

Question 1 (ca. 14 marks)

Decide whether the following statements are true or false, and justify your answers.

- a) There exists a solution $y(t)$ of $y' = y^2 - 2$ satisfying $y(0) = 1$ and $y(1) = 2$.
- b) The maximal solution of the initial value problem $y' = y^2 \cos t$, $y(0) = 1$ exists at time $t = 2$.
- c) Every solution $y(t)$ of $t^2 y'' - ty' + y = 0$, $t > 0$, satisfies $\lim_{t \downarrow 0} y(t) = 0$.
- d) The initial value problem $(\cos x)y'' + (\sin x)y' + y = 0$, $y(0) = y'(0) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ which is defined at $x = 1$.
- e) If $\mathbf{A} \in \mathbb{R}^{n \times n}$ satisfies $\mathbf{A}^2 = \mathbf{I}_n$ (the $n \times n$ identity matrix) then $e^{\mathbf{A}t} = (\cosh t)\mathbf{I}_n + (\sinh t)\mathbf{A}$.
- f) Every solution $\mathbf{y}(t)$ of the system $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix} \mathbf{y}$ satisfies $\lim_{t \rightarrow \infty} \mathbf{y}(t) = (0, 0)^T$.
- g) There exist real numbers b_1, b_2, b_3, \dots such that $x - x^2 = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$ for all $x \in [0, 1]$.

Question 2 (ca. 7 marks)

Consider the differential equation

$$(x-1)^2 y'' + 2(x^2-1)y' - 4y = 0. \quad (\text{DE})$$

- a) Show that $x_0 = 1$ is a regular singular point of (DE).
- b) Determine the general solution of (DE) on $(1, \infty)$.
Hint: It turns out that $r_1 - r_2 \in \mathbb{Z}$, but the more complicated machinery developed in the lecture/textbook for this case is not needed.
- c) Using the result of b), discuss the general solution of (DE) on $(-\infty, 1)$ and on \mathbb{R} .

Question 3 (ca. 5 marks)

Consider the ODE

$$y' = t^3 + \frac{2}{t}y - \frac{1}{t}y^2, \quad t > 0. \quad (\text{R})$$

- a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = t^r$.
- b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE, and explain the precise correspondence between solutions of (R) and solutions of the linear ODE.
- c) Solve the linear ODE in b) and use the result to determine the general solution of (R).

Question 4 (ca. 9 marks)

Consider $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 \\ -4 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$.

- a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.
- b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0, 0, 0)^\top$.

Question 5 (ca. 6 marks)

Consider the differential equation

$$(3xy + 2y^2) dx + (3x^2 + 6xy + 3y^2) dy = 0. \quad (\text{DF})$$

- a) Show that $(0, 0)$ is the only singular point of (DF).
- b) Transform (DF) into an exact equation and determine the general solution in implicit form.
- c) Is every point of \mathbb{R}^2 on a unique integral curve of (DF)?

Question 6 (ca. 7 marks)

Determine all real solutions $y(t)$ of

$$2y^{(5)} - y''' + y'' = 1 + t - 2\sin t.$$

Solutions

- 1 a) False. There is the constant solution $y(t) \equiv \sqrt{2}$. By the Intermediate Value Theorem, a solution satisfying $y(0) = 1$, $y(1) = 2$ would attain the value $\sqrt{2}$ at some $t_0 \in (0, 1)$, contradicting the Uniqueness Theorem. 2
- b) False. Solving this separable ODE in the standard way, we obtain $dy/y^2 = \cos t dt$, $-1/y = \sin t + C$, $y = -\frac{1}{\sin t + C}$ as general solution. The initial condition $y(0) = 1$ gives $C = -1$, so that $y(t) = \frac{1}{1 - \sin t}$, $t \in (-\frac{3\pi}{2}, \frac{\pi}{2})$. At $t = 2$ this solution is not defined. 2
- c) True. This is an Euler equation with parameters $\alpha = -1$, $\beta = 1$, indicial equation $r^2 + (\alpha - 1)r + \beta = r^2 - 2r + 1 = (r - 1)^2$ and general solution $y(t) = c_1 t + c_2 t \ln t$ on $(0, \infty)$. Since $\lim_{t \downarrow 0} t = \lim_{t \downarrow 0} t \ln t = 0$, it follows that $\lim_{t \downarrow 0} y(t) = 0$. 2
- d) True. The explicit form of this homogeneous linear 2nd-order ODE is

