# Math 286
# Introduction to Differential Equations

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# Outline

# Today's Lecture:

# The Uniqueness Theorem

We will prove the Existence and Uniqueness Theorem for solutions of 1st-order ODE's in a more general form than [BDM17, Theorem 2.8.1]. The generalization to $n$-dimensional ODE systems $\mathbf{y}' = f(t, \mathbf{y})$ will enable us to conclude from it a corresponding theorem for higher-order scalar ODE's. Further, we will relax the assumption "$f(t, \mathbf{y})$ has continuous partial derivatives with respect to the variables in $\mathbf{y}$" to "$f(t, \mathbf{y})$ satisfies locally a so-called Lipschitz condition with respect to $\mathbf{y}$". This will allow us to apply the Existence and Uniqueness Theorem to certain ODE's that are not covered by [BDM17, Theorem 2.8.1] but still important in Engineering Mathematics.

## Definition

Suppose $f\colon D \to \mathbb{R}^n$, $D \subseteq \mathbb{R} \times \mathbb{R}^n$, is a map.

1. We say that $f = f(t, \mathbf{y})$ satisfies a *Lipschitz condition with respect to* $\mathbf{y}$ if there exists a constant $L \geq 0$ (the corresponding *Lipschitz constant*) such that

$$|f(t, \mathbf{y}_1) - f(t, \mathbf{y}_2)| \leq L\,|\mathbf{y}_1 - \mathbf{y}_2| \quad \text{for all } (t, \mathbf{y}_1), (t, \mathbf{y}_2) \in D.$$

2. We say that $f$ satisfies *locally* a Lipschitz condition with respect to $\mathbf{y}$ if every point $(t, \mathbf{y}) \in D$ has a neighborhood $D' \subseteq D$ in which (1) holds with a constant $L$ (which may depend on the particular point).

Condition (2), together with continuity of $f$ as a function of $n + 1$ variables, will be taken as the premise of the Existence and Uniqueness Theorem. The next proposition shows that these conditions imply those used in [BDM17, Theorem 2.8.1], so that our Existence and Uniqueness Theorem covers that in the textbook.

## Proposition

*Suppose $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is an open set and $f \colon D \to \mathbb{R}^n$ has continuous (as $(n+1)$-variable functions!) partial derivatives with respect to the variables $\mathbf{y} = (y_1, \dots, y_n)$. Then $f$ satisfies locally a Lipschitz condition with respect to $\mathbf{y}$.*

## Proof.

Let $(a, \mathbf{b}) \in D$. Since $D$ is open, there exists $r > 0$ such that

$$V = \bigl\{(t, \mathbf{y}); |t-a| \leq r \wedge |\mathbf{y} - \mathbf{b}| \leq r\bigr\} \subseteq D.$$

$V$ is a compact subset of $D$.

The Mean Value Theorem (integral form) of Calculus III, applied to $\mathbf{y} \mapsto f(t, \mathbf{y})$, gives for $(t, \mathbf{y}_1), (t, \mathbf{y}_2) \in V$ the identity

$$f(t, \mathbf{y}_1) - f(t, \mathbf{y}_2) = \left(\int_0^1 \mathbf{J}_{f,\mathbf{y}}\bigl(t, \mathbf{y}_1 + s(\mathbf{y}_2 - \mathbf{y}_1)\bigr)\,\mathrm{d}s\right)(\mathbf{y}_1 - \mathbf{y}_2)$$

with $\mathbf{J}_{f,\mathbf{y}}(t, \mathbf{y}) = \left(\frac{\partial f_i}{\partial y_j}(t, \mathbf{y})\right)_{1 \leq i,j \leq n}$ ("partial Jacobi matrix" of $f$).

## Proof cont'd.

Since the entries of the $n \times n$ matrix $\left(\frac{\partial f_i}{\partial y_j}(t, \mathbf{y})\right)$ are continuous functions of $(t, \mathbf{y})$, there exists a constant $M$ such that $\left|\frac{\partial f_i}{\partial y_j}(t, \mathbf{y})\right| \leq M$ for all $(t, \mathbf{y}) \in V$ and all $i, j$. This implies $\left\|\mathbf{J}_{f,\mathbf{y}}(t, \mathbf{y})\right\|_{\mathrm{F}} \leq nM$ for all $(t, \mathbf{y}) \in V$.

The matrix $\mathbf{A} = \mathbf{A}(t, \mathbf{y}_1, \mathbf{y}_2)$ appearing in the Mean Value Theorem is obtained by averaging $\mathbf{J}_{f,\mathbf{y}}(t, \mathbf{y})$ over the line segment $[\mathbf{y}_1, \mathbf{y}_2]$ and is subject to the same bound. More precisely, we have

$$\begin{aligned}
|f(t, \mathbf{y}_1) - f(t, \mathbf{y}_2)| &\leq \|\mathbf{A}\| \, |\mathbf{y}_1 - \mathbf{y}_2| \\
&\leq \|\mathbf{A}\|_{\mathrm{F}} \, |\mathbf{y}_1 - \mathbf{y}_2| && \text{(cf. exercise)} \\
&\leq \sqrt{n^2 M^2} \, |\mathbf{y}_1 - \mathbf{y}_2| = nM \, |\mathbf{y}_1 - \mathbf{y}_2| \, ,
\end{aligned}$$

since $|a_{ij}| = \left| \int_0^1 \frac{\partial f_i}{\partial y_j}\big(t, \mathbf{y}_1 + s(\mathbf{y}_2 - \mathbf{y}_1)\big) \, \mathrm{d}s \right| \leq (1 - 0) \cdot M = M$.

Thus we can take $L = nM$ as the desired local Lipschitz constant and the corresponding neighborhood of $(a, \mathbf{b})$ as $V$.

$\square$

## Remark (Lipschitz-continuity of 1-variable functions)

$f\colon [a,b] \to \mathbb{R}$ is said to be *Lipschitz-continuous* or to satisfy a *Lipschitz condition* with *Lipschitz constant* $L$ if there exists $L > 0$ such that

$$|f(x) - f(y)| \leq L\,|x - y| \quad \text{for all } x, y \in [a,b].$$

The name "Lipschitz-continuity" comes from the fact that this property implies that $f$ is uniformly continuous (take $\delta = \epsilon/L$ as response to $\epsilon$).

