** Construct some weighted Sobolev spaces to translate the Floer-Gromov topology.
- **Step 2.** Show that gradient staircases are solutions of some Fredholm operator between the Sobolev spaces constructed in Step 1.
- **Step 3.** Construct a smooth approximation of the gradient staircase.
- **Step 4.** Apply the Newton-Raphson method to the pregluing constructed in Step 3.

Before concluding this section, let us make some further remarks on the above blueprint:

- Proposition 4.2 plays a crucial role for Step 1.
- Working with weighted Sobolev spaces are needed to ensure that the Fredholm operator constructed in Step 3 has a right-inverse.
- Applying the Newton-Raphson method in Step 4 requires that the linearised Fredholm operator constructed in Step 2 has a right-inverse.

Showing the gluing conjecture, and thus the Henry-Rutherford conjecture, is the main follow-up.

2. Algebraic structure of the generating family homology

The main motivation behind gradient staircases and the associated Henry-Rutherford conjecture was to allow efficient computations of generating family (co)homology on concrete examples. Therefore, the next further questions raised by this dissertation concern homological invariants for generating families of Legendrian submanifolds.

2.1. Geography questions for the generating family homology.

With the constructions from [BG19], the geography result from [BST15, Theorems 1.1 and 1.2] should generalise to the mixed version of generating family homology, as follows:

Conjecture. *The following assertions hold true.*

(1) *Let $\Lambda^n \subset (J^1B, \xi_B)$ be a closed and connected Legendrian submanifold, and let f_1 and f_2 be two linear-at-infinity generating families of Λ. If f_1 and f_2 are non-equivalent, then there exist nonnegative integer coefficients polynomial q and Laurent polynomial p such that*

$$\Gamma_{f_1,f_2}(t) = q(t) + p(t),$$

where q has degree at most $n-1$ and $q(0) = 1$, and p further satisfies

$$\begin{cases} p(-1) \equiv 0 \bmod 2, & \text{if } n = 1, \\ p(-1) \leqslant \dfrac{1-q(-1)}{2}, & \text{if } n = 2, \end{cases} .$$

A Laurent polynomial satisfying all conditions above is mGFH-admissible.

(2) *Let $P(t)$ be a* mGFH*-admissible Laurent polynomial, then there exist an integer $n \geqslant 1$, a closed and connected Legendrian submanifold Λ of $(J^1\mathbf{R}^n, \xi_{\mathbf{R}^n})$, and two non-equivalent linear-at-infinity generating families f_1 and f_2 of Λ such that $P(t) = \Gamma_{f_1,f_2}(t)$.*

Chapter 5 already contains analogues of [BG19, Propositions 4.10 and 4.12].

According to [Mye18, Theorem 1.1], there exists a non-trivial associative product

$$\mu_2 \colon \mathrm{GFH}^k(f) \times \mathrm{GFH}^\ell(f) \to \mathrm{GFH}^{k+\ell}(f),$$

and [BST15, Theorems 1.1 and 1.2] already provides the graded vector space geography of $\mathrm{GFH}^\bullet$. Therefore, it is also natural to ask the following question.

Question. *What is the ring structure geography of* $\mathrm{GFH}^\bullet$*?*

2.2. Complete invariant for equivalence of generating families.

Recall from Theorem 2.4 that the generating family homology fits in the long exact sequence

$$\cdots \to \mathrm{GFH}_k(f_1, f_2) \xrightarrow{\tau_k} H_k(\Lambda; \mathbf{F}_2) \xrightarrow{\sigma_k} \mathrm{GFH}^{n-k}(f_2, f_1) \xrightarrow{\rho_k} \mathrm{GFH}_{k-1}(f_1, f_2) \to \cdots,$$

and according to [BST15, Theorem 6.1], if f_1 and f_2 are equivalent, then τ_n is surjective.

