Math 286
Introduction to Differential Equations

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Outline

1. General Linear Differential Equations

2. Second-Order Linear ODE's

Today's Lecture:

First-Order Linear Systems

Definition

A (possibly *time-dependent*) *first-order linear system of ODE's* has the form

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t), \tag{LS}$$

where $\mathbf{A}\colon I \to \mathbb{C}^{n\times n}$, $t \mapsto \mathbf{A}(t) = \big(a_{ij}(t)\big)$ and $\mathbf{b}\colon I \to \mathbb{C}^n$, $t \mapsto \mathbf{b}(t) = \big(b_i(t)\big)$ are continuous (i.e., all component functions of $\mathbf{A}$ and $\mathbf{b}$ are continuous).

The domain I must be an interval contained in the domains of all component functions. The cases $I = \emptyset$ and $I = \{a\}$ are excluded.

As usual, the system (LS) is said to be *homogeneous* if $\mathbf{b}(t) \equiv 0$ and *inhomogeneous* otherwise.

A solution of (LS) is a differentiable map $\mathbf{y}\colon J \to \mathbb{C}^n$ (i.e., a parametric curve) defined on some subinterval $J \subseteq I$ and satisfying $\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \mathbf{b}(t)$ for all $t \in J$.

Existence and Uniqueness of Solutions

$f(t,\mathbf{y})=\mathbf{A}(t)\mathbf{y}+\mathbf{b}(t)$ satisfies

$$\begin{aligned}|f(t,\mathbf{y}_1)-f(t,\mathbf{y}_2)|&=|\mathbf{A}(t)(\mathbf{y}_1-\mathbf{y}_2)|\leq\|\mathbf{A}(t)\|\,|\mathbf{y}_1-\mathbf{y}_2|\\&\leq L\,|\mathbf{y}_1-\mathbf{y}_2|\,,\end{aligned}$$

provided we restrict t to compact (closed and bounded) subintervals of I, and hence in particular a local Lipschitz condition with respect to $\mathbf{y}$.
$\Longrightarrow$ The Existence and Uniqueness Theorem applies and gives the local solvability and uniqueness of solutions of any IVP $\mathbf{y}'=\mathbf{A}(t)\mathbf{y}+\mathbf{b}(t)\wedge\mathbf{y}(t_0)=\mathbf{y}_0$ ($t_0\in I$, $\mathbf{y}_0\in\mathbb{C}^n$).

Remark

If you are uncomfortable with complex-valued linear ODE systems, note that any such system is equivalent to a real valued system with twice as many equations/component functions in the following sense: $\mathbf{y}(t)$ solves the complex $n\times n$ system $\mathbf{y}'=\mathbf{A}(t)\mathbf{y}+\mathbf{b}(t)$ iff $\begin{pmatrix}\operatorname{Re}\mathbf{y}(t)\\ \operatorname{Im}\mathbf{y}(t)\end{pmatrix}$ solves $\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix}'=\begin{pmatrix}\operatorname{Re}\mathbf{A}(t)&-\operatorname{Im}\mathbf{A}(t)\\ \operatorname{Im}\mathbf{A}(t)&\operatorname{Re}\mathbf{A}(t)\end{pmatrix}\begin{pmatrix}\mathbf{y}_1\\ \mathbf{y}_2\end{pmatrix}+\begin{pmatrix}\operatorname{Re}\mathbf{b}(t)\\ \operatorname{Im}\mathbf{b}(t)\end{pmatrix}$.

But More Is True

Theorem

Solutions to any IVP $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$ *exist on the whole domain I (and are unique).*

Compare this with the nonlinear case, e.g., $y' = y^2$ whose solutions $y = 1/(C - x)$ do not exist on all of $\mathbb{R}$.

Proof.

We estimate the difference of successive Picard-Lindelöf iterates $\mathbf{y}_k(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}(s)\mathbf{y}_{k-1}(s) + \mathbf{b}(s)\,\mathrm{d}s$, $k = 1, 2, 3, \dots$

$$\begin{aligned}
|\mathbf{y}_{k+1}(t) - \mathbf{y}_k(t)| &= \left| \int_{t_0}^{t} \mathbf{A}(s)\big(\mathbf{y}_k(s) - \mathbf{y}_{k-1}(s)\big)\,\mathrm{d}s \right| \\
&\leq \pm L \int_{t_0}^{t} |\mathbf{y}_k(s) - \mathbf{y}_{k-1}(s)|\,\mathrm{d}s,
\end{aligned}$$

provided t is restricted to a compact subinterval $J \subseteq I$ with $t_0 \in J$ and $L = \max\{\|\mathbf{A}(t)\|\,;\, t \in J\}$. (Recall that "$\pm$" is necessary to include the case $t < t_0$.)

Proof cont'd.

$$|\mathbf{y}_1(t) - \mathbf{y}_0| \leq K := \max\{|\mathbf{y}_1(t) - \mathbf{y}_0|\,;\, t \in J\}\,,$$
$$|\mathbf{y}_2(t) - \mathbf{y}_1(t)| \leq \pm L \int_{t_0}^{t} K\,\mathrm{d}s = LK\,|t - t_0|\,,$$
$$|\mathbf{y}_3(t) - \mathbf{y}_2(t)| \leq \pm L \int_{t_0}^{t} LK\,|s - t_0|\,\mathrm{d}s = L^2 K\,\frac{|t - t_0|^2}{2!},$$

and in general, using mathematical induction,

$$|\mathbf{y}_{k+1}(t) - \mathbf{y}_k(t)| \leq L^k K\,\frac{|t - t_0|^k}{k!}.$$

Setting $J = [a, b]$ we can bound the (vectorial) function series $\sum_{k=0}^{\infty}(\mathbf{y}_{k+1} - \mathbf{y}_k)$ independently of t by the convergent series

$$\sum_{k=0}^{\infty} K\,\frac{L^k (b - a)^k}{k!} = K\mathrm{e}^{L(b-a)}.$$

Proof cont'd.

By Weierstrass's Criterion, this implies that the function sequence $\big(\mathbf{y}_k(t)\big)$ converges uniformly on J, the limit function $\mathbf{y}_\infty(t) = \lim_{k\to\infty} \mathbf{y}(t)$ is continuous on J and satisfies the integral equation

$$\mathbf{y}_\infty(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}(s)\mathbf{y}_\infty(s) + \mathbf{b}(s)\,\mathrm{d}s, \quad t \in J.$$

(The fixed-point property $T\mathbf{y}_\infty = \mathbf{y}_\infty$ requires continuity of the operator $(T\phi)(t) = \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{A}(s)\phi(s) + \mathbf{b}(s)\,\mathrm{d}s$ in the metric of uniform convergence on $[a, b]$. From the previous estimate we can infer $\|T\phi_1 - T\phi_2\|_\infty \leq L(b-a)\,\|\phi_1 - \phi_2\|_\infty$, i.e., T is Lipschitz-continuous; cf. also the 1-dimensional example $y' = 2t\,y$ discussed after the Existence Theorem.)

$\Longrightarrow$ The Fundamental Theorem of Calculus gives $\mathbf{y}_\infty'(t) = \mathbf{A}(t)\mathbf{y}_\infty(t) + \mathbf{b}(t)$ for $t \in J$, and of course $\mathbf{y}_\infty(t_0) = \mathbf{y}_0$.

Finally, since an arbitrary interval I can be exhausted by compact intervals, i.e., $I = \bigcup_{m=1}^{\infty} J_m$ with $J_m = [a_m, b_m]$ and $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots$, we obtain that $\big(\mathbf{y}_k(t)\big)$ converges on I, and the limit function $\mathbf{y}_\infty \colon I \to \mathbb{C}^n$ solves the IVP as well. $\square$

Note on the proof

You can see what goes wrong with the proof in the nonlinear case by computing the Picard-Lindelöf iterates for the IVP $y' = y^2 \wedge y(0) = 1$, whose solution is $y(t) = 1/(1-t)$.

$$\begin{aligned}
\phi_0(t) &= 1, \\
\phi_1(t) &= 1 + \int_0^t 1^2 \, \mathrm{d}s = t + 1, \\
\phi_2(t) &= 1 + \int_0^t (s+1)^2 \, \mathrm{d}s = 1 + \left[\tfrac{1}{3}s^3 + s^2 + s\right]_0^t = \tfrac{1}{3}t^3 + t^2 + t + 1, \\
\phi_3(t) &= 1 + \int_0^t \left(\tfrac{1}{3}s^3 + s^2 + s + 1\right)^2 \mathrm{d}s = \cdots \\
&= 1 + t + t^2 + t^3 + \tfrac{2}{3}t^4 + \tfrac{1}{3}t^5 + \tfrac{1}{9}t^6 + \tfrac{1}{63}t^7
\end{aligned}$$

With some effort one can show in general that $\phi_k(t)$ is a polynomial in t which starts with $1 + t + t^2 + \cdots + t^k$ and has the remaining coefficients in $[0,1)$. It follows that $\sum_{j=0}^k t^j \leq \phi_k(t) \leq \sum_{j=0}^\infty t^j = \frac{1}{1-t}$ and $\phi_k(t) \to 1/(1-t)$ for $k \to \infty$ if $0 \leq t < 1$. For $t \geq 1$ the sequence $\bigl(\phi_k(t)\bigr)$ does not converge, and hence nothing prevents the solution from blowing up at $t = 1$.

The Link with Linear Algebra

Theorem

The solutions of a homogeneous linear system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ *with* $\mathbf{A}\colon I \to \mathbb{C}^{n\times n}$ *form an n-dimensional vector space over* $\mathbb{C}$. *For solutions* $\mathbf{y}_1, \dots, \mathbf{y}_k \colon I \to \mathbb{C}^n$ *of* $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ *the following are equivalent:*

1. *The functions* $\mathbf{y}_1, \dots, \mathbf{y}_k \in (\mathbb{C}^n)^I$ *are linearly independent.*
2. *For some* $t_0 \in I$ *the vectors* $\mathbf{y}_1(t_0), \dots, \mathbf{y}_k(t_0) \in \mathbb{C}^n$ *are linearly independent.*
3. *For all* $t_0 \in I$ *the vectors* $\mathbf{y}_1(t_0), \dots, \mathbf{y}_k(t_0)$ *are linearly independent.*

Proof.