The preceding Proposition is a multi-variable generalization of the following fact:

> *Every* $\mathrm{C}^1$*-function on a compact intervall* $[a,b] \subset \mathbb{R}$ *is Lipschitz-continuous.*

For the (much easier) proof define $L = \max\{|f'(x)|\,; a \leq x \leq b\}$ and use the Mean Value Theorem:

$$|f(x) - f(y)| = |f'(\xi)(x - y)| \leq L\,|x - y|\,.$$

## Remark (cont'd)

A $\mathrm{C}^1$-function $f$ on an arbitrary interval $I \subseteq \mathbb{R}$ need not be Lipschitz-continuous, but we still get Lipschitz-continuity on every compact subinterval $[a, b] \subseteq I$. Equivalently, every point $x \in I$ has a neighborhood $(x - \delta, x + \delta)$, $\delta > 0$, such that $f$ is Lipschitz-continuous on $I \cap (x - \delta, x + \delta)$.

The property of uniform continuity is weaker than Lipschitz-continuity. For example, $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, \infty)$, but not Lipschitz-continuous.

## Remark (Metric spaces and Lipschitz-continuity)

The concept of Lipschitz-continuity makes sense for maps between arbitrary metric spaces. If $(M, d)$ and $(M', d')$ are metric spaces and $T\colon (M, d) \to (M', d')$ is a map, we call $T$ *Lipschitz-continuous* if there exists $L > 0$ such that

$$d'\bigl(T(x), T(y)\bigr) \leq L\, d(x, y) \quad \text{for all } x, y \in M.$$

In fact much of the preceding discussion, including Banach's Fixed Point Theorem, is related to this concept. As an example, observe that $T\colon M \to M$ is a contraction iff it satisfies a Lipschitz condition with Lipschitz constant $L < 1$.

Now we can state and prove the Existence and Uniqueness Theorems for explicit 1st-order ODE systems $\mathbf{y}' = f(t, \mathbf{y})$.
For implicit ODE's there are no such Theorems, and hence an implicit ODE must first be converted to explicit form before drawing any conclusions about existence/uniqueness of solutions.

A crucial ingredient for both proofs will be the observation made earlier, that solutions of the differential equation $\mathbf{y}' = f(t, \mathbf{y})$ can be characterized as solutions of a related integral equation:

## Observation (recalled)

Suppose $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open, $f\colon D \to \mathbb{R}^n$ continuous and $(t_0, \mathbf{y}_0) \in D$. A continuous function (curve) $\phi\colon I \to \mathbb{R}^n$ with $\big(t, \phi(t)\big) \in D$ for $t \in I$ solves the IVP $\mathbf{y}' = f(t, \mathbf{y})) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$ iff

$$\phi(t) = \mathbf{y}_0 + \int_{t_0}^{t} f\big(\tau, \phi(\tau)\big)\mathrm{d}\tau \quad \text{for } t \in I.$$

By the Fundamental Theorem of Calculus, this *integral equation* implies that $\phi$ is differentiable with $\phi'(t) = f\big(t, \phi(t)\big)$, and of course $\phi(t_0) = \mathbf{y}_0$. For the converse integrate $\phi'(t) = f\big(t, \phi(t)\big)$.

## Theorem (Uniqueness Theorem)

*Suppose that $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and that $f \colon D \to \mathbb{R}^n$ is continuous and satisfies locally a Lipschitz condition with respect to $\mathbf{y}$. If $\phi, \psi \colon I \to \mathbb{R}^n$ are solutions of an IVP*

$$\mathbf{y}' = f(t, \mathbf{y}) \wedge \mathbf{y}(t_0) = \mathbf{y}_0, \qquad (t_0, \mathbf{y}_0) \in D,$$

*then $\phi(t) = \psi(t)$ for all $t \in I$.*

The key step in the proof is the following lemma, which says that the set $A \subseteq I$ of arguments $t$ where $\phi$ and $\psi$ agree is open in $I$.

## Lemma

*Suppose $a \in I$ is such that $\phi(a) = \psi(a)$ (e.g., take $a = t_0$). Then there exists $\epsilon > 0$ such that $\phi(t) = \psi(t)$ for all $t \in I \cap [a-\epsilon, a+\epsilon]$.*

## Proof.

Integrating the two equations $\phi'(t) = f\bigl(t, \phi(t)\bigr)$, $\psi'(t) = f\bigl(t, \psi(t)\bigr)$ and using $\phi(a) = \psi(a) = \mathbf{b}$, say, we obtain

$$\begin{aligned}\phi(t) - \psi(t) &= \mathbf{b} + \int_a^t f\bigl(\tau, \phi(\tau)\bigr)\mathrm{d}\tau - \mathbf{b} - \int_a^t f\bigl(\tau, \psi(\tau)\bigr)\mathrm{d}\tau \\ &= \int_a^t f\bigl(\tau, \phi(\tau)\bigr) - f\bigl(\tau, \psi(\tau)\bigr)\mathrm{d}\tau.\end{aligned}$$

By assumption, there exists a neighborhood $V$ of $(a, \mathbf{b})$ on which $f$ satisfies a Lipschitz condition with respect to $\mathbf{y}$. Further, since $\phi$ and $\psi$ are continuous in $a$, there exists $\delta > 0$ such that $\bigl(\tau, \phi(\tau)\bigr) \in V$ and $\bigl(\tau, \psi(\tau)\bigr) \in V$ for all $\tau \in I \cap [a-\delta, a+\delta]$. Thus we have, denoting the Lipschitz constant by $L$ as usual,

$$\left|f\bigl(\tau, \phi(\tau)\bigr) - f\bigl(\tau, \psi(\tau)\bigr)\right| \leq L\,|\phi(\tau) - \psi(\tau)| \quad \text{for } \tau \in I \cap [a-\delta, a+\delta].$$

# Proof cont'd.

$$\implies |\phi(t)-\psi(t)| \leq \begin{cases} L\displaystyle\int_a^t |\phi(\tau)-\psi(\tau)|\,\mathrm{d}\tau & \text{for } t\in I\cap[a,a+\delta], \\ L\displaystyle\int_t^a |\phi(\tau)-\psi(\tau)|\,\mathrm{d}\tau & \text{for } t\in I\cap[a-\delta,a]. \end{cases}$$

Now we set $M(t) := \sup\{|\phi(\tau)-\psi(\tau)|\,;\tau \text{ between } a \text{ and } t\}$ for $t\in I\cap[a-\delta,a+\delta]$.

