Technical appendix for the article "New Bounds and Constraint Programming Models for the Weighted Vertex Coloring Problem"

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This document provides proofs for the reduction rules presented in Section 2, the upper bounds presented in Section 3, and the CP models presented in Section 4. It also introduces the complete and detailed experimental results obtained on the whole set of instances discussed in Section 5.

1 Definitions and Notations

This section introduces the notations and definitions used throughout the paper and in this appendix.

Let $P = (G, w)$ be a WVCP instance. G is an undirected graph with vertex set $V = \{v_1, \dots, v_n\}$ and edge set $E = \{e_1, \dots, e_p\}$. w is a function $V \to \mathbb{N}^*$ that assigns to each vertex in V a strictly positive weight $w(v) \in \mathbb{N}^*$. Solving the WVCP instance P consists in finding a partition of V into k disjoint non-empty independent sets $S = \{V_1, \dots, V_k\}$ $(1 \leq k \leq n)$ such that the score $f(S) = \sum_{i=1}^{k} w(V_i)$ is minimal, where $w(V_i) = \max_{v \in V_i} w(v)$ denotes the maximum weight of the vertices in V_i. An independent set V_i, is a non-empty subset of V such that all vertices in V_i are not connected by any edge.

We denote by $N(v) = \{u \in V \mid \{u, v\} \in E\}$ the neighborhood of a vertex v and $\Delta = \max_{v \in V} |N(v)|$ the maximum degree of the graph G. A clique $C = \{c_1, \dots, c_q\}$ of the graph G is a subset of V such that all vertices in C are pairwise adjacent. By convention, the vertices $c_1, c_2, \dots, c_q$ appearing in C are always sorted by decreasing order of their weight: $w(c_1) \geq w(c_2) \geq \dots \geq w(c_q)$.

2 Reduction Rules

Given a vertex $u \in V$, with the set of neighbors $N(u)$ and a clique C of G, such that $u \notin C$ let us first define the following procedure POSICLIQUE taking as input (u, C) and returning a strictly positive number d described with Algorithm 1.

Rule 1. *(R1) Given a* WVCP *instance $P = (G, w)$, with $G = (V, E)$, a vertex $u \in V$ and a clique C, such that $u \notin C$, $d =$ POSICLIQUE(u, C) and $d \leq |C|$, if $w(u) \leq w(c_d)$, then the optimal* WVCP *score of G is unchanged after removing vertex u from G.*

Proof. Let us assume that S' is an optimal WVCP coloring of the subgraph of G induced by $V \backslash \{u\}$. The vertices of the clique C are all in S' (because $u \notin C$), and are colored with $|C|$ different colors $k_1, k_2, \dots, k_{|C|}$.

Algorithm 1 POSICLIQUE

Input: WVCP **instance** $P = (G, w)$**, vertex** $u \in V$ **and clique** $C = \{c_1, \dots, c_{|C|}\}$ **s.t.** $w(c_i) \geq w(c_j)$ $(1 \leq i < j \leq |C|)$.
Output: $d \in \mathbb{N}^*$
$d \leftarrow |N(u)| + 1$ **;** $l_C \leftarrow |C| - 1$
for i **from** 0 **to** l_C **do**
  if $c_{|C|-i} \in N(u)$ **and** $|C| - i \geq d$ **then**
    $d \leftarrow d - 1$
  end if
end for
return d

Let us use the procedure POSICLIQUE defined above to compute $d =$ POSICLIQUE(u, C) and to show that the following claim $H(d)$ is true: "there is at most $d - 1$ neighbors of u taking a color in $\{k_1, k_2, \dots, k_d\}$".

Let us initialize a variable d to the value $|N(u)| + 1$ as it is done in the procedure POSICLIQUE. For this value of d, the claim $H(d)$ is true as there are at most $|N(u)|$ neighbors of u taking a color in the set $\{k_1, k_2, \dots, k_{|N(u)|+1}\}$.

Now when applying the procedure POSICLIQUE, for all i from 0 to $|C| - 1$, if $c_{|C|-i} \in N(u)$, $|C| - i \geq d$ and if the claim $H(d)$ is true, then we know that a neighbor of u, namely $c_{|C|-i}$, has received a color which is not in the set $\{k_1, k_2, \dots, k_{d-1}\}$, therefore the claim $H(d-1)$ is true, and for the next iteration d is replaced by the value $d - 1$.

When the iterative procedure POSICLIQUE ends it returns a value d, and we know that the claim $H(d)$ is true for this value d, that is to say that there is at most $d - 1$ neighbors of u taking a color in $\{k_1, k_2, \dots, k_d\}$. Therefore, S' can be extended to an optimal and legal WVCP coloring S^* of the original graph G, by assigning a color in $\{k_1, \dots, k_d\}$ to the node u.

If $d \leq |C|$, this operation can always be done without increasing the optimal cost because $w(u) \leq w(c_d)$, and then $w(u) \leq w(c_i)$ for $i = 1, \dots, d$, as we have $w(c_1) \geq w(c_2) \geq \dots \geq w(c_q)$. □

Rule 2. *(R2) Given an* WVCP *$P = (G, w)$, with $G = (V, E)$, and two vertices $u, v \in V$, such that $N(u) \subset N(v)$ and $w(u) \leq w(v)$ then the optimal score of P is unchanged after removing vertex u from G.*

Proof. Let us consider two vertices $u, v \in V$, such that

$N(u) \subset N(v)$ and $w(u) \leq w(v)$. Suppose S' is an optimal WVCP coloring of the subgraph of G induced by $V \backslash \{u\}$. As $N(u) \subset N(v)$, S' can be extended to an optimal WVCP coloring S^* of the original graph G, by placing the node u in the independent set containing v, without increasing the score because $w(u) \leq w(v)$. □

3 Upper Bounds on Score and Number of Colors

Let us denote $W = \{w(v), v \in V\}$, the set of the different weight values given by the function w when applied on each vertex of V. We note m the value of the maximum weight in W.

For a weight value w, a subgraph of G of weight w is denoted as $G_w = (V_w, E_w)$, where V_w is the subset of V of all the vertices having the same weight w, $V_w = \{v \in V, w(v) = w\}$, and E_w is the maximal subset of edges in E connecting the vertices in V_w, $E_w = \{e = \{u, v\} \in E, u \in V_w, v \in V_w\}$. The chromatic number of each subgraph G_w is written χ_{G_w}. If there are no vertices in V with a weight equal to w, G_w is an empty graph and by convention, in this case, we set $\chi_{G_i} = 0$.

Given a solution S, corresponding to a partition of the vertices in V into k non-empty independent sets $S = \{V_1, \ldots, V_k\}$, the number of independent sets in S with a weight equal to w is denoted as $k_w^S = |\{w(V_i) = w, i = 1, \ldots, k\}|$.

Let us first derive the following lemma.

Lemma 1. *Given a vertex weighted graph $G = (V, E, w)$ and an optimal solution S^* of the WVCP, for $j = 1, \ldots m$, we have*

$$\sum_{i=1}^{j} i \times (\chi_{G_i} - k_i^{S^*}) \geq 0. \tag{1}$$

Proof. Let us define an operator g_w taking as input a legal solution S and giving as output a legal solution S' where all the vertices having the weight w in S have been removed from their current color groups and placed in χ_w new disjoint non-empty independent sets uniquely composed of vertices having this same weight w. This operation is always possible by definition of the chromatic number of a graph. By convention, if there is no weight in W with the value w, this operation has no effect. When iteratively applying this operation for $w = 1, \ldots, m$ and starting from the solution S^*, it produces m new legal solutions: $S^1 = g_1(S^*)$, $S^2 = g_2 \circ g_1(S^*)$, $\ldots$, $S^m = g_m \circ \cdots \circ g_1(S^*)$. The score of each of the new legal solutions S^j for $j = 1, \ldots m$ is equal to

$$f(S^j) = f(S^*) + \sum_{i=1}^{j} i \times (\chi_{G_i} - k_i^{S^*}). \tag{2}$$

As S^* is an optimal solution, we have for $j = 1, \ldots m$, $f(S^j) \geq f(S^*)$ and then

$$\sum_{i=1}^{j} i \times (\chi_{G_i} - k_i^{S^*}) \geq 0 \text{ for } j = 1, \ldots m. \tag{3}$$

□

Now, using Lemma 1, we derive the following bounds for WVCP:

Theorem 1. *Given a WVCP instance $P = (G, w)$, with a graph $G = (V, E)$, and an optimal solution S^* of this instance P, corresponding to a partition of the vertices in V into k non-empty independent sets $S^* = \{V_1, \ldots, V_k\}$, then $f(S^*) \leq \sum_{w \in W} w \times \chi_{G_w}$ and $k \leq \sum_{w \in W} \chi_{G_w}$.*

Proof. Let $S^* = \{V_1, \ldots, V_k\}$ be an optimal solution of the instance $P = (G, w)$ corresponding to a partition of the vertices in V into k non-empty independent sets.

Using Lemma 1 for $j = m$, we have

$$\sum_{i=1}^{m} i \times (\chi_{G_i} - k_i^{S^*}) \geq 0. \tag{4}$$

Thus, by definition of the WVCP score as $f(S^*) = \sum_{i=1}^{m} i \times k_i^{S^*}$, we obtain

$$f(S^*) \leq \sum_{i=1}^{m} i \times \chi_{G_i}, \tag{5}$$

which proves the first bound on the objective value.

Now, let us prove the bound on the number of colors required to build an optimal solution. Let us assume that $k > \sum_{w \in W} \chi_{G_w}$ and let us show that we arrive at a contradiction. Using the fact that $k = \sum_{w \in W} k_w^{S^*}$, it gives $\sum_{w \in W} (k_w^{S^*} - \chi_{G_w}) > 0$ and thus

$$\sum_{i=1}^{m} (k_i^{S^*} - \chi_{G_i}) > 0. \tag{6}$$

Now, let us show by recurrence that for all $c = 1, \ldots, m-1$

$$\sum_{i=1}^{m-c} (i + c - m - 1)(\chi_{G_i} - k_i^{S^*}) > 0. \tag{7}$$

By summing both terms of the inequalities of Equation (1) given by Lemma 1 for $j = m$ and the inequality given by Equation (6) multiplied by m (strictly positive number), it gives

$$\sum_{i=1}^{m} i \times (\chi_{G_i} - k_i^{S^*}) + m \times \sum_{i=1}^{m} (k_i^{S^*} - \chi_{G_i}) > 0, \tag{8}$$

and then

$$\sum_{i=1}^{m-1} (i - m)(\chi_{G_i} - k_i^{S^*}) > 0. \tag{9}$$

Therefore the property given by Equation (7) is true when $c = 1$. Now we suppose that this property is true at rank c and show that it will remain true at rank $c + 1$, for $c = 1, \ldots, m - 2$.

By summing both terms of the inequalities of Equation (1) for $j = m - c$ given by Lemma 1 and the inequality given by Equation (7) multiplied by $m-c$ (strictly positive number because $c<m$), it gives

$$\sum_{i=1}^{m-c} i(\chi_{G_i}-k_i^{S^*})+(m-c)\sum_{i=1}^{m-c}(i+c-m-1)(\chi_{G_i}-k_i^{S^*})>0.$$

$$\sum_{i=1}^{m-c}(i+mi+mc-m^2-m-ci-c^2+cm+c)\times(\chi_{G_i}-k_i^{S^*})>0.$$

By rearranging terms, and factorizing by $(i+c-m)$, it gives

$$\sum_{i=1}^{m-c}(i+c-m)(m+1-c)(\chi_{G_i}-k_i^{S^*})>0.$$

Then, by dividing both terms of this inequality by $m+1-c$ (strictly positive number), we obtain

$$\sum_{i=1}^{m-c}(i+c-m)(\chi_{G_i}-k_i^{S^*})>0,$$

and then

$$\sum_{i=1}^{m-c-1}(i+c-m)(\chi_{G_i}-k_i^{S^*})>0,$$

which prove the property given by Equation (7) at rank $c+1$.

Now using Equation (7) for $c=m-1$, we obtain

$$(\chi_1-k_1^{S^*})<0,$$

which contradicts Equation (1) given by Lemma 1 for j = 1 and ends the proof. □

4 Constraint Programming Models

We first introduce notations and terminology. Given a set S, $[S]$ denotes the range $\{1,\ldots,|S|\}$. For a function $f:X\mapsto Y$ and $X'\subseteq X$, $f(X')$ denotes the image of X' by f and $f^{-1}:Y\mapsto 2^X$ the function defined by $f^{-1}(y)=\{x\in X|f(x)=y\}$. For a vertex v of a graph G, $N(v)$, $\Delta(v)$ and $\Delta(G)$ denote respectively the set of neighbours of v, its degree and the maximum vertex degree in G.

Let $\kappa\in\mathbb{N}^*$ and P be a WVCP instance of graph $G=(V,E)$ and weight function w. P_κ denotes the problem of determining the existence of a solution to P that uses a number of colors smaller than or equal to κ. $\geq_w$ is a total ordering over V which is consistent with the descending order of weights ($u\geq_w v\rightarrow w(u)\geq w(v)$ for $u,v\in V$). $\geq_w$ is encoded by a consistent indexing of vertices over $[V]$ ($v_i\geq_w v_j\leftrightarrow i\leq j$ for $i,j\in[V]$). The dominant vertex v_i in a set of vertices $\{v_{i_1},\ldots,v_{i_n}\}$ is identified by $i=\min\{i_1,\ldots,i_n\}$.

A solution to P_κ is a map $s:[V]\mapsto K$ where $K=\{1,\ldots,\kappa\}$ such that $s^{-1}(k)$ is an independent set of vertices in G (possibly empty). A solution to P_κ is d-sorted if non-empty colors start from rank 1 and are sorted consistently with the ordering $\geq_w$ of their dominant vertices. Formally, $s:[V]\mapsto K$ is d-sorted if $s([V])=[s([V])]$ and $\min(s^{-1}(j))<\min(s^{-1}(k))$ for $1\leq j<k\leq|s([V])|$. $\mathcal{S}_{P_\kappa}$ denotes the set of d-solutions to P_κ.

Definition 1. *Let P_κ be a satisfiable* WVCP *instance and $\mu_{P_\kappa}:\mathcal{S}_{P_\kappa}\times V\mapsto K$ such that, for all $s\in\mathcal{S}_{P_\kappa}$, $v\in V$, $\mu_{P_\kappa}(s,v)=\min_{k=1..\Delta(v)+1}(\{k\mid\forall u\in N(v):s(u)\neq k\})$. $s\in\mathcal{S}_{P_\kappa}$ is compact if $\mu_{P_\kappa}(s,v)=s(v)$ for all $v\in V$.*

Lemma 2. *Let $s\in\mathcal{S}_{P_\kappa}$, $v\in V$, and $t:V\rightarrow K$ such that $t(v)=\mu_{P_\kappa}(s,v)$ and $t(u)=s(u)$ if $u\neq v$. t is a solution to P_κ and $f(t)\leq f(s)$.*

Proof. We show that (1) μ_{P_κ} exists and is uniquely defined, and given a d-solution s to P_κ, $v\in V$ and t defined by the conditions of the lemma, (2) t exists, (3) t is a solution to P and (4) $f(t)\leq f(s)$.

(1) Let $s\in\mathcal{S}_{P_\kappa}$ and $v_i\in V$. Since the neighbors of v_i may only use a maximum number of $\Delta(v_i)$ colors in any solution, there necessarily exists some color j in $\{1..\Delta(v_i)+1\}$ that is not empty in s and includes no neighbors of v_i. That is, $j\in\{k\mid\forall u\in N(v_i):s(u)\neq k\}$. This ensures μ_{P_κ} exists and, based on its definition, is uniquely defined.

Let $s\in\mathcal{S}_{P_\kappa}$, $v\in V$, and $t:V\rightarrow K$ satisfying the conditions of the lemma.

(2) By definition of μ_{P_κ}, $\mu_{P_\kappa}(s,v_i)\leq s(v_i)$ for all $v_i\in V$. Hence $t(v_i)\leq s(v_i)\leq|K|$ for all $v_i\in V$. So t exists and is uniquely defined.

(3) By definition, $t(v)$ is the lowest-ranked color that is not empty in s and that includes no neighbors of v so t satisfies all coloring constraints involving v. Since s is a solution and $t(u)=s(u)$ if $u\neq v$, t also satisfies all coloring constraints induced by (V,E) on each vertex $u\neq v$. Hence t is a solution to P and since its co-domain is a subset of K ($t(V)\subseteq K$), it is also a solution to P_κ (not necessarily d-sorted).

