

Title

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Abstract

We study the confluence property for rewriting systems whose underlying set of terms admits a vector space structure. For that, we use deterministic reduction strategies. These strategies are based on the choice of standard reductions applied to basis elements. We provide a sufficient condition of confluence in terms of the kernel of the operator which computes standard normal forms. We present a local criterion which enables us to check the confluence property in this framework. We show how this criterion is related to the Diamond Lemma for terminating rewriting systems.

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	plan:	
	• Section I: IWC theorem	
	• Section 2: rew on rationnal Weyl algebras	
	– definition of rew. rules and rew. steps	
	– proposition: convergent implies general form of solution to PDE	
	– Janet bases: Janet complete implies existence of a strategy, passivity implies h -confluence criterion, as a consequence we recover formal solutions to PDE	
	– $y' = xy$ and Janet example	

Example $y' = xy$, main steps:

- general solution is given by $\langle u \mid x^{2n+1} \rangle = 0$ and $\langle u \mid x^{2n} \rangle = u(0)/(2^n n!)$
- we recover this by rew: we need $(\partial_x)^n x = x(\partial_x)^n + n(\partial_x)^{n-1}$ (using Leibniz identity and induction) and $\langle u \mid x^n \rangle = 1/(n!)(\partial_x)^n(u)|_0$ and we prove the previous two formulas by induction (use this example as a running example?)

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1 Introduction

The fact that local confluence together with termination implies confluence has been known for abstract rewriting systems since Newman's work [10]. For rewriting on noncommutative polynomials, a similar result known as the Diamond lemma was introduced by Bergman [3] nearly 30 years later, in order to compute normal forms in noncommutative algebras using rewriting theory. It asserts that for terminating rewriting systems, the local confluence property can be checked on monomials.

One difficulty of rewriting polynomials is that the naive notion of rewriting path (obtained as the closure of the generating rewriting relations under reflexivity, transitivity, sum and product by a scalar) does not terminate. Instead, one needs to first consider well-formed rewriting steps before forming the reflexive transitive closure.

Nevertheless the Diamond lemma has proved to be very useful: together with the works of Bokut [4], it has given birth the theory of noncommutative Gröbner bases [9]. The latter have provided applications to various areas of noncommutative algebra such as the study of embedding problems (which appear in the works of Bokut and Bergman), homological algebra [6, 7] or Koszul duality [2, 11].

Computation of normal forms in noncommutative algebra is also used to provide formal solutions to partial differential equations. In this framework, a confluence criterion analogous to the Diamond Lemma is given by Janet bases [12], which specify a deterministic way to reduce each polynomial into normal form using standard reductions [8]. The confluence criterion may then be asserted as follows: for each monomial m and each non-standard reduction $m \rightarrow f$, f is reducible into \hat{m} , where the latter is obtained from m using only standard reductions.

In the presented paper, we propose an extension of the Diamond Lemma which offers two improvements over the one from Bergman: first it allows the treatment of non-terminating rewriting relations, and second it does not presuppose a notion of well-formed rewriting steps. This last property seems particularly promising in order to extend the Diamond Lemma to other structures.

Instead of supposing that the rewriting relation studied is terminating, we suppose given an ordering on the monomials, independent of the rewriting relation. We then use methods based on standard reductions: for every monomial m , we select exactly one reduction with left-hand side m , which is decreasing for the ordering chosen. Such choices induce a deterministic way to reduce each polynomial, obtained by applying simultaneously standard reductions on every monomial appearing in its decomposition. When these deterministic reductions terminate, one defines an operator which maps every polynomial to its unique standard normal form.

From this operator, we define a suitable notion of confluence in our setting, and show in Proposition ?? that it implies the usual notion of confluence for the rewriting system studied. We then provide an effective method for checking this criterion in Theorem 2.10. This method is based on a local analysis corresponding to checking local confluence on monomials. In particular, when the rewriting system is terminating, we show (Theorem 2.12) that we recover the Diamond Lemma as a particular case of Theorem 2.10.

Conventions and notations. Throughout the paper, we fix some conventions and notations that we present now. We will always work with a tuple $(\mathbb{K}, V, \mathcal{B}, <)$, where

- \mathbb{K} is a field
- V is a \mathbb{K} -vector space and \mathcal{B} is a basis of V
- $<$ is a well-founded order over \mathcal{B} .

