Math 286
Introduction to Differential Equations

Thomas Honold

ZJU-UIUC Institute

Fall Semester 2021

Outline

Today's Lecture:

Definition

An *analytic solution* (or *power series solution*) of a scalar ODE is a "power series function"

$$y(t)=\sum_{n=0}^{\infty}a_n(t-t_0)^n,\quad t\in I,$$

for some interval $I\subseteq\mathbb{R}$ of positive length and some $t_0\in I$.

Notes

- The name "analytic" comes from *analytic function*, which refers to a function $f\colon D\to\mathbb{C}$, $D\subseteq\mathbb{C}$, which locally admits a power series representation $f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$ (either for all $z_0\in D$, in which case f is said to be holomorphic, or only for a fixed point $z_0\in D$).
- The (w.l.o.g. open) interval I must be contained in the interval of convergence of $\sum_{n=0}^{\infty}a_n(t-t_0)^n$, which is of the form $(t_0-\rho,t_0+\rho)=\mathrm{B}_\rho(t_0)\cap\mathbb{R}$ with $\mathrm{B}_\rho(t_0)$ denoting the open disk of convergence of $\sum_{n=0}^{\infty}a_n(z-t_0)^n$. In particular the radius of convergence ρ of the power series must be positive.

Notes cont'd

- There is an "analytic" version of the Existence and Uniqueness Theorem, which roughly says that if $f(t, y_0, y_1, \ldots, y_{n-1})$ is analytic then the solution of an IVP $y^{(n)} = f(t, y, y', \ldots, y^{(n-1)})$, $y^{(i)}(t_0) = c_i$ for $0 \leq i \leq n-1$ must be analytic at t_0 and solve the ODE wherever it is defined.

As a consequence of this theorem we can solve ODE's in the analytic case by a power series „Ansatz".

In what follows, we will switch notation from $y(t)$ to $y(x)$, because for power series the variable symbol 'x' is more common in view of the link $z = x + y\,\mathrm{i}$ with the complex case.

Example

Determine a solution of the IVP $y' = x^2 + y^2$, $y(0) = 1$.

Because $f(x, y) = x^2 + y^2$ is analytic, the solution must be analytic as well, i.e., of the form $y(x) = \sum_{n=0}^{\infty} a_n x^n$ with $a_n = y^{(n)}(0)/n!$.

Method 1: Determine $y^{(n)}(0)$ from the ODE.

$$\begin{aligned}
y' &= x^2 + y^2 &&\implies & y'(0) &= 0^2 + y(0)^2 = 1,\\
y'' &= 2x + 2yy' &&\implies & y''(0) &= 2y(0)y'(0) = 2,\\
y''' &= 2 + 2y'^2 + 2yy'' &&\implies & y'''(0) &= 2 + 2 + 4 = 8,\\
y^{(4)} &= 6y'y'' + 2yy''' &&\implies & y^{(4)}(0) &= 12 + 16 = 28
\end{aligned}$$

$$\begin{aligned}
\implies y(x) &= 1 + x + \tfrac{2}{2!}x^2 + \tfrac{8}{3!}x^2 + \tfrac{28}{4!}x^4 + \cdots\\
&= 1 + x + x^2 + \tfrac{4}{3}x^3 + \tfrac{7}{6}x^4 + \cdots
\end{aligned}$$

This method is cumbersome, and it does not tell us anything about the radius of convergence of the resulting power series, and hence about the domain of the solution (except that by the general theory it must be an interval of positive length).

Example (cont'd)

Method 2: Substitute $y(x)=\sum_{n=0}^{\infty} a_n x^n$ into the ODE and equate coefficients.

$$y'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n,$$

$$x^2+y(x)^2=x^2+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right)x^n$$

$$\Longrightarrow\ (n+1)a_{n+1}=\begin{cases}\sum_{k=0}^{n} a_k a_{n-k} & \text{if } n\neq 2,\\ 1+2a_0a_2+a_1^2 & \text{if } n=2\end{cases}$$

For $n\geq 1$ this determines a_n from a_k, $k<n$, and thus together with the initial value $a_0=y(0)=1$ provides a recursion formula for a_n, which can easily be programmed:

$$y(x)=1+x+x^2+\tfrac{4}{3}x^3+\tfrac{7}{6}x^4+\tfrac{6}{5}x^5+\tfrac{37}{30}x^6+\tfrac{404}{315}x^7+\tfrac{369}{280}x^8+\tfrac{428}{315}x^9+ \\ +\tfrac{1961}{1400}x^{10}+\tfrac{75092}{51975}x^{11}+\tfrac{1238759}{831600}x^{12}+\tfrac{9884}{6435}x^{13}+\cdots$$

Example (cont'd)

The radius of convergence:
It is clear that all a_n are positive. Using induction, we can easily show that $a_n \geq 1$ for all n:

$$\Longrightarrow a_n = \frac{a_0 a_{n-1} + a_1 a_{n-2} + \cdots + a_{n-1} a_0}{n} \geq \frac{1^2 + 1^2 + \cdots + 1^2}{n} = 1$$

$\Longrightarrow \rho \leq 1$

Conversely, suppose $a_k \leq c^k$ for all $k < n$ and some constant c.

$$a_n \leq \frac{c^0 c^{n-1} + c^1 c^{n-2} + \cdots + c^{n-1} c^0}{n} = c^{n-1} \leq c^n,$$

provided that $c \geq 1$ and $n \geq 4$.
$\Longrightarrow a_N \leq c^N$ for all $N \geq n$ (using induction)
$\Longrightarrow \rho \geq 1/c$.

For example we can take $n = 4$ and $c = \sqrt[3]{\frac{4}{3}}$.

$\Longrightarrow \rho \geq \sqrt[3]{\frac{3}{4}} = 0.9085\cdots > 0.9$

Example

Determine the general solution of $y' = y^2$ using the power series „Ansatz".

Since $y' = y^2$ is autonomous, we can restrict ourselves to the case $x_0 = 0$, i.e., make again the „Ansatz" $y(x) = \sum_{n=0}^{\infty} a_n x^n$. The recursion formula of the previous example changes to

$$a_n = \frac{1}{n} \sum_{k=0}^{n-1} a_k a_{n-1-k} \quad \text{for all } n \geq 1.$$

$\Longrightarrow a_1 = a_0^2$, $a_2 = \frac{1}{2}(a_0 a_1 + a_1 a_0) = a_0^3$,
$a_3 = \frac{1}{3}(a_0 a_2 + a_1^2 + a_2 a_0) = a_0^4$, etc., and in general $a_n = a_0^{n+1}$.

$$\Longrightarrow \; y(x) = \sum_{n=0}^{\infty} a_0^{n+1} x^n = a_0 \sum_{n=0}^{\infty} (a_0 x)^n = \frac{a_0}{1 - a_0 x}, \quad |x| < \frac{1}{|a_0|} = \rho$$

Thus $y(x) = 1/(C - x)$ with $C = 1/a_0$, recovering the previously determined general solution of $y' = y^2$.

Example (cont'd)

Notes

- Two solutions, which are not defined at $x_0 = 0$, were missed. These are

$$y_1(x) = -\frac{1}{x} \quad \text{for } x < 0,$$
$$y_2(x) = -\frac{1}{x} \quad \text{for } x > 0.$$

  They can be formally included in $y(x) = a_0/(1 - a_0 x)$ if we permit $a_0 = \infty$.

- The solution $y(x) = a_0/(1 - a_0 x)$ is defined on the unbounded interval containing 0 and having $1/a_0$ as one endpoint (the other endpoint is $+\infty$ or $-\infty$). But the power series representation is valid only on the bounded subinterval $|x| < \frac{1}{|a_0|}$.

Two Euler-Like Equations

Example

We consider simultaneously the ODE's

$$xy'' + y' + y = 0, \tag{E1}$$

$$x^2y'' + y' + y = 0, \tag{E2}$$

which have a singularity at $x = 0$ like the general Euler equation. Note that, in a way, (E2) is more singular at $x = 0$ than (E1).

Making the usual power series „Ansatz“ $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and using

$$y'(x) = \sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1} x^n,$$

$$x\, y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n,$$

$$x^2\, y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^n,$$

Example (cont'd)

the ODE's become

$$a_0 + a_1 + \sum_{n=1}^{\infty}\bigl((n+1)na_{n+1} + (n+1)a_{n+1} + a_n\bigr)x^n = 0,$$

$$a_0 + a_1 + (a_1 + 2a_2)x + \sum_{n=2}^{\infty}\bigl(n(n-1)a_n + (n+1)a_{n+1} + a_n\bigr)x^n = 0.$$

Equating coefficients gives the recursion formulas

$$a_{n+1} = -\frac{1}{(n+1)^2}\,a_n \quad \text{for } n = 0, 1, 2, \dots, \tag{E1}$$

$$a_{n+1} = -\frac{n^2 - n + 1}{n+1}\,a_n \quad \text{for } n = 0, 1, 2, \dots \tag{E2}$$

$\Longrightarrow \rho = \infty$ for (E1), giving the analytic solution

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\,x^n \quad \text{for } x \in \mathbb{R}.$$

$\Longrightarrow \rho = 0$ for (E2), except in the trivial case $a_0 = 0$, giving no nonzero analytic solution.

Recall that a power series is a series of the form

$$a(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad \text{with } z, z_0, a_n \in \mathbb{C}.$$

The complex number z_0 (*center* of the power series) can often be assumed to be zero, since we can make the translation $z \mapsto z + z_0$.

Definition

1. A function $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, is *analytic* in $z_0 \in D$, if z_0 is an inner point of D and there exist $a_n \in \mathbb{C}$ and $\delta > 0$ such that
$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for } z \in \mathbb{C} \text{ with } |z - z_0| < \delta,$$
and *analytic* per se (or *analytic in D*) if f is analytic in every point of D.
2. A function $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, is *analytic* in $x_0 \in D$, if x_0 is an inner point of D and there exist $a_n \in \mathbb{R}$ and $\delta > 0$ such that
$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad \text{for } x \in \mathbb{R} \text{ with } x_0 - \delta < x < x_0 + \delta.$$

Properties of Analytic Functions

1. A real function f satisfying Part (2) of the definition can be extended to a complex function satisfying Part (1) with $z_0 = x_0$ and the same δ, since the radius of convergence of $\sum_{n=0}^{\infty} a_n(z - x_0)^n$ must be $\geq \delta$ and hence the power series converges for all z in the disk $\mathrm{B}_\delta(x_0) = \{z \in \mathbb{C}; |z - x_0| < \delta\}$. For this reason it is hardly necessary to discuss the case of real analytic functions separately. Just consider $\frac{1}{1-z}$, e^z, $\cos z$, $\sin z$, etc. in place of $\frac{1}{1-x}$, e^x, $\cos x$, $\sin x$, etc.

2. For every power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ there exists $0 \leq \rho \leq +\infty$ (*radius of convergence*) such that the power series converges for $|z - z_0| < \rho$ and diverges for $|z - z_0| > \rho$. The number ρ is given by

$$\rho = \frac{1}{L}, \quad \text{where} \quad L = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

If $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$ exists, it must be equal to L, giving another formula for ρ. But the latter is not directly applicable to power series with gaps such that $\sum_{k=1}^{\infty} z^{k^2}$; cf. our earlier discussion.

Properties cont'd

3. Power series $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ can be differentiated termwise within their open disc of convergence, and the power series

$$f'(z)=\sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(z-z_0)^n$$

has the same radius of convergence. Iterating, we obtain

$$f^{(k)}(z)=\sum_{n=0}^{\infty}(n+k)(n+k-1)\cdots(n+1)a_{n+k}(z-z_0)^n,$$
$$f^{(k)}(z_0)=k!\,a_k,$$

so that $a_k=f^{(k)}(z_0)/k!$ and $\quad f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n.$

In other words, a function f is analytic at z_0 if all derivatives $f^{(n)}(z_0)$ exist, so that we can form the Taylor series of f in z_0, and the Taylor series converges and represents f in some neighborhood of z_0.

Properties cont'd

4. Suppose $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ has radius of convergence $\rho>0$ and $z_1\in \mathrm{B}_\rho(z_0)$. Then $f(z)$ can be expanded into a power series with center z_1 in the disk $|z-z_1|<\rho-|z_1-z_0|$ (the disk inside $\mathrm{B}_\rho(z_0)$ that is centered at z_1 and touches the circle $|z-z_0|=\rho$). In particular f is analytic in the whole disk $\mathrm{B}_\rho(z_0)=\{z\in\mathbb{C};|z-z_0|<\rho\}$.
(If $\rho=\infty$ then the new power series has radius of convergence ∞ as well, and both series represent f everywhere in $\mathbb{C}$.

Proof.

$$\begin{aligned}
f(z)&=\sum_{n=0}^{\infty} a_n(z-z_0)^n=\sum_{n=0}^{\infty} a_n(z-z_1+z_1-z_0)^n\\
&=\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_n\binom{n}{k}(z-z_1)^k(z_1-z_0)^{n-k}\\
&=\sum_{k=0}^{\infty}\left(\sum_{n=k}^{\infty} a_n\binom{n}{k}(z_1-z_0)^{n-k}\right)(z-z_1)^k.
\end{aligned}$$

Since $\sum_{n=k}^{\infty} a_n\binom{n}{k}(z_1-z_0)^{n-k}=\frac{f^{(k)}(z_1)}{k!}$, we see from this that the new series is the Taylor series of f in z_1, which comes as no surprise.

Proof cont'd.

The reordering is valid if the double series converges absolutely. Since

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left|a_n\binom{n}{k}(z-z_0)^k(z_1-z_0)^{n-k}\right| = \sum_{n=0}^{\infty}|a_n|\left(|z-z_1|+|z_1-z_0|\right)^n,$$

this is the case provided that $|z-z_1|+|z_1-z_0|) < \rho$ (because $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely in $\mathrm{B}_\rho(z_0)$). □

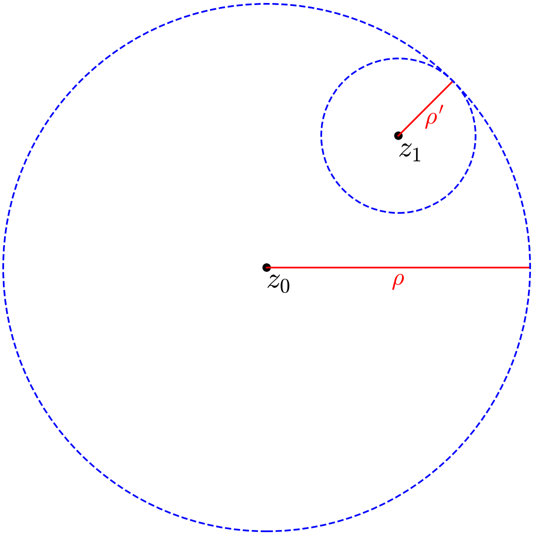

Figure: The Taylor series of f in z_1 converges at least in the open disk $|z - z_1| < \rho'$, $\rho' = \rho - |z_1 - z_0|$, and represents f in this disk.

