1 Preliminaries

Notations. We fix the following notations:

- given a positive integer $n \in \mathbb{N} \setminus \{0\}$, we let $\overline{n} := n - 1$.
- e_i^n denotes the i-th vector of the canonical basis of the free module $\mathbf{D}^{1\times n}$.
- We denote by M^T the transpose matrix of M,
- $p,\ p',\ q$ and q' denote fixed integers, and we let $m := p + p'$ and $n := q + p' + p + q'$.

Effective Fitting's Theorem. Let $\mathbf{D}$ be a ring, $R \in \mathbf{D}^{p\times q}$, $R \in \mathbf{D}^{p'\times q'}$ be two matrices and $\mathbf{M}$ and $\mathbf{M}'$ be the two left modules finitely presented by R and R', respectively:

$$\mathbf{D}^{1\times q} \xrightarrow{\quad R \quad} \mathbf{D}^{1\times p} \longrightarrow \mathbf{M} \longrightarrow 0$$

$$\mathbf{D}^{1\times q'} \xrightarrow[\quad R' \quad]{} \mathbf{D}^{1\times p'} \longrightarrow \mathbf{M}' \longrightarrow 0$$

Assume that there exists an isomorphism $\varphi : \mathbf{M} \to \mathbf{M}'$, so that their exist matrices $P \in \mathbf{D}^{p\times p'}$, $P' \in \mathbf{D}^{p'\times p}$, $Q \in \mathbf{D}^{q\times q'}$ and $Q' \in \mathbf{D}^{q'\times q}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbf{D}^{1\times q} \arrow[r, "R"] \arrow[d, shift left, "Q"] & \mathbf{D}^{1\times p} \arrow[r] \arrow[d, shift left, "P"] & \mathbf{M} \arrow[r] \arrow[d, shift left, "\varphi"] & 0 \\
\mathbf{D}^{1\times q'} \arrow[r, "R'"'] \arrow[u, shift left, "Q'"] & \mathbf{D}^{1\times p'} \arrow[r] \arrow[u, shift left, "P'"] & \mathbf{M}' \arrow[r] \arrow[u, shift left, "\varphi^-"] & 0
\end{tikzcd}$$

Let us consider the two matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \in \mathbf{D}^{(q+p'+p+q')\times(p+p')}$$

Recall from [1] that their exist matrices $Z \in \mathbf{D}^{p\times q}$ $Z' \in \mathbf{D}^{p'\times q'}$, $R_2 \in \mathbf{D}^{r\times q}$, $R_2' \in \mathbf{D}^{r'\times q'}$, $Z_2 \in \mathbf{D}^{q\times r}$ and $Z_2' \in \mathbf{D}^{q'\times r'}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{D}^{1\times(q+p'+p+q')} \arrow[r, "L"] \arrow[d, shift left, "Y_F"] & \mathbf{D}^{1\times(p+p')} \arrow[r] \arrow[d, shift left, "X_F"] & \mathbf{M} \arrow[r] \arrow[d, shift left, "\varphi"] & 0 \\
\mathbf{D}^{1\times(q+p'+p+q')} \arrow[r, "L'"'] \arrow[u, shift left, "Y_F^-"] & \mathbf{D}^{1\times(p+p')} \arrow[r] \arrow[u, shift left, "X_F^-"] & \mathbf{M}' \arrow[r] \arrow[u, shift left, "\varphi^-"] & 0
\end{tikzcd}$$

where

$$X_F := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix}, \quad Y_F := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z_2'R_2' \end{pmatrix}$$

$$X_F^- := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix}, \quad Y_F^- := \begin{pmatrix} Z_2R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix}.$$

2 Reduction of the zero bloc

Assume that $q + p' \leq p + q'$ and $\mathrm{sr}\,(\mathbf{D}) \leq \overline{p + q'}$. We let

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{D}^{\overline{n} \times m} \text{ and } \tilde{L}' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix} \in \mathbf{D}^{\overline{n} \times m}.$$

