Calculus III (Math 241)

H1 Solve [Ste16], Section 12.1, Exercises 44 and 48.

H2 Solve [Ste16], Section 12.2, Exercises 32, 34, 36 and 38; cp. Example 7 on p. 804.

H3 Solve [Ste16], Section 12.3, Exercises 40, 42, 46, 48. For the definition of $\operatorname{orth}_{\mathbf{a}}\mathbf{b}$ see Exercise 45.

H4 *Optional Exercise*

For each of the following real numbers a, find a nonzero polynomial with integer coefficients having a as a zero: $\sqrt{2}+\sqrt[3]{5}$, $\sqrt{2}-\sqrt[3]{5}$, $\sqrt{2}\cdot\sqrt[3]{5}$, and $\sqrt{2}/\sqrt[3]{5}$.

Hint: In each case there is a polynomial of degree 6 having this property. You are required to produce the coefficients of the polynomial (and not merely a representation as a product of certain polynomials, say).

H5 *Optional Exercise*

The goal of this exercise is to construct a bijection (i.e., a one-to-one correspondence) between $\mathbb{R}$ and $\mathbb{R}^2$. For this you may use without proof the following fact already used in the lecture (in CANTOR's diagonal argument): Every real number $a \in (0,1]$ has a unique "non-terminating" decimal expansion $a = 0.a_1a_2a_3\dots$ with $a_i \in \{0,1,\dots,9\}$ and $a_i \neq 0$ for infinitely many i.

a) Find a bijection from $\mathbb{R}$ to $(0,1]$.

   Hint: Use an elementary one-variable function to map $\mathbb{R}$ bijectively onto $(0,1)$, and then adjust.

b) Find a bijection from $(0,1]\times(0,1] = \{(a,b)\in\mathbb{R}^2; 0<a\leq 1, 0<b\leq 1\}$ to $(0,1]$.

   Hint: This is tricky. Interleaving $(0.a_1a_2\dots,0.b_1b_2\dots)\to(0.a_1b_1a_2b_2\dots)$ of non-terminating decimal fractions can't be used, since this wouldn't produce, e.g., $0.101010\dots$ Work with subwords instead of single digits.

c) Use a) and b) to find the desired bijection from $\mathbb{R}$ to $\mathbb{R}^2$.

H6 *Optional Exercise*

Show that the set $\mathbb{A}$ of all real algebraic numbers is countable.

Hint: Every $\alpha\in\mathbb{A}$ is a zero of some polynomial $a(X) = a_0 + a_1X + \cdots + a_dX^d$ with coefficients $a_i\in\mathbb{Z}$ and degree $d\geq 1$ (i.e, $a_d \neq 0$). Define the *height* of $a(X)$ as $d+|a_0|+|a_1|+\cdots+|a_d|$, and show that for every integer $h\in\mathbb{Z}^+$ there exist only finitely many polynomials of height h.

Instructions For your homework it is best to maintain 2 notebooks, which are handed in on alternate Wednesdays. You may also use A4 sheets, provided they are firmly stapled together.

Don't forget to write your name (English and Chinese) and your student ID on the first page.

Homework is handed in on Wednesdays before the discussion session starts (late homework won't be accepted!) and will be returned on the next Wednesday.

Answers to exercises must be justified; it is not sufficient to state only the final result of a computation.

Answers must be written in English.

For a full homework score it is sufficient to solve ca. 80 % of the mandatory homework exercises. Optional exercises contribute to the homework score, but they are usually more difficult and you should work on them only if you have sufficient spare time.

Due on Wed Sep 22, 6 pm

The optional exercises can be handed in until Wed Sep 29, 6 pm.