Math 286
Introduction to Differential Equations

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Outline

Today's Lecture:

Phase Space

We consider an $n \times n$ ODE system

$$\mathbf{y}' = f(\mathbf{y}) \quad \text{with } f\colon D \to \mathbb{R}^n,\, D \subseteq \mathbb{R}^n \text{ open.} \tag{A}$$

Such a system is said to be *autonomous*, because f doesn't depend on t.

Observations

1. Solutions of (A) are parametric curves $\mathbf{y}(t) = \big(y_1(t), \dots, y_n(t)\big)$, $t \in I$, contained in D. (More precisely, the range (or trace) of the associated non-parametric curve is contained in D.)
2. $\mathbf{y}(t)$, $t \in I$ is a solution iff $t \mapsto \mathbf{y}(t - t_0)$, $t \in I + t_0$ is a solution, where $I + t_0 = \{t + t_0; t \in I\}$. This holds for all $t_0 \in \mathbb{R}$.
3. If f is continuous and satisfies on D locally a Lipschitz condition, then for any point $\mathbf{y}^{(0)} \in \mathbb{R}^n$ there exists precisely one maximal solution of the IVP $\mathbf{y}' = f(\mathbf{y}) \wedge \mathbf{y}(0) = \mathbf{y}^{(0)}$, and this solution is defined on a certain open interval I containing $t = 0$ as an inner point (by the Existence and Uniqueness Theorem).

Definition

1. The ambient space $\mathbb{R}^n$ containing D and the solution curves $\mathbf{y}(t)$ is called *phase space* of the autonomous system $\mathbf{y}' = f(\mathbf{y})$.
2. The non-parametric maximal solution curves $\{\mathbf{y}(t), t \in I\}$ (ranges/traces of $t \mapsto \mathbf{y}(t)$) are called *trajectories* or *orbits* of $\mathbf{y}' = f(\mathbf{y})$.

Corollary

Suppose f is continuous and satisfies on D locally a Lipschitz condition. Then every point of D is contained in a unique orbit of $\mathbf{y}' = f(\mathbf{y})$. In other words, the orbits form a partition of D.

Proof.

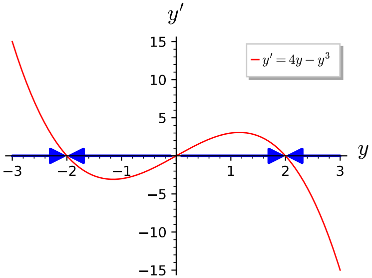
Let $\mathbf{y}^{(0)} \in D$. As already observed, $\mathbf{y}^{(0)} \in D$ is contained in an orbit of a maximal solution curve $\mathbf{y}(t)$, $t \in I$ that is defined at $t = 0$. Now suppose $\mathbf{z}(t)$, $t \in J$ is another maximal solution satisfying $\mathbf{z}(t_0) = \mathbf{y}^{(0)}$. Replacing $\mathbf{z}(t)$ by $t \mapsto \mathbf{z}(t + t_0)$, $t \in J - t_0$, which is a maximal solution as well and has the same orbit as $\mathbf{z}(t)$, we may assume $0 \in J$ and $\mathbf{z}(0) = \mathbf{y}^{(0)}$. But then the Uniqueness Theorem gives $\mathbf{y}(t) = \mathbf{z}(t)$ for $t \in I \cap J$, and maximality forces $I = J$ (because the two curves have a common extension to $I \cup J$). Thus the parametric curves and in particular their orbits are equal. $\square$

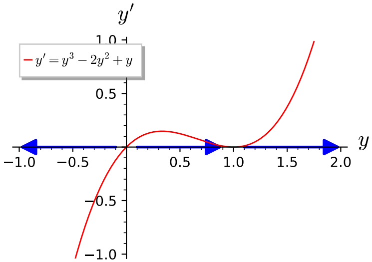
Note

Actually the proof shows more: Suppose we know a family of (parametric) maximal solutions whose associated orbits partition D. Then every further maximal solution has the form $t \mapsto \mathbf{y}(t - t_0)$ for some solution $\mathbf{y}(t)$ in the known family and some $t_0 \in \mathbb{R}$.

