Calculus III (Math 241)

H21 Do Exercise 52 in [Ste16], Ch. 13.1.

H22 Do Exercises 32 and 56 in [Ste16], Ch. 13.2.

H23 Do Exercises 4 and 12 in [Ste16], Ch. 13.3.

H24 (Continuation of Exercise W16 on Worksheet 4)

Determine at which time t^* the particles are closest to each other, and the corresponding minimal distance d^*.

Hint: Work with the squared distance of $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, which is easier to handle, and use Calculus I. The quantities t^*, d^* may be computed numerically.

H25 (Continuation of Exercise W18)

a) As shown in W18, $R(\phi)^{\mathsf{T}}R(\phi) = S(\phi)^{\mathsf{T}}S(\phi) = \mathbf{I}_2$. Show that, conversely, a matrix $\mathbf{A} \in \mathbb{R}^{2\times 2}$ satisfying $\mathbf{A}^{\mathsf{T}}\mathbf{A} = \mathbf{I}_2$ is necessarily a rotation or reflection matrix.

b) Express the matrix products $R(\phi_1)R(\phi_2)$, $S(\phi_1)S(\phi_2)$, $R(\phi_1)S(\phi_2)$, $S(\phi_1)R(\phi_2)$, which describe the compositions of the corresponding rotations/reflections, as either $R(\phi)$ or $S(\phi)$ for some angle ϕ.

Hint: $S(\phi)$ can be expressed in terms of $R(\phi)$ and $S(0)$, simplifying the computations.

c) Find all rotation and reflection matrices that leave the quadrangle in $\mathbb{R}^2$ with vertices $(\pm 1, 0)^{\mathsf{T}}$, $(0, \pm 1)^{\mathsf{T}}$ invariant.

d) Repeat c) for the regular hexagon (6-gon) in $\mathbb{R}^2$ with center $(0,0)$ and one vertex at $(1,0)$.

H26 Show that the rank of $\mathbf{A} \in \mathbb{R}^{m\times n}$ is the largest integer $k \geq 0$ such that $\mathbf{A}$ has an invertible $k \times k$ submatrix.

Note: "*Submatrix of* $\mathbf{A}$" refers to any matrix obtained from $\mathbf{A}$ by deleting zero or more rows and/or columns. The remaining rows/columns are re-indexed but their relative order is preserved.

H27 *Optional exercise*

a) For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ show: $(\mathbf{A}+\mathbf{B})^{\mathsf{T}} = \mathbf{A}^{\mathsf{T}} + \mathbf{B}^{\mathsf{T}}$.

b) For $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times p}$ show: $(\mathbf{AB})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$.

c) For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ show: If $\mathbf{AB} = \mathbf{I}_n$ then $\mathbf{BA} = \mathbf{I}_n$ as well.

Hint: $\mathbf{AB} = \mathbf{I}_n$ implies $\mathbf{ABA} = \mathbf{A}$.

d) For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ show: If $\mathbf{AB}$ is invertible then so are $\mathbf{A}$ and $\mathbf{B}$. If applicable, express the inverses of $\mathbf{A}$, $\mathbf{B}$ in terms of $\mathbf{C} = (\mathbf{AB})^{-1}$.

e) True or false?

i) If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$ are symmetric then so are $\mathbf{A}+\mathbf{B}$ and $\mathbf{AB}$.

ii) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric and invertible then $\mathbf{A}^{-1}$ is symmetric.

H28 *Optional exercise*

a) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ have rank m and the same row space. Show that there exists an invertible matrix $\mathbf{S} \in \mathbb{R}^{m\times m}$ such that $\mathbf{B} = \mathbf{SA}$.

Hint: Show first that there exist matrices $\mathbf{S}, \mathbf{T} \in \mathbb{R}^{m\times m}$, not necessarily invertible, such that $\mathbf{B} = \mathbf{SA}$, $\mathbf{A} = \mathbf{TB}$. Then use the assumption on the rank to show that $\mathbf{ST} = \mathbf{TS} = \mathbf{I}_m$.

b) Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m\times n}$ are related by $\mathbf{B} = \mathbf{SA}$ for some invertible matrix $\mathbf{S} \in \mathbb{R}^{m\times m}$. Show that $\mathbf{A}$ can be transformed into $\mathbf{B}$ using a sequence of elementary row operations.

Hint: Apply Gaussian elimination to $\mathbf{S}$.

H29 *Optional exercise*

This exercise outlines a proof of the multiplication formula for determinants,
$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$.

a) Prove the formula in the case $\mathrm{rk}(\mathbf{A}) < n$.

b) Considering $\mathbf{A}$ with $\mathrm{rk}(\mathbf{A}) = n$ as fixed, show that there exists a constant $c = c(\mathbf{A}) \in \mathbb{R} \setminus \{0\}$ such that $\det(\mathbf{AX}) = c\det(\mathbf{X})$ for all $\mathbf{X} \in \mathbb{R}^{n\times n}$.

Hint: The sequence of elementary row operations used to compute $\mathbf{A}^{-1}$ transforms $\mathbf{AX}$ into $\mathbf{X}$.

c) Conclude from b) that $c = \det(\mathbf{A})$.

Due on Wed Oct 20, 6 pm

The optional exercises can be handed in until Wed Oct 27, 6 pm.

Exercise H23 requires the concept of arc length, which will be discussed in the lecture on Fri Oct 15.