In the sequel, such a tuple is called a **data**. Given a **data**, we simply say ground field, vectors and basis elements for \mathbb{K} , elements of V and \mathcal{B} , respectively. Moreover, every vector v admits a unique finite decomposition with respect to the basis \mathcal{B} and coefficients in the ground field:

$$v = \sum \lambda_i e_i, \quad \lambda_i \neq 0. \quad (1)$$

The set of basis elements which appear in the decomposition (1) is called the *support* of v , and is written $\text{supp}(v)$. We extend $<$ into the multiset order on V : for any $u, v \in V$, we have $u < v$ if $\text{supp}(u) \neq \text{supp}(v)$ and for any $e \in \text{supp}(u) \setminus \text{supp}(v)$, there exists $e' \in \text{supp}(v) \setminus \text{supp}(u)$ such that $e' > e$. Notice that this order coincides with the initial one on basis elements, so that we can denote it by $<$ without ambiguity.

Finally, we use the standard terminology of rewriting theory, see [1]. **Compléter.**

2 A weak version of the Diamond Lemma

In this section, we fix a **data** $(\mathbb{K}, V, \mathcal{B}, <)$ as well as a subset R of $\mathcal{B} \times V$, whose elements are called rewriting rules. Such a rewriting rule is denoted $e \rightarrow_R r$, where e and r are the natural projections of this rule of R on \mathcal{B} and V , respectively. Our objective is to provide a confluence criterion for the rewriting system on V which extends R by linearity.

Throughout the section, an element of V which is minimal for the multiset order $<$ on V is simply called minimal.

2.1 Local strategies of linear extensions

In this section, we first introduce the notion of linear extension, that is the rewriting relation on V obtained by extending R by linearity. Then, we define local strategies for this linear extension, that we use in Section ?? for introducing our confluence criterion.

The linear extension of R enables us to reduce many basis elements at once. Hence, we consider elementary reductions of the following form:

$$\sum_i \lambda_i e_i + v \rightarrow \sum_i \lambda_i r_i + v, \quad (2)$$

where v is a vector, λ_i is a scalar and $e_i \rightarrow_R r_i$ are rewriting rules.

Definition 2.1. The *linear extension* of R is the rewriting relation on V defined such as in (2), where each λ_i is nonzero and each e_i does not belong to $\text{supp}(v)$.

The linear extension of R is still denoted by \rightarrow_R , and its closures under transitivity, symmetry and sum and transitivity, symmetry, reflexivity and sum are denoted by $\xrightarrow{*}_R$ and \leftrightarrow_R^* , respectively. Finally, a normal form for \rightarrow_R is called an R -normal form.

In the following definition, we introduce the notion of local strategy, which is the selection of exactly one rewriting rule for each non minimal basis element. Moreover, the selected rewriting rule as to be compatible with the order $<$.

Definition 2.2. A *local strategy* is a subset S of R such that the following two conditions hold:

- the left-hand sides of elements of S are not minimal basis elements,

- for every non minimal basis element e , there exists exactly one rewriting rule $e \rightarrow_R r_e$ in S , moreover we have $r_e < e$.

Notice that a local strategy may not exist. The elements of such a local strategy S are denoted $e \rightarrow_{\hat{S}} r_e$ ¹. Moreover, instead of working with the full linear extension of S , we consider the rewriting relation, also denoted by $\rightarrow_{\hat{S}}$, which is defined such as the linear extension of S with the additional property that there is no rewriting rule $e \rightarrow_{\hat{S}} r_e$ such that $e \in \text{supp}(v)$. In other words, a vector u admits a unique decomposition

$$u = \sum_i \lambda_i e_i + v,$$

where $\lambda_i \neq 0$, $e' \in \text{supp}(v)$ implies that e' is minimal, and $e_i \in \text{supp}(u) \setminus \text{supp}(v)$ is not, and if $e_i \rightarrow_{\hat{S}} r_i$ are elements of the local strategy, we have

$$u \rightarrow_{\hat{S}} \sum_i \lambda_i r_i + v.$$

The closure of $\rightarrow_{\hat{S}}$ under transitivity and symmetry is denoted by $\xrightarrow{*}_{\hat{S}}$. Notice that $u \xrightarrow{*}_{\hat{S}} v$ implies $u \xrightarrow{*}_R v$. Moreover, a normal form for $\rightarrow_{\hat{S}}$ is called an S -normal form. In particular, every minimal basis element is a S -normal form. More generally, S -normal forms can be easily characterised as follows.

