Math 286
Introduction to Differential Equations

Thomas Honold

ZJU-UIUC Institute

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First-Order Linear Equations
The Complex Case
A Brief Introduction to Complex Numbers
Complex First-Order Linear Equations
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Today's Lecture: First-Order Linear Equations

The Linear Case

Definition

An (explicit) first-order *linear* ODE has the form

$$y' = a(t)y + b(t).$$

If $b(t) \equiv 0$, the linear ODE is called *homogeneous*; if $b(t) \neq 0$ for at least one t, it is called *inhomogeneous*.

Compare this with the theory of linear recurring sequences.

Theorem (homogeneous case)

If $a(t)$ is continuous, the general solution of $y' = a(t)y$ is given by

$$y(t) = c\,\mathrm{e}^{\int_{t_0}^{t} a(s)\,\mathrm{d}s} = y(t_0)\mathrm{e}^{\int_{t_0}^{t} a(s)\,\mathrm{d}s}, \quad c \in \mathbb{R}.$$

The domain of $y(t)$ is that of $a(t)$. (If the domain T of $a(t)$ is not an interval, there exists a solution of the stated form on every connected component of T.)

Proof.

The chain rule and the Fundamental Theorem of Calculus give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c\,\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}\right) = c\,\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^t a(s)\,\mathrm{d}s = c\,\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}\cdot a(t),$$

showing that $t \mapsto c\,\mathrm{e}^{\int_{t_0}^t a(s)\,\mathrm{d}s}$ is a solution of $y' = a(t)y$.

Now let $y(t)$ be any solution and consider the function $f(t) = y(t)\mathrm{e}^{-A(t)}$, where $A(t)$ is an antiderivative of $a(t)$, say $A(t) = \int_{t_0}^t a(s)\,\mathrm{d}s$.

$$f'(t) = y'(t)\mathrm{e}^{-A(t)} + y(t)\mathrm{e}^{-A(t)}\big(-a(t)\big) = \big(y'(t) - a(t)y(t)\big)\mathrm{e}^{-A(t)} \equiv 0$$

$\Longrightarrow f(t) = c$ is constant, and hence $y(t) = c\,\mathrm{e}^{A(t)}$ as claimed. (The choice of $A(t)$ does not matter, since the additive constant K involved in the choice turns into a positive constant e^{-K}, which is "eaten up" by c.) □

The Inhomogeneous Case

Solved by variation of parameters

Variation of parameters

Idea: The homogeneous ODE $y' = a(t)y$ is solved by $y(t) = c\,\mathrm{e}^{A(t)}$. In order to solve $y' = a(t)y + b(t)$, make $c = c(t)$ variable; that is we set $y_p(t) = c(t)\mathrm{e}^{A(t)} = c(t)y_h(t)$, where $y_h(t)$ denotes a solution of the homogeneous ODE.

$$y_p' = (cy_h)' = c'y_h + cy_h' = c'y_h + cay_h = a(cy_h) + b \iff c' = by_h^{-1}$$

Theorem

Suppose $a(t)$ and $b(t)$ are continuous.

1. *A particular solution of $y' = a(t)y + b(t)$ is*

$$y_p(t) = \mathrm{e}^{A(t)}\int_{t_0}^{t} b(s)\mathrm{e}^{-A(s)}\,\mathrm{d}s, \quad \text{where} \quad A(t) = \int_{t_0}^{t} a(s)\,\mathrm{d}s\,.$$

2. *The general solution of $y' = a(t)y + b(t)$ is*

$$y(t) = c\,\mathrm{e}^{A(t)} + y_p(t) = y(t_0)\mathrm{e}^{A(t)} + y_p(t), \quad c \in \mathbb{R}.$$

The remark about the domain of solutions made in the homogeneous case applies, except that now the maximal domain is the intersection of the domains of $a(t)$ and $b(t)$.

Proof.

(1) should be clear from the preceding consideration. Continuity of $b(t)$ is needed for $\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} b(s)\mathrm{e}^{-A(s)}\,\mathrm{d}s = b(t)\mathrm{e}^{-A(t)}$; cf. the proof of the Fundamental Theorem of Calculus.

(2) One needs to show that the difference $t \mapsto y_1(t) - y_2(t)$ of two solutions of $y' = a(t)y + b(t)$ is a solution of $y' = a(t)y$, which is straightforward. $\implies y(t) = \underbrace{y(t) - y_p(t)}_{\text{solves } y' = a(t)y} + y_p(t)$. □

Further Notes

- W.l.o.g. we could have assumed that $t = 0$. This assumption is justified, since the "time shift" $z(t) = y(t - t_0)$ transforms the IVP $y' = a(t)y + b(t) \wedge y(0) = y_0$ into $z'(t) = a(t - t_0)z(t) + b(t - t_0) \wedge z(t_0) = y_0$, which is also 1st-order linear with slightly changed coefficient functions.
- The preceding considerations apply, more generally, to functions $a(t)$, $b(t)$ with finitely many discontinuities of the first kind (i.e., the one-sided limits exist but are different).

Further Notes (cont'd)

- (cont'd)
  In this case the preceding formula gives all continuous functions $y\colon I \to \mathbb{R}$ that satisfy $y'(t) = a(t)y(t) + b(t)$ at every point t at which $a(t)$ and $b(t)$ are continuous.

  This follows from a more general version of the Fundamental Theorem of Calculus, which states that $F(t) = \int_a^t f(s)\,\mathrm{d}s$ satisfies $F'(t) = f(t)$ at every t at which f is continuous and has one-sided derivatives equal to $\lim_{s\uparrow t} f(s)$, $\lim_{s\downarrow t} f(s)$ at discontinuities of f of the first kind.
- The following alternative representation of $y_p(t)$ is sometimes useful: Since $A(t) - A(s) = \int_s^t a(\tau)\,\mathrm{d}\tau$, we have

$$y_p(t) = \int_{t_0}^t b(s)\mathrm{e}^{A(t)-A(s)}\,\mathrm{d}s = \int_{t_0}^t G(s,t)b(s)\,\mathrm{d}s$$

with $G(s,t) = \exp\left(\int_s^t a(\tau)\,\mathrm{d}\tau\right)$.

Examples

1. $y' = 2y + 3$

   In this case $a(t) = 2$, $b(t) = 3$ are constant, and the general solution is
   $$y(t) = -\tfrac{3}{2} + c\,\mathrm{e}^{2t}, \quad c \in \mathbb{R},$$
   because the associated homogeneous ODE $y' = 2y$ is solved by $y_h(t) = c\,\mathrm{e}^{2t}$ and $y' = 2y + 3$ has the constant solution $y_p(t) \equiv -\frac{3}{2}$.

   Solving $y(t_0) = -\frac{3}{2} + c\,\mathrm{e}^{2t_0}$ for c gives the solution of any corresponding IVP:
   $$c = \bigl(y(t_0) + \tfrac{3}{2}\bigr)\mathrm{e}^{-2t_0} \implies \boxed{y(t) = \bigl(y(t_0) + \tfrac{3}{2}\bigr)\mathrm{e}^{2(t-t_0)} - \tfrac{3}{2}}.$$

   We can also solve it by variation of parameters:
   $$y_p(t) = \mathrm{e}^{2t}\int_{t_0}^{t} \mathrm{e}^{-2s}\cdot 3\,\mathrm{d}s = \mathrm{e}^{2t}\left[-\frac{3}{2}\mathrm{e}^{-2s}\right]_{t_0}^{t} = -\tfrac{3}{2} + \frac{3\,\mathrm{e}^{-2t_0}}{2}\,\mathrm{e}^{2t},$$
   which is the constant $-\frac{3}{2}$ plus a solution of $y' = 2y$.

