Calculus III (Math 241)

H7 Solve [Ste16], Section 12.5, Exercises 12, 70, 72, 74, 76, 80.
Hint: The cross product need not be used when solving Exercise 70.

H8 Determine the maximum value of the function $f(x_1,x_2,x_3)=x_1+2x_2-x_3$ on the sphere $S=\{\mathbf{x}\in\mathbb{R}^3; x_1^2+x_2^2+x_3^2=3\}$.
Hint: Use the Cauchy-Schwarz Inequality.

H9 a) Compute the powers $\mathbf{A}^2$, $\mathbf{A}^3$, $\mathbf{B}^2$, $\mathbf{B}^3$ of

$$\mathbf{A}=\begin{pmatrix}2&0&0\\0&-1&0\\0&0&3\end{pmatrix},\qquad \mathbf{B}=\begin{pmatrix}2&1&-1\\0&-1&2\\0&0&3\end{pmatrix}$$

b) Based on the observations in a), state without proof rules for computing powers of *diagonal matrices* ($a_{ij}=0$ for $i\neq j$) and *upper triangular matrices* ($a_{ij}=0$ for $i>j$).

c) Prove that the product $\mathbf{B}_1\mathbf{B}_2$ of two upper triangular $n\times n$ matrices $\mathbf{B}_1$, $\mathbf{B}_2$ is again upper triangular. (For diagonal matrices the same is true, as is easily seen.)

d) Compute the inverse matrices of $\mathbf{A}$ and $\mathbf{B}$, and try to formulate general rules for inverses of upper triangular matrices (and inverses of diagonal matrices).

H10 *Optional Exercise*

The dot product of two complex vectors $\mathbf{z}=(z_1,\dots,z_n)\in\mathbb{C}^n$ and $\mathbf{w}=(w_1,\dots,w_n)\in\mathbb{C}^n$ is defined as $\mathbf{z}\cdot\mathbf{w}=z_1\overline{w}_1+z_2\overline{w}_2+\cdots+z_n\overline{w}_n$ and the length of $\mathbf{z}\in\mathbb{C}^n$ as $|\mathbf{z}|=\sqrt{\mathbf{z}\cdot\mathbf{z}}$.

a) Derive properties of the complex dot product that are analogous to Properties (D1)–(D4) of the real dot product; cf. the lecture.

b) Show that the Cauchy-Schwarz Inequality generalizes to $\mathbb{C}^n$ (where of course linear dependence of $\mathbf{z},\mathbf{w}$ over $\mathbb{C}$ is involved in the 2nd part).

H11 *Optional Exercise*

Prove the follwing quantitative version of the Cauchy-Schwarz Inequality:

$$\left(\sum_{i=1}^n a_i^2\right)\left(\sum_{i=1}^n b_i^2\right)-\left(\sum_{i=1}^n a_ib_i\right)^2=\sum_{1\leq i<j\leq n}(a_ib_j-a_jb_i)^2$$

for any choice of real (or complex) numbers $a_1,\dots,a_n,b_1,\dots,b_n$.

Due on Wed Sep 29, 6 pm

The optional exercises can be handed in until Wed Oct 13, 6 pm.