Math 286
Introduction to Differential Equations

Thomas Honold

ZJU-UIUC Institute

Fall Semester 2021

Outline

Today’s Lecture: Introduction

Teaching Staff

Lecturer

Prof. Dr. Thomas Honold
ZJU-UIUC Institute
Zhejiang University
International Campus, Haining
Office: Room C415, ZJUI Building
Office hours: t.b.a.
Email: `honold@zju.edu.cn`

Teaching Assistants

Niu Yiqun	`Yiqun.17@intl.zju.edu.cn`
Yu Chengting	`Chengting.17@intl.zju.edu.cn`

Weekly Timetable

Lecture

Mon, Fri,	16–17,	LTN-A 301/302
Wed,	15–17,	LTN-A 301/302

Discussion with TA's

There will be an informal weekly discussion group on Thu, 4–5 pm. (???)

Details will be announced later.

Homework

Homework is assigned on Thursdays (???) and must be handed in in the following week during the discussion session. Late homework will not be accepted.

Textbook

[BDM17] William E. Boyce, Richard C. DiPrima, Douglas B. Meade, *Elementary Differential Equations and Boundary Value Problems*, 11th global edition, Wiley, 2017.

Course Contents (tentative)

Week	Topics	[BDM17] Sections
1	Introduction to ODE's	Ch. 1
2–4	1st Order ODE's	Ch. 2
5,6	2nd-Order ODE's	Ch. 3
7	Higher Order ODE's	Ch. 4
8	Series Solutions	Ch. 5
9	Laplace Transform	Ch. 6
10–12	Linear Algebra continued, 1st Order ODE Systems	Ch. 7
13,14	PDE's	Ch. 10

Course material

Textbook + Lecture Slides + Exercises

The Weekly schedule is only approximately true.
The lecture won't follow the textbook strictly.

Examination Regulations

Calculation of the final score

- 45 % final exam (3 hours, closed book)
- 15 % 1 midterm exam (1 hour, closed book)
- 25 % homework
- 15 % lab project
- 5 % extra credit for presenting solutions of exercises

Exam dates will be announced in due course.

The lab projects will be assigned after the midterm, and details will be fixed at this time.

Course Website

As usual, lecture slides, homework assignments, and other accompanying material will be made available through Blackboard `https://c.zju.edu.cn`

Further details regarding homework submission, TA office hours, etc., will be announced later.

Some Advice Before We Start

- Attend each class!
- Solve (well, at least try hard to solve) each exercise!
- Don't hesitate to ask (stupid) questions!

The Subject of the Course

Informally, an *ordinary differential equation* (or ODE, for short) is an equation for a one-variable function $f(t)$ and its derivatives $f'(t)$, $f''(t)$, etc.
This is in contrast to *partial differential equations* (or PDE, for short), which involve multi-variable functions $f(x_1,\dots,x_n)$ and their partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, etc.

Definition (Ordinary Differential Equation, ODE)

An ODE of order n has the form

$$F(t,\mathbf{y},\mathbf{y}',\mathbf{y}'',\dots,\mathbf{y}^{(n)})=0, \tag{$\star$}$$

where F has domain $D\subseteq\mathbb{R}\times\underbrace{\mathbb{R}^m\times\cdots\times\mathbb{R}^m}_{n+1}$ and depends on the last variable (otherwise the order is $<n$).

A *solution* to ($\star$) is a function (curve) $f\colon I\to\mathbb{R}^m$, defined on an interval $I\subseteq\mathbb{R}$, which is n times differentiable and satisfies $\quad F\bigl(t,f(t),f'(t),f''(t),\dots,f^{(n)}(t)\bigr)=0\quad$ for all $t\in I$.

Definition (Initial Value Problem, IVP)

Suppose that an ODE as above is given and $t_0 \in \mathbb{R}$, $\mathbf{y}_0, \dots, \mathbf{y}_n \in \mathbb{R}^m$ are such that $F(t_0, \mathbf{y}_0, \dots, \mathbf{y}_n) = 0$. A solution to the *initial value problem*

$$F(t, \mathbf{y}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^{(n)}) = 0, \quad \mathbf{y}^{(i)}(t_0) = \mathbf{y}_i \ \textit{ for } 0 \leq i \leq n$$

is any function (curve) $f \colon I \to \mathbb{R}^m$ solving ($\star$) on the previous slide and satisfying $f(t_0) = \mathbf{y}_0$, $f'(t_0) = \mathbf{y}_1$, ..., $f^{(n)}(t_0) = \mathbf{y}_n$.

Notes

- It is custom to use "no-name notation" $y = y(t)$ if $m = 1$ (resp., $\mathbf{y} = \mathbf{y}(t)$ if $m > 1$) for solutions of ODE's.
- Denoting the "independent" variable by t reflects the virtually zillions of applications in Physics, where $\mathbf{y}(t)$ models the state of a physical system at time t. Be prepared, however, that many texts on ODE's use x in place of t, i.e., $y(x)$ or $\mathbf{y}(x)$ for the solution function of an ODE.

Notes cont'd

- While our definition of ODE's and IVP's is the most general, the following *explicit* form of an n-th order ODE occurs most frequently:
$$\mathbf{y}^{(n)} = G(t_0, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n-1)}).$$
A corresponding implicit form is $F(t_0, \mathbf{y}, \mathbf{y}', \dots, \mathbf{y}^{(n)}) = 0$ with $F(t_0, \mathbf{y}_0, \dots, \mathbf{y}_n) = \mathbf{y}_n - G(t_0, \mathbf{y}_0, \dots, \mathbf{y}_{n-1})$.
An IVP in explicit form needs to specify only $\mathbf{y}^{(i)}(t_0) = \mathbf{y}_i$ for $0 \leq i \leq n-1$, since the last condition
$$\begin{aligned}\mathbf{y}^{(n)}(t_0) = \mathbf{y}_n &= G(t_0, \mathbf{y}_0, \mathbf{y}_1, \dots, \mathbf{y}_{n-1}) \\ &= G(t_0, \mathbf{y}(t_0), \mathbf{y}'(t_0), \dots, \mathbf{y}^{(n-1)}(t_0))\end{aligned}$$
is a particular case of the explicit ODE.
- Sometimes it is easy to find solutions (or a family of solutions) to a given ODE, and the question arises whether there are further solutions. Since, trivially, restricting a solution $y\colon I \to \mathbb{R}$ to a subinterval $J \subset I$ yields another solution, we are only interested in *maximal solutions*, i.e., those which do not arise by (proper) restriction from another solution.

