# A constructive version of Warfield's Theorem

**Cyrille Chenavier** * **Alban Quadrat** **

* *Inria Lille - Nord Europe, Villeneuve d'Ascq, France (e-mail: cyrille.chenavier@inria.fr).*
** *Inria Paris, Université Pierre et Marie Curie, Paris, France (e-mail: alban.quadrat@inria.fr).*

**Abstract:**

*Keywords:*

## 1. INTRODUCTION

## 2. ISOMORPHISMS AND EQUIVALENT MATRICES

In this section, we recall the characterization of morphisms between finitely presented left $D$-modules, as well as results of Fitting and Warfield which rely isomorphic left $D$-modules to matrix conjugation.

### 2.1 Effective version of Fitting's Theorem

Consider two left $D$-modules $M$ and $M'$ with finite presentations:

$$D^{1\times q} \xrightarrow{.R} D^{1\times p} \xrightarrow{\pi} M \longrightarrow 0,$$

$$D^{1\times q'} \xrightarrow{.R'} D^{1\times p'} \xrightarrow{\pi'} M' \longrightarrow 0,$$

namely, *exact sequences* (see Rotman (2009)), where $R \in D^{q\times p}$, $(.R)(\mu) = \mu R$, for every $\mu \in D^{1\times q}$ and $\pi$ is the natural projection on $M = D^{1\times p}/(D^{1\times q}R)$ (similarly for $R'$ and $\pi'$).

From Rotman (2009), there exists $f \in \hom_D(M, M')$ if and only if there exist matrices $P \in D^{p\times p'}$ and $Q \in D^{q\times q'}$ such that $RP = QR'$ and

$$\forall \lambda \in D^{1\times p},\ f(\pi(\lambda)) = \pi'(\lambda\pi).$$

Hence, the following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times q} \arrow[r, ".R"] \arrow[d, ".Q"] & D^{1\times p} \arrow[r, "\pi"] \arrow[d, ".P"] & M \arrow[r] \arrow[d, "f"] & 0 \\
D^{1\times q'} \arrow[r, ".R'"] & D^{1\times p'} \arrow[r, "\pi'"] & M' \arrow[r] & 0
\end{tikzcd}$$

We let $n := q + p' + p + q'$ and $m := p + p'$. The two $n \times m$ matrices

$$L := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad L' := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix}, \tag{1}$$

induce finite presentations: $M \simeq D^{1\times m}/(D^{1\times n}L)$ and $M' \simeq D^{1\times m}/(D^{1\times n}L')$. In Cluzeau and Quadrat (2011), an effective version of a result due to Fitting (1936) is given. If $f$ is an isomorphism, then $L$ and $L'$ are equivalent: there exist 6 matrices $R_2 \in D^{r\times q}$, $R'_2 \in D^{r'\times q'}$, $Z_2 \in D^{p\times q}$, $Z_2 \in D^{q\times r}$, $Z'_2 \in D^{q'\times r'}$, $Z \in D^{p\times q}$ and $Z' \in D^{p'\times q'}$ and two invertible matrices of size $m$ and $n$

$$X_F := \begin{pmatrix} \mathrm{id}_p & P \\ -P' & \mathrm{id}_{p'} - P'P \end{pmatrix} \text{ and } Y_F := \begin{pmatrix} \mathrm{id}_q & 0 & R & Q \\ 0 & \mathrm{id}_{p'} & -P' & Z' \\ -Z & P & 0 & PZ' - ZQ \\ -Q' & -R' & 0 & Z'_2R'_2 \end{pmatrix}, \tag{2}$$

with inverses

$$X_F^- := \begin{pmatrix} \mathrm{id}_p - PP' & -P \\ P' & \mathrm{id}_{p'} \end{pmatrix} \text{ and } Y_F^- := \begin{pmatrix} Z_2R_2 & 0 & -R & -Q \\ P'Z - Z'Q' & 0 & P' & -Z' \\ Z & -P & \mathrm{id}_p & 0 \\ Q' & R' & 0 & \mathrm{id}_{q'} \end{pmatrix}, \tag{3}$$

such that

$$L' = Y_F^- L X_F. \tag{4}$$

In other words, the following diagram is exact commutative

$$\begin{tikzcd}
D^{1\times n} \arrow[r, ".L"] \arrow[d, shift right, ".Y_W"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_W"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\
D^{1\times n} \arrow[r, ".L'"] \arrow[u, shift right, ".Y_W^-"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_W^-"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0
\end{tikzcd}$$

### 2.2 Warfield's Theorem

A result due to Warfield (1978) asserts that that the size of 0 an id blocs in (1) can be reduced, whereas the new matrices are still equivalent. This result is based on the notion of *stable rank*. The definition of the latter requires to introduce various notions that we present now.