Interlude on Double Series

Roughly speaking, infinite double series bear to doubly-infinite matrices ("doubly-infinite sequences") the same relation as infinite series do to infinite sequences.

Theorem (Fubini's Theorem for double series)

Suppose $a\colon \mathbb{N}\times\mathbb{N}\to\mathbb{C}$, $(m,n)\mapsto a(m,n)=a_{mn}$ is any function (called a doubly-infinite matrix), and there exists $B>0$ such that $\sum_{m=0}^{M}\sum_{n=0}^{N}|a_{mn}|\leq B$ for all $(M,N)\in\mathbb{N}\times\mathbb{N}$. Then

$$\sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty}a_{mn}\right)=\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}}a_{mn}=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{\infty}a_{mn}\right),$$

where it is understood that $\mathbb{N}=\{0,1,2,\dots\}$ and all series and double series involved converge in $\mathbb{C}$.

Of course the same theorem holds mutatis mutandis for functions $a\colon\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ and with $\{1,2,3,\dots\}$ in place of $\mathbb{N}$.

The assumption of the theorem implies that $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}}|a_{mn}|$ converges in $\mathbb{C}$ (resp., $\mathbb{R}$) as well and is often stated as "the double series $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}}a_{mn}$ converges absolutely".

Interlude on Double Series Cont'd

The theorem says in particular that for a doubly-infinite matrix (a_{mn}) whose partial sums over finite rectangle satisfy a uniform bound as stated we can compute the total sum of the matrix either row-wise or column-wise:

$m\backslash n$	0	1	2	$\dots$	$\sum$
0	a_{00}	a_{01}	a_{02}	$\dots$	r_0
1	a_{10}	a_{11}	a_{12}	$\dots$	r_1
2	a_{20}	a_{21}	a_{22}	$\dots$	r_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\sum$	c_0	c_1	c_2	$\dots$	s

If r_m denotes the sum of Row m and c_n the sum of Column n, we have $\sum_{m=0}^{\infty} r_m = \sum_{n=0}^{\infty} c_n$ (denoted by s in the matrix).

For ordinary matrices (i.e., with a finite number of rows and columns) this property is a rather trivial consequence of the commutative and associative laws for addition in $\mathbb{C}$, resp., $\mathbb{R}$.

Interlude on Double Series Cont'd

As a concrete example consider $a_{mn} = \frac{1}{2^m 3^n}$. Here we obtain

$m\backslash n$	0	1	2	$\dots$	$\sum$
0	1	1/3	1/9	$\dots$	3/2
1	1/2	1/6	1/18	$\dots$	3/4
2	1/4	1/12	1/36	$\dots$	3/8
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\sum$	2	2/3	2/9	$\dots$	3

The column sums arise from the geometric series evaluation $1 + 1/2 + 1/4 + \cdots = \frac{1}{1-1/2} = 2$, the row sums from $1 + 1/3 + 1/9 + \cdots = \frac{1}{1-1/3} = 3/2$, and we have indeed

$$2 + \frac{2}{3} + \frac{2}{9} + \cdots = 3 = \frac{3}{2} + \frac{3}{4} + \frac{3}{8} + \cdots .$$

In fact the identity

$$\sum_{m,n=0}^{\infty} \frac{1}{2^m 3^n} = \left(\sum_{m=0}^{\infty} \frac{1}{2^m}\right)\left(\sum_{n=0}^{\infty} \frac{1}{3^n}\right) = 2 \cdot \frac{3}{2} = 3$$

is a discrete analogue of $\int g(x)h(y)\,\mathrm{d}^2(x,y) = \left(\int g(x)\,\mathrm{d}x\right)\left(\int h(y)\,\mathrm{d}y\right)$.

Interlude on Double Series Cont'd

But what is $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn}$ in the first place?

It is possible to define "$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn} = A$" as "for every $\epsilon > 0$ there exist $M_\epsilon, N_\epsilon \in \mathbb{N}$ such that $\left|\left(\sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}\right) - A\right| < \epsilon$ for all $M > M_\epsilon$ and $N > N_\epsilon$". But this definition doesn't imply that $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn}$ is preserved under permutations of $\mathbb{N}\times\mathbb{N}$, as the notation "$\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn}$" suggests.

Modern definition: $\sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn} = A$ if for every $\epsilon > 0$ there exists a finite set $F \subset \mathbb{N}\times\mathbb{N}$ such that for every finite set E with $F \subset E \subset \mathbb{N}\times\mathbb{N}$ we have $\left|\left(\sum_{(m,n)\in E} a_{mn}\right) - A\right| < \epsilon$.

Note that finite sums $\sum_{(m,n)\in E} a_{mn}$ are well defined, since it doesn't matter in which order we add the elements a_{mn} (by the commutative and associative laws in $\mathbb{C}$).

Interlude on Double Series Cont'd

Notes

- The modern definition applies mutatis mutandis to every complex-valued (or real-valued) function and yields a definition of $\sum_{i\in I} a_i$ for any domain ("index set") I and function $a\colon I \to \mathbb{C}$, $i \mapsto a_i$.
- Only countably infinite domains I are of interest, since $\sum_{i\in I} a_i = A \in \mathbb{C}$ implies that $\{i \in I; a_i \neq 0\}$ is either finite or countably infinite.
- $\sum_{i\in I} a_i$ exists (in $\mathbb{C}$) iff $\sum_{i\in I} |a_i|$ exists (in $\mathbb{R}$). In other words, there is no difference between convergence and absolute convergence; cp. with the Lebesgue integral, of which the modern definition of infinite summation is actually a special case.
- If $\pi\colon I \to I$ is any permutation (i.e., bijection) then $\sum_{i\in I} a_i = \sum_{i\in I} a_{\pi(i)}$, a trivial consequence of the definition.
- If $\sum_{i\in I} a_i$ exists and $J \subset I$, the sum $\sum_{i\in J} a_i$ exists as well.
- If $\sum_{i\in I} a_i$ exists and $\mathcal{P}$ is a partition of I then $\sum_{i\in I} a_i = \sum_{J\in\mathcal{P}} \left(\sum_{i\in J} a_i\right)$. Fubini's Theorem for double series is a special case of this.

Interlude on Double Series Cont'd

More precisely, Fubini's Theorem for double series is the statement

$$\sum_{R\in\mathcal{R}}\left(\sum_{(m,n)\in R} a_{mn}\right) = \sum_{(m,n)\in\mathbb{N}\times\mathbb{N}} a_{mn} = \sum_{C\in\mathcal{C}}\left(\sum_{(m,n)\in C} a_{mn}\right),$$

where $\mathcal{R}$, $\mathcal{C}$ are the partitions of $\mathbb{N}\times\mathbb{N}$ into "rows" resp. "columns"; i.e., the members of $\mathcal{R}$ are $\{0\}\times\mathbb{N}$, $\{1\}\times\mathbb{N}$, $\{2\}\times\mathbb{N}$, ..., and the members of $\mathcal{C}$ are $\mathbb{N}\times\{0\}$, $\mathbb{N}\times\{1\}$, $\mathbb{N}\times\{2\}$, ...

The last property is also meaningful for ordinary series. For example, it tells us that for an absolutely convergent series we have

$$a_1 + a_2 + a_3 + \cdots = (a_1 + a_3 + a_5 + \dots) + (a_2 + a_4 + a_6 + \cdots),$$

and also the rather fancy

$$\sum_{n=1}^{\infty} a_n = a_1 + (a_2 + a_3 + a_5 + a_7 + a_{11} + a_{13} + \cdots) + (a_4 + a_6 + a_9 + a_{10} + a_{14} + \cdots) + (a_8 + a_{12} + \cdots) + \cdots.$$

Inner sums are taken over all n with a fixed number of prime factors.

Interlude on Double Series Cont'd

Proofs of these properties can be found in some texts on Real Analysis. Walter Rudin's Principles of Mathematical Analysis that I have recommended as background reference doesn't include it, but Terence Tao's Analysis I (3rd edition, Springer 2015) has it in Ch. 8.2, for example. The notation there is slightly different from our's, and yet different from the one in my source (a not so well-known German textbook).

Properties cont'd

5. *Equating coefficients*
   Suppose f and g are analytic in some common connected domain D (i.e., analytic at every point of D) and that $E = \{z \in D; f(z) = g(z)\}$ has an accumulation point in D. Then $f(z) = g(z)$ for all $z \in D$, and consequently the power series expansions of f and g at any point $z_0 \in D$ must be the same.

Sketch of proof.

Call the accumulation point z_0 and suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ are represented by different power series at z_0, i.e., $a_n \neq b_n$ for some n. If N is the least such n, we have

$$
\begin{aligned}
f(z) - g(z) &= (a_N - b_N)(z - z_0)^N + (a_{N+1} - b_{N+1})(z - z_0)^{N+1} + \cdots \\
&= (z - z_0)^N \bigl(a_N - b_N + (a_{N+1} - b_{N+1})(z - z_0) + \cdots\bigr) \\
&= (z - z_0)^N h(z),
\end{aligned}
$$

where h is analytic at z_0 and $h(z_0) = a_N - b_N \neq 0$.
$\Longrightarrow h(z) \neq 0$ in some disk $|z - z_0| < \delta$ (since h is continuous)
$\Longrightarrow f(z) \neq g(z)$ in the punctured disk $0 < |z - z_0| < \delta$.
This contradicts the assumption that z_0 is an accumulation point of E.

Proof cont'd.

We have thus proved that the set $E_1 \subseteq E$ consisting of all points $z_0 \in D$ where f and g are represented by the same power series is non-empty.

E_1 is closed in D, since to any limit point z_0 of E_1 in D we can apply the preceding argument to show that $z_0 \in E_1$.

But E_1 is also open, since for $z_0 \in E_1$ the functions f and g are represented by the same power series in some disk $|z - z_0| < \delta$, which must be contained in E_1 by Property 4. (For this note that both f and g are represented by a power series $\sum_{k=0}^{\infty} c_k (z-a)^k$ at any point $a \in \mathrm{B}_\delta(z_0)$; the coefficients c_k can be computed from the power series representation at z_0, viz. $c_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} (a - z_0)^{n-k}$, and hence must be the same for f and g.)

Since D is connected, this implies $D = E_1$ and in particular that $f(z) = g(z)$ for all $z \in D$. $\square$

Remark

Property 5 holds a forteriori for real analytic functions defined on an open interval $D \subseteq \mathbb{R}$. For C^∞-functions on $\mathbb{R}$ it grossly fails: There exists, e.g., a C^∞-function $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f(x) = 0$ for $x \leq 0$ and $f(x) > 0$ for $x > 0$; cf. also the subsequent example of a "bell-shaped" function.

Properties cont'd

6. *Algebraic Operations on Power Series*

We assume for the following w.l.o.g. that the centers of the power series involved are equal to 0.

Power series functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ can be added/subtracted/multiplied by scalars termwise,

$$f(z) \pm g(z) = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n,$$

$$c\, f(z) = \sum_{n=0}^{\infty} (c a_n) z^n,$$

and multiplied according to Cauchy's multiplication formula

$$f(z) g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

The radius of convergence of the resulting power series is at least the minimum of the radii of convergence of f and g. In particular, sums and products of analytic functions are again analytic.

Properties cont'd

6. *Algebraic Operations on Power Series cont'd*
   Moreover, if $b_0 = g(0) \neq 0$ then the quotient $h(z) = f(z)/g(z)$ is represented in some neighborhood of 0 by a power series $\sum_{n=0}^{\infty} c_n z^n$ as well, which can be obtained by solving

$$\sum_{n=0}^{\infty} a_n z^n = f(z) = g(z)h(z) = \left(\sum_{n=0}^{\infty} b_n z^n\right)\left(\sum_{n=0}^{\infty} c_n z^n\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k}c_k\right) z^n,$$

   i.e. $c_0 = a_0/b_0$, $c_1 = (a_1 - b_1c_0)/b_0$,
   $c_2 = (a_2 - b_1c_1 - b_2c_0)/b_0$, etc.
   Thus quotients of per se analytic functions are analytic wherever they are defined.

   Finally, there is a "chain rule" for analytic functions: If f is analytic at z_0 and g is analytic at $w_0 = f(z_0)$ then the composition $g \circ f\colon z \mapsto g\bigl(f(z)\bigr)$ is analytic at z_0. Thus compositions of per se analytic functions are analytic as well.

Properties cont'd

6. *Algebraic Operations on Power Series cont'd*
The power series representation of $g \circ f$ at z_0 can be computed from those of f and g, $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $g(w) = \sum_{n=0}^{\infty} b_n(w-w_0)^n$, as follows: In

$$g\bigl(f(z)\bigr) = \sum_{n=0}^{\infty} b_n\bigl(f(z)-w_0\bigr)^n = \sum_{n=0}^{\infty} b_n\left(\sum_{k=1}^{\infty} a_k(z-z_0)^k\right)^n$$

expand for each n the power $\bigl(\sum_{k=1}^{\infty} a_k(z-z_0)^k\bigr)^n$ into a power series $\sum_{l=n}^{\infty} A_{nl}(z-z_0)^l$, and rearrange the resulting double series $\sum_{n,l=0}^{\infty} b_n A_{nl}(z-z_0)^l$ into a power series $\sum_{n=0}^{\infty} c_l(z-z_0)^l$, i.e., $c_l = \sum_{n=0}^{l} b_n A_{nl}$. The required absolute convergence of the double series can be shown to hold in a neighboorhood of z_0.
It should be noted that the computation of the coefficients c_l in $g\bigl(f(z)\bigr) = \sum_{n=0}^{\infty} c_l(z-z_0)^l$ doesn't require taking any limits but uses only the arithmetic of the base field (which is $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ in our case).

