Math 286
Introduction to Differential Equations

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Outline

Today's Lecture: Uniform Convergence

The concept of uniform convergence arises from the question whether the limit function of a sequence or series of functions inherits properties like continuity or differentiability from the terms of the sequence/series. For ordinary (point-wise) convergence the answer is notably false.

Uniform convergence was not discussed in Calculus I/II/III, but is needed to understand the existence theorem for solutions of ODE's, the theory of Fourier series, and many other important topics in Real Analysis.

As background reference for the material on uniform convergence I recommend the respective chapters in

[Bre07] David Bressoud, *A Radical Approach to Real Analysis*, 2nd edition, Mathematical Association of America 2007;

[Ru76] Walter Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill 1976.

[Bre07] is very accessible and retraces the historical development of Calculus in the 19th century and the mathematicians involved in it. [Ru76] is a good reference also for other important concepts not covered by Stewart's book (e.g., the Implicit Function Theorem), but be warned that it is pretty advanced.

The following 3 slides show plots of 3 sequences of functions to be defined and discussed later. These function sequences converge point-wise but not uniformly, and serve as counterexamples to the naive belief, prevalent until the beginning of the 19th century, that point-wise limits of sequences of continuous/differentiable/integrable functions inherit the respective property.

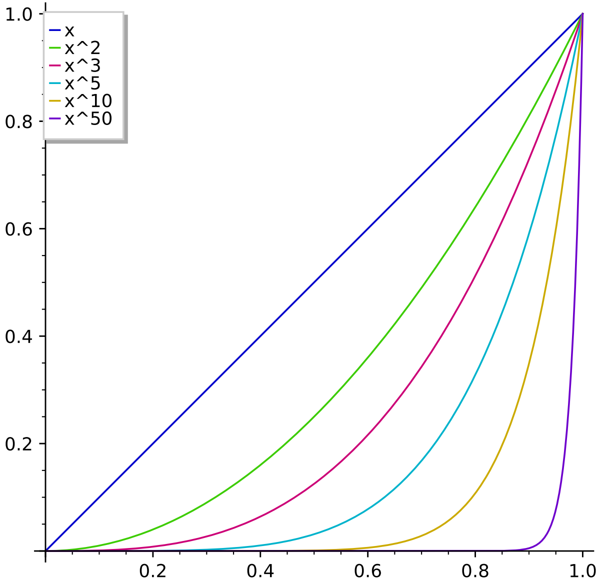


Figure: $f_n(x) = x^n$ for $n = 1, 2, 3, 5, 10, 50$

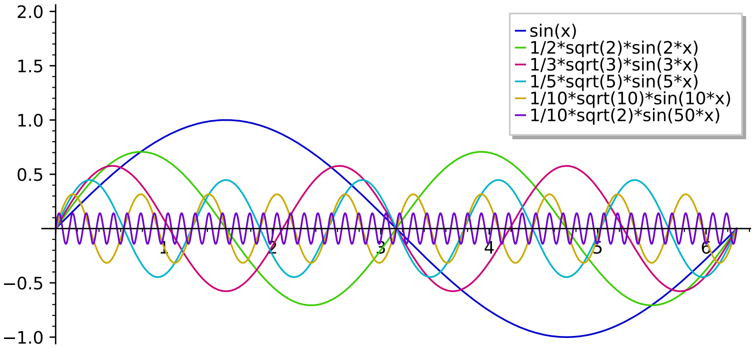


Figure: $g_n(x) = \frac{\sin(nx)}{\sqrt{n}}$ for $n = 1, 2, 3, 5, 10, 50$

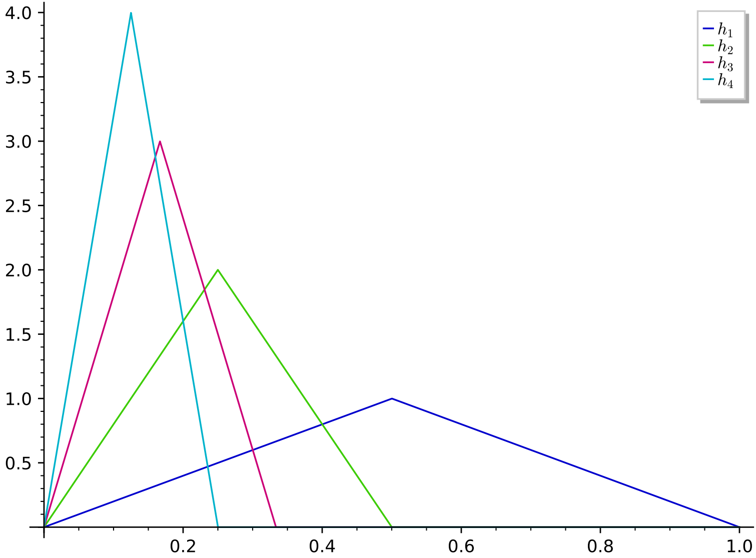


Figure: $h_n(x)$ for $n = 1, 2, 3, 4$

Point-wise vs Uniform Convergence

Definition

Let $I \subseteq \mathbb{R}$ be an interval and $(f_n)_{n=0}^{\infty}$ a sequence of functions $f_n \colon I \to \mathbb{R}$.

