Math 286
Introduction to Differential Equations

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Outline

Today's Lecture:

The goal of this section is to examine the qualitative behaviour, including the asymptotics for $t \to \infty$ of solutions $\mathbf{y}\colon \mathbb{R} \to \mathbb{R}^2$, $t \mapsto \mathbf{y}(t) = \bigl(y_1(t), y_2(t)\bigr)^{\mathsf{T}}$, of ODE systems of the form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b} \quad \text{with } \mathbf{A} \in \mathbb{R}^{2\times 2}, \mathbf{b} \in \mathbb{R}^2.$$

and corresponding IVP's $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $y(t_0) = \mathbf{y}_0$.

Facts already known

1. All maximal solutions are parametric plane curves with domain $\mathbb{R}$.
2. The corresponding non-parametric curves $\{\mathbf{y}(t); t \in \mathbb{R}\}$ (*orbits*, *trajectories*) partition $\mathbb{R}^2$ (the *phase plane*).
3. Solutions $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$ determine the same orbit iff $\mathbf{y}_2(t) = \mathbf{y}_1(t + t_0)$ for some $t_0 \in \mathbb{R}$.

These hold, mutatis mutandis, also for higher-dimensional autonomous linear systems and are consequences of the sharpened version of the EUT in the linear case.

In view of (3) we only need to consider IVP's with $t_0 = 0$.

Definition

A point $\mathbf{y}_0 \in \mathbb{R}^2$ is said to be a *critical point* of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ if there is the constant solution $\mathbf{y}(t) \equiv \mathbf{y}_0$; equivalently, $\{\mathbf{y}_0\}$ is an orbit.

Clearly this equivalent to $\mathbf{A}\mathbf{y}_0 + \mathbf{b} = \mathbf{0}$.

Non-constant solutions $\mathbf{y}(t)$ satisfy $\mathbf{y}'(t) \neq \mathbf{0}$ for all $t \in \mathbb{R}$ and hence are smooth.

Tangent vectors (velocity vectors) at non-critical points are uniquely determined (from Property (3) or directly from $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$).
$\Longrightarrow$ Orbits have a direction of traverse at each of their points. This is indicated in a plot of the *direction field* (the vector field $\mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} + \mathbf{b}$) and in a *phase portrait* (representative collection of orbits) by corresponding arrows.

Since the critical points of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ are the solutions of $\mathbf{A}\mathbf{x} = -\mathbf{b}$, they form an affine subspace of $\mathbb{R}^2$. The possible cases are:

1. No critical point ($\iff \operatorname{rk} \mathbf{A} \leq 1 \wedge \mathbf{b} \notin \operatorname{csp} \mathbf{A}$);
2. exactly one critical point ($\iff \operatorname{rk} \mathbf{A} = 2$);
3. a critical line ($\iff \operatorname{rk} \mathbf{A} = 1 \wedge \mathbf{b} \in \operatorname{csp} \mathbf{A}$).
4. All points of $\mathbb{R}^2$ are critical ($\iff \mathbf{A} = \mathbf{0} \wedge \mathbf{b} = \mathbf{0}$).

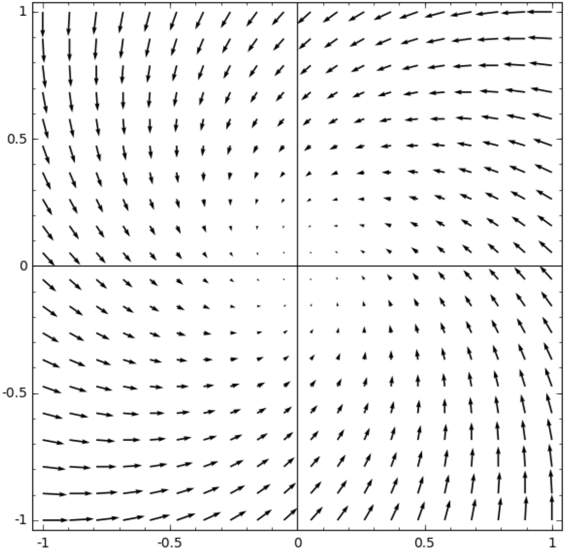

Figure: Direction field of $\mathbf{y}' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}$

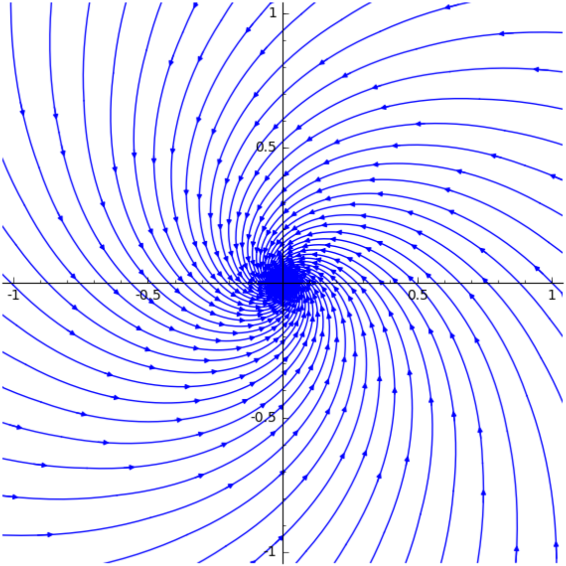

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}$

Observation

An affine coordinate change $\mathbf{y} = \mathbf{S}\mathbf{z} + \mathbf{v}$ ($\mathbf{S} \in \mathbb{R}^{2\times 2}$ invertible, $\mathbf{v} \in \mathbb{R}^2$) preserves most of the qualitative properties of the solutions (and of a phase portrait). If $\mathbf{S}$ is not orthogonal, however, it may distort angles and distances.

The effect on the ODE of such a coordinate transformation is:

$$\mathbf{z} = \mathbf{S}^{-1}(\mathbf{y} - \mathbf{v}) = \mathbf{S}^{-1}\mathbf{y} - \mathbf{S}^{-1}\mathbf{v},$$

$$\mathbf{z}' = \mathbf{S}^{-1}\mathbf{y}' = \mathbf{S}^{-1}(\mathbf{A}\mathbf{y} + \mathbf{b}) = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z} + \mathbf{S}^{-1}(\mathbf{A}\mathbf{v} + \mathbf{b}),$$

i.e., the transformed system is $\mathbf{z}' = \widehat{\mathbf{A}}\mathbf{z} + \widehat{\mathbf{b}}$ with $\widehat{\mathbf{A}} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ and $\widehat{\mathbf{b}} = \mathbf{S}^{-1}(\mathbf{A}\mathbf{v} + \mathbf{b})$.