Example (The Fibonacci generating function)

Consider the rational function

$$f(z)=\frac{1}{1-z-z^2},\quad z\in\mathbb{C}\setminus\left\{\tfrac{-1\pm\sqrt{5}}{2}\right\}.$$

f is analytic at $z_0=0$ and hence has a power series expansion $f(z)=\sum_{n=0}^{\infty}f_nz^n$ for small z. Equating coefficents in

$$\begin{aligned}1&=\left(f_0+f_1z+f_2z^2+f_3z^3+\cdots\right)(1-z-z^2)\\&=f_0+(f_1-f_0)z+(f_2-f_1-f_0)z^2+(f_3-f_2-f_1)z^3+\cdots,\end{aligned}$$

we see that $f_0=f_1=1$, $f_n=f_{n-1}+f_{n-2}$ for $n\geq 2$, i.e., f_n is the n-th Fibonacci number (with the convention that $f_0=f_1=1$).

The closed form of f_n can be obtained by $\frac{1}{1-z-z^2}$ into a power series:

$$\begin{aligned}\sum_{n=0}^{\infty}f_nz^n&=\frac{1}{(1-\alpha z)(1-\beta z)}=\frac{1}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha z}-\frac{\beta}{1-\beta z}\right)\\&=\sum_{n=0}^{\infty}\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}z^n\quad\text{with}\quad\alpha=\frac{1+\sqrt{5}}{2},\ \beta=\frac{1-\sqrt{5}}{2}.\end{aligned}$$

Example (cont'd)

As a simple example for the composition of power series we consider the series expansion

$$
\begin{aligned}
f(z) &= \frac{1}{1-z-z^2} = \sum_{n=0}^{\infty}(z+z^2)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}z^{n+k} \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n}{k}z^{n+k} = \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n-k}{k}z^{n} \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n-k}{k}\right)z^n,
\end{aligned}
$$

which is valid for $|z|+|z|^2<1$, i.e, $|z|<\left(\sqrt{5}-1\right)/2$.
Equating coefficients of z^n shows

$$
f_n = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k} = \binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots,
$$

evaluating the SW-NO diagonal sums of Pascal's Triangle $\left(\binom{n}{k}\right)_{n,k=0}^{\infty}$.

Example (cont'd)

But the example shows more: Since

$$(z+z^2)^n = \sum_{(i_1,i_2,\dots,i_n)\in\{1,2\}^n} z^{i_1+i_2+\cdots+i_n},$$

the coefficient of z^n in $\sum_{n=0}^{\infty}(z+z^2)^n$ is equal to the number of ordered partitions of n into one's and two's. The power series identity $\sum_{n=0}^{\infty}(z+z^2)^n = \frac{1}{1-z-z^2}$ shows that these numbers are just the Fibonacci numbers. For example, we have

$$\begin{aligned}
1 &= 1,\\
2 &= 2 = 1+1,\\
3 &= 2+1 = 1+2 = 1+1+1,\\
4 &= 2+2 = 2+1+1 = 1+2+1 = 1+1+2 = 1+1+1+1,\\
5 &= 2+2+1 = 2+1+2 = 1+2+2\\
&= 2+1+1+1 = 1+2+1+1 = 1+1+2+1 = 1+1+1+2\\
&= 1+1+1+1+1.
\end{aligned}$$

Compare this with

n	0	1	2	3	4	5
f_n	1	1	2	3	5	8

Example (EULER Numbers)

The *Euler numbers* (or *secant numbers*) $\mathrm{E}_0, \mathrm{E}_1, \mathrm{E}_2, \dots$ are defined in the such a way that the corresponding exponential generating function is $1/\cos x = \sec x$:

$$\frac{1}{\cos x} = \sum_{n=0}^{\infty} \frac{\mathrm{E}_n}{n!} x^n.$$

Since $\cos x = \cos(-x)$, we must have $\mathrm{E}_1 = \mathrm{E}_3 = \mathrm{E}_5 = \cdots = 0$ (equate coefficients!), so that we can write the defining equation as

$$\left(\sum_{n=0}^{\infty} \frac{\mathrm{E}_{2n}}{(2n)!} x^{2n}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}\right) = 1$$

Expanding the product and equating coefficients gives the recurrence relation

$$\mathrm{E}_0 = 1, \quad \sum_{k=0}^{n} \mathrm{E}_{2k} \frac{(-1)^{n-k}}{(2k)!(2n-2k)!} = 0 \quad \text{for } n \geq 1.$$

Multiplying by $(2n)!$ puts it into the more convenient integral form

Example (cont'd)

$\sum_{k=0}^{n}(-1)^{n-k}\binom{2n}{2k}\mathrm{E}_{2k}=0$, or

$$\mathrm{E}_{2n}=\binom{2n}{2}\mathrm{E}_{2n-2}-\binom{2n}{4}\mathrm{E}_{2n-4}+\binom{2n}{6}\mathrm{E}_{2n-6}\mp\cdots$$

The first few even Euler numbers are $\mathrm{E}_0=1$,

$$\begin{aligned}
\mathrm{E}_2&=\binom{2}{2}\mathrm{E}_0=1,\\
\mathrm{E}_4&=\binom{4}{2}\mathrm{E}_2-\binom{4}{4}\mathrm{E}_0=5,\\
\mathrm{E}_6&=\binom{6}{2}\mathrm{E}_4-\binom{6}{4}\mathrm{E}_2+\binom{6}{6}\mathrm{E}_0=15\cdot 5-15\cdot 1+1=61,\\
\mathrm{E}_8&=\binom{8}{2}\mathrm{E}_6-\binom{8}{4}\mathrm{E}_4+\binom{8}{6}\mathrm{E}_2-\binom{8}{8}\mathrm{E}_0\\
&=28\cdot 61-70\cdot 5+28\cdot 1-1=1385,\\
\mathrm{E}_{10}&=50\,521,\\
\mathrm{E}_{12}&=2\,702\,765,\\
\mathrm{E}_{14}&=199\,360\,981.
\end{aligned}$$

Example (BERNOULLI Numbers)

The *Bernoulli numbers* $\mathrm{B}_0, \mathrm{B}_1, \mathrm{B}_2, \dots$ are defined in the such a way that the corresponding exponential generating function is $x/(\mathrm{e}^x - 1)$:

$$\frac{x}{\mathrm{e}^x - 1} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots} = \sum_{n=0}^{\infty} \frac{\mathrm{B}_n}{n!} x^n.$$

This gives the recurrence relation

$$\mathrm{B}_0 = 1, \quad \sum_{k=0}^{n} \frac{\mathrm{B}_k}{k!(n-k+1)!} = 0 \quad \text{for } n \geq 1,$$

which can also be written as

$$\mathrm{B}_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} \mathrm{B}_k = 0 \quad \text{for } n \geq 1.$$

The first few Bernoulli numbers are:

n	0	1	2	3	4	5	6	7	8	9	10
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

Example (cont'd)

It is no coincidence that the odd Bernoulli numbers $\mathrm{B}_3, \mathrm{B}_5, \mathrm{B}_7, \dots$ are zero:

$$\begin{aligned}\sum_{n\neq 1} \frac{\mathrm{B}_n}{n!}\,x^n &= \frac{x}{\mathrm{e}^x-1} + \frac{x}{2} = \frac{2x + x(\mathrm{e}^x-1)}{2(\mathrm{e}^x-1)} = \frac{x(\mathrm{e}^x+1)}{2(\mathrm{e}^x-1)}\\ &= \frac{x}{2}\,\frac{\mathrm{e}^{x/2}+\mathrm{e}^{-x/2}}{\mathrm{e}^{x/2}-\mathrm{e}^{-x/2}} = \frac{x}{2}\,\frac{\cosh(x/2)}{\sinh(x/2)} = \frac{x}{2}\coth\frac{x}{2}\end{aligned}$$

is an even function of x, and hence has all odd coefficients equal to zero.

As a by-product, we obtain the power series expansions at $x_0 = 0$ of $x\coth x$, $x\cot x = \mathrm{i}x\coth(\mathrm{i}x)$, $\tan x = \cot x - 2\cot(2x)$, viz.,

$$x\coth x = \sum_{n=0}^{\infty} \frac{\mathrm{B}_{2n}\,2^{2n}}{(2n)!}\,x^{2n}, \qquad x\cot x = \sum_{n=0}^{\infty} \frac{(-1)^n\,\mathrm{B}_{2n}\,2^{2n}}{(2n)!}\,x^{2n},$$

$$\tan x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,\mathrm{B}_{2n}\,2^{2n}(2^{2n}-1)}{(2n)!}\,x^{2n-1} = x + \tfrac{1}{3}\,x^3 + \tfrac{2}{15}\,x^5 + \tfrac{17}{315}\,x^7 + \cdots.$$

Properties cont'd

7. *Zeros and Poles*
In general a quotient $h = f/g$ of nonzero functions f, g that are analytic at z_0 is only defined in a punctured disk $0 < |z - z_0| < \delta$. We can write

$$f(z) = (z - z_0)^{m_1} f_1(z), \quad g(z) = (z - z_0)^{m_2} g_1(z)$$

with $m_1, m_2 \in \mathbb{N}$, f_1, g_1 analytic at z_0 and $f_1(z_0) \neq 0$, $g_1(z_0) \neq 0$. (The exponents m_1, m_2 are those of the smallest powers $(z - z_0)^n$ appearing with a nonzero coefficient in the power series representation of f and g at z_0; cf. Property 5).
$\Longrightarrow$ h has the representation

$$h(z) = (z - z_0)^m h_1(z)$$

with $m = m_1 - m_2 \in \mathbb{Z}$, $h_1 = f_1/g_1$ analytic at z_0, and $h_1(z_0) = f_1(z_0)/g_1(z_0) \neq 0$.
In this case we say that h has *order* m at z_0. If $m > 0$, we call z_0 a *zero of h of order m*; if $m < 0$, we call z_0 a *pole of h of order $-m$* (note that $-m > 0$ in this case).

Properties cont'd

7. *Zeros and Poles cont'd*

   Thus h has a pole of order m at z_0 iff $h_1(z) = (z - z_0)^m h(z)$ is analytic at z_0 and $h_1(z_0) \neq 0$; equivalently, the "power series expansion" of h at z_0 (valid for $0 < |z - z_0| < \delta$) starts with the negative power $(z - z_0)^{-m}$:

$$h(z) = \sum_{n=-m}^{\infty} c_n (z - z_0)^n, \quad c_{-m} \neq 0.$$

   The concept of "pole" applies to any analytic function defined on a punctured disk, e.g., $z \mapsto 1/(\mathrm{e}^z - 1)$ has a pole of order 1 in $z_0 = 0$.

   In Complex Anlaysis it is shown that a bounded analytic function h on a punctured disk $0 < |z - z_0| < \delta$ can be extended to an analytic function on the whole disk (in particular $\lim_{z \to z_0} h(z)$ exists in this case). This gives a characterization of poles of h in terms of boundedness of $z \mapsto (z - z_0)^m h(z)$ and implies that other types of isolated singularities (called *essential* singularities) must be tied to high local fluctuations of the values of the function.

Properties cont'd

We close with two properties whose proofs require the more advanced machinery of Complex Analysis.

8. We have seen that per se analytic functions $f\colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$, are differentiable, i.e., $\lim_{h\to 0,\, h\in\mathbb{C}} \frac{1}{h}\big(f(z+h)-f(z)\big)$ exists for all $z \in D$, and that the derivative $f'(z)$ is again analytic. Conversely, it can be shown that differentiable functions are analytic (and thus have derivatives of all orders). This is in sharp contrast with the real case: A differentiable function
   $f\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, can have a derivative which is not differentiable, and f can be C^∞ without being analytic; cf. examples.

   Differentiable per se (in the above sense) complex functions are also called *holomorphic*.

Properties cont'd

9. It can be shown that the power series expansion $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ of a holomorphic function is valid in the largest open circle around z_0 on which f is defined. In other words, if f is defined on the whole complex plane (a so-called *entire* function) then every power series representing f has radius of convergence $\rho = \infty$, and if $\rho < \infty$ then the circle $|z - z_0| = \rho$ must contain a singularity of f (i.e., a point where f is not defined).

   The proof in the general case requires Cauchy's Integral Formula from Complex Analysis, but in the special case of a rational function $f = P/Q$ (P, Q polynomials) we can see it rather quickly using the partial fractions decomposition.

   Suppose $Q(X) = \prod_{i=1}^{r}(X - \lambda_i)^{m_i}$ is the prime factorization of $Q(X)$ in $\mathbb{C}[X]$ and $0 < |\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_r|$. Then λ_1 is a singularity of f closest to the origin.
   The partial fractions decomposition of f has the form

$$f(z) = R(z) + \sum_{i=1}^{r} \sum_{s=1}^{m_i} \frac{c_{is}}{(z - \lambda_i)^s} \text{ with } R \in \mathbb{C}[X],\ c_{is} \in \mathbb{C},\ c_{i,m_i} \neq 0.$$

Properties cont'd

9. (continued)
The non-polynomial summands can be expanded into a power series using the generalized binomial theorem:

$$\frac{1}{(z-\lambda_i)^s} = \frac{(-1)^s\lambda_i^{-s}}{(1-\lambda_i^{-1}z)^s} = (-1)^s\lambda_i^{-s}\sum_{n=0}^{\infty}\binom{n+s-1}{s-1}\lambda_i^{-n}z^n.$$

The series converges precisely for $\left|\lambda_i^{-1}z\right| < 1$, i.e., for $|z| < |\lambda_i|$.
$\Longrightarrow$ For $|z| < |\lambda_1|$ all such expansions converge, showing that f has a power series expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in the circle $|z| < |\lambda_1|$.
The radius of convergence ρ of $\sum_{n=0}^{\infty} a_n z^n$ cannot be larger than $|\lambda_1|$, because $|z| < \rho$ cannot include a singularity of f.
$\Longrightarrow \rho = |\lambda_1|$
Finally, the change of variables $w = z - z_0$, which transforms $f(z)$ into another rational function, shows that the preceding statement holds for the power series expansion of f at an arbitrary point $z_0 \notin \{\lambda_1, \dots, \lambda_r\}$.

Example

Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \mathrm{e}^{-\frac{1}{1-x^2}} & \text{for } |x| < 1, \\ 0 & \text{for } |x| \geq 1. \end{cases}$$

Then f satisfies $f^{(n)}(\pm 1) = 0$ for all $n \geq 0$, since on $(-1, 1)$ all derivatives $f^{(n)}(x)$ exist and have the form $f^{(n)}(x) = R_n(x)\mathrm{e}^{-\frac{1}{1-x^2}}$ for some rational function $R_n(x)$. It follows that $\lim_{x \to \pm 1} f^{(n)}(x) = 0$, and this is enough to prove by induction that $f^{(n)}(\pm 1)$ exists and equals 0. Thus f is a C^∞-function on $\mathbb{R}$.

