Title

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Abstract

1 Introduction

2 Well-formed rewriting steps

We fix a commutative field $\mathbb{K}$ as well as a well-founded ordered set $(X, <)$. We denote by $\mathbb{K}X$ the vector space spanned by X: an element $v \in \mathbb{K}X$ is a finite formal linear combination of elements of X with coefficients in $\mathbb{K}$. In particular, for every $v \in \mathbb{K}X$, there exists a unique finite set $\mathrm{supp}(v) \subset X$, called the *support* of v, such that

$$v = \sum_{x \in \mathrm{supp}(v)} \lambda_x x \text{ and } x \in \mathrm{supp}(v) \Rightarrow \lambda_x \neq 0. \tag{1}$$

We denote by $\mathrm{supp}(v)^c = X \setminus \mathrm{supp}(v)$. The sum of $u = \sum \lambda_x x$ and $v = \sum \mu_x x$ equals $\sum(\lambda_x + \mu_x)x$ and the product of $\lambda \in \mathbb{K}$ by v equals $\sum(\lambda\lambda_x)x$. We extend the order $<$ into the multiset order, still written $<$, on $\mathbb{K}X$: we have $u < v$ if for every $x \in \mathrm{supp}(u) \cap \mathrm{supp}(v)^c$, there exists $y \in \mathrm{supp}(v) \cap \mathrm{supp}(u)^c$ such that $y > x$.

We fix a set $R \subseteq X \times \mathbb{K}X$ which represents rewrite rules of the form $x \underset{R}{\longrightarrow} r$. The set R induces the rewriting relation on $\mathbb{K}X$, still written $\underset{R}{\longrightarrow}$, defined as follows:

$$\sum \lambda_x x + v \underset{R}{\longrightarrow} \sum \lambda_x r_x + v, \tag{2}$$

whenever $\lambda_x \neq 0$, $x \underset{R}{\longrightarrow} r_x \in R$ and $x \notin \mathrm{supp}(v)$. We assume that for every $x \in X$, not minimal for $<$, there exists $x \underset{R}{\longrightarrow} r \in R$ such that $r < x$. We choose such a rule h_x for every non-minimal x. Any vector v can be decomposed in a unique way as $\sum \lambda_x x + v'$, where $y \in \mathrm{supp}(v')$ implies that y is minimal for $<$, and $x \in \mathrm{supp}(v) \cap \mathrm{supp}(v')^c$ is not. We define a rewriting relation $\underset{h}{\longrightarrow}$ as follows:

$$\sum \lambda_x x + v' \underset{h}{\longrightarrow} \sum \lambda_x r_x + v', \tag{3}$$

where for every x, $h_x = x \underset{R}{\longrightarrow} r_x$.

Definition 2.1. A vector v is said to be a *h-normal form* if it is a normal form for $\underset{h}{\longrightarrow}$.

Example 2.2. We let $X = \{x, y, z, t\}$, $x \xrightarrow[R]{} z$, $x \xrightarrow[R]{} y$, $y \xrightarrow[R]{} z$ and $z \xrightarrow[R]{} t$. We choose the order $x > z$ $y > z$, $h_x = x \xrightarrow[h]{} z$ and $h_y = y \xrightarrow[h]{} z$. Then z is a h-normal form but is not a normal form for $\xrightarrow[R]{}$.

Lemma 2.3. *Let v be a vector in $\mathbb{K}X$. Either v is minimal for $<$, or there exists $v' < v$ such that $v \xrightarrow[h]{} v'$. In particular, h-normal forms are precisely the minimal elements of $\mathbb{K}X$ for $<$.*

For each $v \in \mathbb{K}X$, there exists at most one v' such that $v \xrightarrow[h]{} v'$, and $\xrightarrow[h]{}$ is compatible with the termination order $<$. As a consequence, any $v \in \mathbb{K}X$ is sent by multiple applications of $\xrightarrow[h]{}$ to a unique h-normal form that we denote by $H(v)$. This defines a map $H : \mathbb{K}X \to \mathbb{K}X$.