Ten Examples

1. $y' = a(t)$; more generally, $\mathbf{y}' = \mathbf{a}(t)$
   $F(t, y_0, y_1) = a(t) - y_1$ resp. $F(t, \mathbf{y}_0, \mathbf{y}_1) = \mathbf{a}(t) - \mathbf{y}_1$
2. $y' = y$; more generally, $\mathbf{y}' = \mathbf{y}$
   $F(t, y_0, y_1) = y_1 - y_0$ resp. $F(t, \mathbf{y}_0, \mathbf{y}_1) = \mathbf{y}_1 - \mathbf{y}_0$
3. $y' = ay$ with $a \in \mathbb{R}$; more generally, $\mathbf{y}' = \mathbf{A}\mathbf{y}$ with $\mathbf{A} \in \mathbb{R}^{n \times n}$
   $F(t, y_0, y_1) = y_1 - ay_0$ resp. $F(t, \mathbf{y}_0, \mathbf{y}_1) = \mathbf{y}_1 - \mathbf{A}\mathbf{y}_0$
4. $y' = ay + b$ with $a, b \in \mathbb{R}$; more generally, $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$
   $F(t, y_0, y_1) = y_1 - ay_0 - b$
5. $y' = 2ty$; more generally, $\mathbf{y}' = 2t\,\mathbf{y}$
   $F(t, y_0, y_1) = y_1 - 2ty_0$
6. $y' = y^2$
   $F(t, y_0, y_1) = y_1 - y_0^2$
7. $y' = \sqrt{|y|}$
   $F(t, y_0, y_1) = y_1 - \sqrt{|y_0|}$

Ten Examples Cont'd

8. $(\mathrm{e}^{x-y} + 2x - \mathrm{e})\,\mathrm{d}x - (\mathrm{e}^{x-y} + 2y)\,\mathrm{d}y = 0$
   $F(t, x_0, y_0, x_1, y_1) = (\mathrm{e}^{x_0 - y_0} + 2x_0 - \mathrm{e})x_1 - (\mathrm{e}^{x_0 - y_0} + 2y_0)y_1$
9. $y' = -x/y$
   $F(x, y_0, y_1) = y_1 + x/y_0$
10. $y'' + y = 0$ (or, in explicit form, $y'' = -y$)
    $F(t, y_0, y_1, y_2) = y_2 + y_0$ (resp., $G(t, y_0, y_1) = -y_0$)

The first nine examples have order 1. The last example has order 2.

Examples 8 and 10 are implicit ODE's. The remaining examples are explicit ODE's.

Reading assignment for Week 1

[BDM17], Chapter 1

Ten Examples Cont'd

Solutions

1. $y' = a(t)$: Assuming that $t \mapsto a(t)$ is continuous, the general solution is $y(t) = \int a(t)\,\mathrm{d}t$ (by the Fundamental Theorem of Calculus).

   The solution of the IVP $y' = a(t)$, $y(t_0) = y_0$ is $y(t) = y_0 + \int_{t_0}^{t} a(\tau)\,\mathrm{d}\tau$.

   The solution of the IVP $\mathbf{y}' = \mathbf{a}(t)$, $\mathbf{y}(t_0) = \mathbf{y}^{(0)}$ is

$$\mathbf{y}(t) = \begin{pmatrix} y_1^{(0)} + \int_{t_0}^{t} a_1(\tau)\,\mathrm{d}\tau \\ \vdots \\ y_m^{(0)} + \int_{t_0}^{t} a_m(\tau)\,\mathrm{d}\tau \end{pmatrix},$$

   where $\mathbf{a}(t) = \big(a_1(t), \dots, a_m(t)\big)^{\mathsf{T}}$.

Solutions cont'd

2. $y' = y$: A solution is $y(t) = \mathrm{e}^t$, or more generally $y(t) = c\,\mathrm{e}^t$ with $c \in \mathbb{R}$ (all with maximal domain $\mathbb{R}$).

   Using this family of solutions, we can solve any IVP $y' = y \wedge y(t_0) = y_0$: Just solve $c\mathrm{e}^{t_0} = y_0$ for c, i.e., $c = y_0\mathrm{e}^{-t_0}$ and $y(t) = y_0\mathrm{e}^{t-t_0}$.

   Are there other (maximal) solutions?
   No there aren't, since $y' = y$ implies

   $$(y\mathrm{e}^{-t})' = y'\mathrm{e}^{-t} + y(-\mathrm{e}^{-t}) = (y' - y)\mathrm{e}^{-t} = 0.$$

   $\implies y\mathrm{e}^{-t} = c \in \mathbb{R}$ is a constant.

   These results imply that through any point (t_0, y_0) in the plane $\mathbb{R}^2$ there passes exactly one solution of $y' = y$. In other words, the graphs of the family of functions $y(t) = c\mathrm{e}^t$, $c \in \mathbb{R}$ (with domain $\mathbb{R}$) partition the plane.

   It also follows that the general solution of $\mathbf{y}' = \mathbf{y}$ is $\mathbf{y}(t) = (c_1\mathrm{e}^t, \dots, c_m\mathrm{e}^t) = \mathrm{e}^t\mathbf{c}$ with $\mathbf{c} = (c_1, \dots, c_m) \in \mathbb{R}^m$ or, linking it to the corresponding IVP, $\mathbf{y}(t) = \mathrm{e}^{t-t_0}\mathbf{y}_0$ with $\mathbf{y}_0 \in \mathbb{R}^m$.