1. (f_n) *converges point-wise* (on I) if for every $x \in I$ the sequence $\bigl(f_n(x)\bigr)$, an ordinary sequence of real numbers, converges. If this is the case then $f(x) = \lim_{n\to\infty} f_n(x)$ defines a function $f \colon I \to \mathbb{R}$, called "limit function" or "point-wise limit" of the sequence (f_n).
2. (f_n) *converges uniformly* (on I) if it converges point-wise and the limit function $f \colon I \to \mathbb{R}$ has the following property: For every $\epsilon > 0$ there is a "uniform" response $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon$ for all $n > N$ and all $x \in I$.

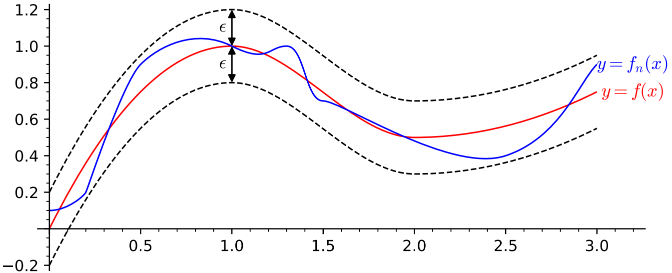
If (1), resp., (2) hold, we also say that (f_n) converges to f point-wise, resp., uniformly.

Notes

- Uniform convergence requires the response $N = N_\epsilon$ to be independent of $x \in I$, while point-wise convergence allows $N = N_{\epsilon,x}$ to depend on x (and ϵ).

Notes cont'd

- Geometrically speaking, (f_n) converges uniformly to f iff for every $\epsilon > 0$ all except finitely many of the graphs of f_n are contained in the strip of vertical width 2ϵ around the graph of f; see picture.



Looking at the preceding plots, you can see that the sequences (f_n) and (h_n) fail to have this property for any $\epsilon < 1$, while (g_n) seems to have it. (At least we can see that the graph of g_{50} is within 0.2 of the graph of the point-wise limit, viz., the x-axis.)

Notes cont'd

- The definition generalizes to functions $f_n\colon X \to \mathbb{R}$ with arbitrary domain X. Further we can replace the codomain $\mathbb{R}$ by $\mathbb{R}^k$, because the concept of convergence for the corresponding vectorial sequences $f_0(x), f_1(x), f_2(x), \dots$ is well-defined (cf. Calculus III) and $|f(x) - f_n(x)| < \epsilon$ can be read as an inequality for the Euclidean length of the vector $f(x) - f_n(x) \in \mathbb{R}^k$. Even more generally, we can take the codomain of f_n as any set M with a distance function $d\colon M \times M \to \mathbb{R}$ (replacing, e.g., "$|f(x) - f_n(x)| < \epsilon$" by "$d\bigl(f(x), f_n(x)\bigr) < \epsilon$"), i.e., by a (generalized) metric space (M, d).
- In the definition of convergence it does not matter whether $<$ or $\leq$ is used. Using the latter has the advantage that the condition "$|f(x) - f_n(x)| \leq \epsilon$ for all $x \in I$" can be succintly stated as $\sup\bigl\{|f(x) - f_n(x)|\,; x \in I\bigr\} \leq \epsilon$. We can view the left-hand side of this inequality as a measure for the distance between the functions f and f_n. More precisely, if we define $\mathrm{d}_\infty(f, g) = \sup\bigl\{|f(x) - g(x)|\,; x \in I\bigr\}$ for $f, g \in \mathbb{R}^I$ (referred to as *metric of uniform convergence* or L^∞*-metric*) then uniform convergence amounts to ordinary convergence in the generalized metric space $(\mathbb{R}^I, \mathrm{d}_\infty)$.

Questions

1. *Is the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ of a point-wise/uniformly convergent sequence of continuous functions itself continuous?*

   A suggestive reformulation of this property is obtained by recalling that a function g is continuous at x iff $x_k \to x$ implies $g(x_k) \to g(x)$. Applying this to f and f_n above gives that f is continuous iff

$$\lim_{k\to\infty} \lim_{n\to\infty} f_n(x_k) = \lim_{k\to\infty} f(x_k) = f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \lim_{k\to\infty} f_n(x_k).$$

2. *How about the related problem of interchanging limits with differentiation? Under which conditions does $f'(x) = (\lim f_n)'(x) = \lim f_n'(x)$ hold?*

3. *How about integration in this regard?*

Three Counterexamples

The following examples show that point-wise convergence is not sufficient for any of the three properties.

Example (continuity)

Consider the sequence of functions $f_n(x) = x^n$, $x \in [0,1]$. The functions f_n are continuous and converge point-wise to

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1, \end{cases}$$

but f has a discontinuity at $x = 1$.

Example (differentiation)

Consider the sequence of functions $g_n(x) = \frac{\sin(nx)}{\sqrt{n}}$, $x \in [0, 2\pi]$. We have $g_n(x) \to g(x) \equiv 0$ (the all-zero function on $[0, 2\pi]$) point-wise (even uniformly!), and g is differentiable with $g'(x) \equiv 0$. But

$$g_n'(x) = \sqrt{n}\cos(nx), \quad x \in [0, 2\pi],$$

and $\lim_{n\to\infty} g_n'(x)$ doesn't exist for $0 < x < 2\pi$.

