Math 286
Introduction to Differential Equations

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Outline

Today’s Lecture:

Separable Equations

Definition

An (explicit) first-order ODE $y' = f(x, y)$ is said to be *separable* if $f(x, y)$ factors as $f(x, y) = f_1(x)f_2(y)$

We assume that the domains I, J of f_1, resp., f_2 are open intervals and that f_2 has no zero in J. Then $N(y) = 1/f_2(y)$ is well-defined and has no zero in J as well.

Writing $M = f_1$, we can rewrite $y' = f_1(x)f_2(y)$ as

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{M(x)}{N(y)} \quad \text{or} \quad M(x)\,\mathrm{d}x - N(y)\,\mathrm{d}y = 0.$$

Theorem

Suppose $M\colon I \to \mathbb{R}$ and $N\colon J \to \mathbb{R}$ are continuous and N has no zero in J. Let $(x_0, y_0) \in I \times J$, and define $H_1\colon I \to \mathbb{R}$, $H_2\colon J \to \mathbb{R}$ by

$$H_1(x) = \int_{x_0}^{x} M(\xi)\mathrm{d}\xi, \qquad H_2(y) = \int_{y_0}^{y} N(\eta)\mathrm{d}\eta.$$

Let further $I' \subseteq I$ be an interval with $x_0 \in I'$ and $H_1(I') \subseteq H_2(J)$. Then there exists a unique solution $y\colon I' \to \mathbb{R}$ of the IVP $y' = M(x)/N(y) \wedge y(x_0) = y_0$, viz. $y(x) = H_2^{-1}\big(H_1(x)\big)$ for $x \in I'$.

Remark

The subsequent proof (cf. the notes thereafter) shows that for sufficiently small $\delta > 0$ the interval $I' = (x_0 - \delta, x_0 + \delta)$ has the required property and hence that the IVP $y' = M(x)/N(y) \wedge y(x_0) = y_0$ has locally near (x_0, y_0) a (unique) solution.

Proof of the theorem.

Since N is continuous and has no zero in J, we have either $N > 0$ or $N < 0$ on J and hence that H_2 is either strictly increasing or strictly decreasing on J. In particular, $H_2\colon J \to H_2(J)$ is bijective and $y\colon I' \to \mathbb{R}$, $x \mapsto H_2^{-1}\big(H_1(x)\big)$ is well-defined.

$$y'(x) = (H_2^{-1})'\big(H_1(x)\big) \cdot H_1'(x) = \frac{H_1'(x)}{H_2'\big(H_2^{-1}(H_1(x))\big)} = \frac{M(x)}{N\big(y(x)\big)},$$

i.e., $y(x)$ satisfies $y' = M(x)/N(y)$
$H_1(x_0) = 0 = H_2(y_0) \Longrightarrow y(x_0) = H_2^{-1}\big(H_1(x_0)\big) = y_0$
It remains to show that any solution $y\colon I' \to \mathbb{R}$ of the IVP must satisfy $H_2\big(y(x)\big) = H_1(x)$ for $x \in I'$.

Proof cont'd.

To this end we write the ODE in the form $y'(x)N\big(y(x)\big) = M(x)$ and integrate:

$$\int_{x_0}^{x} y'(\xi)N\big(y(\xi)\big)\mathrm{d}\xi = \int_{x_0}^{x} M(\xi)\mathrm{d}\xi = H_1(x)$$

Making the substitution $\eta = y(\xi)$, $\mathrm{d}\eta = y'(\xi)\mathrm{d}\xi$ on the left-hand side gives

$$\int_{y(x_0)}^{y(x)} N(\eta)\mathrm{d}\eta = \int_{y_0}^{y(x)} N(\eta)\mathrm{d}\eta = H_2\big(y(x)\big),$$

as desired. $\square$

Notes

- The proof has shown that $H_2(J)$ is an open interval containing $0 = H_2(y_0)$. Since H_1 is continuous and $H_1(x_0) = 0$, there exists $\delta > 0$ such that $H_1(x) \in H_2(J)$ for $x_0 - \delta < x < x_0 + \delta$, justifying the remark made before the proof.

Notes cont'd

- If the integrals in $H_2(y)=\int_{y_0}^{y} N(\eta)\mathrm{d}\eta=\int_{x_0}^{x} M(\xi)\mathrm{d}\xi=H_1(x)$ can be evaluated, we obtain $y=y(x)$ in impicit form $H_2(y)=H_1(x)$. The condition $H_1(I')\subseteq H_2(J)$ guarantees that this equation has a solution $y\in J$ for each $x\in I'$. If we are lucky, we may be able to solve for y and obtain an explicit formula for $y(x)$.
- The notation used in [BDM17], Ch. 2.2 is the same except that N, H_2 are replaced by $-N$, $-H_2$ to put the implicit ODE into the more symmetric form $M(x)\,\mathrm{d}x+N(y)\,\mathrm{d}y=0$.

Example

We determine all solutions of the ODE $y'=dy/dt=t\,y^2$, which is separable with $f_1(t)=t$, $f_2(y)=y^2$.

One solution is the steady-state solution $y\equiv 0$.

For f_1 there is no restriction, and hence $I=\mathbb{R}$ in the theorem. Since $f_2(y)=0$ is not allowed in the theorem, we split the domain of f_2 into the intervals $J_1=(-\infty,0)$, $J_2=(0,+\infty)$. This corresponds to initial values $y(t_0)<0$ and $y(t_0)>0$, respectively.

Example (cont'd)

Rewriting the ODE formally as $\mathrm{d}y/y^2 = t\,\mathrm{d}t$ and integrating gives

$$\int_{y_0}^{y} \frac{\mathrm{d}\eta}{\eta^2} = \int_{t_0}^{t} \tau \mathrm{d}\tau$$

$$\frac{1}{y_0} - \frac{1}{y} = \left[-\frac{1}{\eta}\right]_{y_0}^{y} = \left[\tfrac{1}{2}\tau^2\right]_{t_0}^{t} = \tfrac{1}{2}(t^2 - t_0^2)$$

$$\Longrightarrow y(t) = \frac{1}{1/y_0 - \frac{1}{2}(t^2 - t_0^2)} = \frac{2}{2/y_0 + t_0^2 - t^2}$$

The non-constant solutions of $y' = t\,y^2$ are therefore $y(t) = y_C(t) = 2/(C - t^2)$, $C \in \mathbb{R}$. The solution y_C

- is defined for all $t \in \mathbb{R}$ if $C < 0$ or, equivalently, $-2/t_0^2 < y_0 < 0$ ($y_0 < 0$ for $t_0 = 0$);
- is defined only on the finite interval $(-\sqrt{C}, \sqrt{C})$ if $C > 0 \wedge |t_0| < \sqrt{C}$ or, equivalently, $y_0 > 0$;
- is defined only on $(-\infty, -\sqrt{C})$ if $C \geq 0 \wedge t_0 < -\sqrt{C}$ or, equivalently, $t_0 < 0 \wedge y_0 \leq -2/t_0^2$;
- is defined only on $(\sqrt{C}, +\infty)$ if $C \geq 0 \wedge t_0 > \sqrt{C}$ or, equivalently, $t_0 > 0 \wedge y_0 \leq -2/t_0^2$.

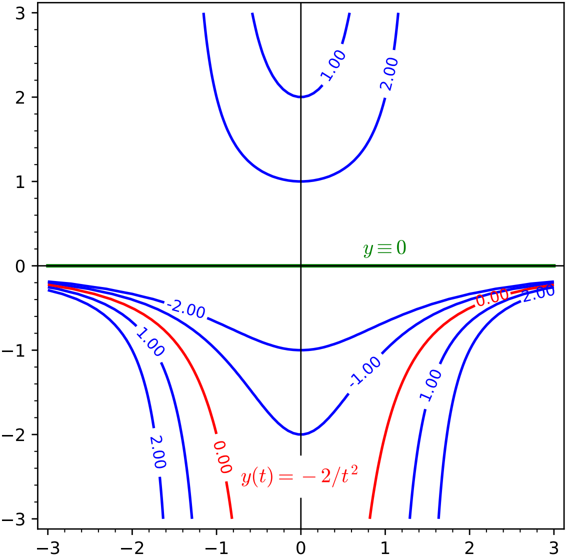


Figure: Solution curves $y_C(t) = 2/(C - t^2)$ of $y' = t\,y^2$

Simplified Variant (but keep the derivation in mind!)

