Math 286
Introduction to Differential Equations

Thomas Honold

ZJU-UIUC Institute

Fall Semester 2021

Outline

Today's Lecture: BESSEL's Differential Equation

Definition (Bessel's Differential Equation)

The 2nd-order linear ODE

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0,$$

with parameter $\nu \geq 0$ is known as *Bessel's Differential Equation*. For $\nu \in \mathbb{Z}$ solutions are called *cylinder functions of order* ν.

Rewriting Bessel's ODE as

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

shows that $x_0 = 0$ is a regular singular point with $p_0 = 1$, $q_0 = -\nu^2$, and that the corresponding indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \nu^2 = (r - \nu)(r + \nu).$$

$\Longrightarrow$ The exponents at the singularity $x_0 = 0$ are $r_1 = \nu$, $r_2 = -\nu$. This means we are in Case 1 (for $\nu \notin \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, Case 3 (for $\nu = 0$), or Case 4 (for $\nu \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, with $N = 2\nu$) of the theorem.

Part 2 of the theorem guarantees that one solution is obtained by the fractional power series „Ansatz“ $y(x) = \sum_{n=0}^{\infty} a_n x^{n+\nu}$.

$$\begin{aligned}
\implies L[y] &= x^2y'' + xy' + (x^2 - \nu^2)y = \\
&= \sum_{n=0}^{\infty}(n+\nu)(n+\nu-1)a_n x^{n+\nu} + \sum_{n=0}^{\infty}(n+\nu)a_n x^{n+\nu} \\
&\quad + \sum_{n=0}^{\infty} a_n x^{n+\nu+2} - \sum_{n=0}^{\infty} \nu^2 a_n x^{n+\nu} \\
&= x^{\nu}\left(0a_0 + (2\nu+1)a_1 x + \sum_{n=2}^{\infty}(n(n+2\nu))a_n + a_{n-2})x^n\right),
\end{aligned}$$

since $(n+\nu)(n+\nu-1)+(n+\nu)-\nu^2 = (n+\nu)^2 - \nu^2 = n(n+2\nu)$.
$L[y] = 0$ implies $a_1 = 0$ (since $2\nu + 1$ is ≥ 1 and hence nonzero) and

$$a_n = -\frac{a_{n-2}}{n(n+2\nu)} \quad \text{for } n \geq 2.$$

$$\begin{aligned}
\implies a_{2m+1} &= 0, \\
a_{2m} &= -\frac{a_{2(m-1)}}{4m(m+\nu)} = \cdots = \frac{(-1)^m}{m!\,4^m(\nu+1)(\nu+2)\cdots(\nu+m)}\,a_0.
\end{aligned}$$

Normalizing by $a_0 = 1$ gives the solution

$$y_1(x) = x^\nu \sum_{m=0}^\infty \frac{(-1)^m}{m!\,2^{2m}(\nu+1)(\nu+2)\cdots(\nu+m)}\,x^{2m} \quad \text{on } (0,\infty).$$

For $\nu \in \mathbb{N}_0$ a different normalization, which gives the coefficients a slightly simpler form, is $a_0 = \frac{1}{2^\nu \nu!}$. The corresponding solution is

$$\begin{aligned}\mathrm{J}_\nu(x) &= x^\nu \sum_{m=0}^\infty \frac{(-1)^m}{m!\,2^{2m+\nu}\nu!(\nu+1)(\nu+2)\cdots(\nu+m)}\,x^{2m}\\ &= \sum_{m=0}^\infty \frac{(-1)^m}{m!(m+\nu)!}\left(\frac{x}{2}\right)^{\nu+2m}.\end{aligned}$$

This makes also sense for non-integral ν, provided we interpret $(m+\nu)!$ as $\Gamma(m+\nu+1)$ (which is true for $\nu \in \mathbb{N}_0$).
In Exercise H62 of HW10 it is shown that $1/\Gamma$ can be continuously extended to $\mathbb{R}$. $\implies 1/\Gamma(m+\nu+1)$ is defined for all $m \in \mathbb{N}_0$ and $\nu \in \mathbb{R}$.

