# Math 286
# Introduction to Differential Equations

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# Outline

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# Today's Lecture:

# Introduction

A first-order autonomous (time-independent) linear ODE system has the form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}, \quad \text{with } \mathbf{A} \in \mathbb{C}^{n \times n}, \mathbf{b} \in \mathbb{C}^n.$$

As in the case of higher-order scalar ODE's, we will include in the discussion the case of a time-dependent continuous "source" $\mathbf{b}(t)$, i.e., consider more generally $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}(t)$ or, written out in full,

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} + \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}.$$

# Motivation

ODE systems of the form just described often occur when modeling physical systems with a number of separate but interconnected ("coupled") components. Examples are provided by spring-mass systems and LRC electric circuits. We just reproduce the two introductory examples from [BDM17], Ch. 7.1.

## Example ([BDM17], p. 279)

A 1-dimensional *two-mass, three-spring system* under the influence of external forces is described by the 2nd-order ODE system

$$\begin{aligned} m\,x_1''(t) &= -(k_1 + k_2)x_1 + k_2 x_2 + F_1(t), \\ m\,x_2''(t) &= k_2 x_1 - (k_2 + k_3)x_2 + F_2(t), \end{aligned}$$

where $x_1, x_2$ denote the coordinates of the masses, $k_1$, $k_2$, $k_3$ the spring constants, and $F_1(t)$, $F_2(t)$ the (time-dependent) external forces.

This $2 \times 2$ linear system can by reduced to a $4 \times 4$ first-order linear system by the the usual method of order reduction, i.e., we introduce two further variables $x_3 = x_1'$, $x_4 = x_2'$.

## Example (cont'd)

This gives the four equations

$$\begin{aligned}
x_1' &= x_3, \\
x_2' &= x_4, \\
x_3' &= x_1'' = -(k_1 + k_2)/m \cdot x_1 + (k_2/m)x_2 + F_1(t)/m, \\
x_4' &= x_2'' = (k_2/m)x_1 - (k_2 + k_3)/m \cdot x_2 + F_2(t)/m,
\end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}' = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{F_1(t)}{m} \\ \frac{F_2(t)}{m} \end{pmatrix}.$$

## Example ([BDM17], p. 280)

The current $I(t)$ and voltage $V(t)$ in a *parallel LRC circuit* satisfy the $2 \times 2$ first-order homogeneous linear system

$$\begin{aligned} I'(t) &= \frac{V}{L}, \\ V'(t) &= -\frac{I}{C} - \frac{V}{RC}, \end{aligned}$$

where $L$, $R$, $C$ denote the inductance/resistance/capacitance of the inductor/resistor/capacitor.
In matrix form this system is

$$\begin{pmatrix} I \\ V \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{pmatrix} \begin{pmatrix} I \\ V \end{pmatrix}.$$

# Facts Already Known

1. Any IVP $\mathbf{y}' = \mathbf{Ay} + \mathbf{b}(t) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$ has a unique maximal solution, which is defined whereever all coordinate functions of $\mathbf{b}(t)$ are defined. In particular, if $\mathbf{b}(t) \equiv \mathbf{b}$ is constant then the solution of the IVP is defined on $\mathbb{R}$.
2. The solutions of any homogeneous system $\mathbf{y}' = \mathbf{Ay}$ form an $n$-dimensional subspace $S$ of the vectorial function space $(\mathbb{C}^n)^{\mathbb{R}}$ (consisting of all maps $f \colon \mathbb{R} \to \mathbb{C}^n$).
3. If $\Phi(t)$ is a fundamental matrix of $\mathbf{y}' = \mathbf{Ay}$ (i.e., the columns of $\Phi(t)$ form a basis of the solution space $S$ of $\mathbf{y}' = \mathbf{Ay}$), the general solution of an associated inhomogeneous system $\mathbf{y}' = \mathbf{Ay} + \mathbf{b}(t)$ is
$$\mathbf{y}(t) = \Phi(t)\left(\mathbf{c}_0 + \int_{t_0}^{t} \Phi(s)^{-1}\mathbf{b}(s)\,\mathrm{d}s\right), \quad \mathbf{c}_0 \in \mathbb{C}^n.$$
Alternatively, if a particular solution $\mathbf{y}_{\mathrm{p}}(t)$ of $\mathbf{y}' = \mathbf{Ay} + \mathbf{b}(t)$ is known, the general solution of $\mathbf{y}' = \mathbf{Ay} + \mathbf{b}(t)$ is $\mathbf{y}(t) = \Phi(t)\mathbf{c}_0 + \mathbf{y}_{\mathrm{p}}(t)$, $\mathbf{c}_0 \in \mathbb{C}^n$.

# Facts Already Known (Cont'd)

While the preceding properties hold more generally for time-dependent systems $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t)$ (provided all coefficient functions $a_{ij}(t)$, $b_i(t)$ are considered for Property 1), the next property is a special feature of the time-independent case.

4. The matrix exponential function $t \mapsto \mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{A}^k$ satisfies $\Phi'(t) = \mathbf{A}\Phi(t)$, $\Phi(0) = \mathbf{I}_n$, and hence provides a fundamental matrix for the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

## Problem

*How to find an explicit fundamental matrix for $\mathbf{y}' = \mathbf{A}\mathbf{y}$ ? Equivalently, how to actually compute $\mathrm{e}^{\mathbf{A}t}$ ?*

Any two fundamental matrices $\Phi_1, \Phi_2$ are related by $\Phi_1(t) = \Phi_2(t)\mathbf{C}$ for some invertible $\mathbf{C} \in \mathbb{C}^{n\times n}$. The matrix $\mathbf{C}$ is the change-of-basis matrix from the ordered basis of $S$ formed by the columns of $\Phi_1(t)$ to that formed by the columns of $\Phi_2(t)$. It is given by $\mathbf{C} = \Phi_2(t_0)^{-1}\Phi_1(t_0)$ for any $t_0 \in \mathbb{R}$.

Hence one fundamental matrix is as good as any other, and for any fundamental matrix $\Phi(t)$ we have $\Phi(t) = \mathrm{e}^{\mathbf{A}t}\Phi(0)$, or $\Phi(t)\Phi(0)^{-1} = \mathrm{e}^{\mathbf{A}t}$.

# A conceptual approach to solve the problem

First we look for instances of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ that are easy to solve directly.

If $\mathbf{A}$ is a diagonal matrix, say with entries $\lambda_1, \dots, \lambda_n$, then

$$\begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda_1 y_1 \\ \vdots \\ \lambda_n y_n. \end{pmatrix}$$

So the system is "decoupled" into the $n$ scalar ODE's $y_i' = \lambda_i y_i$, $1 \leq i \leq n$.

$\Longrightarrow$ The general solution is

$$\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \begin{pmatrix} c_1 \mathrm{e}^{\lambda_1 t} \\ \vdots \\ c_n \mathrm{e}^{\lambda_n t} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n t} \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

From this we see that the diagonal matrix with $i$-th entry $\mathrm{e}^{\lambda_i t}$ (considered as a matrix function of $t \in \mathbb{R}$) is a fundamental matrix. Setting $t = 0$ gives the identity matrix. $\Longrightarrow$ This must be $\mathrm{e}^{\mathbf{A}t}$ !

# Independent verification

$$\exp\left[t\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\right] = \sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}^k$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}\begin{pmatrix}\lambda_1^k & & \\ & \ddots & \\ & & \lambda_n^k\end{pmatrix}$$

$$= \begin{pmatrix}\sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty}\frac{t^k}{k!}\lambda_n^k\end{pmatrix} = \begin{pmatrix}e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n
$$

Thus the problem is solved for diagonal matrices.

# A conceptual approach (cont'd)

For a general matrix $\mathbf{A}$ we would like to find a coordinate transformation $\mathbf{y}(t) = \mathbf{S}\,\mathbf{z}(t)$, or $\mathbf{y} = \mathbf{S}\mathbf{z}$ for short, which puts $\mathbf{y}' = \mathbf{A}\mathbf{y}$ into simpler form (diagonal form, if possible). Of course, we must check whether the transformed system has the form $\mathbf{z}' = \mathbf{B}\mathbf{z}$ at all.

The matrix $\mathbf{S}$ must be invertible, i.e., the columns of $\mathbf{S}$ must form an (ordered) basis of $\mathbb{C}^n$. The matrix $\mathbf{S}$ then switches from this basis to the standard basis of $\mathbb{C}^n$.

$$\mathbf{y} = \mathbf{S}\mathbf{z} \implies \mathbf{y}' = \mathbf{S}\mathbf{z}' \implies \mathbf{z}' = \mathbf{S}^{-1}\mathbf{y}' = \mathbf{S}^{-1}\mathbf{A}\mathbf{y} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{z}$$

$\implies$ The new system has the desired form $\mathbf{z}' = \mathbf{B}\mathbf{z}$ with $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

## Definition

Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{n\times n}$ are said to be *similar* if there exists an invertible matrix $\mathbf{S} \in \mathbb{C}^{n\times n}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

It is easily checked that similarity defines an equivalence relation on $\mathbb{C}^{n\times n}$, which therefore is partitioned into *similarity classes* (the corresponding equivalence classes).

# A conceptual approach (cont'd)

*Conclusion:* If **A** is similar to a diagonal matrix

$$\mathbf{B} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and, secondly, we can compute the corresponding transform matrix **S**, then we can solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$ completely.
The general solution will be

$$\mathbf{y}(t) = \mathbf{S} \begin{pmatrix} c_1 \mathrm{e}^{\lambda_1 t} \\ \vdots \\ c_n \mathrm{e}^{\lambda_n t} \end{pmatrix} = c_1 \mathrm{e}^{\lambda_1 t} \mathbf{v}_1 + \cdots + c_n \mathrm{e}^{\lambda_n t} \mathbf{v}_n,$$

where $\mathbf{S} = (\mathbf{v}_1 | \dots | \mathbf{v}_n)$, and a fundamental system of solutions will be $t \mapsto \mathrm{e}^{\lambda_1 t} \mathbf{v}_1, \dots, t \mapsto \mathrm{e}^{\lambda_n t} \mathbf{v}_n$.
The vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form the ordered basis of $\mathbb{C}^n$ corresponding to the coordinate transformation $\mathbf{y} = \mathbf{S}\mathbf{z}$ (which should be viewed as a coordinate transformation of $\mathbb{C}^n$ that gives rise to a corresponding transformation of functions).

# A conceptual approach (cont'd)

The fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ just determined is

$$\Phi(t) = \left(\mathrm{e}^{\lambda_1 t}\mathbf{v}_1 \middle| \dots \middle| \mathrm{e}^{\lambda_n t}\mathbf{v}_n\right) = \mathbf{S}\begin{pmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n t} \end{pmatrix}.$$

It follows that

$$\mathrm{e}^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1} = \mathbf{S}\begin{pmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n t} \end{pmatrix}\mathbf{S}^{-1}.$$

This can also be verified directly from the series representation:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \implies \mathbf{A} = \mathbf{S}\underbrace{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}}_{\mathbf{B}}\mathbf{S}^{-1}$$

$$\implies \mathbf{A}^k = (\mathbf{S}\mathbf{B}\mathbf{S}^{-1})(\mathbf{S}\mathbf{B}\mathbf{S}^{-1})\cdots(\mathbf{S}\mathbf{B}\mathbf{S}^{-1}) = \mathbf{S}\mathbf{B}^k\mathbf{S}^{-1}$$

# A conceptual approach (cont'd)

$$
\begin{aligned}
\implies \quad e^{\mathbf{A}t} &= \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{S}\mathbf{B}^k\mathbf{S}^{-1} \\
&= \mathbf{S}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\,\mathbf{B}^k\right)\mathbf{S}^{-1} \qquad (\star) \\
&= \mathbf{S}e^{\mathbf{B}t}\mathbf{S}^{-1} = \mathbf{S}\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}\mathbf{S}^{-1}
\end{aligned}
$$

For the step tagged $(\star)$ we have used continuity of matrix multiplication, which implies that for a convergent sequence $\mathbf{X}_k \to \mathbf{X}$ of matrices $\mathbf{X}_k \in \mathbb{C}^{n\times n}$ we have $\mathbf{S}\mathbf{X}_k \to \mathbf{S}\mathbf{X}$, and similarly $\mathbf{X}_k\mathbf{S}^{-1} \to \mathbf{X}\mathbf{S}^{-1}$.

## New problem

Which matrices in $\mathbb{C}^{n\times n}$ are similar to a diagonal matrix?

Unfortunately not all matrices, as it turns out.

## Definition

$\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be *diagonalizable* if $\mathbf{A}$ is similar to a diagonal matrix.

The matrix equation $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$ can be rewritten as $\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{B}$, and for a diagonal matrix $\mathbf{B}$ takes the form

$$\begin{aligned}\mathbf{A}\big(\mathbf{v}_1|\dots|\mathbf{v}_n\big) = \big(\mathbf{A}\mathbf{v}_1|\dots|\mathbf{A}\mathbf{v}_n\big) &= \big(\mathbf{v}_1|\dots|\mathbf{v}_n\big)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\\ &= \big(\lambda_1\mathbf{v}_1|\dots|\lambda_n\mathbf{v}_n\big)\end{aligned}$$

$\Longrightarrow$ $\mathbf{A}$ is diagonalizable iff there exists a basis $\mathbf{v}_1,\dots,\mathbf{v}_n$ of $\mathbb{C}^n$ which satisfies

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i \quad \text{for some } \lambda_i \in \mathbb{C},\ 1 \le i \le n.$$

Using the terminology introduced later, $\mathbf{A}$ is diagonalizable iff there exists a basis of $\mathbb{C}^n$ which consists of eigenvectors of $\mathbf{A}$.

## Example

After introducing the matrix exponential function we had seen that the 1st-order system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

which arises from $y'' + y = 0$, has matrix exponential function

$$\exp\left[t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

Now we use the present approach to rederive this result.

For this we need to solve the equation

$$\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

$x_2 = \lambda x_1 \wedge x_1 = -\lambda x_2 \implies x_1 = -\lambda^2 x_1$
There is the trivial solution $x_1 = x_2 = 0$, $\lambda$ arbitrary, which we cannot use, since we need basis vectors of $\mathbb{C}^2$.
Nontrivial solutions must have $x_1 \neq 0$ and $\lambda^2 = -1$, i.e., $\lambda = \pm\mathrm{i}$, $\mathbf{x} = (x_1, \pm\mathrm{i}\, x_1)^\mathsf{T} = x_1(1, \pm\mathrm{i})^\mathsf{T}$.

## Example (cont'd)

Normalizing $x_1 = 1$ gives two linearly independent solutions.
$\Longrightarrow$ $\mathbf{S} = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}$ diagonalizes $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Indeed we have

$$\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} = \frac{1}{-2\mathrm{i}} \begin{pmatrix} -\mathrm{i} & -1 \\ -\mathrm{i} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{i} & -\mathrm{i} \\ -1 & -1 \end{pmatrix}$$
$$= \frac{\mathrm{i}}{2} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$

Our previous discussion yields that

$$\mathbf{y}_1(t) = \mathrm{e}^{\mathrm{i}t} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-\mathrm{i}t} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$$

form a fundamental system of solutions of $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathbf{y}$.
The corresponding matrix exponential is

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{-\mathrm{i}t} \\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}t} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}^{-1} = \frac{1}{-2\mathrm{i}} \begin{pmatrix} \mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{-\mathrm{i}t} \\ \mathrm{i}\,\mathrm{e}^{\mathrm{i}t} & -\mathrm{i}\,\mathrm{e}^{-\mathrm{i}t} \end{pmatrix} \begin{pmatrix} -\mathrm{i} & -1 \\ -\mathrm{i} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

# A Curious Thing

There is the simple matrix identity

$$\exp\left[t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] = \cos t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},$$

which expresses $\mathrm{e}^{\mathbf{A}t}$ in this special case as a linear combination of $\mathbf{I}$ and $\mathbf{A}$. There is no magic in this—it is just the observation that

$$\begin{pmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{pmatrix} = \cos t\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \sin t\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

However, we will see later (if time permits) that in general $n \times n$ matrix exponentials can be expressed as finite sums

$$\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{n-1} c_k(t)\mathbf{A}^k.$$

This is surprising at the first glance, since $\mathrm{e}^{\mathbf{A}t}$ was defined by an infinite sum, and it does not mean that the matrix exponential series terminates after a finite number of summands.

# Remark about Coordinate Maps

Suppose $V$ is an $n$-dimensional vector space over $F$ and $f\colon V \to F^n$ a bijective linear map (vector space isomorphism). Then there exists a unique ordered basis $B = \{b_1, \dots, b_n\}$ of $V$ such that $f = \phi_B$ viz., the basis of $V$ mapped by $f$ to the standard basis of $F^n$. (This is clear from the fact that among the linear maps from $V$ to $F^n$ the coordinate map $\phi_B$ is characterized by $\phi_B(b_j) = \mathbf{e}_j$, $1 \leq j \leq n$.)

Writing $f = (f_1, \dots, f_n)$, we then have

$$v = f_1(v)b_1 + f_2(v)b_2 + \cdots + f_n(v)b_n.$$

This important representation has occured in several different contexts in the lecture.

1. *Polynomial interpolation*
   Here $V = P_{n-1}$ (space of polynomials in $\mathbb{R}[X]$ of degree $\leq n-1$, $f\big(a(X)\big) = \big(a(x_1), \dots, a(x_n)\big)$ ("evaluation map") and $B = \mathrm{L}$ (Lagrange basis), leading to the representation of $a(X) \in P_{n-1}$ in terms of the Lagrange polynomials:

$$a(X) = a(x_1)\ell_1(X) + \cdots + a(x_n)\ell_n(X).$$

2. *Higher-order scalar linear ODE's*
   Here $V$ is the solution space (over $\mathbb{R}$, say) of $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ (with $a_i \in \mathbb{R}$), $f(\phi) = \big(\phi(t_0), \phi'(t_0), \dots, \phi^{(n-1)}(t_0)\big)$, and $B = \{\phi_0, \dots, \phi_{n-1}\}$ the special basis of $V$ defined by

$$\phi_j^{(i)}(t_0) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

   This leads to the representation $y(t) = \sum_{j=0}^{n-1} y_j \phi_j(t)$ of the solution of the associated IVP with initial conditions $y^{(j)}(t_0) = y_j$, $0 \leq j \leq n-1$.

3. *First-order linear ODE systems*
   Here $V$ is the solution space of $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $f(\mathbf{y}) = \mathbf{y}(0)$, and $B = \{\phi_1, \dots, \phi_n\}$ are the columns of the matrix exponential $\mathrm{e}^{\mathbf{A}t}$.
   This gives the representation $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{y}_0$ of the solution of the associated IVP $\mathbf{y}' = \mathbf{A}\mathbf{y} \wedge \mathbf{y}(0) = \mathbf{y}_0$.

4. *Linear recurring sequences*
Here $V$ is the solution space (over $\mathbb{C}$, say) of a homogeneous linear recurrence relation $x_{k+n} + a_{n-1}x_{k+n-1} + \cdots + a_1 x_{k+1} + a_0 x_k = 0$ (with $a_i \in \mathbb{C}$), $f(a_0, a_1, a_2, \dots) = (a_0, a_1, \dots, a_{n-1})$, and $B = \{\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ is the special basis obtained by prescribing $(x_0, x_1, \dots, x_{n-1})$ as the standard unit vectors, i.e., $\mathbf{e}_0 = (1, 0, \dots, 0, *, *, \dots)$, $\mathbf{e}_1 = (0, 1, 0, \dots, 0, *, *, \dots)$, etc.

This gives the representation

$$(a_0, a_1, a_2, \dots) = a_0\mathbf{e}_0 + a_1\mathbf{e}_1 + \cdots + a_{n-1}\mathbf{e}_{n-1}$$

of an arbitrary solution of the recurrence relation.

Recall the following facts about the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with $\mathbf{A} \in \mathbb{C}^{n\times n}$ from the previous lecture:

1. $\mathbf{A} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \implies \mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n t} \end{pmatrix}$ is a fundamental matrix of the system (and $t \mapsto \mathrm{e}^{\lambda_1 t}\mathbf{e}_1, \dots, t \mapsto \mathrm{e}^{\lambda_n t}\mathbf{e}_n$ form a fundamental system of solutions).
2. If there exists an invertible matrix $\mathbf{S} \in \mathbb{C}^{n\times n}$ such that $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$, the matrix function $\Phi(t) = \mathbf{S} \begin{pmatrix} \mathrm{e}^{\lambda_1 t} & & \\ & \ddots & \\ & & \mathrm{e}^{\lambda_n t} \end{pmatrix}$ is a fundamental matrix, as well as $\mathrm{e}^{\mathbf{A}t} = \Phi(t)\mathbf{S}^{-1}$.
3. $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ iff the columns $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbf{S}$ are linearly independent and satisfy $\boxed{\mathbf{A}\mathbf{v}_j = \lambda_j \mathbf{v}_j \text{ for } 1 \leq j \leq n.}$
If this is the case, the vector functions $t \mapsto \mathrm{e}^{\lambda_1 t}\mathbf{v}_1, \dots, t \mapsto \mathrm{e}^{\lambda_n t}\mathbf{v}_n$ form a fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

# Eigenvectors and Eigenvalues

## Definition

Suppose $V$ is a vector space over $F$ and $f\colon V \to V$ a linear map. A vector $v \in V \setminus \{0_V\}$ is said to be an *eigenvector* of $f$ if $f(v) = \lambda v$ for some $\lambda \in F$. If this is the case then $\lambda$ is called an *eigenvalue* (or *characteristic value*) of $f$.

## Notes

- The word “eigenvalue” is a half-way translation of the German word „Eigenwert“ into English, and similarly for “eigenvector”; the full translation might be “own value/vector of $f$”.
- The zero vector $0_V$ is excluded form the definition, since it satisfies $f(0_V) = 0_V = \lambda 0_V$ for any $\lambda \in F$.
- If $f(v_1) = \lambda v_1$ and $f(v_2) = \lambda v_2$ then $f(a_1 v_1 + a_2 v_2) = a_1 f(v_1) + a_2 f(v_2) = a_1 \lambda v_1 + a_2 \lambda v_2 = \lambda(a_1 v_1 + a_2 v_2)$. This shows that the set of eigenvectors of $f$ (this time including $0 = 0_V$) corresponding to a fixed eigenvalue $\lambda$ form a subspace $V_\lambda$ of $V$, the so-called *eigenspace* of $f$ corresponding to $\lambda$. If $\lambda$ is not an eigenvalue of $f$ then $V_\lambda = \{0\}$.

# Notes cont'd

- The condition $f(v) = 0\,v = 0_V$ is equivalent to $v \in \ker f$. Hence $f$ has the eigenvalue $\lambda = 0$ iff $\ker f \neq \{0_V\}$ (i.e., $f$ is not a bijection), and in general we have that the eigenspace $V_0$ corresponding to $\lambda = 0$ is just the kernel of $f$.
- The definition of eigenvectors and eigenvalues is extended to square matrices by calling $\mathbf{x} \in F^n \setminus \{\mathbf{0}\}$ a (right) eigenvector of $\mathbf{A} \in F^{n\times n}$ with corresponding eigenvalue $\lambda$ if $\boxed{\mathbf{A}\mathbf{x} = \lambda\mathbf{x}}$ (i.e., $f_{\mathbf{A}}(\mathbf{x}) = \lambda\mathbf{x}$). In sync with the previous note, we have $V_0 = \{\mathbf{x} \in F^n; \mathbf{A}\mathbf{x} = \mathbf{0}\} = \operatorname{rker}(\mathbf{A})$ in this case. (Left eigenvectors of $\mathbf{A}$ would be defined by the condition $\mathbf{x}^{\mathsf{T}}\mathbf{A} = \lambda\mathbf{x}^{\mathsf{T}}$, or $\mathbf{A}^{\mathsf{T}}\mathbf{x} = \lambda\mathbf{x}$.)
- The eigenvalues of $f$ and any matrix ${}_B(f)_B$ representing $f$ are the same. More precisely, $v \in V$ is an eigenvector of $f$ corresponding to the eigenvalue $\lambda$ iff $\phi_B(v)$ is an eigenvector of ${}_B(f)_B$ corresponding to $\lambda$. This is immediate from $\phi_B\bigl(f(v)\bigr) = \mathbf{A}\phi_B(v)$ and shows, more generally, that the coordinate map $\phi_B$ maps every eigenspace $V_\lambda$ isomorphically onto the corresponding eigenspace $\{\mathbf{x} \in F^n; \mathbf{A}\mathbf{x} = \lambda\mathbf{x}\}$ of $\mathbf{A}$. (For the definition of ${}_B(f)_B$ see the subsection "Matrices of Linear Maps" in `lecture42-45`.)

# Examples

1. The differentation operator $\mathrm{D}\colon V \to V$ (with $V/\mathbb{C}$ denoting the vector space of complex-valued $\mathrm{C}^\infty$-functions on $\mathbb{R}$) satisfies $\mathrm{D}(\mathrm{e}^{\lambda t}) = \lambda\,\mathrm{e}^{\lambda t}$ for any $\lambda \in \mathbb{C}$.
   $\Longrightarrow$ All complex numbers are eigenvalues of $\mathrm{D}$.

   The eigenspace $V_\lambda$ of $\mathrm{D}$ corresponding to the eigenvalue $\lambda \in \mathbb{C}$ is the solution space of the ODE $\mathrm{D}y = y' = \lambda y$. We know well that the general solution of $y' = \lambda y$ is $y(t) = y(0)\mathrm{e}^{\lambda t}$.
   $\Longrightarrow$ $V_\lambda = \langle e^{\lambda t}\rangle$ is 1-dimensional.

2. The matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ has an obvious eigenvector,, viz. $\mathbf{x} = (1,1)^\mathsf{T}$ with corresponding eigenvalue $\lambda = 3$:

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+2 \\ 2+1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

   *Note:* More generally, if $\mathbf{A} \in F^{n\times n}$ has all row sums equal to a fixed constant $c \in F$ then $c$ is an eigenvalue of $\mathbf{A}$ and $(1,1,\dots,1)^\mathsf{T} \in F^n$ a corresponding eigenvector.

# Example (cont'd)

Are there further eigenvalues/eigenvectors of $\mathbf{A}$ ?