Consequences

- If there are critical points ($\iff$ there exists $\mathbf{v} \in \mathbb{R}^2$ such that $\mathbf{A}\mathbf{v} + \mathbf{b} = \mathbf{0}$), we can transform the system into a homogeneous system $\mathbf{z}' = \widehat{\mathbf{A}}\mathbf{z}$ (which has at least one critical point, viz. the origin $\mathbf{0}$). The critical points of $\mathbf{z}' = \widehat{\mathbf{A}}\mathbf{z}$ are those in $\operatorname{rker}(\widehat{\mathbf{A}})$.
- When classifying homogeneous systems $\mathbf{y}' = \mathbf{A}\mathbf{y}$, we only need to consider one matrix from each complex similarity class that contains a real matrix.

Consequences cont'd

Reason: If $\mathbf{A}$, $\mathbf{B}$ are real and similar over $\mathbb{C}$, they must be similar over $\mathbb{R}$; see subsequent exercise. Thus we can find a linear coordinate transformation $\mathbf{y} = \mathbf{Sz}$, $\mathbf{S} \in \mathbb{R}^{2\times 2}$, which transforms $\mathbf{y}' = \mathbf{Ay}$ into $\mathbf{z}' = \mathbf{Bz}$.

Exercise

Suppose that $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{T} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{B} = \mathbf{T}^{-1}\mathbf{AT}$.

a) Setting $\mathbf{T} = \mathbf{P} + \mathrm{i}\mathbf{Q}$ with $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$, show that $\mathbf{P}, \mathbf{Q}$ satisfy the matrix equation $\mathbf{AX} = \mathbf{XB}$.

b) Show that any matrix $\mathbf{S} = \mathbf{P} + \lambda\mathbf{Q}$, $\lambda \in \mathbb{R}$, satisfies this matrix equation as well.

c) Show that that there exists $\lambda \in \mathbb{R}$ such that $\mathbf{P} + \lambda\mathbf{Q}$ is invertible.

   Hint: Show that $\lambda \mapsto \det(\mathbf{P} + \lambda\mathbf{Q})$ is a polynomial function.

d) Use a), b), c) to show that there exists $\mathbf{S} \in \mathbb{R}^{n\times n}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{AS}$.

The Classification

First we consider the genuinely inhomogeneous systems $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, i.e., those with $\mathbf{b} \notin \operatorname{csp}(\mathbf{A})$ and hence without a critical point/stationary solution.

Up to similarity, there are three cases to consider: (1) $\mathbf{A} = \mathbf{0}$; (2) $\operatorname{rk} \mathbf{A} = 1$ and $\mathbf{A}$ is diagonalisable, i.e., similar to $\begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R} \setminus \{0\}$; (3) $\operatorname{rk} \mathbf{A} = 1$ and $\mathbf{A}$ is not diagonalisable, i.e., $\mathbf{A}$ has the eigenvalue 0 with algebraic multiplicity 2 and geometric multiplicity 1, and hence is similar to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

Case 1: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The general solution is $\mathbf{y}(t) = t\,\mathbf{b} + \mathbf{c}$, i.e., parallel lines with direction vector $\mathbf{b}$ parametrized in the usual way.
A linear coordinate transformation $\mathbf{y} = \mathbf{S}\mathbf{z}$ (which doesn't change $\mathbf{A}$) could be applied to move $\mathbf{b}$ to any fixed nonzero vector, e.g., $\mathbf{e}_1 = (1, 0)^{\mathsf{T}}$, but this is hardly necessary in this case.

Case 2: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with $b_1 \neq 0$.

We can apply a coordinate change of the form $\mathbf{y} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \mathbf{z} + \mathbf{v} = s\,\mathbf{z} + \mathbf{v}$ with $s \neq 0$ (which does not change $\mathbf{A}$) to move $\mathbf{b}$ into $\mathbf{e}_1$. (The explicit transformation is $z_1 = y_1/b_1$, $z_2 = (y_2 + b_2/\lambda)/b_1$.) The new system is equivalent to the scalar equations $y_1' = 1$, $y_2' = \lambda y_2$, which has the general solution

$$\mathbf{y}(t) = \begin{pmatrix} t + c_1 \\ c_2 \mathrm{e}^{\lambda t} \end{pmatrix}.$$

The orbits, written as graphs $y(x)$ ($x \triangleq y_1$, $y \triangleq y_2$) are $y(x) = c\,\mathrm{e}^{\lambda x}$, $c \in \mathbb{R}$ (integral curves of $y' = \lambda y$). This can be seen by taking the solution $t \mapsto \mathbf{y}(t - c_1) = \begin{pmatrix} t \\ c_2 \mathrm{e}^{-\lambda c_1} \mathrm{e}^{\lambda t} \end{pmatrix} = \begin{pmatrix} t \\ c\mathrm{e}^{\lambda t} \end{pmatrix}$ with $c = c_2 \mathrm{e}^{-\lambda c_1}$), which has the same orbit as $\mathbf{y}(t)$ and is just the standard parametrization of the graph of $y(x) = c\,\mathrm{e}^{\lambda x}$.

Case 3: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ with $b_1 \neq 0$.

Again we may assume $\mathbf{b} = (1, 0)^{\mathsf{T}}$, and in this case obtain the system $y_1' = 1$, $y_2' = y_1$, which has general solution

$$\mathbf{y}(t) = \begin{pmatrix} t + c_1 \\ t^2/2 + c_1 t + c_2 \end{pmatrix}.$$

From $\mathbf{y}(t - c_1) = \begin{pmatrix} t \\ t^2/2 + c_1^2/2 + c_2 \end{pmatrix}$ we see (using a further scaling $\mathbf{z} = \frac{1}{2}\mathbf{y}$) that the orbits, written as graphs $y(x)$ ($x \triangleq y_1$, $y \triangleq y_2$) are $y(x) = x^2 + c$, $c \in \mathbb{R}$ (integral curves of $y' = 2x$).

