Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) There exists a solution $y(t)$ of $y' = 2y - y^2$ satisfying $y(0) = y(1) = 1$.

b) The maximal solution of the initial value problem $y' = y^2 - t$, $y(0) = \frac{1}{2}$ exists at time $t = 2021$.

c) Every solution $y\colon (0, \infty) \to \mathbb{R}$ of $t^2y'' + 3t\,y' + 2y = 0$ has infinitely many zeros.

d) The initial value problem $(x^2 + 4)y'' + (x + 4)y' - 4y = 0$, $y(1) = y'(1) = 1$ has a power series solution $y(x) = \sum_{n=0}^{\infty} a_n(x - 1)^n$ which is defined at $x = 3$.

e) Suppose $\mathbf{A} \in \mathbb{R}^{2\times 2}$ satisfies $\mathbf{A}^3 = \mathbf{I}$ (the 2×2 identity matrix), but $\mathbf{A} \neq \mathbf{I}$. Then every solution $\mathbf{y}(t)$ of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$ must satisfy $\lim_{t\to\infty} \mathbf{y}(t) = (0, 0)^\mathsf{T}$.

f) Suppose $f, g\colon (-1, 1) \to \mathbb{R}$ are C^1-functions. Then the IVP $y' = f(t)g(y)$, $y(0) = 0$ has a solution $y(t)$ that is defined for all $t \in (-1, 1)$.

Question 2 (ca. 10 marks)

Consider the differential equation

$$2x^2\,y'' + x(1-x)y' - 6\,y = 0. \tag{DE}$$

a) Verify that $x_0 = 0$ is a regular singular point of (DE).

b) Determine the general solution of (DE) on $(0, \infty)$.

c) Using the result of b), state the general solution of (DE) on $(-\infty, 0)$ and on $\mathbb{R}$.

Question 3 (ca. 7 marks)

Consider the ODE

$$y' = y^2 + \frac{5}{t}\,y + \frac{5}{t^2}, \qquad t > 0. \tag{R}$$

a) Show that there exists a solution $y_1(t)$ of the form $y_1(t) = c\,t^r$ with constants c, r.

b) Show that the substitution $y = y_1 + 1/z$ transforms (R) into a first-order linear ODE.

c) Using b), determine all maximal solutions of (R) and their domain.

Question 4 (ca. 6 marks)

For the matrix $\mathbf{A} = \begin{pmatrix} -8 & 0 & 5 & -2 \\ 5 & -1 & -4 & 1 \\ -10 & 0 & 7 & -2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$ determine the general solution of the linear system $\mathbf{y}' = \mathbf{A}\mathbf{y}$.

Question 5 (ca. 7 marks)

Consider the differential equation

$$x(3y^2 - 1)\,\mathrm{d}x + y\,\mathrm{d}y = 0. \tag{DF}$$

a) Determine the general solution of (DF) in implicit form.

b) Determine the maximal solution $y(x)$ satisfying $y(1) = \frac{1}{3}$ and its domain.
Hint: $\ln\left(\frac{3}{2}\right) \approx 0.4$

c) Is every point of $\mathbb{R}^2$ on a unique integral curve of (DF)?

Question 6 (ca. 8 marks)

a) Determine a real fundamental system of solutions of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 0.$$

Hint: The characteristic polynomial is divisible by the square of a quadratic polynomial.

b) Determine the general real solution of

$$y^{(5)} + 4\,y^{(4)} + 24\,y''' + 40\,y'' + 100\,y' = 200\,t - \mathrm{e}^{-t}.$$

c) Find the Laplace transform $Y(s)$ of the solution of the ODE in b) with initial values $y(0) = y'(0) = y''(0) = y'''(0) = y^{(4)}(0) = 0$.