

LIMITS AND DERIVATIVES

13.1 Overview

13.1.1 Limits of a function

Let f be a function defined in a domain which we take to be an interval, say, I . We shall study the concept of limit of f at a point ' a ' in I .

We say $\lim_{x \rightarrow a^-} f(x)$ is the expected value of f at $x = a$ given the values of f near to the left of a . This value is called the *left hand limit* of f at a .

We say $\lim_{x \rightarrow a^+} f(x)$ is the expected value of f at $x = a$ given the values of f near to the right of a . This value is called the *right hand limit* of f at a .

If the right and left hand limits coincide, we call the common value as the limit of f at $x = a$ and denote it by $\lim_{x \rightarrow a} f(x)$.

Some properties of limits

Let f and g be two functions such that both $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then

- (i) $\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
- (ii) $\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$
- (iii) For every real number α

$$\lim_{x \rightarrow a} (\alpha f)(x) = \alpha \lim_{x \rightarrow a} f(x)$$

- (iv) $\lim_{x \rightarrow a} [f(x) g(x)] = [\lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)]$

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \text{ provided } g(x) \neq 0$$

Limits of polynomials and rational functions

If f is a polynomial function, then $\lim_{x \rightarrow a} f(x)$ exists and is given by

$$\lim_{x \rightarrow a} f(x) = f(a)$$

An Important limit

An important limit which is very useful and used in the sequel is given below:

$$\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Remark The above expression remains valid for any rational number provided ‘ a ’ is positive.

Limits of trigonometric functions

To evaluate the limits of trigonometric functions, we shall make use of the following limits which are given below:

$$(i) \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \quad (ii) \lim_{x \rightarrow 0} \cos x = 1 \quad (iii) \lim_{x \rightarrow 0} \sin x = 0$$

13.1.2 Derivatives Suppose f is a real valued function, then

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad \dots (1)$$

is called the **derivative** of f at x , provided the limit on the R.H.S. of (1) exists.

Algebra of derivative of functions Since the very definition of derivatives involve limits in a rather direct fashion, we expect the rules of derivatives to follow closely that of limits as given below:

Let f and g be two functions such that their derivatives are defined in a common domain. Then :

- (i) Derivative of the sum of two function is the sum of the derivatives of the functions.

$$\frac{d}{dx} [f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

- (ii) Derivative of the difference of two functions is the difference of the derivatives of the functions.

$$\frac{d}{dx} [f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)$$

- (iii) Derivative of the product of two functions is given by the following *product rule*.

$$\frac{d}{dx} [f(x) \cdot g(x)] = \left(\frac{d}{dx} f(x) \right) \cdot g(x) + f(x) \cdot \left(\frac{d}{dx} g(x) \right)$$

This is referred to as Leibnitz Rule for the product of two functions.

- (iv) Derivative of quotient of two functions is given by the following *quotient rule* (wherever the denominator is non-zero).

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{\left(\frac{d}{dx} f(x) \right) \cdot g(x) - f(x) \cdot \left(\frac{d}{dx} g(x) \right)}{(g(x))^2}$$

13.2 Solved Examples

Short Answer Type

Example 1 Evaluate $\lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x} \right]$

Solution We have

$$\begin{aligned} \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x^3-3x^2+2x} \right] &= \lim_{x \rightarrow 2} \left[\frac{1}{x-2} - \frac{2(2x-3)}{x(x-1)(x-2)} \right] \\ &= \lim_{x \rightarrow 2} \left[\frac{x(x-1) - 2(2x-3)}{x(x-1)(x-2)} \right] \\ &= \lim_{x \rightarrow 2} \left[\frac{x^2 - 5x + 6}{x(x-1)(x-2)} \right] \\ &= \lim_{x \rightarrow 2} \left[\frac{(x-2)(x-3)}{x(x-1)(x-2)} \right] \quad [x-2 \neq 0] \\ &= \lim_{x \rightarrow 2} \left[\frac{x-3}{x(x-1)} \right] = \frac{-1}{2} \end{aligned}$$

Example 2 Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x}$

Solution Put $y = 2 + x$ so that when $x \rightarrow 0$, $y \rightarrow 2$. Then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2}}{x} &= \lim_{y \rightarrow 2} \frac{y^{\frac{1}{2}} - 2^{\frac{1}{2}}}{y - 2} \\ &= \frac{1}{2} (2)^{\frac{1}{2}-1} = \frac{1}{2} \cdot 2^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}} \end{aligned}$$

Example 3 Find the positive integer n so that $\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = 108$.