(4) Since $t(v)\leq s(v)$ and colors are sorted in s in descending order of dominant vertices, $w(\min(t(v)))\geq w(\min(s(v)))$ holds true. Besides, $w(\min(s(v)))\geq w(v)$ since color $s(v)$ includes v in s. Hence $w(\min(t(v)))\geq w(v)$. Any increase in the score of t compared to s may only be caused by the re-coloring of v with $t(v)$ so $f(t)-f(s)\leq\max(w(v),w(\min(t(v))))-w(\min(t(v)))$, that is, $f(t)-f(s)\leq 0$. □

Algorithm 2 turns any d-solution into a compact d-solution with no score increase by combining vertex left-shifting (μ_{P_κ}) with color left-shifting and color swap operations. Given a d-sorted solution, Algorithm 2 first builds the set U containing every vertex that must be left-shifted (line 4). It then iterates by gradually emptying U (lines 5-26). Each iteration proceeds in two stages and targets the highest-ranked color I that has vertices in U (line 6). The first stage applies μ_{P_κ} to left-shift every vertex of I that is in U (lines 7-10). The second stage processes color ranks in ascending order starting from rank i of I (lines 11-25). Each step starts by swapping the current color with a higher-ranked color J only if the latter has the dominant vertex amongst all colors ranked in $i..n$ (lines 12-13). The current color is then expanded by iteratively left-shifting any compatible vertex placed in a higher color (lines 14-18). Since vertex relocation operations may break the color ordering (eg. empty colors) and the first stage may also leave I empty, each step terminates by left-shifting colors so that the part of the solution beyond the current color remains compact (lines 19-23).

Each iteration hence ensures the set of vertices left in $U \setminus I$ are all placed in the lowest-ranked possible color. Therefore traversing U by descending color ranks guarantees a compact d-solution is eventually returned with no score increase. Notice that different solutions may actually be generated depending on the order in which function *choose_in* picks vertices for left-shifting (line 8) and relocation (line 16). Notice also that Algorithm 2 may be used as a local search method in its own right to solve WVCP instances or embedded in metaheuristics for intensification purposes.

Algorithm 2 MAXLEFTSHIFTCOLORING

1: **Input:** **WVCP instance** $P = ((V, E), w)$ **and d-sorted solution** $s : V \to K$
2: **Output:** $t : V \to K$
3: $t \leftarrow s$; $n \leftarrow \max(t(V))$
4: $U \leftarrow \{v \in V \mid t(v) > \mu_{P_\kappa}(t, v)\}$
5: **while** $U \neq \emptyset$ **do**
6: $\quad i \leftarrow \max(t(U))$; $I \leftarrow t^{-1}(i)$
7: $\quad$ **repeat**
8: $\quad\quad u \leftarrow choose\_in(U \cap I)$; $t(u) \leftarrow \mu_{P_\kappa}(t, u)$
9: $\quad\quad U \leftarrow U \setminus \{u\}$; $I \leftarrow I \setminus \{u\}$
10: $\quad$ **until** $U \cap I = \emptyset$
11: $\quad$ **while** $i < n$ **do**
12: $\quad\quad j \leftarrow \min(\{v \in V \mid t(v) \geq i\})$; $J \leftarrow t^{-1}(j)$
13: $\quad\quad t^{-1}(j) \leftarrow t^{-1}(i)$; $t^{-1}(i) \leftarrow J$
14: $\quad\quad S \leftarrow \{v \in V \mid t(v) > i \wedge t^{-1}(i) \cap N(v) = \emptyset\}$
15: $\quad\quad$ **while** $S \neq \emptyset$ **do**
16: $\quad\quad\quad v \leftarrow choose\_in(S)$; $t(v) \leftarrow i$;
17: $\quad\quad\quad S \leftarrow \{v \in V \mid t(v) > i \wedge t^{-1}(i) \cap N(v) = \emptyset\}$
18: $\quad\quad$ **end while**
19: $\quad\quad j \leftarrow i + 1$
20: $\quad\quad$ **while** $j < n$ **do**
21: $\quad\quad\quad m \leftarrow |\{l \mid j \leq l \leq n \wedge \forall\, j \leq k \leq l : t^{-1}(k) = \emptyset\}|$
22: $\quad\quad\quad t^{-1}(j) \leftarrow t^{-1}(j + m)$; $j \leftarrow j + 1 + m$
23: $\quad\quad$ **end while**
24: $\quad\quad n \leftarrow \max(t(V))$; $i \leftarrow i + 1$
25: $\quad$ **end while**
26: **end while**
27: **return t**

Theorem 2. *Let P_κ be a satisfiable* WVCP *instance. There exists $g_{P_\kappa} : \mathcal{S}_{P_\kappa} \mapsto \mathcal{S}_{P_\kappa}$ such that, for all $s \in \mathcal{S}_{P_\kappa}$, $g_{P_\kappa}(s)$ is compact, $f(g_{P_\kappa}(s)) \leq f(s)$ and $g_{P_\kappa}(g_{P_\kappa}(s)) = g_{P_\kappa}(s)$.*

Proof. We show that Algorithm 2 terminates on any input and implements a map g_{P_κ} satisfying the conditions of the theorem. Let $t_0 = s$ denote the value t computed on line 3 and U_0 the set U computed on line 4.

(1) Assume U_0 is empty. Then the algorithm returns s since $s(v) \leq \mu_{P_\kappa}(s, v)$ for all $v \in V$ which, by definition of μ_{P_κ}, implies $s(v) = \mu_{P_\kappa}(s, v)$. So s is compact and $f(s) \leq f(s)$.

(2) Assume U_0 is not empty. For any iteration $p \geq 1$ of the main loop, let i_p denote the value of color rank i set on line 6 at the start of the p-th iteration, U_p the value of set U on line 11 which results from the first inner loop, t_p the solution t computed at the end of the p-th iteration, and i_{p+1} the value $\max(t_p(U_p))$ if U_p is not empty.

We first show that the following invariants hold true at each iteration p: (2A) t_p is consistent, d-sorted and satisfies $f(t_p) \leq f(t_{p-1})$, and (2B) $U_p \subsetneq U_{p-1}$, $i_{p+1} < i_p$, and $\mu_{P_\kappa}(t_p, v) = t_p(v)$ for all $v \in V$ s.t. $t_p(v) \geq i_p$.

(2A) The first inner loop (lines 7-10) only applies operation μ_{P_κ} to t_{p-1} which, by Lemma 2, is consistency-preserving and cannot increase the score of t_{p-1}. μ_{P_κ} does not change either the dominance-based ranking between the first i_p colors since it only recolors vertices of the i_p-th color I. The second inner loop (lines 11-25) processes the colors whose rank is greater than or equal to i_p. At each iteration, the dominant vertex is identified amongst all the vertices colored over $i_p \ldots n_{p-1}$ (line 12) where n_{p-1} is the number of colors of t_{p-1} computed at the end of the previous iteration (line 24). If the color of rank j of this dominant vertex is different from the color of rank i_p, the two are effectively swapped in the current solution otherwise the swap leaves the solution unchanged (lines 13-14). In either case, the i_p-th color dominates any higher-ranked color $j > i_p$ in the resulting solution and the latter is consistent. The next inner loop (lines 14-18) relocates to the i_p color any compatible vertex $v \in S$ located beyond i_p until no such move is possible. By definition of S, the resulting solution is consistent. The iteration concludes by removing empty colors and left-shifting the remaining colors (lines 19-23) which again preserves consistency. The color swap, vertex relocation, and color shifting operations clearly terminate and preserve consistency. Besides, the former and the latter leave solution scores unchanged and so does vertex relocation unless a color gets emptied in which case the score of the resulting solution is necessarily lowered. As for color ordering, only vertex relocations may break the dominance-based ordering between colors over ranks $i_p + 1 \ldots n_{p-1}$. Since these colors will eventually be visited in the loop (lines 11-25), any necessary swap will be performed to gradually re-order the solution. Therefore, the solution t_p produced after each iteration p of the main loop is consistent, d-sorted and satisfies $f(t_p) \leq f(t_{p-1})$.

(2B) We show by induction that $U_p \subsetneq U_{p-1}$, $i_{p+1} < i_p$, and $\mu_{P_\kappa}(t_p, v) = t_p(v)$ for all $v \in V$ s.t. $t_p(v) \geq i_p$. At the start of iteration $p = 1$, the only vertices of solution $t_{p-1} = t_0$ that may be left shifted to a color ranked strictly lower than i_1 as computed in line 6 are included in color I_1 of rank i_1 by definition of U_0 and I. Any vertex left in the color of rank i_1 at the end of the first inner-loop cannot be shifted to a color of rank $< i_1$. The second inner loop therefore ensures that every vertex in a color of rank $\geq i_1$ occupies its leftmost color in the resulting solution t_1. Besides, the first inner loop ensures that $U_1 = U_0 \setminus (U_0 \cap I_1)$. Since $U_0 \cap I_1 \neq \emptyset$, $U_1 \subsetneq U_0$, and, by definition of I_1, $i_2 < i_1$ also holds true. Therefore the induction hypothesis holds true for $p = 1$. Assume it holds true for $p > 1$. The same reasoning used for the base case applies and allows to conclude that the property holds true at iteration $p + 1$.

It follows from (2B) that the algorithm terminates at some iteration p such that $i_p = \min(t_{p-1}(U_{p-1}))$. Therefore $\mu_{P_\kappa}(t_p, v) = t_p(v)$ for all $v \in V$. We hence conclude from (1) and (2) that Algorithm 2 terminates on any input and implements a map g_{P_κ} verifying the conditions of the theorem. $\square$

Corollary 1. *Let P be a* WVCP *instance. $P_{\Delta(G)+1}$ has an optimal and compact d-solution which is optimal for P.*

Proof. Theorem 2 clearly ensures that any optimal d-solution can be mapped to a compact optimal d-solution. Since the image of μ is necessarily a subset of $\{1, \ldots, \Delta(G)+1\}$, the result follows. $\square$

5 Detailed Experimental Results

This section reports the results of the reduction and execution of all the different models for all instances of the benchmark. In the tables presented in the following pages, a score is written in bold when it corresponds to the BKS. When the instance is solved exactly, a star is added and the time in seconds required to prove this result is reported, otherwise "tl" is indicated, meaning that the time limit of one hour has been reached. The score is underlined if this optimality has never been proved before in the literature. When the instance has not been resolved due to memory limitation, it is indicated by "-". For the sake of reproducibility, full spreadsheets of results are also available in the code source folder along with raw results from the solver and scripts to convert them.

Tables 1, 2 and 3 display the number of vertices removed by the different reduction rules for all the instances (cf. experimental section 5.1).

Tables 1, 2 and 3 display the number of vertices removed by the different reduction rules for all the instances (cf. experimental section 5.1). Tables 4, 5 and 6 display the impact of the reduction for all the instances and the different models: primal, dual, and joint.

Tables 7, 8 and 9 show the lower and upper bounds on the score and the number of colors required to build an optimal solution for all the instances of the benchmark (cf. section 5.2).

Tables 10, 11 and 12 present the impacts of all precomputed lower and upper bounds on the objective value and the number of colors for the resolution with the primal model for all instances (see section 5.3).

Tables 13, 14 and 15 correspond to the detailed results of the different CP models. (see section 5.4).

Tables 16, 17 and 18 are the results of the different CP models launched in parallel with 10 threads (see section 5.5).

instance	$\|V\|$	R0 # vertices removed	R0 t(s)	R1 # vertices removed	R1 t(s)	R1+R2 # vertices removed	R1+R2 t(s)	Iterated # vertices removed	Iterated t(s)
C2000.5	2000	0	260	0	261	0	387	0	406
C2000.9	2000	0	5356	0	5331	0	5794	0	5776
DSJC125.1g	125	0	0	0	0	0	0	0	0
DSJC125.1gb	125	0	0	0	0	0	0	0	0
DSJC125.5g	125	0	0	0	0	0	0	0	0
DSJC125.5gb	125	0	0	0	0	0	0	0	0
DSJC125.9g	125	0	1	0	1	0	1	0	1
DSJC125.9gb	125	0	1	0	1	0	1	0	1
DSJC250.1	250	0	0	0	0	0	0	0	0
DSJC250.5	250	0	0	0	1	0	1	0	1
DSJC250.9	250	0	7	0	7	0	7	0	7
DSJC500.1	500	0	0	0	0	0	0	0	0
DSJC500.5	500	0	4	0	4	0	5	0	5
DSJC500.9	500	0	72	0	72	0	76	0	75
DSJC1000.1	1000	0	1	0	1	0	1	0	1
DSJC1000.5	1000	0	30	0	31	0	40	0	40
DSJC1000.9	1000	0	617	0	633	0	653	0	649
DSJR500.1	500	78	0	80	0	80	0	256	1
flat1000_50_0	1000	0	28	0	29	0	39	0	39
flat1000_60_0	1000	0	28	0	30	0	39	0	39
flat1000_76_0	1000	0	29	0	30	0	40	0	38
GEOM20	20	13	0	13	0	13	0	13	0
GEOM20a	20	9	0	9	0	11	0	14	0
GEOM20b	20	5	0	5	0	5	0	7	0
GEOM30	30	14	0	14	0	14	0	20	0
GEOM30a	30	6	0	6	0	6	0	10	0
GEOM30b	30	4	0	4	0	4	0	4	0
GEOM40	40	13	0	13	0	13	0	21	0
GEOM40a	40	7	0	7	0	7	0	23	0
GEOM40b	40	5	0	6	0	6	0	13	0
GEOM50	50	9	0	9	0	9	0	26	0
GEOM50a	50	8	0	9	0	9	0	22	0
GEOM50b	50	5	0	7	0	7	0	12	0
GEOM60	60	9	0	12	0	12	0	21	0
GEOM60a	60	8	0	8	0	8	0	16	0
GEOM60b	60	7	0	8	0	8	0	11	0
GEOM70	70	7	0	7	0	7	0	8	0
GEOM70a	70	8	0	8	0	8	0	17	0
GEOM70b	70	6	0	8	0	8	0	15	0
GEOM80	80	16	0	16	0	16	0	37	0
GEOM80a	80	5	0	6	0	6	0	9	0
GEOM80b	80	9	0	9	0	9	0	10	0
GEOM90	90	8	0	9	0	9	0	17	0
GEOM90a	90	3	0	3	0	3	0	4	0
GEOM90b	90	8	0	10	0	10	0	21	0
GEOM100	100	8	0	8	0	8	0	25	0
GEOM100a	100	0	0	1	0	1	0	1	0
GEOM100b	100	9	0	9	0	9	0	12	0
GEOM110	110	6	0	9	0	9	0	23	0
GEOM110a	110	0	0	0	0	0	0	0	0
GEOM110b	110	11	0	11	0	11	0	20	0
GEOM120	120	12	0	12	0	12	0	20	0
GEOM120a	120	0	0	0	0	1	0	1	0
GEOM120b	120	7	0	7	0	7	0	10	0
inithx.i.1	864	469	5	574	6	596	6	683	8
inithx.i.2	645	174	2	319	2	373	2	435	3
inithx.i.3	621	153	2	284	2	339	2	385	3
latin_square_10	900	0	259	0	263	0	279	0	267
le450_15a	450	28	0	28	0	28	0	30	1
le450_15b	450	20	0	20	0	20	0	20	1
le450_15c	450	0	0	0	1	0	1	0	1
le450_15d	450	0	0	0	1	0	1	0	1
le450_25a	450	92	0	97	0	97	0	104	1
le450_25b	450	90	0	90	0	90	1	105	1
le450_25c	450	9	1	10	1	10	1	10	1
le450_25d	450	7	1	8	1	8	1	8	1

Table 1: Number of vertices removed by the different reduction rules (1/3).

instance	$\|V\|$	R0 # vertices removed	R0 t(s)	R1 # vertices removed	R1 t(s)	R1+R2 # vertices removed	R1+R2 t(s)	Iterated # vertices removed	Iterated t(s)
miles250	128	53	0	53	0	53	0	96	0
miles500	128	48	0	51	0	51	0	80	0
miles1000	128	17	0	21	0	21	0	35	1
miles1500	128	19	2	22	2	22	2	35	2
mulsol.i.5	186	28	0	53	0	75	0	82	0
myciel5g	47	0	0	0	0	0	0	0	0
myciel5gb	47	0	0	0	0	0	0	0	0
myciel6g	95	0	0	0	0	0	0	0	0
myciel6gb	95	0	0	0	0	0	0	0	0
myciel7g	191	0	0	0	0	0	0	0	0
myciel7gb	191	0	0	0	0	0	0	0	0
queen8_8g	64	0	0	0	0	0	0	0	0
queen8_8gb	64	0	0	0	0	0	0	0	0
queen9_9g	81	0	0	0	0	0	0	0	0
queen9_9gb	81	0	0	0	0	0	0	0	0
queen10_10	100	0	0	0	0	0	0	0	0
queen10_10g	100	0	0	0	0	0	0	0	0
queen10_10gb	100	0	0	0	0	0	0	0	0
queen11_11	121	0	0	0	0	0	0	0	0
queen11_11g	121	0	0	0	0	0	0	0	0
queen11_11gb	121	0	0	0	0	0	0	0	0
queen12_12	144	0	0	0	0	0	0	0	0
queen12_12g	144	0	0	0	0	0	0	0	0
queen12_12gb	144	0	0	0	0	0	0	0	0
queen13_13	169	0	0	0	0	0	0	0	0
queen14_14	196	0	0	0	0	0	0	0	0
queen15_15	225	0	0	0	0	0	0	0	0
queen16_16	256	0	0	0	0	0	0	0	0
R50_1g	50	5	0	6	0	7	0	8	0
R50_1gb	50	6	0	7	0	8	0	8	0
R50_5g	50	0	0	0	0	0	0	0	0
R50_5gb	50	0	0	0	0	0	0	0	0
R50_9g	50	0	0	0	0	0	0	0	0
R50_9gb	50	0	0	0	0	0	0	0	0
R75_1g	70	1	0	1	0	1	0	1	0
R75_1gb	70	1	0	1	0	1	0	1	0
R75_5g	75	0	0	0	0	0	0	0	0
R75_5gb	75	0	0	0	0	0	0	0	0
R75_9g	75	0	0	0	0	0	0	0	0
R75_9gb	75	0	0	0	0	0	0	0	0
R100_1g	100	2	0	2	0	2	0	2	0
R100_1gb	100	1	0	1	0	1	0	1	0
R100_5g	100	0	0	0	0	0	0	0	0
R100_5gb	100	0	0	0	0	0	0	0	0
R100_9g	100	0	0	0	1	0	1	0	0
R100_9gb	100	0	0	0	0	0	0	0	1
wap01a	2368	134	20	137	29	137	30	211	107
wap02a	2464	161	22	165	32	165	33	249	103
wap03a	4730	131	50	133	90	133	92	136	188
wap04a	5231	244	61	244	99	244	110	321	318
wap05a	905	71	8	76	9	76	10	81	13
wap06a	947	67	8	72	8	72	10	89	14
wap07a	1809	79	14	80	17	81	20	84	31
wap08a	1870	82	15	83	19	83	22	85	33
zeroin.i.1	211	95	1	111	1	123	1	130	1
zeroin.i.2	211	67	0	82	0	109	0	111	0
zeroin.i.3	206	59	0	75	0	105	0	111	0

Table 2: Number of vertices removed by the different reduction rules (2/3).

