Math 286 Differential Equations Plus (Prof. Honold) **Midterm Exam** 2021/11/10

Name: ____________________ Student No.: ____________________ Group B

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2\colon (0,2) \to \mathbb{R}$ satisfying $y_1(1) = y_2(1)$?

   ☐ $y' = \sqrt{t}\,|y|$ ☐ $y'' = yy'$ ☐ $y' = t\ln y$ ☐ $y' = y\ln t$ ☐ $yy' = 0$

2. The ODE $(x - 3y^2/x)\,\mathrm{d}x - 3y\,\mathrm{d}y = 0$ has the integrating factor

   ☐ 0 ☐ x ☐ y ☐ x^2 ☐ y^2

3. For the solution $y(t)$ of the IVP $y' = y^4 - 1,\ y(2021) = 0$ the limit $\lim_{t\to+\infty} y(t)$ equals

   ☐ $-\infty$ ☐ -1 ☐ 0 ☐ 1 ☐ $+\infty$

4. For the solution $y(t)$ of the IVP $y' = e^{t-2y},\ y(0) = 0$ the value $y(1)$ is contained in

   ☐ $[0,\frac{1}{2}]$ ☐ $[\frac{1}{2},1]$ ☐ $[1,\frac{3}{2}]$ ☐ $[\frac{3}{2},2]$ ☐ $[2,\frac{5}{2}]$

5. For the solution $y\colon [0,\infty) \to \mathbb{R}$ of the IVP $(t+1)(y'+1) = 2y,\ y(0) = 0$ the value $y(2)$ is equal to

   ☐ -2 ☐ -4 ☐ -6 ☐ -8 ☐ -10

6. $y'' - 3y' + 2y = 2 + t\,\mathrm{e}^t$ has a particular solution $y_p(t)$ of the form

   ☐ $c_0 + c_1\mathrm{e}^t + c_2 t\,\mathrm{e}^t$ ☐ $c_0 + c_1 t\,\mathrm{e}^t + c_2 t^2\mathrm{e}^t$ ☐ $c_0 + c_1 t^2\mathrm{e}^t$

   ☐ $c_0 t + c_1 t\,\mathrm{e}^t + c_2 t^2\mathrm{e}^t$ ☐ $c_0 t + c_1 t^2\mathrm{e}^t$

7. Maximal solutions of $y' = y^3 + 1$ satisfying $y(0) = 0$ are defined on an interval of the form

   ☐ (a,b) ☐ $[a,b]$ ☐ $(a,+\infty)$ ☐ $(-\infty,b)$ ☐ $(-\infty,+\infty)$

   with $a,b \in \mathbb{R}$.

8. For $\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$, the matrix $\mathrm{e}^{\mathbf{A}t}$ is equal to

   ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{2t} - \mathrm{e}^{-t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$ ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$ ☐ $\begin{pmatrix} \mathrm{e}^{-t} & 0 \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$

   ☐ $\begin{pmatrix} -\mathrm{e}^{t} & \mathrm{e}^{t} \\ 0 & 2\mathrm{e}^{t} \end{pmatrix}$ ☐ $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^{-t} - \mathrm{e}^{2t} \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$

9. The matrix norm of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ (subordinate to the Euclidean length on $\mathbb{R}^2$) is equal to

   ☐ 0 ☐ 1 ☐ $\sqrt{2}$ ☐ 2 ☐ $1+\sqrt{2}$

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10. A contraction $T: M \to M$, $M \subseteq \mathbb{R}^2$, has a fixed point if M is

☐ infinite ☐ connected ☐ open ■ closed ■ bounded

11. The power series $z + z^2 + z^4 + z^8 + z^{16} + \cdots$ has radius of convergence

☐ 0 ☐ $\frac{1}{2}$ ■ 1 ☐ 2 ☐ ∞

12. The smallest integer a such that $f_a(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^a}$ is differentiable on $\mathbb{R}$ is equal to