$$\implies |\phi(t)-\psi(t)| \leq L\,|t-a|\,M(t)$$

for all such $t$. Replacing $t$ by any $t'$ between $a$ and $t$, we also get

$$|\phi(t')-\psi(t')| \leq L\,|t'-a|\,M(t') \leq L\,|t-a|\,M(t).$$

Taking the supremum over all $t'$ between $a$ and $t$ gives

$$M(t) \leq L\,|t-a|\,M(t) \quad \text{for } t\in I\cap[a-\delta,a+\delta].$$

Setting $\epsilon = \min\left\{\delta,\frac{1}{2L}\right\}$, this implies $M(t) \leq \frac{1}{2}\,M(t)$ for all $t\in I\cap[a-\epsilon,a+\epsilon]$. Clearly this can hold only if $M(t)=0$ for all $t\in I\cap[a-\epsilon,a+\epsilon]$, and the proof of the lemma is complete. □

## Note

It was necessary to use "$t \in I \cap [a-\delta, a+\delta]$", etc., throughout the proof, because $a$ may be an endpoint of $I$ (left or right endpoint). In such a case solutions are defined only on one of the intervals $[a-\delta, a]$, $[a, a+\delta]$, etc.

## Proof of the Uniqueness Theorem.

We prove the theorem by contradiction.
Let $A = \{t \in I; \phi(t) = \psi(t)\}$, $N = I \setminus A$, and suppose that $N \neq \emptyset$. Since $\phi(t_0) = \psi(t_0)$, we have $t_0 \notin N$. Hence there are the following two cases to consider.

*Case 1:* There exists $t_1 \in N$ with $t_1 > t_0$.
Define $t_2$ as the infimum of the (non-empty and bounded from below) set $N \cap [t_0, +\infty)$. Then obviously $t_2 \in [t_0, t_1] \subseteq I$.
We claim that $\phi(t_2) = \psi(t_2)$. If $t_2 = t_0$ this is trivial. Otherwise $t_2 > t_0$ and $\phi(t) = \psi(t)$ for all $t \in [t_0, t_2)$. Continuity of $\phi$, $\psi$ then implies $\phi(t_2) = \lim_{t \uparrow t_2} \phi(t) = \lim_{t \uparrow t_2} \psi(t) = \psi(t_2)$.
Now the lemma yields $\epsilon > 0$ such that $[t_2, t_2 + \epsilon] \subseteq A$. This obviously contradicts the definition of $t_2$.

*Case 2:* There exists $t_1 \in N$ with $t_1 < t_0$.
For this case a contradiction is derived in a similar way. $\square$

## Example

Consider the ODE $y' = \sqrt{|y|}$.

We have seen earlier that this ODE has, among others, the solutions $y_1 \equiv 0$ and

$$y_2(t) = \begin{cases} \frac{1}{4}(t-t_0)^2 & \text{if } t \geq t_0, \\ -\frac{1}{4}(t-t_0)^2 & \text{if } t \leq t_0, \end{cases}$$

where $t_0 \in \mathbb{R}$ is arbitrary. We have $y_1(t_0) = y_2(t_0) = 0$, but $y_1(t) \neq y_2(t)$ for $t \neq t_0$.

This doesn't contradict the Uniqueness Theorem, since $f(t,y) = \sqrt{|y|}$ has partial derivative

$$\frac{\partial f}{\partial y}(t,y) = \begin{cases} \frac{1}{2\sqrt{y}} & \text{if } y > 0, \\ -\frac{1}{2\sqrt{-y}} & \text{if } y < 0, \end{cases}$$

and hence doesn't satisfy locally a Lipschitz condition at any point $(t_0, 0)$.

On the other hand, $f$ satisfies locally a Lipschitz condition at every point $(t_0, y_0)$ with $y_0 \neq 0$, and hence solutions of the IVP $y' = \sqrt{|y|} \wedge y(t_0) = y_0 \neq 0$ are unique as long as they stay away from the $t$-axis $y = 0$.

The next example, which a student contributed, shows that $f(t,\mathbf{y})$ need not satisfy a local Lipschitz condition with respect to $\mathbf{y}$ in order for solutions of IVP's $\mathbf{y}' = f(t,\mathbf{y}) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$ to be unique.

## Example

Consider the ODE $\quad y' = \begin{cases} y \ln|y| & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

The solutions are $\mathbf{y}(t) \equiv 0$ and $y(t) = \pm \mathrm{e}^{c\,\mathrm{e}^t}$, $c \in \mathbb{R}$, as is easily derived using the standard machinery for autonomous/separable equations and observing that no non-constant solution can attain a value $0, \pm 1$ (the constant solutions). If it did, there would exist $t_1 \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{0\}$ such that $\lim_{t \to t_1} \mathrm{e}^{c\,\mathrm{e}^t} = \mathrm{e}^{c\,\mathrm{e}^{t_1}} \in \{0, \pm 1\}$, which is impossible. Thus all associated IVP's have a unique solution.

But $f(t,y) = y \ln|y|$ doesn't satisfy a local Lipschitz condition at any point on the $t$-axis $y = 0$, because, e.g., for $0 < y_1 < y_2$ we have

$$f(t,y_2) - f(t,y_1) = \frac{\partial f}{\partial y}(t,\eta)(y_2 - y_1) = (1 + \ln\eta)(y_2 - y_1)$$

for some $\eta \in (y_1, y_2)$ by the Mean Value Theorem of Calculus I, and $1 + \ln\eta \to -\infty$ for $\eta \downarrow 0$.

## Remark

"Continuity of $f(t,\mathbf{y})$" and "local Lipschitz condition with respect to $\mathbf{y}$" has been adopted as premise in both the Uniqueness Theorem and the Existence Theorem (cf. subsequent slide), because under these assumptions the theorems are fairly easy to prove and the assumptions are sufficiently general to cover most applications. At the cost of more difficult proofs, the assumptions can be relaxed. For example, the conclusion of the Existence Theorem remains true if one merely stipulates that $f(t,\mathbf{y})$ is continuous (PEANO*'s Existence Theorem*), and the conclusion of the Uniqueness Theorem remains true if the Lipschitz condition is relaxed to

$$|f(t,\mathbf{y}_1)-f(t,\mathbf{y}_2)|\leq L\,|\mathbf{y}_1-\mathbf{y}_2|\ln|\mathbf{y}_1-\mathbf{y}_2|$$

(a consequence of OSGOOD*'s Condition*; cf. the literature).

