Math 286
Introduction to Differential Equations

Thomas Honold

ZJU-UIUC Institute

Fall Semester 2021

Outline

Today's Lecture: The Laplace Transform

Integral Transforms

Integral transforms are maps $f \mapsto F$, which assign to a function f from a certain domain (e.g., the set of integrable functions $f\colon [a,b] \to \mathbb{C}$) another function F of the form

$$F(s) = \int_a^b K(s,t) f(t)\,\mathrm{d}t\,. \tag{IT}$$

Here $K(s,t)$ is a two-variable function called the *kernel* of the integral transform, and $F(s)$ is defined for all $s \in \mathbb{C}$ for which the integral in (IT) exists.

Definition

The *Laplace transform* is the integral transform with $(a,b) = (0,\infty)$ and $K(s,t) = \mathrm{e}^{-st}$, i.e.,

$$F(s) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t\,.$$

The Laplace transform will be denoted by $\mathcal{L}$; we will write $F = \mathcal{L}f$ or, making explicit reference to the arguments, $F(s) = \mathcal{L}\{f(t)\}$.

An Appropriate Domain for $\mathcal{L}$

Definition

Let $f\colon [0,\infty) \to \mathbb{C}$ be a function.

1. f is said to be *piecewise continuous* if (i) the set Δ of discontinuities of f is discrete (i.e., has no accumulation point in $\mathbb{R}$), (ii) f is continuous on each connected component of $[0,\infty) \setminus \Delta$ (which must be an interval), and (iii) for every $\alpha \in \Delta$ the one-sided limits $f(\alpha+) = \lim_{t\downarrow\alpha} f(t)$, $f(\alpha-) = \lim_{t\uparrow\alpha} f(t)$ exist (with the obvious adjustment in the case $\alpha = 0$).
2. f is said to be of (at most) *exponential order* for $t \to \infty$ if there exist constants $a \in \mathbb{R}$, $K > 0$, $M > 0$ such that $|f(t)| \leq K\mathrm{e}^{at}$ whenever $t \geq M$.

Piecewise continuous functions of exponential order on $[0,\infty)$ form an appropriate domain for the Laplace transform, i.e., for every such function f the function $\mathcal{L}f$ is well-defined.

Changing the values of f on a discrete subset of $[0,\infty)$ doesn't change $\mathcal{L}f$. It is even sufficient if f is undefined for some $t \in \Delta$.

Notes

- Informally speaking, a piecewise continuous function may have only jump discontinuities; there can be infinitely many jump discontinuities in $[0,\infty)$ (as in the case $t \mapsto \lfloor t \rfloor$), but only finitely many in every bounded subinterval $[0,R]$.
- The condition in Part (2) is equivalent to $f(t) = \mathrm{O}(\mathrm{e}^{at})$ for $t \to \infty$ and should be viewed as a property depending on a. It doesn't necessarily mean "f grows exponentially" (since $a < 0$ is allowed and, moreover, only an upper bound for $|f(t)|$ is given), and becomes stronger if we decrease a. (In fact $a < b$ implies $\mathrm{e}^{at} = \mathrm{o}(\mathrm{e}^{bt})$ for $t \to \infty$.)
- If $f\colon [0,\infty) \to \mathbb{R}$ is of exponential order (i.e., there exists $a \in \mathbb{R}$ such that $f(t) = \mathrm{O}(\mathrm{e}^{at})$ for $t \to \infty$), we can define the *exact exponential order* of f as

$$\mathrm{eo}(f) = \inf\left\{a \in \mathbb{R};\, f(t) = \mathrm{O}(\mathrm{e}^{at}) \text{ for } t \to \infty\right\}.$$

Using the infimum is necessary, since, e.g., $t^n = \mathrm{O}(\mathrm{e}^{at})$ for $t \to \infty$ whenever $a > 0$, but $t^n \neq \mathrm{O}(1)$. (Thus all nonzero polynomials have exact exponential order 0.)

Examples

1. All nonzero polynomials in $\mathbb{R}[t]$ have exact exponential order 0. The same is true of nonzero rational functions $f(t) = p(t)/q(t)$, $p(t), q(t) \in \mathbb{R}[t] \setminus \{0\}$.
2. $t \mapsto c_1 \mathrm{e}^{a_1 t} + c_2 \mathrm{e}^{a_2 t} + \cdots + c_n \mathrm{e}^{a_n t}$ ($a_1 < a_2 < \cdots < a_n$, $c_i \neq 0$ for $1 \leq i \leq n$) has exact exponential order a_n.
3. $\sin(at)$, $\cos(at)$ for $a \neq 0$ (more generally, non-vanishing trigonometric polynomials) have exact exponential order 0.
4. $t \mapsto \mathrm{e}^{t^2}$ is not of exponential order (or of exact exponential order $+\infty$), because for $t \to \infty$ it increases faster than any exponential function e^{at}.

   On the other hand, the reciprocal function $t \mapsto \mathrm{e}^{-t^2}$ is $\mathrm{O}(\mathrm{e}^{at})$ (and $\mathrm{o}(\mathrm{e}^{at})$ as well) for every $a \in \mathbb{R}$, and hence according to the definition has exact exponential order $-\infty$.

Theorem

Suppose $f\colon [0,\infty) \to \mathbb{C}$ is piecewise continuous and of exact exponential order $a \in \mathbb{R} \cup \{\pm\infty\}$.

1. *If $a = +\infty$ then $\mathcal{L}(f)$ need not be defined for any $s \in \mathbb{C}$.*
2. *If $a \in \mathbb{R}$ then $\mathcal{L}(f)$ is defined and analytic at least for all s in the open half plane $\mathrm{Re}(s) > a$.*
3. *If $a = -\infty$ then $\mathcal{L}(f)$ is defined and analytic for all $s \in \mathbb{C}$ (a so-called entire function).*

Moreover, in Cases 2 and 3 the Laplace transform $F = \mathcal{L}(f)$ can be differentiated under the integral sign':

$$F'(s) = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = -\int_0^\infty t\,f(t)\mathrm{e}^{-st}\,\mathrm{d}t = -\mathcal{L}\{t\,f(t)\}.$$

Notes

- Differentiating repeatedly gives $F^{(n)}(s) = (-1)^n\mathcal{L}\{t^n f(t)\}$ for $n \in \mathbb{N}$.
- In Case 2 it is possible that $\mathcal{L}(f)$ is defined and analytic in a larger region than $\mathrm{Re}(s) > a$; cf. exercises.

Proof.

Since $f = u + \mathrm{i}v$ implies $\mathcal{L}f = \mathcal{L}u + \mathrm{i}\mathcal{L}v$, we may assume w.l.o.g. that f is real-valued.

Suppose $|f(t)| \leq K\,\mathrm{e}^{at}$ for $t \geq M$. We claim that $\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ converges uniformly (and absolutely) in every closed half plane $\mathrm{Re}(s) \geq a + \delta$, $\delta > 0$.
Indeed, for such s and $t \geq M$ we have

$$\left|f(t)\mathrm{e}^{-st}\right| = |f(t)|\,\mathrm{e}^{-\mathrm{Re}(s)t} \leq K\,\mathrm{e}^{(a-\mathrm{Re}\,s)t} \leq K\,\mathrm{e}^{-\delta t}.$$

Since this bound is independent of s and $\int_M^\infty \mathrm{e}^{-\delta t}\,\mathrm{d}t$ converges, we can apply the Weierstrass test for uniform convergence of improper parameter integrals to conclude that the convergence of $F(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ in $\mathrm{Re}(s) \geq a + \delta$ is uniform. In particular $F(s)$ is defined for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > a$.

Since $t \mapsto t\,f(t)$ is $\mathrm{O}(\mathrm{e}^{at})$ for $t \to \infty$ as well, the integral $\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = -\int_0^\infty t\,f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ also converges uniformly in $\mathrm{Re}(s) \geq a + \delta$ for every $\delta > 0$, so that the necessary assumptions for differentiating $F(s)$ under the integral sign are satisfied.
$\Longrightarrow$ F is complex differentiable (and hence analytic) in $\mathrm{Re}(s) > a$. $\square$

Note on the proof

Writing $s = x + \mathrm{i}y$ we have

$$\begin{aligned}F(s) = F(x+\mathrm{i}y) &= \int_0^\infty f(t)\mathrm{e}^{-xt-\mathrm{i}yt}\,\mathrm{d}t\\ &= \int_0^\infty f(t)\mathrm{e}^{-xt}\big(\cos(yt) - \mathrm{i}\sin(yt)\big)\,\mathrm{d}t\\ &= \int_0^\infty f(t)\mathrm{e}^{-xt}\cos(yt)\,\mathrm{d}t + \mathrm{i}\int_0^\infty -f(t)\mathrm{e}^{-xt}\sin(yt)\,\mathrm{d}t\\ &= u(x,y) + \mathrm{i}\,v(x,y), \quad \text{say.}\end{aligned}$$

Using this formula and the results on partial differentiation of real-variable functions under the integral sign, one can show that u, v are partially differentiable and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$. From this it follows without resort to Complex Analysis that F is complex differentiable; cf. our discussion of real vs. complex differentiability in Calculus III.

Examples

1. $\mathcal{L}\{1\} = \int_0^\infty 1\,\mathrm{e}^{-st}\,\mathrm{d}t = \left[-\frac{1}{s}\mathrm{e}^{-st}\right]_0^\infty = \frac{1}{s}$ for $s > 0$.

   More generally, this holds for $\mathrm{Re}(s) > 0$ since for $s = x + \mathrm{i}y$, $x > 0$, we still have $\mathrm{e}^{-st} = \mathrm{e}^{-xt}\mathrm{e}^{-\mathrm{i}yt} \to 0$ for $t \to \infty$.

2. $\mathcal{L}\{\mathrm{e}^t\} = \int_0^\infty \mathrm{e}^t\,\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{-(s-1)t}\,\mathrm{d}t = \frac{1}{s-1}$ for $\mathrm{Re}(s) > 1$.

3. $\mathcal{L}\{\cos t\} = \int_0^\infty \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}\right)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{2}\int_0^\infty \mathrm{e}^{-(s-\mathrm{i})t} + \mathrm{e}^{-(s+\mathrm{i})t}\,\mathrm{d}t = \frac{1}{2}\left[\frac{1}{s-\mathrm{i}} + \frac{1}{s+\mathrm{i}}\right] = \frac{1}{2}\,\frac{2s}{(s-\mathrm{i})(s+\mathrm{i})} = \frac{s}{s^2+1}$ for $\mathrm{Re}(s) > 0$.

4. $\mathcal{L}\{\sin t\} = \int_0^\infty \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}\right)\mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{2\mathrm{i}}\left[\frac{1}{s-\mathrm{i}} + \frac{1}{s+\mathrm{i}}\right] = \frac{1}{2\mathrm{i}}\,\frac{2\mathrm{i}}{(s-\mathrm{i})(s+\mathrm{i})} = \frac{1}{s^2+1}$ for $\mathrm{Re}(s) > 0$.