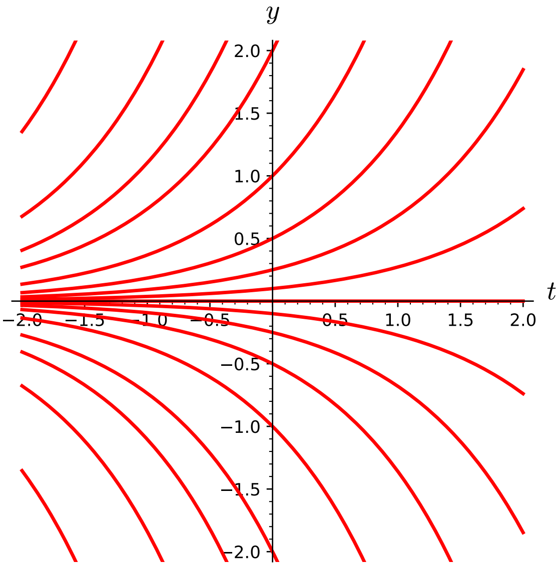

Figure: The solutions $y(t) = c\,\mathrm{e}^t$ for various constants c

Solutions cont'd

3. $\underline{y' = ay}$: This is almost the same as Equation 2. The solutions are $y(t) = c\mathrm{e}^{at}$, $c \in \mathbb{R}$, and the corresponding IVP has a unique solution for each (t_0, y_0). That there are no more solutions is proved in the same way, working with $t \mapsto y(t)\mathrm{e}^{-at}$.

   For the vectorized form a keen guess is that the solution has the form $\mathbf{y}(t) = \mathrm{e}^{\mathbf{A}t}\mathbf{c}$ with $\mathbf{c} \in \mathbb{R}^n$. This is in fact true, as we will see later. For now let us only note that in order to make sense of this $\mathrm{e}^{\mathbf{A}t}$ should be an $n \times n$ matrix as well.

4. $\underline{y' = ay + b}$: If y_1 and y_2 are solutions of $y' = ay + b$ then $y_1 - y_2$ is a solution of the associated *homogeneous* linear ODE $y' = ay$, since

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(y_1(t) - y_2(t)\big) &= \frac{\mathrm{d}y_1(t)}{\mathrm{d}t} - \frac{\mathrm{d}y_2(t)}{\mathrm{d}t} = a\,y_1(t) + b - a\,y_2(t) - b \\ &= a\big(y_1(t) - y_2(t)\big).\end{aligned}$$

   A particular solution of $y' = ay + b$ is $y(t) \equiv -b/a$, and hence the general solution is $y(t) = -b/a + c\mathrm{e}^{at}$, $c \in \mathbb{R}$ (with domain $\mathbb{R}$). From this we see as before that every IVP $y' = ay + b \wedge y(t_0) = y_0$, has a unique solution.

Solutions cont'd

5. $y' = 2ty$: The solutions are $y(t) = c\,\mathrm{e}^{t^2}$, $c \in \mathbb{R}$, and the observed general picture continues to prevail. In particular, we can use the function $t \mapsto y(t)\mathrm{e}^{-t^2}$ to show that there are no more solutions. (If $y(t)$ is a solution, this function must again be constant.) You should also compare this with the solution to Exercise H71 in Homework 12 of Calculus III.

   Clearly something more general works behind the scene in the examples discussed so far. All these are instances of 1st-order linear ODE's, and we shall present a unified treatment of this class of ODE's later.

Solutions cont'd

6. $\underline{y' = y^2}$: It is not hard to guess a solution. One solution is $y(t) = -1/t$, since $(-1/t)' = 1/t^2 = (-1/t)^2$.
An even more obvious one is $y(t) \equiv 0$.
Since $y' = G(y)$ with G not depending on t (such ODE's are called *autonomous*), we can make a translation in the argument t to obtain further solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, y(t-c) = y'(t-c) = y(t-c)^2,$$

i.e., $t \mapsto y(t-c)$ is a solution for any $c \in \mathbb{R}$ (provided that $y(t)$ is). In the case under consideration this gives the family of solutions

$$y(t) = -\frac{1}{t-c} = \frac{1}{c-t}, \quad c \in \mathbb{R}.$$

Since we can solve $\frac{1}{c-t_0} = y_0$ uniquely for c if (t_0, y_0) is not on the t-axis (i.e., $y_0 \neq 0$), the solutions found so far (including $y \equiv 0$) partition the plane $\mathbb{R}^2$, and any IVP $y' = y^2 \wedge y(t_0) = y_0$ has a unique solution within this family.

Solutions cont'd

6. (cont'd)
In contrast with the preceding examples, $y(t) = 1/(c-t)$ is not defined for all $t \in \mathbb{R}$. In fact, according to our definition of "solution of an ODE" we rather have two maximal solutions corresponding to a fixed c,

$$\begin{aligned} y_1(t) &= 1/(c-t), \quad t \in (-\infty, c), \\ y_2(t) &= 1/(c-t), \quad t \in (c, +\infty). \end{aligned}$$

Unique solvabilty of the corresponding IVP is not affected by this change of viewpoint. (The solutions $y_1(t)$ are obtained for initial values $y(t_0) > 0$, the solutions $y_2(t)$ for $y(t_0) < 0$.)

Now we show that there are no further (maximal) solutions.