Lemma 2.3. *Given a vector v , either v is minimal, or there exists $v' < v$ such that $v \rightarrow_{\hat{S}} v'$. In particular, S -normal forms are precisely the minimal elements of V .*

In order to illustrate our notions, we consider the following running example.

Example 2.4. Let $\mathcal{B} = \{a, b, c, d\}$ and consider the set R composed of the following rewriting rules:

$$a \rightarrow_R b, \quad b \rightarrow_R c + d, \quad c \rightarrow_R b - d.$$

Note that the linear extension of R is not terminating since there is a rewriting loop due to $b \rightarrow_R c + d$ and $c + d \rightarrow_R (b - d) + d = b$. Moreover, we choose the order such that d is minimal and other basis elements are not comparable:

$$a > d, \quad b > d, \quad c > d.$$

Finally, as a local strategy, we select S defined by:

$$a \rightarrow_{\hat{S}} b, \quad b \rightarrow_{\hat{S}} c + d.$$

Then, the R -normal forms are the elements of the form $\lambda_d d$, while the h -normal forms are all the expressions of the form $\lambda_d d + \lambda_c c$.

We finish this section by introducing the normalisation operator associated with a local strategy S . By definition of such a strategy, for every $v \in V$, there exists at most one v' such that $v \rightarrow_{\hat{S}} v'$. Moreover, since $\rightarrow_{\hat{S}}$ is compatible with the well-founded order $<$, v is sent by multiple applications of $\rightarrow_{\hat{S}}$ to a unique S -normal form that we denote by $\text{NF}(v)$. This defines a map $\text{NF} : V \rightarrow V$, that is fundamental for our confluence criterion of Section ??.

¹The hat on S is here to mark the difference between the local strategy and rewriting rules.

Proposition 2.5. *The map NF is a linear projector.*

Proof. The S -normal forms are closed under sum, so that $\text{NF}(\text{NF}(v)) = \text{NF}(v)$ for every v , that is NF is a projector. Moreover, if $u \rightarrow_{\hat{S}} u'$ and $v \rightarrow_{\hat{S}} v'$, then we have $u + v \rightarrow_{\hat{S}} u' + v'$. Iterating $\rightarrow_{\hat{S}}$, we get $\text{NF}(u + v) = \text{NF}(\text{NF}(u) + \text{NF}(v)) = \text{NF}(u) + \text{NF}(v)$, which proves linearity of NF . \square

2.2 Confluence relative to a strategy

In this section, we first introduce S -confluence and show that this property implies confluence of \rightarrow_R . Then, we introduce local S -confluence which turns out to imply S -confluence: our confluence criterion follows, and we prove it Theorem 2.10. We finish this section by a new proof of the Diamond Lemma, based on local S -confluence.

Definition 2.6. Given a local strategy S , we say that \rightarrow_R is *S -confluent* if for every rewriting rule $e \rightarrow_{\hat{S}} r_e$, we have $\text{NF}(e - r_e) = 0$.

In the following proposition, we show that S -confluence implies confluence of \rightarrow_R .

Proposition 2.7. *Let S be a local strategy such that \rightarrow_R is S -confluent. We have $u \leftrightarrow_R^* v$ if and only if $\text{NF}(u - v) = 0$. Moreover, \rightarrow_R is confluent.*