$$y'' + \frac{\sin x}{\cos x} y' + \frac{1}{\cos x} y = 0.$$

$x_0 = 0$ is an ordinary point and the coefficient functions $p(x) = \frac{\sin x}{\cos x}$, $q(x) = \frac{1}{\cos x}$ are analytic in the disk $|z| < \pi/2$, because the complex cosine function $z \mapsto \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ has the same zeros $z = (2k + 1)\pi/2$, $k \in \mathbb{Z}$, as the real cosine function.

For this note that $\cos z = 0$ is equivalent to $e^{2iz} = -1$ and, writing $z = x + iy$, in turn to $e^{-2y+2ix} = e^{-2y}(\cos(2x) + i \sin(2x)) = -1$. This implies $\sin(2x) = 0$, i.e., $x = m\pi/2$ with $m \in \mathbb{Z}$ and $\cos(2x) = (-1)^m$, which in turn forces that $m = 2k + 1$ is odd and $y = 0$. Thus $z = x = (2k + 1)\pi/2$.

According to the lecture this guarantees that every associated IVP has a power series solution $\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $\geq \pi/2$ and hence is defined at $x = 1$. 2

- e) True. $\Phi(t) = (\cosh t)\mathbf{I}_n + (\sinh t)\mathbf{A}$ satisfies

$$\begin{aligned}\Phi'(t) &= (\sinh t)\mathbf{I}_n + (\cosh t)\mathbf{A}, \\ \mathbf{A}\Phi(t) &= (\cosh t)\mathbf{A} + (\sinh t)\mathbf{A}^2 = (\cosh t)\mathbf{A} + (\sinh t)\mathbf{I}_n = \Phi'(t), \\ \Phi(0) &= (\cosh 0)\mathbf{I}_n + (\sinh 0)\mathbf{A} = \mathbf{I}_n.\end{aligned}$$

These properties characterize the matrix exponential function of \mathbf{A} uniquely, so that we must have $\Phi(t) = e^{\mathbf{A}t}$. 2

- f) True. Denoting the matrix in question by \mathbf{A} , we have $\chi_{\mathbf{A}}(X) = X^2 + 2X + 2 = (X + 1 - i)(X + 1 + i)$, so that the eigenvalues of \mathbf{A} are $\lambda_1 = -1 - i$, $\lambda_2 = -1 + i$. If $\mathbf{v}_1, \mathbf{v}_2$ are corresponding eigenvectors, every solution of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ has the form

$$\mathbf{y}(t) = c_1 e^{(-1+i)t} \mathbf{v}_1 + c_2 e^{(-1-i)t} \mathbf{v}_2 = e^{-t} (c_1 e^{it} \mathbf{v}_1 + c_2 e^{-it} \mathbf{v}_2)$$

for some constants c_1, c_2 and clearly satisfies $\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{0}$.

Alternatively, the system is asymptotically stable since $\text{trace}(\mathbf{A}) = -2 < 0$ and $\det(\mathbf{A}) = 2 > 0$. So there exists $\delta > 0$ such that every solution $\mathbf{y}(t)$ with $|\mathbf{y}(0)| < \delta$ satisfies $\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{0}$. An arbitrary nonzero solution $\mathbf{y}(t)$ can be scaled by the constant $c = \frac{\delta}{2\mathbf{y}(0)}$ to obtain $|c\mathbf{y}(0)| < \delta$. Since $t \mapsto c\mathbf{y}(t)$ is a solution as well, it follows that $\lim_{t \rightarrow \infty} c\mathbf{y}(t) = \mathbf{0}$ and hence $\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{0}$. 2