Note that families of parabolas, in order to qualify for the orbits of a system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, must partition the plane. Up to a scaling factor this leaves only the possibility of the family which arises from a fixed parabola by shifting it along its axis.

In the remaining cases $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has at least one critical point. After a translation, one of these may be taken as the origin $(0,0)$, implying that the transformed system is homogeneous.

We consider the cases with non-invertible **A** first.

Case 4: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The general solution is $\mathbf{y}(t) \equiv \mathbf{c}$. All points of $\mathbb{R}^2$ are critical.

Case 5: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The system is equivalent to $y_1' = 0$, $y_2' = \lambda y_2$ and has general solution

$$\mathbf{y}(t) = \begin{pmatrix} c_1 \\ c_2\,\mathrm{e}^{\lambda t} \end{pmatrix}.$$

The orbits are the points with $c_2 = 0$ (i.e. the points on the x-axis, which forms the critical line in this case), and the vertical rays $(c_1, 0) \pm \mathbb{R}^+(0,1)$. The direction of traverse of the rays depends on the sign of λ.

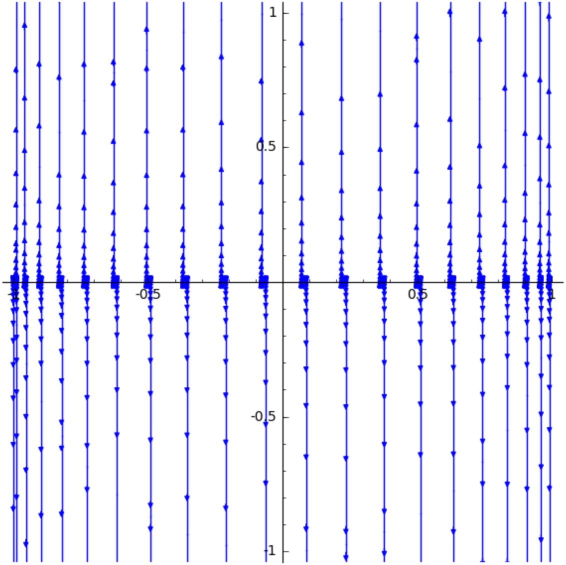

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$

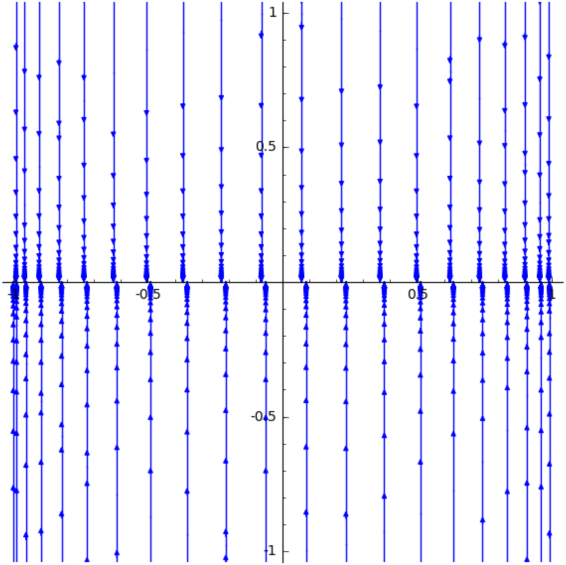

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}$

Case 6: $\mathbf{A} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

The system is equivalent to $y_1' = 0$, $y_2' = y_1$ and has general solution

$$\mathbf{y}(t) = \begin{pmatrix} c_1 \\ c_1 t + c_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + t \begin{pmatrix} 0 \\ c_1 \end{pmatrix}.$$

The orbits are the points with $c_1 = 0$ (i.e. the points on the y-axis, which forms the critical line in this case), and the vertical lines $(c_1, c_2) + \mathbb{R}(0, c_1) = (c_1, 0) + \mathbb{R}(0, 1)$. The direction of traverse of the lines is determined by the sign of c_1. It is N (northward) in the right half plane $x > 0$ and S (southward) in the left half plane $x < 0$.

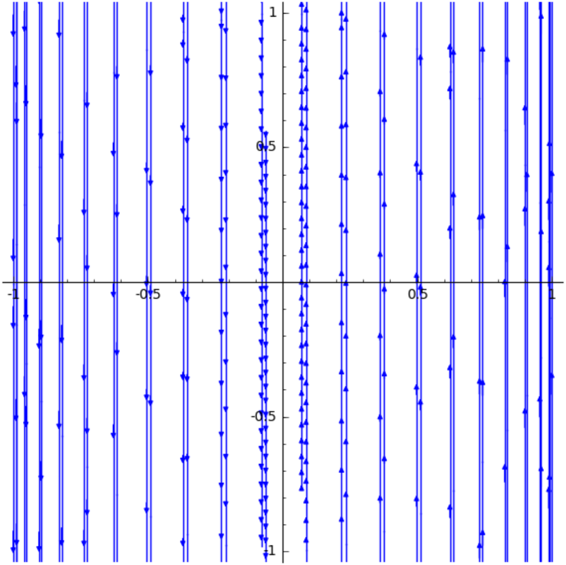

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mathbf{y}$

It remains to consider representatives of the systems with invertible **A**. These have exactly one critical point, which can be taken as the origin $(0,0)$, making the system homogeneous.

If **A** has two distinct eigenvalues λ_1, λ_2 then **A** is diagonalizable and can be taken as the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. If $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ then necessarily $\lambda_2 = \overline{\lambda}_1$.

If **A** has only one eigenvalue λ of multiplicity 2 then necessarily $\lambda \in \mathbb{R}$, and either **A** is diagonalizable and equal to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \mathbf{I}_2$ or **A** is not diagonalizable and similar to $\begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$.

These cases can be easily distinguished: If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $\chi_{\mathbf{A}}(X) = X^2 - (a+d)X + ad - bc$

$$\begin{aligned} \lambda_{1/2} &= \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right) \\ &= \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right). \end{aligned}$$

For example, **A** has purely imaginary eigenvalues iff the trace is zero (i.e., $d = -a$) and $a^2 + bc < 0$; in this case $\lambda_{1/2} = \pm \mathrm{i}\sqrt{-a^2 - bc}$.