		R0		R1		R1+R2		Iterated	
instance	$\|V\|$	# vertices removed	t(s)	# vertices removed	t(s)	# vertices removed	t(s)	# vertices removed	t(s)
p06	16	0	0	0	0	0	0	0	0
p07	24	0	0	0	0	0	0	0	0
p08	24	0	0	0	0	0	0	0	0
p09	25	0	0	0	0	0	0	0	0
p10	16	0	0	0	0	0	0	0	0
p11	18	0	0	0	0	0	0	0	0
p12	26	0	0	0	0	0	0	0	0
p13	34	0	0	0	0	0	0	0	0
p14	31	0	0	0	0	0	0	0	0
p15	34	0	0	0	0	0	0	0	0
p16	34	0	0	0	0	0	0	0	0
p17	37	0	0	0	0	0	0	0	0
p18	35	0	0	0	0	0	0	0	0
p19	36	0	0	0	0	0	0	0	0
p20	37	0	0	0	0	0	0	0	0
p21	38	1	0	1	0	1	0	1	0
p22	38	0	0	0	0	0	0	0	0
p23	44	0	0	0	0	0	0	0	0
p24	34	3	0	3	0	3	0	3	0
p25	36	2	0	2	0	2	0	3	0
p26	37	2	0	2	0	2	0	4	0
p27	44	0	0	0	0	0	0	0	0
p28	44	5	0	5	0	5	0	7	0
p29	53	13	0	14	0	14	0	39	0
p30	60	0	0	0	0	0	0	0	0
p31	47	15	0	15	0	15	0	38	0
p32	51	6	0	6	0	6	0	17	0
p33	56	1	0	1	0	1	0	2	0
p34	74	0	0	0	0	0	0	0	0
p35	86	3	0	3	0	3	0	3	0
p36	101	5	0	5	0	5	0	8	0
p38	87	1	0	1	0	1	0	2	0
p40	86	2	0	2	0	2	0	2	0
p41	116	1	0	1	0	1	0	1	0
p42	138	1	0	1	0	1	0	3	0
r01	144	0	0	0	0	0	0	0	0
r02	142	0	0	0	0	0	0	0	0
r03	139	0	0	0	0	0	0	0	0
r04	151	0	0	0	0	0	0	0	0
r05	142	1	0	1	0	1	0	1	0
r06	148	1	0	1	0	1	0	1	0
r07	141	3	0	3	0	3	0	3	0
r08	138	0	0	0	0	0	0	0	0
r09	129	0	0	1	0	1	0	1	0
r10	150	1	0	1	0	1	0	1	0
r11	208	0	0	0	0	0	0	0	0
r12	199	0	0	0	0	0	0	0	0
r13	217	0	0	0	0	0	0	0	0
r14	214	0	0	0	0	0	0	0	0
r15	198	0	0	0	0	0	0	0	0
r16	188	2	0	2	0	2	0	2	0
r17	213	0	0	0	0	0	0	0	0
r18	200	0	0	0	0	0	0	0	0
r19	185	0	0	0	0	0	0	0	0
r20	217	0	0	0	0	0	0	0	0
r21	281	0	0	0	0	0	0	0	0
r22	285	0	0	0	0	0	0	0	0
r23	288	0	0	0	0	0	0	0	0
r24	269	0	0	0	0	0	0	0	0
r25	266	0	0	0	0	0	0	0	0
r26	284	0	0	0	0	0	0	0	0
r27	259	0	0	0	0	0	0	0	0
r28	288	0	0	0	0	0	0	0	0
r29	281	0	0	0	0	0	0	0	0
r30	301	0	0	0	0	0	0	0	0

Table 3: Number of vertices removed by the different reduction rules (3/3).

		primal original			primal reduced			dual original			dual reduced			joint original			joint reduced		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
C2000.5	2144	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
C2000.9	5477	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJC125.1gb	90	90	34	tl	90	34	tl	122	20	tl	122	20	tl	112	20	tl	113	20	tl
DSJC125.1g	23	**23***	23	847	**23***	23	862	26	5	tl	26	5	tl	24	10	tl	26	10	tl
DSJC125.5gb	240	270	59	tl	270	59	tl	273	92	tl	273	92	tl	271	92	tl	271	92	tl
DSJC125.5g	71	78	25	tl	78	25	tl	84	33	tl	84	33	tl	80	25	tl	79	25	tl
DSJC125.9gb	604*	635	100	tl	635	100	tl	**604***	604	140	**604***	604	149	**604***	604	650	**604***	604	675
DSJC125.9g	169*	176	50	tl	176	50	tl	**169***	169	56	**169***	169	56	**169***	169	178	**169***	169	170
DSJC250.1	127	143	19	tl	143	19	tl	161	19	tl	161	19	tl	145	19	tl	145	19	tl
DSJC250.5	392	466	57	tl	466	57	tl	472	19	tl	472	19	tl	467	57	tl	467	57	tl
DSJC250.9	934*	1057	152	tl	1058	152	tl	1001	918	tl	1007	921	tl	1034	521	tl	1047	545	tl
DSJC500.1	184	220	19	tl	222	19	tl	226	19	tl	226	19	tl	227	19	tl	-		tl
DSJC500.5	685	828	38	tl	826	38	tl	827	19	tl	827	19	tl	-		tl	-		tl
DSJC500.9	1662	1859	152	tl	1859	152	tl	1914	19	tl	1914	19	tl	1855	152	tl	1851	152	tl
DSJC1000.1	300	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJC1000.5	1185	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJC1000.9	2836	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJR500.1	169	-		tl	187	19	tl	-		tl	187	19	tl	-		tl	187	19	tl
flat1000_50_0	924	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
flat1000_60_0	1162	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
flat1000_76_0	1165	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
GEOM20a	33*	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0
GEOM20b	8*	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0
GEOM20	33*	**33***	33	0	**33***	33	0	**33***	33	27	**33***	33	0	**33***	33	0	**33***	33	0
GEOM30a	42*	**42***	42	1	**42***	42	0	**42***	42	172	**42***	42	1	**42***	42	3	**42***	42	0
GEOM30b	12*	**12***	12	1	**12***	12	1	**12***	12	2	**12***	12	1	**12***	12	7	**12***	12	4
GEOM30	32*	**32***	32	1	**32***	32	0	**32***	32	3	**32***	32	0	**32***	32	3	**32***	32	0
GEOM40a	49*	**49***	49	5	**49***	49	0	**49***	49	1	**49***	49	0	**49***	49	20	**49***	49	0
GEOM40b	16*	**16***	16	9	**16***	16	2	**16***	16	3	**16***	16	0	**16***	16	9	**16***	16	2
GEOM40	37*	**37***	37	9	**37***	37	1	37	36	tl	**37***	37	0	**37***	37	48	**37***	37	1
GEOM50a	65*	**65***	65	15	**65***	65	4	65	64	tl	**65***	65	0	**65***	65	31	**65***	65	5
GEOM50b	18*	**18***	18	9	**18***	18	8	**18***	18	4	**18***	18	2	**18***	18	29	**18***	18	9
GEOM50	40*	**40***	40	7	**40***	40	0	40	10	tl	**40***	40	1	**40***	40	16	**40***	40	1
GEOM60a	73*	**73***	73	719	**73***	73	98	74	10	tl	**73***	73	3206	**73***	73	30	**73***	73	19
GEOM60b	23*	**23***	23	236	**23***	23	54	24	3	tl	**23***	23	8	**23***	23	44	**23***	23	35
GEOM60	43*	**43***	43	18	**43***	43	4	**43***	43	11	**43***	43	2	**43***	43	45	**43***	43	9
GEOM70a	73*	**73***	73	1800	**73***	73	252	**73***	73	25	**73***	73	12	**73***	73	96	**73***	73	27
GEOM70b	24*	**24***	24	24	**24***	24	17	25	3	tl	25	15	tl	**24***	24	60	**24***	24	36
GEOM70	47*	**47***	47	158	**47***	47	44	51	10	tl	51	10	tl	**47***	47	53	**47***	47	34
GEOM80a	76*	76	48	tl	76	48	tl	81	10	tl	81	30	tl	**76***	76	235	**76***	76	90
GEOM80b	27*	**27***	27	521	**27***	27	74	30	3	tl	29	3	tl	**27***	27	79	**27***	27	88
GEOM80	66*	**66***	66	1280	**66***	66	36	68	10	tl	66	61	tl	**66***	66	125	**66***	66	11
GEOM90a	73*	73	10	tl	73	10	tl	75	67	tl	75	64	tl	**73***	73	595	**73***	73	327
GEOM90b	30*	30	17	tl	**30***	30	1378	**30***	30	125	**30***	30	16	**30***	30	187	**30***	30	80
GEOM90	61*	61	20	tl	**61***	61	2485	62	10	tl	66	10	tl	**61***	61	525	**61***	61	55
GEOM100a	89*	91	10	tl	91	10	tl	98	10	tl	94	69	tl	**89***	89	575	**89***	89	425
GEOM100b	32*	**32***	32	1873	**32***	32	781	**32***	32	86	**32***	32	72	**32***	32	158	**32***	32	180
GEOM100	65*	65	10	tl	65	29	tl	65	10	tl	67	39	tl	**65***	65	478	**65***	65	110
GEOM110a	97*	104	20	tl	104	20	tl	106	30	tl	106	30	tl	97	67	tl	97	48	tl
GEOM110b	37*	**37***	37	365	**37***	37	137	**37***	37	115	**37***	37	30	**37***	37	304	**37***	37	122
GEOM110	68*	**68***	68	2544	69	30	tl	72	10	tl	73	10	tl	**68***	68	1350	**68***	68	422
GEOM120a	105*	113	20	tl	113	20	tl	119	10	tl	119	10	tl	112	25	tl	112	25	tl
GEOM120b	35*	37	15	tl	37	15	tl	37	3	tl	37	3	tl	**35***	35	1135	**35***	35	560
GEOM120	72*	72	10	tl	72	20	tl	77	10	tl	**72***	72	1454	72	10	tl	**72***	72	554
inithx.i.1	569*	-		tl	569	57	tl	-		tl	569	19	tl	-		tl	**569***	569	2432
inithx.i.2	329*	330	19	tl	329	19	tl	330	19	tl	329	19	tl	-		tl	329	19	tl
inithx.i.3	337*	337	57	tl	337	76	tl	339	19	tl	339	19	tl	339	38	tl	337	38	tl
latin_square_10	1480	1873	95	tl	-		tl	-		tl	-		tl	-		tl	-		tl
le450_15a	212	247	19	tl	245	19	tl	250	19	tl	250	19	tl	252	19	tl	251	19	tl
le450_15b	216	243	19	tl	249	19	tl	245	19	tl	251	19	tl	245	19	tl	253	19	tl
le450_15c	275	338	38	tl	336	38	tl	338	19	tl	338	19	tl	344	38	tl	344	38	tl
le450_15d	272	325	19	tl	329	19	tl	332	19	tl	332	19	tl	328	19	tl	328	19	tl
le450_25a	306	-		tl	323	38	tl	319	19	tl	317	19	tl	328	38	tl	323	38	tl
le450_25b	307	307	19	tl	307	19	tl	312	19	tl	314	19	tl	311	19	tl	310	19	tl
le450_25c	342	396	38	tl	398	38	tl	400	19	tl	397	19	tl	401	38	tl	406	38	tl
le450_25d	330	387	57	tl	388	57	tl	393	19	tl	393	19	tl	397	57	tl	396	57	tl

Table 4: Impact of the reduction for all the instances on the primal, dual and joint CP models (1/3).

instance	BKS	primal original			primal reduced			dual original			dual reduced			joint original			joint reduced		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
miles250	102*	**102***	102	474	**102***	102	16	114	19	tl	**102***	102	3	**102***	102	1228	**102***	102	15
miles500	260*	260	38	tl	260	77	tl	263	19	tl	**260***	260	1	260	38	tl	**260***	260	18
miles1000	431*	445	38	tl	445	38	tl	433	379	tl	**431***	431	45	**431***	431	1155	**431***	431	210
miles1500	797*	797	304	tl	797	304	tl	799	19	tl	**797***	797	0	**797***	797	474	**797***	797	27
mulsol.i.5	367*	367	114	tl	367	114	tl	368	19	tl	367	19	tl	**367***	367	713	**367***	367	130
myciel5gb	69*	**69***	69	11	**69***	69	12	**69***	69	3502	**69***	69	3498	**69***	69	26	**69***	69	45
myciel5g	22*	**22***	22	17	**22***	22	19	23	5	tl	23	5	tl	**22***	22	183	**22***	22	176
myciel6gb	94	**94***	94	928	**94***	94	998	97	20	tl	97	20	tl	94	20	tl	94	20	tl
myciel6g	26	**26***	26	1772	**26***	26	1724	27	5	tl	27	5	tl	26	10	tl	26	10	tl
myciel7gb	109	109	20	tl	109	20	tl	112	20	tl	112	20	tl	109	20	tl	109	20	tl
myciel7g	29	31	5	tl	31	5	tl	32	5	tl	32	5	tl	31	5	tl	31	5	tl
queen8_8gb	132*	**132***	132	2326	**132***	132	2447	135	125	tl	135	125	tl	**132***	132	1392	**132***	132	1241
queen8_8g	36*	36	24	tl	36	24	tl	38	33	tl	38	33	tl	**36***	36	947	**36***	36	1127
queen9_9gb	159	163	40	tl	163	40	tl	164	139	tl	164	139	tl	163	100	tl	163	103	tl
queen9_9g	41	42	10	tl	42	10	tl	44	38	tl	44	38	tl	42	19	tl	42	16	tl
queen10_10	162	170	19	tl	170	19	tl	177	19	tl	177	19	tl	172	89	tl	172	120	tl
queen10_10gb	164	177	40	tl	177	40	tl	182	79	tl	182	79	tl	181	69	tl	181	70	tl
queen10_10g	43	45	15	tl	45	15	tl	46	26	tl	46	26	tl	46	15	tl	46	15	tl
queen11_11	172	182	38	tl	182	38	tl	189	38	tl	189	38	tl	184	38	tl	184	51	tl
queen11_11gb	176	187	40	tl	187	40	tl	189	40	tl	189	40	tl	187	40	tl	187	40	tl
queen11_11g	47	50	10	tl	50	10	tl	53	20	tl	53	20	tl	51	18	tl	51	12	tl
queen12_12	185	203	19	tl	203	19	tl	211	19	tl	211	19	tl	203	38	tl	203	38	tl
queen12_12gb	191	208	20	tl	208	20	tl	224	20	tl	224	20	tl	213	34	tl	213	50	tl
queen12_12g	50	55	15	tl	55	15	tl	63	5	tl	63	5	tl	55	15	tl	55	15	tl
queen13_13	194	214	19	tl	214	19	tl	220	19	tl	220	19	tl	215	29	tl	215	19	tl
queen14_14	215	230	19	tl	230	19	tl	236	19	tl	236	19	tl	233	19	tl	233	19	tl
queen15_15	223	246	38	tl	246	38	tl	257	19	tl	257	19	tl	247	38	tl	247	38	tl
queen16_16	234	263	19	tl	263	19	tl	276	19	tl	276	19	tl	263	19	tl	263	19	tl
R50_1gb	53*	**53***	53	9	**53***	53	5	**53***	53	2847	**53***	53	42	**53***	53	86	**53***	53	10
R50_1g	14*	**14***	14	7	**14***	14	4	**14***	14	9	**14***	14	4	**14***	14	14	**14***	14	9
R50_5gb	135*	**135***	135	374	**135***	135	384	**135***	135	100	**135***	135	101	**135***	135	283	**135***	135	283
R50_5g	37*	**37***	37	146	**37***	37	134	**37***	37	45	**37***	37	42	**37***	37	100	**37***	37	101
R50_9gb	262*	**262***	262	195	**262***	262	196	**262***	262	0	**262***	262	0	**262***	262	29	**262***	262	26
R50_9g	74*	**74***	74	101	**74***	74	104	**74***	74	0	**74***	74	0	**74***	74	19	**74***	74	18
R75_1gb	70*	**70***	70	80	**70***	70	56	83	20	tl	83	20	tl	**70***	70	1044	**70***	70	1251
R75_1g	18*	**18***	18	14	**18***	18	13	21	5	tl	21	5	tl	**18***	18	190	**18***	18	244
R75_5gb	186*	200	111	tl	200	111	tl	198	161	tl	198	161	tl	194	145	tl	194	145	tl
R75_5g	51*	54	27	tl	54	27	tl	54	44	tl	54	44	tl	52	29	tl	52	27	tl
R75_9gb	396*	400	144	tl	400	144	tl	**396***	396	1	**396***	396	1	**396***	396	50	**396***	396	51
R75_9g	110*	112	30	tl	112	30	tl	**110***	110	2	**110***	110	2	**110***	110	40	**110***	110	40
R100_1gb	81*	**81***	81	331	**81***	81	283	93	20	tl	95	20	tl	81	43	tl	82	40	tl
R100_1g	21*	**21***	21	225	**21***	21	261	25	5	tl	23	5	tl	22	5	tl	22	5	tl
R100_5gb	220	235	40	tl	235	40	tl	242	95	tl	242	95	tl	235	99	tl	235	96	tl
R100_5g	59	65	15	tl	66	15	tl	67	34	tl	67	44	tl	66	25	tl	66	25	tl
R100_9gb	518*	531	80	tl	531	80	tl	**518***	518	56	**518***	518	59	**518***	518	228	**518***	518	217
R100_9g	141*	146	50	tl	146	50	tl	**141***	141	14	**141***	141	14	**141***	141	160	**141***	141	154
wap01a	545	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap02a	538	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap03a	562	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap04a	563	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap05a	542	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap06a	516	-		tl	-		tl	588	19	tl	-		tl	-		tl	-		tl
wap07a	555	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap08a	529	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
zeroin.i.1	511*	518	57	tl	511	57	tl	518	19	tl	511	503	tl	518	57	tl	**511***	511	21
zeroin.i.2	336*	336	57	tl	336	57	tl	336	19	tl	336	19	tl	336	57	tl	**336***	336	90
zeroin.i.3	298*	299	76	tl	298	76	tl	299	19	tl	302	19	tl	299	76	tl	**298***	298	97