Solution We have

$$\lim_{x \rightarrow 3} \frac{x^n - 3^n}{x - 3} = n(3)^{n-1}$$

Therefore,

$$n(3)^{n-1} = 108 = 4(27) = 4(3)^{4-1}$$

Comparing, we get

$$n = 4$$

Example 4 Evaluate $\lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x)$

Solution Put $y = \frac{\pi}{2} - x$. Then $y \rightarrow 0$ as $x \rightarrow \frac{\pi}{2}$. Therefore

$$\begin{aligned} \lim_{x \rightarrow \frac{\pi}{2}} (\sec x - \tan x) &= \lim_{y \rightarrow 0} [\sec(\frac{\pi}{2} - y) - \tan(\frac{\pi}{2} - y)] \\ &= \lim_{y \rightarrow 0} (\operatorname{cosec} y - \cot y) \\ &= \lim_{y \rightarrow 0} \left(\frac{1}{\sin y} - \frac{\cos y}{\sin y} \right) \\ &= \lim_{y \rightarrow 0} \left(\frac{1 - \cos y}{\sin y} \right) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{2 \sin \frac{y}{2} \cos \frac{y}{2}} \quad \left(\begin{array}{l} \text{since, } \sin^2 \frac{y}{2} = \frac{1 - \cos y}{2} \\ \sin y = 2 \sin \frac{y}{2} \cos \frac{y}{2} \end{array} \right) \\
 &= \lim_{\frac{y}{2} \rightarrow 0} \tan \frac{y}{2} = 0
 \end{aligned}$$

Example 5 Evaluate $\lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x}$

Solution (i) We have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin(2+x) - \sin(2-x)}{x} &= \lim_{x \rightarrow 0} \frac{2 \cos \frac{(2+x+2-x)}{2} \sin \frac{(2+x-2+x)}{2}}{x} \\
 &= \lim_{x \rightarrow 0} \frac{2 \cos 2 \sin x}{x} \\
 &= 2 \cos 2 \lim_{x \rightarrow 0} \frac{\sin x}{x} = 2 \cos 2 \left(\text{as } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right)
 \end{aligned}$$

Example 6 Find the derivative of $f(x) = ax + b$, where a and b are non-zero constants, by first principle.

Solution By definition,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{a(x+h) + b - (ax+b)}{h} = \lim_{h \rightarrow 0} \frac{bh}{h} = b
 \end{aligned}$$

Example 7 Find the derivative of $f(x) = ax^2 + bx + c$, where a , b and c are non-zero constant, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{a(x+h)^2 + b(x+h) + c - ax^2 - bx - c}{h} \\
&= \lim_{h \rightarrow 0} \frac{bh + ah^2 + 2axh}{h} = \lim_{h \rightarrow 0} ah + 2ax + b = b + 2ax
\end{aligned}$$

Example 8 Find the derivative of $f(x) = x^3$, by first principle.

Solution By definition,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3xh(x+h) - x^3}{h} \\
&= \lim_{h \rightarrow 0} (h^2 + 3x(x+h)) = 3x^2
\end{aligned}$$

Example 9 Find the derivative of $f(x) = \frac{1}{x}$ by first principle.