If $\mathbf{y}_1, \mathbf{y}_2 \colon I \to \mathbb{C}^n$ are solutions then
$(\mathbf{y}_1 + \mathbf{y}_2)' = \mathbf{y}_1' + \mathbf{y}_2' = \mathbf{A}(t)\mathbf{y}_1 + \mathbf{A}(t)\mathbf{y}_2 = \mathbf{A}(t)(\mathbf{y}_1 + \mathbf{y}_2)$,
i.e., $\mathbf{y}_1 + \mathbf{y}_2$ is a solution as well. Similarly, scalar multiples of solutions are again solutions, and of course the all-zero function is a solution. This proves that the solutions form a vector space over $\mathbb{C}$ (subspace of the vectorial function space $(\mathbb{C}^n)^I$).

Proof cont'd.

Next we prove the equivalences. The implications (3)$\Longrightarrow$(2)$\Longrightarrow$(1) are trivial and it remains to show (1)$\Longrightarrow$(3).
Suppose $\mathbf{y}_1, \dots, \mathbf{y}_k$ are linearly independent and that $c_1\mathbf{y}_1(t_0) + \cdots + c_k\mathbf{y}_k(t_0) = \mathbf{0} \in \mathbb{C}^n$. Then the two solutions $c_1\mathbf{y}_1 + \cdots + c_k\mathbf{y}_k$ and $\mathbf{y} \equiv 0$ agree at $t = t_0$.
$\Longrightarrow$ By the Existence and Uniqueness Theorem, they must agree everywhere, i.e., $c_1\mathbf{y}_1 + \cdots + c_k\mathbf{y}_k = 0$ in $(\mathbb{C}^n)^I$.
$\Longrightarrow$ $c_1 = \cdots = c_k = 0$, since $\mathbf{y}_1, \dots, \mathbf{y}_k$ are linearly independent.
This proves (1)$\Longrightarrow$(3).

Finally we show that the solution space V has dimension n.
Fix $t_0 \in I$ and consider the evaluation map $V \to \mathbb{C}^n$, $\mathbf{y} \mapsto \mathbf{y}(t_0)$, which is obviously linear.
Since (1)$\Longrightarrow$(2), this map is injective.
Since every IVP $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} \wedge \mathbf{y}(t_0) = \mathbf{y}_0 \in \mathbb{C}^n$ is solvable, the map is surjective.
$\Longrightarrow$ The map is a vector space isomorphism and $\dim V = \dim \mathbb{C}^n = n$. $\square$

Remarks

1. The same Theorem holds, mutatis mutandis, for real-valued solutions of "real" linear systems $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ with $\mathbf{A} \colon I \to \mathbb{R}^{n\times n}$. This can be proved in the same way or inferred from the complex case.

2. A basis $\mathbf{y}_1, \dots, \mathbf{y}_n$ of the solution space of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ is called a *fundamental system of solutions*. The equivalence (1)$\Longrightarrow$(2) yields the following handy test for fundamental systems:

   Writing $\mathbf{y}_j(t) = \big(y_{1j}(t), \dots, y_{nj}(t)\big)^{\mathsf{T}}$ as columns of a matrix $\Phi(t)$ (so-called *fundamental matrix*), we have that $\{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ is a fundamental system of solutions iff $\Phi(t_0)$ has rank n for some (and hence all) $t_0 \in I$.

3. With $\Phi(t)$ as in (2) we have, using matrix-vector multiplication for functions, the matrix version $\Phi'(t) = \mathbf{A}(t)\Phi(t)$ of the homogeneous ODE system and the representation

$$\mathbf{y}(t) = c_1\mathbf{y}_1(t) + \dots + c_n\mathbf{y}_n(t) = \Phi(t)\mathbf{c}$$

   with $\mathbf{c} = (c_1, \dots, c_n)^{\mathsf{T}} \in \mathbb{C}^n$ for the general solution of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$.

The Inhomogeneous Case

Theorem

1. *Every inhomogeneous linear system* $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t)$ *is solvable. A particular solution is* $\mathbf{y}_\mathrm{p}\colon I \to \mathbb{C}^n$ *defined by*

$$\mathbf{y}_\mathrm{p}(t) = \Phi(t)\mathbf{c}(t) \quad \text{with} \quad \mathbf{c}(t) = \int_{t_0}^{t} \Phi(s)^{-1}\mathbf{b}(s)\,\mathrm{d}s,$$

   where $t_0 \in I$ *can be arbitrarily chosen.*

2. *The general solution of* $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t)$ *is obtained by adding to the particular solution* $\mathbf{y}_\mathrm{p}$ *from (1) the general solution of the associated homogeneous linear system* $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$*, i.e.,*

$$\mathbf{y}(t) = \Phi(t)\mathbf{c}(t) + \Phi(t)\mathbf{c}_0$$

   with $\mathbf{c}(t)$ *as in (1) and* $\mathbf{c}_0 \in \mathbb{C}^n$*.*

Note that $\mathbf{c}_0$ and $\mathbf{y}(t_0) = \mathbf{y}_0$ determine each other via $\mathbf{y}_0 = \Phi(t_0)\mathbf{c}_0$, and that the general solution can also be obtained by using $\mathbf{c}(t) = \int \Phi(t)^{-1}\mathbf{b}(t)\,\mathrm{d}t = \mathbf{c}_0 + \int_{t_0}^{t} \Phi(s)^{-1}\mathbf{b}(s)\,\mathrm{d}s$ in (1).

Proof.

(1) is proved by a higher-dimensional analogue of "variation of parameters". Any fundamental solution matrix $\Phi(t)$ satisfies $\Phi' = \mathbf{A}\Phi$ and hence

$$(\Phi\mathbf{c})' = \Phi'\mathbf{c} + \Phi\mathbf{c}' = \mathbf{A}\Phi\mathbf{c} + \Phi\mathbf{c}'$$

$\Longrightarrow (\Phi\mathbf{c})' = \mathbf{A}\Phi\mathbf{c} + \mathbf{b}$ is equivalent to $\Phi\mathbf{c}' = \mathbf{b}$, i.e., to $\mathbf{c}' = \Phi^{-1}\mathbf{b}$. Since $t \mapsto \Phi(t)^{-1}\mathbf{b}(t)$ is continuous, the Fundamental Theorem of Calculus applies and $\mathbf{c}(t) = \int_{t_0}^{t} \Phi(s)^{-1}\mathbf{b}(s)\,\mathrm{d}s$ solves $\mathbf{c}' = \Phi^{-1}\mathbf{b}$.

(2) is proved as in the one-dimensional case: If $\mathbf{y}_1, \mathbf{y}_2$ are solutions of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t)$ then $\mathbf{y}_1 - \mathbf{y}_2$ solves the associated homogeneous system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$. □

Note

An important step in the proof is the observation that $t \mapsto \Phi(t)^{-1}$ is continuous. Why is this true?

Reason: The entries of $\Phi(t)^{-1}$ are obtained from the entries of $\Phi(t)$ (which are continuous) by applying the four basic arithmetic operations. This follows from $\Phi(t)^{-1} = \frac{1}{\det \Phi(t)} \operatorname{Adj} \Phi(t)$ (see Linear Algebra part), which expresses the entries of $\Phi(t)^{-1}$ in terms of certain subdeterminants of $\Phi(t)$.

Example

We consider the 1st-order system

$$\begin{aligned} y_1' &= y_1 + t\,y_2 + 1, \\ y_2' &= t\,y_1 + y_2. \end{aligned}$$

This is a time-dependent inhomogeneous linear system, with standard form

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

i.e., $\mathbf{A}(t) = \left(\begin{smallmatrix} 1 & t \\ t & 1 \end{smallmatrix}\right)$, $\mathbf{b}(t) = \mathbf{b} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$.

The associated homogeneous system $y_1' = y_1 + t\,y_2$, $y_2' = t\,y_1 + y_2$ can be solved using the observation that $s = y_1 + y_2$, $d = y_1 - y_2$ satisfy the "decoupled" system

$$\begin{aligned} s' &= y_1' + y_2' = y_1 + t\,y_2 + t\,y_1 + y_2 = (1+t)(y_1 + y_2) = (1+t)s, \\ d' &= y_1' - y_2' = y_1 + t\,y_2 - t\,y_1 - y_2 = (1-t)(y_1 - y_2) = (1-t)d. \end{aligned}$$

The solution is $s(t) = c_1\,\mathrm{e}^{t+t^2/2}$, $d(t) = c_2\,\mathrm{e}^{t-t^2/2}$ with $c_1, c_2 \in \mathbb{R}$, say.

Example (cont'd)

$$\implies y_1(t) = \frac{s(t)+d(t)}{2} = \frac{1}{2}\left(c_1\,\mathrm{e}^{t+t^2/2} + c_2\,\mathrm{e}^{t-t^2/2}\right),$$
$$y_2(t) = \frac{s(t)-d(t)}{2} = \frac{1}{2}\left(c_1\,\mathrm{e}^{t+t^2/2} - c_2\,\mathrm{e}^{t-t^2/2}\right).$$

The corresponding matrix-vector form is

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{2}\underbrace{\begin{pmatrix} \mathrm{e}^{t+t^2/2} & \mathrm{e}^{t-t^2/2} \\ \mathrm{e}^{t+t^2/2} & -\mathrm{e}^{t-t^2/2} \end{pmatrix}}_{\Phi(t)}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

(The factor $1/2$ doesn't matter for the fundamental matrix.) Thus every solution of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ is a linear combination of

$$\mathbf{y}_1(t) = \begin{pmatrix} \mathrm{e}^{t+t^2/2} \\ e^{t+t^2/2} \end{pmatrix} = e^{t+t^2/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{t-t^2/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

which therefore form a fundamental system of solutions.