Proof. We show the first assertion. The relation \leftrightarrow_R^* is the closure of \rightarrow_R under transitivity, symmetry and sum. Since the relation $\text{NF}(u - v) = 0$ is closed under these operations, we get one implication. Reciprocally, if $\text{NF}(u - v) = 0$, we have $\text{NF}(u) = \text{NF}(v)$. Moreover, by definition of NF , we have $u \xrightarrow{\hat{S}} \text{NF}(u)$ and $v \xrightarrow{\hat{S}} \text{NF}(v)$, and since $v_1 \xrightarrow{\hat{S}} v_2$ implies $v_1 \xrightarrow{R} v_2$, we get $u \leftrightarrow_R v$. For the second assertion, let us consider three vectors $v, v_1, v_2 \in V$ such that for $i = 1, 2$, we have $v \xrightarrow{R} v_i$. By the first part of the proposition, we have $\text{NF}(v_1) = \text{NF}(v_2)$. From $v_i \xrightarrow{\hat{S}} \text{NF}(v_i)$, we get $v_i \xrightarrow{R} \text{NF}(v_i)$, so that \rightarrow_R is confluent. \square

Note that the previous proposition is a sufficient but not a necessary condition. Indeed, with \mathcal{B} the set of integers and the rewriting rules $n \rightarrow_R n + 1$, the linear extension of R is confluent, but there is no local strategy such that \rightarrow_R is confluent relative to this strategy.

Example 2.8. Let us continue Example 2.4. The following identities hold:

$$\text{NF}(a) = c + d = \text{NF}(b), \quad \text{NF}(b) = c + d = \text{NF}(c + d), \quad \text{NF}(c) = c = \text{NF}(b - d),$$

so that \rightarrow_R is S -confluent, and hence confluent. Notice that if we replace the rule $c \rightarrow_R b - d$ by $c \rightarrow_R b$, we get $\text{NF}(c) = c$ and $\text{NF}(b) = c + d$, so \rightarrow_R is not S -confluent anymore.

Now, we introduce local S -confluence, which is a decreasingness property with respect to a well-founded order on rewriting rules.

Definition 2.9. Let \prec be a well-founded order on R and let S be a local strategy. We say that \rightarrow_R is *locally S -confluent with respect to \prec* if for every $e \in \mathcal{B}$ and for every rewriting rule $e \rightarrow_R r$, we have a confluence diagram:

$$\begin{array}{ccc} e & \xrightarrow{R} & r \\ \hat{S} \downarrow & & \downarrow \hat{*} \\ r_e & \xleftarrow{\hat{*}} & u \end{array}$$

where each rewriting step occurring in the dotted arrows belongs to R (we removed the subscript R for simplicity) and is strictly smaller than the rewriting rule $e \rightarrow_R r$ relative to \prec .

The main result of this section is the following.

Theorem 2.10. *Let R be a set of rewriting rules. Assume that there exists a local strategy S and a well-founded order \prec on R such that \rightarrow_R is locally S -confluent with respect to \prec . Then, \rightarrow_R is confluent.*

Proof. It is sufficient to show that \rightarrow_R is S -confluent.

We reason by induction on rewriting rules f according to the order \prec . Looking at the square corresponding to f :

$$\begin{array}{ccc} e & \xrightarrow{f} & v \\ s_e \downarrow & & \uparrow^* \\ r_e & \xrightarrow[*]{} & v' \end{array}$$

we have $H(e) = H(r_e)$ by definition of H , and $H(r_e) = H(v') = H(v)$ by induction hypothesis, which concludes the proof. **Ce n'est pas hyper clair pour moi. Peux-tu reprendre la fin de la preuve?.** \square

Local h -confluence implies that the pair of rewriting relations $(\rightarrow_{\hat{S}}, \rightarrow_R)$ is decreasing with respect to conversions (see [13, Definition 3]), using the order \prec on R and the discrete ordering on $\rightarrow_{\hat{S}}$. By [13, Theorem 3], this implies that $(\rightarrow_{\hat{S}}, \rightarrow_R)$ commute. Using the fact that $\rightarrow_{\hat{S}} \subseteq \rightarrow_R$, one can then recover that \rightarrow_R is confluent. **Rajouter un phrase au début faisant comprendre que cette remarque s'adresse à des spécialistes de réécriture.**

Example 2.11. Using local S -confluence, we way recover that the rewriting relation of Example 2.4 is confluent. Indeed, let us consider the following order \prec on rewriting rules:

$$(a \rightarrow_R b) \prec (c \rightarrow_R b - d), \quad (b \rightarrow_R c + d) \prec (c \rightarrow_R b - d).$$

This choice is guided by the heuristic that rules advancing towards an S -normal form should be favored over rules that do not: here c is an S -normal form so its rewriting rule is for \prec . The following diagrams show that \rightarrow_R is locally S -confluent:

$$\begin{array}{ccc} a \xrightarrow{R} b & b \xrightarrow{R} c + d & c \xrightarrow{R} b - d \\ \hat{S} \downarrow & \hat{S} \downarrow & \hat{S} \downarrow \\ b = b & c + d = c + d & c = c \end{array}$$

We finish this section by showing how the Diamond Lemma fits as a particular case of our setup.