## Exercise

Does the Uniqueness Theorem apply to the ODE $y'=|y|$ ?
If you are unsure, solve the ODE directly.

# The Existence Theorem

## Theorem (Picard-Lindelöf)

*Suppose $D \subseteq \mathbb{R} \times \mathbb{R}^n$ is open and $f\colon D \to \mathbb{R}^n$, $(t, \mathbf{y}) \mapsto f(t, \mathbf{y})$ is a continuous function which satisfies on $D$ locally a Lipschitz condition with respect to $\mathbf{y}$. Then for every $(t_0, \mathbf{y}_0) \in D$ there exists an interval $I$ containing $t_0$ as an inner point and a solution $\phi\colon I \to \mathbb{R}^n$ of the IVP $\quad \mathbf{y}' = f(t, \mathbf{y}) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$.*

## Proof.

By our previous observation it suffices to construct a continuous function $\phi^*\colon [t_0 - \epsilon, t_0 + \epsilon] \to \mathbb{R}^n$ satisfying $T\phi^* = \phi^*$, where $T$ is the "operator"

$$(T\phi)(t) = \mathbf{y}_0 + \int_{t_0}^{t} f\big(\tau, \phi(\tau)\big)\,\mathrm{d}\tau.$$

Right now $T$ is not well-defined, because we haven't yet specified a suitable domain from which the function $\phi$ is taken. But this will be cured in a moment.

## Proof cont'd.

Our goal is to apply Banach's Fixed Point Theorem to $T$.

By assumption there exists $r > 0$ such that the compact set

$$V = \{(t, \mathbf{y}) \in \mathbb{R} \times \mathbb{R}^n; |t - t_0| \leq r, |\mathbf{y} - \mathbf{y}_0| \leq r\}$$

is contained in $D$ and $f$ satisfies a Lipschitz condition with respect to $\mathbf{y}$ on $V$. Denote the corresponding Lipschitz constant by $L$. Further, since $f$ is continuous, there exists $M > 0$ such that $|f(t, \mathbf{y})| \leq M$ on $V$.

Now let $\epsilon = \min\{r, r/M, 1/(2L)\}$ and define $\mathscr{M}$ as the set of all continuous functions $\phi\colon [t_0 - \epsilon, t_0 + \epsilon] \to \mathbb{R}^n$ satisfying $|\phi(t) - \mathbf{y}_0| \leq r$ for all $t \in [t_0 - \epsilon, t_0 + \epsilon]$), and hence $\big(t, \phi(t)\big) \in V$ for such $t$.

$\mathscr{M}$ is equipped with the metric of uniform convergence, i.e.

$$d_\infty(\phi, \psi) = \max\big\{|\phi(t) - \psi(t)|\,; t_0 - \epsilon \leq t \leq t_0 + \epsilon\big\} = \|\phi - \psi\|_\infty\,,$$

where $\|\phi\|_\infty = \max\big\{|\phi(t)|\,; t_0 - \epsilon \leq t \leq t_0 + \epsilon\big\}$.
The metric space $(\mathscr{M}, d_\infty)$ is complete, since it is a closed subspace of $\mathrm{C}\big([t_0 - \epsilon, t_0 + \epsilon]\big)$ (in fact the closed ball $\overline{\mathrm{B}_r(\mathbf{y}_0)}$).

# Proof cont'd.

In order to apply Banach's Theorem, it remains to show $T(\mathscr{M}) \subseteq \mathscr{M}$ and that $T$ defines a contraction of $(\mathscr{M}, d_\infty)$.

Let $\phi \in \mathscr{M}$ and $\psi = T\phi$. For $t_0 - \epsilon \leq t \leq t_0 + \epsilon$ we have

$$\begin{aligned} |\psi(t) - \mathbf{y}_0| &= \left| \int_{t_0}^{t} f\bigl(\tau, \phi(\tau)\bigr) \mathrm{d}\tau \right| \leq \pm \int_{t_0}^{t} \left| f\bigl(\tau, \phi(\tau)\bigr) \right| \mathrm{d}\tau \\ &\leq \epsilon M \leq r. \end{aligned}$$

(The minus sign is necessary to account for the case $t < t_0$, in which we rather mean $\int_t^{t_0} f\bigl(\tau, \phi(\tau)\bigr)\mathrm{d}\tau$.) This shows $T(\mathscr{M}) \subseteq \mathscr{M}$.

Let $\phi_1, \phi_2 \in \mathscr{M}$ and $\psi_1 = T\phi_1$, $\psi_2 = T\phi_2$.

$$\begin{aligned} |\psi_1(t) - \psi_2(t)| &= \left| \int_{t_0}^{t} f\bigl(\tau, \phi_1(\tau)\bigr) - f\bigl(\tau, \phi_2(\tau)\bigr) \mathrm{d}\tau \right| \\ &\leq \pm \int_{t_0}^{t} \left| f\bigl(\tau, \phi_1(\tau)\bigr) - f\bigl(\tau, \phi_2(\tau)\bigr) \right| \mathrm{d}\tau \\ &\leq \pm \int_{t_0}^{t} L \left| \phi_1(\tau) - \phi_2(\tau) \right| \mathrm{d}\tau \leq \epsilon L \left\| \phi_1 - \phi_2 \right\|_\infty \\ &\leq \tfrac{1}{2} \left\| \phi_1 - \phi_2 \right\|_\infty \end{aligned}$$

## Proof cont'd.

Maximizing over $t \in [t_0 - \epsilon, t_0 + \epsilon]$ gives

$$\|\psi_1 - \psi_2\|_\infty \leq \frac{1}{2} \|\phi_1 - \phi_2\|_\infty , \quad \text{i.e.,} \quad d_\infty\bigl(T\phi_1, T\phi_2\bigr) \leq \frac{1}{2} d_\infty(\phi_1, \phi_2).$$

This shows that $T \colon \mathscr{M} \to \mathscr{M}$ is a contraction with $C = 1/2$.
Now Banach's Theorem can be applied and yields $\phi^* \in \mathscr{M}$ with $T\phi^* = \phi^*$. This function $\phi^*$ is the desired solution of the given IVP. $\square$

