A constructive version of Warfield's Theorem

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Abstract:

Keywords:

1. INTRODUCTION

2. ISOMORPHISMS AND EQUIVALENT MATRICES

In this section, we recall the characterization of morphisms between finitely presented left D-modules, as well as results of Fitting and Warfield which rely isomorphic left D-modules to matrix conjugation.

2.1 Effective version of Fitting's Theorem

Consider two left D-modules M and M' with finite presentations:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

namely, *exact sequences* (see Rotman (2009)), where $R \in D^{q\times p}$, $(.R)(\mu) = \mu R$, for every $\mu \in D^{1\times q}$ and π is the natural projection on $M = D^{1\times p}/(D^{1\times q}R)$ (similarly for R' and π').

From Rotman (2009), there exists $f \in \hom_D(M, M')$ if and only if there exist matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ such that $RP = QR'$ and

$$\forall \lambda \in D^{1\times p},\ f(\pi(\lambda)) = \pi'(\lambda\pi).$$

Hence, the following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

We let $n := q + p' + p + q'$ and $m := p + p'$. The two $n \times m$ matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix}, \tag{1}$$

induce finite presentations: $M \simeq D^{1\times m}/(D^{1\times n}L)$ and $M' \simeq D^{1\times m}/(D^{1\times n}L')$. In Cluzeau and Quadrat (2011), an effective version of a result due to Fitting (1936) is given. If f is an isomorphism, then L and L' are equivalent: there exist 6 matrices $R_2 \in D^{r\times q}$, $R'_2 \in D^{r'\times q'}$, $Z_2 \in D^{p\times q}$, $Z_2 \in D^{q\times r}$, $Z'_2 \in D^{q'\times r'}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ and two invertible matrices of size m and n

$$X_F := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix} \quad \text{and} \quad Y_F := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z'_2R'_2 \end{pmatrix}, \tag{2}$$

with inverses

$$X_F^- := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix} \quad \text{and} \quad Y_F^- := \begin{pmatrix} Z_2R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix}, \tag{3}$$

such that

$$L' = Y_F^- L X_F. \tag{4}$$

In other words, the following diagram is exact commutative

$$\begin{tikzcd}
D^{1\times n} \arrow[r, ".L"] \arrow[d, shift right, ".Y_W"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_W"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\
D^{1\times n} \arrow[r, ".L'"] \arrow[u, shift right, ".Y_W^-"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_W^-"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0
\end{tikzcd}$$

2.2 Warfield's Theorem

A result due to Warfield (1978) asserts that that the size of 0 an id blocs in (1) can be reduced, whereas the new matrices are still equivalent. This result is based on the notion of *stable rank*. The definition of the latter requires to introduce various notions that we present now.

A column vector $u := (u_1 \cdots u_k)^T \in D^{k\times 1}$ is called *unimodular* if there exists a line $v \in D^{1\times k}$ such that $vu = 1$. Moreover, u is said to be *stable* if there exist $d_1, \cdots, d_{k-1} \in D$ such that $(u_1 + d_1u_k \cdots u_{k-1} + d_{k-1}u_k)$ is unimodular. An integer r is said to be in the stable rank of D if whenever $k > r$, every column $u \in D^{k\times 1}$ is stable. The stable rank $\mathrm{sr}(D)$ of D is the smallest integer in the stable rank of D.

Assume that the two matrices (1) are equivalent, then Warfield's Theorem asserts that if there exist two integers r and s such that

$$\begin{cases} s \leq \min(p+q', q+p'), \\ \mathrm{sr}(D) \leq \max(p+q'-s, q+p'-s), \\ r \leq \min(p, p'), \\ \mathrm{sr}(D) \leq \max(p-r, p'-r), \end{cases} \tag{5}$$

then the following $(n-r-s) \times (m-r)$ matrices are equivalent

$$\overline{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-r} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{L'} := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix}, \tag{6}$$

and induce finite presentations of M and M', respectively.

In the next section, we introduce a procedure which computes invertible matrices X_W and Y_W such that

$$\overline{L'} = Y_W^{-} \overline{L} X_W.$$

3. EFFECTIVE WARFIELD'S THEOREM

Throughout this Section, we fix some notations. Let M and M' be two left D-modules, isomorphic with $f : M \xrightarrow{\sim} M'$, finitely presented by matrices $R \in D^{q\times p}$ and $R' \in D^{q'\times p'}$, respectively, and let L, L', X_F, Y_F, X_F^- and Y_f^- be the matrices defined in (1) (2) and (3). Let $m := q+p'+p+q'$ and $n := p+p'$. Given a nonzero integer k, we let $\overline{k} := k-1$. For $1 \leq i \leq k$, the i-th vector of the canonical basis of $D^{1\times k}$ is written e_i^k.