A column vector $u := (u_1 \cdots u_k)^T \in D^{k\times 1}$ is called *unimodular* if there exists a line $v \in D^{1\times k}$ such that $vu = 1$. Moreover, $u$ is said to be *stable* if there exist $d_1, \cdots, d_{k-1} \in D$ such that $(u_1 + d_1u_k \cdots u_{k-1} + d_{k-1}u_k)$ is unimodular. An integer $r$ is said to be in the stable rank of $D$ if whenever $k > r$, every column $u \in D^{k\times 1}$ is stable. The stable rank $\mathrm{sr}(D)$ of $D$ is the smallest integer in the stable rank of $D$.

Assume that the two matrices (1) are equivalent, then Warfield's Theorem asserts that if there exist two integers $r$ and $s$ such that

$$\begin{cases} s \leq \min(p+q', q+p'), \\ \mathrm{sr}(D) \leq \max(p+q'-s, q+p'-s), \\ r \leq \min(p, p'), \\ \mathrm{sr}(D) \leq \max(p-r, p'-r), \end{cases} \tag{5}$$

then the following $(n-r-s) \times (m-r)$ matrices are equivalent

$$\overline{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'-r} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \overline{L'} := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{p-r} & 0 \\ 0 & R' \end{pmatrix}, \tag{6}$$

and induce finite presentations of $M$ and $M'$, respectively.

In the next section, we introduce a procedure which computes invertible matrices $X_W$ and $Y_W$ such that

$$\overline{L'} = Y_W^{-} \overline{L} X_W.$$

## 3. EFFECTIVE WARFIELD'S THEOREM

Throughout this Section, we fix some notations. Let $M$ and $M'$ be two left $D$-modules, isomorphic with $f : M \xrightarrow{\sim} M'$, finitely presented by matrices $R \in D^{q \times p}$ and $R' \in D^{q' \times p'}$, respectively, and let $L$, $L'$, $X_F$, $Y_F$, $X_F^-$ and $Y_f^-$ be the matrices defined in (1) (2) and (3). Let $n := q+p'+p+q'$ and $m := p+p'$. Given a nonzero integer $k$, we let $\overline{k} := k-1$. For $1 \leq i \leq k$, the $i$-th vector of the canonical basis of $D^{1 \times k}$ is written $e_i^k$.

### *3.1 Reduction of the zero bloc*

In this section, we present the procedure for removing one 0 in $L$ and $L'$. Inductive applications of this procedure enables us to remove many 0. Without lost of generalities, we may suppose that $q+p' \leq p+q'$, and we assume that $\mathrm{sr}(D) \leq \overline{p+q'}$, so that the hypotheses of (5) are fullfilled for $s = 1$. Our purpose is to show that the following $\overline{n} \times m$ matrices are equivalent:

$$\tilde{L} := \begin{pmatrix} R & 0 \\ 0 & \mathrm{id}_{p'} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{L}' := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_p & 0 \\ 0 & R' \end{pmatrix}.$$

*Proposition 1.* There exist $\mathbf{c},\ \mathbf{u} \in D^{1 \times p}$ and $\mathbf{d},\ \mathbf{v} \in D^{1 \times \overline{q'}}$ such that

$$(0\ \mathbf{c}\ \mathbf{d}) \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F (e^n_{q+p'})^T = 1. \tag{7}$$

**Proof.** Getting terms of the $p'$-th column and $p'$-th line in the relation $\mathrm{id}_{p'} = P'P + Z'R'$, we get the following:

$$\sum_{k=1}^{p} P'_{p'k} P_{kp'} + \sum_{k=1}^{q'} Z'_{p'k} R'_{kp'} = 1.$$

From $\mathrm{sr}(D) \leq \overline{p+q'}$, we deduce that there exist $c_1, \cdots, c_p,\ d_1, \cdots, d_{\overline{q'}},\ u_1, \cdots, u_p,\ v_1, \cdots, v_{\overline{q'}} \in D$ such that

$$\sum_{k=1}^{p} c_k \left(P_{kp'} + u_k R'_{q'p'}\right) + \sum_{k=1}^{\overline{q'}} d_k \left(R'_{kp'} + v_k R'_{q'p'}\right) = 1. \tag{8}$$

Letting $\mathbf{c} := (c_1, \cdots, c_p)$, $\mathbf{d} := \left(-d_1, \cdots, -d_{\overline{q'}}\right)$, $\mathbf{u} := (-u_1, \cdots, -u_p)$ and $\mathbf{v} := \left(v_1, \cdots, v_{\overline{q'}}\right)$, the hand side of (8) is the left hand side of (7), which proves Proposition 1.