Theorem 2.12 ([3]). *Let R be a set of rewriting rules such that \rightarrow_R is terminating, and for every $e \in \mathcal{B}$ such that $e \rightarrow_R r$ and $e \rightarrow_R r'$, r and r' are joinable. Then, \rightarrow_R is confluent.*

Proof. We define an ordering $e > e'$ on \mathcal{B} as the transitive closure of the relation “there exists a vector v such that $e \rightarrow_R v$ and $e' \in \text{supp}(v)$ ”. This is a well-founded order since \rightarrow_R is assumed to be terminating. Let us define the local strategy S as follows. First, notice that by definition of the order $<$, if the basis element e is not minimal, then it is not an R -normal form. Then, S is composed of elements $e \rightarrow_S r$, where $e \rightarrow_R r$ is an arbitrary rewriting rules reducing e . It remains to define \prec : we let $(e \rightarrow_R r) \prec (e' \rightarrow_R r')$ if $e < e'$, which is well-founded since $<$ is. Finally, the hypothesis r and r' are joinable when $e \rightarrow_R r$ and $e \rightarrow_R r'$ implies that \rightarrow_R is locally S -confluent with respect to \prec . Hence, by Theorem 2.10, \rightarrow_R is confluent. \square

3 Rewriting and partial derivative equations

3.1 Rewriting systems on rational Weyl algebras

In this section, we introduce rewriting systems in rational Weyl algebras, and use them to compute formal solutions to systems of partial derivative equations. For that, we need to define a suitable **data** on rational Weyl algebras.

We first recall the definitions of rational Weyl algebras as well as of monomial orders. We fix a set $X = \{x_1, \dots, x_n\}$ of indeterminates and we denote by $\mathbb{Q}(X) = \mathbb{Q}(x_1, \dots, x_n)$ the field of fractions of the polynomial algebra $\mathbb{Q}[x_1, \dots, x_n]$ over \mathbb{Q} . Let us introduce another set of variables $\Delta = \{\partial_1, \dots, \partial_n\}$.

Definition 3.1. The *rational Weyl algebra* over $\mathbb{Q}(X)$ is the set of polynomials over the indeterminates Δ and coefficients in $\mathbb{Q}(X)$, subject to the following commutation rule:

$$\partial_i f = f \partial_i + \frac{d}{dx_i}(f), \quad f \in \mathbb{Q}(X), \quad 1 \leq i \leq n,$$

where $d/dx_i : \mathbb{Q}(X) \rightarrow \mathbb{Q}(X)$ is the partial derivative operator with respect to the indeterminate x_i . This algebra is denoted by $B_n(\mathbb{Q})$.

Given $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, we denote by $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Notice that $B_n(\mathbb{Q})$ is a vector space on $\mathbb{Q}(X)$ and admits as a basis the set

$$\Delta^c = \{\partial^\alpha \mid \alpha \in \mathbb{N}^n\},$$

which is the free commutative monoid over Δ . Hence, elements of $B_n(\mathbb{Q})$ should be thought as differential operators with rational functions coefficients, and for this reason, a generic element of this algebra is denoted by \mathcal{D} and is called a differential operator. In the following example, we illustrate how these operators naturally appear in the study of linear systems of ordinary differential and partial derivative equations.

Example 3.2.