Solution By definition,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) \\
&= \lim_{h \rightarrow 0} \frac{-h}{h(x+h)x} = \frac{-1}{x^2}.
\end{aligned}$$

Example 10 Find the derivative of $f(x) = \sin x$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2}}{2 \cdot \frac{h}{2}} \\
&= \lim_{h \rightarrow 0} \cos \frac{(2x+h)}{2} \cdot \lim_{h \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \\
&= \cos x \cdot 1 = \cos x
\end{aligned}$$

Example 11 Find the derivative of $f(x) = x^n$, where n is positive integer, by first principle.

Solution By definition,

$$\begin{aligned}
f'(x) &= \frac{f(x+h) - f(x)}{h} \\
&= \frac{(x+h)^n - x^n}{h}
\end{aligned}$$

Using Binomial theorem, we have $(x+h)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} h + \dots + {}^nC_n h^n$

$$\begin{aligned}
\text{Thus, } f'(x) &= \lim_{h \rightarrow 0} \frac{(x+h)^n - x^n}{h} \\
&= \lim_{h \rightarrow 0} \frac{h(nx^{n-1} + \dots + h^{n-1})}{h} = nx^{n-1}.
\end{aligned}$$

Example 12 Find the derivative of $2x^4 + x$.

Solution Let $y = 2x^4 + x$

Differentiating both sides with respect to x , we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(2x^4) + \frac{d}{dx}(x) \\
&= 2 \times 4x^{4-1} + 1x^0
\end{aligned}$$

$$= 8x^3 + 1$$

Therefore, $\frac{d}{dx}(2x^4 + x) = 8x^3 + 1$.

Example 13 Find the derivative of $x^2 \cos x$.

Solution Let $y = x^2 \cos x$

Differentiating both sides with respect to x , we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(x^2 \cos x) \\ &= x^2 \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2) \\ &= x^2 (-\sin x) + \cos x (2x) \\ &= 2x \cos x - x^2 \sin x\end{aligned}$$

Long Answer Type

Example 14 Evaluate $\lim_{x \rightarrow \frac{\pi}{6}} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1}$

Solution Note that

$$\begin{aligned}2 \sin^2 x + \sin x - 1 &= (2 \sin x - 1)(\sin x + 1) \\ 2 \sin^2 x - 3 \sin x + 1 &= (2 \sin x - 1)(\sin x - 1)\end{aligned}$$

$$\begin{aligned}\text{Therefore, } \lim_{x \rightarrow \frac{\pi}{6}} \frac{2 \sin^2 x + \sin x - 1}{2 \sin^2 x - 3 \sin x + 1} &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{(2 \sin x - 1)(\sin x + 1)}{(2 \sin x - 1)(\sin x - 1)} \\ &= \lim_{x \rightarrow \frac{\pi}{6}} \frac{\sin x + 1}{\sin x - 1} \quad (\text{as } 2 \sin x - 1 \neq 0) \\ &= \frac{1 + \sin \frac{\pi}{6}}{\sin \frac{\pi}{6} - 1} = -3\end{aligned}$$

Example 15 Evaluate $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x}$

Solution We have

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan x - \sin x}{\sin^3 x} &= \lim_{x \rightarrow 0} \frac{\sin x \left(\frac{1}{\cos x} - 1 \right)}{\sin^3 x} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{\cos x \sin^2 x} \\ &= \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{\cos x \left(4 \sin^2 \frac{x}{2} \cdot \cos^2 \frac{x}{2} \right)} = \frac{1}{2}. \end{aligned}$$

Example 16 Evaluate $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$

Solution We have $\lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}}$

$$\begin{aligned} &= \lim_{x \rightarrow a} \frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \times \frac{\sqrt{a+2x} + \sqrt{3x}}{\sqrt{a+2x} + \sqrt{3x}} \\ &= \lim_{x \rightarrow a} \frac{a+2x-3x}{(\sqrt{3a+x}-2\sqrt{x})(\sqrt{a+2x}+\sqrt{3x})} \\ &= \lim_{x \rightarrow a} \frac{(a-x)(\sqrt{3a+x}+2\sqrt{x})}{(\sqrt{a+2x}+\sqrt{3x})(\sqrt{3a+x}-2\sqrt{x})(\sqrt{3a+x}+2\sqrt{x})} \\ &= \lim_{x \rightarrow a} \frac{(a-x) \sqrt{3a+x}+2\sqrt{x}}{(\sqrt{a+2x}+\sqrt{3x})(3a+x-4x)} \end{aligned}$$

$$= \frac{4\sqrt{a}}{3 \times 2\sqrt{3a}} = \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9}.$$