Examples Cont'd

1. (cont'd)
   Note that any solution $y(t)$ with $y(t_0) \neq -\frac{3}{2}$ grows exponentially for $t \to +\infty$.
2. $y' = -2y + 3$
   Here the general solution is

$$\boxed{y(t) = \left(y(t_0) - \tfrac{3}{2}\right)\mathrm{e}^{-2(t-t_0)} + \tfrac{3}{2}},$$

and every solution (regardless of the initial value $y(t_0)$) converges for $t \to +\infty$ towards the constant (equilibrium, steady-state) solution $y(t) \equiv \frac{3}{2}$.

Examples Cont'd

3. $y' = -2y + t$
The associated homogeneous ODE remains the same, and a particular solution is

$$\begin{aligned} y_p(t) &= \mathrm{e}^{-2t} \int t\,\mathrm{e}^{2t}\,\mathrm{d}t \\ &= \mathrm{e}^{-2t}\left(\tfrac{1}{2}t\,\mathrm{e}^{2t} - \tfrac{1}{2}\int \mathrm{e}^{2t}\,\mathrm{d}t\right) = \tfrac{1}{2}t - \tfrac{1}{4}. \end{aligned}$$

$\Longrightarrow$ The general solution is

$$\begin{aligned} y(t) &= \tfrac{1}{2}t - \tfrac{1}{4} + c\,\mathrm{e}^{-2t} \quad (c \in \mathbb{R}) \\ &= \tfrac{1}{2}t - \tfrac{1}{4} + \left(y(t_0) - \tfrac{1}{2}t_0 + \tfrac{1}{4}\right)\mathrm{e}^{-2(t-t_0)}. \end{aligned}$$

For $t \to +\infty$ every solution quickly approaches the particular solution $y_p(t) = \frac{1}{2}t - \frac{1}{4}$.

Examples Cont'd

4. $y' = -ty + 1$
   The associated homogeneous ODE $y' = -ty$ has the solution $y(t) = c\,\mathrm{e}^{-t^2/2}$, $c \in \mathbb{R}$.
   A particular solution of $y' = -ty + 1$ is

$$y_p(t) = \mathrm{e}^{-t^2/2} \int_{t_0}^{t} \mathrm{e}^{s^2/2}\,\mathrm{d}s,$$

   and the general solution of $y' = -ty + 1$ is

$$y(t) = \mathrm{e}^{-t^2/2} \left(c + \int_{t_0}^{t} \mathrm{e}^{s^2/2}\,\mathrm{d}s \right), \quad c = y(t_0)\mathrm{e}^{t_0^2/2}.$$

Examples Cont'd

5. $y' = -ty + t$
   The associated homogeneous ODE remains unchanged, so that we only need to find one particular solution.
   Using variation of parameters we get

$$\begin{aligned} y_p(t) &= \mathrm{e}^{-t^2/2} \int_{t_0}^{t} s\,\mathrm{e}^{s^2/2}\,\mathrm{d}s = \mathrm{e}^{-t^2/2}\left[\mathrm{e}^{s^2/2}\right]_{t_0}^{t} \\ &= \mathrm{e}^{-t^2/2}\left(\mathrm{e}^{t^2/2} - \mathrm{e}^{t_0^2/2}\right) = 1 - e^{t_0^2/2}\mathrm{e}^{-t^2/2}, \end{aligned}$$

   so that the general solution is

$$y(t) = 1 - e^{t_0^2/2}\mathrm{e}^{-t^2/2} + c\,\mathrm{e}^{-t^2/2} = 1 + c'\mathrm{e}^{-t^2/2} \quad (c, c' \in \mathbb{R}).$$

   Surprise?
   Not really, because $y' = -ty + t = -t(y-1)$ has the solution $y(t) \equiv 1$.

Examples Cont'd

6. $ty' + 2y = 4t^2$ (cf. [BDM17])
This is an example of an implicit 1st-order linear ODE. First we rewrite it in explicit form:

$$y' = -\frac{2}{t}\,y + 4t.$$

Note that this splits the original domain $\mathbb{R}$ (for t) into the two subintervals $I_1 = (-\infty, 0)$ and $I_2 = (0, +\infty)$. In what follows we consider only I_2 and choose $t_0 = 1$.
The usual method yields

$$y_h(t) = \exp\left(\int_1^t (-2/s)\,\mathrm{d}s\right) = \mathrm{e}^{-2\ln t} = \frac{1}{t^2},$$

$$y_p(t) = \frac{1}{t^2}\int_1^t s^2 \cdot 4s\,\mathrm{d}s = \frac{1}{t^2}\left[s^4\right]_1^t = t^2 - \frac{1}{t^2}.$$

It follows that another particular solution is $y_p(t) = t^2$, and the general solution is

$$y(t) = t^2 + \frac{c}{t^2} \text{ for } t > 0, \quad \text{with parameter } c \in \mathbb{R}.$$

Examples Cont'd

6. (cont'd) Note that exactly one of these solutions, viz. $y(t) = t^2$, is defined also for $t = 0$.

   The solutions on I_1 are the mirror images w.r.t. to the y-axis of the solutions on I_1, and $y(t) = t^2$ is the only solution of the original ODE $ty' + 2y = 4t^2$ that is defined in a neighborhood of $t = 0$.
   In other words, the IVP $ty' + 2y = 4t^2 \wedge y(0) = y_0$ has a solution precisely for $y_0 = 0$ (and this solution is defined on all of $\mathbb{R}$).

   Plotting solutions:
   Rewriting $y = t^2 + c/t^2$ as $t^2y - t^4 = c$, we see that the integral curves (solution curves) of $ty' + 2y = 4t^2$ are the contours of $F(t, y) = t^2y - t^4$.
   $\Longrightarrow$ We can use the contour plot facilities, e.g., of SageMath to plot the solutions.

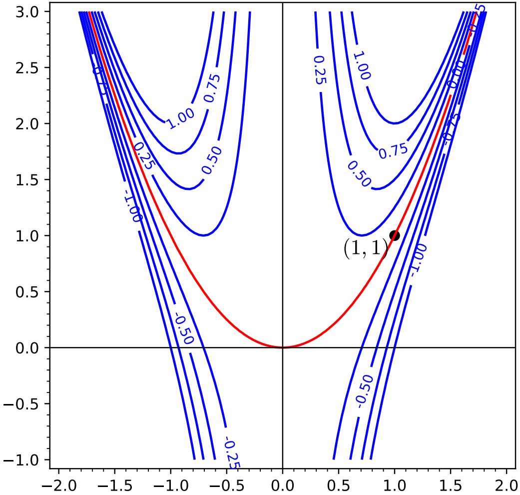

Figure: Graphs of $y_c(t) = t^2 + c/t^2$ for various values of c (including the branches for $t < 0$)

Note on the picture

The empty regions in the plot (due to laziness of your professor) are misleading. Since we can solve $t_0^2 + c/t_0^2 = y_0$ for c provided only that $t_0 \neq 0$, there passes a solution curve through any point of the (t, y)-plane that is not on the y-axis.

