

**Question 1** (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

- a) There exists a solution  $y(t)$  of  $y' = 2y - y^2$  satisfying  $y(0) = y(1) = 1$ .
- b) The maximal solution of the initial value problem  $y' = y^2 - t$ ,  $y(0) = \frac{1}{2}$  exists at time  $t = 2021$ .
- c) Every solution  $y: (0, \infty) \rightarrow \mathbb{R}$  of  $t^2 y'' + 3t y' + 2y = 0$  has infinitely many zeros.
- d) The initial value problem  $(x^2 + 4)y'' + (x + 4)y' - 4y = 0$ ,  $y(1) = y'(1) = 1$  has a power series solution  $y(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$  which is defined at  $x = 3$ .
- e) Suppose  $\mathbf{A} \in \mathbb{R}^{2 \times 2}$  satisfies  $\mathbf{A}^3 = \mathbf{I}$  (the  $2 \times 2$  identity matrix), but  $\mathbf{A} \neq \mathbf{I}$ . Then every solution  $\mathbf{y}(t)$  of the linear system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$  must satisfy  $\lim_{t \rightarrow \infty} \mathbf{y}(t) = (0, 0)^T$ .
- f) Suppose  $f, g: (-1, 1) \rightarrow \mathbb{R}$  are  $C^1$ -functions. Then the IVP  $y' = f(t)g(y)$ ,  $y(0) = 0$  has a solution  $y(t)$  that is defined for all  $t \in (-1, 1)$ .

**Question 2** (ca. 10 marks)

Consider the differential equation

$$2x^2 y'' + x(1-x)y' - 6y = 0. \quad (\text{DE})$$

- a) Verify that  $x_0 = 0$  is a regular singular point of (DE).
- b) Determine the general solution of (DE) on  $(0, \infty)$ .
- c) Using the result of b), state the general solution of (DE) on  $(-\infty, 0)$  and on  $\mathbb{R}$ .

**Question 3** (ca. 7 marks)

Consider the ODE

$$y' = y^2 + \frac{5}{t}y + \frac{5}{t^2}, \quad t > 0. \quad (\text{R})$$

- a) Show that there exists a solution  $y_1(t)$  of the form  $y_1(t) = ct^r$  with constants  $c, r$ .
- b) Show that the substitution  $y = y_1 + 1/z$  transforms (R) into a first-order linear ODE.
- c) Using b), determine all maximal solutions of (R) and their domain.

**Question 4** (ca. 6 marks)

For the matrix  $\mathbf{A} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$  determine the general solution of

the linear system  $\mathbf{y}' = \mathbf{A}\mathbf{y}$ .

**Question 5** (ca. 7 marks)

Consider the differential equation

$$x(3y^2 - 1) dx + y dy = 0. \quad (\text{DF})$$

- a) Determine the general solution of (DF) in implicit form.
- b) Determine the maximal solution  $y(x)$  satisfying  $y(1) = \frac{1}{3}$  and its domain.  
*Hint:*  $\ln\left(\frac{3}{2}\right) \approx 0.4$
- c) Is every point of  $\mathbb{R}^2$  on a unique integral curve of (DF)?

**Question 6** (ca. 8 marks)

- a) Determine a real fundamental system of solutions of

$$y^{(5)} + 4y^{(4)} + 24y''' + 40y'' + 100y' = 0.$$

*Hint:* The characteristic polynomial is divisible by the square of a quadratic polynomial.

- b) Determine the general real solution of

$$y^{(5)} + 4y^{(4)} + 24y''' + 40y'' + 100y' = 200t - e^{-t}.$$

- c) Find the Laplace transform  $Y(s)$  of the solution of the ODE in b) with initial values  $y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 0$ .

## Solutions

- 1 a) False. Since  $y' = y(2 - y)$  is positive if  $0 < y < 2$ , and any solution starting in the strip  $0 < y < 2$  is confined to this strip (e.g., because the strip is bounded by the constant solutions  $y(t) \equiv 0$  and  $y(t) \equiv 2$ ), the solution with  $y(0) = 1$  must be strictly increasing and hence satisfy  $y(1) > 1$ . 2

Alternatively one can argue that this ODE is of the form  $y' = ay^2 + by + c$  and the corresponding canonical form, viz.  $z' = -z^2 + 1$ , is the same as for the Logistic Equation (cf. our discussion in the lecture and H16 of HW2). Hence solutions starting between the two equilibrium solutions must be monotonically increasing.

