The Diamond Lemma for non-terminating rewriting systems using deterministic reduction strategies (Long Version)

Cyrille Chenavier[1] and Maxime Lucas[2]

[1] Inria Lille - Nord Europe, Équipe Valse
cyrille.chenavier@inria.fr
[2] Inria Rennes - Bretagne Atlantique, Équipe Gallinette
maxime.lucas@inria.fr

Abstract

We study the confluence property for rewriting systems whose underlying set of terms admits a vector space structure. For that, we use deterministic reduction strategies. These strategies are based on the choice of standard reductions applied to basis elements. We provide a sufficient condition of confluence in terms of the kernel of the operator which computes standard normal forms. We present a local criterion which enables us to check the confluence property in this framework. We show how this criterion is related to the Diamond Lemma for terminating rewriting systems.

Contents

plan:

- **Section I: IWC theorem**
- **Section 2: rew on rationnal Weyl algebras**
  - **definition of rew. rules and rew. steps**
  - **proposition: convergent implies general form of solution to PDE**
  - **Janet bases: Janet complete implies existence of a strategy, passivity implies h-confluence criterion, as a consequence we recover formal solutions to PDE**
  - **$y' = xy$ and Janet example**

Example $y' = xy$, main steps:

- **general solution is given by $\langle u \mid x^{2n+1}\rangle = 0$ and $\langle u \mid x^{2n}\rangle = u(0)/(2^n n!)$**

- **we recover this by rew: we need $(\partial_x)^n x = x(\partial_x)^n + n(\partial_x)^{n-1}$ (using Leibniz identity and induction) and $\langle u \mid x^n \rangle = 1/(n!)(\partial_x)^n(u)_{|0}$ and we prove the previous two formulas by induction (use this example as a running example?)**

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1 Introduction

The fact that local confluence together with termination implies confluence has been known for abstract rewriting systems since Newman's work [9]. For rewriting on noncommutative polynomials, a similar result known as the Diamond lemma was introduced by Bergman [3] nearly 30 years later, in order to compute normal forms in noncommutative algebras using rewriting theory. It asserts that for terminating rewriting systems, the local confluence property can be checked on monomials.

One difficulty of rewriting polynomials is that the naive notion of rewriting path (obtained as the closure of the generating rewriting relations under reflexivity, transitivity, sum and product by a scalar) does not terminate. Instead, one needs to first consider well-formed rewriting steps before forming the reflexive transitive closure.

Nevertheless the Diamond lemma has proved to be very useful: together with the works of Bokut [4], it has given birth the theory of noncommutative Gröbner bases [8]. The latter have provided applications to various areas of noncommutative algebra such as the study of embedding problems (which appear in the works of Bokut and Bergman), homological algebra [5, 6] or Koszul duality [2, 10].

Computation of normal forms in noncommutative algebra is also used to provide formal solutions to partial differential equations. In this framework, a confluence criterion analogous to the Diamond Lemma is given by Janet bases [11], which specify a deterministic way to reduce each polynomial into normal form using standard reductions [7]. The confluence criterion may then be asserted as follows: for each monomial m and each non-standard reduction $m \to f$, f is reducible into $\widehat{m}$, where the latter is obtained from m using only standard reductions.

In the presented paper, we propose an extension of the Diamond Lemma which offers two improvements over the one from Bergman: first it allows the treatment of non-terminating rewriting relations, and second it does not presuppose a notion of well-formed rewriting steps. This last property seems particularly promising in order to extend the Diamond Lemma to other structures.

Instead of supposing that the rewriting relation studied is terminating, we suppose given an ordering on the monomials, independent of the rewriting relation. We then use methods based on standard reductions: for every monomial m, we select exactly one reduction with left-hand side m, which is decreasing for the ordering chosen. Such choices induce a deterministic way to reduce each polynomial, obtained by applying simultaneously standard reductions on every monomial appearing in its decomposition. When these deterministic reductions terminate, one defines an operator which maps every polynomial to its unique standard normal form.

