Differential Equations Plus (Math 286)

H57 For each of the following ODE's, find two linearly independent real solutions.

a) $4xy'' + 3y' - 3y = 0, \quad x \lessgtr 0;$

b) $x^2y'' - x(1+x)y' + y = 0, \quad x \lessgtr 0;$

c) $x^2y'' + xy' - (1+x)y = 0, \quad x > 0;$

d) $x^2y'' + xy' + (1+x)y = 0, \quad x > 0.$

H58 Consider the ODE

$$xy'' + 3y' - 3y = 0, \quad x > 0.$$

a) Show that the roots of the indicial equation are $r = 0$ and $r = -2$.

b) Find a solution $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$.

c) Find a second solution $y_2(x) = a\, y_1(x) \ln x + x^{-2}\,(1 + \sum_{n=1}^{\infty} c_n x^n)$.

H59 Do Exercises 5, 6, 9 in [BDM17], Ch. 5.7 (Exercises 6, 7, 10 in the 11th edition). Additionally show that $\mathrm{Y}_0'(x) = -\mathrm{Y}_1(x)$ for $x > 0$; see p. 236 (p. 238 in the 11th edition) for the definition of $\mathrm{Y}_1(x)$. The solution $y_2(x)$ appearing in the definition of $\mathrm{Y}_1(x)$ is the same as that you obtain in Exercise 9 (resp., Exercise 10).

H60 The Γ function is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$, and for non-integral $x < 0$ by choosing an integer $n > -x$ and setting

$$\Gamma(x) := \frac{\Gamma(x+n)}{x(x+1)\cdots(x+n-1)}.$$

a) Show that $\Gamma(x)$ is well-defined for $x < 0$, $x \notin \mathbb{Z}$, and satisfies $\Gamma(x+1) = x\,\Gamma(x)$ for all $x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}$.

Hint: Recall from Calculus III that $\Gamma(x+1) = x\,\Gamma(x)$ for $x > 0$.

b) Show $\lim_{x \to -n} \frac{1}{\Gamma(x)} = 0$ for $n \in \mathbb{N} = \{0, 1, 2, \dots\}$.
This shows that $1/\Gamma$ can be continuously extended to $\mathbb{R}$ by defining $1/\Gamma(-n) := 0$ for $n \in \mathbb{N}$.

c) The Bessel function of order $\nu \in \mathbb{R}$ is defined as (cf. the lecture)

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{\nu+2m} m!\,\Gamma(\nu+m+1)}\, x^{\nu+2m} \quad \text{for } x \in \mathbb{R},$$

cf. b) for the definition of $1/\Gamma(\nu+m+1)$.
Show $\quad \mathrm{J}_{-\nu} = (-1)^\nu \mathrm{J}_\nu \quad$ for $\nu \in \mathbb{N}$.

Hint: Show first that the coefficients of x^n in the expansion of $\mathrm{J}_{-\nu}(x)$ are zero for $n < \nu$.

 PLEASE TURN OVER

H61 *Optional Exercise*

For $x \in \mathbb{R} \setminus \{0\}$, $\nu \in \mathbb{R}$ show:

a) $\mathrm{J}_{\nu+1}(x) = \dfrac{2\nu}{x} \mathrm{J}_\nu(x) - \mathrm{J}_{\nu-1}(x)$;

b) $\mathrm{J}'_\nu(x) = -\mathrm{J}_{\nu+1}(x) + \dfrac{\nu}{x} \mathrm{J}_\nu(x)$.

Remark: a) Provides a recurrence relation to determine J_ν for $\nu \in \mathbb{N}$ from J_0, J_1. The Neumann functions Y_ν, $\nu \in \mathbb{N}$, satisfy the same recurrence relation and provide a 2nd solution of $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, which is linearly independent of J_ν. Thus in order to determine Y_ν for $\nu \in \mathbb{N}$ (the only case of interest) it suffices to know Y_0 and Y_1.

H62 *Optional Exercise*

Show $\quad \mathrm{J}_0(x) = \dfrac{1}{\pi} \displaystyle\int_0^\pi \cos(x \sin\theta) \mathrm{d}\theta \quad$ for $x \in \mathbb{R}$.

Due on Thu Dec 2, 7:30 pm

Exercises H57 c), H58, and H59, Ex. 9 are similar, and it suffices to do one of them. (But we recommend to do more, if you have time.)

The optional exercises and H60 can be handed in one week later.