- b) True. For  $t \geq 0$  we have  $-t \leq y^2 - t \leq y^2$ , so that the solution  $\phi_1(t)$  of  $y' = y^2 \wedge y(0) = \frac{1}{2}$  is an upper bound for  $y(t)$  and the solution  $\phi_2(t)$  of  $y' = -t \wedge y(0) = \frac{1}{2}$  is a lower bound for  $y(t)$ . Solving the two auxiliary IVP's gives  $\phi_1(t) = 1/(2 - t)$ ,  $\phi_2(t) = (1 - t^2)/2$ . Hence  $y(t)$  is defined at least on  $[0, 2)$ . 1

Since  $y(1) \leq \phi_1(1) = 1$  and  $y^2 - t \leq y^2 - 1$  for  $t \geq 1$ , the solution  $\phi_3(t)$  of  $y' = y^2 - 1 \wedge y(1) = 1$  is an upper bound for  $y(t)$ . Solving the auxiliary IVP gives  $\phi_3(t) \equiv 1$ . Since  $\phi_2(t)$  and  $\phi_3(t)$  are defined on  $[1, \infty)$ , the same must be true of  $y(t)$ . +1

- c) True. This is an Euler equation with parameters  $\alpha = 3$ ,  $\beta = 2$ , indicial equation  $r^2 + (\alpha - 1)r + \beta = r^2 + 2r + 2 = (r + 1 + i)(r + 1 - i) = 0$ , complex fundamental system  $t^{-1 \pm i}$ , and general real solution  $y(t) = t^{-1} (c_1 \cos \ln t + c_2 \sin \ln t)$  on  $(0, \infty)$ . 1

Since  $c_1 \cos x + c_2 \sin x = A \sin(x - \alpha)$  for some  $A \geq 0$ ,  $\alpha \in [0, 2\pi)$ , we can write the solution in the form  $y(t) = A t^{-1} \sin(\ln t - \alpha)$  and conclude that  $t_k = e^{k\pi + \alpha}$ ,  $k = 0, 1, 2, \dots$ , are solutions. 1

- d) True. The explicit form of this homogeneous linear 2nd-order ODE is

$$y'' + \frac{x+4}{x^2+4} y' - \frac{4}{x^2+4} y = 0.$$

$x_0 = 1$  is an ordinary point and the coefficient functions  $p(x) = \frac{x+4}{x^2+4}$ ,  $q(x) = -\frac{4}{x^2+4}$  are analytic (even rational) in the disk  $|z - 1| < \sqrt{5}$  (the disk with center 1 that has the singularities  $\pm 2i$  of  $p(x)$  and  $q(x)$  on its boundary). Hence (referring to a theorem proved in the lecture) there exists a power series solution  $y(x) = \sum_{n=0}^{\infty} a_n (x - 1)^n$  of the IVP with radius of convergence  $\geq \sqrt{5}$ . Since  $\sqrt{5} > 2$ , this solution is defined at  $x = 3$ . 2

- e) True. If  $\mathbf{v} \in \mathbb{C}^2$  is an eigenvector of  $\mathbf{A}$  with eigenvalue  $\lambda \in \mathbb{C}$ , we have  $\mathbf{v} = \mathbf{A}^3 \mathbf{v} = \lambda^3 \mathbf{v}$ , and hence  $\lambda^3 = 1$ . Thus  $\lambda \in \left\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\right\}$ . If  $\mathbf{A}$  has an eigenvalue  $\neq 1$ , the eigenvalues must be  $\lambda_{1/2} = \frac{-1 \pm i\sqrt{3}}{2}$  (since complex eigenvalues of real matrices occur in conjugate pairs). Since their real part is  $-\frac{1}{2} < 0$ , the matrix  $\mathbf{A}$  is asymptotically stable. If  $\lambda_1 = \lambda_2 = 1$  then  $(\mathbf{A} - \mathbf{I})^2 = \mathbf{0}$  (by the Cayley-Hamilton Theorem), which gives  $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$ ,  $\mathbf{I} = \mathbf{A}^3 = 2\mathbf{A}^2 - \mathbf{A} = 3\mathbf{A} - 2\mathbf{I}$ , and hence  $\mathbf{A} = \mathbf{I}$ ; contradiction. +2