From this operator, we define a suitable notion of confluence in our setting, and show in Proposition **??** that it implies the usual notion of confluence for the rewriting system studied. We then provide an effective method for checking this criterion in Theorem 2.10. This method is based on a local analysis corresponding to checking local confluence on monomials. In particular, when the rewriting system is terminating, we show (Theorem 2.12) that we recover the Diamond Lemma as a particular case of Theorem 2.10.

Conventions and notations. Throughout the paper, we fix some conventions and notations that we present now. We will always work with a tuple $(\mathbb{K}, V, \mathscr{B}, <)$, where

- $\mathbb{K}$ is a field
- V is a $\mathbb{K}$-vector space and $\mathscr{B}$ is a basis of V
- $<$ is a well-founded order over $\mathscr{B}$.

In the sequel, such a tuple is called a **data**. Given a **data**, we simply say ground field, vectors and basis elements for $\mathbb{K}$, elements of V and $\mathscr{B}$, respectively. Moreover, every vector v admits a unique finite decomposition with respect to the basis $\mathscr{B}$ and coefficients in the ground field:

$$v = \sum \lambda_i e_i, \quad \lambda_i \neq 0. \tag{1}$$

The set of basis elements which appear in the decomposition (1) is called the *support* of v, and is written $\mathrm{supp}(v)$. We extend $<$ into the multiset order on V: for any $u, v \in V$, we have $u < v$ if $\mathrm{supp}(u) \neq \mathrm{supp}(v)$ and for any $e \in \mathrm{supp}(u) \setminus \mathrm{supp}(v)$, there exists $e' \in \mathrm{supp}(v) \setminus \mathrm{supp}(u)$ such that $e' > e$. Notice that this order coincides with the initial one on basis elements, so that we can denote it by $<$ without ambiguity.

Finally, we use the standard terminology of rewriting theory, see [1]. **Compléter.**

2 A weak version of the DiamondLlemma

In this section, we fix a **data** $(\mathbb{K}, V, \mathscr{B}, <)$ as well as a subset R of $\mathscr{B} \times V$, whose elements are called rewriting rules. Such a rewriting rule is denoted $e \to_R r$, where e and r are the natural projections of this rule of R on $\mathscr{B}$ and V, respectively. Our objective is to provide a confluence criterion for the rewriting system on V which extends R by linearity.

Throughout the section, an element of V which is minimal for the multiset order $<$ on V is simply called minimal.

2.1 Local strategies of linear extensions

In this section, we first introduce the notion of linear extension, that is the rewriting relation on V obtained by extending R by linearity. Then, we define local strategies for this linear extension, that we use in Section 2.2 for introducing our confluence criterion.

The linear extension of R enables us to reduce many basis elements at once. Hence, we consider elementary reductions of the following form:

$$\sum_i \lambda_i e_i + v \to \sum_i \lambda_i r_i + v, \tag{2}$$

where v is a vector, λ_i is a scalar and $e_i \to_R r_i$ are rewriting rules.

Definition 2.1. The *linear extension* of R is the rewriting relation on V defined such as in (2), where each λ_i is nonzero and each e_i does not belong to $\mathrm{supp}(v)$.

The linear extension of R is still denoted by $\to_R$, and its closures under transitivity, symmetry and sum and transitivity, symmetry, reflexivity and sum are denoted by $\overset{*}{\to}_R$ and $\overset{*}{\leftrightarrow}_R$, respectively. Finally, a normal form for $\to_R$ is called an R-normal form.

In the following definition, we introduce the notion of local strategy, which is the selection of exactly one rewriting rule for each non minimal basis element. Moreover, the selected rewriting rule as to be compatible with the order $<$.

Definition 2.2. A *local strategy* is a subset S of R such that the following two conditions hold:

- the left-hand sides of elements of S are not minimal basis elements,
- for every non minimal basis element e, there exists exactly one rewriting rule $e \to_R r_e$ in S, moreover we have $r_e < e$.