Afternote

Our derivation of the general solution of $y' = (-2/t)y + 4t$ illustrates another important point: For solving an inhomogeneous linear 1st-order ODE it suffices to compute 1 nonzero solution $y_h(t)$ of the associated homogeneous ODE and 1 solution $y_p(t)$ of the given inhomogeneous ODE, because the general solution is then $y(t) = y_p(t) + c\,y_h(t)$, $c \in \mathbb{R}$. For the determination of $y_h(t)$, $y_p(t)$ one may integrate from any $t_0 \in I$.

However, it is not necessarily true that varying t_0 over all of I yields all solutions $y_p(t)$. (In the homogeneous case it never does.) In our example this produces $(1/t^2) \int_{t_0}^{t} s^2 \cdot 4s \, \mathrm{d}s = t^2 - t_0^4/t^2$, missing all solutions $t^2 + c/t^2$ with $c \geq 0$.

The correct way to obtain all solutions by integration is to fix t_0 and add an arbitrary constant to the factor $c(t)$ in $y_p(t) = c(t)\mathrm{e}^{A(t)}$, i.e., $y_p(t) = (1/t^2) \left(c_0 + \int_1^t s^2 \cdot 4s \, \mathrm{d}s \right)$, $c_0 \in \mathbb{R}$.

Examples Cont'd

7. $mv' = mg - kv$ (2nd model for a falling object)
This ODE has the form $v' = av + b$ with $a = -k/m$, $b = g$.
The general solution is $v(t) = mg/k + c\mathrm{e}^{-kt/m}$, $c \in \mathbb{R}$.
Suppose the object is released at time $t = 0$.
$\Longrightarrow v(0) = 0 \Longrightarrow c = -mg/k$

$$\Longrightarrow v(t) = \frac{mg}{k}\left(1 - \mathrm{e}^{-kt/m}\right) \quad \text{for } 0 \leq t \leq T,$$

where T is the time when the object hits the ground.
The "limiting velocity" is $v_\infty = mg/k$.

Suppose the object is released at height x_0 above ground. For the distance traveled by the object we obtain by integrating and using $x(0) = 0$

$$x(t) = \frac{mg}{k}\left(t + \frac{m}{k}\mathrm{e}^{-kt/m}\right) + C, \quad C = -\frac{m^2 g}{k^2}.$$

$\Longrightarrow$ T can be found by (numerically) solving the equation

$$\frac{mg}{k}\left(T + \frac{m}{k}\mathrm{e}^{-kT/m}\right) - \frac{m^2 g}{k^2} = x_0.$$

Integrating Factors

There is an alternative way to solve $y' = a(t)y + b(t)$ using a so-called integrating factor. We can rewrite the ODE as

$$y'(t) - a(t)y(t) = b(t).$$

This equation can be multiplied by any function $m(t)$ with domain I to yield the equivalent form

$$m(t)y'(t) - a(t)m(t)y(t) = m(t)b(t), \qquad (\star)$$

provided that $m(t) \neq 0$ for all $t \in I$.
The goal is to choose $m(t)$ in such a way that the left-hand side can be integrated to yield $y(t)$ ($\rightarrow$ *integrating factor*).
Here $m(t) = \mathrm{e}^{-A(t)}$, $A(t) = \int a(t)\,\mathrm{d}t$, does the job, since $m'(t) = -a(t)m(t)$ and hence the left-hand side of $(\star)$ is $m(t)y'(t) + m'(t)y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(m(t)y(t)\right)$:

$$\mathrm{e}^{-A(t)}y'(t) - a(t)\mathrm{e}^{-A(t)}y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-A(t)}y(t)\right).$$

$$\implies \mathrm{e}^{-A(t)}y(t) = \int \mathrm{e}^{-A(t)}b(t)\,\mathrm{d}t \implies y(t) = \mathrm{e}^{A(t)}\int \mathrm{e}^{-A(t)}b(t)\,\mathrm{d}t$$

The Linear Algebra Aspect

The set of real-valued functions on a given domain I (i.e., maps $f\colon I \to \mathbb{R}$) is often denoted by $\mathbb{R}^I$. It forms a vector space over $\mathbb{R}$ with respect to the "point-wise" operations

$$(f+g)(t) = f(t) + g(t) \quad \text{for } f, g \in \mathbb{R}^I,$$
$$(cf)(t) = c\,f(t) \quad \text{for } f \in \mathbb{R}^I,\ c \in \mathbb{R}.$$

The general definition of subspaces of an abstract vector space specializes to:

Definition

A set of functions $S \subseteq \mathbb{R}^I$ is called a *subspace* if $S \neq \emptyset$ and S is closed w.r.t. the vector space operations, i.e., $f, g \in S$ implies $f + g \in S$ and $f \in S$ implies $cf \in S$ for all $c \in \mathbb{R}$.

Linear independence, generating set (spanning set), basis, and dimension of subspaces of $\mathbb{R}^I$ are defined in the same way as for $\mathbb{R}^n$ (and are special cases of the corresponding definitions for abstract vector spaces).

From now on we assume that I is an interval of positive length (and thus in particular an infinite set).

Remark

The most important difference between $\mathbb{R}^n$ and $\mathbb{R}^I$ is that $\mathbb{R}^I$ is infinite-dimensional (i.e., does not have a finite basis). For $I = \mathbb{R}$ this can be inferred from the following exercise.

Exercise

Let $f_\lambda(t) = \mathrm{e}^{\lambda t}$ for $\lambda \in \mathbb{R}$. Show that $\{f_\lambda; \lambda \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}^\mathbb{R}$.

Hint: Suppose there exists $r \in \mathbb{Z}^+$ and distinct numbers $\lambda_1, \dots, \lambda_r, c_1, \dots, c_r \in \mathbb{R}$ such that

$$c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} + \cdots + c_r \mathrm{e}^{\lambda_r t} = 0 \quad \text{for all } t \in \mathbb{R}.$$

Assuming $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $c_r \neq 0$, divide this equation by $\mathrm{e}^{\lambda_r t}$ and let $t \to +\infty$ to obtain a contradiction.

Proposition

Assume that $t \mapsto a(t)$ is continuous on I. Then the solution set S of $y' = a(t)y$ forms a 1-dimensional subspace of $\mathbb{R}^I$ and, for any choice of $t_0 \in I$, is generated by the function $I \to \mathbb{R}$, $t \mapsto \exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right)$.

Note that we assume that all solutions have maximal domain I.

Proof.

We have $S \neq \emptyset$, since the all-zero function $I \to \mathbb{R}$, $t \mapsto 0$ is a solution of $y' = a(t)y$. Further, it is easy to verify that sums and scalar multiples of solutions of $y' = a(t)y$ are again solutions.
$\Longrightarrow$ S is a subspace of $\mathbb{R}^n$.
The fact that S is 1-dimensional is less trivial; it follows from our theorem on solutions of homogeneous linear 1st-order ODE's, which says that every solution is a scalar multiple of
$t \mapsto \exp\left(\int_{t_0}^{t} a(s)\,\mathrm{d}s\right)$. $\square$

Note

In a way it is surprising that $\dim(S) = 1$, because S is defined by a single linear differential equation. Looking at the case of $\mathbb{R}^n$, where solution spaces of single (nontrivial) linear equations have dimension $n - 1$, one would rather expect $\dim(S) = \infty - 1 = \infty$.