☐ 1 ☐ 2 ■ 3 ☐ 4 ☐ 5

13. For which choice of $f_n(x)$ does the function sequence (f_n) converge uniformly on $\mathbb{R}$?

☐ $f_n(x) = e^{-nx^2}$ ■ $f_n(x) = x/(1+nx^2)$ ☐ $f_n(x) = 1/(1+nx^2)$

■ $f_n(x) = x/(1+n^2)$ ☐ $f_n(x) = e^{-n^2x}$

14. If $y(t)$ solves $y' = 1 + y/t - y^2/t^2$ then $z = 1/\big(y(t) - t\big)$ solves

☐ $z' = z/t - 1/t^2$ ☐ $z' = tz + t^2$ ☐ $z' = tz - t^2$ ☐ $z' = t^2z + t$

■ $z' = z/t + 1/t^2$

15. The sequence $\phi_0, \phi_1, \phi_2, \ldots$ of Picard-Lindelöf iterates for the IVP $y' = 2y + 2$, $y(0) = 1$ has $\phi_2(t)$ equal to

☐ $2t + 6t^2$ ☐ $2t + 5t^2$ ☐ $2 + 6t + 6t^2$ ☐ $2t + 4t^2$ ■ $1 + 4t + 4t^2$

Time allowed: 50 min CLOSED BOOK ***Good luck!***

Notes

1. The correct answer is $y'' = yy'$, since this is an explicit 2nd-order ODE and in addition to $y(1)$ we can also prescribe $y'(1)$ freely. Of course, one also needs to check that (maximal) solutions of such IVP's are defined on $(0,2)$. This can be done, but you can also check that $y_1(t) \equiv -2$ and $y_2(t) = -2/t$ form a pair of solutions with the required property.

   The implicit ODE $yy' = 0$ doesn't satisfy the assumptions of the EUT, but has only the constant solutions. The remaining three ODE's satisfy all assumptions of the EUT.

2. Multiplying the ODE with x^2 gives the ODE $(x^3 - 3xy^2)\,\mathrm{d}z - 3x^2y\,\mathrm{d}y = 0$, which is of the form $P\,\mathrm{d}x + Q\,\mathrm{d}y = 0$ with $P_y = -6xy = Q_x$, and hence exact. The other nonzero factors do not yield an ODE with $P_y = Q_x$. Although $0\,\mathrm{d}x + 0\,\mathrm{d}y = 0$ is trivially exact, 0 is not an integrating factor, because integrating factors are required to be nonzero everywhere.

3. The phaseline should be used to answer this question. We have $f(y) = y^4 - 1 = (y^2 - 1)(y^2 + 1)$ with roots $z_1 = -1$, $z_2 = 1$. Since $t_0 = 0 \in (z_1, z_2)$ and f is negative in (z_1, z_2), we must have $\lim_{\to +\infty} y(t) = z_1 = -1$; cf. lecture.

4. This is a separable ODE and can be solved with the standard method: $\mathrm{e}^{2y}\,\mathrm{d}y = \mathrm{e}^t \Longrightarrow \frac{1}{2}\mathrm{e}^{2y} = \mathrm{e}^t + C \Longrightarrow y = \frac{1}{2}\ln(2\mathrm{e}^t + 2C)$ with $C \in \mathbb{R}$. The initial value $y(0) = 0$ gives $y(t) = \frac{1}{2}\ln(2\mathrm{e}^t - 1)$, $y(1) = \frac{1}{2}\ln(2\mathrm{e} - 1)$. Since trivially $\mathrm{e} < 2\mathrm{e} - 1 < \mathrm{e}^2$, we have $1 < \ln(2\mathrm{e} - 1) < 2$, and hence $y(1) \in \left[\frac{1}{2}, 1\right]$.

5. The ODE has explicit form $y' = \frac{2y}{t+1} - 1$, which is 1st-order linear. The general solution of the associated homogeneous ODE $y' = \frac{2y}{t+1}$ is $y(t) = c(t+1)^2$, a particular solution of the inhomogeneous ODE is $y_p(t) = (t+1)^2 \int \frac{-1}{(t+1)^2}\,\mathrm{d}t = t + 1$, and hence the general solution of the inhomogeneous ODE is $y(t) = c(t+1)^2 + t + 1$. The initial value $y(0) = 0$ gives $c = -1$, so that $y(t) = -(t+1)^2 + t + 1 = -t(t+1)$ and $y(1) = -2$, $y(2) = -6$.