Notice that a local strategy may not exist. The elements of such a local strategy S are denoted $e \to_{\hat{S}} r_e$[1]. Moreover, instead of working with the full linear extension of S, we consider the rewriting relation, also denoted by $\to_{\hat{S}}$, which is defined such as the linear extension of S with the additional property that there is no rewriting rule $e \to_{\hat{S}} r_e$ such that $e \in \operatorname{supp}(v)$. In other words, a vector u admits a unique decomposition

$$u = \sum_i \lambda_i e_i + v,$$

where $\lambda_i \neq 0$, $e' \in \operatorname{supp}(v)$ implies that e' is minimal, and $e_i \in \operatorname{supp}(u) \setminus \operatorname{supp}(v')$ is not, and if $e_i \to_{\hat{S}} r_i$ are elements of the local strategy, we have

$$u \to_{\hat{S}} \sum_i \lambda_i r_i + v.$$

The closure of $\to_{\hat{S}}$ under transitivity and symmetry is denoted by $\overset{*}{\to}_{\hat{S}}$. Notice that $u \overset{*}{\to}_{\hat{S}} v$ implies $u \overset{*}{\to}_R v$. Moreover, a normal form for $\to_{\hat{S}}$ is called an S-normal form. In particular, every minimal basis element is a S-normal form. More generally, S-normal forms can be easily characterised as follows.

Lemma 2.3. *Given a vector v, either v is minimal, or there exists $v' < v$ such that $v \to_{\hat{S}} v'$. In particular, S-normal forms are precisely the minimal elements of V.*

In order to illustrate our notions, we consider the following running example.

Example 2.4. Let $\mathscr{B} = \{a, b, c, d\}$ and consider the set R composed of the following rewriting rules:

$$a \to_R b, \quad b \to_R c + d, \quad c \to_R b - d.$$

Note that the linear extension of R is not terminating since there is a rewriting loop due to $b \to_R c + d$ and $c + d \to_R (b - d) + d = b$. Moreover, we choose the order such that d is minimal and other basis elements are not comparable:

$$a > d, \quad b > d, \quad c > d.$$

[1]The hat on S is here to mark the difference between the local strategy and rewriting rules.

Finally, as a local strategy, we select S defined by:

$$a \to_{\hat{S}} b, \quad b \to_{\hat{S}} c + d.$$

Then, the R-normal forms are the elements of the form $\lambda_d d$, while the h-normal forms are all the expressions of the form $\lambda_d d + \lambda_c c$.

We finish this section by introducing the normalisation operator associated with a local strategy S. By definition of such a strategy, for every $v \in V$, there exists at most one v' such that $v \to_{\hat{S}} v'$. Moreover, since $\to_{\hat{S}}$ is compatible with the well-founded order $<$, v is sent by multiple applications of $\to_{\hat{S}}$ to a unique S-normal form that we denote by $\mathrm{NF}(v)$. This defines a map $\mathrm{NF} : V \to V$, that is fundamental for our confluence criterion of Section 2.2.

Proposition 2.5. *The map* NF *is a linear projector.*

Proof. The S-normal forms are closed under sum, so that $\mathrm{NF}(\mathrm{NF}(v)) = \mathrm{NF}(v)$ for every v, that is NF is a projector. Moreover, if $u \to_{\hat{S}} u'$ and $v \to_{\hat{S}} v'$, then we have $u + v \to_{\hat{S}} u' + v'$. Iterating $\to_{\hat{S}}$, we get $\mathrm{NF}(u+v) = \mathrm{NF}(\mathrm{NF}(u)+\mathrm{NF}(v)) = \mathrm{NF}(u)+\mathrm{NF}(v)$, which proves linearity of NF. □

2.2 Confluence relative a strategy

In this section, we first introduce S-confluence and show that this property implies confluence of $\to_R$. Then, we introduce local S-confluence which turns out to imply S-confluence: our confluence criterion follows, and we prove it Theorem 2.10. We finish this section by a new proof of the Diamond Lemma, based on local S-confluence.

Definition 2.6. Given a local strategy S, we say that $\to_R$ is *S-confluent* if for every rewriting rule $e \to_{\hat{S}} r_e$, we have $\mathrm{NF}(e - r_e) = 0$.

In the following proposition, we show that S-confluence implies confluence of $\to_R$.