Case 7: Two distinct nonzero real eigenvalues

If the eigenvalues are λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ then the general real solution is

$$\mathbf{y}(t) = c_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + c_2 \mathrm{e}^{\lambda_2 t} \mathbf{v}_2, \quad c_1, c_2 \in \mathbb{R}.$$

The four rays $\mathbb{R}^+(\pm\mathbf{v}_1)$, $\mathbb{R}^+(\pm\mathbf{v}_2)$ form orbits (arising from solutions with $c_1 = 0$ or $c_2 = 0$).

1. $\lambda_1 > \lambda_2 > 0$: Non-stationary solutions $\mathbf{y}(t)$ are unbounded for $t \to \infty$ and satisfy $\lim_{t\to-\infty} \mathbf{y}(t) = \mathbf{0}$. In this case the critical point $\mathbf{0}$ is called a *nodal source*.
2. $\lambda_1 < \lambda_2 < 0$: Compared with (1), the roles of $\pm\infty$ are reversed. In particular we have $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$ for every non-constant solution. Here $\mathbf{0}$ is called a *nodal sink*.
3. $\lambda_1 > 0 > \lambda_2$: Here we have $\lim_{t\to-\infty} \mathrm{e}^{\lambda_1 t}(\pm\mathbf{v}_1) = \lim_{t\to+\infty} \mathrm{e}^{\lambda_2 t}(\pm\mathbf{v}_2) = \mathbf{0}$, i.e., the arrows point outwards (inwards) on the two rays contained in $\mathbb{R}\mathbf{v}_1$ (resp., in $\mathbb{R}\mathbf{v}_2$). For $t \to \infty$ all non-constant solutions approach a ray solution containd in $\mathbb{R}\mathbf{v}_1$ (resp., for $t \to -\infty$ emerge from a ray solution containd in $\mathbb{R}\mathbf{v}_2$). In this case $\mathbf{0}$ is called a *saddle point*.

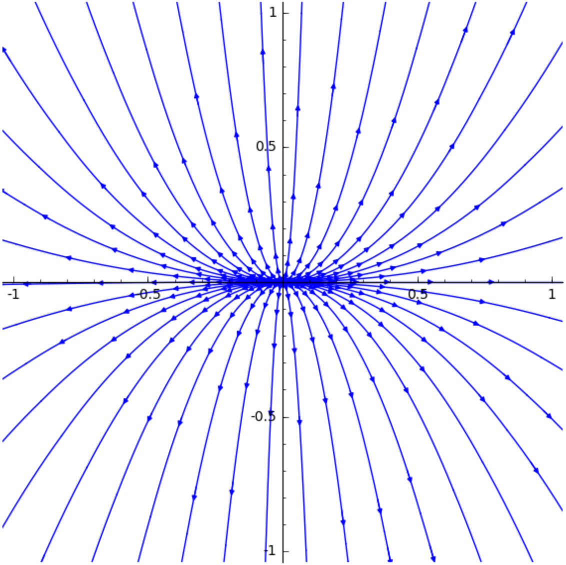

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \mathbf{y}$

The origin is a nodal source.

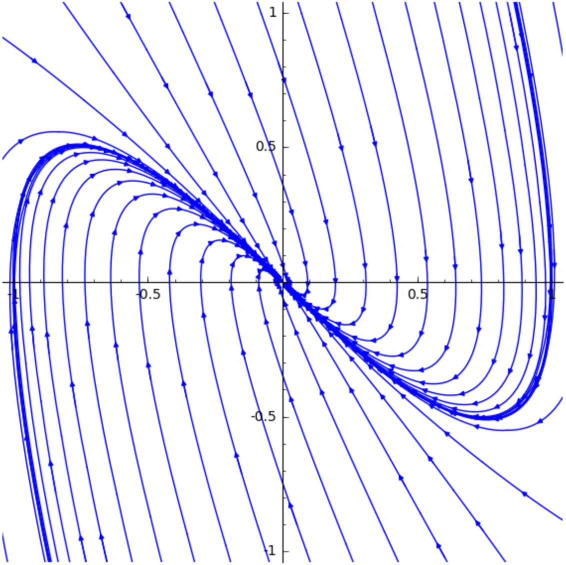

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{y}$

The origin is a nodal sink.

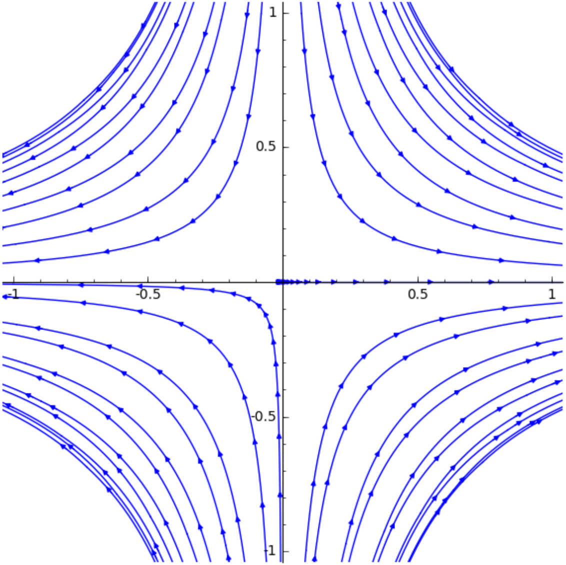

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{y}$

The origin is a saddle point.

Exercise

a) Suppose that $\mathbf{A}$ has two distinct real eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Show that for $t \to \pm\infty$ the tangent unit vector $\frac{\mathbf{y}'(t)}{|\mathbf{y}'(t)|}$ of any non-constant solution $\mathbf{y}(t)$ approaches the direction of one of the four rays $\mathbb{R}^+(\pm\mathbf{v}_1)$, $\mathbb{R}^+(\pm\mathbf{v}_2)$ (i.e., $\pm\frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\pm\frac{\mathbf{v}_2}{|\mathbf{v}_2|}$).

b) Work out the four possible cases (including the explicit determination of the rays) for the system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{y}.$$

Case 8: Two equal (real) eigenvalues, diagonalisable

In this case $\mathbf{A} = \lambda \mathbf{I}_2$ and the general solution is

$$\mathbf{y}(t) = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix} = e^{\lambda t} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

$\Longrightarrow$ $\mathbf{y}(t)$ moves on the ray $\mathbb{R}^+(c_1, c_2)$, provided that $(c_1, c_2) \neq (0, 0)$. The direction of traverse is determined by the sign of λ.