Example (integration)

Consider the sequence of functions $h_n\colon [0,1] \to \mathbb{R}$ defined by

$$h_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \leq x \leq 1/2n, \\ 2n - 2n^2 x & \text{if } 1/2n \leq x \leq 1/n, \\ 0 & \text{if } 1/n \leq x \leq 1. \end{cases}$$

The graph of h_n and the x-axis determine an (isosceles) triangle with vertices $(0,0)$, $(1/2n, n)$, $(1/n, 0)$, and h_n vanishes on $[1/n, 1]$.

It follows that $h_n(x) \to h(x) \equiv 0$ (the all-zero function on $[0,1]$) point-wise, and that the area under the graph of h_n is $1/2 \cdot 1/n \cdot n = 1/2$ for all n. This gives

$$\lim_{n\to\infty} \int_0^1 h_n(x)\,\mathrm{d}x = \frac{1}{2} \neq 0 = \int_0^1 \lim_{n\to\infty} h_n(x)\,\mathrm{d}x\,.$$

Three Theorems

The purpose of introducing the concept of uniform convergence is to prevent such “counterexamples”. The answer to all three questions will be positive, provided we require the sequence of functions (f_n) and/or the sequence of its derivatives (f_n') to be uniformly convergent.

Theorem (continuity)

If all functions f_n are continuous at $x_0 \in I$ and (f_n) converges uniformly on I then $f(x) = \lim f_n(x)$, $x \in I$, is continuous at x_0 as well. In particular, the limit function of a uniformly convergent sequence of continuous functions is itself continuous.

Proof.

Let $\epsilon > 0$ be given. Then there exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < \epsilon/3$ for $n > N$ and $x \in I$. Further, since f_{N+1} is continuous at x_0, there exists $\delta > 0$ such that $|f_{N+1}(x) - f_{N+1}(x_0)| < \epsilon/3$ for $x \in I$ with $|x - x_0| < \delta$. For such x we then have

$$\begin{aligned}|f(x) - f(x_0)| &\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(x_0)| + |f_{N+1}(x_0) - f(x_0)| \\ &< \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} + \tfrac{\epsilon}{3} = \epsilon.\end{aligned}$$

$\square$

Theorem (differentiation)

If all functions f_n are C^1-functions, (f_n) converges point-wise on I, and (f_n') converges uniformly on I, then $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in I$, is a C^1-function as well and satisfies $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Proof.

Choosing an arbitary point $a \in I$, the Fundamental Theorem of Calculus gives

$$f_n(x) = f_n(a) + \int_a^x f_n'(t)\,\mathrm{d}t \quad \text{for } x \in I.$$

Since (f_n') converges uniformly to $g\colon I \to \mathbb{R}$, say, we can find an $N \in \mathbb{N}$ such that $|f_n'(t) - g(t)| < 1$ for all $n > N$ and $t \in I$.
By the preceding theorem, g is continuous, and the inequality implies

$$|f_n'(t)| \leq 1 + |g(t)| \quad \text{for } n > N \text{ and } t \in I.$$

Thus $\Phi(t) = 1 + |g(t)|$ is an integrable bound for $(f_n')_{n>N}$ on $[a,x]$, and we can apply Lebesgue's Bounded Convergence Theorem to conclude that $\lim_{n\to\infty} \int_a^x f_n'(t)\,\mathrm{d}t = \int_a^x g(t)\,\mathrm{d}t$. □

Proof cont'd.

Hence, letting $n \to \infty$ in the first identity we obtain

$$f(x) = f(a) + \int_a^x g(t)\,\mathrm{d}t \quad \text{for } x \in I.$$

Finally, applying the Fundamental Theorem of Calculus a second time gives that f is differentiable with $f'(x) = g(x) = \lim f_n'(x)$. □

Notes

1. The proof also shows that

$$f(x) - f_n(x) = f(a) - f_n(a) + \int_a^x [g(t) - f_n'(t)]\,\mathrm{d}t\,.$$

Since $f_n' \to g$ uniformly, given $\epsilon > 0$, we can find a response N such that

$$|f(x) - f_n(x)| \leq \epsilon\,(1 + |x - a|) \quad \text{for all } n > N \text{ and } x \in I.$$

This shows that (f_n) converges not only point-wise to f but uniformly on every bounded subinterval of I. (If I is unbounded, however, we don't get uniform convergence of (f_n) on I, as the example $I = \mathbb{R}$, $f_n(x) = x/n$ shows.)

Notes cont'd

2. The key assumption in the Differentiation Theorem is that the *sequence of derivatives* (f_n') converges uniformly (and not, as one might think in the first place, the sequence (f_n)). For (f_n) the weaker assumption of point-wise convergence is enough. (In fact it would even be sufficient to require only that $\big(f_n(a)\big)$ converges.) But at least some assumption on (f_n) is clearly necessary, because we can add arbitrary constants to f_n without affecting f_n'.