Table 5: Impact of the reduction for all the instances on the primal, dual and joint ÷cp models (2/3).

instance	BKS	primal original			primal reduced			dual original			dual reduced			joint original			joint reduced		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
p06	565*	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0
p07	3771*	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	1	**3771***	3771	1
p08	4049*	**4049***	4049	1	**4049***	4049	1	**4049***	4049	0	**4049***	4049	0	**4049***	4049	2	**4049***	4049	2
p09	3388*	**3388***	3388	1	**3388***	3388	1	**3388***	3388	3	**3388***	3388	2	**3388***	3388	5	**3388***	3388	5
p10	3983*	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0
p11	3380*	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0
p12	657*	**657***	657	1	**657***	657	1	**657***	657	0	**657***	657	0	**657***	657	3	**657***	657	3
p13	3220*	**3220***	3220	3	**3220***	3220	3	**3220***	3220	7	**3220***	3220	8	**3220***	3220	17	**3220***	3220	16
p14	3157*	**3157***	3157	2	**3157***	3157	1	**3157***	3157	82	**3157***	3157	89	**3157***	3157	4	**3157***	3157	4
p15	341*	**341***	341	10	**341***	341	10	**341***	341	2	**341***	341	2	**341***	341	9	**341***	341	9
p16	2343*	**2343***	2343	4	**2343***	2343	4	**2343***	2343	15	**2343***	2343	15	**2343***	2343	10	**2343***	2343	11
p17	3281*	**3281***	3281	8	**3281***	3281	10	**3281***	3281	5	**3281***	3281	8	**3281***	3281	23	**3281***	3281	22
p18	3228*	**3228***	3228	3	**3228***	3228	3	**3228***	3228	2	**3228***	3228	2	**3228***	3228	7	**3228***	3228	7
p19	3710*	**3710***	3710	3	**3710***	3710	3	**3710***	3710	0	**3710***	3710	0	**3710***	3710	5	**3710***	3710	6
p20	1830*	**1830***	1830	9	**1830***	1830	10	**1830***	1830	6	**1830***	1830	5	**1830***	1830	16	**1830***	1830	15
p21	3660*	**3660***	3660	9	**3660***	3660	9	**3660***	3660	1	**3660***	3660	1	**3660***	3660	10	**3660***	3660	11
p22	1912*	**1912***	1912	6	**1912***	1912	5	**1912***	1912	9	**1912***	1912	9	**1912***	1912	15	**1912***	1912	15
p23	3770*	**3770***	3770	38	**3770***	3770	40	**3770***	3770	17	**3770***	3770	17	**3770***	3770	40	**3770***	3770	38
p24	661*	**661***	661	2	**661***	661	2	**661***	661	1	**661***	661	0	**661***	661	5	**661***	661	7
p25	504*	**504***	504	3	**504***	504	2	**504***	504	348	**504***	504	36	**504***	504	6	**504***	504	4
p26	520*	**520***	520	9	**520***	520	5	560	488	tl	**520***	520	0	**520***	520	9	**520***	520	6
p27	216*	**216***	216	8	**216***	216	8	**216***	216	8	**216***	216	8	**216***	216	24	**216***	216	28
p28	1729*	**1729***	1729	8	**1729***	1729	6	**1729***	1729	2	**1729***	1729	1	**1729***	1729	10	**1729***	1729	9
p29	3470*	**3470***	3470	15	**3470***	3470	0	**3470***	3470	2	**3470***	3470	0	**3470***	3470	15	**3470***	3470	0
p30	4891*	**4891***	4891	15	**4891***	4891	12	**4891***	4891	10	**4891***	4891	11	**4891***	4891	20	**4891***	4891	26
p31	620*	**620***	620	7	**620***	620	0	**620***	620	1	**620***	620	0	**620***	620	10	**620***	620	0
p32	2480*	**2480***	2480	209	**2480***	2480	50	2480	800	tl	**2480***	2480	0	**2480***	2480	12	**2480***	2480	5
p33	3018*	**3018***	3018	57	**3018***	3018	28	3018	2288	tl	3018	987	tl	**3018***	3018	22	**3018***	3018	29
p34	1980*	**1980***	1980	414	**1980***	1980	396	**1980***	1980	18	**1980***	1980	16	**1980***	1980	44	**1980***	1980	43
p35	2140*	**2140***	2140	2061	**2140***	2140	1719	**2140***	2140	27	**2140***	2140	38	**2140***	2140	121	**2140***	2140	116
p36	7210*	7210	4200	tl	7210	4200	tl	**7210***	7210	25	**7210***	7210	20	**7210***	7210	218	**7210***	7210	99
p38	2130*	2130	1390	tl	2130	1390	tl	2150	700	tl	2150	700	tl	**2130***	2130	159	**2130***	2130	205
p40	4984*	**4984***	4984	2203	**4984***	4984	3005	4984	4647	tl	4984	4890	tl	**4984***	4984	206	**4984***	4984	217
p41	2688*	2718	1136	tl	2688	1136	tl	**2688***	2688	213	**2688***	2688	417	**2688***	2688	489	**2688***	2688	1258
p42	2466*	2480	568	tl	2480	568	tl	2517	568	tl	2517	568	tl	2466	568	tl	**2466***	2466	430
r01	6724*	6865	1402	tl	6865	1402	tl	7138	703	tl	7138	703	tl	**6724***	6724	2582	**6724***	6724	2835
r02	6771*	6771	1403	tl	6771	1403	tl	6786	704	tl	6786	704	tl	6771	1403	tl	6771	1403	tl
r03	6473*	6513	698	tl	6513	698	tl	6627	698	tl	6627	698	tl	6627	698	tl	**6473***	6473	1386
r04	6342*	6342	1396	tl	6342	1396	tl	6388	702	tl	6388	702	tl	**6342***	6342	3459	6342	1396	tl
r05	6408*	6435	1404	tl	6435	1404	tl	6495	703	tl	6495	703	tl	**6408***	6408	2205	**6408***	6408	1021
r06	7550*	7550	702	tl	7550	702	tl	7550	702	tl	7550	702	tl	**7550***	7550	2751	**7550***	7550	2319
r07	6889*	6889	701	tl	6889	932	tl	7201	701	tl	7201	701	tl	**6889***	6889	454	**6889***	6889	736
r08	6057*	6084	703	tl	6084	703	tl	6146	703	tl	6146	703	tl	**6057***	6057	1116	**6057***	6057	1050
r09	6358*	6407	701	tl	6407	701	tl	6542	701	tl	6542	701	tl	**6358***	6358	1777	**6358***	6358	1299
r10	6508*	6540	2042	tl	6540	2042	tl	6540	694	tl	6540	694	tl	**6508***	6508	1578	**6508***	6508	1398
r11	7654*	7788	703	tl	7788	703	tl	8074	703	tl	8074	703	tl	7788	703	tl	7788	703	tl
r12	7690*	7711	704	tl	7711	704	tl	7930	704	tl	7930	704	tl	7711	704	tl	7711	704	tl
r13	7500*	7726	702	tl	7722	702	tl	7749	702	tl	7749	702	tl	7536	702	tl	7536	702	tl
r14	8254*	8259	1404	tl	8259	1404	tl	8259	703	tl	8259	703	tl	8259	1404	tl	8259	1404	tl
r15	8021*	8021	702	tl	8021	702	tl	8021	702	tl	8021	702	tl	8021	702	tl	8021	702	tl
r16	7755*	7764	703	tl	7764	703	tl	7764	703	tl	7764	703	tl	7764	703	tl	7764	703	tl
r17	7979*	7988	1397	tl	7988	1397	tl	8214	700	tl	8214	700	tl	8001	1397	tl	8001	1397	tl
r18	7232*	7279	1048	tl	7279	1048	tl	7455	704	tl	7455	704	tl	7332	704	tl	7332	704	tl
r19	6826*	6846	1397	tl	6846	1397	tl	7296	703	tl	7296	703	tl	6826	5653	tl	6826	5653	tl
r20	8023*	8027	704	tl	8027	704	tl	8055	704	tl	8055	704	tl	8055	704	tl	8055	704	tl
r21	9284*	9290	704	tl	9290	704	tl	9293	704	tl	9293	704	tl	9290	704	tl	9290	704	tl
r22	8887*	8981	702	tl	8981	702	tl	9014	702	tl	9014	702	tl	9036	702	tl	9036	702	tl
r23	9136*	9162	703	tl	9162	703	tl	9239	703	tl	9239	703	tl	9201	703	tl	9201	703	tl
r24	8464*	8464	703	tl	8464	703	tl	8484	703	tl	8484	703	tl	8464	703	tl	8464	703	tl
r25	8426*	8544	701	tl	8544	701	tl	8583	701	tl	8583	701	tl	8583	701	tl	8583	701	tl
r26	8819*	9005	1400	tl	9005	1400	tl	9013	704	tl	9013	704	tl	9008	1400	tl	9008	1400	tl
r27	7975*	7975	1398	tl	7975	1398	tl	8003	701	tl	8003	701	tl	8003	1398	tl	8003	1398	tl
r28	9407*	9407	1407	tl	9407	1407	tl	9407	704	tl	9407	704	tl	9407	1407	tl	9407	1407	tl
r29	8693*	8693	703	tl	8693	703	tl	8973	703	tl	8973	703	tl	8959	703	tl	8959	703	tl
r30	9816*	9831	704	tl	9831	704	tl	9831	704	tl	9831	704	tl	9831	704	tl	9831	704	tl
nb bks		97/188			101/188			67/188			79/188			108/188			113/188		
nb optim		71/188			72/188			56/188			68/188			94/188			101/188		

Table 6: Impact of the reduction for all the instances on the primal, dual and joint ÷cp÷ models (3/3).

instance	$\|V'\|$	density	$\|W\|/\|V\|$	$\Delta+1$	colors bounds lb	colors bounds ub	BKS	score bounds lb	score bounds ub
C2000.5	2000	0.5	0.01	1075	15	301	2144	243	3032
C2000.9	2000	0.9	0.01	1849	69	744	5477	1013	7620
DSJC125.1gb	125	0.1	0.16	24	4	24	90	70	351
DSJC125.1g	125	0.1	0.04	24	4	14	23	19	42
DSJC125.5gb	125	0.5	0.16	76	10	58	240	142	589
DSJC125.5g	125	0.5	0.04	76	10	34	71	42	105
DSJC125.9gb	125	0.9	0.16	121	33	104	604*	453	1102
DSJC125.9g	125	0.9	0.04	121	32	72	169*	124	220
DSJC250.1	250	0.1	0.08	39	4	39	127	71	460
DSJC250.5	250	0.5	0.08	148	11	88	392	164	860
DSJC250.9	250	0.9	0.08	235	40	162	934*	543	1615
DSJC500.1	500	0.1	0.04	69	5	56	184	82	548
DSJC500.5	500	0.5	0.04	287	13	127	685	205	1288
DSJC500.9	500	0.9	0.04	472	52	274	1662	719	2716
DSJC1000.1	1000	0.1	0.02	128	6	72	300	90	708
DSJC1000.5	1000	0.5	0.02	552	15	190	1185	219	1906
DSJC1000.9	1000	0.9	0.02	925	61	457	2836	846	4391
DSJR500.1	244	0.03	0.08	26	12	26	169	166	477
flat1000_50_0	1000	0.49	0.02	521	13	184	924	215	1823
flat1000_60_0	1000	0.49	0.02	525	14	186	1162	224	1877
flat1000_76_0	1000	0.49	0.02	533	14	189	1165	218	1892
GEOM20a	6	0.2	0.83	6	5	6	33*	33	40
GEOM20b	13	0.17	0.23	6	3	6	8*	8	11
GEOM20	7	0.1	0.71	5	5	5	33*	33	48
GEOM30a	20	0.19	0.5	11	6	11	42*	42	75
GEOM30b	26	0.19	0.12	11	5	9	12*	12	18
GEOM30	10	0.12	0.7	6	6	6	32*	32	43
GEOM40a	17	0.19	0.59	9	7	9	49*	49	83
GEOM40b	27	0.2	0.11	14	7	11	16*	16	22
GEOM40	19	0.1	0.42	7	6	7	37*	36	71
GEOM50a	28	0.19	0.36	15	9	15	65*	64	113
GEOM50b	38	0.2	0.08	18	8	11	18*	18	23
GEOM50	24	0.1	0.38	8	6	8	40*	40	81
GEOM60a	44	0.19	0.23	18	10	18	73*	73	132
GEOM60b	49	0.21	0.06	21	9	17	23*	23	35
GEOM60	39	0.1	0.26	10	6	10	43*	42	106
GEOM70a	53	0.19	0.19	21	11	21	73*	73	110
GEOM70b	55	0.2	0.05	25	10	18	24*	24	36
GEOM70	62	0.11	0.16	14	8	14	47*	47	105
GEOM80a	71	0.19	0.14	24	12	24	76*	75	160
GEOM80b	70	0.21	0.04	30	12	18	27*	27	36
GEOM80	43	0.11	0.23	15	8	15	66*	66	108
GEOM90a	86	0.2	0.12	26	12	26	73*	70	139
GEOM90b	69	0.21	0.04	35	15	19	30*	30	36
GEOM90	73	0.11	0.14	16	8	16	61*	57	128
GEOM100a	99	0.2	0.1	29	13	29	89*	87	169
GEOM100b	88	0.21	0.03	38	15	20	32*	32	41
GEOM100	75	0.11	0.13	18	9	18	65*	65	123
GEOM110a	110	0.2	0.09	33	14	33	97*	96	190
GEOM110b	90	0.21	0.03	40	15	25	37*	37	52
GEOM110	87	0.11	0.11	20	9	20	68*	65	151
GEOM120a	119	0.2	0.08	36	16	34	105*	102	192
GEOM120b	110	0.21	0.03	44	16	24	35*	35	46
GEOM120	100	0.11	0.1	22	11	22	72*	71	146
inithx.i.1	181	0.05	0.1	169	54	78	569*	569	800
inithx.i.2	210	0.07	0.09	194	31	62	329*	329	615
inithx.i.3	236	0.07	0.08	220	31	65	337*	336	669
latin_square_10	900	0.76	0.02	684	85	297	1480	993	3058
le450_15a	420	0.08	0.05	99	15	61	212	206	628
le450_15b	430	0.08	0.04	95	15	61	216	213	638
le450_15c	450	0.17	0.04	140	15	71	275	208	680
le450_15d	450	0.17	0.04	139	15	70	272	199	676
le450_25a	346	0.08	0.05	128	25	69	306	304	720
le450_25b	345	0.08	0.06	108	25	73	307	307	735
le450_25c	440	0.17	0.04	178	25	83	342	330	805
le450_25d	442	0.17	0.04	158	25	80	330	309	750

