Math 286
Introduction to Differential Equations

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Outline

Today's Lecture: Introduction to PDE's

Motivation

FOURIER's Problem

Describe the heat flow in a long and thin rectangular plate, when some known temperature function is applied to one of the short sides and the long sides are kept at constant temperature.

For simplicity, the plate is assumed to be infinitely long and thin, and given by the region

$$P = \{(x, y) \in \mathbb{R}^2; -1 \leq x \leq 1, y \geq 0\}.$$

For a stationary solution, the temperature $z(x, y)$ at $(x, y) \in P^\circ$ (i.e., for $-1 < x < 1$, $y > 0$) has to satisfy

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \qquad \text{(Laplace's Equation)}$$

The boundary conditions will be

$$\begin{aligned} z(x, 0) &= f(x) \quad \text{for } -1 \leq x \leq 1, \\ z(-1, y) = z(1, y) &= 0 \quad \text{for } y > 0, \end{aligned}$$

where $f\colon [-1, 1] \to \mathbb{R}$ gives the temperature on the short side (the other short side is considered as infinitely far away).

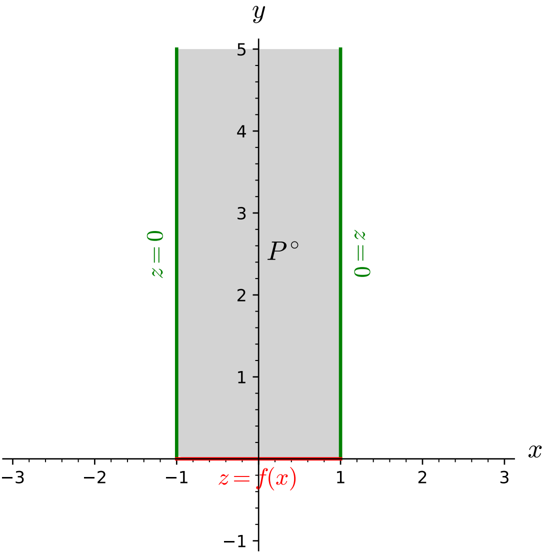

Figure: Fourier's Problem

FOURIER assumed further that $f(x) = f(-x)$.
He was interested especially in the case of a constant temperature > 0 ("heating"), which we can take w.l.o.g. as $f(x) \equiv 1$.

Fourier's Solution

We start with the "separation ansatz" $z(x, y) = a(x)b(y)$.

Plugging this into Laplace's Equation gives

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = a''(x)b(y) + a(x)b''(y) = 0.$$

At points (x, y) with $z(x, y) \neq 0$ we can rewrite this as

$$\frac{a''(x)}{a(x)} = -\frac{b''(y)}{b(y)}.$$

$\Longrightarrow$ Both sides must be constant, i.e., there exists $C \in \mathbb{R}$ such that

$$a''(x) = -C\,a(x), \quad b''(y) = C\,b(y) \quad \text{for all } (x, y) \in P^\circ.$$

Assuming $z(x, y)$ is not identically zero, the first boundary condition implies $a(-1) = a(1) = 0$.

Fourier's Solution cont'd

$\Longrightarrow$ $a(x)$ and $a''(x)$ must have opposite signs
$\Longrightarrow$ $C > 0$, and we can set $C = K^2$ with $K > 0$.

The general real solution of the two resulting ODE's is

$$\begin{aligned} a(x) &= c_1 \cos(Kx) + c_2 \sin(Kx), && c_1, c_2 \in \mathbb{R}, \\ b(y) &= c_3 \mathrm{e}^{Ky} + c_4 \mathrm{e}^{-Ky}, && c_3, c_4 \in \mathbb{R}. \end{aligned}$$

Since $f(x) = z(x, 0) = a(x)b(0)$ should be an even function, we must have $c_2 = 0$.
Since the temperature should drop to zero for $y \to +\infty$ (from physics or just common sense) we must have $c_3 = 0$.
Since $a(1) = a(-1) = 0$, K must be an odd multiple of $\pi/2$.

$$\Longrightarrow \quad z(x, y) = a\,\mathrm{e}^{-\frac{(2k-1)\pi y}{2}} \cos\left(\frac{(2k-1)\pi x}{2}\right), \quad a \in \mathbb{R},\ k = 1, 2, \dots.$$

Since superposition preserves solutions, any function of the form

$$\begin{aligned} z(x, y) = a_1\,\mathrm{e}^{-\pi y/2} \cos(\pi x/2) + a_2\,\mathrm{e}^{-3\pi y/2} \cos(3\pi x/2) \\ + \dots + a_n\,\mathrm{e}^{-(2n-1)\pi y/2} \cos\bigl((2n-1)\pi x/2\bigr) \end{aligned}$$

Fourier's Solution cont'd

with $a_1, \dots, a_n \in \mathbb{R}$ will then also be a solution of Laplace's Equation and satisfy the boundary conditions $z(-1, y) = z(1, y) = 0$, as well as

$$f(x) = z(x, 0) = \sum_{k=1}^{n} a_k \cos\left(\frac{(2k-1)\pi x}{2}\right).$$

The function $f(x) \equiv 1$, however, is not of this form, since any (finite) linear combination of the functions $\cos\left(\frac{(2k-1)\pi x}{2}\right)$ vanishes at $x = \pm 1$.

Question: What to do?

FOURIER assumed the existence of an infinite series representation

$$f(x) = 1 = \sum_{k=1}^{\infty} a_k \cos\left(\frac{(2k-1)\pi x}{2}\right) \quad \text{for } -1 < x < 1,$$

and showed how to compute a_k from this and the additional assumption that this series can be integrated termwise.

Fourier's Solution cont'd

Lemma

For $k, l \in \mathbb{Z}^+$ we have

$$\int_{-1}^{1} \cos\left(\frac{(2k-1)\pi x}{2}\right) \cos\left(\frac{(2l-1)\pi x}{2}\right) \mathrm{d}x = \begin{cases} 0 & \text{if } k \neq l, \\ 1 & \text{if } k = l. \end{cases}$$

Proof.

Making the substitution $t = \pi x/2$, $\mathrm{d}t = (\pi/2)\,\mathrm{d}x$, the integral becomes

$$\begin{aligned} &\frac{2}{\pi} \int_{-\pi/2}^{\pi/2} \cos\bigl((2k-1)t\bigr) \cos\bigl((2l-1)t\bigr)\,\mathrm{d}t \\ &= \frac{2}{\pi} \int_{0}^{\pi} \cos\bigl((2k-1)t\bigr) \cos\bigl((2l-1)t\bigr)\,\mathrm{d}t \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \cos\bigl((2k-1)t\bigr) \cos\bigl((2l-1)t\bigr)\,\mathrm{d}t\,. \end{aligned}$$

The latter integral is well-known to have the value 0 for $k \neq l$ and π for $k = l$; cf. exercises. $\square$

Fourier's Solution cont'd

If $f(x)=\sum_{k=1}^{\infty} a_k \cos\left(\frac{(2k-1)\pi x}{2}\right)$ can be integrated termwise, the lemma implies

$$\begin{aligned}
&\int_{-1}^{1} f(x)\cos\left(\frac{(2l-1)\pi x}{2}\right)\mathrm{d}x\\
&\quad=\int_{-1}^{1}\sum_{k=1}^{\infty} a_k\cos\left(\frac{(2k-1)\pi x}{2}\right)\cos\left(\frac{(2l-1)\pi x}{2}\right)\mathrm{d}x\\
&\quad=\sum_{k=1}^{\infty} a_k\int_{-1}^{1}\cos\left(\frac{(2k-1)\pi x}{2}\right)\cos\left(\frac{(2l-1)\pi x}{2}\right)\mathrm{d}x=a_l.
\end{aligned}$$

In particular for $f(x)\equiv 1$ we obtain (the values $f(\pm 1)$ do not matter here)

$$a_l=\int_{-1}^{1}\cos\left(\frac{(2l-1)\pi x}{2}\right)\mathrm{d}x=\left[\frac{2}{(2l-1)\pi}\sin\left(\frac{(2l-1)\pi x}{2}\right)\right]_{-1}^{1}=\frac{4(-1)^{l-1}}{(2l-1)\pi}.$$

Fourier's Solution cont'd

Thus FOURIER arrived at the series representation

$$1 = \frac{4}{\pi}\left(\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{3\pi x}{2} + \frac{1}{5}\cos\frac{5\pi x}{2} - \frac{1}{7}\cos\frac{7\pi x}{2} \pm \cdots\right)$$

and concluded from this that

$$\begin{aligned} z(x,y) &= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}\mathrm{e}^{-(2k-1)\pi y/2}\cos\frac{(2k-1)\pi x}{2} \\ &= \frac{4}{\pi}\Bigg(\mathrm{e}^{-\pi y/2}\cos\frac{\pi x}{2} - \frac{1}{3}\,\mathrm{e}^{-3\pi y/2}\cos\frac{3\pi x}{2} \\ &\qquad + \frac{1}{5}\,\mathrm{e}^{-5\pi y/2}\cos\frac{5\pi x}{2} - \frac{1}{7}\,\mathrm{e}^{-7\pi y/2}\cos\frac{7\pi x}{2} + \cdots\Bigg) \end{aligned}$$

solves the Laplace equation in P° (assuming that $z(x,y)$ can be differentiated termwise with respect to x, y) and satisfies the boundary conditions $z(\pm 1, y) = 0$ for $y \geq 0$, $z(x,0) = 1$ for $-1 < x < 1$.