Example 17 Evaluate $\lim_{x \rightarrow 0} \frac{\cos ax - \cos bx}{\cos cx - 1}$

Solution We have $\lim_{x \rightarrow 0} \frac{2 \sin \left(\frac{(a+b)x}{2} \right) \sin \frac{(a-b)x}{2}}{2 \frac{\sin^2 cx}{2}}$

$$= \lim_{x \rightarrow 0} \frac{2 \sin \frac{(a+b)x}{2} \cdot \sin \frac{(a-b)x}{2}}{x^2} \cdot \frac{x^2}{\sin^2 \frac{cx}{2}}$$

$$= \lim_{x \rightarrow 0} \frac{\sin \frac{(a+b)x}{2}}{\frac{(a+b)x}{2} \cdot \left(\frac{2}{a+b} \right)} \cdot \frac{\sin \frac{(a-b)x}{2}}{\frac{(a-b)x}{2} \cdot \frac{2}{a-b}} \cdot \frac{\left(\frac{cx}{2} \right)^2 \times \frac{4}{c^2}}{\sin^2 \frac{cx}{2}}$$

$$= \left(\frac{a+b}{2} \times \frac{a-b}{2} \times \frac{4}{c^2} \right) = \frac{a^2 - b^2}{c^2}$$

Example 18 Evaluate $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

Solution We have $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

$$= \lim_{h \rightarrow 0} \frac{(a^2 + h^2 + 2ah) [\sin a \cos h + \cos a \sin h] - a^2 \sin a}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{a^2 \sin a (\cos h - 1)}{h} + \frac{a^2 \cos a \sin h}{h} + (h + 2a) (\sin a \cos h + \cos a \sin h) \right]$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[\frac{a^2 \sin a \left(-2 \sin^2 \frac{h}{2}\right) \cdot \frac{h}{2}}{\frac{h^2}{2}} + \lim_{h \rightarrow 0} \frac{a^2 \cos a \sin h}{h} + \lim_{h \rightarrow 0} (h + 2a) \sin (a + h) \right] \\
&= a^2 \sin a \times 0 + a^2 \cos a (1) + 2a \sin a \\
&= a^2 \cos a + 2a \sin a.
\end{aligned}$$

Example 19 Find the derivative of $f(x) = \tan (ax + b)$, by first principle.

Solution We have $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{\tan (a(x+h) + b) - \tan (ax + b)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\frac{\sin (ax + ah + b)}{\cos (ax + ah + b)} - \frac{\sin (ax + b)}{\cos (ax + b)}}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin (ax + ah + b) \cos (ax + b) - \sin (ax + b) \cos (ax + ah + b)}{h \cos (ax + b) \cos (ax + ah + b)} \\
&= \lim_{h \rightarrow 0} \frac{a \sin (ah)}{a \cdot h \cos (ax + b) \cos (ax + ah + b)} \\
&= \lim_{h \rightarrow 0} \frac{a}{\cos (ax + b) \cos (ax + ah + b)} \lim_{ah \rightarrow 0} \frac{\sin ah}{ah} \quad [\text{as } h \rightarrow 0 \text{ } ah \rightarrow 0] \\
&= \frac{a}{\cos^2 (ax + b)} = a \sec^2 (ax + b).
\end{aligned}$$

Example 20 Find the derivative of $f(x) = \sqrt{\sin x}$, by first principle.