3. One can use a variant of this theorem to prove that analytic functions of a complex variable, i.e., functions $f\colon D \to \mathbb{C}$ defined on some open disk $D = \mathrm{B}_R(a) \subseteq \mathbb{C}$ ($a \in \mathbb{C}$, $R > 0$) by a convergent power series $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$, are holomorphic. For this the following two key observations are needed: (1) Power series converge uniformly on any closed disk $\overline{\mathrm{B}_{R'}(a)}$, $R' < R$, where R denotes the radius of convergence. (2) The series $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$ of derivatives has the same radius of convergence as the original series and hence converges uniformly on $\overline{\mathrm{B}_{R'}(a)}$ as well.

Theorem (integration)

If I is a bounded interval, all functions f_n are (Lebesgue) integrable over I and (f_n) converges uniformly on I then the limit function $f(x) = \lim_{n\to\infty} f_n(x)$ is integrable as well, and we have $\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x$.

Proof.

This follows by using Lebesgue's Theorem in a similar way as in the preceding proof:

There exists $N \in \mathbb{N}$ such that $|f(x) - f_n(x)| < 1$ for all $n > N$ and $x \in I$.

$\Longrightarrow |f_n(x) - f_{N+1}(x)| < 2$ for $n > N$ and $x \in I$.

$\Longrightarrow$ An integrable bound for $(f_n)_{n>N}$ is $\Phi(x) := |f_{N+1}(x)| + 2$. $\square$

Note on the proof

If you wonder where the assumption "I is bounded" is needed in the proof: It is hidden in the definition of $\Phi(x)$: The function f_{N+1} (and hence $|f_{N+1}|$) is integrable by assumption, but the constant function 2 is integrable only if I is bounded. Therefore, Φ is integrable only if I is bounded.

Notes on the Integration Theorem

1. For a sequence of continuous functions on a compact intervall $I = [a, b]$ (or any other sequence of functions for which it is known in advance that the limit function is integrable) we can alternatively argue as follows:

$$\begin{aligned}\left|\int_a^b f(x)\,\mathrm{d}x - \int_a^b f_n(x)\,\mathrm{d}x\right| &= \left|\int_a^b f(x) - f_n(x)\,\mathrm{d}x\right| \\ &\leq \int_a^b |f(x) - f_n(x)|\,\mathrm{d}x \\ &\leq (b-a)\sup\bigl\{|f(x) - f_n(x)|\,;\,a \leq x \leq b\bigr\}\end{aligned}$$

   Hence, if $f_n \to f$ uniformly on $[a, b]$ then $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$.

2. The assumption that I is bounded is essential. Without this assumption, the conclusion generally fails to hold. For example, define $f_n \colon \mathbb{R} \to \mathbb{R}$ by $f_n(x) = 1/n$ if $0 \leq x \leq n$ and $f_n(x) = 0$ otherwise. Then $f_n \to 0$ uniformly, but $\int_{\mathbb{R}} f_n(x)\,\mathrm{d}x = 1 \not\to 0 = \int_{\mathbb{R}} 0\,\mathrm{d}x$.

Notes cont'd

3. Using the integration theorem, we can give a simpler proof of the differentiation theorem, which avoids reference to the rather deep theory of Lebesgue integration:

   In the previous proof the key step is the implication

$$
\begin{aligned}
& f_n(x) = f_n(a) + \int_a^x f_n'(t)\,\mathrm{d}t \quad (n \in \mathbb{N},\ x \in I) \\
\Longrightarrow\ & f(x) = f(a) + \int_a^x g(t)\,\mathrm{d}t \quad (x \in I),
\end{aligned}
$$

   where g denotes the uniform limit of the sequence of derivatives (f_n').

   Since $f_n \to f$ point-wise, we have $f_n(x) \to f(x)$ and $f_n(a) \to f(a)$. Since $I = [a, x]$ is bounded, f_n' is integrable over I (since it is continuous), and $f_n' \to g$ uniformly, we can apply the integration theorem to conclude $\int_a^x f_n'(t)\,\mathrm{d}t \to \int_a^x g(t)\,\mathrm{d}t$. This provides an alternative proof of the key step.

   In fact the special case of the integration theorem considered in Note 1, valid also for the Riemann integral, is sufficient.

Weierstrass's Criterion

A handy test for the uniform convergence of function series

Theorem (Weierstrass's Criterion)

Suppose $f_n\colon D \to \mathbb{R}$ $(n = 0, 1, 2, \dots)$, are functions with common domain D and there exist "uniform" bounds $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all $n \in \mathbb{R}$ and $x \in D$. If the series $\sum_{n=0}^{\infty} M_n$ converges in $\mathbb{R}$ (i.e., $\sum_{n=0}^{\infty} M_n < \infty$) then the function series $\sum_{n=0}^{\infty} f_n$ converges uniformly.

Proof.

First we show that $\sum_{n=0}^{\infty} f_n$ converges point-wise.
Fix $x \in D$. Since $\sum_{n=0}^{\infty} M_n$ is convergent and $|f_n(x)| \leq M_n$, the comparison test yields that $\sum_{n=0}^{\infty} f_n(x)$ is absolutely convergent and hence convergent.
Thus $\sum_{n=0}^{\infty} f_n$ converges point-wise and has a limit function $F\colon D \to \mathbb{R}$, $x \mapsto \sum_{n=0}^{\infty} f_n(x)$.