Conjecture. *Let $\Lambda^n \subset (J^1B, \xi_B)$ be a closed and connected Legendrian submanifold, let f_1 and f_2 be two linear-at-infinity generating families of Λ, then f_1 and f_2 are in the same equivalence class if, and only if, the map $\tau_n \colon \mathrm{GFH}_n(f_1, f_2) \to H_n(\Lambda; \mathbf{F}_2)$ is surjective.*

According to [BG19, Corollary 1.2], the pseudo-holomorphic version of this conjecture is true, since bilinearized Legendrian contact homology has been shown to be a complete invariant for DGA-homotopy of augmentations of the Chekanov-Eliashberg algebra.

2.3. Operations on generating family homology.

Following [Lim16], an addition and a multiplication are naturally defined on generating families, but the effect in homology of these operations has not yet been studied.

Question. *How do these operations descend in generating family homology?*

Theorem 4.1 does not apply to the Legendrian submanifolds constructed from these operations, since they generically have non-generic singularities, thus the above question requires more work than it could be first expected.

3. Generating families and augmentations

Understanding the correspondence between generating family and pseudo-holomorphic curves invariants of Legendrian submanifolds has been a fruitful ground for long-standing conjectures, whose most optimistic version states a derived category equivalence.

In particular, [FR11, Theorem 5.1] provides a partial answer to this question in dimension three, which is still conjectured to be true in higher dimension.

Conjecture. *Let f be a generating family of some connected Legendrian submanifold $\Lambda \subset (J^1B, \xi_B)$. If f is linear-at-infinity, then there exists an augmentation ε of Λ such that*

$$\mathrm{LCH}^{\varepsilon}_{\bullet}(\Lambda; \mathbf{F}_2) \simeq \mathrm{GFH}_{\bullet}(f),$$

as graded $\mathbf{F}_2$-vector spaces.

Provided the Henry-Rutherford conjecture holds, gradient staircases offers a very promising way to tackle this conjecture, since [HR13, Theorems 5.4 and 5.5] then recover [FR11, Theorem 5.1], but without relying too heavily on combinatorial arguments that are not robust to the dimension. In particular, the remaining difficulty is just showing that the candidate provided by gradient staircases is indeed an augmentation of the Chekanov-Eliashberg algebra.

Bibliography

[AB95] **D. M. Austin** and **P. J. Braam**, *Morse–Bott theory and equivariant cohomology*. In *The Floer memorial volume* (**H. Hofer**, **C. H. Taubes**, **A. Weinstein**, and **E. Zehnder**, eds.), Birkhäuser (1995), pp. 123–183.

[AD10] **M. Audin** and **M. Damian**, *Théorie de Morse et homologie de Floer*. EDP Sciences (2010).

[AG18] **D. Álvarez-Gavela**, *The simplification of singularities of Lagrangian and Legendrian fronts*. Invent. Math., vol. 214 (2018), pp. 641–737.

[AGI21] **D. Álvarez-Gavela** and **K. Igusa**, *A Legendrian Turaev torsion via generating families*. J. Éc. Polytech. - Math., vol. 8 (2021), pp. 57–119.

[AGZV85] **V. I. Arnol'd**, **S. M. Gusein-Zade**, and **A. N. Varchenko**, *Singularities of Differentiable Maps, Volume I: The Classification of Critical Points, Caustics and Wave Fronts*. Birkhäuser (1985).

[Arn86] **V. I. Arnol'd**, *First steps in symplectic topology*. Russian Mathematical Surveys, vol. 41 (1986), pp. 1–21.

[Arn89] ———, *Mathematical Methods of Classical Mechanics*. Springer (1989).

[BC14] **F. Bourgeois** and **B. Chantraine**, *Bilinearized Legendrian contact homology and the augmentation category*. J. Symplectic Geom., vol. 12 (2014), pp. 553–583,.

[BG19] **F. Bourgeois** and **D. Galant**, *Geography of bilinearized Legendrian contact homology*. arXiv preprint arXiv:1905.12037 (2019).

[BH13] **A. Banyaga** and **D. E. Hurtubise**, *Cascades and perturbed Morse–Bott functions*. Algebraic Geom. Topol., vol. 13 (2013), pp. 237–275.