Solution By definition,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{\sqrt{\sin(x+h)} - \sqrt{\sin x}}{h} \\
&= \lim_{h \rightarrow 0} \frac{(\sqrt{\sin(x+h)} - \sqrt{\sin x})(\sqrt{\sin(x+h)} + \sqrt{\sin x})}{h(\sqrt{\sin(x+h)} + \sqrt{\sin x})} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h(\sqrt{\sin(x+h)} + \sqrt{\sin x})} \\
&= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin \frac{h}{2}}{2 \cdot \frac{h}{2} (\sqrt{\sin(x+h)} + \sqrt{\sin x})} \\
&= \frac{\cos x}{2\sqrt{\sin x}} = \frac{1}{2} \cot x \sqrt{\sin x}
\end{aligned}$$

Example 21 Find the derivative of $\frac{\cos x}{1 + \sin x}$.

Solution Let $y = \frac{\cos x}{1 + \sin x}$

Differentiating both sides with respects to x , we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx} \left(\frac{\cos x}{1 + \sin x} \right) \\
&= \frac{(1 + \sin x) \frac{d}{dx} (\cos x) - \cos x \frac{d}{dx} (1 + \sin x)}{(1 + \sin x)^2} \\
&= \frac{(1 + \sin x) (-\sin x) - \cos x (\cos x)}{(1 + \sin x)^2}
\end{aligned}$$

$$\begin{aligned}
 &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\
 &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} = \frac{-1}{1 + \sin x}
 \end{aligned}$$

Objective Type Questions

Choose the correct answer out of the four options given against each Example 22 to 28 (M.C.Q.).

Example 22 $\lim_{x \rightarrow 0} \frac{\sin x}{x(1 + \cos x)}$ is equal to

- (A) 0 (B) $\frac{1}{2}$ (C) 1 (D) -1

Solution (B) is the correct answer, we have

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\sin x}{x(1 + \cos x)} &= \lim_{x \rightarrow 0} \frac{2 \sin \frac{x}{2} \cos \frac{x}{2}}{x \left(2 \cos^2 \frac{x}{2} \right)} \\
 &= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\tan \frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}
 \end{aligned}$$

Example 23 $\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x}$ is equal to

- (A) 0 (B) -1 (C) 1 (D) does not exist

Solution (A) is the correct answer, since

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} = \lim_{y \rightarrow 0} \left[\frac{1 - \sin \left(\frac{\pi}{2} - y \right)}{\cos \left(\frac{\pi}{2} - y \right)} \right] \left(\text{taking } \frac{\pi}{2} - x = y \right)$$

$$\begin{aligned}
 &= \lim_{y \rightarrow 0} \frac{1 - \cos y}{\sin y} = \lim_{y \rightarrow 0} \frac{2 \sin^2 \frac{y}{2}}{2 \sin \frac{y}{2} \cos \frac{y}{2}} \\
 &= \lim_{y \rightarrow 0} \tan \frac{y}{2} = 0
 \end{aligned}$$

Example 24 $\lim_{x \rightarrow 0} \frac{|x|}{x}$ is equal to

- (A) 1 (B) -1 (C) 0 (D) does not exist

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x \rightarrow 0^+} \frac{|x|}{x} = \frac{x}{x} = 1$$

and

$$\text{L.H.S} = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \frac{-x}{x} = -1$$

Example 25 $\lim_{x \rightarrow 1} [x - 1]$, where $[.]$ is greatest integer function, is equal to

- (A) 1 (B) 2 (C) 0 (D) does not exist

Solution (D) is the correct answer, since

$$\text{R.H.S} = \lim_{x \rightarrow 1^+} [x - 1] = 0$$

and

$$\text{L.H.S} = \lim_{x \rightarrow 1^-} [x - 1] = -1$$

Example 26 $\lim_{x \rightarrow 0} x \sin \frac{1}{x}$ is equals to

- (A) 0 (B) 1 (C) $\frac{1}{2}$ (D) does not exist

Solution (A) is the correct answer, since

$\lim_{x \rightarrow 0} x = 0$ and $-1 \leq \sin \frac{1}{x} \leq 1$, by Sandwich Theorem, we have

$$\lim_{x \rightarrow 0} x \sin \frac{1}{x} = 0$$

Example 27 $\lim_{n \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2}$, $n \in \mathbb{N}$, is equal to