With the notations of Proposition 1, we introduce the lines $\tilde{\ell} \in D^{1 \times \overline{n}}$ and $\ell \in D^{1 \times n}$ defined as follows:

$$\tilde{\ell} := (0\ \mathbf{c}\ \mathbf{d}) \quad \text{and} \quad \ell := (0\ \mathbf{c}\ \mathbf{d}\ 0),$$

as well as the matrices $U \in D^{n \times n}$, $F \in D^{\overline{n} \times n}$ defined as follows:

$$U := \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad F := \left(\mathrm{id}_{\overline{n}}\ 0\right) U Y_F.$$

From Relation (7) and

$$F = \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & \mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & \mathbf{v}^T \end{pmatrix} Y_F,$$

we get:

$$1 = \tilde{\ell} F (e^n_{q+p'})^T = \ell U Y_F (e^n_{q+p'})^T. \tag{9}$$

We consider the matrices $\mathrm{pr},\ \mathrm{pr}' \in D^{n \times \overline{n}}$

$$\mathrm{pr} := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell} \\ \tilde{\ell} \end{pmatrix},$$

$$\mathrm{pr}' := \left(\mathrm{id}_n - (e^n_{q+p'})^T \ell U Y_F\right) \begin{pmatrix} \mathrm{id}_{\overline{q+p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p+q'} \end{pmatrix},$$

and $\iota,\ \iota' \in D^{\overline{n} \times n}$

$$\iota := \left(\mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell}\ \ F(e^n_{q+p'})^T\right),\ \iota' := \begin{pmatrix} \mathrm{id}_{\overline{q+p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p+q'} \end{pmatrix}.$$

*Proposition 2.* We have the following relations:

(1) $\iota \mathrm{pr} = \mathrm{id}_{\overline{n}}$,
(2) $\iota' \mathrm{pr}' = \mathrm{id}_{\overline{n}}$,
(3) $\ker(.\mathrm{pr}) = D\ell$,
(4) $\ker(.\mathrm{pr}') = D\ell U Y_F$.

**Proof.**

(1) From (9), we have $\left(F(e^n_{q+p'})^T \tilde{\ell}\right)^2 = F(e^n_{q+p'})^T \tilde{\ell}$, from which we deduce $\iota \mathrm{pr} = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
(2) We have $\iota'(e^n_{q+p'})^T = 0$, from which we deduce $\iota' \mathrm{pr}' = \mathrm{id}_{\overline{n}}$ by computing the matrix product.
(3) Considering the isomorphism $D^{1 \times n} \simeq D^{1 \times \overline{n}} \oplus D$, we have $\mathrm{pr} = \mathrm{pr}_1 \mathrm{pr}_2$, where $\mathrm{pr}_1 \in D^{n \times n}$ and $\mathrm{pr}_2 \in D^{n \times \overline{n}}$ are defined as follows:

$$\mathrm{pr}_1 := \begin{pmatrix} \mathrm{id}_{\overline{n}} - F(e^n_{q+p'})^T \tilde{\ell} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{pr}_2 := \begin{pmatrix} \mathrm{id}_{\overline{n}} \\ \tilde{\ell} \end{pmatrix}.$$

From (9), $\operatorname{im}(.\mathrm{pr}_1)$ is included in $\ker(.F(e^n_{q+p'})^T) \oplus D$ and the restriction of $.\mathrm{pr}_2$ to the latter is injective: for $(u,x) \in \left(\ker(.F(e^n_{q+p'})^T) \oplus D\right) \cap \ker(.\mathrm{pr}_2)$, we have $u + x\tilde{\ell} = 0$, which gives $x = 0$ and $u = 0$ by (9). Hence, we have $\ker(.\mathrm{pr}) = \ker(.\mathrm{pr}_1)$, that is $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) \oplus 0$. We conclude by showing $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right) = D\tilde{\ell}$: the right to left inclusion is due to (9), and the other one is due to the relation $x = \left(xF(e^n_{q+p'})^T\right)\tilde{\ell}$, for every $x$ in $\ker\left(\mathrm{id}_{\overline{n}} - .F(e^n_{q+p'})^T\tilde{\ell}\right)$.

(4) From (9), we have $D\ell UY_F \subseteq \ker(.\mathrm{pr}')$. The converse inclusion is due to the relation $x = x_{q+p'}\ell UY_F$, for every $x \in \ker(.\mathrm{pr}')$. Indeed, the first $q + \overline{p'}$ and the last $p+q'$ columns of $x$ and $x_{q+p'}\ell UY_F$ are equal since $x \in \ker(.\mathrm{pr}')$ implies

$$x\begin{pmatrix}\mathrm{id}_{q+\overline{p'}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{p+q'}\end{pmatrix} = x_{q+p'}\ell UY_F\begin{pmatrix}\mathrm{id}_{q+\overline{p'}} & 0\\ 0 & 0\\ 0 & \mathrm{id}_{p+q'}\end{pmatrix}.$$

Moreover, the $q+p'$-th colums of $x_{q+p'}\ell UY_F$ is computed by right multiplication by $(e^n_{q+p'})^T$ and is equal to $x_{q+p'}$ from (9).