The Case $n = 1$

In this case $y' = f(y)$ for some one-variable function f. It is convenient to graph y' versus y, i.e., the function f.





The *phase line* is the y-axis (horizontal axis). The blue arrows indicate whether $y(t)$ is increasing/decreasing in the respective interval. *Caution:* This property depends on $y(t)$ rather than t!

Theorem

Suppose f is analytic on D, i.e., $f(y)$ is a polynomial ($D = \mathbb{R}$) or a power series in $y - y_0$ ($D = (y_0 - R, y_0 + R)$ for some $y_0 \in \mathbb{R}$ and $0 < R \leq \infty$), and $Z \subset D$ denotes the (discrete) set of zeros of f.

1. *The orbits of $y' = f(y)$ are the singleton sets $\{z\}$ for $z \in Z$ and the connected components of $D \setminus Z$, which in the polynomial case are the open intervals determined by adjacent zeros and intervals of the form $(-\infty, z)$, $(z, +\infty)$).*
2. *For $z \in Z$, $y' = f(y)$ has the equilibrium solution $y(t) \equiv z$.*
3. *If $f'(z) < 0$ then $y(t) \equiv z$ is asymptotically stable.*
   More generally, if f has a zero of odd multiplicity $m = 2k + 1$ at z and $f^{(2k+1)}(z) < 0$ then $y(t) \equiv z$ is asymptotically stable.
4. *If $f'(z) > 0$ then $y(t) \equiv z$ is unstable.*
   More generally, if f has a zero of odd multiplicity $m = 2k + 1$ at z and $f^{(2k+1)}(z) > 0$ then $y(t) \equiv z$ is unstable.
5. *If f has a zero of even multiplicity $m = 2k$ at z then $y(t) \equiv z$ is semistable (asymptotically stable from below if $f^{(2k)}(z) > 0$, respectively, from above if $f^{(2k)}(z) < 0$).*

Sketch of proof.

(2) is by now well-known and implies that for $z \in Z$ the set $\{z\}$ forms an orbit (arising from $y(t) \equiv z$).
Regarding (1), we prove only that if $z_1 < z_2$ are adjacent zeros of f and $f(y) > 0$ for $z_1 < y < z_2$ then (z_1, z_2) forms an orbit of $y' = f(y)$. (The other cases are similar.)

If suffices to show that a maximal solution $y(t)$ of $y' = f(y)$ with $y(0) = y_0 \in (z_1, z_2)$ exists for all $t \in \mathbb{R}$, is strictly increasing, and satisfies

$$\lim_{t\to-\infty} y(t) = z_1, \quad \lim_{t\to+\infty} y(t) = z_2,$$

because then clearly $y(\mathbb{R}) = (z_1, z_2)$.

Let I be the (open) interval on which $y(t)$ is defined. We can write $I = (a, b)$, where $a = -\infty$ and/or $b = +\infty$ is possible.

First we show that $y(t) \in (z_1, z_2)$ for all $t \in I$.
This is true for $t = 0$ and can fail for some t only if there exists t_0 such that $y(t_0) = z_1$ or $y(t_0) = z_2$ (by the Intermediate Value Theorem). This, however, would contradict the Uniqueness Theorem, because we also have the constant solutions $y(t_0) \equiv z_1$ and $y(t_0) \equiv z_2$.

$\Longrightarrow$ $y(t)$ is strictly increasing on I and bounded from above by z_2.
$\Longrightarrow$ $y_2 := \lim_{t\uparrow b} y(t)$ exists and satisfies $y_0 < y_2 \leq z_2$.

Proof cont'd.

Now we distinguish two cases:

Case 1: $b \in \mathbb{R}$

In this case $y(t)$ can be extended to $(a, b]$ by setting $y(b) = y_2$, and one verifies easily that the extension solves $y' = f(y)$ also in $t = b$. This contradicts the maximality of $y(t)$.

