Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = y^4 + y$ satisfying $y(0) = 1$, $y(1) = 0$.

b) There exists a solution $y(t)$ of $y' = ty^2 - t^2 y$ satisfying $\lim_{t\to+\infty} y(t) = 2021$.

c) Every maximal solution of $(x^2+1)y'' + (x+1)y' + y = 1$ has domain $\mathbb{R}$.

d) The initial value problem $(x^2+1)y'' + (x+1)y' + y = 1$, $y(1) = y'(1) = 0$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n (x-1)^n$ which is defined at $x = 3$.

e) Every solution of the system $\mathbf{y}' = \begin{pmatrix} -3 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{y}$ satisfies $\lim\limits_{t\to+\infty} \mathbf{y}(t) = (0,0)^\mathsf{T}$.

f) If $\mathbf{A} \in \mathbb{R}^{3\times3}$ satisfies $\mathbf{A}^\mathsf{T} = -\mathbf{A}$ then $\mathrm{e}^{\mathbf{A}t}$ is an orthogonal matrix for all $t \in \mathbb{R}$.

Question 2 (ca. 9 marks)

Consider the differential equation

$$2x^2y'' + 3x\,y' + (2x-1)y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 6 marks)

Determine all maximal solutions (including their domains) of

$$y' = \frac{y}{t+1} + y^4, \qquad t > -1. \tag{B}$$

Hint: A substitution of the form $z(t) = y(t)^r$ with $r \in \mathbb{R}$ may help. When translating (B) into an ODE for $z(t)$, you will see how r should be chosen.

Question 4 (ca. 6 marks)

Consider $\mathbf{A} = \begin{pmatrix} 2 & 12 & -32 \\ -4 & -14 & 32 \\ -1 & -3 & 6 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

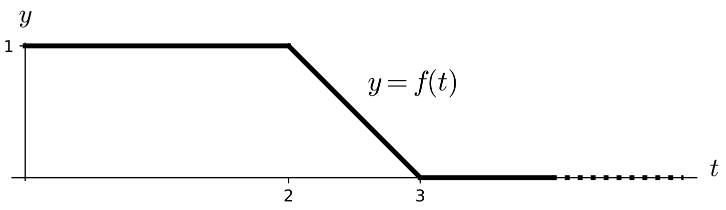
b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0,0,0)^\mathsf{T}$.

Question 5 (ca. 6 marks)

For the function f sketched below, solve the initial value problem

$$y'' + y' - 2y = f(t), \quad y(0) = y'(0) = 0$$

with the Laplace transform.



Question 6 (ca. 6 marks)

a) Determine a real fundamental system of solutions of

$$4\,y^{(4)} - 4\,y^{(3)} + 17\,y'' - 16\,y' + 4\,y = 0.$$

b) Determine the general real solution of

$$4\,y^{(4)} - 4\,y^{(3)} + 17\,y'' - 16\,y' + 4\,y = (3 - \cos t)(3 + \sin t).$$

Solutions

1 a) False. There is the constant solution $\mathbf{y}(t) \equiv 0$, and hence the existence of such a solution would contradict the Uniqueness Theorem. Alternatively, use the phase line: The function $f(y) = y^4 + y = y(y+1)(y^2 - y + 1)$ has zeros -1, 0 and is positive in $(0, \infty)$. Hence the solution with $y(0) = 1$ will be strictly increasing in its domain and, provided it is defined at $t = 1$ satisfy $y(1) > 1$. $\boxed{2}$

In fact the solution is not even defined at $t = 1$, since it blows up at time $t_\infty = \int_1^\infty \frac{dy}{y^4+y} = \frac{1}{3}\ln 2 < 1$.

b) False. The derivative $y' = ty(y-t)$ would tend to $(+\infty)2021(-\infty) = -\infty$ for $t \to +\infty$, which is utterly incompatible with the existence of $\lim_{t\to+\infty} y(t)$. $\boxed{2}$

c) True. The explicit form of this linear ODE is $y'' + \frac{x+1}{x^2+1}\,y' + \frac{1}{x^2+1}\,y = \frac{1}{x^2+1}$. The coefficient functions (including the right-hand side) have domain $\mathbb{R}$, and hence the same is true of every maximal solution on account of the sharpened version of the EUT for linear ODE's. $\boxed{2}$

d) False. The distance from the center 1 to the singular points $\pm\mathrm{i}$ (the roots of the coefficient function of y'' in the original ODE or, alternatively, the poles of the coefficient functions in the explicit ODE) is $\sqrt{2} < 3 - 1$. Hence the power series solution of the IVP is not guaranteed to exist at $x = 3$. $\boxed{2}$

In fact, making the substitution $t = x-1$, the ODE becomes $(t^2+2t+2)y''+(t+2)y'+y = 1$, and plugging in $y(t) = \sum_{n=0}^\infty a_n t^n$ and the initial conditions gives $a_0 = a_1 = 0$, $a_2 = 1/4$ and the recurrence relation $a_{n+2} = -\frac{n+1}{n+2}\,a_{n+1} - \frac{n^2+1}{2(n+1)(n+2)}\,a_n$, $n = 1, 2, 3, \dots$ From this one can see with some effort that the radius of convergence is exactly $\sqrt{2}$ and the solution is not defined at $t = 2$.