Table 7: Lower and upper bounds on the score and the number of colors required to build an optimal solution. (1/3)

instance	$\|V'\|$	density	$\|W\|/\|V\|$	$\Delta+1$	colors bounds lb	ub	BKS	score bounds lb	ub
miles250	32	0.1	0.44	17	8	17	102*	100	207
miles500	48	0.29	0.31	37	20	32	260*	260	377
miles1000	93	0.79	0.2	85	42	61	431*	412	600
miles1500	93	1.28	0.2	92	73	82	797*	797	880
mulsol.i.5	104	0.23	0.18	88	31	58	367*	367	574
myciel5gb	47	0.22	0.43	24	2	24	69*	38	274
myciel5g	47	0.22	0.11	24	2	11	22*	10	36
myciel6gb	95	0.17	0.21	48	2	32	94	39	339
myciel6g	95	0.17	0.05	48	2	15	26	10	45
myciel7gb	191	0.13	0.1	96	2	41	109	40	438
myciel7g	191	0.13	0.03	96	2	17	29	10	52
queen8_8gb	64	0.72	0.31	28	8	28	132*	120	365
queen8_8g	64	0.72	0.08	28	8	21	36*	31	62
queen9_9gb	81	0.65	0.25	33	9	33	159	147	486
queen9_9g	81	0.65	0.06	33	9	24	41	38	75
queen10_10	100	0.59	0.19	36	10	36	162	153	420
queen10_10gb	100	0.59	0.2	36	10	36	164	149	473
queen10_10g	100	0.59	0.05	36	10	23	43	40	70
queen11_11	121	0.55	0.16	41	11	41	172	157	468
queen11_11gb	121	0.55	0.16	41	11	41	176	151	458
queen11_11g	121	0.55	0.04	41	11	26	47	43	81
queen12_12	144	0.5	0.13	44	12	44	185	173	482
queen12_12gb	144	0.5	0.14	44	12	44	191	177	515
queen12_12g	144	0.5	0.03	44	12	25	50	45	76
queen13_13	169	0.47	0.11	49	13	49	194	179	550
queen14_14	196	0.44	0.1	52	14	52	215	206	530
queen15_15	225	0.41	0.08	57	15	56	223	209	590
queen16_16	256	0.39	0.07	60	16	60	234	218	650
R50_1gb	42	0.09	0.43	9	3	9	53*	48	199
R50_1g	42	0.09	0.12	9	3	8	14*	12	21
R50_5gb	50	0.5	0.38	37	7	33	135*	102	347
R50_5g	50	0.5	0.1	37	7	19	37*	28	59
R50_9gb	50	0.89	0.36	48	20	44	262*	237	452
R50_9g	50	0.89	0.1	48	20	38	74*	66	112
R75_1gb	69	0.1	0.29	13	3	13	70*	55	301
R75_1g	69	0.1	0.07	13	3	11	18*	14	34
R75_5gb	75	0.51	0.27	49	8	41	186*	126	447
R75_5g	75	0.51	0.07	49	8	23	51*	34	70
R75_9gb	75	0.91	0.27	72	25	69	396*	318	702
R75_9g	75	0.91	0.07	72	25	51	110*	87	144
R100_1gb	99	0.1	0.2	21	4	21	81*	62	368
R100_1g	98	0.1	0.05	21	4	14	21*	16	39
R100_5gb	100	0.5	0.2	62	8	54	220	137	596
R100_5g	100	0.5	0.05	62	8	30	59	36	90
R100_9gb	100	0.9	0.2	97	29	88	518*	414	960
R100_9g	100	0.9	0.05	97	29	59	141*	114	186
wap01a	2157	0.04	0.01	285	41	121	545	533	1229
wap02a	2215	0.04	0.01	289	40	115	538	529	1153
wap03a	4594	0.03	0	343	40	134	562	523	1308
wap04a	4910	0.02	0	348	40	136	563	538	1339
wap05a	824	0.1	0.02	220	50	110	542	540	1100
wap06a	858	0.1	0.02	222	40	110	516	503	1074
wap07a	1725	0.06	0.01	299	40	123	555	531	1307
wap08a	1785	0.06	0.01	305	40	118	529	510	1167
zeroin.i.1	81	0.18	0.23	74	49	59	511*	511	629
zeroin.i.2	100	0.16	0.19	84	30	57	336*	336	594
zeroin.i.3	95	0.17	0.2	79	30	51	298*	297	487

Table 8: Lower and upper bounds on the score and the number of colors required to build an optimal solution. (2/3)

instance	$\lvert V'\rvert$	density	$\lvert W\rvert/\lvert V\rvert$	$\Delta+1$	colors bounds lb	colors bounds ub	BKS	score bounds lb	score bounds ub
p06	16	0.32	0.44	7	4	7	565*	560	764
p07	24	0.33	1	9	5	9	3771*	3771	10527
p08	24	0.33	0.79	9	5	9	4049*	4049	11567
p09	25	0.33	1	9	5	9	3388*	3314	11857
p10	16	0.27	0.62	5	4	5	3983*	3983	4668
p11	18	0.31	0.5	9	5	9	3380*	3380	3590
p12	26	0.28	0.5	9	5	9	657*	630	1295
p13	34	0.28	0.62	11	6	11	3220*	3220	8444
p14	31	0.24	0.94	10	6	10	3157*	3157	11821
p15	34	0.24	0.5	11	6	11	341*	334	741
p16	34	0.24	0.88	10	6	10	2343*	2321	8210
p17	37	0.24	0.89	12	7	12	3281*	3259	12454
p18	35	0.24	0.91	11	6	11	3228*	3228	12201
p19	36	0.25	0.42	13	7	13	3710*	3710	4300
p20	37	0.21	0.7	10	6	10	1830*	1800	6630
p21	37	0.22	0.68	13	7	13	3660*	3620	9040
p22	38	0.22	0.82	11	6	11	1912*	1912	5699
p23	44	0.22	0.68	13	7	13	3770*	3670	10155
p24	31	0.18	0.68	10	6	10	661*	661	1845
p25	33	0.19	0.94	11	6	11	504*	504	2079
p26	33	0.2	0.55	11	7	11	520*	520	1418
p27	44	0.18	1	10	6	10	216*	211	990
p28	37	0.17	0.7	11	7	11	1729*	1729	5685
p29	14	0.18	0.79	13	10	12	3470*	3470	4760
p30	60	0.18	0.52	15	8	15	4891*	4891	7118
p31	9	0.17	1	9	8	9	620*	620	709
p32	34	0.17	0.74	15	8	15	2480*	2480	7660
p33	54	0.17	0.61	13	7	13	3018*	3018	9463
p34	74	0.16	0.36	19	10	19	1980*	1980	5880
p35	83	0.15	0.3	18	10	18	2140*	2140	5660
p36	93	0.16	0.3	20	12	20	7210*	7210	16740
p38	85	0.14	0.32	18	10	18	2130*	2130	6634
p40	84	0.14	0.77	18	10	18	4984*	4984	23165
p41	115	0.14	0.54	23	13	23	2688*	2688	13255
p42	135	0.12	0.46	25	14	25	2466*	2466	8108
r01	144	0.12	0.89	24	13	24	6724*	6707	61764
r02	142	0.12	0.89	24	13	24	6771*	6771	59350
r03	139	0.12	0.86	24	13	24	6473*	6461	54638
r04	151	0.12	0.87	24	13	24	6342*	6337	56085
r05	141	0.13	0.87	24	13	24	6408*	6401	55854
r06	147	0.13	0.86	26	14	26	7550*	7550	59051
r07	138	0.13	0.86	26	14	26	6889*	6889	52931
r08	138	0.13	0.93	24	13	24	6057*	6052	55037
r09	128	0.12	0.88	24	13	24	6358*	6352	53939
r10	149	0.13	0.83	24	13	24	6508*	6497	55615
r11	208	0.1	0.86	27	15	27	7654*	7640	85543
r12	199	0.1	0.83	26	14	26	7690*	7690	77611
r13	217	0.1	0.8	28	15	28	7500*	7499	77738
r14	214	0.1	0.81	28	15	28	8254*	8254	81419
r15	198	0.1	0.86	29	16	29	8021*	8021	75659
r16	186	0.11	0.84	29	16	29	7755*	7755	70935
r17	213	0.11	0.77	29	15	29	7979*	7979	76125
r18	200	0.1	0.82	27	14	27	7232*	7222	75066
r19	185	0.11	0.85	25	13	25	6826*	6814	71137
r20	217	0.1	0.81	30	16	30	8023*	8018	82395
r21	281	0.09	0.78	33	17	33	9284*	9284	105117
r22	285	0.09	0.77	35	18	35	8887*	8887	100442
r23	288	0.09	0.77	34	18	34	9136*	9136	102883
r24	269	0.09	0.8	32	17	32	8464*	8464	97105
r25	266	0.09	0.76	31	17	31	8426*	8423	95884
r26	284	0.09	0.75	34	18	34	8819*	8819	99182
r27	259	0.09	0.78	31	16	31	7975*	7975	91594
r28	288	0.09	0.78	35	19	35	9407*	9407	105198
r29	281	0.09	0.76	33	17	33	8693*	8693	97545
r30	301	0.09	0.76	35	19	35	9816*	9816	104285

Table 9: Lower and upper bounds on the score and the number of colors required to build an optimal solution. (3/3)

		primal			primal lb color			primal ub color			primal lb score			primal ub score			primal all bounds		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
C2000.5	2144	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
C2000.9	5477	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJC125.1gb	90	90	34	tl	90	37	tl	91	59	tl	91	70	tl	90	34	tl	91	70	tl
DSJC125.1g	23	**23***	23	862	**23***	23	777	**23***	23	435	**23***	23	1031	**23***	23	1020	**23***	23	450
DSJC125.5gb	240	270	59	tl	270	59	tl	270	59	tl	270	142	tl	270	59	tl	270	142	tl
DSJC125.5g	71	78	25	tl	78	25	tl	78	35	tl	78	42	tl	78	25	tl	78	42	tl
DSJC125.9gb	604*	635	100	tl	635	100	tl	635	100	tl	635	453	tl	635	100	tl	635	453	tl
DSJC125.9g	169*	176	50	tl	176	50	tl	176	50	tl	176	124	tl	176	50	tl	176	124	tl
DSJC250.1	127	143	19	tl	143	19	tl	142	19	tl	143	71	tl	143	19	tl	138	71	tl
DSJC250.5	392	466	57	tl	466	57	tl	465	57	tl	466	164	tl	466	57	tl	465	164	tl
DSJC250.9	934*	1058	152	tl	1057	152	tl	1057	152	tl	1057	543	tl	1058	152	tl	1058	543	tl
DSJC500.1	184	222	19	tl	222	19	tl	214	19	tl	222	82	tl	222	19	tl	214	82	tl
DSJC500.5	685	826	38	tl	828	38	tl	822	38	tl	828	205	tl	826	38	tl	824	205	tl
DSJC500.9	1662	1859	152	tl	1858	152	tl	1859	152	tl	1859	719	tl	1856	152	tl	1856	719	tl
DSJC1000.1	300	-		tl	-		tl	360	38	tl	-		tl	-		tl	360	90	tl
DSJC1000.5	1185	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJC1000.9	2836	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
DSJR500.1	169	187	19	tl	187	19	tl	177	19	tl	187	166	tl	187	19	tl	169	166	tl
flat1000_50_0	924	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
flat1000_60_0	1162	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
flat1000_76_0	1165	-		tl	-		tl	-		tl	-		tl	-		tl	1386	218	tl
GEOM20a	33*	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0
GEOM20b	8*	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0
GEOM20	33*	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0
GEOM30a	42*	**42***	42	0	**42***	42	0	**42***	42	0	**42***	42	0	**42***	42	0	**42***	42	0
GEOM30b	12*	**12***	12	1	**12***	12	1	**12***	12	0	**12***	12	1	**12***	12	1	**12***	12	0
GEOM30	32*	**32***	32	0	**32***	32	0	**32***	32	0	**32***	32	0	**32***	32	0	**32***	32	0
GEOM40a	49*	**49***	49	0	**49***	49	0	**49***	49	0	**49***	49	0	**49***	49	0	**49***	49	0
GEOM40b	16*	**16***	16	2	**16***	16	0	**16***	16	0	**16***	16	0	**16***	16	2	**16***	16	0
GEOM40	37*	**37***	37	1	**37***	37	0	**37***	37	0	**37***	37	1	**37***	37	1	**37***	37	0
GEOM50a	65*	**65***	65	4	**65***	65	2	**65***	65	2	**65***	65	4	**65***	65	4	**65***	65	1
GEOM50b	18*	**18***	18	8	**18***	18	4	**18***	18	2	**18***	18	3	**18***	18	8	**18***	18	0
GEOM50	40*	**40***	40	0	**40***	40	0	**40***	40	0	**40***	40	0	**40***	40	0	**40***	40	0
GEOM60a	73*	**73***	73	98	**73***	73	100	**73***	73	95	**73***	73	6	**73***	73	79	**73***	73	2
GEOM60b	23*	**23***	23	54	**23***	23	8	**23***	23	21	**23***	23	7	**23***	23	51	**23***	23	2
GEOM60	43*	**43***	43	4	**43***	43	4	**43***	43	1	**43***	43	4	**43***	43	5	**43***	43	0
GEOM70a	73*	**73***	73	252	**73***	73	191	**73***	73	206	**73***	73	17	**73***	73	286	**73***	73	10
GEOM70b	24*	**24***	24	17	**24***	24	11	**24***	24	9	**24***	24	9	**24***	24	13	**24***	24	3
GEOM70	47*	**47***	47	44	**47***	47	17	**47***	47	13	**47***	47	25	**47***	47	46	**47***	47	9
GEOM80a	76*	76	48	tl	76	55	tl	76	52	tl	76	75	tl	76	48	tl	76	75	tl
GEOM80b	27*	**27***	27	74	**27***	27	33	**27***	27	28	**27***	27	65	**27***	27	74	**27***	27	10
GEOM80	66*	**66***	66	36	**66***	66	26	**66***	66	9	**66***	66	5	**66***	66	39	**66***	66	1
GEOM90a	73*	73	10	tl	73	10	tl	73	28	tl	73	70	tl	73	10	tl	73	70	tl
GEOM90b	30*	**30***	30	1378	**30***	30	28	**30***	30	107	**30***	30	64	**30***	30	1020	**30***	30	12
GEOM90	61*	**61***	61	2485	**61***	61	2712	**61***	61	260	**61***	61	2607	**61***	61	2593	**61***	61	111
GEOM100a	89*	91	10	tl	91	10	tl	89	29	tl	90	87	tl	91	10	tl	89	87	tl
GEOM100b	32*	**32***	32	781	**32***	32	1093	**32***	32	77	**32***	32	355	**32***	32	686	**32***	32	12
GEOM100	65*	65	29	tl	**65***	65	1103	**65***	65	276	**65***	65	97	65	29	tl	**65***	65	24
GEOM110a	97*	104	20	tl	104	20	tl	104	39	tl	104	96	tl	104	20	tl	104	96	tl
GEOM110b	37*	**37***	37	137	**37***	37	143	**37***	37	26	**37***	37	66	**37***	37	141	**37***	37	7
GEOM110	68*	69	30	tl	69	30	tl	**68***	68	1892	69	65	tl	69	30	tl	**68***	68	1728
GEOM120a	105*	113	20	tl	113	20	tl	113	49	tl	113	102	tl	113	20	tl	112	102	tl
GEOM120b	35*	37	15	tl	37	15	tl	37	22	tl	37	35	tl	37	15	tl	**35***	35	171
GEOM120	72*	72	20	tl	72	20	tl	72	40	tl	72	71	tl	72	20	tl	72	71	tl
inithx.i.1	569*	569	57	tl	569	57	tl	569	57	tl	**569***	569	73	569	57	tl	**569***	569	54
inithx.i.2	329*	329	19	tl	329	19	tl	329	19	tl	**329***	329	150	329	19	tl	**329***	329	65
inithx.i.3	337*	337	76	tl	337	76	tl	337	76	tl	337	336	tl	337	76	tl	337	336	tl
latin_square_10	1480	-		tl	-		tl	1873	95	tl	-		tl	-		tl	1873	993	tl
le450_15a	212	245	19	tl	245	19	tl	234	19	tl	235	206	tl	237	19	tl	234	206	tl
le450_15b	216	249	19	tl	243	19	tl	234	19	tl	242	213	tl	242	19	tl	233	213	tl
le450_15c	275	336	38	tl	338	38	tl	325	38	tl	338	208	tl	338	38	tl	325	208	tl
le450_15d	272	329	19	tl	329	19	tl	318	19	tl	-		tl	322	19	tl	324	199	tl
le450_25a	306	323	38	tl	323	38	tl	323	38	tl	323	304	tl	323	38	tl	323	304	tl
le450_25b	307	307	19	tl	315	19	tl	307	19	tl	**307***	307	1372	307	19	tl	**307***	307	321
le450_25c	342	398	38	tl	400	38	tl	389	38	tl	397	330	tl	398	38	tl	389	330	tl
le450_25d	330	388	57	tl	389	57	tl	383	57	tl	397	309	tl	398	57	tl	383	309	tl