One sees that the orbits of non-constant solutions are precisely the rays $\mathbb{R}^+\mathbf{c}$, $\mathbf{c} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Accordingly, the orrigin is called a *star point* in this case.

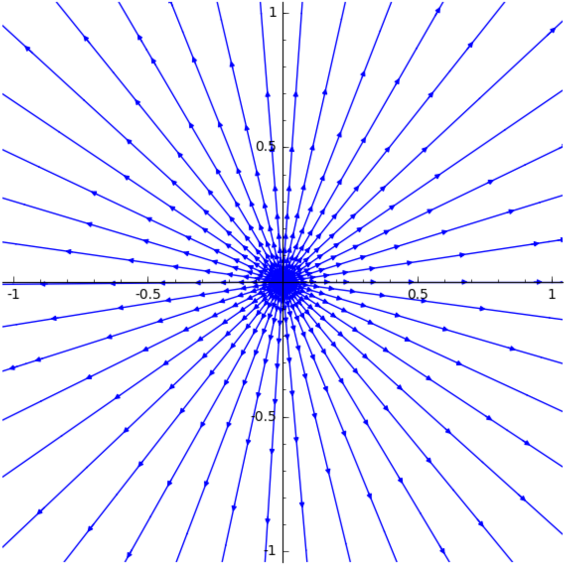

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y}$

The origin is a star point.

Exercise

Suppose $\mathbf{A} = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$ with $\lambda_1\lambda_2 \neq 0$. (This is essentially Case 7 with $\mathbf{v}_1 = \mathbf{e}_1$, $\mathbf{v}_2 = \mathbf{e}_2$, with the possibility $\lambda_1 = \lambda_2$ included; cf. Case 8.)

a) Which one-parameter family of functions $y(x)$ gives rise to the orbits of $\mathbf{y}' = \mathbf{A}\mathbf{y}$? From which scalar ODE does this family arise?

   Hint: Eliminate t.

b) Under which conditions on λ_1, λ_2 do the orbits form straight lines, respectively, hyperbolas?

c) What can you conclude from a), b) about the form of the orbits in Case 7 and Case 8 in general?

Case 9: Two equal (real) eigenvalues, not diagonalisable

In this case the general solution is

$$\begin{aligned}\mathbf{y}(t) &= c_1\left(\mathrm{e}^{\lambda t}\mathbf{v} + t\,\mathrm{e}^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}\right) + c_2\mathrm{e}^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}\\ &= c_1\mathrm{e}^{\lambda t}\mathbf{v} + (c_1 t + c_2)\mathrm{e}^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v},\end{aligned}$$

where λ denotes the eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ is any nonzero vector in $\mathbb{R}^2$ that is not an eigenvector of $\mathbf{A}$ ($\mathbf{v}$ can be taken as $\mathbf{e}_1$ or $\mathbf{e}_2$).

There are only two ray orbits, viz. $\pm\mathbb{R}^+(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}$ (contained in the line through the origin spanned by the eigenvector).

The tangent direction of every non-constant solution approaches one of the two ray directions for $t \to \pm\infty$, as can be seen from

$$\begin{aligned}\mathbf{y}'(t) &= c_1\lambda\,\mathrm{e}^{\lambda t}\mathbf{v} + c_2\lambda\,\mathrm{e}^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + c_1(1+\lambda t)\mathrm{e}^{\lambda t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v}\\ &= t\,\mathrm{e}^{\lambda t}\left[c_1\lambda(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + \frac{c_1+c_2\lambda}{t}(\mathbf{A}-\lambda\mathbf{I})\mathbf{v} + \frac{c_1\lambda}{t}\mathbf{v}\right].\end{aligned}$$

Accordingly, $\mathbf{0}$ is called an *improper node* in this case.

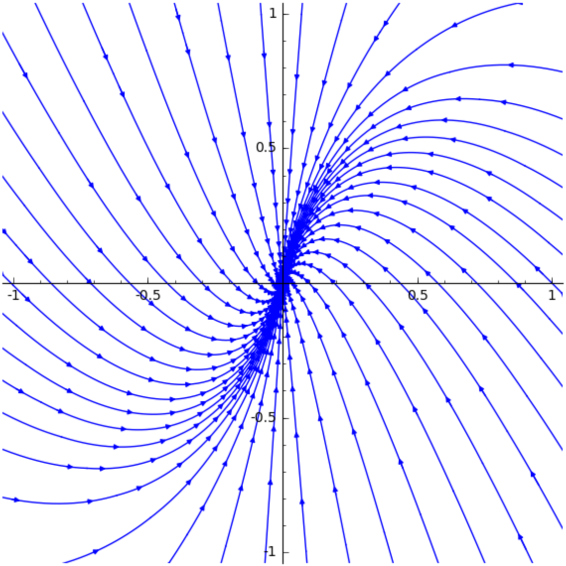

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \mathbf{y}$

The origin is an improper node.

Case 10: A pair of conjugate complex eigenvalues

If the eigenvalues are $\lambda_1 = \mu + \mathrm{i}\,\alpha$, $\lambda_2 = \mu - \mathrm{i}\,\alpha$ and $\mathbf{v}_1, \mathbf{v}_2$ are corresponding eigenvectors satisfying $\overline{\mathbf{v}}_1 = \mathbf{v}_2$, a complex fundamental system of solutions is $\mathbf{y}_1(t) = \mathrm{e}^{(\mu+\mathrm{i}\,\alpha)t}\mathbf{v}_1$, $\mathbf{y}_2(t) = \overline{\mathbf{y}_1(t)} = \mathrm{e}^{(\mu-\mathrm{i}\,\alpha)t}\,\overline{\mathbf{v}}_1$.

