Differential Equations Plus (Math 286)

H27 Use the phase line to investigate the stability of the equilibrium solutions of the following autonomous ODE's.

a) $y' = 2(1-y)(1-\mathrm{e}^y)$; b) $y' = (1-y^2)(4-y^2)$; c) $y' = \sin^2 y$.

H28 For the following ODE's $y' = f(y)$, use the Existence and Uniqueness Theorem to determine the points $(t_0, y_0) \in \mathbb{R}^2$ such that the initial value problem $y' = f(y) \wedge y(t_0) = y_0$ has a unique solution near (t_0, y_0). Then solve the ODE, sketch the integral curves, and compare with your prediction.

a) $y' = |y|$; b) $y' = \sqrt{|y - y^2|}$.

H29 Use Picard-Lindelöf iteration to compute the solution $\phi = (\phi_1, \phi_2)^\mathsf{T}$ of the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

with initial condition $\phi(0) = (1, 0)^\mathsf{T}$.

H30 Suppose that $f\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous and satisfies locally a Lipschitz condition, and that

$$f(-t, y) = -f(t, y) \quad \text{for all } (t, y) \in \mathbb{R}^2.$$

Show that any solution $\phi\colon [-r, r] \to \mathbb{R}$, $r > 0$, of $y' = f(t, y)$ is its own mirror image with respect to the y-axis.

H31 Compute the norms $\|\mathbf{A}\|$ of the following matrices $\mathbf{A} \in \mathbb{R}^{2\times 2}$ and compare them with their Frobenius norms $\|\mathbf{A}\|_\mathrm{F}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{2} & \pm 1 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

H32 Solve the initial value problem

$$y'' + |y| = 0, \quad y(0) = 0,\ y'(0) = 1.$$

Your solution should have the (maximal) domain $\mathbb{R}$.

Does the Existence and Uniqueness Theorem apply to this ODE?

 PLEASE TURN OVER

H33 *Optional Exercise*

Let M be a set and $d\colon M \times M \to \mathbb{R}$ a function satisfying $d(a,a) = 0$ for $a \in M$, $d(a,b) \neq 0$ for $a, b \in M$ with $a \neq b$, and $d(a,b) \leq d(b,c)) + d(c,a)$ for $a, b, c \in M$.

a) Show that d is a metric.

b) Does this conclusion also hold if $d(a,b) \leq d(b,c)) + d(c,a)$ is replaced by the ordinary triangle inequality $d(a,b) \leq d(a,c)) + d(c,b)$?

H34 *Optional Exercise*

Let (M,d) be a metric space and $(a,b) \in M \times M$.

a) Show that the metric d is *continuous* in the following sense:
For every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x,a) < \delta \wedge d(y,b) < \delta$ implies $|d(x,y) - d(a,b)| < \epsilon$.
Hint: First derive the so-called *quadrangle inequality* $|d(x,y) - d(a,b)| \leq d(x,a) + d(y,b)$.

b) Using a), show in detail that $x_n \to a$ and $y_n \to b$ implies $d(x_n, y_n) \to d(a,b)$. (A special case of this, viz. $d(x_n, b) \to d(a,b)$, was used in the proof of Part (2) of Banach's Fixed-Point Theorem.)

H35 *Optional Exercise*

a) Show that a closed subset N of a complete metric space (M,d) is complete in the induced metric $d|_N\colon N \times N \to \mathbb{R}$, $(x,y) \mapsto d(x,y)$.

b) Conversely, show that a subset of a metric space that is complete in the induced metric must be closed.

H36 *Optional Exercise*

a) Prove that $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|$ satisfies (N1)–(N4).

b) Repeat a) for the Frobenius norm $\mathbb{R}^{n\times n} \to \mathbb{R}$, $\mathbf{A} \mapsto \|\mathbf{A}\|_{\mathrm{F}}$.

c) Show that $\|\mathbf{A}\| \leq \|\mathbf{A}\|_{\mathrm{F}}$ for all matrices $\mathbf{A} \in \mathbb{R}^{n\times n}$ or, equivalently, $|\mathbf{A}\mathbf{x}| \leq \|\mathbf{A}\|_{\mathrm{F}}\,|\mathbf{x}|$ for all $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.
Hint: Use $\|\mathbf{A}\| = \max\bigl\{|\mathbf{A}\mathbf{x}|\,; \mathbf{x} \in \mathbb{R}^n, |\mathbf{x}| = 1\bigr\}$ and the Cauchy-Schwarz Inequality for vectors in $\mathbb{R}^n$.

Due on Fri Oct 29, 6 pm

The phase line of an autonomous ODE (required for H27) will be discussed in the lecture on Wed Oct 27 (cf. also [BDM17], Ch. 2.5); Picard-Lindelöf iteration (required for H29) in the lecture on Mon Oct 25 (cf. also [BDM17], Ch. 2.8). The optional exercises can be handed in until Fri Nov 5, 6 pm.