Example (cont'd)

In order to solve the original inhomogeneous system, we compute

$$\Phi(t)^{-1}\mathbf{b}(t) = \frac{1}{-2\mathrm{e}^{2t}}\begin{pmatrix} -e^{t-t^2/2} & -e^{t-t^2/2} \\ -e^{t+t^2/2} & e^{t+t^2/2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{-t-t^2/2} \\ e^{-t+t^2/2} \end{pmatrix},$$

$$\mathbf{c}(t) = \int_0^t \frac{1}{2}\begin{pmatrix} e^{-s-s^2/2} \\ e^{-s+s^2/2} \end{pmatrix}\mathrm{d}s = \frac{1}{2}\begin{pmatrix} \int_0^t e^{-s-s^2/2}\,\mathrm{d}s \\ \int_0^t e^{-s+s^2/2}\,\mathrm{d}s \end{pmatrix}.$$

A particular solution of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}$ is therefore

$$\begin{aligned} \mathbf{y}_p(t) = \Phi(t)\mathbf{c}(t) &= \frac{1}{2}\begin{pmatrix} \mathrm{e}^{t+t^2/2} & \mathrm{e}^{t-t^2/2} \\ \mathrm{e}^{t+t^2/2} & -\mathrm{e}^{t-t^2/2} \end{pmatrix}\begin{pmatrix} \int_0^t e^{-s-s^2/2}\,\mathrm{d}s \\ \int_0^t e^{-s+s^2/2}\,\mathrm{d}s \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} \mathrm{e}^{t+t^2/2}\int_0^t e^{-s-s^2/2}\,\mathrm{d}s + \mathrm{e}^{t-t^2/2}\int_0^t e^{-s+s^2/2}\,\mathrm{d}s \\ \mathrm{e}^{t+t^2/2}\int_0^t e^{-s-s^2/2}\,\mathrm{d}s - \mathrm{e}^{t-t^2/2}\int_0^t e^{-s+s^2/2}\,\mathrm{d}s \end{pmatrix}. \end{aligned}$$

Of course this is a toy example. You can check that $s(t)$ and $d(t)$, defined as in the homogeneous case, solve the decoupled system $s' = (1+t)s + 1$, $d' = (1-t)d + 1$, and that ordinary variation of parameters for these two ODE's and the backwards substitution $\mathbf{y}_p = \frac{1}{2}\binom{s_p+d_p}{s_p-d_p}$ leads to the same result.

The Exponential of a Matrix

Providing a solution in the time-independent case $\mathbf{y}' = \mathbf{A}\mathbf{y}$

Definition

The *matrix exponential function* $\exp\colon \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ is defined by

$$\exp \mathbf{A} := \mathrm{e}^{\mathbf{A}} := \sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k = \mathbf{I}_n + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 + \frac{1}{6}\mathbf{A}^3 + \cdots .$$

Since convergence of a sequence/series in $\mathbb{C}^{n\times n}$ is equivalent to entry-wise convergence, this limit is well-defined if the (i,j)-entries of the partial sum $\sum_{k=0}^{n} \frac{1}{k!}\mathbf{A}^k$, which are $\sum_{k=0}^{n} \frac{1}{k!}(\mathbf{A}^k)_{ij}$, converge in $\mathbb{C}$.
Let $a = \max\{|a_{ij}|\,; 1 \leq i,j \leq n\}$.
$\Longrightarrow$ The entries of $\mathbf{A}^k$ are bounded by $n^{k-1}a^k$.

$$\Longrightarrow \quad \sum_{k=0}^{n} \left|\frac{1}{k!}(\mathbf{A}^k)_{ij}\right| \leq 1 + a + \frac{na^2}{2!} + \frac{n^2a^3}{3!} + \cdots \leq \mathrm{e}^{na} < \infty$$

By the comparison test, the series formed by the (i,j)-entries of the partial sums converges (even absolutely!), and hence the limit defining $\mathrm{e}^{\mathbf{A}}$ is well defined.

Lemma

If $\mathbf{AB} = \mathbf{BA}$ *then* $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$.

Proof.

The preceding argument shows that the series defining $e^{\mathbf{A}}$ converges absolutely. Hence we can freely rearrange the summands in the following double series:

$$\begin{aligned} e^{\mathbf{A}}e^{\mathbf{B}} &= \left(\sum_{k=0}^{\infty} \frac{1}{k!}\mathbf{A}^k\right)\left(\sum_{l=0}^{\infty} \frac{1}{l!}\mathbf{B}^l\right) = \sum_{m=0}^{\infty} \sum_{\substack{k,l \\ k+l=m}} \frac{1}{k!\,l!}\mathbf{A}^k\mathbf{B}^l \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{k=0}^{m} \binom{m}{k} \mathbf{A}^k\mathbf{B}^{m-k} \\ &= \sum_{m=0}^{\infty} \frac{1}{m!}(\mathbf{A}+\mathbf{B})^m = e^{\mathbf{A}+\mathbf{B}} \qquad \text{(Binomial Theorem)} \end{aligned}$$

For the Binomial Theorem to hold we need the assumption $\mathbf{AB} = \mathbf{BA}$. As an example consider the case $m = 2$: $(\mathbf{A}+\mathbf{B})^2 = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ iff $\mathbf{AB} = \mathbf{BA}$. □

Note

Since $\pm\mathbf{A}$ commute with each other, the lemma gives in particular $\mathrm{e}^{\mathbf{A}}\mathrm{e}^{-\mathbf{A}} = \mathrm{e}^{\mathbf{A}-\mathbf{A}} = \mathrm{e}^{\mathbf{0}} = \mathbf{I}_n$, i.e., $\mathrm{e}^{\mathbf{A}}$ is invertible with $(\mathrm{e}^{\mathbf{A}})^{-1} = \mathrm{e}^{-\mathbf{A}}$.

Example

$\mathbf{A} = \mathbf{E}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\mathbf{B} = \mathbf{E}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\mathbf{AB} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,
$\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{AB}$;

$\mathrm{e}^{\mathbf{A}} = \begin{pmatrix} \mathrm{e} & 0 \\ 0 & 1 \end{pmatrix}$, $\mathrm{e}^{\mathbf{B}} = \mathbf{I}_2 + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (since $\mathbf{B}^2 = \mathbf{0}$),
$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $(\mathbf{A} + \mathbf{B})^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathbf{A} + \mathbf{B}$;

$$\begin{aligned}
\mathrm{e}^{\mathbf{A}}\mathrm{e}^{\mathbf{B}} &= \begin{pmatrix} \mathrm{e} & \mathrm{e} \\ 0 & 1 \end{pmatrix}, \\
\mathrm{e}^{\mathbf{B}}\mathrm{e}^{\mathbf{A}} &= \begin{pmatrix} \mathrm{e} & 1 \\ 0 & 1 \end{pmatrix}, \\
\mathrm{e}^{\mathbf{A}+\mathbf{B}} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=1}^{\infty} \frac{1}{k!} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \mathrm{e}-1 & \mathrm{e}-1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{e} & \mathrm{e}-1 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Exercise

Show that for any diagonal matrix

$$\mathbf{D} = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{pmatrix} \quad \text{we have} \quad \mathrm{e}^{\mathbf{D}} = \begin{pmatrix} \mathrm{e}^{d_1} & & & \\ & \mathrm{e}^{d_2} & & \\ & & \ddots & \\ & & & \mathrm{e}^{d_n} \end{pmatrix}.$$

Exercise

Which condition should a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfy in order to conclude that $\mathrm{e}^{\mathbf{A}}$ is

a) symmetric;

b) orthogonal.

Exercise

1. Does $\mathrm{e}^{\mathbf{A}} = \mathrm{e}^{\mathbf{B}}$ imply $\mathbf{A} = \mathbf{B}$?
2. Is every invertible matrix $\mathbf{B} \in \mathbb{R}^{n \times n}$ in the range of $\mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$, $\mathbf{A} \mapsto \mathrm{e}^{\mathbf{A}}$? What if $\mathbb{R}$ is replaced by $\mathbb{C}$ in this problem?

Now consider for a fixed $n \times n$ matrix $\mathbf{A}$ the matrix function

$$t \mapsto \mathrm{e}^{\mathbf{A}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\,\mathbf{A}^k = \mathbf{I}_n + t\,\mathbf{A} + \frac{t^2}{2}\,\mathbf{A}^2 + \frac{t^3}{6}\,\mathbf{A}^3 + \cdots.$$

The (i,j)-entry of $\mathrm{e}^{\mathbf{A}t}$, viz. $\sum_{k=0}^{\infty} \frac{(\mathbf{A}^k)_{ij}}{k!}\,t^k$, is a power series, which converges for all $t \in \mathbb{R}$. Termwise differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{\mathbf{A}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\,\mathbf{A}^k\right) = \sum_{k=1}^{\infty} \frac{k\,t^{k-1}}{k!}\,\mathbf{A}^k = \mathbf{A}\,\mathrm{e}^{\mathbf{A}t}.$$

Theorem

The columns of $\mathrm{e}^{\mathbf{A}t}$ form a fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$, and the general solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\,\mathbf{y}(0)$.

Proof.

Since $\Phi(t) := \mathrm{e}^{\mathbf{A}t}$ satisfies the matrix ODE $\Phi'(t) = \mathbf{A}\Phi(t)$, its columns solve $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

$$\Phi(0) = \mathbf{I}_n + 0\,\mathbf{A} + \frac{0^2}{2!}\,\mathbf{A}^2 + \cdots = \mathbf{I}_n$$

Hence $\Phi(0)$ is invertible and the assertion follows. $\square$

Example

Consider the system $y_1' = y_2$, $y_2' = -y_1$, which arises from the 2nd-order ODE $y'' + y = 0$ by setting $(y_1, y_2) = (y, y')$. We are interested in the solution with initial values $y_1(0) = 6$, $y_2(0) = 2$, (or $y(0) = 6$, $y'(0) = 2$ for the 2nd-order ODE).

Of course this solution is $y(t) = 6\cos t + 2\sin t$, which we can use to verify our computation.

The system has the form $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
Since $\mathbf{A}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\mathbf{I}_2$, $\mathbf{A}^3 = -\mathbf{A}$, $\mathbf{A}^4 = \mathbf{I}_2$, we get

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \begin{pmatrix} & t \\ -t & \end{pmatrix} + \begin{pmatrix} -\frac{t^2}{2!} & \\ & -\frac{t^2}{2!} \end{pmatrix} + \begin{pmatrix} & -\frac{t^3}{3!} \\ +\frac{t^3}{3!} & \end{pmatrix} + \begin{pmatrix} \frac{t^4}{4!} & \\ & \frac{t^4}{4!} \end{pmatrix} + \cdots \\ &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned}$$

Hence the solution of our IVP is

$$\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{y}(0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 6\cos t + 2\sin t \\ -6\sin t + 2\cos t \end{pmatrix},$$

in accordance with the known solution.