That the convergence is uniform is shown on the next slide. First recall that $\sum_{n=0}^{\infty} f_n$ refers to the sequence of partial sums $F_n = \sum_{k=0}^{n} f_k$, i.e., $F_n\colon D \to \mathbb{R}$, $x \mapsto \sum_{k=0}^{n} f_k(x)$.

Proof cont'd.

We estimate as follows:

$$|F(x) - F_n(x)| = \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sum_{k=n+1}^{\infty} |f_k(x)| \leq \sum_{k=n+1}^{\infty} M_k$$

Since $\sum_{n=0}^{\infty} M_n$ converges, we can find, for every $\epsilon > 0$, an index N such that $\sum_{k=N+1}^{\infty} M_k < \epsilon$. Using the above estimate and $M_k \geq 0$ then shows $|F(x) - F_n(x)| < \epsilon$ for all $n \geq N$ and $x \in D$. This completes the proof. □

Application to trigonometric series

The function series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converge uniformly on $\mathbb{R}$ (and hence represent continuous functions of x with domain $\mathbb{R}$).

To prove this, e.g., for the first series, use the estimate $\left| \frac{\cos(nx)}{n^2} \right| \leq \frac{1}{n^2}$. Since the series $\sum_{n=1}^{\infty} 1/n^2$ is convergent, Weierstrass's Criterion can be applied with $M_n = 1/n^2$.

Application to Power Series

A (complex) power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ with radius of convergence $R>0$ (including the possibility $R=\infty$) represents a differentiable (holomorphic) function $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$ on the open disk $\mathrm{B}_R(a)=\{z\in\mathbb{C};|z-a|<R\}$ (respectively, on $\mathbb{C}$ if $R=\infty$) and can be differentiated term-wise:

$$f'(z)=\sum_{n=1}^{\infty} na_n(z-a)^{n-1}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(z-a)^n.$$

Moreover, the radius of convergence of the derived series is again R.
$\Longrightarrow$ We can iterate the argument, showing that f has derivatives of all orders explicitly given by

$$\begin{aligned}f^{(k)}(z)&=\sum\nolimits_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(z-a)^{n-k}\\&=\sum\nolimits_{n=0}^{\infty}(n+1)(n+2)\cdots(n+k)a_{n+k}(z-a)^n,\quad k\in\mathbb{N}.\end{aligned}$$

These facts are proved on the next slides. The key step is to show that power series converge uniformly on all strictly smaller disks $\mathrm{B}_{R'}(a)$, $R'<R$ (but not necessarily on $\mathrm{B}_R(a)$).

Power Series cont'd

For the proof of the key step we choose $z_1 = a + (R' + R)/2$, so that $R' < |z_1 - a| < R$. (In the case $R = \infty$, in which R' may be any positive radius, we can take $z_1 = a + 2R'$.)

For $z \in \mathrm{B}_{R'}(a)$ (in fact $|z - a| \leq R'$ suffices) we then have

$$\begin{aligned} |a_n(z-a)^n| &= |a_n(z_1-a)^n| \left|\frac{z-a}{z_1-a}\right|^n \leq |a_n(z_1-a)^n| \left(\frac{2R'}{R'+R}\right)^n \\ &= |a_n(z_1-a)^n|\,\theta^n \end{aligned}$$

with $\theta := \frac{2R'}{R'+R} < 1$.

Since $|z_1 - a| < R$, the series $\sum_{n=0}^{\infty} a_n(z_1 - a)^n$ converges. Hence we have $|a_n(z_1 - a)^n| \leq M$ for some constant M and $|a_n(z - a)^n| \leq M\theta^n$ on $\mathrm{B}_{R'}(a)$. Since $\sum_{n=0}^{\infty} M\theta^n = \frac{M}{1-\theta}$ converges, we can apply Weierstrass's Criterion to conclude that $\sum_{n=0}^{\infty} a_n(z - a)^n$ converges uniformly on $\mathrm{B}_{R'}(a)$.

Note

If a power series $\sum_{n=0}^{\infty} a_n(z - a)^n$ converges for some $z_1 \in \mathbb{C}$, it necessarily converges for all $z \in \mathbb{C}$ with $|z - a| < |z_1 - a|$ (i.e., in the open disk with center a and z_1 on its boundary). For the proof we can use the same estimate as above with $\theta := \frac{|z-a|}{|z_1-a|}$.

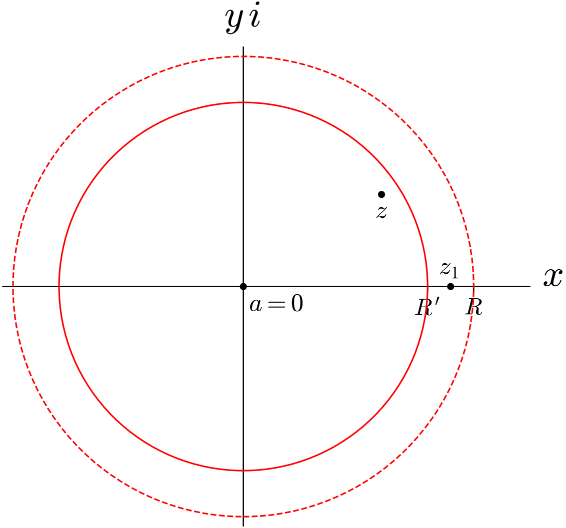


Figure: The geometry behind the proof of the key step (assuming $a=0$): If $|z-a|\leq R'$ then $\frac{|z-a|}{|z_1-a|}\leq\frac{R'}{(R+R')/2}=\frac{2R'}{R+R'}<1$.