Examples (cont'd)

5

$$
\begin{aligned}
\mathcal{L}\{t^n\} &= \int_0^\infty t^n \mathrm{e}^{-st}\,\mathrm{d}t \\
&= \int_0^\infty \left(\frac{\tau}{s}\right)^n \mathrm{e}^{-\tau}\,\frac{\mathrm{d}\tau}{s} \qquad \text{(Subst. } \tau = st,\ \mathrm{d}\tau = s\,\mathrm{d}t) \\
&= \frac{1}{s^{n+1}} \int_0^\infty \tau^n \mathrm{e}^{-\tau}\,\mathrm{d}\tau = \frac{n!}{s^{n+1}}
\end{aligned}
$$

for $\mathrm{Re}(s) > 0$.

More generally, we have

$$
\mathcal{L}\{t^r\} = \frac{1}{s^{r+1}} \int_0^\infty \tau^r \mathrm{e}^{-\tau}\,\mathrm{d}\tau = \frac{\Gamma(r+1)}{s^{r+1}}
$$

for $\mathrm{Re}(s) > 0$, $r > -1$. (It doesn't matter here that $t \mapsto t^r$ isn't defined at $t = 0$ for $-1 < r < 0$.)

In particular, $\mathcal{L}\{t^{-1/2}\} = \Gamma(1/2)s^{-1/2} = \sqrt{\pi}\,s^{-1/2}$, i.e., $t \mapsto 1/\sqrt{t}$ is an eigenfunction of $\mathcal{L}$ for the eigenvalue $\sqrt{\pi}$.

Examples

6. We compute the Laplace transform of the "staircase" function $t \mapsto \lfloor t \rfloor$, which is defined for $\mathrm{Re}(s) > 0$.
   Since $\lfloor t \rfloor = n$ for $t \in [n, n+1)$, we obtain

$$
\begin{aligned}
\mathcal{L}\{\lfloor t \rfloor\} &= \int_0^\infty \lfloor t \rfloor \mathrm{e}^{-st}\,\mathrm{d}t = \sum_{n=0}^\infty \int_n^{n+1} n\,\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \sum_{n=0}^\infty n\left[-\frac{1}{s}\,\mathrm{e}^{-st}\right]_n^{n+1} = \frac{1}{s}\sum_{n=0}^\infty n\left(\mathrm{e}^{-ns} - \mathrm{e}^{-(n+1)s}\right) \\
&= \frac{1}{s}\left(\mathrm{e}^{-s} - \mathrm{e}^{-2s} + 2\,\mathrm{e}^{-2s} - 2\,\mathrm{e}^{-3s} + 3\,\mathrm{e}^{-3s} - 3\,\mathrm{e}^{-4s} + \cdots\right) \\
&= \frac{1}{s}\left(\mathrm{e}^{-s} + \mathrm{e}^{-2s} + \mathrm{e}^{-3s} + \cdots\right) \\
&= \frac{1}{s}\,\frac{\mathrm{e}^{-s}}{1 - \mathrm{e}^{-s}} = \frac{1}{s\,(\mathrm{e}^{s} - 1)}.
\end{aligned}
$$

Exercise

Suppose that $f\colon [0,\infty) \to \mathbb{C}$ is piecewise continuous. Show:

1. If $\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ converges absolutely for $s = s_0$, it converges absolutely for $\mathrm{Re}(s) \geq \mathrm{Re}(s_0)$.
2. If $\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ converges for $s = s_0$, it converges for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$ and for such s satisfies
$$\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = (s - s_0)\int_0^\infty \phi(t)\mathrm{e}^{-(s-s_0)t}\,\mathrm{d}t$$
with $\phi(t) = \int_0^t f(\tau)\mathrm{e}^{-s\tau}\,\mathrm{d}\tau$. Moreover, the integral $\int_0^\infty \phi(t)\mathrm{e}^{-(s-s_0)t}\,\mathrm{d}t$ converges absolutely for $\mathrm{Re}(s) > \mathrm{Re}(s_0)$.
3. There exist numbers $-\infty \leq \beta \leq \alpha \leq \infty$, such that $\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ diverges for $\mathrm{Re}(s) < \beta$, converges conditionally (i.e., not absolutely) for $\beta < \mathrm{Re}(s) < \alpha$ and converges absolutely for $\mathrm{Re}(s) > \alpha$. Moreover, on the line $\mathrm{Re}(s) = \alpha$ the Laplace integral converges absolutely either for all s or for no s.
4. $F(s) := \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ is analytic in $\mathrm{Re}(s) > \beta$.

The numbers α, β defined in the preceding exercise are called *abscissa of absolute convergence*, resp., *abscissa of convergence* of the Laplace integral $\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$, and the corresponding lines $\mathrm{Re}(s) = \alpha$, $\mathrm{Re}(s) = \beta$ *line of absolute convergence*, resp., *line of convergence*.

If f has exact exponential order a, we must have $\beta \leq \alpha \leq a$. Both inequalities may be strict. For the second inequality this is shown in the following exercise.

Exercise

Let $f\colon [0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} \mathrm{e}^n & \text{if } |t - n| < \mathrm{e}^{-2n} \text{ for some } n \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

Show that f has exact exponential order 1, but $\int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t$ converges (absolutely) for $\mathrm{Re}(s) = 0$.

Linearity

Suppose $\mathcal{L}f_1$ is defined for $\mathrm{Re}(s) > a_1$ and $\mathcal{L}f_2$ for $\mathrm{Re}(s) > a_2$. Then for any $c_1, c_2 \in \mathbb{C}$ the function $\mathcal{L}(c_1 f_1 + c_2 f_2)$ is defined for $\mathrm{Re}(s) > \max\{a_1, a_2\}$ and satisfies

$$\mathcal{L}\left\{c_1\, f_1(t) + c_2\, f_2(t)\right\} = c_1\mathcal{L}\left\{f_1(t)\right\} + c_2\mathcal{L}\left\{f_2(t)\right\}.$$

The proof is trivial.

As an application of linearity, we get from $\mathcal{L}\{t^n\} = n!/s^{n+1}$ the Laplace transform of any polynomial:

$$\mathcal{L}\left\{c_0 + c_1 t + c_2 t^2 + \dots + c_d t^d\right\} = \frac{c_0}{s} + \frac{c_1}{s^2} + \frac{c_2\, 2!}{s^3} + \dots + \frac{c_d\, d!}{s^{d+1}}$$

or, writing $a_n = n!c_n$,

$$\mathcal{L}\left\{\frac{a_0}{0!} + \frac{a_1}{1!}\, t + \dots + \frac{a_d}{d!}\, t^d\right\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \dots + \frac{a_d}{s^{d+1}},$$

valid in the right half plane $\mathrm{Re}(s) > 0$.

Exercise

1. Suppose $f_i\colon [0,\infty) \to \mathbb{C}$ are piecewise continuous and of exponential order a_i ($i = 1, 2$). Show that the product $f_1 f_2\colon [0,\infty) \to \mathbb{R}$, $t \mapsto f_1(t)f_2(t)$ is piecewise continuous and of exponential order $a_1 + a_2$ (and hence $\mathcal{L}(f_1 f_2)$ is defined for $\mathrm{Re}(s) > a_1 + a_2$).

2. Suppose $f\colon [0,\infty) \to \mathbb{C}$ is piecewise continuous and of exponential order a. Show that $g\colon [0,\infty) \to \mathbb{R}$ defined by

$$g(t) = \int_0^t f(\tau)\mathrm{d}\tau$$

is continuous and of exponential order $\max\{a, 0\}$.

Hint: Show first f satisfies a bound $|f(t)| \leq K\,\mathrm{e}^{at}$ for all $t \geq 0$.

3. Show, by way of a counterexample, that a piecewise continuous function $f\colon [0,\infty) \to \mathbb{C}$ of exponential order may have a piecewise continuous derivative f' that is not of exponential order.

Hint: Compose a suitable function with $t \mapsto \mathrm{e}^{t^2}$.

Dilations in the argument

Suppose $F(s) = \mathcal{L}\{f(t)\} f$ is defined for $\mathrm{Re}(s) > a$ and $r > 0$. Then $\mathcal{L}\{f(rt)\}$ is defined for $\mathrm{Re}(s) > ra$ and satisfies

$$\mathcal{L}\{f(rt)\} = \frac{1}{r} F\left(\frac{s}{r}\right)$$

Proof.

$$\begin{aligned}
\mathcal{L}\{f(rt)\} &= \int_0^\infty f(rt)\mathrm{e}^{-st}\,\mathrm{d}t \\
&= \frac{1}{r}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau/r}\mathrm{d}\tau \qquad (\text{Subst. } \tau = rt,\ \mathrm{d}\tau = r\,\mathrm{d}t) \\
&= \frac{1}{r} F\left(\frac{s}{r}\right).
\end{aligned}$$

$\square$

Example

From $\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$, $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$ we get

$$\mathcal{L}\{\cos(\omega t)\} = \frac{1}{\omega}\frac{s/\omega}{(s/\omega)^2+1} = \frac{s}{s^2+\omega^2},$$

$$\mathcal{L}\{\sin(\omega t)\} = \frac{1}{\omega}\frac{1}{(s/\omega)^2+1} = \frac{\omega}{s^2+\omega^2}.$$

Remark

The dilation formula can also be stated as $F(rs) = \mathcal{L}\{\frac{1}{r} f(t/r)\}$ for $\mathrm{Re}(s) > a/r$. To see this, multiply the original dilation formula by r, use linearity of $\mathcal{L}$, and replace r by $1/r$.

The next example combines the dilation property with linearity.

Example

Find $\mathcal{L}\{\cos^2 t\}$ and $\mathcal{L}\{\sin^2 t\}$.

Solution: We have $\cos(2t) = \cos^2 t - \sin^2 t = 2\cos^2 t - 1$and hence $\cos^2 t = \frac{1+\cos(2t)}{2}$.

$$\Longrightarrow \mathcal{L}\{\cos^2 t\} = \frac{1}{2}\left(\mathcal{L}\{1\} + \mathcal{L}\{\cos(2t)\}\right) = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+4}\right)$$

$$= \frac{s^2+2}{s(s^2+4)},$$

$$\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{1\} - \mathcal{L}\{\cos^2 t\} = \frac{1}{s} - \frac{s^2+2}{s(s^2+4)}$$

$$= \frac{2}{s(s^2+4)}.$$

The Laplace transform is also well-behaved w.r.t. translations of the argument t, but the corresponding property is more technical to state. For $c \in \mathbb{R}$ define the "unit step function" $\mathrm{u}_c \colon \mathbb{R} \to \mathbb{R}$ by

$$\mathrm{u}_c(t) = \mathrm{u}(t-c) \begin{cases} 0 & \text{for } t < c, \\ 1 & \text{for } t \geq c, \end{cases}$$

($\mathrm{u}_0(t) = \mathrm{u}(t)$ is the familiar *Heaviside function*), and use this to define, for any function $f \colon [0,\infty) \to \mathbb{C}$ a new function $g \colon \mathbb{R} \to \mathbb{C}$ by

$$g(t) = \mathrm{u}_c(t) f(t-c) = \begin{cases} 0 & \text{for } t < c, \\ f(t-c) & \text{for } t \geq c. \end{cases}$$

Here we use the convention "$0 \times \text{undefined} = 0$".