But f is not analytic at $x_0 = \pm 1$, since the Taylor series at ± 1 vanishes but f does not vanish in any neighborhood of ± 1.

Moreover, f vanishes on a large subset of $\mathbb{R}$ but not entirely. This cannot happen for (per se) analytic functions (cf. Property 5): If a real analytic function g vanishes on an interval of positive length, it must vanish entirely. Similarly, if g is analytic, defined at zero, and $g(1/n) = 0$ for all sufficiently large integers n then g must vanish entirely.

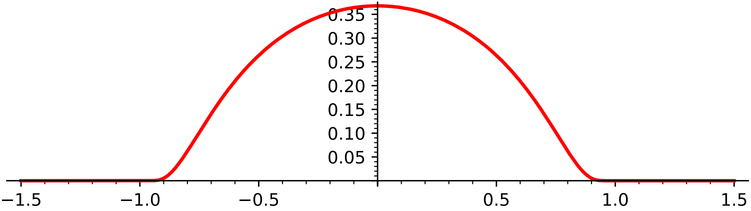

Figure: Graph of $f(x) = \mathrm{e}^{-\frac{1}{1-x^2}}$ for $|x| < 1$, $f(x) = 0$ for $|x| \geq 1$

Example (Geometric series)

The function $f\colon \mathbb{C}\setminus\{1\} \to \mathbb{C}$, $z \mapsto \frac{1}{1-z}$ is holomorphic with $f'(z) = \frac{1}{(1-z)^2}$. At $z_0 = 0$ it has the well-known series representation

$$1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

The radius of convergence of the geometric series is $\rho = 1$. (It cannot be larger since on the circle $|z| = 1$ there is a singularity of f.

We can expand f into a power series at any point $a \in \mathbb{C}\setminus\{1\}$ by the following computational trick. (There is no need to compute the derivatives $f^{(n)}(a)$.)

$$\begin{aligned} f(z) &= \frac{1}{1-z} = \frac{1}{1-a-(z-a)} = \frac{\frac{1}{1-a}}{1-\frac{z-a}{1-a}} \\ &= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(1-a)^{n+1}}. \end{aligned}$$

Example (Geometric series cont'd)

The radius of convergence of the new power series, which except for the factor $(1-a)^{-1}$ is a geometric series as well and converges for $|z-a|<|1-a|$, is $\rho'=|1-a|$ (the distance from a to the singularity of f, as predicted by Property 9).

The derivatives of f can now be read off from the power series expansion:

$$f^{(n)}(a)=\frac{n!}{(1-a)^{n+1}},\quad a\in\mathbb{C}\setminus\{1\}.$$

Of course you can also prove by induction that $f^{(n)}(z)=\frac{n!}{(1-z)^{n+1}}$.

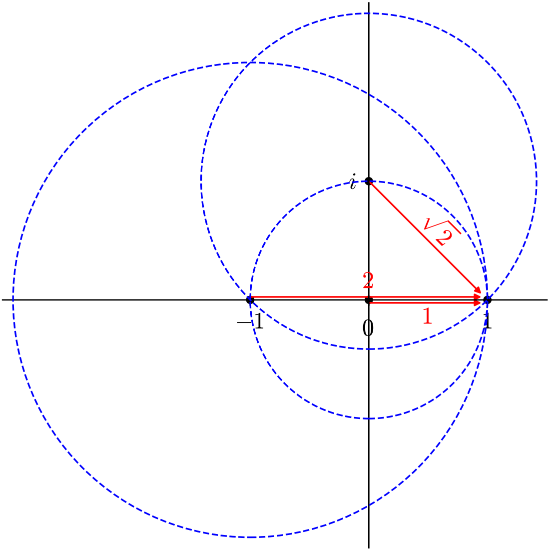

Figure: Disks of convergence of the Taylor series of $z \mapsto \frac{1}{1-z}$ at $a \in \{0, -1, \mathrm{i}\}$

Example

The function $g\colon \mathbb{C}\setminus\{\pm\mathrm{i}\}\to\mathbb{C}$, $z\mapsto\frac{1}{z^2+1}$ is holomorphic with $g'(z)=-\frac{2z}{(z^2+1)^2}$. At $z_0=0$ it has the series representation

$$1-z^2+z^4-z^6\pm\cdots=\frac{1}{1+z^2},\quad |z|<1,$$

which is also an instance of the geometric series.

Restricting g to $\mathbb{R}$ gives the function $\mathbb{R}\to\mathbb{R}$, $x\mapsto\frac{1}{x^2+1}$, which is real analytic everywhere. But unlike the exponential series, its Taylor series $1-x^2+x^4-x^6\pm\cdots$ at $x_0=0$ doesn't converge on all of $\mathbb{R}$, but only for $|x|<1$.

The same is true for $\mathbb{R}\to\mathbb{R}$, $x\mapsto\arctan x$, which has Taylor series $x-x^3/3+x^5/5-x^7/7\pm\cdots$ at $x_0=0$ and represents an antiderivative of $x\mapsto\frac{1}{x^2+1}$.

This example vividly explains why we need to look at the complex extensions of real analytic functions to determine their more subtle properties.

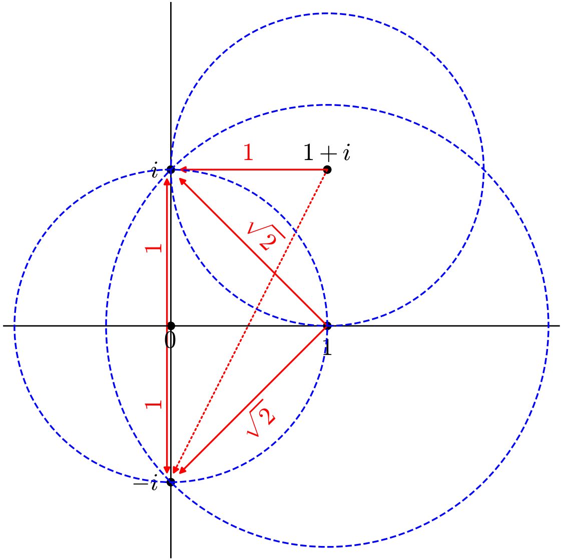

Figure: Disks of convergence of the Taylor series of $z \mapsto \frac{1}{z^2+1}$ at $a \in \{0, 1, 1 + \mathrm{i}\}$. For $a = 1 + \mathrm{i}$ the nearest singularity is i, and hence $\rho = 1$.

Exercise

Compute the Taylor series of $z \mapsto \frac{1}{z^2+1}$ at $a = 1$ and $a = 1 + \mathrm{i}$.

Hint: Proceed as for $z \mapsto \frac{1}{1-z}$ and then use partial fractions.

Regular and Singular Points

We consider implicit 2nd-order homogeneous linear time-dependent ODE's with analytic coefficients, i.e.,

$$P(x)y'' + Q(x)y' + R(x)y = 0 \qquad (1)$$

with real analytic functions $P \neq 0$, Q, and R defined on some common interval I.

At points $x_0 \in I$ with $P(x_0) \neq 0$ we can put (1) into the explicit form

$$y'' + p(x)y' + q(x)y = 0, \quad p(x) = \frac{Q(x)}{P(x)},\ q(x) = \frac{R(x)}{P(x)}, \qquad (2)$$

and we know from the discussion of quotients of analytic functions (see Property 6) that p and q are analytic at x_0.

If $P(x_0) = 0$, let $P(x) = P_1(x)(x - x_0)^m$ with P_1 analytic at x_0 and $P_1(x_0) \neq 0$. (The integer $m \geq 1$ is the multiplicity of x_0 as a zero of P or, equivalently, the smallest index of a nonzero coefficient in the power series expansion of P at x_0; it exists in view of $P \neq 0$.) We can assume that one of $Q(x_0)$, $R(x_0)$ is $\neq 0$, since otherwise we can divide (1) by $x - x_0$, which doesn't change solutions (why?) and reduces the multiplicity of x_0 as a zero of P by one.

Then (1) becomes

$$(x-x_0)^m y''+p_1(x)y'+q_1(x)y=0, \quad p_1(x)=\frac{Q(x)}{P_1(x)},\ q_1(x)=\frac{R(x)}{P_1(x)}, \tag{3}$$

and again p_1, q_1 are analytic at x_0.

Finally we can put (3) formally into an "explicit form", which is not defined for $x=x_0$, if we admit the coefficients of y' and y to have poles at x_0:

$$y''+\underbrace{\frac{p_1(x)}{(x-x_0)^m}}_{p(x)}\,y'+\underbrace{\frac{q_1(x)}{(x-x_0)^m}}_{q(x)}\,y=0. \tag{4}$$

Since one of $p_1(x_0)$, $q_1(x_0)$ is nonzero, either $p(x)$ or $q(x)$ (or both) have a pole of exact order m at x_0.

Definition

1. $x_0\in I$ is called a *singular point* of (1) if $P(x_0)=0$, and an *ordinary point* otherwise.
2. A singular point $x_0\in I$ of (1) is called a *regular singular point* if $\lim_{x\to x_0}(x-x_0)p(x)$ and $\lim_{x\to x_0}(x-x_0)^2q(x)$ exist in $\mathbb{R}$; equivalently, $m\in\{1,2\}$ in (4) and $p_1(x_0)=0$ if $m=2$.

Notes on the definition

1. x_0 is a singular point iff at least one of $p(x)$, $q(x)$ has a pole at x_0.
2. A singular point x_0 is a regular singular point iff the order of the pole(s) in Note 1 is ≤ 1 for $p(x)$ and ≤ 2 for $q(x)$.
3. The condition for a regular singular point may also be rephrased as: $f(x) := (x - x_0)p(x)$ and $g(x) := (x - x_0)^2 q(x)$ can be made analytic at x_0 by setting $f(x_0) = \lim_{x \to x_0}(x - x_0)p(x)$, $g(x_0) = \lim_{x \to x_0}(x - x_0)^2 q(x)$. (For an ordinary point x_0 the functions f, g are trivially analytic at x_0.)

From now on we will assume w.l.o.g. that $x_0 = 0$.

If $x_0 = 0$ is an ordinary point or a regular singular point of the ODE (1) then (4) can be rewritten as

$$y'' + \underbrace{\left(\frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + \cdots\right)}_{p(x)} y' + \underbrace{\left(\frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + q_4 x^2 + \cdots\right)}_{q(x)} y = 0.$$

The case of an ordinary point corresponds to $p_0 = q_0 = q_1 = 0$.

After multiplication by x^2 this takes the more convenient "analytic" form

$$x^2y''+x(p_0+p_1x+p_2x^2+\dots)y'+(q_0+q_1x+q_2x^2+\dots)y=0, \quad (5)$$

which is of course still equivalent to (1).
Here $f(x)=p_0+p_1x+p_2x^2+\cdots$, $g(x)=q_0+q_1x+q_2x^2+\cdots$ are the analytic functions defined in the previous Note 3.

Caution: Don't confuse $f(x)$, $g(x)$ with $p(x)$, $q(x)$, which may contain negative powers of x and whose cofficients are indexed in a non-standard way (cf. previous slide). Also don't confuse the real numbers p_1, q_1 in (5) with the analytic functions $p_1(x)$, $q_1(x)$ appearing in (3).

Example

The Euler equation $x^2y''+\alpha xy'+\beta y=0$, $(\alpha,\beta)\neq(0,0)$, has a regular singular point at $x=0$. The corresponding analytic functions are the constant functions $f(x)=\alpha$, $g(x)=\beta$.

We also see that truncating the coefficient functions of y' and y in the general form (5) after the first term yields an Euler equation, viz. $x^2y''+p_0xy'+q_0y=0$. This Euler equation will play an important role when solving (5) in the case of a regular singular point. But first we consider the case of an ordinary point.

The Analytic Case

Theorem

Suppose that x_0 is an ordinary point of

$$P(x)y'' + Q(x)y' + R(x)y = 0, \tag{1}$$

and that the power series representing $p(x) = Q(x)/P(x)$, $q(x) = R(x)/P(x)$ at x_0 converge for $|x - x_0| < \rho$. Then for any pair $(a_0, a_1) \in \mathbb{R}^2$ (or $\mathbb{C}^2$) there exists an analytic solution of (1) of the form $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ for $|x - x_0| < \rho$.

Notes

- Since $y(x_0) = a_0$ and $y'(x_0) = a_1$, this says in particular that every IVP associated with (1) locally at x_0 has an analytic solution.
- The best choice of ρ in the theorem is the minimum of the radii of convergence of the power series representing $p(x)$ and $q(x)$ at x_0, and for this choice the theorem says that the radius of convergence of $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $\geq \rho$.

Proof of the theorem.

We assume w.l.o.g. $x_0 = 0$ and use the equivalent form (5) of (1) with $p_0 = q_0 = q_1 = 0$ for the proof. Plugging

$$x^2 y'' = x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) a_n x^n,$$

$$xy' = x \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^n$$

into (5) gives

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \left(\sum_{n=1}^{\infty} n a_n x^n \right) \left(\sum_{n=1}^{\infty} p_n x^n \right) + \\ + \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=2}^{\infty} q_n x^n \right) = 0,$$

or, equivalently,

$$n(n-1) a_n + \sum_{k=1}^{n-1} k a_k p_{n-k} + \sum_{k=0}^{n-2} a_k q_{n-k} = 0 \quad \text{for } n = 2, 3, 4, \dots$$

Proof cont'd.

It is clear that this recurrence relation for the sequence $(a_0, a_1, a_2, \dots)$ has a unique solution for any given a_0, a_1.

It remains to show that the so-defined power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < \rho$. For this we use the following

Lemma

The radius of convergence of any power series $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ *is given by*

$$R := \sup\bigl\{r \in \mathbb{R};\ \textit{the sequence } (|b_n|\, r^n) \textit{ is bounded}\bigr\}.$$

Proof of the lemma.

We show that $\sum_{n=0}^{\infty} b_n(z - z_0)^n$ converges for $|z - z_0| < R$ and diverges for $|z - z_0| > R$. (Notably this also proves the existence of the radius of convergence.) W.l.o.g. we assume $z_0 = 0$.

$|z| < R$: There exists $r > |z|$ such that $(|b_n|\, r^n)$ is bounded, say $|b_n|\, r^n \leq M$ for all n. $\Longrightarrow |b_n z^n| = |b_n|\, r^n (|z|/r)^n \leq M q^n$ with $q := |z|/r < 1$. Since $\sum M q^n$ converges, we can apply the comparison test and conclude that $\sum b_n z^n$ converges.