[BO09a] **F. Bourgeois** and **A. Oancea**, *An exact sequence for contact- and symplectic homology*. Invent. Math., vol. 175 (2009), pp. 611–680.

[BO09b] ———, *Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces*. Duke Math. J., vol. 146 (2009), pp. 71–174.

[Boa67] **J. M. Boardman**, *Singularities of differentiable maps*. Publ. Math. Inst. Hautes Études Sci, vol. 33 (1967), pp. 383–419.

[Bou02] **F. Bourgeois**, *A Morse–Bott approach to contact homology*. Ph.D. thesis, Stanford University (2002).

[BST15] **F. Bourgeois**, **J. M. Sabloff**, and **L. Traynor**, *Lagrangian cobordisms via generating family: Construction and geography*. Algebraic Geom. Topol., vol. 15 (2015), pp. 2439–2477.

[CGH09] **V. Colin**, **E. Giroux**, and **K. Honda**, *Finitude homotopique et isotopique des structures de contact tendues*. Publ. Math. Inst. Hautes Études Sci., vol. 109 (2009), pp. 245–293.

[Cha84] **M. Chaperon**, *Une idée de type géodésique brisée pour systèmes hamiltoniens*. C.R. Acad. Sci. Paris Sér. I Math., vol. 298 (1984), pp. 293–296.

[Cha95] ———, *On generating families*. In *The Floer memorial volume* (**H. Hofer**, **C. H. Taubes**, **A. Weinstein**, and **E. Zehnder**, eds.), Birkhäuser (1995), pp. 123–183.

[Cha18] **A. Charel**, *Gradient flow line near birthdeath critical points*. J. Topol. Anal., vol. 12 (2018), pp. 1–45.

[Che96] **Y. V. Chekanov**, *Critical points of quasi-functions and generating families of Legendrian manifolds*. Funct. Anal. its Appl., vol. 30 (1996), pp. 118–128.

[Che02] ———, *Differential algebra of Legendrian links*. Invent. math., vol. 150 (2002), pp. 441–483.

[EES05a] **T. Ekholm**, **J. Etnyre**, and **M. G. Sullivan**, *The contact homology of Legendrian submanifolds in* $\mathbf{R}^{2n+1}$. J. Differential Geom., vol. 71 (2005), pp. 177–305.

[EES05b] ———, *Non-isotopic Legendrian submanifolds in* $\mathbf{R}^{2n+1}$. J. Differ. Geom., vol. 71 (2005), pp. 85–128.

[EES07] ———, *Legendrian contact homology in* $P \times \mathbf{R}$. Trans. Amer. Math. Soc., vol. 359 (2007), p. 33013335.

[EF09] **Y. Eliashberg** and **M. Fraser**, *Topologically trivial Legendrian knots*. J. Symplectic Geom., vol. 7 (2009), pp. 77–127.

[EGH00] **Y. Eliashberg**, **A. Givental**, and **H. Hofer**, *Introduction to Symplectic Field Theory*. In *Visions in Mathematics* (**N. Alon**, **J. Bourgain**, **A. Connes**, **M. Gromov**, and **V. D. Milman**, eds.), Birkhäuser (2000), pp. 560–673.

[EH01] **J. B. Etnyre** and **K. Honda**, *Knots and contact geometry I: torus knots and the figure eight knot*. J. Symplectic Geom., vol. 1 (2001), pp. 63–620.

[Ekh07] **T. Ekholm**, *Morse flow trees and Legendrian contact homology in* 1*-jet spaces*. Geom. Topol., vol. 11 (2007), pp. 1083–1224.

[EM02] **Y. M. Eliashberg** and **N. M. Mishachev**, *Introduction to the h-principle*. American Mathematical Society (2002).

[Fen71] **N. Fenichel**, *Persistence and Smoothness of Invariant Manifolds for Flows*. Indiana Univ. Math. J., vol. 21 (1971), pp. 193–226.

[Fen74] ———, *Asymptotic Stability with Rate Conditions*. Indiana Univ. Math. J., vol. 23 (1974), pp. 1109–1137.

