Title

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Abstract

1 Introduction

2 Well-formed rewriting steps

We fix a commutative field $\mathbb{K}$ as well as a well-founded ordered set $(X, <)$. We denote by $\mathbb{K}X$ the vector space spanned by X: an element $v \in \mathbb{K}X$ is a finite formal linear combination of elements of X with coefficients in $\mathbb{K}$. The sum of $u = \sum \lambda_x x$ and $v = \sum \mu_x x$ is equal to $\sum (\lambda_x + \mu_x)x$ and the product of $\lambda \in \mathbb{K}$ by v is equal to $\sum (\lambda\lambda_x)x$. We extend the order $<$ into the multiset order, still written $<$, on $\mathbb{K}X$: we have $u < v$ if for every element $x \in X$ which occures in the decomposition of u but not in the one of v, there exists $y > x$ occuring in the decomposition of v but not in the one of u.

We fix a set $R \subseteq X \times \mathbb{K}X$ which represents rewrite rules of the form $x \underset{R}{\longrightarrow} v$. The set R induces the rewriting relation on $\mathbb{K}X$, still written $\underset{R}{\longrightarrow}$, defined as follows:

$$\sum \lambda_x x + u' \underset{R}{\longrightarrow} \sum \lambda_x r_x + u', \tag{1}$$

whenever $\lambda_x \neq 0$ and $x \underset{R}{\longrightarrow} r_x$ are rewrite rules. We assume that for every $x \in X$, not minimal for $<$, there exists a rewrite rule in R with left-hand side x. We choose such a rule h_x for every non-minimal x.

Definition 2.1. A rewriting step as in (1) is said to be *well-formed* if each rewrite rule $x \underset{R}{\longrightarrow} r_x$ is equal to h_x and if there does not exist any minimal y in the decomposition of u'. A vector $v \in \mathbb{K}X$ is a *well-formed normal form* if there is no well-formed rewriting step with left-hand side v.

Example 2.2. Donner un exemple et un contre-exemple.

For each $v \in \mathbb{K}X$, there exists at most one well-formed rewriting step with left-hand side v. Moreover, the well-formed rewriting steps are compatible with the termination order, so that we have a well-defined map $H : \mathbb{K}X \to \mathbb{K}X$, such that v rewrites into the unique well-formed normal form $H(v)$ using only well-formed rewriting steps.

Proposition 2.3. *The map H is linear.*

Sketch of proof. TODO. ∎

3 A confluence criterion

In this section, we assume that R satisfies the following property:

Property 3.1. For every rewrite rule $x \xrightarrow[R]{} v \in R$, we have $x - v \in \ker(H)$.

Example 3.2. Donner un exemple et un contre-exemple.

Proposition 3.3. *Assume that Property 3.1 holds, then $u \xleftrightarrow[R]{*} v$ if and only if $u - v \in \ker(H)$.*

Sketch of proof. Il faut montrer que les gens qui vérifient la propriété de standardisation sont clos par les opérations somme, composition et inverse. □

In Theorem 3.5, we introduce a confluence criterion when R satisfies 3.1. For that, we assume that R is equipped with a well-founded order $\prec$ satisfying the following decreasingness property:

Property 3.4. For every $x \in X$ and $r = x \to v$, if x is not minimal for $<$, then letting $h_x = x \xrightarrow[R]{} r_x$, we have the confluence diagram

$$\begin{tikzcd}
& r_x \arrow[dr, dotted, bend left, "*"] & \\
x \arrow[ur, bend left, "h_x"] \arrow[dr, bend right, "r"'] & & v' \\
& v \arrow[ur, dotted, bend right, "*"'] &
\end{tikzcd}$$

where each rewriting step occuring in the dotted arrows are strictly smaller than r for $\prec$.

Theorem 3.5. *Assume that R satisfies Properties 3.1 and 3.4. Then $\xrightarrow[R]{}$ is confluent.*

Proof. Adapter le cas ensembliste. □