e) True. The characteristic polynomial of $\mathbf{A} = \left(\begin{smallmatrix}-3 & -1\\ 1 & -1\end{smallmatrix}\right)$ is $X^2 + 4X + 4 = (X+2)^2$, so that $\mathbf{A}$ has the eigenvalue $\lambda = -2$ with algebraic multiplicity 2 (and geometric multiplicity 1). Then every solution is of the form $\mathbf{y}(t) = \mathrm{e}^{-2t}\,\mathbf{v}_0 + t\,\mathrm{e}^{-2t}\,\mathbf{v}_1$ for some $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{R}^2$, and clearly $\lim_{t\to\infty} \mathbf{y}(t) = \mathbf{0}$. $\boxed{2}$

f) True. $\mathbf{A}^\mathsf{T} = -\mathbf{A}$ implies $(\mathbf{A}t)^\mathsf{T} = -\mathbf{A}t$ for all $t \in \mathbb{R}$, and hence

$$\begin{aligned}
\mathrm{e}^{\mathbf{A}t}\left(\mathrm{e}^{\mathbf{A}t}\right)^\mathsf{T} &= \mathrm{e}^{\mathbf{A}t}\mathrm{e}^{(\mathbf{A}t)^\mathsf{T}} = \mathrm{e}^{\mathbf{A}t}\mathrm{e}^{-\mathbf{A}t}\\
&= \mathrm{e}^{\mathbf{A}t-\mathbf{A}t} && \text{(since } \mathbf{A}t \text{ and } -\mathbf{A}t \text{ commute)}\\
&= \mathrm{e}^{\mathbf{0}} = \mathbf{I}. && \boxed{2}
\end{aligned}$$

Remarks:

a) was solved by most students in one of the two indicated ways.

b) Here several students argued that $\lim_{t\to\infty} y(t) = 2021$ implies $\lim_{t\to\infty} y'(t) = 0$. This is not true, and a counterexample is $y(t) = 2021 + \sin(t^2)/t$, whose derivative $y'(t) = 2\cos(t^2) - \sin(t^2)/t^2$ oszillates like $2\cos(t^2)$ for large t. I mentioned this phenomenon in the lecture when proving the main theorem about the phase line: If $\lim_{t\to\infty} y'(t)$ exists, it must be zero. But the limit need not exist.

c) A few students misunderstood this question as one about analytic solutions and argued with power series expansions. The solution is in fact analytic on $\mathbb{R}$, but it is not possible to prove this using power series expansions. Regardless of the particular center a of the power series chosen, this gives only a solution on some finite interval $(a-R, a+R)$. That all these power series expansions come from the same solution, which is defined on $\mathbb{R}$, must still be proved.

d) Strictly speaking, justification of the answer "False" requires a proof that the radius of convergence of the power series is smaller than 2. However, full marks were assigned for a proof of the weaker statement that the solution is not guaranteed to exist at $x = 3$; see above.

e) For the justification one can also quote a theorem from the lecture that matrices all of whose eigenvalues have negative real part are asymptotically stable.

f) During the examination I had explained that $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is orthogonal iff $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$. Unfortunately, I used the same letter $\mathbf{A}$ as in the statement of the question. Some students misinterpreted this and assumed that $\mathbf{A}$ satisfies both $\mathbf{A}^\mathsf{T} = -\mathbf{A}$ and $\mathbf{A}^\mathsf{T}\mathbf{A} = \mathbf{I}$. Both together imply $\mathbf{A}^2 = -\mathbf{I}$, but that doesn't help with the proof. My apologies!

$$\sum_1 = 12$$

2 a) The explicit form of (DE) is

$$y'' + \frac{3}{2x}\,y' + \left(\frac{1}{x} - \frac{1}{2x^2}\right) y = 0$$

$p(x) := \frac{3}{2x}$ has a pole of order 1 at 0, and $q(x) := \frac{1}{x} - \frac{1}{2x^2}$ has a pole of order 2 at 0. This shows that 0 is a regular singular point of (DE). $\boxed{1}$

Alternatively, use that the limits defining p_0, q_0 below are finite.

b) From a) we have $p_0 = \lim_{x\to 0} x\,p(x) = 3/2$, $q_0 = \lim_{x\to 0} x^2\,q(x) = -1/2$. (These coefficients can just be read off from the explicit form.)
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + \frac{1}{2}\,r - \frac{1}{2} = (r+1)(r-1/2) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = 1/2$, $r_2 = -1$. Since $r_1 - r_2 \notin \mathbb{Z}$, there exist two fundamental solutions y_1, y_2 of the form

$$\begin{aligned} y_1(x) &= x^{1/2}\sum_{n=0}^\infty b_n x^n = \sum_{n=0}^\infty b_n x^{n+1/2}, \\ y_2(x) &= x^{-1}\sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty a_n x^{n-1} \end{aligned} \qquad \boxed{1}$$

with normalization $a_0 = b_0 = 1$.