Proposition 2.4. *The map H is a linear projector.*

Proof. The h-normal forms are closed under sums, so that $H(H(v)) = H(v)$ for every v, that is H is a projector. Moreover, if $u \xrightarrow[h]{} u'$ and $v \xrightarrow[h]{} v'$, then we have $u + v \xrightarrow[h]{} u' + v'$. Iterating $\xrightarrow[h]{}$, we get $H(u + v) = H(H(u) + H(v)) = H(u) + H(v)$. $\square$

3 A confluence criterion

In this section, we assume that R satisfies the following property:

Definition 3.1. We say that R is h-confluent if for every rewrite rule $x \xrightarrow[R]{} v \in R$, we have $x - v \in \ker(H)$.

Example 3.2. We let $X = \{x, y, z, t\}$, $x \xrightarrow[R]{} z$, $x \xrightarrow[R]{} y$ and $y \xrightarrow[R]{} z$. We choose the order $x > z$ $y > z$, $h_x = x \xrightarrow[h]{} z$ and $h_y = y \xrightarrow[h]{} z$. We have $H(x) = H(y) = H(z) = z$, so that R is h-confluent. However, if we had the rewrite rule $z \xrightarrow[R]{} t$ as in Example 2.2, we have $H(z) = z$ and $H(t) = t$, so that R is not h-confluent.

Proposition 3.3. *If R is h-confluent, then $u \overset{*}{\underset{R}{\longleftrightarrow}} v$ if and only if $u - v \in \ker(H)$.*

Proof. The relation $\overset{*}{\underset{R}{\longleftrightarrow}}$ is the closure of $\xrightarrow[R]{}$ under transitivity, symmetry and sum. Since the relation $u - v \in \ker(H)$ is closed under these operations, we get one implication.

Reciprocally, if $u - v \in \ker(H)$ then by definition of H we have $u \overset{*}{\underset{h}{\longleftrightarrow}} v$, and in particular $u \overset{*}{\underset{R}{\longleftrightarrow}} v$. $\square$

Proposition 3.4. *If R is h-confluent, then $\xrightarrow[R]{}$ is confluent.*

Proof. Let $v,\ v_1,\ v_2 \in \mathbb{K}X$ be such that $v \xrightarrow[R]{*} v_i$, for $i = 1,\ 2$. From Proposition 3.3, $v_1 - v_2$ belongs to $\ker(H)$, that is $H(v_1) = H(v_2)$. Hence, we get

$$\begin{tikzcd}
 & v_1 \arrow[dr, bend left, "*"] & \\
v \arrow[ur, bend left, "*"] \arrow[dr, bend right, "*"'] & & H(v_i) \\
 & v_2 \arrow[ur, bend right, "*"'] &
\end{tikzcd}$$

□

In Theorem 3.6, we introduce a confluence criterion when R satisfies 3.1. For that, we assume that R is equipped with a well-founded order $\prec$ satisfying the following decreasingness property:

Definition 3.5. We say that R is *locally h-confluent* if for every non minimal $x \in X$ and $r = x \underset{R}{\longrightarrow} v$, then letting $h_x = x \underset{h}{\longrightarrow} r_x$, we have the confluence diagram:

$$\begin{tikzcd}
 & r_x \arrow[dr, dotted, "*"] & \\
x \arrow[ur, bend left, "h_x"] \arrow[dr, bend right, "r"'] & & v' \\
 & v \arrow[ur, dotted, "*"'] &
\end{tikzcd}$$

where each rewriting step occurring in the dotted arrows are strictly smaller than r for $\prec$.

Theorem 3.6. *If R is locally h-confluent, then R is h-confluent. In particular, $\underset{R}{\longrightarrow}$ is confluent.*

Proof. Adapter le cas ensembliste. □