Further Notes

- The theorem also gives that the solution set of an inhomogeneous ODE $y' = a(t)y + b(t)$ forms a *line* in the corresponding space $\mathbb{R}^I$, which does not pass through the origin (the all-zero function $I \to \mathbb{R}$). As in our Linear Algebra crash course you may check that any affine combination $t \mapsto \lambda y_1(t) + (1-\lambda)y_2(t)$, $\lambda \in \mathbb{R}$, of two solutions y_1, y_2 of $y' = a(t)y + b(t)$ is again a solution.
- In Example 10 of the introduction we found that the solutions of $y'' + y = 0$ on $\mathbb{R}$ form a 2-dimensional subspace of $\mathbb{R}^{\mathbb{R}}$ with basis $\sin t$, $\cos t$. (We had proved that every solution has the form $A\cos t + B\sin t$, i.e., is in the span of $\{\cos t, \sin t\}$, and it only remains to observe that $\cos t$, $\sin t$ are not constant multiples of each other.)

  The evaluation map $S \to \mathbb{R}^2$, $y \mapsto \bigl(y(0), y'(0)\bigr)$, which sends a solution to the corresponding initial values at $t = 0$, is a linear bijection with inverse map $\mathbb{R}^2 \to S$, $(A, B) \mapsto A\cos t + B\sin t$.
- Linear Algebra will play a much more prominent role when we analyze higher-order linear ODE's and 1st-order ODE systems later.

Special Cases of $y' = a(t)y + b(t)$

1. $a(t) = a$ and $b(t) = b$ are constants.

   In this case we have

$$y_p(t) = e^{at}\int_0^t b\mathrm{e}^{-as}\,\mathrm{d}s = be^{at}\left[-\frac{1}{a}\mathrm{e}^{-as}\right]_0^t = \frac{b}{a}\mathrm{e}^{at}(1-\mathrm{e}^{-at}) = \frac{b}{a}(\mathrm{e}^{at}-1)$$

   and hence as solution of the IVP $y' = ay + b \wedge y(t_0) = y_0$ the function

$$\boxed{y(t) = y(t_0)\mathrm{e}^{a(t-t_0)} + \frac{b}{a}\left(\mathrm{e}^{a(t-t_0)} - 1\right)}$$

   Setting $y(t_0) = -b/a$ gives the particular solution $y_p(t) \equiv -b/a$ noted earlier.

   For $a < 0$ we have $\lim_{t\to+\infty} y(t) = -b/a = y_\infty$, say, and

$$\boxed{y(t) = y_\infty + (y_0 - y_\infty)\mathrm{e}^{a(t-t_0)}, \quad y_0 = y(t_0)}.$$

   In other words, every solution $y(t)$ approaches the *steady-state* y_∞ exponentially fast.

Special Cases (cont'd)

2. $a(t) = a$, $b(t) = \mathrm{e}^{ct}$.

In this case we have

$$y_p(t) = e^{at}\int_{t_0}^{t} \mathrm{e}^{(c-a)s}\,\mathrm{d}s$$

$$= \begin{cases} \mathrm{e}^{at}\left[\frac{1}{c-a}\mathrm{e}^{(c-a)s}\right]_{t_0}^{t} = \frac{\mathrm{e}^{ct}-\mathrm{e}^{at+(c-a)t_0}}{c-a} & \text{if } c \neq a, \\ (t-t_0)\mathrm{e}^{at} & \text{if } c = a. \end{cases}$$

and hence

$$\boxed{y(t) = \begin{cases} y(t_0)\mathrm{e}^{a(t-t_0)} + \mathrm{e}^{ct_0}\cdot\frac{\mathrm{e}^{c(t-t_0)}-\mathrm{e}^{a(t-t_0)}}{c-a} & \text{if } c \neq a, \\ y(t_0)\mathrm{e}^{a(t-t_0)} + \mathrm{e}^{at_0}\cdot(t-t_0)\mathrm{e}^{a(t-t_0)} & \text{if } c = a. \end{cases}}$$

In the second case (a type of *resonance*) the solution may grow initially (i.e., for $t \downarrow t_0$) even if $a < 0$. This happens precisely for $y'(t_0) = ay(t_0) + \mathrm{e}^{at_0} > 0$, i.e., $y(t_0) < -\frac{1}{a}\mathrm{e}^{at_0}$.

Special Cases (cont'd)

3. $a(t) = a$, $b(t) = \begin{cases} 0 & \text{if } t < T, \\ b & \text{if } t \geq T. \end{cases}$

Assuming that $t_0 = T$, we have $y_p(t) = 0$ for $t \leq T$ and

$$y_p(t) = \mathrm{e}^{at} \int_T^t b\mathrm{e}^{-as}\,\mathrm{d}s = b\mathrm{e}^{at}\left[-\frac{1}{a}\mathrm{e}^{-as}\right]_T^t = \frac{b}{a}\left(e^{a(t-T)} - 1\right)$$

for $t \geq T$. This gives the general solution as

$$\boxed{y(t) = \begin{cases} y(T)\mathrm{e}^{a(t-T)} & \text{for } t \leq T, \\ y(T)\mathrm{e}^{a(t-T)} + \frac{b}{a}\left(e^{a(t-T)} - 1\right) & \text{for } t \geq T. \end{cases}}$$

We can verify that

$$\lim_{h\uparrow 0} \frac{y(T+h) - y(T)}{h} = a\,y(T), \quad \lim_{h\downarrow 0} \frac{y(T+h) - y(T)}{h} = a\,y(T) + b,$$

in accordance with the preceding note about discontinuities of $b(t)$. The solutions $y(t)$ simply arise by continuously gluing a solution of $y'(t) = a\,y(t)$ for $t \leq T$ with the corresponding solution of $y'(t) = a\,y(t) + b$ for $t \geq T$.

Special Cases (cont'd)

4. $a(t) = a$, $b(t) = \begin{cases} +\infty & \text{if } t = T, \\ 0 & \text{if } t \neq T. \end{cases}$

In the special case $t = 0$ this function is called *delta function* and denoted by $\delta(t)$, so that in general $b(t) = \delta(t - T)$.

$\delta(t)$ is not an ordinary function but represents a so-called *distribution*, which acts by integration on functions. The precise definition is

$$\int_{\mathbb{R}} f(t)\delta(t)\,\mathrm{d}t = \lim_{h\downarrow 0} \int_{\mathbb{R}} f(t)\delta_h(t),$$

where $\delta_h(t) = \frac{1}{2h} \times$ characteristic function of $[-h, h]$. If f is continuous at $t = 0$, this definition gives $\int_{\mathbb{R}} f(t)\delta(t)\,\mathrm{d}t = f(0)$ and in particular $\int_{\mathbb{R}} \delta(t)\,\mathrm{d}t = 1$.

Substituting $\delta(t - T)$ into the solution formula gives $y_p(t) = 0$ for $t < T$ and, assuming $t_0 < T$,

$$y_p(t) = \mathrm{e}^{at} \int_{t_0}^{t} \delta(T - s)\mathrm{e}^{-as}\,\mathrm{d}s = \mathrm{e}^{at}\mathrm{e}^{-aT} = \mathrm{e}^{a(t-T)} \quad \text{for } t > T.$$

Special Cases (cont'd)

4. (cont'd)
The general solution of $y'(t) = a\,y(t) + \delta(t - T)$ is then

$$\boxed{y(t) = \begin{cases} y(t_0)\mathrm{e}^{a(t-t_0)} & \text{if } t < T, \\ y(t_0)\mathrm{e}^{a(t-t_0)} + \mathrm{e}^{a(t-T)} & \text{if } t \geq T. \end{cases}}$$

We have $\lim_{t\uparrow T} y(t) = y(t_0)\mathrm{e}^{a(t-t_0)}$,
$\lim_{t\downarrow T} y(t) = y(T) = y(t_0)\mathrm{e}^{a(t-t_0)} + 1$, $y'(t) = a\,y(t)$ for $t \neq T$, and $y(t)$ arises from gluing together solutions of $y' = ay$ on $(-\infty, T)$ and $(T, +\infty)$ which differ by a unit step at $t = T$.