In order to answer this question, we have to solve

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

$$\left[\begin{array}{cc|c} 1-\lambda & 2 & 0 \\ 2 & 1-\lambda & 0 \end{array}\right] \xrightarrow[R1=R2]{R2=R1-\frac{1-\lambda}{2}R2} \left[\begin{array}{cc|c} 2 & 1-\lambda & 0 \\ 0 & -\frac{1}{2}(\lambda^2-2\lambda-3) & 0 \end{array}\right]$$

The matrix on the right-hand side has rank 1 if $\lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1) = 0$, and rank 2 otherwise.
$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are 3 and $-1$.

For $\lambda = 3$ the solution is $\mathbb{R}(1,1)^{\mathsf{T}}$ and we recover the previously determined eigenvector $\mathbf{v}_1 = (1,1)^{\mathsf{T}}$ as essentially unique solution.
For $\lambda = -1$ the solution is $\mathbb{R}(1,-1)^{\mathsf{T}}$ and a corresponding basis vector is $\mathbf{v}_2 = (1,-1)^{\mathsf{T}}$. $\Longrightarrow$ $\mathbf{A}$ is diagonalized by $\mathbf{S} = (\mathbf{v}_1|\mathbf{v}_2) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}.$$

# Example (cont'd)

Keep in mind that the preceding equation is equivalent to

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

In this form it says that multiplication by $\mathbf{A}$ scales the columns of $\mathbf{S} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$ by 3 and $-1$, respectively. This is exactly what $f_{\mathbf{A}}$ is supposed to do!

Since $\mathbf{A}$ is diagonalizable, the theory of 1st-order linear ODE systems developed so far applies and tells us that the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$y(t) = c_1 \mathrm{e}^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{C},$$

and that the canonical fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \exp\left[\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} t\right] = \begin{pmatrix} \mathrm{e}^{3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{3t} & -\mathrm{e}^{-t} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \mathrm{e}^{3t} & \mathrm{e}^{-t} \\ \mathrm{e}^{3t} & -\mathrm{e}^{-t} \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathrm{e}^{3t} + \mathrm{e}^{-t} & \mathrm{e}^{3t} - \mathrm{e}^{-t} \\ \mathrm{e}^{3t} - \mathrm{e}^{-t} & \mathrm{e}^{3t} + \mathrm{e}^{-t} \end{pmatrix}. \end{aligned}$$

## Example (Rotation and reflection matrices)

We determine the eigenvalues and eigenvectors of

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \in \mathbb{R}^{2\times 2}.$$

Since lines through $(0,0)^\mathsf{T}$ spanned by eigenvectors are mapped to itself, $R(\phi)$ has real eigenvalues/eigenvectors only for $\phi \in \{0, \pi\}$, in which case $R(\phi) = \pm\mathbf{I}_2$ and all vectors in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ are eigenvectors (with eigenvalue $\pm 1$).

The geometric interpretation gives similarly that $S(\phi)$ has eigenvalues 1 and $-1$ with corresponding eigenvectors $\mathbf{v}_1 = \big(\cos(\phi/2), \sin(\phi/2)\big)^\mathsf{T}$ (spanning the reflecting line), respectively, $\mathbf{v}_2 = \big(-\sin(\phi/2), \cos(\phi/2)\big)^\mathsf{T}$ (spanning the line orthogonal to the reflecting line).

These properties can also be verified through direct computation.

As a by-product we obtain the matrix identity

$$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix} \begin{pmatrix} \cos(\frac{\phi}{2}) & -\sin(\frac{\phi}{2}) \\ \sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) \end{pmatrix} = \begin{pmatrix} \cos(\frac{\phi}{2}) & -\sin(\frac{\phi}{2}) \\ \sin(\frac{\phi}{2}) & \cos(\frac{\phi}{2}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which reflects the addition theorems for cos, sin.

# Example (cont'd)

The characteristic polynomials of $R(\phi)$, $S(\phi)$ are

$$\chi_{R(\phi)}(X) = X^2 - 2\cos\phi\, X + 1 = \left(X - \mathrm{e}^{\mathrm{i}\phi}\right)\left(X - \mathrm{e}^{-\mathrm{i}\phi}\right),$$
$$\chi_{S(\phi)}(X) = X^2 - 1 = (X-1)(X+1),$$

in accordance with the previous considerations.

In addition we see that the complex eigenvalues of $R(\phi)$, $\phi \notin \{0,\pi\}$, are $\lambda_1 = \mathrm{e}^{\mathrm{i}\phi}$, $\lambda_2 = \mathrm{e}^{-\mathrm{i}\phi}$. Corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ are determined from

$$R(\phi) - \mathrm{e}^{\mathrm{i}\phi}\,\mathbf{I}_2 = \begin{pmatrix} \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}\right) & \frac{\mathrm{i}}{2}\left(\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}\right) \\ -\frac{\mathrm{i}}{2}\left(\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}\right) & \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\phi} - \mathrm{e}^{-\mathrm{i}\phi}\right) \end{pmatrix},$$

which has kernel $\mathbb{C}(1,-\mathrm{i})^{\mathsf{T}}$, i.e., we can take $\mathbf{v}_1 = (1,-\mathrm{i})^{\mathsf{T}}$ and $\mathbf{v}_2 = (1,\mathrm{i})^{\mathsf{T}}$. (Note that for a real matrix $\mathbf{A}$ the equation $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ implies $\mathbf{A}\overline{\mathbf{v}} = \overline{\mathbf{A}}\overline{\mathbf{v}} = \overline{\mathbf{A}\mathbf{v}} = \overline{\lambda\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$, so that $\overline{\mathbf{v}}$ is an eigenvector of $\mathbf{A}$ for the eigenvalue $\overline{\lambda}$.) The corresponding "diagonalizing equation" is then

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi} & \\ & \mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix}.$$

# Example

*Problem:* Determine the number $s(n)$ of bit strings of length $n$ with Hamming weight (i.e., number of 1's) divisible by 3.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $s(n)$ | 1 | 1 | 1 | 2 | 5 | 11 | 22 | 43 | 85 | 170 | 341 |

*Solution*: We write $s_j(n) = \left|\left\{\mathbf{x} \in \mathbb{F}_2^n; \mathrm{w}_{\mathrm{Ham}}(\mathbf{x}) \equiv j \bmod 3\right\}\right|$ for $j = 0, 1, 2$, so that $s(n) = s_0(n)$.

The quantities $s_j(n)$ satisfy the initial conditions $s_0(0) = 1$, $s_1(0) = s_2(0) = 0$, and the coupled system of recurrence relations

$$\begin{pmatrix} s_0(n+1) \\ s_1(n+1) \\ s_2(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} s_0(n) \\ s_1(n) \\ s_2(n) \end{pmatrix}.$$

This is easily proved by induction.

Clearly it follows that

$$\begin{pmatrix} s_0(n) \\ s_1(n) \\ s_2(n) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} s_0(0) \\ s_1(0) \\ s_2(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

## Example (cont'd)

... but this does not yet solve the problem in a satisfactory way. Having a closed formula for $s_j(n)$ would be much better!

In terms of binomial coefficients, we have $s_0(n) = \sum_{k \leq \lfloor n/3 \rfloor} \binom{n}{3k}$ and similarly for $s_1(n)$, $s_2(n)$, but this isn't explicit either.

In vector form the solution reads $\mathbf{s}(n) = \mathbf{A}^n \mathbf{s}(0)$ with a certain matrix $\mathbf{A} \in \mathbb{C}^{3\times 3}$. Let us assume for the moment that we have found 3 linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbf{A}$ with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Then $\mathbf{s}(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$ for some $c_1, c_2, c_3 \in \mathbb{C}$ and

$$\begin{aligned}
\mathbf{s}(n) &= \mathbf{A}^n(c_1\mathbf{v}_1) + \mathbf{A}^n(c_2\mathbf{v}_2) + \mathbf{A}^n(c_3\mathbf{v}_3) \\
&= \mathbf{A}^{n-1}(\lambda_1 c_1\mathbf{v}_1) + \mathbf{A}^{n-1}(\lambda_2 c_2\mathbf{v}_2) + \mathbf{A}^{n-1}(\lambda_3 c_3\mathbf{v}_3) \\
&\ \,\vdots \\
&= \lambda_1^n(c_1\mathbf{v}_1) + \lambda_2^n(c_2\mathbf{v}_2) + \lambda_3^n(c_3\mathbf{v}_3) \quad \text{for } n \in \mathbb{N},
\end{aligned}$$

a 3-dimensional analogue of the solution formula for linear recurrence relations.

## Example (cont'd)

In order to determine $\lambda_j$ and $\mathbf{v}_j$, we exploit the *circulant* structure of $\mathbf{A}$, i.e., the special form $\mathbf{A} = \begin{pmatrix} a_0 & a_1 & a_2 \\ a_2 & a_0 & a_1 \\ a_1 & a_2 & a_0 \end{pmatrix}$ with $(a_0, a_1, a_2) = (1, 0, 1)$.
If $\mathbf{A} \in \mathbb{C}^n$ is circulant with first row $(a_0, a_1, \dots, a_{n-1})$, it can be checked that for any $n$-th root of unity $\omega \in \mathbb{C}$ the vector $(1, \omega, \omega^2, \dots, \omega^{n-1})^\mathsf{T}$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue $a_0 + a_1\omega + \dots + a_{n-1}\omega^{n-1}$.
Here we have for $\omega \in \{1, \mathrm{e}^{2\pi\mathrm{i}/3}, \mathrm{e}^{4\pi\mathrm{i}/3}\}$ the identity

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} = \begin{pmatrix} 1+\omega^2 \\ 1+\omega \\ \omega+\omega^2 \end{pmatrix} = (1+\omega^2) \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix}.$$

Switching notation to $\omega = \mathrm{e}^{2\pi\mathrm{i}/3}$ (so that $\omega^2 = \mathrm{e}^{4\pi\mathrm{i}/3} = \overline{\omega}$) and writing $\mathbf{v}_1 = (1,1,1)^\mathsf{T}$, $\mathbf{v}_2 = (1, \omega, \omega^2)^\mathsf{T}$, $\mathbf{v}_3 = (1, \omega^2, \omega)^\mathsf{T}$, we have $(1,0,0)^\mathsf{T} = \frac{1}{3}(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)$ and hence, using $1 + \omega^2 = -\omega$,

$$\begin{pmatrix} s_0(n) \\ s_1(n) \\ s_2(n) \end{pmatrix} = \frac{1}{3}\left[ 2^n \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + (-\omega)^n \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} + (-\omega^2)^n \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix} \right].$$

## Example (cont'd)

$$\begin{aligned}\implies s_0(n) &= \frac{1}{3}\left[2^n + 2\,\operatorname{Re}(\mathrm{e}^{n\pi\mathrm{i}/3})\right]\\ &= \begin{cases} \frac{1}{3}(2^n+2) & \text{if } n \equiv 0 \pmod 6,\\ \frac{1}{3}(2^n-2) & \text{if } n \equiv 3 \pmod 6,\\ \frac{1}{3}(2^n-1) & \text{if } n \equiv 2,4 \pmod 6,\\ \frac{1}{3}(2^n+1) & \text{if } n \equiv 1,5 \pmod 6.\end{cases}\end{aligned}$$

In the first line we have used that $-\omega = \mathrm{e}^{5\pi\mathrm{i}/3}$ and $-\omega^2 = \mathrm{e}^{\pi\mathrm{i}/3}$ form a complex conjugate pair and hence their sum is twice their real part; the same is then true of $(-\omega)^n$ and $(-\omega^2)^n$.

Perhaps more important than the exact formula is the observation that $|s_0(n) - 2^n/3| < 1$ for all $n$. This holds for $s_1(n)$ and $s_2(n)$ as well, so that the relative frequencies $s_j(n)/2^n$, $j = 0, 1, 2$, are very close to $1/3$.

## Example

Now consider the linear 1st-order ODE system

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

which is the continuous analogue of the linear recurrence relation considered in the previous example.

Since we have already determined a basis of $\mathbb{C}^3$ consisting of eigenvectors of $\mathbf{A}$, we can write down a fundamental system of solutions immediately. But let us recall why this is so.

The continuous analogue of the sequence $(\lambda^n)_{n\in\mathbb{N}}$ is the function $t \mapsto \mathrm{e}^{\lambda t}$. If $\mathbf{v} = (v_1, v_2, v_3)^\mathsf{T} \in \mathbb{C}^3$ is an eigenvector of $\mathbf{A}$ with corresponding eigenvalue $\lambda$ then $\mathbf{y}(t) = \mathrm{e}^{\lambda t}\mathbf{v}$ satisfies

$$\mathbf{y}'(t) = \begin{pmatrix} \mathrm{e}^{\lambda t} v_1 \\ \mathrm{e}^{\lambda t} v_2 \\ \mathrm{e}^{\lambda t} v_3 \end{pmatrix}' = \begin{pmatrix} \lambda\mathrm{e}^{\lambda t} v_1 \\ \lambda\mathrm{e}^{\lambda t} v_2 \\ \lambda\mathrm{e}^{\lambda t} v_3 \end{pmatrix} = \lambda\,\mathbf{y}(t) = \mathbf{A}\,\mathbf{y}(t),$$

because $\mathrm{e}^{\lambda t}\mathbf{v} \in V_\lambda$ as well.

## Example (cont'd)

Inspecting the previous example, we obtain the general solution of the given system as

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ y_3'(t) \end{pmatrix} = c_1 \mathrm{e}^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-\omega t} \begin{pmatrix} 1 \\ \omega \\ \omega^2 \end{pmatrix} + c_3 \mathrm{e}^{-\omega^2 t} \begin{pmatrix} 1 \\ \omega^2 \\ \omega \end{pmatrix}$$

with $c_1, c_2, c_3 \in \mathbb{C}$, where $\omega = \mathrm{e}^{2\pi \mathrm{i}/3} = \frac{-1+\mathrm{i}\sqrt{3}}{2}$.

A fundamental matrix is

$$\Phi(t) = \left(\mathrm{e}^{2t}\mathbf{v}_1 | \mathrm{e}^{-\omega t}\mathbf{v}_2 | \mathrm{e}^{-\omega^2 t}\mathbf{v}_3\right) = \begin{pmatrix} \mathrm{e}^{2t} & \mathrm{e}^{-\omega t} & \mathrm{e}^{-\omega^2 t} \\ \mathrm{e}^{2t} & \omega \mathrm{e}^{-\omega t} & \omega^2 \mathrm{e}^{-\omega^2 t} \\ \mathrm{e}^{2t} & \omega^2 \mathrm{e}^{-\omega t} & \omega \mathrm{e}^{-\omega^2 t} \end{pmatrix},$$

and the canonical fundamental matrix is

$$\mathrm{e}^{\mathbf{A}t} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} \mathrm{e}^{2t} & & \\ & \mathrm{e}^{-\omega t} & \\ & & \mathrm{e}^{-\omega^2 t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}^{-1}.$$

# Example (cont'd)

The matrix

$$\mathbf{S} = \left(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3\right) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} = \left(\mathbf{v}_1|\mathbf{v}_2|\overline{\mathbf{v}}_2\right)$$

satisfies $\mathbf{S}\overline{\mathbf{S}} = 3\,\mathbf{I}_3$ (check it!), and hence (using $\omega^2 = \omega^{-1} = \overline{\omega}$)

$$\mathbf{S}^{-1} = \frac{1}{3}\overline{\mathbf{S}} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \mathbf{v}_1^{\mathsf{T}} \\ \overline{\mathbf{v}}_2^{\mathsf{T}} \\ \mathbf{v}_2^{\mathsf{T}} \end{pmatrix}.$$

This gives

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} \mathrm{e}^{2t} & & \\ & \mathrm{e}^{-\omega t} & \\ & & \mathrm{e}^{-\omega^2 t} \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \\ &= \frac{1}{3}\left(\mathrm{e}^{2t}\,\mathbf{v}_1\mathbf{v}_1^{\mathsf{T}} + \mathrm{e}^{-\omega t}\,\mathbf{v}_2\overline{\mathbf{v}}_2^{\mathsf{T}} + \mathrm{e}^{-\omega^2 t}\,\overline{\mathbf{v}}_2\mathbf{v}_2^{\mathsf{T}}\right) \\ &= \frac{1}{3}\left[\mathrm{e}^{2t}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \mathrm{e}^{-\omega t}\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega & 1 & \omega^2 \\ \omega^2 & \omega & 1 \end{pmatrix} + \mathrm{e}^{-\omega^2 t}\begin{pmatrix} 1 & \omega & \omega^2 \\ \omega^2 & 1 & \omega \\ \omega & \omega^2 & 1 \end{pmatrix}\right]. \end{aligned}$$

## Example (cont'd)

Alternative representations are

$$\begin{aligned}
e^{\mathbf{A}t} &= \frac{1}{3}\left(e^{2t}+e^{-\omega t}+e^{-\omega^2 t}\right)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\frac{1}{3}\left(e^{2t}+\omega e^{-\omega t}+\omega^2 e^{-\omega^2 t}\right)\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}+\frac{1}{3}\left(e^{2t}+\omega^2 e^{-\omega t}+\omega e^{-\omega^2 t}\right)\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\\
&= \frac{1}{3}\left(e^{2t}-2\omega e^{-\omega t}-2\omega^2 e^{-\omega^2 t}\right)\mathbf{I}_3+\frac{1}{3}\left(-e^{2t}+(2+3\omega)e^{-\omega t}+(2+3\omega^2)e^{-\omega^2 t}\right)\mathbf{A}+\frac{1}{3}\left(e^{2t}+\omega^2 e^{-\omega t}+\omega e^{-\omega^2 t}\right)\mathbf{A}^2.
\end{aligned}$$

Finally, note that the matrix $\mathbf{S}$ simultaneously diagonalizes $\mathbf{A}$ and $e^{\mathbf{A}t}$. So we also have

$$\begin{aligned}
\mathbf{A} = \begin{pmatrix}1&0&1\\1&1&0\\0&1&1\end{pmatrix} &= \mathbf{S}\begin{pmatrix}2&&\\&-\omega&\\&&-\omega^2\end{pmatrix}\mathbf{S}^{-1}\\
&= \frac{1}{3}\left[2\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}-\omega\begin{pmatrix}1&\omega^2&\omega\\\omega&1&\omega^2\\\omega^2&\omega&1\end{pmatrix}-\omega^2\begin{pmatrix}1&\omega&\omega^2\\\omega^2&1&\omega\\\omega&\omega^2&1\end{pmatrix}\right].
\end{aligned}$$

## Exercise

1. Using the notation of the previous example, show that $\mathbf{S}$ diagonalizes $\mathbf{A}^k$ for every integer $k$.
2. Determine all matrices in $\mathbb{C}^{3\times 3}$ diagonalized by $\mathbf{S}$.

## Exercise

Suppose that $F$ is a field and $x_1, \dots, x_n \in F$ are distinct. Show that the matrix

$$\begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix} \in F^{n\times n}$$

has full rank $n$.

*Hint:* Relate this matrix to the Lagrange polynomials for $x_1, \dots, x_n$. (The Lagrange polynomials make sense for any field $F$, provided that $n$ distinct elements in $F$ can be found, i.e., $|F| \geq n$.)

## Definition (characteristic polynomial)

1. The *characteristic polynomial* of a square matrix $\mathbf{A} \in F^{n\times n}$ is defined as the polynomial $\chi_{\mathbf{A}}(X) = \det(X\,\mathbf{I}_n - \mathbf{A}) \in F[X]$.
2. The *characteristic polynomial* of an endomorphism $f\colon V \to V$ of a finite-dimensional vector space $V/F$ is defined as the characteristic polynomial of any matrix ${}_B(f)_B$ representing $f$.

## Notes

1. In terms of $\mathbf{A} = (a_{ij})$ we have

$$\chi_{\mathbf{A}}(X) = \begin{vmatrix} X - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & X - a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_{n-1,n} \\ -a_{n1} & \dots & -a_{n,n-1} & X - a_{nn} \end{vmatrix}$$
$$= X^n - (a_{11} + a_{22} + \dots + a_{nn})X^{n-1} + \dots + (-1)^n \det \mathbf{A},$$

as follows, e.g., from Leibniz's formula; cf. the subsection on determinants in `lecture42-45`.

# Notes cont'd

1. The quantity $a_{11} + a_{22} + \cdots + a_{nn}$ (sum of the entries of $\mathbf{A}$ on the main diagonal) is called *trace* of $\mathbf{A}$ and denoted by $\mathrm{tr}(\mathbf{A})$. Thus we have

$$\chi_{\mathbf{A}}(X) = X^n - \mathrm{tr}(\mathbf{A})X^{n-1} + c_{n-2}X^{n-2} + \cdots + c_1 X + (-1)^n \det(\mathbf{A})$$

for certain $c_i \in F$.

2. $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ implies $X\,\mathbf{I}_n - \mathbf{B} = \mathbf{S}^{-1}(X\,\mathbf{I}_n - \mathbf{A})\mathbf{S}$ and further $\chi_{\mathbf{B}}(X) = \chi_{\mathbf{A}}(X)$ (exercise). Thus similar matrices have the same characteristic polynomial.
In particular the characteristic polynomial of an endomorphism, whose representing matrices form a similarity class, is well-defined.

3. $\lambda \in F$ is an eigenvalue of $\mathbf{A}$ iff $\chi_{\mathbf{A}}(\lambda) = 0$.
*Reason:* $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is equivalent to $(\mathbf{A} - \lambda\,\mathbf{I}_n)\mathbf{x} = \mathbf{0}$.
This homogeneous linear system has a nontrivial solution iff $\mathbf{A} - \lambda\,\mathbf{I}_n$ is singular iff $\det(\mathbf{A} - \lambda\,\mathbf{I}_n) = 0$.
The analogous statement holds of course for the characteristic polynomial $\chi_f(X)$ of an endomorphism $f\colon V \to V$, $\dim V < \infty$.

# Notes cont'd

4. If $\lambda_1, \dots, \lambda_n \in F$ are the eigenvalues of $\mathbf{A}$ counted with their multiplicities as roots of $\chi_{\mathbf{A}}(X)$ (so-called *algebraic multiplicities*), we have

$$\begin{aligned} \operatorname{tr}(\mathbf{A}) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n, \\ \det(\mathbf{A}) &= \lambda_1 \lambda_2 \cdots \lambda_n. \end{aligned}$$

This follows by expanding

$$\begin{aligned} \chi_{\mathbf{A}}(X) &= (X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_n) \\ &= X^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) X^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \end{aligned}$$

and comparing with the expression for $\chi_{\mathbf{A}}(X)$ in Note 1.

Here we assume that $\chi_{\mathbf{A}}(X)$ splits into linear factors over $F$ (i.e., in the polynomial ring $F[X]$). If this is not the case, we must replace $F$ by an extension field $E$, over which $\chi_{\mathbf{A}}(X)$ splits into linear factors. Such an extension field always exists. In the case $F = \mathbb{R}$ we can take $E = \mathbb{C}$ (by the Fundamental Theorem of Algebra).

## Example

For the previously considered matrix $\mathbf{A} = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ we have

$$\chi_{\mathbf{A}}(X) = X^2 - (1+1)X + 1^2 - 2^2 = X^2 - 2X - 3 = (X-3)(X+1).$$

$\Longrightarrow$ The eigenvalues of $\mathbf{A}$ are $\lambda_1 = 3$, $\lambda_2 = -1$ (as we already know, of course).

The characteristic polynomial of a "generic" $2 \times 2$ matrix $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is $\chi_{\mathbf{A}}(X) = X^2 - (a+d)X + ad - bc$.

## Example

We determine the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & -2 & -16 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

$$\chi_{\mathbf{A}}(X) = \begin{vmatrix} X-2 & 2 & 16 \\ 0 & X-1 & -6 \\ 0 & 0 & X+2 \end{vmatrix} = (X-2)(X-1)(X+2) = X^3 - X^2 - 4X + 4.$$

$\Longrightarrow \lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -2$.

## Example (cont'd)

Note the general fact that the eigenvalues of an upper-triangular (or lower-triangular) matrix are the entries on the main diagonal.

Now we determine the corresponding eigenspaces:

$\underline{\lambda_1 = 2:}$

$$\mathbf{A} - 2\,\mathbf{I} = \begin{pmatrix} 0 & -2 & -16 \\ 0 & -1 & 6 \\ 0 & 0 & -4 \end{pmatrix}$$

This matrix has rank 2 and right kernel $\mathbb{R}(1,0,0)^{\mathsf{T}}$.

$\underline{\lambda_2 = 1:}$

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 1 & -2 & -16 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{pmatrix}$$

This matrix has rank 2 and right kernel $\mathbb{R}(2,1,0)^{\mathsf{T}}$.

$\underline{\lambda_3 = -2:}$

$$\mathbf{A} + 2\,\mathbf{I} = \begin{pmatrix} 4 & -2 & -16 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 2 & -1 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has rank 2 and right kernel $\mathbb{R}(-3,2,-1)^{\mathsf{T}}$.

## Example (cont'd)

We see that the three generators $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis of $\mathbb{R}^3$, which has therefore a basis consisting of eigenvectors of $\mathbf{A}$.

$$\Longrightarrow \quad \mathbf{S} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{diagonalizes } \mathbf{A}$$

The corresponding matrix equation is

$$\underbrace{\begin{pmatrix} 2 & -2 & -16 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}}_{\mathbf{S}} = \underbrace{\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}}_{\mathbf{B}}.$$

A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\mathbf{y}_1(t) = \mathrm{e}^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{t} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = \mathrm{e}^{-2t} \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}.$$

## Example

We repeat the preceding example for the matrix

$$\mathbf{A} = \begin{pmatrix} -26 & 49 & 74 \\ -8 & 16 & 25 \\ -4 & 7 & 10 \end{pmatrix}.$$

$$\begin{aligned}
\chi_{\mathbf{A}}(X) &= \begin{vmatrix} X+26 & -49 & -74 \\ 8 & X-16 & -25 \\ 4 & -7 & X-10 \end{vmatrix} = \begin{vmatrix} X-2 & 0 & -7X-4 \\ 0 & X-2 & -2X-5 \\ 4 & -7 & X-10 \end{vmatrix} \\
&= (X-2)^2(X-10) + 4(X-2)(7X+4) - 7(X-2)(2X+5) \\
&= X^3 - 3X - 2 \\
&= (X-2)(X+1)^2.
\end{aligned}$$

Then, as before we compute the corresponding eigenspaces.