- g) True. The function $f(x) = x - x^2$, $x \in [0, 1]$, can be extended to an odd 2-periodic function on \mathbb{R} , since $f(0) = f(1) = 0$. The extension is piecewise C^1 (in fact even C^1) and hence represented by its Fourier series everywhere. Since $L = 2$ and the extension is odd, its Fourier series is a pure sine series $\sum_{k=1}^{\infty} b_k \sin(k\pi x)$. +2

$$\sum_1 = 12 + 2$$

- 2 a) The explicit form of (DE) is

$$y'' + \frac{2(x+1)}{x-1} y' - \frac{4}{(x-1)^2} y = 0$$

$$\iff y'' + \left(\frac{4}{x-1} + 2 \right) y' - \frac{4}{(x-1)^2} y = 0$$

One sees that the coefficient of y' has a pole of order 1 at $x_0 = 1$, and the coefficient of y has a pole of order 2 at $x_0 = 1$. This implies that $x_0 = 1$ is a regular singular point. 1

- b) From a) we have, using the notation in the lecture, that $p_0 = 4$, $p_1 = 2$, $q_0 = -4$ and all other coefficients p_i , q_i are zero. \implies The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 + 3r - 4 = (r - 1)(r + 4) = 0.$$

\implies The exponents at the singularity $x_0 = 1$ are $r_1 = 1$, $r_2 = -4$. We have $r_1 - r_2 = 5 \in \mathbb{Z}$, but we will show that nevertheless two fundamental solutions y_1 , y_2 of the form

$$y_1(x) = (x-1) \sum_{n=0}^{\infty} a_n (x-1)^n, \quad y_2(x) = (x-1)^{-4} \sum_{n=0}^{\infty} b_n (x-1)^n \quad \text{span style="float: right; border: 1px solid black; padding: 2px;">1}$$

exist. For this it will be convenient to use the abbreviation $t = x - 1$, which turns (DE) into

$$t^2 y'' + 2t(t+2)y' - 4y = t^2 y'' + (2t^2 + 4t)y' - 4y = 0.$$

First we determine $y_1(x)$. We have

$$y_1 = \sum_{n=0}^{\infty} a_n t^{n+1},$$

$$t^2 y_1''(x) = \sum_{n=0}^{\infty} (n+1) n a_n t^{n+1},$$

$$t^2 y_1'(x) = \sum_{n=0}^{\infty} (n+1) a_n t^{n+2} = \sum_{n=1}^{\infty} n a_{n-1} t^{n+1},$$

$$t y_1'(x) = \sum_{n=0}^{\infty} (n+1) a_n t^{n+1}.$$

Substituting these into (DE) gives, setting $a_{-1} = 0$,

$$\sum_{n=0}^{\infty} [(n+1) n a_n + 2n a_{n-1} + 4(n+1) a_n - 4a_n] t^{n+1} = \sum_{n=0}^{\infty} [(n+5) n a_n + 2n a_{n-1}] t^{n+1} = 0.$$

Equating coefficients gives the recurrence relation

$$a_n = -\frac{2}{n+5}a_{n-1} \quad \text{for } n = 1, 2, \dots$$

The coefficient a_0 can be chosen freely. Using the normalization $a_0 = 1$ we obtain

$$\begin{aligned} y_1(x) &= (x-1) - \frac{2}{6}(x-1)^2 + \frac{2^2}{6 \cdot 7}(x-1)^3 - \frac{2^3}{6 \cdot 7 \cdot 8}(x-1)^4 \pm \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{6 \cdot 7 \cdot 8 \cdots (n+5)} (x-1)^{n+1}. \end{aligned} \quad \boxed{2}$$

For the determination of $y_2(x)$ we repeat the computation with exponents decreased by 5. The result is

$$\begin{aligned} \sum_{n=0}^{\infty} [(n-4)(n-5)b_n + 2(n-5)b_{n-1} + 4(n-4)b_n - 4b_n] t^{n-4} = \\ = \sum_{n=0}^{\infty} [n(n-5)b_n + 2(n-5)b_{n-1}] t^{n-4} = 0. \end{aligned}$$

Here we obtain the recurrence relation

$$b_n = -\frac{2}{n} b_{n-1} \quad \text{for } n = 1, 2, 3, 4, 6, \dots$$

and, setting $b_0 = 1$, $b_5 = 0$

$$y_2(x) = (x-1)^{-4} - \frac{2}{1!}(x-1)^{-3} + \frac{2^2}{2!}(x-1)^{-2} - \frac{2^3}{3!}(x-1)^{-1} + \frac{2^4}{4!}, \quad \boxed{1\frac{1}{2}}$$

which is a finite sum!