A real fundamental system of solutions is obtained by extracting real and imaginary part of one complex fundamental solution, say $\mathbf{y}_1(t)$, and is given by

$$t \mapsto \mathrm{e}^{\mu t}\left[\cos(\alpha t)\,\mathrm{Re}\,\mathbf{v}_1 - \sin(\alpha t)\,\mathrm{Im}\,\mathbf{v}_1\right],$$
$$t \mapsto \mathrm{e}^{\mu t}\left[\cos(\alpha t)\,\mathrm{Im}\,\mathbf{v}_1 + \sin(\alpha t)\,\mathrm{Re}\,\mathbf{v}_1\right].$$

1. $\mu > 0$: In this case non-constant solutions are unbounded for $t \to \infty$ and spiral towards $\mathbf{0}$ for $t \to -\infty$. The critical point $\mathbf{0}$ is called a *spiral source*.
2. $\mu < 0$: The roles of $\pm\infty$ are reversed and solutions spiral into $\mathbf{0}$ for $t \to +\infty$. The critical point $\mathbf{0}$ is called a *spiral sink*.
3. $\mu = 0$: Orbits of non-constant solutions are closed curves (in fact ellipses), and are traversed periodically (with period of revolution $2\pi/\alpha$). The critical point $\mathbf{0}$ is called a *center*.

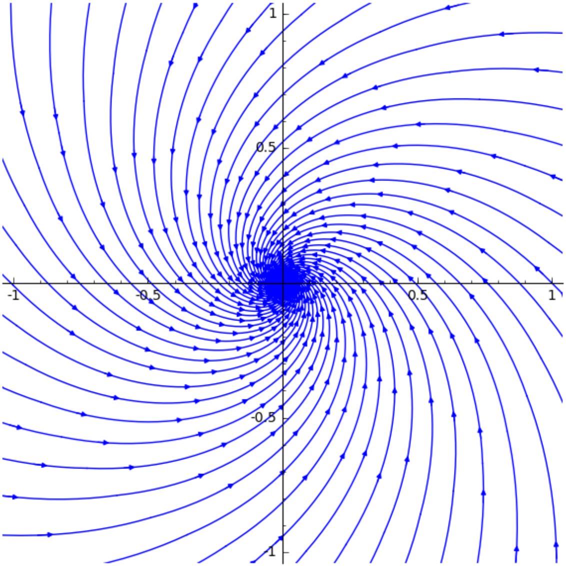

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}$

The origin is a spiral sink.

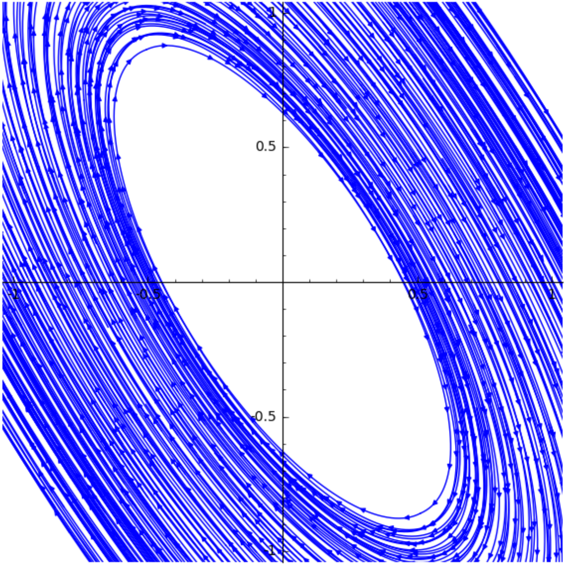

Figure: Phase portrait of $\mathbf{y}' = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix} \mathbf{y}$

The origin is a center.

Stability of Autonomous Linear Systems $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$

Recall that a steady-state solution (equilibrium solution, constant solution) $y(t) \equiv y_0$ of a scalar autonomous ODE $y' = f(y)$ was called *asymptotically stable* if there exists $\delta > 0$ such that any solution $y(t)$ satisfying $|y(0) - y_0| < \delta$ exists for all $t > 0$ and satisfies $\lim_{t\to\infty} y(t) = y_0$ (and "unstable" otherwise).

Definition

Suppose $\mathbf{y}_0$ is a critical point of the autonomous linear system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ (i.e., $\mathbf{A}\mathbf{y}_0 + \mathbf{b} = \mathbf{0}$ and $\mathbf{y}(t) \equiv \mathbf{y}_0$ is a solution).

1. $\mathbf{y}_0$ is said to be *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that every solution $\mathbf{y}(t)$ of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ with $|\mathbf{y}(0) - \mathbf{y}_0| < \delta$ satisfies
$$|\mathbf{y}(t) - \mathbf{y}_0| < \epsilon \quad \text{for all } t \geq 0;$$
2. $\mathbf{y}_0$ is said to be *asymptotically) stable* if there exists $\delta > 0$ such that every solution $\mathbf{y}(t)$ of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ with $|\mathbf{y}(0) - \mathbf{y}_0| < \delta$ satisfies $\lim_{t\to+\infty} \mathbf{y}(t) = \mathbf{y}_0$;
3. $\mathbf{y}_0$ is said to *unstable* otherwise.

Notes

1. The definitions are stated in a manner that makes sense for not necessarily linear autonomous systems $\mathbf{y}' = f(\mathbf{y})$. For linear autonomous systems (1) simplifies to "every solution is bounded on $[0, \infty)$" and (2) to "every solution satisfies $\lim_{t \to +\infty} \mathbf{y}(t) = \mathbf{y}_0$".
2. A critical point $\mathbf{y}_0$ of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is stable/asymptotically stable/unstable iff the critical point $\mathbf{0}$ of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ has this property. (This follows from the fact that $\mathbf{y}(t)$ solves $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ iff $\mathbf{y}(t) - \mathbf{y}_0$ solves $\mathbf{y}' = \mathbf{A}\mathbf{y}$.) Hence (1)–(3) are actually properties of the matrix $\mathbf{A}$. Accordingly we also say that $\mathbf{A}$ is stable/asymptotically stable/unstable in the respective cases.
3. A system with more than one critical point cannot be asymptotically stable. For this note that the critical points form an affine subspace C of $\mathbb{R}^n$. If the dimension of C is ≥ 1 then there are constant solutions arbitrarily close to but different from a given constant solution $\mathbf{y}(t) \equiv \mathbf{y}_0$, and these don't converge to $\mathbf{y}_0$.

Notes (cont'd)

4. The definition can be extended to non-constant solutions $\mathbf{y}_0(t)$ in the obvious way. For linear systems this doesn't yield anything substantially new, however, because for any further solution $\mathbf{y}(t)$ the difference is a solution of the associated homogeneous system.