# Example (cont'd)

$\Longrightarrow \lambda_1 = 2$, $\lambda_2 = -1$ (with multiplicity 2).

$\underline{\lambda_1 = 2:}$

$$\mathbf{A} - 2\,\mathbf{I} = \begin{pmatrix} -28 & 49 & 74 \\ -8 & 14 & 25 \\ -4 & 7 & 8 \end{pmatrix} \to \begin{pmatrix} 4 & -7 & -8 \\ 0 & 0 & 9 \\ 0 & 0 & 18 \end{pmatrix}$$

This matrix has rank 2 and right kernel $\mathbb{R}(7,4,0)^{\mathsf{T}}$.

$\underline{\lambda_2 = -1:}$

$$\mathbf{A}+\mathbf{I} = \begin{pmatrix} -25 & 49 & 74 \\ -8 & 17 & 25 \\ -4 & 7 & 11 \end{pmatrix} \to \begin{pmatrix} 4 & -7 & -11 \\ 0 & 3 & 3 \\ 3 & 0 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix has rank 2 and right kernel $\mathbb{R}(-1,1,-1)^{\mathsf{T}}$.

$\Longrightarrow$ The eigenvectors of $\mathbf{A}$ span only a 2-dimensional subspace of $\mathbb{R}^3$, and hence $\mathbf{A}$ is not diagonalizable.

The best we can do in this case is to take as a basis of $\mathbb{R}^3$ the two eigenvectors $\mathbf{v}_1 = (7,4,0)^{\mathsf{T}}$, $\mathbf{v}_2 = (-1,1,-1)^{\mathsf{T}}$, and a third vector $\mathbf{v}_3$ solving $(\mathbf{A}+\mathbf{I})\mathbf{v}_3 = \mathbf{v}_2$, and hence $(\mathbf{A}+\mathbf{I})^2\mathbf{v}_3 = \mathbf{0}$.

# Example (cont'd)

$$\left(\begin{array}{ccc|c} -25 & 49 & 74 & -1 \\ -8 & 17 & 25 & 1 \\ -4 & 7 & 11 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 4 & -7 & -11 & 1 \\ 0 & 3 & 3 & 3 \\ 3 & 0 & -3 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A solution is $\mathbf{v}_3 = (2, 1, 0)^{\mathsf{T}}$.

By definition of $\mathbf{v}_3$ we have $\mathbf{A}\mathbf{v}_3 = -\mathbf{v}_3 + \mathbf{v}_2$ and hence for

$$\mathbf{S} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

the matrix equation

$$\underbrace{\begin{pmatrix} -26 & 49 & 74 \\ -8 & 16 & 25 \\ -4 & 7 & 10 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)} = \underbrace{\begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{(\mathbf{v}_1|\mathbf{v}_2|\mathbf{v}_3)} \underbrace{\left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right)}_{\mathbf{B}}$$

$\mathbf{v}_3$ is called a *generalized eigenvector* of $\mathbf{A}$ and the block diagonal matrix on the right is the so-called *Jordan canonical form* of $\mathbf{A}$.

# Example (cont'd)

As before, we obtain from the two eigenvectors two fundamental solutions of $\mathbf{y}' = \mathbf{Ay}$:

$$\mathbf{y}_1(t) = \mathrm{e}^{2t}\mathbf{v}_1 = \mathrm{e}^{2t}\begin{pmatrix}7\\4\\0\end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-t}\mathbf{v}_2 = \mathrm{e}^{-t}\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

A third fundamental solution (linearly independent of the first two) is

$$\mathbf{y}_3(t) = \mathrm{e}^{-t}\mathbf{v}_3 + t\,\mathrm{e}^{-t}\mathbf{v}_2 = \mathrm{e}^{-t}\begin{pmatrix}2\\1\\0\end{pmatrix} + t\,\mathrm{e}^{-t}\begin{pmatrix}-1\\1\\-1\end{pmatrix}.$$

For now this can be checked directly:

$$\begin{aligned}\mathbf{y}_3'(t) &= -\mathrm{e}^{-t}\mathbf{v}_3 + (1-t)\mathrm{e}^{-t}\mathbf{v}_2 = \mathrm{e}^{-t}(-\mathbf{v}_3 + \mathbf{v}_2) - t\mathrm{e}^{-t}\mathbf{v}_2,\\ \mathbf{Ay}_3(t) &= \mathrm{e}^{-t}\mathbf{Av}_3 + t\,\mathrm{e}^{-t}\mathbf{Av}_2 = \mathrm{e}^{-t}(-\mathbf{v}_3 + \mathbf{v}_2) - t\mathrm{e}^{-t}\mathbf{v}_2.\end{aligned}$$

The canonical fundamental matrix is (observe that $\mathbf{y}_3(0) = \mathbf{v}_3$ still holds!)

$$\mathrm{e}^{\mathbf{A}t} = \big(\mathbf{y}_1(t)|\mathbf{y}_2(t)|\mathbf{y}_3(t)\big)\,\mathbf{S}^{-1} = \cdots$$

# The Link with Representing Matrices

In the previous example the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbb{C}^3$ together with the information how $f_\mathbf{A}$ acts on this basis, i.e.,

$$\mathbf{A}\mathbf{v}_1 = 2\,\mathbf{v}_1, \quad \mathbf{A}\mathbf{v}_2 = -\mathbf{v}_2, \quad \mathbf{A}\mathbf{v}_3 = -\mathbf{v}_3 + \mathbf{v}_2,$$

specifies the linear map $f_\mathbf{A} \colon \mathbb{C}^3 \to \mathbb{C}^3$, $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ completely.

In general, if $\mathbf{A} \in F^{n\times n}$ and $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an ordered basis of $F^n$, we can express the images $f_\mathbf{A}(\mathbf{v}_j)$ of the basis vectors in terms of the basis, i.e.,

$$f_\mathbf{A}(\mathbf{v}_j) = \mathbf{A}\mathbf{v}_j = \sum_{i=1}^n b_{ij}\mathbf{v}_i \quad \text{for some } b_{ij} \in F.$$

This is equivalent to the matrix equation

$$\mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{B} \quad \text{with } \mathbf{S} = \big(\mathbf{v}_1|\dots|\mathbf{v}_n\big) \text{ and } \mathbf{B} = (b_{ij}) \in F^{n\times n}$$

or, alternatively, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{B}$.

The matrix $\mathbf{B}$ is called the *matrix representing $f_\mathbf{A}$ w.r.t. the ordered basis $B$*; notation: $\mathbf{B} = {}_B(f_\mathbf{A})_B$. It satisfies $\phi_B(\mathbf{A}\mathbf{v}) = \mathbf{B}\phi_B(\mathbf{v})$ for all $\mathbf{v} \in F^n$; cf. also the more general discussion of matrices representing linear maps in `lecture42-45`.

## Example (Markov matrices)

A *Markov matrix* is a square matrix whose entries are non-negative real numbers and which has all column sums equal to 1. Markov matrices are used to represent so-called *Markov chains*. An example is

$$\mathbf{A} = \begin{pmatrix} 0.75 & 0.50 \\ 0.25 & 0.50 \end{pmatrix},$$

which arises from the following
*Problem:* For a certain city (not necessarily Haining) it is known that sunny days are followed on average by 50 % sunny and 50 % rainy days; days following a rainy day have only a 25 % change to be sunny (and a 75 % chance to be likewise rainy). Determine the percentage of sunny days observed in the long run.

We can model this problem by a system with two states 'r' (for "rainy day") and 's' (for "sunny day") and transition probabilities $\mathrm{p}(\mathrm{s}|\mathrm{s}) = \mathrm{p}(\mathrm{r}|\mathrm{s}) = 0.5$, $\mathrm{p}(\mathrm{r}|\mathrm{r}) = 0.75$, $\mathrm{p}(\mathrm{s}|\mathrm{r}) = 0.25$ and arrange these into the matrix above:

$$\mathbf{A} = \begin{pmatrix} \mathrm{p}(\mathrm{r}|\mathrm{r}) & \mathrm{p}(\mathrm{r}|\mathrm{s}) \\ \mathrm{p}(\mathrm{s}|\mathrm{r}) & \mathrm{p}(\mathrm{s}|\mathrm{s}) \end{pmatrix} = \begin{pmatrix} 0.75 & 0.50 \\ 0.25 & 0.50 \end{pmatrix}$$

## Example (cont'd)

Suppose today it is rainy with probability $\mathrm{p(r)}$ and sunny with probability $\mathrm{p(s)} = 1 - \mathrm{p(r)}$. Then, according to the laws of probability, the corresponding probabilities for tomorrow are

$$\begin{aligned} \mathrm{p}^{(1)}(\mathrm{r}) &= \mathrm{p(r|r)p(r)} + \mathrm{p(r|s)p(s)}, \\ \mathrm{p}^{(1)}(\mathrm{s}) &= \mathrm{p(s|r)p(r)} + \mathrm{p(s|s)p(s)} \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} \mathrm{p}^{(1)}(\mathrm{r}) \\ \mathrm{p}^{(1)}(\mathrm{s}) \end{pmatrix} = \begin{pmatrix} \mathrm{p(r|r)} & \mathrm{p(r|s)} \\ \mathrm{p(s|r)} & \mathrm{p(s|s)} \end{pmatrix} \begin{pmatrix} \mathrm{p(r)} \\ \mathrm{p(s)} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathrm{p(r)} \\ \mathrm{p(s)} \end{pmatrix}.$$

It follows that the corresponding probabilities for the $n$-th day of observation (starting with today as day zero) are

$$\begin{pmatrix} \mathrm{p}^{(n)}(\mathrm{r}) \\ \mathrm{p}^{(n)}(\mathrm{s}) \end{pmatrix} = \mathbf{A}^n \begin{pmatrix} \mathrm{p(r)} \\ \mathrm{p(s)} \end{pmatrix}.$$

Thus we need to compute the powers $\mathbf{A}^n$, $n = 1, 2, 3, \dots$

## Example (cont'd)

This computation is cheap for a diagonal matrix **D** (just raise the diagonal elements of **D** to the corresponding power), and it becomes cheap for any matrix **A** once we know how to diagonalize **A**:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D} \quad \Longrightarrow \quad \mathbf{A} = \mathbf{S}\mathbf{D}\mathbf{S}^{-1}$$

$$\begin{aligned} \Longrightarrow \quad \mathbf{A}^n &= \underbrace{(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})\cdots(\mathbf{S}\mathbf{D}\mathbf{S}^{-1})}_{n \text{ factors}} \\ &= \mathbf{S}\mathbf{D}(\mathbf{S}^{-1}\mathbf{S})\mathbf{D}(\mathbf{S}^{-1}\mathbf{S})\cdots\mathbf{D}\mathbf{S}^{-1} \\ &= \mathbf{S}\mathbf{D}^n\mathbf{S}^{-1} \end{aligned}$$

In the case under consideration we have $\mathbf{A} = \begin{pmatrix} a & b \\ 1-a & 1-b \end{pmatrix}$ for some $0 \leq a, b \leq 1$ and hence $(1,1)\mathbf{A} = (1,1)$, i.e., $(1,1)$ is a left eigenvector of **A** with corresponding eigenvalue 1.
$\Longrightarrow$ The eigenvalues of **A** are $\lambda_1 = 1$, $\lambda_2 = a - b$ (the latter, because $\lambda_1 + \lambda_2 = \mathrm{tr}(\mathbf{A}) = a + 1 - b$).

## Example (cont'd)

Next, as usual, we compute the eigenvectors of **A**. Since we have already one left eigenvector, it is more convenient to work entirely with left eigenvectors (but with a little more effort you can also do it in the standard "right-handed" way).

$$\mathbf{A} - (a-b)\mathbf{I} = \begin{pmatrix} b & b \\ 1-a & 1-a \end{pmatrix}$$

has rank 1 and left kernel $\mathbb{R}(a-1, b)$.

$\implies$ For $\mathbf{T} = \begin{pmatrix} 1 & 1 \\ a-1 & b \end{pmatrix}$ we have the matrix equation

$$\mathbf{TA} = \begin{pmatrix} 1 & \\ & a-b \end{pmatrix}\mathbf{T} \quad \text{or} \quad \mathbf{TAT}^{-1} = \begin{pmatrix} 1 & \\ & a-b \end{pmatrix},$$

which expresses the fact that the rows of **T** are left eigenvectors of **A** corresponding to the eigenvalues 1 and $a-b$.

$$\implies \mathbf{A} = \mathbf{T}^{-1}\begin{pmatrix} 1 & \\ & a-b \end{pmatrix}\mathbf{T} \quad \implies \mathbf{A}^n = \mathbf{T}^{-1}\begin{pmatrix} 1^n & \\ & (a-b)^n \end{pmatrix}\mathbf{T}$$

When taking the limit, we must distinguish the cases $|a-b| = 1$ and $|a-b| < 1$.

# Example (cont'd)

Case 1: $(a, b) = (1, 0)$ or $(0, 1)$

Here $\mathbf{A} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ (the system remains in the initial state forever) and $\mathbf{A} = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ (the system alternates between the two states), respectively. This case is rather uninteresting.

Case 2: $(a, b) \neq (1, 0)$ and $(0, 1)$

This case includes in particular the cases $0 < a < 1$ and $0 < b < 1$. Here we have $|a - b| < 1$ and hence

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{A}^n &= \mathbf{T}^{-1} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \mathbf{T} \\
&= \frac{1}{b + 1 - a} \begin{pmatrix} b & -1 \\ 1 - a & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ a - 1 & b \end{pmatrix} \\
&= \frac{1}{b + 1 - a} \begin{pmatrix} b & b \\ 1 - a & 1 - a \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\implies \lim_{n\to\infty} \begin{pmatrix} \mathrm{p}^{(n)}(r) \\ \mathrm{p}^{(n)}(s) \end{pmatrix} &= \lim_{n\to\infty} \mathbf{A}^n \begin{pmatrix} \mathrm{p}(r) \\ \mathrm{p}(s) \end{pmatrix} = \frac{1}{b + 1 - a} \begin{pmatrix} b & b \\ 1 - a & 1 - a \end{pmatrix} \begin{pmatrix} \mathrm{p}(r) \\ \mathrm{p}(s) \end{pmatrix} \\
&= \frac{1}{b + 1 - a} \begin{pmatrix} b \\ 1 - a \end{pmatrix}
\end{aligned}$$

## Example (cont'd)

Thus regardless of the initial state the system will converge to the stationary distribution

$$\left(\mathrm{p}^{(\infty)}(r), \mathrm{p}^{(\infty)}(s)\right) = \left(\frac{b}{b+1-a}, \frac{1-a}{b+1-a}\right).$$

In our case we have $b = 0.5$, $1 - a = 0.25$, and hence in the long run there will be about 66.6 % rainy days and 33.3 % sunny days.

## Notes

- In the example the stationary distribution $\mathbf{p}^{\infty} = (p_1^{\infty}, p_2^{\infty})^{\mathsf{T}}$ is the unique solution of $\mathbf{A}\mathbf{x} = \mathbf{x} \wedge x_1 + x_2 = 1$. (Although Banach's Fixed Point Theorem does not apply here, the proof of the fixed point property of the limit remains valid.) Thus we can also compute $\mathbf{p}^{\infty}$ by determining a right eigenvector of $\mathbf{A}$ for the eigenvalue 1 and normalizing.
- The PERRON-FROBENIUS Theorem implies that, more generally, every so-called irreducible and aperiodic finite Markov chain exhibits convergence to a unique stationary probability distribution for every choice of initial distribution.

# Afternote

Please note the recurring theme of computing the powers $\mathbf{A}^n$, $n = 1, 2, 3, \dots$, of a given square matrix $\mathbf{A}$ efficiently by diagonalizing it or, equivalently, compute its eigenvalues/eigenvectors. By now we have seen three different instances of this:

1. the example involving a Markov matrix;
2. the bit string example;
3. the example discussed in H79 of Homework 12, which can also be interpreted as a solution of the coupled system of recurrence relations

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \quad \text{i.e.,} \quad \begin{aligned} x_{n+1} &= 2x_n - y_n, \\ y_{n+1} &= 2y_n - x_n. \end{aligned}$$

# General Results on Diagonalizable Matrices

In the following $F$ denotes an arbitrary field and $n$ a positive integer.

## Theorem

*For $\mathbf{A} \in F^{n \times n}$ the following are equivalent:*

1. *$\mathbf{A}$ is diagonalizable, i.e., there exists an invertible matrix $\mathbf{S} \in F^{n \times n}$ such that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ is a diagonal matrix.*
2. *There exists a basis of $F^n$ consisting of eigenvectors of $\mathbf{A}$.*

## Proof.

(1)$\Longrightarrow$(2): Write $\mathbf{S} = (\mathbf{v}_1|\dots|\mathbf{v}_n)$. Then, since $\mathbf{S}$ is invertible, the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ form a basis of $F^n$, and

$$\begin{aligned}(\mathbf{A}\mathbf{v}_1|\dots|\mathbf{A}\mathbf{v}_n) = \mathbf{A}\mathbf{S} = \mathbf{S}\mathbf{D} &= (\mathbf{v}_1|\dots|\mathbf{v}_n)\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}\\ &= (\lambda_1\mathbf{v}_1|\dots|\lambda_n\mathbf{v}_n).\end{aligned}$$

$\Longrightarrow \mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$ for $1 \leq i \leq n$.

## Proof cont'd.

(2)$\Longrightarrow$(1): Call the basis vectors $\mathbf{v}_i$ and suppose $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Then $\mathbf{S} := (\mathbf{v}_1| \dots |\mathbf{v}_n)$ is invertible and satisfies $\mathbf{AS} = \mathbf{SD}$, where $\mathbf{D}$ is the diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$ (in this order); cf. the first part of the proof.

$\Longrightarrow$ $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$, and the proof is complete. □

The theorem has been proved already in Lecture 46 (albeit in an implicit manner), the only difference being that the diagonal matrix was then denoted by $\mathbf{B}$.

From the diagonalizability of $\mathbf{A}$ we can infer the diagonalizability of other matrices related to $\mathbf{A}$.

## Theorem

*Suppose $\mathbf{A} \in F^{n\times n}$ is diagonalizable.*

1. *$\mathbf{A}^\mathsf{T}$ is diagonalizable.*
2. *If $\mathbf{A}$ is invertible then $\mathbf{A}^{-1}$ is diagonalizable.*
3. *Any matrix polynomial in $\mathbf{A}$, i.e. $p(\mathbf{A}) = p_0\mathbf{I}_n + p_1\mathbf{A} + p_2\mathbf{A}^2 + \cdots + p_d\mathbf{A}^d$ with $d \geq 0$, $p_i \in F$, is diagonalizable.*

## Proof.

*(1) If $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ is diagonal then*

$$\mathbf{D} = \mathbf{D}^\mathsf{T} = (\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^\mathsf{T} = \mathbf{S}^\mathsf{T}\mathbf{A}^\mathsf{T}(\mathbf{S}^\mathsf{T})^{-1} = \mathbf{T}^{-1}\mathbf{A}^\mathsf{T}\mathbf{T}$$

*with $\mathbf{T} = (\mathbf{S}^\mathsf{T})^{-1} = (\mathbf{S}^{-1})^\mathsf{T}$. $\Longrightarrow$ $\mathbf{A}^\mathsf{T}$ is diagonalizable.*
*We also see from this that the eigenvalues of $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$ are the same, and that a basis of $F^n$ consisting of eigenvectors of $\mathbf{A}^\mathsf{T}$ is formed by the (transposed) rows of $\mathbf{S}^{-1}$. (Alternatively, $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ implies that the rows of $\mathbf{S}^{-1}$ are left eigenvectors of $\mathbf{A}$ and hence, after transposing, right eigenvectors of $\mathbf{A}^\mathsf{T}$.)*

## Proof cont'd.

(2) $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ implies $\mathbf{S}^{-1}\mathbf{A}^{-1}\mathbf{S} = (\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^{-1} = \mathbf{D}^{-1}$, which is also diagonal. More precisely the eigenvalues of $\mathbf{A}^{-1}$ are the reciprocals of the eigenvalues of $\mathbf{A}$ and the eigenvectors of $\mathbf{A}$ and $\mathbf{A}^{-1}$ are the same. (One can also see directly that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, $\mathbf{v} \neq \mathbf{0}$, $\lambda \neq 0$, implies $\mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$.)

(3) $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ implies

$$\begin{aligned}
\mathbf{S}^{-1}p(\mathbf{A})\mathbf{S} &= \mathbf{S}^{-1}\left(p_0\mathbf{I}_n + p_1\mathbf{A} + p_2\mathbf{A}^2 + \cdots + p_d\mathbf{A}^d\right)\mathbf{S} \\
&= p_0\mathbf{I}_n + p_1\mathbf{S}^{-1}\mathbf{A}\mathbf{S} + p_2\mathbf{S}^{-1}\mathbf{A}^2\mathbf{S} + \cdots + p_d\mathbf{S}^{-1}\mathbf{A}^d\mathbf{S} \\
&= p_0\mathbf{I}_n + p_1\mathbf{S}^{-1}\mathbf{A}\mathbf{S} + p_2(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^2 + \cdots + p_d(\mathbf{S}^{-1}\mathbf{A}\mathbf{S})^d \\
&= p(\mathbf{D}),
\end{aligned}$$

which is also a diagonal matrix. More precisely, if the eigenvalues of $\mathbf{A}$ (diagonal entries of $\mathbf{D}$) are $\lambda_1, \dots, \lambda_n$, then the eigenvalues of $p(\mathbf{A})$ (diagonal entries of $p(\mathbf{D})$) are $p(\lambda_1), \dots, p(\lambda_n)$, and the eigenvectors of $\mathbf{A}$ and $p(\mathbf{A})$ are the same. $\square$

# Notes on Matrix Polynomials

1. For a polynomial $p(X) = \sum_{i=0}^{d} p_i X^i \in F[X]$ and an $n \times n$ matrix $\mathbf{A} \in F^{n\times n}$ ($\mathbf{A}$ must be square!), the *matrix polynomial* $p(\mathbf{A}) = \sum_{i=0}^{d} p_i \mathbf{A}^i$ (with $\mathbf{A}^0 = \mathbf{I}_n$) is an $n \times n$ matrix as well.
2. For fixed $\mathbf{A}$ the collection $F[\mathbf{A}] = \{p(\mathbf{A}); p(X) \in F[X]\}$ of matrix polynomials in $\mathbf{A}$ inherits the algebraic operations for polynomials (and thus forms a commutative ring) in the same way as the polynomial differential operators $p(\mathrm{D})$ do; in particular we have $(p+q)(\mathbf{A}) = p(\mathbf{A}) + q(\mathbf{A})$ and $(pq)(\mathbf{A}) = p(\mathbf{A})q(\mathbf{A})$ for $p = p(X)$, $q = q(X) \in F[X]$. But there is one fundamental difference: $p(X) \neq 0$ implies $p(\mathrm{D}) \neq 0$ but not $p(\mathbf{A}) \neq \mathbf{0}$; cf. next item.
3. For $\mathbf{A} \in F^{n\times n}$ there exists a nonzero polynomial $p(X) \in F[X]$ of degree at most $n^2$ such that $p(\mathbf{A}) = \mathbf{0}$ (i.e., $p(\mathbf{A})$ is the all-zero matrix in $F^{n\times n}$).
   *Reason*: Since $\dim(F^{n\times n}) = n^2$, the $n^2 + 1$ matrices $\mathbf{I}_n$, $\mathbf{A}$, $\mathbf{A}^2, \dots, \mathbf{A}^{n^2}$ must be linearly dependent, and thus there exist $p_0, p_1, \dots, p_{n^2} \in F$, not all zero, such that $\sum_{i=0}^{n^2} p_i \mathbf{A}^i = \mathbf{0}$.

# Notes on Matrix Polynomials Cont'd

4. The monic polynomial $p(X) \in F[X] \setminus \{0\}$ of least degree satisfying $p(\mathbf{A}) = \mathbf{0}$ is called *minimum polynomial* of $\mathbf{A}$ and denoted by $\mu_{\mathbf{A}}(X)$. The polynomial $\mu_{\mathbf{A}}(X)$ is uniquely determined and divides every polynomial $p(X) \in F[X]$ that satisfies $p(\mathbf{A}) = \mathbf{0}$. The proof of these facts is the same as for polynomial differential operators annihilating a given function; cf. Exercise H42 of Homework 6.
5. The *Cayley-Hamilton Theorem* asserts that $\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$ for $\mathbf{A} \in F^{n\times n}$.
   $\Longrightarrow$ $\mu_{\mathbf{A}}(X)$ divides $\chi_{\mathbf{A}}(X)$, and in particular has degree $\leq n$ (a much stronger bound than the one in (3)). For the proof one shows first that every upper-triangular matrix $\mathbf{B} \in F^{n\times n}$ with diagonal entries $\lambda_1, \dots, \lambda_n$ (the eigenvalues of $\mathbf{B}$) satisfies
   $$\chi_{\mathbf{B}}(\mathbf{B}) = (\mathbf{B} - \lambda_1 \mathbf{I}_n)(\mathbf{B} - \lambda_2 \mathbf{I}_n) \cdots (\mathbf{B} - \lambda_n \mathbf{I}_n) = \mathbf{0}.$$
   The general case is then reduced to this special case by showing that $\mathbf{A} \in F^{n\times n}$ is similar to an upper-triangular matrix over an extension field of $F$ (which in the case $F = \mathbb{R}$ can be taken as $\mathbb{C}$).

# Notes on Matrix Polynomials Cont'd

6. If $\mathbf{A} \in F^{n \times n}$ is invertible, there are polynomial relations of the form $p_d\mathbf{A}^d + p_{d-1}\mathbf{A}^{d-1} + \cdots + p_1\mathbf{A} + \mathbf{I}_n = 0$. (Just take $\chi_{\mathbf{A}}(\mathbf{A}) = 0$ or $\mu_{\mathbf{A}}(\mathbf{A}) = 0$, and normalize the nonzero constant coefficient of the polynomial to 1.)
   The relation can be rewritten as
   $$\mathbf{A}(p_d\mathbf{A}^{d-1} + p_{d-1}\mathbf{A}^{d-2} + \cdots + p_1\mathbf{I}_n) = -\mathbf{I}_n$$
   and implies that $\mathbf{A}^{-1} = -p_d\mathbf{A}^{d-1} - p_{d-1}\mathbf{A}^{d-2} - \cdots - p_1\mathbf{I}_n$ is actually a matrix polynomial in $\mathbf{A}$. Thus Part (2) of the preceding theorem is a consequence of Part (3).