Proposition 1. *There exist lines* $\mathbf{c},\ \mathbf{u} \in \mathbf{D}^{1\times p}$ *and* $\mathbf{d},\ \mathbf{v} \in \mathbf{D}^{1\times \overline{q'}}$ *such that*

$$\begin{pmatrix} 0 & \mathbf{c} & \mathbf{d} \end{pmatrix} \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F (e^n_{q+p'})^T = 1. \tag{1}$$

Proof. Getting terms of the p'-th column and p'-th line in the relation $\mathrm{id}_{p'} = P'P + Z'R'$, we get the following:

$$\sum_{k=1}^{p} P'_{p'k} P_{kp'} + \sum_{k=1}^{q'} Z'_{p'k} R'_{kp'} = 1.$$

From $\mathrm{sr}\,(\mathbf{D}) \leq \overline{p + q'}$, we deduce that there exist $c_1, \cdots, c_p,\ d_1, \cdots, d_{\overline{q'}},\ u_1, \cdots, u_p,\ v_1, \cdots, v_{\overline{q'}} \in \mathbf{D}$ such that

$$\sum_{k=1}^{p} c_k \left(P_{kp'} + u_k R'_{q'p'}\right) + \sum_{k=1}^{\overline{q'}} d_k \left(R'_{kp'} + v_k R'_{q'p'}\right) = 1. \tag{2}$$

Letting $\mathbf{c} := (c_1, \cdots, c_p)$, $\mathbf{d} := \left(-d_1, \cdots, -d_{\overline{q'}}\right)$, $\mathbf{u} := (-u_1, \cdots, -u_p)$ and $\mathbf{v} := \left(v_1, \cdots, v_{\overline{q'}}\right)$, the hand side of (2) is the left hand side of (1), which proves Proposition 1. $\square$

With the notations of Proposition 1, we introduce the lines $\tilde{\ell} \in \mathbf{D}^{1\times\overline{n}}$ and $\ell \in \mathbf{D}^{1\times n}$ defined as follows:

$$\tilde{\ell} := \begin{pmatrix} 0 & \mathbf{c} & \mathbf{d} \end{pmatrix} \text{ and } \ell := \begin{pmatrix} 0 & \mathbf{c} & \mathbf{d} & 0 \end{pmatrix},$$

as well as the matrices $U \in \mathbf{D}^{n\times n}$, $F \in \mathbf{D}^{\overline{n}\times n}$ defined as follows:

$$U := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } F := \begin{pmatrix} \mathrm{id}_{\overline{n}} & 0 \end{pmatrix} U Y_F.$$

Corollary 1. *We have:*

$$1 = \tilde{\ell} F (e^n_{q+p'})^T = \ell U Y_F (e^n_{q+p'})^T, \tag{3}$$

Proof. The first equality is a consequence of Relation (1) and the following relation:

$$F = \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F.$$

The second equality is due to the definitions of $\tilde{\ell}$, ℓ and F. $\square$

We consider the matrices $\pi,\ \pi' \in \mathbf{D}^{n\times\overline{n}}$ and $\iota,\ \iota' \in \mathbf{D}^{\overline{n}\times n}$ defined as follows:

$$\pi := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T\tilde{\ell} \\ \tilde{\ell} \end{pmatrix}, \qquad \pi' := \left(\mathrm{id}_n - (e^n_{q+p'})^T \ell U Y_F\right) \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix},$$

$$\iota := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T\tilde{\ell} & F(e^n_{q+p'})^T \end{pmatrix}, \quad \iota' := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

Proposition 2. *We have the following relations*

1. $\iota\pi = \mathrm{id}_{\overline{n}}$,
2. $\iota'\pi' = \mathrm{id}_{\overline{n}}$,
3. $\ker(.\pi) = \mathbf{D}\ell$,
4. $\ker(.\pi') = \mathbf{D}\ell U Y_F$.