Definition

For $\nu \in \mathbb{R}$, the function

$$\mathrm{J}_\nu(x) = \sum_{m=0}^\infty \frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{\nu+2m}, \quad x \in (0,\infty),$$

is called *Bessel function of order* ν.

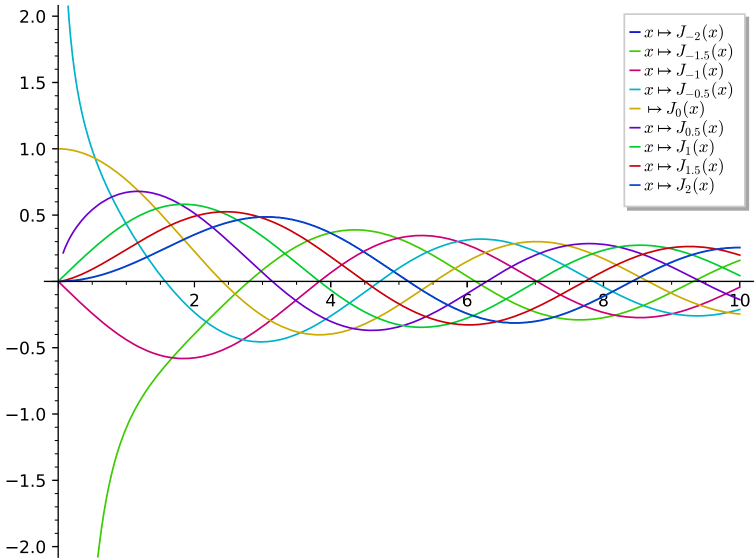

Figure: Bessel functions of orders $\nu = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ with domain $(0, \infty)$

For the analysis of the different cases of Bessel's Differential Equation (depending on ν) we switch back to the convention $\nu \geq 0$ adopted earlier.

Though it is not needed for most cases, we first determine the rational functions $a_n(r)$ arising from the condition $L[\phi] = L\left[\sum_{n=0}^{\infty} a_n(r)x^{r+n}\right] = F(r)x^r$.
Since $p(x) = 1/x$, $q(x) = 1 - \nu^2/x^2$, all coefficients p_i, q_i are zero except for $p_0 = 1$, $q_0 = -\nu^2$ and $q_2 = 1$. This gives

$$
\begin{aligned}
L\left[\phi\right] &= \sum_{n=0}^{\infty}\left(F(r+n)a_n(r) + \sum_{k=0}^{n-1}\left[(r+k)p_{n-k} + q_{n-k}\right]a_k(r)\right)x^{r+n} \\
&= F(r)a_0(r)x^r + F(r+1)a_1(r)x^{r+1} + \sum_{n=2}^{\infty}\left[F(r+n)a_n(r) + a_{n-2}(r)\right]x^{r+n}
\end{aligned}
$$

$$\implies a_0(r) = 1,\ a_1(r) = 0,\ a_n(r) = -\tfrac{a_{n-2}(r)}{F(r+n)} = -\tfrac{a_{n-2}(r)}{(r+n-\nu)(r+n+\nu)}$$

$$\implies a_{2m+1}(r) = 0, \quad a_{2m}(r) = \frac{(-1)^m}{\prod_{i=1}^{m}\left[(r+2i-\nu)(r+2i+\nu)\right]}.$$

(Check that for $r = \nu$ this reduces to the previous formula $a_{2m} = a_{2m}(\nu) = \frac{(-1)^m}{2^{2m}m!(\nu+1)(\nu+2)\cdots(\nu+m)}$.)

