The Diamond Lemma for non-terminating rewriting systems using deterministic reductions

Cyrille Chenavier[1] and Maxime Lucas[2]

[1] Inria Lille - Nord Europe, équipe Valse
cyrille.chenavier@inria.fr
[2] Inria Rennes - Bretagne Atlantique, équipe Gallinette
maxime.lucas@inria.fr

Abstract

We study the confluence property for rewriting systems over vector spaces using deterministic reduction strategies. These strategies are based on the choice of standard reductions applied to basis elements. We show that the confluence property is interpreted in terms of tke kernel of the operator which computes standard normal forms. We present a local criterion which enables us to check the confluence property in this framework. We show how this criterion is related to the Diamond Lemma for terminating rewriting systems.

1 Introduction

The Diamond lemma for noncommutative polynomials was introduced by Bergman [2] for computing normal forms in noncommutative algebras using rewriting theory. It asserts that for terminating rewriting systems, the local confluence property can be checked on monomials. The Diamond lemma together with the works of Bokut [3] gave birth the theory of noncommutative Gröbner bases [7]. The latter provide applications to various areas of noncommutative algebra: study of embedding problems, which appears in the works of Bokut and Bergman, homological algebra [4, 5] or Koszul duality [1, 8], for instance.

Computation of normal forms in noncommutative algebra is also used to provide formal solutions to partial differential equations. In this framework, a confluence criterion analogous to the Diamond Lemma is given by Janet bases [9]. Tha latter specify a determinitic way to reduce each polynomial into normal form using standard reductions [6]. Then, the confluence criterion maybe asserted as follows: for each monomial m and each non-standard reduction $m \to f$, f is reducible into $\widehat{m}$, where the latter is obtained from m using only standard reductions.

In the presented paper, we propose an extension of the Diamond Lemma, where we do not assume termination of the underlying rewriting relation. For that, we use methods based on standard reductions. Indeed, for every monomial m, we select exactly one reduction with left-hand side m. Such choices induce a determinitic way to reduce each polynomial: it is done by applying simultenaously standard reductions on the monomials appearing in its decomposition.

When these deterministic reductions terminate, one define an operator which maps every polynomial to its unique standard normal form. From this operator, we deduce in Proposition 3.4 a theoritical criterion for confluence. In Theorem 3.7, we provide an effective method for checking this criterion. This method is based on a local analysis corresponding to checking local confluence on monomials. In particular, in the case where the rewriting system is terminating, we show in Theorem 3.8 that the Diamond Lemma is a consequence of the criterion presented in Theorem 3.7.

2 Local strategies and h-normal forms

We fix a commutative field $\mathbb{K}$ as well as a well-founded ordered set $(X, <)$. We denote by $\mathbb{K}X$ the vector space spanned by X: an element $v \in \mathbb{K}X$ is a finite formal linear combination of elements of X with coefficients in $\mathbb{K}$. In particular, for every $v \in \mathbb{K}X$, there exists a unique finite set $\mathrm{supp}(v) \subset X$, called the *support* of v, such that

$$v = \sum_{x \in \mathrm{supp}(v)} \lambda_x x \text{ and } x \in \mathrm{supp}(v) \Rightarrow \lambda_x \neq 0. \tag{1}$$

We denote by $\mathrm{supp}(v)^c = X \setminus \mathrm{supp}(v)$. The sum of $u = \sum \lambda_x x$ and $v = \sum \mu_x x$ equals $\sum(\lambda_x + \mu_x)x$ and the product of $\lambda \in \mathbb{K}$ by v equals $\sum(\lambda\lambda_x)x$. We extend the order $<$ into the multiset order, still written $<$, on $\mathbb{K}X$: we have $u < v$ if for every $x \in \mathrm{supp}(u) \cap \mathrm{supp}(v)^c$, there exists $y \in \mathrm{supp}(v) \cap \mathrm{supp}(u)^c$ such that $y > x$.