[Fen77] ———, *Asymptotic Stability with Rate Conditions II*. Indiana Univ. Math. J., vol. 26 (1977), pp. 81–93.

[Fen79] ———, *Geometric singular perturbation theory for ordinary differential equations*. J. Differ. Equ., vol. 31 (1979), pp. 53–98.

[Flo89] **A. Floer**, *Witten's complex and infinite dimensional Morse theory*. J. Differ. Geom., vol. 30 (1989), pp. 207–221.

[FR11] **D. Fuchs** and **D. Rutherford**, *Generating families and Legendrian contact homology in the standard contact space*. J. Topol., vol. 4 (2011), pp. 190–226.

[Fra04] **U. Frauenfelder**, *The Arnold-Givental conjecture and moment Floer homology*. Int. Math. Res. Not., vol. 2004 (2004), pp. 2179–2269.

[Gei08] **H. Geiges**, *An Introduction to Contact Topology*. Cambridge University Press (2008).

[GKS12] **S. Guillermou**, **M. Kashiwara**, and **P. Schapira**, *Sheaf quantization of Hamiltonian isotopies and applications to nondisplaceability problems*. Duke Math. J., vol. 161 (2012), pp. 201–245.

[Gro86] **M. Gromov**, *Partial Differential Relations*. Springer (1986).

[Hir76] **M. W. Hirsch**, *Differential topology*. Springer (1976).

[HR13] **M. B. Henry** and **D. Rutherford**, *A combinatorial DGA for Legendrian knots from generating families*. Commun. Contemp. Math., vol. 15 (2013), p. 1250 059.

[HW73] **A. E. Hatcher** and **J. B. Wagoner**, *Pseudo-isotopies of compact manifolds*. Société mathématique de France (1973).

[HWZ96] **H. Hofer**, **K. Wysocki**, and **E. Zehnder**, *Properties of pseudoholomorphic curves in symplectizations IV: Asymptotics with degeneracies*. In *Contact and symplectic geometry* (**C. B. Thomas**, ed.), Cambridge University Press (1996), pp. 78–117.

[Kho13] **A. L. Kholodenko**, *Applications of contact geometry and topology in physics*. World Scientific (2013).

[Lim16] **M. Limouzineau**, *Construction de sous-variétés legendriennes dans les espaces de jet d'ordre un de fonctions et fonctions génératrices*. Ph.D. thesis, Université Pierre et Marie Curie (2016).

[LS85] **F. Laudenbach** and **J.-C. Sikorav**, *Persistance d'intersection avec la section nulle au cours d'une isotopie hamiltonienne dans un fibré cotangent*. Invent. Math., vol. 82 (1985), pp. 349–357.

[MS95] **D. McDuff** and **D. Salamon**, *Introduction to symplectic topology*. Oxford University Press (1995).

[MS04] ———, *J-holomorphic curves and symplectic topology*. American Mathematical Society (2004).

[Mye18] **Z. Myer**, *A product structure on Generating Family Cohomology for Legendrian Submanifolds*. arXiv preprint arXiv:1712.00364 (2018).

[Nic11] **L. Nicolaescu**, *An invitation to Morse theory*. Springer (2011).

[NRS+20] **L. Ng**, **D. Rutherford**, **V. Shende**, **S. Sivek**, and **E. Zaslow**, *Augmentations are sheaves*. Geom. Topol., vol. 24 (2020), pp. 2149–2286.

[RS18] **D. Rutherford** and **M. G. Sullivan**, *Generating families and augmentations for Legendrian surfaces*. Algebr. Geom. Topol, vol. 18 (2018), pp. 1675–1731.

[RS21] ———, *Sheaves via augmentations of Legendrian surfaces*. J. Homotopy Relat. Struct., vol. 16 (2021), pp. 703–752.

[Sch93] **M. Schwarz**, *Morse homology*. Birkhäuser (1993).