Example (cont'd)

Continuing with the example, we use the opportunity to illustrate the solution method for an inhomogeneous system. The new task is to determine the general solution of $y_1' = y_2$, $y_2' = -y_1 + t$, which arises from the 2nd-order equation $y'' + y = t$.

This inhomogeneous system has the form $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}(t)$ with $\mathbf{A} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $\mathbf{b}(t) = \left(\begin{smallmatrix} 0 \\ t \end{smallmatrix}\right)$.

A particular solution is $\mathbf{y}_{\mathrm{p}}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t)$ with

$$\begin{aligned}
\mathbf{c}(t) &= \int_0^t \mathrm{e}^{-\mathbf{A}s}\mathbf{b}(s)\,\mathrm{d}s = \int_0^t \begin{pmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{pmatrix} \begin{pmatrix} 0 \\ s \end{pmatrix} \mathrm{d}s \\
&= \int_0^t \begin{pmatrix} -s\sin s \\ s\cos s \end{pmatrix} \mathrm{d}s = \left[\begin{pmatrix} s\cos s - \sin s \\ s\sin s + \cos s \end{pmatrix}\right]_0^t \\
&= \begin{pmatrix} t\cos t - \sin t \\ t\sin t + \cos t - 1 \end{pmatrix},
\end{aligned}$$

so that

$$\mathbf{y}_{\mathrm{p}}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} t\cos t - \sin t \\ t\sin t + \cos t - 1 \end{pmatrix} = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}.$$

Example (cont'd)

Since $t \mapsto \left(\begin{smallmatrix}\sin t\\ \cos t\end{smallmatrix}\right)$ is a solution of the associated homogeneous system, we may also take

$$\mathbf{y}_{\mathrm{p}}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}.$$

Well, this solution could have been guessed without going through the rather tedious computation!

The general solution of $y_1' = y_2$, $y_2' = -y_1 + t$ is therefore

$$\mathbf{y}(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \cos t + c_2 \sin t + t \\ -c_1 \sin t + c_2 \cos t + 1 \end{pmatrix}$$

with $c_1, c_2 \in \mathbb{C}$ (or $\mathbb{R}$), and that of $y'' + y = t$ the 1st coordinate function $y_1(t) = c_1 \cos t + c_2 \sin t + t$ (and $y_2(t) = y_1'(t)$, of course).

Note that solving the homogeneous system with the matrix exponential has produced (and for real systems always produces) a real fundamental system of solutions, whereas diagonalizing $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ (cf. Linear Algebra part) gives the complex fundamental system

$$\mathbf{y}_1(t) = \mathrm{e}^{\mathrm{i}t} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix}, \quad \mathbf{y}_2(t) = \mathrm{e}^{-\mathrm{i}t} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}.$$

Higher-Order Linear ODE's

The general time-dependent case

We consider only scalar ODE's, that is

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_1(t)y' + a_0(t)y = b(t)$$

with continuous functions $a_0, a_1, \dots, a_{n-1}, b\colon I \to \mathbb{C}$.

Order reduction $(y_1, y_2, \dots, y_n) = (y, y', \dots, y^{(n-1)})$ transforms such an ODE into the 1st-order system

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0(t) & -a_1(t) & \dots & -a_{n-2}(t) & -a_{n-1}(t) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}.$$

The coefficient matrix $\mathbf{A}(t)$ is the transposed companion matrix (cf. Linear Algebra part) of the polynomial $X^n + a_{n-1}(t)X^{n-1} + \dots + a_1(t)X + a_0(t) \in \mathbb{C}[X]$ (when $t \in I$ is considered as fixed).

The sharpened version of the Existence and Uniqueness Theorem for solutions of linear 1st-order ODE systems has the following

Corollary

1. *The solutions of any homogeneous nth-order ODE* $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = 0$ *exist on the whole interval I and form an n-dimensional subspace S of the function space* $\mathbb{C}^I$.
2. *Solutions* $y_1(t), \dots, y_n(t)$ *form a basis of the solution space S iff for some (and hence all)* $t \in I$ *the matrix*

$$\mathbf{W}(t) = \begin{pmatrix} y_1(t) & y_2(t) & \dots & y_n(t) \\ y_1'(t) & y_2'(t) & \dots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{pmatrix} \quad \textit{is invertible.}$$

3. *Any inhomogeneous nth-order ODE* $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = b(t)$ *is solvable. Solutions exist on the whole interval I, and they form a coset* $\{y_\mathrm{p}(t) + y_\mathrm{h}(t); y_\mathrm{h}(t) \in S\}$ *of S.*

Corollary (cont'd)

4. *Any IVP* $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = b(t) \wedge y^{(i)}(t_0) = c_i$ *for* $0 \leq i \leq n-1$ *has a unique solution, which is defined on the whole interval I.*

For real nth-order ODE's mutatis mutandis the same assertions hold (in particular such an ODE has a fundamental system consisting of real-valued solutions).

Proof of the corollary.

All assertions follow from the said theorem and the observation that solutions $\mathbf{y}(t) = \big(y_1(t), \dots, y_n(t)\big)^{\mathsf{T}}$ of the reduced 1st-order system must satisfy $y_2(t) = y_1'(t)$, $y_3(t) = y_2'(t) = y_1''(t)$, ..., $y_n(t) = y_1^{(n-1)}(t)$ and hence $y_n'(t) = y_1^{(n)}(t)$, so that $y_1(t)$ is a solution of the nth-order ODE. The map $\mathbf{y}(t) \mapsto \mathbf{y}_1(t)$ ("strip off all components of $\mathbf{y}(t)$ except the first") is then a vector space isomorphism from the solution space of the reduced 1st-order system onto S. $\square$

Note

In the statement of the corollary and its proof $y_1(t), \dots, y_n(t)$ have a different meaning (solutions of the n-th order scalar ODE versus coordinate functions of a solution of the 1st-order ODE system).

Example

Let us consider the ODE

$$y''' - y'' - 2y' = \frac{1}{1-t}, \quad t \in (-\infty, 1).$$

For the associated homogeneous ODE $y''' - y'' - 2y' = 0$ we had determined a fundamental system of solutions earlier, viz. $y_1(t) = 1$, $y_2(t) = \mathrm{e}^{-t}$, $y_3(t) = \mathrm{e}^{2t}$.

The corresponding 1st-order system is $y_1' = y_2$, $y_2' = y_3$, $y_3' = y_1''' = y_1'' + 2y_1' = y_3 + 2y_2$ or, in matrix form,

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}.$$

Reading the proof of the corollary backwards, we see that

$$\mathbf{W}(t) = \begin{pmatrix} y_1(t) & y_2(t) & y_3(t) \\ y_1'(t) & y_2'(t) & y_3'(t) \\ y_1''(t) & y_2''(t) & y_3''(t) \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{e}^{-t} & \mathrm{e}^{2t} \\ 0 & -\mathrm{e}^{-t} & 2\mathrm{e}^{2t} \\ 0 & \mathrm{e}^{-t} & 4\mathrm{e}^{2t} \end{pmatrix}$$

is a fundamental matrix of this system.

Example (cont'd)

Variation of parameters requires to determine $\mathbf{W}(t)^{-1}$. This is done using Gaussian elimination as usual:

$$\left(\begin{array}{ccc|ccc} 1 & e^{-t} & e^{2t} & 1 & 0 & 0 \\ 0 & -e^{-t} & 2e^{2t} & 0 & 1 & 0 \\ 0 & e^{-t} & 4e^{2t} & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 3e^{2t} & 1 & 1 & 0 \\ 0 & -e^{-t} & 2e^{2t} & 0 & 1 & 0 \\ 0 & 0 & 6e^{2t} & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -e^{-t} & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 6e^{2t} & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & -\frac{2}{3}e^{t} & \frac{1}{3}e^{t} \\ 0 & 0 & 1 & 0 & \frac{1}{6}e^{-2t} & \frac{1}{6}e^{-2t} \end{array}\right)$$

$$\implies \mathbf{W}(t)^{-1}\mathbf{b}(t) = \begin{pmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{2}{3}e^{t} & \frac{1}{3}e^{t} \\ 0 & \frac{1}{6}e^{-2t} & \frac{1}{6}e^{-2t} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{1}{1-t} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2(1-t)} \\ \frac{e^{t}}{3(1-t)} \\ \frac{e^{-2t}}{6(1-t)} \end{pmatrix}$$

$$\implies \mathbf{c}(t) = \int_0^t \begin{pmatrix} -\frac{1}{2(1-s)} \\ \frac{e^{s}}{3(1-s)} \\ \frac{e^{-2s}}{6(1-s)} \end{pmatrix} ds = \begin{pmatrix} -\int_0^t \frac{1}{2(1-s)}\, ds \\ \int_0^t \frac{e^{s}}{3(1-s)}\, ds \\ \int_0^t \frac{e^{-2s}}{6(1-s)}\, ds \end{pmatrix}$$

Example (cont'd)

$\Longrightarrow$ One particular solution is

$$
\begin{aligned}
y_p(t) &= c_1(t)y_1(t) + c_2(t)y_2(t) + c_3(t)y_3(t)\\
&= -\int_0^t \frac{1}{2(1-s)}\,\mathrm{d}s + \left(\int_0^t \frac{\mathrm{e}^{s}}{3(1-s)}\,\mathrm{d}s\right)\mathrm{e}^{-t} + \left(\int_0^t \frac{\mathrm{e}^{-2s}}{6(1-s)}\,\mathrm{d}s\right)\mathrm{e}^{2}
\end{aligned}
$$

$\Longrightarrow$ The general solution is

$$
\begin{aligned}
y(t) &= y_p(t) + \gamma_1 y_1(t) + \gamma_2 y_2(t) + \gamma_3 y_3(t)\\
&= \big(c_1(t)+\gamma_1\big)y_1(t) + \big(c_2(t)+\gamma_2\big)y_2(t) + \big(c_3(t)+\gamma_3\big)y_3(t)
\end{aligned}
$$

with constants $\gamma_1, \gamma_2, \gamma_3$.