*Theorem 3.* With the previous notations, we let

$$X_W := X_F, \quad Y_W := \iota UY_F\mathrm{pr}',$$

$$X_W^- := X_F, \quad Y_W^- := \iota' Y_F^- U^-\mathrm{pr}.$$

The following diagram is exact and commutative:

$$\begin{tikzcd}
D^{1\times\overline{n}} \arrow[r, ".\tilde{L}"] \arrow[d, shift right, ".Y_W"'] & D^{1\times m} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_W"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\
D^{1\times\overline{n}} \arrow[r, ".\tilde{L}'"] \arrow[u, shift right, ".Y_W^-"'] & D^{1\times m} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_W^-"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0
\end{tikzcd}$$

In particular, we have

$$\tilde{L}' = Y_W^-\tilde{L}X_W.$$

**Proof.** We only have to show that the diagram is commutative.

First, we show that $Y_W$ and $Y_W^-$ are inverse to each other. From Proposition 2, the lines of the following diagram are exact

$$\begin{tikzcd}
D \arrow[r, ".\ell"] \arrow[d, "\mathrm{id}_D"'] & D^{1\times n} \arrow[r, "\mathrm{pr}"] \arrow[d, shift right, ".UY_F"'] & D^{1\times\overline{n}} \arrow[r] \arrow[d, shift right, ".Y_W"'] & 0 \\
D \arrow[r, ".\ell UY_F"'] & D^{1\times n} \arrow[r, "\mathrm{pr}'"'] \arrow[u, shift right, ".Y_F^- U^-"'] & D^{1\times\overline{n}} \arrow[r] \arrow[u, shift right, ".Y_W^-"'] & 0
\end{tikzcd} \tag{10}$$

Moreover, it is also commutative. Indeed, $\mathrm{pr}Y_W$ is equal to $\mathrm{pr}\iota UY_F\mathrm{pr}'$ and from 1 of Proposition 2, we have $\operatorname{im}(.\mathrm{pr}\iota - \mathrm{id}_n) \subseteq \ker(.\mathrm{pr})$. By commutativity of the left rectangle and by exactness of the lines of (10), we have $(\mathrm{pr}\iota - \mathrm{id}_n)UY_F\mathrm{pr}' = 0$, so that $\mathrm{pr}Y_W = UY_F\mathrm{pr}'$. In the same manner, we show that $Y_F^-U^-\mathrm{pr} = \mathrm{pr}'Y_W^-$. By commutativity and exactness of (10) and from the equations $UY_FY_F^-U^- = Y_F^-U^-UY_F = \mathrm{id}_n$, we get $Y_WY_W^- = Y_W^-Y_W = \mathrm{id}_{\overline{n}}$.

Moreover, $Y_W\tilde{L}' = \tilde{L}X_F$ and $Y_W^-\tilde{L} = \tilde{L}'X_F^-$ follow from the following commutative diagram:

$$\begin{tikzcd}
D^{1\times\overline{n}} \arrow[r, shift left, "\iota"] \arrow[d, "\tilde{L}"'] & D^{1\times n} \arrow[l, shift left, "\mathrm{pr}"] \arrow[r, shift left, "U"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "U^-"] \arrow[r, shift left, "Y_F"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "Y_F^-"] \arrow[r, shift left, "\mathrm{pr}'"] \arrow[d, "L'"] & D^{1\times\overline{n}} \arrow[l, shift left, "\iota'"] \arrow[d, "\tilde{L}'"] \\
D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, "\mathrm{id}_m"] & D^{1\times m} \arrow[r, shift left, "X_F"] & D^{1\times m} \arrow[l, shift left, "X_F^-"] \arrow[r, "\mathrm{id}_m"] & D^{1\times m}
\end{tikzcd}$$

Indeed by computing the matrix products and from (4), we have the following relations, proven in Section 4.1:

$$\begin{aligned} &\iota L = \tilde{L}, \quad && UL = L, \quad && Y_FL' = LX_F, \quad && \mathrm{pr}'\tilde{L}' = L', \\ &\iota' L' = \tilde{L}', \quad && Y_F^-L = L'X_F^-, \quad && U^-L = L, \quad && \mathrm{pr}\tilde{L} = L. \end{aligned} \tag{11}$$