It is often easier to use indefinite integration to determine the general solution of a separable 1st-order ODE as a 1-parameter family and then adapt the constant to satisfy a given initial condition:

$$
\begin{aligned}
y'(t)N\big(y(t)\big) &= M(t) \\
\implies \int y'(t)N\big(y(t)\big)\,\mathrm{d}t &= \int M(t)\,\mathrm{d}t + C, \quad C \in \mathbb{R} \\
\implies \int N(y)\,\mathrm{d}y &= \int M(t)\,\mathrm{d}t + C, \quad C \in \mathbb{R},\ y = y(t)
\end{aligned}
$$

Memorizing the ODE as $\mathrm{d}y \,/\, \mathrm{d}t = M(t)/N(y)$ and formally rewriting it as $N(y)\,\mathrm{d}y = N(t)\,\mathrm{d}t$, we can directly short-circuit to the previous line:

$$
\begin{aligned}
N(y)\,\mathrm{d}y &= M(t)\,\mathrm{d}t \\
\implies \int N(y)\,\mathrm{d}y &= \int M(t)\,\mathrm{d}t + C
\end{aligned}
$$

In our present example: "$\mathrm{d}y \,/y^2 = t\,\mathrm{d}t \implies -1/y = t^2/2 + C$", leading again to $y = -\frac{1}{C+t^2/2} = \frac{2}{-2C-t^2} = \frac{2}{C'-t^2}$, $C' \in \mathbb{R}$.

Example

We determine the solution of the IVP $mv' = mg - kv^2 \wedge v(0) = 0$ (best of the three models for a falling object).

Separating the variables gives

$$\frac{v'}{g-(k/m)v^2} = 1$$

$$\implies \int_0^v \frac{\mathrm{d}\eta}{g-(k/m)\eta^2} = \int_0^t \mathrm{d}\tau = t$$

Since $\int \frac{\mathrm{d}x}{1-x^2} = \operatorname{artanh}(x) + C$, using the substitution $x = \sqrt{k/(mg)}\,\eta$ and $\tanh(y) = \frac{\mathrm{e}^y-\mathrm{e}^{-y}}{\mathrm{e}^y+\mathrm{e}^{-y}} = 1 - \frac{2}{\mathrm{e}^{2y}+1}$, we obtain

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(t\sqrt{\frac{gk}{m}}\right) = \sqrt{\frac{mg}{k}}\left(1-\frac{2}{\mathrm{e}^{2t\sqrt{gk/m}}+1}\right),$$

for $0 \leq t \leq T$ (the time when the object hits the ground).
Setting $v_\infty = \sqrt{\frac{mg}{k}}$ (*limiting velocity*), this can also be written as

$$v(t) = v_\infty\left(1-\frac{2}{\mathrm{e}^{2t\,g/v_\infty}+1}\right), \quad 0 \leq t \leq T.$$

Example (cont'd)

Reasonable values for a skydiver (S) and a parachutist (P) with round canopy are $v_\infty = 50\,\mathrm{m/s}$ and $v_\infty = 5\,\mathrm{m/s}$, respectively, which gives

$$v_S(t) = 50\left(1 - \frac{2}{\mathrm{e}^{0.4\,t}+1}\right) \quad [\mathrm{m/s}]\,,$$

$$v_P(t) = 5\left(1 - \frac{2}{\mathrm{e}^{4\,t}+1}\right) \quad [\mathrm{m/s}]\,,$$

when time is measured in seconds.

This agrees well with experimentally found data.

Remark

Here, in contrast with the 2nd model, we can compute T in closed form: Denoting by $s(t)$ the distance traveled at time t, we have

$$s(t) = \frac{m}{k}\log\cosh\left(t\sqrt{\frac{gk}{m}}\right),$$

$$t(s) = \sqrt{\frac{m}{gk}}\,\operatorname{arcosh}\left(\mathrm{e}^{sk/m}\right),$$

and $T = t(s_0)$ if the object is released at height s_0.

Exercise

a) Show that in the 3rd model for a falling object released at height s_0 the terminal velocity v_T of the object at time of impact is given by

$$v_T = \sqrt{\frac{mg}{k}} \cdot \sqrt{1 - \mathrm{e}^{-2ks_0/m}}.$$

Hint: Consider the velocity as a function $v(s)$ of the distance s traveled. Show that $y(s) = v(s)^2$ satisfies the ODE $my' = 2mg - 2ky$.

b) The limiting velocity of a falling basketball ($m = 620\,\mathrm{g}$) has been estimated at $20\,\mathrm{m/s}$. Using this data, graph v_T as a function of s_0. For which heights s_0 does the basketball reach $50\,\%$, $90\,\%$, and $99\,\%$ of its limiting velocity?

Example

We solve $y' = y^2$, which is autonomous and hence separable with $f_1(t) = 1$, $f_2(y) = y^2$, using the simplified solution method.

There is the constant solution $y \equiv 0$.
Otherwise we can rewrite $\mathrm{d}y \,/\, \mathrm{d}t = y^2$ as $\mathrm{d}y \,/ y^2 = \mathrm{d}t$ and obtain

$$\begin{aligned} \frac{\mathrm{d}y}{y^2} &= \mathrm{d}t \\ \implies -\frac{1}{y} &= \int \frac{\mathrm{d}y}{y^2} = \int \mathrm{d}t = \int 1 \,\mathrm{d}t = t + C \\ \implies y &= \frac{1}{-C-t} = \frac{1}{C'-t}, \quad C, C' \in \mathbb{R}. \end{aligned}$$

This recovers the already known general solution.

But don't forget: The informal computation is justified by rewriting it in terms of $y(t)$ and using the subsitution $\eta = y(t)$:

$$\begin{aligned} \frac{y'(t)}{y(t)^2} &= 1 \\ \iff \quad \frac{-1}{y(t)} = \frac{-1}{\eta} = \int \frac{\mathrm{d}\eta}{\eta^2} &= \int \frac{y'(t)}{y(t)^2} \,\mathrm{d}t = \int \mathrm{d}t = t + C \end{aligned}$$

Example (cont'd)

This tells us that the solutions $y\colon I \to \mathbb{R}$ of $y' = y^2$ with $y(t) \neq 0$ for all $t \in I$ are precisely the functions whose graph is contained in a contour of $F(t, y) = -1/y - t$, i.e., satisfy $F\bigl(t, y(t)\bigr) = C$ for some $C \in \mathbb{R}$.

It doesn't tell us whether such functions actually exist.

However, in this particular case we can solve for y to show that precisely the functions $y(t) = 1/(C - t)$, $C \in \mathbb{R}$ (defined on an appropriate interval I) have this property.

In the case of a general separable ODE we can't solve for y and must invoke the theorem on separable ODE's to conclude the local existence and uniqueness of solutions for any prescribed initial value $y(t_0) \neq 0$. (The Implicit Function Theorem also yield this, cf. subsequent remark.)

Example

The ODE $y' = \sqrt{|y|}$ can of course also be solved by the new method:

A solution $y \colon I \to \mathbb{R}$ with $y(t) \neq 0$ for all $t \in I$ must satisfy either $y > 0$ on I or $y < 0$ on I.

$\underline{y > 0:}$ In this case $y' = \sqrt{y}$ and we get

$$\frac{dy}{\sqrt{y}} = 1\,\mathrm{d}t \iff 2\sqrt{y} = t + C \iff y = \frac{(t+C)^2}{4}, \quad C \in \mathbb{R}.$$

Because of the middle equation, we must have $t > -C$, i.e., $I \subseteq (-C, +\infty)$.

$\underline{y < 0:}$ Here $y' = \sqrt{-y}$ and we get

$$\frac{dy}{\sqrt{-y}} = 1\,\mathrm{d}t \iff -2\sqrt{-y} = t+C \iff y = -\frac{(t+C)^2}{4},\ C \in \mathbb{R},$$

and $t < -C$, i.e., $I \subseteq (-\infty, -C)$.

The guaranteed uniqueness of solutions applies only to the regions $y > 0$ and $y < 0$ in the (t, y)-plane and doesn't exlude the observed branching of solutions on the t-axis.

General Remarks on $y' = f_1(x)f_2(y)$

Extracted from the previous examples

We assume that $f_1 \colon I \to \mathbb{R}$, $f_2 \colon J \to \mathbb{R}$ are continuous functions on open intervals $I, J \subseteq \mathbb{R}$. Thus $I \times J$ is an open rectangle with possibly infinite sides.