Proof.
1. From (3), we have $\left(F(e^n_{q+p'})^T\tilde{\ell}\right)^2 = F(e^n_{q+p'})^T\tilde{\ell}$, from which we deduce $\iota\pi = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
2. We have $\iota'(e^n_{q+p'})^T = 0$, from which we deduce $\iota'\pi' = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
3. By considering the natural isomorphism $\mathbf{D}^{1\times n} \simeq \mathbf{D}^{1\times\overline{n}} \oplus \mathbf{D}$, we have $\pi = \pi_1\pi_2$, where $\pi_1 \in \mathbf{D}^{n\times n}$ and $\pi_2 \in \mathbf{D}^{n\times\overline{n}}$ are defined as follows:

$$\pi_1 := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T\tilde{\ell} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pi_2 := \begin{pmatrix} \mathrm{id}_{\overline{n}} \\ \tilde{\ell} \end{pmatrix}.$$

   From (3), $\mathrm{im}\,(.\pi_1)$ is included in $\ker(.F(e^n_{q+p'})^T) \oplus \mathbf{D}$ and the restriction of $.\pi_2$ to the latter is injective: for $(u,x) \in \left(\ker(.F(e^n_{q+p'})^T) \oplus \mathbf{D}\right) \cap \ker(.\pi_2)$, we have $u + x\tilde{\ell} = 0$, which gives $x = 0$ and $u = 0$ by (3). Hence, we have $\ker(.\pi) = \ker(.\pi_1) = \ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) \oplus 0$. We conclude by showing $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) = \mathbf{D}\tilde{\ell}$: the right to left inclusion is due to (3), and the other one is due to the relation $x = \left(xF(e^n_{q+p'})^T\right)\tilde{\ell}$, for every $x \in \ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right)$.

4. From (3), we have $\mathbf{D}\ell U Y_F \subseteq \ker(.\pi')$. The converse inclusion is due to the relation $x = x_{q+p'}\ell U Y_F$, for every $x \in \ker(.\pi')$. Indeed, the first $q+\overline{p'}$ and the last $p+q'$ columns of x and $x_{q+p'}\ell U Y_F$ are equal since $x \in \ker(.\pi')$ implies

$$x \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix} = x_{q+p'}\ell U Y_F \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix}. \tag{4}$$

   Moreover, the $q+p'$-th colums of $x_{q+p'}\ell U Y_F$ is computed by right multiplication by $(e^n_{q+p'})^T$ and is equal to $x_{q+p'}$ from (3).

$\square$

Theorem 1. *With the previous notations, we let*

$$Y_W := \iota U Y_F \pi' \quad \textit{and} \quad Y_W^- := \iota' Y_F^- U^- \pi.$$

The following diagram is exact and commutative:

$$\begin{tikzcd}
\mathbf{D}^{1\times\overline{n}} \arrow[r, ".\tilde{L}"] \arrow[d, shift left, ".Y_W"] & \mathbf{D}^{1\times m} \arrow[r] \arrow[d, shift left, ".X_F"] & \mathbf{M} \arrow[r] \arrow[d, shift left, "\varphi"] & 0 \\
\mathbf{D}^{1\times\overline{n}} \arrow[r, ".\tilde{L}'"'] \arrow[u, shift left, ".Y_W^-"] & \mathbf{D}^{1\times m} \arrow[r] \arrow[u, shift left, ".X_F^-"] & \mathbf{M}' \arrow[r] \arrow[u, shift left, "\varphi^-"] & 0
\end{tikzcd}$$

Proof. We only have to show that the diagram is commutative.

First, we show that Y_W and Y_W^- are inverse to each other. From Proposition 2, the lines of the following diagram are exact

$$\begin{tikzcd}
\mathbf{D} \arrow[r, ".\ell"] \arrow[d, "\mathrm{id}_{\mathbf{D}}"'] & \mathbf{D}^{1\times n} \arrow[r, ".\pi"] \arrow[d, shift left, ".UY_F"] & \mathbf{D}^{1\times\overline{n}} \arrow[r] \arrow[d, shift left, ".Y_W"] & 0 \\
\mathbf{D} \arrow[r, ".\ell U Y_F"'] & \mathbf{D}^{1\times n} \arrow[r, ".\pi'"'] \arrow[u, shift left, ".Y_F^- U^-"] & \mathbf{D}^{1\times\overline{n}} \arrow[r] \arrow[u, shift left, ".Y_W^-"] & 0
\end{tikzcd} \tag{5}$$