Using the reduction to the homogeneous case $\mathbf{y}' = \mathbf{A}\mathbf{y}$ in Note 2, we now show the simplifications announced in Note 1:

(1) If $\mathbf{0}$ is a stable point of $\mathbf{y}' = \mathbf{A}\mathbf{y}$, there exists $\delta > 0$ such that $|\mathbf{y}(0)| < \delta$ implies $|\mathbf{y}(t)| < 1$ for $t \geq 0$. If $\mathbf{y}(t)$ is an arbitrary nonzero solution, there exists $\lambda > 0$ such that $|\lambda\,\mathbf{y}(0)| < \delta$ (take $\lambda = \frac{\delta}{2\,\mathbf{y}(0)}$). Since $\lambda\,\mathbf{y}(t)$ is a solution as well, this implies $|\lambda\,\mathbf{y}(t)| < 1$ and hence $|\mathbf{y}(t)| < 1/\lambda$ for all $t \geq 0$.
Conversely, suppose every solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is bounded on $[0, \infty)$ and let $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ be a fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Then there exist $M, m > 0$ such that $|\mathbf{y}_j(t)| \leq M$ for $t \geq 0$ and $1 \leq j \leq n$ (boundedness assumption) and $|c_1\mathbf{y}_1(0) + \cdots + c_n\mathbf{y}_n(0)| \geq m$ for all $\mathbf{c} = (c_1, \dots, c_n) \in \mathbb{R}^n$ with $|\mathbf{c}| = 1$ (compactness of the unit sphere $\mathbb{S}^{n-1} = \mathbb{S}_1(\mathbf{0})$).

Notes (cont'd)

Now we claim that in the definition of stability of $\mathbf{0}$ we can choose $\delta = \frac{\epsilon m}{\sqrt{n}\,M}$. Indeed, assume $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + \cdots + c_n\mathbf{y}_n(t)$ is a nonzero solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ satisfying $|c_1\mathbf{y}_1(0) + \cdots + c_n\mathbf{y}_n(0)| < \frac{\epsilon m}{\sqrt{n}\,M}$. Then we must have $|\mathbf{c}| < \frac{\epsilon}{\sqrt{n}\,M}$ (since $\mathbf{c} = |\mathbf{c}|\,\mathbf{s}$ with $|\mathbf{s}| = 1$), and hence

$$\begin{aligned} |\mathbf{y}(t)| &\leq |c_1|\,|\mathbf{y}_1(t)| + \cdots + |c_n|\,|\mathbf{y}_n(t)| \\ &\leq M\,(|c_1| + \cdots + |c_n|) \leq M\sqrt{n}\,|\mathbf{c}| < \epsilon \quad \text{for } t \geq 0. \end{aligned}$$

(2) Suppose $\mathbf{0}$ is an asymptotically stable point of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and $\delta > 0$ is such that $|\mathbf{y}(0)| < \delta$ implies $\lim_{t\to+\infty} \mathbf{y}(t) = \mathbf{0}$. For an arbitary nonzero solution $\mathbf{y}(t)$ of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ there exists $\lambda > 0$ such that $|\lambda\,\mathbf{y}(0)| < \delta$ (cf. earlier argument) and hence $\lim_{t\to+\infty} \lambda\,\mathbf{y}(t) = \mathbf{0}$. But then clearly $\lim_{t\to+\infty} \mathbf{y}(t) = \mathbf{0}$ as well.

Theorem

Suppose $\mathbf{y}_0$ *is a critical point of* $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$.

1. $\mathbf{y}_0$ *is asymptotically stable iff all eigenvalues of* $\mathbf{A}$ *have real part* < 0.
2. $\mathbf{y}_0$ *is stable iff (i) all eigenvalues of* $\mathbf{A}$ *have real part* ≤ 0 *and (ii) the geometric multiplicity of each purely imaginary eigenvalue (including zero) equals the algebraic multiplicity.*

Consequently, $\mathbf{y}_0$ *is unstable iff* $\mathbf{A}$ *has at least one eigenvalue with positive real part or a purely imaginary eigenvalue with geom.mult* $<$ *alg.mult.*

Proof.

We may assume that $\mathbf{b} = \mathbf{0}$.

(1) Suppose $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \dots, \lambda_r$ with algebraic multiplicities $m_1, \dots, m_r$.
We use the fact that the entries of $\mathrm{e}^{\mathbf{A}t}$, and likewise of any solution $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{y}(0)$, are linear combinations of functions of the form $t^k \mathrm{e}^{\lambda_i t}$ with $0 \leq k < m_i$. This can be worked out directly for matrices in JCF and then follows for general $\mathbf{A}$ from the equation $\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}$; or use the "new method" to compute $\mathrm{e}^{\mathbf{A}t}$.

$$\left|t^k \mathrm{e}^{\lambda t}\right| = t^k \mathrm{e}^{(\mathrm{Re}\,\lambda + \mathrm{i}\ \mathrm{Im}\,\lambda)t} = t^k \mathrm{e}^{(\mathrm{Re}\,\lambda)t} \to 0 \quad \text{for } t \to +\infty$$

holds iff $\mathrm{Re}\,\lambda < 0$. From this the if-part is immediate.

For the converse, assume $\mathrm{Re}\,\lambda_1 \geq 0$ and let $\mathbf{v}_1$ be a corresponding eigenvector.
$\Longrightarrow \mathbf{y}(t) = \mathrm{e}^{\lambda_1 t}\mathbf{v}_1$ is a solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
Since $\mathrm{e}^{\lambda_1 t} \not\to 0$ for $t \to \infty$, the same holds for at least one coordinate function of $\mathbf{y}(t)$ (in fact for each coordinate in which $\mathbf{v}_1$ has a nonzero entry). Thus there exists a solution $\mathbf{y}(t)$ with $\mathbf{y}(t) \not\to \mathbf{0}$ for $t \to \infty$. As noted previously, this implies that the system cannot be asymptotically stable.

Proof cont'd.

(2) If the stated conditions are satisfied, eigenspaces corresponding to purely imaginary eigenvalues $\lambda = \mathrm{i}\alpha$ have a basis consisting of eigenvectors and account only for fundamental solutions of the form $\mathbf{y}(t) = \mathrm{e}^{\mathrm{i}\alpha t}\mathbf{v}$ with $\mathbf{v} \in \mathbb{C}^n$. Since such solutions are bounded, and all other eigenvalues have negative real part (accounting for fundamental solutions with limit $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$), all solutions are bounded on $[0, \infty)$. As noted previosuly, this is equivalent to stability of the system.