It remains yet to prove:

1. the identity

$$1 = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos \frac{(2k+1)\pi x}{2}, \quad -1 < x < 1;$$

2. the function

$$z(x,y) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \, \mathrm{e}^{-(2k+1)\pi y/2} \cos \frac{(2k+1)\pi x}{2}, \; (x,y) \in P$$

   is well-defined for all $(x,y) \in \mathbb{R}^2$ with $y > 0$ and satisfies $\Delta z(x,y) = 0$ for those (x,y).

Property (2) is quite easy and will be proved right now.

Property (1) is more difficult and a proof was only found by DIRICHLET some 20 years after FOURIER had submitted his manuscript *Theory of the Propagation of Heat in Solid Bodies*. It is a consequence of a general theorem on the point-wise convergence of Fourier series, which we will derive later.

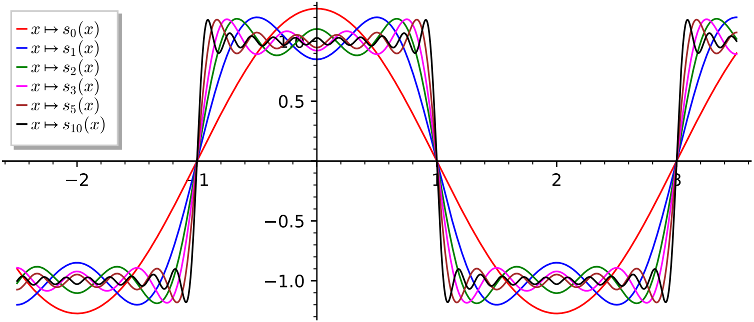

Figure: Partial sums of FOURIER's cosine series

Note that all functions $f_k(x) = \cos\frac{(2k+1)\pi x}{2}$, $x \in \mathbb{R}$ satisfy $f_k(x+2) = -f_k(x)$ and hence $f_k(x+4) = f_k(x)$. Hence the same is true of the limit function f. In particular, the series does not represent the function $f(x) \equiv 1$ outside $[-1, 1]$ (only for about half of the points $x \in \mathbb{R}$).

Proof of Property (2).

It suffices to show that the series defining $z(x, y)$ and the corresponding series of partial derivatives up to order 2 converge uniformly on every subset $H_\delta = \{(x, y) \in \mathbb{R}^2; y \geq \delta\}$, $\delta > 0$, of the domain $H = \{(x, y) \in \mathbb{R}^2; y > 0\}$ (the open "upper half-plane"). This shows that $z(x, y)$ is well-defined, and the Differentiation Theorem gives that $\partial^2 z/\partial x^2$, $\partial^2 z/\partial y^2$ (and the Laplacian as well) can be computed term-wise. Since the terms $z_k(x, y) = \pm\frac{4}{(2k+1)\pi}\mathrm{e}^{-(2k+1)\pi y/2}\cos\frac{(2k+1)\pi x}{2}$ satisfy $\Delta z_k(x, y) = 0$ by construction, the same is then true of $z(x, y)$.

We give the proof only for the series representing $\partial^2 z/\partial x^2$. (The remaining proofs are virtually the same.)

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\partial^2 z_k(x,y)}{\partial x^2} &= \sum_{k=0}^{\infty}\frac{4(-1)^k}{(2k+1)\pi}\,\mathrm{e}^{-(2k+1)\pi y/2}\cos\frac{(2k+1)\pi x}{2}\left(-\frac{(2k+1)^2\pi^2}{4}\right)\\ &= \sum_{k=0}^{\infty}(-1)^{k+1}(2k+1)\pi\,\mathrm{e}^{-(2k+1)\pi y/2}\cos\frac{(2k+1)\pi x}{2}\end{aligned}$$

Proof cont'd.

$\implies$ For $(x,y) \in H_\delta$ we have

$$\sum_{k=0}^{\infty} \left| \frac{\partial^2 z_k(x,y)}{\partial x^2} \right| \leq \pi \sum_{k=0}^{\infty} (2k+1) \mathrm{e}^{-(2k+1)\pi\delta/2} < \infty,$$

since the exponentials decrease faster than $(2k+1)^{-3}$, say.

Hence $\sum_{k=0}^{\infty} \frac{\partial^2 z_k(x,y)}{\partial x^2}$ is majorized on H_δ by a convergent series, which is independent of x and y. This implies uniform convergence on H_δ, as asserted (by Weierstrass's Criterion). $\square$

Note

In the case under consideration it would have been sufficient to show that the majorizing series doesn't depend on x, because $\partial^2 z / \partial x^2$ is computed by considering y as a fixed parameter. To make the same argument work for both $\partial^2 z / \partial x^2$ and $\partial^2 z / \partial y^2$, one needs indpendence of x and y.

The Vibrating String Problem

A homogeneous string of length $L > 0$ is stretched along the line segment $0 \leq x \leq L$, $y = 0$ of the (x, y)-plane and fixed at both ends. At time $t = 0$ it is displaced from this "equilibrium position" to an "initial position" $y = f(x)$, $0 \leq x \leq L$, which satisfies $f(0) = f(L) = 0$, and an "initial velocity" $y = g(x)$ in the y-direction is applied to it. The function g should also satisfy $g(0) = g(L) = 0$. From then the string is left at the disposal of the elastic forces acting on it and '"vibrates" around the equilibrium position.

Problem

Determine the "elongation function" $y(x, t)$, $0 \leq x \leq L$, $t \geq 0$ describing the movement of the string over time.

$y(x, t)$ must be a solution of the 1-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

with $c > 0$ a physically determined constant (tension-to-density ratio of the string).

$y(x,t)$ must satisfy the boundary conditions

$$y(0,t) = y(L,t) = 0, \quad t \geq 0,$$
$$y(x,0) = f(x), \qquad y_t(x,0) = g(x).$$

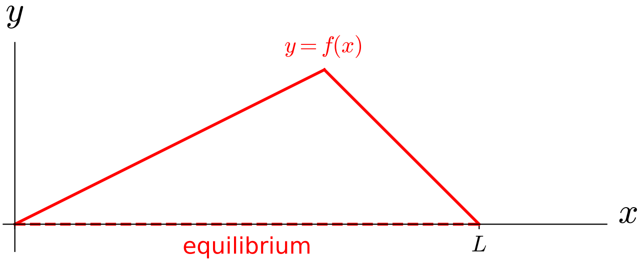

Figure: Vibrating string problem with $g(x) \equiv 0$

D. BERNOULLI's solution

First we determine the solutions of the special form $y(x,t) = a(x)b(t)$.

Proceeding as in FOURIER's solution, we obtain the pair of 2nd-order ODE's

$$\frac{a''(x)}{a(x)} = \frac{1}{c^2}\,\frac{b''(t)}{b(t)} = -K^2,$$

where $K \geq 0$ is a real constant.

$$\begin{aligned} \implies \quad a(x) &= c_1\cos(Kx) + c_2\sin(Kx), \\ b(t) &= c_3\cos(cKt) + c_4\sin(cKt) \quad \text{with } c_1, c_2, c_3, c_4 \in \mathbb{R}. \end{aligned}$$

The boundary condition $y(0,t) = y(L,t) = 0$ translates into $a(0) = a(L) = 0$ and implies

$$c_1 = 0 \quad \text{and} \quad K \in \{\pi/L, 2\pi/L, 3\pi/L, \dots\}.$$

D. BERNOULLI's solution cont'd

Strictly speaking, we only obtain $K \in \mathbb{Z}(\pi/L)$, but the all-zero solution ($K = 0$) can be omitted, and $\pm K$ give scalar multiples of the same solution.

$\Longrightarrow$ All functions of the form

$$y(x,t) = \sum_{k=1}^{n} \sin\frac{k\pi x}{L}\left(a_k \cos\frac{ck\pi t}{L} + b_k \sin\frac{ck\pi t}{L}\right), \quad a_k, b_k \in \mathbb{R},$$

are solutions of $y_{tt} = c^2\, y_{xx}$ satisfying the first boundary condition $y(0,t) = y(L,t) = 0$ and

$$y(x,0) = \sum_{k=1}^{n} a_k \sin\frac{k\pi x}{L},$$

$$y_t(x,0) = \frac{c\pi}{L}\sum_{k=1}^{n} k b_k \sin\frac{k\pi x}{L}.$$

D. BERNOULLI then claimed (without providing the necessary justification for convergence and term-wise differentiability of the series) that the solution to the vibrating string problem is

D. BERNOULLI's solution cont'd

$$y(x,t) = \sum_{k=1}^{\infty} \sin \frac{k\pi x}{L} \left(a_k \cos \frac{ck\pi t}{L} + b_k \sin \frac{ck\pi t}{L} \right),$$

where a_k, b_k are determined from $f(x) = \sum_{k=1}^{\infty} a_k \sin \frac{k\pi x}{L}$ and $g(x) = \frac{c\pi}{L} \sum_{k=1}^{n} k b_k \sin \frac{k\pi x}{L}$, respectively.

Question

Does every continuous function $f\colon [0,L] \to \mathbb{R}$ with $f(0) = f(L) = 0$ have a representation as a sine series of the above form?

Note that the problem for $g(x)$ is the same.

Requiring f and g to be continuous comes from the physical interpretation. Note, however, that at least f need not be differentiable, since we want to model situations like plucking a string, which corresponds to a piece-wise linear function $f(x)$.