*3.2 Reduction of the identity bloc*

In this section, we present the procedure for removing on 1 in $L$ and $L'$, many 1 are removing by induction. Without lost of generality, we assume that $p \le p'$ and $\mathrm{sr}(D) \le \overline{p'}$, hence (5) is fullfilled for $r = 1$. Our purpose is to show that the following $\overline{n}\times\overline{m}$ matrices are equivalent:

$$\tilde{L} := \begin{pmatrix} R & 0\\ 0 & \mathrm{id}_{\overline{p'}}\\ 0 & 0\end{pmatrix} \quad\text{and}\quad \tilde{L}' := \begin{pmatrix} 0 & 0\\ \mathrm{id}_{\overline{p}} & 0\\ 0 & R'\end{pmatrix}.$$

*Proposition 4.* We let $k := q+p'+p$ and $l := p+q'$. There exist $\mathbf{c} \in D$ and lines $\mathbf{d},\ \mathbf{u} \in D^{1\times\overline{p'}}$ such that

$$\left(\mathbf{c}e^n_kY_F^-\begin{pmatrix}\mathrm{id}_q & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \mathrm{id}_l\end{pmatrix} + \begin{pmatrix}0 & \mathbf{d} & 0\end{pmatrix}\right)\begin{pmatrix}\mathrm{id}_q & 0 & 0 & 0\\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T & 0\\ 0 & 0 & 0 & \mathrm{id}_l\end{pmatrix}Y_F(e^n_k)^T = 1, \tag{12}$$

and

$$\left(\mathbf{c}e^m_pX_F^-\begin{pmatrix}\mathrm{id}_p & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}0 & \mathbf{d}\end{pmatrix}\right)\begin{pmatrix}\mathrm{id}_p & 0 & 0\\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T\end{pmatrix}X_F(e^m_p)^T = 1. \tag{13}$$

**Proof.** We have

$$\left(1 - PP'_{pp}\right) + \sum_{i=1}^{p'} P_{pi}P'_{ip} = 1.$$

By projecting this equality on the finitely presented left module $N := D/D\left(1 - PP'_{pp}\right)$, the latter is spanned by $[P'_{1p}]_N, \cdots, [P'_{p'p}]_N$. From McConnell and Robson (2001), $\mathrm{sr}(D)$ is in the stable range of $N$, so that there exists $\mathbf{u} := \left(u_1, \cdots, u_{\overline{p'}}\right) \in D^{1\times\overline{p'}}$ such that $N$ is spanned by $[P'_{1p'} + u_1P'_{p'p}]_{\mathbf{N}}$. Hence, $\mathbf{c} \in D$ and $d_1, \cdots, d_{\overline{p'}} \in D^{1\times\overline{p'}}$ exist such that

$$\mathbf{c}\left(1 - PP'_{pp}\right) + \sum_{k=1}^{\overline{p'}} d_k\left(P'_{kp} + u_kP'_{p'p}\right) = 1. \tag{14}$$

Letting $\mathbf{d} := \left(-d_1, \cdots, -d_{\overline{p'}}\right) \in D^{1\times\overline{p'}}$ and from the relation $\mathrm{id}_p = ZR + PP'$, the left hand sides of (12) and (13) are both equal to the left hand side of (14), which proves Proposition 4.

With the notations of Proposition 4, we introduce the lines $\tilde{\ell}_r \in D^{1\times\overline{n}}$, $\ell_r \in D^{1\times n}$, $\tilde{\ell}_g \in D^{1\times\overline{m}}$ and $\ell_g \in D^{1\times m}$ defined as follows:

$$\tilde{\ell}_r := \mathbf{c}e_k^n Y_F^- \begin{pmatrix} \mathrm{id}_q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} & 0 \end{pmatrix}, \qquad \ell_r := \tilde{\ell}_r \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix},$$

$$\tilde{\ell}_g := \mathbf{c}e_p^m X_F^- \begin{pmatrix} \mathrm{id}_p & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{d} \end{pmatrix}, \qquad \ell_g := \begin{pmatrix} \tilde{\ell}_g & 0 \end{pmatrix}.$$

as well as the matrices $U_r \in D^{n\times n}$, $U_g \in D^{m\times m}$, $F_r \in D^{\overline{n}\times n}$ and $F_g \in D^{\overline{m}\times m}$ defined as follows:

$$U_r := \begin{pmatrix} \mathrm{id}_q & 0 & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{id}_l \end{pmatrix}, \qquad F_r := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix} U_r Y_F,$$

$$U_g := \begin{pmatrix} \mathrm{id}_p & 0 & 0 \\ 0 & \mathrm{id}_{\overline{p'}} & \mathbf{u}^T \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_g := \begin{pmatrix} \mathrm{id}_{\overline{m}} & 0 \end{pmatrix} U_g X_F.$$

Explicitly, we have:

$$\tilde{\ell}_r = \begin{pmatrix} \mathbf{c}Z_{p.} & \mathbf{d} & \mathbf{c}e_p^l \end{pmatrix} \qquad \ell_r = \begin{pmatrix} \mathbf{c}Z_{p.} & \mathbf{d} & 0 & \mathbf{c}e_p^l \end{pmatrix},$$

$$\tilde{\ell}_g = \begin{pmatrix} \mathbf{c}\left(\mathrm{id}_p - PP'\right)_{p.} & \mathbf{d} \end{pmatrix} \qquad \ell_g = \begin{pmatrix} \mathbf{c}\left(\mathrm{id}_p - PP'\right)_{p.} & \mathbf{d} & 0 \end{pmatrix},$$

where the index $p.$ denotes the $p$-th line of the considered matrix.