Note: If instead we stipulate the recurrence relation $b_n = (-2/n)b_{n-1}$ for all n , then we obtain

$$\tilde{y}_2(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!} (x-1)^{n-4} = (x-1)^{-4} e^{-2(x-1)} = y_2(x) - \frac{2^5}{5!} y_1(x).$$

Alternative solution: We use the general recurrence relation for the functions $a_n(r)$, viz. $a_0(r) = 1$ and

$$\begin{aligned} a_n(r) &= -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}] a_{n-1}(r) \\ &= -\frac{1}{(r+n-1)(r+n+4)} [(r+n-1)2] a_{n-1}(r) = -\frac{2a_{n-1}(r)}{r+n+4}. \end{aligned}$$

Thus the coefficients $a_n(1)$ of $y_1(x)$ satisfy the recurrence relation $a_n(1) = -\frac{2a_{n-1}(1)}{n+5}$ (the same as for a_n above) and the coefficients $a_n(-4)$ of $y_1(x)$ satisfy the recurrence relation $a_n(-4) = -\frac{2a_{n-1}(-4)}{n}$ (the same as for b_n above, except that it holds also for

$n = 5$). The rest of the computation remains the same and leads to $y_1(x)$ and the nonterminating series solution $\tilde{y}_2(x)$.

The general (real) solution on $(1, \infty)$ is then $y(x) = c_1 y_1(x) + c_2 y_2(x)$, $c_1, c_2 \in \mathbb{R}$. 1/2

That solutions are defined on the whole of $(1, \infty)$, is guaranteed by the analyticity of $p(x)$, $q(x)$ in $\mathbb{C} \setminus \{1\}$, but follows also readily from the easily established fact that the radius of convergence of both power series is ∞ .

- c) The solution on $(-\infty, 1)$ is exactly the same (i.e., the formulas for $y_1(x)$, $y_2(x)$ remain unchanged), since according to the general theory we have to replace $(x - 1)^r$ by $|x - 1|^r$. But this leaves $y_2(x)$ unchanged and changes only the sign of $y_1(x)$. The linear span of $y_1(x)$, $y_2(x)$ remains the same. 1

Since $y_1(x)$ is analytic everywhere and $\lim_{x \rightarrow 1} y_2(x) = \infty$, the solution on \mathbb{R} is $y(x) = c_1 y_1(x)$, $c_1 \in \mathbb{R}$. +1

$$\sum_2 = 7 + 1$$

- 3 a) Substituting $y_1(t) = t^r$ into the ODE gives

$$r t^{r-1} = t^3 + 2 t^{r-1} - t^{2r-1},$$

so that we can take $r = 2$ and $y_1(t) = t^2$. 1

- b) $y = t^2 + 1/z \implies y' = 2t - z'/z^2$
Substituting this into (R) gives

$$\begin{aligned} 2t - \frac{z'}{z^2} &= t^3 + \frac{2}{t} \left(t^2 + \frac{1}{z} \right) - \frac{1}{t} \left(t^2 + \frac{1}{z} \right)^2 \\ &= t^3 + 2t + \frac{2}{tz} - t^3 - \frac{2t}{z} - \frac{1}{tz^2} \\ \iff z' &= \left(2t - \frac{2}{t} \right) z + \frac{1}{t}. \end{aligned}$$