Table 10: Impacts of all pre-computed lower and upper bounds on the objective value and the number of colors for the resolution with the primal model. (1/3)

instance	BKS	primal			primal lb color			primal ub color			primal lb score			primal ub score			primal all bounds		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
miles250	102*	**102***	102	16	**102***	102	8	**102***	102	11	**102***	102	17	**102***	102	15	**102***	102	6
miles500	260*	260	77	tl	260	191	tl	260	80	tl	**260***	260	12	260	77	tl	**260***	260	4
miles1000	431*	445	38	tl	445	38	tl	445	38	tl	445	412	tl	445	38	tl	445	412	tl
miles1500	797*	797	304	tl	797	304	tl	797	304	tl	**797***	797	31	797	304	tl	**797***	797	28
mulsol.i.5	367*	367	114	tl	367	114	tl	367	114	tl	**367***	367	51	367	114	tl	**367***	367	30
myciel5gb	69*	**69***	69	12	**69***	69	11	**69***	69	17	**69***	69	11	**69***	69	12	**69***	69	17
myciel5g	22*	**22***	22	19	**22***	22	16	**22***	22	6	**22***	22	18	**22***	22	17	**22***	22	7
myciel6gb	94	**94***	94	998	**94***	94	1171	**94***	94	1358	**94***	94	1016	**94***	94	1187	**94***	94	1936
myciel6g	26	**26***	26	1724	**26***	26	1787	**26***	26	1011	**26***	26	1756	**26***	26	1907	**26***	26	1026
myciel7gb	109	109	20	tl	109	20	tl	109	20	tl	109	40	tl	109	20	tl	109	40	tl
myciel7g	29	31	5	tl	31	5	tl	30	15	tl	31	10	tl	31	5	tl	30	15	tl
queen8_8gb	132*	**132***	132	2447	**132***	132	1872	**132***	132	3589	**132***	132	2199	**132***	132	2159	132	120	tl
queen8_8g	36*	36	24	tl	36	26	tl	36	24	tl	36	31	tl	36	24	tl	36	31	tl
queen9_9gb	159	163	40	tl	163	40	tl	163	92	tl	163	147	tl	163	40	tl	163	147	tl
queen9_9g	41	42	10	tl	42	10	tl	42	24	tl	42	38	tl	42	10	tl	42	38	tl
queen10_10	162	170	19	tl	170	19	tl	169	71	tl	170	153	tl	170	19	tl	169	153	tl
queen10_10gb	164	177	40	tl	177	40	tl	176	87	tl	177	149	tl	177	40	tl	176	149	tl
queen10_10g	43	45	15	tl	45	15	tl	45	24	tl	45	40	tl	45	15	tl	45	40	tl
queen11_11	172	182	38	tl	183	38	tl	181	38	tl	183	157	tl	183	57	tl	181	157	tl
queen11_11gb	176	187	40	tl	187	40	tl	187	40	tl	187	151	tl	187	40	tl	187	151	tl
queen11_11g	47	50	10	tl	50	10	tl	50	24	tl	50	43	tl	50	10	tl	50	43	tl
queen12_12	185	203	19	tl	203	19	tl	203	19	tl	203	173	tl	203	19	tl	203	173	tl
queen12_12gb	191	208	20	tl	208	20	tl	204	20	tl	208	177	tl	208	20	tl	204	177	tl
queen12_12g	50	55	15	tl	55	15	tl	55	24	tl	55	45	tl	55	15	tl	55	45	tl
queen13_13	194	214	19	tl	214	19	tl	213	19	tl	214	179	tl	214	19	tl	214	179	tl
queen14_14	215	230	19	tl	230	19	tl	225	19	tl	230	206	tl	230	19	tl	225	206	tl
queen15_15	223	246	38	tl	246	38	tl	246	38	tl	246	209	tl	246	38	tl	246	209	tl
queen16_16	234	263	19	tl	263	19	tl	261	19	tl	263	218	tl	263	19	tl	261	218	tl
R50_1gb	53*	**53***	53	5	**53***	53	6	**53***	53	0	**53***	53	5	**53***	53	5	**53***	53	0
R50_1g	14*	**14***	14	4	**14***	14	5	**14***	14	0	**14***	14	4	**14***	14	4	**14***	14	0
R50_5gb	135*	**135***	135	384	**135***	135	348	**135***	135	333	**135***	135	379	**135***	135	377	**135***	135	291
R50_5g	37*	**37***	37	134	**37***	37	157	**37***	37	178	**37***	37	137	**37***	37	138	**37***	37	149
R50_9gb	262*	**262***	262	196	**262***	262	164	**262***	262	202	**262***	262	188	**262***	262	191	**262***	262	148
R50_9g	74*	**74***	74	104	**74***	74	71	**74***	74	113	**74***	74	99	**74***	74	79	**74***	74	83
R75_1gb	70*	**70***	70	56	**70***	70	54	**70***	70	48	**70***	70	54	**70***	70	54	**70***	70	50
R75_1g	18*	**18***	18	13	**18***	18	14	**18***	18	5	**18***	18	13	**18***	18	13	**18***	18	5
R75_5gb	186*	200	111	tl	200	112	tl	198	118	tl	200	126	tl	200	111	tl	195	126	tl
R75_5g	51*	54	27	tl	54	28	tl	54	27	tl	54	34	tl	54	27	tl	54	34	tl
R75_9gb	396*	400	144	tl	400	142	tl	400	119	tl	400	318	tl	400	144	tl	400	318	tl
R75_9g	110*	112	30	tl	112	30	tl	112	68	tl	112	87	tl	112	30	tl	112	87	tl
R100_1gb	81*	**81***	81	283	**81***	81	403	**81***	81	297	**81***	81	331	**81***	81	326	**81***	81	374
R100_1g	21*	**21***	21	261	**21***	21	203	**21***	21	162	**21***	21	263	**21***	21	239	**21***	21	191
R100_5gb	220	235	40	tl	235	40	tl	233	40	tl	235	137	tl	235	40	tl	233	137	tl
R100_5g	59	66	15	tl	67	15	tl	66	29	tl	67	36	tl	66	15	tl	66	36	tl
R100_9gb	518*	531	80	tl	531	80	tl	531	80	tl	531	414	tl	531	80	tl	531	414	tl
R100_9g	141*	146	50	tl	146	50	tl	146	50	tl	146	114	tl	146	50	tl	146	114	tl
wap01a	545	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap02a	538	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap03a	562	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap04a	563	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap05a	542	-		tl	-		tl	588	57	tl	-		tl	-		tl	592	540	tl
wap06a	516	-		tl	-		tl	595	76	tl	599	503	tl	-		tl	598	503	tl
wap07a	555	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
wap08a	529	-		tl	-		tl	-		tl	-		tl	-		tl	-		tl
zeroin.i.1	511*	511	57	tl	511	57	tl	511	57	tl	**511***	511	20	511	57	tl	**511***	511	10
zeroin.i.2	336*	336	57	tl	336	57	tl	336	57	tl	**336***	336	41	336	57	tl	**336***	336	28
zeroin.i.3	298*	298	76	tl	298	76	tl	298	76	tl	298	297	tl	298	76	tl	298	297	tl

Table 11: Impacts of all pre-computed lower and upper bounds on the objective value and the number of colors for the resolution with the primal model. (2/3)

instance	BKS	primal			primal lb color			primal ub color			primal lb score			primal ub score			primal all bounds		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
p06	565*	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0
p07	3771*	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0
p08	4049*	**4049***	4049	1	**4049***	4049	1	**4049***	4049	0	**4049***	4049	1	**4049***	4049	1	**4049***	4049	0
p09	3388*	**3388***	3388	1	**3388***	3388	1	**3388***	3388	0	**3388***	3388	1	**3388***	3388	2	**3388***	3388	0
p10	3983*	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0
p11	3380*	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0
p12	657*	**657***	657	1	**657***	657	1	**657***	657	0	**657***	657	1	**657***	657	1	**657***	657	0
p13	3220*	**3220***	3220	3	**3220***	3220	3	**3220***	3220	1	**3220***	3220	3	**3220***	3220	3	**3220***	3220	0
p14	3157*	**3157***	3157	1	**3157***	3157	2	**3157***	3157	0	**3157***	3157	2	**3157***	3157	1	**3157***	3157	0
p15	341*	**341***	341	10	**341***	341	7	**341***	341	3	**341***	341	9	**341***	341	10	**341***	341	3
p16	2343*	**2343***	2343	4	**2343***	2343	4	**2343***	2343	1	**2343***	2343	3	**2343***	2343	4	**2343***	2343	1
p17	3281*	**3281***	3281	10	**3281***	3281	7	**3281***	3281	3	**3281***	3281	8	**3281***	3281	8	**3281***	3281	2
p18	3228*	**3228***	3228	3	**3228***	3228	3	**3228***	3228	1	**3228***	3228	3	**3228***	3228	3	**3228***	3228	0
p19	3710*	**3710***	3710	3	**3710***	3710	2	**3710***	3710	0	**3710***	3710	2	**3710***	3710	0	**3710***	3710	0
p20	1830*	**1830***	1830	10	**1830***	1830	8	**1830***	1830	4	**1830***	1830	5	**1830***	1830	8	**1830***	1830	1
p21	3660*	**3660***	3660	9	**3660***	3660	7	**3660***	3660	6	**3660***	3660	4	**3660***	3660	7	**3660***	3660	1
p22	1912*	**1912***	1912	5	**1912***	1912	5	**1912***	1912	2	**1912***	1912	4	**1912***	1912	6	**1912***	1912	1
p23	3770*	**3770***	3770	40	**3770***	3770	32	**3770***	3770	16	**3770***	3770	11	**3770***	3770	36	**3770***	3770	5
p24	661*	**661***	661	2	**661***	661	3	**661***	661	0	**661***	661	1	**661***	661	2	**661***	661	0
p25	504*	**504***	504	2	**504***	504	2	**504***	504	0	**504***	504	2	**504***	504	2	**504***	504	0
p26	520*	**520***	520	5	**520***	520	3	**520***	520	2	**520***	520	2	**520***	520	3	**520***	520	0
p27	216*	**216***	216	8	**216***	216	8	**216***	216	2	**216***	216	8	**216***	216	8	**216***	216	2
p28	1729*	**1729***	1729	6	**1729***	1729	4	**1729***	1729	2	**1729***	1729	3	**1729***	1729	6	**1729***	1729	0
p29	3470*	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0
p30	4891*	**4891***	4891	12	**4891***	4891	11	**4891***	4891	4	**4891***	4891	10	**4891***	4891	12	**4891***	4891	4
p31	620*	**620***	620	0	**620***	620	0	**620***	620	0	**620***	620	0	**620***	620	0	**620***	620	0
p32	2480*	**2480***	2480	50	**2480***	2480	22	**2480***	2480	8	**2480***	2480	2	**2480***	2480	29	**2480***	2480	0
p33	3018*	**3018***	3018	28	**3018***	3018	16	**3018***	3018	8	**3018***	3018	10	**3018***	3018	48	**3018***	3018	1
p34	1980*	**1980***	1980	396	**1980***	1980	352	**1980***	1980	27	**1980***	1980	28	**1980***	1980	212	**1980***	1980	7
p35	2140*	**2140***	2140	1719	**2140***	2140	3145	**2140***	2140	732	**2140***	2140	30	**2140***	2140	2108	**2140***	2140	10
p36	7210*	7210	4200	tl	7210	4200	tl	7210	6000	tl	**7210***	7210	19	7210	4200	tl	**7210***	7210	8
p38	2130*	2130	1390	tl	**2130***	2130	3385	**2130***	2130	567	**2130***	2130	2653	**2130***	2130	2734	**2130***	2130	43
p40	4984*	**4984***	4984	3005	4984	3909	tl	**4984***	4984	957	**4984***	4984	60	**4984***	4984	3401	**4984***	4984	18
p41	2688*	2688	1136	tl	2688	1136	tl	2688	1925	tl	**2688***	2688	29	2688	1136	tl	**2688***	2688	13
p42	2466*	2480	568	tl	2503	568	tl	2466	1317	tl	2480	2466	tl	2480	568	tl	**2466***	2466	2907
r01	6724*	6865	1402	tl	6847	1402	tl	6847	3289	tl	6865	6707	tl	6865	1402	tl	6727	6707	tl
r02	6771*	6771	1403	tl	6771	1403	tl	6771	3188	tl	**6771***	6771	141	6771	1403	tl	**6771***	6771	98
r03	6473*	6513	698	tl	6475	698	tl	6475	2029	tl	6475	6461	tl	6475	698	tl	6475	6461	tl
r04	6342*	6342	1396	tl	6342	1396	tl	6342	2518	tl	6342	6337	tl	6342	1396	tl	6342	6337	tl
r05	6408*	6435	1404	tl	6435	1404	tl	6435	2558	tl	6435	6401	tl	6435	1404	tl	6435	6401	tl
r06	7550*	7550	702	tl	7550	702	tl	7550	702	tl	**7550***	7550	43	7550	702	tl	**7550***	7550	16
r07	6889*	6889	932	tl	6889	701	tl	6889	2036	tl	**6889***	6889	202	6889	932	tl	**6889***	6889	151
r08	6057*	6084	703	tl	6084	703	tl	6084	1977	tl	6084	6052	tl	6084	703	tl	6081	6052	tl
r09	6358*	6407	701	tl	6407	701	tl	6407	2490	tl	6407	6352	tl	6407	701	tl	6367	6352	tl
r10	6508*	6540	2042	tl	6540	2042	tl	6540	3212	tl	6540	6497	tl	6540	2042	tl	6540	6497	tl
r11	7654*	7788	703	tl	7788	703	tl	7747	703	tl	7788	7640	tl	7788	703	tl	7747	7640	tl
r12	7690*	7711	704	tl	7711	704	tl	7690	704	tl	7711	7690	tl	7711	704	tl	**7690***	7690	1352
r13	7500*	7722	702	tl	7517	702	tl	7500	702	tl	7507	7499	tl	7507	702	tl	7500	7499	tl
r14	8254*	8259	1404	tl	8259	1404	tl	8259	1404	tl	8259	8254	tl	8259	1404	tl	8259	8254	tl
r15	8021*	8021	702	tl	8021	702	tl	8021	702	tl	**8021***	8021	83	8021	702	tl	**8021***	8021	18
r16	7755*	7764	703	tl	7764	703	tl	7764	703	tl	7764	7755	tl	7764	703	tl	7764	7755	tl
r17	7979*	7988	1397	tl	8001	1397	tl	7988	1397	tl	7988	7979	tl	7988	1397	tl	7988	7979	tl
r18	7232*	7279	1048	tl	7279	704	tl	7279	704	tl	7279	7222	tl	7279	1048	tl	7279	7222	tl
r19	6826*	6846	1397	tl	6846	1397	tl	6846	1397	tl	6846	6814	tl	6846	1397	tl	6846	6814	tl
r20	8023*	8027	704	tl	8027	704	tl	8027	704	tl	8027	8018	tl	8027	704	tl	8027	8018	tl
r21	9284*	9290	704	tl	9290	704	tl	9290	704	tl	9290	9284	tl	9290	704	tl	9287	9284	tl
r22	8887*	8981	702	tl	8932	702	tl	8981	702	tl	8981	8887	tl	8981	702	tl	8932	8887	tl
r23	9136*	9162	703	tl	9162	703	tl	9162	703	tl	9162	9136	tl	9162	703	tl	9154	9136	tl
r24	8464*	8464	703	tl	8464	703	tl	8464	703	tl	**8464***	8464	219	8464	703	tl	**8464***	8464	50
r25	8426*	8544	701	tl	8544	701	tl	8544	701	tl	8544	8423	tl	8544	701	tl	8523	8423	tl
r26	8819*	9005	1400	tl	8989	1400	tl	9005	1400	tl	9005	8819	tl	9005	1400	tl	8941	8819	tl
r27	7975*	7975	1398	tl	7975	1398	tl	7975	1398	tl	**7975***	7975	906	7975	1398	tl	**7975***	7975	250
r28	9407*	9407	1407	tl	9407	1407	tl	9407	1407	tl	**9407***	9407	283	9407	1407	tl	**9407***	9407	60
r29	8693*	8693	703	tl	8693	703	tl	8693	703	tl	**8693***	8693	1695	8693	703	tl	**8693***	8693	329
r30	9816*	9831	704	tl	9831	704	tl	9831	704	tl	9831	9816	tl	9831	704	tl	9831	9816	tl
nb bks		101/188			100/188			105/188			100/188			101/188			107/188		
nb optim		72/188			73/188			75/188			92/188			73/188			95/188		