- (A) 0 (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{4}$

Solution (C) is the correct answer. As $\lim_{x \rightarrow \infty} \frac{1+2+3+\dots+n}{n^2}$

$$= \lim_{n \rightarrow \infty} \frac{n(n+1)}{2n^2} = \lim_{x \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{n} \right) = \frac{1}{2}$$

Example 28 If $f(x) = x \sin x$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

- (A) 0 (B) 1 (C) -1 (D) $\frac{1}{2}$

Solution (B) is the correct answer. As $f'(x) = x \cos x + \sin x$

So,
$$f'\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} = 1$$

13.3 EXERCISE

Short Answer Type

Evaluate :

1. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$

2. $\lim_{x \rightarrow \frac{1}{2}} \frac{4x^2 - 1}{2x - 1}$

3. $\lim_{h \rightarrow 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$

4. $\lim_{x \rightarrow 0} \frac{(x+2)^{\frac{1}{3}} - 2^{\frac{1}{3}}}{x}$

5. $\lim_{x \rightarrow 1} \frac{(1+x)^6 - 1}{(1+x)^2 - 1}$

6. $\lim_{x \rightarrow a} \frac{(2+x)^{\frac{5}{2}} - (a+2)^{\frac{5}{2}}}{x - a}$

$$7. \lim_{x \rightarrow 1} \frac{x^4 - \sqrt{x}}{\sqrt{x} - 1}$$

$$8. \lim_{x \rightarrow 2} \frac{x^2 - 4}{\sqrt{3x - 2} - \sqrt{x + 2}}$$

$$9. \lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2x} - 8}$$

$$10. \lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$$

$$11. \lim_{x \rightarrow 0} \frac{\sqrt{1 + x^3} - \sqrt{1 - x^3}}{x^2}$$

$$12. \lim_{x \rightarrow -3} \frac{x^3 + 27}{x^5 + 243}$$

$$13. \lim_{x \rightarrow \frac{1}{2}} \left(\frac{8x - 3}{2x - 1} - \frac{4x^2 + 1}{4x^2 - 1} \right)$$

$$14. \text{Find 'n', if } \lim_{x \rightarrow 2} \frac{x^n - 2^n}{x - 2} = 80, n \in \mathbb{N}$$

$$15. \lim_{x \rightarrow a} \frac{\sin 3x}{\sin 7x}$$

$$16. \lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 4x}$$

$$17. \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$$

$$18. \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3}$$

$$19. \lim_{x \rightarrow 0} \frac{1 - \cos mx}{1 - \cos nx}$$

$$20. \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sqrt{1 - \cos 6x}}{\sqrt{2} \left(\frac{\pi}{3} - x \right)}$$

$$21. \lim_{x \rightarrow \frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$

$$22. \lim_{x \rightarrow \frac{\pi}{6}} \frac{\sqrt{3} \sin x - \cos x}{x - \frac{\pi}{6}}$$

$$23. \lim_{x \rightarrow 0} \frac{\sin 2x + 3x}{2x + \tan 3x}$$

$$24. \lim_{x \rightarrow a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$$

$$25. \lim_{x \rightarrow \frac{\pi}{6}} \frac{\cot^2 x - 3}{\operatorname{cosec} x - 2}$$

$$26. \lim_{x \rightarrow 0} \frac{\sqrt{2} - \sqrt{1 + \cos x}}{\sin^2 x}$$

$$27. \lim_{x \rightarrow 0} \frac{\sin x - 2 \sin 3x + \sin 5x}{x}$$

$$28. \text{If } \lim_{x \rightarrow 1} \frac{x^4 - 1}{x - 1} = \lim_{x \rightarrow k} \frac{x^3 - k^3}{x^2 - k^2}, \text{ then find the value of } k.$$