6. A solution of $y'' - 3y' + 2y = 2$ is $y_1(t) = 1$. The correct „Ansatz" for obtaining solution of $y'' - 3y' + 2y = t\,\mathrm{e}^t$ is $y_2(t) = (c_1 t + c_2 t^2)\mathrm{e}^t$, giving after a short computation $y_2(t) = (-t - t^2/2)\mathrm{e}^t$. Thus a particular solution of $y'' - 3y' + 2y = 2 + t\,\mathrm{e}^t$ is $y_p(t) = y_1(t) + y_2(t) = 1 - (t + t^2/2)\mathrm{e}^t$, and the general (real) solution of $y'' - 3y' + 2y = 2 + t\,\mathrm{e}^t$ is
$$a\,\mathrm{e}^t + b\,\mathrm{e}^{2t} + 1 - (t + t^2/2)\mathrm{e}^t, \quad \text{with } a, b \in \mathbb{R}.$$
   Hence none of the other forms offered can yield a solution.

7. The only equilibrium solution of this autonomous ODE is $y \equiv -1$. Integrating gives
$$\int_0^y \frac{\mathrm{d}\eta}{\eta^3 + 1} = \int_0^t \mathrm{d}\tau = t.$$
   Since the integral $\int_0^{+\infty} \frac{\mathrm{d}\eta}{\eta^3+1}$ is finite, the solution is only defined for $t < t_\infty = \int_0^{+\infty} \frac{\mathrm{d}\eta}{\eta^3+1}$. Since $\lim_{y \downarrow -1} \int_0^y \frac{\mathrm{d}\eta}{\eta^3+1} = -\lim_{y \downarrow -1} \int_y^0 \frac{\mathrm{d}\eta}{\eta^3+1} = -\infty$, there is no further restriction on the domain of y. Hence y is defined on $(-\infty, t_\infty)$.

8. Substituting $t = 0$ should produce $\mathrm{e}^{\mathbf{A}0} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. This excludes $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^t \\ 0 & \mathrm{e}^{2t} \end{pmatrix}$ and $\begin{pmatrix} \mathrm{e}^{-t} & \mathrm{e}^t \\ 0 & 2\mathrm{e}^t \end{pmatrix}$. For the remaining three matrices we check whether the 2nd column solves $\mathbf{y}' = \mathbf{A}\mathbf{y}$. (The 1st column of all matrices is a solution.)
$$\begin{pmatrix} 0 \\ \mathrm{e}^{2t} \end{pmatrix}' = \begin{pmatrix} 0 \\ 2\mathrm{e}^{2t} \end{pmatrix}, \qquad \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ \mathrm{e}^{2t} \end{pmatrix} = \begin{pmatrix} 3\mathrm{e}^{2t} \\ 2\mathrm{e}^{2t} \end{pmatrix};$$
$$\begin{pmatrix} \mathrm{e}^{2t} - \mathrm{e}^{-t} \\ \mathrm{e}^{2t} \end{pmatrix}' = \begin{pmatrix} 2\mathrm{e}^{2t} + \mathrm{e}^{-t} \\ 2\mathrm{e}^{2t} \end{pmatrix}, \qquad \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{2t} - \mathrm{e}^{-t} \\ \mathrm{e}^{2t} \end{pmatrix} = \begin{pmatrix} 2\mathrm{e}^{2t} + e^{-t} \\ 2\mathrm{e}^{2t} \end{pmatrix};$$
$$\begin{pmatrix} \mathrm{e}^{-t} - \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{pmatrix}' = \begin{pmatrix} -\mathrm{e}^{-t} - 2\mathrm{e}^{2t} \\ 2\mathrm{e}^{2t} \end{pmatrix}, \qquad \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} \mathrm{e}^{-t} - \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{pmatrix} = \begin{pmatrix} -e^{-t} + 4\mathrm{e}^{2t} \\ 2\mathrm{e}^{2t} \end{pmatrix}.$$

Thus the correct answer is $\begin{pmatrix} e^{-t} & e^{2t}-e^{-t} \\ 0 & e^{2t} \end{pmatrix}$.

9. $\|\mathbf{A}\| = \max\{|\mathbf{Ax}|; |\mathbf{x}| = 1\}$ cannot be smaller than the length of a column of $\mathbf{A}$, which is the image of a standard unit vector under $\mathbf{x} \mapsto \mathbf{Ax}$. Since the first column of the given matrix has length $\sqrt{5} > 2$, the correct answer must be $1 + \sqrt{2}$.