   As an example consider $2 \times 2$ matrices. In this case the Cayley Hamilton Theorem says $\mathbf{A}^2 - (a+d)\mathbf{A} + (ad-bc)\mathbf{I}_2 = 0$, or
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
   in full, and gives
   $$\mathbf{A}^{-1} = -\frac{1}{ad-bc}(\mathbf{A} - (a+d)\mathbf{I}_2) = -\frac{1}{ad-bc}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix}\right].$$

# Diagonalizability and $\chi_{\mathbf{A}}(X)$

First recall that $\lambda \in F$ is an eigenvalue of **A** iff $\chi_{\mathbf{A}}(\lambda) = 0$.

## Definition

Suppose $\lambda \in F$ is an eigenvalue of $\mathbf{A} \in F^{n\times n}$.

1. The dimension of the corresponding eigenspace $V_\lambda = \{\mathbf{x} \in F^n; \mathbf{A}\mathbf{x} = \lambda\mathbf{x}\}$ is referred to as the *geometric multiplicity* of the eigenvalue $\lambda$.
2. The multiplicity as a root of $\chi_{\mathbf{A}}(X)$, i.e., the largest positive integer $m$ such that $(X-\lambda)^m$ divides $\chi_{\mathbf{A}}(X)$, is referred to as the *algebraic multiplicity* of $\lambda$.

## Theorem

1. *If* **A** *is diagonalizable (over $F$) then $\chi_{\mathbf{A}}(X)$ splits into linear factors in $F[X]$.*
2. *For any eigenvalue of* **A** *the geometric multiplicity is less than or equal to the algebraic multiplicity.*
3. **A** *is diagonalizable if and only if $\chi_{\mathbf{A}}(X)$ splits into linear factors in $F[X]$ and for each eigenvalue of* **A** *the geometric multiplicity is equal to the algebraic multiplicity.*

# Proof of the theorem.

(1) If $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ is diagonal with diagonal entries $d_1, \dots, d_n$ then

$$\chi_{\mathbf{A}}(X) = \chi_{\mathbf{D}}(X) = \begin{vmatrix} X - d_1 & & \\ & \ddots & \\ & & X - d_n \end{vmatrix} = \prod_{i=1}^{n}(X - d_i).$$

(2) Let $k = \dim V_\lambda$, choose a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $F^n$ such that $\mathbf{v}_1, \dots, \mathbf{v}_k$ form a basis of $V_\lambda$, and let $\mathbf{S} = (\mathbf{v}_1 | \dots | \mathbf{v}_n)$.

$$\Longrightarrow \mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \left(\begin{array}{c|c} \lambda\mathbf{I}_k & \mathbf{A}_{12} \\ \hline \mathbf{0} & \mathbf{A}_{22} \end{array}\right) \quad \text{for some matrices } \mathbf{A}_{12}, \mathbf{A}_{22}$$

$$\Longrightarrow \chi_{\mathbf{A}}(X) = \chi_{\lambda\mathbf{I}_k}(X)\chi_{\mathbf{A}_{22}}(X) = (X - \lambda)^k \chi_{\mathbf{A}_{22}}(X),$$

since the determinant/characteristic polynomial of a block triangular matrix is equal to the product of the determinants/characteristic polynomials of the diagonal blocks (exercise).

(3) First we show that eigenvectors corresponding to distinct eigenvalues of $\mathbf{A}$ are linearly independent:

# Proof cont'd.

Suppose $\mathbf{A}$ has $r$ distinct eigenvalues $\lambda_1, \dots, \lambda_r$, and $\mathbf{v}_1, \dots, \mathbf{v}_r$ are corresponding eigenvectors satisfying a relation $c_1\mathbf{v}_1 + \dots + c_r\mathbf{v}_r = \mathbf{0}$ with $c_i \in F$.

$\implies \mathbf{0} = \mathbf{A}(c_1\mathbf{v}_1 + \dots + c_r\mathbf{v}_r) = (\lambda_1 c_1)\mathbf{v}_1 + \dots + (\lambda_r c_r)\mathbf{v}_r$

$\implies \mathbf{0} = \sum_{i=1}^r \lambda_i c_i \mathbf{v}_i - \lambda_1 \sum_{i=1}^r c_i\mathbf{v}_i = (\lambda_2 - \lambda_1)c_2\mathbf{v}_2 + \dots + (\lambda_r - \lambda_1)c_r\mathbf{v}_r$

This is a relation of the form $c_2'\mathbf{v}_2 + \dots + c_r'\mathbf{v}_r = \mathbf{0}$, which can be similarly reduced to one involving only $\mathbf{v}_3, \dots, \mathbf{v}_r$, etc., arriving finally at $(\lambda_r - \lambda_{r-1}) \cdots (\lambda_r - \lambda_1)c_r\mathbf{v}_r = \mathbf{0}$.

$\implies c_r = 0$ and, inductively, $c_{r-1} = \dots = c_1 = 0$. This completes the proof of our claim.

In terms of direct sums, this result says

$$V_{\lambda_1} + \dots + V_{\lambda_r} = V_{\lambda_1} \oplus \dots \oplus V_{\lambda_r},$$

and the union of bases of $V_{\lambda_1}, \dots, V_{\lambda_r}$ is a basis of the sum. Consequently, $F^n$ has a basis consisting of eigenvectors of $\mathbf{A}$ iff the sum is equal to $F^n$. This in turn is equivalent to $k_1 + \dots + k_r = n$, where $k_i = \dim V_{\lambda_i}$ denote the geometric multiplicities. For the corresponding algebraic multiplicities $m_1, \dots, m_r$ we have $k_i \leq m_i$ on account of (1), and $m_1 + \dots + m_r \leq \deg \chi_{\mathbf{A}}(X) = n$ with equality iff $\chi_{\mathbf{A}}(X)$ splits into linear factors. From this the assertion in (2) follows. $\square$

# Notes on the theorem

- In the special case $F = \mathbb{C}$ the necessary condition (1) for diagonalizability is always satisfied (by the Fundamental Theorem of Algebra).

  For $F = \mathbb{R}$ this is not always the case (cf. the example of rotation matrices, whose characteristic polynomials have non-real roots), but we can view $\mathbf{A} \in \mathbb{R}^{n\times n}$ as a matrix in $\mathbb{C}^{n\times n}$ and try to diagonalize it over $\mathbb{C}$. For this modified problem, Condition (1) is again satisfied.
- Part (3) of the theorem implies in particular that any matrix $\mathbf{A} \in F^{n\times n}$ whose characteristic polynomial has $n$ distinct roots in $F$ is diagonalizable.
- Another useful characterization of diagonalizable matrices is the following: $\mathbf{A}$ is diagonalizable iff the minimum polynomial $\mu_{\mathbf{A}}(X)$ splits into distinct linear factors; equivalently, $\mathbf{A}$ satisfies an equation

  $$(\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I})\cdots(\mathbf{A} - \lambda_r\mathbf{I}) = \mathbf{0}$$

  with pairwise distinct $\lambda_1, \dots, \lambda_r \in F$.

# General Solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$

Recall that the solution space of an $n$-th order scalar homogeneous linear ODE $a(\mathrm{D})y = 0$ (with constant coefficients) is generated by the exponential polynomials $t^k \mathrm{e}^{\lambda t}$ with $\lambda \in \mathbb{C}$ a root of $a(X)$ and $k$ a non-negative integer less than the (algebraic) multiplicity of $\lambda$.

Order reduction gives the 1st-order $n \times n$ system

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{n-2} & -a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}.$$

The coefficient matrix $\mathbf{A}$ is called *companion matrix* of the polynomial $a(X) = X^n + a_{n-1}X^{n-1} + \dots + a_1 X + a_0$, since its characteristic polynomial is equal to $a(X)$.
This can verified by expanding $\det(X\,\mathbf{I}_n - \mathbf{A})$ along the last column, obtaining $\chi_{\mathbf{A}}(X) = (X + a_{n-1})X^{n-1} - (-1)\det(\mathbf{B})$ with $\mathbf{B}$ of similar form, and using induction.

In the special case under consideration there is a fundamental system of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ consisting of vectorial functions of the form

$$\mathbf{y}(t) = \left(t^k \mathrm{e}^{\lambda t}, (t^k \mathrm{e}^{\lambda t})', \dots, (t^k \mathrm{e}^{\lambda t})^{(n-1)}\right),$$

which have exponential polynomials $p(t)\mathrm{e}^{\lambda t}$ with $\lambda$ an eigenvalue of $\mathbf{A}$ and polynomial factor $p(t)$ of degree less than the algebraic multiplicity of $\lambda$ as entries.

This motivates the „Ansatz“

$$\mathbf{y}(t) = \mathrm{e}^{\lambda t}\mathbf{v}_0 + t\,\mathrm{e}^{\lambda t}\mathbf{v}_1 + \cdots + t^{m-1}\,\mathrm{e}^{\lambda t}\mathbf{v}_{m-1}, \quad \mathbf{v}_j \in \mathbb{C}^n,$$

for eigenvalues $\lambda$ of $\mathbf{A}$ of algebraic multiplicity $m$ to solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

$$\begin{aligned}
\mathbf{y}'(t) &= \lambda \mathrm{e}^{\lambda t}\mathbf{v}_0 + (1+\lambda t)\mathrm{e}^{\lambda t}\mathbf{v}_1 + (2t + \lambda t^2)\mathrm{e}^{\lambda t}\mathbf{v}_2 + \cdots + \\
&\qquad + \left((m-1)t^{m-2} + \lambda t^{m-1}\right)\mathrm{e}^{\lambda t}\mathbf{v}_{m-1} \\
&= (\lambda\mathbf{v}_0 + \mathbf{v}_1)\mathrm{e}^{\lambda t} + (\lambda\mathbf{v}_1 + 2\mathbf{v}_2)t\,\mathrm{e}^{\lambda t} + \cdots + \\
&\qquad + \left(\lambda\mathbf{v}_{m-2} + (m-1)\mathbf{v}_{m-1}\right)t^{m-2}\,\mathrm{e}^{\lambda t} + \lambda\mathbf{v}_{m-1}t^{m-1}\,\mathrm{e}^{\lambda t} \\
\mathbf{A}\mathbf{y}(t) &= \mathrm{e}^{\lambda t}\mathbf{A}\mathbf{v}_0 + t\,\mathrm{e}^{\lambda t}\mathbf{A}\mathbf{v}_1 + \cdots + t^{m-1}\,\mathrm{e}^{\lambda t}\mathbf{A}\mathbf{v}_{m-1}
\end{aligned}$$

$\Longrightarrow$ $\mathbf{y}(t)$ solves $\mathbf{y}' = \mathbf{A}\mathbf{y}$ iff

$$\begin{aligned}
\mathbf{A}\mathbf{v}_0 &= \lambda\mathbf{v}_0 + \mathbf{v}_1,\\
\mathbf{A}\mathbf{v}_1 &= \lambda\mathbf{v}_1 + 2\mathbf{v}_2,\\
&\vdots\\
\mathbf{A}\mathbf{v}_{m-2} &= \lambda\mathbf{v}_{m-2} + (m-1)\mathbf{v}_{m-1},\\
\mathbf{A}\mathbf{v}_{m-1} &= \lambda\mathbf{v}_{m-1}.
\end{aligned}$$

This can be rewritten as $(\mathbf{A}-\lambda\mathbf{I}_n)\mathbf{v}_0 = \mathbf{v}_1$, $(\mathbf{A}-\lambda\mathbf{I}_n)\mathbf{v}_1 = 2\mathbf{v}_2, \dots,$ $(\mathbf{A}-\lambda\mathbf{I}_n)\mathbf{v}_{m-2} = (m-1)\mathbf{v}_{m-1}$, $(\mathbf{A}-\lambda\mathbf{I}_n)\mathbf{v}_{m-1} = \mathbf{0}$ and is equivalent to

$$\mathbf{v}_k = \frac{1}{k!}(\mathbf{A}-\lambda\mathbf{I}_n)^k\mathbf{v}_0 \quad \text{for } 1 \leq k \leq m-1, \quad (\mathbf{A}-\lambda\mathbf{I}_n)^m\mathbf{v}_0 = \mathbf{0}.$$

## Definition

Suppose $\lambda$ is an eigenvalue of $\mathbf{A} \in F^{n\times n}$ of algebraic multiplicity $m$. A vector $\mathbf{v} \in F^n \setminus \{\mathbf{0}\}$ is said to be a *generalized eigenvector* of $\mathbf{A}$ associated to $\lambda$ if $(\mathbf{A}-\lambda\mathbf{I}_n)^m\mathbf{v} = \mathbf{0}$.

Thus the generalized eigenvectors of $\mathbf{A}$ associated to $\lambda$ are precisely the nonzero elements in the right kernel of $(\mathbf{A}-\lambda\mathbf{I}_n)^m$. The subspace $\operatorname{rker}(\mathbf{A}-\lambda\mathbf{I}_n)^m$ of $F^n$ is called *generalized eigenspace* of $\mathbf{A}$ associated to $\lambda$ and will be denoted by $W_\lambda$. (Recall that $V_\lambda$ denotes the corresponding eigenspace of $\mathbf{A}$.)

## Notes on the definition

- If $\lambda$ is a simple root of $\chi_{\mathbf{A}}(X)$, eigenvectors and generalized eigenvectors of $\mathbf{A}$ associated to $\lambda$ are the same thing (and thus $W_\lambda = V_\lambda$).
- In general we have the chain of subspaces

$$V_\lambda = \operatorname{rker}(\mathbf{A} - \lambda \mathbf{I}_n) \subseteq \operatorname{rker}(\mathbf{A} - \lambda \mathbf{I}_n)^2 \subseteq \cdots \subseteq \operatorname{rker}(\mathbf{A} - \lambda \mathbf{I}_n)^m = W_\lambda.$$

## Theorem

*Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ has distinct eigenvalues $\lambda_1, \dots, \lambda_r$ with corresponding algebraic multiplicities $m_1, \dots, m_r$, i.e., the characteristic polynomial of $\mathbf{A}$ factors in $\mathbb{C}[X]$ as $\chi_{\mathbf{A}}(X) = (X - \lambda_1)^{m_1} \cdots (X - \lambda_r)^{m_r}$.*

1. *$\mathbb{C}^n$ has a basis $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ consisting of generalized eigenvectors of $\mathbf{A}$.*
2. *If $\mathbf{v}_j \in B$ is associated to the eigenvalue $\lambda_i$ of $\mathbf{A}$ and $\mathbf{y}_j \colon \mathbb{R} \to \mathbb{C}^n$ is defined by*
$$\mathbf{y}_j(t) = \sum_{k=0}^{m_i-1} \frac{1}{k!} t^k \, \mathrm{e}^{\lambda_i t} (\mathbf{A} - \lambda_i \mathbf{I}_n)^k \mathbf{v}_j,$$
*then $\mathbf{y}_1, \dots, \mathbf{y}_n$ form a fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.*
3. *The matrix exponential function of $\mathbf{A}$ is*
$$t \mapsto \mathrm{e}^{\mathbf{A}t} = \big(\mathbf{y}_1(t)| \dots |\mathbf{y}_n(t)\big)\big(\mathbf{v}_1| \dots |\mathbf{v}_n\big)^{-1}.$$

## Example

We determine a fundamental system of solutions of

$$\mathbf{y}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{y}; \qquad \text{cf. [BDM17], p. 336.}$$

Here the characteristic polynomial is $\chi_{\mathbf{A}}(X) = X^2 - 4X + 4 = (X-2)^2$, so that $\mathbf{A} = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 3 \end{smallmatrix}\right)$ has the single eigenvalue $\lambda = 2$ with algebraic multiplicity $m = 2$.
$\Longrightarrow$ The corresponding generalized eigenspace must be $\mathbb{C}^2$, and, using for $B$ the standard basis of $\mathbb{C}^2$ the theorem gives

$$\mathbf{y}_1(t) = \mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
$$\mathbf{y}_2(t) = \mathrm{e}^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathrm{e}^{2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\,\mathrm{e}^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

as fundamental system of solutions.

We must have $\Phi(t) := \big(\mathbf{y}_1(t)|\mathbf{y}_2(t)\big) = \mathrm{e}^{\mathbf{A}t}$, since $\Phi(0) = \big(\mathbf{y}_1(0)|\mathbf{y}_2(0)\big) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\mathrm{e}^{\mathbf{A}t}$ is characterized by this condition among the fundamental matrices of $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

## Example (cont'd)

$$\Longrightarrow \exp\left[t\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\right] = e^{2t}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t\,e^{2t}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} e^{2t} - t\,e^{2t} & -t\,e^{2t} \\ t\,e^{2t} & e^{2t} + t\,e^{2t} \end{pmatrix}.$$

The eigenspace $V_2$ is 1-dimensional and generated by $(1,-1)^{\mathsf{T}}$, so that we can replace one of $\mathbf{y}_1, \mathbf{y}_2$, say $\mathbf{y}_2$, by the "simpler" solution $\mathbf{y}(t) = e^{2t}(1,-1)^{\mathsf{T}}$. (This amounts to applying the theorem to the basis $(1,0)^{\mathsf{T}}$, $(1,-1)^{\mathsf{T}}$ of $\mathbb{C}^2$ instead.)

The corresponding fundamental matrix is

$$\begin{pmatrix} e^{2t} - t\,e^{2t} & -t\,e^{2t} \\ t\,e^{2t} & e^{2t} + t\,e^{2t} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{2t} - t\,e^{2t} & e^{2t} \\ t\,e^{2t} & -e^{2t} \end{pmatrix}.$$

# Proof of the theorem.

First we show (2) and (3) under the assumption that (1) holds.
We have already seen that the functions $\mathbf{y}_j$ solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
It remains to show that they are linearly independent. We have

$$\mathbf{y}_j(t) = \mathrm{e}^{\lambda_i t}\mathbf{v}_j + t\,\mathrm{e}^{\lambda_i t}\mathbf{w}_1 + \cdots + t^{m_i-1}\,\mathrm{e}^{\lambda_i t}\mathbf{w}_{m_i-1}$$

for certain vectors $\mathbf{w}_1, \dots, \mathbf{w}_{m_i-1} \in \mathbb{C}^n$.
$\Longrightarrow \mathbf{y}_j(0) = \mathbf{v}_j$.
Since $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent, so are $\mathbf{y}_1, \dots, \mathbf{y}_n$.
This proves (2); (3) is an instance of the formula $\mathrm{e}^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}$.

(1) Since $\sum_{i=1}^r m_i = \deg \chi_{\mathbf{A}}(X) = n$, it suffices to show that (i) the generalized eigenspace $W_{\lambda_i}$ of $\mathbf{A}$ associated to $\lambda_i$ has dimension $\geq m_i$, and that (ii) the sum $W_{\lambda_1} + \cdots + W_{\lambda_r}$ is direct.
Then for reasons of dimension we must have $\dim(W_{\lambda_i}) = m_i$ and $W_{\lambda_1} \oplus \cdots \oplus W_{\lambda_r} = V$.

Write $\lambda_i = \lambda$, $m_i = m$, and choose $\mathbf{v}$ as an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda$. Then there exists an invertible matrix $\mathbf{S} \in \mathbb{C}^{n\times n}$ with first column $\mathbf{v}$, and the condition $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ gives

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \left(\begin{array}{c|c} \lambda & \mathbf{v}_1 \\ \hline \mathbf{0} & \mathbf{A}_1 \end{array}\right) \text{ with } \mathbf{A}_1 \in \mathbb{C}^{(n-1)\times(n-1)},\ \mathbf{0} \in \mathbb{C}^{(n-1)\times 1},\ \mathbf{v}_1 \in \mathbb{C}^{1\times(n-1)}.$$

# Proof cont'd.

$\Longrightarrow \chi_{\mathbf{A}}(X) = \chi_{\mathbf{S}^{-1}\mathbf{A}\mathbf{S}}(X) = (X-\lambda)\chi_{\mathbf{A}_1}(X)$.
If $m \geq 2$ then $\lambda$ must be an eigenvalue of $\mathbf{A}_1$ and $\mathbf{A}_1$ must be similar to a matrix of the form $\begin{pmatrix} \lambda & \mathbf{v}_2 \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}$ with $\mathbf{A}_2 \in \mathbb{C}^{(n-2)\times(n-2)}$, $\mathbf{0} \in \mathbb{C}^{(n-1)\times 1}$, $\mathbf{v}_2 \in \mathbb{C}^{1\times(n-2)}$. This implies in turn that $\mathbf{A}$ is similar to a $2\times 2$ block matrix $\begin{pmatrix} \mathbf{U}_2 & \mathbf{V}_2 \\ \mathbf{0} & \mathbf{A}_2 \end{pmatrix}$ with top-right block $\mathbf{U}_2$ of the form $\begin{pmatrix} \lambda & * \\ 0 & \lambda \end{pmatrix}$.
Using induction, we obtain that $\mathbf{A}$ is similar to a matrix of the form

$$\left(\begin{array}{c|c} \mathbf{U}_m & \mathbf{V}_m \\ \hline \mathbf{0} & \mathbf{A}_m \end{array}\right), \qquad (\star)$$

where $\mathbf{U}_m \in \mathbb{C}^{m\times m}$ is upper-triangular and has all entries on the main diagonal equal to $\lambda$, and the remaining blocks satisfy $\mathbf{0} \in \mathbb{C}^{(n-m)\times m}$, $\mathbf{V}_m \in \mathbb{C}^{m\times(n-m)}$, $\mathbf{A}_m \in \mathbb{C}^{(n-m)\times(n-m)}$.
$\Longrightarrow$ $\mathbf{A} - \lambda\mathbf{I}_n$ is similar to a matrix of the same form as $(\star)$ but with the entries on the main diagonal of $\mathbf{U}$ equal to zero.
Now it is easy to see that a strictly upper-triangular $n\times n$ matrix $\mathbf{U}$ (i.e., $u_{ij} = 0$ for $i \geq j$) satisfies $\mathbf{U}^n = \mathbf{0}$.
It follows that $\mathbf{U}_m^m = \mathbf{0}$ and hence that $(\mathbf{A} - \lambda\mathbf{I}_n)^m$ is similar to

# Proof cont'd.

$$\left(\begin{array}{c|c}\mathbf{U}_m & \mathbf{V}_m \\ \hline \mathbf{0} & \mathbf{A}_m\end{array}\right)^m = \left(\begin{array}{c|c}\mathbf{U}_m^m & * \\ \hline \mathbf{0} & \mathbf{A}_m^m\end{array}\right) = \left(\begin{array}{c|c}\mathbf{0} & * \\ \hline \mathbf{0} & \mathbf{A}_m^m\end{array}\right).$$

$\Longrightarrow \operatorname{rk}(\mathbf{A} - \lambda\mathbf{I}_n)^m \leq n - m$
$\Longrightarrow \dim(W_{\lambda_i}) = \dim\left(\operatorname{rker}(\mathbf{A} - \lambda\mathbf{I}_n)^m\right) \geq m$, completing the proof of (i).

As a by-product of the preceding proof we see that ($\star$) can be transformed further if we use eigenvalues of $\mathbf{A}$ distinct from $\lambda$, which must be eigenvalues of $\mathbf{A}_m$. This process stops with an upper-triangular matrix and provides a proof of the fact mentioned earlier that every matrix in $\mathbb{C}^{n\times n}$ is similar to an upper-triangular matrix.

For the proof of (ii) suppose that $\mathbf{w}_i \in W_{\lambda_i}$ are such that

$$\mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_r = \mathbf{0}.$$

Our task is to show that necessarily $\mathbf{w}_1 = \cdots = \mathbf{w}_r = 0$.
We rewrite the identity as $\mathbf{w}_1 = -\mathbf{w}_2 - \cdots - \mathbf{w}_r$ and use matrix polynomials to finish the proof.

# Proof cont'd.

Write $\chi_{\mathbf{A}}(X) = (X - \lambda_1)^{m_1} q(X)$ with $q(X) = \prod_{i=2}^{r}(X - \lambda_i)^{m_i}$. Since the polynomials $(X - \lambda_1)^{m_1}$ and $q(X)$ are relatively prime, there exist polynomials $a(X), b(X) \in \mathbb{C}[X]$ such that

$$a(X)(X - \lambda_1)^{m_1} + b(X)q(X) = 1.$$

Substituting $\mathbf{A}$ for $X$ gives the matrix equation

$$a(\mathbf{A})(\mathbf{A} - \lambda_1\mathbf{I}_n)^{m_1} + b(\mathbf{A})q(\mathbf{A}) = \mathbf{I}_n.$$

$$\Longrightarrow \mathbf{w}_1 = \mathbf{I}_n\mathbf{w}_1 = a(\mathbf{A})(\mathbf{A} - \lambda_1\mathbf{I}_n)^{m_1}\mathbf{w}_1 + b(\mathbf{A})q(\mathbf{A})\mathbf{w}_1.$$

Since $\mathbf{w}_1 \in W_1$, we have $a(\mathbf{A})(\mathbf{A} - \lambda_1\mathbf{I}_n)^{m_1}\mathbf{w}_1 = \mathbf{0}$. On the other hand, for $i \in \{2, \dots, r\}$ the matrix $q(\mathbf{A}) = \prod_{j=2}^{r}(\mathbf{A} - \lambda_j\mathbf{I}_n)^{m_j}$ is a multiple of $(\mathbf{A} - \lambda_i\mathbf{I}_n)^{m_i}$ and hence $q(\mathbf{A})\mathbf{w}_i = \mathbf{0}$.

$$\Longrightarrow b(\mathbf{A})q(\mathbf{A})\mathbf{w}_1 = -\sum_{i=2}^{r} b(\mathbf{A})q(\mathbf{A})\mathbf{w}_i = \mathbf{0}.$$

In all it follows that $\mathbf{w}_1 = \mathbf{0}$.
In the same way one proves that $\mathbf{w}_2 = \cdots = \mathbf{w}_r = \mathbf{0}$, completing the proof of the theorem. $\square$

# Notes on the theorem

1. The required basis $B$ can be calculated by determining, for each $i \in \{1, \dots, r\}$, a basis of the solution space of $(\mathbf{A} - \lambda_i \mathbf{I}_n)^{m_i}\mathbf{x} = \mathbf{0}$. This is done with the usual algorithm based on Gaussian elimination.