First we determine $y_2(x)$. We have

$$\begin{aligned}
0 &= 2x^2\,y_2'' + 3x\,y_2' + (2x-1)y_2\\
&= 2x^2\sum_{n=0}^{\infty}(n-1)(n-2)a_nx^{n-3} + 3x\sum_{n=0}^{\infty}(n-1)a_nx^{n-2} + (2x-1)\sum_{n=0}^{\infty}a_nx^{n-1}\\
&= \sum_{n=0}^{\infty}\left[2(n-1)(n-2)+3(n-1)-1\right]a_nx^{n-1} + \sum_{n=0}^{\infty}2a_nx^n\\
&= \sum_{n=0}^{\infty}(2n^2-3n)a_nx^{n-1} + \sum_{n=1}^{\infty}2a_{n-1}x^{n-1}\\
&= \sum_{n=1}^{\infty}\left[n(2n-3)a_n+2a_{n-1}\right]x^{n-1}.
\end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{2}{n(2n-3)}a_{n-1} \quad \text{for } n=1,2,3,\dots. \qquad \boxed{1}$$

and with $a_0=1$ further $a_n = (-1)^n\frac{2^n}{n!(-1)1\cdot3\cdot5\cdots(2n-3)} = (-1)^{n-1}\frac{2^n}{n!\,1\cdot3\cdot5\cdots(2n-3)}$ for $n\geq 1$. (For $n=1$ the product in the denominator is empty.)

$$\begin{aligned}
\implies y_2(x) &= \sum_{n=0}^{\infty}\frac{(-1)^{n-1}2^n}{n!\,1\cdot3\cdot5\cdots(2n-3)}\,x^{n-1} \qquad \boxed{1\tfrac{1}{2}}\\
&= x^{-1}+2-2x+\frac{2^3}{3!\,3}\,x^2-\frac{2^4}{4!\,3\cdot5}\,x^3+\frac{2^5}{5!\,3\cdot5\cdot7}\,x^4\mp\cdots
\end{aligned}$$

For the determination of $y_1(x)$ we repeat the process with exponents increased by 1.5 :

$$\begin{aligned}
0 &= 2x^2\,y_1'' + 3x\,y_1' + (2x-1)y_1\\
&= 2x^2\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)b_nx^{n-3/2} + 3x\sum_{n=0}^{\infty}\left(n+\tfrac{1}{2}\right)b_nx^{n-1/2} + (2x-1)\sum_{n=0}^{\infty}b_nx^{n+1/2}\\
&= \sum_{n=0}^{\infty}\left[2\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right)+3\left(n+\tfrac{1}{2}\right)-1\right]b_nx^{n+1/2} + \sum_{n=0}^{\infty}2b_nx^{n+3/2}\\
&= \sum_{n=0}^{\infty}(2n^2+3n)b_nx^{n+1/2} + \sum_{n=1}^{\infty}2b_{n-1}x^{n+1/2}\\
&= \sum_{n=1}^{\infty}\left[n(2n+3)b_n+2b_{n-1}\right]x^{n+1/2}.
\end{aligned}$$

Here we obtain the recurrence relation

$$b_n = -\frac{2}{n(2n+3)}\,b_{n-1} \quad \text{for } n=1,2,3,\dots, \qquad \boxed{1}$$

and with $b_0=1$ further $b_n = \frac{(-1)^n2^n}{n!\,5\cdot7\cdots(2n+3)}$ for $n\geq 0$. (For $n=0$ the expression reduces to 1, as needed.)

$$\Longrightarrow\ y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!\,5\cdot 7\cdots(2n+3)}\, x^{n+1/2} \qquad \boxed{1\tfrac{1}{2}}$$

$$= x^{1/2} - \frac{2}{5}\,x^{3/2} + \frac{2}{5\cdot 7}\,x^{5/2} - \frac{2^3}{3!\,5\cdot 7\cdot 9}\,x^{7/2} + \frac{2^4}{4!\,5\cdot 7\cdot 9\cdot 11}\,x^{9/2} \mp \cdots$$

Alternative solution: We use the general recurrence relation for the rational functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_{n-1}(r) \quad \text{for } n \geq 1.$$

Since $F(r) = (r+1)(r-1/2)$ and all coefficients p_i, q_i except for p_0, q_0 and $q_1 = 1$ are zero, we obtain

$$a_n(r) = -\frac{1}{(r+n+1)(r+n-1/2)}\, a_{n-1}(r) \quad \text{for } n \geq 1.$$

Thus the coefficients $a_n(-1)$ of $y_2(x)$ satisfy the recurrence relation $a_n(-1) = -\frac{1}{n(n-3/2)}\, a_{n-1}(-1)$ $= -\frac{2}{n(2n-3)}\, a_{n-1}(-1)$ (the same as for a_n above) and the coefficients $a_n(1/2)$ of $y_1(x)$ satisfy the recurrence relation $a_n(1/2) = -\frac{1}{(n+3/2)n}\, a_{n-1}(1/2) = -\frac{2}{(2n+3)n}$ (the same as for b_n above). The rest of the computation remains the same.