Now suppose we want to solve the IVP

$$
y''' - y'' - 2y' = \frac{1}{1-t}, \quad y(0) = 1,\ y'(0) = y''(0) = 0, \quad \text{say.}
$$

It is possible to do this from the general solution by determining the constants γ_i from the given initial conditions.

However, there is a more conceptional approach using the matrix exponential function $\mathrm{e}^{\mathbf{A}t}$.

Example (cont'd)

In terms of the initial conditions $\mathbf{y}(0) = \big(y(0), y'(0), y''(0)\big) = \mathbf{y}_0$, the general solution of the associated 1st-order system is also given by

$$\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}(\mathbf{c}(t) + \mathbf{y}_0),$$

where $\mathbf{y}_p(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}(t)$ is the particular solution satisfying $\mathbf{c}(0) = \mathbf{0}$. For this note that any solution $\mathbf{y} = (y_1, y_2, y_3)^\mathsf{T}$ of the inhomogeneous system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \mathbf{y} + \begin{pmatrix} 0 \\ 0 \\ \frac{1}{1-t} \end{pmatrix}$$

still satisfies $y_2 = y_1'$, $y_3 = y_2' = y_1''$.

It is not necessary to compute the matrix exponential $\mathrm{e}^{\mathbf{A}t}$ directly from the series representation. Instead we can use that $\Phi(t) = \mathrm{e}^{\mathbf{A}t}$ is the unique solution of the matrix IVP $\Phi'(t) = \mathbf{A}\Phi(t) \wedge \Phi(0) = \mathbf{I}_3$.

Claim: $\mathrm{e}^{\mathbf{A}t} = \mathbf{W}(t)\mathbf{W}(0)^{-1}$, where $\mathbf{W}(t)$ denotes the fundamental matrix determined earlier. (In fact, $\mathbf{W}(t)$ can also be any other fundamental matrix of the given system.)

Example (cont'd)

Proof of the claim: An arbitrary fundamental matrix $\Phi(t) = (\mathbf{y}_1(t)|\mathbf{y}_2(t)|\mathbf{y}_3(t))$ has the form $\Phi(t) = \mathbf{W}(t)\mathbf{S}$ for some invertible matrix $\mathbf{S}$, since its columns must be linear combinations of $\mathbf{y}_1(t)$, $\mathbf{y}_2(t)$, $\mathbf{y}_3(t)$, and vice versa.
For $\Phi(t) = \mathrm{e}^{\mathbf{A}t}$ we have $\Phi(0) = \mathbf{I}_3$ and hence $\mathbf{S} = \mathbf{W}(0)^{-1}$.

$$\Longrightarrow \quad \mathrm{e}^{\mathbf{A}t} = \mathbf{W}(t)\mathbf{W}(0)^{-1} = \begin{pmatrix} 1 & \mathrm{e}^{-t} & \mathrm{e}^{2t} \\ 0 & -\mathrm{e}^{-t} & 2\mathrm{e}^{2t} \\ 0 & \mathrm{e}^{-t} & 4\mathrm{e}^{2t} \end{pmatrix} \begin{pmatrix} 1 & 1/2 & -1/2 \\ 0 & -2/3 & 1/3 \\ 0 & 1/6 & 1/6 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{1}{2} - \frac{2}{3}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{2t} & -\frac{1}{2} + \frac{1}{3}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{2t} \\ 0 & \frac{2}{3}\mathrm{e}^{-t} + \frac{1}{3}\mathrm{e}^{2t} & -\frac{1}{3}\mathrm{e}^{-t} + \frac{1}{3}\mathrm{e}^{2t} \\ 0 & -\frac{2}{3}\mathrm{e}^{-t} + \frac{2}{3}\mathrm{e}^{2t} & \frac{1}{3}\mathrm{e}^{-t} + \frac{2}{3}\mathrm{e}^{2t} \end{pmatrix}$$

$$\Longrightarrow \quad \mathbf{c}(t) = \int_0^t \mathrm{e}^{-\mathbf{A}s}\mathbf{b}(s)\,\mathrm{d}s = \begin{pmatrix} \int_0^t \frac{-\frac{1}{2}+\frac{1}{3}\mathrm{e}^{s}+\frac{1}{6}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \\ \int_0^t \frac{-\frac{1}{3}\mathrm{e}^{s}+\frac{1}{3}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \\ \int_0^t \frac{\frac{1}{3}\mathrm{e}^{s}+\frac{2}{3}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \end{pmatrix},$$

from which the general solution of any IVP is given as

Example (cont'd)

$$
\begin{aligned}
y(t) = 1 &\left(y(0) + \int_0^t \frac{-\frac{1}{2} + \frac{1}{3}\mathrm{e}^{s} + \frac{1}{6}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \right) \\
&+ \left(\frac{1}{2} - \frac{2}{3}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{2t} \right) \left(y'(0) + \int_0^t \frac{-\frac{1}{3}\mathrm{e}^{s} + \frac{1}{3}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \right) \\
&+ \left(-\frac{1}{2} + \frac{1}{3}\mathrm{e}^{-t} + \frac{1}{6}\mathrm{e}^{2t} \right) \left(y''(0) + \int_0^t \frac{\frac{1}{3}\mathrm{e}^{s} + \frac{2}{3}\mathrm{e}^{-2s}}{1-s}\,\mathrm{d}s \right).
\end{aligned}
$$

Example

We determine all solutions of the 2nd-order ODE

$$y'' - \frac{1}{2t}\,y' + \frac{1}{2t^2}\,y = 0 \qquad \text{on } I = (0, +\infty).$$

Since the coefficients are polynomials in t, it is reasonable to guess that there are solutions of the form $y(t) = t^k$. Substituting this into the ODE gives

$$k(k-1)t^{k-2} - \frac{kt^{k-1}}{2t} + \frac{t^k}{2t^2} = \left(k^2 - \frac{3}{2}k + \frac{1}{2}\right)t^{k-2} = 0.$$

The solutions of the quadratic are $k = 1$ and $k = \frac{1}{2}$, giving the solutions

$$y_1(t) = t \quad \text{and} \quad y_2(t) = \sqrt{t}.$$

From the theory we know that the solution space is 2-dimensional. Hence $y_1(t)$, $y_2(t)$ form a basis (fundamental system of solutions) iff they are linearly independent.

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} t & \sqrt{t} \\ 1 & \frac{1}{2\sqrt{t}} \end{vmatrix} = \frac{1}{2}\sqrt{t} - \sqrt{t} = -\frac{1}{2}\sqrt{t} \neq 0$$

Example (cont'd)

$\Longrightarrow y_1(t)$, $y_2(t)$ form a basis and the general solution is

$$y(t) = c_1 t + c_2 \sqrt{t}, \quad c_1, c_2 \in \mathbb{C}.$$

The general real solution is then of course $y(t) = c_1 t + c_2 \sqrt{t}$, $c_1, c_2 \in \mathbb{R}$.

The Wronskian

In the preceding example, the determinant $\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = -\frac{1}{2}\sqrt{t}$ is called the Wronskian of $y_1(t)$, $y_2(t)$. More generally we define

Definition

1. Suppose $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ are solutions of the 1st-order linear ODE system $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ with $\mathbf{A}\colon I \to \mathbb{R}^{n\times n}$. The function
$$\mathrm{W}(t) = \det\bigl(\mathbf{y}_1(t)|\dots|\mathbf{y}_n(t)\bigr)$$
is called the *Wronskian* (*Wronski determinant*) of $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$.
2. Suppose $y_1(t), \dots, y_n(t)$ are solutions of the n-th order scalar ODE $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_0(t)y = 0$. The function
$$\mathrm{W}(t) = \det\begin{pmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \vdots & & \vdots \\ y_1^{(n-1)}(t) & \dots & y_n^{(n-1)}(t) \end{pmatrix}$$
is called the *Wronskian* of $y_1(t), \dots, y_n(t)$.

Note

Earlier we had denoted the matrix in Part 2 of the definition by $\mathbf{W}(t)$, anticipating its name *Wronski matrix*. The matrix appearing in Part 1 is also called a *Wronski matrix* (though less frequently). Note that "Wronski matrix" refers to the matrix formed from any set of n solutions, while "fundamental matrix" requires the solutions to be linearly independent.

Theorem (Abel's Theorem)

$\mathrm{W}(t)$ satisfies a homogeneous 1st-order linear ODE $\mathrm{W}'(t) = a(t)\mathrm{W}(t)$. The function $a(t)$ is the sum of the main diagonal entries of $\mathbf{A}(t)$ in Case (1) and equal to $-a_{n-1}(t)$ in Case (2).

Corollary

There exists a constant $c \in \mathbb{C}$ such that $\mathrm{W}(t) = c\,\mathrm{e}^{\int_{t_0}^{t} a(s)\,\mathrm{d}s}$ for $t \in I$.
In particular $\mathrm{W}(t) \equiv 0$ iff $c = 0$ iff $\mathrm{W}(t_0) = 0$.

Note

This explains in a different way the criterion for solutions $\mathbf{y}_1(t), \dots, \mathbf{y}_n(t)$ to form a basis of the solution space of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ established earlier.

Proof of Abel's Theorem.

We give the proof only in the case $n = 2$. The proof of the general case requires properties of the determinant we haven't developed yet. Moreover, it suffices to prove Case (1), because Case (2) then follows by inspecting the form of $\mathbf{A}(t)$ in the order reduction formula.

