# Differential Equations Plus (Math 286)

**H6** Solve the initial value problem $y' + 4y = 8e^{-4t} + 20,\ y(0) = 0$ and determine $y_\infty = \lim_{t\to\infty} y(t)$ for the solution.

**H7** Solve $y' - 2y = e^{ct},\ y(0) = 1$ and graph the solution for

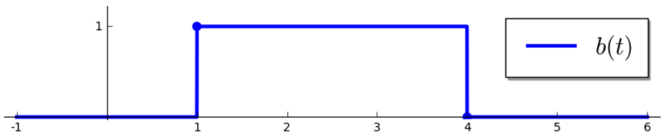
a) $c = 2$; b) $c = 2.01$.

What do you observe?

**H8** The Heaviside function $\mathrm{u}\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\mathrm{u}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Express $b(t)$ (cf. picture) in terms of $\mathrm{u}(t)$, solve the initial value problem $y' + 2y = b(t),\ y(0) = 0$, and determine $y_\infty$ (cf. H6).




**H9** a) Write the following complex numbers in polar form:

(i) $\sqrt{3}\,i + 1$; (ii) $\sqrt{3}\,i - 1$; (iii) $i - \sqrt{3}$.

b) Determine the general solution of the following ODE's:

(i) $y' + y = \cos\left(\sqrt{3}\,t\right)$; (ii) $y' - y = \cos\left(\sqrt{3}\,t\right)$;

(iii) $y' - \sqrt{3}\,y = \cos t + \sin t$.

c) Suppose $A\colon I \to \mathbb{C},\ t \mapsto A_1(t) + i\,A_2(t)$ is differentiable (i.e., $A_1 = \operatorname{Re} A$ and $A_2 = \operatorname{Im} A$ are differentiable). Show that $I \to \mathbb{C},\ t \mapsto e^{A(t)}$ is differentiable as well, and

$$\frac{d}{dt} e^{A(t)} = A'(t) e^{A(t)}.$$

*Hint:* Start with $e^{A(t)} = e^{A_1(t) + i\,A_2(t)} = e^{A_1(t)} e^{i\,A_2(t)} = e^{A_1(t)} \cos A_2(t) + i\,e^{A_1(t)} \sin A_2(t)$.

**H10** a) Show that in the 3rd model $mv' = mg - kv^2$ for a falling object released at height $s_0$ the terminal velocity $v_T$ of the object at time of impact is given by

$$v_T = \sqrt{\frac{mg}{k}} \cdot \sqrt{1 - e^{-2ks_0/m}}.$$

*Hint:* Consider the velocity as a function $v(s)$ of the distance $s$ traveled. Show that $y(s) = v(s)^2$ satisfies the ODE $my' = 2mg - 2ky$.

b) The limiting velocity of a falling basketball ($m = 620\,\mathrm{g}$) has been estimated at $20\,\mathrm{m/s}$. Using this data, graph $v_T$ as a function of $s_0$. For which heights $s_0$ does the basketball reach $50\,\%$, $90\,\%$, and $99\,\%$ of its limiting velocity?

**H11** a) Let $f_\lambda(t) = \mathrm{e}^{\lambda t}$ for $\lambda \in \mathbb{R}$. Show that $\{f_\lambda; \lambda \in \mathbb{R}\}$ is linearly independent in $\mathbb{R}^\mathbb{R}$.

*Hint:* Suppose there exists $r \in \mathbb{Z}^+$ and distinct numbers $\lambda_1, \dots, \lambda_r, c_1, \dots, c_r \in \mathbb{R}$ such that

$$c_1 \mathrm{e}^{\lambda_1 t} + c_2 \mathrm{e}^{\lambda_2 t} + \cdots + c_r \mathrm{e}^{\lambda_r t} = 0 \quad \text{for all } t \in \mathbb{R}. \tag{$\star$}$$

Assuming $\lambda_1 < \lambda_2 < \cdots < \lambda_r$ and $c_r \neq 0$, divide this equation by $\mathrm{e}^{\lambda_r t}$ and let $t \to +\infty$ to obtain a contradiction.

b) For $\lambda \in \mathbb{C}$ the functions $f_\lambda(t) = \mathrm{e}^{\lambda t}$ belong to the vector space $\mathbb{C}^\mathbb{R}$ of all complex-valued functions on $\mathbb{R}$ (with scalar multiplication by complex numbers). Show that $\{f_\lambda; \lambda \in \mathbb{C}\}$ is linearly independent in $\mathbb{C}^\mathbb{R}$.

*Hint:* The proof outlined in a) breaks down in the complex case. Instead differentiate the identity in $(\star)$ $j$ times, $0 \leq j < r$, and set $t = 0$.

c) Let $c_\lambda(t) = \cos(\lambda t)$, $s_\lambda(t) = \sin(\lambda t)$. Show that $\{c_\lambda; \lambda \in \mathbb{R}, \lambda \geq 0\} \cup \{s_\lambda; \lambda \in \mathbb{R}, \lambda > 0\}$ is linearly independent in $\mathbb{R}^\mathbb{R}$.

## Due on Fri Oct 8, 6 pm

Exercises H11 b) and H11 c) are optional.