The general (real) solution on $(0,\infty)$ is then $y(x) = c_1\, y_1(x) + c_2\, y_2(x)$, $c_1, c_2 \in \mathbb{R}$. $\boxed{\tfrac{1}{2}}$

That solutions are defined on the whole of $(0,\infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C}\setminus\{0\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞. $\boxed{\tfrac{1}{2}}$

c) The solution on $(-\infty,0)$ is $y(x) = c_1\, y_1^-(x) + c_2\, y_2(x)$ with the same power series $y_2(x)$ as in b) and

$$y_1^-(x) = (-x)^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!\,5\cdot 7\cdots(2n+3)}\, x^n = \sum_{n=0}^{\infty} \frac{2^n}{n!\,5\cdot 7\cdots(2n+3)} (-x)^{n+1/2} \qquad \boxed{1}$$

(This is <u>not</u> the same as $y_1(-x)$, which has alternating coefficients when written in terms of powers of $-x$.)

Since none of $y_1(x)$, $y_2(x)$ is analytic at zero, the only solution on $\mathbb{R}$ is $y(x) \equiv 0$. $\boxed{1}$

Remarks: In b) several students forgot to state the general solution. In c) almost all students obtained a nontrivial (and hence wrong) solution space. Just like $x \mapsto \sqrt{x}$, nonzero multiples of $y_1(x)$ are not differentiable at $x = 0$, and hence cannot be extended to a solution on $[0,\infty)$ (let alone on $\mathbb{R}$).

$$\sum_2 = 10$$

3 Suppressing the argument t as usual, we have

$$z = y^r \implies z' = ry^{r-1}y' = ry^{r-1}\left(\frac{y}{t+1} + y^4\right) = \frac{r}{t+1}\,y^r + r\,y^{r+3}.$$

If we choose $r = -3$, i.e., $z(t) = 1/y(t)^3$, the new ODE for $z(t)$ becomes particularly simple, viz.

$$z' = \frac{r}{t+1}\,z + r = -\frac{3}{t+1}\,z - 3. \qquad \boxed{2}$$

Next we determine the general solution of this inhomogeneous linear ODE. The general solution of the associated homogeneous ODE is

$$z_h(t) = c\,\exp\int -\frac{3}{t+1}\,\mathrm{d}t = c(t+1)^{-3}, \quad c \in \mathbb{R}. \qquad \boxed{1}$$

Variation of the constant c then yields a particular solution of the inhomogeneous ODE:

$$z_p(t) = c(t)(t+1)^{-3} = (t+1)^{-3}\int (t+1)^3(-3)\,\mathrm{d}t = (t+1)^{-3}\left(-\frac{3}{4}\right)(t+1)^4 = -\frac{3}{4}(t+1). \qquad \boxed{1}$$

The general solution of $z' = -\frac{3}{t+1}\,z - 3$ is therefore

$$z(t) = z_p(t) + z_h(t) = -\frac{3}{4}(t+1) + c(t+1)^{-3}, \quad c \in \mathbb{R}. \qquad \boxed{1}$$

The general solution of (B) is then

$$\begin{aligned} y(t) = z(t)^{-1/3} &= \left(-\frac{3}{4}(t+1) + c(t+1)^{-3}\right)^{-1/3} \\ &= (t+1)\left(c - \frac{3}{4}(t+1)^4\right)^{-1/3}, \quad c \in \mathbb{R}\cup\{\infty\}, \end{aligned}$$

with $A^{-1/3}$ for negative A interpreted as $-(-A)^{1/3}$ and $c = \infty$ corresponding to $\mathbf{y} \equiv 0$, which is also a solution but went missing through the subsitution. $\boxed{1}$

For $c \leq 0$ or $c = \infty$ maximal solutions have domain $(-1, \infty)$. For $c > 0$ the expression above provides two maximal solutions, one defined on $\big(-1, (4c/3)^{1/4} - 1\big)$ and the other on $\big((4c/3)^{1/4} - 1, \infty\big)$. $\boxed{1}$

Remarks: In my original solution the trivial solution $\mathbf{y} \equiv 0$ was missing. If students found this solution but didn't discuss the meaning of $A^{-1/3}$ for $A < 0$, I have assigned 6 marks. If they also found the correct domains, I have assigned 7 marks, because this indicates the required knowledge of $A^{-1/3}$ for $A < 0$.

$$\sum_3 = 7$$

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4 a) The characteristic polynomial of $\mathbf{A}$ is

$$\begin{aligned} \chi_{\mathbf{A}}(X) &= \begin{vmatrix} X-2 & -12 & 32 \\ 4 & X+14 & -32 \\ 1 & 3 & X-6 \end{vmatrix} = \begin{vmatrix} 1 & 3 & X-6 \\ 0 & -3X-6 & -X^2+8X+20 \\ 0 & X+2 & -4X-8 \end{vmatrix} \\ &= -\begin{vmatrix} 1 & 3 & X-6 \\ 0 & X+2 & -4X-8 \\ 0 & 0 & -X^2-4X-4 \end{vmatrix} = (X+2)^3. \end{aligned}$$