Case 2: $b = +\infty$

Here we use that the limit

$$\lim_{t \to +\infty} y'(t) = \lim_{t \to +\infty} f\bigl(y(t)\bigr) = f(y_2)$$

exists. Since $\lim_{t\to+\infty}\bigl(y(t+1) - y(t)\bigr) = y_2 - y_2 = 0$, for sufficiently large t the quantity

$$0 < y(t+1) - y(t) = y'(\tau), \quad \tau \in (t, t+1),$$

is smaller than any given $\epsilon > 0$. Together with the existence of $\lim_{t\to+\infty} y'(t)$ this implies $\lim_{t\to+\infty} y'(t) = 0$, i.e., $f(y_2) = 0$ and hence $y_2 = \lim_{t\to+\infty} y(t) = z_2$.

In the same way one proves $a = -\infty$ and $\lim_{t\to-\infty} y(t) = z_1$.

Proof cont'd.

(3), (4), (5) follow from (2) and the known characterization of sign changes/non-changes at zeros of f in terms of the first non-vanishing derivative. □

Example ($y' = 4y - y^3$)

The preceding theorem gives immediately that the equilibrium solutions $y(t) \equiv \pm 2$ are asymptotically stable and $y(t) \equiv 0$ is unstable; cf. picture.

Example ($y' = y^3 - 2y^2 + y$)

$y(t) \equiv 0$ is unstable and $y(t) \equiv 1$ is semistable (more precisely, asymptotically stable from below and unstable from above); cf. picture.

Example ($y' = y - y^2$)

This is the logistic equation with $a = b = 1$. The graph of $f(y) = y - y^2 = -(y - 1/2)^2 + 1/4$ is the standard parabola upside down. It has zeros 0 and 1.
$\Longrightarrow y(t) \equiv 0$ (corresponding to the left zero) is unstable, and $y(t) \equiv 1$ (corresponding to the right zero) is asymptotically stable.

Remark

Solutions of scalar autonomous ODE's are best viewed as functions $t(y)$.

$$\begin{aligned} y' &= f(y) \\ \mathrm{d}y\,/f(y) &= \mathrm{d}t \\ \int \frac{\mathrm{d}y}{f(y)} &= t = t(y). \end{aligned}$$

$\Longrightarrow$ $y' = f(y)$ can be solved by a single integration (just like $y' = f(t)$, only the roles of t and y are interchanged.

For example, in the case of $y' = y - y^2$ we obtain

$$t(y) = \int \frac{\mathrm{d}y}{y - y^2} = \int \left(\frac{1}{y} + \frac{1}{1-y}\right) \mathrm{d}y = \ln\left|\frac{y}{1-y}\right| + C.$$

The plot on the next slide shows 5 particular representative solutions for the 5 orbits of $y' = y - y^2$. The 3 branches of $y \mapsto \ln\left|\frac{y}{1-y}\right|$ represent the non-constant solutions. They can be independently shifted vertically to produce the remaining solutions.

