

Title

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Abstract

1 Introduction

2 Well-formed rewriting steps

We fix a commutative field \mathbb{K} as well as a well-founded ordered set $(X, <)$. We denote by $\mathbb{K}X$ the vector space spanned by X : an element $v \in \mathbb{K}X$ is a finite formal linear combination of elements of X with coefficients in \mathbb{K} . In particular, for every $v \in \mathbb{K}X$, there exists a unique finite set $\text{supp}(v) \subset X$, called the *support* of v , such that

$$v = \sum_{x \in \text{supp}(v)} \lambda_x x \text{ and } x \in X \Rightarrow \lambda_x \neq 0. \quad (1)$$

We denote by $\text{supp}(v)^c = X \setminus \text{supp}(v)$. The sum of $u = \sum \lambda_x x$ and $v = \sum \mu_x x$ equals $\sum (\lambda_x + \mu_x) x$ and the product of $\lambda \in \mathbb{K}$ by v equals $\sum (\lambda \lambda_x) x$. We extend the order $<$ into the multiset order, still written $<$, on $\mathbb{K}X$: we have $u < v$ if for every $x \in \text{supp}(u) \cap \text{supp}(v)^c$, there exists $y \in \text{supp}(v) \cap \text{supp}(u)^c$ such that $y > x$.

We fix a set $R \subseteq X \times \mathbb{K}X$ which represents rewrite rules of the form $x \xrightarrow{R} r$. The set R induces the rewriting relation on $\mathbb{K}X$, still written \xrightarrow{R} , defined as follows:

$$\sum \lambda_x x + v \xrightarrow{R} \sum \lambda_x r_x + v, \quad (2)$$

whenever $\lambda_x \neq 0$, $x \xrightarrow{R} r_x \in R$ and $x \notin \text{supp}(v)$. We assume that for every $x \in X$, not minimal for $<$, there exists $x \xrightarrow{R} r \in R$ such that $r < x$. We choose such a rule h_x for every non-minimal x . Any vector v can be decomposed in a unique way as $\sum \lambda_x x + v'$, where $y \in \text{supp}(v')$ implies that y is minimal for $<$, and $x \in \text{supp}(v)$ is not. We define a rewriting relation \xrightarrow{h} as follows:

$$\sum \lambda_x x + v' \xrightarrow{h} \sum \lambda_x r_x + v', \quad (3)$$

where for every x , $h_x = x \xrightarrow{R} r_x$.

Definition 2.1. A vector v is said to be an *h-normal form* if it is a normal form for \xrightarrow{h} .

Example 2.2. Donner un exemple et un contre-exemple.

Lemma 2.3. *Let v be a vector in $\mathbb{K}X$, and suppose $v \xrightarrow[h]{} v'$. Then, either v is minimal for $<$, or $v' > v$. In particular, h -normal forms are precisely the minimal elements of $\mathbb{K}X$ for $<$.*

For each $v \in \mathbb{K}X$, there exists at most one v' such that $v \xrightarrow[h]{} v'$, and $\xrightarrow[h]{}$ is compatible with the termination order $<$. As a consequence, any $v \in \mathbb{K}X$ is sent by multiple applications of $\xrightarrow[h]{}$ to a unique h -normal form that we denote by $H(v)$. This defines a map $H : \mathbb{K}X \rightarrow \mathbb{K}X$.

Proposition 2.4. *The map H is linear.*

Sketch of proof. **TODO.**

□

3 A confluence criterion

In this section, we assume that R satisfies the following property:

Property 3.1. For every rewrite rule $x \xrightarrow[R]{} v \in R$, we have $x - v \in \ker(H)$.

Example 3.2. **Donner un exemple et un contre-exemple.**

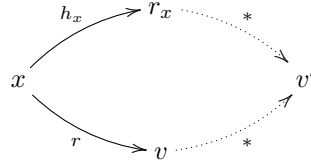
Proposition 3.3. *Assume that Property 3.1 holds, then $u \xleftarrow[R]{*} v$ if and only if $u - v \in \ker(H)$.*

Sketch of proof. **Il faut montrer que les gens qui vérifient la propriété de standardisation sont clos par les opérations somme, composition et inverse.**

□

In Theorem 3.5, we introduce a confluence criterion when R satisfies 3.1. For that, we assume that R is equipped with a well-founded order \prec satisfying the following decreasingness property:

Property 3.4. For every $x \in X$ and $r = x \rightarrow v$, if x is not minimal for $<$, then letting $h_x = x \xrightarrow[R]{} r_x$, we have the confluence diagram



where each rewriting step occuring in the dotted arrows are strictly smaller than r for \prec .

Theorem 3.5. *Assume that R satisfies Properties 3.1 and 3.4. Then $\xrightarrow[R]{}$ is confluent.*

Proof. **Adapter le cas ensembliste.**

□