# Notes

- In the proof of the Existence Theorem (and similarly in the proof of the key lemma to the Uniqueness Theorem) we have used estimates of the form $\dots \leq \pm \int_{t_0}^{t} |\dots| \, \mathrm{d}\tau$, where the minus sign is chosen in the case $t < t_0$ to make the right-hand side non-negative. This rather awkward notation, or the even more awkward $\dots \leq \left| \int_{t_0}^{t} |\dots| \, \mathrm{d}\tau \right|$ used in some books, can be avoided if we interpret $\int_{t_0}^{t}$ in these cases as the Lebesgue integral over the interval with endpoints $t_0$ and $t$ (which can be either $[t_0, t]$ or $[t, t_0]$).
- Likewise, in the proof of both theorems we have used estimates of the form $\left| \int_a^b \phi(t) \, \mathrm{d}t \right| \leq \int_a^b |\phi(t)| \, \mathrm{d}t$ with $\phi \colon [a, b] \to \mathbb{R}^n$ continuous. For $n > 1$ the vertical bars refer to the Euclidean length in $\mathbb{R}^n$ rather than the absolute value on $\mathbb{R}$, and the inequality does not follow from the 1-dimensional integration theory developed in Calculus II. A proof can be found in Exercise W22 of Worksheet 5, Calculus III. Alternatively, approximate $\phi$ by vector-valued step functions and check that for such functions the inequality reduces to the triangle inequality for the Euclidean length in $\mathbb{R}^n$.

# Notes cont'd

- Banach's Theorem also gives that a solution of the IVP can be obtained as the limit function $\phi(t) = \lim_{k\to\infty} \phi_k(t)$ of the "Picard-Lindelöf iteration"

$$\phi_0(t) \equiv \mathbf{y}_0, \quad \phi_{k+1}(t) = \mathbf{y}_0 + \int_{t_0}^{t} f\big(\tau, \phi_k(\tau)\big)\mathrm{d}\tau, \quad k = 0, 1, 2, \dots,$$

  because certainly the constant function $\phi_0(t) \equiv \mathbf{y}_0$ is in $\mathscr{M}$ (whatever the chosen domain $[t_0 - \epsilon, t_0 + \epsilon]$ is). This is illustrated in the following example.

## Example

We apply Picard-Lindelöf iteration to construct a solution of the IVP $y' = 2ty \wedge y(0) = y_0$; cf. our introductory Example **??**.

Here the iteration takes the form

$$\phi_{k+1}(t) = y_0 + 2\int_0^t \tau\phi_k(\tau)\mathrm{d}\tau.$$

We obtain

$$\phi_1(t) = y_0 + 2\int_0^t \tau y_0 \mathrm{d}\tau = y_0 + 2y_0\int_0^t \tau\mathrm{d}\tau = y_0(1+t^2),$$
$$\phi_2(t) = y_0 + 2\int_0^t \tau y_0(1+\tau^2)\mathrm{d}\tau = y_0(1+t^2+t^4/2)$$

and in general, using induction,

$$\phi_k(t) = y_0\left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^{2k}}{k!}\right).$$

The limit function is $\phi(t) = y_0\sum_{k=0}^{\infty}\frac{t^{2k}}{k!} = y_0\mathrm{e}^{t^2}$, the already known solution.

# Example (cont'd)

Since the functions $\phi_k$ have maximal domain $\mathbb{R}$ and converge uniformly to $\phi(t) = y_0\mathrm{e}^{t^2}$ on every compact subinterval of $\mathbb{R}$ (known from the theory of power series), we can conclude without checking the assumptions of the Existence Theorem, or direct verification, that $\phi$ solves the given IVP on $\mathbb{R}$. This is done as follows:

For any $R > 0$ and any continuous functions $\psi_1, \psi_2 \colon [-R, R] \to \mathbb{R}$ we have

$$\left|(T\psi_1)(t) - (T\psi_2)(t)\right| = \left|2\int_0^t \tau\big(\psi_1(\tau) - \psi_2(\tau)\big)\mathrm{d}\tau\right| \leq R^2 \|\psi_1 - \psi_2\|_\infty$$

on $[-R, R]$ and hence $\|T\psi_1 - T\psi_2\|_\infty \leq R^2 \|\psi_1 - \psi_2\|_\infty$, where $\|\psi\|_\infty = \|\psi\|_{\infty,R} = \max\big\{|\psi(t)|\,; -R \leq t \leq R\big\}$. This shows that $T$ defines a continuous operator on $\mathrm{C}\big([-R, R]\big)$ and implies

$$T\phi = T\left(\lim_{k\to\infty} \phi_k\right) = \lim_{k\to\infty} T\phi_k = \lim_{k\to\infty} \phi_{k+1} = \phi,$$

which in turn implies that $\phi$ solves the given IVP on $I = [-R, R]$, as we have seen. Letting $R \to +\infty$ then shows the same for $I = \mathbb{R}$.

## Exercise

Use Picard-Lindelöf iteration to compute the solution $\phi = (\phi_1, \phi_2)^\mathsf{T}$ of the system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}$$

with initial condition $\phi(0) = (1, 0)^\mathsf{T}$.

## Exercise

Suppose that $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies locally a Lipschitz condition, and that

$$f(-t, y) = -f(t, y) \quad \text{for all } (t, y) \in \mathbb{R}^2.$$

Show that any solution $\phi\colon [-r, r] \to \mathbb{R}$, $r > 0$, of $y' = f(t, y)$ is its own mirror image with respect to the $y$-axis.

## Exercise (hard)

Suppose $I \subseteq \mathbb{R}$ is an interval and $f\colon I \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and satisfies (globally) a Lipschitz condition with Lipschitz constant $L$. For any two solutions $\phi, \psi\colon I \to \mathbb{R}^n$ of $y' = f(t, y)$ and $t_0 \in I$ show that $\quad |\phi(t) - \psi(t)| \leq \delta \mathrm{e}^{L|t-t_0|} \quad$ on $I$, $\quad$ where $\delta = |\phi(t_0) - \psi(t_0)|$.

# Maximal Solutions

Recall that domains of solutions of ODE's must be intervals in $\mathbb{R}$ of positive (possibly infinite) length and may or may not contain their boundary point(s).