1. The zeros of f_2 (if any) partition J into open subintervals on which f_2 has no zeros. If J' is such a subinterval then on the rectangle $I \times J'$ we have local existence and uniqueness of solutions of IVP's $\quad y' = f_1(x)f_2(y) \wedge y(x_0) = y_0 \quad$ at any point $(x_0, y_0) \in I \times J'$.
2. Rewriting $y' = f_1(x)f_2(y)$ as $y' = M(x)/N(y)$ and denoting by $F \colon I \times J' \to \mathbb{R}$ an antiderivative of $M(x)\,\mathrm{d}x - N(y)\,\mathrm{d}y$ (i.e., $\partial F/\partial x = M \wedge \partial F/\partial y = -N$), the solutions $y(x)$ with graph $\mathrm{G}_y \subset I \times J'$ are given in implicit form as $F(x, y) = C$, $C \in \mathbb{R}$.

The function F in (2) can be chosen as

$$F(x, y) = \int_{x_0}^{x} M(\xi)\mathrm{d}\xi - \int_{y_0}^{y} N(\eta)\mathrm{d}\eta, \quad (x_0, y_0) \in I \times J'.$$

In particular the differential 1-form $M(x)\,\mathrm{d}x - N(y)\,\mathrm{d}y$ is exact on $I \times J'$ (which also follows form $M_y = N_x = 0$ and the shape of $I \times J'$).

Remarks on $y' = f_1(x)f_2(y)$ Cont'd

3. For any zero y_0 of f_2 there is the steady-state solution $y(x) \equiv y_0$ on I. Together with (1) this shows that all IVP's $y' = f_1(x)f_2(y) \wedge y(x_0) = y_0$ with $(x_0, y_0) \in I \times J$ are solvable.

Linear Versus Separable 1st-Order ODE's

Note the following important differences between the two cases.

1. Domains of $y' = a(x)y + b(x)$ are of the form $I \times \mathbb{R}$; domains of $y' = f_1(x)f_2(y)$ are of the form $I \times J$, where J may be a proper subinterval of $\mathbb{R}$.
2. Solutions of $y' = a(x)y + b(x)$ can be extended to I (i.e., maximal solutions have domain I); solutions of $y' = f_1(x)f_2(y)$ may be defined only on proper subintervals $I' \subset I$, which depend on the solution and are not visible in the ODE.
3. Solutions of IVP's $y' = a(x)y + b(x) \wedge y(x_0) = y_0$ are unique in the sense that if $y_1 \colon I_1 \to \mathbb{R}$, $y_2 \colon I_2 \to \mathbb{R}$, solve the IVP then $y_1(x) = y_2(x)$ for all $x \in I_1 \cap I_2$; solutions of IVP's $y' = f_1(x)f_2(y) \wedge y(x_0) = y_0$ are unique only at points (x_0, y_0) with $f_2(y_0) \neq 0$, and only if their ranges don't contain zeros of f_2.

The Logistic Equation

Definition

The ODE $y' = ay - by^2$ with constants $a, b > 0$ is called *logistic equation*.

The logistic equation was introduced by the Belgian mathematician P. VERHULST (1804–1849) in 1837 as a mathematical model for population growth. It provides a more accurate model of population growth than the exponential model $y' = ay$, adding a term $-by^2$, which accounts for the competition between individuals if resources are limited.

The logistic equation has the form $y' = f_1(t)f_2(y)$ with $f_1(t) = 1$, $f_2(y) = ay - by^2$, and hence is separable (even autonomous).

Since $ay - by^2 = y(a - by)$ the steady-state solutions are $y \equiv 0$ and $y \equiv a/b$.

We determine the general solution by the usual method. Since

$$\frac{1}{y(a-by)}=\frac{1/a}{y}+\frac{b/a}{a-by},$$

e.g., by the method of partial fractions, we obtain

$$\begin{aligned}
\int \frac{1}{y}+\frac{b}{a-by}\,\mathrm{d}y &= \int a\,\mathrm{d}t+C\\
\ln|y|-\ln|a-by| &= at+C\\
\ln\left|\frac{y}{a-by}\right| &= at+C\\
\pm\frac{y}{a-by} &= \mathrm{e}^{at+C}\\
\pm y &= \mathrm{e}^{at+C}(a-by)\\
y &= \frac{a\,\mathrm{e}^{at+C}}{\pm 1+b\,\mathrm{e}^{at+C}}=\frac{a}{\pm\mathrm{e}^{-C}\mathrm{e}^{-at}+b}.
\end{aligned}$$

Setting $d=\pm\mathrm{e}^{-C}$, we obtain the solution $\boxed{y(t)=\dfrac{a}{d\mathrm{e}^{-at}+b},\ \ d\in\mathbb{R}.}$

$d=0$ gives the steady-state $y\equiv a/b$ (and $d=\infty$ gives $y\equiv 0$).

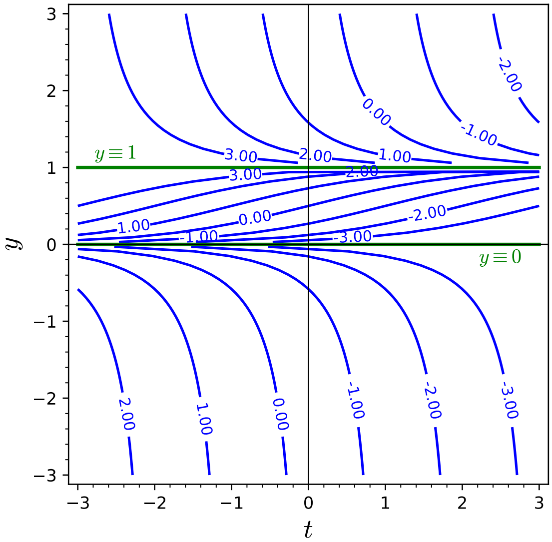


Figure: Solution curves of $y' = y - y^2$, represented as level sets $F(t,y) = \ln\left|\frac{y}{1-y}\right| - t = C$

Asymptotic behaviour

Observation

For every $d \in \mathbb{R}$ we have

$$\lim_{t \to +\infty} \frac{a}{d e^{-at} + b} = \frac{a}{b}, \qquad \lim_{t \to -\infty} \frac{a}{d e^{-at} + b} = 0.$$

Caution

This does not imply that all solutions $y(t)$ to the logistic equation exist at any time t and have the indicated limits for $t \to \pm\infty$.

The precise asymptotics are given on the next slide.

Since $y' = ay - by^2$ is autonomous, horizontal shifts $t \mapsto y(t - t_0)$ of solutions $y(t)$ are again solutions and we can assume w.l.o.g. that $y(t)$ is defined at $t_0 = 0$. As usual, we set $y(0) = y_0$.

In terms of y_0, the parameter d is given by

$$\frac{a}{d+b} = y_0, \quad \text{i.e.,} \quad d = \frac{a}{y_0} - b.$$

The solution with $d = -b$ is not defined at $t = 0$.

Asymptotic behaviour cont'd

1. Solutions $y(t)$ with $d > 0$ or, equivalently, $0 < y_0 < a/b$ exist at any time t (i.e., have maximal domain $\mathbb{R}$) and for $t \to \pm\infty$ have the limits indicated on the previous slide.
2. Solutions $y(t)$ with $d < 0$ have two branches and a vertical asymptote at $t_\infty = \bigl(\ln(-d) - \ln b\bigr)/a$, which is the solution of $d\mathrm{e}^{-at} + b = 0$.
   - (2.1) If $-b < d < 0$, we have $t_\infty < 0$ and the branch defined at $t = 0$ has domain $(t_\infty, +\infty)$; moreover, $\lim_{t\downarrow t_\infty} = +\infty$, $\lim_{t\to+\infty} y(t) = a/b$. All solutions satisfying $y_0 > a/b$ arise in this way (with $d = a/y_0 - b$).
   - (2.2) If $d < -b$, we have $t_\infty > 0$ and the branch defined at $t = 0$ has domain $(-\infty, t_\infty)$; moreover, $\lim_{t\to-\infty} y(t) = 0$, $\lim_{t\uparrow t_\infty} y(t) = -\infty$. All solutions satisfying $y_0 < 0$ arise in this way (with $d = a/y_0 - b$).

The remaining solutions defined at $t = 0$ are the two steady-state solutions $y(t) \equiv 0$ ($d = \infty$) and $y(t) \equiv a/b$ ($d = 0$).

$\Longrightarrow$ The solutions (single branches!) defined at $t = 0$ are in 1-1 correspondence with $d \in \mathbb{R} \setminus \{-b\} \cup \{\infty\}$.

But there are further solutions (the 2nd branches of the solutions for $d < 0$, $d \neq -b$, and both branches for $d = -b$).