2. If the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are indexed in such a way that $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{m_1}$ form a basis of $W_{\lambda_1}$, $\mathbf{v}_{m_1+1}, \mathbf{v}_{m_1+2}, \dots, \mathbf{v}_{m_1+m_2}$ a basis of $W_{\lambda_2}$, etc., then $\mathbf{S} = (\mathbf{v}_1 | \dots | \mathbf{v}_n)$ "block-diagonalizes" $\mathbf{A}$ in the following sense:

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{pmatrix} \mathbf{A}_1 & & & \\ & \mathbf{A}_2 & & \\ & & \ddots & \\ & & & \mathbf{A}_r \end{pmatrix} \quad \text{with } \mathbf{A}_i \in \mathbb{C}^{m_i \times m_i}.$$

Moreover, the characteristic polynomial of $\mathbf{A}_i$ is $(X - \lambda_i)^{m_i}$. The block-diagonal form expresses the fact that $f_{\mathbf{A}}$ maps the generalized eigenspaces of $\mathbf{A}$ to itself:

$$(\mathbf{A} - \lambda \mathbf{I}_n)^m \mathbf{v} = \mathbf{0} \implies (\mathbf{A} - \lambda \mathbf{I}_n)^m \mathbf{A}\mathbf{v} = \mathbf{A}(\mathbf{A} - \lambda \mathbf{I}_n)^m \mathbf{v} = \mathbf{0}.$$

# Notes on the theorem cont'd

3. A different way to calculate $e^{\mathbf{A}t}$ is as follows: From the preceding note we have

$$\mathbf{A} = \mathbf{S}\begin{pmatrix}\mathbf{A}_1 & & \\ & \ddots & \\ & & \mathbf{A}_r\end{pmatrix}\mathbf{S}^{-1} \Longrightarrow e^{\mathbf{A}t} = \mathbf{S}\begin{pmatrix}e^{\mathbf{A}_1 t} & & \\ & \ddots & \\ & & e^{\mathbf{A}_r t}\end{pmatrix}\mathbf{S}^{-1}.$$

Further, since $(\mathbf{A}_i - \lambda_i\,\mathbf{I})^{m_i} = \mathbf{0}$, where $\mathbf{I} = \mathbf{I}_{m_i}$, we have

$$\begin{aligned} e^{\mathbf{A}_i t} &= e^{\lambda_i t\,\mathbf{I}} e^{\mathbf{A}_i t - \lambda_i t\,\mathbf{I}} = (e^{\lambda_i t}\mathbf{I}) e^{t(\mathbf{A}_i - \lambda_i\,\mathbf{I})} \\ &= e^{\lambda_i t}\sum_{k=0}^{m_i-1}\frac{1}{k!}t^k(\mathbf{A}_i - \lambda_i\,\mathbf{I})^k = \sum_{k=0}^{m_i-1}\frac{1}{k!}t^k e^{\lambda_i t}(\mathbf{A}_i - \lambda_i\,\mathbf{I})^k. \end{aligned}$$

The exponential series terminates, since $(\mathbf{A}_i - \lambda_i\,\mathbf{I})^k = \mathbf{0}$ for $k \geq m_i$.

The solutions $\mathbf{y}_j(t)$ in Part 2 of the theorem are in fact the columns of $e^{\mathbf{A}t}\mathbf{S}$, as is clear from Part 3 of the theorem.

## Notes on the theorem cont'd

3. (cont'd)
   This can also be seen directly as follows: If $\mathbf{v}_j$ is the $j$-th column of $\mathbf{S}$ and belongs to the eigenvalue $\lambda_i$, we have

$$\begin{aligned}
\mathrm{e}^{\mathbf{A}t}\mathbf{v}_j &= \mathrm{e}^{\lambda_i t\,\mathbf{I}}\mathrm{e}^{(\mathbf{A}-\lambda_i\mathbf{I})t}\mathbf{v}_j = \mathrm{e}^{\lambda_i t}\sum_{k=0}^{\infty}\frac{1}{k!}t^k(\mathbf{A}-\lambda_i\mathbf{I})^k\mathbf{v}_j \\
&= \mathrm{e}^{\lambda_i t}\sum_{k=0}^{m_i-1}\frac{1}{k!}t^k(\mathbf{A}-\lambda_i\mathbf{I})^k\mathbf{v}_j. \quad \text{(since } (\mathbf{A}-\lambda_i\mathbf{I})^{m_i}\mathbf{v}_j = \mathbf{0}\text{)}
\end{aligned}$$

   This is precisely $\mathbf{y}_j(t)$, as defined in Part 2 of the theorem.

# Notes on the theorem cont'd

4. There are other proofs of parts of the theorem, e.g., a proof using matrix polynomials that the generalized eigenvectors of $\mathbf{A} \in \mathbb{C}^{n \times n}$ generate $\mathbb{C}^n$.

   For this set $q_i(X) = \chi_{\mathbf{A}}(X)/(X - \lambda_i)^{m_i}$, i.e., $q_i(X)$ is the product of all "prime powers" $(X - \lambda_j)^{m_j}$, $j \neq i$. Since the polynomials $q_i(X)$are relatively prime (i.e., their polynomial g.c.d. is 1), there exist polynomials $b_i(X) \in \mathbb{C}[X]$ such that

   $$b_1(X)q_1(X) + \cdots + b_r(X)q_r(X) = 1.$$

   Substituting $\mathbf{A}$ for $X$ gives

   $$b_1(\mathbf{A})q_1(\mathbf{A}) + \cdots + b_r(\mathbf{A})q_r(\mathbf{A}) = \mathbf{I}_n.$$

   This implies for $\mathbf{v} \in \mathbb{C}^n$ that

   $$\mathbf{v} = \mathbf{I}_n\mathbf{v} = b_1(\mathbf{A})q_1(\mathbf{A})\mathbf{v} + \cdots + b_r(\mathbf{A})q_r(\mathbf{A})\mathbf{v}.$$

   Since $(\mathbf{A} - \lambda_i\mathbf{I}_n)^{m_i} q_i(\mathbf{A})\mathbf{v} = \chi_{\mathbf{A}}(\mathbf{A})\mathbf{v} = \mathbf{0}$ (by the Cayley-Hamilton Theorem), we have $q_i(\mathbf{A})\mathbf{v} \in \operatorname{rker}(\mathbf{A} - \lambda_i\mathbf{I}_n)^{m_i} = W_{\lambda_i}$, and hence $b_i(\mathbf{A})q_i(\mathbf{A})\mathbf{v} \in W_{\lambda_i}$ as well; thus $\mathbb{C}^n = W_{\lambda_1} + \cdots + W_{\lambda_r}$.

# Notes on the theorem cont'd

5. The vectors $\mathbf{w}_k = (\mathbf{A} - \lambda_i \mathbf{I}_n)^k \mathbf{v}_j$, $0 \leq k \leq m_i - 1$ (with $\mathbf{w}_0 = \mathbf{v}_j$), which need to be calculated in order to obtain $\mathbf{y}_j(t)$, are itself generalized eigenvectors associated to $\lambda_i$ and hence can serve as members of the basis $B$, provided they are nonzero. (If you find the reasoning circular, change to "serve as members of another basis $B'$ consisting of generalized eigenvectors of $\mathbf{A}$".)

   Suppose that the sum defining $\mathbf{y}_j(t)$ terminates with the summand $\frac{1}{k!} t^k \mathrm{e}^{\lambda_i t} \mathbf{w}_k$, i.e., $\mathbf{w}_k \neq \mathbf{0}$, $\mathbf{w}_{k+1} = \mathbf{0}$. Then the vectors in the *chain* $\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_k$ are linearly independent. This can be seen as follows: Writing $\mathbf{N} = \mathbf{A} - \lambda_i \mathbf{I}_n$, we have $\mathbf{N}\mathbf{w}_s = \mathbf{w}_{s+1}$. If $\sum_{s=0}^{k} c_s \mathbf{w}_s = \mathbf{0}$, we can apply $\mathbf{N}^k$ to this sum and from $\sum_{s=0}^{k} c_s \mathbf{w}_{s+k} = c_0 \mathbf{w}_k = \mathbf{0}$ conclude that $c_0 = 0$. Then we apply $\mathbf{N}^{k-1}$ and obtain $c_1 = 0$, etc.

   If $k = m_i - 1$, the chain forms a basis of $W_{\lambda_i}$. If $k < m_i - 1$, this is not the case, but we can use several such chains, starting with other vectors $\mathbf{v}_l \in W_{\lambda_i}$.

# Notes on the theorem cont'd

5. (cont'd)
   The following facts, which are readily proved, provide the key to success of this approach.

   - The last nonzero vector of each chain is an eigenvector corresponding to the eigenvalue $\lambda_i$.
   - The vectors in a union of chains (belonging to the same $\lambda_i$) are linearly independent iff the corresponding eigenvectors (last vectors of the chains) are linearly independent.
   - There exists a basis of $W_{\lambda_i}$ that is a union of chains, and the number and lengths of the chains in such a basis are uniquely determined.

   The number of chains is equal to the geometric multiplicity of $\lambda_i$, and the lengths of the chains can be determined from the dimensions of $\operatorname{rker}(\mathbf{A} - \lambda_i \mathbf{I})^k$, $1 \leq k \leq m_i$.

   The matrix respresenting $f_{\mathbf{A}}$ w.r.t. such a basis is in Jordan Canonical Form (see subsequent section), with the number/sizes of the Jordan blocks equal to the number/lengths of the chains.

# Notes on the theorem cont'd

5. (cont'd)
The preceding observation motivates the following "depth-first" strategy for obtaining a basis of $W_{\lambda_i}$:

First determine the smallest non-negative integer $k$ such that $\operatorname{rker}\big((\mathbf{A}-\lambda_i\mathbf{I}_n)^k\big) = W_{\lambda_i}$ (equivalently, $\operatorname{rker}\big((\mathbf{A}-\lambda_i\mathbf{I}_n)^k\big)$ has dimension $m_i$), and a vector $\mathbf{w} \in W_{\lambda_i}$ satisfying $(\mathbf{A}-\lambda_i\mathbf{I}_n)^{k-1}\mathbf{w} \neq \mathbf{0}$. Include the vectors $\mathbf{w}_0 = \mathbf{w}, \mathbf{w}_1, \dots, \mathbf{w}_{k-1}$ as defined above in the basis. If $k < m_i$, start over and determine the largest non-negative integer $k'$ for which there exists a vector $\mathbf{w}' \in W_{\lambda_i}$ such that $(\mathbf{A}-\lambda_i\mathbf{I}_n)^{k'-1}\mathbf{w}'$ is linearly independent of $(\mathbf{A}-\lambda_i\mathbf{I}_n)^{k-1}\mathbf{w}$. Include $\mathbf{w}'_0 = \mathbf{w}', \mathbf{w}'_1, \dots, \mathbf{w}'_{k'-1}$ as defined above in the basis; etc. Clearly the procedure terminates, and it can be shown that it yields a basis of $W_{\lambda_i}$. The corresponding fundamental solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ have the simple form

$$\begin{aligned}
\mathbf{y}_0(t) &= \mathrm{e}^{\lambda_i t}\mathbf{w}_0 + t\,\mathrm{e}^{\lambda_i t}\mathbf{w}_1 + \dots + \frac{1}{(k-1)!}\,t^{k-1}\mathrm{e}^{\lambda_i t}\mathbf{w}_{k-1},\\
\mathbf{y}_1(t) &= \mathrm{e}^{\lambda_i t}\mathbf{w}_1 + t\,\mathrm{e}^{\lambda_i t}\mathbf{w}_2 + \dots + \frac{1}{(k-2)!}\,t^{k-2}\mathrm{e}^{\lambda_i t}\mathbf{w}_{k-1},\\
&\ \vdots\\
\mathbf{y}_{k-1}(t) &= \mathrm{e}^{\lambda_i t}\mathbf{w}_{k-1}, \quad \text{etc.}
\end{aligned}$$

## Notes on the theorem cont'd

5. (cont'd)
   Thus a chain of length $k$ gives rise to $k$ fundamental solutions $\mathbf{y}_0(t), \mathbf{y}_1(t), \dots, \mathbf{y}_{k-1}(t)$ having, in order, $k$ summands, $k-1$ summands, ..., and finally 1 summand. A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ obtained from a union of such chains is essentially the simplest possible.

## Example

Determine the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ for

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 9 & 0 & 1 & 19 \\ -1 & -2 & -23 & 0 & 0 & -46 \\ 0 & 0 & -8 & 0 & 0 & -12 \\ -6 & 1 & 9 & 4 & 1 & 19 \\ 1 & 0 & -1 & 0 & -2 & -2 \\ 0 & 0 & 6 & 0 & 0 & 10 \end{pmatrix}.$$

$\mathbf{A}$ has the eigenvalue 4, since $\mathbf{A}\mathbf{e}_4 = 4\mathbf{e}_4$. Thus $\chi_{\mathbf{A}}(X)$ is divisible by $X - 4$, which also follows immediately from expanding $\det(X\,\mathbf{I}_6 - \mathbf{A})$ along the 4th column:

$$\chi_{\mathbf{A}}(X) = (4-X) \begin{vmatrix} -2-X & 1 & 9 & 1 & 19 \\ -1 & -2-X & -23 & 0 & -46 \\ 0 & 0 & -8-X & 0 & -12 \\ 1 & 0 & -1 & -2-X & -2 \\ 0 & 0 & 6 & 0 & 10-X \end{vmatrix}$$

Next we add $X + 2$ times first row to the second row in order to obtain a column with only one nonzero entry.

## Example (cont'd)

$$\begin{aligned}
\chi_{\mathbf{A}}(X) &= (4-X)\begin{vmatrix} -X-2 & 1 & 9 & 1 & 19 \\ -X^2-4X-5 & 0 & 9X-5 & X+2 & 19X-8 \\ 0 & 0 & -X-8 & 0 & -12 \\ 1 & 0 & -1 & -X-2 & -2 \\ 0 & 0 & 6 & 0 & -X+10 \end{vmatrix} \\
&= (X-4)\begin{vmatrix} -X^2-4X-5 & 9X-5 & X+2 & 19X-8 \\ 0 & -X-8 & 0 & -12 \\ 1 & -1 & -X-2 & -2 \\ 0 & 6 & 0 & -X+10 \end{vmatrix} \\
&= (X-4)(X+2)\begin{vmatrix} -X^2-4X-5 & 9X-5 & 1 & 19X-8 \\ 0 & -X-8 & 0 & -12 \\ 1 & -1 & -1 & -2 \\ 0 & 6 & 0 & -X+10 \end{vmatrix} \\
&= (X-4)(X+2)\begin{vmatrix} -X^2-4X-4 & 9X-5 & 1 & 19X-8 \\ 0 & -X-8 & 0 & -12 \\ 0 & -1 & -1 & -2 \\ 0 & 6 & 0 & -X+10 \end{vmatrix} \\
&= (X-4)(X+2)^3\begin{vmatrix} -X-8 & -12 \\ 6 & -X+10 \end{vmatrix}
\end{aligned}$$

## Example (cont'd)

The final result is

$$\chi_{\mathbf{A}}(X) = (X-4)(X+2)^3(X^2-2X-8) = (X-4)^2(X+2)^4.$$

$\Longrightarrow$ The eigenvalues of **A** are $\lambda_1 = 4$ with multiplicity 2 and $\lambda_2 = -2$ with multiplicity 4.

$\underline{\lambda_1 = 4:}$

$$\mathbf{A} - 4\,\mathbf{I} = \begin{pmatrix} -6 & 1 & 9 & 0 & 1 & 19 \\ -1 & -6 & -23 & 0 & 0 & -46 \\ 0 & 0 & -12 & 0 & 0 & -12 \\ -6 & 1 & 9 & 0 & 1 & 19 \\ 1 & 0 & -1 & 0 & -6 & -2 \\ 0 & 0 & 6 & 0 & 0 & 6 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 & -6 & -2 \\ 0 & 1 & 3 & 0 & -35 & 7 \\ 0 & -6 & -24 & 0 & -6 & -48 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 0 & -6 & -2 \\ 0 & 1 & 3 & 0 & -35 & 7 \\ 0 & 0 & -6 & 0 & -216 & -6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 & -6 & -2 \\ 0 & 1 & 3 & 0 & -35 & 7 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Longrightarrow \mathbf{V}_4 = \left\langle \mathbf{v}_1 = (0,0,0,1,0,0)^{\mathsf{T}},\ \mathbf{v}_2 = (1,-4,-1,0,0,1)^{\mathsf{T}} \right\rangle$$

# Example (cont'd)

$\underline{\lambda_2 = -2:}$

$$\mathbf{A} + 2\,\mathbf{I} = \begin{pmatrix} 0 & 1 & 9 & 0 & 1 & 19 \\ -1 & 0 & -23 & 0 & 0 & -46 \\ 0 & 0 & -6 & 0 & 0 & -12 \\ -6 & 1 & 9 & 6 & 1 & 19 \\ 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 0 & 6 & 0 & 0 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 9 & 0 & 1 & 19 \\ 0 & 1 & 3 & 6 & 1 & 7 \\ 0 & 0 & -24 & 0 & 0 & -48 \\ 0 & 0 & 6 & 0 & 0 & 12 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 9 & 0 & 1 & 19 \\ 0 & 0 & -6 & 6 & 0 & -12 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -2 \\ 0 & 1 & 9 & 0 & 1 & 19 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \mathbf{V}_{-2} = \left\langle \mathbf{v}_3 = (0,-1,0,0,1,0)^{\mathsf{T}},\ \mathbf{v}_4 = (0,-1,-2,0,0,1)^{\mathsf{T}} \right\rangle$$

Thus $\mathbf{A}$ has only 4 linearly independent eigenvectors and is not diagonalizable.

The theory developed tells us that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ can be extended to a basis of $\mathbb{C}^6$ by two generalized eigenvectors $\mathbf{v}_5, \mathbf{v}_6$ associated to $\lambda_2 = -2$.

## Example (cont'd)

Therefore we compute

$$(\mathbf{A}+2\,\mathbf{I})^2 = \begin{pmatrix} 0 & 0 & 36 & 0 & 0 & 72 \\ 0 & -1 & -147 & 0 & -1 & -295 \\ 0 & 0 & -36 & 0 & 0 & -72 \\ -36 & 0 & 36 & 36 & 0 & 72 \\ 0 & 1 & 3 & 0 & 1 & 7 \\ 0 & 0 & 36 & 0 & 0 & 72 \end{pmatrix},$$

$$(\mathbf{A}+2\,\mathbf{I})^3 = \begin{pmatrix} 0 & 0 & 216 & 0 & 0 & 432 \\ 0 & 0 & -864 & 0 & 0 & -1728 \\ 0 & 0 & -216 & 0 & 0 & -432 \\ -216 & 0 & 216 & 216 & 0 & 432 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 216 & 0 & 0 & 432 \end{pmatrix}.$$

We see that $(\mathbf{A}+2\,\mathbf{I})^3$ has rank 2 and a 4-dimensional right kernel. $\implies W := \operatorname{rker}\left((\mathbf{A}+2\,\mathbf{I})^3\right)$ is the generalized eigenspace for $\lambda_2 = -2$ and we don't need to compute $(\mathbf{A}+2\,\mathbf{I})^4$.

# Example (cont'd)

A "nice" basis of $W$ is obtained by selecting a vector $\mathbf{w}_1 \in W$ satisfying $(\mathbf{A} + 2\,\mathbf{I})^2\mathbf{w}_1 \neq \mathbf{0}$, e.g., $\mathbf{w}_1 = \mathbf{e}_2 = (0,1,0,0,0,0)^\mathsf{T}$.

$$\Longrightarrow \mathbf{w}_1 = \mathbf{e}_2,\ \mathbf{w}_2 = (\mathbf{A}+2\,\mathbf{I})\mathbf{e}_2 = \begin{pmatrix}1\\0\\0\\1\\0\\0\end{pmatrix},\ \mathbf{w}_3 = (\mathbf{A}+2\,\mathbf{I})^2\mathbf{e}_2 = \begin{pmatrix}0\\-1\\0\\0\\1\\0\end{pmatrix}$$

can be taken as the first 3 basis vectors.

The 4th vector can be taken as an eigenvector linearly independent from $\mathbf{w}_3$, e.g., $\mathbf{w}_4 = \mathbf{v}_4 = (0,-1,-2,0,0,1)^\mathsf{T}$.

For $\mathbf{S} = (\mathbf{v}_1|\mathbf{v}_2|\mathbf{w}_1|\mathbf{w}_2|\mathbf{w}_3|\mathbf{w}_4)$ we then have

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \left(\begin{array}{cc|ccc|c} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -2 \end{array}\right), \dots$$

## Example (cont'd)

... reflecting that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_3, \mathbf{w}_4$ are eigenvectors of $\mathbf{A}$ and $\mathbf{A}\mathbf{w}_1 = -2\mathbf{w}_1 + \mathbf{w}_2$, $\mathbf{A}\mathbf{w}_2 = -2\mathbf{w}_2 + \mathbf{w}_3$.

A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is

$$\begin{aligned}
\mathbf{y}_1(t) &= \mathrm{e}^{4t}\mathbf{v}_1 = \mathrm{e}^{4t}(0,0,0,1,0,0)^\mathsf{T},\\
\mathbf{y}_2(t) &= \mathrm{e}^{4t}\mathbf{v}_2 = \mathrm{e}^{4t}(1,-4,-1,0,0,1)^\mathsf{T},\\
\mathbf{y}_3(t) &= \mathrm{e}^{-2t}\mathbf{w}_1 + t\,\mathrm{e}^{-2t}\mathbf{w}_2 + \tfrac{1}{2}t^2\,\mathrm{e}^{-2t}\mathbf{w}_3\\
&= \left(t\,\mathrm{e}^{-2t}, \mathrm{e}^{-2t} - \tfrac{1}{2}t^2\mathrm{e}^{-2t}, 0, t\,\mathrm{e}^{-2t}, \tfrac{1}{2}t^2\mathrm{e}^{-2t}, 0\right)^\mathsf{T},\\
\mathbf{y}_4(t) &= \mathrm{e}^{-2t}\mathbf{w}_2 + t\,\mathrm{e}^{-2t}\mathbf{w}_3\\
&= \left(\mathrm{e}^{-2t}, -t\,\mathrm{e}^{-2t}, 0, \mathrm{e}^{-2t}, t\,\mathrm{e}^{-2t}, 0\right)^\mathsf{T},\\
\mathbf{y}_5(t) &= \mathrm{e}^{-2t}\mathbf{w}_3 = \mathrm{e}^{-2t}(0,-1,0,0,1,0)^\mathsf{T},\\
\mathbf{y}_6(t) &= \mathrm{e}^{-2t}\mathbf{w}_4 = \mathrm{e}^{-2t}(0,-1,-2,0,0,1)^\mathsf{T}.
\end{aligned}$$

Note that only the vectors in the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$ or, equivalently, the matrix $\mathbf{S}$ is required to compute the fundamental system.

# Example (cont'd)

Finally we determine the canonical fundamental matrix of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ in the usual way (with the help of SageMath):

$$
e^{\mathbf{A}t} = \begin{pmatrix}
0 & e^{4t} & te^{-2t} & e^{-2t} & 0 & 0 \\
0 & -4e^{4t} & e^{-2t} - \frac{1}{2}t^2e^{-2t} & -te^{-2t} & -e^{-2t} & -e^{-2t} \\
0 & -e^{4t} & 0 & 0 & 0 & -2e^{-2t} \\
e^{4t} & 0 & te^{-2t} & e^{-2t} & 0 & 0 \\
0 & 0 & \frac{1}{2}t^2e^{-2t} & te^{-2t} & e^{-2t} & 0 \\
0 & e^{4t} & 0 & 0 & 0 & e^{-2t}
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & -4 & 1 & 0 & -1 & -1 \\
0 & -1 & 0 & 0 & 0 & -2 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}^{-1}
$$

$$
= \begin{pmatrix}
e^{-2t} & te^{-2t} & 3te^{-2t} + e^{4t} - e^{-2t} & 0 & te^{-2t} & 7te^{-2t} + 2e^{4t} - 2e^{-2t} \\
-te^{-2t} & -\frac{1}{2}t^2e^{-2t} + e^{-2t} & -\frac{3}{2}t^2e^{-2t} + te^{-2t} - 4e^{4t} + 4e^{-2t} & 0 & -\frac{1}{2}t^2e^{-2t} & -\frac{7}{2}t^2e^{-2t} + 2te^{-2t} - 8e^{4t} + 8e^{-2t} \\
0 & 0 & -e^{4t} + 2e^{-2t} & 0 & 0 & -2e^{4t} + 2e^{-2t} \\
-e^{4t} + e^{-2t} & te^{-2t} & 3te^{-2t} + e^{4t} - e^{-2t} & e^{4t} & te^{-2t} & 7te^{-2t} + 2e^{4t} - 2e^{-2t} \\
te^{-2t} & \frac{1}{2}t^2e^{-2t} & \frac{3}{2}t^2e^{-2t} - te^{-2t} & 0 & \frac{1}{2}t^2e^{-2t} + e^{-2t} & \frac{7}{2}t^2e^{-2t} - 2te^{-2t} \\
0 & 0 & e^{4t} - e^{-2t} & 0 & 0 & 2e^{4t} - e^{-2t}
\end{pmatrix}.
$$

# The Spectral Theorem for Real Symmetric Matrices

The set of eigenvalues of a square matrix $\mathbf{A}$ or an endomorphism $f\colon V \to V$ is known as the *spectrum* of $\mathbf{A}$, resp., of $f$.