$|z| > R$: If $\sum b_n z^n$ converges then $b_n z^n \to 0$ and $|b_n z^n| = |b_n|\, |z|^n$ is bounded. This contradicts the definition of R. □

Remark

The sequence $(|b_n|\,R^n)$ can be bounded or unbounded. For example, $\sum_{n=1}^{\infty} nz^n$ has $R = 1$ and $|b_n|\,R^n = n \to \infty$, whereas $\sum_{n=1}^{\infty}(1/n)z^n$ has $R = 1$ and $|b_n|\,R^n = 1/n \to 0$.

Proof of the theorem cont'd.

Let $r < \rho$ be given. Then, by the lemma, the sequences $(|p_n|\,r^n)$ and $(|q_n|\,r^n)$ are bounded, say by M.

Using the recurrence relation for a_n, we now try to bound $(|a_n|\,r^n)$.

Guided by one of our introductory examples, we find

$$\begin{aligned} n(n-1)\,|a_n|\,r^n &= \left|\sum_{k=1}^{n-1} k(a_k r^k)(p_{n-k} r^{n-k}) + \sum_{k=0}^{n-2}(a_k r^k)(q_{n-k} r^{n-k})\right| \\ &\leq M\left(\sum_{k=1}^{n-1} k\,|a_k|\,r^k + \sum_{k=0}^{n-2}|a_k|\,r^k\right) \\ \implies |a_n|\,r^n &\leq \sum_{k=0}^{n-1}\frac{M(k+1)}{n(n-1)}\,|a_k|\,r^k \quad (n \geq 2) \end{aligned}$$

This is a recursive bound for the sequence $(|a_n|\,r^n)$, which unfortunately is not as simple to handle as the former one.

Proof of the theorem cont'd.

Now we proceed as follows: We define an auxiliary sequence (u_n) by $u_0 = |a_0|$, $u_1 = |a_1|\, r$, and

$$u_n = \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)}\, u_k \quad \text{for } n \geq 2.$$

One can show by induction that

$$|a_n|\, r^n \leq u_n \quad \text{for all } n \quad \text{and} \quad \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1;$$

cf. exercises. It follows that for any positive $r_1 < r$ we have

$$|a_n|\, r_1^n \leq u_n (r_1/r)^n \quad \text{with } q := r_1/r < 1,$$

and that the series $\sum_{n=0}^{\infty} u_n q^n$ converges (because $\sum_{n=0}^{\infty} u_n x^n$ has radius of convergence 1).

$\Longrightarrow$ We can apply the comparison test to conclude that $\sum_{n=0}^{\infty} |a_n|\, r_1^n$ converges as well (or that $(|a_n|\, r_1^n)$ is bounded, whatever you prefer!).

Finally, since r and r_1 were chosen arbitrarily subject only to $r_1 < r < \rho$, it is clear that $\sum_{n=0}^{\infty} |a_n|\, r_1^n$ converges for all $r_1 < \rho$, and hence $\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < \rho$. $\square$

Remark

If y_1, y_2 are solutions of (1) satisfying the particular initial conditions $y_1(x_0) = 1$, $y_1'(x_0) = 0$ and $y_2(x_0) = 0$, $y_2'(x_0) = 1$ then the general solution of (1) is $y(x) = a_0 y_1(x) + a_1 y_2(x)$. In particular, y_1, y_2 form a fundamental system of solutions of (1).

Exercise

Suppose (α_n) and (u_n) are sequences of nonnegative real numbers satisfying

$$\begin{aligned} \alpha_n &\leq \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} \alpha_k \quad (n \geq 2), \\ u_n &= \sum_{k=0}^{n-1} \frac{M(k+1)}{n(n-1)} u_k \quad (n \geq 2), \\ u_0 &= \alpha_0, \; u_1 = \alpha_1 \end{aligned}$$

for some constant $M > 0$.

a) Show $\alpha_n \leq u_n$ for all n.

b) Show $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$.
Hint: Express u_{n+1} in terms of u_n.

c) Is the sequence (u_n) (and hence (α_n) as well) necessarily bounded from above?

Example (Airy's Equation)

The ODE $y'' - xy = 0$ is known as *Airy's Equation*. The theorem predicts that its general solution is analytic everywhere. Making the usual power series „Ansatz“ $y(x) = \sum_{n=0}^{\infty} a_n x^n$ and substituting $y(x)$ into Airy's Equation, we obtain

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n,$$

$$x\,y(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$$

$$2a_2 + \sum_{n=1}^{\infty} \left((n+2)(n+1)a_{n+2} - a_{n-1}\right) x^n = 0$$

$$\implies a_2 = 0 \quad \text{and} \quad a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)} \quad \text{for } n = 1, 2, 3, \dots$$

$$\implies a_{3n+2} = 0,\ a_{3n} = a_0 \prod_{k=1}^{n} \frac{1}{(3k)(3k-1)},\ a_{3n+1} = a_1 \prod_{k=1}^{n} \frac{1}{(3k+1)(3k)}.$$

Example (cont'd)

$\Longrightarrow$ A fundamental system of solutions of Airy's Equation is

$$y_1(x) = 1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \frac{x^9}{9\cdot 8\cdot 6\cdot 5\cdot 3\cdot 2} + \cdots,$$
$$y_2(x) = x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 5\cdot 3} + \frac{x^{10}}{10\cdot 9\cdot 8\cdot 6\cdot 5\cdot 3} + \cdots.$$

The radius of convergence of these power series is $\rho = \infty$, as you can check by applying the ratio test to the series with the gaps removed (or substitute $z = x^3$).

$\Longrightarrow$ The general solution of Airy's Equation, which is

$$\begin{aligned} y(x) &= a_0 y_1(x) + a_1 y_2(x) \\ &= a_0 + a_1 x + \frac{a_0}{3\cdot 2}x^3 + \frac{a_1}{4\cdot 3}x^4 + \frac{a_0}{6\cdot 5\cdot 3\cdot 2}x^6 + \frac{a_1}{7\cdot 6\cdot 5\cdot 3}x^7 + \cdots, \end{aligned}$$

also has radius of convergence $\rho = \infty$.

This direct proof of $\rho = \infty$ is instructive but not necessary, since the theorem implies $\rho = \infty$ (as for any explicit linear 2nd-order ODE with polynomial coefficients).

Example (Legendre's Equation)

The Legendre equation (or family of equations)

$$(1-x^2)y'' - 2x\,y' + n(n+1)y = 0 \qquad (\mathrm{Le}_n)$$

has regular singular points in $x_0 = \pm 1$, but an ordinary point in $x_0 = 0$.
$\Longrightarrow$ We can solve it in $(-1, 1)$ with the usual power series „Ansatz" $y(x) = \sum_{k=0}^{\infty} a_k x^k$ ("k" is necessary, since (Le_n) is indexed by n).
Since the coefficients are polynomials, it is better not to rewrite it in explicit form (which would produce the power series $\frac{2x}{1-x^2}$ and $\frac{n(n+1)}{1-x^2}$) but solve it directly.
We obtain

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} - 2x\sum_{k=1}^{\infty} k a_k x^{k-1} + n(n+1)\sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)a_k x^k - \sum_{k=1}^{\infty} 2k a_k x^k + \sum_{k=0}^{\infty} n(n+1)a_k x^k$$

$$= \sum_{k=0}^{\infty}\bigl((k+2)(k+1)a_{k+2} - (k^2+k-n^2-n)a_k\bigr)x^k = 0.$$

Example (cont'd)

$$\implies a_{k+2}=\frac{k^2+k-n^2-n}{(k+2)(k+1)}\,a_k=\frac{(k-n)(k+n+1)}{(k+2)(k+1)}\,a_k \quad (k\in\mathbb{N});$$

$$\implies a_{2m}=a_0\frac{(-1)^m}{(2m)!}\prod_{i=0}^{m-1}\left[(n-2i)(n+2i+1)\right],$$

$$a_{2m+1}=a_1\frac{(-1)^m}{(2m+1)!}\prod_{i=0}^{m-1}\left[(n-2i-1)(n+2i+2)\right]$$

A fundamental system of solutions of the Legendre equation is therefore

$$y_1(x)=\sum_{m=0}^{\infty}(-1)^m\frac{n(n-2)\cdots(n-2m+2)(n+1)(n+3)\cdots(n+2m-1)}{(2m)!}\,x^{2m},$$

$$y_2(x)=\sum_{m=0}^{\infty}(-1)^m\frac{(n-1)(n-3)\cdots(n-2m+1)(n+2)(n+4)\cdots(n+2m)}{(2m+1)!}\,x^{2m+1}.$$

Example (cont'd)

Notes

- One of the two fundamental solutions (y_1 if n is even, y_2 if n is odd) is a polynomial function of degree n and hence analytic everywhere. The other solution is analytic in $(-1, 1)$, since $p(x) = \frac{-2x}{1-x^2}$, $q(x) = \frac{n(n+1)}{1-x^2}$ have power series expansions at $x_0 = 0$ with $\rho = 1$. In fact the ratio test applied to the non-polynomial solution shows that its radius of convergence is precisely 1.
- Since the polynomial solutions are scalar multiples of the polynomial fundamental solution, this must also hold for the n-th Legendre polynomial $\mathrm{P}_n(x)$. Hence up to a normalizing factor $\mathrm{P}_n(x)$ is equal to $y_1(x)$ if n is even and to $y_2(x)$ if n is odd. The normalizing factor can be determined from the leading coefficients of $\mathrm{P}_n(x)$ and the polynomial fundamental solution, which are

$$\frac{(2n)(2n-1)\cdots(n+1)}{2^n n!} = \frac{1}{2^n}\binom{2n}{n} \text{ resp. } (-1)^{\lfloor n/2 \rfloor} \frac{\prod_{k=n+1,\,k\text{ odd}}^{2n} k}{\prod_{k=1,\,k\text{ odd}}^{n} k}.$$

Example (cont'd)

Notes cont'd

- (cont'd)
  For the latter observe that for even n the signless coefficient of x^n n $y_1(x)$ is

$$\frac{n(n-2)\cdots 2(n+1)(n+3)\cdots(2n-1)}{n!}=\frac{(n+1)(n+3)\cdots(2n-1)}{(n-1)(n-3)\cdots 3\cdot 1},$$

  and similarly for odd n. It follows that

$$\mathrm{P}_n(x)=\frac{(-1)^{\lfloor n/2\rfloor}\prod_{k=n+1,\,k\text{ even}}^{2n}k}{2^n\prod_{k=1,\,k\text{ even}}^{n}k}\times\begin{cases}y_1(x) & \text{if n is even,}\\ y_2(x) & \text{if n is odd,}\end{cases}$$

  which together with the formulas for $y_1(x)$, $y_2(x)$ determines the coefficients of $\mathrm{P}_n(x)$.
  Alternatively (and less cumbersome), differentiate $(x^2-1)^n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}x^{2n-2k}$ exactly n times to obtain the coefficients of $\mathrm{P}_n(x)=\frac{1}{2^n n!}\,\mathrm{D}^n\left[(x^2-1)^n\right]$ directly.

Inhomogeneous Equations

x_0 is said to be an ordinary point (resp., a regular singular point) of

$$P(x)y'' + Q(x)y' + R(x)y = S(x), \tag{2}$$

if the same is true of the associated homogeneous equation and $S(x)$ is analytic at x_0 as well.

Corollary

If x_0 is an ordinary point of (2) then solutions $y_p(x)$ of (2) are analytic in x_0, and the power series representation $y_p(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ is valid (at least) for $|x - x_0| < \rho$, where ρ denotes the minimum of the radii of convergence of the three power series representing $p(x) = Q(x)/P(x)$, $q(x) = R(x)/P(x)$, and $r(x) = S(x)/P(x)$ at x_0.

Proof of the corollary.

Let I be an open interval containing x_0 and not containing any zero of P. In terms of a fundamental system $y_1(x), y_2(x)$ of solutions of (1) on I, any particular solution of (2) on I has the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$ with

$$c_1(x) = \gamma_1 + \int_{x_0}^{x} \frac{-y_2(t)r(t)}{\mathrm{W}(t)}\,\mathrm{d}t, \quad c_2(x) = \gamma_2 + \int_{x_0}^{x} \frac{y_1(t)r(t)}{\mathrm{W}(t)}\,\mathrm{d}t,$$

where γ_1, γ_2 are constants and $\mathrm{W}(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian of $y_1(x), y_2(x)$; cf. the vectorial variation-of-parameters formula. By Abel's Theorem, the Wronskian has the form $\mathrm{W}(x) = \gamma \exp \int_{x_0}^{x} -p(t)\,\mathrm{d}t$ for some constant $\gamma \neq 0$. Since $p(x)$ is analytic in the disk $|z - x_0| < \rho$ and integration doesn't change the radius of convergence of a power series, the function $\mathrm{W}(x)^{-1} = \gamma^{-1} \exp \int_{x_0}^{x} p(t)\,\mathrm{d}t$ is analytic in $|z - x_0| < \rho$ as well. The same is true of $y_1(x)$, $y_2(x)$ (by the theorem) and $r(x)$ (by assumption). Since $y_p(x)$ is obtained from these functions by a finite number of additions, multiplications and integrations, it must also be analytic in $|z - x_0| < \rho$. $\square$

Example

We solve the IVP

$$y'' + y = \frac{1}{1-x} \wedge y(0) = y'(0) = 0 \quad \text{on } (-1,1).$$

Notably, the machinery developed for higher-order linear ODE's with constant coefficients can't be used to solve this ODE (since $\frac{1}{1-x}$ is not an exponential polynomial), but order reduction and vectorial variation of parameters can be, of course.

Expanding the right-hand side into a geometric series and making the usual power series „Ansatz“ turns the ODE into $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} + a_n\right] x^n = \sum_{n=0}^{\infty} x^n.$$

$$\Longrightarrow \quad a_{n+2} = \frac{1-a_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \dots$$

Example (cont'd)

Together with $a_0 = a_1 = 0$ this gives

$$y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{24}x^5 + \frac{23}{720}x^6 + \frac{23}{1008}x^7 + \frac{697}{40320}x^8 + \\ + \frac{985}{72576}x^9 + \frac{39623}{3628800}x^{10} + \cdots$$

By the corollary, the series is guaranteed to converge for $|x| < 1$ and solves the ODE in $(-1, 1)$.