Proposition 2.7. *Let S be a local strategy such that $\to_R$ is S-confluent. We have $u \overset{*}{\leftrightarrow}_R v$ if and only if $\mathrm{NF}(u - v) = 0$. Moreover, $\to_R$ is confluent.*

Proof. We show the first assertion. The relation $\overset{*}{\leftrightarrow}_R$ is the closure of $\to_R$ under transitivity, symmetry and sum. Since the relation $\mathrm{NF}(u - v) = 0$ is closed under these operations, we get one implication. Reciprocally, if $\mathrm{NF}(u - v) = 0$, we have $\mathrm{NF}(u) = \mathrm{NF}(v)$. Moreover, by definition of NF, we have $u \overset{*}{\to}_{\hat{S}} \mathrm{NF}(u)$ and $v \overset{*}{\to}_{\hat{S}} \mathrm{NF}(v)$, and since $v_1 \overset{*}{\to}_{\hat{S}} v_2$ implies $v_1 \overset{*}{\to}_R v_2$, we get $u \overset{*}{\leftrightarrow}_R v$. For the second assertion, let us consider three vectors $v, v_1, v_2 \in V$ such that for $i = 1, 2$, we have $v \overset{*}{\to}_R v_i$. By the first part of the proposition, we have $\mathrm{NF}(v_1) = \mathrm{NF}(v_2)$. From $v_i \overset{*}{\to}_{\hat{S}} \mathrm{NF}(v_i)$, we get $v_i \overset{*}{\to}_R \mathrm{NF}(v_i)$, so that $\to_R$ is confluent. □

Note that the previous proposition is a sufficient but not a necessary condition. Indeed, with $\mathscr{B}$ the set of integers and the rewriting rules $n \to_R n + 1$, the linear extension of R is confluent, but there is no local strategy such that $\to_R$ is confluent relative to this strategy.

Example 2.8. Let us continue Example 2.4. The following identities hold:

$$\mathrm{NF}(a) = c + d = \mathrm{NF}(b), \qquad \mathrm{NF}(b) = c + d = \mathrm{NF}(c + d), \qquad \mathrm{NF}(c) = c = \mathrm{NF}(b - d),$$

so that $\to_R$ is S-confluent, and hence confluent. Notice that if we replace the rule $c \to_R b - d$ by $c \to_R b$, we get $\mathrm{NF}(c) = c$ and $\mathrm{NF}(b) = c + d$, so $\to_R$ is not S-confluent anymore.

Now, we introduce local S-confluence, which is a decreasingness property with respect to a well-founded order on rewriting rules.

Definition 2.9. Let $\prec$ be a well-founded order on R and let S be a local strategy. We say that $\to_R$ is *locally S-confluent with respect to* $\prec$ if for every $e \in \mathscr{B}$ and for every rewriting rule $e \to_R r$, we have a confluence diagram:

$$\begin{tikzcd} e \arrow[r, "R"'] \arrow[d, "\hat{S}"'] & r \arrow[d, dotted, leftrightarrow, "*"] \\ r_e \arrow[r, dotted, leftrightarrow, "*"] & v \end{tikzcd}$$

where each rewriting step occurring in the dotted arrows belongs to R (we removed the subscript R for simplicity) and is strictly smaller than the rewriting rule $e \to_R r$ relative to $\prec$.

The main result of this section is the following.

Theorem 2.10. *Let R be a set of rewriting rules. Assume that there exists a local strategy S and a well-founded order $\prec$ on R such that $\to_R$ is locally S-confluent with respect to $\prec$. Then, $\to_R$ is confluent.*

Proof. It is sufficient to show that $\to_R$ is S-confluent.