The case $\nu \notin \mathbb{Z}$

In this case we claim that there exists a fundamental system of solutions of the form

$$y_1(x) = x^{\nu} \sum_{n=0}^{\infty} a_n(\nu) x^n, \quad y_2(x) = x^{-\nu} \sum_{n=0}^{\infty} a_n(-\nu) x^n$$

with $a_0(\nu) = a_0(-\nu) = 1$.

We have already computed $y_1(x)$ and observed that $\mathrm{J}_\nu(x)$ is a constant multiple of $y_1(x)$.

For $r = -\nu$ we have $a_{2m+1}(-\nu) = 0$,

$$\begin{aligned} a_{2m}(-\nu) &= \frac{(-1)^m}{2 \cdot 4 \cdots 2m(2-2\nu)(4-2\nu)\cdots(2m-2\nu)} \\ &= \frac{(-1)^m}{2^{2m} m!(1-\nu)(2-\nu)\cdots(m-\nu)}, \end{aligned}$$

which is defined for all m, since $\nu \notin \mathbb{Z}$. Since $F(-\nu) = 0$, the function $y_2(x)$ defined in this way must then satisfy $L[y_2] = 0$. Moreover, y_1 and y_2 are linearly independent since $y_1(x) \simeq x^{\nu}$, $y_2(x) \simeq x^{-\nu}$ for $x \downarrow 0$.

The case $\nu \notin \mathbb{Z}$ cont'd

Multiplication of $y_2(x)$ with $\frac{2^\nu}{\Gamma(1-\nu)}$ yields $\mathrm{J}_{-\nu}(x)$ (use the functional equation $\Gamma(x+1) = x\,\Gamma(x)$ repeatedly) and shows that in this case the two Bessel functions J_ν, $\mathrm{J}_{-\nu}$ form a fundamental system of solutions.

Remark

For $\nu \in \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\right\}$ the number $N = r_1 - r_2 = 2\nu$ is a nonzero integer and Case 4 of our "big theorem" (Case 3 in [BDM17], Th. 5.6.1) applies. Thus it is rather surprising that there is such a simple formula for $y_2(x)$ (the same as in Case 1 of the theorem).

Explanation: Since $N = 2\nu$ is odd in this case, we have $a_N(r) = 0$ and hence $a = \lim_{r \to r_2}(r - r_2)a_N(r) = 0$. Thus the formula for $y_2(x)$ in Case 4 of the theorem contains no logarithmic term. Moreover, all functions $a_n(r)$ are analytic at $r_2 = -\nu$ and hence

$$\begin{aligned}\alpha_n'(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left[(r - r_2)a_n(r)\right] = a_n(r) + (r - r_2)a_n'(r),\\ \alpha_n'(r_2) &= a_n(r_2),\end{aligned}$$

reducing the formula for $y_2(x)$ to that in Case 1.

The case $\nu = 0$

In this case

$$\mathrm{J}_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2}\, x^{2m} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \left(\frac{x}{2}\right)^{2m}$$

is a solution.

$\mathrm{J}_0(x)$ is defined for $x \in \mathbb{R}$, as is easily shown using the ratio test. This is also guaranteed by the theorem, because $p(x) = 1/x$ and $q(x) = 1$ have no singularity except $x_0 = 0$.)

Note

J_0 solves the IVP

$$xy'' + y' + xy = 0,\ y(0) = 1,\ y'(0) = 0.$$

According to Case 3 (the case $r_1 = r_2$) of our theorem, there exists a 2nd fundamental solution (linearly independent of J_0) of the form

$$y_2(x) = \mathrm{J}_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n, \quad b_n = a_n'(0).$$

For $\nu = 0$ the coefficient functions $a_n(r)$ specialize to $a_{2m+1}(r) = 0$ and

$$a_{2m}(r) = \frac{(-1)^m}{(r+2)^2(r+4)^2\cdots(r+2m)^2}.$$

It follows that $a_n'(0) = 0$ for odd n, so that the 2nd summand in $y_2(x)$ is an even function of x, just like $\mathrm{J}_0(x)$.