In fact, it is not difficult to see that

$$a_n \simeq \frac{1}{n(n-1)} \quad \text{for } n \to \infty,$$

i.e., $\lim_{n\to\infty}\bigl(n(n-1)a_n\bigr) = 1$.

This follows, e.g., from $\frac{n-3}{n(n-1)(n-2)} < a_n < \frac{1}{n(n-1)}$ for $n \geq 3$, which can be shown by induction.

$\Longrightarrow$ The radius of convergence of $\sum_{n=0}^{\infty} a_n x^n$ is exactly 1.

From the general theory we know, however, that $y(x)$ has a (unique) extension to $(-\infty, 1)$, which is analytic as well.

Example (cont'd)

Question: How to find the extension of $y(x)$ to $(-\infty, 1)$?

Answer: With power series we cannot do this in one fell swoop, but we can use a different center to enlarge the domain.

Let us consider this for $x_0 = -1$, i.e., we make the powers series „Ansatz" $y(x) = \sum_{n=0}^{\infty} b_n(x+1)^n$.

$$\Longrightarrow \sum_{n=0}^{\infty} \left[(n+2)(n+1)b_{n+2} + b_n\right](x+1)^n = \frac{1}{2-(x+1)} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}}$$

$$\Longrightarrow \quad b_{n+2} = \frac{1/2^{n+1} - b_n}{(n+2)(n+1)} \quad \text{for } n = 0, 1, 2, \dots$$

This gives the general solution as

$$\begin{aligned} y(x) = b_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x+1)^{2n} + b_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(x+1)^{2n+1} + \\ + \tfrac{1}{4}(x+1)^2 + \tfrac{1}{24}(x+1)^3 - \tfrac{1}{96}(x+1)^4 + \tfrac{1}{960}(x+1)^5 + \\ + \tfrac{1}{720}(x+1)^6 + \tfrac{1}{2880}(x+1)^7 + \tfrac{37}{322560}(x+1)^8 + \cdots \end{aligned}$$

Example (cont'd)

This expansion is valid (at least) for $-3 < x < 1$, because the distance from $x_0 = -1$ to the singularity of $1/(1-x)$ is 2.

Since we are interested in the extension of the solution on $(-1, 1)$ determined before, we need to solve the same IVP $y(0) = y'(0) = 0$. This gives the two equations

$$
\begin{aligned}
b_0 \cos 1 + b_1 \sin 1 + \tfrac{1}{4} + \tfrac{1}{24} - \tfrac{1}{96} + \tfrac{1}{960} + \tfrac{1}{720} + \cdots &= 0, \\
-b_0 \sin 1 + b_1 \cos 1 + \tfrac{2}{4} + \tfrac{3}{24} - \tfrac{4}{96} + \tfrac{5}{960} + \tfrac{6}{720} + \cdots &= 0,
\end{aligned}
$$

from which b_0, b_1 can be determined. (Likely the two series involved can't be evaluated in closed form, but we can use numerical approximations instead.)

Note: Alternatively, one could determine $b_0 = y(-1)$, $b_1 = y'(-1)$ directly from the series representation in $(-1, 1)$. But this is not advisable, since the resulting alternating series converge slowly (because -1 is on the boundary of the disk of convergence).

Similarly, we can obtain power series solutions of $y'' + y = \frac{1}{1-x}$ that are defined for $x > 1$ by choosing a center $x_0 > 1$, but we stop the discussion here.

The Case of a Regular Singular Point

Now suppose that x_0 is a regular singular point of

$$P(x)y'' + Q(x)y' + R(x)y = 0. \tag{1}$$

As discussed before we can assume $x_0 = 0$, in which case the ODE has the equivalent form

$$x^2y''+x(p_0+p_1x+p_2x^2+\dots)y'+(q_0+q_1x+q_2x^2+\dots)y = 0 \tag{5}$$

with $x \mapsto f(x) = \sum_{n=0}^{\infty} p_nx^n$ and $x \mapsto g(x) = \sum_{n=0}^{\infty} q_nx^n$ analytic in $\mathrm{B}_\rho(0)$ for some $\rho > 0$.

Since truncating $f(x)$, $g(x)$ after their constant term yields the Euler equation $x^2y'' + xp_0y' + q_0y = 0$, it is reasonable to try the generalized power series „Ansatz“

$$y(x) = \sum_{n=0}^{\infty} a_nx^{r+n} = x^r \sum_{n=0}^{\infty} a_nx^n = x^r \times \text{analytic}$$

for finding a solution of (1).

Substituting

$$y'(x) = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1},$$

$$y''(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$$

into the explicit form of (1), we obtain the power series equation

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n} + \left(\sum_{n=0}^{\infty}(r+n)a_n x^{r+n}\right)\left(\sum_{n=0}^{\infty}p_n x^n\right) + \\ + \left(\sum_{n=0}^{\infty}a_n x^{r+n}\right)\left(\sum_{n=0}^{\infty}q_n x^n\right) = 0,$$

which (take the factor x^r out!) is equivalent to

$$(r+n)(r+n-1)a_n + \sum_{k=0}^{n}(r+k)a_k p_{n-k} + \sum_{k=0}^{n}a_k q_{n-k} = 0, \\ n = 0, 1, 2, \dots$$

This can be rewritten as

$$[(r+n)(r+n-1)+(r+n)p_0+q_0]\,a_n+\sum_{k=0}^{n-1}[(r+k)p_{n-k}+q_{n-k}]\,a_k=0$$

for $n=0,1,2,\dots$

Observations

- The first equation is

$$[r(r-1)+rp_0+q_0]\,a_0=0.$$

  It has the form $F(r)a_0=0$ with the quadratic polynomial $F(r)=r(r-1)+rp_0+q_0=r^2+(p_0-1)r+q_0$.

- In each of the subsequent equations, a_n appears with the coefficient

$$(r+n)(r+n-1)+(r+n)p_0+q_0=F(r+n).$$

  Apart from a_n only numbers a_k with $k<n$ appear in such an equation.

Observations cont'd

- Regarding r as a variable, we can solve the equations for $n = 1, 2, 3, \dots$ by defining $a_n = a_n(r)$ recursively as

$$a_0(r) = 1,$$
$$a_n(r) = -\frac{1}{F(r+n)} \sum_{k=0}^{n-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_k(r) \quad \text{for } n \geq 1.$$

The so-defined $r \mapsto a_n(r) = P_n(r)/Q_n(r)$ is a rational function of r, whose denominator can be taken as the polynomial $Q_n(r) = F(r+1)F(r+2)\cdots F(r+n)$.

For example, we have

$$a_1(r) = -\frac{1}{F(r+1)}[rp_1 + q_1]a_0(r) = -\frac{rp_1 + q_1}{F(r+1)},$$
$$a_2(r) = -\frac{1}{F(r+2)}\left([rp_2 + q_2]a_0(r) + [(r+1)p_1 + q_1]a_1(r)\right)$$
$$= \frac{F(r+1)[rp_2 + q_2] - [(r+1)p_1 + q_1][rp_1 + q_1]}{F(r+1)F(r+2)}.$$

These observations remain mutatis mutandis true for regular singular points $x_0 \neq 0$. In the general case, $Q(x)/P(x)$ and $R(x)/P(x)$ must be expanded into powers of $x - x_0$, viz.,

$$p(x) = \frac{Q(x)}{P(x)} = \frac{p_0}{x - x_0} + p_1 + p_2(x - x_0) + p_3(x - x_0)^2 + \cdots,$$

$$q(x) = \frac{R(x)}{P(x)} = \frac{q_0}{(x - x_0)^2} + \frac{q_1}{x - x_0} + q_2 + q_3(x - x_0) + \cdots$$

and $F(r) = r^2 + (p_0 - 1)r + q_0$ formed from the numbers p_0, q_0 appearing in these expansions.

Definition

The quadratic equation $F(r) = 0$ is called *indicial equation* associated with the regular singular point x_0 of (1). Its roots r_1, r_2 (in the case $r_1, r_2 \in \mathbb{R}$ ordered as $r_1 \geq r_2$) are called *exponents at the singularity* x_0.

Note

$F(r) = 0$ is exactly the equation that r should satisfy in order for $y(x) = x^r$ to form a solution of the Euler equation $x^2y'' + p_0xy' + q_0y = 0$ obtained by truncating (5).

Theorem (cf. [BDM17], Th. 5.6.1)

Suppose that $x_0 = 0$ is a regular singular point of

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

and that

$$p(x) = \frac{Q(x)}{P(x)} = \frac{p_0}{x} + p_1 + p_2 x + p_3 x^2 + p_4 x^3 + \cdots,$$
$$q(x) = \frac{R(x)}{P(x)} = \frac{q_0}{x^2} + \frac{q_1}{x} + q_2 + q_3 x + q_4 x^2 + \cdots$$

holds for $0 < |x| < \rho$ (i.e., $\sum_{n=0}^{\infty} p_n x^n$, $\sum_{n=0}^{\infty} q_n x^n$ converge for $|x| < \rho$).

1. *If the exponents r_1, r_2 at x_0 are distinct and $r_1 - r_2 \notin \mathbb{Z}$, then (1) has two linearly independent solutions on $(0, \rho)$, viz.*

$$y_i(x) = x^{r_i}\left(1 + \sum_{n=1}^{\infty} a_n(r_i)x^n\right), \quad i = 1, 2.$$

These are obtained by setting $a_0(r_i) = 1$ and for $n \geq 1$ determining $a_n(r_i)$ recursively from

$$a_n(r_i) = -\frac{1}{F(r_i + n)} \sum_{k=0}^{n-1} \left[(r_i + k)p_{n-k} + q_{n-k}\right] a_k(r_i).$$

Theorem (cont'd)

2. *If $r_1 - r_2 \in \mathbb{Z}$, the larger root r_1 (respectively, the double root $r_1 = r_2$) yields one solution of* (1) *of the form $y_1(x) = x^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right)$ on* $(0, \rho)$. *The coefficients $a_n(r_1)$ are determined in the same way as in Case (1).*
3. *If $r_1 = r_2$, a second solution of* (1) *on* $(0, \rho)$ *that is linearly independent of $y_1(x)$ is*

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n(r_1)x^n.$$

   with $b_n(r_1) = a_n'(r_1)$.
4. *If $r_1 - r_2 = N \in \mathbb{Z}^+$, a second solution of* (1) *on* $(0, \rho)$ *that is linearly independent of $y_1(x)$ is*

$$y_2(x) = a\, y_1(x) \ln x + x^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right)$$

   with $a = \lim_{r \to r_2} (r - r_2)a_N(r)$ *and* $c_n(r_2) = \frac{\mathrm{d}}{\mathrm{d}r}\left[(r - r_2)a_n(r)\right]\Big|_{r=r_2}$.

Notes on the theorem

- The theorem also holds for regular singular points $x_0 \neq 0$, provided one replaces x by $x - x_0$ and $(0, \rho)$ by $(x_0, x_0 + \rho)$ everywhere in its statement.
- Solutions on $(-\rho, 0)$ (resp., on $(x_0 - \rho, x_0)$ can be obtained by making the substitution $z(x) = y(-x)$ in (1), which gives $P(-x)z'' - Q(-x)z' + R(-x)z = 0$. The corresponding equation (5) is

  $$x^2 z'' + x(p_0 - p_1 x + p_2 x^2 \mp \cdots)z' + (q_0 - q_1 x + q_2 x^2 \mp \cdots)z = 0,$$

  which has the same indicial equation as (1). Further one can show by induction that the coefficients $a_n(r)$ change to $(-1)^n a_n(r)$ when using the "alternating" sequences $(p_0, -p_1, p_2, \dots)$, $(q_0, -q_1, q_2, \dots)$ instead of $(p_0, p_1, p_2, \dots)$, $(q_0, q_1, q_2, \dots)$. This implies that solutions on $(-\rho, 0)$ have the same form as in the theorem (with the same $a_n(r)$, $b_n(r)$, $c_n(r)$, a, because the change $a_n(r) \to (-1)^n a_n(r)$ is undone by the the back substitution $y(x) = z(-x)$), except that x^{r_i} is replaced by $(-x)^{r_i} = |x|^{r_i}$ and $\ln x$ by $\ln(-x) = \ln|x|$. Thus, if we write $|x|^{r_i}$ and $\ln|x|$ in the formulas then both cases $0 < x < \rho$ and $-\rho < x < 0$ are covered; cf. [BDM17], Th. 5.6.1.

Notes on the theorem cont'd

- It is usually difficult to obtain an explicit formula for the functions $a_n(r)$ from the recurrence relation. Hence, instead of computing $a_n(r)$ and the expressions for $b_n(r)$, $c_n(r)$ and a in terms of $a_n(r)$, it is often better to use the postulated form of the solution as an „Ansatz" and try to determine the coefficients $a_n(r_i)$, $b_n(r_i)$, $c_n(r_i)$ and a by substituting it into (1).
- Since the roots of the indicial equation are $r_{1,2} = \frac{1}{2}\left(1 - p_0 \pm \sqrt{(p_0-1)^2 - 4q_0}\right)$, Case 4 ($r_1 - r_2 \in \mathbb{Z}^+$) occurs iff $(p_0-1)^2 - 4q_0 = N^2$ is a perfect square, and then $r_1 = (1-p_0+N)/2$, $r_2 = (1-p_0-N)/2$.
- In Case 1 ($r_1 - r_2 \notin \mathbb{Z}$) it is possible that $r_1, r_2 \in \mathbb{C} \setminus \mathbb{R}$. Then $r_2 = \overline{r_1}$, the two indicated solutions satisfy $y_2(x) = \overline{y_1(x)}$, and the real and imaginary part of one of them provide a real fundamental system.
- The subsequent proof of the theorem shows that the solution $y_2(x)$ in Case 4 is obtained in the same way as in Case 3 except that the exponent r_1 is replaced by r_2 and the rational functions $a_n(r)$ by $\alpha_n(r) = (r - r_2)a_n(r)$.

Notes on the theorem cont'd

- The theorem is essentially due to GEORG FERDINAND FROBENIUS (1849–1917), and the method developed to solve 2nd-order time-dependent linear ODE's at regular singular points is commonly referred to as the *method of Frobenius*.

Proof of the theorem.

(1) For the first equation to be satisfied, we must choose r as a root of the indicial equation., i.e., $r = r_1$ or $r = r_2$.
Since $r_1 - r_2 \notin \mathbb{Z}$, we have $F(r_i + n) \neq 0$ for all $n \in \mathbb{Z}^+$.
$\implies a_n(r_i)$ is defined for all $n \in \mathbb{N}$ and yields a solution of (1).
The solutions $y_1(x)$, $y_2(x)$ obtained in this way are linearly independent, since $y_i(x) \simeq x^{r_i}$ for $x \downarrow 0$ and certainly not $x^{r_1} \simeq c\,x^{r_2}$ for $x \downarrow 0$.