3.1 Reduction of the zero bloc

In this section, we present the procedure for removing one 0 in L and L'. Inductive applications of this procedure enables us to remove many 0. Without lost of generalities, we may suppose that $q+p' \leq p+q'$, and we assume that $\mathrm{sr}(D) \leq \overline{p+q'}$, so that the hypotheses of (5) are fullfilled for $s=1$. Our purpose is to show that the following $\overline{n} \times m$ matrices are equivalent:

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix}.$$

Proposition 1. There exist $\mathbf{c},\ \mathbf{u} \in D^{1\times p}$ and $\mathbf{d},\ \mathbf{v} \in D^{1\times \overline{q'}}$ such that

$$(0\ \mathbf{c}\ \mathbf{d}) \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F (e^n_{q+p'})^T = 1. \tag{7}$$

Proof. Getting terms of the p'-th column and p'-th line in the relation $\mathrm{id}_{p'} = P'P + Z'R'$, we get the following:

$$\sum_{k=1}^{p} P'_{p'k} P_{kp'} + \sum_{k=1}^{q'} Z'_{p'k} R'_{kp'} = 1.$$

From $\mathrm{sr}(D) \leq \overline{p+q'}$, we deduce that there exist $c_1, \cdots, c_p, d_1, \cdots, d_{\overline{q'}}, u_1, \cdots, u_p, v_1, \cdots, v_{\overline{q'}} \in D$ such that

$$\sum_{k=1}^{p} c_k \left(P_{kp'} + u_k R'_{q'p'}\right) + \sum_{k=1}^{\overline{q'}} d_k \left(R'_{kp'} + v_k R'_{q'p'}\right) = 1. \tag{8}$$

Letting $\mathbf{c} := (c_1, \cdots, c_p)$, $\mathbf{d} := \left(-d_1, \cdots, -d_{\overline{q'}}\right)$, $\mathbf{u} := (-u_1, \cdots, -u_p)$ and $\mathbf{v} := \left(v_1, \cdots, v_{\overline{q'}}\right)$, the hand side of (8) is the left hand side of (7), which proves Proposition 1.

With the notations of Proposition 1, we introduce the lines $\tilde{\ell} \in D^{1\times\overline{n}}$ and $\ell \in D^{1\times n}$ defined as follows:

$$\tilde{\ell} := (0\ \mathbf{c}\ \mathbf{d}) \quad \text{and} \quad \ell := (0\ \mathbf{c}\ \mathbf{d}\ 0),$$

as well as the matrices $U \in D^{n\times n}$, $F \in D^{\overline{n}\times n}$ defined as follows:

$$U := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F := \left(\mathrm{id}_{\overline{n}}\ 0\right) U Y_F.$$

From Relation (7) and

$$F = \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F,$$

we get:

$$1 = \tilde{\ell} F (e^n_{q+p'})^T = \ell U Y_F (e^n_{q+p'})^T. \tag{9}$$

We consider the matrices $\mathrm{pr},\ \mathrm{pr}' \in D^{n\times\overline{n}}$

$$\mathrm{pr} := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell} \\ \tilde{\ell} \end{pmatrix},$$

$$\mathrm{pr}' := \left(\mathrm{id}_n - (e^n_{q+p'})^T \ell U Y_F\right) \begin{pmatrix} \mathrm{id}_{\overline{q+p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix},$$

and $\iota,\ \iota' \in D^{\overline{n}\times n}$

$$\iota := \left(\mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell}\ \ F(e^n_{q+p'})^T\right),\ \iota' := \begin{pmatrix} \mathrm{id}_{\overline{q+p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

Proposition 2. We have the following relations:

(1) $\iota \mathrm{pr} = \mathrm{id}_{\overline{n}}$,
(2) $\iota' \mathrm{pr}' = \mathrm{id}_{\overline{n}}$,
(3) $\ker(.\mathrm{pr}) = D\ell$,
(4) $\ker(.\mathrm{pr}') = D\ell U Y_F$.

Proof.

(1) From (9), we have $\left(F(e^n_{q+p'})^T \tilde{\ell}\right)^2 = F(e^n_{q+p'})^T \tilde{\ell}$, from which we deduce $\iota \mathrm{pr} = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
(2) We have $\iota'(e^n_{q+p'})^T = 0$, from which we deduce $\iota' \mathrm{pr}' = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
(3) Considering the isomorphism $D^{1\times n} \simeq D^{1\times\overline{n}} \oplus D$, we have $\mathrm{pr} = \mathrm{pr}_1 \mathrm{pr}_2$, where $\mathrm{pr}_1 \in D^{n\times n}$ and $\mathrm{pr}_2 \in D^{n\times\overline{n}}$ are defined as follows:

$$\mathrm{pr}_1 := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{pr}_2 := \begin{pmatrix} \mathrm{id}_{\overline{n}} \\ \tilde{\ell} \end{pmatrix}.$$

From (9), $\mathrm{im}(.\mathrm{pr}_1)$ is included in $\ker(.F(e^n_{q+p'})^T) \oplus D$ and the restriction of $.\mathrm{pr}_2$ to the latter is injective: for $(u, x) \in \left(\ker(.F(e^n_{q+p'})^T) \oplus D\right) \cap \ker(.\mathrm{pr}_2)$, we have $u + x\tilde{\ell} = 0$, which gives $x = 0$ and $u = 0$ by (9). Hence, we have $\ker(.\mathrm{pr}) = \ker(.\mathrm{pr}_1)$, that is $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) \oplus 0$. We conclude by showing $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) = D\tilde{\ell}$: the right to left inclusion is due to (9), and the other one is due to the relation $x = \left(xF(e^n_{q+p'})^T\right)\tilde{\ell}$, for every x in $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right)$.