We fix a set $R \subseteq X \times \mathbb{K}X$ which represents rewrite rules of the form $x \underset{R}{\longrightarrow} r$. The set R induces the rewriting relation on $\mathbb{K}X$, still written $\underset{R}{\longrightarrow}$, defined as follows:

$$\sum \lambda_x x + v \underset{R}{\longrightarrow} \sum \lambda_x r_x + v, \tag{2}$$

whenever $\lambda_x \neq 0$, $x \underset{R}{\longrightarrow} r_x \in R$ and $x \notin \mathrm{supp}(v)$.

Definition 2.1. A *local strategy* for R is the choice, for every $x \in X$ not minimal for $<$, of a rewriting rule $h_x = x \underset{R}{\longrightarrow} r_x$ such that $r_x < x$.

Suppose chosen such a local strategy h. Any vector v can be decomposed in a unique way as $\sum \lambda_x x + v'$, where $y \in \mathrm{supp}(v')$ implies that y is minimal for $<$, and $x \in \mathrm{supp}(v) \cap \mathrm{supp}(v')^c$ is not. We define a rewriting relation $\underset{h}{\longrightarrow}$ as follows:

$$\sum \lambda_x x + v' \underset{h}{\longrightarrow} \sum \lambda_x r_x + v', \tag{3}$$

where for every x, $h_x = x \underset{R}{\longrightarrow} r_x$.

Definition 2.2. A vector v is said to be a *h-normal form* if it is a normal form for $\underset{h}{\longrightarrow}$.

Example 2.3. Let $X = \{x, y, z, t\}$, $x \underset{R}{\longrightarrow} y$, $y \underset{R}{\longrightarrow} z + t$, $z \underset{R}{\longrightarrow} y - t$. Note that this is not terminating since we have the infinite loop $y \underset{R}{\longrightarrow} z + t \underset{R}{\longrightarrow} (y - t) + t = y$. We choose the order $x > y > z$, and the following distinguished rewrite rules: $h_x = x \underset{h}{\longrightarrow} y$, $h_y = y \underset{h}{\longrightarrow} z + t$. Then the R-normal forms are the $\lambda_t t$, while the h-normal form are all the $\lambda_t t + \lambda_z z$.

Lemma 2.4. *Let v be a vector in $\mathbb{K}X$. Either v is minimal for $<$, or there exists $v' < v$ such that $v \underset{h}{\longrightarrow} v'$. In particular, h-normal forms are precisely the minimal elements of $\mathbb{K}X$ for $<$.*

For each $v \in \mathbb{K}X$, there exists at most one v' such that $v \underset{h}{\longrightarrow} v'$, and $\underset{h}{\longrightarrow}$ is compatible with the termination order $<$. As a consequence, any $v \in \mathbb{K}X$ is sent by multiple applications of $\underset{h}{\longrightarrow}$ to a unique h-normal form that we denote by $H(v)$. This defines a map $H : \mathbb{K}X \to \mathbb{K}X$.

Proposition 2.5. *The map H is a linear projector.*

Proof. The h-normal forms are closed under sums, so that $H(H(v)) = H(v)$ for every v, that is H is a projector. Moreover, if $u \xrightarrow[h]{} u'$ and $v \xrightarrow[h]{} v'$, then we have $u + v \xrightarrow[h]{} u' + v'$. Iterating $\xrightarrow[h]{}$, we get $H(u + v) = H(H(u) + H(v)) = H(u) + H(v)$. □

3 A confluence criterion

Definition 3.1. We say that R is *h-confluent* if for every rewrite rule $x \xrightarrow[R]{} v \in R$, we have $x - v \in \ker(H)$.

Example 3.2. Let us take the same example as in 2.3. Then $H(x) = H(y) = z + t$, with $H(z) = z = H(y - t)$, and so R is h-confluent. Replacing the rule $z \xrightarrow[R]{} y - t$ by $z \xrightarrow[R]{} y$, we get $H(z) = z$ and $H(y) = z + t$, so R is not h-confluent anymore.

Proposition 3.3. *If R is h-confluent, then $u \xleftrightarrow[R]{*} v$ if and only if $u - v \in \ker(H)$.*

Proof. The relation $\xleftrightarrow[R]{*}$ is the closure of $\xrightarrow[R]{}$ under transitivity, symmetry and sum. Since the relation $u - v \in \ker(H)$ is closed under these operations, we get one implication.