Translations in the argument (cf. [BDM17], Th. 6.3.1)

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$ and $c > 0$. Then

$$\mathcal{L}\{\mathrm{u}_c(t) f(t-c)\} = \mathrm{e}^{-cs} F(s) \quad \text{for } \mathrm{Re}(s) > a.$$

The assumption $c > 0$ guarantees that $g(t) = \mathrm{u}_c(t) f(t-c)$ vanishes on $(-\infty, 0)$, i.e., we can view it as a function on $[0, \infty)$.

Proof.

$$\begin{aligned}\mathcal{L}\{\mathrm{u}_c(t)f(t-c)\} &= \int_c^\infty f(t-c)\mathrm{e}^{-st}\,\mathrm{d}t \\ &= \int_0^\infty f(\tau)\mathrm{e}^{-s(\tau+c)}\mathrm{d}\tau \\ &\qquad\qquad (\text{Subst. } \tau = t-c,\ \mathrm{d}\tau = \mathrm{d}t) \\ &= \mathrm{e}^{-sc}\int_0^\infty f(\tau)\mathrm{e}^{-s\tau}\mathrm{d}\tau = \mathrm{e}^{-cs}F(s). \quad \square\end{aligned}$$

Remark

The corresponding translation formula for $F(s)$ is

$$F(s-c) = \mathcal{L}\{e^{ct}f(t)\} \quad \text{for } \mathrm{Re}(s) > a + \mathrm{Re}(c).$$

Here c can be any complex number. This follows immediately from $\mathcal{L}\{e^{ct}f(t)\} = \int_0^\infty e^{ct}f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty f(t)\mathrm{e}^{-(s-c)t}\,\mathrm{d}t$.

Example

For $c > 0$ we have $\mathcal{L}\{\mathrm{u}_c(t)\} = \mathrm{e}^{-cs}/s$, valid for $\mathrm{Re}(s) > 0$. This follows by taking $f(t) \equiv 1$ in the first translation formula.

Example

The "ceiling" function $f(t) = \lceil t \rceil$ and the "floor" function $g(t) = \lfloor t \rfloor$ are related by $g(t) = u_1(t) f(t-1)$ (picture?). It follows that their Laplace transforms $F(s)$, resp., $G(s)$ are related by $G(s) = \mathrm{e}^{-s} F(s)$.
$\Longrightarrow$ The Laplace transform of the "ceiling" function is $F(s) = \mathrm{e}^{s} G(s) = \frac{\mathrm{e}^{s}}{s(\mathrm{e}^{s}-1)}$; cf. previous example.

Example

Earlier we have shown that $\mathcal{L}\{t^n\} = n!/s^{n+1}$ for $n \in \mathbb{N}$. The preceding remark gives, for $\mathrm{Re}(s) > \mathrm{Re}(c)$,

$$\mathcal{L}\{t^n \mathrm{e}^{ct}\} = \frac{n!}{(s-c)^{n+1}} \quad \text{or} \quad \frac{1}{(s-c)^{n+1}} = \mathcal{L}\left\{\frac{t^n}{n!}\,\mathrm{e}^{ct}\right\}.$$

Together with the partial fraction expansion of rational functions and linearity of $\mathcal{L}$ this shows (at least in principle) how to find for any rational function $F(s) = P(s)/Q(s)$ without polynomial part (i.e., $\deg P < \deg Q$) a corresponding function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. In fact, the function $f(t)$ obtained in this way will be an exponential polynomial (and, conversely, the Laplace transform of any exponential polynomial is a rational function without polynomial part).

Term-wise Integration of Laplace Integrals

From $\mathcal{L}\{t^n\} = n!/s^{n+1}$ and linearity of $\mathcal{L}$ it follows that

$$\mathcal{L}\left\{\frac{a_0}{0!} + \frac{a_1}{1!}\,t + \cdots + \frac{a_d}{d!}\,t^d\right\} = \frac{a_0}{s} + \frac{a_1}{s^2} + \cdots + \frac{a_d}{s^{d+1}}.$$

Under a suitable assumption on the growth of the coefficients, this can be extended to power series (i.e., functions $f(t)$ analytic at $t = 0$). Writing power series $\sum_{n=0}^{\infty} b_n t^n$ as exponential generating functions (i.e., $b_n = a_n/n!$ or $a_n = b_n n!$) makes it apparent that the Laplace transform of a power series in t is a power series in $1/s$.

Theorem

Suppose $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$. Then $f(t) = \sum_{n=0}^{\infty}(a_n/n!)t^n$ is defined for all $t \geq 0$ and we have

$$\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{a_n}{n!}\,t^n\right\} = \sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}} \quad \textit{for} \quad \mathrm{Re}(s) > 1/R.$$

Note that, by definition of R, the series $\sum_{n=0}^{\infty} \frac{a_n}{s^{n+1}}$ converges for $\mathrm{Re}(s) > 1/R$ (even for all $s \in \mathbb{C}$ with $|s| > 1/R$).

Proof.

Since $R = \sup\{r \geq 0; |a_n|\, r^n \text{ is bounded}\} > 0$, there exists for any $r \in (0, R)$ a corresponding constant K such that $|a_n|\, r^n \leq K$ for all n, i.e., $\sqrt[n]{|a_n|} \leq \sqrt[n]{K}/r$ for all n. But then we must have $\sqrt[n]{|a_n|/n!} \to 0$ for $n \to \infty$, so that $\sum_{n=0}^{\infty}(a_n/n!)z^n$ has radius of convergence ∞ and $f(t)$ is defined in particular for all $t \geq 0$.

Moreover,

$$|f(t)| \leq \sum_{n=0}^{\infty} \frac{|a_n|}{n!}\, t^n \leq \sum_{n=0}^{\infty} \frac{K\, r^{-n}}{n!}\, t^n \leq K\, \mathrm{e}^{t/r} \quad \text{for } 0 < r < R,$$

showing that f has exponential order at most $1/R$, so that $\mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > 1/R$.

Writing $f_n(t) = \sum_{k=0}^{n}(a_n/n!)t^n$, the claimed identity takes the form

$$\int_0^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_0^{\infty} \lim_{n\to\infty} f_n(t)\mathrm{e}^{-st}\,\mathrm{d}t = \lim_{n\to\infty} \int_0^{\infty} f_n(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

for $\mathrm{Re}(s) > 1/R$.

Proof cont'd.

We prove it directly without resorting to convergence theorems for Lebesgue or improper Riemann integrals. Writing $s = x + \mathrm{i}y$, we have

$$\begin{aligned}
&\left|\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t - \int_0^\infty f_n(t)\mathrm{e}^{-st}\,\mathrm{d}t\right| = \left|\int_0^\infty \bigl(f(t) - f_n(t)\bigr)\mathrm{e}^{-st}\,\mathrm{d}t\right| \\
&\quad= \left|\int_0^\infty \sum_{k=n+1}^\infty \frac{a_k}{k!}\,t^k\mathrm{e}^{-st}\,\mathrm{d}t\right| \leq \sum_{k=n+1}^\infty \frac{|a_k|}{k!}\int_0^\infty t^k\mathrm{e}^{-xt}\,\mathrm{d}t \\
&= \sum_{k=n+1}^\infty \frac{|a_k|}{k!}\,\frac{k!}{x^{k+1}} = \sum_{k=n+1}^\infty \frac{|a_k|}{x^{k+1}}.
\end{aligned}$$

As long as $x = \mathrm{Re}(s) > 1/R$ this converges to zero for $n \to \infty$, since $\sum_{n=0}^\infty |a_n|\, z^n$ has the same radius of convergence as $\sum_{n=0}^\infty a_n z^n$. This completes the proof of the theorem. □

Example

Consider the function $f(t) = \frac{\sin t}{t}$, $t \in [0, \infty)$ (extended continuously to $t = 0$ by defining $f(0) = 1$).

The Laplace transform of f is

$$F(s) = \mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_0^\infty \frac{\sin t}{t}\,\mathrm{e}^{-st}\,\mathrm{d}t, \quad \mathrm{Re}(s) > 0.$$

We have met $F(s)$ before (in our Calculus III final exam) and evaluated it using integration by parts.

Using the preceding theorem and the Taylor series of $\frac{\sin t}{t}$, we can determine F in a more conceptual way:

$$\begin{aligned} F(s) &= \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, t^{2n}\right\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,\frac{t^{2n}}{(2n)!}\right\} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}\,\frac{1}{s^{2n+1}} = \frac{1}{s} - \frac{1}{3\,s^3} + \frac{1}{5\,s^5} \mp \dots = \arctan(1/s), \end{aligned}$$

for $\mathrm{Re}(s) > 1$, since the arctan series has radius of convergence 1.

Example (cont'd)

The extension of the identity $F(s) = \arctan(1/s) = \operatorname{arccot}(s)$ to the whole right half plane $H_0 = \{s \in \mathbb{C}; \operatorname{Re}(s) > 0\}$ is then a consequence of the fact that both $F(s)$ and $\arctan(s)$ are analytic in H_0 and coincide on a subset of H_0, viz. $H_1 = \{s \in \mathbb{C}; \operatorname{Re}(s) > 1\}$, which has an accumulation point in H_0 (in fact all points of H_1 are accumulation points).

However, the delicate argument required to evaluate $\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t$ (using continuity of F in $s = 0$, which can't be derived from the results on the Laplace transform established so far) is not facilitated in any way by the present discussion.

Exercise

Find the Laplace transform of the Bessel function J_0.

Hint: The power series expansion

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n, \quad \text{valid for } |x| < 1/4,$$

may help (but you should prove it first).

Exercise

1. Show that
$$\int_0^\infty \ln t \, \mathrm{e}^{-t} \, \mathrm{d}t = -\gamma = -0.577\dots$$
For this recall that the Euler-Mascheroni constant γ was defined as $\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$
Hint: Relate the integral to the Gamma function. Gauss's formula
$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\, n^x}{x(x+1)\cdots(x+n)} \qquad (x \neq 0, -1, -2, \dots),$$
which you don't need to prove, may help.

2. Use a) to find the Laplace transform of $t \mapsto \ln t$ and the inverse Laplace transform of $s \mapsto \frac{\ln s}{s}$ $(\mathrm{Re}\, s > 0)$.