Firstly, suppose $y\colon I \to \mathbb{R}$ is a solution with $y(t) \neq 0$ for $t \in I$. Then $y'(t)/y^2(t) = 1$ on I. Fixing some $t_0 \in I$ and integrating, we get

$$t - t_0 = \int_{t_0}^{t} \frac{y'(\tau)}{y(\tau)^2}\,\mathrm{d}\tau = \int_{y(t_0)}^{y(t)} \frac{\mathrm{d}\eta}{\eta^2} = \left[-\frac{1}{\eta}\right]_{y(t_0)}^{y(t)} = \frac{1}{y(t_0)} - \frac{1}{y(t)}.$$

Solutions cont'd

6. (cont'd)

This holds for $t \in I$ and can be solved for $y(t)$:

$$y(t) = \frac{1}{t_0 + y(t_0)^{-1} - t} = \frac{1}{c - t} \quad \text{with } c = t_0 + y(t_0)^{-1}.$$

Hence I is contained in either $(-\infty, c)$ or $(c, +\infty)$, and $y(t)$ coincides with one of the corresponding solutions $y_1(t)$ or $y_2(t)$ on I.

Secondly suppose there exists $t_0, t_1 \in I$ such that $y(t_0) \neq 0$, $y(t_1) = 0$. We may assume $t_1 < t_0$ (the case that $y(t)$ "branches from $y \equiv 0$", the other case being similar) and $y(t) > 0$ for $t \in (t_1, t_0]$ (since $y(t)$ is continuous, it has a largest zero in I, which we can choose as t_1).

On one hand we now have $\lim_{t \downarrow t_1} y(t) = y(t_1) = 0$. On the other hand, the first case applies to $t \in (t_1, t_0]$ and yields

$$y(t) = \frac{1}{t_0 + y(t_0)^{-1} - t} \geq \frac{1}{t_0 + y(t_0)^{-1} - t_1} > 0 \quad \text{for } t \in (t_1, t_0].$$

This contradiction shows that the second case does not occur.

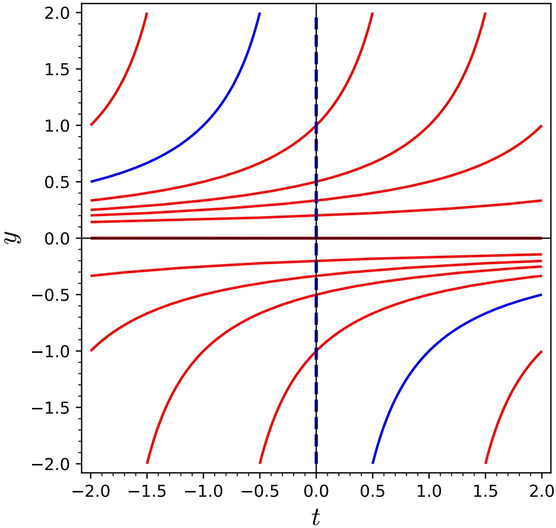

Figure: The solutions $y(t) = 1/(c - t)$ for various constants c (including ∞), with the one for $c = 0$ and its asymptote colored blue

Remark on Notation

No-name notation $y(t)$ for solutions of ODE's can easily cause confusion, when more than one solution is considered, e.g., when we say “all solutions are horizontal shifts of a particular solution”. At least in such cases we should follow good mathematical practice and specify solutions, which are mathematical functions, in full—like this:

Three particular maximal solutions of $y' = y^2$ are

$$\begin{aligned} f_1 &\colon (-\infty, 0) \to \mathbb{R},\ t \mapsto -1/t, \\ f_2 &\colon (0, +\infty) \to \mathbb{R},\ t \mapsto -1/t, \\ f_3 &\colon \mathbb{R} \to \mathbb{R},\ t \mapsto 0. \end{aligned}$$

Every further solution arises from one of these solutions by a horizontal shift and/or restriction to a subinterval, i.e., it is zero or has the form

$$g\colon I \to \mathbb{R},\ t \mapsto -1/(t-c)$$

for some $c \in \mathbb{R}$ and some interval I contained in either $(-\infty, c)$ or $(c, +\infty)$.

Solutions cont'd

7. $\underline{y' = \sqrt{|y|}}$: Again $y \equiv 0$ is an obvious solution, and further solutions may be guessed: Since $y(t) = t^2$ satisfies $y'(t) = 2t = 2\sqrt{y}$ on $[0, +\infty)$, we can scale this "approximate solution" by an appropriate constant, viz. $\frac{1}{4}$, to obtain the real solution $y(t) = \frac{1}{4}t^2$, $t \in [0, +\infty)$. Note that both $y(t) \equiv 0$ and $y(t) = \frac{1}{4}t^2$ solve the IVP $y' = \sqrt{|y|}$, $y(0) = 0$.

   Since $y' = \sqrt{|y|}$ is autonomous, $y(t) = \frac{1}{4}(t-c)^2$, $t \in [c, +\infty)$, is a solution for any $c \in \mathbb{R}$, and so is $y(t) = -\frac{1}{4}(t-c)^2$, $t \in (-\infty, c]$.

   Since on the t-axis all solutions have horizontal tangents ($y = 0 \implies y' = \sqrt{|y|} = 0$), we can "glue together" the different types of solutions and obtain the following 2-parameter family of solutions with domain $\mathbb{R}$:

$$y(t) = \begin{cases} -\frac{1}{4}(t-c_1)^2 & \text{if } t \leq c_1, \\ 0 & \text{if } c_1 \leq t \leq c_2, \\ \frac{1}{4}(t-c_2)^2 & \text{if } t \geq c_2. \end{cases}$$

   Here $-\infty \leq c_1 \leq c_2 \leq +\infty$; equality in either of these indicates that the corresponding section is omitted.

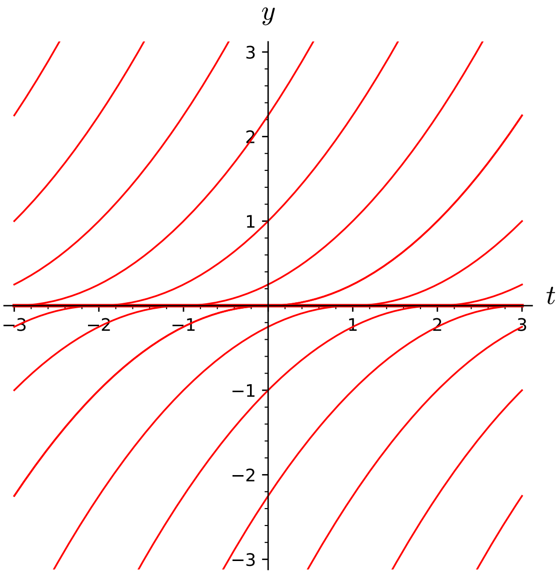

Figure: Solutions of $y' = \sqrt{|y|}$

Solutions cont'd

7. (cont'd)
   It can be shown that these are all maximal solutions.