We reason by induction on rewriting rules f according to the order $\prec$. Looking at the square corresponding to f:

$$\begin{tikzcd} e \arrow[r, "f"] \arrow[d, "s_e"'] & v \arrow[d, dotted, leftrightarrow, "*"] \\ r_e \arrow[r, dotted, leftrightarrow, "*"'] & v' \end{tikzcd}$$

we have $H(e) = H(r_e)$ by definition of H, and $H(r_e) = H(v') = H(v)$ by induction hypothesis, which concludes the proof. **Ce n'est pas hyper clair pour moi. Peux-tu reprendre la fin de la preuve?.** $\square$

Local h-confluence implies that the pair of rewriting relations $(\to_{\hat{S}}, \to_R)$ is decreasing with respect to conversions (see [12, Definition 3]), using the order $\prec$ on R and the discrete ordering on $\to_{\hat{S}}$. By [12, Theorem 3], this implies that $(\to_{\hat{S}}, \to_R)$ commute. Using the fact that $\to_{\hat{S}} \subseteq \to_R$, one can then recover that $\to_R$ is confluent. **Rajouter un phrase au début faisant comprendre que cette remarque s'adresse à des spécialistes de réécriture.**

Example 2.11. Using local S-confluence, we way recover that the rewriting relation of Example 2.4 is confluent. Indeed, let us consider the following order $\prec$ on rewriting rules:

$$(a \to_R b) \prec (c \to_R b - d), \quad (b \to_R c + d) \prec (c \to_R b - d).$$

This choice is guided by the heuristic that rules advancing towards an S-normal form should be favored over rules that do not: here c is an S-normal form so its rewriting rule is for $\prec$. The following diagrams show that $\to_R$ is locally S-confluent:

$$\begin{tikzcd} a \arrow[r, "R"'] \arrow[d, "\hat{S}"'] & b \arrow[d, equal] \\ b \arrow[r, equal] & b \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, "R"'] \arrow[d, "\hat{S}"'] & c + d \arrow[d, equal] \\ c + d \arrow[r, equal] & c + d \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r, "R"'] \arrow[d, "\hat{S}"'] & b - d \arrow[d, "R"] \\ c \arrow[r, equal] & c \end{tikzcd}$$

We finish this section by showing how the Diamond Lemma fits as a particular case of our setup.

Theorem 2.12 ([3])**.** *Let R be a set of rewriting rules such that $\to_R$ is terminating, and for every $e \in \mathscr{B}$ such that $e \to_R r$ and $e \to_R r'$, r and r' are joinable. Then, $\to_R$ is confluent.*

Proof. We define an ordering $e > e'$ on $\mathscr{B}$ as the transitive closure of the relation "there exists a vector v such that $e \to_R v$ and $e' \in \operatorname{supp}(v)$". This is a well-founded order since $\to_R$ is assumed to be terminating. Let us define the local strategy S as follows. First, notice that by definition of the order $<$, if the basis element e is not minimal, then it is not an R-normal form. Then, S is composed of elements $e \to_{\hat{S}} r$, where $e \to_R r$ is an arbitrary rewriting rules reducing e. It remains to define $\prec$: we let $(e \to_R r) \prec (e' \to_R r')$ if $e < e'$, which is well-founded since $<$ is. Finally, the hypothesis r and r' are joinable when $e \to_R r$ and $e \to_R r'$ implies that $\to_R$ is locally S-confluent with respect to $\prec$. Hence, by Theorem 2.10, $\to_R$ is confluent. $\square$

3 Rewriting and partial derivative equations

3.1 Rewriting systems on rational Weyl algebras

In this section, we introduce rewriting systems in rational Weyl algebras and relate them to the framework introduced in Section 2. Before that, we first recall the definitions of Weyl algebras and of monomial orders.

We fix a set $X = \{x_1, \cdots, x_n\}$ of indeterminates and we denote by $\mathbb{Q}(X) = \mathbb{Q}(x_1, \cdots, x_n)$ the field of fractions of the polynomial algebra $\mathbb{Q}[x_1, \cdots, x_n]$ over $\mathbb{Q}$. Let us introduce another set of variables $\Delta := \{\partial_1, \cdots, \partial_n\}$.

Definition 3.1. The *rational Weyl algebra* over $\mathbb{Q}(X)$ is the set of polynomials over the inderminates Δ and coefficients in $\mathbb{Q}(X)$, subject to the following commutation rule:

$$\partial_i f = f\partial_i + \frac{d}{dx_i}(f), \quad f \in \mathbb{Q}(X), \quad 1 \leq i \leq n,$$

where $d/dx_i : \mathbb{Q}(X) \to \mathbb{Q}(X)$ is the partial derivative operator with respect to the indeterminate x_i. This algebra is denoted by $B_n(\mathbb{Q})$.