Conversely, suppose that the stated conditions are not satisfied. Clearly an eigenvalue λ with $\mathrm{Re}(\lambda) > 0$ yields a solution that is unbounded on $[0, \infty)$. A purely imaginary eigenvalue $\lambda = \mathrm{i}\alpha$ with geom.mult $<$ alg.mult has an associated generalized eigenvector $\mathbf{w}$ with $(\mathbf{A} - \lambda\mathbf{I})\mathbf{w} \neq \mathbf{0}$, $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{w} = \mathbf{0}$, and hence yields a solution of the form

$$\mathbf{y}(t) = \mathrm{e}^{\mathrm{i}\alpha t}\mathbf{w} + t\,\mathrm{e}^{\mathrm{i}\alpha t}\underbrace{(\mathbf{A} - \mathrm{i}\alpha\,\mathbf{I})\mathbf{w}}_{\neq\mathbf{0}}.$$

Since such solutions are unbounded on $[0, \infty)$ (because of the factor t), the system cannot be stable in either case. $\square$

The case $n = 2$

A nonzero real (!) matrix $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{R}^{2\times 2}$ is asymptotically stable iff $\mathrm{tr}(\mathbf{A}) = a + d < 0 \wedge \det(\mathbf{A}) = ad - bc > 0$, and stable but not asymptotically stable iff $\mathrm{tr}(\mathbf{A}) = 0 \wedge \det(\mathbf{A}) \geq 0$.

This follows from $a + d = \lambda_1 + \lambda_2$, $ad - bc = \lambda_1\lambda_2$ and

$$\lambda_{1,2} = \frac{1}{2}\left(a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}\right) = \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4bc}\right)$$

by inspection. (Consider the two possible cases (i) $\lambda_1, \lambda_2 \in \mathbb{R}$ and (ii) $\lambda_1 = \overline{\lambda}_2 \in \mathbb{C} \setminus \mathbb{R}$.)

Thus nodal and spiral sinks are asymptotically stable; nodal and spiral sources and saddle points are unstable; centers are stable but not asymptotically stable; and star points and improper nodes can be either asymptotically stable or unstable, depending on the sign of the (real) eigenvalue λ.

A Digression

Solving $\mathbf{y}' = \mathbf{A}\mathbf{y}$ by a Change-of-Variables

Recall that the general solution of an autonomous (= time-independent) homogeneous linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is $y(t) = e^{\mathbf{A}t}\mathbf{y}(0)$, where $\mathbf{y}(0) \in \mathbb{C}^n$ can be arbitrarily chosen.

Change of Variables

Suppose $\mathbf{y}(t)$ solves $\mathbf{y}' = \mathbf{A}\mathbf{y}$ and $\mathbf{z}(t) = \mathbf{T}\mathbf{y}(t)$ for some $\mathbf{T} \in \mathbb{C}^{n\times n}$.

$$\mathbf{z}'(t) = \mathbf{T}\mathbf{y}'(t) = \mathbf{T}\mathbf{A}\mathbf{y}(t) = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}\mathbf{z}(t)$$

In terms of the inverse $\mathbf{S} = \mathbf{T}^{-1}$ this says that the change of variables $\mathbf{y}(t) = \mathbf{S}\mathbf{z}(t)$ transforms $\mathbf{y}' = \mathbf{A}\mathbf{y}$ into $\mathbf{z}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z}$. The initial values transform as $\mathbf{y}(0) = \mathbf{S}\mathbf{z}(0)$, $\mathbf{z}(0) = \mathbf{T}\mathbf{y}(0)$.

Example

We solve the initial value problem

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{pmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}.$$

This system arises from the scalar 3rd-order equation $y''' = -6y'' - 12y' - 8y$ by order reduction, i.e., the substitution $(y_1, y_2, y_3) = (y, y', y'')$. The initial value $\mathbf{y}(0) = (1, 0, -2)^\mathsf{T}$ corresponds to $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

From previous considerations we know that $\chi_\mathbf{A}(X) = (X+2)^3$ and $\mathbf{N} = \mathbf{A} + 2\mathbf{I}_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -8 & -12 & -4 \end{pmatrix}$ satisfies $\mathbf{N}^3 = \mathbf{0}$, $\mathbf{N}^2 \neq \mathbf{0}$.

One finds that $\mathbf{N}^2\mathbf{e}_1 = (4, -8, 16)^\mathsf{T} \neq \mathbf{0}$.

$\Longrightarrow$ The matrix $\mathbf{J}$ of $f_\mathbf{A}$ with respect to the basis $\{\mathbf{e}_1, \mathbf{N}\mathbf{e}_1, \mathbf{N}^2\mathbf{e}_1\}$ is in JCF, i.e.,

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \\ 0 & -8 & 16 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -12 & -6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \\ 0 & -8 & 16 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}.$$

Example (cont'd)

The system $\mathbf{z}' = \mathbf{J}\mathbf{z}$, i.e., $z_1' = -2z_1$, $z_2' = z_1 - 2z_2$, $z_3' = z_2 - 2z_3$, is solved by

$$\mathbf{z}(t) = \mathrm{e}^{-2t} \exp\left[\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} t\right] \mathbf{z}(0) = \mathrm{e}^{-2t} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{pmatrix} \mathbf{z}(0).$$

$\mathbf{z}(0)$ is determined from

$$\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} = \mathbf{y}(0) = \mathbf{S}\mathbf{z}(0) = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \\ 0 & -8 & 16 \end{pmatrix} \mathbf{z}(0),$$

i.e., $\mathbf{z}(0) = (\frac{1}{2}, \frac{1}{4}, 0)^{\mathsf{T}}$.

$$\Longrightarrow \mathbf{y}(t) = \mathbf{S}\mathbf{z}(t) = \mathrm{e}^{-2t} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \\ 0 & -8 & 16 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ \frac{1}{2}t^2 & t & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \\ 0 \end{pmatrix}$$

$$= \mathrm{e}^{-2t} \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -8 \\ 0 & -8 & 16 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ \frac{t}{2} + \frac{1}{4} \\ \frac{t^2}{4} + \frac{t}{4} \end{pmatrix} = \mathrm{e}^{-2t} \begin{pmatrix} t^2 + 2t + 1 \\ -2t^2 - 2t \\ 4t^2 - 2 \end{pmatrix}$$