Moreover, it is also commutative. Indeed, we have $\pi Y_W = \pi\iota U Y_F \pi'$ and from 1 of Proposition 2, we have $\mathrm{im}\,(.\pi\iota - \mathrm{id}_n) \subseteq \ker(.\pi)$. By commutativity of the left rectangle and by exactness of the lines of (5), we have $(\pi\iota - \mathrm{id}_n)UY_F\pi' = 0$, so that $\pi Y_W = UY_F\pi'$. In the same manner, we show that $Y_F^- U^- \pi = \pi' Y_W^-$. By commutativity and exactness of (5) and from the equations $UY_F Y_F^- U^- = Y_F^- U^- U Y_F = \mathrm{id}_n$, we get $Y_W Y_W^- = Y_W^- Y_W = \mathrm{id}_{\overline{n}}$.

Moreover, $Y_W \tilde{L}' = \tilde{L} X_F$ and $Y_W^- \tilde{L} = \tilde{L}' X_F^-$ follow from the following commutative diagram:

$$\begin{tikzcd}
\mathbf{D}^{1\times\overline{n}} \arrow[r, shift left, "\iota"] \arrow[d, "\tilde{L}"'] & \mathbf{D}^{1\times n} \arrow[l, shift left, "\pi"] \arrow[r, shift left, "U"] \arrow[d, "L"] & \mathbf{D}^{1\times n} \arrow[l, shift left, "U^-"] \arrow[r, shift left, "Y_F"] \arrow[d, "L"] & \mathbf{D}^{1\times n} \arrow[l, shift left, "Y_F^-"] \arrow[r, shift left, "\pi'"] \arrow[d, "L'"] & \mathbf{D}^{1\times\overline{n}} \arrow[l, shift left, "\iota'"] \arrow[d, "\tilde{L}'"] \\
\mathbf{D}^{1\times m} \arrow[r, "\mathrm{id}_m"] & \mathbf{D}^{1\times m} \arrow[r, "\mathrm{id}_m"] & \mathbf{D}^{1\times m} \arrow[r, shift left, "X_F"] & \mathbf{D}^{1\times m} \arrow[l, shift left, "X_F^-"] \arrow[r, "\mathrm{id}_m"] & \mathbf{D}^{1\times m}
\end{tikzcd} \tag{6}$$

Indeed by computing the matrix products and from [1], we have the following relations:

$$\begin{aligned}
&\iota L = \tilde{L}, \quad && UL = L, \quad && Y_F L' = L X_F, \quad && \pi' \tilde{L}' = L', \\
&\iota' L' = \tilde{L}', \quad && Y_F^- L = L' X_F^-, \quad && U^- L = L, \quad && \pi \tilde{L} = L.
\end{aligned} \tag{7}$$

More details are given in Section 4.1. $\square$

3 Reduction of the identity bloc

We assume that $p \leq p'$ and $\mathrm{sr}\,(\mathbf{D}) \leq \overline{p'}$. We let

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{\overline{p'}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbf{D}^{\overline{n} \times \overline{m}} \quad \text{and} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_{\overline{p}} & 0 \\ 0 & R' \end{pmatrix} \in \mathbf{D}^{\overline{n} \times \overline{m}}.$$

Proposition 3. *There exist* $\mathbf{c} \in \mathbf{D}$ *and lines* $\mathbf{d},\ \mathbf{u} \in \mathbf{D}^{1 \times \overline{p'}}$ *such that*

$$\left(\mathbf{c} e^n_{q+p'+p} Y_F^- \begin{pmatrix} \mathrm{id}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} & 0 \end{pmatrix} \right) \begin{pmatrix} \mathrm{id}_q & 0 & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T & 0 \\ 0 & 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix} Y_F (e^n_{q+p'+p})^T = 1, \tag{8}$$

and

$$\left(\mathbf{c} e^m_p X_F^- \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} \end{pmatrix} \right) \begin{pmatrix} \mathrm{id}_p & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T \end{pmatrix} X_F (e^m_p)^T = 1. \tag{9}$$