- f) False. Separable ODE's  $y' = f(t)g(y)$  may have maximal solutions with strictly smaller domain than  $f(t)$ . 1

This can happen even for autonomous ODE's, and we can take  $y' = 2(y+1)^2$ , i.e.,  $f(t) = 1$ ,  $g(y) = 2(y+1)^2$  as counterexample. The general solution of this ODE is  $y(t) = \frac{1}{C-2t} - 1$ ,  $C \in \mathbb{R}$ , and  $y(0) = 0$  gives  $C = 1$ . But  $y(t) = \frac{1}{1-2t} - 1$  is not defined for  $t \in [\frac{1}{2}, 1)$ . +2

$$\sum_1 = 8 + 5$$

- 2 a) The explicit form of (DE) is

$$y'' + \frac{1-x}{2x} y' - \frac{3}{x^2} y = 0$$

Using the notation of the lecture/textbook,  $p(x) = \frac{1-x}{2x} = \frac{1}{2}x^{-1} - \frac{1}{2}$  has a pole of order 1 at 0, and  $q(x) = -\frac{3}{x^2}$  has a pole of order 2 at 0. This implies that 0 is a regular singular point of (DE). 1

Alternatively, use that the limits defining  $p_0, q_0$  below are finite.

- b) From a) we have  $p_0 = \lim_{x \rightarrow 0} x p(x) = 1/2$ ,  $q_0 = \lim_{x \rightarrow 0} x^2 q(x) = -3$ .  
 $\Rightarrow$  The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \frac{1}{2}r - 3 = (r - 2)(r + 3/2) = 0.$$

$\Rightarrow$  The exponents at the singularity  $x_0 = 0$  are  $r_1 = 2$ ,  $r_2 = -3/2$ . Since  $r_1 - r_2 \notin \mathbb{Z}$ , there exist two fundamental solutions  $y_1, y_2$  of the form

$$\begin{aligned} y_1(x) &= x^2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+2}, \\ y_2(x) &= x^{-3/2} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} b_n x^{n-3/2} \end{aligned} \quad \text{1}$$

with normalization  $a_0 = b_0 = 1$ .

First we determine the analytic solution  $y_1(x)$ . We have

$$\begin{aligned} 0 &= 2x^2 y_1'' + x(1-x)y_1' - 6y_1 \\ &= 2x^2 \sum_{n=0}^{\infty} (n+2)(n+1)a_n x^n + (x-x^2) \sum_{n=0}^{\infty} (n+2)a_n x^{n+1} - 6 \sum_{n=0}^{\infty} a_n x^{n+2} \\ &= \sum_{n=0}^{\infty} [2(n+2)(n+1) + n+2-6] a_n x^{n+2} - \sum_{n=0}^{\infty} (n+2)a_n x^{n+3} \\ &= \sum_{n=0}^{\infty} (2n^2 + 7n)a_n x^{n+2} - \sum_{n=1}^{\infty} (n+1)a_{n-1} x^{n+2} \\ &= \sum_{n=1}^{\infty} [n(2n+7)a_n - (n+1)a_{n-1}] x^{n+2}. \end{aligned}$$

Equating coefficients gives the recurrence relation

$$a_n = \frac{n+1}{n(2n+7)} a_{n-1} \quad \text{for } n = 1, 2, 3, \dots \quad [1]$$

and with  $a_0 = 1$  further  $a_n = \frac{(n+1)!}{n! \cdot 9 \cdot 11 \cdot 13 \cdots (2n+7)} = \frac{n+1}{9 \cdot 11 \cdot 13 \cdots (2n+7)}$ .