   Brief sketch: For solutions not meeting the t-axis use a similar argument as for $y' = y^2$. A solution meeting the t-axis must meet it in an interval, since it can only flow in from below and branch off above, hence never return. Since the solutions are continuous, the interval must be closed. Denote it by $[c_1, c_2]$ and show that the solution is as stated on the previous slide.

Solutions cont'd

8. $(\mathrm{e}^{x-y} + 2x - \mathrm{e})\,\mathrm{d}x - (\mathrm{e}^{x-y} + 2y)\,\mathrm{d}y = 0$

   By a solution of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ we mean a smooth curve $\gamma(t) = \bigl(x(t), y(t)\bigr)$, $t \in I$, satisfying

   $$M\bigl(x(t),y(t)\bigr)x'(t) + N\bigl(x(t),y(t)\bigr)y'(t) = 0 \quad \text{for all } t \in I;$$

   equivalently,

   $$\begin{pmatrix} M\bigl(\gamma(t)\bigr) \\ N\bigl(\gamma(t)\bigr) \end{pmatrix} \cdot \gamma'(t) = 0 \quad \text{for all } t \in I.$$

   Geometrically this says that the tangents to γ at every regular point should be orthogonal to the vector field (M, N) (the vector field corresponding to $\omega = M\,\mathrm{d}x + N\,\mathrm{d}y$) at this point.

   In the case under consideration ω is exact, $\omega = \mathrm{d}f$ for $f(x,y) = \mathrm{e}^{x-y} + x^2 - \mathrm{e}x - y^2$, and hence the parametrized contours $\mathrm{e}^{x-y} + x^2 - \mathrm{e}x - y^2 = c$, $c \in \mathbb{R}$, of f provide solutions to $(\mathrm{e}^{x-y} + 2x - \mathrm{e})\,\mathrm{d}x - (\mathrm{e}^{x-y} + 2y)\,\mathrm{d}y = 0$.

   Accordingly, ODE's of the special form $f_x\,\mathrm{d}x + f_y\,\mathrm{d}y = 0$ are said to be *exact*.

Solutions cont'd

8. The Implicit Function Theorem gives that every IVP

$$(\mathrm{e}^{x-y}+2x-\mathrm{e})\,\mathrm{d}x-(\mathrm{e}^{x-y}+2y)\,\mathrm{d}y=0,$$
$$\gamma(t_0)=(x_0,y_0)\neq\left(\frac{\mathrm{e}-\mathrm{e}^{\mathrm{e}/2}}{2},-\frac{\mathrm{e}^{\mathrm{e}/2}}{2}\right)$$

(the unique critical point of f) has a solution. The point curve $\gamma(t)\equiv\left(\frac{\mathrm{e}-\mathrm{e}^{\mathrm{e}/2}}{2},-\frac{\mathrm{e}^{\mathrm{e}/2}}{2}\right)$, as well as any other point curve $\gamma(t)\equiv(x_0,y_0)$, trivially satisfies the ODE but isn't counted as a solution, since it is not smooth.

Solutions are highly non-unique, since we can choose the parametrization of the contours freely.

Further, the Implicit Function Theorem gives that at every point (x_0,y_0) with $f_y(x_0,y_0)=-(\mathrm{e}^{x_0-y_0}+2y_0)\neq 0$ the corresponding contour admits locally a parametrization $y(x)$, which must be a solution of the ODE

$$y'=\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{f_x}{f_y}=\frac{\mathrm{e}^{x-y}+2x-\mathrm{e}}{\mathrm{e}^{x-y}+2y}.$$

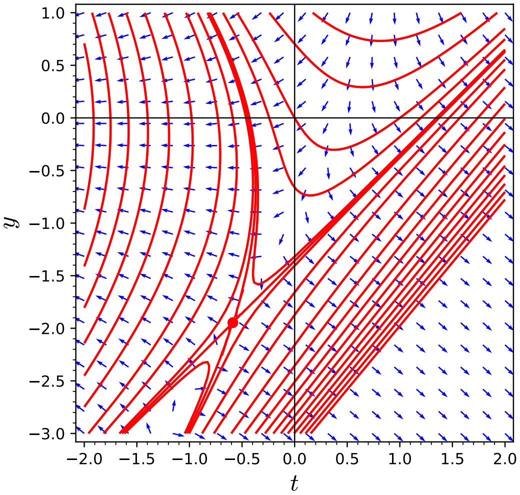

Figure: Solutions of $(\mathrm{e}^{x-y}+2x-\mathrm{e})\,\mathrm{d}x-(\mathrm{e}^{x-y}+2y)\,\mathrm{d}y=0$ in implicit form (red), and the gradient field of $f(x,y)=\mathrm{e}^{x-y}+x^2-\mathrm{e}x-y^2$ normalized to unit length (blue) The critical point $\left(\frac{\mathrm{e}-\mathrm{e}^{\mathrm{e}/2}}{2},-\frac{\mathrm{e}^{\mathrm{e}/2}}{2}\right)\approx(-0.59,-1.95)$ is on two intersecting solution curves.

Remark on the plot

If you look carefully at the two solution curves through the critical point $\mathbf{p}_0 = (x_0, y_0)$, you can see that they have been approximated by line segments, the reason being that the contour plot function of SageMath doesn't draw the correct picture near $\mathbf{p}_0$.