Up to horizontal shifts, there are only 3 essentially different solutions:

$$
\begin{aligned}
y_1(t) &= \frac{a}{b(1 + \mathrm{e}^{-at})}, && t \in \mathbb{R}, \\
y_2(t) &= \frac{a}{b(1 - \mathrm{e}^{-at})}, && t \in (-\infty, 0), \\
y_3(t) &= \frac{a}{b(1 - \mathrm{e}^{-at})}, && t \in (0, +\infty).
\end{aligned}
$$

We also see that the corresponding graphs ("integral curves") depend only on the quotient a/b.

The Case $d > 0$

For applications to population growth only Cases 1 and 2 are interesting. Information about the solution graphs can easily be obtained from the logistic equation:

$$y' = ay - by^2 = y(a - by),$$
$$y'' = ay' - 2byy' = y'(a - 2by)$$

1. Solutions $y(t)$ with $0 < y(0) < a/b$ are strictly increasing (since they satisfy $0 < y(t) < a/b$ for all $t \in \mathbb{R}$). Denoting by t_h the unique solution of $d\mathrm{e}^{-at} = b$, i.e. $t_h = (\ln d - \ln b)/a$, we have $y(t_h) = \frac{a}{d\mathrm{e}^{-at_h}+b} = a/2b$ and further that $y(t)$ is convex in $[-\infty, t_h]$ (since $0 < y(t) < a/2b$ in this interval) and concave in $[t_h, +\infty]$. In particular $y(t)$ has a (unique) inflection point in $(t_h, a/2b)$.
2. Solutions $y(t)$ with $y(0) > a/b$ are strictly decreasing and convex in their domain $[t_\infty, +\infty)$.

Since the logistic equation is autonomous, in Case 1 every solution arises from the solution with $t_h = 0$ (i.e., $d = b$) by a time shift. This is visible in $y(t) = a/(d\mathrm{e}^{-at} + b) = a/(b\,\mathrm{e}^{-a(t-t_h)} + b)$.

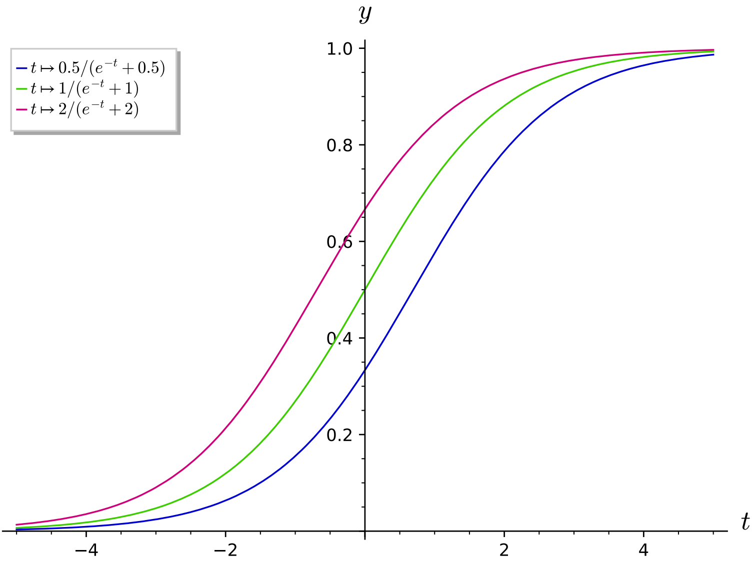


Figure: Three S-curves following the Logistic Law with $a/b = 1$ and $d > 0$

Population of the Earth

The US Department of Commerce estimated in 1965 the world's population at 3.34 billion people, with an annual increase of 2 % per year. Using the exponential model $y' = ay$, this gives

$$y(t) = 3.34 \cdot 10^9 \times \mathrm{e}^{0.02(t-1965)}.$$

In this model the population would double every $\frac{\ln 2}{0.02} \approx 34.6$ years.

The logistic model $y' = ay - by^2$ with the reasonable parameter $a = 0.029$ (natural reproduction rate, if unlimited resources are available) and b, d' computed from

$$\frac{y'(1965)}{y(1965)} = a - b\,y(1965) = a - b \times 3.34 \cdot 10^9 = 0.02,$$

$$y(1965) = \left.\frac{a}{d'\mathrm{e}^{-a(t-1965)} + b}\right|_{t=1965} = \frac{a}{d' + b}$$

i.e. $b = 2.695 \cdot 10^{-12}$, $d' = 5.988 \cdot 10^{-12}$, gives

$$y(t) = \frac{0.029 \cdot 10^{12}}{5.988\,\mathrm{e}^{-0.029(t-1965)} + 2.695}, \quad y(2020) = 7.42 \cdot 10^9, \quad \frac{a}{b} = 10.76 \cdot 10^9.$$

Uniqueness of Solutions

So far we have proved uniqueness of solutions of initial value problems $y' = G(t, y) \wedge y(t_0) = y_0$ in the following two ways:

1. Derive the general solution of $y' = G(t, y)$ and observe that it is a 1-parameter family of functions $y_C(t)$ depending on a constant C; plug in $y_C(t_0) = y_0$ to determine C, and hence the solution, uniquely.
2. If the solution to $y' = G(t, y)$ involves more than one parameter, show additionally that an initial condition $y(t_0) = y_0$ cannot be satisfied by solutions corresponding to different parameters.

Way (1) applies to 1st-order linear ODE's (homogeneous or inhomogeneous) and to separable ODE's without steady-state solutions.

Way (2) applies to separable ODE's with steady-state solutions, such as $y' = y^2$, $y' = ty^2$, $y' = ay - by^2$.
"Different parameters" refers to both continuous 1-parameter families of solutions and "exceptional" steady-state solutions.

Neither way applies to $y' = \sqrt{|y|}$.

Uniqueness of Solutions Cont'd

Example

The logistic equation $y' = ay - by^2$ has the solutions $y_\infty(t) \equiv 0$ and $y_d(t) = \frac{a}{d\mathrm{e}^{-at}+b}$, $d \in \mathbb{R}$. We assume that the solutions are maximal, i.e., the domains are $\mathbb{R}$ for $d \geq 0$ and $\mathbb{R} \setminus \{t_\infty\}$ for $d < 0$. For $d < 0$ we count the two branches $y_d^\pm(t)$ as different solutions, according to our requirement that domains of ODE solutions should be intervals.

For $t_0 \in \mathbb{R}$ and $y_0 \neq 0$ we can solve $\frac{a}{d\mathrm{e}^{-at_0}+b} = y_0$ uniquely for d, showing that (t_0, y_0) is on precisely one solution curve (graph) $y_d(t)$, $d \in \mathbb{R}$. Moreover, since $\frac{a}{d\mathrm{e}^{-at}+b} \neq 0$, these solution curves don't intersect the steady-state solution $y_\infty(t) \equiv 0$. This implies that the solution curves $y_d(t)$, $d \in \mathbb{R} \cup \{\infty\}$, partition the (t, y)-plane, which is equivalent to the unique solvability of all IVP's $y' = ay - by^2 \wedge y(t_0) = y_0$ within the given class of functions.

However, this doesn't exclude the existence of further solutions. In fact there are no further solutions, and a rigorous proof is given on the next slide.

Example (cont'd)

The theorem on separable ODE's implies that there can't be two distinct solutions through a point (t_0, y_0) with $y_0 \notin \{0, a/b\}$, and hence all solutions not intersecting the lines $y = 0$, $y = a/b$ are known.

Now suppose there is a non-constant solution $y(t)$ satisfying $y(t_0) = 0$, say, for some $t_0 \in \mathbb{R}$. (The case $y(t_0) = a/b$ is done in the same way.)

W.l.o.g. we can assume that $0 < y(t) < a/b$ for $t_0 < t < t_0 + \delta$, where δ is some positive number. (By symmetry, we can assume that there exists $t_1 > t_0$ satisfying $y(t_1) > 0$. Since $y(t)$ is continuous, there exists a largest zero t^* of $y(t)$ in $[t_0, t_1]$. Then $y(t) > 0$ for $t^* < t < t_1$, and hence our assumption is satisfied if we replace t_0 by t^* and set $\delta = t_1 - t^*$.)

Now, by continuity, we must have $\lim_{t \downarrow t_0} y(t) = 0$, but none of the solutions that are defined for $t \in (t_0, t_0 + \delta)$ and attain small positive values there (these must be of the form $y_d(t)$ with $d > 0$) has this property, since $y_d(t) = \frac{a}{d\mathrm{e}^{-at} + b} \to \frac{a}{d\mathrm{e}^{-at_0} + b} \neq 0$ for $t \downarrow t_0$.

This contradiction completes the proof.