Reciprocally, if $u - v \in \ker(H)$ then by definition of H we have $u \xleftrightarrow[h]{*} v$, and in particular $u \xleftrightarrow[R]{*} v$. □

Proposition 3.4. *If R is h-confluent, then $\xrightarrow[R]{}$ is confluent.*

Proof. Let $v,\ v_1,\ v_2 \in \mathbb{K}X$ be such that $v \xrightarrow[R]{*} v_i$, for $i = 1,\ 2$. From Proposition 3.3, $v_1 - v_2$ belongs to $\ker(H)$, that is $H(v_1) = H(v_2)$. Denoting by u the common value, we get

$$\begin{tikzcd} v \arrow[r, "*"] \arrow[d, "*"'] & v_1 \arrow[d, "*"] \\ v_2 \arrow[r, "*"'] & u. \end{tikzcd}$$

□

In Theorem 3.7, we introduce a confluence criterion when R satisfies 3.1. For that, we assume that R is equipped with a well-founded order $\prec$ satisfying the following decreasingness property:

Definition 3.5. We say that R is *locally h-confluent* if for every non minimal $x \in X$ and $r = x \xrightarrow[R]{} v$, then letting $h_x = x \xrightarrow[h]{} r_x$, we have the confluence diagram:

$$\begin{tikzcd} x \arrow[r, "r"] \arrow[d, "h_x"'] & v \arrow[d, dotted, "*"] \\ r_x \arrow[r, dotted, "*"] & v', \end{tikzcd}$$

where each rewriting step occurring in the dotted arrows are strictly smaller than r for $\prec$.

Example 3.6. Continuing with Example 2.3, let us define an order $\prec$ on R as follows:

$$(x \xrightarrow[R]{} y), (y \xrightarrow[R]{} z + t) \prec (z \xrightarrow[R]{} y - t).$$

This is guided by the heuristic that rules advancing towards an h-normal form should be favored over rules that do not: here z is an h-normal form so the rule rewriting it is large for $\prec$. The following diagrams show that R is locally h-confluent:

$$\begin{tikzcd} x \arrow[r, "R"'] \arrow[d, "h_x"'] & y \arrow[d, equal] \\ y \arrow[r, equal] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "R"'] \arrow[d, "h_y"'] & z+t \arrow[d, equal] \\ z+t \arrow[r, equal] & z+t \end{tikzcd} \qquad \begin{tikzcd} z \arrow[r, "R"'] \arrow[d, equal, "h_z"'] & y-t \arrow[d, "R"'] \\ z \arrow[r, equal] & z \end{tikzcd}$$

Our main result is the following.

Theorem 3.7. *If R is locally h-confluent, then R is h-confluent. In particular, $\xrightarrow[R]{}$ is confluent.*

Proof. We reason by induction on r. Looking at the square corresponding to r:

$$\begin{tikzcd} x \arrow[r, "r"] \arrow[d, "h_x"'] & v \arrow[d, dotted, "*"] \\ r_x \arrow[r, dotted, "*"] & v', \end{tikzcd}$$

we have $H(x) = H(r_x)$ by definition of H, and $H(r_x) = H(v') = H(v)$ by induction hypothesis, which concludes the proof. $\square$

Let us show how the Diamond Lemma fits as a particular case of our set up.

Theorem 3.8 ([2])**.** *Assume that $\xrightarrow[R]{}$ is terminating and that for every $x \in X$, $x \xrightarrow[R]{} r$ and $x \xrightarrow[R]{} r' \in R$, r and r' are joinable. Then, $\xrightarrow[R]{}$ is confluent.*

Proof. We define the relation $x > y$ on X whenever $x \xrightarrow[R]{} v$ exists such that $y \in \mathrm{supp}(v)$. The induced order relation is well-founded. By definition, if $x \in X$ is not minimal for $>$, then x is not an R-normal form. Let us fix an arbitrary rewriting step $h_x = x \xrightarrow[h]{} r_x$. By definition of $>$, $r_x < x$ so that h is a local strategy. Ordering the rewrite rules by their left hand sides makes R locally h-confluent. Theorem 3.7 finally shows that R is confluent. $\square$

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