$$\begin{aligned} \Rightarrow y_1(x) &= \sum_{n=0}^{\infty} \frac{n+1}{9 \cdot 11 \cdot 13 \cdots (2n+7)} x^{n+2} \\ &= x^2 + \frac{2}{9} x^3 + \frac{3}{9 \cdot 11} x^4 + \frac{4}{9 \cdot 11 \cdot 13} x^5 + \frac{5}{9 \cdot 11 \cdot 13 \cdot 15} x^7 + \dots \end{aligned} \quad [1\frac{1}{2}]$$

For the determination of  $y_2(x)$  we repeat the process with exponents decreased by 3.5:

$$\begin{aligned} 0 &= 2x^2 y_2'' + x(1-x)y_2' - 6y_2 \\ &= 2x^2 \sum_{n=0}^{\infty} \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) b_n x^{n-7/2} + (x-x^2) \sum_{n=0}^{\infty} \left(n - \frac{3}{2}\right) b_n x^{n-5/2} - 6 \sum_{n=0}^{\infty} b_n x^{n-3/2} \\ &= \sum_{n=0}^{\infty} \left[2 \left(n - \frac{3}{2}\right) \left(n - \frac{5}{2}\right) + n - \frac{3}{2} - 6\right] b_n x^{n-3/2} - \sum_{n=0}^{\infty} \left(n - \frac{3}{2}\right) b_n x^{n-1/2} \\ &= \sum_{n=0}^{\infty} (2n^2 - 7n) b_n x^{n-3/2} - \sum_{n=1}^{\infty} \left(n - \frac{5}{2}\right) b_{n-1} x^{n-3/2} \\ &= \sum_{n=1}^{\infty} \left[n(2n-7)b_n - \left(n - \frac{5}{2}\right) b_{n-1}\right] x^{n-3/2}. \end{aligned}$$

Here we obtain the recurrence relation

$$b_n = \frac{n-5/2}{n(2n-7)} b_{n-1} = \frac{2n-5}{2n(2n-7)} b_{n-1} \quad \text{for } n = 1, 2, 3, \dots, \quad [1]$$

and with  $b_0 = 1$  further  $b_n = \frac{(-3)(-1)\cdots(2n-5)}{2 \cdot 4 \cdots 2n(-5)(-3)\cdots(2n-7)} = \frac{2n-5}{2 \cdot 4 \cdots 2n(-5)}$ .

$$\begin{aligned} y_2(x) &= -\frac{1}{5} \sum_{n=0}^{\infty} \frac{2n-5}{2 \cdot 4 \cdots 2n} x^{n-3/2} \\ &= x^{-3/2} + \frac{3}{5 \cdot 2} x^{-1/2} + \frac{1}{5 \cdot 2 \cdot 4} x^{1/2} - \frac{1}{5 \cdot 2 \cdot 4 \cdot 6} x^{3/2} - \frac{3}{5 \cdot 2 \cdot 4 \cdot 6 \cdot 8} x^{5/2} - \frac{5}{5 \cdot 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10} x^{7/2} - \dots \end{aligned} \quad [1\frac{1}{2}]$$

*Alternative solution:* We use the general recurrence relation for the rational functions  $a_n(r)$ , viz.  $a_0(r) = 1$  and

$$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} [(r+k)p_{n-k} + q_{n-k}] a_{n-1}(r) \quad \text{for } n \geq 1.$$

Since  $F(r) = (r-2)(r+3/2)$ ,  $p_1 = -1/2$ ,  $p_2 = p_3 = \dots = q_1 = q_2 = \dots = 0$ , we obtain

$$\begin{aligned} a_n(r) &= -\frac{1}{(r+n-2)(r+n+3/2)} [(r+n-1)(-1/2)] a_{n-1}(r) \\ &= \frac{r+n-1}{(r+n-2)(2r+2n+3)} a_{n-1}(r), \quad n \geq 1. \end{aligned}$$

Thus the coefficients  $a_n(2)$  of  $y_1(x)$  satisfy the recurrence relation  $a_n(2) = \frac{n+1}{n(2n+7)} a_{n-1}(2)$  (the same as for  $a_n$  above) and the coefficients  $a_n(-3/2)$  of  $y_2(x)$  satisfy the recurrence relation  $a_n(-3/2) = \frac{n-5/2}{(n-7/2)2n} a_{n-1}(-3/2)$  (the same as for  $b_n$  above). The rest of the computation remains the same.