[Sik86] **J.-C. Sikorav**, *Sur les immersions lagrangiennes dans un fibré cotangent admettant une phase génératrice globale*. C.R. Acad. Sci. Paris Sér. I Math., vol. 302 (1986), pp. 119–122.

[SS16] **J. M. Sabloff** and **M. G. Sullivan**, *Families of Legendrian submanifolds via generating families*. Quantum Topol., vol. 7 (2016), pp. 639–668.

[ST13] **J. M. Sabloff** and **L. Traynor**, *Obstructions to Lagrangian cobordisms between Legendrians via generating families*. Algebraic Geom. Topol., vol. 13 (2013), pp. 2733–2797.

[STZ17] **V. Shende**, **D. Treumann**, and **E. Zaslow**, *Legendrian knots and constructible sheaves*. Invent. math., vol. 207 (2017), pp. 1031–1133.

[Swi92] **J. Swiatkowski**, *On the Isotopy of Legendrian Knots*. Ann. Glob. Anal. Geom., vol. 10 (1992), pp. 195–207.

[Tho56] **R. Thom**, *Les singularités des applications differentiables*. Ann. Inst. Fourier (Grenoble), vol. 6 (1956), pp. 43–87.

[Tra01] **L. Traynor**, *Generating function polynomials for Legendrian links*. Geom. Topol., vol. 5 (2001), pp. 719–760.

[Vit92] **C. Viterbo**, *Symplectic topology as the geometry of generating functions*. Math. Ann., vol. 292 (1992), pp. 685–710.

[Wit82] **E. Witten**, *Supersymmetry and Morse theory*. J. Differ. Geom., vol. 17 (1982), pp. 661–692.

ÉCOLE DOCTORALE
de mathématiques
Hadamard (EDMH)

université PARIS-SACLAY

Titre : Homologies des familles génératrices de sous-variétés legendriennes
et espaces de modules d'escaliers de gradient

Mots clefs : Topologie symplectique et de contact, Sous-variétés legendriennes,
Théories de Morse–Bott–Cerf, Familles génératrices, Analyse des espaces de modules, Homologie.

Résumé : Pour pallier les difficultés techniques substantielles qui se posent dans l'utilisation pratique des familles génératrices pour explorer la diversité des sous-variétés legendriennes en topologie de contact, Henry et Rutherford proposent dans un article de 2013 une perturbation singulière du flot de gradient qui est utilisé pour définir l'homologie de la famille génératrice. Ils conjecturent alors que lorsqu'ils sont finis, les espaces de modules qui interviennent dans la définition de l'opérateur de bord de cette homologie sont en bijection avec certains espaces de modules de trajectoires brisées en escalier.
Cette thèse réalise le premier pas vers une démonstration complète par compacité et recollement de cette conjecture. Il y est démontré que si certaines hypothèses géométriques et conditions de transversalité sont satisfaites, alors dans la limite adiabatique de Henry et Rutherford, les trajectoires de gradient authentiques « convergent », à extraction près, vers des chaînes d'escaliers de gradient.

Title: Generating family homologies of Legendrian submanifolds
and moduli spaces of gradient staircases

Key words: Symplectic and contact topology, Legendrian submanifolds,
Morse–Bott–Cerf theories, Generating families, Moduli spaces analysis, Homology.

Abstract: Willing to overcome the substantial technical difficulties arising in the practical use of generating families to explore the diversity of Legendrian submanifolds in contact topology, Henry and Rutherford introduce in a 2013 paper a singular perturbation of the gradient flow, used to define the homology of the generating family. They conjecture that when finite, the moduli spaces used to define the boundary operator of this homology are in one-to-one correspondence with some moduli spaces of staircase trajectories.
This thesis takes the first step towards a complete proof by compactness-gluing of this conjecture. It is shown that provided some geometric assumptions and transversality conditions are satisfied, then in the Henry and Rutherford adiabatic limit, genuine gradient trajectories "accumulate" on gradient staircases chains.

Université Paris-Saclay
Espace Technologique / Immeuble Discovery
Route de l'Orme aux Merisiers RD 128 / 91190 Saint-Aubin, France