Table 12: Impacts of all pre-computed lower and upper bounds on the objective value and the number of colors for the resolution with the primal model. (3/3)

		primal			primal + P11			dual			joint + J4		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
C2000.5	2144	-		tl	-		tl	-		tl	-		tl
C2000.9	5477	-		tl	-		tl	-		tl	-		tl
DSJC125.1gb	90	90	34	tl	90	62	tl	122	20	tl	92	41	tl
DSJC125.1g	23	**23***	23	862	**23***	23	627	26	5	tl	24	10	tl
DSJC125.5gb	240	270	59	tl	270	59	tl	273	92	tl	271	76	tl
DSJC125.5g	71	78	25	tl	78	25	tl	84	33	tl	78	25	tl
DSJC125.9gb	604*	635	100	tl	644	134	tl	**604***	604	149	**604***	604	1570
DSJC125.9g	169*	176	50	tl	176	50	tl	**169***	169	56	**169***	169	380
DSJC250.1	127	143	19	tl	141	19	tl	161	19	tl	150	29	tl
DSJC250.5	392	466	57	tl	466	57	tl	472	19	tl	469	57	tl
DSJC250.9	934*	1058	152	tl	1054	152	tl	1007	921	tl	-		tl
DSJC500.1	184	222	19	tl	-		tl	226	19	tl	-		tl
DSJC500.5	685	826	38	tl	-		tl	827	19	tl	-		tl
DSJC500.9	1662	1859	152	tl	-		tl	1914	19	tl	-		tl
DSJC1000.1	300	-		tl	-		tl	-		tl	-		tl
DSJC1000.5	1185	-		tl	-		tl	-		tl	-		tl
DSJC1000.9	2836	-		tl	-		tl	-		tl	-		tl
DSJR500.1	169	187	19	tl	173	19	tl	187	19	tl	186	19	tl
flat1000_50_0	924	-		tl	-		tl	-		tl	-		tl
flat1000_60_0	1162	-		tl	-		tl	-		tl	-		tl
flat1000_76_0	1165	-		tl	-		tl	-		tl	-		tl
GEOM20a	33*	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0
GEOM20b	8*	**8***	8	0	**8***	8	0	**8***	8	0	**8***	8	0
GEOM20	33*	**33***	33	0	**33***	33	0	**33***	33	0	**33***	33	0
GEOM30a	42*	**42***	42	0	**42***	42	0	**42***	42	1	**42***	42	1
GEOM30b	12*	**12***	12	1	**12***	12	0	**12***	12	1	**12***	12	3
GEOM30	32*	**32***	32	0	**32***	32	0	**32***	32	0	**32***	32	0
GEOM40a	49*	**49***	49	0	**49***	49	0	**49***	49	0	**49***	49	1
GEOM40b	16*	**16***	16	2	**16***	16	0	**16***	16	0	**16***	16	3
GEOM40	37*	**37***	37	1	**37***	37	0	**37***	37	0	**37***	37	1
GEOM50a	65*	**65***	65	4	**65***	65	1	**65***	65	0	**65***	65	9
GEOM50b	18*	**18***	18	8	**18***	18	3	**18***	18	2	**18***	18	14
GEOM50	40*	**40***	40	0	**40***	40	0	**40***	40	1	**40***	40	5
GEOM60a	73*	**73***	73	98	**73***	73	3	**73***	73	3206	**73***	73	26
GEOM60b	23*	**23***	23	54	**23***	23	7	**23***	23	8	**23***	23	80
GEOM60	43*	**43***	43	4	**43***	43	1	**43***	43	2	**43***	43	11
GEOM70a	73*	**73***	73	252	**73***	73	44	**73***	73	12	**73***	73	37
GEOM70b	24*	**24***	24	17	**24***	24	14	25	15	tl	**24***	24	42
GEOM70	47*	**47***	47	44	**47***	47	8	51	10	tl	**47***	47	66
GEOM80a	76*	76	48	tl	76	52	tl	81	30	tl	**76***	76	197
GEOM80b	27*	**27***	27	74	**27***	27	70	29	3	tl	**27***	27	192
GEOM80	66*	**66***	66	36	**66***	66	6	66	61	tl	**66***	66	24
GEOM90a	73*	73	10	tl	73	30	tl	75	64	tl	**73***	73	684
GEOM90b	30*	**30***	30	1378	**30***	30	732	**30***	30	16	**30***	30	122
GEOM90	61*	**61***	61	2485	**61***	61	88	66	10	tl	**61***	61	259
GEOM100a	89*	91	10	tl	89	29	tl	94	69	tl	**89***	89	742
GEOM100b	32*	**32***	32	781	**32***	32	136	**32***	32	72	**32***	32	229
GEOM100	65*	65	29	tl	**65***	65	73	67	39	tl	**65***	65	257
GEOM110a	97*	104	20	tl	104	39	tl	106	30	tl	97	78	tl
GEOM110b	37*	**37***	37	137	**37***	37	103	**37***	37	30	**37***	37	426
GEOM110	68*	69	30	tl	**68***	68	52	73	10	tl	**68***	68	741
GEOM120a	105*	113	20	tl	113	49	tl	119	10	tl	112	35	tl
GEOM120b	35*	37	15	tl	37	21	tl	37	3	tl	**35***	35	840
GEOM120	72*	72	20	tl	**72***	72	148	**72***	72	1454	**72***	72	1230
inithx.i.1	569*	569	57	tl	569	57	tl	569	19	tl	**569***	569	1922
inithx.i.2	329*	329	19	tl	329	19	tl	329	19	tl	329	19	tl
inithx.i.3	337*	337	76	tl	337	76	tl	339	19	tl	337	38	tl
latin_square_10	1480	-		tl	-		tl	-		tl	-		tl
le450_15a	212	245	19	tl	235	19	tl	250	19	tl	-		tl
le450_15b	216	249	19	tl	231	19	tl	251	19	tl	-		tl
le450_15c	275	336	38	tl	334	38	tl	338	19	tl	-		tl
le450_15d	272	329	19	tl	331	19	tl	332	19	tl	-		tl
le450_25a	306	323	38	tl	323	38	tl	317	19	tl	336	38	tl
le450_25b	307	307	19	tl	310	19	tl	314	19	tl	-		tl
le450_25c	342	398	38	tl	390	38	tl	397	19	tl	-		tl
le450_25d	330	388	57	tl	385	57	tl	393	19	tl	-		tl

Table 13: Results of the different constraint-programming models. (1/3)

		primal			primal + P11			dual			joint + J4		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
miles250	102*	**102***	102	16	**102***	102	8	**102***	102	3	**102***	102	31
miles500	260*	260	77	tl	260	119	tl	**260***	260	1	**260***	260	26
miles1000	431*	445	38	tl	456	38	tl	**431***	431	45	**431***	431	372
miles1500	797*	797	304	tl	797	304	tl	**797***	797	0	**797***	797	86
mulsol.i.5	367*	367	114	tl	367	156	tl	367	19	tl	**367***	367	202
myciel5gb	69*	**69***	69	12	**69***	69	6	**69***	69	3498	**69***	69	30
myciel5g	22*	**22***	22	19	**22***	22	6	23	5	tl	**22***	22	108
myciel6gb	94	**94***	94	998	**94***	94	908	97	20	tl	94	37	tl
myciel6g	26	**26***	26	1724	**26***	26	600	27	5	tl	26	11	tl
myciel7gb	109	109	20	tl	109	20	tl	112	20	tl	109	39	tl
myciel7g	29	31	5	tl	30	6	tl	32	5	tl	31	5	tl
queen8_8gb	132*	**132***	132	2447	**132***	132	3515	135	125	tl	**132***	132	1848
queen8_8g	36*	36	24	tl	36	24	tl	38	33	tl	**36***	36	1388
queen9_9gb	159	163	40	tl	163	95	tl	164	139	tl	163	87	tl
queen9_9g	41	42	10	tl	42	24	tl	44	38	tl	42	18	tl
queen10_10	162	170	19	tl	170	71	tl	177	19	tl	172	115	tl
queen10_10gb	164	177	40	tl	177	88	tl	182	79	tl	182	90	tl
queen10_10g	43	45	15	tl	45	24	tl	46	26	tl	46	15	tl
queen11_11	172	182	38	tl	186	38	tl	189	38	tl	187	80	tl
queen11_11gb	176	187	40	tl	187	40	tl	189	40	tl	187	40	tl
queen11_11g	47	50	10	tl	50	24	tl	53	20	tl	50	15	tl
queen12_12	185	203	19	tl	202	19	tl	211	19	tl	204	38	tl
queen12_12gb	191	208	20	tl	204	20	tl	224	20	tl	213	40	tl
queen12_12g	50	55	15	tl	55	15	tl	63	5	tl	55	15	tl
queen13_13	194	214	19	tl	213	19	tl	220	19	tl	215	19	tl
queen14_14	215	230	19	tl	226	19	tl	236	19	tl	232	19	tl
queen15_15	223	246	38	tl	246	38	tl	257	19	tl	247	38	tl
queen16_16	234	263	19	tl	262	19	tl	276	19	tl	264	19	tl
R50_1gb	53*	**53***	53	5	**53***	53	1	**53***	53	42	**53***	53	6
R50_1g	14*	**14***	14	4	**14***	14	0	**14***	14	4	**14***	14	5
R50_5gb	135*	**135***	135	384	**135***	135	367	**135***	135	101	**135***	135	394
R50_5g	37*	**37***	37	134	**37***	37	253	**37***	37	42	**37***	37	167
R50_9gb	262*	**262***	262	196	**262***	262	160	**262***	262	0	**262***	262	46
R50_9g	74*	**74***	74	104	**74***	74	100	**74***	74	0	**74***	74	31
R75_1gb	70*	**70***	70	56	**70***	70	31	83	20	tl	**70***	70	854
R75_1g	18*	**18***	18	13	**18***	18	6	21	5	tl	**18***	18	38
R75_5gb	186*	200	111	tl	198	118	tl	198	161	tl	194	140	tl
R75_5g	51*	54	27	tl	53	27	tl	54	44	tl	52	26	tl
R75_9gb	396*	400	144	tl	398	127	tl	**396***	396	1	**396***	396	123
R75_9g	110*	112	30	tl	111	30	tl	**110***	110	2	**110***	110	80
R100_1gb	81*	**81***	81	283	**81***	81	214	95	20	tl	82	47	tl
R100_1g	21*	**21***	21	261	**21***	21	182	23	5	tl	22	10	tl
R100_5gb	220	235	40	tl	233	40	tl	242	95	tl	235	106	tl
R100_5g	59	66	15	tl	66	15	tl	67	44	tl	66	29	tl
R100_9gb	518*	531	80	tl	529	138	tl	**518***	518	59	**518***	518	407
R100_9g	141*	146	50	tl	150	50	tl	**141***	141	14	**141***	141	332
wap01a	545	-		tl	-		tl	-		tl	-		tl
wap02a	538	-		tl	-		tl	-		tl	-		tl
wap03a	562	-		tl	-		tl	-		tl	-		tl
wap04a	563	-		tl	-		tl	-		tl	-		tl
wap05a	542	-		tl	-		tl	-		tl	-		tl
wap06a	516	-		tl	-		tl	-		tl	-		tl
wap07a	555	-		tl	-		tl	-		tl	-		tl
wap08a	529	-		tl	-		tl	-		tl	-		tl
zeroin.i.1	511*	511	57	tl	511	159	tl	511	503	tl	**511***	511	48
zeroin.i.2	336*	336	57	tl	346	123	tl	336	19	tl	-		tl
zeroin.i.3	298*	298	76	tl	298	161	tl	302	19	tl	**298***	298	190

Table 14: Results of the different constraint-programming models. (2/3)

instance	BKS	primal			primal + P11			dual			joint + J4		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
p06	565*	**565***	565	0	**565***	565	0	**565***	565	0	**565***	565	0
p07	3771*	**3771***	3771	0	**3771***	3771	0	**3771***	3771	0	**3771***	3771	2
p08	4049*	**4049***	4049	1	**4049***	4049	0	**4049***	4049	0	**4049***	4049	3
p09	3388*	**3388***	3388	1	**3388***	3388	0	**3388***	3388	2	**3388***	3388	13
p10	3983*	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0
p11	3380*	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0
p12	657*	**657***	657	1	**657***	657	0	**657***	657	0	**657***	657	5
p13	3220*	**3220***	3220	3	**3220***	3220	1	**3220***	3220	8	**3220***	3220	52
p14	3157*	**3157***	3157	1	**3157***	3157	1	**3157***	3157	89	**3157***	3157	4
p15	341*	**341***	341	10	**341***	341	3	**341***	341	2	**341***	341	19
p16	2343*	**2343***	2343	4	**2343***	2343	1	**2343***	2343	15	**2343***	2343	13
p17	3281*	**3281***	3281	10	**3281***	3281	3	**3281***	3281	8	**3281***	3281	90
p18	3228*	**3228***	3228	3	**3228***	3228	1	**3228***	3228	2	**3228***	3228	8
p19	3710*	**3710***	3710	3	**3710***	3710	2	**3710***	3710	0	**3710***	3710	5
p20	1830*	**1830***	1830	10	**1830***	1830	4	**1830***	1830	5	**1830***	1830	40
p21	3660*	**3660***	3660	9	**3660***	3660	4	**3660***	3660	1	**3660***	3660	42
p22	1912*	**1912***	1912	5	**1912***	1912	3	**1912***	1912	9	**1912***	1912	75
p23	3770*	**3770***	3770	40	**3770***	3770	14	**3770***	3770	17	**3770***	3770	162
p24	661*	**661***	661	2	**661***	661	0	**661***	661	0	**661***	661	4
p25	504*	**504***	504	2	**504***	504	5	**504***	504	36	**504***	504	4
p26	520*	**520***	520	5	**520***	520	2	**520***	520	0	**520***	520	9
p27	216*	**216***	216	8	**216***	216	4	**216***	216	8	**216***	216	36
p28	1729*	**1729***	1729	6	**1729***	1729	1	**1729***	1729	1	**1729***	1729	24
p29	3470*	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0
p30	4891*	**4891***	4891	12	**4891***	4891	5	**4891***	4891	11	**4891***	4891	26
p31	620*	**620***	620	0	**620***	620	0	**620***	620	0	**620***	620	0
p32	2480*	**2480***	2480	50	**2480***	2480	2	**2480***	2480	0	**2480***	2480	16
p33	3018*	**3018***	3018	28	**3018***	3018	6	3018	987	tl	**3018***	3018	61
p34	1980*	**1980***	1980	396	**1980***	1980	53	**1980***	1980	16	**1980***	1980	52
p35	2140*	**2140***	2140	1719	**2140***	2140	72	**2140***	2140	38	**2140***	2140	129
p36	7210*	7210	4200	tl	7210	6000	tl	**7210***	7210	20	**7210***	7210	143
p38	2130*	2130	1390	tl	**2130***	2130	86	2150	700	tl	**2130***	2130	390
p40	4984*	**4984***	4984	3005	**4984***	4984	136	4984	4890	tl	**4984***	4984	129
p41	2688*	2688	1136	tl	2688	1925	tl	**2688***	2688	417	**2688***	2688	783
p42	2466*	2480	568	tl	2480	1317	tl	2517	568	tl	**2466***	2466	672
r01	6724*	6865	1402	tl	6847	3289	tl	7138	703	tl	6724	6148	tl
r02	6771*	6771	1403	tl	6771	3188	tl	6786	704	tl	6771	1403	tl
r03	6473*	6513	698	tl	6475	2029	tl	6627	698	tl	**6473***	6473	788
r04	6342*	6342	1396	tl	6342	2039	tl	6388	702	tl	**6342***	6342	1423
r05	6408*	6435	1404	tl	6435	2558	tl	6495	703	tl	**6408***	6408	536
r06	7550*	7550	702	tl	7550	702	tl	7550	702	tl	7550	702	tl
r07	6889*	6889	932	tl	6889	2378	tl	7201	701	tl	**6889***	6889	445
r08	6057*	6084	703	tl	6081	1977	tl	6146	703	tl	**6057***	6057	777
r09	6358*	6407	701	tl	6407	2546	tl	6542	701	tl	**6358***	6358	1100
r10	6508*	6540	2042	tl	6540	3212	tl	6540	694	tl	**6508***	6508	821
r11	7654*	7788	703	tl	7747	703	tl	8074	703	tl	7803	703	tl
r12	7690*	7711	704	tl	7690	704	tl	7930	704	tl	7711	704	tl
r13	7500*	7722	702	tl	7501	702	tl	7749	702	tl	7716	702	tl
r14	8254*	8259	1404	tl	8259	1404	tl	8259	703	tl	8259	1404	tl
r15	8021*	8021	702	tl	8021	702	tl	8021	702	tl	8021	7738	tl
r16	7755*	7764	703	tl	7764	703	tl	7764	703	tl	7764	703	tl
r17	7979*	7988	1397	tl	7988	1397	tl	8214	700	tl	8008	1397	tl
r18	7232*	7279	1048	tl	7279	704	tl	7455	704	tl	7332	704	tl
r19	6826*	6846	1397	tl	6846	1397	tl	7296	703	tl	**6826***	6826	2741
r20	8023*	8027	704	tl	8027	704	tl	8055	704	tl	8055	704	tl
r21	9284*	9290	704	tl	9287	704	tl	9293	704	tl	9290	704	tl
r22	8887*	8981	702	tl	8981	702	tl	9014	702	tl	9036	702	tl
r23	9136*	9162	703	tl	9162	703	tl	9239	703	tl	9201	703	tl
r24	8464*	8464	703	tl	8464	703	tl	8484	703	tl	8464	703	tl
r25	8426*	8544	701	tl	8537	701	tl	8583	701	tl	8583	701	tl
r26	8819*	9005	1400	tl	9005	1400	tl	9013	704	tl	9005	1400	tl
r27	7975*	7975	1398	tl	7975	1398	tl	8003	701	tl	8003	1398	tl
r28	9407*	9407	1407	tl	9407	1407	tl	9407	704	tl	9407	1407	tl
r29	8693*	8693	703	tl	8693	703	tl	8973	703	tl	8985	703	tl
r30	9816*	9831	704	tl	9831	704	tl	9831	704	tl	9831	704	tl
nb bks		101/188			102/188			79/188			112/188		
nb optim		72/188			76/188			68/188			100/188		

