Differential Equations Plus (Math 286)

H68 Find the following convolutions and their Laplace transforms (three answers suffice):

a) $t^2 * t^3$; b) $\mathrm{J}_0 * \mathrm{J}_0$; c) $\sin t * \cos(2t)$; d) $\mathrm{u}(t-1) * t$.

H69 Suppose $F(s) = \mathcal{L}\{f(t)\}$ is defined for $\mathrm{Re}(s) > a$, $a \in [-\infty, \infty)$. Show that $\lim_{s\to+\infty} F(s) = 0$; cp. Exercise 24 in [BDM17], Ch. 6.1.
Hint: Use the uniform convergence of $\int_0^\infty f(t)\mathrm{e}^{-st}$ on $\mathrm{Re}(s) \geq a+1$ (resp., for $a = -\infty$ on $\mathrm{Re}(s) \geq 0$).

H70 Solve the following IVP's with the Laplace transform:

a) $y'' + y' + y = \mathrm{u}_\pi(t) - \mathrm{u}_{2\pi}(t), \quad y(0) = 1,\ y'(0) = 0;$

b) $y'' + 2y' + y = \begin{cases} \sin(2t) & \text{if } 0 \leq t \leq \pi/2, \\ 0 & \text{if } t > \pi/2, \end{cases} \quad y(0) = 1,\ y'(0) = 0.$

H71 Do Exercise 18 in [BDM17], Ch. 6.5.

H72 *Optional Exercise*

Repeat Exercises 20, 21 in [BDM17], Ch. 6.6, for the integro-differential equation

$$\phi'(t) = \sin t + \int_0^t \phi(t-\xi)\cos\xi\,\mathrm{d}\xi, \quad \phi(0) = 2.$$

Hint: It may be helpful to use the commutativity of the convolution product.

H73 Do Exercises 11 and 20 in [BDM17], Ch. 7.1.

H74 Find $\mathbf{S}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix for

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Show that $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$ for $k \in \mathbb{N}$, and use this to obtain explicit formulas for the entries of $\mathbf{A}^k$.

H75 *Optional Exercise*

a) Show that $\displaystyle\int_0^\infty \ln t\,\mathrm{e}^{-t}\,\mathrm{d}t = -\gamma = -0.577\dots.$ For this recall that the Euler-Mascheroni constant γ was defined as $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n\right)$
Hint: Relate the integral to the Gamma function. Gauss's formula

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\cdots(x+n)} \qquad (x \neq 0, -1, -2, \dots),$$

which you don't need to prove, may help.

b) Use a) to find the Laplace transform of $t \mapsto \ln t$ and the inverse Laplace transform of $s \mapsto \frac{\ln s}{s}$ ($\operatorname{Re} s > 0$).

H76 *Optional Exercise*

Suppose V is a vector space over a field F.

a) Using the vector space axioms, prove the *scalar zero law*

$$0_F\, v = 0_V \quad \text{for all } v \in V.$$

b) Similarly, prove the *vector zero law*

$$a\, 0_V = 0_V \quad \text{for all } a \in F.$$

c) Prove that $(-1)x = -x$ for all $x \in V$.

H77 *Optional Exercise*

In each of the following cases, let S be the set of vectors $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ satisfying the given condition. Decide whether S is a subspace of $\mathbb{C}^3/\mathbb{C}$ and, if so, determine the dimension of S.

a) $\alpha = 0$;
b) $\alpha\beta = 0$;
c) $\alpha + \beta = 1$;
d) $\alpha + \beta = 0$;
e) $\alpha = 3\beta \wedge \beta = (2 - \mathrm{i})\gamma$;
f) $\alpha \in \mathbb{R}$.

H78 *Optional Exercise*

Let P_3 be the vector space (over $\mathbb{R}$) of polynomials $p(X) \in \mathbb{R}[X]$ of degree at most 3. Repeat the previous exercise for the sets $S \subseteq P_3$ defined by each of the following conditions:

a) $p(X)$ has degree 3;
b) $2p(0) = p(1)$;
c) $p(t) \geq 0$ for $0 \leq t \leq 1$;
d) $p(t) = p(1 - t)$ for all $t \in \mathbb{R}$.

H79 *Optional Exercise*

a) Write down the linear system of equations satisfied by a classical 3×3 magic square and transform this system into row-echelon form. (What is the magic number in this case?)

b) Use the equations in a) to show that up to obvious symmetries there exists exactly one classical 3×3 magic square.

Due on Thu Dec 16, 7:30 pm

The optional exercises can be handed in until Wed Dec 22, 6 pm.