The Laplace Transform and Differentiation

The formula for $F'(s)$ was already stated and proved:

Differentiation in the Codomain

If $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, it is analytic (complex differentiable) for $\mathrm{Re}(s) > a$ with

$$F'(s) = \mathcal{L}\{-t\,f(t)\}.$$

Example

We use this formula to give an alternative derivation of $\mathcal{L}\{t^k\,\mathrm{e}^{ct}\} = k!/(s-c)^{k+1}$ for $k \in \mathbb{N}$:

$$\begin{aligned} \frac{1}{s-c} &= \mathcal{L}\{\mathrm{e}^{ct}\}, && \text{(from } \mathrm{e}^{ct}\mathrm{e}^{-st} = \mathrm{e}^{-(s-c)t}\text{)} \\ \implies \frac{1}{(s-c)^2} &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{s-c} = -\mathcal{L}\{-t\,\mathrm{e}^{ct}\} = \mathcal{L}\{t\,\mathrm{e}^{ct}\}, \\ \implies \frac{2}{(s-c)^3} &= -\frac{\mathrm{d}}{\mathrm{d}s}\frac{1}{(s-c)^2} = -\mathcal{L}\{-t^2\,\mathrm{e}^{ct}\} = \mathcal{L}\{t^2\,\mathrm{e}^{ct}\}, \end{aligned}$$

etc.

The following formula provides the key to applying the Laplace transform to the solution of (time-independent) linear ODE's.

Theorem (Differentiation in the Domain)

Suppose that $f\colon [0,\infty) \to \mathbb{C}$ *is continuous with piece-wise continuous derivative* f'*, and* $F(s) = \mathcal{L}\{f(t)\}$ *is defined for* $\mathrm{Re}(s) > a$*. Then we have*

$$\mathcal{L}\{f'(t)\} = s\,F(s) - f(0) \quad \text{for } \mathrm{Re}(s) > a.$$

If f is continuous in $(0,\infty)$ but discontinuous in 0, the formula still holds with $f(0)$ replaced by $f(0+)$, i.e., $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0+)$.

Proof.

Assume first that f is a C^1-function. Then integration by parts gives

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= \int_0^\infty f'(t)\mathrm{e}^{-st}\,\mathrm{d}t = \left[f(t)\mathrm{e}^{-st}\right]_0^\infty + s\int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \\ &= s\,F(s) - f(0),\end{aligned}$$

since $|f(t)| \leq K\,\mathrm{e}^{at}$ for $t \geq M$ and hence $\left|f(t)\mathrm{e}^{-st}\right| \leq K\,\mathrm{e}^{-(\mathrm{Re}\,s-a)t}$, which tends to zero for $t \to \infty$ on account of $\mathrm{Re}(s) - a > 0$.

Proof cont'd.

Next assume that f' has finitely many discontinuities $t_1 < t_2 < \cdots < t_n$. Then we can apply integration by parts to the C^1-functions $f|_{[0,t_1]}$, $f|_{[t_{k-1},t_k]}$ for $2 \leq k \leq n$, $f_{[t_n,\infty)}$, and obtain

$$\int_0^{t_1} f'(t)\mathrm{e}^{-st}\,\mathrm{d}t = f(t_1)\mathrm{e}^{-st_1} - f(0) + s\int_0^{t_1} f(t)\mathrm{e}^{-st}\,\mathrm{d}t,$$

$$\int_{t_{k-1}}^{t_k} f'(t)\mathrm{e}^{-st}\,\mathrm{d}t = f(t_k)\mathrm{e}^{-st_k} - f(t_{k-1})\mathrm{e}^{-st_{k-1}} + s\int_{t_{k-1}}^{t_k} f(t)\mathrm{e}^{-st}\,\mathrm{d}t,$$

$$\int_{t_n}^{\infty} f'(t)\mathrm{e}^{-st}\,\mathrm{d}t = \lim_{t\to\infty} f(t)\mathrm{e}^{-st} - f(t_n)\mathrm{e}^{-st_n} + s\int_{t_n}^{\infty} f(t)\mathrm{e}^{-st}\,\mathrm{d}t\,.$$

Since $\lim_{t\to\infty} f(t)\mathrm{e}^{-st} = 0$ (as shown above), summing these identities yields again $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$.

Finally, if f' has countably many discontinuities $t_1 < t_2 < \dots$, the preceding argument remains valid (now involving an infinite summation).

The generalization to functions f discontinuous at $t = 0$ follows by changing $f(0)$ to $f(0+)$, which makes f continuous in 0 but doesn't change $\mathcal{L}f$. $\square$

Corollary

Suppose that $f\colon [0,\infty) \to \mathbb{C}$ is a C^{n-1}-function with piece-wise continuous n-th derivative $f^{(n)}$, and $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$. Then we have

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s^0 f^{(n-1)}(0)$$

for $\mathrm{Re}(s) > a$.

Again the continuity assumption on f, f', ..., $f^{(n-1)}$ at $t = 0$ can be dropped, if one uses $f^{(k)}(0+)$ in place of $f^{(k)}(0)$ in the formula.

Proof.

Use the theorem and induction on n. □

Remarks

In the theorem and its corollary, the derivatives f' resp. $f^{(n)}$ may be undefined on a discrete subset $\Delta \subset [0,\infty)$; cf. the previous note about this generalization of piecewise continuity. In fact one can show that for a differentiable 1-variable function g the derivative g' cannot have jump discontinuities. Hence if the one-sided limits $g'(t_0\pm)$ exist but are different, $g'(t_0)$ cannot exist. Also, if f is of exponential order a, the derivatives need not be of exponential order, but their Laplace integrals nevertheless exist for $\mathrm{Re}(s) > a$.

The Laplace Transform and Integration

If $f\colon [0,\infty) \to \mathbb{C}$ is piecewise continuous then $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$ is defined for $t \in [0,\infty)$, continuous on $[0,\infty)$, differentiable everywhere except for the discontinuities of f, and at discontinuities t_k of f the one-sided derivatives $g'(t_k\pm)$ still exist.

Integration in the Domain

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$. Then

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \frac{F(s)}{s} \quad \text{for } \mathrm{Re}(s) > \max\{a, 0\}.$$

The possible additional singularity of $\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\}$ at $s = 0$ can be explained as follows: Consider $f(t) = \mathrm{e}^t$. Then $F(s) = \frac{1}{s-1}$ and $\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \mathcal{L}\left\{\int_0^t \mathrm{e}^\tau\mathrm{d}\tau\right\} = \mathcal{L}\left\{\mathrm{e}^\tau - 1\right\} = \frac{1}{s-1} - \frac{1}{s} = \frac{1}{(s-1)s}$. A new singularity at $s = 0$ is introduced, since a constant C of integration has Laplace transform C/s.

Proof.

Let $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$ and $G(s) = \mathcal{L}\{g(t)\}$. The function g is continuous on $[0, \infty)$, and from $|f(t)| \leq K\,\mathrm{e}^{at}$ for $t \geq M$ we obtain

$$\begin{aligned}
|g(t)| &= \left| \int_0^M f(\tau)\mathrm{d}\tau + \int_M^t f(\tau)\mathrm{d}\tau \right| \leq |g(M)| + \int_M^t |f(\tau)|\,\mathrm{d}\tau \\
&\leq |g(M)| + \int_M^t K\,\mathrm{e}^{a\tau}\mathrm{d}\tau = |g(M)| + \frac{K}{a}\left(\mathrm{e}^{at} - \mathrm{e}^{aM}\right) \\
&= |g(M)| - \frac{K}{a}\,\mathrm{e}^{aM} + \frac{K}{a}\,\mathrm{e}^{at}
\end{aligned}$$

for $t \geq M$. Clearly this implies that $g(t)$ is of exponential order at most $\max\{a, 0\}$, so that $G(s)$ isdefined for $\mathrm{Re}(s) > \max\{a, 0\}$.

Applying differentiation in the domain gives

$$F(s) = \mathcal{L}\{f(t)\} = \mathcal{L}\{g'(t)\} = s\,G(s) - g(0) = s\,G(s),$$

i.e., $G(s) = F(s)/s$. □

Integration in the Codomain

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$ and $\int_0^1 \frac{f(t)}{t}\,\mathrm{d}t$ exists. Then

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\mathrm{d}\sigma \quad \text{for } s > a.$$

The condition on the existence of $\int_0^1 \frac{f(t)}{t}\,\mathrm{d}t$ is satisfied in particular if $f(0) = 0$ and $f'(0+) = \lim_{t\downarrow 0} \frac{f(t)}{t}$ exists, but also if there exists $r > 0$ such that $f(t) \simeq t^r$ for $t \downarrow 0$.

The formula remains true for complex numbers s with $\mathrm{Re}(s) > a$, provided we replace $\int_s^\infty F(\sigma)\mathrm{d}\sigma$ by $\int_0^\infty F(s+\sigma)\mathrm{d}\sigma$ (or, more generally, as the complex line integral of $F(s)$ along any ray emanating from s and contained in the half plane $\mathrm{Re}(s) > a$).

Proof.

Since $g(t) = f(t)/t$ has the same exponential order as f, the Laplace transform $G(s)$ of g is defined for $\mathrm{Re}(s) > a$ if $\int_0^1 g(t)\mathrm{e}^{-st}\,\mathrm{d}t$ exists for such s. The latter is equivalent to the existence of $\int_0^1 g(t)\,\mathrm{d}t$, which is true by assumption.

The formula an then be proved as follows:

$$G'(s) = -\mathcal{L}\{t\,g(t)\} = -\mathcal{L}\{f(t)\} = -F(s)$$

$$\Longrightarrow\ G(s) = G(s_0) - \int_{s_0}^{s} F(\sigma)\mathrm{d}\sigma = G(s_0) + \int_{s}^{s_0} F(\sigma)\mathrm{d}\sigma$$

for $s_0, s > a$. Letting $s_0 \to \infty$, we obtain $G(s) = \int_s^\infty F(\sigma)\mathrm{d}\sigma$ using the known fact $\lim_{s_0\to\infty} G(s_0) = 0$; cf. exercise. □

Exercise

Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, $a \in [-\infty, 0)$. Show that $\lim_{s\to\infty} F(s) = 0$; cp. Exercise 24 in [BDM17], Ch. 6.1. This implies, e.g., that no nonzero polynomial can be a Laplace transform.

Hint: Use the uniform convergence of $\int_0^\infty f(t)\mathrm{e}^{-st}$ on $\mathrm{Re}(s) \geq a+1$ (resp., for $a = -\infty$ on $\mathrm{Re}(s) \geq 0$).

Example

From $\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$, using integration in the codomain, we find again

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{\mathrm{d}\sigma}{\sigma^2+1} = \frac{\pi}{2} - \arctan s = \operatorname{arccot} s \quad \text{for } s > 0.$$

From this in turn, using integration in the domain, we can compute the Laplace transform of the sine integral:

$$\mathcal{L}\{\operatorname{Si} t\} = \mathcal{L}\left\{\int_0^t \frac{\sin\tau}{\tau}\,\mathrm{d}\tau\right\} = \frac{\operatorname{arccot} s}{s} \quad \text{for } s > 0.$$

As remarked before, these formulas also hold for $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 0$.