The general (real) solution on  $(0, \infty)$  is then  $y(x) = c_1 y_1(x) + c_2 y_2(x)$ ,  $c_1, c_2 \in \mathbb{R}$ . 1/2

That solutions are defined on the whole of  $(0, \infty)$ , is guaranteed by the analyticity of  $p(x)$ ,  $q(x)$  in  $\mathbb{C} \setminus \{0\}$ , but follows also readily from the easily established fact that the radius of convergence of both power series is  $\infty$ . 1/2

- c) The solution on  $(-\infty, 0)$  is  $y(x) = c_1 y_1(x) + c_2 y_2^-(x)$  with the same power series  $y_1(x)$  as in b) and

$$y_2^-(x) = -\frac{1}{5|x|^{3/2}} \sum_{n=0}^{\infty} \frac{2n-5}{2 \cdot 4 \cdots 2n} x^n = -\frac{1}{5(-x)^{3/2}} \sum_{n=0}^{\infty} \frac{2n-5}{2 \cdot 4 \cdots 2n} x^n. \quad 1$$

(This is not the same as  $y_2(-x)$ , whose coefficients have an additional factor  $(-1)^n$ .)

Since  $y_1(x)$  is analytic everywhere and  $\lim_{x \downarrow 0} y_2(x) = \infty$ , the general solution on  $\mathbb{R}$  is  $y(x) = c_1 y_1(x)$ ,  $c_1 \in \mathbb{R}$ . 1

---


$$\sum_2 = 10$$


---

- 3 a) Substituting  $y_1(t) = c t^r$  into the ODE gives

$$c r t^{r-1} = c^2 t^{2r} + 5c t^{r-1} + 5 t^{-2},$$

which holds if  $r = -1$  and  $-c = c^2 + 5c + 5$ , i.e.,  $c^2 + 6c + 5 = 0$ , which has solutions  $c \in \{-1, -5\}$ . Thus we can take  $y_1(t) = -t^{-1}$  or  $y_1(t) = -5 t^{-1}$ . 1

- b) Taking  $y_1(t) = -t^{-1}$  in a), the substitution becomes  $y = -t^{-1} + 1/z$ ,  $y' = 1/t^2 - z'/z^2$ . Substituting this into (R) gives

$$\begin{aligned} \frac{1}{t^2} - \frac{z'}{z^2} &= \left(-\frac{1}{t} + \frac{1}{z}\right)^2 + \frac{5}{t} \left(-\frac{1}{t} + \frac{1}{z}\right) + \frac{5}{t^2} = \frac{1}{t^2} - \frac{2}{tz} + \frac{1}{z^2} - \frac{5}{t^2} + \frac{5}{tz} + \frac{5}{t^2} \\ \iff -\frac{z'}{z^2} &= \frac{3}{tz} + \frac{1}{z^2} \\ \iff z' &= -\frac{3}{t} z - 1. \end{aligned}$$

This is of the form  $z' = a(t)z + b(t)$ , hence first-order (inhomogeneous) linear. 2

- c) The general solution of  $z' = (-3/t)z$  is

$$z(t) = c \exp \int -\frac{3}{t} dt = \frac{c}{t^3}, \quad c \in \mathbb{R}. \quad 1$$

Variation of parameters then yields a particular solution  $z_p$  of  $z' = (-3/t) - 1$ :

$$z_p(t) = t^{-3} \int t^3(-1) dt = -\frac{t}{4}. \quad 1$$

$\Rightarrow$  The general solution of  $z' = (-3/t)z - 1$  is

$$z(t) = -\frac{t}{4} + \frac{c}{t^3}, \quad c \in \mathbb{R}.$$

$\Rightarrow$  The general solution of (R) is

$$y(t) = -\frac{1}{t} + \frac{1}{-t/4 + c/t^3} = -\frac{1}{t} + \frac{t^3}{c - t^4/4}, \quad c \in \mathbb{R} \cup \{\infty\}, \quad \boxed{1}$$

where  $c = \infty$  represent the solution  $y_1$ .