$\Longrightarrow$ The only eigenvalue of $\mathbf{A}$ is $\lambda = -2$ with algebraic multiplicity 3. $\boxed{2}$

$$\mathbf{A} + 2\,\mathbf{I} = \begin{pmatrix} 4 & 12 & -32 \\ -4 & -12 & 32 \\ -1 & -3 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 3 & -8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow$ The eigenspace corresponding to $\lambda = -2$ is two-dimensional and generated by $\mathbf{v}_1 = (-3, 1, 0)^\mathsf{T}$, $\mathbf{v}_2 = (8, 0, 1)^\mathsf{T}$.

Since the generalized eigenspace corresponding to $\lambda = -2$ is the whole of $\mathbb{R}^3$, we can take

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{e}_1 = (1,0,0)^\mathsf{T}, \\ \mathbf{w}_2 &= (\mathbf{A} + 2\,\mathbf{I})\mathbf{e}_1 = (4,-4,-1)^\mathsf{T} = -4\,\mathbf{v}_1 - \mathbf{v}_2, \\ \mathbf{w}_3 &= \mathbf{v}_1 = (-3,1,0)^\mathsf{T} \end{aligned}$$

as convenient basis and obtain as corresponding fundamental system of solutions:

$$\begin{aligned} \mathbf{y}_1(t) &= \mathrm{e}^{-2t}\begin{pmatrix}1\\0\\0\end{pmatrix} + t\,\mathrm{e}^{-2t}\begin{pmatrix}4\\-4\\-1\end{pmatrix} = \mathrm{e}^{-2t}\begin{pmatrix}1+4t\\-4t\\-t\end{pmatrix}, \\ \mathbf{y}_2(t) &= \mathrm{e}^{-2t}\begin{pmatrix}4\\-4\\-1\end{pmatrix}, \\ \mathbf{y}_3(t) &= \mathrm{e}^{-2t}\begin{pmatrix}-3\\1\\0\end{pmatrix}. \end{aligned} \qquad \boxed{3}$$

In place of $\mathbf{w}_2$ we can of course also use the earlier determined eigenvector $\mathbf{v}_2$, i.e., $\mathbf{y}_2(t) = \mathrm{e}^{-2t}(8,0,1)^\mathsf{T}$.

b) Since $\mathbf{A}$ is invertible, the system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has a unique constant solution (critical point) $\mathbf{y}(t) \equiv (y_1, y_2, y_3)^\mathsf{T}$, which is obtained by solving $\mathbf{A}\mathbf{y} + \mathbf{b} = \mathbf{0}$.

$$\left[\begin{array}{rrr|r} 2 & 12 & -32 & 0 \\ -4 & -14 & 32 & 0 \\ -1 & -3 & 6 & -2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -6 & 2 \\ 0 & 6 & -20 & -4 \\ 0 & -2 & 8 & 8 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 3 & -6 & 2 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$\Longrightarrow y_3 = 5,\ y_2 = -4 + 4\,y_3 = 16,\ y_1 = 2 - 3y_2 + 6y_3 = -16.$ $\boxed{1}$

$\Longrightarrow$ The general (real) solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is

$$\mathbf{y}(t) = \begin{pmatrix}-16\\16\\5\end{pmatrix} + c_1\,\mathrm{e}^{-2t}\begin{pmatrix}1+4t\\-4t\\-t\end{pmatrix} + c_2\,\mathrm{e}^{-2t}\begin{pmatrix}4\\-4\\-1\end{pmatrix} + c_3\,\mathrm{e}^{-2t}\begin{pmatrix}-3\\1\\0\end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

The initial condition $\mathbf{y}(0) = \mathbf{0}$ gives for c_1, c_2, c_3 the system of linear equations

$$\left[\begin{array}{rrr|r} 1 & 4 & -3 & 16 \\ 0 & -4 & 1 & -16 \\ 0 & -1 & 0 & -5 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 4 & -3 & 16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right],$$

which has the solution $c_3 = 4$, $c_2 = 5$, $c_1 = 16 - 4c_2 + 3c_3 = 8$.

$\Longrightarrow$ The solution of the given IVP is

$$\begin{aligned}\mathbf{y}(t) &= \begin{pmatrix}-16\\16\\5\end{pmatrix} + 8\,\mathrm{e}^{-2t}\begin{pmatrix}1+4t\\-4t\\-t\end{pmatrix} + 5\,\mathrm{e}^{-2t}\begin{pmatrix}4\\-4\\-1\end{pmatrix} + 4\,\mathrm{e}^{-2t}\begin{pmatrix}-3\\1\\0\end{pmatrix}\\ &= \begin{pmatrix}-16\\16\\5\end{pmatrix} + 8\,\mathrm{e}^{-2t}\begin{pmatrix}1+4t\\-4t\\-t\end{pmatrix} + \mathrm{e}^{-2t}\begin{pmatrix}8\\-16\\-5\end{pmatrix}. \qquad \boxed{2}\end{aligned}$$

Remarks:

In a) it is of course also possible to use $\mathbf{y}_2(t) = \mathrm{e}^{-2t}(8,0,1)^\mathsf{T}$ as part of the fundamental system and/or compute $\mathbf{y}_1(t)$ using a different chain $\mathbf{w}_1', \mathbf{w}_2'$; e.g., one can take $\mathbf{w}_2' = \mathbf{v}_1$ and determine $\mathbf{w}_1'$ by solving $(\mathbf{A} + 2\,\mathbf{I})\mathbf{w}_1' = \mathbf{v}_1$.