Given $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, we denote by $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. Notice that $B_n(\mathbb{Q})$ is a vector space on $\mathbb{Q}(X)$ and admits as a basis the set $\{\partial^\alpha \mid \alpha \in \mathbb{N}^n\}$. Hence, elements of $B_n(\mathbb{Q})$ should be thought as differential operators with rational functions coefficients, and for this reason, a generic element of this algebra is denoted by $\mathcal{D}$ and is called a differential operator.

Definition 3.2. A *monomial order* $\preceq$ on $\mathscr{B}$ is a well-founded total order that is compatible with multiplication of monomials, *i.e.*, $\partial^\alpha \preceq \partial^\beta$ implies $\partial^{\alpha+\gamma} \preceq \partial^{\beta+\gamma}$, for every $\alpha, \beta, \gamma \in \mathbb{N}^n$. For every differential operator $\mathcal{D} \in B_n(\mathbb{Q})$, let us denote by $\operatorname{lm}(\mathcal{D})$ and $\operatorname{lc}(\mathcal{D})$ the leading monomial and the leading coefficient of $\mathcal{D}$ with respect to $\preceq$, respectively, and let us write $r(\mathcal{D}) := \operatorname{lc}(\mathcal{D})\operatorname{lm}(\mathcal{D}) - \mathcal{D}$.

Hence, we get a **data** $(\mathbb{Q}(X), B_n(\mathbb{Q}), \Delta^c, \preceq)$

The framework developed in Section **??** can be applied to rational Weyl algebras by taking $\mathbb{K} := \mathbb{Q}(X)$ and $\mathscr{B} := \{\partial^\alpha \mid \alpha \in \mathbb{N}^n\}$. In particular, the support of a differential operator $\mathcal{D}$

is the product of partial derivative operators which occur in $\mathcal{D}$. It remains to introduce a well-founded order on $\mathscr{B}$, and for that, we consider monomial orders.

Given an operator $\mathcal{D} \in B_n(\mathbb{Q})$, consider the rewriting relation on $B_n(\mathbb{Q})$ defined as follows. Letting $\mathrm{lm}(\mathcal{D}) = \partial^\alpha$, we first have the rewriting steps on monomials defined by

$$\forall \beta \in \mathbb{N}^n, \quad \partial^{\alpha+\beta} \to_{\mathcal{D}} \partial^\beta r(\mathcal{D}), \tag{3}$$

and then extend (3) by linearity. More generally, for every set $\Theta = \{\mathcal{D}_1, \cdots, \mathcal{D}_r\}$ of differential operators, we consider the rewriting system $(B_n(\mathbb{Q}), \to_\Theta)$ defined by $\mathcal{D} \to_\Theta \mathcal{D}'$, whenever $\mathcal{D} \to_{\mathcal{D}_i} \mathcal{D}'$, for some $1 \le i \le r$.

Il faudra remonter tout ça au cadre abstrait.

3.2 Janet bases and h-confluence

In this section, we take $\mathbb{K} = \mathbb{R}(x_1, x_2, \ldots, x_n)$ the field of real rational functions in n variables, and look at the $\mathbb{K}$-algebra $A = \mathbb{K}[\partial_1, \ldots, \partial_n]$. We suppose given an ordering $\partial_n > \partial_{n-1} > \ldots > \partial_1$. By monomial, we mean unitary monomial.

Example 3.3. Elements of A should be thought of as systems of linear partial derivative equations. For example, the equation $x_1 \frac{\partial u}{\partial x_1} - u = 0$ corresponds to the element $x_1\partial_1 - 1$ in A.

Similarly, the 1-dimensional heat equation $\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$ corresponds (taking $x_1 = t$ and $x_2 = x$) to the element $\partial_1 - \kappa\partial_2^2$.