Example

The equation $y = \sqrt{|y|}$ has the steady-state solution $y(t) \equiv 0$ and the two 1-parameter families

$$y_c^-(t) = -\tfrac{1}{4}(t-c)^2, \qquad t \in (-\infty, c),$$
$$y_c^+(t) = \tfrac{1}{4}(t-c)^2, \qquad t \in (c, +\infty),$$

as solutions, where $c \in \mathbb{R}$ is arbitrary.

Collectively, these solutions partition the (t, y)-plane, so that every point $(t_0, y_0) \in \mathbb{R}^2$ is on exactly one solution curve of this kind. (This follows, e.g., from the theorem on separable ODE's.) However, there are further (maximal) solutions obtained by glueing together $y_c^{\pm}(t)$ at $t = c$ (and other combinations as well), which leads to non-uniqueness of solutions of all IVP's $y = \sqrt{|y|} \wedge y(t_0) = y_0$. (The indicated combination shows this only for the points $(c, 0)$ on the t-axis, through which we have the solution combined from $y_c^{\pm}(t)$ and also the steady-state solution $y(t) \equiv 0$.)

Remark

The general Existence and Uniqueness Theorem for solutions of 1st-order ODE's (to be proved later) will explain the observed fundamental difference between the two examples and give a more conceptual proof of the uniqueness of solutions of all IVP's $y' = ay - by^2 \wedge y(t_0) = y_0$ (and, similarly, of the uniqueness of solutions of all IVP's corresponding to the harvesting equation discussed subsequently).

The Harvesting Equation

Suppose a population follows the logistic law of growth but additionally individuals are removed ("harvested") at a constant rate $h > 0$.

Definition

The ODE $y' = ay - by^2 - h$ $(a, b, h > 0)$ is called *harvesting equation*.

Changes

- For $h < a^2/4b$ the quadratic $-by^2 + ay - h = 0$, whose discriminant is $\Delta = a^2 - 4bh$, still has two zeros, viz.

  $$y_1 = \left(a - \sqrt{a^2 - 4bh}\right)/2b, \quad y_2 = \left(a + \sqrt{a^2 - 4bh}\right)/2b,$$

  which satisfy $0 < y_1 < y_2$ and provide two steady-state solutions.
- For $h = a^2/4b$ the quadratic has a double root, which provides one steady-state solution $y \equiv a/2b$.
- $h > a^2/4b$ the quadratic has no real zeros, and the harvesting equation has no steady-state solutions.

For a more detailed analysis we transform the harvesting equation into canonical form.

Lemma

We can transform the harvesting equation by means of a substitution $y(t) = u\,z(mt) + v$ *with* $u, v, m \in \mathbb{R}$ *and* $u, m > 0$ *into one of the three canonical forms*

$$z' = -z^2 + 1, \quad z' = -z^2, \quad z' = -z^2 - 1.$$

Proof.

Writing $s = mt$ have $y'(t) = mu\,z'(mt) = mu\,z'(s)$ and hence, using the usual shorthands

$$\begin{aligned} z' = \frac{y'}{mu} &= \frac{-b(uz+v)^2 + a(uz+v) - h}{mu} \\ &= -\frac{bu}{m}\,z^2 + \frac{a-2bv}{m}\,z + \frac{-bv^2 + av - h}{mu}. \end{aligned}$$

With $m = bu$, $v = a/2b$ this becomes

$$z' = -z^2 + \frac{\frac{-a^2}{4b} + \frac{a^2}{2b} - h}{bu^2} = -z^2 + \frac{a^2 - 4bh}{4b^2u^2} = -z^2 + \frac{\Delta}{4b^2u^2}.$$

Proof cont'd.

If $\Delta > 0$ ($\Delta < 0$) then $u = \sqrt{\Delta}/(2b)$ (resp., $u = \sqrt{-\Delta}/(2b)$) gives $z' = -z^2 + 1$ (resp., $z' = -z^2 - 1$). $\square$

Notes

- Substitutions of the form $y(t) = u\,z(mt) + v$ ($u, v, m \in \mathbb{R}$, $u, v > 0$) arise from changing the units of measurement on both the t-axis and the y-axis and an additional vertical shift of the graph of $t \mapsto y(t)$. They do not change the overall shape of the solution graphs.
- Substitutions of this form do not change the number of steady-state solutions, and hence the corresponding canonical form is also determined by the number of zeros of $-by^2 + ay - h = 0$.
- The logistic equation $y' = ay - by^2$ has canonical form $z' = -z^2 + 1$ (regardless of the particular choice of $a, b > 0$).

Analysis of the Canonical Forms

In the following we will test "stability" of solutions $y(t)$ of the harvesting equation—a concept that describes their asymptotic behaviour for $t \to +\infty$.

Definition (Stability)

A steady-state solution $y \equiv y_0$ of an autonomous first-order ODE $y' = f(y)$ (i.e., $f(y_0) = 0$) is said to be *(asymptotically) stable* if there exists $\delta > 0$ such that every solution $y(t)$ of $y' = f(y)$ with initial value $y(0) \in [y_0 - \delta, y_0 + \delta]$ is defined for sufficiently large t and satisfies $\lim_{t \to +\infty} y(t) = y_0$, and *unstable* otherwise.

1. $z' = -z^2 + 1$.
   This is the logistic equation (without harvesting), with steady states $z \equiv \pm 1$.
   Our previous analysis shows that

$$\lim_{s \to +\infty} z(s) = \begin{cases} 1 & \text{if } z_0 > -1, \\ \text{undefined} & \text{if } z_0 < -1. \end{cases}$$

   Thus $z \equiv 1$ is stable and $z \equiv -1$ is unstable.

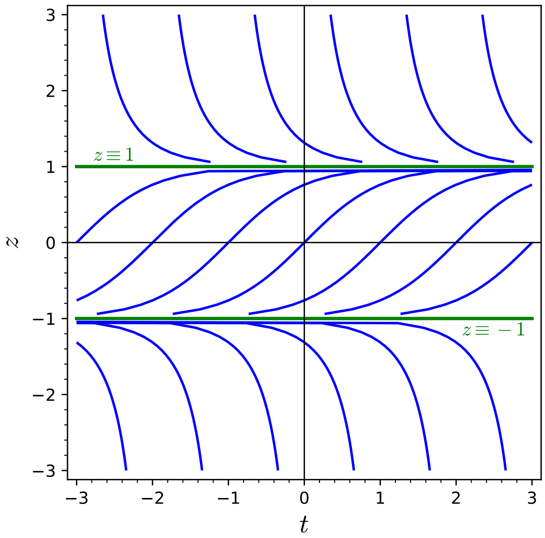


Figure: Solution curves of $z' = 1 - z^2$

Analysis of the Canonical Forms Cont'd

2. $z' = -z^2$.
   The standard solution method gives

$$\frac{1}{z} - \frac{1}{z_0} = \int_{z_0}^{z} -\frac{\mathrm{d}\zeta}{\zeta^2} = \int_{s_0}^{s} \mathrm{d}\sigma = s - s_0,$$

   i.e., $z(s) = 1/(s - C)$ with $C = s_0 - 1/z_0$.

   This tells us:

   Solutions $z(s)$ with $z(s_0) = z_0 > 0$ (equivalently, $s_0 > C$) exist forever and satisfy $\lim_{s\to+\infty} z(s) = 0$.

   Solutions $z(s)$ with $z(s_0) = z_0 < 0$ (equivalently, $s_0 < C$) exist only on $(-\infty, C)$ and satisfy $\lim_{s\uparrow C} z(s) = -\infty$.

   In other words, the steady-state solution $z \equiv 0$ is *one-sided stable* (*stable from above*).

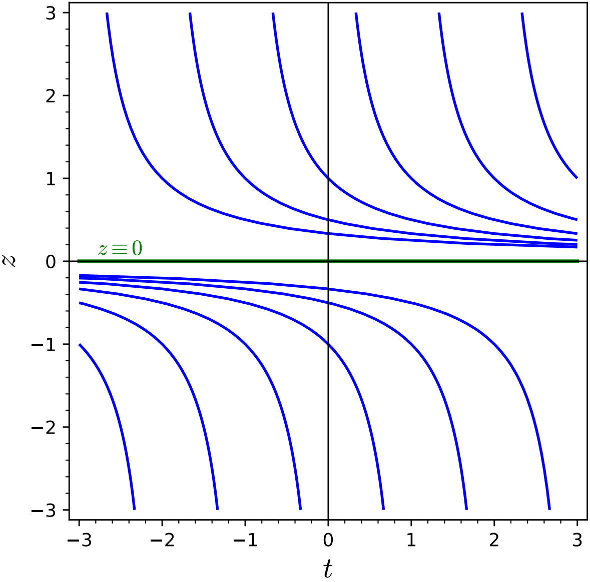


Figure: Solution curves of $z' = -z^2$

Analysis of the Canonical Forms Cont'd

3. $z' = -z^2 - 1$.
Here the standard solution method gives

$$\arctan z_0 - \arctan z = \int_{z_0}^{z} -\frac{\mathrm{d}\zeta}{\zeta^2+1} = \int_{s_0}^{s} \mathrm{d}\sigma = s - s_0,$$

i.e., $z(s) = \tan(C - s)$ with $C = s_0 + \arctan z_0$.