For even n we use the fact that the *logarithmic derivative* $\mathrm{ld}(f) = f'/f$ (which in the case $f > 0$ coincides with $ln(f')$) satisfies

$$\frac{(f^a g^b)'}{f^a g^b} = \frac{(af^{a-1}f')g^b + f^a(bg^{b-1}g')}{f^a g^b} = a\,\frac{f'}{f} + b\,\frac{g'}{g} \quad \text{for } a, b \in \mathbb{Z}.$$

In particular $\mathrm{ld}(fg) = \mathrm{ld}(f) + \mathrm{ld}(g)$ and $\mathrm{ld}(f^a) = a\,\mathrm{ld}(f)$, relations that resemble those of the logarithm.

$$\implies \frac{a_{2m}'(r)}{a_{2m}(r)} = m\,\mathrm{ld}(-1) - 2\,\mathrm{ld}(r+2) - 2\,\mathrm{ld}(r+4) - \cdots - 2\,\mathrm{ld}(r+2m)$$
$$= 0 - \frac{2}{r+2} - \frac{2}{r+4} - \cdots - \frac{2}{r+2m} \quad \text{for } m \geq 1$$
$$\implies \frac{a_{2m}'(0)}{a_{2m}(0)} = -\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)$$

The numbers $\mathrm{H}_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$ are called *harmonic numbers*, because they form the partial sums of the harmonic series. In all we obtain, using $a_{2m}(0) = \frac{(-1)^m}{2^{2m}(m!)^2}$,

$$y_2(x) = \mathrm{J}_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m}(m!)^2}\, x^{2m}$$

Another choice for the 2nd fundamental solution is

$$\mathrm{Y}_0(x) = \frac{2}{\pi}\big(y_2(x) + (\gamma - \ln 2)\mathrm{J}_0(x)\big)$$
$$= \frac{2}{\pi}\left[\left(\ln\frac{x}{2} + \gamma\right)\mathrm{J}_0(x) + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}\mathrm{H}_m}{2^{2m}(m!)^2}\, x^{2m}\right],$$

where $\gamma = \lim_{n\to\infty}(\mathrm{H}_n - \ln n) \approx 0.577$ is the *Euler-Mascheroni constant*.

Definition

Y_0 is called *Neumann function* of order 0.

Other names are *Weber function* or *Bessel function of the 2nd kind* of order 0.

In contrast with J_0, the function Y_0 is not analytic at $x = 0$ (not even defined there) and satisfies

$$\mathrm{Y}_0(x) \simeq \frac{2}{\pi} \ln x \quad \text{for } x \downarrow 0.$$

If you want to learn more about J_0 and Y_0 (as well as about Bessel functions in general and many further so-called *special functions*), look for the *Handbook of Mathematical Functions* edited by M. Abramowitz and I. A. Stegun, the classic reference on this topic.

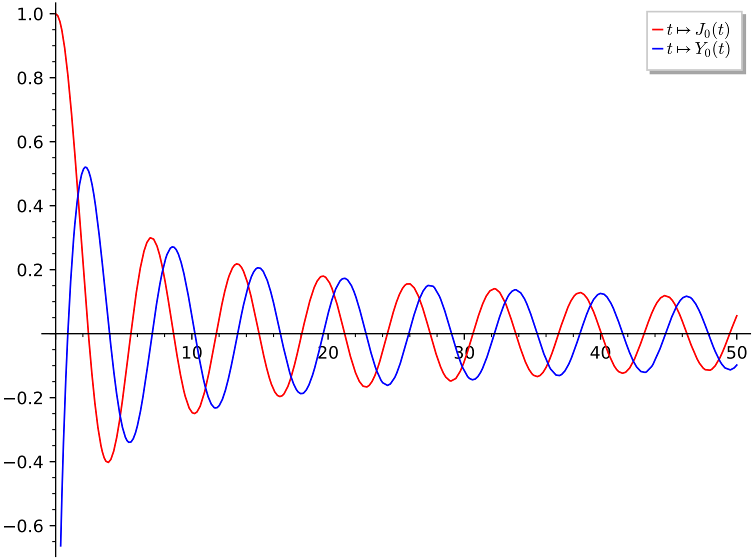

Figure: The Bessel and Neumann functions of order $\nu = 0$ with domain $\mathbb{R}^+$.