Writing $\big(\mathbf{y}_1(t)|\mathbf{y}_2(t)\big) = \Phi(t) = \begin{pmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{pmatrix}$ and using $\Phi' = \mathbf{A}\Phi = \begin{pmatrix} a_{11}\phi_{11}+a_{12}\phi_{21} & a_{11}\phi_{12}+a_{12}\phi_{22} \\ a_{21}\phi_{11}+a_{22}\phi_{21} & a_{21}\phi_{12}+a_{22}\phi_{22} \end{pmatrix}$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{W}(t) &= \frac{\mathrm{d}}{\mathrm{d}t}\begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{vmatrix} = (\phi_{11}\phi_{22} - \phi_{21}\phi_{12})' \\
&= \phi_{11}'\phi_{22} + \phi_{11}\phi_{22}' - \phi_{21}'\phi_{12} - \phi_{21}\phi_{12}' \\
&= (a_{11}\phi_{11} + a_{12}\phi_{21})\phi_{22} + \phi_{11}(a_{21}\phi_{12} + a_{22}\phi_{22}) \\
&\quad - (a_{21}\phi_{11} + a_{22}\phi_{21})\phi_{12} - \phi_{21}(a_{11}\phi_{12} + a_{12}\phi_{22}) \\
&= (a_{11} + a_{22})(\phi_{11}\phi_{22} - \phi_{21}\phi_{12}) \\
&= (a_{11} + a_{22})\mathrm{W}(t).
\end{aligned}$$

$\square$

Example (cont'd)

We continue our previous example

$$y'' - \frac{1}{2t}\,y' + \frac{1}{2t^2}\,y = 0 \qquad t \in (0, +\infty).$$

Order reduction reduces this 2nd-order ODE to the 2×2-system

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2t^2} & \frac{1}{2t} \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

$\Longrightarrow \mathrm{W}'(t) = \frac{1}{2t}\,\mathrm{W}(t)$ by Abel's Theorem.
The solution of this ODE is

$$\mathrm{W}(t) = c \exp\left(\int \frac{\mathrm{d}t}{2t}\right) = c\,|t|^{1/2} = c\sqrt{t},$$

since $t > 0$.
For the fundamental system $y_1(t) = t$, $y_2(t) = \sqrt{t}$ the constant is $c = \mathrm{W}(1) = \begin{vmatrix} 1 & 1 \\ 1 & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$, as determined earlier.

Example (cont'd)

Now we consider the inhomogeneous ODE

$$y'' - \frac{1}{2t}\,y' + \frac{1}{2t^2}\,y = b(t), \quad t \in (0, +\infty).$$

For the case $b(t) = t^\ell$ our previous computation suggests a solution. In terms of the linear differential operator $L = \mathrm{D}^2 - \frac{1}{2t}\mathrm{D} + \frac{1}{2t^2}\mathrm{id}$ the ODE can be concisely written as $Ly = b(t)$, and we had found earlier that

$$L[t^k] = \left(k^2 - \frac{3}{2}\,k + \frac{1}{2}\right) t^{k-2}.$$

Since L is linear, it follows that $L\left[\frac{1}{k^2 - \frac{3}{2}\,k + \frac{1}{2}}\,t^k\right] = t^{k-2}$.
Substituting $\ell = k - 2$ gives
$k^2 - \frac{3}{2}\,k + \frac{1}{2} = (\ell+2)^2 - \frac{3}{2}(\ell+2) + \frac{1}{2} = \ell^2 + \frac{5}{2}\ell + \frac{3}{2} = (\ell+1)(\ell+\frac{3}{2})$
and hence

$$L\left[\frac{1}{(\ell+1)(\ell+\frac{3}{2})}\,t^{\ell+2}\right] = t^\ell, \qquad \text{valid for } \ell \notin \{-1, -\tfrac{3}{2}\}.$$

Example (cont'd)

Solutions for $\ell \in \{-1, -\frac{3}{2}\}$ can be determined using variation of parameters. We consider $b(t) = t^{-1}$ as an example.

It suffices to extract the first coordinate function of the vectorial solution (cf. the Theorem on Slide 13 of the handout version) of the corresponding 2×2 system:

$$y_{\mathrm{p}}(t) = c_1(t)y_1(t) + c_2(t)y_2(t) \qquad \text{with}$$

$$\begin{aligned}\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} &= \int \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ b(t) \end{pmatrix} \mathrm{d}t = \int \frac{1}{\mathrm{W}(t)} \begin{pmatrix} -y_2(t)b(t) \\ y_1(t)b(t) \end{pmatrix} \mathrm{d}t \\ &= \int \frac{1}{-\frac{1}{2}\sqrt{t}} \begin{pmatrix} -\sqrt{t}\,t^{-1} \\ t\,t^{-1} \end{pmatrix} \mathrm{d}t = \int \begin{pmatrix} 2t^{-1} \\ -2t^{-1/2} \end{pmatrix} \mathrm{d}t = \begin{pmatrix} 2\ln t \\ -4\sqrt{t} \end{pmatrix}.\end{aligned}$$

$\Longrightarrow$ $y_{\mathrm{p}}(t) = 2t\ln t - 4t$ is a particular solution, and (since $t \mapsto 4t$ solves the homogeneous ODE) $t \mapsto 2t \ln t$ as well.

The same formula works for any continuous right-hand side $b(t)$, except that it may not be integrable in closed form; cp. also [BDM17], Theorem 3.6.1.

Caveats when working with time-dependent differential operators

Polynomial differential operators with coefficients depending on t, like $L = \mathrm{D}^2 - \frac{1}{2t}\mathrm{D} + \frac{1}{2t^2}\mathrm{id}$ (or $L = \mathrm{D}^2 - \frac{1}{2t}\mathrm{D} + \frac{1}{2t^2}$, for short) do not satisfy the usual algebraic rules for working with polynomials! In particular we must not transpose time-dependent coefficients from right to left.

Example

$L = \mathrm{D}^2 - \frac{1}{2t}\mathrm{D} + \frac{1}{2t^2} \neq \mathrm{D}^2 - \mathrm{D}\frac{1}{2t} + \frac{1}{2t^2} = L^*$.

You can work out Lf and $L^* f$ for suitable functions f and see that they differ.

But the following easier example also tells you what's going on:

$$\begin{aligned}(t\mathrm{D})[f] &= t(\mathrm{D}f) = tf',\\ (\mathrm{D}t)[f] &= \mathrm{D}(tf) = f + tf'.\end{aligned}$$

Thus $\mathrm{D}t - t\mathrm{D} = \mathrm{id}$ rather than $\mathrm{D}t - t\mathrm{D} = 0$.

Today's Lecture:

Second-Order Linear ODE's

Additional Remarks

First we state three famous (time-dependent) 2nd-order linear ODE's. Actually these are one-parameter families of ODE's providing one ODE for every $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.

1. LEGENDRE's Differential Equation is

$$(1 - t^2)y'' - 2ty' + n(n+1)y = 0 \qquad (\mathrm{Le}_n)$$

with time domain $-1 < t < 1$.

2. HERMITE's Differential Equation is

$$y'' - 2ty' + 2ny = 0 \qquad (\mathrm{He}_n)$$

with time domain $t \in \mathbb{R}$.

3. LAGUERRE's Differential Equation is

$$ty'' + (1 - t)y' + ny = 0 \qquad (\mathrm{La}_n)$$

with time domain $t > 0$.

Theorem

The following polynomials solve these ODE's:

1. *(Le_n) is solved by the Legendre Polynomial $\mathrm{P}_n(X)$ of order n, which is defined by*

$$\mathrm{P}_n(t) = \frac{1}{2^n n!}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left((t^2-1)^n\right).$$

2. *(He_n) is solved by the Hermite Polynomial $\mathrm{H}_n(X)$ of order n, which is defined by*

$$\mathrm{H}_n(t) = (-1)^n \mathrm{e}^{t^2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \mathrm{e}^{-t^2}.$$

3. *(La_n) is solved by the Laguerre Polynomial $\mathrm{L}_n(X)$ of order n, which is defined by*

$$\mathrm{L}_n(t) = \mathrm{e}^{t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n (t^n \mathrm{e}^{-t}).$$

Notes

- Since $|\mathbb{R}| = \infty$, polynomials $a(X) = \sum_{i=0}^{n} a_i X^i \in \mathbb{R}[X]$ and polynomial functions $\mathbb{R} \to \mathbb{R}$, $t \mapsto a(t) = \sum_{i=0}^{n} a_i t^i$ determine each other uniquely, validating the preceding definitions.

  This follows from the *degree bound* for polynomials with coefficients in a field F, which implies that polynomial functions $t \mapsto a(t)$ and $t \mapsto b(t)$ arising from distinct polynomials $a(X), b(X) \in F[X]$ can have at most $\deg\bigl(a(X) - b(X)\bigr)$ equal values.

  For a finite field F the corresponding proposition is no longer true. For example, the polynomials $0 \in \mathbb{F}_2[X]$ and $X^2 + X \in \mathbb{F}_2[X]$ both determine the all-zero function $\mathbb{F}_2 \to \mathbb{F}_2$, as follows from the identities $0^2 + 0 = 1^2 + 1 = 0$ in $\mathbb{F}_2$. However, the proposition remains true under the additional assumption that $a(X)$ and $b(X)$ have degree less than $|F|$ (again by the degree bound).
- The normalization factors, $\frac{1}{2^n n!}$ for $\mathrm{P}_n(X)$ and $(-1)^n$ for $\mathrm{H}_n(X)$, do not matter for the solution of the ODE (since it is linear). We could as well have assumed that all three families consist of monic polynomials, obtained by dividing the non-normalized polynomials by their leading coefficients.

Proof.

We prove the assertion only for the Legendre polynomials (with different normalization factors). Writing as usual $\mathrm{D}=\frac{\mathrm{d}}{\mathrm{d}t}$, we evaluate $\mathrm{D}^{n+1}\left[(t^2-1)\mathrm{D}\big((t^2-1)^n\big)\right]$ in two different ways.