As we did for Relation (9), we obtain the following relations:

$$1 = \tilde{\ell}_r F_r (e_k^n)^T = \ell_r U_r Y_F (e_k^n)^T, \tag{15}$$

and

$$1 = \tilde{\ell}_g F_g (e_p^m)^T = \ell_g U_g X_F (e_p^m)^T. \tag{16}$$

Let us consider $\mathrm{pr}_r$, $\mathrm{pr}'_r \in D^{n\times\overline{n}}$, $\mathrm{pr}_g$, $\mathrm{pr}'_g \in D^{m\times\overline{m}}$, $\iota_r$, $\iota'_r \in D^{\overline{n}\times n}$ and $\iota_g$, $\iota'_g \in D^{\overline{m}\times m}$ defined as follows:

$$\mathrm{pr}_r := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_l \end{pmatrix} \left(\mathrm{id}_{\overline{n}} - F_r \left(e_k^n\right)^T \tilde{\ell}_r\right) + (e_k^n)^T \tilde{\ell}_r,$$

$$\mathrm{pr}'_r := \left(\mathrm{id}_n - (e_k^n)^T \ell_r U_r Y_F\right) \begin{pmatrix} \mathrm{id}_{\overline{k}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{q'} \end{pmatrix},$$

$$\iota_r := \left(\mathrm{id}_{\overline{n}} - F_r \left(e_k^n\right)^T \tilde{\ell}_r\right) \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix} + F_r \left(e_k^n\right)^T e_{q+p'}^n,$$

$$\iota'_r := \begin{pmatrix} \mathrm{id}_{\overline{k}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{q'} \end{pmatrix},$$

$$\mathrm{pr}_g := \begin{pmatrix} \mathrm{id}_{\overline{m}} - F_g (e_p^m)^T \tilde{\ell}_g \\ \tilde{\ell}_g \end{pmatrix},$$

$$\mathrm{pr}'_g := \left(\mathrm{id}_m - (e_p^m)^T \ell_g U_g X_F\right) \begin{pmatrix} \mathrm{id}_{\overline{p}} & 0 \\ 0 & 0 \\ 0 & \mathrm{id}_{p'} \end{pmatrix},$$

$$\iota_g := \begin{pmatrix} \mathrm{id}_{\overline{m}} - F_g (e_p^m)^T \tilde{\ell}_g & F_g (e_p^m)^T \end{pmatrix},$$

$$\iota'_g := \begin{pmatrix} \mathrm{id}_{\overline{p}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_{p'} \end{pmatrix}.$$

By adapting the arguments of the proof of Proposition 2, we get the following.

*Proposition 5.* We have the following relations

| | |
|---|---|
| (1) $\iota_r \mathrm{pr}_r = \mathrm{id}_{\overline{n}}$, | (5) $\ker(.\mathrm{pr}_r) = D\ell_r$, |
| (2) $\iota_r{}' \mathrm{pr}_r{}' = \mathrm{id}_{\overline{n}}$, | (6) $\ker(.\mathrm{pr}_r{}') = D\ell_r U_r Y_F$, |
| (3) $\iota_g \mathrm{pr}_g = \mathrm{id}_{\overline{m}}$, | (7) $\ker(.\mathrm{pr}_g) = D\ell_g$, |
| (4) $\iota_g{}' \mathrm{pr}_g{}' = \mathrm{id}_{\overline{m}}$, | (8) $\ker(.\mathrm{pr}_g{}') = D\ell_g U_g X_F$. |

*Theorem 6.* With the previous notations, we let

$$Y_W := \iota_r U_r Y_F \mathrm{pr}'_r, \qquad X_W := \iota_g U_g X_F \mathrm{pr}'_g,$$

$$Y_W^- := \iota'_r Y_F^- U_r^- \mathrm{pr}_r, \qquad X_W^- := \iota'_g U_g X_F^- \mathrm{pr}_g.$$

The following diagram is exact and commutative:

$$\begin{tikzcd} D^{1\times\overline{n}} \arrow[r, ".\tilde{L}"] \arrow[d, shift right, ".Y_W"'] & D^{1\times\overline{m}} \arrow[r, "\pi\oplus 0"] \arrow[d, shift right, ".X_W"'] & M \arrow[r] \arrow[d, shift right, "f"'] & 0 \\ D^{1\times\overline{n}} \arrow[r, ".\tilde{L'}"] \arrow[u, shift right, ".Y_W^-"'] & D^{1\times\overline{m}} \arrow[r, "0\oplus\pi'"] \arrow[u, shift right, ".X_W^-"'] & M' \arrow[r] \arrow[u, shift right, "f^-"'] & 0 \end{tikzcd}$$

In particular, we have

$$\tilde{L'} = Y_W^- \tilde{L} X_W.$$

**Proof.** We only have to show that the diagram is commutative.

We show that $Y_W$ and $Y_W^-$ (respectively, $X_W$ and $X_W^-$) are inverse to each over in the same manner that we did in the proof of Theorem 3.