The case $\nu \in \mathbb{Z}^+$

In this case the Bessel function J_ν of order ν provides one solution, valid on the whole of $\mathbb{R}$. It is characterized as the unique solution that is analytic at $x_0 = 0$ and has normalization constant $a_0 = \frac{1}{2^\nu \nu!}$.

$$\mathrm{J}_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m+\nu)!} \left(\frac{x}{2}\right)^{\nu+2m} \quad \text{for } x \in \mathbb{R}.$$

Observe that $\mathrm{J}_\nu(0) = \mathrm{J}_\nu'(0) = \cdots = \mathrm{J}_\nu^{(\nu-1)}(0) = 0$ and $\mathrm{J}_\nu^{(\nu)}(0) = \nu! a_0 = \frac{1}{2^\nu}$.

A second solution $\mathrm{Y}_\nu(x)$, linearly independent of $\mathrm{J}_\nu(x)$, can be obtained in a similar (but increasingly more complicated) way as for $\nu = 0$. Since $N = 2\nu \in \mathbb{Z}^+$, Case 4 of our "big theorem" (Case 3 in [BDM17], Th. 5.6.1) applies, and there is no simplification this time. The case $\nu = 1$ is discussed as part of HW9, Ex. H59.

Remark

The function $\mathrm{J}_{-\nu}$ also solves the Bessel ODE on $\mathbb{R}$, but for $\nu \in \mathbb{Z}^+$ is linearly dependent on J_ν; cf. HW10, Ex. H62 c).

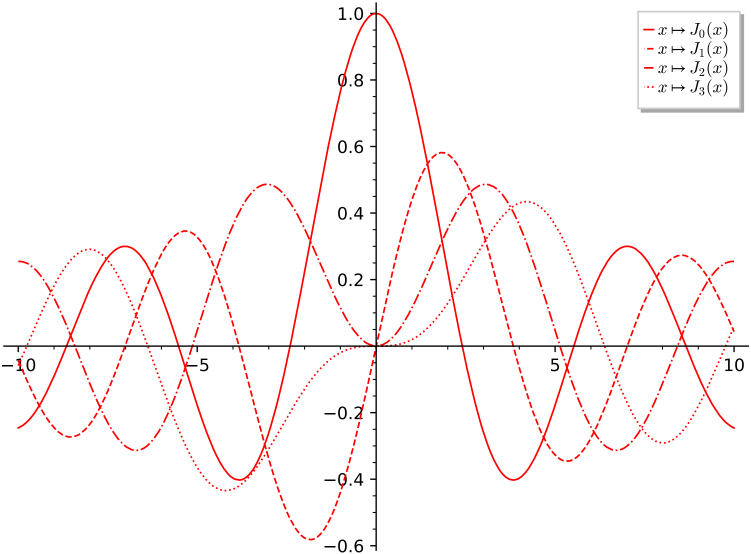

Figure: Bessel functions of various integral orders $\nu \geq 0$ with domain $\mathbb{R}$

The case $\nu = \frac{1}{2}$

This case is in a way special: The fractional power series „Ansatz" $y(x) = x^{-1/2} \sum_{n=0}^{\infty} a_n x^n$ yields two linearly independent solutions, since a_0 and a_1 can be chosen freely. For this recall that $L\left[\sum_{n=0}^{\infty} a_n x^{n\pm\nu}\right] = x^{\pm\nu}\left(0a_0 + (\pm 2\nu + 1)a_1 x + \cdots\right)$.
For $(a_0, a_1) = (1, 0)$ the recursion $a_n = -\frac{a_{n-2}}{n(n+2\nu)} = -\frac{a_{n-2}}{n(n-1)}$ yields $a_{2m-1} = 0$, $a_{2m} = \frac{(-1)^m}{(2m)!}$, and hence

$$y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m-1/2} = \frac{\cos x}{\sqrt{x}}.$$