Negative answer

There exist continuous functions which are not represented by their Fourier series at every point. Piece-wise C^1-functions, however, and hence virtually all physically meaningful functions, are represented everywhere by their Fourier series.

D'ALEMBERT's solution

This solution is completely different from BERNOULLI's and starts by making the variable substitution

$$\xi = x + ct,\ \eta = x - ct, \quad \text{i.e.,} \quad z(\xi,\eta) = y\left(\tfrac{1}{2}(\xi+\eta), \tfrac{1}{2c}(\xi-\eta)\right).$$

$$\begin{aligned}
\implies \quad z_\xi &= \frac{1}{2}y_x + \frac{1}{2c}y_t, \\
z_{\xi\eta} = (z_\xi)_\eta &= \frac{1}{2}z_{\xi x} - \frac{1}{2c}z_{\xi t} \\
&= \frac{1}{2}\left(\frac{1}{2}y_{xx} + \frac{1}{2c}y_{tx}\right) - \frac{1}{2c}\left(\frac{1}{2}y_{xt} + \frac{1}{2c}y_{tt}\right) \\
&= \frac{1}{4}\left(y_{xx} - \frac{1}{c^2}y_{tt}\right),
\end{aligned}$$

provided that y, and hence z, are C^2-functions.
Hence $y(x,t)$ solves the 1-dimensional wave equation iff $z_{\xi\eta}(\xi,\eta) \equiv 0$.

D'ALEMBERT's solution cont'd

The solution of $z_{\xi\eta} = 0$ is

$$\begin{aligned} z_\xi(\xi,\eta) &= \phi(\xi), \\ z(\xi,\eta) &= \Phi(\xi) + \Psi(\eta) \qquad \text{(with } \Phi'(\xi) = \phi(\xi)\text{)} \\ &= \Phi(x+ct) + \Psi(x-ct). \end{aligned}$$

$$\Longrightarrow y(x,t) = \Phi(x+ct) + \Psi(x-ct)$$

with arbitrary C^2-functions $\Phi\colon [0,+\infty) \to \mathbb{R}$ and $\Psi\colon (-\infty, L] \to \mathbb{R}$.

The boundary conditions for $y(x,t)$ translate into

$$\begin{aligned} \Phi(t) + \Psi(-t) = \Phi(L+t) + \Psi(L-t) &= 0 \quad \text{for } t \geq 0, \\ \Phi(x) + \Psi(x) = f(x),\ \Phi'(x) - \Psi'(x) &= g(x)/c \quad \text{for } 0 \leq t \leq L. \end{aligned}$$

The first set of equations imply

$$\begin{aligned} \Phi(t+2L) = \Phi(L+t+L) = -\Psi(L-t-L) &= -\Psi(-t) = \Phi(t), \\ \Psi(-t-2L) = -\Phi(t+2L) = -\Phi(t) &= \Psi(-t) \qquad \text{for } t \geq 0. \end{aligned}$$

$\Longrightarrow$ We can extend Φ, Ψ to $2L$-periodic functions with domain $\mathbb{R}$.

D'ALEMBERT's solution cont'd

With this definition the first set of equations then hold for all $t \in \mathbb{R}$. On account of periodicity, it suffices to verify this for $t \in [-L, 0)$:

$$\Phi(t) = \Phi(t+2L) = \Phi(L+t+L) = -\Psi(L-t-L) = -\Psi(-t),$$
$$\Phi(L+t) = -\Psi(-L-t) = -\Psi(L-t),$$

where $t + L = L + t \geq 0$ was used.

Further, since Φ and Ψ are determined only up to an additive constant and $\Phi(0) + \Psi(0) = 0$, we can normalize to $\Phi(0) = \Psi(0) = 0$.

Then the second set of equations gives

$$\begin{aligned} \Phi(x) - \Psi(x) &= \frac{1}{c}\int_0^x g(\xi)\mathrm{d}\xi, \\ \Phi(x) &= \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(\xi)\mathrm{d}\xi\right), \\ \Psi(x) &= \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(\xi)\mathrm{d}\xi\right) \qquad \text{for } 0 \leq x \leq L. \end{aligned}$$

These identities can be made to hold for all $x \in \mathbb{R}$, provided ...

D'ALEMBERT's solution cont'd

... we extend f, g first to odd functions on $[-L, L]$ (possible, since $f(0) = g(0) = 0$) and then $2L$-periodically to the whole of $\mathbb{R}$.
Reason: With this definition we have for $x \in [-L, 0)$

$$\begin{aligned}\Phi(x) &= -\Psi(-x) = -\frac{1}{2}\left(f(-x) - \frac{1}{c}\int_0^{-x} g(\xi)\mathrm{d}\xi\right) \\ &= \frac{f(x)}{2} + \frac{1}{c}\int_0^x g(-\eta)(-\mathrm{d}\eta) = \frac{f(x)}{2} + \frac{1}{c}\int_0^x g(\eta)\mathrm{d}\eta,\end{aligned}$$

as asserted. For general x the second identity then follows from the $2L$-periodicity of both sides, using $\int_x^{x+2L} g(\xi)\mathrm{d}\xi = \int_{-L}^{L} g(\xi)\mathrm{d}\xi = 0$. The third identity (and hence the first) is proved similarly.

$$\begin{aligned}\implies y(x,t) &= \Phi(x+ct) + \Psi(x-ct) \\ &= \frac{1}{2}\left(f(x+ct) + f(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} g(\xi)\mathrm{d}\xi\right).\end{aligned}$$

for $0 \leq x \leq L$, $t \geq 0$. This is D'ALEMBERT's solution.

Notes

- BERNOULLI's solution also has the form $y(x,t) = \Phi(x+ct) + \Psi(x-ct)$, as can be seen by rewriting it using the formulas

$$\sin\phi_1 \cos\phi_2 = \frac{1}{2}\bigl(\sin(\phi_1+\phi_2) - \sin(\phi_1-\phi_2)\bigr),$$
$$\sin\phi_1 \sin\phi_2 = -\frac{1}{2}\bigl(\cos(\phi_1+\phi_2) - \cos(\phi_1-\phi_2)\bigr)$$

in the following way:

$$\begin{aligned} y(x,t) &= \sum_{k=1}^{\infty} \sin\frac{k\pi x}{L}\left(a_k \cos\frac{ck\pi t}{L} + b_k \sin\frac{ck\pi t}{L}\right) \\ &= \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k \sin\frac{k\pi(x+ct)}{L} - b_k \cos\frac{k\pi(x+ct)}{L}\right) \\ &\quad + \frac{1}{2}\sum_{k=1}^{\infty}\left(a_k \sin\frac{k\pi(x-ct)}{L} + b_k \cos\frac{k\pi(x-ct)}{L}\right). \end{aligned}$$

Notes cont'd

- Although in the derivation of D'ALEMBERT's formula we have implicitly assumed that f is a C^2-function and g is a C^1-function, the formula remains true for functions f, g whose derivatives have jump discontinuities, such as piece-wise linear functions.
- D'ALEMBERT's formula can be physically interpreted as the superposition of two waves with initial states $\Phi(x)$ and $\Psi(x)$ moving at constant speed in opposite directions.
- BERNOULLI's Fourier series solution, although conceptually more complicated than D'ALEMBERT's solution, has the additional benefit of revealing the "harmonic analysis" of the vibrating string.

The Linear Algebra of Fourier Series

Among others, the following two variants of the definition of a Fourier series are most commonly found in the literature.

Definition

1. The *Fourier series* of an absolutely integrable 2π-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ is the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(kt) + b_k \sin(kt) \right),$$

where a_k $(k = 0, 1, 2, \dots)$ and b_k $(k = 1, 2, 3, \dots)$, the so-called *Fourier coefficients* of the series, are defined by

$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \cos(kt) \, \mathrm{d}t, \quad b_k = \frac{1}{\pi} \int_0^{2\pi} f(t) \sin(kt) \, \mathrm{d}t.$$

Definition (cont'd)

2. The *Fourier series* of an absolutely integrable 2π-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ is the series

$$c_0 + \sum_{k=1}^{\infty} \left(c_k \mathrm{e}^{\mathrm{i}kt} + c_{-k} \mathrm{e}^{-\mathrm{i}kt} \right),$$

where c_k ($k = 0, \pm 1 \pm 2, \dots$), the complex *Fourier coefficients* of the series, are defined by

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-\mathrm{i}kt} \, \mathrm{d}t .$$

Notes

- $f\colon \mathbb{R} \to \mathbb{C}$ has *period* L if $f(t+L) = f(t)$ for all $t \in \mathbb{R}$. The substitution $s = 2\pi t/L$, transforming L-periodic functions into 2π-periodic functions, can be used to define Fourier series (and develop the corresponding theory) for complex-valued functions of any period $L > 0$. The corresponding L-periodic cosine, sine and exponential functions are $\cos(2\pi kt/L)$, $\sin(2\pi kt/L)$ and $\mathrm{e}^{2\pi \mathrm{i} kt/L}$, respectively, and the formulas for the Fourier coefficients are
$$\begin{aligned} a_k &= \frac{2}{L}\int_0^L f(t)\cos(2\pi kt/L)\,\mathrm{d}t,\\ b_k &= \frac{2}{L}\int_0^L f(t)\sin(2\pi kt/L)\,\mathrm{d}t,\\ c_k &= \frac{1}{L}\int_0^L f(t)\mathrm{e}^{-2\pi \mathrm{i} kt/L}\,\mathrm{d}t\,. \end{aligned}$$
- The integral of an L-periodic function over any interval of length L is the same; cf. exercises. Thus, for example, we can obtain the Fourier coefficients a_k, b_k, c_k in the previous definition also by integrating over $[-\pi, \pi]$ instead of $[0, 2\pi]$.