The maximal domain of  $y(t)$  is  $(0, \infty)$  for  $c \in \{0, \infty\}$  and  $c < 0$  ( $c = 0$  corresponds to the 2nd solution  $y(t) = -5t^{-1}$  discovered in a).) For  $c > 0$  the expression for  $y(t)$  defines two maximal solutions on  $(0, \sqrt[4]{4c})$  and  $(\sqrt[4]{4c}, \infty)$ .  $\boxed{1}$

$$\sum_3 = 7$$

**4** The characteristic polynomial of  $\mathbf{A}$  is

$$\begin{aligned} \chi_{\mathbf{A}}(X) &= \begin{vmatrix} X+8 & 0 & -5 & 2 \\ -5 & X+1 & 4 & -1 \\ 10 & 0 & X-7 & 2 \\ 0 & 0 & -3 & X-2 \end{vmatrix} = (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 10 & X-7 & 2 \\ 0 & -3 & X-2 \end{vmatrix} \\ &= (X+1) \begin{vmatrix} X+8 & -5 & 2 \\ 2-X & X-2 & 0 \\ 0 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+8 & -5 & 2 \\ -1 & 1 & 0 \\ 0 & -3 & X-2 \end{vmatrix} \\ &= (X+1)(X-2) \begin{vmatrix} X+3 & -5 & 2 \\ 0 & 1 & 0 \\ -3 & -3 & X-2 \end{vmatrix} = (X+1)(X-2) \begin{vmatrix} X+3 & 2 \\ -3 & X-2 \end{vmatrix} \\ &= (X+1)(X-2) [(X+3)(X-2) - (-3)2] \\ &= X(X+1)^2(X-2). \end{aligned}$$

$\Rightarrow$  The eigenvalues of  $\mathbf{A}$  are  $\lambda_1 = 0$  with algebraic multiplicity 1,  $\lambda_2 = -1$  with algebraic multiplicity 2,  $\lambda_3 = 2$  with algebraic multiplicity 1.  $\boxed{2}$

$\lambda_1 = 0$ :

$$\mathbf{A} - 0\mathbf{I} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -2 & 0 & 2 & 0 \\ -8 & 0 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The latter is in “permuted” echelon form with  $x_1$  as a free variable, say. Setting  $x_1 = 1$  gives  $x_3 = 1$ ,  $x_4 = -3/2$ ,  $x_2 = -1/2$ .

$\Rightarrow$  The eigenspace corresponding to  $\lambda_1 = 0$  is generated by  $\mathbf{v}_1 = (2, -1, 2, -3)^\top$ .

$\lambda_2 = -1$ :

$$\mathbf{A} + \mathbf{I} = \begin{pmatrix} -7 & 0 & 5 & -2 \\ 5 & 0 & -4 & 1 \\ -10 & 0 & 8 & -2 \\ 0 & 0 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & -3 & 0 \\ 5 & 0 & -4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 3 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$  The eigenspace corresponding to  $\lambda_2 = -1$  is generated by  $\mathbf{v}_2 = (0, 1, 0, 0)^\top$  and  $\mathbf{v}_3 = (-1, 0, -1, 1)^\top$ .

$\lambda_3 = 2$ :

$$\mathbf{A} - 2\mathbf{I} = \begin{pmatrix} -10 & 0 & 5 & -2 \\ 5 & -3 & -4 & 1 \\ -10 & 0 & 5 & -2 \\ 0 & 0 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & -3 & -4 & 1 \\ 0 & -6 & -3 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Setting  $x_4 = 1$  gives  $x_3 = 0$ ,  $x_2 = 0$ ,  $x_1 = -1/5$ .

$\Rightarrow$  The eigenspace corresponding to  $\lambda_3 = 2$  is generated by  $\mathbf{v}_4 = (-1, 0, 0, 5)^\top$ .