Note cont'd

This observation implies that the number

$$R := \sup\{r \in \mathbb{R}; \sum_{n=0}^{\infty} a_n r^n \text{ converges in } \mathbb{C}\}$$

has the property that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges for $|z-a| < R$ and diverges for $|z-a| > R$. (For, if $|z-a| < R$ then there exists $r > |z-a|$ for which $\sum_{n=0}^{\infty} a_n r^n$ converges, and hence the observation with $z_1 := a + r$ yields that $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges. Similarly, if the power series would converge for some z_1 with $|z_1 - a| > R$, it would necessarily converge for all $z = a + r$ with $R < r < |z_1 - a|$, contradicting the definition of R.)

Thus we have proved that a complex power series has a *radius of convergence* in the first place; cf. our Calculus textbook [Ste16], Theorem 11.8.4. (The proof of Th. 11.8.4 given in Appendix F generalizes to complex numbers x and is essentially the same as our argument.)

Power Series cont'd

The radius of convergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$ is given by the CAUCHY-HADAMARD formula

$$R = \frac{1}{L}, \quad \text{where} \quad L = \limsup \sqrt[n]{|a_n|}. \tag{CH}$$

Here "lim sup" (limit superior) refers to the largest accumulation point of a sequence (including the possibilities $\pm\infty$ for sequences which are unbounded from above/below) and coincides with the ordinary limit if the limit exists. It is necessary to use "lim sup", because for lacunary power series (power series with "gaps") such as $\sum_{k=0}^{\infty} z^{2k}$ or $\sum_{k=1}^{\infty} z^{k^2}$ the ordinary limit $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ doesn't exist, but the limit superior is 1 and gives the correct value $R = 1$. For a proof of (CH) see HW3, Ex. H19, and for the said special case see also [Ste16], Ch. 11.8, Ex. 39.

Since $\sqrt[n]{n} \to 1$ for $n \to \infty$, its follows that a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ and its derived series $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$ have the same radius of convergence, as asserted earlier. Hence the derived series converges uniformly for $|z-a| \leq R' < R$ as well.

Power Series cont'd

A different formula for the radius of convergence can be obtained from the ratio test for ordinary series: $R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$, provided that this limit exists; cf. [Ste16], Ch. 11.8, Ex. 40. This formula, often called *ratio test for power series*, also fails for lacunary power series.

Sometimes one can apply the ordinary ratio test to the series obtained by omitting the gaps. For example, in the case of $\sum_{k=1}^{\infty} z^{k^2}$ we can set $b_k = z^{k^2}$ and obtain

$$\frac{|b_{k+1}|}{|b_k|} = |z|^{(k+1)^2-k^2} = |z|^{2k+1} \to \begin{cases} 0 & \text{if } |z| < 1, \\ \infty & \text{if } |z| > 1, \end{cases}$$

showing together with the ratio test for ordinary series that $\sum_{k=1}^{\infty} z^{k^2}$ has radius of convergence $R = 1$.
But even in this modified form the ratio test is weaker than the Cauchy-Hadamard formula, since, e.g., it can't be applied to

$$z + 2\,z^2 + z^3 + 2\,z^4 + z^5 + 2\,z^6 + \cdots,$$

which clearly satisfies $\sqrt[n]{a_n} \to 1$ for $n \to \infty$ and hence has $R = 1$.

Power Series cont'd

The term-wise differentiabilty of complex power series can be proved by generalizing the Differentiation Theorem to complex derivatives. This requires the concept of complex line integrals and will be discussed on the next two slides. Here is a direct proof of this fact (w.l.o.g. we can assume $a = 0$):

$$\begin{aligned}\frac{f(z)-f(z_0)}{z-z_0} &= \sum_{n=0}^{\infty} a_n \frac{z^n - z_0^n}{z-z_0} \\ &= \sum_{n=1}^{\infty} a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1}) \\ &\to \sum_{n=1}^{\infty} n a_n z_0^{n-1} \quad \text{for } z \to z_0,\end{aligned}$$

provided we can interchange the two limits. This is precisely what the Continuity Theorem asserts. So we have to prove uniform convergence of the above series in some neighborhood of z_0, which can be done using the Weierstrass criterion and the estimate

$$\left| a_n(z^{n-1} + z^{n-2}z_0 + \cdots + z_0^{n-1} \right| \leq n\left|a_n\right| \left(\max\{|z|, |z_0|\}\right)^{n-1}.$$

Power Series cont'd

Given z_0 with $|z|_0 < R$, we set $R' := (R + |z_0|)/2$. Then $z_0 \in \mathrm{B}_{R'}(0)$, and for $z \in \mathrm{B}_{R'}(0)$ the series terms are bounded in absolute value by $M_n = n\,|a_n|\,(R')^{n-1}$. Since $R' < R$, the series $\sum_{n=1}^{\infty} M_n$ converges, and hence the above series converges uniformly on $\mathrm{B}_{R'}(0)$. Thus $\mathrm{B}_{R'}(0)$ provides the desired neighborhood of z_0, and the proof is complete.