A Brief Introduction to Complex Numbers

For a more gentle introduction see [Ste16], Appendix H

A complex number is a point in the Euclidean plane $\mathbb{R}^2$. Complex numbers are added and multiplied according to the rules

$$\begin{aligned} (a,b)+(c,d) &:= (a+c,b+d), && \text{(Vector addition)} \\ (a,b)(c,d) &:= (ac-bd,ad+bc). && \text{(Well, just fancy)} \end{aligned}$$

In particular we have

$$(a,0)+(c,0)=(a+c,0),\ (a,0)(c,0)=(ac,0) \quad \text{for } a,c\in\mathbb{R}$$

(the numbers on the real axis are multiplied as usual), and

$$(0,1)^2=(0,1)(0,1)=(-1,0)$$

(the square of the "imaginary unit" $\mathrm{i}=(0,1)$ is the point on the real axis corresponding to -1).

Making the identification $(a,0)\triangleq a$, we obtain

$$(a,b)=(a,0)+(0,b)=(a,0)+(b,0)(0,1)=a+b\,\mathrm{i}, \quad \mathrm{i}^2=-1.$$

The complex numbers form a field, i.e., their addition/multiplication follows the usual laws of arithmetic. Thus it suffices to memorize only $\mathrm{i}^2 = -1$: Any complex number z has the form $z = a + b\,\mathrm{i}$ for some unique real numbers a, b, and

$$\begin{aligned} z + w &= (a + b\,\mathrm{i}) + (c + d\,\mathrm{i}) = a + c + (b + d)\mathrm{i}, \\ zw &= (a + b\,\mathrm{i})(c + d\,\mathrm{i}) = ac + ad\,\mathrm{i} + bc\,\mathrm{i} + bd\,\mathrm{i}^2 \\ &= ac - bd + (ad + bc)\mathrm{i}. \qquad\qquad (\text{Using } \mathrm{i}^2 = -1) \end{aligned}$$

Complex variables are commonly denoted by $z, w, \dots$ (cp. with $x, y, \dots$ for real variables), and the field of complex numbers is denoted by $\mathbb{C}$. (But keep in mind that $a + b\,\mathrm{i}$ is just the point (a, b), i.e., $\mathbb{C}$ is just $\mathbb{R}^2$ equipped with a fancy multiplication.)

The key property that distinguishes fields from commutative rings such as $\mathbb{Z}$ is that every element $z \neq 0$ has a “multiplicative inverse w” satisfying $zw = 1$. One writes z^{-1} or $1/z$ for w and z_1/z_2 for $z_1 z_2^{-1}$.

For a complex number $z = a + b\,\mathrm{i} \neq 0$ (i.e., at least one of a, b is nonzero) the multiplicative inverse is easily obtained by rationalizing the denominator:

$$\frac{1}{z} = \frac{1}{a + b\,\mathrm{i}} = \frac{a - b\,\mathrm{i}}{(a + b\,\mathrm{i})(a - b\,\mathrm{i})} = \frac{a - b\,\mathrm{i}}{a^2 + b^2} = \frac{a}{a^2 + b^2} + \frac{-b}{a^2 + b^2}\,\mathrm{i}.$$

The analogy with $\mathbb{R}^2$ goes further: The *absolute value* $|z|$ of a complex number z is its Euclidean length, i.e.,

$$|z| = |a + b\,\mathrm{i}| := \sqrt{a^2 + b^2}.$$

It satisfies $|z + w| \leq |z| + |w|$ (triangle inequality for the Euclidean length/distance) and $|zw| = |z|\,|w|$. For the latter check that

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

The *complex conjugate* of $z = a + b\,\mathrm{i} \in \mathbb{C}$ is $\overline{z} = a - b\,\mathrm{i}$. Geometrically, the map $z = (a, b) \mapsto \overline{z} = (a, -b)$ is reflection at the x-axis ("real axis"). Algebraically, it satisfies $\overline{z + w} = \overline{z} + \overline{w}$, $\overline{zw} = \overline{z}\,\overline{w}$, i.e., forms an automorphism of $\mathbb{C}$.

The coordinates a, b of $z = (a, b) = a + b\,\mathrm{i} \in \mathbb{C}$ are called *real part* resp. *imaginary part* of z, notation $a = \mathrm{Re}(z)$, $b = \mathrm{Im}(z)$.

Since $\mathbb{C} = \mathbb{R}^2$ as a set, we can do analysis in $\mathbb{C}$ as you have learned in Calculus III. For example, a sequence (z_n) of complex numbers converges to $z \in \mathbb{C}$ if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|z_n - z| < \epsilon$ for all $n > N$. Writing $z_n = a_n + b_n\,\mathrm{i}$, $z = a + b\,\mathrm{i}$ ($a_n, b_n, a, b \in \mathbb{R}$), the convergence $z_n \to z$ is equivalent to $a_n \to a \wedge b_n \to b$ (coordinate-wise convergence).

A series $\sum_{n=0}^{\infty} z_n$ of complex numbers converges (i.e., the associated sequence $s_n = z_1 + z_2 + \cdots + z_n$ of partial sums converges), provided it *converges absolutely*, i.e., the $\sum_{n=0}^{\infty} |z_n|$ (an ordinary series of real numbers) converges. This follows by applying the absolute convergence test for real series [Ste16, Ch. 11.6, Th. 3] to $\sum_{n=0}^{\infty} \operatorname{Re}(z_n)$, $\sum_{n=0}^{\infty} \operatorname{Im}(z_n)$. The details are left as an exercise. (One should note that there is nothing special about complex numbers here. The analogous statement holds for series of points in $\mathbb{R}^n$.)

Polar Form for complex numbers

Using polar coordinates in $\mathbb{R}^2$ we can write every nonzero complex number z in the form

$$z = (r\cos\phi, r\sin\phi) = r\cos\phi + r\sin\phi\,\mathrm{i} = r(\cos\phi + \mathrm{i}\sin\phi).$$

Here $r = |z|$ and $\phi \in [0, 2\pi)$ are uniquely determined by z.The complex exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined by the same power series as in the real case (and extends $x \mapsto \mathrm{e}^x$ to $\mathbb{C}$):

$$\mathrm{e}^z := \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Polar Form for complex numbers cont'd

That the exponential series converges for all $z \in \mathbb{C}$, can be proved using the absolute convergence test mentioned above.

The functional equation $\mathrm{e}^{z+w} = \mathrm{e}^z\mathrm{e}^w$ holds for all $z, w \in \mathbb{C}$. This can be proved by rearranging the double series representing $\mathrm{e}^z\mathrm{e}^w$ according to $z^i w^j$ with $i + j$ fixed and using the Binomial Theorem; cf. exercise.