Notes cont'd

- Integrating over $[-\pi, \pi]$ instead of $[0, 2\pi]$ has the advantage that it yields immediately the following result:

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(kt) = 0 \qquad \text{if } f(-t) = -f(t),$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(kt) = 0 \qquad \text{if } f(-t) = f(t);$$

  i.e., the Fourier series of an odd periodic function is a pure sine series, and the Fourier series of an even periodic function is a pure cosine series.

Notes cont'd

- Variant (1) of the definition also makes sense for complex valued functions f, and the Fourier series obtained by both definitions are in fact the same!

  In order to see this, recall that a series $\sum_{n=0}^{\infty} f_n$ is defined as the sequence (g_n) of partial sums $g_n = \sum_{k=0}^{n} f_k$. Hence it suffices to verify $c_0 = a_0/2$ and $c_k \mathrm{e}^{\mathrm{i}kt} + c_{-k}\mathrm{e}^{-\mathrm{i}kt} = a_k \cos(kt) + b_k \sin(kt)$. We have

$$\begin{aligned} c_k &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi} f(t)\bigl(\cos(kt) - \mathrm{i}\sin(kt)\bigr)\,\mathrm{d}t \\ &= \frac{1}{2}(a_k - \mathrm{i}b_k), \\ c_{-k} &= \frac{1}{2\pi}\int_0^{2\pi} f(t)\mathrm{e}^{\mathrm{i}kt}\,\mathrm{d}t = \frac{1}{2}(a_k + \mathrm{i}b_k), \end{aligned}$$

  and hence

$$\begin{aligned} c_k \mathrm{e}^{\mathrm{i}kt} + c_{-k}\mathrm{e}^{-\mathrm{i}kt} &= \frac{a_k}{2}(\mathrm{e}^{\mathrm{i}kt} + \mathrm{e}^{-\mathrm{i}kt}) - \frac{\mathrm{i}b_k}{2}(\mathrm{e}^{\mathrm{i}kt} - \mathrm{e}^{-\mathrm{i}kt}) \\ &= a_k\cos(kt) + b_k\sin(kt), \qquad \text{as desired.} \end{aligned}$$

Example

We compute the Fourier series of the putative limit function of FOURIER's cosine series, which has period 4 and is defined on $[0,4)$ by

$$f(x) = \begin{cases} 1 & \text{if } -1 < x < 1, \\ -1 & \text{if } 1 \leq x < 3, \end{cases}$$

and verify that both series coincide (which is a nontrivial fact!). Here $L = 4$, and the Fourier series of f has the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x/2) + b_k \sin(k\pi x/2).$$

Since f is even, we have $b_k = 0$ for all k. For $k \in \mathbb{Z}^+$ we have

$$\begin{aligned} a_k &= \frac{2}{4}\int_{-1}^{3} f(x)\cos(k\pi x/2)\,\mathrm{d}x \\ &= \frac{1}{2}\int_{-1}^{1}\cos(k\pi x/2)\,\mathrm{d}x + \frac{1}{2}\int_{1}^{3} -\cos(k\pi x/2)\,\mathrm{d}x \\ &= \frac{1}{2}\left[\frac{2}{k\pi}\sin(k\pi x/2)\right]_{-1}^{1} - \frac{1}{2}\left[\frac{2}{k\pi}\sin(k\pi x/2)\right]_{1}^{3} \end{aligned}$$

Example (cont'd)

It follows that

$$a_k = \begin{cases} 0 & \text{if } k \equiv 0 \pmod 2, \\ \frac{4}{k\pi} & \text{if } k \equiv 1 \pmod 4, \\ -\frac{4}{k\pi} & \text{if } k \equiv 3 \pmod 4. \end{cases}$$

This also holds for $k = 0$, since $a_0 = \frac{1}{2}\int_{-1}^{3} f(x)\,\mathrm{d}x = 0$.
$\Longrightarrow$ The Fourier series of f is

$$\frac{4}{\pi}\cos\frac{\pi x}{2} - \frac{4}{3\pi}\cos\frac{3\pi x}{2} + \frac{4}{5\pi}\cos\frac{5\pi x}{2} - \frac{4}{7\pi}\cos\frac{7\pi x}{2} \pm \dots,$$

the same as FOURIER's cosine series.

Exercise

Suppose that $f\colon \mathbb{R} \to \mathbb{C}$ is L-periodic and integrable over $[0, L]$. Show that f is integrable over any interval $[a, a + L]$, $a \in \mathbb{R}$, and

$$\int_a^{a+L} f(x)\,\mathrm{d}x = \int_0^L f(x)\,\mathrm{d}x\,.$$

Exercise

Show that the Fourier coefficients a_k, b_k, c_k of any function f are related by

$$c_0 = \frac{a_0}{2} \quad \text{and} \quad |c_k|^2 + |c_{-k}|^2 = \frac{1}{2}\left(|a_k|^2 + |b_k|^2\right) \quad \text{for } k \geq 1.$$

Exercise

What can you say about the Fourier coefficients a_k, b_k, c_k of an L-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ that satisfies

a) $f(x + L/2) = f(x)$ for all $x \in \mathbb{R}$?

b) $f(x + L/2) = -f(x)$ for all $x \in \mathbb{R}$?

Exercise

Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is L-periodic and satisfies one of the symmetry properties $f(x_0 - x) = f(x_0 + x)$ or $f(x_0 - x) = -f(x_0 + x)$ for some $x_0 \in \mathbb{R}$. What can you say about x_0 and the Fourier coefficients of f?

Approximation by Trigonometric Polynomials

We consider the vector space V (over $\mathbb{C}$) formed by all 2π-periodic functions $f\colon \mathbb{R} \to \mathbb{C}$, which are (Lebesgue-)integrable over $[0, 2\pi]$ (and hence over all intervals of length 2π), and satisfy $\int_0^{2\pi} |f(x)|^2 \,\mathrm{d}x < \infty$ (so-called *square-integrable* periodic functions).

V comes with the "inner product"

$$\langle f, g \rangle = \int_0^{2\pi} \overline{f(x)} g(x) \,\mathrm{d}x \,.$$

Strictly speaking, V is not an inner product space, since there are 2π-periodic functions $f \neq 0$ satisfying $\langle f, f \rangle = 0$, for example the characteristic function of $2\pi\mathbb{Z}$. However, we can identify $f, g \in V$ if $f(x) = g(x)$ almost everywhere and consider the vector space $\overline{V}$ formed by the resulting equivalence classes $[f]$. Setting $\langle [f], [g] \rangle = \langle f, g \rangle$, the space $\overline{V}$ becomes a "real" inner product space, since $\langle f, f \rangle = 0$ implies $f = 0$ almost everywhere and hence $[f] = [0]$. The space $\overline{V}$ is also denoted by $\mathrm{L}^2([0, 2\pi])$.

Square-integrability of f, g is needed for showing that $\int_0^{2\pi} \overline{f(x)} g(x)\,\mathrm{d}x$ exists.

We have seen that V (strictly speaking, $\overline{V}$) forms a metric space relative to the mean-square distance (also called L^2-distance) defined by

$$\mathrm{d}_2(f, g) = \|f - g\|_2 = \sqrt{\langle f - g, f - g\rangle} = \sqrt{\int_0^{2\pi} |f(x) - g(x)|^2\,\mathrm{d}x}.$$

Definition

A function $g \in V$ of the form

$$g(x) = \sum_{k=0}^{n} \big(a_k \cos(kx) + b_k \sin(kx)\big) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx}$$

with $a_k, b_k, c_k \in \mathbb{C}$ is called a *trigonometric polynomial* of *degree* at most n (exactly n, if one of a_n, b_n or one of c_n, c_{-n} is nonzero).

Note that the 2π-periodic trigonometric polynomials are precisely the functions in the span TP of 1, $\cos x$, $\sin x$, $\cos(2x)$, $\sin(2x)$, … or, alternatively, in the span of $\{\mathrm{e}^{\mathrm{i}kx}; k \in \mathbb{Z}\}$. Likewise, the 2π-periodic trigonometric polynomials of degree $\leq n$ form a subspace TP_n of V (and of TP).

For each function $f \in V$, the Fourier coefficients a_k, b_k, c_k, and hence the Fourier series of f, n are well-defined. The partial sums of the Fourier series,

$$\mathrm{S}_n f = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} \in \mathrm{TP}_n,$$

are called *Fourier polynomials* and have the following "best-approximation" property:

Theorem

Suppose $f \in V$ and $n \in \mathbb{N}$.

1. *The Fourier polynomial $\mathrm{S}_n f$ is the unique trigonometric polynomial in TP_n minimizing the mean-square distance to f :*

$$\|f - \mathrm{S}_n f\|_2 < \|f - g\|_2 \quad \text{for all } g \in \mathrm{TP}_n \setminus \{\mathrm{S}_n f\}.$$

2. *We have*

$$\|f - \mathrm{S}_n f\|_2^2 = \|f\|_2^2 - \|\mathrm{S}_n f\|_2^2 = \|f\|_2^2 - 2\pi \sum_{k=-n}^{n} |c_k|^2 .$$

Proof.