1. The ordinary differential equation $y'(x) = xy(x)$ is written in the form $\mathcal{D}y = 0$, where the operator $\mathcal{D} = \partial - x$ belongs to $B_1(\mathbb{Q}) = \mathbb{Q}(x)[\partial]$.
2. Consider the Janet example [5], that is the system of partial derivative equations with 3 variables and one unknown function: $y_{33} = x_2 y_{11}$ and $y_{22} = 0$, where y_{ij} denotes the second order derivative of the unknown function y with respect to the variables x_i and x_j . Then, these equations are written $\mathcal{D}_1 y = 0$ and $\mathcal{D}_2 y = 0$, where \mathcal{D}_1 and \mathcal{D}_2 are the following elements of $B_3(\mathbb{Q})$:

$$\mathcal{D}_1 = \partial_3^2 - x_2 \partial_1^2, \quad \mathcal{D}_2 = \partial_2^2.$$

Now, we recall the definition of monomial orders.

Definition 3.3. A *monomial order* \prec on \mathcal{B} is a well-founded total order that is compatible with multiplication of monomials, i.e., $\partial^\alpha \prec \partial^\beta$ implies $\partial^{\alpha+\gamma} \prec \partial^{\beta+\gamma}$, for every $\alpha, \beta, \gamma \in \mathbb{N}^n$. Moreover, the \prec -**data** on $B_n(\mathbb{Q})$ is the following **data**:

$$(\mathbb{Q}(X), B_n(\mathbb{Q}), \Delta^c, \prec).$$

Let \prec be a fixed monomial order and let \mathcal{D} be a differential operator. As usual, we denote by $\text{lm}(\mathcal{D})$ and $\text{lc}(\mathcal{D})$ the leading monomial and the leading coefficient of \mathcal{D} with respect to \prec , respectively, and we also let $r(\mathcal{D}) = \text{lc}(\mathcal{D})\text{lm}(\mathcal{D}) - \mathcal{D}$.

Now, we introduce the rewriting system on $B_n(\mathbb{Q})$ associated with a set $\Theta = \{\mathcal{D}_1, \dots, \mathcal{D}_r\}$ of differential operators, as being the linear extension of the following set of rewriting rules:

$$R_\Theta = \left\{ \text{lm}(\mathcal{D}_i) \rightarrow_{R_\Theta} r(\mathcal{D}_i) \mid 1 \leq i \leq r \right\}.$$

For simplicity, we write rewriting rules and the the linear extension of these rules $\mathcal{D} \rightarrow_\Theta \mathcal{D}'$ instead of $\mathcal{D} \rightarrow_{R_\Theta} \mathcal{D}'$.

Example 3.4. We continue Example 3.2.

1. Let $\mathcal{D} = \partial - x \in B_1(\mathbb{Q})$. Since ∂ is greater than 1 for every monomial order, the rewriting system associated with $\Theta = \{\mathcal{D}\}$ extends the rewriting rule $\partial \rightarrow_\Theta x$.
2. Extending $\partial_3 \succ \partial_2 \succ \partial_1$ into the deglex order, the rewriting system associated with the pair $\Theta = \{\mathcal{D}_1, \mathcal{D}_2\}$ of the two operators defined in Example 3.2 extend the two rewriting rules $\partial_3^2 \rightarrow_\Theta x_2 \partial_1^2$ and $\partial_2^2 \rightarrow_\Theta 0$.

3.2 Janet bases and h -confluence

In this section, we take $\mathbb{K} = \mathbb{R}(x_1, x_2, \dots, x_n)$ the field of real rational functions in n variables, and look at the \mathbb{K} -algebra $A = \mathbb{K}[\partial_1, \dots, \partial_n]$. We suppose given an ordering $\partial_n > \partial_{n-1} > \dots > \partial_1$. By monomial, we mean unitary monomial.

Example 3.5. Elements of A should be thought of as systems of linear partial derivative equations. For example, the equation $x_1 \frac{\partial u}{\partial x_1} - u = 0$ corresponds to the element $x_1 \partial_1 - 1$ in A .

Similarly, the 1-dimensional heat equation $\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$ corresponds (taking $x_1 = t$ and $x_2 = x$) to the element $\partial_1 - \kappa \partial_2^2$.

Definition 3.6. For any monomial m and $1 \leq k \leq n$ we denote by $\nu_k(m)$ the power of ∂_k in m . For any $\vec{j} = (j_1, \dots, j_n) \in \mathbb{N}^n$, we denote by $\partial^{\vec{j}}$ the monomial m such that $\nu_k(m) = j_k$ for all k . For any subset $E \subset \{1, \dots, n\}$, we denote by $\text{Mon}(E)$ the set of monomials m such that $\nu_k(m) = 0$ for all $k \notin E$. In particular, $\text{Mon}(\{1, \dots, n\})$ is the set of all monomials of A which we denote simply by Mon , while $\text{Mon}(\emptyset)$ is reduced to $\{1\}$.