Finally, extracting real and imaginary part of $\mathrm{e}^{\mathrm{i}\phi} = \sum_{n=0}^{\infty}(\mathrm{i}\phi)^n/n!$ and using the known Taylor series of cos, sin, one arrives at *Euler's Identity*

$$\mathrm{e}^{\mathrm{i}\phi} = \cos\phi + \mathrm{i}\sin\phi, \quad \phi \in \mathbb{R}.$$

Combining this with polar coordinates in $\mathbb{R}^2$, we see that every $z \in \mathbb{C} \setminus \{0\}$ admits a unique representation

$$z = r(\cos\phi + \mathrm{i}\sin\phi) = r\mathrm{e}^{\mathrm{i}\phi} \quad \text{with } r = |z| > 0,\ \phi \in [0, 2\pi).$$

This is the so-called *polar form* of z. The angle ϕ is called the *argument* of z, notation $\phi = \mathit{Arg}(z)$. Analytically, for $z = x + y\,\mathrm{i}$ we have $\phi = \arctan(y/x)$ if $x > 0$, $\phi = \arctan(y/x) + \pi$ if $x < 0$, and $\phi = \pm\pi/2$ if $x = 0 \wedge y \gtrless 0$.

The polar form easily shows the geometric meaning of complex multiplication. For $z = r\mathrm{e}^{\mathrm{i}\phi}$, $w = s\mathrm{e}^{\mathrm{i}\psi}$ in polar form, we have

$$zw = rs\,\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\psi} = rs\,\mathrm{e}^{\mathrm{i}(\phi+\psi)}$$

(using the functional equation for $z \mapsto \mathrm{e}^z$). This is the polar form of zw, except that $\phi + \psi$ is not necessarily reduced modulo 2π. From it we see that multiplication by z is composed of a rotation with angle $\phi = Arg(z)$ (the map $w \mapsto \mathrm{e}^{\mathrm{i}\phi}w$ and a scaling map (the map $w \mapsto |z|\,w$.

For example, multiplication by the imaginary unit $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$ just rotates every $w \in \mathbb{C} \setminus \{0\}$ around the origin by $90°$, and multiplication by $1 + \mathrm{i} = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}$ rotates $w \in \mathbb{C} \setminus \{0\}$ by $45°$ and scales it by the factor $\sqrt{2}$.

Roots of Unity

A complex number z is said to be an *n-th root of unity* if $z^n = 1$. Writing this equation in polar form, $z^n = r^n\mathrm{e}^{\mathrm{i}n\phi} = 1\,\mathrm{e}^{0\,\phi}$, shows that the n-th roots of unity are precisely the n numbers $\mathrm{e}^{2\pi\mathrm{i}k/n}$ with $k \in \{0, 1, \dots, n-1\}$. These form the vertices of the regular n-gon centered at 0 and with one vertex at 1, which is inscribed into the unit circle. Writing $\zeta_n = \mathrm{e}^{2\pi\mathrm{i}/n}$, the n-th roots of unity are $1, \zeta_n, \zeta_n^2, \dots, \zeta_n^{n-1}$.

The Fundamental Theorem of Algebra

That the polynomial $X^n - 1$ has n distinct roots in $\mathbb{C}$ and hence splits in $\mathbb{C}[X]$ (the polynomial ring in one indeterminate over $\mathbb{C}$) into linear factors, viz.

$$X^n - 1 = \prod_{k=0}^{n-1} \left(X - \mathrm{e}^{2\pi\mathrm{i}k/n}\right),$$

is a special case of the so-called *Fundamental Theorem of Algebra*:

> *Every polynomial $p(X) = p_0 + p_1X + \cdots + p_dX^d$ with coefficients $p_i \in \mathbb{C}$ and degree $d \geq 1$ (i.e., $p_d \neq 0$) has at least one root in $\mathbb{C}$.*

Since $p(c) = 0$ implies $p(X) = (X - c)q(X)$ for some polynomial $q(X)$of degree $d - 1$, it follows by induction that $p(X)$ splits into linear factors in $\mathbb{C}[X]$.

No easy proof of the Fundamental Theorem of Algebra is known. A rather elementary, but still quite intricate proof due to ARGAND (1814) is within the scope of a Calculus III course. One assumes, by contradiction, that $p(X)$ has no root in $\mathbb{C}$. Then $f\colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \frac{1}{|p(z)|}$ is well defined, and one can easily show that f attains a maximum at some point $z_0 \in \mathbb{C}$. Algebraic properties of $\mathbb{C}$ are then used to derive a contradiction from this.

Exercises on Complex Numbers

1. Show $\overline{zw} = \overline{z}\,\overline{w}$ and $|zw| = |z|\,|w|$ for $z, w \in \mathbb{C}$.
2. Show $z\,\overline{z} = |z|^2$ for $z \in \mathbb{C}$, and give a geometric interpretation of the inversion map $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$, $z \mapsto \frac{1}{z} = \frac{\overline{z}}{z\,\overline{z}}$.
3. Show that a series $\sum_{n=0}^{\infty} z_n$ of complex numbers converges if it converges absolutely, i.e., $\sum_{n=0}^{\infty} |z_n|$ converges in $\mathbb{R}$.
4. The complex exponential function is defined by
$$\mathrm{e}^z := \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{for } z = x + \mathrm{i}\,y \in \mathbb{C}.$$
Show that this series converges for all $z \in \mathbb{C}$.
5. Evaluate $\sum_{n=0}^{\infty} \left(\frac{1+\mathrm{i}}{2}\right)^n$ and graph the first few partial sums of this series in the complex plane (i.e., in $\mathbb{R}^2$).

Exercises on Complex Numbers Cont'd

6. Prove Euler's identity $\mathrm{e}^{\mathrm{i}\phi} + \cos\phi + \mathrm{i}\sin\phi$.
   Hint: $\mathrm{i}^2 = 1$, $\mathrm{i}^3 = -\mathrm{i}$, $\mathrm{i}^4 = 1$, $\mathrm{i}^5 = \mathrm{i}$, etc.
7. Prove the functional equation for the complex exponential function: $\mathrm{e}^{z+w} = \mathrm{e}^z \mathrm{e}^w$ for $z, w \in \mathbb{C}$.

   Hint: For two absolutely convergent series $\sum_{k=0}^{\infty} c_k$, $\sum_{l=0}^{\infty} d_l$ the identity
   $$\left(\sum_{k=0}^{\infty} c_k\right)\left(\sum_{l=0}^{\infty} d_l\right) = \sum_{n=0}^{\infty}(c_0 d_n + c_1 d_{n-1} + \cdots + c_n d_0) \quad \text{holds.}$$
8. For $z = x + \mathrm{i}y$ show that $\operatorname{Re}(\mathrm{e}^z) = \mathrm{e}^x \cos y$, $\operatorname{Im}(\mathrm{e}^z) = \mathrm{e}^x \sin y$.
9. Show that the range of the complex exponential function is $\mathbb{C} \setminus \{0\}$ and that $\mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^z$ for $z \in \mathbb{C}$.

Exercises on Complex Numbers Cont'd

10. Suppose $c = a + \mathrm{i}b \in \mathbb{C}$ is nonzero. Show without recourse to Euler's Identity (cf. previous exercise) that the equation $z^2 = c$ has exactly two solutions in $\mathbb{C}$.

    Hint: For $z = x + \mathrm{i}y$ the equation $z^2 = c$ is equivalent to $x^2 - y^2 = a \wedge 2xy = b$. Express $x^2 + y^2$ in terms of a, b.