The Laplace Transform and Convolution

We have seen that the Lapace transform of a sum of two functions is the sum of their Lapace transforms. How about their product? For the product it is not true, since
$1/s = \mathcal{L}\{1\} = \mathcal{L}\{1^2\} \neq \mathcal{L}\{1\}^2 = 1/s^2$ as functions.

But we can try to determine a different product $(f, g) \mapsto f * g$ that satisfies $\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}$. Suppose $F = \mathcal{L}\{f\}$, $G = \mathcal{L}\{g\}$.

$$
\begin{aligned}
F(s)G(s) &= \left(\int_0^\infty f(t_1)\mathrm{e}^{-st_1}\,\mathrm{d}t_1\right)\left(\int_0^\infty g(t_2)\mathrm{e}^{-st_2}\,\mathrm{d}t_2\right) \\
&= \int_{t_1=0}^\infty \int_{t_2=0}^\infty f(t_1)g(t_2)\mathrm{e}^{-s(t_1+t_2)}\,\mathrm{d}t_2\,\mathrm{d}t_1 \\
&= \int_{t_1=0}^\infty \int_{\tau=t_1}^\infty f(t_1)g(\tau-t_1)\mathrm{e}^{-s\tau}\mathrm{d}\tau\,\mathrm{d}t_1 \\
&\qquad\qquad\qquad \text{(Subst. } \tau = t_1 + t_2,\ \mathrm{d}\tau = \mathrm{d}t_2\text{)} \\
&= \int_{\tau=0}^\infty \int_{t_1=0}^\tau f(t_1)g(\tau-t_1)\mathrm{e}^{-s\tau}\,\mathrm{d}t_1\,\mathrm{d}\tau \\
&\qquad\qquad\qquad \text{(Fubini's Theorem)} \\
&= \mathcal{L}\{h(\tau)\}
\end{aligned}
$$

with $h\colon [0,\infty) \to \mathbb{C}$, $\tau \mapsto \int_0^\tau f(t_1)g(\tau - t_1)\,\mathrm{d}t_1$.

Definition (convolution on $\mathbb{C}^{[0,\infty)}$)

Suppose $f, g\colon [0,\infty) \to \mathbb{C}$ are piecewise continuous. The *convolution (product)* of f and g is the function $f * g\colon [0,\infty) \to \mathbb{C}$ defined by

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\mathrm{d}\tau.$$

Remark

In Real Analysis there are several different types of convolutions in use. The present definition is tailored to the Laplace transform.

Clearly the convolution product is bilinear (i.e., linear in each argument).

Exercise

Show that the convolution product is commutative and associative, i.e. $f * g = g * f$ and $(f * g) * h = f * (g * h)$ hold for all piecewise continuous functions f, g, h on $[0,\infty)$.

Theorem

If $F(s) = \mathcal{L}f$ *exists for* $\operatorname{Re}(s) > a$ *and* $G(s) = \mathcal{L}g$ *exists for* $\operatorname{Re}(s) > b$ *then* $H(s) = \mathcal{L}(f * g)$ *exists for* $\operatorname{Re}(s) > \max\{a, b\}$ *and satisfies*

$$H(s) = F(s)G(s) \quad \text{for } \operatorname{Re}(s) > \max\{a, b\}.$$

Proof.

The identity $H(s) = F(s)G(s)$, $\mathrm{Re}(s) > \max\{a, b\}$, is true by definition of H, provided we can show the existence of $\mathcal{L}\{f * g\}$ for $\mathrm{Re}(s) > \max\{a, b\}$ and justify the use of Fubini's Theorem.

Clearly $f * g$ is piece-wise continuous as well (even continuous).

Since piecewise continuous functions are bounded on every finite interval $[0, M]$, there exist constants K, L such that $|f(t)| \leq K\,\mathrm{e}^{at}$ and $|g(t)| \leq L\,\mathrm{e}^{bt}$ for all $t \geq 0$.

$$\begin{aligned}
\implies |(f * g)(t)| &\leq \int_0^t |f(\tau)|\,|g(t-\tau)|\,\mathrm{d}\tau \leq \int_0^t KL\,\mathrm{e}^{a\tau}\mathrm{e}^{b(t-\tau)}\mathrm{d}\tau \\
&= KL\,\mathrm{e}^{bt}\int_0^t \mathrm{e}^{(a-b)\tau}\mathrm{d}\tau \\
&= \begin{cases} KL\,t\,\mathrm{e}^{bt} & \text{if } a = b, \\ KL\,\mathrm{e}^{bt}\,\frac{\mathrm{e}^{(a-b)t}-1}{a-b} = KL\,\frac{\mathrm{e}^{at}-\mathrm{e}^{bt}}{a-b} & \text{if } a \neq b. \end{cases}
\end{aligned}$$

From this it is clear that $f * g$ has exponential order at most $\max\{a, b\}$, and hence $H(s) = \mathcal{L}\{f * g\}$ exists for $\mathrm{Re}(s) > \max\{a, b\}$.

Proof cont'd.

Regarding Fubini's Theorem, it suffices to show that the 2-dimensional Lebesgue integral

$$\int_{\mathbb{R}^2} f(t_1)g(t_2)\mathrm{e}^{-s(t_1+t_2)}\mathrm{d}^2(t_1,t_2)$$

exists. (The integral to which we have applied Fubini's Theorem differs only by a change-of-variables from this.) Since f and g are piecewise continuous, the corresponding finite integrals over $[0,R]^2$ exist for every $R>0$, and as shown in Calclus III it then suffices to find a universal bound for

$$\int_{[0,R]^2} |f(t_1)g(t_2)|\,\mathrm{e}^{-s(t_1+t_2)}\mathrm{d}^2(t_1,t_2) = \left(\int_0^R |f(t_1)|\,\mathrm{e}^{-st_1}\,\mathrm{d}t_1\right)\left(\int_0^R |g(t_2)|\,\mathrm{e}^{-st_2}\,\mathrm{d}t_2\right)$$

("integration by exhaustion"). Since the Laplace integrals of f and g converge absolutely for $\mathrm{Re}(s)>\max\{a,b\}$, this is trivial: Just take the product of the corresponding limits for $R\to\infty$. Using $|f(t_1)|\leq K\,\mathrm{e}^{at_1}$, $|g(t_2)|\leq L\,\mathrm{e}^{bt_2}$ we can also derive the explicit bound $\frac{KL}{(s-a)(s-b)}$. □

Example

The convolution of exponentials is given by

$$e^{at} * e^{bt} = \begin{cases} t\,e^{at} & \text{if } a = b, \\ \frac{e^{at}-e^{bt}}{a-b} & \text{if } a \neq b. \end{cases}$$

This follows from the preceding computation.

Example

Find the inverse Laplace transform of $\quad F(s) = \dfrac{s}{(s^2+1)^2}$.

Solution: One way to solve this problem is to use the convolution theorem and the known Laplace transforms of sin, cos:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\,\frac{s}{s^2+1}\right\} = \sin t * \cos t$$

$$= \int_0^t \sin\tau\cos(t-\tau)\mathrm{d}\tau = \cos t\int_0^t \sin\tau\cos\tau\mathrm{d}\tau + \sin t\int_0^t \sin^2\tau\mathrm{d}\tau$$

$$= \cos t\left[-\frac{1}{4}\cos(2\tau)\right]_0^t + \sin t\left[\frac{\tau}{2} - \frac{1}{4}\sin(2\tau)\right]_0^t = \frac{t\sin t}{2}.$$

Inversion of the Laplace Transform

Changing a function $f\colon [0,\infty) \to \mathbb{C}$ on a discrete subset of $[0,\infty)$, which must be countable, doesn't affect $\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$.
$\Longrightarrow$ The Laplace transform cannot be one-to-one.

However, we have the following

Theorem

Suppose $f_1, f_2\colon [0,\infty) \to \mathbb{C}$ are piecewise continuous and $F_i(s) = \mathcal{L}\{f_i(t)\}$ is defined for $\mathrm{Re}(s) > a_i$ ($i = 1, 2$). If there exists $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \max\{a_1, a_2\}$ and $x > 0$ such that $F_1(s + kx) = F_2(s + kx)$ for all $k \in \mathbb{N}$, then $f_1(t-) = f_2(t-)$ and $f_1(t+) = f_2(t+)$ for all $t \geq 0$, and hence f_2 arises from f_1 by changing the values on some discrete subset of $[0,\infty)$.

Notes

- The conlusion of the theorem implies $a_1 = a_2$ and $F_1 = F_2$.
- The assumptions of the theorem are satisfied in particular if F_1 and F_2 coincide on their common domain $\mathrm{Re}(s) > \max\{a_1, a_2\}$.
- If f_1, f_2 satisfy the assumptions of the theorem and are continuous, we must have $f_1 = f_2$.

The proof uses the following lemma, whose proof is a bit technical and omitted.

Lemma

If f_1, f_2 satisfy the assumptions of the theorem then

$$\int_0^t f_1(\tau)\mathrm{d}\tau = \int_0^t f_2(\tau)\mathrm{d}\tau \quad \textit{for all } t \in [0,\infty).$$

Proof of the theorem.

The lemma implies that $g(t) := \int_0^t f_1(\tau)\mathrm{d}\tau = \int_0^t f_2(\tau)\mathrm{d}\tau$ for $t \in [0,\infty)$. The function g is continuous, and a straightforward generalization of the Fundamental Theorem of Calculus implies

$$\begin{aligned} f_1(t+) &= \lim_{h\downarrow 0} \frac{g(t+h)-g(t)}{h} = f_2(t+) \quad \text{for } t \geq 0, \\ f_1(t-) &= \lim_{h\uparrow 0} \frac{g(t+h)-g(t)}{h} = f_2(t-) \quad \text{for } t > 0, \end{aligned}$$

completing the proof of the theorem. □

Remark

There is also an explicit inversion formula for the Laplace transform known, but this formula uses complex line integrals and is of practical use only when combined with the residue theorem of Complex Analysis. For now we omit it.

The Basic Idea

The Laplace transform is particularly helpful for solving IVP's corresponding to linear ODE's with constant coefficients and a right-hand side ("forcing function") $f(t)$, whose Laplace transform exists (i.e., f need not be continuous, let alone be an exponential polynomial). If the ODE has order 2 (the most important case for applications in physics/electrotechnics), the IVP looks like

$$y'' + b\,y' + c\,y = f(t), \quad y(0) = y_0,\ y'(0) = y_1,$$

where $b, c, y_0, y_1 \in \mathbb{R}$ are given constants.

The solution method consists of 3 steps:

1. Using differentiation in the domain, translate the IVP for $y(t)$ into an algebraic equation for the Laplace transform $Y(s) = \mathcal{L}\{y(t)\}$.
2. Determine $Y(s)$ by solving this algebraic equation.
3. Use Laplace Transform inversion to find $y(t) = \mathcal{L}^{-1}\{Y(s)\}$.

The solution $y(t)$ is the unique continuous Laplace-inverse of $Y(s)$ and hence well-determined in Step 3.

Example

Solve the IVP $y' + 2\,y = -1$, $y(0) = 1$ with the Laplace Transform.

