1 HW 3-1

Problem 1: By definition of continuity, for every sequence x_n which converges to $c \in A$, $g(x_n) = g(c)$. Let's define the sequence $g_n = g(x_n)$. If $g(c) > 0$, then $|g(x_n) - g(c)| < \epsilon$, for $n \geq k(\epsilon)$, $\forall \epsilon > 0$. Expanding the equation we find $g(c) - \epsilon < g(x_n) < g(c) + \epsilon$. Choosing $\epsilon < g(c)$, we have $g(x_n) > 0$, for $n \geq k(\epsilon)$. Thus, a tail of g_n is strictly positive and converges to $g(c)$, and since $h(c) = g(c)$ for $g > 0$, h must also converge to $g(c) = h(c)$, hence h is continuous at c. A similar argument shows if $g(c) \leq 0$, then there is a strictly negative tail which forces h to converge to 0.

Final case, $g(c) = 0$. Consider the set of terms in $h(g_n)$ such that $g_n > 0$ or $g_n \leq 0$. If either set has a finite size, then there exists a strictly positive or negative tail of g_n, and hence by the above arguments, g is continuous at c. If both sets are infinite, then there exists sub-sequences such that $g_n > 0$ and $g_n \leq 0$. For any $\epsilon > 0$, we can choose $k(\epsilon)$ to be the $k(\epsilon)$ associated with g_n. Then we write $|h(g_{n_1}) - 0| = |g_{n_1} - 0| < \epsilon$ and $|h(g_{n_2}) - 0| = |0 - 0| = 0 < \epsilon$, $\forall n_1, n_2 \geq k(\epsilon)$. Since each term in g_n is either positive or non-positive, $|h(g_n) - 0| < \epsilon$, and thus, $h(g_n)$ is convergent and equal to $h(g(0))$. Thus, in all cases g is continuous at c.

Problem 2: To be uniformly continuous, a function must satisfy, for any $\epsilon > 0$, $|f(x) - f(u)| < \epsilon$, for all $x, u \in A$ satisfying $|x - u| < \delta(\epsilon)$. For $f = \frac{1}{x}$, we have $|f(x) - f(u)| = |\frac{u-x}{ux}| = |x - u||\frac{1}{ux}| < \epsilon$. When the domain is $(1, \infty)$, $\frac{1}{ux} < 1$, thus if we choose $|x - u| < \delta = \epsilon$, then $|x - u||\frac{1}{ux}| < |x - u| = \epsilon$, which proves uniform continuity. If the domain is $(0, \infty)$, then suppose we've found some valid value for δ. Then $\frac{\delta}{ux} < \epsilon$, and $ux > \frac{\delta}{\epsilon}$. WLOG we choose $x = u$, and setting $u < \sqrt{\frac{\delta}{\epsilon}}$, we find $u^2 < \frac{\delta}{\epsilon}$, a contradiction. Thus, over the domain $(0, \infty)$ the function is not uniformly continuous.

Problem 3: To be uniformly continuous, a function must satisfy, for any $\epsilon > 0$, $|f(x) - f(u)| < \epsilon$, for all $x, u \in A$ satisfying $|x - u| < \delta(\epsilon)$. Suppose we've found some δ that satisfies these conditions for $\epsilon < 1$. Then, using the Archimiddean property, we can find $0 < \frac{1}{n_1} < |x - u| < \delta$, for $n_1 \in \mathbf{N}$. Furthermore, $0 < \frac{1}{n_1+1} < \frac{1}{n_1} < \delta$. Since n_1 and $n_1 + 1$ have differing parity and their reciprocal are contained in our interval, $|f(\frac{1}{n_1}) - f(\frac{1}{n_1+1})| = 1$, which violates our conditions, and thus our function cannot be uniformly continuous.

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Problem 4: A point c is a cluster point of a set $A \subset \mathbf{R}$ if for all $\delta > 0$, $\exists x \in A$, S.T $|x - c| < \delta$. By the definition of a limit, we know for all $\epsilon > 0$, $|a_n - c| < \epsilon$ for $n \geq k(\epsilon)$. Choosing $\epsilon = \delta$, we see for any δ, $\exists a_n$, S.T $|a_n - c| < \delta$. Since $a_n \in A$, this completes the proof.

Problem 5: Consider the set $S = \frac{1}{n} : n \in \mathbf{N}$. This set has a cluster point at 0. To prove this, we need to show for δ, $\exists x$, S.T $|x - 0| < \delta$. By the Archimiddean principle, $\exists n$, S.T $\frac{1}{n} < \delta$, where δ is any number. Thus, there will always exist an x in our set which meets this criterion and so 0 is a cluster point. Conversely, there cannot be any other cluster points. To show this there are four cases. Let our candidate cluster point be a.

