Calculus III (Math 241)

W10 Prove in a direct way the following:

a) The solution of any solvable system of linear equations $\mathbf{Ax} = \mathbf{b}$ ($\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$) has the form $\mathbf{x}_0 + S = \{\mathbf{x}_0 + \mathbf{s}; \mathbf{s} \in S\}$, where S denotes the solution of the associated homogeneous system $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x}_0$ is any particular ("special") solution of $\mathbf{Ax} = \mathbf{b}$;

b) S is a linear subspace of $\mathbb{R}^n$ (and hence the solution of $\mathbf{Ax} = \mathbf{b}$ an affine subspace of $\mathbb{R}^n$).

W11 Solve

$$\begin{array}{rcrcrcrcrcrcl}
x_1 & + & 2x_2 & + & 3x_3 & - & 2x_4 & + & x_5 & & & = & 0\\
2x_1 & + & 4x_2 & + & x_3 & & & - & 2x_5 & + & x_6 & = & 0\\
-x_1 & - & 2x_2 & + & 2x_3 & + & 3x_4 & - & 3x_5 & - & 2x_6 & = & 0\\
 & & & & 6x_3 & - & 4x_4 & + & 3x_5 & + & 2x_6 & = & 0\\
3x_1 & + & 6x_2 & + & 10x_3 & - & 6x_4 & + & 2x_5 & + & 3x_6 & = & 0
\end{array}$$

W12 Repeat W11 with the same coefficient matrix $\mathbf{A}$ but with right-hand side $\mathbf{b} = (0, 4, 1, -4, 0)^\mathsf{T}$ in place of $(0, 0, 0, 0, 0)^\mathsf{T}$.

Hint: There is a shortcut using W10.

W13 Find the (unique) polynomial function of degree at most 3 whose graph passes through the points $(1, 0)$, $(2, 1)$, $(3, 2)$, and $(4, 1)$.

W14 Compute $\mathbf{FF}^\mathsf{T}$ and $\mathbf{F}^\mathsf{T}\mathbf{F}$ for the 7×7 matrix

$$\mathbf{F} = \begin{pmatrix}
1 & 1 & 0 & 1 & 0 & 0 & 0\\
0 & 1 & 1 & 0 & 1 & 0 & 0\\
0 & 0 & 1 & 1 & 0 & 1 & 0\\
0 & 0 & 0 & 1 & 1 & 0 & 1\\
1 & 0 & 0 & 0 & 1 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 1\\
1 & 0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

Which combinatorial property of $\{013, 124, 235, 346, 450, 561, 602\}$ ("translates of 013 modulo 7") does the result reflect?

Hint: The entries of $\mathbf{FF}^\mathsf{T}$ are the pairwise dot products of the rows of $\mathbf{F}$, and similarly for $\mathbf{F}^\mathsf{T}\mathbf{F}$. The "circulant" structure of $\mathbf{F}$ simplifies the computation.