Proof. We have

$$\left(1 - PP'_{pp}\right) + \sum_{k=1}^{p'} P_{pk} P'_{kp} = 1.$$

By projecting this equality on the left module $\mathbf{N} := \mathbf{D}/\mathbf{D}\left(1 - PP'_{pp}\right)$, the latter is spanned by $[P'_{1p}]_{\mathbf{N}}, \cdots,\ [P'_{p'p}]_{\mathbf{N}}$. Moreover, we have $\mathrm{sr}\,(\mathbf{N}) = \mathrm{sr}\,(\mathbf{D}) \geq p' - 1$, so that $\mathbf{u} := \left(u_1, \cdots,\ u_{\overline{p'}}\right) \in \mathbf{D}^{1 \times \overline{p'}}$ exists such that $\mathbf{N}$ is spanned by $[P'_{1p'} + u_1 P'_{p'p}]_{\mathbf{N}}$. Hence, $\mathbf{c} \in \mathbf{D}$ and $d_1, \cdots,\ d_{\overline{p'}} \in \mathbf{D}^{1 \times \overline{p'}}$ exist such that

$$\mathbf{c}\left(1 - PP'_{pp}\right) + \sum_{k=1}^{\overline{p'}} d_k \left(P'_{kp} + u_k P'_{p'p}\right) = 1. \tag{10}$$

Letting $\mathbf{d} := \left(-d_1, \cdots,\ -d_{\overline{p'}}\right) \in \mathbf{D}^{1 \times \overline{p'}}$ and from the relation $\mathrm{id}_p = ZR + PP'$, the left hand sides of (8) and (9) are both equal to the left hand side of (10), which proves Proposition 3. $\square$

With the notations of Proposition 3, we introduce the lines $\tilde{\ell}_r \in \mathbf{D}^{1 \times \overline{n}}$, $\ell_r \in \mathbf{D}^{1 \times n}$, $\tilde{\ell}_g \in \mathbf{D}^{1 \times \overline{m}}$ and $\ell_g \in \mathbf{D}^{1 \times m}$ defined as follows:

$$\tilde{\ell}_r := \mathbf{c} e^n_{q+p'+p} Y_F^- \begin{pmatrix} \mathrm{id}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} & 0 \end{pmatrix}, \quad \ell_r := \tilde{\ell}_r \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix},$$

$$\tilde{\ell}_g := \mathbf{c} e^m_p X_F^- \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} \end{pmatrix}, \qquad \ell_g := \begin{pmatrix} \tilde{\ell}_g & 0 \end{pmatrix}.$$

as well as the matrices $U_r \in \mathbf{D}^{n \times n}$, $U_g \in \mathbf{D}^{m \times m}$, $F_r \in \mathbf{D}^{\overline{n} \times n}$ and $F_g \in \mathbf{D}^{\overline{m} \times m}$ defined as follows:

$$U_r := \begin{pmatrix} \mathrm{id}_q & 0 & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}, \quad F_r := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix} U_r Y_F,$$

$$U_g := \begin{pmatrix} \mathrm{id}_p & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_g := \begin{pmatrix} \mathrm{id}_{\overline{m}} & 0 \end{pmatrix} U_g X_F.$$

Explicitly, we have:

$$\tilde{\ell}_r = \begin{pmatrix}\mathbf{c}Z_{p.} & \mathbf{d} & \mathbf{c}e_p^{p+q'}\end{pmatrix} \qquad \ell_r = \begin{pmatrix}\mathbf{c}Z_{p.} & \mathbf{d} & 0 & \mathbf{c}e_p^{p+q'}\end{pmatrix},$$

$$\tilde{\ell}_g = \begin{pmatrix}\mathbf{c}\left(\mathrm{id}_p - PP'\right)_{p.} & \mathbf{d}\end{pmatrix} \quad \ell_g = \begin{pmatrix}\mathbf{c}\left(\mathrm{id}_p - PP'\right)_{p.} & \mathbf{d} & 0\end{pmatrix},$$

Using arguments analogs to the ones in the proof of Corollary 1, we get the following.