The tangent directions at $\mathbf{p}_0$ of the two curves can be found from the Taylor approximation

$$f(\mathbf{p}_0 + \mathbf{h}) = f(\mathbf{p}_0) + \tfrac{1}{2}\mathbf{h}^{\mathsf{T}}\mathbf{H}_f(\mathbf{p}_0)\mathbf{h} + \mathrm{o}(|\mathbf{h}|^2) \quad \text{for } \mathbf{h} \to \mathbf{0}.$$

From this one sees using *Morse's Lemma* that the contour of f through $\mathbf{p}_0$ is, after translation of $\mathbf{p}_0$ into the origin, locally well-approximated by the 0-contour of the Hesse quadratic form

$$\begin{aligned} q(\mathbf{h}) = \mathbf{h}^{\mathsf{T}}\mathbf{H}_f(\mathbf{p}_0)\mathbf{h} &= (\mathrm{e}^{e/2} + 2)h_1^2 - 2\mathrm{e}^{e/2}h_1h_2 + (\mathrm{e}^{e/2} - 2)h_2^2 \\ &= (\mathrm{e}^{e/2} - 2)(h_2 - h_1)\left(h_2 - \tfrac{\mathrm{e}^{e/2}+2}{\mathrm{e}^{e/2}-2}h_1\right), \end{aligned}$$

which is the union of two lines (expressing the fact that $\mathbf{p}_0$ is a saddle point of f). The said tangent directions are the directions of these lines, i.e., have slopes 1 and $\frac{\mathrm{e}^{e/2}+2}{\mathrm{e}^{e/2}-2} \approx 3.11$.

Solutions cont'd

9. $y' = -x/y$

This ODE can be written as $y'y + x = 0$ and integrated to yield $\frac{y^2}{2} + \frac{x^2}{2} = C = \frac{C'}{2} \in \mathbb{R}$. The solutions are therefore the half-circle parametrizations

$$y(x) = \pm\sqrt{r^2 - x^2}, \quad x \in (-r, r) \quad (\text{with } r > 0).$$

Every IVP $y' = -x/y \wedge y(x_0) = y_0 \neq 0$ has a unique solution, as is easily seen.

Alternatively, we can rewrite

$$y' = \frac{\mathrm{d}y}{dx} = -\frac{x}{y} \quad \text{as} \quad x\,\mathrm{d}x + y\,\mathrm{d}y = 0,$$

which is exact with anti-derivative $f(x, y) = \frac{x^2+y^2}{2}$. This gives whole-circles centered as $(0, 0)$, which are the contours of f, as implicit solutions. The exceptional role of the x-axis, visible in the original explicit ODE, has gone away.

Solutions cont'd

10. $y'' + y = 0$

Two particular solutions of $y'' = -y$ are $y_1(t) = \cos t$ and $y_2(t) = \sin t$ (with domain $\mathbb{R}$). For $A, B \in \mathbb{R}$, since

$$\begin{aligned}(A\cos t + B\sin t)'' &= A(\cos t)'' + B(\sin t)'' = -A\cos t - B\sin t\\ &= -(A\cos t + B\sin t),\end{aligned}$$

we obtain further solutions $y(t) = A\cos t + B\sin t$ (also with domain $\mathbb{R}$).

Since $y(0) = A\cos 0 + B\sin 0 = A$, $y'(0) = -A\sin 0 + B\cos 0 = B$, there is exactly one solution of any IVP $\quad y'' = -y \wedge y(0) = A \wedge y'(0) = B$ $(A, B \in \mathbb{R})$.

Similarly, one can show that the IVP

$$y'' = -y, \quad y(t_0) = y_0,\ y'(t_0) = y_1 \quad (t_0, y_0, y_1 \in \mathbb{R}).$$

has exactly one solution of the said form $y(t) = A\cos t + B\sin t$ (Exercise). This means that the graphs of the maps $t \mapsto \big(y(t), y'(t)\big)$, or traces of the curves $t \mapsto \big(t, y(t), y'(t)\big)$, with $y(t) = A\cos t + B\sin t$, $A, B \in \mathbb{R}$, partition the space $\mathbb{R}^3$. (Can you imagine that?).

Solutions cont'd

10. (con't)

Now we show that every solution $y(t)$ of $y'' = -y$ has the form $A\cos t + B\sin t$.

A trick that will do the job is considering the function $t \mapsto y(t)^2 + y'(t)^2$. Since

$$(y^2 + y'^2)' = 2yy' + 2y'y'' = 2y'(y + y'') = 0,$$

this function must be constant for any solution of $y'' = -y$.

Now let $y(t)$ be any solution and set $A = y(0)$, $B = y'(0)$. We have already a solution in our family with these initial values, viz. $z(t) = A\cos t + B\sin t$. The difference $d(t) = y(t) - z(t)$ is then also a solution and satisfies $d(0) = d'(0) = 0$. Since $d^2 + d'^2$ is constant, we have

$$d(t)^2 + d'(t)^2 = d(0)^2 + d'(0)^2 = 0^2 + 0^2 = 0.$$

This obviously implies $d(t) \equiv 0$, i.e., $y(t) = z(t) = A\cos t + B\sin t$.

Exercise

We have considered the ODE $y' = -x/y$ as an example. Actually there are four ODE's $y' = \pm x/y$ and $y' = \pm y/x$, which look very similar. Draw direction fields for the other three ODE's and determine their solutions in both implicit and explicit form (if possible).

Exercise

Let $t_0, y_0, y_1 \in \mathbb{R}$. Show that the IVP

$$y'' = -y, \quad y(t_0) = y_0,\ y'(t_0) = y_1$$

has a unique solution.