Since eigenvectors corresponding to different eigenvalues are linearly independent,  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$  form a basis of  $\mathbb{R}^4$  (and  $\mathbf{A}$  is diagonalizable).

$\Rightarrow$  A fundamental system of solutions of  $\mathbf{y}' = \mathbf{A}\mathbf{y}$  is

$$\mathbf{y}_1(t) \equiv \begin{pmatrix} 2 \\ -1 \\ 2 \\ -3 \end{pmatrix}, \quad \mathbf{y}_2(t) = e^{-t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{y}_3(t) = e^{-t} \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{y}_4(t) = e^{2t} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 5 \end{pmatrix},$$

4

and the general (real) solution is  $\mathbf{y}(t) = c_1 \mathbf{y}_1(t) + c_2 \mathbf{y}_2(t) + c_3 \mathbf{y}_3(t) + c_4 \mathbf{y}_4(t)$ ,  $c_1, c_2, c_3, c_4 \in \mathbb{R}$ .

$$\sum_4 = 6$$

5 a) Dividing (DF) by  $3y^2 - 1$  gives the exact (even separable) equation

$$x \, dx + \frac{y}{3y^2 - 1} \, dy = 0.$$

A function  $F$  with  $dF = x \, dx + \frac{y}{3y^2 - 1} \, dy$  is  $F(x, y) = \frac{1}{2} x^2 + \frac{1}{6} \ln |3y^2 - 1|$ , and hence the general solution of (DF) in implicit form is

$$\frac{1}{2} x^2 + \frac{1}{6} \ln |3y^2 - 1| = C, \quad C \in \mathbb{R}. \quad 2$$

This must be complemented by the horizontal lines  $y = \pm 1/\sqrt{3}$ , which have been lost when dividing by  $3y^2 - 1$ . Since  $y = \text{const.}$  implies  $dy = 0$ , these are indeed solutions (even explicit solutions  $y(x) \equiv \pm 1/\sqrt{3}$ ). 1

b)  $y(1) = 1/3$  requires  $C = \frac{1}{2} + \frac{1}{6} \ln \frac{2}{3}$ . Then we solve the corresponding contour equation



for  $y$ :

$$\begin{aligned}\frac{1}{6} \ln |3y^2 - 1| &= \frac{1}{2}(1 - x^2) + \frac{1}{6} \ln \frac{2}{3} \\ \ln |3y^2 - 1| &= 3(1 - x^2) + \ln \frac{2}{3} \\ |3y^2 - 1| &= \frac{2}{3} e^{3(1-x^2)} \\ 3y^2 - 1 &= -\frac{2}{3} e^{3(1-x^2)} \quad (\text{since } 3y(1) - 1 = -\frac{2}{3}) \\ y = y(x) &= \sqrt{\frac{1 - \frac{2}{3} e^{3(1-x^2)}}{3}}\end{aligned}\quad \boxed{2}$$

The domain  $I$  of  $y(x)$  is determined by

$$\begin{aligned}1 - \frac{2}{3} e^{3(1-x^2)} &\geq 0 \\ \iff 3(1 - x^2) &\leq \ln \frac{3}{2} \\ \iff x^2 &\geq 1 - \frac{1}{3} \ln \frac{3}{2} \approx 0.87\end{aligned}$$

Since  $I$  must be an interval containing 1, we obtain  $I = [\sqrt{0.87}, \infty)$ .  $\boxed{1}$

More precisely  $I = (a, \infty)$  with  $a = \sqrt{1 - \frac{1}{3} \ln \frac{3}{2}} \approx 0.929970410262577$ . Since  $y(x)$  is not differentiable at  $a$ , we exclude  $a$  from the domain.