Note

The series representing $\frac{f(z)-f(z_0)}{z-z_0}$ is not a power series, but like $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$ converges uniformly on every disk $\mathrm{B}_{R'}(0)$ with $R' < R$. Whereas uniform convergence of $\sum_{n=0}^{\infty} a_n z^n$ yields only the continuity of f and that of $\sum_{n=1}^{\infty} n a_n z^{n-1}$ requires reasoning beyond the ordinary Differentation Theorem to yield the differentiability of f (see below), the present argument yields both properties (recall that differentiable functions are automatically continuous) in the most economic way.

Power Series cont'd

Finally, we transfer our proof of the Differentiation Theorem to the present complex setting. Writing $f_n(z) = \sum_{k=0}^{n} a_k(z-a)^k$, we have

$$f_n(z) = f_n(a) + \int_a^z f_n'(w)\,\mathrm{d}w\,.$$

Here $\int_a^z f_n'(w)\,\mathrm{d}w$ is the (path-independent) complex line integral of the (closed) differential 1-form $f_n'(z)\,\mathrm{d}z$ from a to z, which can be computed using the straight line path $\gamma(t) = a + t(z-a)$, $t \in [0,1]$, as $\int_0^1 f_n'\bigl(\gamma(t)\bigr)\gamma'(t)\,\mathrm{d}t$.

From the preceding slide we know that (f_n') converges uniformly on $[a,z]$ (the line segment joining a zu z, which is contained in a suitable disk $\mathrm{B}_{R'}(a)$) to $g(z) = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}$. Together with the estimate

$$\begin{aligned}\left|\int_a^z f_n'(w)\,\mathrm{d}w - \int_a^z g(w)\,\mathrm{d}w\right| &= \left|\int_a^z f_n'(w) - g(w)\,\mathrm{d}w\right| \\ &\leq \left(\max_{w\in[a,z]} \left|f_n'(w) - g(w)\right|\right)|z-a|\end{aligned}$$

this shows $\lim_{n\to\infty} \int_a^z f_n'(w)\,\mathrm{d}w = \int_a^z g(w)\,\mathrm{d}w$ und further,

Power Series cont'd

letting $n \to \infty$ in the above identity,

$$f(z) = f(a) + \int_a^z g(w)\,\mathrm{d}w \quad \text{for } z \in \mathrm{B}_R(a).$$

Finally, as in the proof of a theorem in Calculus III ("independence of path of $\int_\gamma \omega$ implies exactness of ω") we obtain from this

$$\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) = \frac{1}{z - z_0} \int_{z_0}^z g(w) - g(z_0)\,\mathrm{d}w$$

for $z_0, z \in \mathrm{B}_R(a)$ with $z \neq z_0$, which for $z \to z_0$ tends to zero on account of the continuity of g; cp. the estimate for $\int_a^z f_n'(w) - g(w)\,\mathrm{d}w$ on the previous slide.

This shows that f is differentiable with $f' = g$.

Remarks

- The Diffentiation Theorem holds more generally for sequences of uniformly convergent holomorphic functions $f_n \colon D \to \mathbb{C}$, $D \subseteq \mathbb{C}$; it is even sufficient that every point $z \in D$ has a neighborhood on which the convergence is uniform.

  If you think we should rather have required uniform convergence of the sequence (f_n')—true, but surprisingly this is equivalent to uniform convergence of (f_n) in the complex case!

- The uniform convergence of power series on proper subdisks (with the same center) of their open disk $\mathrm{B}_R(a)$ of convergence also implies that power series may be integrated term-wise along any path γ contained in $\mathrm{B}_R(a)$. This follows from the analogue of the Integration Theorem for line integrals in the plane, which can be deduced from the Integration Theorem and (assuming the parameter intervall of γ is $[0,1]$) the explicit formula $\int_\gamma f(z)\,\mathrm{d}z = \int_0^1 f\bigl(\gamma(t)\bigr)\gamma'(t)\,\mathrm{d}t$. The factor $\gamma'(t)$ doesn't affect uniform convergence, since it is bounded.

  In particular this holds for ordinary integrals of real power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, $a_n, a \in \mathbb{R}$, over compact intervals $[\alpha, \beta]$ that are contained in $\mathrm{B}_R(a)$; cp. subsequent example.

Remarks (cont'd)

- As an interesting fact, note that the formula for the derivatives of a power series, viz. $f^{(k)}(z) = \sum_{n=0}^{\infty}(n+1)\dots(n+k)a_{n+k}(z-a)^n$, implies $f^{(k)}(a) = k!\,a_k$ for $k = 0, 1, 2, \dots$, i.e., a power series is its own Taylor series, and knowledge of f in an arbitrarily small neighborhood of a (and hence of its derivatives $f^{(k)}(a)$) determines the coefficients $a_k = f^{(k)}(a)/k!$ and hence f uniquely.
- Power series with coefficients $a_n \in \mathbb{R}$, center $a \in \mathbb{R}$ and radius of convergence $R > 0$ define an ordinary real function $f\colon (a-R, a+R) \to \mathbb{R}$, $x \mapsto \sum_{n=0}^{\infty} a_n(x-a)^n$. Since these are discussed in our Calculus textbook [Ste16], Ch. 11.8–11.10, I suppose you are at least familiar with this more restricted view of power series, which doesn't reveal some important aspects of the theory, though, for example why are we saying "radius of convergence"? For understanding Math 286 the restricted view will be mostly enough, because power series solutions of ODE's, to be discussed later, will only involve real power series. Holomorphic functions, complex differential forms $f(z)\,\mathrm{d}z$ and their properties, and complex line integrals $\int_\gamma f(z)\,\mathrm{d}z$ won't be needed in the sequel.