For $(a_0, a_1) = (0, 1)$ the recursion similarly yields $a_{2m} = 0$, $a_{2m+1} = \frac{(-1)^m}{(2m+1)!}$, and hence

$$y(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} x^{2m+1/2} = \frac{\sin x}{\sqrt{x}}.$$

It follows that $\frac{\cos x}{\sqrt{x}}, \frac{\sin x}{\sqrt{x}}$ form a fundamental system of solutions of $x^2 y'' + xy' + (x^2 - \frac{1}{4})y = 0$, which can also be verified directly; cf. also HW8, Ex. H51.

The case $\nu = \frac{1}{2}$ cont'd

This case is of course contained in the case $\nu \notin \mathbb{Z}$ considered earlier, which tells us that the Bessel functions $\mathrm{J}_{1/2}$ and $\mathrm{J}_{-1/2}$ form a fundamental system of solutions. The link is best illustrated by computing $\mathrm{J}_{1/2}$ and $\mathrm{J}_{-1/2}$ from the general formula for J_ν :

$$\begin{aligned}
\mathrm{J}_{\frac{1}{2}}(x) &= \sqrt{\frac{x}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{m!\,\Gamma(m+\frac{3}{2})2^{2m}}\\
&= \sqrt{\frac{x}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{m!\,\Gamma(\frac{1}{2})\frac{3}{2}\frac{5}{2}\cdots\frac{2m+1}{2}2^{2m}}\\
&= \sqrt{\frac{x}{2}}\sum_{m=0}^{\infty}\frac{2(-1)^m x^{2m}}{(2m+1)!\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\,\sin x,\\
\mathrm{J}_{-\frac{1}{2}}(x) &= \sqrt{\frac{2}{x}}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m}}{m!\,\Gamma(m+\frac{1}{2})2^{2m}} = \cdots = \sqrt{\frac{2}{\pi x}}\,\cos x,
\end{aligned}$$

using $\Gamma(x+1) = x\,\Gamma(x)$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Thus $\mathrm{J}_{1/2}$ and $\mathrm{J}_{-1/2}$ are just scalar multiples of the fundamental solutions previously determined.

An Application of Bessel Functions

Solutions of the 2-dimensional wave equation

Theorem

Suppose $f\colon \mathbb{R}^+ \to \mathbb{C}$ is a C^2-function, $\nu \in \mathbb{Z}$, $\lambda, c > 0$, $D = \{(x,y,t) \in \mathbb{R}^3; x^2 + y^2 > 0\}$, and $u\colon D \to \mathbb{C}$ is defined by

$$u(x,y,t) = f(\lambda r)\mathrm{e}^{\mathrm{i}(\nu\phi \pm \lambda ct)}, \quad x = r\cos\phi,\ y = r\sin\phi.$$

Then u solves the 2-dimensional wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(x,y,t) = 0 \qquad \text{on } D,$$

iff f solves the Bessel ODE with parameter ν,

$$s^2 f''(s) + s\,f'(s) + \left(s^2 - \nu^2\right) f(s) = 0, \qquad s \in \mathbb{R}^+.$$

