Differential Equations Plus (Math 286)

H80 Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Use this to solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y}$, $\mathbf{y}(0) = (0,1,0)^{\mathsf{T}}$, and determine $\lim_{t\to+\infty} \mathbf{y}(t)$ for the solution.

H81 Determine a fundamental system of solutions of $\mathbf{y}' = \mathbf{B}\mathbf{y}$ for the matrix

$$\mathbf{B} = \begin{pmatrix} 0 & 1 & -2 & -1 & 2 \\ 5 & -2 & -3 & -2 & 3 \\ 14 & 3 & -12 & -5 & 9 \\ 13 & 3 & -8 & -8 & 8 \\ 16 & 3 & -10 & -6 & 7 \end{pmatrix}$$

H82 Consider again the matrix $\mathbf{A}$ from H80. Determine the matrix eponential function $\mathrm{e}^{\mathbf{A}t}$in two ways,

a) using the fundamental matrix $\mathbf{\Phi}(t)$ obtained in H80 and the formula $\mathrm{e}^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$;

b) using the “new method” for computing $\mathrm{e}^{\mathbf{A}t}$ discussed in Lecture 51.

H83 Consider the matrix

$$\mathbf{M} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{pmatrix}.$$

a) Show, without using the characteristic polynomial, that the eigenvalues of $\mathbf{M}$ are purely imaginary and have absolute value 1 (hence must be i or $-$i).
Hint: Compute $\mathbf{M}^2$ first.

b) Determine the eigenvalues of $\mathbf{M}$ and their multiplicities from the trace of $\mathbf{M}$.

H84 a) If $\mathbf{A} \in F^{n\times n}$ is invertible, show that

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det \mathbf{A}}.$$

b) Show that similar matrices have the same characteristic polynomial (and hence the same trace, the same determinant, and the same eigenvalues with the same multiplicities).

H85 *Optional Exercise*

a) Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \ell_i(\mathbf{A}),$$

where $\ell_i(X) = \prod_{j=1, j\neq i}^{n} \frac{X-\lambda_j}{\lambda_i-\lambda_j}$ are the corresponding Lagrange polynomials.

b) Suppose that $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric and $\mathbf{v}_1, \dots, \mathbf{v}_n$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$. Show that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{i=1}^{n} \mathrm{e}^{\lambda_i t} \mathbf{v}_i \mathbf{v}_i^{\mathsf{T}},$$

where λ_i is the eigenvalue corresponding to $\mathbf{v}_i$. (Note that the vectors $\mathbf{v}_i$ are column vectors, and hence $\mathbf{v}_i \mathbf{v}_i^{\mathsf{T}}$ are $n \times n$ matrices of rank 1.)

c) The matrix considered in H 80 and H 83 satisfies both conditions. Use a) and b) to give two further evaluations of its matrix exponential function.

H86 *Optional Exercise*

Suppose that $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ and $\mathbf{T} \in \mathbb{C}^{n\times n}$ satisfy $\mathbf{B} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

a) Setting $\mathbf{T} = \mathbf{P} + \mathrm{i}\mathbf{Q}$ with $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{n\times n}$, show that $\mathbf{P}, \mathbf{Q}$ satisfy the matrix equation $\mathbf{AX} = \mathbf{XB}$.

b) Show that any matrix $\mathbf{S} = \mathbf{P} + \lambda\mathbf{Q}$, $\lambda \in \mathbb{R}$, satisfies this matrix equation as well.

c) Show that that there exists $\lambda \in \mathbb{R}$ such that $\mathbf{P} + \lambda\mathbf{Q}$ is invertible.
Hint: Show that $\lambda \mapsto \det(\mathbf{P} + \lambda\mathbf{Q})$ is a polynomial function.

d) Use a), b), c) to show that there exists $\mathbf{S} \in \mathbb{R}^{n\times n}$ such that $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}$.

H87 *Optional Exercise*

a) Suppose that $\mathbf{A} \in \mathbb{R}^{2\times 2}$ has two distinct real eigenvalues λ_1, λ_2 with corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Show that for $t \to \pm\infty$ the tangent unit vector $\frac{\mathbf{y}'(t)}{|\mathbf{y}'(t)|}$ of any non-constant solution $\mathbf{y}(t)$ of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ approaches the direction of one of the four rays $\mathbb{R}^+(\pm\mathbf{v}_1)$, $\mathbb{R}^+(\pm\mathbf{v}_2)$ (i.e., $\pm\frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\pm\frac{\mathbf{v}_2}{|\mathbf{v}_2|}$).

b) Work out the four possible cases (including the explicit determination of the rays) for the system

$$\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \mathbf{y}.$$

Due on Wed Dec 22, 6 pm

Optional exercises this time should be handed in simultaneously with the mandatory exercises, if you want them graded. Alternatively, you can do optional exercises later and use the published solution to validate your own solution.