- c) No. The integral curves of (DF) are the contours of  $F(x, y) = \frac{1}{2} x^2 + \frac{1}{6} \ln |3y^2 - 1|$ . From  $dF(x, y) = x dx + \frac{y}{3y^2-1} dy$  we get  $F_x = x$ ,  $F_y = \frac{y}{3y^2-1}$ ,  $F_{xx} = 1$ ,  $F_{xy} = F_{yx} = 0$ , and  $F_{yy} = -\frac{3y^2+1}{(3y^2-1)^2}$ . Since  $F_x(0, 0) = F_y(0, 0) = 0$ ,  $F_{xx}(0, 0) = 1$ ,  $F_{yy}(0, 0) = -1$ , the origin  $(0, 0)$  is a saddle point of  $F$  and hence contained in two distinct integral curves.  $\boxed{+2}$

$$\sum_5 = 6 + 2$$

**6** a) The characteristic polynomial is

$$\begin{aligned}a(X) &= X^5 + 4X^4 + 24X^3 + 40X^2 + 100X \\ &= X(X^4 + 4X^3 + 24X^2 + 40X + 100) \\ &= X(X^2 + 2X + 10)^2 \\ &= X(X + 1 - 3i)^2(X + 1 + 3i)^2\end{aligned}$$

with zeros  $\lambda_1 = 0$  of multiplicity 1 and  $\lambda_2 = -1 + 3i$ ,  $\lambda_3 = -1 - 3i$  of multiplicity 2.  
 $\implies$  A complex fundamental system of solutions is  $1$ ,  $e^{(-1+3i)t}$ ,  $t e^{(-1+3i)t}$ ,  $e^{(-1-3i)t}$ ,  $t e^{(-1-3i)t}$  and the corresponding real fundamental system is

$$1, \quad e^{-t} \cos(3t), \quad e^{-t} \sin(3t), \quad t e^{-t} \cos(3t), \quad t e^{-t} \sin(3t).\quad \boxed{2}$$

b) In order to obtain a particular solution  $y_p(t)$  of the inhomogeneous equation, we solve the two systems  $a(D)y_i = b_i(t)$  for  $b_1(t) = 200t$ ,  $b_2(t) = e^{-t}$ . Superposition then yields the particular solution  $y_p(t) = y_1(t) - y_2(t)$ .

(1) Since  $\mu = 0$  is a zero of multiplicity 1 of  $a(X)$ , the proper Ansatz in this case is  $y_1(t) = c_0t + c_1t^2$ . Substituting it into the ODE we get

$$40(2c_1) + 100(c_0 + 2c_1t) = 200t.$$

$$\implies c_1 = 1, c_0 = -\frac{4}{5}, \text{ and } y_1(t) = t^2 - \frac{4}{5}t. \quad [1]$$

(2) Since  $\mu = -1$  is not a root of  $a(X)$ , we can take  $y_2(t) = \frac{1}{a(-1)}e^{-t} = -\frac{1}{81}e^{-t}$ . [1]

$$\implies y_p(t) = t^2 - \frac{4}{5}t + \frac{1}{81}e^{-t} \text{ is a particular solution.} \quad [1]$$

The general real solution is then

$$y(t) = c_1 + c_2 e^{-t} \cos(3t) + c_3 e^{-t} \sin(3t) + c_4 t e^{-t} \cos(3t) + c_5 t e^{-t} \sin(3t) \\ + t^2 - \frac{4}{5}t + \frac{1}{81}e^{-t} \quad \text{with } c_1, c_2, c_3, c_4, c_5 \in \mathbb{R}. \quad [1]$$

c) The Laplace transform of the right-hand side of the ODE in b) is

$$200 \mathcal{L}\{t\} - \mathcal{L}\{e^{-t}\} = \frac{200}{s^2} - \frac{1}{s+1} = \frac{-s^2 + 200s + 200}{s^2(s+1)}. \quad [1]$$

Using the formulas for the Laplace transform of the derivatives of  $y(t)$  and the given initial conditions, this implies  $s^5 Y(s) + 4s^4 Y(s) + 24s^3 Y(s) + 40s^2 Y(s) + 100s Y(s) = \frac{-s^2 + 200s + 200}{s^2(s+1)}$ , i.e.,

$$Y(s) = \frac{-s^2 + 200s + 200}{s^3(s+1)(s^2 + 2s + 10)^2}. \quad [1]$$

$$\sum_6 = 8$$

$$\sum = 45 + 7$$

Final Exam