Moreover, $Y_W \tilde{L'} = \tilde{L} X_W$ and $Y_W^- \tilde{L} = \tilde{L'} X_W^-$ follow from the following commutative diagram:

$$\begin{tikzcd} D^{1\times\overline{n}} \arrow[r, shift left, "\iota_r"] \arrow[d, "\tilde{L}"'] & D^{1\times n} \arrow[l, shift left, "\mathrm{pr}_r"] \arrow[r, shift left, "U_r"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "U_r^-"] \arrow[r, shift left, "Y_F"] \arrow[d, "L"] & D^{1\times n} \arrow[l, shift left, "Y_F^-"] \arrow[r, shift left, "\mathrm{pr}'_r"] \arrow[d, "L'"] & D^{1\times\overline{n}} \arrow[l, shift left, "\iota'_r"] \arrow[d, "\tilde{L'}"] \\ D^{1\times\overline{m}} \arrow[r, shift left, "\iota_g"] & D^{1\times m} \arrow[l, shift left, "\mathrm{pr}_g"] \arrow[r, shift left, "U_g"] & D^{1\times m} \arrow[l, shift left, "U_g^-"] \arrow[r, shift left, "X_F"] & D^{1\times m} \arrow[l, shift left, "X_F^-"] \arrow[r, shift left, "\mathrm{pr}'_g"] & D^{1\times\overline{m}} \arrow[l, shift left, "\iota'_g"] \end{tikzcd}$$

Indeed by computing the matrix products and from (4), we have the following relations, proven in Section 4.2:

$$\begin{aligned} &\iota_r L = \tilde{L}\iota_g, \quad U_r L = L U_g, \quad Y_F L' = L X_F, \quad \mathrm{pr}_r \tilde{L} = L \mathrm{pr}_g, \\ &\iota'_r L' = \tilde{L'}\iota'_g, \quad Y_F^- L = L' X_F^-, \quad U_r^- L = L U_g^-, \quad \mathrm{pr}'_r \tilde{L'} = L' \mathrm{pr}'_g. \end{aligned} \tag{17}$$

## 4. ANNEX: PROOFS

### *4.1 Proof of Formulas* (11)

We have to show the following relations

$$\iota L = \tilde{L}, \tag{18}$$

$$U L = L, \tag{19}$$

$$Y_F L' = L X_F, \tag{20}$$

$$\mathrm{pr}' \tilde{L'} = L', \tag{21}$$

$$\iota' L' = \tilde{L'}, \tag{22}$$

$$Y_F^- L = L' X_F^-, \tag{23}$$

$$U^- L = L, \tag{24}$$

$$\mathrm{pr} \tilde{L} = L. \tag{25}$$

The two relations (20) and (23) come from Cluzeau and Quadrat (2011), (19) and (22) are proven by direct computations, (18) and (24) are proven by direct computations using respectively the following two relations:

$$\ell L = 0 \text{ and } U^{-} = \begin{pmatrix} \mathrm{id}_{q+p'} & 0 & 0 & 0 \\ 0 & \mathrm{id}_p & 0 & -\mathbf{u}^T \\ 0 & 0 & \mathrm{id}_{\overline{q'}} & -\mathbf{v}^T \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In order to prove (21), we first show $\mathrm{pr}'\iota' L' = L'$: from 2 and 4 of Proposition 2, $\mathrm{im}(\mathrm{pr}'\iota' - \mathrm{id}_n)$ is included $D\ell UY_F$ and using (19), (20) and $\ell L = 0$, we have $\ell UY_F L' = \ell LUX_F = 0$, which proves the desired relation. Moreover, from (22), we get $\mathrm{pr}'\iota' L' = \mathrm{pr}'\tilde{L}'$ which, with $\mathrm{pr}'\iota' L' = L'$, gives (21). We show (25) in the same manner using (18).