## Theorem

*Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a symmetric matrix (i.e., $\mathbf{A} = \mathbf{A}^\mathsf{T}$).*

1. *The eigenvalues of $\mathbf{A}$ are real.*
2. *Eigenvectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ corresponding to different eigenvalues of $\mathbf{A}$ are orthogonal (i.e., $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n x_i y_i = 0$).*
3. *$\mathbf{A}$ is diagonalizable, and there exists an orthogonal matrix $\mathbf{Q}$ (i.e., $\mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}_n$) such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ is a diagonal matrix.*

# Inner Products

## Definition

Let $V$ be a vector space over $\mathbb{R}$. A map $\sigma\colon V \times V \to \mathbb{R}$ is called *inner product* on $V$ if it satisfies the following axioms:

(IP1) $\sigma(x, y + y') = \sigma(x, y) + \sigma(x, y')$ and $\sigma(x, cy) = c\,\sigma(x, y)$ for all $x, y, y' \in V$ and $c \in \mathbb{R}$, i.e., $\sigma$ is linear in the second argument;

(IP2) $\sigma(y, x) = \sigma(x, y)$ for all $x, y \in V$;

(IP3) $\sigma(x, x) \geq 0$ for all $x \in V$ with equality iff $x = 0_V$.

Note that (IP1) and (IP2) imply that $\sigma$ is also linear in the first argument, i.e., *bilinear*. The standard example of an inner product space is $\mathbb{R}^n$ with the dot product.

## Definition

Let $V$ be a vector space over $\mathbb{C}$. A map $\sigma\colon V \times V \to \mathbb{C}$ is called *inner product* on $V$ if it satisfies Axioms (IP1), (IP3) above and the following replacement for (IP2):

(IP2') $\sigma(y, x) = \overline{\sigma(x, y)}$ for all $x, y \in V$.

## Example

The standard example of a complex inner product space is $\mathbb{C}^n$ with

$$\sigma(\mathbf{x}, \mathbf{y}) = \overline{x}_1 y_1 + \overline{x}_2 y_2 + \cdots + \overline{x}_n y_n.$$

Writing $\mathbf{A}^* = \overline{\mathbf{A}}^{\mathsf{T}} = (\overline{a}_{ji})$ for the conjugate transpose of $\mathbf{A}$, we have $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x}^*\mathbf{y}$ and in particular $\sigma(\mathbf{x}, \mathbf{x}) = \mathbf{x}^*\mathbf{x} = \sum_{i=1}^n |x_i|^2$.

## Example

Let $V = \mathrm{C}^0([a, b])$ be the vector space (over $\mathbb{R}$) of continuous real-valued functions on $[a, b]$ and

$$\sigma(f, g) = \int_a^b f(x)g(x)\,\mathrm{d}x \quad \text{for } f, g \in V.$$

Then $(V, \sigma)$ is an inner product space.

The only property requiring justification is the second part of (IP3). If $f \neq 0$, there exists $x_0 \in (a, b)$ and $\delta > 0$ such that $[x_0 - \delta, x_0 + \delta] \subseteq [a, b]$ and $|f(x)| \geq \frac{1}{2}|f(x_0)| \neq 0$ for $x \in [x_0 - \delta, x_0 + \delta]$.

$$\Longrightarrow \sigma(f, f) = \int_a^b f(x)^2\,\mathrm{d}x \geq \tfrac{1}{4} f(x_0)^2 \cdot 2\delta > 0.$$

## Example

Replacing in the previous example "real-valued" by "complex-valued" and defining $\sigma\colon V \to V$ by

$$\sigma(f, g) = \int_a^b \overline{f(x)} g(x) \, \mathrm{d}x \quad \text{for } f, g \in V$$

gives a further example of a complex inner product space.

## Further Notes

- Inner products on complex vector spaces are not bilinear but rather *sesqui-linear* ($1\frac{1}{2}$-fold linear) in the following sense:

$$\begin{aligned}\sigma(x, c_1 y_1 + c_2 y_2) &= c_1 \, \sigma(x, y_1) + c_2 \, \sigma(x, y_2), \\ \sigma(c_1 x_1 + c_2 x_2, y) &= \overline{c}_1 \, \sigma(x_1, y) + \overline{c}_2 \, \sigma(x_2, y).\end{aligned}$$

- Inner products $\sigma(x, y)$ are often denoted without a name as $\langle x, y\rangle$ or $(x, y)$. For the standard inner product on $\mathbb{C}^n$ it is safe to use the same notation as for $\mathbb{R}^n$, because then $\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^n \overline{x}_i y_i$ restricts to the usual dot product on $\mathbb{R}^n$. The same applies to the length $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^n |x_i|^2} = \sqrt{\sum_{i=1}^n (\operatorname{Re} x_i)^2 + \sum_{i=1}^n (\operatorname{Im} x_i)^2}$.

# Inner Products—The Basic Theory

The basic theory of inner product spaces (both real and complex) very much parallels that of the dot product on $\mathbb{R}^n$.

- The *Cauchy-Schwarz Inequality* generalizes to $|\sigma(x, y)|^2 \leq \sigma(x, x)\sigma(y, y)$ and implies that $d(x, y) = \sqrt{\sigma(x - y, x - y)}$ defines a metric on $V$.
- in $(V, \sigma)$ *orthogonality* of vectors $x, y$ (denoted as usual by $x \perp y$) is defined by $\sigma(x, y) = 0$. For a subspace $U$ of $V$ the orthogonal subspace $U^\perp = \{y \in V; \sigma(x, y) = 0 \text{ for all } x \in U\}$ satisfies $U \cap U^\perp = \{0\}$ by Axiom (IP3) and, provided $V$ is finite-dimensional, $U + U^\perp = V$ (thus $V = U \oplus U^\perp$), $U^{\perp\perp} = U$.

- Assuming $n = \dim(V) < \infty$, given any basis $v_1, \dots, v_n$ of $V$ there exists another basis $u_1, \dots, u_n$ of $V$ satisfying $\langle u_1, \dots, u_i \rangle = \langle v_1, \dots, v_i \rangle$ for $1 \leq i \leq n$ and

$$\sigma(u_i, u_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{ONB}$$

  This basis is defined recursively by

$$u_i' = v_i - \sum_{j=1}^{i-1} \sigma(u_j, v_i) u_j, \quad u_i = \frac{1}{\sqrt{\sigma(u_i', u_i')}} u_i' \quad (1 \leq i \leq n).$$

  A basis satisfying (ONB) is said to be an *orthonormal basis*, and the computation of an orthonormal basis using the stated formulas is called *Gram-Schmidt* orthogonalization.

- A matrix $\mathbf{U} \in \mathbb{C}^{n \times n}$ is said to be *unitary* if $\mathbf{U}^*\mathbf{U} = \mathbf{I}_n$. Equivalently, the columns of $\mathbf{U}$ (or the rows of $\mathbf{U}$) form an orthonormal basis of $\mathbb{C}^n$ relative to the standard inner product $\sigma(\mathbf{x}, \mathbf{y}) = \mathbf{x} \cdot \mathbf{y} = \mathbf{x}^*\mathbf{y}$.

## Exercise

Show that the set $\mathrm{O}(n)$ of real orthogonal $n \times n$ matrices, as well as the set $\mathrm{U}(n)$ of complex unitary $n \times n$ matrices, are closed under matrix multiplication/inversion.

*Note:* Together with the trivial fact that the (rational) $n \times n$ identity matrix is both orthogonal and unitary, this implies that each of $\mathrm{O}(n)$ and $\mathrm{U}(n)$ forms a group under matrix multiplication.

## Exercise

Consider the vector space $P_3$ of polynomials in $\mathbb{R}[X]$ of degree at most 3, equipped with the inner product

$$\sigma(f,g) = \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x\,.$$

Compute an orthonormal basis of $(P_3,\sigma)$ by applying the Gram-Schmidt algorithm to the power basis $\{1, X, X^2, X^3\}$.

## Proof of the Spectral Theorem.

(1) Let $\lambda$ be an eigenvalue of $\mathbf{A}$ in $\mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ a corresponding eigenvector. We need to show $\lambda = \overline{\lambda}$. For this we compute $\mathbf{x}^*\mathbf{A}\mathbf{x}$ in two ways:

$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\lambda\mathbf{x} = \lambda\,\mathbf{x}^*\mathbf{x},$$
$$\mathbf{x}^*\mathbf{A}\mathbf{x} = \mathbf{x}^*\mathbf{A}^*\mathbf{x} = (\mathbf{A}\mathbf{x})^*\mathbf{x} = (\lambda\mathbf{x})^*\mathbf{x}) = \overline{\lambda}\,\mathbf{x}^*\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$, we have $\mathbf{x}^*\mathbf{x} \neq 0$ by (IP3), and hence $\lambda = \overline{\lambda}$.

(2) Assume $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ satisfy $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbf{A}\mathbf{y} = \mu\mathbf{y}$ and $\lambda \neq \mu$. Proceeding in a similar way, we have

$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{y} = \mathbf{x}^{\mathsf{T}}\mu\mathbf{y} = \mu\,\mathbf{x}^{\mathsf{T}}\mathbf{y},$$
$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{y} = \mathbf{x}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{y} = (\mathbf{A}\mathbf{x})^{\mathsf{T}}\mathbf{y} = (\lambda\mathbf{x})^{\mathsf{T}}\mathbf{y} = \lambda\,\mathbf{x}^{\mathsf{T}}\mathbf{y}$$

Since $\lambda \neq \mu$, this implies $\mathbf{x}^{\mathsf{T}}\mathbf{y} = \mathbf{0}$, i.e., $\mathbf{x} \perp \mathbf{y}$.

# Proof cont'd.

(3) Let $\lambda \in \mathbb{R}$ be an eigenvalue of $\mathbf{A}$ and $\mathbf{x}$ a corresponding eigenvector. W.l.o.g. we may assume $|\mathbf{x}| = 1$. Then there exists an orthonormal basis $\mathbf{u}_1, \dots, \mathbf{u}_n$ of $\mathbb{R}^n$ with $\mathbf{u}_1 = \mathbf{x}$ and hence an orthogonal matrix $\mathbf{Q}_1 \in \mathbb{R}^{n \times n}$ with first column $\mathbf{x}$, e.g., $\mathbf{Q}_1 = (\mathbf{u}_1 | \dots | \mathbf{u}_n)$.

$$\implies \quad \mathbf{Q}_1^{-1}\mathbf{A}\mathbf{Q}_1 = \mathbf{Q}_1^{\mathsf{T}}\mathbf{A}\mathbf{Q}_1 = \left(\begin{array}{c|c} \lambda & \mathbf{v}^{\mathsf{T}} \\ \hline \mathbf{0} & \mathbf{A}' \end{array}\right)$$

for some $\mathbf{A}' \in \mathbb{R}^{(n-1)\times(n-1)}$ and $\mathbf{v} \in \mathbb{R}^{n-1}$.
Since $(\mathbf{Q}_1^{\mathsf{T}}\mathbf{A}\mathbf{Q}_1)^{\mathsf{T}} = \mathbf{Q}_1^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}\mathbf{Q}_1^{\mathsf{TT}} = \mathbf{Q}_1^{\mathsf{T}}\mathbf{A}\mathbf{Q}_1$, the indicated block matrix is symmetric, and hence $\mathbf{v} = \mathbf{0}$, $\mathbf{A}' = \mathbf{A}'^{\mathsf{T}}$.
Using induction on $n$, we may suppose $\mathbf{Q}'^{-1}\mathbf{A}'\mathbf{Q}' = \mathbf{D}'$ for some diagonal matrix $\mathbf{D}'$ and orthogonal matrix $\mathbf{Q}'$. It is then readily checked that

$$\mathbf{Q}_2 = \left(\begin{array}{c|c} 1 & \mathbf{0}^{\mathsf{T}} \\ \hline \mathbf{0} & \mathbf{Q}' \end{array}\right) \quad \text{satisfies} \quad \mathbf{Q}_2^{-1}(\mathbf{Q}_1^{-1}\mathbf{A}\mathbf{Q}_1)\mathbf{Q}_2 = \left(\begin{array}{c|c} \lambda & \mathbf{0}^{\mathsf{T}} \\ \hline \mathbf{0} & \mathbf{D}' \end{array}\right).$$

$\implies \mathbf{Q} = \mathbf{Q}_1\mathbf{Q}_2$ (which is orthogonal as well) has the required property. □

## Note

The matrix $\mathbf{Q}$ in the Spectral Theorem can be computed by first determining for each eigenvalue of $\mathbf{A}$ a basis of the corresponding eigenspace and then applying Gram-Schmidt orthogonalization to these bases. Since eigenvectors corresponding to different eigenvalues of $\mathbf{A}$ are orthogonal, the union of these bases is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. Arranging the basis vectors as columns of a matrix then gives the desired orthogonal matrix $\mathbf{Q}$.

## Example

We diagonalize the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix} \in \mathbb{R}^{3\times 3}.$$

$$\chi_{\mathbf{A}}(X) = \begin{vmatrix} X-1 & -4 & 2 \\ -4 & X-1 & 2 \\ 2 & 2 & X+2 \end{vmatrix} = X^3 - 27X - 54 = (X-6)(X+3)^2$$

$\implies \lambda_1 = 6$, $\lambda_2 = -3$ (the latter with algebraic multiplicity 2).

$\underline{\lambda_1 = 6:}$

$$\mathbf{A} - 6\,\mathbf{I} = \begin{pmatrix} -5 & 4 & -2 \\ 4 & -5 & -2 \\ -2 & -2 & -8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 4 \\ 0 & -9 & -18 \\ 0 & 9 & 18 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 2 and right kernel $\mathbb{R}(-2,-2,1)^{\mathsf{T}}$.
A corresponding unit eigenvector is $\mathbf{u}_1 = \frac{1}{3}(-2,-2,1)^{\mathsf{T}}$.

## Example (cont'd)

$\underline{\lambda_2 = -3:}$

$$\mathbf{A} + 3\,\mathbf{I} = \begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has rank 1 and right kernel $\mathbb{R}(0,1,2)^{\mathsf{T}} + \mathbb{R}(1,0,2)^{\mathsf{T}}$. Orthonormalization of the basis vectors yields

$$\mathbf{u}_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \mathbf{u}_3' = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \frac{4}{5}\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}, \ \mathbf{u}_3 = \frac{1}{\sqrt{21}}\begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}.$$

$$\Longrightarrow \begin{pmatrix} -\frac{2}{3} & 0 & \frac{1}{\sqrt{21}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{21}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{21}} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{2}{3} & 0 & \frac{1}{\sqrt{21}} \\ -\frac{2}{3} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{21}} \\ \frac{1}{3} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{21}} \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

# Example (con't)

A nicer orthogonal matrix that does the same job is

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}.$$

Why?

It's because the last column is $\mathbf{u}_1$, the eigenvector corresponding to $\lambda_1 = 6$, and the first two columns span the orthogonal space $(\mathbb{R}\mathbf{u}_1)^\perp$, which is the eigenspace corresponding to $\lambda_2 = -3$, as we know from the Spectral Theorem.

Since $\mathbf{u}_1$ has moved into the last column and $\mathbf{Q} = \mathbf{Q}^\mathsf{T} = \mathbf{Q}^{-1}$, we have

$$\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

From this you can perhaps guess how the lecturer designed this example. (Compare with the earlier example of a non-diagonalizable matrix.)

# The Principal Axes Theorem

## Definition

A *quadratic form* on $F^n$ is a function $q\colon F^n \to F$ of the form

$$q(\mathbf{x}) = q(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n}^{n} q_{ij} x_i x_j \quad \text{with } q_{ij} \in F.$$

In the cases $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$ (more generally, for fields $F$ in which $2 = 1 + 1 \neq 0$) there exists a unique symmetric matrix $\mathbf{A} \in F^{n \times n}$ such that $q(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ for $\mathbf{x} \in F^n$. In terms of $q_{ij}$ the matrix $\mathbf{A} = (a_{ij})$ is defined by $a_{ii} = q_{ii}$ and $a_{ij} = a_{ji} = q_{ij}/2$ for $i \neq j$.

## Observation

Changing coordinates in $F^n$, i.e., $\mathbf{x} = \mathbf{S}\mathbf{x}'$ for some invertible matrix $\mathbf{S} \in F^{n \times n}$, changes $q(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ into

$$(\mathbf{S}\mathbf{x}')^\mathsf{T}\mathbf{A}\mathbf{S}\mathbf{x}' = \mathbf{x}'^\mathsf{T}(\mathbf{S}^\mathsf{T}\mathbf{A}\mathbf{S})\mathbf{x}',$$

i.e., transforms the representing matrix $\mathbf{A}$ into $\mathbf{A}' = \mathbf{S}^\mathsf{T}\mathbf{A}\mathbf{S}$.

Hence the Spectral Theorem has the following

## Corollary (Principal Axes Theorem)

*Every quadratic form on $\mathbb{R}^n$ can be transformed into a "diagonal form" $\lambda_1 {x'_1}^2 + \cdots + \lambda_n {x'_n}^2$ by means of an Euclidean motion (isometry) that fixes the origin.*

## Proof.

The Euclidean motions fixing the origin are precisely the "orthogonal" linear transformations $\mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x}' \mapsto \mathbf{Q}\mathbf{x}'$ with $\mathbf{Q} \in \mathbb{R}^{n \times n}$ satisfying $\mathbf{Q}^\mathsf{T}\mathbf{Q} = \mathbf{I}_n$, and the Spectral Theorem on account of $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{Q}^\mathsf{T}\mathbf{A}\mathbf{Q}$ gives the result. □

The directions determined by the columns of $\mathbf{Q}$ are called *principal axes* of $q$ (in analogy with the semi-major/minor axes of two-dimensional ellipses/hyperbolas).

If $\mathbf{A}$ has $n$ distinct eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ (equivalently, all eigenspaces are 1-dimensional) then the principal axes $\mathbb{R}\mathbf{u}_i$ of $q$ are uniquely determined and characterized by

$$q(x'_1\mathbf{u}_1 + \cdots + x'_n\mathbf{u}_n) = \lambda_1 {x'_1}^2 + \cdots + \lambda_n {x'_n}^2.$$

The classification of quadratic forms yields a geometric classification of the corresponding *quadrics* (level sets of quadratic forms).

## Example

The quadratic form corresponding to our example symmetric matrix is

$$\begin{aligned} q(x_1, x_2, x_3) &= \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 4 & -2 \\ 4 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= x_1^2 + x_2^2 - 2x_3^2 + 8x_1x_2 - 4x_1x_3 - 4x_2x_3. \end{aligned}$$

Now consider a quadric $Q$ associated to $q$, say the one with equation $q(x_1, x_2, x_3) = 1$. The reasoning in the preceding example implies that in coordinates w.r.t. the (ordered) orthonormal basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ given by

$$\mathbf{u}_1 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{3}\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}, \quad \mathbf{u}_3 = \frac{1}{3}\begin{pmatrix} -2 \\ -2 \\ 1 \end{pmatrix}$$

the quadric has the equation $-3x_1'^2 - 3x_2'^2 + 6x_3'^2 = 1$. (Alternatively, the linear isometry of $\mathbb{R}^3$ sending $\mathbf{e}_i$ to $\mathbf{u}_i$, $i = 1, 2, 3$, maps the quadric $-3x_1'^2 - 3x_2'^2 + 6x_3'^2 = 1$ to $Q$.)

# Example (cont'd)

As we know from Calculus III, $Q$ is a hyperboloid of two sheets. The type of $Q$ can also be obtained by determining the Sylvester canonical form of $q$, but the present calculation provides more information:

For example, since $\lambda_1 = -3 = \lambda_2$, $Q$ is rotation-symmetric with respect to the 3rd principal axis $\mathbb{R}\mathbf{u}_3$. Also we can see that the vertices of $Q$ on $\mathbb{R}\mathbf{u}_3$ have coordinates $x_1' = x_2' = 0$, $x_3' = \pm\frac{1}{\sqrt{6}}$ and hence are the points on $\mathbb{R}\mathbf{u}_3$ at distance $\frac{1}{\sqrt{6}}$ from the origin (center).

*Last but not least:* The transforming matrix

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$$

satisfies $\mathbf{Q}^2 = \mathbf{Q}\mathbf{Q}^\mathsf{T} = \mathbf{I}_3$, since it is both orthogonal and symmetric. One eigenvalue of $\mathbf{Q}$ is $-1$ (the constant row/column sum, with corresponding eigenvector $(1,1,1)^\mathsf{T}$). The other two eigenvalues are 1, as applying $f_\mathbf{Q}$ to a basis of the orthogonal space (the plane $x_1 + x_2 + x_3 = 0$) shows. Thus $f_\mathbf{Q}(\mathbf{x}) = \mathbf{Q}\mathbf{x}$ is the reflection at this plane.

# A Spectral Theorem for Complex Matrices

The spectral theorem for real symmetric matrices can be fomulated as follows: $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A} = \mathbf{A}^\mathsf{T}$ iff there exists an orthogonal matrix $\mathbf{Q} \in \mathbb{R}^{n\times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n\times n}$ such that $\mathbf{Q}^{-1}\mathbf{A}\mathbf{Q} = \mathbf{D}$. The yet unproved if-part is easy:

$$\mathbf{A}^\mathsf{T} = (\mathbf{Q}\mathbf{D}\mathbf{Q}^{-1})^\mathsf{T} = (\mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T})^\mathsf{T} = \mathbf{Q}\mathbf{D}^\mathsf{T}\mathbf{Q}^\mathsf{T} = \mathbf{Q}\mathbf{D}\mathbf{Q}^\mathsf{T} = \mathbf{A}.$$

## Theorem (Spectral Theorem for Normal Matrices)

*For $\mathbf{A} \in \mathbb{C}^{n\times n}$ the following properties are equivalent:*

1. $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^*$*, i.e.,* $\mathbf{A}$ *commutes with its conjugate transpose.*
2. *There exists a unitary matrix* $\mathbf{U} \in \mathbb{C}^{n\times n}$ *and a diagonal matrix* $\mathbf{D} \in \mathbb{C}^{n\times n}$ *such that* $\mathbf{U}^{-1}\mathbf{A}\mathbf{U} = \mathbf{D}$*.*

Matrices satisfying Condition (1) of this theorem are said to be *normal*. Examples include real symmetric and orthogonal matrices, real *skew-symmetric matrices* ($\mathbf{A}^\mathsf{T} = -\mathbf{A}$), *hermitean* and *skew-hermitean* matrices ($\mathbf{A}^* = \mathbf{A}$, resp., $\mathbf{A}^* = -\mathbf{A}$), and unitary matrices.

## Exercise

Prove the if-part "(2)$\Longrightarrow$(1)" of the spectral theorem for normal matrices.

## Exercise

Show that $x^2 + xy + y^2 = 1$ defines an ellipse in $\mathbb{R}^2$ and compute its principal axes and the lengths $a$, $b$ of the semi-major and semi-minor axis, respectively.

## Exercise

Diagonalize the rotation and reflection matrices

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}, \quad S(\phi) = \begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}$$

over $\mathbb{C}$. Use the characteristic polynomial approach for both types of matrices; don't use geometric properties of reflections as we have done in the lecture.

## Exercise

Let $\mathbf{Q} \in \mathbb{R}^{n\times n}$ be an orthogonal matrix.

1. Show that every eigenvalue $\lambda \in \mathbb{C}$ of $\mathbf{Q}$ satisfies $|\lambda| = 1$.

   *Hint:* Suppose $\mathbf{Qx} = \lambda\mathbf{x}$ and evaluate $(\mathbf{Qx})^*(\mathbf{Qx})$ in two ways.

2. Show that eigenvectors in $\mathbb{C}^n$ corresponding to different eigenvalues of $\mathbf{Q}$ are orthogonal with respect to the standard inner product on $\mathbb{C}^n$.

   *Hint:* Suppose $\mathbf{Qx} = \lambda\mathbf{x}$, $\mathbf{Qy} = \mu\mathbf{y}$ and evaluate $(\mathbf{Qx})^*(\mathbf{Qy})$ in two ways.

3. Now assume $n = 3$. Show that either $\mathbf{Q}$ or $-\mathbf{Q}$ represents a rotation of $\mathbb{R}^3$ with axis through the origin, and hence that the Euclidean motion in the Principal Axes Theorem can be taken as a rotation.

   *Hint:* Checking the possibilities for the roots of $\chi_{\mathbf{Q}}(X)$, show that $\det(\mathbf{Q}) = \pm 1$ is an eigenvalue of $\mathbf{Q}$.

## Exercise

Show that for a matrix $\mathbf{Q} \in \mathbb{R}^{3\times 3}$ the following are equivalent:

1. $f_{\mathbf{Q}}(\mathbf{x}) = \mathbf{Qx}$ is a reflection at some plane;

2. $\mathbf{Q} = \mathbf{Q}^{\mathsf{T}}$, $\mathbf{Q}^2 = \mathbf{I}_3$, $\mathbf{Q} \neq \pm\mathbf{I}_3$, and $\det \mathbf{Q} = -1$.

# Matrix Polynomials

## The High Road

In any polynomial $a(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_d X^d \in F[X]$ we can substitute a square matrix $\mathbf{A} \in F^{n\times n}$ in the following way:

$$a(A) = a_0 \mathbf{I}_n + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \cdots + a_d \mathbf{A}^d,$$

which is in $F^{n\times n}$ as well.

For fixed $\mathbf{A}$ the substitution map $F[X] \to F[\mathbf{A}]$, $a(X) \mapsto a(\mathbf{A})$ is a ring homomorphism, i.e., it sends $a(X) + b(X)$ to $a(\mathbf{A}) + b(\mathbf{A})$, the polynomial product $a(X)b(X)$ to the matrix product $a(\mathbf{A})b(\mathbf{A})$, and the polynomial 1 to the identity matrix. (For those who wonder how $F[\mathbf{A}]$ is defined—it's the set of all matrix polynomials in $\mathbf{A}$, which forms a subring of the ring $F^{n\times n}$.)

Unlike the polynomial ring $F[X]$, which has infinite dimension over $F$, the ring $F[\mathbf{A}]$ is finite-dimensional: $\dim F[\mathbf{A}] \leq n^2$, since $F[\mathbf{A}]$ is a subspace of $F^{n\times n}$.