(1) From the general theory of inner product spaces we know that the distance between f and the vectors in a finite-dimensional subspace $W \subseteq V$ is minimized by the orthogonal projection of f onto W, which relative to any basis $b_1, \dots, b_r$ of W is obtained by solving

$$\left\langle b_i, f - \sum_{j=1}^{r} \lambda_j b_j \right\rangle = 0 \quad \text{for } 1 \leq i \leq r.$$

The solution is uniquely determined, if the Gram matrix $\big(\langle g_i, g_j \rangle\big)_{1 \leq i,j \leq r}$ is invertible (which need not be the case for arbitrary subspaces W).
If the basis vectors are orthogonal and have length > 0, we further get

$$\left\langle b_i, f - \sum_{j=1}^{r} \lambda_j b_j \right\rangle = \langle b_i, f \rangle - \sum_{j=1}^{r} \lambda_j \langle b_i, b_j \rangle = \langle b_i, f \rangle - \lambda_i \langle b_i, b_i \rangle,$$

i.e.,

$$\lambda_i = \frac{\langle b_i, f \rangle}{\langle b_i, b_i \rangle}.$$

Proof cont'd.

The proof is finished by showing that the Fourier polynomial $\mathrm{S}_n f$ is equal to the orthogonal projection of f onto $W = \mathrm{TP}_n$.
For the proof we use the basis of TP_n consisting of the exponentials $\mathrm{e}^{\mathrm{i}kx}$, $-n \leq k \leq n$, which turns out to be the most convenient:

$$\langle \mathrm{e}^{\mathrm{i}kx}, \mathrm{e}^{\mathrm{i}lx} \rangle = \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(l-k)x}\,\mathrm{d}x = \begin{cases} 0 & \text{for } l \neq k, \\ 2\pi & \text{for } l = k. \end{cases}$$

$\Longrightarrow$ The exponentials are mutually orthogonal, and the coefficient of $\mathrm{e}^{\mathrm{i}kx}$ in the orthogonal projection of f onto TP_n is equal to

$$\frac{\langle \mathrm{e}^{\mathrm{i}kx}, f \rangle}{\langle \mathrm{e}^{\mathrm{i}kx}, \mathrm{e}^{\mathrm{i}kx} \rangle} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x = c_k.$$

Hence the orthogonal projection is $\sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} = \mathrm{S}_n f$, as asserted.

(2) Since $f - \mathrm{S}_n f \perp \mathrm{S}_n f$, Pythagoras' Theorem gives
$\|f\|_2^2 = \|f - \mathrm{S}_n f\|_2^2 + \|\mathrm{S}_n f\|_2^2$ and
$\|\mathrm{S}_n f\|_2^2 = \sum_{k,l=-n}^{n} \overline{c}_k c_l \langle \mathrm{e}^{\mathrm{i}kx}, \mathrm{e}^{\mathrm{i}lx} \rangle = 2\pi \sum_{k=-n}^{n} |c_k|^2$. □

Corollary (Bessel's Inequality)

For any $f \in V$ we have

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x \, .$$

Notes

- Bessel's Inequality implies in particular that $\lim_{k\to+\infty} c_k = \lim_{k\to+\infty} c_{-k} = 0$ (or, equivalently, $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$).
- In fact equality holds in Bessel's Inequality, as we will see subsequently. However, Bessel's Inequality generalizes to any sequence $b_1, b_2, b_3, \dots$ of mutually orthogonal functions $b_n \in V$ with $\|b_n\|_2^2 = \langle b_n, b_n\rangle > 0$ in the form

$$\sum_{n=1}^{\infty} \frac{\langle b_n, f\rangle\langle f, b_n\rangle}{\langle b_n, b_n\rangle^2} \leq \langle f, f\rangle = \int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x,$$

and for such sequences equality need no longer hold.

Notes cont'd

- It goes without saying that the theorem and it's corollary (Bessel's Inequality) hold in the more general setting of L-periodic functions. The inner product space view provides in fact the best mnemonic for the various Fourier coefficient formulas. We illustrate this for the orthogonal system of L-periodic functions formed by $c_k(x) = \cos(2k\pi x/L)$, $k \geq 0$, and $s_k(x) = \sin(2k\pi x/L)$, $k \geq 1$.

  The Fourier series of an L-periodic function $f\colon \mathbb{R} \to \mathbb{C}$ is

$$\frac{\langle c_0, f\rangle}{\langle c_0, c_0\rangle} c_0(x) + \sum_{k=1}^{\infty} \left(\frac{\langle c_k, f\rangle}{\langle c_k, c_k\rangle} c_k(x) + \frac{\langle s_k, f\rangle}{\langle s_k, s_k\rangle} s_k(x) \right).$$

$$\Longrightarrow \quad \frac{a_0}{2} = \frac{\langle c_0, f\rangle}{\langle c_0, c_0\rangle} = \frac{\langle 1, f\rangle}{\langle 1, 1\rangle} = \frac{\int_0^L f(x)\,\mathrm{d}x}{\int_0^L 1\,\mathrm{d}x} = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x,$$

$$a_k = \frac{\langle c_k, f\rangle}{\langle c_k, c_k\rangle} = \frac{\int_0^L f(x)\cos(2k\pi x/L)\,\mathrm{d}x}{\int_0^L \cos^2(2k\pi x/L)\,\mathrm{d}x} = \frac{2}{L}\int_0^L f(x)\cos(2k\pi x/L)\,\mathrm{d}x,$$

$$b_k = \frac{\langle s_k, f\rangle}{\langle s_k, s_k\rangle} = \frac{\int_0^L f(x)\sin(2k\pi x/L)\,\mathrm{d}x}{\int_0^L \sin^2(2k\pi x/L)\,\mathrm{d}x} = \frac{2}{L}\int_0^L f(x)\sin(2k\pi x/L)\,\mathrm{d}x.$$

L^2-Convergence of Fourier Series

The following functions are used in the convergence proofs of Fourier series (for both point-wise convergence and L^2-convergence).

$$\begin{aligned} \mathrm{D}_n(x) &= \sum_{k=-n}^{n} \mathrm{e}^{\mathrm{i}kx} = 1 + 2\sum_{k=1}^{n} \cos(kx) \\ &= \frac{\sin\bigl((n+\frac{1}{2})x\bigr)}{\sin\frac{1}{2}x}, && \text{(DIRICHLET kernel)} \\ \mathrm{F}_n(x) &= \frac{1}{n}\bigl(\mathrm{D}_0(x) + \mathrm{D}_1(x) + \dots + \mathrm{D}_{n-1}(x)\bigr) \\ &= \frac{1}{n}\left(\frac{\sin\bigl(\frac{1}{2}nx\bigr)}{\sin\frac{1}{2}x}\right)^2. && \text{(FEJÉR kernel)} \end{aligned}$$

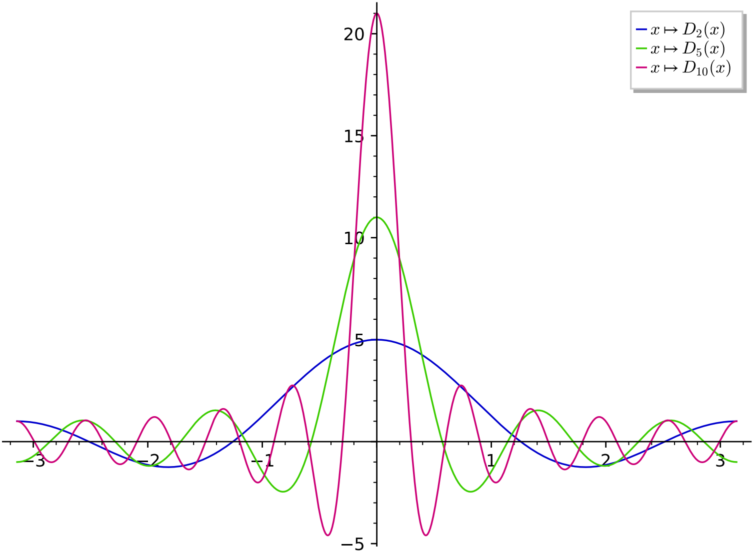

Figure: Some Dirichlet kernels

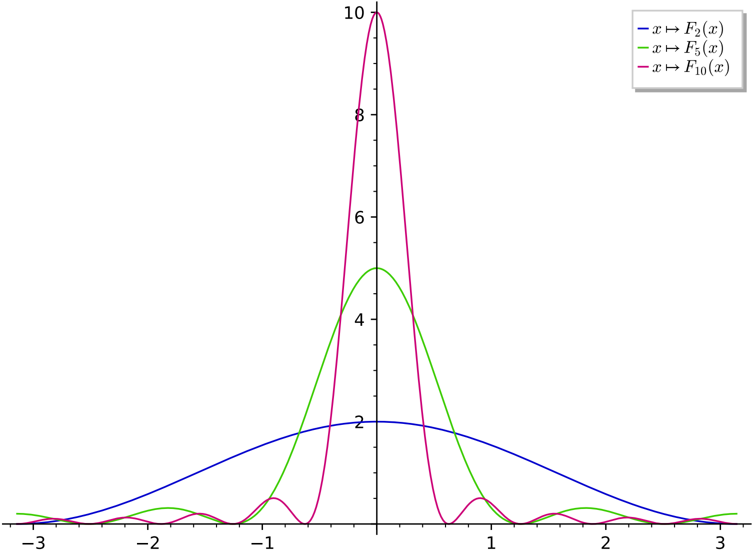

Figure: Some Fejér kernels

Proofs of the formulas

$$\begin{aligned}
\mathrm{D}_n(x) &= \mathrm{e}^{-\mathrm{i}nx}\sum_{k=0}^{2n}(\mathrm{e}^{\mathrm{i}x})^k = \mathrm{e}^{-\mathrm{i}nx}\,\frac{\mathrm{e}^{\mathrm{i}(2n+1)x}-1}{\mathrm{e}^{\mathrm{i}x}-1} = \frac{\mathrm{e}^{\mathrm{i}(n+1)x}-\mathrm{e}^{-\mathrm{i}nx}}{\mathrm{e}^{\mathrm{i}x}-1}\\
&= \frac{\mathrm{e}^{\mathrm{i}(n+1/2)x}-\mathrm{e}^{-\mathrm{i}(n+1/2)x}}{\mathrm{e}^{\mathrm{i}x/2}-\mathrm{e}^{-\mathrm{i}x/2}} = \frac{\sin\bigl((n+\frac{1}{2})x\bigr)}{\sin\frac{1}{2}x},
\end{aligned}$$

