Differential Equations Plus (Math 286)

H48 Determine the general solution of the following ODE's (two answers suffice):

a) $(2t+1)y'' + (4t-2)y' - 8y = (6t^2+t-3)\mathrm{e}^t, \quad t > -1/2;$

b) $t^2(1-t)y'' + 2t(2-t)y' + 2(1+t)y = t^2, \quad 0 < t < 1;$

c) $(t^2-4t+4)y'' + (3t-6)y' + 2y = t^2+1, \quad t > 2.$

Hints: The associated homogeneous ODE in a) has a solution of the form $y(t) = \mathrm{e}^{\alpha t}$ and that in b) a solution of the form $y(t) = t^\beta$ with constants α, β. In both cases a particular solution of the inhomogeneous ODE can be determined by reducing it to a first-order system and using variation of parameters (though this may not be the most economic solution). The ODE in c) is an inhomogeneous Euler equation in disguise.

H49 Determine a fundamental system of solutions for Bessel's ODE with $p = \frac{1}{2}$,

$$y'' + \frac{1}{t}y' + \left(1 - \frac{1}{4t^2}\right)y = 0,$$

using the „Ansatz" $z = \sqrt{t}\,y$.

H50 *On Hermite Polynomials*

In the lecture the Hermite polynomials $\mathrm{H}_n(X) \in \mathbb{R}[X]$ are defined by $\mathrm{H}_n(t) = (-1)^n \mathrm{e}^{t^2} \mathrm{D}^n[\mathrm{e}^{-t^2}]$ for $t \in \mathbb{R}$ $(n = 0, 1, 2, \dots)$.

a) Show that $t \mapsto \mathrm{H}_n(t)$ is a polynomial function, justifying the definition.

b) Show that $\deg \mathrm{H}_n(X) = n$ and the leading coefficient of $\mathrm{H}_n(X)$ is 2^n.

c) Show that $\mathrm{H}_n(X)$ satisfies the recurrence relation $\mathrm{H}_{n+1}(X) = 2X\,\mathrm{H}_n(X) - 2n\,\mathrm{H}_{n-1}(X)$, and compute $\mathrm{H}_n(X)$ for $n \leq 6$.

d) Show that $t \mapsto \mathrm{H}_n(t)$ solves Hermite's differential equation $y'' - 2ty' + 2ny = 0$.
Hint: The equation is equivalent to $Ly = 0$, where $L = \mathrm{D}^2 - 2t\mathrm{D} + 2n\,\mathrm{id}$. Express $L\big[\mathrm{H}_n(t)\big]$ in terms of $\mathrm{D}^n[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+1}[\mathrm{e}^{-t^2}]$, $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}]$, and rewrite the latter using $\mathrm{D}^{n+2}[\mathrm{e}^{-t^2}] = \mathrm{D}^{n+1}[-2t\,\mathrm{e}^{-t^2}]$.

H51 *Optional Exercise*

The function e^t has no zero and satisfies $y' = y$. The function $\sin t$ has no zero in common with its derivative $\cos t$ and satisfies $y'' = -y$. Generalizing this observation, show that a nonzero C^n-function $f\colon I \to \mathbb{R}$ on an interval $I \subseteq \mathbb{R}$ of positive length satisfies an explicit (possibly time-dependent) homogeneous linear ODE of order n if and only if $y, y', \dots, y^{(n-1)}$ have no common zero.
Hint: For the if-part work with the function $t \mapsto f(t)^2 + f'(t)^2 + \cdots + f^{(n-1)}(t)^2$.

H52 *On Legendre Polynomials (optional exercise)*

In the lecture the Legendre polynomials $\mathrm{P}_n(X) \in \mathbb{R}[X]$ were defined by $\mathrm{P}_n(t) = \frac{1}{2^n n!}\mathrm{D}^n\left[(t^2-1)^n\right]$, $n = 0, 1, 2, \ldots$.

a) Compute $\mathrm{P}_n(X)$ for $n \leq 6$.

b) Show that

$$\int_{-1}^{1} \mathrm{P}_m(t)\mathrm{P}_n(t) = \begin{cases} 0 & \text{if } m \neq n, \\ \frac{2}{2n+1} & \text{if } m = n. \end{cases}$$

Hint: Use partial integration and the fact that $(t^2-1)^n$ has a zero of multiplicity n at $t = \pm 1$. For the case $m = n$ it may be helpful to recall from Calculus III that $\int_0^{\pi/2} \sin^{2n+1} t \,\mathrm{d}t = \frac{(2n)(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3}$.

c) Show that $\mathrm{P}_n(X)$ has n distinct zeros $\alpha_1^{(n)} < \alpha_2^{(n)} < \cdots < \alpha_n^{(n)}$ in $[-1, 1]$.

d) Suppose $n \in \mathbb{Z}^+$ and $x_1, \ldots, x_n \in \mathbb{R}$ are such that $-1 \leq x_1 < x_2 < \cdots < x_n \leq 1$. Show that there are uniquely determined constants ("weights") $c_1, \ldots, c_n \in \mathbb{R}$ such that

$$\int_{-1}^{1} f(t)\,\mathrm{d}t \approx c_1 f(x_1) + \cdots + c_n f(x_n) \tag{GQ$_n$}$$

is exact for all polynomial functions $f(t)$ of degree $\leq n-1$.

e) Show that for the particular choice $x_i = \alpha_i^{(n)}$, cf. c), Formula (GQ$_n$) is exact for all polynomial functions $f(t)$ of degree $\leq 2n-1$.

Hint: Long division of $f(t)$ by $\mathrm{P}_n(t)$.

f) Determine (GQ$_n$) for $n = 1, 2, 3$ and the special choice $x_i = \alpha_i^{(n)}$.

Due on Thu Nov 18, 7:30 pm

The optional exercises can be handed in one week later.