Corollary 2. *We have*

$$1 = \tilde{\ell}_r F_r (e^n_{q+p'+p})^T = \ell_r U_r Y_F (e^n_{q+p'+p})^T, \tag{11}$$

and

$$1 = \tilde{\ell}_g F_g (e^m_p)^T = \ell_g U_g X_F (e^m_p)^T. \tag{12}$$

Let us consider the matrices $\pi_r,\ \pi_r' \in \mathbf{D}^{n\times\overline{n}}$, $\pi_g,\ \pi_g' \in \mathbf{D}^{m\times\overline{m}}$, $\iota_r,\ \iota_r' \in \mathbf{D}^{\overline{n}\times n}$ and $\iota_g,\ \iota_g' \in \mathbf{D}^{\overline{m}\times m}$ defined as follows:

$$\pi_r := \begin{pmatrix}\mathrm{id}_{q+\overline{p'}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{p+q'}\end{pmatrix}\left(\mathrm{id}_{\overline{n}} - F_r\left(e^n_{q+p'+p}\right)^T\tilde{\ell}_r\right) + (e^n_{q+p'+p})^T\tilde{\ell}_r \qquad \pi_r' := \left(\mathrm{id}_n - (e^n_{q+p'+p})^T\ell_r U_r Y_F\right)\begin{pmatrix}\mathrm{id}_{q+p'+\overline{p}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{q'}\end{pmatrix}$$

$$\iota_r := \left(\mathrm{id}_{\overline{n}} - F_r\left(e^n_{q+p'+p}\right)^T\tilde{\ell}_r\right)\begin{pmatrix}\mathrm{id}_{q+\overline{p'}} & 0 & 0\\ 0 & 0 & \mathrm{id}_{q+p'}\end{pmatrix} + F_r\left(e^n_{q+p'+p}\right)^T e^n_{q+p'}, \quad \iota_r' := \begin{pmatrix}\mathrm{id}_{q+p'+\overline{p}} & 0 & 0\\ 0 & 0 & \mathrm{id}_{q'}\end{pmatrix},$$

$$\pi_g := \begin{pmatrix}\mathrm{id}_{\overline{m}} - F_g(e^m_p)^T\tilde{\ell}_g\\ \tilde{\ell}_g\end{pmatrix}, \qquad \pi_g' := \left(\mathrm{id}_m - (e^m_p)^T\ell_g U_g X_F\right)\begin{pmatrix}\mathrm{id}_{\overline{p}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{p'}\end{pmatrix},$$

$$\iota_g := \begin{pmatrix}\mathrm{id}_{\overline{m}} - F_g(e^m_p)^T\tilde{\ell}_g & F_g(e^m_p)^T\end{pmatrix} \qquad \iota_g' := \begin{pmatrix}\mathrm{id}_{\overline{p}} & 0 & 0\\ 0 & 0 & \mathrm{id}_{p'}\end{pmatrix}.$$

By adapting the arguments of the proof of Proposition 2, we get the following.

Proposition 4. *We have the following relations*

1. $\iota_r\pi_r = \mathrm{id}_{\overline{n}}$,
2. ${\iota_r}'{\pi_r}' = \mathrm{id}_{\overline{n}}$,
3. $\ker(.\pi_r) = \mathbf{D}\ell_r$,
4. $\ker(.{\pi_r}') = \mathbf{D}\ell_r U_r Y_F$,
5. $\iota_g\pi_g = \mathrm{id}_{\overline{m}}$,
6. ${\iota_g}'{\pi_g}' = \mathrm{id}_{\overline{m}}$,
7. $\ker(.\pi_g) = \mathbf{D}\ell_g$,
8. $\ker(.{\pi_g}') = \mathbf{D}\ell_g U_g X_F$.

Theorem 2. *With the previous notations, we let*

$$Y_W := \iota_r U_r Y_F \pi_r', \quad X_W := \iota_g U_g X_F \pi_g', \quad Y_W^- := \iota_r' Y_F^- U_r^- \pi_r, \quad X_W^- := \iota_g' U_g X_F^- \pi_g.$$

The following diagram is exact and commutative:

$$\begin{array}{ccccccc}
\mathbf{D}^{1\times\overline{n}} & \xrightarrow{.\tilde{L}} & \mathbf{D}^{1\times\overline{m}} & \longrightarrow & \mathbf{M} & \longrightarrow & 0\\
{\scriptstyle .Y_W^-}\big\uparrow\big\downarrow{\scriptstyle .Y_W} & & {\scriptstyle .X_W^-}\big\uparrow\big\downarrow{\scriptstyle .X_W} & & {\scriptstyle \varphi^-}\big\uparrow\big\downarrow{\scriptstyle \varphi} & & \\
\mathbf{D}^{1\times\overline{n}} & \xrightarrow[.\tilde{L}']{} & \mathbf{D}^{1\times\overline{m}} & \longrightarrow & \mathbf{M}' & \longrightarrow & 0
\end{array}$$

ι_r,

Proof. We only have to show that the diagram is commutative.