$a > 1$: In this case, we choose $\delta = a - 1$. Then $|x - a| < a - 1$. Thus, $1 < x < 2a - 1$, which is not possible as $\sup(S) = 1$.

$a < 0$: We choose $\delta = -a$. Then $|x - a| < \delta - a$. Thus, $2a < x < 0$, which is not possible as $\inf(S) = 0$.

$a \in S$: Let $a = \frac{1}{n}$. Then we'll choose δ, S.T $|x - a| < \delta = \frac{1}{n} - \frac{1}{n+1}$. Thus, $\frac{1}{n+1} < \frac{1}{n} < x < \frac{2}{n} - \frac{1}{n-1} < \frac{1}{n-1}$. This interval doesn't contain any elements in the set, as all other elements in the set are larger or smaller then $\frac{1}{n+1}$ and $\frac{1}{n-1}$. Hence, a is not a cluster point.

$a \in (0, 1] \notin S$: a must exist in some interval $[\frac{1}{n}, \frac{1}{n+1}]$, as $\inf(S) < a \leq \sup(S)$, and since the set is ordered there there exists a minimum element in the set greater then a, the element after this cannot be equal to a, and so it must be less then a. Let's call those elements $\frac{1}{n}$ and $\frac{1}{n+1}$. Then, by the same reasoning in the above case, a is not a cluster point.

Hence, the subset of $\frac{1}{n} : n \in \mathbf{N}$ has precisely one cluster point at 0. Now consider the set $k + \frac{1}{n} : n \in \mathbf{N}, k \in \mathbf{Z}$. For any value x in our original set, we can find $k+x$ in our new set. Thus, for the value k, $\exists k+x$, S.T $|k+x-k| = |x| < \delta$. We've already proved this as true, and so for any value k in our new set we have a cluster point. Since our set contains all $k \in \mathbf{Z}$, it must contain cluster points at all $k \in \mathbf{Z}$.

Problem 6: If a point r is rational the statement obviously holds, we just need to prove the case where r is irrational. We know there is a rational sequence of numbers which converges to any real number. Calling this sequence x_n, we have $\forall \epsilon > 0$, $|r - x_n| < \epsilon$, when $n \geq k(\epsilon)$. By the definition of a cluster point, we know there exists a point y S.T, $|y - x_n| < \sigma$. Choosing $\sigma < \epsilon$, we can unravel our inequality to obtain $x_n - \epsilon < y < x_n + \epsilon$. This point is within the interval $|r - x_n| < \epsilon$ as expanding it yields $x_n - \epsilon < r < x_n + \epsilon$. Thus, we've show there always exists a y S.T $|r - y| < \epsilon$, and that completes the proof.

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Problem 7: For any real number, we can construct a sequence of entirely rational or entirely irrational numbers which converges to it. Let's call these sequences R_n, and I_n. Then by the definition of D $\lim D(R_n) = 1$, and $\lim D(I_n) = 0$. Hence, the limit does not exist.

Problem 8: Applying the continuity criterion, for all sequences x_n which converge to an element $a \in A$, $\lim f(x_n) = f(a)$. Or, $\forall \epsilon > 0$, $\exists \delta$, S.T $|x_n - a| < \delta$, $|f(x_n) - f(a)| < \epsilon$. In a finite set, there is a minimum distance between distinct elements, let's call this D. If we choose $\delta < D$, then x_n can only be a. Thus, $|f(x_n) - f(a)| = |f(a) - f(a)| = 0 < \epsilon$. Thus, f is continuous. Furthermore, our value for δ does not depend on a, and so we also satisfy the definition for uniform convergence.

Problem 9: Let's divide the numerator and denominator by x^3. This doesn't change the value of the function since x is non-zero. The numerator is now $2 - \frac{3}{x} + \frac{1}{x^3}$. The absolute value of this, is bounded by the triangle inequality: $|2 - \frac{3}{x} + \frac{1}{x^3}| \leq |2| + | - \frac{3}{x} + \frac{1}{x^3} \leq |2| + |\frac{3}{x}| + |\frac{1}{x^3}|$. In general, $\frac{C}{x^n} \leq \frac{C}{x}$ for $x \geq 1$ which by squeeze theorem means such functions must have a limit of 0. Applying limit laws yields $|2 + 0 + 0| = 2$. We can do a similar procedure with the denominator. $|4 + \frac{1}{x^2} - \frac{6}{x^3}| \leq |4| + |\frac{1}{x^2}| + |\frac{6}{x^3}|$, applying limit laws yields 4. Thus, the limit of the numerator is 2, and the limit of the denominator is 4. Since the limits both exist and are non-zero, we can apply the division limit law, yielding $\frac{2}{4} = \frac{1}{2}$.