Title

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Abstract

1 Introduction

2 Well-formed rewriting steps

We fix a commutative field $\mathbb{K}$ as well as a well-founded ordered set $(X, <)$. We denote by $\mathbb{K}X$ the vector space spanned by X: an element $v \in \mathbb{K}X$ is a finite formal linear combination of elements of X with coefficients in $\mathbb{K}$. In particular, for every $v \in \mathbb{K}X$, there exists a unique finite set $\operatorname{supp}(v) \subset X$, called the *support* of v, such that

$$v = \sum_{x \in \operatorname{supp}(v)} \lambda_x x \text{ and } x \in X \Rightarrow \lambda_x \neq 0. \tag{1}$$

We denote by $\operatorname{supp}(v)^c = X \setminus \operatorname{supp}(v)$. The sum of $u = \sum \lambda_x x$ and $v = \sum \mu_x x$ equals $\sum (\lambda_x + \mu_x) x$ and the product of $\lambda \in \mathbb{K}$ by v equals $\sum (\lambda \lambda_x) x$. We extend the order $<$ into the multiset order, still written $<$, on $\mathbb{K}X$: we have $u < v$ if for every $x \in \operatorname{supp}(u) \cap \operatorname{supp}(v)^c$, there exists $y \in \operatorname{supp}(v) \cap \operatorname{supp}(u)^c$ such that $y > x$.

We fix a set $R \subseteq X \times \mathbb{K}X$ which represents rewrite rules of the form $x \underset{R}{\longrightarrow} r$. The set R induces the rewriting relation on $\mathbb{K}X$, still written $\underset{R}{\longrightarrow}$, defined as follows:

$$\sum \lambda_x x + v \underset{R}{\longrightarrow} \sum \lambda_x r_x + v, \tag{2}$$

whenever $\lambda_x \neq 0$, $x \underset{R}{\longrightarrow} r_x \in R$ and $x \notin \operatorname{supp}(v)$. We assume that for every $x \in X$, not minimal for $<$, there exists $x \underset{R}{\longrightarrow} r \in R$ such that $r < x$. We choose such a rule h_x for every non-minimal x. Any vector v can be decomposed in a unique way as $\sum \lambda_x x + v'$, where $y \in \operatorname{supp}(v')$ implies that y is minimal for $<$, and $x \in \operatorname{supp}(v)$ is not. We define a rewriting relation $\underset{h}{\longrightarrow}$ as follows:

$$\sum \lambda_x x + v' \underset{h}{\longrightarrow} \sum \lambda_x r_x + v', \tag{3}$$

where for every x, $h_x = x \underset{R}{\longrightarrow} r_x$.

Definition 2.1. A vector v is said to be an *h-normal form* if it is a normal form for $\underset{h}{\longrightarrow}$.

Example 2.2. Donner un exemple et un contre-exemple.

Lemma 2.3. *Let v be a vector in $\mathbb{K}X$, and suppose $v \xrightarrow[h]{} v'$. Then, either v is minimal for $<$, or $v' > v$. In particular, h-normal forms are precisely the minimal elements of $\mathbb{K}X$ for $<$.*

For each $v \in \mathbb{K}X$, there exists at most one v' such that $v \xrightarrow[h]{} v'$, and $\xrightarrow[h]{}$ is compatible with the termination order $<$. As a consequence, any $v \in \mathbb{K}X$ is sent by multiple applications of $\xrightarrow[h]{}$ to a unique h-normal form that we denote by $H(v)$. This defines a map $H : \mathbb{K}X \to \mathbb{K}X$.

Proposition 2.4. *The map H is linear.*

Sketch of proof. TODO. □

3 A confluence criterion

In this section, we assume that R satisfies the following property:

Property 3.1. For every rewrite rule $x \xrightarrow[R]{} v \in R$, we have $x - v \in \ker(H)$.

Example 3.2. Donner un exemple et un contre-exemple.

Proposition 3.3. *Assume that Property 3.1 holds, then $u \overset{*}{\underset{R}{\longleftrightarrow}} v$ if and only if $u - v \in \ker(H)$.*

Sketch of proof. Il faut montrer que les gens qui vérifient la propriété de standardisation sont clos par les opérations somme, composition et inverse. □

In Theorem 3.5, we introduce a confluence criterion when R satisfies 3.1. For that, we assume that R is equipped with a well-founded order $\prec$ satisfying the following decreasingness property:

Property 3.4. For every $x \in X$ and $r = x \to v$, if x is not minimal for $<$, then letting $h_x = x \xrightarrow[R]{} r_x$, we have the confluence diagram

$$\begin{tikzcd}
 & r_x \arrow[dr, dotted, "*"] & \\
x \arrow[ur, bend left, "h_x"] \arrow[dr, bend right, "r"'] & & v' \\
 & v \arrow[ur, dotted, "*"'] &
\end{tikzcd}$$

where each rewriting step occuring in the dotted arrows are strictly smaller than r for $\prec$.

Theorem 3.5. *Assume that R satisfies Properties 3.1 and 3.4. Then $\xrightarrow[R]{}$ is confluent.*

Proof. Adapter le cas ensembliste. □