Supplementary Remarks on the Material in [BDM17], Chapter 1

Direction fields

Let $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$, be a function

Solving the 1st-order ODE $y' = f(t, y)$ amounts to finding a function $y = y(t)$, defined on some interval $I \subseteq \mathbb{R}$, and such that for all $t \in I$

1. $\bigl(t, y(t)\bigr) \in D$;
2. the slope of the graph of y at the point $\bigl(t, y(t)\bigr)$ equals $f\bigl(t, y(t)\bigr)$. Alternatively, the tangent direction to the graph at $\bigl(t, y(t)\bigr)$ is represented by the vector $\bigl(1, f(t, y(t))\bigr)$.

We can illustrate this by attaching to sample points $(t, y) \in D$ a small line segment with direction $\bigl(1, f(t, y)\bigr)$ (or a positive multiple of this vector). This is called a *slope field* or *direction field* of $y' = f(t, y)$. Solving $y' = f(t, y)$ graphically then amounts to finding a function $y = y(t)$ such that the tangent direction of the graph of y at every sample point encountered is given by the corresponding line segment.

Examples

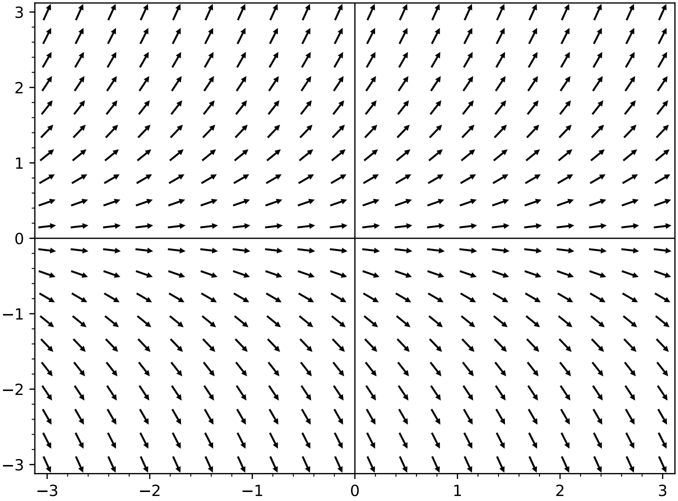

Figure: Direction field of $y' = y$

Examples Cont'd

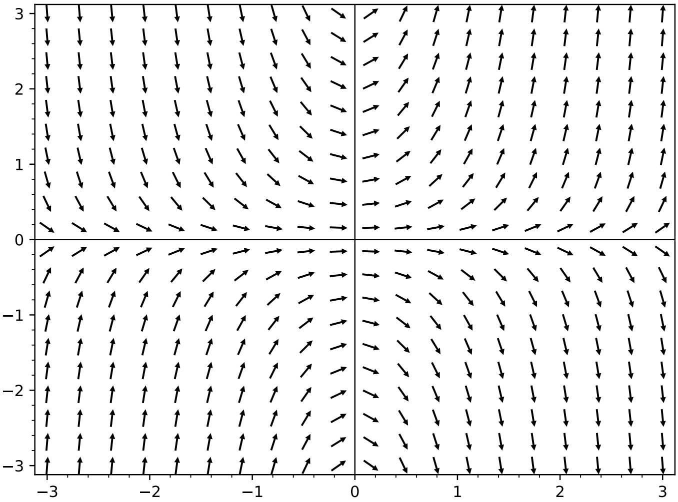

Figure: Direction field of $y' = 2ty$

Examples Cont'd

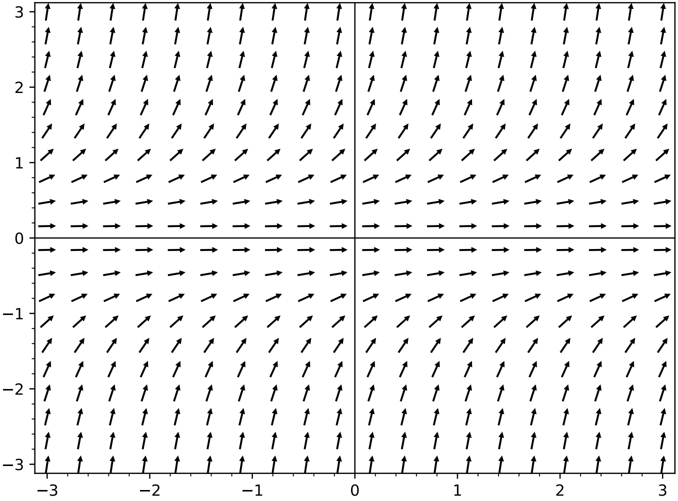

Figure: Direction field of $y' = y^2$

Examples Cont'd

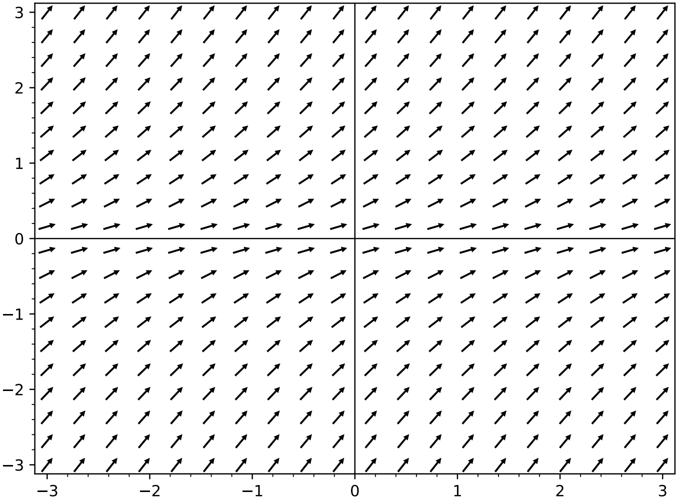

Figure: Direction field of $y' = \sqrt{|y|}$

Examples Cont'd

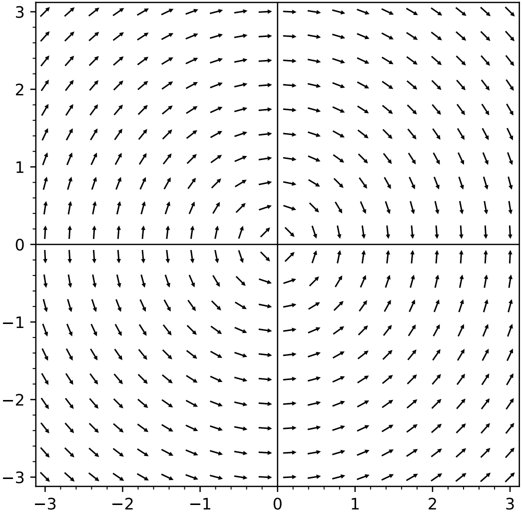

Figure: Direction field of $y' = -t/y$

Examples Cont'd

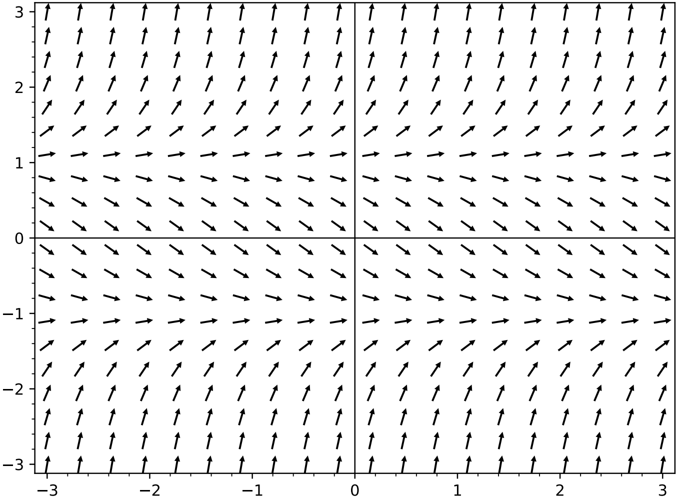

Figure: Direction field of $y' = y^2 - 1$

Mathematical Modelling with ODE's and PDE's

Differential equations are ubiquitous in Physics and Engineering, because fundamental laws of Physics can be expressed in terms of differential equations, and hence physicists and engineers need to solve such equations in order to describe the quantities involved in a physical process or system.

We consider, without actually solving the differential equations, a few examples.

Modeling with Initial Value Problems

Falling objects

$v(t)$ denotes the speed of the falling object;
initial condition $v(0) = 0$ (object is released at time $t = 0$);
Newton's 2nd Law $F(t) = m\,a(t) = m\,\mathrm{d}v/\mathrm{d}t$ and assumptions on the *drag force* F_D due to air resistance give the following models:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg \qquad F_D = 0 \qquad \text{(no air resistance)}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - k_1 v \qquad F_D = k_1 v \qquad \text{(very small objects)}$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - k_2 v^2 \qquad F_D = k_2 v^2 \qquad \text{(most common case)}$$

The first model is realistic only for short distances, the second for very small objects like dust particels, and the third applies in all other cases (assuming that air density and gravitational acceleration are approximately the same as on the surface of the earth).

Modeling with Initial Value Problems Cont'd

Oscillating pendulum