Solution: Setting $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\begin{aligned}\mathcal{L}\{y'(t)\} + 2\,\mathcal{L}\{y(t)\} &= -\mathcal{L}\{1\},\\ s\,Y(s) - y(0) + 2\,Y(s) &= -1/s,\\ s\,Y(s) - 1 + 2\,Y(s) &= -1/s.\end{aligned}$$

$$\implies Y(s) = \frac{1 - 1/s}{s+2} = \frac{s-1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

with $A = \frac{s-1}{s+2}\Big|_{s=0} = -1/2$, $B = \frac{s-1}{s}\Big|_{s=-2} = 3/2$.

$$\implies Y(s) = -\frac{1}{2}\,\frac{1}{s} + \frac{3}{2}\,\frac{1}{s+2}$$

$$\implies y(t) = -\frac{1}{2} + \frac{3}{2}\,\mathrm{e}^{-2t}.$$

Example

Solve the IVP $y'' + y = \sin(\omega t)$, $y(0) = y'(0) = 1$ with the Laplace Transform.

Solution: Setting $Y(s) = \mathcal{L}\{y(t)\}$, we have

$$\begin{aligned} s^2\,Y(s) - s\,y(0) - y'(0) + Y(s) &= \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}, \\ s^2\,Y(s) - s - 1 + Y(s) &= \mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}. \end{aligned}$$

$$\implies Y(s) = \frac{s+1}{s^2+1} + \frac{\omega}{(s^2+1)(s^2+\omega^2)}$$

Now there are two cases to consider:

$\underline{\omega \neq \pm 1}$:

$$\begin{aligned} \implies Y(s) &= \frac{s+1}{s^2+1} + \frac{\omega}{\omega^2-1}\,\frac{1}{s^2+1} - \frac{\omega}{\omega^2-1}\,\frac{1}{s^2+\omega^2} \\ \implies y(t) &= \cos t + \sin t + \frac{\omega}{\omega^2-1}\sin t - \frac{1}{\omega^2-1}\sin(\omega t) \\ &= \cos t + \frac{\omega^2+\omega-1}{\omega^2-1}\sin t - \frac{1}{\omega^2-1}\sin(\omega t). \end{aligned}$$

Example (cont'd)

$\underline{\omega = \pm 1}$: Here we have

$$Y(s) = \frac{s+1}{s^2+1} + \frac{1}{(s^2+1)^2} = \frac{s+1}{s^2+1} + \frac{1}{2}\,\frac{1}{s^2+1} - \frac{1}{2}\,\frac{s^2-1}{(s^2+1)^2}$$

$$\Longrightarrow \quad y(t) = \cos t + \sin t + \frac{1}{2}\sin t - \frac{1}{2}\,t\cos t$$

$$= \cos t + \frac{3}{2}\sin t - \frac{1}{2}\,t\cos t.$$

Explanation: The above decomposition of $1/(s^2+1)^2$ and its inverse Laplace transform were found by playing around with the known Laplace transforms $\mathcal{L}\{\cos t\} = s/(s^2+1)$, $\mathcal{L}\{\sin t\} = 1/(s^2+1)$. Use $\mathcal{L}\{t\cos t\} = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{s}{s^2+1} = \frac{s^2-1}{(s^2+1)^2} = \frac{1}{s^2+1} - \frac{2}{(s^2+1)^2}$, from which it is obvious.

The standard way to compute $\mathcal{L}\left\{\frac{1}{(s^2+1)^2}\right\}$ is to use complex partial fractions $\frac{1}{(s^2+1)^2} = \frac{1}{(s-\mathrm{i})^2(s+\mathrm{i})^2} = \frac{A}{s-\mathrm{i}} + \frac{B}{(s-\mathrm{i})^2} + \frac{C}{s+\mathrm{i}} + \frac{D}{(s+\mathrm{i})^2}$ together with $\frac{1}{s\mp\mathrm{i}} = \mathcal{L}\left\{\mathrm{e}^{\pm\mathrm{i}t}\right\}$, $\frac{1}{(s\mp\mathrm{i})^2} = \mathcal{L}\left\{t\,\mathrm{e}^{\pm\mathrm{i}t}\right\}$. One obtains $B = D = -1/4$, $A = -\mathrm{i}/4$, $C = +\mathrm{i}/4$, ...

Exercise (advanced)

It appears that we can obtain the solution for $\omega = 1$ by viewing the solution for $\omega \neq \pm 1$ as a two-variable function $y(\omega, t)$ and computing $\lim_{\omega \to 1} y(\omega, t)$ with the aid of L'Hospital's Rule. Can you prove this rigorously? (Compare also with the proof of Part 3 of our big theorem on fractional power series solutions of 2nd-order linear ODE's near regular singular points.)

Continuous Forcing

The preceding two examples had a continuous forcing function $f\colon [0,\infty) \to \mathbb{C}$ (viz. $f(t) = -1$, resp., $f(t) = \sin(\omega t)$). In such a case the sharpened version of the Existence and Uniqueness Theorem for linear ODE's applies and guarantees that there exists a unique solution $y(t)$ on $[0,\infty)$ satisfying any given initial conditions $y(0) = y_0$, $y'(0) = y_1$ (and, similarly, for initial times $t_0 > 0$). As argued before the examples, the solution method using the Laplace transform produces this solution, provided $y(t)$ has a Laplace transform $Y(s)$ and $\mathcal{L}^{-1}\{Y(s)\}$ can be found. For this it is sufficient that $f(t)$ has a Laplace transform; see the subsequent theorem.

If the forcing function f satisfies $f(0) = 0$, its trivial extension to $\mathbb{R}$ (by setting $f(t) < 0$ for $t < 0$) is continuous as well, and hence maximal solutions of corresponding initial value problems are defined on $\mathbb{R}$. Such solutions do not necessarily vanish on $(-\infty, 0)$; this is the case iff the initial values at $t_0 = 0$ are $y_0 = y_1 = 0$.

Theorem

Suppose $f\colon [0,\infty) \to \mathbb{C}$ *is continuous and of exponential order. Then the same is true of any solution* $y\colon [0,\infty) \to \mathbb{C}$ *of* $y'' + by' + cy = f(t)$*, and hence* $Y(s) = \mathcal{L}\{y(t)\}$ *is defined in some half plane* $\mathrm{Re}(s) > a$*.*

Proof.

In the homogeneous case $f(t) \equiv 0$ solutions are exponential polynomials and the assertion is obvious. In the general case it follows by inspecting the variation-of-parameters formula for the solution (cf. our earlier discussion of analytic solutions of 2nd-order inhomogeneous linear ODE's) and using that "exponential order" is inherited by products and integrals. Since the ODE's considered here have constant coefficients, the Wronskian appearing in the formula is a nonzero multiple of e^{-bt}.) □

Discontinuous Forcing

Now consider the more general IVP

$$y'' + b\,y' + c\,y = f(t), \quad y(0) = y_0,\ y'(0) = y_1 \qquad (\star)$$

with constants $b, c, y_0, y_1 \in \mathbb{R}$ and forcing function $f\colon [0, \infty) \to \mathbb{C}$, whose Laplace transform $F(s)$ exists in some half plane $\operatorname{Re}(s) > a$. For the following discussion we assume that f is piecewise continuous and of at most exponential order.

Because y'' cannot exist at discontinuities of f (as can be shown), we must adapt the definition of a solution to this more general situation.

Definition

By a *solution* of $(\star)$ we mean a C^1-function $y\colon [0, \infty) \to \mathbb{C}$ with the following properties:

1. $y''(t)$ exists at all points $t \in [0, \infty)$ where f is continuous, and $y''(t) + b\,y'(t) + c\,y(t) = f(t)$ holds for those points t;
2. $y(0) = y_0$, $y'(0) = y_1$.

Notes

- In the definition the values of f at its discontinuities do not matter, and hence need not even be defined.
- For a solution y the derivative y'' must be piecewise continuous (with the same exceptional set as f), as the representation $y''(t) = f(t) - b\,y'(t) - c\,y(t)$ shows together with the assumption that y is a C^1-function.
$\Longrightarrow \mathcal{L}\{y'' + by' + cy\}$ can be computed using the integration-in-the-domain formulas.
- A priori a solution y determines solutions in the original sense only on the open intervals (t_{k-1}, t_k) between adjacent discontinuities of f (including $(0, t_1)$ and, if there is a largest discontinuity t_n, also (t_n, ∞)). However, the endpoints can be included since y'' has one-sided derivatives in the endpoints, viz. $y''_+(t_{k-1}) = \lim_{t \downarrow t_{k-1}} y''(t)$, $y''_-(t_k) = \lim_{t \uparrow t_k} y''(t)$, which satisfy the ODE as well.

Theorem

1. *If the forcing function f is piecewise continuous, the IVP* $(\star)$ *has a unique solution y.*
2. *If the Laplace transform of f exists then the Laplace transform method can be applied and produces the solution y.*

Proof.

(1) The Existence and Uniqueness Theorem first gives a unique solution y_1 of the IVP $(\star)$ on $[0, t_1]$, then a unique solution y_2 of the ODE on $[t_1, t_2]$ with initial values $y_2(t_1) = y_1(t_1)$, $y_2'(t_1) = y_1'(t_1)$, and so forth. Defining y as y_k on $[t_{k-1}, t_k]$ yields the desired solution of $(\star)$ on $[0, \infty)$. Conversely, the requirement that y be a C^1-function forces the initial conditions of y_k and y_{k+1} at t_k to match and hence determines y uniquely.

(2) Piecewise continuity of y'' ensures that $Y(s) = \mathcal{L}\{y(t)\}$ can be computed as usual from the given data:

$$s^2 Y(s) - s\,y_0 - y_1 + b\big(s\,Y(s) - y_0\big) + c\,Y(s) = F(s)$$

$$\implies \quad Y(s) = \frac{F(s) + s\,y_0 + b\,y_0 + y_1}{s^2 + b\,s + c} = G(s), \quad \text{say.}$$

$\implies y(t) = \mathcal{L}^{-1}\{G(s)\}$ (i.e., the unique continuous preimage). $\square$

Remarks

$Y(s) = Y_p(s) + Y_h(s)$ with $Y_p(s) = \frac{F(s)}{s^2+b\,s+c}$, $Y_h(s) = \frac{s\,y_0+b\,y_0+y_1}{s^2+b\,s+c}$.

1. $Y_p(s)$ is the Laplace transform of the solution $y_p(t)$ of $(\star)$ with initial values $y_p(0) = y_p'(0) = 0$.
2. $Y_h(s)$ is the Laplace transform of the solution $y_h(t)$ of the associated homogeneous ODE with initial values y_0, y_1.
3. The denominator of $Y_p(s)$, $Y_h(s)$, viewed as a polynomial in s, is precisely the characteristic polynomial of $(\star)$.
4. $Y_h(s)$ is a rational function of s, and hence $y_h(t) = \mathcal{L}^{-1}\{Y_h(s)\}$ can be determined from the partial fraction decomposition of $Y_h(s)$. This provides an alternative method to determine the general solution in the homogeneous case.
5. If $f(t) = \sum_{i=1}^{r} c_i t^{m_i} \mathrm{e}^{\mu_i t}$ is an exponential polynomial, $Y_p(s)$ and $Y(s)$ are rational functions of s as well, so that $y_p(t)$, $y(t)$ can be determined in the same way using partial fractions. If $f(t)$ is not an exponential polynomial, the Laplace-inverse of $Y_p(s)$ may nevertheless be known, providing a method to solve additional instances of such ODE's.