(4) From (9), we have $D\ell UY_F \subseteq \ker(.\mathrm{pr}')$. The converse inclusion is due to the relation $x = x_{q+p'}\ell UY_F$, for every $x \in \ker(.\mathrm{pr}')$. Indeed, the first $q + \overline{p'}$ and the last $p+q'$ columns of x and $x_{q+p'}\ell UY_F$ are equal since $x \in \ker(.\mathrm{pr}')$ implies

$$x \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix} = x_{q+p'}\ell UY_F \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

Moreover, the $q + p'$-th colums of $x_{q+p'}\ell UY_F$ is computed by right multiplication by $(e^n_{q+p'})^T$ and is equal to $x_{q+p'}$ from (9).

Theorem 3. With the previous notations, we let

$$X_W := X_F, \quad Y_W := \iota UY_F\mathrm{pr}',$$

$$X_W^- := X_F, \quad Y_W^- := \iota' Y_F^- U^- \mathrm{pr}.$$

The following diagram is exact and commutative:

$$\begin{tikzcd} D^{1\times\overline{n}} \arrow[r, ".\tilde{L}"] \arrow[d, shift right, ".Y_W"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_W"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times\overline{n}} \arrow[r, ".\tilde{L}'"] \arrow[u, shift right, ".Y_W^-"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_W^-"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0 \end{tikzcd}$$

In particular, we have

$$\tilde{L}' = Y_W^- \tilde{L} X_W.$$

Proof. We only have to show that the diagram is commutative.

First, we show that Y_W and Y_W^- are inverse to each other. From Proposition 2, the lines of the following diagram are exact

$$\begin{tikzcd} D \arrow[r, ".\ell"] \arrow[d, "\mathrm{id}_D"'] & D^{1\times n} \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, ".UY_F"'] & D^{1\times\overline{n}} \arrow[r] \arrow[d, shift right, ".Y_W"'] & 0 \\ D \arrow[r, ".\ell UY_F"'] & D^{1\times n} \arrow[r, "\mathrm{pr}'"'] \arrow[u, shift right, ".Y_F^-U^-"'] & D^{1\times\overline{n}} \arrow[r] \arrow[u, shift right, ".Y_W^-"'] & 0 \end{tikzcd} \tag{10}$$

Moreover, it is also commutative. Indeed, $\mathrm{pr}Y_W$ is equal to $\mathrm{pr}\iota UY_F\mathrm{pr}'$ and from 1 of Proposition 2, we have $\mathrm{im}(.\mathrm{pr}\iota - \mathrm{id}_n) \subseteq \ker(.\mathrm{pr})$. By commutativity of the left rectangle and by exactness of the lines of (10), we have $(\mathrm{pr}\iota - \mathrm{id}_n)UY_F\mathrm{pr}' = 0$, so that $\mathrm{pr}Y_W = UY_F\mathrm{pr}'$. In the same manner, we show that $Y_F^-U^-\mathrm{pr} = \mathrm{pr}'Y_W^-$. By commutativity and exactness of (10) and from the equations $UY_FY_F^-U^- = Y_F^-U^-UY_F = \mathrm{id}_n$, we get $Y_WY_W^- = Y_W^-Y_W = \mathrm{id}_{\overline{n}}$.

Moreover, $Y_W\tilde{L}' = \tilde{L}X_F$ and $Y_W^-\tilde{L} = \tilde{L}'X_F^-$ follow from the following commutative diagram:

$$\begin{tikzcd} D^{1\times\overline{n}} \arrow[r, shift left, "\iota"] \arrow[d, "\tilde{L}"'] & D^{1\times n} \arrow[l, shift left, "\mathrm{pr}"] \arrow[r, shift left, "U"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "U^-"] \arrow[r, shift left, "Y_F"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "Y_F^-"] \arrow[r, shift left, "\mathrm{pr}'"] \arrow[d, "L'"] & D^{1\times\overline{n}} \arrow[l, shift left, "\iota'"] \arrow[d, "\tilde{L}'"] \\ D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, shift left, "X_F"] & D^{1\times m} \arrow[l, shift left, "X_F^-"] \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \end{tikzcd} \tag{11}$$

Indeed by computing the matrix products and from (4), we have the following relations:

$$\iota L = \tilde{L}, \quad UL = L, \quad Y_FL' = LX_F, \quad \mathrm{pr}'\tilde{L}' = L',$$

$$\iota'L' = \tilde{L}', \quad Y_F^-L = L'X_F^-, \quad U^-L = L, \quad \mathrm{pr}\tilde{L} = L. \tag{12}$$

More details are given in Section **??**.

3.2 Reduction of the identity bloc

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