(2) Here we have $F(r_1 + n) \neq 0$ for all $n \in \mathbb{Z}^+$.
$\implies r_1$ gives rise to a solution of (1) as in Case 1.

(3) We work with the two-variable function

$$\phi(r, x) = x^r \sum_{n=0}^{\infty} a_n(r) x^n = \sum_{n=0}^{\infty} a_n(r) x^{r+n},$$

which is defined for $|x| < \rho$, $r \notin \{r_1 - 1, r_1 - 2, \dots\}$ (see below), and the differential operator

$$\begin{aligned} L &= x^2 \mathrm{D}^2 + x\,f(x)\mathrm{D} + g(x) \\ &= x^2 \mathrm{D}^2 + x(p_0 + p_1 x + \cdots)\mathrm{D} + (q_0 + q_1 x + \cdots)\mathrm{id}, \end{aligned}$$

which in the following acts like a partial derivative (i.e., $\mathrm{D} \triangleq \frac{\partial}{\partial x}$).

Proof cont'd.

By definition of the coefficients $a_n(r)$, we have

$$L[\phi] = F(r)a_0(r)x^r + \sum_{n=1}^{\infty} 0\,x^{r+n} = (r-r_1)^2 x^r.$$

Since L involves only $\frac{\partial}{\partial x}$, Clairaut's Theorem gives $L \circ \frac{\partial}{\partial r} = \frac{\partial}{\partial r} \circ L$ (provided we apply it to a C^2-function).

$$\Longrightarrow \; L\left[\frac{\partial \phi}{\partial r}\right] = \frac{\partial}{\partial r} L[\phi] = 2(r-r_1)x^r + (r-r_1)^2(\ln x)x^r,$$

$$L\left[\frac{\partial \phi}{\partial r}(r_1, x)\right] = \frac{\partial}{\partial r} L[\phi]\Big|_{r=r_1} = 0.$$

It follows that a second solution of (1) is

$$\frac{\partial \phi}{\partial r}(r_1, x) = (\ln x)x^{r_1} \sum_{n=0}^{\infty} a_n(r_1)x^n + x^{r_1} \sum_{n=0}^{\infty} a_n'(r_1)x^n$$

$$= (\ln x)y_1(x) + x^{r_1} \sum_{n=1}^{\infty} a_n'(r_1)x^n.$$

Clearly this solution is linearly independent of $y_1(x)$.

Proof cont'd.

The proof of (3) is not yet finished, because the termwise differentiation used in the computation needs to be justified. Also we need to show that the generalized power series series solutions in Parts (1)–(4) actually converge for $|x| < \rho$.

The latter can be done by a slight modification of the method used in the analytic case. For $0 < \rho_1 < \rho$ we have a recursive bound

$$|a_n(r)|\,\rho_1^n \leq \sum_{k=0}^{n-1} \frac{(|r| + k + 1)\,M\,|a_k(r)|\,\rho_1^k}{|F(r+n)|}$$

for the coefficients of $\phi(r, x)$, obtained from the recurrence relation for $a_n(r)$ in the same way as before (and with the same meaning of M). The auxiliary sequence $\big(u_n(r)\big)$ defined by $u_0(r) = 1$ and

$$u_n(r) = \sum_{k=0}^{n-1} \frac{(|r| + k + 1)\,Mu_k(r)}{|F(r+n)|} \quad \text{for } n \geq 1$$

still satisfies $\lim_{n\to\infty} \frac{u_{n+1}(r)}{u_n(r)} = 1$ (independently of r), so that the proof can be finished in the same way as before.

Proof cont'd.

The preceding argument can be modified to yield uniform convergence of $\phi(r,x) = \sum_{n=0}^{\infty} a_n(r)x^{r+n}$ and its partial derivatives up to certain orders (order 1 for $\frac{\partial}{\partial r}$ and order 2 for $\frac{\partial}{\partial x}$) on compact subsets of their domain, justifying termwise differentiation. The arguments in Parts (1), (2), (4) are similar.

(4) Here we set

$$\phi(r,x) = (r-r_2)x^r \sum_{n=0}^{\infty} a_n(r)x^n = x^r \sum_{n=0}^{\infty} (r-r_2)a_n(r)x^n.$$

The functions $\alpha_n(r) = (r-r_2)a_n(r)$ satisfy the same recurrence relation as $a_n(r)$, but start with $\alpha_0(r) = r - r_2$.

$$\Longrightarrow \alpha_N(r) = -\frac{1}{(r+N-r_1)(r+N-r_2)} \sum_{k=0}^{N-1} \left[(r+k)p_{n-k} + q_{n-k}\right](r-r_2)a_k($$

$$= -\frac{1}{r+N-r_2} \sum_{k=0}^{N-1} \left[(r+k)p_{n-k} + q_{n-k}\right] a_k(r),$$

since $r_1 = r_2 + N$.

Proof cont'd.

$\implies \alpha_N(r)$ is analytic at r_2.
The recurrence relation then implies that $\alpha_n(r)$ are analytic at r_2 for $n > N$. Clearly this also holds for $n < N$, in which case $\alpha_n(r_2) = 0$.
As in (3) it then follows that $\phi(r, x)$ defined for $|x| < \rho$, $r \notin \{r_1 - 1, r_1 - 2, \dots, r_1 - N + 1, r_1 - N - 1, r_1 - N - 2, \dots\}$, and satisfies

$$L[\phi] = (r - r_1)(r - r_2)\alpha_0(r) = (r - r_1)(r - r_2)^2 x^r,$$

$$L\left[\frac{\partial\phi}{\partial r}(r_2, x)\right] = \frac{\partial}{\partial r} L[\phi]\bigg|_{r=r_2} = 0.$$

$$\implies \frac{\partial\phi}{\partial r}(r_2, x) = (\ln x)x^{r_2}\sum_{n=0}^{\infty}\alpha_n(r_2)x^n + x^{r_2}\sum_{n=0}^{\infty}\alpha_n'(r_2)x^n$$

is a second solution of (1).
It remains to verify that this solution has the form stated in the theorem. For the 2nd summand this is true by definition of $\alpha_n(r)$.

Proof cont'd.

The first summand can be rewritten as

$$(\ln x)\sum_{n=N}^{\infty}\alpha_n(r_2)x^{n+r_2}=(\ln x)\sum_{n=0}^{\infty}\alpha_{n+N}(r_2)x^{n+r_1},$$

which is equal to $a\,y_1(x)=\alpha_N(r_2)y_1(x)$ iff $\alpha_{n+N}(r_2)=\alpha_N(r_2)\alpha_n(r_1)$ for $n\in\mathbb{N}_0$. This in turn can be proved by induction on n (the case $n=0$ being trivial):

$$\begin{aligned}
\alpha_{n+N}(r_2)&=-\frac{1}{F(r_2+n+N)}\sum_{k=0}^{n+N-1}\left[(r_2+k)p_{n+N-k}+q_{n+N-k}\right]\alpha_k(r_2)\\
&=-\frac{1}{F(r_1+n)}\sum_{k=N}^{n+N-1}\left[(r_2+k)p_{n+N-k}+q_{n+N-k}\right]\alpha_k(r_2)\\
&=-\frac{1}{F(r_1+n)}\sum_{k=0}^{n-1}\left[(r_2+k+N)p_{n-k}+q_{n-k}\right]\alpha_{k+N}(r_2)\\
&=-\frac{1}{F(r_1+n)}\sum_{k=0}^{n-1}\left[(r_1+k)p_{n-k}+q_{n-k}\right]\alpha_N(r_2)\alpha_k(r_1)\\
&=\alpha_N(r_2)\alpha_n(r_1).
\end{aligned}$$

□

Example

Find two linearly independent solutions of the ODE

$$2xy'' + y' + xy = 0, \quad x > 0.$$

Rewriting the ODE as

$$y'' + \frac{1}{2x}\,y' + \frac{1}{2}\,y = 0,$$

we see that $x = 0$ is a regular singular point and $p_0 = 1/2$ ($p_1 = p_2 = \cdots = 0$), $q_0 = 0$ ($q_1 = 0$, $q_2 = 1/2$, $q_3 = q_4 = \cdots = 0$).
$\Longrightarrow$ The indicial equation is

$$r^2 + (p_0 - 1)r + q_0 = r^2 - \tfrac{1}{2}\,r = r\left(r - \tfrac{1}{2}\right) = 0.$$

$\Longrightarrow$ The exponents at the singularity $x = 0$ are $r_1 = 1/2$, $r_2 = 0$. Thus we are in Case (1) of the theorem and there must be solutions $y_1(x)$, $y_2(x)$ of the form

$$y_1(x) = \sqrt{x}\sum_{n=0}^{\infty} a_n(1/2)x^n, \qquad y_2(x) = \sum_{n=0}^{\infty} a_n(0)x^n.$$

Example (cont'd)

$r_1 = 1/2$:

Instead of using the general recurrence relation for $a_n(r)$ at $r = 1/2$, we determine it directly from the ODE, writing a_n in place of $a_n(1/2)$.

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+1/2}$$

$$2x\,y''(x) = 2x \sum_{n=0}^{\infty} \left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) a_n x^{n-3/2}$$

$$= \sum_{n=0}^{\infty} 2\left(n+\tfrac{1}{2}\right)\left(n-\tfrac{1}{2}\right) a_n x^{n-1/2}$$

$$y'(x) = \sum_{n=0}^{\infty} \left(n+\tfrac{1}{2}\right) a_n x^{n-1/2}$$

$$x\,y(x) = \sum_{n=0}^{\infty} a_n x^{n+3/2} = \sum_{n=2}^{\infty} a_{n-2} x^{n-1/2}$$

$$\Longrightarrow 0 \cdot a_0 + 3a_1 x + \sum_{n=2}^{\infty} \left[(2n+1)na_n + a_{n-2}\right] x^n = 0$$

Example (cont'd)

$\Longrightarrow a_1 = 0$ and $a_n = -\dfrac{a_{n-2}}{n(2n+1)}$ for $n \geq 2$

$\Longrightarrow$ All odd coefficients a_{2n+1} are zero, and

$$\begin{aligned} y_1(x) &= x^{1/2}\left(1 - \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} - \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 5 \cdot 9 \cdot 13} \pm \cdots\right) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+\frac{1}{2}}}{2^n n!\, 5 \cdot 9 \cdot 13 \cdots (4n+1)}. \end{aligned}$$

The theorem predicts $\rho = \infty$ for the power series (without the factor $x^{1/2}$), which can also be seen with the aid of the ratio test.

Example (cont'd)

$\underline{r_2 = 0:}$ Writing again a_n in place of $a_n(0)$, we obtain

$$
\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} a_n x^n \\
2x\,y''(x) &= 2x \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\
&= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-1} \\
y'(x) &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\
x\,y(x) &= \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=2}^{\infty} a_{n-2} x^{n-1}
\end{aligned}
$$

$$
\Longrightarrow 0 \cdot a_0 + a_1 x + \sum_{n=2}^{\infty} \left[n(2n-1)a_n + a_{n-2} \right] x^n = 0
$$

Example (cont'd)

$\implies a_1 = 0$ and $a_n = -\dfrac{a_{n-2}}{n(2n-1)}$ for $n \geq 2$

$\implies$ Again all odd coefficients a_{2n+1} are zero, and

$$\begin{aligned} y_2(x) &= 1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 4 \cdot 3 \cdot 7} - \frac{x^6}{2 \cdot 4 \cdot 6 \cdot 3 \cdot 7 \cdot 11} \pm \cdots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!\, 3 \cdot 7 \cdot 11 \cdots (4n-1)}. \end{aligned}$$

The theorem predicts $\rho = \infty$, which again can also be easily found with the ratio test.

In all we have shown that $y_1(x)$, $y_2(x)$ form a fundamental system of solutions of $2xy'' + y' + xy = 0$ on $(0, \infty)$. A fundamental system $y_1^-(x)$, $y_2^-(x)$ of solutions on $(-\infty, 0)$ is then obtained by changing the fractional part of $y_1(x)$ to $\sqrt{-x}$.

In the special case under consideration, since only even powers x^{2n} appear in $y_1(x)$, $y_2(x)$, this is equivalent to setting $y_1^-(x) = y_1(-x)$, $y_2^-(x) = y_2(-x)$ for $x < 0$.

$y_2(x)$ and its constant multiples are defined and solve the ODE on $\mathbb{R}$.

Example (cont'd)

Finally we compare our method of determining the fundamental solutions $y_1(x)$, $y_2(x)$ directly from the ODE with that of employing the rational functions $a_n(r)$. Since the only nonzero coefficients among p_i, q_i are $p_0 = q_2 = 1/2$, the general recurrence relation becomes

$$\begin{aligned}a_n(r) &= -\frac{q_2\, a_{n-2}(r)}{F(r+n)} = -\frac{a_{n-2}(r)}{2(r+n)(r+n-\frac{1}{2})}\\ &= -\frac{a_{n-2}(r)}{(r+n)(2r+2n-1)},\end{aligned}$$

supplemented by $a_0(r) = 1$, $a_1(r) = 0$.
$\Longrightarrow a_{2n+1}(r) = 0$ and

$$a_{2n}(r) = \frac{(-1)^n}{(r+2)(r+4)\cdots(r+2n)(2r+3)(2r+7)\dots(2r+4n-1)}.$$

For $r \in \left\{0, \frac{1}{2}\right\}$ this coincides with the formula determined for the coefficients of $y_1(x)$, $y_2(x)$ earlier. (For $r = 1/2$ the denominator of $a_{2n}(1/2)$ is $2^{-n}\, 5 \cdot 9 \cdots (4n+1) 4 \cdot 8 \cdots 4n = 2^n n!\, 5 \cdot 9 \cdots (4n+1)$.)

Example

We solve the Legendre equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0, \qquad n \in \mathbb{N},$$

near the singular point $x_0 = 1$.

Since $x_0 = 1$ is a simple zero of $P(x) = 1 - x^2$ and not a zero of $Q(x) = -2x$ (or not a zero of $R(x) = n(n+1)$), $x_0 = 1$ is a regular singular point.

