Differential Equations Plus (Math 286)

H37 Determine a real fundamental system of solutions for the following ODE's:

a) $y'' - 4y' + 4y = 0$;
b) $y''' - 2y'' - 5y' + 6y = 0$;
c) $y''' - 2y'' + 2y' - y = 0$;
d) $y''' - y = 0$;
e) $y^{(4)} + y = 0$;
f) $y^{(8)} + 4y^{(6)} + 6y^{(4)} + 4y'' + y = 0$.

Four answers suffice.

H38 Determine the general real solution of

a) $y'' + 3y' + 2y = 2$;
b) $y'' + y' - 12y = 1 + t^2$;
c) $y'' - 5y' + 6y = 4t\mathrm{e}^t - \sin t$;
d) $y''' - 2y'' + y' = 1 + \mathrm{e}^t \cos(2t)$;
e) $y^{(4)} + 2y'' + y = 25\mathrm{e}^{2t}$;
f) $y^{(n)} = t\mathrm{e}^t$, $n \in \mathbb{N}$.

Four answers suffice.

H39 a) Suppose $\phi\colon \mathbb{R} \to \mathbb{C}$ solves a homogeneous linear ODE

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \quad a_i \in \mathbb{C}, \tag{H}$$

but no such ODE of order $< n$. Show that ϕ, ϕ', ϕ'', ..., $\phi^{(n-1)}$ form a fundamental system of solutions of (H).

b) Find a fundamental system of solutions of the form ϕ, ϕ', ϕ'', ϕ''' for the ODE $y^{(4)} - y^{(3)} - y' + y = 0$.

H40 Do three of the four Exercises 4, 6, 14, 16 in our Calculus textbook [Ste16], Ch. 17.3. You may need to study the relevant material in [Ste16], Ch. 17, or [BDM17], Ch. 3.7, 3.8 first.

H41 *Optional Exercise*

For the following functions ϕ_i, find the homogeneous linear ODE $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ ($a_i \in \mathbb{C}$) of smallest order having ϕ_i as a solution; cf. H42.

a) $\phi_1(t) = 2\sin t - 3\cos(3t)$;
b) $\phi_2(t) = \sin t \cos(3t)$;
c) $\phi_3(t) = -1 + t\mathrm{e}^{-2t}\cos t$;
d) $\phi_4(t) = \mathrm{e}^t + t^{1949} + t^{2019}$.

H42 *Optional Exercise*

Suppose that $y\colon \mathbb{R} \to \mathbb{C}$ solves some homogeneous linear ODE $a(D)y = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ with coefficients $a_i \in \mathbb{C}$ (i.e., y is an exponential polynomial). Show:

a) There is a unique monic polynomial $m(X) \in \mathbb{C}[X]$ of smallest degree satisfying $m(\mathrm{D})y = 0$.

b) If $b(X) \in \mathbb{C}[X]$ satisfies $b(\mathrm{D})y = 0$ then $m(X)$ divides $b(X)$.

Hint: Using long division of polynomials, show that any polynomial $m(X)$ with the properties in a) must satisfy b), and from this conclude that $m(X)$ is uniquely determined.

H43 *Optional Exercise*

In the lecture we have found that the ODE $y'' - y' - y = 1$ and its discrete "analogue" $y_{i+2} - y_{i+1} - y_i = 1$ both have the constant function $y(t) \equiv -1$ as a solution (of course, with different domains $\mathbb{R}$ resp. $\mathbb{N}$). Is this a pure coincidence or an instance of a more general correspondence between the continuous and discrete case?

Hint: It may help to identify the discrete analogue of the exponential function e^t first.

H44 *Optional Exercise*

a) Determine (directly or using the theory you have learned in Discrete Mathematics) the homogeneous linear recurrence relation of order (degree) 3 satisfied by the squares sequence $(n^2)_{n\in\mathbb{N}} = (0, 1, 4, 9, 16, \dots)$.

b) Suppose $\mathbf{x} = (x_0, x_1, x_2 \dots)$ satisfies $a(\mathrm{S})\mathbf{x} = \mathbf{0}$. Show that the sequence of partial sums $\mathbf{y} = (x_0, x_0 + x_1, x_0 + x_1 + x_2, \dots)$ satisfies $(\mathrm{S} - 1)a(\mathrm{S})\mathbf{y} = \mathbf{0}$.

c) Derive an explicit formula for $s_n = \sum_{k=1}^{n} k^2$ by representing the sequence as solution of a 4th-order homogeneous linear recurrence relation and solving this recurrence relation. (The solution is of course well-known, but the exercise provides a conceptual approach which also works for higher powers and other sequences.)

Due on Fri Nov 5, 6 pm

Solution methods for homogenenous linear ODE's with constant coefficients (required for H37, H39, H40) will be discussed in the lecture on Mon Nov 1, and those for the inhomogeneous case (required for H38 and part of H40) in the lecture on Wed Nov 3.

The optional exercises and H40 can be handed in until Fri Nov 12, 6 pm.