$$\begin{aligned}
n\,\mathrm{F}_n(x) &= \sum_{k=0}^{n-1}\frac{\mathrm{e}^{\mathrm{i}(k+1)x}-\mathrm{e}^{-\mathrm{i}kx}}{\mathrm{e}^{\mathrm{i}x}-1} = \frac{\mathrm{e}^{\mathrm{i}x}(\mathrm{e}^{\mathrm{i}nx}-1)}{(\mathrm{e}^{\mathrm{i}x}-1)^2} - \frac{\mathrm{e}^{-\mathrm{i}(n-1)x}(\mathrm{e}^{\mathrm{i}nx}-1)}{(\mathrm{e}^{\mathrm{i}x}-1)^2}\\
&= \frac{(1-\mathrm{e}^{-\mathrm{i}nx})(\mathrm{e}^{\mathrm{i}nx}-1)}{(\mathrm{e}^{\mathrm{i}x/2}-\mathrm{e}^{-\mathrm{i}x/2})^2} = \frac{\mathrm{e}^{\mathrm{i}nx}+\mathrm{e}^{-\mathrm{i}nx}-2}{(\mathrm{e}^{\mathrm{i}x/2}-\mathrm{e}^{-\mathrm{i}x/2})^2}\\
&= \frac{(\mathrm{e}^{\mathrm{i}nx/2}-\mathrm{e}^{-\mathrm{i}nx/2})^2}{(\mathrm{e}^{\mathrm{i}x/2}-\mathrm{e}^{-\mathrm{i}x/2})^2} = \left(\frac{\sin\bigl(\frac{1}{2}nx\bigr)}{\sin\frac{1}{2}x}\right)^2
\end{aligned}$$

Lemma

The Fejér kernels have the following properties:

1. $\mathrm{F}_n(x) \geq 0$ *for all* $n \in \mathbb{N}$, $x \in \mathbb{R}$;
2. $\int_0^{2\pi} \mathrm{F}_n(x)\,\mathrm{d}x = 2\pi$ *for all* $n \in \mathbb{N}$;
3. $\lim_{n\to\infty} \int_r^{2\pi-r} \mathrm{F}_n(x)\,\mathrm{d}x = 0$ *for all* r *in the range* $0 < r < \pi$.

Property (3) is equivalent to $\lim_{n\to\infty} \int_{-r}^{r} \mathrm{F}_n(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} \mathrm{F}_n(x)\,\mathrm{d}x = 2\pi$ for $0 < r < \pi$ and says that for large n the mass with density function $x \mapsto \mathrm{F}_n(x)\,\mathrm{d}x$ on $[-\pi, \pi]$ is concentrated near $x = 0$.

Proof.

(1) is clear from the closed formula for $\mathrm{F}_n(x)$; (2) follows from

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \begin{cases} 2\pi & \text{if } k = 0, \\ 0 & \text{if } k \neq 0, \end{cases}$$

which shows $\int_0^{2\pi} \mathrm{D}_n(x)\,\mathrm{d}x = 2\pi$ and implies the corresponding result for F_n; (3) follows from the estimate $\mathrm{F}_n(x) \leq \frac{1}{\sin^2(r/2)}$ for $r \leq x \leq 2\pi - r$ (or $-r < x < r$). □

Theorem

Suppose $f \in V$ (i.e., f is 2π-periodic and square-integrable over $[0, 2\pi]$.

1. *For every $\epsilon > 0$ there exists a trigonometric polynomial $g \in V$ such that $\|f - g\|_2 < \epsilon$.*
2. $\lim_{n\to\infty} \|f - S_n f\|_2 = 0$*, i.e., f is the limit of its Fourier polynomials in the metric space (V, d_2) ("L^2-limit", "mean-square" limit).*
3. *Bessel's Inequality holds with equality, i.e.,*

$$\sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x \,. \qquad \text{(PLANCHEREL's Identity)}$$

Remark

The following more general form of Plancherel's Identity is also true:

$$\sum_{k\in\mathbb{Z}} \overline{c}_k d_k = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(x)} g(x) \, \mathrm{d}x \text{ for } f, g \in V, \quad \text{(PARSEVAL's Identity)}$$

where c_k, d_k denote the Fourier coefficients of f and g, respectively.

Proof.

First we show that (1), (2) and (3) are equivalent. The key to this is the formula

$$\|f-g\|_2^2 \geq \|f-\mathrm{S}_n f\|_2^2 = \|f\|_2^2 - 2\pi \sum_{k=-n}^{n} |c_k|^2\,,$$

which holds for all $g \in \mathrm{TP}_n$. (Recall that $\mathrm{S}_n f$ provides the best approximation to f in TP_n in the L^2-metric.)

If $\|f-g\| < \epsilon$ and $g \in \mathrm{TP}_N$, then $\|f-\mathrm{S}_n f\|_2^2 < \epsilon$ for all $n \geq N$, so that N can be taken as the response to ϵ in the proof of (2). Hence (1) implies (2). The converse is trivial, and that (2) and (3) are equivalent is immediate from the formula.

For the proof of (1) we use the concept of *(periodic) convolution* of two functions, which for $f, g \in V$ is defined by

$$(f * g)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y)g(x-y)\,\mathrm{d}y\,.$$

It is easy to see that the function $f * g \colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and square-integrable as well, and that the convolution operation $V \times V \to V$, $(f, g) \mapsto f * g$ is bilinear, associative, and commutative.

Proof cont'd.

In particular we have

$$f * \mathrm{e}^{\mathrm{i}kx} = \frac{1}{2\pi}\int_0^{2\pi} f(y)\mathrm{e}^{\mathrm{i}k(x-y)}\,\mathrm{d}y = \mathrm{e}^{\mathrm{i}kx}\frac{1}{2\pi}\int_0^{2\pi} f(y)\mathrm{e}^{-\mathrm{i}ky}\,\mathrm{d}y = c_k\,\mathrm{e}^{\mathrm{i}kx}.$$

This shows that the Fourier coefficients of f are eigenvalues of the "multiplication map" $V \to V$, $g \mapsto f * g$ and implies

$$f * \mathrm{D}_n = \sum_{k=-n}^{n} f * \mathrm{e}^{\mathrm{i}kx} = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx} = \mathrm{S}_n f,$$

$$f * \mathrm{F}_n = \frac{1}{n}\sum_{k=0}^{n-1} f * \mathrm{D}_k = \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{S}_k f$$

The function $\sigma_n f = \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{S}_n f$, which is obviously a trigonometric polynomial, is called n-th *Fejér polynomial* of f. The preceding computation shows that the Fejér polynomials have the integral representation

$$\sigma_n f(x) = f * \mathrm{F}_n = \mathrm{F}_n * f = \frac{1}{2\pi}\int_0^{2\pi} f(x-y)\mathrm{F}_n(y)\,\mathrm{d}y\,.$$

Proof cont'd.

Since $\frac{1}{2\pi}\int_0^{2\pi} \mathrm{F}_n(y)\,\mathrm{d}y = 1$, this gives

$$f(x) - \sigma_n f(x) = \frac{1}{2\pi}\int_0^{2\pi} \big(f(x) - f(x-y)\big)\mathrm{F}_n(y)\,\mathrm{d}y\,.$$

Now assume first that f is continuous. Then there exists $M > 0$ such that $|f(x)| \leq M$ for $x \in [0, 2\pi]$. For $0 < r < \pi$ we then have

$$\begin{aligned}
|f(x) - \sigma_n f(x)| &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(x) - f(x-y)|\,\mathrm{F}_n(y)\,\mathrm{d}y \\
&\leq \frac{1}{2\pi}\int_{-r}^{r} |f(x) - f(x-y)|\,\mathrm{F}_n(y)\,\mathrm{d}y + \frac{2M}{2\pi}\int_r^{2\pi-r} \mathrm{F}_n(y)\,\mathrm{d}y.
\end{aligned}$$

(Since the integrand is 2π-periodic, integrating over $[0, r]$ and $[2\pi - r, 2\pi]$ amounts to integrating over $[-r, r]$.)
The 1st summand can be made arbitrarily small (i.e., $< \epsilon/2$) by choosing r sufficiently small, since f is uniformly continuous and $\frac{1}{2\pi}\int_{-r}^{r} \mathrm{F}_n(y)\,\mathrm{d}y \leq 1$.
By Property 3 Fejér kernels, the 2nd summand can then be made arbitrarily small by choosing n sufficiently large.

Proof cont'd.

$\Longrightarrow$ $\sigma_n f$ converges uniformly to f, since this estimate holds independently of x.