Using Leibniz's Formula $\mathrm{D}^n(fg)=\sum_{i=0}^{n}\binom{n}{i}(\mathrm{D}^i f)(\mathrm{D}^{n-i}g)$ for the n-th derivative of a product with $f=t^2-1$, $g=\mathrm{D}\big((t^2-1)^n\big)$, we have

$$\begin{aligned}
&\mathrm{D}^{n+1}\left[(t^2-1)\mathrm{D}\big((t^2-1)^n\big)\right]=\\
&\quad=(t^2-1)\mathrm{D}^{n+2}\big((t^2-1)^n\big)+(n+1)(2t)\mathrm{D}^{n+1}\big((t^2-1)^n\big)+n(n+1)\mathrm{D}^n\big((t^2-1)^n\big)\\
&\quad=(t^2-1)\mathrm{P}_n''(t)+2(n+1)t\,\mathrm{P}_n'(t)+n(n+1)\mathrm{P}_n(t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathrm{D}^{n+1}\left[(t^2-1)\mathrm{D}\big((t^2-1)^n\big)\right]=\\
&\quad=\mathrm{D}^{n+1}\left[(t^2-1)2nt(t^2-1)^{n-1}\right]\\
&\quad=2n\,\mathrm{D}^{n+1}\left[t(t^2-1)^n\right]\\
&\quad=2n\left[t\,\mathrm{D}^{n+1}\big((t^2-1)^n\big)+(n+1)\mathrm{D}^n\big((t^2-1)^n\big)\right]\\
&\quad=2nt\,\mathrm{P}_n'(t)+2n(n+1)\mathrm{P}_n(t).
\end{aligned}$$

$$\Longrightarrow (1-t^2)\mathrm{P}_n''(t)-2t\,\mathrm{P}_n'(t)+n(n+1)\mathrm{P}_n(t)=0 \qquad \square$$

Order Reduction

A different way

Theorem

Suppose $I \subseteq \mathbb{R}$ is an interval and $a, b\colon I \to \mathbb{C}$ are continuous. Further, suppose that $\phi\colon I \to \mathbb{C}$ is a nonzero solution of

$$y'' + a(t)y' + b(t)y = 0, \tag{$\star$}$$

and $J \subseteq I$ is a subinterval (of length > 0) such that $\phi(t) \neq 0$ for all $t \in J$. Then a second fundamental solution of ($\star$) on J (i.e., linearly independent of the restriction $\phi|_J$), is obtained as $\psi(t) = \phi(t)u(t)$, where $u(t)$ is any non-constant solution of

$$u'' + \left(2\frac{\phi'(t)}{\phi(t)} + a(t)\right) u' = 0. \tag{R}$$

Note

(R) is a 1st-order linear ODE for u', solved as usual by

$$u'(t) = \exp\left(-\int_{t_0}^{t} 2\frac{\phi'(s)}{\phi(s)} + a(s)\,\mathrm{d}s\right) = \frac{1}{\phi(t)^2}\exp\left(-\int_{t_0}^{t} a(s)\,\mathrm{d}s\right).$$

A further integration then yields $u(t)$.

Proof of the theorem.

We have

$$\begin{aligned}\psi &= \phi u,\\ \psi' &= \phi' u + \phi u',\\ \psi'' &= \phi'' u + 2\phi' u' + \phi u''.\end{aligned}$$

$$\begin{aligned}\implies \psi'' + a\psi' + b\psi &= (\phi'' + a\phi' + b\phi)u + (2\phi' + a\phi)u' + \phi u''\\ &= (2\phi' + a\phi)u' + \phi u'',\end{aligned}$$

since ϕ solves $y'' + ay' + by = 0$.
Hence we have

$$\psi'' + a\psi' + b\psi = 0 \iff u'' + (2\phi'/\phi + a)u' = 0. \qquad \square$$

Example

We compute a fundamental system of solutions of Legendre's ODE for $n = 1$, which has the explicit form

$$y'' - \frac{2t}{1-t^2}\,y' + \frac{2}{1-t^2}\,y = 0, \quad -1 < t < 1. \tag{Le$_1$}$$

As we have seen, one solution is $\mathrm{P}_1(t) = \frac{1}{2}\mathrm{D}(t^2-1) = t$. Hence, by the theorem, a second linearly independent (of the first) solution on $J = (0,1)$ is $\psi(t) = t\,u(t)$, where $u'(t)$ solves

$$u''(t) + \left(2\,\frac{\mathrm{P}_1'(t)}{\mathrm{P}_1(t)} - \frac{2t}{1-t^2}\right)u'(t) = u''(t) + \left(\frac{2}{t} - \frac{2t}{1-t^2}\right)u'(t) = 0.$$

A nonzero solution is

$$\begin{aligned} u'(t) &= \exp\left(\int -\frac{2}{t} + \frac{2t}{1-t^2}\,\mathrm{d}t\right) = \exp\bigl(-2\ln t - \ln(1-t^2)\bigr) \\ &= \frac{1}{t^2(1-t^2)} \end{aligned}$$

Example (cont'd)

$$\implies u(t) = \int \frac{\mathrm{d}t}{t^2(1-t^2)} = \int \frac{1}{t^2} + \frac{\frac{1}{2}}{1+t} + \frac{\frac{1}{2}}{1-t}\,\mathrm{d}t = -\frac{1}{t} + \frac{1}{2}\ln\frac{1+t}{1-t}$$

$$\implies \psi(t) = t\,u(t) = \frac{t}{2}\ln\frac{1+t}{1-t} - 1$$

The solution $\psi(t)$ was guaranteed to exist only on $(0,1)$, but clearly it is defined on the whole interval $(-1,1)$ and solves (Le_1).

$\implies$ A fundamental system of solutions of (Le_1) is

$$t, \quad \frac{t}{2}\ln\frac{1+t}{1-t} - 1.$$

Euler Equations

cf. [BDM17], Ch. 5.4

Definition

The ODE

$$t^2 y'' + \alpha t y' + \beta y = 0 \tag{E}$$

is called *Euler equation* with parameters α, β.

We will assume $\alpha, \beta \in \mathbb{R}$ and consider only real solutions. (The complex case is easily reduced to the real case.)

(E) is homogeneous linear, time-dependent, of order 2.

Except for the trivial case $\alpha = \beta = 0$, (E) has a singular point in $t = 0$, where the corresponding explicit equation $y'' + (\alpha/t)y' + (\beta/t^2)y = 0$ is not defined.
$\Longrightarrow$ Solutions exist "independently" on $(-\infty, 0)$, $(0, +\infty)$ and form a 2-dimensional real vector space in both cases.

For Part (2) of the following theorem, recall that solutions of (E) are twice differentiable functions $y\colon I \to \mathbb{R}$ satisfying (E) for every $t \in I$. For $I = \mathbb{R}$ and $t = 0$ the ODE reduces to $\beta y(0) = 0$, which for $\beta \neq 0$ requires $y(0) = 0$ (in particular associated IVP's with $y(0) = y_0 \neq 0$ are not solvable).

Reflection Principle

1. A solution $\phi\colon (0,+\infty) \to \mathbb{R}$ yields a solution $\psi\colon (-\infty,0) \to \mathbb{R}$ by reflecting the graph of ϕ at the *y*-axis (and vice versa).
2. ϕ can be extended to a solution on $\mathbb{R}$ whose graph is symmetric w.r.t. the *y*-axis if $\lim_{t\downarrow 0} \phi'(t) = 0$ and $\lim_{t\downarrow 0} \phi''(t)$ exists in $\mathbb{R}$.

As sketched in the proof, the existence of $\lim_{t\downarrow 0} \phi''(t)$ implies that of $\lim_{t\downarrow 0} \phi'(t)$ and $\lim_{t\downarrow 0} \phi(t)$, and the first condition requires that $\lim_{t\downarrow 0} \phi'(t)$ is zero.

Proof.

(1) For $t < 0$ let $\psi(t) = \phi(-t) = \phi(|t|)$. Then $\psi'(t) = -\phi(-t)$, $\psi''(t) = \phi''(-t)$, and hence

$$\begin{aligned} t^2\psi''(t) + \alpha t\psi'(t) + \beta\psi(t) &= t^2\phi''(-t) - \alpha t\phi'(-t) + \beta\phi(-t) \\ &= (-t)^2\phi''(-t) + \alpha(-t)\phi'(-t) + \beta\phi(-t) \\ &= 0, \end{aligned}$$

since $s = -t$ runs through $(0,+\infty)$ if t runs through $(-\infty,0)$. This proves Part (1).

Proof cont'd.

(2) The existence of $\lim_{t\downarrow 0}\phi''(t)$ implies that ϕ, which is C^∞ on $(0,+\infty)$, can be extended to a C^2-function on $[0,+\infty)$ (with one-sided derivatives in $t=0$). This follows from

$$\phi'(t)=\phi'(t_0)+\int_{t_0}^{t}\phi''(s)\,\mathrm{d}s,$$
$$\phi(t)=\phi(t_0)+\int_{t_0}^{t}\phi'(s)\,\mathrm{d}s,\qquad t_0,t>0,$$

which makes also sense for $t=0$ and gives continuous extensions of ϕ', ϕ (first for ϕ', then for ϕ) to $[0,+\infty)$.
It is then not difficult to show that ϕ is twice differentiable at $t=0$ from the right and that $\phi'(0)=\lim_{t\downarrow 0}\phi'(t)$, $\phi''(0)=\lim_{t\downarrow 0}\phi''(t)$.

The reflected function $\psi(t)=\phi(-t)$, $t\in(-\infty,0]$, is C^2 as well and satisfies $\psi(0)=\phi(0)$, $\psi'(0)=-\phi'(0)$, $\psi''(0)=\phi''(0)$.
Hence, provided that $\phi'(0)=0$ (and only then) we obtain a consistent extension of ϕ to $\mathbb{R}$.

