Differential Equations Plus (Math 286)

H12 Determine all maximal solutions of $t^2\,y' = y^2$ and decide for which points $(t_0, y_0) \in \mathbb{R}^2$ the IVP $t^2\,y' = y^2 \wedge y(t_0) = y_0$ has no solution/exactly one solution/more than one solution.

H13 Determine the general solution of the following ODE's in terms of $y(0)$ (three answers suffice).

a) $\mathrm{d}y/\mathrm{d}t = e^{y+t}$;

b) $\mathrm{d}y/\mathrm{d}t = ty + y + t$;

c) $\mathrm{d}y/\mathrm{d}t = (\cos t)y + 4\cos t$;

d) $\mathrm{d}y/\mathrm{d}t = t^m y^n$ $(m, n \in \mathbb{Z})$.

H14 For the following ODE's, solve the corresponding IVP with $y(0) = 1$.

a) $\mathrm{d}y/\mathrm{d}t = -4ty$;

b) $\mathrm{d}y/\mathrm{d}t = t\,y^3$;

c) $(1+t)\mathrm{d}y/\mathrm{d}t = 4y$.

H15 Show that the graph of $y(t) = a/(d\,\mathrm{e}^{-at} + b)$ $(a, b, d > 0)$ is point-symmetric to its inflection point.

Hint: A superb way to solve this exercise is to observe that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

H16

a) Explain how to adapt the analysis of the harvesting equation in the lecture to $y' = ay^2 + by + c$ with $a, b, c \in \mathbb{R}$ and $a > 0$.

b) Sketch the solution curves of (i) $y' = y^2 - y + 1$, (ii) $y' = y^2 + 2y + 1$, (iii) $y' = y^2 + y - 2$ without actually computing solutions. Steady-state solutions and inflection points (if any) should be drawn exactly.

H17 The ODE $y' = a(t)y - b(t)y^n$, $n \in \mathbb{R} \setminus \{0, 1\}$ is called *Bernoulli's differential equation.*

a) Show that for an appropriate choice of $\beta \in \mathbb{R}$ the substitution $z = y^\beta$ turns Bernoulli's differential equation into a linear 1st-order ODE (which can be solved by the usual methods).

b) Solve the IVP $y' = 4y - y^3 \wedge y(0) = 1$ by the method suggested in a).

c) Investigate the asymptotic stability of the steady-state solutions of the ODE in b).

H18 *Optional exercise*

a) Show that the general (real) solution of $y'' = y$ is $y(x) = c_1 e^x + c_2 e^{-x}$, $c_1, c_2 \in \mathbb{R}$.
Hint: For a solution y the functions $y + y'$ and $y - y'$ satisfy linear 1st-order ODE's.

b) For $x \in \mathbb{R}$ let
$$F(x) = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t\,.$$
Show that
$$F'(x) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin(xt)}{t(t^2+1)}\,\mathrm{d}t \quad \text{for } x > 0.$$
Hint: Differentiate F under the integral sign and use $\int_0^\infty \sin(xt)/t\,\mathrm{d}t = \int_0^\infty \sin(t)/t\,\mathrm{d}t = \pi/2$ for $x > 0$.

c) Show that F solves $y'' = y$ on $(0, \infty)$.

d) Determine F from a), c) and $F(0)$, $F'(0+)$, and use the result to evaluate the integral
$$\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t\,.$$

H19 *Optional exercise*

The task of this exercise is to show the Cauchy-Hadamard formula
$$R = \frac{1}{L}, \quad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$$
(with the conventions $1/0 = \infty$, $1/\infty = 0$) for the radius of convergence R of a (complex) power series $\sum_{n=0}^\infty a_n(z-a)^n$. Here $L = \limsup_{n\to\infty} x_n \in [-\infty, +\infty]$ (*limit superior*) denotes the largest accumulation point of a real sequence (x_n), i.e., for every $\epsilon > 0$ there are only finitely many indexes n satisfying $x_n \geq L + \epsilon$ but no real number $L' < L$ has this property (with suitable modifications for $L = \pm\infty$).

a) If $L = \infty$ (i.e., $\sqrt[n]{|a_n|}$ is unbounded), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges only for $z = a$.

b) If $L = 0$ (i.e., $\sqrt[n]{|a_n|}$ converges to zero), show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for all $z \in \mathbb{C}$.

c) If $0 < L < \infty$, show that $\sum_{n=0}^\infty a_n(z-a)^n$ converges for $|z-a| < 1/L$ and diverges for $|z-a| > 1/L$.

Due on Fri Oct 15, 6 pm

The optional exercises can be handed in until Fri Oct 22, 6 pm.