But then $\sigma_n f$ converges to f also in the L^2-metric, as the estimate

$$\begin{aligned} \|f - \sigma_n f\|_2 &= \sqrt{\int_0^{2\pi} |f(x) - \sigma_n f(x)|^2 \, \mathrm{d}x} \\ &\leq \sqrt{2\pi} \max\{|f(x) - \sigma_n f(x)| \,; 0 \leq x \leq 2\pi\} \end{aligned}$$

shows.

Finally, it can be shown that an arbitrary function $f \in V$ can be approximated in the L^2-metric by continuous 2π-periodic functions, i.e., given $\epsilon > 0$ there exists a continuous function $h \in V$ such $\|f - h\|_2 < \epsilon/2$. The preceding argument then yields $n \in \mathbb{N}$ such that $\|h - \sigma_n h\| < \epsilon/2$, and the triangle inequality for the L^2-metric further $\|f - \sigma_n h\| < \epsilon$. Since $\sigma_n h$ is a trigonometric polynomial, this concludes the proof of (1). $\square$

Example

We apply the Parseval identity to FOURIER's introductory example. Clearly the 4-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{if } -1 \leq x < 1, \\ -1 & \text{if } 1 \leq x < 3, \end{cases}$$

(the values at ± 1 don't matter for the mean-square approximation, so we can define them in any way) is square-integrable with $\int_{-1}^{3} f(x)^2 \,\mathrm{d}x = 4$. We have seen that the Fourier coefficients a_k, b_k are zero except for $a_{2k+1} = \frac{4(-1)^k}{\pi(2k+1)}$. Since $|c_k|^2 + |c_{-k}|^2 = \frac{1}{2}\left(|a_k|^2 + |b_k|^2\right)$, the Parseval identity gives

$$\frac{1}{2}\sum_{k=0}^{\infty} \frac{16}{\pi^2(2k+1)^2} = \sum_{k\in\mathbb{Z}} |c_k|^2 = \frac{1}{4}\int_{-1}^{3} f(x)^2 \,\mathrm{d}x = 1.$$

It follows that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Example

We compute the Fourier series of the "repeating ramp" function $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = |x|$ for $-\pi \leq x \leq \pi$ and 2π-periodic extension. (Since $|-\pi| = |\pi|$, the 2π-periodic extension is well-defined.)

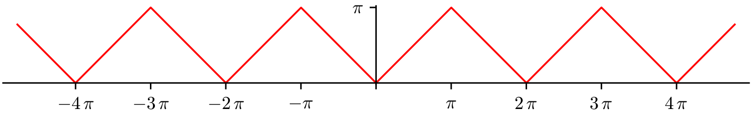

Figure: The repeating-ramp function

Since $g(x) = g(-x)$, the Fourier series of g is likewise a pure cosine series with

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(kx)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\cos(kx) \\ &= \frac{2}{\pi}\left[\frac{x\sin(kx)}{k} + \frac{\cos(kx)}{k^2}\right]_0^{\pi} = \begin{cases} 0 & \text{for } k \geq 2 \text{ even,} \\ -\frac{4}{\pi k^2} & \text{for } k \text{ odd,} \end{cases} \end{aligned}$$

Example (cont'd)

Moreover, $a_0 = \frac{2}{\pi}\int_0^\pi x\,\mathrm{d}x = \pi$.

$\Longrightarrow$ The Fourier series of g is

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2} + \cdots\right).$$

In this case Parseval's identity gives

$$\frac{\pi^2}{4} + \frac{1}{2}\sum_{k=0}^{\infty}\frac{16}{\pi^2(2k+1)^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)^2\,\mathrm{d}x = \frac{1}{\pi}\int_0^\pi x^2\,\mathrm{d}x = \frac{\pi^2}{3}.$$

$$\Longrightarrow \quad \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{\pi^2}{8}\left(\frac{\pi^2}{3} - \frac{\pi^2}{4}\right) = \frac{\pi^4}{96}.$$

From this one can easily derive EULER's formula for the sum of the reciprocals of the 4th powers as follows:

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \sum_{k=1}^{\infty}\frac{1}{(2k)^4} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4} = \frac{1}{16}\sum_{n=1}^{\infty}\frac{1}{n^4} + \sum_{k=0}^{\infty}\frac{1}{(2k+1)^4}$$

Example (cont'd)

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{16}{15} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4} = \frac{16}{15}\,\frac{\pi^4}{96} = \frac{\pi^4}{90}.$$

This complements $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, and similar identities can be derived for the sums $\sum_{n=1}^{\infty} n^{-2r}$, $r = 3, 4, 5, \dots$.

But more is true: Since g is continuous and piece-wise C^1, the Fourier series of g represents g everywhere, i.e., we have

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2} + \cdots\right)$$

for $x \in [-\pi, \pi]$; cf. the subsequent theorems.

You are invited to substitute a few particular values of x into this series and discover further interesting identities.

Theorem (FEJÉR)

Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and the one-sided limits

$$f(x+) = \lim_{x' \downarrow x} f(x'), \quad f(x-) = \lim_{x' \uparrow x} f(x')$$

properly exist for all $x \in \mathbb{R}$. (It suffices to require this for $x \in [0, 2\pi)$, of course.)

1. *For every $x \in \mathbb{R}$ we have $\lim_{n\to\infty} \sigma_n f(x) = \frac{f(x+)+f(x-)}{2}$. In particular, if f is continuous at x then $\lim_{n\to\infty} \sigma_n f(x) = f(x)$.*
2. *If f is continuous everywhere then $(\sigma_n f)$ converges to f uniformly on $\mathbb{R}$.*

Note

The conditions on f imply that f is integrable (in fact even Riemann integrable) and bounded, hence square-integrable. But it is still too weak to conclude point-wise convergence of the Fourier series of f. However, if the Fourier series converges in x, it must have the limit in (1), i.e., $\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sigma_n(x) = \frac{1}{2}\bigl(f(x+) + f(x-)\bigr)$; cf. exercises.

Proof of Fejér's Theorem.

We have already shown (2) in the course of the proof of the previous theorem. The argument to prove (1) is similar. For $0 < r < \pi$ we have

$$\begin{aligned}\sigma_n f(x) &= \frac{1}{2\pi}\int_0^{2\pi} f(x-y)\mathrm{F}_n(y)\,\mathrm{d}y \\ &= \frac{1}{2\pi}\left(\int_0^r + \int_r^{2\pi-r} + \int_{2\pi-r}^{2\pi}\right) = \frac{1}{2\pi}\left(\int_0^r + \int_r^{2\pi-r} + \int_{-r}^{0}\right).\end{aligned}$$

Since f is bounded, the middle integral can be made arbitarily small in absolute value by choosing n sufficiently large (possibly depending on r); cf. Porperty 3 of Fejér kernels.

The left integral can be rewritten as

$$\int_0^r f(x-y)\mathrm{F}_n(y)\,\mathrm{d}y = \int_0^r \big(f(x-y) - f(x-)\big)\mathrm{F}_n(y)\,\mathrm{d}y + f(x-)\int_0^r \mathrm{F}_n(y)\,\mathrm{d}y.$$

Here, the 1st summand can be made arbitrarily small in absolute value by choosing r appropriately (since $\int_0^r \mathrm{F}_n(y)\,\mathrm{d}y \leq \int_0^\pi \mathrm{F}_n(y)\,\mathrm{d}y = \pi$), and the 2nd summand can be made arbitrarily close to $f(x-)\pi$ by choosing n sufficiently large

Proof of Fejér's Theorem cont'd.

(since $\mathrm{F}_n(x) = \mathrm{F}(-x)$ and hence $\int_0^r \mathrm{F}_n(y)\,\mathrm{d}y = \int_{-r}^0 \mathrm{F}_n(y)\,\mathrm{d}y \to \pi$ for $n \to \infty$).

A similar argument applies to the 3rd integral.

In all it follows that

$$\sigma_n f(x) - \frac{f(x-)}{2} - \frac{f(x+)}{2}$$

can be made arbitrarily small in absolute value by choosing n sufficiently large. This completes the proof of (1).

$\square$

Point-wise Convergence of Fourier Series

If $x \in \mathbb{R}$ is such that $f(x+)$ and $f(x-)$ exist, we can define *one-sided derivatives* of f in x as

$$f'(x+) = \lim_{x' \downarrow x} \frac{f(x') - f(x+)}{x' - x}, \quad f'(x-) = \lim_{x' \uparrow x} \frac{f(x') - f(x-)}{x' - x},$$

provided that these limits exist.

Theorem (Dirichlet)

Suppose $f \colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and the one-sided limits $f(x\pm)$ exist for all $x \in \mathbb{R}$.

If $x \in \mathbb{R}$ is such that the one-sided derivatives $f'(x\pm)$ exist as well, then

$$\lim_{n\to\infty} \mathrm{S}_n f(x) = \frac{f(x+) + f(x-)}{2}.$$

In particular, if in addition f is continuous at x then $\lim_{n\to\infty} \mathrm{S}_n f(x) = f(x)$.

The proof of Dirichlet's Theorem is considerably more involved than that of Fejér's Theorem and will not be given in this lecture. Instead we state and prove a weaker version of Dirichlet's Theorem.

Theorem

Suppose $f\colon \mathbb{R} \to \mathbb{C}$ is 2π-periodic and continuous, and there exists a subdivision $0 = x_0 < x_1 < \cdots < x_r = 2\pi$ of $[0, 2\pi]$ such that the restriction of f to $[x_{i-1}, x_i]$ is a C^1-function for $1 \leq i \leq k$. Then the Fourier series of f converges uniformly to f on $\mathbb{R}$.