We show that Y_W and Y_W^- (respectively, X_W and X_W^-) are inverse to each over in the same manner that we did in the proof of Theorem 1.

Moreover, $Y_W \tilde{L}' = \tilde{L} X_W$ and $Y_W^- \tilde{L} = \tilde{L}' X_W^-$ follow from the following commutative diagram:

$$
\begin{tikzcd}
\mathbf{D}^{1\times\overline{n}} \arrow[r, shift left, "\iota_r"] \arrow[d, "\tilde{L}"'] & \mathbf{D}^{1\times n} \arrow[l, shift left, "\pi_r"] \arrow[r, shift left, "U_r"] \arrow[d, "L"] & \mathbf{D}^{1\times n} \arrow[l, shift left, "U_r^-"] \arrow[r, shift left, "Y_F"] \arrow[d, "L"] & \mathbf{D}^{1\times n} \arrow[l, shift left, "Y_F^-"] \arrow[r, shift left, "\pi'_r"] \arrow[d, "L'"] & \mathbf{D}^{1\times\overline{n}} \arrow[l, shift left, "\iota'_r"] \arrow[d, "\tilde{L}'"] \\
\mathbf{D}^{1\times\overline{m}} \arrow[r, shift left, "\iota_g"] & \mathbf{D}^{1\times m} \arrow[l, shift left, "\pi_g"] \arrow[r, shift left, "U_g"] & \mathbf{D}^{1\times m} \arrow[l, shift left, "U_g^-"] \arrow[r, shift left, "X_F"] & \mathbf{D}^{1\times m} \arrow[l, shift left, "X_F^-"] \arrow[r, shift left, "\pi'_g"] & \mathbf{D}^{1\times\overline{m}} \arrow[l, shift left, "\iota'_g"]
\end{tikzcd}
\tag{13}
$$

Indeed by computing the matrix products and from [1], we have the following relations:

$$
\begin{aligned}
&\iota_r L = \tilde{L}\iota_g, \quad && U_r L = L U_g, \quad && Y_F L' = L X_F, \quad && \pi_r \tilde{L} = L \pi_g, \\
&\iota'_r L' = \tilde{L}'\iota'_g, \quad && Y_F^- L = L' X_F^-, \quad && U_r^- L = L U_g^-, \quad && \pi'_r \tilde{L}' = L' \pi'_g.
\end{aligned}
\tag{14}
$$

More details are given in Section 4.2. $\square$

4 Annex: proofs

4.1 Proof of Formulas 7

We have to show the following relations

$$\iota L = \tilde{L}, \tag{15}$$
$$U L = L, \tag{16}$$
$$Y_F L' = L X_F, \tag{17}$$
$$\pi' \tilde{L}' = L', \tag{18}$$
$$\iota' L' = \tilde{L}', \tag{19}$$
$$Y_F^- L = L' X_F^-, \tag{20}$$
$$U^- L = L, \tag{21}$$
$$\pi \tilde{L} = L. \tag{22}$$

The two relations (17) and (20) come from [1], (16) and (19) are proven by direct computations, (15) and (21) are proven by direct computations using respectively the following two relations:

$$\ell L = 0 \ \text{ and } \ U^- = \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & -\mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & -\mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to prove (18), we first show $\pi'\iota' L' = L'$: from 2 and 4 of Proposition 2, $\mathrm{im}\,(\pi'\iota' - \mathrm{id}_n)$ is included $\mathbf{D}\ell U Y_F$ and using (16), (17) and $\ell L = 0$, we have $\ell U Y_F L' = \ell L U X_F = 0$, which proves the desired relation. Moreover, from (19), we get $\pi'\iota' L' = \pi'\tilde{L}'$ which, with $\pi'\iota' L' = L'$, gives (18). We show (22) in the same manner using (15).