This is of the form $z' = a(t)z + b(t)$, hence first-order (inhomogeneous) linear. 2
Since $y = t^2 + 1/z$ is equivalent to $z = 1/(y - t^2)$ and $y(t) \neq t^2$ for all t in the domain of y (by the Uniqueness Theorem), this defines a one-to-one correspondence between solutions of (R) different from y_1 and solutions of $z' = \left(2t - \frac{2}{t} \right) z + 1/t$. 1/2

- c) The general solution of $z' = \left(2t - \frac{2}{t} \right) z$ is

$$z(t) = c \exp \left(\int 2t - \frac{2}{t} dt \right) = c \exp (t^2 - 2 \ln t) = c e^{t^2}/t^2, \quad c \in \mathbb{R} \quad 1$$

Variation of parameters then yields a particular solution z_p of $z' = \left(2t - \frac{2}{t} \right) z + 1/t$:

$$z_p(t) = e^{t^2}/t^2 \int t^2 e^{-t^2} (1/t) dt = e^{t^2}/t^2 \left(-\frac{1}{2} e^{-t^2} \right) = -\frac{1}{2t^2}. \quad 1$$

\Rightarrow The general solution of $z' = (2t - \frac{2}{t})z + 1/t$ is

$$z(t) = -\frac{1}{2t^2} + ce^{t^2}/t^2 = \frac{2ce^{t^2} - 1}{2t^2}, \quad c \in \mathbb{R}.$$

\Rightarrow The general solution of (R) is

$$y_c(t) = t^2 + \frac{2t^2}{2ce^{t^2} - 1} = t^2 \frac{2ce^{t^2} + 1}{2ce^{t^2} - 1}, \quad c \in \mathbb{R} \cup \{\infty\}, \quad \boxed{1\frac{1}{2}}$$

or, slightly simplified,

$$y_c(t) = t^2 \frac{ce^{t^2} + 1}{ce^{t^2} - 1}, \quad c \in \mathbb{R} \cup \{\infty\}.$$

For $c = \infty$ we obtain the special solution $y(t) = t^2$, which would otherwise be missing.

The (maximal) solutions $y_\infty(t) = t^2$, $y_0(t) = -t^2$, and $y_c(t)$ for $c < 0$ are defined on $(0, \infty)$. For $c \geq 1$ the same is true, since then $ce^{t^2} > 1$ for all $t > 0$.

“Solutions” $y_c(t)$ with $0 < c < 1$ do, strictly speaking, not form solutions of their own (because their maximal domain is not an interval) but give rise to two maximal solutions (“branches”) $y_c^{(1)}(t)$, $y_c^{(2)}(t)$ with domains $(0, \sqrt{-\ln c})$ and $(\sqrt{-\ln c}, \infty)$, respectively. $\boxed{+1}$

$$\sum_3 = 7 + 1$$

4 The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} \chi_{\mathbf{A}}(X) &= \begin{vmatrix} X-1 & 2 & 2 \\ 4 & X+1 & -2 \\ 0 & 0 & X+3 \end{vmatrix} = (X+3)[(X-1)(X+1) - 4 \cdot 2] = (X+3)(X^2 - 9) \\ &= (X-3)(X+3)^2. \end{aligned}$$

\Rightarrow The eigenvalues of \mathbf{A} are $\lambda_1 = 3$ with algebraic multiplicity 1 and $\lambda_2 = -3$ with algebraic multiplicity 2. $\boxed{2}$

$\lambda_1 = 3$:

$$\mathbf{A} - 3\mathbf{I} = \begin{pmatrix} -2 & -2 & -2 \\ -4 & -4 & 2 \\ 0 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

\Rightarrow The eigenspace corresponding to $\lambda_1 = 3$ is generated by $\mathbf{v}_1 = (1, -1, 0)^\top$.

$\lambda_2 = -3$:

$$\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 4 & -2 & -2 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

\Rightarrow The eigenspace corresponding to $\lambda_2 = -3$ is generated by $\mathbf{v}_2 = (1, 2, 0)^\top$ and $\mathbf{v}_3 = (1, 0, 2)^\top$.

A fundamental system of solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ is therefore

$$\mathbf{y}_1(t) = e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \quad \boxed{3}$$

\mathbf{A} is invertible, since 0 is not an eigenvalue of \mathbf{A} .