Proof.

Partial integration over $[x_{i-1}, x_i]$, where the functions involved are continuous, yields for $k \in \mathbb{Z} \setminus \{0\}$

$$\int_{x_{i-1}}^{x_i} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{-\mathrm{i}k}\left(f(x)\mathrm{e}^{-\mathrm{i}kx}\Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} f'(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \right).$$

Summing these identities for $1 \leq i \leq r$ and dividing by 2π gives

$$c_k = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \frac{1}{\mathrm{i}k}\int_0^{2\pi} f'(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = -\frac{\mathrm{i}}{k}\,c_k',$$

since $f(2\pi) = f(0)$; here c_k' denote the Fourier coefficients of f'.

Proof cont'd.

The Cauchy-Schwarz Inequality gives further

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |c_k| = \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{|c_k'|}{k} \leq \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} |c_k'|^2\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{k^2}\right)^{\frac{1}{2}}$$

The two sums on the right-hand side are $< \infty$, and hence the same is true of $\sum_{k\in\mathbb{Z}\setminus\{0\}} |c_k|$.

$\Longrightarrow$ The Fourier series $\sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{\mathrm{i}kx}$ converges uniformly (and absolutely) on $\mathbb{R}$, say to $g(x)$, since it is majorized by the convergent series $\sum_{k\in\mathbb{Z}} |c_k|$.

$\Longrightarrow$ We can integrate the Fourier series term-wise and obtain that g has the same Fourier coefficients as f :

$$\begin{aligned}\int_0^{2\pi} g(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x &= \int_0^{2\pi} \sum_{l\in\mathbb{Z}} c_l \mathrm{e}^{\mathrm{i}lx}\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x \\ &= \sum_{l\in\mathbb{Z}} c_l \int_0^{2\pi} \mathrm{e}^{\mathrm{i}(l-k)x}\,\mathrm{d}x = 2\pi\, c_k.\end{aligned}$$

Moreover, by the Continuity Theorem g is continuous as well.

Proof cont'd.

It remains to show that two continuous 2π-periodic functions which have the same Fourier series must be equal.

By a previous theorem, f and g are equal to the L^2-limit of their common Fourier series, and hence $\|f - g\|_2 = 0$ or, equivalently, $f(x) = g(x)$ almost everywhere.
But for continuous functions this can hold only if $f = g$, because $f(x_0) \neq g(x_0)$ implies $f(x) \neq g(x)$ in some interval $(x_0 - \delta, x_0 + \delta)$ of positive length. $\square$

Exercise

The subject of this exercise is a more down-to-earth proof of the fact used in the last step of the proof of the preceding theorem: *Two continuous, 2π-periodic functions f and g having the same Fourier coefficients must be equal.*

1. Reduce the statement to the following: *A continuous, 2π-periodic function f having all Fourier coefficients equal to zero must be the all-zero function.*
2. Show that all Fejér polynomials $\sigma_n f$ of such a function f are zero.
3. Assume w.l.o.g. $f(x_0) = c > 0$ and hence $f(x) > c/2$ in some interval $(x_0 - r, x_0 + r)$ of positive length. Use the convolution representation

$$\sigma_n f(x_0) = \frac{1}{2\pi} \int_0^{2\pi} f(x_0 - y) \mathrm{F}_n(y) \, \mathrm{d}y$$

and the three properties of Fejér kernels stated earlier to derive a contradiction for large n.

Example

The theorem applies to the repeating-ramp function and gives the identity

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \frac{\cos(7x)}{7^2} + \cdots\right), \quad x \in [-\pi, \pi],$$

announced earlier.

Example (Partial fractions of the cotangent)

As a further example we consider the function $f_a\colon \mathbb{R} \to \mathbb{R}$ defined by $f_a(x) = \cos(ax)$ for $x \in [-\pi, \pi]$ and 2π-periodic extension.

For $a = k \in \mathbb{Z}$ the function $f_k(x) = \cos(kx)$ is its own (one-term) Fourier series and nothing interesting can be concluded.

For $a \in \mathbb{C} \setminus \mathbb{Z}$ the situation is more interesting, because f_a is then a "new" function.

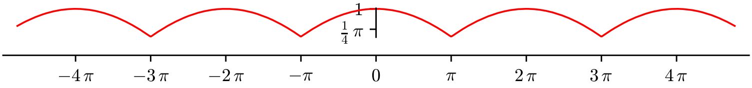

Figure: The function $x \mapsto f_{\frac{1}{4}}(x)$

Example (cont'd)

Since f_a is even, we have $b_k = 0$ for all k.

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} \cos(ax)\cos(kx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\int_0^{\pi} \cos(ax+kx)+\cos(ax-kx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left[\frac{\sin\bigl((a+k)x\bigr)}{a+k}+\frac{\sin\bigl((a-k)x\bigr)}{a-k}\right]_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{(-1)^k\sin(a\pi)}{a+k}+\frac{(-1)^k\sin(a\pi)}{a-k}\right) \end{aligned}$$

$\Longrightarrow$ The Fourier series of f_a is

$$\frac{\sin(a\pi)}{\pi}\left[\frac{1}{a}+\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{a+k}-\frac{1}{a-k}\right)\cos(kx)\right].$$

Since $f_a(x)$ is continuous and piece-wise C^1, the theorem gives that this series is equal to $\cos(ax)$ for $x\in[-\pi,\pi]$.

Example (cont'd)

Thus we have for $a \in \mathbb{C} \setminus \mathbb{Z}$ and $x \in [-\pi, \pi]$ the identity

$$\cos(ax) = \frac{\sin(a\pi)}{\pi} \left[\frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{a+k} + \frac{1}{a-k} \right) \cos(kx) \right].$$

Setting $x = \pi$ gives

$$\begin{aligned} \pi \cot(a\pi) = \frac{\pi \cos(a\pi)}{\sin(a\pi)} &= \frac{1}{a} + \sum_{k=1}^{\infty} (-1)^k \left(\frac{1}{a+k} + \frac{1}{a-k} \right) \\ &= \frac{1}{a} + 2a \sum_{k=1}^{\infty} \frac{(-1)^k}{a^2 - k^2}. \qquad (a \in \mathbb{C} \setminus \mathbb{Z}) \end{aligned}$$

This famous identity, supposedly due to EULER, is known as the *partial fractions decomposition of the cotangent*.

Example

As an example for Dirichlet's Theorem we compute the Fourier series of the function $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) = (\pi - x)/2$ for $0 \leq x < 2\pi$ and extended periodically.

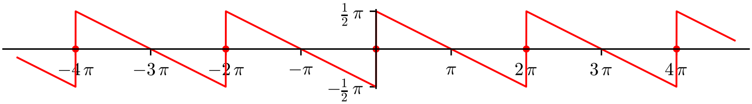

Figure: The function represented by the Fourier series of h

'Since h is odd, we have $a_k = 0$ for all k.

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2}\sin(kx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\left(-\frac{(\pi - x)\cos(kx)}{2k}\bigg|_0^{2\pi} - \frac{1}{2k}\int_0^{2\pi}\cos(kx)\,\mathrm{d}x\right) \\ &= \frac{1}{\pi}\left(\frac{\pi}{2k} + \frac{\pi}{2k} - 0\right) = \frac{1}{k}. \end{aligned}$$

Example (cont'd)

Hence the Fourier series of h is $\sum_{k=1}^{\infty} \frac{\sin(kx)}{k}$, and we obtain from Dirichlet's Theorem the series representation

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k} = \begin{cases} (\pi - x)/2 & \text{if } 0 < x < 2\pi, \\ 0 & \text{if } x = 0 \vee x = 2\pi. \end{cases}$$

Recall that we have derived this result already when discussing uniform convergence.

Exercise

For a sequence (a_n) of complex numbers the associated sequence (c_n) of CESÀRO *means* is defined by

$$c_n = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

a) Show that $\lim_{n\to\infty} a_n = A \in \mathbb{C}$ implies $\lim_{n\to\infty} c_n = A$.

b) Give an example of a divergent sequence (a_n) for which the sequence of Cesàro means converges.

Since the Fejér polynomials $\sigma_n f$ are Cesàro means of the Fourier polynomials, Part a) shows that that the convergence of the Fourier series of f at x implies $\lim_{n\to\infty} \sigma_n f(x) = \lim_{n\to\infty} S_n f(x)$.

Exercise

Prove the properties of the periodic convolution operation $V \times V \to V$, $(f, g) \mapsto f * g$ mentioned in the lecture.

The End

We wish you every success in the final examination!

Final Examination

Date/Time/Venue

Wed Jan 5 2022, 9.00–12.00, in LTW-201

Instructions to candidates

- This examination paper contains six (6) questions.
- Please answer every question and subquestion, and justify your answers.
- For your answers please use the space provided after each question. If this space is insufficient, please continue on the blank sheets provided.
- This is a CLOSED BOOK examination, except that you may bring 1 sheet of A4 paper (hand-written only) and a Chinese-English dictionary (paper copy only) to the examination.

Course Evaluation

`https://forms.office.com/Pages/ResponsePage.aspx?id=xSAmKRCuU0Wsz2jsgDJQCOyRCXXAbFdNl4Oej4tC0c1UNU`

Q15 Which topic in the course did you find most/least interesting?

Q16 Where there any prerequisites for the course that you think you didn't have; if "yes", which are these?