This tells us:

Solutions $z(s)$ with $z(s_0) = z_0$ exist only on $(C - \pi/2, C + \pi/2)$ and satisfy $\lim_{s \uparrow C+\pi/2} z(s) = -\infty$.

The solutions with $z_0 > 0$ have $C > s_0$ and hence exist for a period larger than $\pi/2$, while those with $z_0 < 0$ have $C < s_0$ and exist for a period less than $\pi/2$.

Since there are no steady-state solutions, the question of stability does not arise.

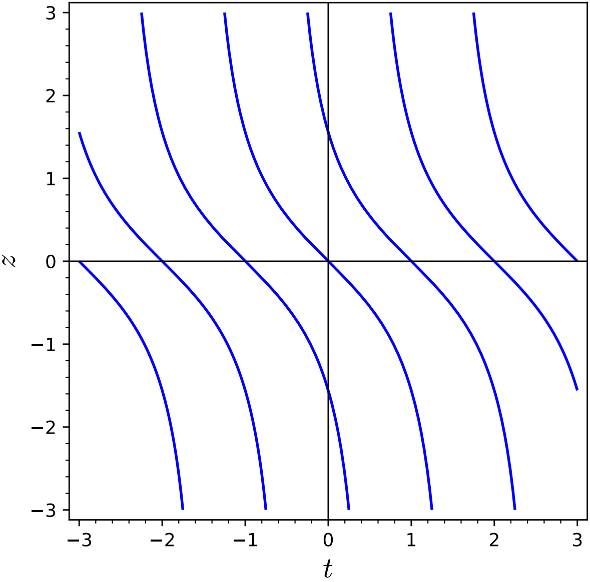


Figure: Solution curves of $z' = -1 - z^2$

Remark

It is instructive to represent the solution curves (except the steady-state solutions) in the preceding examples as function $t(y)$ resp. $s(z)$. This makes sense for any autonomous ODE and (provided the ODE can be integrated in closed form) often yields a simpler formula for the solution curves which better explains their shape.

Exercise

Show that the graph of $y(t) = a/(d\,\mathrm{e}^{-at} + b)$ $(a, b, d > 0)$ is point-symmetric to its inflection point.

Hint: A superb way to solve this exercise is to observe that the mirror image of a solution curve w.r.t. its inflection point represents a solution as well and use the uniqueness of solutions of associated IVP's.

Remark: For $d < 0$ graphs have a similar symmetry, but the meaning of the center of symmetry is different.

Finally we translate the results on the asymptotic behaviour back into the original harvesting equation $y' = ay - by^2 - h$ $(a, b, h > 0)$. Recall that for $\Delta = a^2 - 4bh \geq 0$ there are the steady-sate solutions $y \equiv y_{1/2}$ with $y_1 = \left(a - \sqrt{a^2 - 4bh}\right)/2b$, $y_2 = \left(a + \sqrt{a^2 - 4bh}\right)/2b$, which satisfy $0 < y_1 \leq y_2$.

Analysis of the Harvesting Equation

$h < a^2/4b$ If the initial population $y(t_0)$ satisfies $y_1 < y(t_0) < y_2$ then the population $y(t)$ increases and $\lim_{t\to+\infty} y(t) = y_2$. If $y(t_0) > y_2$ then $y(t)$ decreases and $\lim_{t\to\infty} y(t) = y_2$. If $y(t_0) < y_1$ then $y(t)$ decreases and $y(t_1) = 0$ for some $t_1 > t_0$, i.e., the population dies out.

$h = a^2/4b$ If $y(t_0) > a/2b$, the population decreases and $\lim_{t\to\infty} y(t) = a/2b$. If $y(t_0) < a/2b$, the population decreases and dies out at some time $t_1 > t_0$.

$h > a^2/4b$ Regardless of the initial population, the population dies out in finite time.

Exact First-Order Equations

Definition

A first-order ODE of the form

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{D}$$

with $M, N\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ open, is said to be *exact* if there exists a function $f\colon D \to \mathbb{R}$ satisfying $\mathrm{d}f = M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y$ or, equivalently, $\nabla f = (f_x, f_y) = (M, N)$.

Notes

- Criteria for exactness have been developed in Calculus III. Recall that for C^1-functions $M, N\colon D \to \mathbb{R}$ a necessary condition for exactness is $M_y = N_x$, which is also sufficient if D is simply connected.
- As explained on the next two slides, the “differential-like” form (D) of a first-order ODE is essentially equivalent to the explicit form
$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{M(x,y)}{N(x,y)}.$$
obtained from (D) by pretending that $\mathrm{d}x$, $\mathrm{d}y$ are numbers.

Solutions of (D)

By a *solution curve* (*integral curve*, *parametrized solution*) of (D) we mean a smooth differentiable curve $\gamma\colon I \to D$, $t \mapsto \bigl(x(t), y(t)\bigr)$ satisfying

$$M\bigl(x(t), y(t)\bigr)x'(t) + N\bigl(x(t), y(t)\bigr)y'(t) = 0 \quad \text{for } t \in I. \tag{O}$$

Geometrically, the tangent to the curve at any point must be orthogonal (perpendicular) to the vector of the vector field (M, N) at that point (since (D) is equivalent to $(M, N) \cdot \gamma' = 0$).

By an *(explicit) solution* $y = y(x)$ (resp., $x = x(y)$) we mean a function $y\colon I \to \mathbb{R}$ (resp., $x\colon J \to \mathbb{R}$) with graph contained in D and satisfying

$$\begin{aligned} M\bigl(x, y(x)\bigr) + N\bigl(x, y(x)\bigr)y'(x) &= 0 \quad \text{for } x \in I, \quad \text{resp.,} \\ M\bigl(x(y), y\bigr)x'(y) + N\bigl(x(y), y\bigr) &= 0 \quad \text{for } y \in J. \end{aligned}$$

Notes

- These concepts make sense for any (not necessarily exact) 1st-order ODE in differential-like form.

Notes cont'd

- A point $(x_0, y_0) \in D$ is said to be a *singular point* of the ODE $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ if $M(x_0, y_0) = N(x_0, y_0) = 0$. The orthogonality condition (O) is trivially satisfied in any singular point.
- Suppose (x_0, y_0) is a non-singular point of $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ and satisfies $N(x_0, y_0) = 0$.
  $\Longrightarrow$ Any solution curve $\gamma = (x, y)$ passing through (x_0, y_0) must have $x' = 0$ at (x_0, y_0).
  This says that γ has a vertical tangent at (x_0, y_0) and clearly forms an obstruction to representing it as a function $y(x)$.
  Conversely, if γ satisfies $\gamma(t_0) = (x_0, y_0)$ and $x'(t_0) = 0$ then $y'(t_0) \neq 0$ (since solution curves are smooth) and hence $N(x_0, y_0) = 0$.

The last note helps to clarify the correspondence between solution curves of $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$ and solutions of $y' = -M(x, y)/N(x, y)$; cf. next slide.

Correspondence

Solution curves of $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ and explicit solutions correspond to each other in the following way:

1. Given a solution curve γ, smoothness implies that at each non-singular point $(x_0, y_0) \in \gamma(I)$ we can write the curve locally as graph of a C^1-function $y(x)$ or $x(y)$ (or both), and these functions satisfy

$$y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y\,/\,\mathrm{d}t}{\mathrm{d}x\,/\,\mathrm{d}t} = -\frac{M(x,y)}{N(x,y)}, \quad \text{resp.,}$$
$$x' = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}x\,/\,\mathrm{d}t}{\mathrm{d}y\,/\,\mathrm{d}t} = -\frac{N(x,y)}{M(x,y)},$$

   i.e., are explicit solutions. Note that, e.g., the representation $y(x)$ implies $x'(t) \neq 0$ and hence $N\big(x(t), y(t)\big) \neq 0$, as remarked in the previous note.