\Rightarrow The system $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ has the constant solution $\mathbf{y}(t) \equiv -\mathbf{A}^{-1}\mathbf{b}$, which is found by solving the system $\mathbf{A}\mathbf{x} = -\mathbf{b}$.

$$\left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ -4 & -1 & 2 & 0 \\ 0 & 0 & -3 & -3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & -9 & -6 & 0 \\ 0 & 0 & -3 & -3 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & -2 & -2 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

$$\Rightarrow x_3 = 1, x_2 = -2x_3/3 = -2/3, x_1 = 2x_2 + 2x_3 = 2/3$$

$$\Rightarrow \text{The constant solution is } \mathbf{y}(t) \equiv \left(\frac{2}{3}, -\frac{2}{3}, 1\right)^T.$$

\Rightarrow The general (real) solution of $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$ is

$$\mathbf{y}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + c_3 e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix}, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

Fitting the initial conditions gives for c_1, c_2, c_3 the linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{2}{3} \\ -1 & 2 & 0 & \frac{2}{3} \\ 0 & 0 & 2 & -1 \end{array} \right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & -\frac{2}{3} \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{array} \right)$$

$$\Rightarrow c_3 = -1/2, c_2 = -c_3/3 = 1/6, c_1 = -2/3 - c_2 - c_3 = -1/3$$

\Rightarrow The solution of the IVP is

$$\begin{aligned} \mathbf{y}(t) &= -\frac{1}{3} e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{6} e^{-3t} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} - \frac{1}{2} e^{-3t} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{3} - \frac{1}{3}e^{3t} - \frac{1}{3}e^{-3t} \\ -\frac{2}{3} + \frac{1}{3}e^{3t} + \frac{1}{3}e^{-3t} \\ 1 - e^{-3t} \end{pmatrix}. \end{aligned}$$

$$\sum_4 = 8$$

5 a) $M(x, y) = 3xy + 2y^2 = y(3x + 2y)$, $N(x, y) = 3x^2 + 6xy + 3y^2 = 3(x + y)^2$ have no common zero except $(0, 0)$. $\Rightarrow (0, 0)$ is the only singular point. 1

b) We have

$$M_y - N_x = 3x + 4y - (6x + 6y) = -3x - 2y = M(-1/y).$$

Thus $(M_y - N_x)/M$ depends only on y , and there is an integrating factor of the form $g(y)$.

The integrability condition $(gM)_y = (gN)_x$ then becomes $g'M + gM_y = gN_x$, i.e.,

$$g' = \frac{g(N_x - M_y)}{M} = \frac{g}{y}.$$

The solution of this ODE is $g(y) = cy$, so that we can take $g(y) = y$. 2

\Rightarrow On $\mathbb{R}^2 \setminus x$ -axis the ODE $(3xy + 2y^2) dx + (3x^2 + 6xy + 3y^2) dy = 0$ is equivalent to the exact ODE

$$(3xy^2 + 2y^3) dx + (3x^2y + 6xy^2 + 3y^3) dy = 0. \quad \frac{1}{2}$$

An antiderivative f of the corresponding exact differential is determined in the usual way by “partial integration” with respect to x , say.

$$f(x, y) = \frac{3}{2}x^2y^2 + 2xy^3 + g(y),$$

$$f_y(x, y) = 3x^2y + 6xy^2 + g'(y) \stackrel{!}{=} 3x^2y + 6xy^2 + 3y^3$$

$$\implies g'(y) = 3y^3 \implies g(y) = \frac{3}{4}y^4 + C \implies f(x, y) = \frac{3}{2}x^2y^2 + 2xy^3 + \frac{3}{4}y^4 + C$$

The general implicit solution of the exact ODE is then given by (in slightly simplified form and with a different C)

$$6x^2y^2 + 8xy^3 + 3y^4 = C, \quad C \in \mathbb{R}. \quad \boxed{2}$$

Solutions with $C < 0$ don't exist and for $C = 0$ the x -axis is obtained, since $6x^2y^2 + 8xy^3 + 3y^4 = y^2(6x^2 + 8xy + 3y^2)$ and the quadratic has discriminant $8^2 - 4 \cdot 6 \cdot 3 = -8 < 0$.