Definition 3.4. For any monomial m and $1 \le k \le n$ we denote by $\nu_k(m)$ the power of ∂_k in m. For any $\bar{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$, we denote by $\partial^{\bar{j}}$ the monomial m such that $\nu_k(m) = j_k$ for all k. For any subset $E \subset \{1, \ldots, n\}$, we denote by $Mon(E)$ the set of monomials m such that $\nu_k(m) = 0$ for all $k \notin E$. In particular, $Mon(\{1, \ldots, n\}$ is the set of all monomials of A which we denote simply by Mon, while $Mon(\emptyset)$ is reduced to $\{1\}$.

Definition 3.5. A *cone* is a pair (m, E), where $m \in A$ and $E \subset \{1, \ldots, n\}$. m is called the *origin of the cone*, while E is the *direction*. A cone is *monomial* if m is a monomial. We denote by $(m, E)^*$ the family of monomials of the form mp, where $p \in Mon(E)$.

If S is a family of monomials, a *cone-partition* of S is a family of monomial cones (m_i, E_i) for $i \in I$ such that for $i \neq j$, $(m_i, E_i)^* \cap (m_j, E_j) = \emptyset$ and $S = \coprod_{i \in I}(m_i, E_i)^*$. In other words the $(m_i, E_i)^*$ form a partition of S into monomial cones. It is a finite cone-partition if I is finite.

Definition 3.6. A family of monomial cones (m_i, E_i) with distinct origins is said to be *complete* if it induces a cone-partition of the monomial ideal generated by the m_is.

A family of cones (p_i, E_i) is said to be *complete* if $(lm(p_i), E_i)$ is complete.

Lemma 3.7. *A complete family of cones gives rise to a local strategy on A.*

Sketch. We use the monomial ordering on reducible monomials, and reducible ones are bigger than irreducible ones.

Then we only need to define h on reducible ones. For $x \in Red(R)$, there exists a unique (m_i, E_i) in the cone-partition such that $x = m_i p$ with $p \in E_i$ and m_i corresponds to an element $m_i + r$ of R. Then h is defined by $h_x = x = m_i p \to_R -rp$. □

In order to produce suitable cone-partitions as in the previous lemma, we introduce the notion of multiplicative variables.

Definition 3.8. Let M be a subset of Mon, and let $m \in M$. We define a subset $\mu_M(m)$ of $\{1, \dots, n\}$. Let us write $m = \partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n}$, and take $k \in \{1, \dots, n\}$. Then $k \in \mu_M(m)$ if the following implication is true:

$$\forall m' \in M, \quad (\forall j < k, \nu_j(m') = \nu_j(m) \quad \Rightarrow \quad \nu_k(m') \leq \nu_k(m)).$$

We call $(m, \mu_M(m))$ the cone of m in M. If M is clear then we will just write $\mu(m)$. If $k \in \mu(m)$ we will say that ∂_k is a *multiplicative variable* of m.

Lemma 3.9. *Let $M \subset Mon$ and $m, m' \in M$. If $m \neq m'$, then $(m, \mu(m))^* \cap (m', \mu(m'))^* = \emptyset$.*

Proof. Since $m \neq m'$, there exists k minimal such that $\nu_k(m) \neq \nu_k(m')$. Without loss of generality, we can suppose $\nu_k(m) < \nu_k(m')$. By definition of μ, we therefore have that $k \notin \mu(m)$.

Take now $p \in (m, \mu(m))^* \cap (m', \mu(m'))^*$. Then $p = mq$, with $\nu_k(q) = 0$, and so $\nu_k(p) = \nu_k(m)$. But p is a multiple of m' and so $\nu_k(m) = \nu_k(p) \geq \nu_k(m')$, which is contradictory. □

Example 3.10. **TODO.**

Definition 3.11. A set of monomials M is *Janet-complete* if the family of all $(m, \mu(m))$ is complete.

A set of polynomials R is *Janet-complete* if $lm(R)$ is Janet-complete.

Conclusion. We introduced a sufficient condition, based on deterministic reduction strategies, of confluence for rewriting systems on vector spaces. As a particular case, we recover the Diamond Lemma. This work maybe extended in particular into two main directions. The first one consists in weakening our assumption on the set $\mathbb{K}$ of coefficients, by allowing non invertible coefficients. A second extension consists in characterising Janet bases in this framework, with the objective to develop constructive methods in the analysis and formal resolution of PDE's.

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