### Question

*What can be said about the dimension of $F[\mathbf{A}]$ ?*

## Observation

$\dim F[\mathbf{A}]$ is equal to the smallest integer $r \geq 0$ such that $\mathbf{I}_n, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^r$ are linearly dependent, since a dependency relation $c_0\mathbf{I} + c_1\mathbf{A} + \cdots + c_r\mathbf{A}^r = \mathbf{0}$, $c_r \neq 0$, can be used to write all higher powers of $\mathbf{A}$ as linear combinations of $\mathbf{I}_n, \mathbf{A}, \dots, \mathbf{A}^{r-1}$, e.g.,

$$\begin{aligned}
\mathbf{A}^r &= \sum_{i=0}^{r-1}(-c_i/c_r)\mathbf{A}^i,\\
\mathbf{A}^{r+1} &= \sum_{i=0}^{r-1}(-c_i/c_r)\mathbf{A}^{i+1}\\
&= (-c_{r-1}/c_r)\sum_{j=0}^{r-1}(-c_j/c_r)\mathbf{A}^j + \sum_{i=1}^{r-1}(-c_{i-1}/c_r)\mathbf{A}^i, \quad \text{etc.}
\end{aligned}$$

## Definition

The monic polynomial $a(X) = a_0 + a_1X + \cdots + a_{r-1}X^{r-1} + X^r \in F[X]$ of smallest degree satisfying $a(\mathbf{A}) = \mathbf{0}$ is called *minimum polynomial* of $\mathbf{A}$ and denoted by $\mu_{\mathbf{A}}(X)$.

By the preceding observation, $\dim F[\mathbf{A}] = \deg \mu_{\mathbf{A}}(X)$.

## Note

It can be shown that the minimum polynomial $\mu_{\mathbf{A}}(X)$ of $\mathbf{A}$ is indeed uniquely determined and divides every other polynomial $a(X)$ that satisfies $a(\mathbf{A}) = \mathbf{0}$. ($\Longrightarrow$ The set of polynomials in $F[X]$ having $\mathbf{A}$ as a “zero” is precisely the set of polynomial multiples of $\mu_{\mathbf{A}}(X)$.)

## Theorem (Cayley-Hamilton)

*For any square matrix $\mathbf{A} \in F^{n\times n}$ we have $\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$.*

## Proof.

See any book on Linear Algebra. $\square$

It follows that $\mu_{\mathbf{A}}(X)$ divides $\chi_{\mathbf{A}}(X)$ and $\dim F[\mathbf{A}] = \deg \mu_{\mathbf{A}}(X) \leq n$. In a way this is surprising, since the bound based on $F[\mathbf{A}] \subseteq F^{n\times n}$ guarantees only $\dim F[\mathbf{A}] \leq n^2$.

## Example

For any $2 \times 2$ matrix $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 - (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} + (ad-bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In this case $\mu_{\mathbf{A}}(X) = \chi_{\mathbf{A}}(X) = X^2 - (a+d)X + ad - bc$ except for scalar multiples $\left(\begin{smallmatrix} a & \\ & a \end{smallmatrix}\right)$ of the identity matrix, which have $\mu_{\mathbf{A}}(X) = X - a$.

## Example

The minimum polynomial of $\mathbf{A} \in F^{n\times n}$ has degree 1 iff $\mathbf{A}$ is a scalar multiple of the $n \times n$ identity matrix $\mathbf{I}_n$. This is because substituting $\mathbf{A}$ into $X - \lambda$ gives $\mathbf{A} - \lambda\mathbf{I}_n$.

## Example

The minimum polynomial of the diagonal matrix $\mathbf{D} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$ over $\mathbb{R}$ is $X^2 - 1 = (X-1)(X+1)$.
This follows from $\mathbf{D}^2 = \mathbf{I}_3$ and the fact that $\mathbf{D} \neq \pm\mathbf{I}_3$

## Example

The minimum polynomial of $\mathbf{A} = \begin{pmatrix} 2&0&0\\1&2&0\\0&1&2 \end{pmatrix}$ over $\mathbb{R}$ is $(X-2)^3$.

This follows from $\mathbf{A} - 2\,\mathbf{I}_3 = \begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}$ and $\begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}^3 = \mathbf{0}$ but $\begin{pmatrix} 0&0&0\\1&0&0\\0&1&0 \end{pmatrix}^2 \neq \mathbf{0}$ (check it!).
Note that we only have to check divisors $p(X)$ of $(X-2)^3$ for $p(\mathbf{A}) = \mathbf{0}$.

## Example

The minimum polynomial of

$$\mathbf{J} = \left(\begin{array}{ccc|cc} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

over $\mathbb{R}$ is $(X-2)^3$.

This is because block diagonal matrices $\mathbf{A}$, $\mathbf{B}$ are multiplied like ordinary diagonal matrices (except that not necessarily $\mathbf{AB} = \mathbf{BA}$), and hence that the minimum polynomial of a block diagonal matrix is the least common multiple of the minimum polynomials of its diagonal blocks.

In this example we have $\mathbf{J} = \begin{pmatrix} \mathbf{J}_1 & 0 \\ 0 & \mathbf{J}_2 \end{pmatrix}$, with $\mu_{\mathbf{J}_1}(X) = (X-2)^3$ and $\mu_{\mathbf{J}_2}(X) = (X-2)^2$.

## Example (Companion matrices)

Given any monic polynomial $a(X) = a_0 + a_1 X + \cdots + a_{n-1}X^{n-1} + X^n \in F[X]$ we can find a matrix $\mathbf{A} \in F^{n\times n}$, which has $a(X)$ both as its minimum polynomial and its characteristic polynomial. Such a matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & \ddots & \vdots & -a_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix},$$

called *companion matrix* of $a(X)$.

It is straightforward to verify $\chi_{\mathbf{A}}(X) = a(X)$ (expand $\det(X\,\mathbf{I}_n - \mathbf{A})$ along the last column), and the cyclic nature of $\mathbf{A}$ implies $\mu_{\mathbf{A}}(X) = \chi_{\mathbf{A}}(X)$ (look at the entries of $\mathbf{I}_n, \mathbf{A}, \mathbf{A}^2, \dots, \mathbf{A}^{n-1}$ in the first column).

As a concrete example, the polynomial $X^2 + 1$ (over any field) has companion matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,
which over the binary field reduces to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

## Example

Minimal polynomials are ubiquitous in Abstract Algebra, and here are further examples not related to matrices.

1. The minimum polynomial of $\mathrm{i} \in \mathbb{C}$ over $\mathbb{R}$ is $X^2 + 1$.
   This is so, because the monic polynomial of smallest degree with coefficients in $\mathbb{R}$ having $\mathrm{i}$ as a root is $X^2 + 1$.
2. The minimum polynomial of $\sqrt{2}$ over $\mathbb{Q}$ is $X^2 - 2$.
   For this note that there is no polynomial of degree 1 with rational coefficients having $\sqrt{2}$ as a root.
3. The minimum polynomial of $\sqrt[3]{2}$ over $\mathbb{Q}$ is $X^3 - 2$.
   For this note that
   $$X^3 - 2 = (X - \sqrt[3]{2})(X - \omega\sqrt[3]{2})(X - \omega^2\sqrt[3]{2}),$$
   where $\omega = \mathrm{e}^{2\pi\mathrm{i}/3} = \frac{1}{2}\left(-1 + \mathrm{i}\sqrt{3}\right)$, has no factor of degree 1 or 2 with cofficients in $\mathbb{Q}$.
4. The numbers $\pi$ and $\mathrm{e}$ have no minimum polynomials over $\mathbb{Q}$, because they are *transcendental*, i.e., don't satisfy any polynomial equation with coeffcients in $\mathbb{Q}$.

Recall that a necessary condition for diagonalizing $\mathbf{A} \in F^{n\times n}$ (over $F$) is that $\chi_{\mathbf{A}}(X)$ splits into linear factors in $F[X]$. (This is automatically satisfied for $F = \mathbb{C}$.)

## Theorem

*Suppose that the characteristic polynomial of $\mathbf{A} \in F^{n\times n}$ factors in $F[X]$ as $\chi_{\mathbf{A}}(X) = \prod_{i=1}^{r}(X - \lambda_i)^{m_i}$ with $\lambda_1, \dots, \lambda_r$ distinct and $m_i \geq 1$ (i.e., $\lambda_1, \dots, \lambda_r$ are the eigenvalues of $\mathbf{A}$ and $m_1, \dots, m_r$ the corresponding algebraic multiplicities).*

1. *The minimum polynomial $\mu_{\mathbf{A}}(X)$ of $\mathbf{A}$ (over $F$) divides $\chi_{\mathbf{A}}(X)$ and is a multiple of $(X - \lambda_1)(X - \lambda_2)\cdots(X - \lambda_r)$.*
2. *$\mathbf{A}$ is diagonalizable (over $F$) iff $\mu_{\mathbf{A}}(X) = (X - \lambda_1)(X - \lambda_2)\cdots(X - \lambda_r)$; equivalently, there exist distinct $c_1, \dots, c_s \in F$ such that $(\mathbf{A} - c_1\mathbf{I}_n)\cdots(\mathbf{A} - c_s\mathbf{I}_n) = \mathbf{0}$.*

## Proof.

(1) Since $\chi_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$ (the Cayley-Hamilton Theorem), $\mu_{\mathbf{A}}(X)$ divides $\chi_{\mathbf{A}}(X)$.
Suppose $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue $\lambda$. Since $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ implies $p(\mathbf{A})\mathbf{v} = p(\lambda)\mathbf{v}$ for all polynomials $p(X) \in F[X]$, we can set $p(X) = \mu_{\mathbf{A}}(X)$ and conclude from

# Proof cont'd.

$\mu_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$ that $\mu_{\mathbf{A}}(\lambda) = 0$. Thus $\lambda_1 \dots, \lambda_r$ are roots of $\mu_{\mathbf{A}}(X)$ and $(X - \lambda_1)(X - \lambda_2) \cdots (X - \lambda_r)$ divides $\mu_{\mathbf{A}}(X)$.

(2) $\Longrightarrow$: One can check that $\mathbf{D} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$ implies $p(\mathbf{D}) = \mathbf{S}^{-1}p(\mathbf{A})\mathbf{S}$. Hence $p(\mathbf{D}) = \mathbf{0} \iff p(\mathbf{A}) = \mathbf{0}$ and $\mu_{\mathbf{D}}(X) = \mu_{\mathbf{A}}(X)$. Moreover, a diagonal matrix $\mathbf{D} \in F^{n\times n}$ satisfies $\prod_{i=1}^{r}(\mathbf{D} - \lambda_i\mathbf{I}_n) = \mathbf{0}$ if each diagonal element of $\mathbf{D}$ is equal to some $\lambda_i$. In particular this holds if we take $\lambda_1, \dots, \lambda_r$ as the distinct diagonal elements of $\mathbf{D}$.

$\Longleftarrow$: For $1 \leq i \leq r$ we set $q_i(X) = \mu_{\mathbf{A}}(X)/(X - \lambda_i)$. Since the polynomials $q_i(X)$ are relatively prime, there are polynomials $b_i(X) \in F[X]$ satisfying $b_1(X)q_1(X) + \cdots + b_r(X)q_r(X) = 1$. Substituting $\mathbf{A}$ for $X$ gives $b_1(\mathbf{A})q_1(\mathbf{A}) + \cdots + b_r(\mathbf{A})q_r(\mathbf{A}) = \mathbf{I}_n$ and hence

$$\mathbf{v} = \mathbf{I}_n\mathbf{v} = b_1(\mathbf{A})q_1(\mathbf{A})\mathbf{v} + \cdots + b_r(\mathbf{A})q_r(\mathbf{A})\mathbf{v}$$

for all $\mathbf{v} \in F^n$. Since $(\mathbf{A} - \lambda_i\,\mathbf{I})q_i(\mathbf{A}) = \mu_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$, the vectors $q_i(\mathbf{A})\mathbf{v}$, and hence $b_i(\mathbf{A})q_i(\mathbf{A})\mathbf{v}$ as well, are eigenvectors of $\mathbf{A}$. Thus $F^n$ is spanned by eigenvectors of $\mathbf{A}$, showing that $\mathbf{A}$ is diagonalizable. $\square$

## Example (Projection matrices)

A matrix $\mathbf{P} \in F^{n\times n}$ is called *projection matrix* if $\mathbf{P}^2 = \mathbf{P}$.

This can also be written as $\mathbf{P}^2 - \mathbf{P} = \mathbf{P}(\mathbf{P} - \mathbf{I}_n) = \mathbf{0}$ and implies that $\mu_{\mathbf{P}}(X)$ divides $X^2 - X = X(X-1)$.
Except for the cases $\mu_{\mathbf{P}}(X) = X$ ($\mathbf{P} = \mathbf{0}$) and $\mu_{\mathbf{P}}(X) = X - 1$ ($\mathbf{P} = \mathbf{I}_n$) we have $\mu_{\mathbf{P}}(X) = X^2 - X$.

By the theorem $\mathbf{P}$ is diagonalizable.

Now we show this fact directly: We can write any $\mathbf{v} \in F^n$ as

$$\mathbf{v} = \mathbf{P}\mathbf{v} + (\mathbf{I}_n - \mathbf{P})\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$$

with $\mathbf{v}_1 = \mathbf{P}\mathbf{v}$, $\mathbf{v}_2 = (\mathbf{I}_n - \mathbf{P})\mathbf{v} = \mathbf{v} - \mathbf{P}\mathbf{v}$.
$\mathbf{P}\mathbf{v}_2 = \mathbf{P}(\mathbf{I}_n - \mathbf{P})\mathbf{v} = (\mathbf{P}^2 - \mathbf{P})\mathbf{v} = \mathbf{0}$
$\mathbf{P}\mathbf{v}_1 = \mathbf{P}^2\mathbf{v}_1 = \mathbf{P}\mathbf{v}_1 = \mathbf{v}_1$
These computations show that $F^n$ is the (direct) sum of the kernel of $f_{\mathbf{P}}$ (eigenspace corresponding to $\lambda_1 = 0$) and the range of $f_{\mathbf{P}}$ (equal to the eigenspace corresponding to $\lambda_2 = 1$).

If $\operatorname{rk}\mathbf{P} = r$ and $\mathbf{S} = (\mathbf{v}_1|\dots|\mathbf{v}_n)$ has in its first $r$ columns a basis of $f_{\mathbf{P}}(\mathbb{R}^n)$ and in its last $n-r$ columns a basis of $\ker f_{\mathbf{P}}$, we have

$$\mathbf{S}^{-1}\mathbf{P}\mathbf{S} = \left(\begin{array}{c|c} \mathbf{I}_r & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right).$$

## Example (projection matrices cont'd)

Now assume in addition that $F = \mathbb{R}$ and $\mathbf{P}$ is symmetric.Then, by the Spectral Theorem, $\mathbf{v}_1$ and $\mathbf{v}_2$ must be perpendicular. Indeed,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (\mathbf{P}\mathbf{v})^\mathsf{T}(\mathbf{I}_n - \mathbf{P})\mathbf{v} = \mathbf{v}^\mathsf{T}\mathbf{P}^\mathsf{T}(\mathbf{I}_n - \mathbf{P})\mathbf{v} = \mathbf{v}^\mathsf{T}\mathbf{P}(\mathbf{I}_n - \mathbf{P})\mathbf{v} = 0.$$

$\Longrightarrow$ $\mathbf{P} \in \mathbb{R}^{n\times n}$ with $\mathbf{P}^2 = \mathbf{P}^\mathsf{T} = \mathbf{P}$ represents the orthogonal projection onto the column space $\operatorname{csp}(\mathbf{P})$ of $\mathbf{P}$.

Moreover, if $\mathbf{Q} \in \mathbb{R}^{n\times r}$ contains an orthonormal basis of $\operatorname{csp}(\mathbf{P})$ then $\mathbf{P} = \mathbf{Q}\mathbf{Q}^\mathsf{T}$.

## Exercise

Show that, conversely, the matrix $\mathbf{P}$ of an orthogonal projection $f_\mathbf{P}\colon \mathbb{R}^n \to \mathbb{R}^n$, $\mathbf{x} \mapsto \mathbf{P}\mathbf{x}$ must be symmetric.

## Exercise

Suppose that $\mathbf{A} \in \mathbb{R}^{n\times k}$ has rank $k$. Show that the orthogonal projection from $\mathbb{R}^n$ onto the column space of $\mathbf{A}$ is given by $\mathbf{x} \mapsto \mathbf{A}(\mathbf{A}^\mathsf{T}\mathbf{A})^{-1})\mathbf{A}^\mathsf{T}\mathbf{x}$.

## Exercise (Least Squares)

Consider a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{n \times k}$ of rank $k$ and $\mathbf{b} \in \mathbb{R}^n$.

a) Show that (regardless of whether $\mathbf{A}\mathbf{x} = \mathbf{b}$ is solvable or not) the optimization problem $\min\{|\mathbf{A}\mathbf{x} - \mathbf{b}|\,;\mathbf{x} \in \mathbb{R}^k\}$ has a unique solution $\widehat{\mathbf{x}}$, which is obtained by solving $\mathbf{A}^{\mathsf{T}}\mathbf{A}\mathbf{x} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$.
*Hint:* $\mathbf{A}^{\mathsf{T}}\mathbf{A}\widehat{\mathbf{x}} = \mathbf{A}^{\mathsf{T}}\mathbf{b}$ is equivalent to $\mathbf{A}^{\mathsf{T}}(\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}) = \mathbf{0}$ and says that $\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}$ is orthogonal to the column space of $\mathbf{A}$. Show that this implies $|\mathbf{A}\mathbf{x} - \mathbf{b}|^2 = \left|\mathbf{A}(\mathbf{x} - \widehat{\mathbf{x}})\right|^2 + \left|\mathbf{A}\widehat{\mathbf{x}} - \mathbf{b}\right|^2$.

b) Using the result of a), determine the line $y = a + bx$ that minimizes the mean square interpolation error $\sum_{i=1}^4 (y_i - a - bx_i)^2$ for the 4 points

| $x_i$ | 1 | 2 | 2 | 3 |
|---|---|---|---|---|
| $y_i$ | 0 | 1 | 3 | 3 |

Then compute the vectors $\mathbf{A}\widehat{\mathbf{x}}$, $\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}$ and verify that $\left|\mathbf{A}\widehat{\mathbf{x}}\right|^2 + \left|\mathbf{b} - \mathbf{A}\widehat{\mathbf{x}}\right|^2 = |\mathbf{b}|^2$.

## Exercise

1. Use the Cayley-Hamilton Theorem to express the inverse $\mathbf{A}^{-1}$ of an invertible $2 \times 2$ matrix $\mathbf{A} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in the form $c_0\mathbf{I}_2 + c_1\mathbf{A}$.
2. Suppose that $\mathbf{A} \in F^{n\times n}$ is invertible. Show that $\mathbf{A}^{-1} \in F[\mathbf{A}]$.

## Exercise

Suppose $\mathbf{S}$ diagonalizes $\mathbf{A} \in F^{n\times n}$. Show that $\mathbf{S}$ also diagonalizes every matrix polynomial $a(\mathbf{A}) = a_0 + a_1\mathbf{A} + \cdots + a_d\mathbf{A}^d$, $a(X) \in F[X]$. How are the eigenvalues of $a(\mathbf{A})$ related to the eigenvalues of $\mathbf{A}$?

First-Order Autonomous Linear ODE Systems

Definitions and Examples

The Use of the Characteristic Polynomial

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Diagonalizable Matrices

Generalized Eigenvectors and the General Solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$

The Spectral Theorem for Real Symmetric Matrices

# The Jordan Canonical Form

According to the previous theorem, any matrix whose minimum polynomial is of the form $(X-\lambda)^m$ with $m \geq 2$ is not diagonalizable.

As an example we take the companion matrix of

$$(X+2)^3 = X^3 + 6X^2 + 12X + 8,$$

which is

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & -8 \\ 1 & 0 & -12 \\ 0 & 1 & -6 \end{pmatrix}$$

and has $\mu_{\mathbf{A}}(X) = \chi_{\mathbf{A}}(X) = (X+2)^3$.

*Question:* What can we do in this and similar cases?

*Answer:* Look at the matrix

$$\mathbf{N} = \mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 2\mathbf{I} = \begin{pmatrix} 2 & 0 & -8 \\ 1 & 2 & -12 \\ 0 & 1 & -4 \end{pmatrix}.$$

$\mathbf{N}$ satisfies $\mathbf{N}^3 = \mathbf{0}$, $\mathbf{N}^2 \neq \mathbf{0}$.

## Definition

A square matrix $\mathbf{N} \in F^{n\times n}$ is said to be *nilpotent* if there exists a positive integer $m$ such that $\mathbf{N}^m = \mathbf{0}$.

It turns out that nilpotent matrices are similar to a $(0,1)$-matrix with 1's restricted to the second lower diagonal, i.e., $\mathbf{N}' = (n'_{ij})$ with $n'_{ij} \in \{0,1\}$ and $n'_{ij} = 0$ unless $i = j+1$.

We illustrate this for our $3\times 3$ matrix $\mathbf{N}$.

Since $\mathbf{N}^2 \neq \mathbf{0}$, there exists $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{N}^2\mathbf{v} \neq \mathbf{0}$. We claim that $B = \{\mathbf{v}, \mathbf{N}\mathbf{v}, \mathbf{N}^2\mathbf{v}\}$ is a basis of $\mathbb{R}^3$. Suppose

$$c_0\mathbf{v} + c_1\mathbf{N}\mathbf{v} + c_2\mathbf{N}^2\mathbf{v} = \mathbf{0}.$$

Applying $\mathbf{N}$ to both sides of the equation gives

$$\mathbf{0} = \mathbf{N}\mathbf{0} = c_0\mathbf{N}\mathbf{v} + c_1\mathbf{N}^2\mathbf{v} + c_2\mathbf{N}^3\mathbf{v} = c_0\mathbf{N}\mathbf{v} + c_1\mathbf{N}^2\mathbf{v}.$$

Applying $\mathbf{N}$ once more gives $c_0\mathbf{N}^2\mathbf{v} = \mathbf{0}$.

$\Longrightarrow c_0 = 0$ and further $c_1 = c_2 = 0$, proving the claim.

The matrix of $f_{\mathbf{N}}\colon F^3 \to F^3$, $\mathbf{x} \to \mathbf{N}\mathbf{x}$ with respect to $\mathbf{B}$ is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

$$\implies {}_B(f_{\mathbf{A}})_B = {}_B(f_{\mathbf{N}})_B - 2\mathbf{I} = \begin{pmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \end{pmatrix}$$

Such a matrix which has constant main diagonal $\lambda \in \mathbb{C}$ and constant first lower diagonal 1 is called a *Jordan box* or *Jordan block*.

In general the situation is more complicated, but one can show that for any matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ the space $\mathbb{C}^n$ decomposes into a direct sum of subspaces on which $f_{\mathbf{A}}$ is represented by a Jordan box. The corresponding basis has exactly one eigenvector of $\mathbf{A}$ corresponding to each Jordan box and possibly further generalized eigenvectors, i.e., solutions of $(\mathbf{A} - \lambda\mathbf{I}_n)^k\mathbf{x} = \mathbf{0}$ for powers $k \geq 2$.

The Jordan boxes of a diagonalizable matrix have dimension 1 (i.e., their first lower diagonal is void).

# Theorem (Jordan Canonical Form (JCF) for Complex $n \times n$ Matrices)

1. *Any matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is similar to a block diagonal matrix of the form*

$$\begin{pmatrix} \mathbf{J}_1 & & & \\ & \mathbf{J}_2 & & \\ & & \ddots & \\ & & & \mathbf{J}_r \end{pmatrix} \quad \textit{with} \quad \mathbf{J}_i = \begin{pmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_i \end{pmatrix}.$$

2. *Two matrices of the form (1) are similar iff they differ by a permutation of the diagonal blocks $\mathbf{J}_i$.*

## Notes

- For any fixed eigenvalue $\lambda$ of $\mathbf{A}$ the number of corresponding Jordan boxes in the JCF of $\mathbf{A}$ is equal to the geometric multiplicity and the sum of their dimensions to the algebraic multiplicity.
- The multiplicity of $\lambda$ as a zero of $\mu_{\mathbf{A}}$ is equal to the dimension of the largest Jordan box for the eigenvalue $\lambda$ in the JCF of $\mathbf{A}$.

## Exercise

1. How may similarity classes of nilpotent matrices does $\mathbb{C}^{n \times n}$ have for $1 \leq n \leq 6$ ?
2. How many similarity classes are there in $\mathbb{C}^{6 \times 6}$ whose matrices have characteristic polynomial $(X-1)^6$ and minimum polynomial $(X-1)^3$ ?

## Example (cont'd)

For diagonalizable matrices $\mathbf{A}$ there is an alternative method for obtaining a paricular solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{q}$, which expresses the source $\mathbf{q}(t)$ in terms of eigenvectors of $\mathbf{A}$ and solves the resulting 1-dimensional systems.

In the case under consideration we have

$$\mathbf{q}(t) = \begin{pmatrix} 0 \\ t \end{pmatrix} = \frac{t}{2\mathrm{i}} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} - \frac{t}{2\mathrm{i}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix} = \mathbf{q}_1(t) + \mathbf{q}_2(t),$$

and we can solve $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{q}_i(t)$ entirely in the corresponding eigenspace.

For $\lambda_1 = \mathrm{i}$ one-dimensional variation of parameters gives a particular solution $\mathbf{z}_1(t) = c(t)\mathbf{y}_1(t)$ with

$$c(t) = \int_0^t \mathrm{e}^{-\mathrm{i}s} \frac{s}{2\mathrm{i}} \,\mathrm{d}s = \frac{1}{2\mathrm{i}} \left[\mathrm{i}s\mathrm{e}^{-\mathrm{i}s} + \mathrm{e}^{-\mathrm{i}s}\right]_0^t = \frac{1}{2\mathrm{i}} \left(\mathrm{i}t\mathrm{e}^{-\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t} - 1\right),$$

which simplifies (and changes) to $\mathbf{z}_1(t) = \frac{t-\mathrm{i}}{2} \binom{1}{\mathrm{i}}$. Similarly, for $\lambda_2 = -\mathrm{i}$ we obtain $\mathbf{z}_2(t) = \overline{\mathbf{z}_1(t)} = \frac{t+\mathrm{i}}{2} \binom{1}{-\mathrm{i}}$. Superposing the individual solutions gives $\mathbf{y}_p(t) = \mathbf{z}_1(t) + \mathbf{z}_2(t) = \binom{t}{1}$, as before.