4.2 Proof of Formulas 14

We have to show the following relations

$$\iota_r L = \tilde{L}\iota_g, \tag{23}$$
$$U_r L = L U_g, \tag{24}$$
$$Y_F L' = L X_F, \tag{25}$$
$$\pi'_r \tilde{L}' = L' \pi'_g, \tag{26}$$

$$\iota'_r L' = \tilde{L}'\iota'_g, \qquad (27) \qquad Y_F^- L = L' X_F^-, \qquad (28) \qquad U_r^- L = L U_g^-, \qquad (29) \qquad \pi_r \tilde{L} = L\pi_g. \qquad (30)$$

The two relations (25) and (28) come from [1], (24) and (27) are proven by direct computations, (29) is computed by a direct computation using the inverse formulas for U_g and U_r which are analog to the inverse of U given in Section 4.1.

Let us show (23). For that, we decompose ι_r and ι_g in 3 parts, as follows:

$$\iota_r^1 := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}, \quad \iota_r^2 := F_r(e^n_{q+p'+p})^T \tilde{\ell}_r \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}, \quad \iota_r^3 := F_r(e^n_{q+p'+p})^T e^n_{q+p'},$$

$$\iota_g^1 := \begin{pmatrix} \mathrm{id}_{\overline{m}} & 0 \end{pmatrix}, \qquad \iota_g^2 := F_g(e^m_p)^T \ell_g, \qquad \iota_g^3 := F_g(e^m_p)^T e^m_{p+p'},$$

so that we have $\iota_r = \iota_r^1 - \iota_r^2 + \iota_r^3$ and $\iota_g = \iota_g^1 - \iota_g^2 + \iota_g^3$. By compuring the matrix products, we show that $\iota_r^1 L = \tilde{L}\iota_g^1$, that the first $p + \overline{p'}$ columns of $\iota_r^2 L$ and $\tilde{L}\iota_g^2$ are both equal to 0 and their $p + p'$ column are both equal to

$$\begin{pmatrix} R_{.p} \\ -\left(P'_{ip} + u_i P'_{p'p}\right)_{1\leq i\leq \overline{p'}} \\ 0 \\ 0 \end{pmatrix}.$$

Finally, by computing the matrix products, we show that $\iota_r^3 L$ and $\tilde{L}\iota_g^3$ are respectively equal to

$$\begin{pmatrix} R_{.p} \\ \left(\mathrm{id}_{p'}{}_{ip'} - P'P_{ip'}\right)_{1\leq i\leq p'} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}\,(ZR)_{p.} & \mathbf{d} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} R_{.p} \\ \left(\mathrm{id}_{p'}{}_{ip'} - P'P_{ip'}\right)_{1\leq i\leq p'} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}\,(\mathrm{id}_p - PP')_{p.} & \mathbf{d} & 0 \end{pmatrix}$$

so that they are equal from the relation $\mathrm{id}_p = PP' + ZR$, which proves (23).

Let us show (26). By using Relation (27), and 1 of Proposition 4, we have $\pi'_r \iota'_r L' \pi'_g = \pi'_r \tilde{L}'$. We proceed as in the proof of (18): we only have to show that $\mathrm{im}\,(\pi'_r \iota'_r - \mathrm{id}_n)\, L'$ is included in $\ker\left(\pi'_g\right)$. We have $\mathrm{im}\,(\pi'_r \iota'_r - \mathrm{id}_n) \subseteq \mathbf{D}\ell_r U_r Y_F$, $\ell_r U_r Y_F L' = \ell_r L U_g X_F$ and $\ell_r L = \ell_g$, so that $\ell_r U_r Y_F L' = \ell_g U_g X_F \in \ker(\pi'_g)$. With the same arguments, we show (26).

References

[1] Thomas Cluzeau and Alban Quadrat. A constructive version of fitting's theorem on isomorphisms and equivalences of linear systems. In *Multidimensional (nD) Systems (nDs), 2011 7th International Workshop on*, pages 1–8. IEEE, 2011.