## Definition

A solution $\phi\colon I \to \mathbb{R}$ of $y' = f(t, y)$ is said to be *maximal* (or *non-extendable*) if there is no solution $\psi\colon J \to \mathbb{R}$ with $J \supsetneq I$ and $\psi(t) = \phi(t)$ for $t \in I$.

This definition extends in the obvious way to higher-order ODE's, ODE systems (both explicit and implicit ones).

## Corollary

*Under the assumptions of the Existence and Uniqueness Theorem,*

1. *for every $(t_0, \mathbf{y}_0) \in D$ there exists a unique maximal solution $\phi_0\colon I_0 \to \mathbb{R}^n$ of the IVP $\mathbf{y}' = f(t, \mathbf{y}) \wedge \mathbf{y}(t_0) = \mathbf{y}_0$;*
2. *$I_0$ is open in $\mathbb{R}$, and for every end point $e$ of $I_0$ (if any) the solution curve $\left\{\bigl(t, \phi_0(t)\bigr); t \in I_0\right\}$ comes arbitrarily close to the boundary of $D$ when $t \to e$.*

# Maximal Solutions Cont'd

## Note

The precise mathematical definition of "comes arbitrarily close to the boundary" is that, e.g., if $a$ is the left end point of $I_0$ and $t_0 \in I_0$ then $\{(t, \phi_0(t)); a < t \leq t_0\}$ is not contained in a compact subset of $D$. This means that there exists a sequence $t_k \downarrow a$ such that either $|\phi_0(t_k)| \to \infty$ or there exists $\mathbf{b} \in \mathbb{R}^n$ such that $(a, \mathbf{b}) \notin D$ and $\lim_{k\to\infty} \phi_0(t_k) = \mathbf{b}$.

## Proof of the corollary.

(1) Let $I_0 = \bigcup I$ be the union of all domains of solutions $\phi\colon I \to \mathbb{R}^n$ of the given IVP and define $\phi_0\colon I_0 \to \mathbb{R}^n$ by $\phi_0(t) = \phi(t)$ if $t$ is contained in the domain of $\phi$. Clearly $I_0$ is an interval containing $t_0$. If $t \in I_0$ and $\phi_1$, $\phi_2$ are solutions of the IVP defined at $t$, we must have $\phi_1(t) = \phi_2(t)$ (apply the Uniqueness Theorem with $I = [t_0, t]$ or $[t, t_0]$). Hence $\phi_0$ is well-defined, and it cleary solves the IVP. Since the domain of $\phi_0$ contains the domains of all solutions, $\phi_0$ is maximal. Finally the Uniqueness Theorem gives that there cannot be another maximal solution (whose domain would necessarily be $I_0$).

# Proof cont'd.

(2) First we show that $I_0$ is open.
Suppose by contradiction, e.g., that $I_0$ contains its left end point $a$. Then $\big(a, \phi_0(a)\big) \in D$, and the Existence Theorem provides us with a solution $\phi\colon [a-\epsilon, a+\epsilon] \to \mathbb{R}^n$ of the IVP $\mathbf{y}' = f(t, \mathbf{y}) \wedge y(a) = \phi_0(a)$ for some $\epsilon > 0$. By the Uniqueness Theorem, $\phi(t) = \phi_0(t)$ for $t \in [a, a+\epsilon]$. Hence, using the definition in terms of $\phi$ on $[a-\epsilon, a)$, we can prolong $\phi_0$ to a solution on $[a-\epsilon, a) \cup I_0$, which is an interval strictly containing $I_0$; contradiction.

For a proof of the remaining assertion, assume $e = a$ and by contradiction that $\big\{\big(t, \phi_0(t)\big); a < t \leq t_0\big\}$ is contained in a compact subset $C \subset D$. To derive the desired contradiction, it then suffices to show that $\phi_0$ admits an extension to a solution on $\{a\} \cup I_0$. The integral equation

$$\phi_0(t) = \mathbf{y}_0 - \int_t^{t_0} f\big(\tau, \phi_0(\tau)\big)\mathrm{d}\tau.$$

holds for $t \in (a, t_0]$. Since $f$ is continuous, it is bounded on $C$ and hence $\big|f\big(\tau, \phi_0(\tau)\big)\big| \leq M$ for $\tau \in (a, t_0]$. Using this, it is easy to see that $\phi_0(a) := \mathbf{y}_0 - \int_a^{t_0} f\big(\tau, \phi_0(\tau)\big)\mathrm{d}\tau$ provides the desired extension. $\square$

# Higher-Order ODE's

In order to adapt the Existence and Uniqueness Theorems for 1st-order ODE systems to explicit $n$-th order (scalar) ODE's

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

with $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R} \times \mathbb{R}^n$ open (again there is no analogue for implicit $n$-th order ODE's), we need to relate the property "$f$ satisfies locally a Lipschitz condition w.r.t. $\mathbf{y}$" to that of the corresponding 1st-order system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. (Here, and only here, we are using bold type to distinguish scalar and vectorial functions.)

Inspecting the explicit formula for $\mathbf{f}$ ("order reduction") and writing $\mathbf{y} = (y_0, \dots, y_{n-1})$, $\mathbf{z} = (z_0, \dots, z_{n-1})$, we have

$$\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z}) = \begin{pmatrix} y_1 - z_1 \\ \vdots \\ y_{n-1} - z_{n-1} \\ f(t, \mathbf{y}) - f(t, \mathbf{z}) \end{pmatrix}$$

Now suppose that $|f(t, \mathbf{y}) - f(t, \mathbf{z})| \leq L\,|\mathbf{y} - \mathbf{z}|$. For the squared Euclidean length of $\mathbf{f}(t, \mathbf{y}) - \mathbf{f}(t, \mathbf{z})$ we then obtain the estimate

$$\begin{aligned}
|\mathbf{f}(t,\mathbf{y})-\mathbf{f}(t,\mathbf{z})|^2 &= \sum_{i=1}^{n-1}(y_i-z_i)^2+|f(t,\mathbf{y})-f(t,\mathbf{z})|^2 \\
&\leq \sum_{i=0}^{n-1}(y_i-z_i)^2+L^2\,|\mathbf{y}-\mathbf{z}|^2 \\
&= (1+L^2)\,|\mathbf{y}-\mathbf{z}|^2
\end{aligned}$$

This says that $\mathbf{f}(t,\mathbf{y})$ satisfies a Lipschitz condition w.r.t. $\mathbf{y}$ with Lipschitz constant $\sqrt{1+L^2}$.