2. Conversely, given an explicit solution $y(x)$, we can use, e.g., $x(t) = t$ as parameter to define a curve $\gamma(t) = \big(t, y(t)\big)$, and this curve γ is a solution curve on account of

$$M(x,y)x' + N(x,y)y' = M(x,y) \cdot 1 + N(x,y)y' = 0.$$

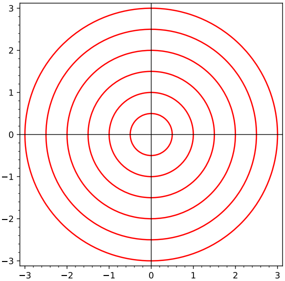
Thus, if we remove from D all points (x, y) with $N(x, y) = 0$, which form a closed set, and call the resulting domain D', we get a 1-1 correspondence between non-parametric solution curves of $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ (or classes of parametric solution curves under the equivalence relation of smooth reparametrization) and explicit solutions of $y' = -M(x, y)/N(x, y)$; and similarly for the case $M(x, y) = 0$.

Example

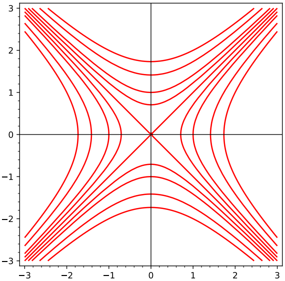
In the lecture and an exercise we have considered the four ODE's $y' = \pm x/y$, $y' = \pm y/x$. Associated differential-like forms are

1. $y' = -x/y \triangleq x\,\mathrm{d}x + y\,\mathrm{d}y = 0$;
2. $y' = x/y \triangleq x\,\mathrm{d}x - y\,\mathrm{d}y = 0$;
3. $y' = -y/x \triangleq x\,\mathrm{d}y + y\,\mathrm{d}x = 0$;
4. $y' = y/x \triangleq x\,\mathrm{d}y - y\,\mathrm{d}x = 0$.

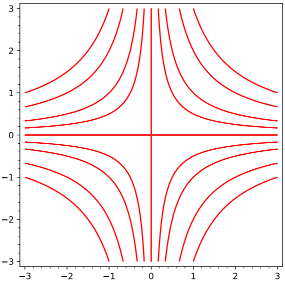
All four differential-like ODE's have exactly one singular point, viz. $(0, 0)$, and we need to remove either the x-axis (1st and 2nd ODE) or the y-axis (3rd and 4th ODE) in order to get a 1-1 correspondence of their solution curves with the solutions of the original explicit ODE. Solution curves are shown on the next slide.



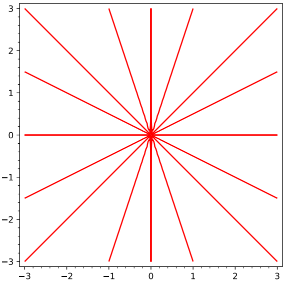
(1) $x\,\mathrm{d}x + y\,\mathrm{d}y = 0$



(2) $x\,\mathrm{d}x - y\,\mathrm{d}y = 0$



(3) $x\,\mathrm{d}y + y\,\mathrm{d}x = 0$



(4) $x\,\mathrm{d}y - y\,\mathrm{d}x = 0$

Theorem

Suppose $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ *is exact with antiderivative (potential function)* F*. Then the solution curves of* $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ *are precisely the parametrized level sets (contours)* $F(x,y) = C$*,* $C \in \mathbb{R}$*, or (sub-)branches thereof.*

Proof.

It suffices to show that any solution $\gamma(t) = \big(x(t), y(t)\big)$, $t \in I$, of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$ is contained in a level set of F.
We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} F\big(\gamma(t)\big) &= \nabla F\big(\gamma(t)\big) \cdot \gamma'(t) \\
&= \begin{pmatrix} M\big(x(t), y(t)\big) \\ N\big(x(t), y(t)\big) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\
&= M\big(x(t), y(t)\big)x'(t) + N\big(x(t), y(t)\big)y'(t) \\
&= 0,
\end{aligned}$$

because $\gamma(t)$ is a solution of $M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$.
This shows that $t \mapsto F\big(\gamma(t)\big)$ is constant on I, i.e., $\big\{\gamma(t); t \in I\big\}$ is contained in a level set of F. $\square$

Example

Consider the ODE

$$(x-y)\,\mathrm{d}x+\left(\frac{1}{y^2}-x\right)\mathrm{d}y=0$$

The domain consists of all $(x,y)\in\mathbb{R}^2$ with $y\neq 0$ and has two simply-connected components (upper half plane and lower half plane).

Since $\frac{\mathrm{d}}{\mathrm{d}y}(x-y)=-1=\frac{\mathrm{d}}{\mathrm{d}x}(y^{-2}-x)$, the ODE is exact.

An antiderivative, determined as usual by partial integration, is

$$f(x,y)=\frac{x^2}{2}-xy-\frac{1}{y}.$$

The general solution in implicit form is therefore

$$x^2y-2y^2x-2-Cy=0,\quad C\in\mathbb{R}.$$

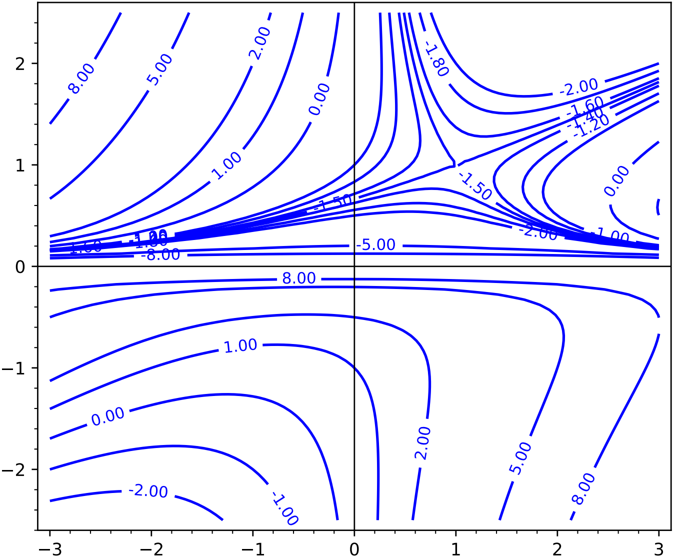


Figure: Solution curves of $(x - y)\,\mathrm{d}x + (y^{-2} - x)\,\mathrm{d}y = 0$, represented as contours of $F(x, y) = x^2/2 - xy - 1/y$

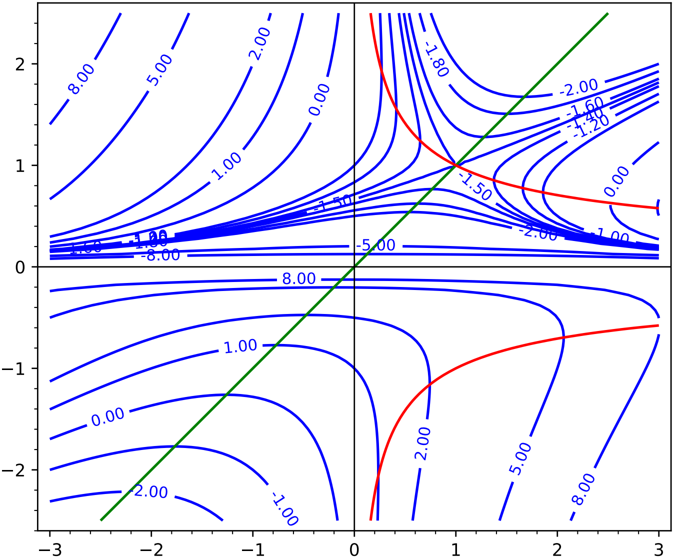


Figure: The same with all points highlighted that satisfy $M(x,y)=0$ or $N(x,y)=0$; removing the red (green) curve leaves solutions $y(x)$ of $y'=\frac{x-y}{x-y^{-2}}$ (resp., solutions $x(y)$ of $x'=\frac{x-y^{-2}}{x-y}$)

Example

Of the four ODE's $y' = \pm x/y$, $y' = \pm y/x$ three are exact, viz.

1. $x\,\mathrm{d}x + y\,\mathrm{d}y = \mathrm{d}F$ for $F(x,y) = \frac{1}{2}(x^2 + y^2)$;
2. $x\,\mathrm{d}x - y\,\mathrm{d}y = \mathrm{d}F$ for $F(x,y) = \frac{1}{2}(x^2 - y^2)$;
3. $x\,\mathrm{d}y + y\,\mathrm{d}x = \mathrm{d}F$ for $F(x,y) = xy$.