Several students exhibited 3 eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ of $\mathbf{A}$ and claimed that the corresponding 3 solutions $\mathbf{y}_i(t) = \mathrm{e}^{-2t}\mathbf{v}_i(t)$, $i = 1,2,3$, form a fundamental system. This is false, because any 3 eigenvectors of $\mathbf{A}$, and hence the corresponding solutions, must be linearly dependent. The eigenspace of $\mathbf{A}$ for $\lambda = -2$ is only 2-dimensional!

In b) a few students, who had determined the special fundamental matrix (or system) $\mathrm{e}^{\mathbf{A}t}$ in a), noticed the following clever solution: After determining the constant solution $\mathbf{y}(t) = (-16,16,5)^\mathsf{T}$ of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, we know that the general solution is $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c} + (-16,16,5)^\mathsf{T}$, $\mathbf{c} \in \mathbb{R}^3$. Since $\mathrm{e}^{\mathbf{A}t}\big|_{t=0} = \mathbf{I}$, we obtain the condition $\mathbf{y}(0) = \mathbf{c} + (-16,16,5)^\mathsf{T} = (0,0,0^\mathsf{T}$, which gives $\mathbf{c} = (16,-16,-5)$.

Direct determination of $\mathrm{e}^{\mathbf{A}t}$ in a) is in fact a cheap alternative: By the Cayley-Hamilton Theorem we have $(\mathbf{A} + 2\,\mathbf{I})^3 = \mathbf{0}$; in fact, even $(\mathbf{A} + 2\,\mathbf{I})^2 = \mathbf{0}$, since there is no chain of generalized eigenvectors of length 3. (This would require the eigenspace for $\lambda = -2$ to be 1-dimensional.) Thus

$$\mathrm{e}^{\mathbf{A}t} = \mathrm{e}^{-2\,\mathbf{I}t}\mathrm{e}^{(\mathbf{A}+2\,\mathbf{I})t} = \mathrm{e}^{-2t}\left(\mathbf{I} + (\mathbf{A} + 2\,\mathbf{I})t\right) = \mathrm{e}^{-2t}\begin{pmatrix}1+4t & 1+12t & -32t\\ -4t & 1-12t & 32t\\ -t & -3t & 1+8t\end{pmatrix}.$$

$$\sum_4 = 8$$

5 Writing $Y(s) = \mathcal{L}\{y(t)\}$ and applying the Laplace transform to both sides of the ODE gives, because the initial values are all zero,

$$\mathcal{L}\{y'' + y' - 2y\} = (s^2 + s - 2)Y(s) = \mathcal{L}\big\{f(t)\big\} = F(s)$$

with $F(s) = \mathcal{L}\big\{f(t)\big\}$.

Further we have

$$\begin{aligned} f(t) &= \mathrm{u}(t) - \mathrm{u}(t-2) + (3-t)\big(\mathrm{u}(t-2) - \mathrm{u}(t-3)\big)\\ &= \mathrm{u}(t) - (t-2)\mathrm{u}(t-2) + (t-3)\mathrm{u}(t-3), && \boxed{1}\\ \Longrightarrow\quad F(s) &= \frac{1}{s} - \frac{\mathrm{e}^{-2s}}{s^2} + \frac{\mathrm{e}^{-3s}}{s^2}, && \boxed{1}\end{aligned}$$

Together with $s^2+s-2=(s-1)(s+2)$ this gives

$$Y(s)=\frac{1}{s(s-1)(s+2)}-\frac{\mathrm{e}^{-2s}}{s^2(s-1)(s+2)}+\frac{\mathrm{e}^{-3s}}{s^2(s-1)(s+2)}. \qquad \boxed{1}$$

The relevant partial fractions expansions are

$$\frac{1}{s(s-1)(s+2)}=-\frac{1}{2s}+\frac{1}{3(s-1)}+\frac{1}{6(s+2)}. \qquad \boxed{1}$$
$$\frac{1}{s^2(s-1)(s+2)}=-\frac{1}{2s^2}-\frac{1}{4s}+\frac{1}{3(s-1)}-\frac{1}{12(s+2)}. \qquad \boxed{2}$$