*Conclusion:* If $f$ satisfies on $D$ locally a Lipschitz condition w.r.t. $\mathbf{y}$ then so does $\mathbf{f}$ (with slightly larger Lipschitz constants).

Of course we also have: If $f$ has continuous partial derivatives $\frac{\partial f}{\mathrm{d}y_0}(t,\mathbf{y}),\dots,\frac{\partial f}{\mathrm{d}y_{n-1}}(t,\mathbf{y})$ then $f$ satisfies on $D$ locally a Lipschitz condition w.r.t. $\mathbf{y}$, and hence so does $\mathbf{f}$.

# Corollary (Existence and Uniqueness Theorem for $n$-th order ODE's)

*Suppose $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R} \times \mathbb{R}^n$, is continuous and satisfies on $D$ locally a Lipschitz condition w.r.t. $\mathbf{y}$. Further, let $(a, \mathbf{b}) = (a, b_0, \dots, b_{n-1}) \in D$.*

1. *If $\phi, \psi\colon I \to \mathbb{R}$ are solutions of the IVP*

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) \wedge y^{(i)}(a) = b_i \text{ for } 0 \leq i \leq n-1, \tag{$\star$}$$

   *then $\phi(t) = \psi(t)$ for all $t \in I$.*

2. *There exists $\epsilon > 0$ and a solution $\phi\colon [a - \epsilon, a + \epsilon] \to \mathbb{R}$ of the IVP ($\star$).*

As remarked before, continuity of $f$, $\frac{\partial f}{y_0}, \dots, \frac{\partial f}{y_{n-1}}$ is sufficient for the assumptions of the corollary to hold.

## Proof.

Use the reduction to a 1st-order system $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$ as discussed earlier (setting $y_0 = y$, $y_1 = y'$, etc.), and apply the Existence and Uniqueness Theorem to $\mathbf{y}' = \mathbf{f}(t, \mathbf{y})$. As we have shown on the previous slide, this system satisfies the necessary assumptions (continuity is clear). $\square$

## Example

We have seen in the introduction that $y'' + y = 0$ has the general (real) solution $y(t) = A\cos t + B\sin t$ with constants $A, B \in \mathbb{R}$.

Here $f(t, y_0, y_1) = -y_0$, which even satisfies a global Lipschitz condition with $L = 1$.

$\Longrightarrow$ The Existence and Uniqueness Theorem applies.

Since $y \mapsto \big(y(0), y'(0)\big) = (A, B)$ produces every vector in $\mathbb{R}^2$ exactly once, the Existence and Uniqueness Theorem gives without any previous knowledge (except, of course, that $t \mapsto A\cos t + B\sin t$ is a solution of $y'' + y = 0$) that locally at $t = 0$ all solutions have this form for unique constants $A$, $B$. Since these solutions are defined on the whole of $\mathbb{R}$, one can then conclude that this remains true globally.

Using the addition theorems for $\cos t$, $\sin t$, one can show that an alternative representation of the general nonzero solution of $y'' + y = 0$ is, e.g., $y(t) = A\sin(t - t_0)$ with $A > 0$ and $0 \leq t_0 < 2\pi$. This follows from the Existence and Uniqueness Theorem as well, since $y \mapsto \big(y(0), y'(0)\big) = (-A\sin t_0, A\cos t_0)$ produces every nonzero vector in $\mathbb{R}^2$ exactly once (by the polar coordinate representation of points in $\mathbb{R}^2$).

# Example

Consider the (rather fancy) 3rd-order ODE

$$y''' = \begin{cases} \sin(\mathrm{e}^y - y') & \text{if } t \leq 0, \\ \sin(\mathrm{e}^y - y' + ty'') & \text{if } t > 0. \end{cases} \tag{$\star$}$$

Here we have $y''' = f(t, y, y', y'')$ with $f\colon \mathbb{R}^4 \to \mathbb{R}$ defined by

$$f(t, y_0, y_1, y_2) = \begin{cases} \sin(\mathrm{e}^{y_0} - y_1) & \text{if } t \leq 0, \\ \sin(\mathrm{e}^{y_0} - y_1 + ty_2) & \text{if } t > 0. \end{cases}$$

$f$ is continuous (check the behaviour near $t = 0$) and partially differentiable w.r.t. $y_0, y_1, y_2$, and $\frac{\partial f}{\partial y_0}$, $\frac{\partial f}{\partial y_1}$, $\frac{\partial f}{\partial y_2}$ are continuous; e.g.,

$$\frac{\partial f}{\partial y_2}(t, y_0, y_1, y_2) = \begin{cases} 0 & \text{if } t \leq 0, \\ t\cos(\mathrm{e}^{y_0} - y_1 + ty_2) & \text{if } t > 0. \end{cases}$$

$\Longrightarrow$ $f$ satisfies on $\mathbb{R}^4$ locally a Lipschitz condition with respect to $\mathbf{y} = (y_0, y_1, y_2)$ (it doesn't matter that $\frac{\partial f}{\partial t}$ doesn't exist at some points).
$\Longrightarrow$ The Existence and Uniqueness Theorem applies, giving unique solvability of ($\star$) for any initial values $y^{(i)}(a) = b_i$, $i = 0, 1, 2$.

## Exercise

Determine all maximal solutions of the 2nd order ODE $y'' = |y|$.

# Integral Curves

As with the concept of a maximal solution, we first have to make precise what we mean by “integral curve”. For simplicity we consider only 1st-order, scalar valued ODE's, including those in “differential-like” form $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$.

## Definition

1. By an *integral curve* of $y' = f(t,y)$, or the more general implicit form $f(t,y,y') = 0$, we mean the graph $\{(t,\phi(t));\, t \in I\}$ of a maximal solution $\phi\colon I \to \mathbb{R}$.
2. By an *integral curve* of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ we mean the range $\gamma(I) \subseteq \mathbb{R}^2$ of a solution $\gamma\colon I \to \mathbb{R}^2$, i.e., $\gamma(t) = \big(x(t), y(t)\big)$ should be smooth and satisfy $M\big(x(t),y(t)\big)x'(t) + N\big(x(t),y(t)\big)y'(t) = 0$ for all $t \in I$, which is maximal with respect to this property, i.e., there must not be a solution with range strictly containing $\gamma(I)$.

