Differential Equations Plus (Math 286)

H45 For $\alpha, \beta \in \mathbb{C}$ consider the explicit so-called Euler equation

$$y'' + \frac{\alpha}{t}\, y' + \frac{\beta}{t^2}\, y = 0 \qquad (t > 0). \tag{1}$$

a) Show that $\phi\colon \mathbb{R}^+ \to \mathbb{C}$ is a solution of (1) iff $\psi\colon \mathbb{R} \to \mathbb{C}$ defined by $\psi(s) = \phi(\mathrm{e}^s)$ is a solution of

$$y'' + (\alpha - 1)y' + \beta y = 0. \tag{2}$$

b) Using a), determine the general solution of (1) for $(\alpha, \beta) = (6, 4)$ and $(3, 1)$.

H46 The solution to this exercise provides an easy method for computing $\mathrm{e}^{\mathbf{A}t}$ for a 2×2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbb{R}^{2\times 2}$ (or $\mathbb{C}^{2\times 2}$). We assume throughout the exercise that $\mathbf{A}$ is not a scalar multiple of the identity matrix $\mathbf{I}_2$.

a) Show $\mathbf{A}^2 - (a + d)\mathbf{A} + (ad - bc)\mathbf{I}_2 = \mathbf{0}$ (the all-zero 2×2 matrix).

b) Use a) to show that there exist uniquely determined functions $c_0, c_1\colon \mathbb{R} \to \mathbb{R}$ such that

$$\mathrm{e}^{\mathbf{A}t} = c_0(t)\mathbf{I}_2 + c_1(t)\mathbf{A} \quad \text{for } t \in \mathbb{R}. \tag{$\star$}$$

Further, show that c_0, c_1 are at least twice differentiable.

c) Show that c_0, c_1 solve the homogeneous linear ODE of order 2 with characteristic polynomial $X^2 - (a + d)X + ad - bc$ and satisfy the initial conditions $c_0(0) = 1$, $c_0'(0) = 0$ and $c_1(0) = 0$, $c_1'(0) = 1$.
Hint: Differentiate ($\star$) twice.

d) By solving the IVP's in c) determine $\mathrm{e}^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 0 & 6 \\ 1 & 1 \end{pmatrix}$.

H47 Solve the initial value problem

$$\mathbf{y}' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \mathbf{y} + \begin{pmatrix} t \\ \sin t \end{pmatrix}, \quad \mathbf{y}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Due on Fri Nov 12, 6 pm

The matrix exponential function (relevant for H46) will be discussed in the lecture on Mon Nov 8. You are advised to do H45 and H46 before the midterm, because Euler equations and simple (2×2) instances of the matrix exponential function can be the subject of a midterm question; cf. the online midterm samples. Exercise H47, which is computationally intensive, can be safely considered only after the midterm.