This shows that the corresponding solution curves are

1. circles centered at the origin (contours of $(x,y) \mapsto x^2 + y^2$);
2. hyperbolas centered at the origin with asymptotes $y = \pm x$ (contours of $(x,y) \mapsto x^2 - y^2$);
3. hyperbolas centered at the origin with asymptotes $x = 0$ and $y = 0$ (contours of $(x,y) \mapsto xy$).

The 4th ODE $x\,\mathrm{d}y - y\,\mathrm{d}x = 0$ (corresponding to the winding form/field) is not exact.

But it can be multiplied by $1/(xy)$ to yield the exact (even separable) ODE $y^{-1}\,\mathrm{d}y - x^{-1}\,\mathrm{d}x = 0$ ($\to$ integrating factors), which has solution curves $\ln|y| - \ln|x| = C$ or, equivalently, $y/x = \pm\mathrm{e}^C$; compare with the previous plot of these curves.

Integrating Factors

The ODE $y\,\mathrm{d}x + (x^2y - x)\,\mathrm{d}y = 0$ is not exact, since $\frac{\partial M}{\partial y} = 1$ and $\frac{\partial N}{\partial x} = 2xy - 1$.

But we can multiply the equation by $1/x^2$, turning it into the exact ODE

$$\frac{y}{x^2}\,\mathrm{d}x + \left(y - \frac{1}{x}\right)\mathrm{d}y = 0$$

with potential $f(x, y) = -y/x + y^2/2$ and general solution $xy^2 - 2y - Cx = 0$.

Since the exact ODE has a strictly smaller domain, viz. $\mathbb{R}^2$ without the y-axis, we also need to check whether the parametrized y-axis $\gamma(t) = (0, t)$ is a solution of $y\,\mathrm{d}x + (x^2y - x)\,\mathrm{d}y = 0$, and indeed it is ($x(t) = x'(t) = 0$). But it is missing in the implicit solution.

Definition

A function $\mu(x, y)$ with domain $D' \subseteq D$ is called an *integrating factor* (or *Euler multiplier*) of $M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0$, if

1. $\mu(x, y) \neq 0$ for all $(x, y) \in D'$;
2. $\mu(x, y)M(x, y)\,\mathrm{d}x + \mu(x, y)N(x, y)\,\mathrm{d}y = 0$ is exact on D'.

Lemma

If an ODE $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ has a general solution of the form $f(x,y) = C$ then it has an integrating factor.

Proof.

Differentiating $f(x,y) = C$ gives $\frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = 0$.

$$\Longrightarrow \frac{\mathrm{d}y}{dx} = -\frac{M}{N} = -\frac{\partial f/\partial x}{\partial f/\partial y},$$

which can be rewritten as

$$\frac{\partial f/\partial x}{M} = \frac{\partial f/\partial y}{N} = \mu(x,y), \quad \text{say.}$$

This says that $\mu M\,\mathrm{d}x + \mu N\,\mathrm{d}y = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y = 0$ is exact. □

Remark

We can multiply an integrating factor μ by any continuous function $F(f)$ of the antiderivative f of the resulting exact equation, thereby obtaining another integrating factor $\mu F(f)$. (Check that a suitable antiderivative is $G(f)$, where $G' = F$.) Hence integrating factors are highly non-unique.

How to Find an Integrating Factor?

The (local) exactness condition for an integrating factor μ is $\partial(\mu M)/\partial y = \partial(\mu N)/\partial x$. This gives

$$\mu\,\frac{\partial M}{\partial y} + M\,\frac{\partial \mu}{\partial y} = \mu\,\frac{\partial N}{\partial x} + N\,\frac{\partial \mu}{\partial x}, \qquad \text{or}$$

$$\mu\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = N\,\frac{\partial \mu}{\partial x} - M\,\frac{\partial \mu}{\partial y}.$$

This partial differential equation (PDE) for μ is not easy to solve in general, but frequently one can make a particular „Ansatz" for μ and solve it in this special case.

Example

Suppose $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = g(x)$ depends only on x but not on y.

Then $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ has the integrating factor $\mu(x) = \mathrm{e}^{\int g(x)\,\mathrm{d}x}$.

Reason: In this case the PDE for $\mu(x, y) = \mu(x)$ is equivalent to $\mu'(x) = g(x)\mu(x)$.

Example (cont'd)

As a concrete example we reconsider $y\,\mathrm{d}x + (x^2y - x)\,\mathrm{d}y = 0$.

Here we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{M_y - N_x}{N} = \frac{1 - (2xy - 1)}{x^2y - x} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x}.$$

An integrating factor is therefore

$$\mu(x) = \mathrm{e}^{\int(-2/x)\,\mathrm{d}x} = \mathrm{e}^{-2\ln x} = \frac{1}{x^2},$$

as we have seen before.

Remark

In particular we can solve the PDE for μ if all of μ, $M_y - N_x$, N depend only on x. But we only need the weaker condition "$(M_y - N_x)/N$ depends only on x". In the example above both $M_y - N_x$ and N depend on both x and y but $(M_y - N_x)/N$ depends only on x.

Theorem

The ODE $M\,\mathrm{d}x + N\,\mathrm{d}y = 0$ has an integrating factor of the form

1. $\mu(x)$ *if* $\frac{M_y - N_x}{N} = g(x)$;
2. $\mu(y)$ *if* $\frac{M_y - N_x}{M} = g(y)$;
3. $\mu(xy)$ *if* $\frac{M_y - N_x}{Ny - Mx} = g(xy)$;
4. $\mu(y/x)$ *if* $\frac{x^2(M_y - N_x)}{Ny + Mx} = g(y/x)$.

Proof.

In each case the PDE $(M_y - N_x)\mu = N\mu_x - M\mu_y$ derived for $\mu(x, y)$ becomes a homogeneous linear 1st-order ODE for the one-variable function $\mu(s)$ (note the slight abuse of notation in the last two cases!), which can be solved using the standard method. The resulting ODE for $\mu(s)$ is
$\mu'(s) = g(s)\mu(s)$ in Cases (1) and (3), and
$\mu'(s) = -g(s)\mu(s)$ in Cases (2) and (4).

Proof cont'd.

We do this explicitly for the last case:

$$\frac{\partial}{\partial x}\,\mu\left(\frac{y}{x}\right)=\mu'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right),$$
$$\frac{\partial}{\partial y}\,\mu\left(\frac{y}{x}\right)=\mu'\left(\frac{y}{x}\right)\frac{1}{x}.$$

Hence $(M_y-N_x)\mu=N\mu_x-M\mu_y$ becomes

$$\begin{aligned}
&& (M_y-N_x)\,\mu\left(\frac{y}{x}\right) &= N\,\mu'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)-M\,\mu'\left(\frac{y}{x}\right)\frac{1}{x}\\
\iff && x^2(M_y-N_x)\,\mu\left(\frac{y}{x}\right) &= -(Ny+Mx)\,\mu'\left(\frac{y}{x}\right)\\
\iff && \mu'\left(\frac{y}{x}\right) &= -\frac{x^2(M_y-N_x)}{Ny+Mx}\,\mu\left(\frac{y}{x}\right).
\end{aligned}$$

If $\frac{x^2(M_y-N_x)}{Ny+Mx}=g(y/x)$ depends only on y/x, we can substitute $s=y/x$ and obtain the equivalent ODE $\mu'(s)=-g(s)\mu(s)$. $\square$

Final Remarks

- In some texts the case of an integrating factor of the form $\mu(x/y)$ is listed as well. But this reduces to Case (4) if we consider $\tilde{\mu}(s) = \mu(1/s)$.

- The PDE $(M_y - N_x)\mu = N\mu_x - M\mu_y$ only guarantees local exactness of $(\mu M)\,\mathrm{d}x + (\mu N)\,\mathrm{d}y$ on D'. To obtain an anti-derivative, it may be necessary to restrict the domain further to simply-connected subsets of D', on which $(\mu M)\,\mathrm{d}x + (\mu N)\,\mathrm{d}y$ then must be exact, and determine solutions there.

  For example, $x\,\mathrm{d}y - y\,\mathrm{d}x = 0$ has the integrating factor $1/(xy)$, as we have seen, whose domain $\mathbb{R}^2$ with the coordinate axes removed consists of 4 simply connected regions (the 4 open quadrants). On each quadrant, an antiderivative of $(xy)^{-1}(x\,\mathrm{d}y - y\,\mathrm{d}x) = y^{-1}\,\mathrm{d}y - x^{-1}\,\mathrm{d}x$ exists and can be taken as $f(x,y) = \ln|y| - \ln|x|$, amounting to 4 different choices of signs of x, y for the 4 regions.