Of these, six coefficients (all except that of $1/s$ in the 2nd formula) are easily obtained by multiplying both sides with the corresponding denominator and setting s equal to the (unique) root of the denominator. The coefficient of $1/s$ in the 2nd formula can be obtained by computing first the coefficient of $1/s^2$, which is $-1/2$, and then applying the analogous reasoning to the function $\frac{1}{s^2(s-1)(s+2)}+\frac{1}{2s^2}=\frac{s+1}{2s(s-1)(s+2)}$. One can also obtain the 2nd expansion from the first like this: $\frac{1}{s^2(s-1)(s+2)}=-\frac{1}{2s^2}+\frac{1}{3s(s-1)}+\frac{1}{6s(s+2)}=-\frac{1}{2s^2}+\frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s}\right)+\frac{1}{12}\left(\frac{1}{s}-\frac{1}{s+2}\right)=-\frac{1}{2s^2}-\frac{1}{4s}+\frac{1}{3(s-1)}-\frac{1}{12(s+2)}$. Using the formulas $\mathcal{L}\{\mathrm{e}^{at}\}=\frac{1}{s-a}$ with $a=0,1,-2$, $\mathcal{L}\{1\}=1/s$, $\mathcal{L}\{t\}=1/s^2$, and $\mathcal{L}\big\{\mathrm{u}(t-c)g(t-c)\big\}=\mathrm{e}^{-cs}G(s)$ with $c=2,3$ in the reverse direction then gives $y(t)=\mathcal{L}^{-1}\big\{Y(s)\big\}$ as a sum of $3+4+4$ functions:

$$\begin{aligned}y(t)=&-\frac{1}{2}+\frac{1}{3}\,\mathrm{e}^{t}+\frac{1}{6}\,\mathrm{e}^{-2t}\\&+\frac{1}{2}\,\mathrm{u}_2(t)(t-2)+\frac{1}{4}\,\mathrm{u}_2(t)-\frac{1}{3}\,\mathrm{u}_2(t)\mathrm{e}^{t-2}+\frac{1}{12}\,\mathrm{u}_2(t)\mathrm{e}^{-2(t-2)}\\&-\frac{1}{2}\,\mathrm{u}_3(t)(t-3)-\frac{1}{4}\,\mathrm{u}_3(t)+\frac{1}{3}\,\mathrm{u}_3(t)\mathrm{e}^{t-3}-\frac{1}{12}\,\mathrm{u}_3(t)\mathrm{e}^{-2(t-3)},\end{aligned} \qquad \boxed{2}$$

where we have written $\mathrm{u}_c(t)$ for $\mathrm{u}(t-c)$ as usual.

A different description of the solution is the following:

$$y(t)=\begin{cases}-\frac{1}{2}+\frac{1}{3}\,\mathrm{e}^{t}+\frac{1}{6}\,\mathrm{e}^{-2t} & \text{if } 0\leq t\leq 2,\\ \frac{t}{2}-\frac{5}{4}+\frac{1-\mathrm{e}^{-2}}{3}\,\mathrm{e}^{t}+\frac{2+\mathrm{e}^{4}}{12}\,\mathrm{e}^{-2t} & \text{if } 2\leq t\leq 3,\\ \frac{1-\mathrm{e}^{-2}+\mathrm{e}^{-3}}{3}\,\mathrm{e}^{t}+\frac{2+\mathrm{e}^{4}-\mathrm{e}^{6}}{12}\,\mathrm{e}^{-2t} & \text{if } t\geq 3.\end{cases}$$

Remarks: Almost all students had problems with this question. Quite a few applied the Laplace transform separately to the restrictions of $f(t)$ to the intervals $[0,2]$, $[2,3]$, and $[3,\infty)$. The solution obtained in this way was usually wrong because of unmatched initial conditions at $t=2$ and $t=3$. For this note that we can compute the solutions on the three intervals separately using the machinery for linear ODE's (cf. Question 6), but then, e.g., the initial conditions at $t=2$ of the solution on $[2,3]$ must be fitted to those of the solution on $[0,2]$, which is determined uniquely from $y(0)=y'(0)=0$. Since $y(2)$, $y'(2)$ turn out to be nonzero, the formula for $Y(s)$ becomes more complicated.

The general fact behind this is that Laplace transforms are only defined for functions with domain $[0,\infty)$. If we want to compute the Laplace transform of the constant function $f(t)\equiv 1$ on $[0,2]$, say, we need to extend it to $[0,\infty)$ first. The unique extension vanishing on $[2,\infty)$ is $t\mapsto \mathrm{u}(t)-\mathrm{u}(t-2)$, and this is the only one respecting addition of forcing functions (and hence suitable for computing $Y(s)$ using the linearity of the Laplace transform).

$$\sum_5 = 8$$

6 a) The characteristic polynomial is

$$\begin{aligned} a(X) &= 4\,X^4 - 4\,X^3 + 17\,X^2 - 16\,X + 4 \\ &= \left(4X^2 - 4X + 1\right)\left(X^2 + 4\right) \\ &= 4\left(X - \tfrac{1}{2}\right)^2 (X - 2\mathrm{i})(X + 2\mathrm{i}). \end{aligned}$$

with zeros $\lambda_1 = \frac{1}{2}$ of multiplicity 2 and $\lambda_2 = 2\mathrm{i}$, $\lambda_3 = -2\mathrm{i}$ of multiplicity 1. $\boxed{1}$