# How to Compute $e^{\mathbf{A}t}$ in General?

1. **A** *is diagonalizable*.
   If $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ then

$$\begin{aligned}\mathbf{S}^{-1}e^{\mathbf{A}t}\mathbf{S} &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{S}^{-1}\mathbf{A}^k\mathbf{S}\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbf{D}^k = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix} = e^{\mathbf{D}t}.\end{aligned}$$

This gives

$$e^{\mathbf{A}t} = \mathbf{S}\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}\mathbf{S}^{-1},\ e^{\mathbf{A}t}\mathbf{y}(0) = \mathbf{S}\begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t}\end{pmatrix}\mathbf{S}^{-1}\mathbf{y}(0).$$

Writing $\mathbf{S} = (\mathbf{v}_1|\dots|\mathbf{v}_n)$, the right-hand identity says that the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is $\mathbf{y}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n$ with $\mathbf{c}$ determined from $\mathbf{S}\mathbf{c} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{y}(0)$.

1. (cont'd)
   This reaffirms our earlier observation that from a basis $\mathbf{v}_1, \dots, \mathbf{v}_n$ of $\mathbb{C}^n$ consisting of eigenvectors of $\mathbf{A}$ one obtains a fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ by multiplying each eigenvector $\mathbf{v}_i$ with the corresponding (scalar) exponential function $\mathrm{e}^{\lambda_i t}$.

2. **A** has only one eigenvalue $\lambda$.
   In this case we have $\chi_{\mathbf{A}}(X) = (X - \lambda)^n$ and $(\mathbf{A} - \lambda\mathbf{I}_n)^k = 0$ for $k \geq n$ by the Cayley-Hamilton Theorem.

$$\implies \mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{(\lambda\mathbf{I}+\mathbf{A}-\lambda\mathbf{I})t} = \mathrm{e}^{\lambda\mathbf{I}t}\mathrm{e}^{(\mathbf{A}-\lambda\mathbf{I})t}$$
$$= \mathrm{e}^{\lambda t}\left[\mathbf{I} + t(\mathbf{A}-\lambda\mathbf{I}) + \frac{t^2}{2!}(\mathbf{A}-\lambda\mathbf{I})^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(\mathbf{A}-\lambda\mathbf{I})^{n-1}\right].$$

3. *The general case*
   One possible solution is to compute the Jordan canonical form $\mathbf{J}$ of $\mathbf{A}$ and a matrix $\mathbf{S}$ satisfying $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$. Then $\mathrm{e}^{\mathbf{A}t} = \mathbf{S}\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1}$, and the computation of $\mathrm{e}^{\mathbf{J}t}$ reduces to that of $\mathrm{e}^{\mathbf{J}_i t}$ for the Jordan blocks $\mathbf{J}_i$. The matrices $\mathrm{e}^{\mathbf{J}_i t}$ in turn can be computed by the method in (2).

## Example (taken from [Str14])

We solve the two systems

$$\mathbf{y}' = \mathbf{A}\mathbf{y} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{y}, \qquad \mathbf{y}' = \mathbf{B}\mathbf{y} = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix} \mathbf{y}$$

and the corresponding IVP's with $\mathbf{y}(0) = (6,2)^\mathsf{T}$.

**A** has characteristic polynomial
$\chi_\mathbf{A}(X) = X^2 + 4X + 3 = (X+1)(X+3)$ and eigenvalues
$\lambda_1 = -1$, $\lambda_2 = -3$.
Corresponding eigenvectors are $\mathbf{v}_1 = (1,1)^\mathsf{T}$, $\mathbf{v}_2 = (1,-1)^\mathsf{T}$.
$\Longrightarrow$ The general solution of the first system is

$$\mathbf{y}(t) = c_1 \mathrm{e}^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \mathrm{e}^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{C}$$

(and the general real solution is obtained by requiring $c_1, c_2 \in \mathbb{R}$).
The coefficients of the special solution with $\mathbf{y}(0) = (6,2)^\mathsf{T}$ are determined by solving

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

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# Example (cont'd)

$$\implies \quad \mathbf{y}(t) = 4\mathrm{e}^{-t}\begin{pmatrix}1\\1\end{pmatrix} + 2\mathrm{e}^{-3t}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}4\mathrm{e}^{-t}+2\mathrm{e}^{-3t}\\4\mathrm{e}^{-t}-2\mathrm{e}^{-3t}\end{pmatrix}.$$

**B** has characteristic polynomial $\chi_{\mathbf{A}}(X) = X^2 + 4X + 5 = (X + 2 - \mathrm{i})(X + 2 + \mathrm{i})$ and eigenvalues $\lambda_1 = -2 + \mathrm{i}$, $\lambda_2 = -2 - \mathrm{i}$.
Corresponding eigenvectors are obtained by solving

$$(\mathbf{B} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix}-\mathrm{i} & 1\\-1 & -\mathrm{i}\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}, \quad \text{e.g.,} \quad \mathbf{v}_1 = \begin{pmatrix}1\\\mathrm{i}\end{pmatrix},$$

and similarly for $\lambda_2$, giving $\mathbf{v}_2 = (1, -\mathrm{i})^{\mathsf{T}}$.
(Note that $\mathbf{v}_2 = \overline{\mathbf{v}}_1$, so that no computation is necessary.)
$\implies$ The general solution of the second system is

$$\mathbf{y}(t) = c_1\mathrm{e}^{(-2+\mathrm{i})t}\begin{pmatrix}1\\\mathrm{i}\end{pmatrix} + c_2\mathrm{e}^{(-2-\mathrm{i})t}\begin{pmatrix}1\\-\mathrm{i}\end{pmatrix}, \quad c_1, c_2 \in \mathbb{C}.$$

## Example (cont'd)

As before, the coefficients of the special solution with $\mathbf{y}(0) = (6,2)^{\mathsf{T}}$ is determined by solving

$$\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \quad \text{which gives} \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3-\mathrm{i} \\ 3+\mathrm{i} \end{pmatrix}.$$

$$\begin{aligned} \Longrightarrow \quad \mathbf{y}(t) &= \mathrm{e}^{(-2+\mathrm{i})t} \begin{pmatrix} 3-\mathrm{i} \\ 1+3\mathrm{i} \end{pmatrix} + \mathrm{e}^{(-2-\mathrm{i})t} \begin{pmatrix} 3+\mathrm{i} \\ 1-3\mathrm{i} \end{pmatrix} \\ &= 2\,\mathrm{Re}\left[ \mathrm{e}^{(-2+\mathrm{i})t} \begin{pmatrix} 3-\mathrm{i} \\ 1+3\mathrm{i} \end{pmatrix} \right] \\ &= \mathrm{e}^{-2t} \begin{pmatrix} 6\cos t + 2\sin t \\ 2\cos t - 6\sin t \end{pmatrix}. \end{aligned}$$

## Note

In these examples it was easier to solve the given ODE system directly without recourse to matrix exponentials. Conversely, we can use the solutions to find the matrix exponentials by means of the formula $\mathrm{e}^{\mathbf{A}t} = \Phi(t)\Phi(0)^{-1}$, which switches any known fundamental system of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ into the standard one represented by the matrix exponential; cf. the exercises.

## Exercise

a) Show that for $\mathbf{A} \in \mathbb{C}^{n\times n}$ the matrix exponential $\mathrm{e}^{\mathbf{A}t}$ and an arbitrary fundamental matrix $\Phi(t)$ are related by $\Phi(t) = \mathrm{e}^{\mathbf{A}t}\Phi(0)$.

b) For $\mathbf{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$ compute $\mathrm{e}^{\mathbf{A}t}$ using a) and the fundamental system determined in the lecture.

c) For the matrix in b), alternatively compute $\mathrm{e}^{\mathbf{A}t}$ using the series representation and the decomposition

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = -2\,\mathbf{I}_2 + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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## Example

As an example of a system with a non-diagonalizable coefficient matrix we consider

$$\mathbf{y}' = \mathbf{C}\mathbf{y} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \mathbf{y}.$$

$\mathbf{C}$ has the eigenvalue $\lambda = -2$ with algebraic multiplicity 2 and geometric multiplicity 1. (The corresponding eigenspace is $\mathbb{C}(1,0)^{\mathsf{T}}$.)

Here we compute $\mathrm{e}^{\mathbf{C}t}$ directly using the method for a single eigenvalue:

$$\begin{aligned}\mathrm{e}^{\mathbf{C}t} &= \mathrm{e}^{-2\mathbf{I}t}\mathrm{e}^{(\mathbf{C}+2\mathbf{I})t} = \mathrm{e}^{-2t}\left[\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + t\begin{pmatrix} & 1 \\ & \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} & 1 \\ & \end{pmatrix}^2 + \cdots\right] \\ &= \mathrm{e}^{-2t}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-2t} & t\mathrm{e}^{-2t} \\ 0 & \mathrm{e}^{-2t} \end{pmatrix}.\end{aligned}$$

It follows that the general solution in this case is

$$\mathbf{y}(t) = \mathrm{e}^{-2t}\begin{pmatrix} c_1 + tc_2 \\ c_2 \end{pmatrix}, \quad c_1, c_2 \in \mathbb{C}.$$

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# A New Method to Compute $\mathrm{e}^{\mathbf{A}t}$

## Theorem

*Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ satisfies*

$$a(\mathbf{A}) = a_0\mathbf{I}_n + a_1\mathbf{A} + \cdots + a_d\mathbf{A}^d = \mathbf{0}$$

*for some $a(X) = a_0 + a_1X + \cdots + a_dX^d \in \mathbb{C}[X]$.*

1. *The entries $e_{ij}(t)$ of the matrix exponential $\mathrm{e}^{\mathbf{A}t} = \big(e_{ij}(t)\big)$ solve the scalar ODE $a(\mathrm{D})y = 0$.*
2. *The matrix exponential $\mathrm{e}^{\mathbf{A}t}$ admits the representation*

$$\mathrm{e}^{\mathbf{A}t} = c_0(t)\mathbf{I}_n + c_1(t)\mathbf{A} + \cdots + c_{d-1}(t)\mathbf{A}^{d-1},$$

*where $c_k(t)$ is the solution of the IVP*

$$a(\mathrm{D})y = 0 \wedge \Big(y(0), y'(0), \dots, y^{(d-1)}(0)\Big) = \mathbf{e}_{k+1},$$

*the standard unit vector in $\mathbb{C}^d$ of the form $(\underbrace{0,\dots,0}_{k},1,0,\dots,0)$.*

In other words, $c_0(t), c_1(t), \dots, c_{d-1}(t)$ is the special fundamental system of solutions of $a(\mathrm{D})y = 0$ whose Wronski matrix $\mathbf{W}(0)$ at $t = 0$ is the $d \times d$ identity matrix. (This also shows $\mathbf{W}(t) = \mathrm{e}^{\mathbf{C}t}$, where $\mathbf{C}$ is the companion matrix of $a(X)$).

## Corollary

*Suppose* $\mathbf{A}$ *has* $r$ *distinct eigenvalues* $\lambda_1, \dots, \lambda_r$ *and* $\mu_{\mathbf{A}}(X) = \prod_{i=1}^{r}(X - \lambda_i)^{m_i} = X^d + \mu_{d-1}X^{d-1} + \cdots + \mu_1 X + \mu_0$. *Then the entries* $\mathrm{e}_{ij}(t)$ *of* $\mathrm{e}^{\mathbf{A}t}$ *and the (uniquely determined) functions* $c_k(t)$ *in the representation* $\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{d-1} c_k(t)\mathbf{A}^k$ *have the form*

$$\sum_{i=1}^{r} f_i(t)\mathrm{e}^{\lambda_i t}$$

*for some polynomials* $f_i(X) \in \mathbb{C}[X]$ *of degree* $\leq m_i - 1$. *The same assertion holds, mutatis mutandis, for* $\chi_{\mathbf{A}}(X)$ *in place of* $\mu_{\mathbf{A}}(X)$ *(except that for* $n > d$ *the functions* $c_k(t)$ *are no longer uniquely determined by the requirement* $\mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{n-1} c_k(t)\mathbf{A}^k$ *and must be defined as in Part (2) of the theorem).*

## Note

If $\mu_{\mathbf{A}}(X)$ properly divides $\chi_{\mathbf{A}}(X)$ then the bound for $\deg f_i(X)$ in terms of $\mu_{\mathbf{A}}(X)$ is stronger.

# Proof of the theorem.

(1) Writing $\Phi(t) = \mathrm{e}^{\mathbf{A}t}$, we infer from $\Phi'(t) = \mathbf{A}\Phi(t)$ that

$$a(\mathrm{D})\Phi(t) = a(\mathbf{A})\Phi(t) = a(\mathbf{A})\mathrm{e}^{\mathbf{A}t}$$

for all polynomials $a(X) \in \mathbb{C}[X]$.
If $a(\mathbf{A}) = 0$ then $a(\mathrm{D})\Phi(t) = \mathbf{0}$ and, since differentiation acts entry-wise on $\Phi(t)$, further $a(\mathrm{D})e_{ij}(t) = 0$ for $1 \leq i, j \leq n$.

(2) Defining $\Phi(t)$ as the indicated representation of $\mathrm{e}^{\mathbf{A}t}$, we have

$$\begin{aligned}
\Phi(t) &= c_0(t)\mathbf{I}_n + c_1(t)\mathbf{A} + \cdots + c_{d-1}(t)\mathbf{A}^{d-1},\\
\Phi'(t) &= c_0'(t)\mathbf{I}_n + c_1'(t)\mathbf{A} + \cdots + c_{d-1}'(t)\mathbf{A}^{d-1},\\
&\vdots\\
\Phi^{(d)}(t) &= c_0^{(d)}(t)\mathbf{I}_n + c_1^{(d)}(t)\mathbf{A} + \cdots + c_{d-1}^{(d)}(t)\mathbf{A}^{d-1}.
\end{aligned}$$

If the functions $c_j(t)$ solve the given IVP's then

$$a(\mathrm{D})\Phi(t) = \mathbf{0} \quad \text{and} \quad \Phi^{(i)}(0) = \mathbf{A}^i \text{ for } 0 \leq i \leq d-1.$$

Since $t \mapsto \mathrm{e}^{\mathbf{A}t}$ satisfies these conditions as well, we must have $\Phi(t) = \mathrm{e}^{\mathbf{A}t}$. $\square$

For the last step of the proof note that the matrix IVP $a(\mathrm{D})\Phi(t) = \mathbf{0} \wedge \Phi^{(k)}(0) = \mathbf{A}^k$ for $0 \leq k \leq d-1$ amounts to $n^2$ scalar IVP's for the entries $\mathrm{e}_{ij}(t)$, which are specified in terms of the entries $(\mathbf{A}^k)_{ij}$.

The corollary is an immediate consequence of the theorem in view of $\mu_{\mathbf{A}}(\mathbf{A}) = \mathbf{0}$ and the known structure of the solution space of $\mu_{\mathbf{A}}(\mathrm{D})y = 0$, and similarly for $\chi_{\mathbf{A}}$.

## Example

We compute again $\mathrm{e}^{\mathbf{A}t}$ for $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Since $\mathbf{A}^2 = -\mathbf{I}_2$, we can take $a(X) = X^2 + 1$, $d = 2$ in the theorem (in fact $X^2 + 1$ is just the characteristic polynomial of $\mathbf{A}$), which yields the 2nd-order ODE $y'' + y = 0$ for $c_0(t)$ and $c_1(t)$.
Since $\cos t$ and $\sin t$ solve this ODE and satisfy the required initial conditions (i.e, the Wronski matrix of $\cos t, \sin t$ is already $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$), we obtain

$$\mathrm{e}^{\mathbf{A}t} = (\cos t)\mathbf{I}_2 + (\sin t)\mathbf{A} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

## Example

Let $\mathbf{P} \in \mathbb{C}^{n \times n}$ be a projection matrix, i.e., $\mathbf{P}^2 = \mathbf{P}$.

Here we can take $a(X) = X^2 - X = X(X-1)$ corresponding to the ODE $y'' - y' = 0$. A fundamental system is $1, \mathrm{e}^t$, and the the required initial conditions are satisfied by $c_0(t) = 1$, $c_1(t) = \mathrm{e}^t - 1$.

$$\Longrightarrow \mathrm{e}^{\mathbf{P}t} = \mathbf{I}_n + (\mathrm{e}^t - 1)\mathbf{P}.$$

This result can also be derived directly from the series representation of $\mathrm{e}^{\mathbf{P}t}$, using the observation that $\mathbf{P}^2 = \mathbf{P}$ imlies $\mathbf{P}^n = \mathbf{P}$ for all $n \geq 1$.

## Note

The characteristic polynomial of $\chi_{\mathbf{P}}(X)$ has degree $n$ and leads to a more complicated formula for $\mathrm{e}^{\mathbf{P}t}$ if $n > 2$. For example, projection matrices $\mathbf{P} \in \mathbb{C}^{3\times 3}$ of rank 1 and 2 have characteristic polynomials $X^2(X-1)$ and $X(X-1)^2$, respectively, which lead to representations

$$\begin{aligned} \mathrm{e}^{\mathbf{P}t} &= \mathbf{I}_3 + t\mathbf{P} + (-1 - t + \mathrm{e}^t)\mathbf{P}^2, \quad \text{resp.,} \\ \mathrm{e}^{\mathbf{P}t} &= \mathbf{I}_3 + (-2 + 2\mathrm{e}^t - t\mathrm{e}^t)\mathbf{P} + (1 - \mathrm{e}^t + t\mathrm{e}^t)\mathbf{P}^2. \end{aligned}$$

Since $\mathbf{P}^2 = \mathbf{P}$, both representations collapse to $\mathrm{e}^{\mathbf{P}t} = \mathbf{I}_n + (\mathrm{e}^t - 1)\mathbf{P}$.

## Example

We compute the matrix eponential of

$$\mathbf{A} = \begin{pmatrix} -26 & 49 & 74 \\ -8 & 16 & 25 \\ -4 & 7 & 10 \end{pmatrix}.$$

This matrix is not diagonalizable, as we have seen earlier, and the example is meant to illustrate the fact that the new method for computing matrix eponentials works just as well for non-diagonalizable matrices.

From the earlier example we use $\chi_{\mathbf{A}}(X) = (X-2)(X+1)^2$ (which happens to coincide with $\mu_{\mathbf{A}}(X)$ in this case, but this fact is not needed for the computation).

A fundamental system of solutions of the corresponding ODE is $y_1(t) = \mathrm{e}^{2t}$, $y_2(t) = \mathrm{e}^{-t}$, $y_3(t) = t\mathrm{e}^{-t}$, which satisfy the initial conditions

$$\Phi(0) = \begin{pmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1'(0) & y_2'(0) & y_3'(0) \\ y_1''(0) & y_2''(0) & y_3''(0) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 2 & -1 & 1 \\ 4 & 1 & -2 \end{pmatrix}.$$

For this observe that $y_3'(t) = (1-t)\mathrm{e}^{-t}$, $y_3''(t) = (t-2)\mathrm{e}^{-t}$.

# Example (cont'd)

The required initial conditions $\Psi(0) = \mathbf{I}_3$ are then satisfied by the transformed system $\Psi(t) = \Phi(t)\Phi(0)^{-1}$, so that we need to invert the matrix $\Phi(0)$. Applying the standard algorithm gives

$$\Phi(0)^{-1} = \frac{1}{9}\begin{pmatrix} 1 & 2 & 1 \\ 8 & -2 & -1 \\ 6 & 3 & -3 \end{pmatrix}.$$

This matrix contains the coefficients of $c_0(t)$, $c_1(t)$, $c_2(t)$ with respect to $\mathrm{e}^{2t}$, $\mathrm{e}^{-t}$, $t\mathrm{e}^{-t}$ in the respective column (look at the 1st row of the matrix equation $\Psi(t) = \Phi(t)\Phi(0)^{-1}$, which is $\big(c_0(t), c_1(t), c_2(t)\big) = \big(e^{2t}, \mathrm{e}^{-t}, t\mathrm{e}^{-t}\big)\Phi(0)^{-1}$), and we finally obtain

$$\begin{aligned}
\mathrm{e}^{\mathbf{A}t} &= \tfrac{1}{9}(\mathrm{e}^{2t} + 8\mathrm{e}^{-t} + 6t\mathrm{e}^{-t})\mathbf{I}_3 + \tfrac{1}{9}(2\mathrm{e}^{2t} - 2\mathrm{e}^{-t} + 3t\mathrm{e}^{-t})\mathbf{A} + \tfrac{1}{9}(\mathrm{e}^{2t} - \mathrm{e}^{-t} - 3t\mathrm{e}^{-t})\mathbf{A}^2 \\
&= \left[\mathrm{e}^{2t}\begin{pmatrix} -7 & 14 & 21 \\ -4 & 8 & 12 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{e}^{-t}\begin{pmatrix} 8 & -14 & -21 \\ 4 & -7 & -12 \\ 0 & 0 & 1 \end{pmatrix} + t\mathrm{e}^{-t}\begin{pmatrix} -4 & 7 & 11 \\ 4 & -7 & -11 \\ -4 & 7 & 11 \end{pmatrix}\right] \\
&= \begin{pmatrix} -7\mathrm{e}^{2t} + 8\mathrm{e}^{-t} - 4t\mathrm{e}^{-t} & 14\mathrm{e}^{2t} - 14\mathrm{e}^{-t} + 7t\mathrm{e}^{-t} & 21\mathrm{e}^{2t} - 21\mathrm{e}^{-t} + 11t\mathrm{e}^{-t} \\ -4\mathrm{e}^{2t} + 4\mathrm{e}^{-t} + 4t\mathrm{e}^{-t} & 8\mathrm{e}^{2t} - 7\mathrm{e}^{-t} - 7t\mathrm{e}^{-t} & 12\mathrm{e}^{2t} - 12\mathrm{e}^{-t} - 11t\mathrm{e}^{-t} \\ -4t\mathrm{e}^{-t} & 7t\mathrm{e}^{-t} & \mathrm{e}^{-t} + 11t\mathrm{e}^{-t} \end{pmatrix}.
\end{aligned}$$

## Example (cont'd)

One should compare the costs of this computation to the one using the JCF. In the earlier example we had computed the JCF

$$\mathbf{J} = \left(\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & -1 \end{array}\right)$$

of $\mathbf{A}$ and $\mathbf{S}$ satisfying $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{J}$. From this we can continue as follows:

$$\begin{aligned} \mathrm{e}^{\mathbf{A}t} &= \mathbf{S}\mathrm{e}^{\mathbf{J}t}\mathbf{S}^{-1} \\ &= \begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \left(\begin{array}{c|cc} \mathrm{e}^{2t} & 0 & 0 \\ \hline 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ 0 & 0 & \mathrm{e}^{-t} \end{array}\right) \begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 7 & -1 & 2 \\ 4 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix} \left(\begin{array}{c|cc} \mathrm{e}^{2t} & 0 & 0 \\ \hline 0 & \mathrm{e}^{-t} & t\mathrm{e}^{-t} \\ 0 & 0 & \mathrm{e}^{-t} \end{array}\right) \begin{pmatrix} -1 & 2 & 3 \\ 0 & 0 & -1 \\ 4 & -7 & -11 \end{pmatrix} \\ &= \cdots \end{aligned}$$

The total costs are certainly no less than those of the new method.

# What Goes Wrong for $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ ?

The exponential matrix

$$\mathrm{e}^{\mathbf{B}(t)} = \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{B}(t)^k$$

is well-defined but does not satisfy $\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathbf{B}(t)} = \mathbf{B}'(t)\mathrm{e}^{\mathbf{B}(t)}$ in general.
$\implies \mathbf{y}(t) = \mathrm{e}^{\int \mathbf{A}(t)\,\mathrm{d}t}$ does not necessarily solve $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$.

*Reason:* When differentiating $\mathrm{e}^{\mathbf{B}(t)}$ termwise, we need the relation $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}(t)^k = k\mathbf{B}(t)^{k-1}\mathbf{B}'(t) = k\mathbf{B}'(t)\mathbf{B}(t)^{k-1}$, but we have only

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}(t)^2 &= \mathbf{B}(t)\mathbf{B}'(t) + \mathbf{B}'(t)\mathbf{B}(t),\\
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{B}(t)^3 &= \mathbf{B}'(t)\mathbf{B}(t)^2 + \mathbf{B}(t)\mathbf{B}'(t)\mathbf{B}(t) + \mathbf{B}(t)^2\mathbf{B}'(t), \quad \text{etc.}
\end{aligned}$$

## A special case

If $\mathbf{A}(t)$ and $\mathbf{B}(t) = \mathbf{B}_0 + \int_{t_0}^{t} \mathbf{A}(s)\,\mathrm{d}s$ commute then $\mathbf{y}(t) = \mathrm{e}^{\mathbf{B}(t)}$ solves $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$.

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## Exercise

Suppose $\mathbf{A} \in \mathbb{C}^{n\times n}$ has $n$ distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \ell_i(A),$$

where $\ell_i(X) = \prod_{j=1, j\neq i}^{n} \frac{X-\lambda_j}{\lambda_i-\lambda_j}$ are the Lagrange polynomials corresponding to $\lambda_1, \dots, \lambda_n$.

*Hint*: Show that $\Phi(t) = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \ell_i(A)$ solves the IVP $\Phi'(t) = \mathbf{A}\Phi(t) \wedge \Phi(0) = \mathbf{I}_n$.

## Exercise

Consider the two time-dependent linear systems

$$\mathbf{y}' = \mathbf{A}_1(t)\mathbf{y} = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \mathbf{y} \quad \text{and} \quad \mathbf{y}' = \mathbf{A}_2(t)\mathbf{y} = \begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \mathbf{y}.$$

Compute the matrix exponentials $\mathbf{E}_i(t) = \exp\left(\int_0^t \mathbf{A}_i(s)\,\mathrm{d}s\right)$, $i = 1, 2$, and show that $\mathbf{E}_1(t)$ forms a fundamental matrix of the corresponding system but $\mathbf{E}_2(t)$ does not.