### 4.2 Proof of Formulas (17)

We have to show the following relations

$$\iota_r L = \tilde{L}\iota_g, \quad (26)$$

$$U_r L = LU_g, \quad (27)$$

$$Y_F L' = LX_F, \quad (28)$$

$$\mathrm{pr}'_r \tilde{L}' = L'\mathrm{pr}'_g, \quad (29)$$

$$\iota'_r L' = \tilde{L}'\iota'_g, \quad (30)$$

$$Y_F^{-} L = L' X_F^{-}, \quad (31)$$

$$U_r^{-} L = LU_g^{-}, \quad (32)$$

$$\mathrm{pr}_r \tilde{L} = L\mathrm{pr}_g. \quad (33)$$

The two relations (28) and (31) come from Cluzeau and Quadrat (2011), (27) and (30) are proven by direct computations, (32) is computed by a direct computation using the inverse formulas for $U_g$ and $U_r$ which are analog to the inverse of $U$ given in Section 4.1.

Let us show (26). For that, we decompose $\iota_r$ and $\iota_g$ in 3 parts, as follows:

$$\iota_r^1 := \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix}, \qquad \iota_g^1 := \begin{pmatrix} \mathrm{id}_{\overline{m}} & 0 \end{pmatrix},$$

$$\iota_r^2 := F_r (e_k^n)^T \tilde{\ell}_r \begin{pmatrix} \mathrm{id}_{q+\overline{p'}} & 0 & 0 \\ 0 & 0 & \mathrm{id}_l \end{pmatrix}, \qquad \iota_g^2 := F_g (e_p^m)^T \ell_g,$$

$$\iota_r^3 := F_r (e_k^n)^T e_{q+p'}^n, \qquad \iota_g^3 := F_g (e_p^m)^T e_{p+p'}^m,$$

so that we have $\iota_r = \iota_r^1 - \iota_r^2 + \iota_r^3$ and $\iota_g = \iota_g^1 - \iota_g^2 + \iota_g^3$. By computing the matrix products, we show that $\iota_r^1 L = \tilde{L}\iota_g^1$, that the first $\overline{m}$ columns of $\iota_r^2 L$ and $\tilde{L}\iota_g^2$ are both equal to 0 and their $m$-th column are both equal to

$$\begin{pmatrix} R_{.p} \\ -\left(P'_{ip} + u_i P'_{p'p}\right)_{1 \le i \le \overline{p'}} \\ 0 \\ 0 \end{pmatrix}.$$

Finally, by computing the matrix products, we show that $\iota_r^3 L$ and $\tilde{L}\iota_g^3$ are respectively equal to

$$\begin{pmatrix} R_{.p} \\ \left(\mathrm{id}_{p'\,ip'} - P'P_{ip'}\right)_{1 \le i \le p'} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}\,(ZR)_{p.} & \mathbf{d} & 0 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} R_{.p} \\ \left(\mathrm{id}_{p'\,ip'} - P'P_{ip'}\right)_{1 \le i \le p'} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \mathbf{c}\left(\mathrm{id}_p - PP'\right)_{p.} & \mathbf{d} & 0 \end{pmatrix}$$

so that they are equal from the relation $\mathrm{id}_p = PP' + ZR$, which proves (26).

Let us show (29). By using Relation (30), and 1 of Proposition 5, we have $\mathrm{pr}'_r \iota'_r L' \mathrm{pr}'_g = \mathrm{pr}'_r \tilde{L}'$. We proceed as in the proof of (21): we only have to show that $\mathrm{im}(\mathrm{pr}'_r \iota'_r - \mathrm{id}_n) L'$ is included in $\ker\left(\mathrm{pr}'_g\right)$. We have $\mathrm{im}(\mathrm{pr}'_r \iota'_r - \mathrm{id}_n) \subseteq D\ell_r U_r Y_F$, $\ell_r U_r Y_F L' = \ell_r L U_g X_F$ and $\ell_r L = \ell_g$, so that $\ell_r U_r Y_F L' = \ell_g U_g X_F \in \ker(\mathrm{pr}'_g)$. With the same arguments, we show (29).

REFERENCES


Cluzeau, T. and Quadrat, A. (2011). A constructive version of Fitting's theorem on isomorphisms and equivalences of linear systems. In *7th International Workshop on Multidimensional Systems (nDs)*, 1–8. Poitiers, France. URL `https://hal.archives-ouvertes.fr/hal-00682765`.

Fitting, H. (1936). Über den Zusammenhang zwischen dem Begriff der Gleichartigkeit zweier Ideale und dem Äquivalenzbegriff der Elementarteilertheorie. *Math. Ann.*, 112(1), 572–582.

McConnell, J.C. and Robson, J.C. (2001). *Noncommutative Noetherian rings*, volume 30 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, revised edition. With the cooperation of L. W. Small.

Rotman, J.J. (2009). *An introduction to homological algebra*. Universitext. Springer, New York, second edition.

Warfield, Jr., R.B. (1978). Stable equivalence of matrices and resolutions. *Comm. Algebra*, 6(17), 1811–1828.