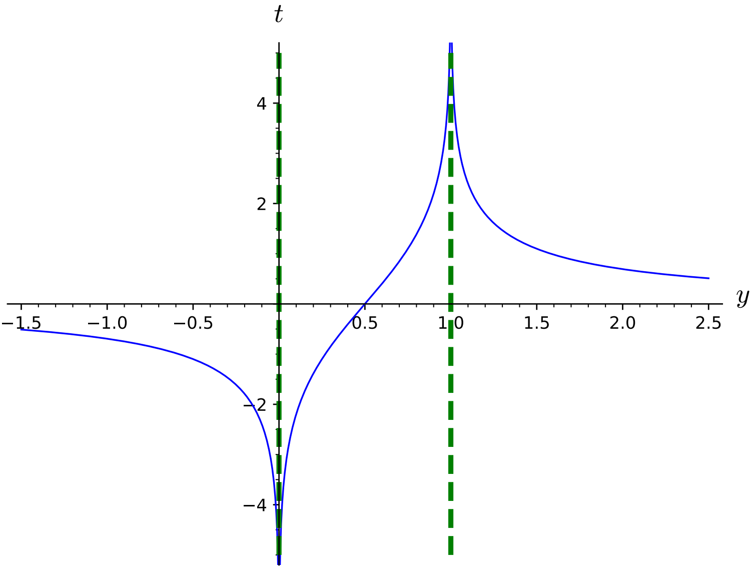


Figure: $t(y) = \ln\left|\frac{y}{1-y}\right|$

Exercise

In the proof of the theorem we have seen that maximal solutions representing orbits of $y' = f(y)$ of the form (z_1, z_2) have domain $\mathbb{R}$. How can we determine from properties of f the domain of solutions representing orbits of the form $(-\infty, z)$ or $(z, +\infty)$? In particular, answer this question for the case of a polynomial $f(y)$.

Exercise (cf. [BDM17], Sect. 2.5, p. 61)

The phase line can also be used to determine the curvature (i.e., whether it is convex or concave) of solutions of $y' = f(y)$. Show that solutions $y(t)$ are strictly convex (concave) in regions of the (t, y)-plane where $f(y)f'(y) > 0$ (respectively, $f(y)f'(y) < 0$). In particular, the inflection points of solutions (if any) are located on lines $y = y_0$ with $f(y_0) \neq 0 \wedge f'(y_0) = 0$ (e.g., for $y' = y - y^2$ on the line $y = 1/2$). What can be said about the number of inflection points of a non-constant solution with domain of the form $(-\infty, a)$, (a, b), or (b, ∞) with $a, b \in \mathbb{R}$?

The Case $n = 2$

Phase planes and planar trajectories/orbits are associated to 2×2 autonomous ODE systems

$$\begin{aligned} y_1' &= f_1(y_1, y_2), \\ y_2' &= f_2(y_1, y_2). \end{aligned}$$

Every maximal solution $\mathbf{y}(t) = \big(y_1(t), y_2(t)\big)$, $t \in I$ of such a system is a parametric plane curve. The orbit of $y(t)$, viz. $\big\{\big(y_1(t), y_2(t)\big); t \in I\big\}$, is the corresponding non-parametric curve.

Here we consider only one important example.

Example (Phase portrait of $y'' + y = 0$)

Order reduction $y_1 = y$, $y_2 = y'$ transforms this 2nd-order ODE into

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The orbit of a nonzero solution

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} A\cos t + B\sin t \\ -A\sin t + B\cos t \end{pmatrix} = \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

is a circle of radius $\sqrt{A^2 + B^2}$ with center $(0, 0)$.

Example (cont'd)

Geometrically this says, that the solution with initial values $y(0) = A$, $y'(0) = B$ has the property that all "state vectors" $\big(y(t), y'(t)\big)$, describing the displacement from the equilibrium position $y = 0$ and its velocity of change at an arbitrary time t, are located on the circle $X^2 + Y^2 = A^2 + B^2$. (Recall that when we first determined the solutions of $y'' + y = 0$ we used this property, viz. $y(t)^2 + y'(t)^2 = A^2 + B^2 = y(0)^2 + y'(0)^2$, as a key fact.)

As predicted by the corollary, the orbits partition the plane if we also include $\big\{(0,0)\big\}$, the orbit of the constant solution $y(t) \equiv 0$.

We have also seen that solutions $y(t)$, $z(t)$ with the same orbit, i.e., the same $\sqrt{A^2 + B^2}$, differ only by a time shift (phase shift) $z(t) = y(t - t_0)$, $t_0 \in \mathbb{R}$. This is visible in the alternative representation

$$y(t) = A\cos t + B\sin t = \operatorname{Re}\left[(A + B\,\mathrm{i})\mathrm{e}^{-\mathrm{i}t}\right] = \sqrt{A^2 + B^2}\,\sin(t + \phi),$$

in which ϕ is determined from $\sin\phi = \frac{A}{\sqrt{A^2+B^2}}$, $\cos\phi = \frac{B}{\sqrt{A^2+B^2}}$.

The collection of all orbits (or a good representative selection of orbits) of a given autonomous ODE system is referred to as a *phase portrait*.