Notes

1. Solutions having the indicated form arise form the separation ansatz $u(x, y, t) = a(r)b(\phi)c(t)$.
2. The theorem can be used to determine the so-called normal modes of a vibrating circular membrane of radius R, for which u must also be defined and continuous at $(0, 0, t)$, and satisfy the boundary condition
$$u(x, y, t) = 0 \quad \text{if } x^2 + y^2 = R^2.$$
This is achieved by choosing f as a scalar multiple of J_ν, $\nu = 0, 1, 2, \dots$, and $\lambda = z_{\nu n}/R$, where $z_{\nu n}$ denotes the n-th positive zero of J_ν. (It can be shown that the positive zeros of J_ν form an infinite sequence $z_{\nu 1} > z_{\nu 2} > z_{\nu 3} > \cdots$.) See
https://commons.wikimedia.org/wiki/File:Vibrating_drum_Bessel_function.gif
for an animation.
3. The case $\nu = 0$ corresponds to rotation-invariant solutions of the 2-dimensional wave equation. Solutions satisfying the boundary conditions in (2) have the form
$u(x, y, t) = \mathrm{J}_0(\lambda r)\bigl(c_1 \mathrm{e}^{\mathrm{i}\lambda ct} + c_2 \mathrm{e}^{-\mathrm{i}\lambda ct}\bigr)$, $\lambda = z_{0n}/R$, $c_1, c_2 \in \mathbb{C}$.

Proof of the theorem.

We use the representation of the Laplace operator in polar coordinates (known from an exercise in Calculus III):

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial \phi^2}.$$

We have

$$\begin{aligned}\Delta u(x,y,t) &= \mathrm{e}^{\mathrm{i}(\nu\phi \pm \lambda ct)}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r}\right) f(\lambda r) + \frac{f(\lambda r)\mathrm{e}^{\pm\mathrm{i}\lambda ct}}{r^2}\,\frac{\partial^2 \mathrm{e}^{\mathrm{i}\nu\phi}}{\partial\phi^2}\\ &= \mathrm{e}^{\mathrm{i}(\nu\phi \pm \lambda ct)}\left(\lambda^2 f''(\lambda r) + \frac{\lambda}{r}\, f'(\lambda r) - \frac{\nu^2}{r^2}\, f(\lambda r)\right),\end{aligned}$$

$$\frac{\partial^2}{\partial t^2} u(x,y,t) = f(\lambda r)\mathrm{e}^{\mathrm{i}\nu\phi}\,\frac{\partial^2 \mathrm{e}^{\pm\mathrm{i}\lambda ct}}{\partial t^2} = -\lambda^2 c^2\, f(\lambda r)\mathrm{e}^{\mathrm{i}(\nu\phi \pm \lambda ct)}.$$

Since $\mathrm{e}^{\mathrm{i}\nu\phi \pm \mathrm{i}\lambda ct} \neq 0$, it follows that $u(x,y,t)$ solves the 2-dimensional wave equation iff

$$\lambda^2 f''(\lambda r) + \frac{\lambda}{r}\, f'(\lambda r) - \frac{\nu^2}{r^2}\, f(\lambda r) = -\lambda^2 f(\lambda r).$$

Multiplying this equation by r^2 and setting $s = \lambda r$ gives the Bessel ODE for $f(s)$, as asserted. $\square$

Exercise

Determine a fundamental system of solutions for Bessel's ODE with $\nu = \frac{1}{2}$,

$$y'' + \frac{1}{t}\,y' + \left(1 - \frac{1}{4t^2}\right) y = 0,$$

using the ansatz $z = \sqrt{t}\,y$. Then compare your result with that of the lecture.

Exercise

Determine the general solution of the following ODE's:

a) $(2t+1)y'' + (4t-2)y' - 8y = (6t^2 + t - 3)\mathrm{e}^t, \quad t > -1/2;$

b) $t^2(1-t)y'' + 2t(2-t)y' + 2(1+t)y = t^2, \quad 0 < t < 1.$

Hints: The associated homogeneous ODE in a) has a solution of the form $y(t) = \mathrm{e}^{\alpha t}$ and that in b) a solution of the form $y(t) = t^\beta$ with constants α, β. In both cases a particular solution of the inhomogeneous ODE can be determined by reducing it to a first-order system and using variation of parameters (though this may not be the most economic solution).