Finally, the ODE is satisfied also for $t=0$ (for $\beta=0$ this is trivial, for $\beta\neq 0$ it follows from $y(0)=\lim_{t\to 0}y(t)=\lim_{t\to 0}\frac{-t^2y''(t)-\alpha t\,y'(t)}{\beta}=0$), completing the proof. $\square$

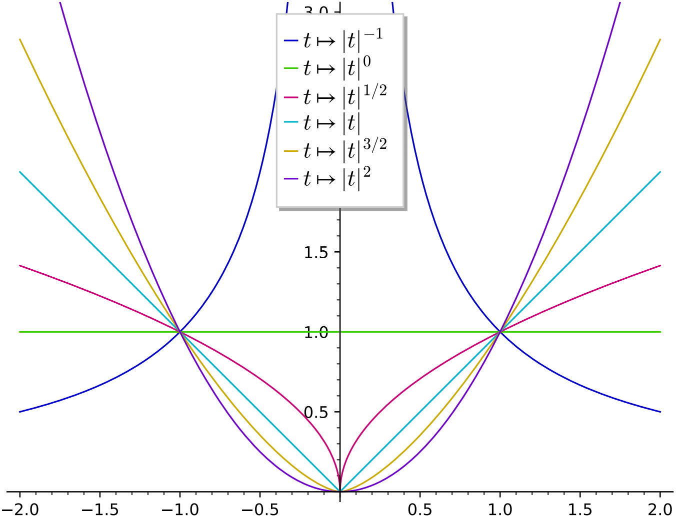

Figure: Illustration of the Reflection Principle: $t \mapsto |t|^r$ defines a C^2-function on $\mathbb{R}$ iff $t = 0 \vee t \geq 2$

Solution of the Euler Equations

Using the reflection principle, we can restrict attention to $t \geq 0$. We have already considered the special case $\alpha = -1/2$, $\beta = 1/2$, in which a fundamental system was given by t and $\sqrt{t}$. The earlier method works with some modifications also in the general case:

Setting $L = t^2\,\mathrm{D}^2 + \alpha t\,\mathrm{D} + \beta\,\mathrm{id}$, the Euler equation becomes $Ly = 0$, and we have

$$L[t^r] = \big(r(r-1) + \alpha r + \beta\big)t^r \equiv 0 \iff r^2 + (\alpha - 1)r + \beta = 0,$$

which is solved by $r_1 = \frac{1}{2}\left(1 - \alpha + \sqrt{(\alpha-1)^2 - 4\beta}\right)$ and $r_2 = \frac{1}{2}\left(1 - \alpha - \sqrt{(\alpha-1)^2 - 4\beta}\right)$.

Case 1: $(\alpha - 1)^2 > 4\beta$

In this case $r_1 > r_2$ are real, so that

$$\phi_1(t) = t^{r_1} \quad \text{and} \quad \phi_2(t) = t^{r_2}$$

(resp., $\phi_1(t) = (-t)^{r_1}$, $\phi_2(t) = (-t)^{r_2}$ for $t < 0$) form a fundamental system of solutions.

Case 1 cont'd

For the following analysis, let S_0 be the solution space with domain $I = \mathbb{R}$, i.e., S_0 consists of all functions $y \colon \mathbb{R} \to \mathbb{R}$ satisfying (E). Because a solution defined on some interval $(-\delta, \delta)$, $\delta > 0$, can be uniquely extended to $\mathbb{R}$, we can alternatively view S_0 as the local solution space at $t = 0$.

Combinatorial counting gives that there are $\binom{6}{2} - 2 = 13$ (???) cases to consider. We consider only a few of them.

If r_1, r_2 are negative (equivalently $\beta > 0$ and $\alpha > 1 + 2\sqrt{\beta}$) then the only solution defined at $t = 0$ is $y(t) \equiv 0$. In other words, $S_0 = \{0\}$ and the only realizable initial values at $t = 0$ are $y(0) = y'(0) = 0$.

If $r_1 = 0$ (equivalently $\beta = 0$ and $\alpha > 1$) then the solutions defined at $t = 0$ are the constant functions $y(t) \equiv c$, $c \in \mathbb{R}$.
$\Longrightarrow \dim(S_0) = 1$, and the (uniquely) realizable initial values at $t = 0$ are $y(0) = c \in \mathbb{R}$ arbitrary, $y'(0) = 0$.

If $0 < r_1 < 1 \lor 1 < r_1 < 2$ and r_2 is either negative or satisfies the same condition as r_1, then again the only solution defined at $t = 0$ is $y(t) \equiv 0$, and consequently $S_0 = \{0\}$.

If $r_1 = 1$, $r_2 = 0$ (the non-singular case $\alpha = \beta = 0$, in which the general solution is $y(t) = c_1 + c_2 t$) then $\dim(S_0) = 2$ and all initial conditions at $t = 0$ are uniquely realizable.

Case 1 cont'd

If $r_1 > 2$ and $0 < r_2 < 1 \vee 1 < r_2 < 2$, the solutions on $(0, +\infty)$ that can be extended to $[0, +\infty)$ have the form $y(t) = c\,t^{r_1}$ and satisfy $y(0) = y'(0) = y''(0) = 0$. Solutions $y(t) = c_1 t^{r_1}$ on $[0, +\infty)$ and $z(t) = c_2(-t)^{r_1}$ on $(-\infty, 0]$ can be combined freely to yield a solution on $\mathbb{R}$.

$\Longrightarrow \dim(S_0) = 2$, and a basis of S_0 (fundamental system of solutions) is formed by

$$y_1(t) = \begin{cases} t^{r_1} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \qquad y_2(t) = \begin{cases} (-t)^{r_1} & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

If $r_1 > r_2 = 2$ then the solutions on $(0, +\infty)$ have the form $y(t) = c_1 t^{r_1} + c_2 t^2$ and can be uniquely extended to $[0, +\infty)$ by setting $y(0) = y'(0) = 0$, $y''(0) = 2c_2$. Solutions $y(t) = c_1 t^{r_1} + c_2 t^2$ on $[0, +\infty)$ and $z(t) = c_3(-t)^{r_1} + c_4(-t)^2$ on $(-\infty, 0]$ can be glued to yield a solution on $\mathbb{R}$ iff $c_2 = c_4$.

$\Longrightarrow \dim(S_0) = 3$, and a basis of S_0 is formed by the functions $y_1(t)$, $y_2(t)$ defined above and $y_3(t) = t^2$.

Case 1 cont'd

If $r_1 > r_2 > 2$ then all solutions $y(t)$ on $(0, +\infty)$ can be uniquely extended to $[0, +\infty)$ by setting $y(0) = y'(0) = y''(0) = 0$. Solutions on $(-\infty, 0]$ and $[0, +\infty)$ can be freely combined to yield solutions on $\mathbb{R}$. $\implies \dim(S_0) = 4$, and a basis of S_0 is formed by

$$y_1(t) = \begin{cases} t^{r_1} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \qquad y_2(t) = \begin{cases} (-t)^{r_1} & \text{if } t \leq 0, \\ 0 & \text{if } t > 0, \end{cases}$$

$$y_3(t) = \begin{cases} t^{r_2} & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases} \qquad y_4(t) = \begin{cases} (-t)^{r_2} & \text{if } t \leq 0, \\ 0 & \text{if } t > 0. \end{cases}$$

Be sure to understand the precise meaning of "basis" here:

1. Every solution $y(t) \in S_0$ is of the form $y(t) = c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) + c_4 y_4(t)$ for some $c_1, c_2, c_3, c_4 \in \mathbb{R}$. This follows from the above discussion and

$$c_1 y_1(t) + c_2 y_2(t) + c_3 y_3(t) + c_4 y_4(t) = \begin{cases} c_1 t^{r_1} + c_3 t^{r_2} & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ c_2(-t)^{r_1} + c_4(-t)^{r_2} & \text{if } t < 0. \end{cases}$$

2. The coefficients c_1, c_2, c_3, c_4 are uniquely determined by $y(t)$.

Case 2: $(\alpha - 1)^2 = 4\beta$

Here $r_1 = r_2 = (1-\alpha)/2$, yielding only one fundamental solution $\phi_1(t) = t^{(1-\alpha)/2}$.

A second fundamental solution $\phi_2(t)$ can be determined using order reduction. The cofactor $u(t)$ in $\phi_2(t) = u(t)\phi_1(t)$ satisfies

$$\begin{aligned} u''(t) + \left(2\,\frac{\phi_1'(t)}{\phi_1(t)} + a(t)\right)u'(t) &= u''(t) + \left(2\cdot\frac{1-\alpha}{2t} + \frac{\alpha}{t}\right)u'(t) \\ &= u''(t) + \frac{1}{t}\,u'(t) = 0. \end{aligned}$$

$$\implies u'(t) = \exp\left(-\int\frac{\mathrm{d}t}{t}\right) = \frac{c_1}{t} \quad \implies \quad u(t) = c_1 \ln t + c_2$$

Hence a fundamental system of solutions on $(0, +\infty)$ in this case is

$$\phi_1(t) = t^{(1-\alpha)/2}, \quad \phi_2(t) = (\ln t)t^{(1-\alpha)/2}.$$

Extendability to solutions on $\mathbb{R}$ is discussed in the same way as before. We omit the general discussion, but one case is worth noting: For $\alpha < -3$ (the case in which $\frac{1-\alpha}{2} > 2$) we have $\lim_{t\downarrow 0}\phi_2(t) = \lim_{t\downarrow 0}\phi_2'(t) = \lim_{t\downarrow 0}\phi_2''(t) = 0$. Hence the analysis done for $r_1 > r_2 > 2$ carries over, and it follows that $\dim(S_0) = 4$.

Case 3: $(\alpha - 1)^2 < 4\beta$

In this case r_1, r_2 are complex,

$$r_1 = \frac{1}{2}\left(1 - \alpha + \mathrm{i}\sqrt{4\beta - (\alpha - 1)^2}\right),$$
$$r_2 = \frac{1}{2}\left(1 - \alpha - \mathrm{i}\sqrt{4\beta - (\alpha - 1)^2}\right) = \overline{r_1}.$$

A complex fundamental system of solutions is again t^{r_1}, t^{r_2}.
A real fundamental system can be obtained by extracting real and imaginary part of one of these. Writing $r_{1/2} = \lambda \pm \mathrm{i}\mu$, we get

$$y_1(t) = \mathrm{Re}(t^{\lambda + \mathrm{i}\mu}) = \mathrm{Re}\left(t^\lambda \mathrm{e}^{\mathrm{i}\mu \ln t}\right) = t^\lambda \cos(\mu \ln t),$$
$$y_2(t) = t^\lambda \sin(\mu \ln t).$$

In this case the determination of S_0 is comparatively easy:
Nonzero solutions on $(0, +\infty)$ are extendable to solutions on $\mathbb{R}$ iff $\lambda > 2$. (For $\lambda = 2$ the 2nd derivative of $y_1(t)$, $y_2(t)$ oszillates wildly near $t = 0$; the same is true of any nonzero linear combination of $y_1(t)$, $y_2(t)$.) If $\lambda > 2$ then all solutions on $[0, +\infty)$ satisfy $y(0) = y'(0) = y''(0) = 0$, and hence we have again $\dim(S_0) = 4$.