$\theta(t)$ denotes the angle between the rod at time t makes with the vertical direction and L the length of the rod;
initial condition $\theta(0) = \theta_0$ (angle when the pendulum is released);
Newton's 2nd Law gives the following ODE for $\theta(t)$:

$$mL\,\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -mg\sin\theta$$

Modeling with Initial Value Problems Cont'd

Predator-Prey Models

$x(t)$, $y(t)$ denote the population sizes of two species. We assume that the second species (the *predator*) preys on the first species (the *prey*), while the prey lives on a different source of food;
initial population size $x(0) = x_0$, $y(0) = y_0$;
reasonable assumptions on the reproduction rates of the two species and their interaction lead to the following system of ODE's:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - \alpha xy,$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = -cy + \gamma xy, \quad \text{with constants } a, \alpha, c, \gamma > 0.$$

These are known as the *Lotka-Volterra equations*.

Modeling with Boundary Value Problems

Heat Conduction

$u(x,t)$ denotes the temperature in a thin solid bar of length L at distance x from one end and at time $t \geq 0$.
The temperature variation is subject to

$$\alpha^2 u_{xx}(x,t) = u_t(x,t) \quad \text{for } 0 < x < L,\ t > 0.$$

(Heat conduction equation)

Boundary conditions:

$$u(x,0) = f(x), \quad u(0,t) = T_1, \quad u(L,t) = T_2.$$

These express the requirements that the initial ($t = 0$) temperature distribution in the bar is some known function $f(x)$, and the ends of the bar are kept at constant temperatures T_1 resp. T_2.

Modeling with Boundary Value Problems Cont'd

Vibrating String

$u(x,t)$ denotes the vertical displacement of an elastic string of length L from its horizontal equilibrium position at distance x from one end and at time $t \geq 0$.
The string's vibration is subject to

$$a^2\, u_{xx}(x,t) = u_{tt}(x,t) \quad \text{for } 0 < x < L,\ t > 0.$$

(Wave equation)

Boundary conditions:

$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

These express the requirements that the string is released at time $t = 0$ from some known position $f(x)$ (i.e., a plucked guitar string) and is fixed at both ends.

Further Notes on Modeling with ODE's and PDE's

- As in the case of falling objects, it is often not clear which mathematical model is appropriate. Solving the model equations and making predictions based on the results must be checked against real-world data.
- Even when using a well-established model there is the problem of estimating numerically the corresponding physical parameters involved. If the model is very sensitive in this regard, small inaccuracies in the input data may lead to completely false predictions by the model.
- ODE's and PDE's used in modeling have at least one undetermined parameter, because physical quantities are relative to the unit of measurement used. For example, the ODE $v'(t) = g$ describing a falling object over a short time has the parameter g, which can take different real values depending on the choice of units, e.g., $g = 9.81\,[\mathrm{m/s^2}]$ vs. $g = 127\,000\,[\mathrm{km/h^2}]$.