These observations generalize mutatis mutandis to higher-order ODE's.

The Laplace transform of the Heaviside function

$$\mathrm{u}(t) = \begin{cases} 1 & \text{if } t \geq 0, \\ 0 & \text{if } t < 0. \end{cases}$$

is $\mathcal{L}\{\mathrm{u}(t)\} = \mathcal{L}\{1\} = 1/s$.

Here we use the convention that the Laplace transform of a function defined on $\mathbb{R}$ (and vanishing on $(-\infty, 0)$) is that of its restriction to $[0, \infty)$. Conversely, we can view any function $f\colon [0, \infty) \to \mathbb{C}$ as a function on $\mathbb{R}$ by setting $f(t) = 0$ for $t < 0$. The extended function is piecewise continuous and of exponential order a iff the original function is.

Now consider a rectangular forcing function of unit height, i.e.,

$$\mathrm{r}_{a,b}(t) = \begin{cases} 1 & \text{if } a \leq t \leq b, \\ 0 & \text{if } t < a \text{ or } t > b, \end{cases}$$

with $a, b \in \mathbb{R}$ satisfying $0 \leq a < b$.

$\mathrm{r}_{a,b}$ can be expressed in terms of the Heaviside function as

$$\mathrm{r}_{a,b}(t) = \mathrm{u}(t-a) - \mathrm{u}(t-b) = \mathrm{u}_a(t) - \mathrm{u}_b(t)$$

(except for $t = b$, where the right-hand side is $\mathrm{u}(b-a) - \mathrm{u}(0)$ $= 1 - 1 = 0$, but this change doesn't affect the Laplace transform).

Using linearity of the Laplace transform and the translation in the argument formula, we obtain

$$\mathcal{L}\{\mathrm{r}_{a,b}\} = \mathcal{L}\{\mathrm{u}_a\} - \mathcal{L}\{\mathrm{u}_b\} = \mathrm{e}^{-as}\mathcal{L}\{\mathrm{u}\} - \mathrm{e}^{-bs}\mathcal{L}\{\mathrm{u}\} = \frac{\mathrm{e}^{-as} - \mathrm{e}^{-bs}}{s}.$$

In particular, if the upward step is at $t = 0$ ($a = 0$) then $\mathrm{r}_{a,b}(t) = \mathrm{r}_{0,b}(t) = (1 - \mathrm{e}^{-bs})/s$.

Example (discontinuous forcing)

Solve the IVP $\quad y''(t) + y(t) = \begin{cases} 1 & \text{for } 0 < t < 1, \\ 0 & \text{for } t > 1, \end{cases}$ with initial conditions $y(0) = y'(0) = 0$ with the Laplace transform.

Solution: Since $y_0 = y_1 = 0$, the Laplace transform of the left-hand side is $(s^2+1)Y(s)$, and that of the right-hand side is $\mathcal{L}\{\mathrm{r}_{0,1}\} = (1 - \mathrm{e}^{-s})/s$.

$$\implies Y(s) = \frac{1 - \mathrm{e}^{-s}}{s(s^2+1)}.$$

Using partial fractions ($1 = 1 \cdot (s^2+1) - s \cdot s$), we obtain

$$Y(s) = \left(1 - \mathrm{e}^{-s}\right)\left(\frac{1}{s} - \frac{s}{s^2+1}\right).$$

Example (cont'd)

Since $\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{s}{s^2+1}\right\}=1-\cos t$, this gives

$$\begin{aligned} y(t) &= 1-\cos t-\mathrm{u}_1(t)\,[1-\cos(t-1)] \\ &= \begin{cases} 1-\cos t & \text{for } 0\leq t\leq 1, \\ \cos(t-1)-\cos t & \text{for } t\geq 1. \end{cases} \end{aligned}$$

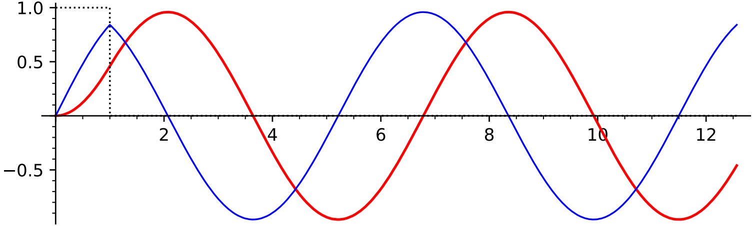

Figure: The solution $y(t)$ (in red), its derivative $y'(t)$ (in blue), and the forcing function $f(t)$ (dotted)

Note that both sections of $y(t)$ are periodic oszillations. For the section on $[1,\infty)$, we alternatively have $\cos(t-1)-\cos t = 2\sin\frac{t+t-1}{2}\sin\frac{t-(t-1)}{2}=2\sin\left(\frac{1}{2}\right)\sin\left(t-\frac{1}{2}\right)\approx 0.96\sin\left(t-\frac{1}{2}\right)$.

Example (discontinuous forcing)

Solve the IVP $\quad y'' + 3\,y' + 2y = \begin{cases} 1 & \text{for } t \in [0,1] \cup [2,3] \cup [4,5], \\ 0 & \text{otherwise}, \end{cases}$,

$y(0) = y'(0) = 0$ with the Laplace transform.

Solution: Since $y_0 = y_1 = 0$, again $Y(s)$ has the simple form

$$Y(s) = \frac{F(s)}{s^2 + 3s + 2} = \frac{F(s)}{(s+1)(s+2)}$$

with $F(s) = \mathcal{L}\{f(t)\}$, where

$$f(t) = [\mathrm{u}_0(t) - \mathrm{u}_1(t)] + [\mathrm{u}_2(t) - \mathrm{u}_3(t)] + [\mathrm{u}_4(t) - \mathrm{u}_5(t)]\,.$$

$$\Longrightarrow\; F(s) = \frac{1}{s} - \frac{\mathrm{e}^{-s}}{s} + \frac{\mathrm{e}^{-2s}}{s} - \frac{\mathrm{e}^{-3s}}{s} + \frac{\mathrm{e}^{-4s}}{s} - \frac{\mathrm{e}^{-5s}}{s},$$

$$\Longrightarrow\; Y(s) = \frac{1 - \mathrm{e}^{-s} + \mathrm{e}^{-2s} - \mathrm{e}^{-3s} + \mathrm{e}^{-4s} - \mathrm{e}^{-5s}}{s(s+1)(s+2)}.$$

Example (cont')

The partial fractions decomposition of the denominator is

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{2}\,\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2} = \mathcal{L}\left\{\frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{-2t}\right\}.$$

Writing $g(t) = \frac{1}{2} - \mathrm{e}^{-t} + \frac{1}{2}\,\mathrm{e}^{-2t}$, the solution is

$$y(t) = g(t) - \mathrm{u}_1(t)g(t-1) + \mathrm{u}_2(t)g(t-2) - \mathrm{u}_3(t)g(t-3) + \\ + \mathrm{u}_4(t)g(t-4) - \mathrm{u}_5(t)g(t-5) = \cdots$$

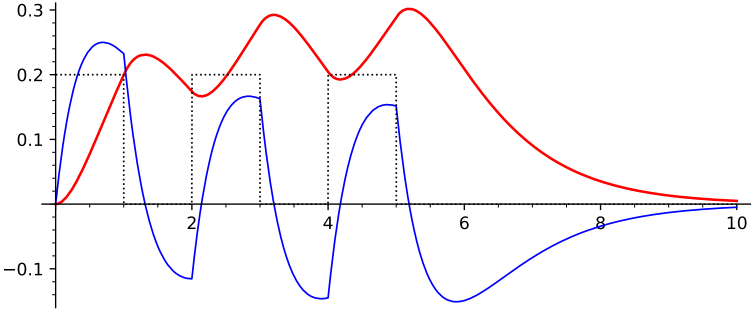

Figure: $y(t)$ (in red), $y'(t)$ (in blue), and $0.2\,f(t)$ (dotted)

Example (continuous forcing)

Solve the IVP $\quad y'' + y = \begin{cases} t & \text{for } t \in [0,1], \\ 2-t & \text{for } t \in [1,2], \\ 0 & \text{otherwise} \end{cases}$

for general initial values $y(0) = y_0$, $y'(0) = y_1$ with the Laplace transform.

Solution: The solution is $y(t) = y_p(t) + y_0 \cos t + y_1 \sin t$, where $y_p(t)$ is the particular solution with $y_p(0) = y_p'(0) = 0$.

As before, $Y(s) = \mathcal{L}\{y_p(t)\}$ has the form $\quad Y(s) = \frac{F(s)}{s^2+1} \quad$ with

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \mathcal{L}\left\{t\big(\mathrm{u}(t) - \mathrm{u}(t-1)\big) + (2-t)\big(\mathrm{u}(t-1) - \mathrm{u}(t-2)\big)\right\} \\ &= \mathcal{L}\left\{t\,\mathrm{u}(t) - 2(t-1)\mathrm{u}(t-1) + (t-2)\mathrm{u}(t-2)\right\} \\ &= \frac{1}{s^2} - \frac{2\,\mathrm{e}^{-s}}{s^2} + \frac{e^{-2s}}{s^2}. \end{aligned}$$

$$\implies Y(s) = \frac{1 - 2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}}{s^2(s^2+1)} = \left(1 - 2\,\mathrm{e}^{-s} + \mathrm{e}^{-2s}\right)\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right).$$

Example (cont'd)

Since $\frac{1}{s^2} - \frac{1}{s^2+1} = \mathcal{L}\{t - \sin t\}$, this gives

$$y_p(t) = t - \sin t - 2\,\mathrm{u}_1(t)\,[t-1-\sin(t-1)] + \mathrm{u}_2(t)\,[t-2-\sin(t-2)]$$
$$= \begin{cases} t - \sin t & \text{if } 0 \le t \le 1, \\ 2 - t + 2\sin(t-1) - \sin t & \text{if } 1 \le t \le 2, \\ 2\sin(t-1) - \sin t - \sin(t-2) & \text{if } t \ge 2. \end{cases}$$

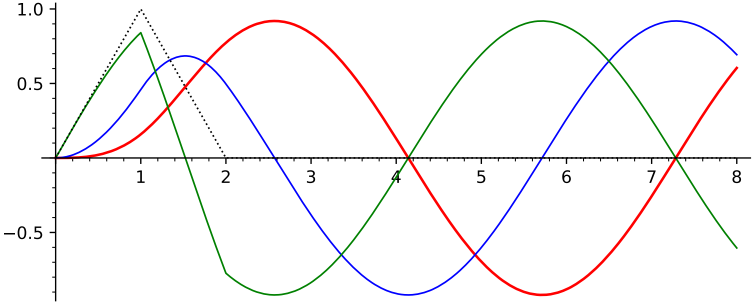

Figure: $y_p(t)$ (in red), $y_p'(t)$ (in blue), $y_p''(t)$ (in green), and $f(t)$ (dotted)

Impulsive forcing

cf. [BDM17], Ch. 6.5

Imagine that in our first example we replace the forcing function $f(t) = \mathrm{r}_{0,1}(t)$ by $f_\epsilon(t) = (1/\epsilon)\mathrm{r}_{0,\epsilon}(t)$, $\epsilon > 0$ (a rectangle with basis ϵ and height $1/\epsilon$, hence still of area 1), and let $\epsilon \downarrow 0$ (or at least consider very small ϵ).