Table 15: Results of the different constraint-programming models. (3/3)

instance	BKS	primal parallel			dual parallel			joint parallel		
		score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
C2000.5	2144	-		tl	-		tl	-		tl
C2000.9	5477	-		tl	-		tl	-		tl
DSJC125.1gb	90	**90***	90	20	93	70	tl	**90***	90	68
DSJC125.1g	23	**23***	23	11	23	19	tl	**23***	23	30
DSJC125.5gb	240	246	181	tl	264	179	tl	256	167	tl
DSJC125.5g	71	72	58	tl	80	54	tl	76	49	tl
DSJC125.9gb	604*	610	469	tl	**604***	604	5	**604***	604	260
DSJC125.9g	169*	172	136	tl	**169***	169	2	**169***	169	258
DSJC250.1	127	130	103	tl	147	71	tl	134	71	tl
DSJC250.5	392	428	188	tl	467	164	tl	446	164	tl
DSJC250.9	934*	1047	543	tl	934	926	tl	-		tl
DSJC500.1	184	191	104	tl	225	82	tl	-		tl
DSJC500.5	685	819	205	tl	824	205	tl	-		tl
DSJC500.9	1662	1858	719	tl	1882	1069	tl	-		tl
DSJC1000.1	300	-		tl	-		tl	-		tl
DSJC1000.5	1185	-		tl	-		tl	-		tl
DSJC1000.9	2836	-		tl	3267	846	tl	-		tl
DSJR500.1	169	**169***	169	66	172	166	tl	**169***	169	387
flat1000_50_0	924	-		tl	1352	215	tl	-		tl
flat1000_60_0	1162	-		tl	-		tl	-		tl
flat1000_76_0	1165	-		tl	-		tl	-		tl
GEOM20a	33*	**33***	33	0	**33***	33	0	**33***	33	0
GEOM20b	8*	**8***	8	0	**8***	8	0	**8***	8	0
GEOM20	33*	**33***	33	0	**33***	33	0	**33***	33	0
GEOM30a	42*	**42***	42	0	**42***	42	0	**42***	42	0
GEOM30b	12*	**12***	12	0	**12***	12	0	**12***	12	1
GEOM30	32*	**32***	32	0	**32***	32	0	**32***	32	0
GEOM40a	49*	**49***	49	0	**49***	49	0	**49***	49	0
GEOM40b	16*	**16***	16	0	**16***	16	0	**16***	16	1
GEOM40	37*	**37***	37	0	**37***	37	0	**37***	37	0
GEOM50a	65*	**65***	65	0	**65***	65	0	**65***	65	1
GEOM50b	18*	**18***	18	0	**18***	18	0	**18***	18	3
GEOM50	40*	**40***	40	0	**40***	40	0	**40***	40	0
GEOM60a	73*	**73***	73	1	**73***	73	0	**73***	73	7
GEOM60b	23*	**23***	23	1	**23***	23	0	**23***	23	7
GEOM60	43*	**43***	43	0	**43***	43	0	**43***	43	2
GEOM70a	73*	**73***	73	2	**73***	73	0	**73***	73	9
GEOM70b	24*	**24***	24	1	**24***	24	1	**24***	24	9
GEOM70	47*	**47***	47	1	**47***	47	1	**47***	47	9
GEOM80a	76*	**76***	76	7	**76***	76	4	**76***	76	54
GEOM80b	27*	**27***	27	3	**27***	27	2	**27***	27	13
GEOM80	66*	**66***	66	1	**66***	66	0	**66***	66	5
GEOM90a	73*	**73***	73	17	**73***	73	21	**73***	73	61
GEOM90b	30*	**30***	30	3	**30***	30	2	**30***	30	13
GEOM90	61*	**61***	61	2	**61***	61	6	**61***	61	16
GEOM100a	89*	**89***	89	24	**89***	89	24	**89***	89	146
GEOM100b	32*	**32***	32	6	**32***	32	5	**32***	32	23
GEOM100	65*	**65***	65	3	**65***	65	4	**65***	65	12
GEOM110a	97*	**97***	97	45	**97***	97	91	**97***	97	343
GEOM110b	37*	**37***	37	7	**37***	37	4	**37***	37	22
GEOM110	68*	**68***	68	5	**68***	68	23	**68***	68	18
GEOM120a	105*	**105***	105	88	**105***	105	79	**105***	105	397
GEOM120b	35*	**35***	35	12	**35***	35	52	**35***	35	39
GEOM120	72*	**72***	72	10	**72***	72	34	**72***	72	26
inithx.i.1	569*	**569***	569	48	**569***	569	19	**569***	569	176
inithx.i.2	329*	**329***	329	47	**329***	329	104	**329***	329	198
inithx.i.3	337*	337	336	tl	**337***	337	672	337	336	tl
latin_square_10	1480	1889	993	tl	1887	993	tl	-		tl
le450_15a	212	229	206	tl	246	206	tl	240	206	tl
le450_15b	216	225	213	tl	248	213	tl	242	213	tl
le450_15c	275	324	208	tl	338	208	tl	-		tl
le450_15d	272	315	199	tl	331	199	tl	-		tl
le450_25a	306	315	304	tl	316	304	tl	330	304	tl
le450_25b	307	310	307	tl	314	307	tl	312	307	tl
le450_25c	342	388	330	tl	396	330	tl	-		tl
le450_25d	330	384	309	tl	391	309	tl	-		tl

Table 16: Results of the different constraint-programming models launched on parallel with 10 threads. (1/3)

		primal parallel			dual parallel			joint parallel		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
miles250	102*	**102***	102	0	**102***	102	0	**102***	102	2
miles500	260*	**260***	260	1	**260***	260	0	**260***	260	10
miles1000	431*	431	412	tl	**431***	431	1	**431***	431	201
miles1500	797*	**797***	797	16	**797***	797	0	**797***	797	68
mulsol.i.5	367*	**367***	367	16	**367***	367	3	**367***	367	66
myciel5gb	69*	**69***	69	1	**69***	69	16	**69***	69	7
myciel5g	22*	**22***	22	0	**22***	22	27	**22***	22	6
myciel6gb	94	**94***	94	17	94	48	tl	**94***	94	87
myciel6g	26	**26***	26	13	26	15	tl	**26***	26	73
myciel7gb	109	**109***	109	543	109	59	tl	**109***	109	2928
myciel7g	29	**29***	29	241	29	15	tl	**29***	29	1544
queen8_8gb	132*	**132***	132	15	**132***	132	451	**132***	132	127
queen8_8g	36*	**36***	36	26	**36***	36	674	**36***	36	207
queen9_9gb	159	159	153	tl	162	152	tl	162	147	tl
queen9_9g	41	**41***	41	508	41	40	tl	41	38	tl
queen10_10	162	162	156	tl	167	153	tl	164	153	tl
queen10_10gb	164	165	157	tl	170	153	tl	166	150	tl
queen10_10g	43	**43***	43	820	45	40	tl	43	41	tl
queen11_11	172	177	163	tl	180	157	tl	179	157	tl
queen11_11gb	176	177	163	tl	187	151	tl	177	152	tl
queen11_11g	47	48	44	tl	50	43	tl	48	43	tl
queen12_12	185	188	175	tl	197	173	tl	193	173	tl
queen12_12gb	191	203	177	tl	208	177	tl	198	177	tl
queen12_12g	50	51	46	tl	55	45	tl	53	45	tl
queen13_13	194	205	185	tl	207	179	tl	205	179	tl
queen14_14	215	224	206	tl	231	206	tl	232	206	tl
queen15_15	223	237	209	tl	252	209	tl	243	209	tl
queen16_16	234	262	218	tl	270	218	tl	252	218	tl
R50_1gb	53*	**53***	53	0	**53***	53	1	**53***	53	2
R50_1g	14*	**14***	14	0	**14***	14	0	**14***	14	1
R50_5gb	135*	**135***	135	5	**135***	135	12	**135***	135	21
R50_5g	37*	**37***	37	2	**37***	37	12	**37***	37	16
R50_9gb	262*	**262***	262	3	**262***	262	0	**262***	262	20
R50_9g	74*	**74***	74	2	**74***	74	0	**74***	74	18
R75_1gb	70*	**70***	70	2	**70***	70	253	**70***	70	9
R75_1g	18*	**18***	18	1	**18***	18	75	**18***	18	15
R75_5gb	186*	186	163	tl	186	170	tl	190	166	tl
R75_5g	51*	**51***	51	2140	51	46	tl	52	44	tl
R75_9gb	396*	396	360	tl	**396***	396	0	**396***	396	42
R75_9g	110*	110	94	tl	**110***	110	0	**110***	110	39
R100_1gb	81*	**81***	81	8	81	67	tl	**81***	81	23
R100_1g	21*	**21***	21	6	22	16	tl	**21***	21	21
R100_5gb	220	227	191	tl	236	191	tl	231	160	tl
R100_5g	59	60	50	tl	61	49	tl	60	47	tl
R100_9gb	518*	520	431	tl	**518***	518	2	**518***	518	229
R100_9g	141*	142	123	tl	**141***	141	1	**141***	141	91
wap01a	545	-		tl	-		tl	-		tl
wap02a	538	-		tl	-		tl	-		tl
wap03a	562	-		tl	-		tl	-		tl
wap04a	563	-		tl	-		tl	-		tl
wap05a	542	598	540	tl	599	540	tl	-		tl
wap06a	516	584	503	tl	584	503	tl	-		tl
wap07a	555	-		tl	-		tl	-		tl
wap08a	529	-		tl	-		tl	-		tl
zeroin.i.1	511*	**511***	511	12	**511***	511	0	**511***	511	37
zeroin.i.2	336*	**336***	336	25	**336***	336	2	**336***	336	36
zeroin.i.3	298*	**298***	298	62	**298***	298	1	**298***	298	59

Table 17: Results of the different constraint-programming models launched on parallel with 10 threads. (2/3)

		primal parallel			dual parallel			joint parallel		
instance	BKS	score	LB	time(s)	score	LB	time(s)	score	LB	time(s)
p06	565*	**565***	565	0	**565***	565	0	**565***	565	0
p07	3771*	**3771***	3771	0	**3771***	3771	0	**3771***	3771	1
p08	4049*	**4049***	4049	0	**4049***	4049	0	**4049***	4049	0
p09	3388*	**3388***	3388	0	**3388***	3388	0	**3388***	3388	1
p10	3983*	**3983***	3983	0	**3983***	3983	0	**3983***	3983	0
p11	3380*	**3380***	3380	0	**3380***	3380	0	**3380***	3380	0
p12	657*	**657***	657	0	**657***	657	0	**657***	657	1
p13	3220*	**3220***	3220	0	**3220***	3220	0	**3220***	3220	2
p14	3157*	**3157***	3157	0	**3157***	3157	0	**3157***	3157	1
p15	341*	**341***	341	0	**341***	341	0	**341***	341	2
p16	2343*	**2343***	2343	0	**2343***	2343	0	**2343***	2343	2
p17	3281*	**3281***	3281	0	**3281***	3281	0	**3281***	3281	3
p18	3228*	**3228***	3228	0	**3228***	3228	0	**3228***	3228	2
p19	3710*	**3710***	3710	0	**3710***	3710	0	**3710***	3710	1
p20	1830*	**1830***	1830	0	**1830***	1830	0	**1830***	1830	2
p21	3660*	**3660***	3660	0	**3660***	3660	0	**3660***	3660	2
p22	1912*	**1912***	1912	0	**1912***	1912	0	**1912***	1912	3
p23	3770*	**3770***	3770	1	**3770***	3770	0	**3770***	3770	5
p24	661*	**661***	661	0	**661***	661	0	**661***	661	1
p25	504*	**504***	504	0	**504***	504	0	**504***	504	1
p26	520*	**520***	520	0	**520***	520	0	**520***	520	1
p27	216*	**216***	216	0	**216***	216	0	**216***	216	3
p28	1729*	**1729***	1729	0	**1729***	1729	0	**1729***	1729	2
p29	3470*	**3470***	3470	0	**3470***	3470	0	**3470***	3470	0
p30	4891*	**4891***	4891	2	**4891***	4891	0	**4891***	4891	9
p31	620*	**620***	620	0	**620***	620	0	**620***	620	0
p32	2480*	**2480***	2480	0	**2480***	2480	0	**2480***	2480	2
p33	3018*	**3018***	3018	1	**3018***	3018	1	**3018***	3018	7
p34	1980*	**1980***	1980	3	**1980***	1980	6	**1980***	1980	13
p35	2140*	**2140***	2140	5	**2140***	2140	8	**2140***	2140	15
p36	7210*	**7210***	7210	9	**7210***	7210	6	**7210***	7210	22
p38	2130*	**2130***	2130	4	**2130***	2130	7	**2130***	2130	16
p40	4984*	**4984***	4984	5	**4984***	4984	5	**4984***	4984	18
p41	2688*	**2688***	2688	15	**2688***	2688	15	**2688***	2688	58
p42	2466*	**2466***	2466	20	**2466***	2466	206	**2466***	2466	31
r01	6724*	**6724***	6724	47	**6724***	6724	1773	**6724***	6724	242
r02	6771*	**6771***	6771	20	**6771***	6771	274	**6771***	6771	100
r03	6473*	**6473***	6473	40	**6473***	6473	411	**6473***	6473	128
r04	6342*	**6342***	6342	48	**6342***	6342	275	**6342***	6342	304
r05	6408*	**6408***	6408	23	**6408***	6408	472	**6408***	6408	86
r06	7550*	**7550***	7550	23	**7550***	7550	33	**7550***	7550	82
r07	6889*	**6889***	6889	24	**6889***	6889	125	**6889***	6889	66
r08	6057*	**6057***	6057	34	**6057***	6057	757	**6057***	6057	81
r09	6358*	**6358***	6358	24	**6358***	6358	328	**6358***	6358	52
r10	6508*	**6508***	6508	54	**6508***	6508	567	**6508***	6508	737
r11	7654*	**7654***	7654	228	7654	7640	tl	**7654***	7654	3000
r12	7690*	**7690***	7690	85	7699	7690	tl	**7690***	7690	1436
r13	7500*	**7500***	7500	196	7504	7499	tl	**7500***	7500	1153
r14	8254*	**8254***	8254	351	**8254***	8254	595	**8254***	8254	335
r15	8021*	**8021***	8021	31	**8021***	8021	74	**8021***	8021	121
r16	7755*	**7755***	7755	81	**7755***	7755	1069	**7755***	7755	380
r17	7979*	**7979***	7979	322	7987	7979	tl	7994	7979	tl
r18	7232*	**7232***	7232	212	7260	7222	tl	**7232***	7232	2801
r19	6826*	**6826***	6826	78	6915	6814	tl	**6826***	6826	689
r20	8023*	**8023***	8023	389	8027	8018	tl	**8023***	8023	1978
r21	9284*	**9284***	9284	743	9290	9284	tl	**9284***	9284	2803
r22	8887*	**8887***	8887	1939	8928	8887	tl	8950	8887	tl
r23	9136*	9141	9136	tl	9183	9136	tl	9158	9136	tl
r24	8464*	**8464***	8464	53	8484	8464	tl	**8464***	8464	238
r25	8426*	**8426***	8426	2165	8501	8423	tl	8426	8423	tl
r26	8819*	**8819***	8819	2362	8943	8819	tl	8938	8819	tl
r27	7975*	**7975***	7975	58	7981	7975	tl	**7975***	7975	246
r28	9407*	**9407***	9407	55	**9407***	9407	159	**9407***	9407	176
r29	8693*	**8693***	8693	2009	8715	8693	tl	8708	8693	tl
r30	9816*	**9816***	9816	2673	9829	9816	tl	9831	9816	tl
nb bks		137/188			122/188			132/188		
nb optim		130/188			111/188			128/188		

Table 18: Results of the different constraint-programming models launched on parallel with 10 threads. (3/3)