Differentiate each of the functions w. r. to x in Exercises 29 to 42.

$$29. \frac{x^4 + x^3 + x^2 + 1}{x}$$

$$30. \left(x + \frac{1}{x} \right)^3$$

$$31. (3x + 5)(1 + \tan x)$$

$$32. (\sec x - 1)(\sec x + 1) \quad 33. \frac{3x+4}{5x^2-7x+9} \quad 34. \frac{x^5 - \cos x}{\sin x}$$

$$35. \frac{x^2 \cos \frac{\pi}{4}}{\sin x} \quad 36. (ax^2 + \cot x)(p + q \cos x)$$

$$37. \frac{a+b \sin x}{c+d \cos x} \quad 38. (\sin x + \cos x)^2 \quad 39. (2x-7)^2(3x+5)^3$$

$$40. x^2 \sin x + \cos 2x \quad 41. \sin^3 x \cos^3 x \quad 42. \frac{1}{ax^2 + bx + c}$$

Long Answer Type

Differentiate each of the functions with respect to 'x' in Exercises 43 to 46 using first principle.

$$43. \cos(x^2 + 1) \quad 44. \frac{ax+b}{cx+d} \quad 45. \frac{2}{x^3}$$

$$46. x \cos x$$

Evaluate each of the following limits in Exercises 47 to 53.

$$47. \lim_{y \rightarrow 0} \frac{(x+y) \sec(x+y) - x \sec x}{y}$$

$$48. \lim_{x \rightarrow 0} \frac{(\sin(\alpha + \beta)x + \sin(\alpha - \beta)x + \sin 2\alpha x)}{\cos 2\beta x - \cos 2\alpha x} \cdot x$$

$$49. \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan^3 x - \tan x}{\cos\left(x + \frac{\pi}{4}\right)} \quad 50. \lim_{x \rightarrow \pi} \frac{1 - \sin \frac{x}{2}}{\cos \frac{x}{2} \left(\cos \frac{x}{4} - \sin \frac{x}{4} \right)}$$

$$51. \text{Show that } \lim_{x \rightarrow 4} \frac{|x-4|}{x-4} \text{ does not exist}$$

52. Let $f(x) = \begin{cases} \frac{k \cos x}{\pi - 2x} & \text{when } x \neq \frac{\pi}{2} \\ 3 & x = \frac{\pi}{2} \end{cases}$ and if $\lim_{x \rightarrow \frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$,

find the value of k .

53. Let $f(x) = \begin{cases} x + 2 & x \leq 1 \\ cx^2 & x > -1 \end{cases}$, find 'c' if $\lim_{x \rightarrow -1} f(x)$ exists.

Objective Type Questions

Choose the correct answer out of 4 options given against each Exercise 54 to 76 (M.C.Q).

54. $\lim_{x \rightarrow \pi} \frac{\sin x}{x - \pi}$ is

- (A) 1 (B) 2 (C) -1 (D) -2

55. $\lim_{x \rightarrow 0} \frac{x^2 \cos x}{1 - \cos x}$ is

- (A) 2 (B) $\frac{3}{2}$ (C) $-\frac{3}{2}$ (D) 1

56. $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$ is

- (A) n (B) 1 (C) $-n$ (D) 0

57. $\lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1}$ is

- (A) 1 (B) $\frac{m}{n}$ (C) $-\frac{m}{n}$ (D) $\frac{m^2}{n^2}$

58. $\lim_{x \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is

(A) $\frac{4}{9}$ (B) $\frac{1}{2}$ (C) $\frac{-1}{2}$ (D) -1

59. $\lim_{x \rightarrow 0} \frac{\operatorname{cosec} x - \cot x}{x}$ is

(A) $\frac{-1}{2}$ (B) 1 (C) $\frac{1}{2}$ (D) 1

60. $\lim_{x \rightarrow 0} \frac{\sin x}{\sqrt{x+1} - \sqrt{1-x}}$ is

(A) 2 (B) 0 (C) 1 (D) -1

61. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\sec^2 x - 2}{\tan x - 1}$ is