10. The correct answer is "closed", since for Banach's Fixed Point Theorem to hold M needs to be a complete metric space, and only closed subspaces of $\mathbb{R}^2$ have this property.

11. As mentioned in the lecture, the radius of convergence of any non-terminating power series with coefficients in $\{0, 1\}$ is equal to 1.

12. For checking the differentiability of $f_a(x)$ one has to look at the series of derivatives, which is
$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{a-1}},$$
and prove that this series converges uniformly on $\mathbb{R}$ (or on all intervals of the form $[-R, R]$, $R > 0$). For $a \geq 3$ we can apply the Weierstrass test with $M_n = 1/n^2$ and conclude that the series of derivatives converges uniformly on $\mathbb{R}$. The Differentiation Theorem then gives that f_3, f_4, f_5 are differentiable. For $a = 2$ we obtain the series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$, which doesn't converge for $x = 0$ and represents the function $x \mapsto -\ln\left(2\sin\frac{x}{2}\right)$ on $(0, 2\pi)$; cf. the lecture. Although the divergence for $x = 0$ doesn't imply that f_2 is not differentiable at 0, the latter is nevertheless true and can be seen as follows: Since $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$ converges uniformly on every interval $[\delta, 2\pi - \delta]$, $\delta > 0$, the function f_2 is differentiable in $(0, 2\pi)$ with derivative $f_2'(x) = -\ln\left(2\sin\frac{x}{2}\right)$. Since f_2 is continuous at 0 (by the Continuity Theorem), we can apply the Mean Value Theorem and obtain
$$\frac{f_2(x)}{x} = \frac{f_2(x) - f_2(0)}{x - 0} = f'(\xi) = -\ln\left(2\sin\frac{\xi}{2}\right) \quad \text{with } \xi \in (0, x) \text{ for small } x > 0.$$
Since the right-hand side tends to $+\infty$ for $\xi \downarrow 0$, f_2 can't be differentiable at 0.

    For $a = 1$ we have seen in the lecture that $f_1(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is discontinuous at 0, let alone differentiable.

13. The correct answer is $f_n(x) = x/(1 + nx^2)$, because we can use the inequality $1 + nx^2 \geq 2\sqrt{n}x$ to conclude that $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ for $x \neq 0$, which is independent of x. Trivially this holds also for $x = 0$, and hence $f_n(x) \to 0$ uniformly.

    Independently, the other 4 answers can be excluded as follows: $e^{-n^2 x}$ doesn't converge at all for $x < 0$, the limit function f of both $1/(1 + nx^2)$ and e^{-nx^2} is discontinuous ($f(x) = 0$ for $x \neq 0$ but $f(0) = 1$), which would contradict the Continuity Theorem, and $f_n(x) = x/(1 + n^2)$ converges point-wise to 0 but satisfies $f_n(1 + n^2) = 1$, implying that for $\varepsilon = 1$ no uniform response can be found.

14. This is a Riccati equation with solution $y_1(t) = t$, and it is known that the substitution $z = 1/(y - y_1)$ transforms it into a 1st-order linear equation. (You are not supposed to know this, and in fact this knowledge doesn't help since a ll offered answers are 1st-order linear.)
$$z = \frac{1}{y - t} \Longrightarrow z' = -\frac{y' - 1}{(y - t)^2} = -\frac{y/t - y^2/t^2}{(y - t)^2} = \frac{y}{t^2(y - t)} = \frac{zy}{t^2} = \frac{z(t + 1/z)}{t^2} = \frac{z}{t} + \frac{1}{t^2}$$

15. $\phi_0(t) = 1$; $\phi_1(t) = 1 + \int_0^t 2\phi_0(s) + 2\,ds = 1 + \int_0^t 4\,ds = 1 + 4t$; $\phi_2(t) = 1 + \int_0^t 2\phi_1(s) + 2\,ds = 1 + \int_0^t 2(1 + 4s) + 2\,ds = 1 + \int_0^t 4 + 8s\,ds = 1 + \left[4s + 4s^2\right]_0^t = 1 + 4t + 4t^2$.