$\Longrightarrow$ A complex fundamental system of solutions is $\mathrm{e}^{t/2}$, $t\,\mathrm{e}^{t/2}$, $\mathrm{e}^{2\mathrm{i}t}$, $\mathrm{e}^{-2\mathrm{i}t}$, and the corresponding real fundamental system is

$$\mathrm{e}^{t/2}, \quad t\,\mathrm{e}^{t/2}, \quad \cos(2t), \quad \sin(2t). \qquad \boxed{2}$$

b) In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, which has right-hand side $(3 - \cos t)(3 + \sin t) = 9 - 3\cos t + 3\sin t - \cos t \sin t = 9 - 3\cos t + 3\sin t - \frac{1}{2}\sin(2t)$, we solve the three equations $a(\mathrm{D})y_i = b_i(t)$ for $b_1(t) = 9$, $b_2(t) = \mathrm{e}^{\mathrm{i}t}$, $b_3(t) = \mathrm{e}^{2\mathrm{i}t}$. Superposition then yields the particular solution $y_p(t) = y_1(t) - 3\,\mathrm{Re}\,y_2(t) + 3\,\mathrm{Im}\,y_2(t) - \frac{1}{2}\,\mathrm{Im}\,y_3(t)$.

(1) Here we can take the constant solution $y_1(t) = 9/4$. $\boxed{1}$

(2) Since $\mu = \mathrm{i}$ is not a zero of $a(X)$, we can take

$$y_2(t) = \frac{1}{a(\mathrm{i})}\,\mathrm{e}^{\mathrm{i}t} = \frac{1}{(4\mathrm{i}^2 - 4\mathrm{i} + 1)(\mathrm{i}^2 + 4)}\,\mathrm{e}^{\mathrm{i}t} = \frac{1}{(-3 - 4\mathrm{i})3}\,\mathrm{e}^{\mathrm{i}t} = \frac{-3 + 4\mathrm{i}}{75}\,\mathrm{e}^{\mathrm{i}t} \qquad \boxed{1}$$

(3) Since $\mu = 2\mathrm{i}$ is a root of multiplicity 1 of $a(X)$, the correct Ansatz is $y_3(t) = ct\,\mathrm{e}^{2\mathrm{i}t}$ with $c \in \mathbb{C}$. We then obtain

$$\begin{aligned} a(\mathrm{D})y_3(t) &= c(4\mathrm{D}^2 - 4\mathrm{D} + 1)(\mathrm{D} + 2\mathrm{i})(\mathrm{D} - 2\mathrm{i})t\,\mathrm{e}^{2\mathrm{i}t} \\ &= c(4\mathrm{D}^2 - 4\mathrm{D} + 1)(\mathrm{D} + 2\mathrm{i})\mathrm{e}^{2\mathrm{i}t} \\ &= c\big((4(2\mathrm{i})^2 - 4(2\mathrm{i}) + 1\big)4\mathrm{i}\,\mathrm{e}^{2\mathrm{i}t} \\ &= c(32 - 60\mathrm{i})\mathrm{e}^{2\mathrm{i}t} \end{aligned}$$

$$\Longrightarrow \quad c = \frac{1}{32 - 60\mathrm{i}} = \frac{1}{4(8 - 15\mathrm{i})} = \frac{8 + 15\mathrm{i}}{4 \cdot 289} \quad \Longrightarrow \quad y_3(t) = \frac{8 + 15\mathrm{i}}{4 \cdot 289}\,t\,\mathrm{e}^{2\mathrm{i}t}. \qquad \boxed{1}$$

Putting things together gives

$$\begin{aligned} y_p(t) &= \frac{9}{4} - \frac{1}{25}(-3\cos t - 4\sin t) + \frac{1}{25}(-3\sin t + 4\cos t) - \frac{1}{8 \cdot 289}\big(8t\sin(2t) + 15t\cos(2t)\big) \\ &= \frac{9}{4} + \frac{7}{25}\cos t + \frac{1}{25}\sin t - \frac{15}{8 \cdot 289}\,t\cos(2t) - \frac{1}{289}\,t\sin(2t). \end{aligned} \qquad \boxed{1}$$

The general real solution is then

$$y_p(t) = y_p(t) + c_1 \mathrm{e}^{t/2} + c_2 t\, \mathrm{e}^{t/2} + c_3 \cos(2t) + c_4 \sin(2t), \quad c_1, c_2, c_3, c_4 \in \mathbb{R}. \qquad \boxed{1}$$

Remarks: Most students solved a) successfully, but had problems with b).

While it is possible to determine $y_p(t)$ in b) using real „Ansätze“, e.g., $t \mapsto A\cos t + B\sin t$ in place of $y_2(t)$, the computations become more complicated, and hence more error prone.

$$\sum_{6} = 8$$

$$\sum_{\text{Final Exam}} = 53$$