(A) 3 (B) 1 (C) 0 (D) $\sqrt{2}$

62. $\lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(2x - 3)}{2x^2 + x - 3}$ is

(A) $\frac{1}{10}$ (B) $\frac{-1}{10}$ (C) 1 (D) None of these

63. If $f(x) = \begin{cases} \frac{\sin[x]}{[x]}, [x] \neq 0 \\ 0, [x] = 0 \end{cases}$, where $[.]$ denotes the greatest integer function,

then $\lim_{x \rightarrow 0} f(x)$ is equal to

(A) 1 (B) 0 (C) -1 (D) None of these

64. $\lim_{x \rightarrow 0} \frac{|\sin x|}{x}$ is

(A) 1 (B) -1 (C) does not exist (D) None of these

65. Let $f(x) = \begin{cases} x^2 - 1, 0 < x < 2 \\ 2x + 3, 2 \leq x < 3 \end{cases}$, the quadratic equation whose roots are $\lim_{x \rightarrow 2^-} f(x)$ and

$\lim_{x \rightarrow 2^+} f(x)$ is

(A) $x^2 - 6x + 9 = 0$

(B) $x^2 - 7x + 8 = 0$

(C) $x^2 - 14x + 49 = 0$

(D) $x^2 - 10x + 21 = 0$

66. $\lim_{x \rightarrow 0} \frac{\tan 2x - x}{3x - \sin x}$ is

(A) 2

(B) $\frac{1}{2}$

(C) $\frac{-1}{2}$

(D) $\frac{1}{4}$

67. Let $f(x) = x - [x]; \in \mathbf{R}$, then $f'\left(\frac{1}{2}\right)$ is

(A) $\frac{3}{2}$

(B) 1

(C) 0

(D) -1

68. If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$ is

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{\sqrt{2}}$

(D) 0

69. If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is

(A) $\frac{5}{4}$

(B) $\frac{4}{5}$

(C) 1

(D) 0

70. If $y = \frac{1 + \frac{1}{x^2}}{1 - \frac{1}{x^2}}$, then $\frac{dy}{dx}$ is

(A) $\frac{-4x}{(x^2-1)^2}$

(B) $\frac{-4x}{x^2-1}$

(C) $\frac{1-x^2}{4x}$

(D) $\frac{4x}{x^2-1}$

71. If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$ is

- (A) -2 (B) 0 (C) $\frac{1}{2}$ (D) does not exist

72. If $y = \frac{\sin(x+9)}{\cos x}$ then $\frac{dy}{dx}$ at $x = 0$ is

- (A) $\cos 9$ (B) $\sin 9$ (C) 0 (D) 1

73. If $f(x) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^{100}}{100}$, then $f'(1)$ is equal to

- (A) $\frac{1}{100}$ (B) 100 (C) does not exist (D) 0

74. If $f(x) = \frac{x^n - a^n}{x - a}$ for some constant ' a ', then $f'(a)$ is

- (A) 1 (B) 0 (C) does not exist (D) $\frac{1}{2}$

75. If $f(x) = x^{100} + x^{99} + \dots + x + 1$, then $f'(1)$ is equal to

- (A) 5050 (B) 5049 (C) 5051 (D) 50051

76. If $f(x) = 1 - x + x^2 - x^3 \dots - x^{99} + x^{100}$, then $f'(1)$ is equal to

- (A) 150 (B) -50 (C) -150 (D) 50

Fill in the blanks in Exercises 77 to 80.

77. If $f(x) = \frac{\tan x}{x - \pi}$, then $\lim_{x \rightarrow \pi} f(x) =$ _____

78. $\lim_{x \rightarrow 0} \left(\sin mx \cot \frac{x}{\sqrt{3}} \right) = 2$, then $m =$ _____

79. if $y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, then $\frac{dy}{dx} =$ _____

80. $\lim_{x \rightarrow 3^+} \frac{x}{[x]} =$ _____

