Calculus III (Math 241)

H12 Use Gaussian elimination to determine the general solution of the following systems of linear equations:

a)
$$\begin{array}{rcrcr} 5x & - & 7y & = & 3 \\ -10x & + & 14y & = & -6 \end{array}$$

b)
$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & x_4 & = & 0 \\ 2x_1 & + & x_2 & + & x_3 & + & 2x_4 & = & 0 \\ x_1 & + & 2x_2 & + & 2x_3 & + & x_4 & = & 0 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 0 \end{array}$$

c)
$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ 4x & + & 5y & + & 6z & = & 0 \\ 7x & + & 8y & + & 9z & = & 4 \end{array}$$

d)
$$\begin{array}{rcrcrcr} 2x_1 & & & + & \mathrm{i}x_3 & = & \mathrm{i} \\ x_1 & - & 3x_2 & - & \mathrm{i}x_3 & = & 2\mathrm{i} \\ \mathrm{i}x_1 & + & x_2 & + & x_3 & = & 1+\mathrm{i} \end{array}$$

Note: Solve a), b), c) as usual over $\mathbb{R}$. The system d) has some coefficients in $\mathbb{C}\setminus\mathbb{R}$ and should be solved over $\mathbb{C}$.

H13 Find a matrix $\mathbf{A}\in\mathbb{R}^{3\times 3}$ with row space $U=\{\mathbf{x}\in\mathbb{R}^3; x_1+2x_2+3x_3=0\}$ and column space $V=\{\mathbf{x}\in\mathbb{R}^3; x_1-x_2+x_3=0\}$.

H14 a) Determine the rank of

$$\begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix} \quad \text{for } a,b\in\mathbb{R}.$$

b) For those $a,b\in\mathbb{R}$ for which the matrix in a) has rank 4, compute its inverse matrix.

Hint: The inverse matrix has the same form with possibly different constants c,d in place of a,b.

H15 Let S,T be subspaces of $\mathbb{R}^n$ satisfying $S\subseteq T$. Prove that $\dim(S)\leq\dim(T)$ with equality iff $S=T$.

Hint: Choose a basis $\mathbf{b}_1,\dots,\mathbf{b}_s$ of S and vectors $\mathbf{b}_{s+1},\dots,\mathbf{b}_t\in T$ such that
(i) $\mathbf{b}_1,\dots,\mathbf{b}_t$ are linearly independent, and (ii) t is maximal subject to this property.
Then prove that $\mathbf{b}_1,\dots,\mathbf{b}_t$ is a basis of T.

H16 Solve at least two of [Ste16], Section 12.4, Exercises 48, 50, 52.

H17 Compute the determinants of the matrices

$$\mathbf{A}=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix},\quad \mathbf{B}=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \\ 0 & 0 & 0 & 10 \end{pmatrix},\quad \mathbf{C}=\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 0 \\ 8 & 9 & 0 & 0 \\ 10 & 0 & 0 & 0 \end{pmatrix},$$

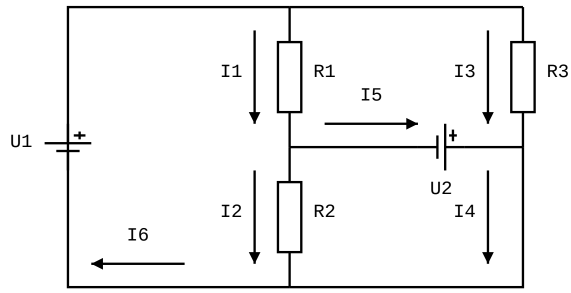
Can you find the general rule for evaluating determinants of “reverse-triangular” matrices like $\mathbf{C}$?

H18 *Optional exercise*

The Kirchhoff Laws for electric circuits assert the following: (i) The sum of the currents in every node is zero; (ii) the sum of the voltage drops along every loop is zero.

a) Derive for the currents $I_1, \ldots, I_6$ of the displayed DC circuit a linear system of equations (signs of the currents according to the arrows; use Ohm's Law $U = R \cdot I$).

b) Determine I_1 bis I_6 for $U_1 = 4{,}5\,V$, $U_2 = 1{,}5\,V$, $R_1 = R_2 = 50\,\Omega$, $R_3 = 100\,\Omega$.



H19 *Optional exercise*

a) Suppose $\mathbf{A} \in \mathbb{R}^{m \times n}$ is transformed into a matrix $\mathbf{A}'$ in row-echelon form by elementary row operations. Show that the columns of $\mathbf{A}$ corresponding to the pivot columns of $\mathbf{A}'$ (i.e., with the same column indexes) form a basis of the column space of $\mathbf{A}$.

b) Let S be the span of $\mathbf{v}_1 = (4,1,1,0,-2)$, $\mathbf{v}_2 = (0,1,4,-1,2)$, $\mathbf{v}_3 = (4,3,9,-2,2)$, $\mathbf{v}_4 = (1,1,1,1,1)$, $\mathbf{v}_5 = (0,-2,-8,2,-4)$ in $\mathbb{R}^5$. Find a subset of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\}$ that forms a basis of S.

H20 *Optional exercise*

Suppose that $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are subspaces with $\dim U = \dim V$. Show that there exists $\mathbf{A} \in \mathbb{R}^{m \times n}$ with row space U and column space V.

Hint: Choose a basis of U and write its vectors as rows of a matrix $\mathbf{U}$; similarly, choose a basis of V and write its vectors as columns of a matrix $\mathbf{V}$. Then consider the matrix product $\mathbf{VU}$.

Due on Wed Oct 13, 6 pm

The optional exercises can be handed in until Wed Oct 20, 6 pm.

For some of the exercises you may have to wait until the corresponding concepts have been discussed in the lecture. (Determinants and cross products will be discussed in Lecture 10 on Sat Oct 9.)