Since the x -axis (equivalently, the function $y(x) \equiv 0$) is a solution of (DF), multiplication by y hasn't introduced any new solution, and $6x^2y^2 + 8xy^3 + 3y^4 = C$, $C \geq 0$ solves (DF) as well. $\boxed{\frac{1}{2}}$

- c) Yes. This is implicit in the preceding discussion. Intersection points of integral curves must be singular, so that the only candidate for such a point is the origin. But the corresponding contour of f , the 0-contour, consists of a single integral curve, viz. the x -axis. $\boxed{+1}$

$$\sum_5 = 6 + 1$$

6 The characteristic polynomial is

$$\begin{aligned} a(X) &= 2X^5 - X^3 + X^2 = X^2(2X^3 - X + 1) \\ &= X^2(X + 1)(2X^2 - 2X + 1) && \text{(since } -1 \text{ is a root)} \\ &= 2X^2(X + 1)(X^2 - X + \tfrac{1}{2}) \\ &= 2X^2(X + 1) \left(X - \frac{1+i}{2}\right) \left(X - \frac{1-i}{2}\right) \end{aligned}$$

with zeros $\lambda_1 = 0$ of multiplicity 2, $\lambda_2 = -1$, $\lambda_3 = (1+i)/2$, $\lambda_4 = (1-i)/2$.

\implies A complex fundamental system of solutions of the associated homogeneous equation is $1, t, e^{-t}, e^{(1+i)t/2}, e^{(1-i)t/2}$, and the corresponding real fundamental system is

$$1, \quad t, \quad e^{-t}, \quad e^{t/2} \cos(t/2), \quad e^{t/2} \sin(t/2). \quad \boxed{2}$$

In order to obtain a particular solution $y_p(t)$ of the inhomogeneous equation, we solve the two systems $a(D)y_i = b_i(t)$ for $b_1(t) = 1 + t$, $b_2(t) = e^{it}$. Superposition and extraction of the real part then yields the particular solution $y_p(t) = y_1(t) - 2\operatorname{Im} y_2(t)$. $\boxed{1}$

- (1) Since $\mu = 0$ is a zero of multiplicity 2 of $a(X)$, the proper Ansatz in this case is $y_1(t) = c_0 t^2 + c_1 t^3$. Substituting it into the ODE we get

$$2 \cdot 0 - 6c_1 + 2c_0 + 6c_1 t = 1 + t.$$

$$\implies c_1 = 1/6, \quad c_0 = 1, \quad \text{and } y_1(t) = t^2 + t^3/6. \quad \boxed{1\frac{1}{2}}$$

- (2) Since $a(D)e^{it} = a(i)e^{it}$ and $a(i) \neq 0$, we can take $y_2(t) = \frac{1}{a(i)}e^{it}$. Since $a(i) = 2i^5 - i^3 + i^2 = 3i - 1$, we obtain

$$y_2(t) = \frac{1}{-1 + 3i} e^{it} = \frac{-1 - 3i}{10} (\cos t + i \sin t),$$

$$\operatorname{Im} y_2(t) = -\frac{3}{10} \cos t - \frac{1}{10} \sin t. \quad \boxed{1\frac{1}{2}}$$

It follows that a particular real solution of the inhomogeneous equation is $y_p(t) = t^2 + \frac{1}{6}t^3 + \frac{3}{5}\cos t + \frac{1}{5}\sin t$ and that the general real solution of the inhomogeneous equation is

$$y(t) = t^2 + \frac{1}{6}t^3 + \frac{3}{5}\cos t + \frac{1}{5}\sin t + c_1 + c_2 t + c_3 e^{-t} + c_4 e^{t/2} \cos(t/2) + c_5 e^{t/2} \sin(t/2),$$

$$c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}. \quad \boxed{1}$$

$$\sum_6 = 7$$

$$\sum = 47 + 5$$

Final Exam