Question 1 (ca. 14 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = y^2 - 2$ satisfying $y(0) = 1$ and $y(1) = 2$.

b) The maximal solution of the initial value problem $y' = y^2 \cos t$, $y(0) = 1$ exists at time $t = 2$.

c) Every solution $y(t)$ of $t^2y'' - ty' + y = 0$, $t > 0$, satisfies $\lim_{t\downarrow 0} y(t) = 0$.

d) The initial value problem $(\cos x)y'' + (\sin x)y' + y = 0$, $y(0) = y'(0) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$ which is defined at $x = 1$.

e) If $\mathbf{A} \in \mathbb{R}^{n\times n}$ satisfies $\mathbf{A}^2 = \mathbf{I}_n$ (the $n \times n$ identity matrix) then $\mathrm{e}^{\mathbf{A}t} = (\cosh t)\mathbf{I}_n + (\sinh t)\mathbf{A}$.

f) Every solution $\mathbf{y}(t)$ of the system $\mathbf{y}' = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}\mathbf{y}$ satisfies $\lim_{t\to\infty} \mathbf{y}(t) = (0,0)^{\mathsf{T}}$.

g) There exist real numbers $b_1, b_2, b_3, \dots$ such that $x - x^2 = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$ for all $x \in [0,1]$.

Question 2 (ca. 7 marks)

Consider the differential equation

$$(x-1)^2y'' + 2(x^2-1)y' - 4y = 0. \tag{DE}$$

a) Show that $x_0 = 1$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(1, \infty)$.

Hint: It turns out that $r_1 - r_2 \in \mathbb{Z}$, but the more complicated machinery developed in the lecture/textbook for this case is not needed.

c) Using the result of b), discuss the general solution of (DE) on $(-\infty, 1)$ and on $\mathbb{R}$.

Question 3 (ca. 5 marks)

Consider the ODE

$$y' = t^3 + \frac{2}{t}\,y - \frac{1}{t}\,y^2, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = t^r$.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE, and explain the precise correspondence between solutions of (R) and solutions of the linear ODE.

c) Solve the linear ODE in b) and use the result to determine the general solution of (R).

Question 4 (ca. 9 marks)

Consider $\mathbf{A} = \begin{pmatrix} 1 & -2 & -2 \\ -4 & -1 & 2 \\ 0 & 0 & -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$.

a) Determine a fundamental system of solutions of the system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

b) Solve the initial value problem $\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{b}$, $\mathbf{y}(0) = (0,0,0)^{\mathsf{T}}$.

Question 5 (ca. 6 marks)

Consider the differential equation

$$(3xy + 2y^2)\,\mathrm{d}x + (3x^2 + 6xy + 3y^2)\,\mathrm{d}y = 0. \tag{DF}$$

a) Show that $(0,0)$ is the only singular point of (DF).

b) Transform (DF) into an exact equation and determine the general solution in implicit form.

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

Question 6 (ca. 7 marks)

Determine all real solutions $y(t)$ of

$$2\,y^{(5)} - y''' + y'' = 1 + t - 2\sin t.$$