11. Show (e.g., by completing the square) that a quadratic equation $Az^2 + Bz + C = 0$, $A, B, C \in \mathbb{C}$, $A \neq 0$, has (exactly) 2 solutions in $\mathbb{C}$ if $B^2 - 4AC \neq 0$ and 1 solution if $B^2 - 4AC = 0$.

12. Euler's Identity and the functional equation for $z \mapsto \mathrm{e}^z$ (cf. previous exercise) imply that the solutions of $z^n = 1$ in $\mathbb{C}$ (*n-th roots of unity*) have the form $\mathrm{e}^{2\pi\mathrm{i}k/n} = \zeta_n^k$ with $k \in \{0, 1, \dots, n-1\}$, $\zeta_n = \mathrm{e}^{2\pi\mathrm{i}/n}$, and form the vertices of a regular n-gon inscribed in the unit circle. Using the result of a), determine ζ_{24} in the form $u + \mathrm{i}v$ and sketch the solutions of $z^{24} = 1$ that are contained in the 1st quadrant of the plane.

Complex 1st-Order Linear ODE's

Definition

An (explicit) first-order linear ODE with (non-constant) complex coefficients has the form

$$z'(t) = a(t)z(t) + b(t) \quad \text{with } a, b \colon D \to \mathbb{C}.$$

A solution of such a complex ODE is a complex-valued function $z(t) = x(t) + \mathrm{i}y(t)$, defined on an interval $I \subseteq D$ and satisfying $z'(t) = x'(t) + \mathrm{i}y'(t) = a(t)z(t) + b(t)$ for all $t \in I$.

Writing $a(t) = a_1(t) + \mathrm{i}a_2(t)$, $b(t) = b_1(t) + \mathrm{i}b_2(t)$, the complex ODE is equivalent to

$$\begin{aligned} x'(t) &= a_1(t)x(t) - a_2(t)y(t) + b_1(t), \\ y'(t) &= a_2(t)x(t) + a_1(t)y(t) + b_2(t); \end{aligned}$$

in matrix form:

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} a_1(t) & -a_2(t) \\ a_2(t) & a_1(t) \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}.$$

General solution

The general solution of $z'(t) = a(t)z(t) + b(t)$ is $z(t) = z_0 e^{A(t)} + z_p(t)$ with $z_0 \in \mathbb{C}$ and $A, z_p \colon I \to \mathbb{C}$ defined by

$$A(t) = \int_{t_0}^{t} a(s)\,\mathrm{d}s, \quad z_p(t) = e^{A(t)} \int_{t_0}^{t} b(s) e^{-A(s)}\,\mathrm{d}s\,.$$

The proof given in the real case carries over—essentially because differentiation/integration of complex-valued functions of a real variable is done component-wise and the formula $\frac{\mathrm{d}}{\mathrm{d}t} e^{A(t)} = A'(t) e^{A(t)}$ also holds for complex-valued functions $A(t)$.

The chosen normalization of $A(t)$, $z_p(t)$ implies $A(t_0) = z_p(t_0) = 0$, showing that $z(t) = z_0 e^{A(t)} + z_p(t)$ is the unique solution of the corresponding IVP $z'(t) = a(t)z(t) + b(t) \wedge z(t_0) = z_0$.

Complexification of real ODE's

In order to solve a real ODE $y'(t) = a(t)y(t) + b(t)$, write $b(t) = \operatorname{Im} B(t)$ and solve the complex ODE $z'(t) = a(t)z(t) + B(t)$.

$$\begin{aligned} z'(t) = x'(t) + \mathrm{i}y'(t) &= a(t)\bigl(x(t) + \mathrm{i}y(t)\bigr) + \operatorname{Re} B(t) + \mathrm{i} \operatorname{Im} B(t) \\ &= a(t)x(t) + \operatorname{Re} B(t) + \mathrm{i}\bigl(a(t)y(t) + b(t)\bigr), \end{aligned}$$

$\Longrightarrow y(t) = \operatorname{Im} z(t)$ will then be a solution of $y'(t) = a(t)y(t) + b(t)$.

Even though it adds additional complexity, complexification can be useful since complex functions are sometimes easier to evaluate/differentiate/integrate than real functions. As an example, recall the computation of $\int_0^{2\pi}\cos(mt)\cos(nt)\,\mathrm{d}t$ by using $\mathrm{e}^{\mathrm{i}x}$ in place of $\cos x$.

Example

We solve $y' = ay + \sin(\omega t)$ with $a, \omega \in \mathbb{R}$.

Complexifying this ODE leads to $z' = az + \mathrm{e}^{\mathrm{i}\omega t}$, which is a complex analogue of $y' = ay + \mathrm{e}^{ct}$ (with $c = \mathrm{i}\omega$).
Now we could recall the corresponding formula derived by variation of parameters, but it is also instructive to solve the complex ODE ad hoc.

Since $(\mathrm{e}^{\mathrm{i}\omega t})' = \mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t}$, it is reasonable to guess that there exists a particular solution of the form $z(t) = A\,\mathrm{e}^{\mathrm{i}\omega t}$ with $A \in \mathbb{C}$.

$$z'(t) = A\mathrm{i}\omega\,\mathrm{e}^{\mathrm{i}\omega t} = a(A\,\mathrm{e}^{\mathrm{i}\omega t}) + \mathrm{e}^{\mathrm{i}\omega t} \iff A\mathrm{i}\omega = aA + 1 \iff A = \frac{1}{\mathrm{i}\omega - a}.$$

$$\Longrightarrow z(t) = \frac{1}{-a + \mathrm{i}\omega}\,\mathrm{e}^{\mathrm{i}\omega t} = \frac{-a - \mathrm{i}\omega}{a^2 + \omega^2}\big(\cos(\omega t) + \mathrm{i}\,\sin(\omega t)\big)$$

Example (cont'd)

$$\implies y(t) = \operatorname{Im} z(t) = -\frac{\omega}{a^2+\omega^2}\cos(\omega t) - \frac{a}{a^2+\omega^2}\sin(\omega t)$$

This function is indeed a solution of $y' = ay + \sin(\omega t)$, as the following double-check shows:

$$\begin{aligned} y'(t) &= \frac{\omega^2}{a^2+\omega^2}\sin(\omega t) - \frac{a\omega}{a^2+\omega^2}\cos(\omega t) \\ &= \sin(\omega t) - \frac{a^2}{a^2+\omega^2}\sin(\omega t) - \frac{a\omega}{a^2+\omega^2}\cos(\omega t) \\ &= \sin(\omega t) + ay(t) \end{aligned}$$

Notes

- Of course we can also complexify using $y(t) = \operatorname{Re} z(t)$.
- Using the polar form $A = R\,\mathrm{e}^{\mathrm{i}\phi}$, the solution of the preceding example can also be expressed as

$$y(t) = \operatorname{Im}\left(R\,\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\omega t}\right) = \operatorname{Im}\left(R\,\mathrm{e}^{\mathrm{i}(\omega t+\phi)}\right) = R\sin(\omega t+\phi).$$

Example (cont'd)