Definition 3.7. A *cone* is a pair (m, E) , where $m \in A$ and $E \subset \{1, \dots, n\}$. m is called the *origin of the cone*, while E is the *direction*. A cone is *monomial* if m is a monomial. We denote by $(m, E)^*$ the family of monomials of the form mp , where $p \in \text{Mon}(E)$.

If S is a family of monomials, a *cone-partition* of S is a family of monomial cones (m_i, E_i) for $i \in I$ such that for $i \neq j$, $(m_i, E_i)^* \cap (m_j, E_j)^* = \emptyset$ and $S = \coprod_{i \in I} (m_i, E_i)^*$. In other words the $(m_i, E_i)^*$ form a partition of S into monomial cones. It is a *finite cone-partition* if I is finite.

Definition 3.8. A family of monomial cones (m_i, E_i) with distinct origins is said to be *complete* if it induces a cone-partition of the monomial ideal generated by the m_i s.

A family of cones (p_i, E_i) is said to be *complete* if $(lm(p_i), E_i)$ is complete.

Lemma 3.9. *A complete family of cones gives rise to a local strategy on A .*

Sketch. We use the monomial ordering on reducible monomials, and reducible ones are bigger than irreducible ones.

Then we only need to define h on reducible ones. For $x \in Red(R)$, there exists a unique (m_i, E_i) in the cone-partition such that $x = m_i p$ with $p \in E_i$ and m_i corresponds to an element $m_i + r$ of R . Then h is defined by $h_x = x = m_i p \rightarrow_R -rp$. \square

In order to produce suitable cone-partitions as in the previous lemma, we introduce the notion of multiplicative variables.

Definition 3.10. Let M be a subset of Mon , and let $m \in M$. We define a subset $\mu_M(m)$ of $\{1, \dots, n\}$. Let us write $m = \partial_1^{i_1} \partial_2^{i_2} \dots \partial_n^{i_n}$, and take $k \in \{1, \dots, n\}$. Then $k \in \mu_M(m)$ if the following implication is true:

$$\forall m' \in M, \quad (\forall j < k, \nu_j(m') = \nu_j(m)) \quad \Rightarrow \quad \nu_k(m') \leq \nu_k(m).$$

We call $(m, \mu_M(m))$ the cone of m in M . If M is clear then we will just write $\mu(m)$. If $k \in \mu(m)$ we will say that ∂_k is a *multiplicative variable* of m .

Lemma 3.11. *Let $M \subset Mon$ and $m, m' \in M$. If $m \neq m'$, then $(m, \mu(m))^* \cap (m', \mu(m'))^* = \emptyset$.*

Proof. Since $m \neq m'$, there exists k minimal such that $\nu_k(m) \neq \nu_k(m')$. Without loss of generality, we can suppose $\nu_k(m) < \nu_k(m')$. By definition of μ , we therefore have that $k \notin \mu(m)$.

Take now $p \in (m, \mu(m))^* \cap (m', \mu(m'))^*$. Then $p = mq$, with $\nu_k(q) = 0$, and so $\nu_k(p) = \nu_k(m)$. But p is a multiple of m' and so $\nu_k(m) = \nu_k(p) \geq \nu_k(m')$, which is contradictory. \square

Example 3.12. TODO.

Definition 3.13. A set of monomials M is *Janet-complete* if the family of all $(m, \mu(m))$ is complete.

A set of polynomials R is *Janet-complete* if $lm(R)$ is Janet-complete.

Conclusion. We introduced a sufficient condition, based on deterministic reduction strategies, of confluence for rewriting systems on vector spaces. As a particular case, we recover the Diamond Lemma. This work maybe extended in particular into two main directions. The first one consists in weakening our assumption on the set \mathbb{K} of coefficients, by allowing non invertible coefficients. A second extension consists in characterising Janet bases in this framework, with the objective to develop constructive methods in the analysis and formal resolution of PDE's.

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