First we put the ODE into explicit form and rewrite the coefficients in terms of $x - 1$:

$$\begin{aligned}
&y'' + \frac{2x}{(x-1)(x+1)}\,y' - \frac{n(n+1)}{(x-1)(x+1)}\,y \\
&= y'' + \left(\frac{1}{x-1} + \frac{1}{2+(x-1)}\right) y' - \frac{n(n+1)}{(x-1)\big(2+(x-1)\big)}\,y \\
&= y'' + \left(\sum_{n=-1}^{\infty} \frac{(x-1)^n}{2^{n+1}}\right) y' - n(n+1)\left(\sum_{n=-1}^{\infty} \frac{(x-1)^n}{2^{n+2}}\right) y,
\end{aligned}$$

from which we can read off $p_0 = 1$, $q_0 = 0$, and $\rho = 2$.

Example (cont'd)

Remark: Since we do not need the full expansion of $p(x)$ and $q(x)$, it is easier in this case to use the formulas

$$p_0 = \lim_{x\to 1}(x-1)p(x) = \lim_{x\to 1}\frac{2x}{x+1} = 1,$$
$$q_0 = \lim_{x\to 1}(x-1)^2 q(x) = \lim_{x\to 1}\left(-\frac{n(n+1)(x-1)}{x+1}\right) = 0,$$

and by Property (9) of power series the radii of convergence of $\sum p_n(x-1)^n$, $\sum q_n(x-1)^n$ are the distances from $x=1$ to the singularity of $\frac{2x}{x+1}$ resp. $-\frac{n(n+1)}{x+1}$, viz. $1-(-1)=2$.

The indicial equation at $x_0 = 1$ is therefore

$$r^2 + (p_0 - 1)r + q_0 = r^2 = 0. \qquad \Longrightarrow r_1 = r_2 = 0.$$

Hence we are in Cases (2) and (3) of the theorem, and there are fundamental solutions of the form

$$y_1(x) = \sum_{k=0}^{\infty} a_k(x-1)^k, \quad y_2(x) = y_1(x)\ln|x-1| + \sum_{k=1}^{\infty} b_k(x-1)^k.$$

Example (cont'd)

For the computation of $y_1(x)$ we make the substitution $t = x - 1$, i.e., $x = t + 1$, which gives $1 - x^2 = -t^2 - 2t$, $-2x = -2t - 2$ and turns the Legendre equation into

$$-(t^2 + 2t)y''(t+1) - (2t+2)y'(t+1) + n(n+1)y(t+1) = 0.$$

Substituting $y_1(x) = y_1(t+1) = \sum_{k=0}^{\infty} a_k t^k$ gives

$$-(t^2+2t)\left(\sum_{k=2}^{\infty} k(k-1)a_k t^{k-2}\right) - (2t+2)\left(\sum_{k=1}^{\infty} k a_k t^{k-1}\right) + n(n+1)\sum_{k=0}^{\infty} a_k t^k$$

$$= \sum_{k=0}^{\infty}\bigl[-k(k-1)a_k - 2(k+1)ka_{k+1} - 2ka_k - 2(k+1)a_{k+1} + n(n+1)a_k\bigr]t^k$$

$$= \sum_{k=0}^{\infty}\Bigl[\bigl(n(n+1) - k(k+1)\bigr)a_k - 2(k+1)^2 a_{k+1}\Bigr]t^k = 0$$

$$\Longrightarrow a_{k+1} = \frac{n(n+1) - k(k+1)}{2(k+1)^2}\,a_k = \frac{(n-k)(n+k+1)}{2(k+1)^2}\,a_k$$

for $k = 0, 1, 2, \ldots$

Example (cont'd)

Setting $a_0 = 1$ gives

$$\begin{aligned} a_k &= \frac{n(n-1)\cdots(n-k+1)(n+1)(n+2)\cdots(n+k)}{2^k(k!)^2} \\ &= \frac{1}{2^k}\binom{n}{k}\binom{n+k}{k}. \end{aligned}$$

The coefficients a_k with $k > n$ vanish, since in this case $n(n-1)\cdots(n-k+1)$ contains the factor 0.

$$\Longrightarrow \quad y_1(x) = \sum_{k=0}^{n} \frac{1}{2^k}\binom{n}{k}\binom{n+k}{k}(x-1)^k.$$

Since $y_1(x)$ is a polynomial, it solves the Legendre equation everywhere, and hence must be a constant multiple of the Legendre polynomial $\mathrm{P}_n(x)$. The leading coefficient of $y_1(x)$ is

$$\frac{1}{2^n}\binom{n}{n}\binom{n+n}{n} = \frac{1}{2^n}\binom{2n}{n},$$

the same as that of $\mathrm{P}_n(x)$!!!

Example (cont'd)

So we have discovered the identity

$$\begin{aligned}\mathrm{P}_n(x) &= \sum_{k=0}^{n} \frac{1}{2^k}\binom{n}{k}\binom{n+k}{k}(x-1)^k \\ &= 1 + \frac{n(n+1)}{2}(x-1) + \frac{(n-1)n(n+1)(n+2)}{16}(x-1)^2 + \cdots\end{aligned}$$

From this we see that $\mathrm{P}_n(1) = 1$, which is not obvious from the original definition of $\mathrm{P}_n(x)$ and explains why the Legendre polynomials are normalized in the strange way $\mathrm{P}_n(x) = \frac{1}{2^n}\binom{2n}{n}x^n +$ smaller powers.

Since $\mathrm{P}_n(x)$ is even (odd) when n is even (resp., odd), this also gives $\mathrm{P}_n(-1) = (-1)^n$, which in turn yields the binomial coefficient identity

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n+k}{k} = (-1)^n, \quad n = 0, 1, 2, \ldots$$

Example (cont'd)

For determining the 2nd fundamental solution $y_2(x)$ it will be convenient to use the associated differential operator $L[y] = (1-x^2)y'' - 2xy' + n(n+1)y$. For $x > 1$ we compute

$$\begin{aligned}
L\big[y_2(x)\big] &= L\big[\mathrm{P}_n(x)\ln(x-1)\big] + L\left[\sum_{k=1}^{\infty} b_k(x-1)^k\right] \\
&= (1-x^2)\left(\mathrm{P}_n''(x)\ln(x-1) + 2\mathrm{P}_n'(x)\frac{1}{x-1} + \mathrm{P}_n(x)\frac{-1}{(x-1)^2}\right) \\
&\quad - 2x\left(\mathrm{P}_n'(x)\ln(x-1) + \mathrm{P}_n(x)\frac{1}{x-1}\right) \\
&\quad + n(n+1)\mathrm{P}_n(x)\ln(x-1) + L\left[\sum_{k=1}^{\infty} b_k(x-1)^k\right] \\
&= -2(x+1)\mathrm{P}_n'(x) - \mathrm{P}_n(x) + L[\dots] \\
&= -2(t+2)\mathrm{P}_n'(t+1) - \mathrm{P}_n(t+1) + \\
&\quad + \sum_{k=0}^{\infty}\Big[\big(n(n+1) - k(k+1)\big)b_k - 2(k+1)^2 b_{k+1}\Big]t^k = 0,
\end{aligned}$$

where we have set $b_0 = 0$.

Example (cont'd)

The resulting inhomogeneous linear recurrence relation for $(b_1, b_2, b_3, \dots)$ clearly has a unique solution. (For $b_0 \neq 0$ it has a unique solution as well, but this amounts to adding $b_0 y_1(x) = b_0 \mathrm{P}_n(x)$ to $y_2(x)$, which gives nothing new.)

If the order n increases, the number of nonzero terms in the inhomogeneous part (which is a polynomial of degree n) will increase as well, making it unlikely that there is a simple formula for b_n in general. For this reason we will consider only the cases $n = 0$ and $n = 1$.

It should be noted here that Frobenius' power series method doesn't provide a convenient way of solving the general Legendre equation completely. A 2nd fundamental solution of the Legendre equation on $(-\infty, -1)$ or $(1, +\infty)$ can be more easily found by other methods; cf. the subsequent remarks.

Example (cont'd)

We consider only the cases $n = 0$ and $n = 1$.

$n = 0$

Since $\mathrm{P}_0(x) = 1$, we obtain $b_1 = -1/2$ and the recurrence relation

$$b_{k+1} = \frac{(0-k)(0+k+1)}{2(k+1)^2}\, b_k = -\frac{k}{2(k+1)}\, b_k.$$

The solution is $b_k = \frac{(-1)^k}{k2^k}$ (obvious from $(k+1)b_{k+1} = -\frac{1}{2}\, kb_k$).

$$\begin{aligned}\Longrightarrow y_2(x) &= \ln(x-1) - \frac{x-1}{2} + \frac{1}{2}\frac{(x-1)^2}{2^2} - \frac{1}{3}\frac{(x-1)^3}{2^3} \pm \dots \\ &= \ln(x-1) - \ln\left(1 + \frac{x-1}{2}\right) = \ln(x-1) - \ln\left(\frac{x+1}{2}\right) \\ &= \ln\frac{x-1}{x+1} + \ln 2 \qquad \text{for } 1 < x < 3.\end{aligned}$$

An equivalent choice is

$$\mathrm{Q}_0(x) = \tfrac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = \tfrac{1}{2}\ln\frac{x+1}{x-1} = -\tfrac{1}{2}\big(y_2(x) - \ln 2\big) \quad \text{for } 1 < x < 3.$$

Example (cont'd)

The function $\mathrm{Q}_0(x) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ is defined on $\mathbb{R} \setminus \{\pm 1\}$ and solves the Legendre equation with $n = 0$, viz. $(1 - x^2)y'' - 2xy' = 0$, on the three intervals into which $\mathbb{R} \setminus \{\pm 1\}$ decomposes. On the middle interval $(-1, 1)$ it is characterized as the solution satisfying the initial conditions $\mathrm{Q}_0(0) = 0$, $\mathrm{Q}_0'(0) = 1$.

Moreover, $\mathrm{Q}_0(x)$ coincides on $(-1, 1)$ with the non-polynomial series solution obtained earlier (and also with $\mathtt{tanh}^{-1}$).

In fact one easily verifies

$$\mathrm{Q}_0(x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots,$$

$$\mathrm{Q}_0'(x) = 1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1 - x^2}$$

for $|x| < 1$. The identity $\mathrm{Q}_0'(x) = \frac{1}{1-x^2}$ holds for all $x \in \mathbb{R} \setminus \{\pm 1\}$.

Finally let us note that $(1 - x^2)y'' - 2xy' = 0$ is 1st-order linear in y' and hence can be solved by the standard method $y'(x) = \exp\left(\int \frac{2x}{1-x^2}\,\mathrm{d}x\right)$ and one further integration.

Example (cont'd)

$n = 1$

Since $\mathrm{P}_1(x) = x$, the condition for y_2 takes the form

$$L\big[y_2(x)\big] = -3t - 5 + \sum_{k=0}^{\infty}\Big[\big(2 - k(k+1)\big)b_k - 2(k+1)^2 b_{k+1}\Big]t^k = 0.$$

$$\Longrightarrow b_1 = -\frac{5}{2},\ b_2 = -\frac{3}{8},\quad b_{k+1} = -\frac{(k-1)(k+2)}{2(k+1)^2}\,b_k \quad \text{for } k \geq 2$$

The solution is $b_k = \frac{(-1)^{k-1}(k+1)}{k(k-1)2^k}$ for $k \geq 2$, as can be seen by writing the recurrence relation in the form $\frac{b_{k+1}}{k+2} = -\frac{k-1}{2(k+1)}\,\frac{b_k}{k+1}$ or,equivalently, $\frac{b_k}{k+1} = -\frac{k-2}{2k}\,\frac{b_{k-1}}{k}$.

$$\Longrightarrow y_2(x) = x\ln(x-1) - 5\,\frac{x-1}{2} + \sum_{k=2}^{\infty}\frac{k+1}{k(k-1)}\,\frac{(x-1)^k}{2^k},$$

valid for $1 < x < 3$. Replacing $\ln(x-1)$ by $\ln|x-1|$, we can extend the range to $-1 < x < 3$, $x \neq 1$.

Example (cont'd)

With some effort one can derive from this that another equivalent choice is

$$\mathrm{Q}_1(x) = \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right| - 1 = -\tfrac{1}{2}y_2(x) - x.$$

The function $\mathrm{Q}_1(x)$ is defined on $\mathbb{R}\setminus\{\pm 1\}$ and solves the Legendre equation for $n = 1$, viz. $(1-x^2)y'' - 2xy' + 2y = 0$, on all three subintervals. On the middle interval $(-1, 1)$ it is the solution characterized by $\mathrm{Q}_1(0) = -1$, $\mathrm{Q}_1'(0) = 0$.

Like $\mathrm{Q}_0(x)$, the solution $\mathrm{Q}_1(x)$ can be found with less effort by other means, for example by using the method of order reduction for linear 2nd-order ODE's (see our earlier example in `lecture27-28`), or by inspecting the non-polynomial series solution obtained earlier for $x \in (-1, 1)$, rewriting it in terms of ln, and extending it to $\mathbb{R}\setminus\{\pm 1\}$.

Remark

The Legendre polynomials (or "Legendre P-functions") are determined by $\mathrm{P}_0(x) = 1$, $\mathrm{P}_1(x) = x$, and the recurrence relation

$$\mathrm{P}_{n+1}(x) = \frac{2n+1}{n+1}\,x\,\mathrm{P}_n(x) - \frac{n}{n+1}\,\mathrm{P}_{n-1}(x), \quad n = 1, 2, 3, \dots.$$

The *Legendre Q-functions* $\mathrm{Q}_n(x)$ are defined for $x \in \mathbb{R} \setminus \{\pm 1\}$ by $\mathrm{Q}_0(x) = \frac{1}{2} \ln \left|\frac{1+x}{1-x}\right|$, $\mathrm{Q}_1(x) = \frac{x}{2} \ln \left|\frac{1+x}{1-x}\right| - 1$ and the same recurrence relation

$$\mathrm{Q}_{n+1}(x) = \frac{2n+1}{n+1}\,x\,\mathrm{Q}_n(x) - \frac{n}{n+1}\,\mathrm{Q}_{n-1}(x), \quad n = 1, 2, 3, \dots.$$

One can show that for each $n \in \mathbb{N}$ the functions P_n and Q_n form a fundamental system of solutions of Legendre's equation $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$ on $(-\infty, -1)$, $(-1, 1)$, and $(1, +\infty)$.

Exercise

a) Determine $\mathrm{P}_n(0)$ for $n \in \mathbb{N}$.

b) Use a) to derive a binomial coefficient identity along the lines of the previous example.