Since

$$\left|\frac{-a-\mathrm{i}\omega}{a^2+\omega^2}\right|=\left|\frac{1}{-a+\mathrm{i}\omega}\right|=\frac{1}{|-a+\mathrm{i}\omega|}=\frac{1}{\sqrt{a^2+\omega^2}},$$

our previously found particular solution of $y'=ay+\sin(\omega t)$ admits the two alternative representations

$$\begin{aligned}y(t)&=-\frac{\omega}{a^2+\omega^2}\cos(\omega t)-\frac{a}{a^2+\omega^2}\sin(\omega t)\\&=\frac{1}{\sqrt{a^2+\omega^2}}\sin(\omega t+\phi)\end{aligned}$$

with

$$\phi=\begin{cases}\arctan(\omega/a) & \text{if } a<0,\\ \arctan(\omega/a)+\pi & \text{if } a>0.\end{cases}$$

In fact any linear combination $A\cos(\omega t)+B\sin(\omega t)$ ($A,B\in\mathbb{R}$) can be brought into such a form (with $\cos$ or $\sin$), since

$$A\cos(\omega t)+B\sin(\omega t)=\mathrm{Re}\big((A-\mathrm{i}B)\mathrm{e}^{\mathrm{i}\omega t}\big)=\mathrm{Im}\big((B+\mathrm{i}A)\mathrm{e}^{\mathrm{i}\omega t}\big).$$

Pure Mathematicians would . . .

cf. the previous set of exercises

- start with the series representation $\mathrm{e}^z = \exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$, $z \in \mathbb{C}$, of the exponential function.
- Derive the functional equation $\exp(z+w) = (\exp z)(\exp w)$ ($z, w \in \mathbb{C}$) from this using the binomial theorem in the form $\frac{(z+w)^n}{n!} = \sum_{k=0}^{n} \frac{z^k}{k!} \frac{w^{n-k}}{(n-k)!}$.
- Define $\cos(x) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}x})$ and $\sin(x) = \mathrm{Im}(\mathrm{e}^{\mathrm{i}x})$, making Euler's Identity a trivial fact.
- Derive the powers series representations
$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}, \quad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$
by separating $\exp(\mathrm{i}x) = \sum_{n=0}^{\infty} \frac{(\mathrm{i}x)^n}{n!}$ into real and imaginary part.
- Derive all the well-known properties of cos, sin from their power series representations and the functional equation for the exponential function. As an example, we have $\cos^2 x + \sin^2 x = \left|\mathrm{e}^{\mathrm{i}x}\right|^2 = \mathrm{e}^{\mathrm{i}x} \cdot \overline{\mathrm{e}^{\mathrm{i}x}} = \mathrm{e}^{\mathrm{i}x} \cdot \mathrm{e}^{-\mathrm{i}x} = 1$.

Pure Mathematicians would … (cont'd)

- Define the famous numbers e and π by $\mathrm{e} = \exp(1)$ and

  $$\pi = 2 \times \text{smallest positive zero of } x \mapsto \cos x.$$

  That this zero is well-defined, follows from continuity of $\cos$ (which requires its own proof, of course) and the intermediate value theorem on account of $\cos(0) = 1 > 0$,

  $$\cos(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - < 1 - 2 + 16/24 = -1/3 < 0,$$

  where we have used the alternating series test for convergence and the corresponding limit estimation. As a by-product, we obtain $0 < \pi/2 < 2$ or $0 < \pi < 4$ (a rather weak estimate, which can be easily improved using, e.g., Newton's Iteration).

- Use $\cos(\pi/2) = 0$, $\sin(\pi/2)^2 + \cos(\pi/2)^2 = 1$ and $\sin' x = \cos x > 0$ for $x \in [0, \pi/2)$ to conclude that $\sin(\pi/2) = 1$, $\mathrm{e}^{\mathrm{i}\pi/2} = \cos(\pi/2) + \mathrm{i}\sin(\pi/2) = \mathrm{i}$, and $\mathrm{e}^{z+w} + \mathrm{e}^z\mathrm{e}^w$ to conclude further that $\mathrm{e}^{z+\mathrm{i}\pi/2} = \mathrm{e}^z\mathrm{e}^{\mathrm{i}\pi/2} = \mathrm{i}\,\mathrm{e}^z$, $\mathrm{e}^{z+\mathrm{i}\pi} = -\mathrm{e}^z$ and $\mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^z$ (hence $\mathrm{e}^{\pi\mathrm{i}} = -1$, $\mathrm{e}^{2\pi\mathrm{i}} = 1$).

Exercise

Suppose $A\colon I \to \mathbb{C}$, $t \mapsto A_1(t) + \mathrm{i}\, A_2(t)$ is differentiable (i.e., $A_1 = \operatorname{Re} A$ and $A_2 = \operatorname{Im} A$ are differentiable). Show that $I \to \mathbb{C}$, $t \mapsto \mathrm{e}^{A(t)}$ is differentiable as well, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{A(t)} = A'(t)\mathrm{e}^{A(t)}.$$

Hint: Start with
$\mathrm{e}^{A(t)} = \mathrm{e}^{A_1(t)+\mathrm{i}\,A_2(t)} = \mathrm{e}^{A_1(t)}\mathrm{e}^{\mathrm{i}\,A_2(t)} = \mathrm{e}^{A_1(t)}\cos A_2(t) + \mathrm{i}\,\mathrm{e}^{A_1(t)}\sin A_2(t)$.

The Analogy with Linear Recurring Sequences

We consider only the case $y' = ay + b$ with constant coefficients a, b, since for the discussion of linear recurring sequences in Discrete Mathematics the same assumption was made.

The discrete analog of $y' = ay + b$ is the 1st-order linear recurrence relation $x_{n+1} = ax_n + b$ (equivalently, $x_n = ax_{n-1} + b$).

Three ways to solve $x_n = ax_{n-1} + b$

1. *Direct solution.*

$$\begin{aligned}
x_1 &= ax_0 + b, \\
x_2 &= a(ax_0 + b) + b = a^2x_0 + (1 + a)b, \\
x_3 &= a\bigl(a^2x_0 + (1 + a)b\bigr) + b = a^3x_0 + (1 + a + a^2)b, \\
&\vdots \\
x_n &= a^nx_0 + (1 + a + \cdots + a^{n-1})b.
\end{aligned}$$

Three ways to solve $x_n = ax_{n-1} + b$ cont'd

2. *Use the theory developed in Discrete Mathematics.*
   The associated homogeneous linear recurrence relation $x_n = ax_{n-1}$ has the solution $x_n = c\,a^n$, $c \in \mathbb{R}$.
   A particular solution $x_n^{(p)}$ can be found by trying a constant $x_n^{(p)} = x$ and solving the resulting equation $x = ax + b$.
   This gives $x = \frac{b}{1-a}$, and the general solution is therefore

   $$x_n = c\,a^n + \frac{b}{1-a} \quad (c \in \mathbb{R}), \quad \text{provided that } a \neq 1.$$

   If $a = 1$ (the "resonance case"), we have $x_n = nb + x_0$.

3. *Use variation of parameters.*
   Setting $x_n = c(n)a^n = c_n a^n$, we have

   $$x_n = c_n a^n = a \cdot c_{n-1} a^{n-1} + b \iff c_n - c_{n-1} = ba^{-n}.$$

   This gives

   $$c_n = \sum_{k=1}^{n} ba^{-k} + c_0 \quad \text{and} \quad x_n = c_0 a^n + \sum_{k=1}^{n} ba^{n-k}.$$

Exercise

Verify that Methods 1 and 2 for solving $x_n = ax_{n-1} + b$ actually yield the same solution (although this is not directly visible in the formulas).

Hint: In the formula derived in Method 2, determine c in terms of x_0.