A constructive version of Warfield's Theorem

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Abstract:

Keywords:

1. INTRODUCTION

2. MODULE ISOMORPHISMS AND EQUIVALENT MATRICES

In this section, we recall the characterization of morphisms between finitely presented left D-modules, as well as results of Fitting and Warfield which rely isomorphic left D-modules to matrix conjugation.

2.1 Effective version of Fitting's Theorem

Consider two left D-modules M and M' with finite presentations:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$
$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0, \tag{1}$$

namely, *exact sequences* (see Rotman (2009)), where $R \in D^{q\times p}$, $(.R)(\mu) = \mu R$, for every $\mu \in D^{1\times q}$ and π is the natural projection on $M = D^{1\times p}/(D^{1\times q}R)$ (similarly for R' and π').

From Rotman (2009), there exists $f \in \hom_D(M, M')$ if and only if there exist matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ such that $RP = QR'$ and

$$\forall \lambda \in D^{1\times p},\ f(\pi(\lambda)) = \pi'(\lambda\pi). \tag{2}$$

Hence, the following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

We let $n := p + p'$ and $m := q + p' + p + q'$. The two $m \times n$ matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix}, \tag{3}$$

induce finite presentations: $M \simeq D^{1\times n}/(D^{1\times m}L)$ and $M' \simeq D^{1\times n}/(D^{1\times m}L')$. In Cluzeau and Quadrat (2011), an effective version of a result due to Fitting (1936) is given. If f is an isomorphism, then L and L' are equivalent: there exist 6 matrices $R_2 \in D^{r\times q}$, $R_2' \in D^{r'\times q'}$, $Z_2 \in D^{p\times q}$, $Z_2 \in D^{q\times r}$, $Z_2' \in D^{q'\times r'}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ and two invertible matrices of size n and m

$$X_F := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix} \quad \text{and} \quad Y_F := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z_2'R_2' \end{pmatrix}, \tag{4}$$

with inverses

$$X_F^- := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix} \quad \text{and} \quad Y_F^- := \begin{pmatrix} Z_2R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix}, \tag{5}$$

such that

$$L' = Y_F^- L X_F. \tag{6}$$

2.2 Warfield's Theorem

A result due to Warfield (1978) asserts that that the size of 0 an id blocs in (3) can be reduced, whereas the new matrices are still equivalent. This result is based on the notion of *stable rank*. The definition of the latter requires to introduce various notions that we present now.

A column vector $u := (u_1 \cdots u_k)^T \in D^{k\times 1}$ is called *unimodular* if there exists a line $v \in D^{1\times k}$ such that $vu = 1$. Moreover, u is said to be *stable* if there exist $d_1, \cdots, d_{k-1} \in D$ such that $(u_1 + d_1u_k \cdots u_{k-1} + d_{k-1}u_k)$ is unimodular. An integer r is said to be in the stable rank of D if whenever $k > r$, every column $u \in D^{k\times 1}$ is stable. The stable rank $\mathrm{sr}(D)$ of D is the smallest integer in the stable rank of D.

Assume that the two matrices (3) are equivalent, then Warfield's Theorem asserts that if there exist two integers r and s such that

$$\begin{cases} s \leq \min(p + q', q + p'), \\ \mathrm{sr}(D) \leq \max(p + q' - s, q + p' - s), \\ s \leq \min(p + q', q + p'), \\ \mathrm{sr}(D) \leq \max(p + q' - s, q + p' - s), \end{cases} \tag{7}$$

then the following $(m - r - s) \times (n - r)$ matrices are equivalent

$$\overline{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-r} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{L'} := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix}, \tag{8}$$

and induce finite presentations of M and M', respectively.

In the next section, we introduce a procedure which computes invertible matrices X_W and Y_W such that

$$\overline{L'} = Y_W^{-}\overline{L} X_W. \tag{9}$$

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