Thus integral curves are smooth non-parametric plane curves describing/representing the solutions of 1st-order, scalar valued ODE's.

# Example

Consider the ODE $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$ and the corresponding explicit form $y' = \mathrm{d}y\,/\,\mathrm{d}x = -x/y$.

Parametric solutions $\gamma(t) = \big(x(t), y(t)\big)$ of the differential-like ODE must satisfy $x(t)x'(t) + y(t)y'(t) \equiv 0$, which after multiplication by 2 becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(x(t)^2 + y(t)^2\big) \equiv 0.$$

Thus $x(t)^2 + y(t)^2 = C$ must be constant, showing that the integral curves of $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$ are precisely the circles $x^2 + y^2 = R^2$, $R > 0$. (For this note that smoothness of $\gamma$ excludes the case $R = 0$, and that the maximality condition excludes proper pieces of circles.)

The explicit ODE $y' = -x/y$ is not defined at $y = 0$. Its integral curves are the half-circles $x^2 + y^2 = R^2$, $y \gtrless 0$ (again excluding $R = 0$), which represent the graphs of its solutions $y(x) = \pm\sqrt{R^2 - x^2}$, $x \in (-R, R)$.

We see from this that integral curves of a differential-like ODE $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ may split into several pieces forming integral curves of the explicit ODE $y' = \mathrm{d}y\,/dx = -M(x, y)/N(x, y)$. This happens at zeros of $N$ (singular points or points with a vertical tangent).

## Corollary (Uniqueness of Integral Curves)

*Suppose $N, M\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ open, are $\mathrm{C}^1$-functions. If $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ has no singular points then through every point of $D$ there passes exactly one integral curve ("solution curve").*

## Proof.

Let $(x_0, y_0) \in D$. By assumption $(x_0, y_0)$ is non-singular, i.e. $M(x_0, y_0) \neq 0$ or $N(x_0, y_0) \neq 0$. Then the tangent direction of an integral curve in $(x_0, y_0)$ is uniquely determined as the direction orthogonal to the vector $\bigl(M(x_0, y_0), N(x_0, y_0)\bigr)$. Since the tangent cannot be horizontal and vertical at the same time, we can parametrize $\gamma$ locally either as $y(x)$ or as $x(y)$, which then must solve the explicit ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x,y)}{N(x,y)} \quad \text{or} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{N(x,y)}{M(x,y)}, \quad \text{respectively.}$$

$\Longrightarrow$ The Existence and Uniqueness Theorem can be applied and yields that an integral curve through $(x_0, y_0)$ exists. Uniqueness follows from the maximality condition (cf. the uniqueness proof for maximal solutions of IVP's). $\square$

## Remark

Since $M$ and $N$ are continuous, the set $S$ of singular points of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$ is closed in $D$. Hence $D' = D \setminus S$ is open and satisfies all assumptions of the corollary. $\Longrightarrow$ Integral curves of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ can intersect only in singular points.

## Afternote

It is not true in general that through every non-singular point of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ (where $M$, $N$ are $\mathrm{C}^1$-functions on some open set $D \subseteq \mathbb{R}^2$) there passes exactly one solution curve.

As a counterexample consider the family of curves $y = Cx^2$, $C \in \mathbb{R}$. Since all these curves have a horizontal tangent in $(0,0)$, it is clear that we can glue branches with different $C$ together at $(0,0)$ to form differentiable functions $y(x)$ on $\mathbb{R}$ other than $y(x) = Cx^2$ (e.g., $y(x) \equiv 0$ for $x \leq 0$ and $y(x) = x^2$ for $x \geq 0$). On the other hand, solving $y = Cx^2$ for $C$ and taking partial derivatives gives the ODE $-2yx^{-3}\,\mathrm{d}x + x^{-2}\,\mathrm{d}y = 0$ or, clearing denominators,

$$2y\,\mathrm{d}x - x\,\mathrm{d}y = 0.$$

At the singular point $(0,0)$ there is no condition for parametric solutions (except differentiability), and hence all curves described above solve the ODE.

## Afternote con't

(You can also check directly that, e.g., $\gamma(t) = (t, 0)$ for $t \leq 0$ and $\gamma(t) = (t, t^2)$ for $t \geq 0$ is differentiable at $t = 0$ and solves the ODE.)

$\Longrightarrow$ Through any point $(x_0, y_0)$ with $x_0 \neq 0$ there are infinitely many integral curves—follow the curve with $C = y_0/x_0^2$ to the origin and from there proceed to the other side of the $y$-axis using any choice for $C$.

The picture is completed by the curve $x = x(y) = 0$ (the $y$-axis), which is the only solution through a point $(0, y_0)$ with $y_0 \neq 0$.

Thus all points in $\mathbb{R}^2 \setminus \{(0, 0)\}$ are non-singular, but only through some of these points passes a unique integral curve.

The situation is similar to that for the ODE $y' = \sqrt{|y|}$, or $\mathrm{d}y - \sqrt{|y|}\,\mathrm{d}x = 0$, which has $M(x, y) = -\sqrt{|y|}$ non-differentiable.

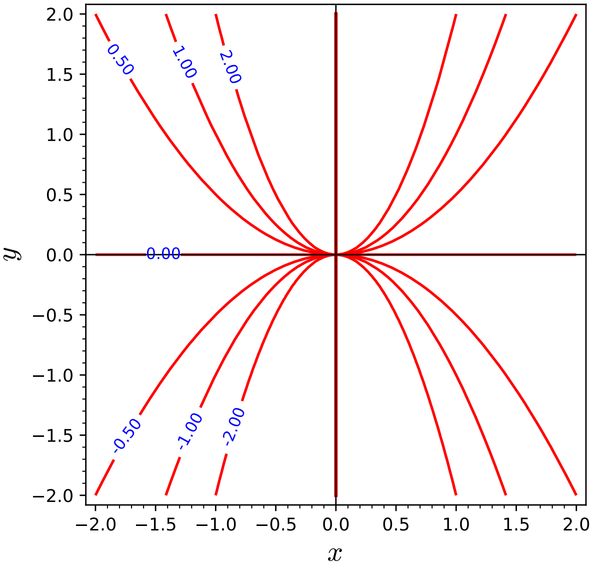


Figure: Integral curves of $2y\,\mathrm{d}x - x\,\mathrm{d}y = 0$, represented (except for $x = 0$) as contours of $F(x, y) = y/x^2$