Such forcing functions are important for applications, where they describe time-dependent forces acting over a short period of time and such that the total impulse of the force is constant (for mechanical systems), or electric impulses of high intensity over a short period such that the total voltage of the impulse is constant (for electric circuits).

The solution of the IVP $y'' + y = (1/\epsilon)\mathrm{r}_{0,\epsilon}$, $y(0) = y'(0) = 0$ is

$$y_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}(1 - \cos t) & \text{if } 0 \leq t \leq \epsilon, \\ \frac{1}{\epsilon}\left[\cos(t-\epsilon) - \cos t\right] & \text{if } t \geq \epsilon. \end{cases}$$

Since $\cos(t-\epsilon) - \cos t = 2\sin\left(t - \frac{\epsilon}{2}\right)\sin\left(\frac{\epsilon}{2}\right)$, the "limiting solution" is

$$y(t) = \lim_{\epsilon\downarrow 0} y_\epsilon(t) = \lim_{\epsilon\downarrow 0} \frac{\sin(t-\epsilon/2)\sin(\epsilon/2)}{\epsilon/2} = \sin t \quad \text{for } t > 0.$$

Since $y_\epsilon(0) = 0$, this also holds at $t = 0$.

Observation

If we assign to the "limit function"

$$\delta(t) = \lim_{\epsilon\downarrow 0} f_\epsilon(t) = \lim_{\epsilon\downarrow 0}(1/\epsilon)\mathrm{r}_{0,\epsilon}(t) = \begin{cases} +\infty & \text{if } t = 0, \\ 0 & \text{if } t \neq 0, \end{cases}$$

the Laplace transform $\mathcal{L}\{\delta(t)\} = 1$, then $y(t) = \sin t = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$ can be obtained directly with the usual solution method.

Of course we know that there is no ordinary function on $[0, \infty)$ with Laplace transform 1. In fact the Laplace integral of $\delta(t)$ is zero for all s, because the single value $\delta(0) = +\infty$ doesn't matter for integration.

But it turns out that we can work with $\delta(t)$ in a meaningful way, provided we leave the definition of $\delta(t)$ as an ordinary function aside, use $f_\epsilon(t)$ in place of $\delta(t)$ in all computations, and obtain the value corresponding to $\delta(t)$ by letting $\epsilon \downarrow 0$. The precise mathematical term for such "generalized functions" is "*distribution*", and the present discussion should be viewed as a simplified (and sometimes non-rigorous) account of Dirac's δ-distribution.

It is custom to use rectangular functions that are symmetric about the origin in the final definition of $\delta(t)$, because then the resulting distribution reflects local properties at zero (on both sides) of the functions it is applied to.

Definition

Dirac's Delta function is the distribution ("generalized function") defined on $\mathbb{R}$ by $\delta(t) = \lim_{\epsilon\downarrow 0} f_\epsilon(t)$ with

$$f_\epsilon(t) = \frac{1}{2\epsilon}\,\mathrm{r}_{-\epsilon,\epsilon}(t) = \frac{\mathrm{u}(t+\epsilon)-\mathrm{u}(t-\epsilon)}{2\epsilon}.$$

As a simple example for the ideas involved in the definition of Dirac's Delta function we prove the following two properties:

1. $\displaystyle\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t = 1.$
2. If $f\colon\mathbb{R}\to\mathbb{C}$ is piecewise continuous then
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = \frac{f(t_0-)+f(t_0+)}{2};$$
in particular, if f is continuous in t_0 then $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = f(t_0)$.

Proof.

(1) Since $\int_{-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = 1$ for every $\epsilon > 0$, we also have

$$\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = \lim_{\epsilon\downarrow 0} \int_{-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = 1.$$

(2) We have

$$\int_{-\infty}^{\infty} f(t) f_\epsilon(t-t_0)\,\mathrm{d}t = \frac{1}{2\epsilon}\int_{t_0-\epsilon}^{t_0+\epsilon} f(t)\,\mathrm{d}t$$

$$= \frac{1}{2\epsilon}\int_{t_0-\epsilon}^{t_0} f(t) - f(t_0-)\,\mathrm{d}t + \frac{1}{2\epsilon}\int_{t_0}^{t_0+\epsilon} f(t) - f(t_0+)\,\mathrm{d}t + \frac{f(t_0-)+f(t_0+)}{2}.$$

Since $\left|\int_{t_0-\epsilon}^{t_0} f(t) - f(t_0-)\,\mathrm{d}t\right| \leq \epsilon \max\bigl\{|f(t)-f(t_0-)|\,;\, t_0-\epsilon \leq t \leq t_0\bigr\}$, the first summand tends to zero for $\epsilon \downarrow 0$, and similarly for the 2nd summand.

$\implies \int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,\mathrm{d}t = \lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} f(t) f_\epsilon(t-t_0)\,\mathrm{d}t = \frac{f(t_0-)+f(t_0+)}{2}.$ □

In order to work with $\delta(t)$ symbolically in the context of the Laplace transform, we need further properties:

3. $\mathcal{L}\{\delta(t-t_0)\} = \mathrm{e}^{-st_0}$ for $t_0 > 0$;
4. $\mathcal{L}\{\delta(t)\} = 1$ (the constant function $s \mapsto 1$);
5. $\mathrm{u}'(t) = \delta(t)$.

Proof.

(3) For $\epsilon \leq t_0$ the function $t \mapsto f_\epsilon(t-t_0)$ vanishes on $(-\infty, 0)$.

$$\Longrightarrow \int_0^\infty f_\epsilon(t-t_0)\mathrm{e}^{-st}\,\mathrm{d}t = \int_{-\infty}^\infty f_\epsilon(t-t_0)\mathrm{e}^{-st}\,\mathrm{d}t,$$

which for $\epsilon \downarrow 0$ converges to e^{-st_0}, since $t \mapsto \mathrm{e}^{-st}$ is continuous.; cf. Property 1 and its proof.

(4) This follows by letting $t_0 \downarrow 0$ in (3).

(5) We have

$$\int_{-\infty}^t f_\epsilon(\tau)\mathrm{d}\tau = \begin{cases} 0 & \text{if } t \leq -\epsilon, \\ \frac{t+\epsilon}{2\epsilon} & \text{if } t \in [-\epsilon, \epsilon], \\ 1 & \text{if } t \geq \epsilon. \end{cases}$$

$\Longrightarrow \int_{-\infty}^t \delta(\tau)\mathrm{d}\tau = \lim_{\epsilon\downarrow 0} \int_{-\infty}^t f_\epsilon(\tau)\mathrm{d}\tau = \mathrm{u}(t)$ (except for $t = 0$). $\square$

Notes

- Since $\lim_{\epsilon\downarrow 0}\mathcal{L}\{f_\epsilon(t)\}=1/2$, Property 4 cannot be concluded in the usual way. (For this one needs to use the one-sided analog of $f_\epsilon(t)$ as in the example.) If we want the translation-in-the-domain formula also hold for $\delta(t)$, we must define $\mathcal{L}\{\delta(t)\}=1$.
- Some people define the value of the Heaviside function at $t=0$ as $\mathrm{u}(0)=1/2$. With this definition, Property 5 holds also at $t=0$.

Example

Find the solution of the initial value problem

$$y''-4y'+4y=3\,\delta(t-1)+\delta(t-2);\quad y(0)=y'(0)=1.$$

Solution: Applying $\mathcal{L}$ to both sides of the ODE and using Property (3) gives

$$s^2\,Y(s)-s-1-4\bigl(s\,Y(s)-1\bigr)+4\,Y(s)=3\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}$$
$$(s^2-4s+4)\,Y(s)=s-3+3\,\mathrm{e}^{-s}+\mathrm{e}^{-2s}$$

Example (cont'd)

$$\Longrightarrow Y(s) = \frac{s-3}{(s-2)^2} + \frac{3\,e^{-s}}{(s-2)^2} + \frac{\mathrm{e}^{-2s}}{(s-2)^2}$$
$$= \frac{1}{s-2} - \frac{1}{(s-2)^2} + \frac{3\,e^{-s}}{(s-2)^2} + \frac{\mathrm{e}^{-2s}}{(s-2)^2}$$

$$y(t) = \mathrm{e}^{2t} - t\,\mathrm{e}^{2t} + 3\,\mathrm{u}_1(t)(t-1)\mathrm{e}^{2(t-1)} + \mathrm{u}_2(t)(t-2)\mathrm{e}^{2(t-2)}.$$

The meaning of this solution is the following: If $y_\epsilon(t)$ denotes the solution of the IVP

$$y'' - 4y' + 4y = 3\,f_\epsilon(t-1) + f_\epsilon(t-2); \quad y(0) = y'(0) = 1, \ (\mathrm{IVP}_\epsilon)$$

we have $\lim_{\epsilon\downarrow 0} y_\epsilon(t) = y(t)$. Hence for small ϵ the solution of (IVP_ϵ) is well approximated by $y(t)$.

The use of the convolution

We may view $Y_p(s) = \frac{F(s)}{s^2+bs+c}$ as a function of $F(s)$, (and hence of the forcing function $f(t)$. This functional relation can be written as

$$Y_p(s) = \mathrm{H}(s)F(s) \quad \text{with} \quad \mathrm{H}(s) = (s^2+bs+c)^{-1}.$$

The function $\mathrm{H}(s)$ is called *transfer function* of the ODE (or the physical system described by the ODE). The name comes from the fact that we can consider the solution $y(t)$ as "output" of the system when the forcing function $f(t)$ (e.g., a mechanical force/electric impulse) is applied as "input".

The convolution theorem gives

$$y_p(t) = \int_0^t \mathrm{h}(t-\tau)f(\tau)\mathrm{d}\tau$$

with $\mathrm{h}(t) = \mathcal{L}^{-1}\{\mathrm{H}(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+bs+c}\right\}$. Thus $\mathrm{h}(t)$ (the so-called "impulse response") is the solution for $f(t) = \delta(t)$ ("unit impulse at time $t=0$"), and the solution $y_p(t)$ in the general case is the convolution of the impulse response and the forcing function.