Examples

Example

In a mathematically rigorous development of Calculus the trigonometric functions sin, cos are defined by their power series expansions, which amounts to taking real and imaginary part in the expansion $\mathrm{e}^{\mathrm{i}x} = \sum_{n=0}^{\infty}(\mathrm{i}x)^n/n!$ (thus giving $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$):

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}, \quad \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1.}$$

Both series have radius of convergence $R = \infty$, and therefore can be differentiated (and integrated) term-wise for every x. This gives the known relations $\sin' = \cos$, $\cos' = -\sin$; e.g.,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\, x^{2k+1.} &= \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(-1)^k}{(2k+1)!}\, x^{2k+1.}\right) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)}{(2k+1)!}\, x^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k} = \cos x.\end{aligned}$$

Example (cont'd)

The preceding dicussion holds more generally for complex arguments (replace $x \in \mathbb{R}$ by $z \in \mathbb{C}$ throughout). In the complex world there is no need to distinhuish between trigonometric and hyperbolic functions:

$$\cos z = \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z}\right) = \cosh(\mathrm{i}z),$$
$$\sin z = \frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right) = -\mathrm{i}\sinh(\mathrm{i}z),$$

and hence sin, cos are obtained from sinh, cosh by 90° rotations in the domain/codomain. Thus, e.g., $\cos(\mathrm{i}y) = \cosh(-y) = \cosh y$, revealing that the complex cosine function on the imaginary axis looks like the real hyperbolic cosine, and $\sin(\mathrm{i}y) = \mathrm{i}\sinh y$, having a similar geometric interpretation.

Exercise

Using $\mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y$, show that the complex cosine and sine functions have the following explicit representation:

$$\cos z = \cos(x + \mathrm{i}y) = \cos x \cosh y - \mathrm{i} \sin x \sinh y,$$
$$\sin z = \sin(x + \mathrm{i}y) = \sin x \cosh y + \mathrm{i} \cos x \sinh y.$$

The next example is for the integration theorem, since this is the one most widely applicable (due to the fact that integration "smoothes" functions, while differentiation "roughens" them).

Example

The *sine integral* (function) is defined as

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,\mathrm{d}t = \int_0^x \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!}\,\mathrm{d}t \quad \text{for } x \in \mathbb{R}.$$

Since the power series defining $\sin x$ and $\sin(x)/x$ have radius of convergence ∞, the function series in the definition of $\mathrm{Si}(x)$ converges uniformly on every interval $[0, x]$ and hence can be integrated termwise:

$$\begin{aligned}\mathrm{Si}(x) &= \sum_{n=0}^{\infty} \int_0^x (-1)^n \frac{t^{2n}}{(2n+1)!}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left[\frac{t^{2n+1}}{2n+1}\right]_0^x \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!(2n+1)} = x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{35280} \pm \cdots .\end{aligned}$$

Of course, the sine series itself can also be integrated term-wise over $[0, x]$, producing the power series of $1 - \cos x$.

Power series are very useful in combinatorial enumeration. Here is one of my favorite examples in this regard. (Students of Discrete Mathematics may have seen it earlier.)

Example

Find a closed formula for $s_n = 1^2 + 2^2 + \cdots + n^2$. In high school you may have seen the formula already and been asked to prove it, but how to discover it in the first place?

Using power series this can be done as follows. Start with the geometric series and differentiate it term-wise (valid for $|x| < 1$):

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

$$\Longrightarrow \sum_{n=1}^{\infty} nx^{n-1} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{1}{(1-x)^2},$$

$$\Longrightarrow \sum_{n=1}^{\infty} nx^{n} = x\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \frac{x}{(1-x)^2}.$$

From this we see that the operator $x(\mathrm{d}/\,\mathrm{d}x)$ ("first differentiate, then mutliply by x") effects the transformation $(a_n) \mapsto (na_n)$ on the corresponding coefficient sequence of a power series.

Example (cont'd)

Hence, applying the operator twice we obtain

$$\sum_{n=1}^{\infty} n^2 x^n = x \frac{\mathrm{d}}{\mathrm{d}x} \frac{x}{(1-x)^2} = \frac{x+x^2}{(1-x)^3}.$$

Further we have

$$\begin{aligned}\frac{1}{1-x}\sum_{n=1}^{\infty} n^2 x^n &= \left(\sum_{n=0}^{\infty} x^n\right)\left(\sum_{n=1}^{\infty} n^2 x^n\right)\\ &= 1^2 x + (1^2+2^2)x^2 + (1^2+2^2+3^2)x^3 + \dots +\\ &= \sum_{n=1}^{\infty} s_n x^n.\end{aligned}$$

In general, the operator "$\times \frac{1}{1-x}$" effects on the corresponding coefficient sequence of a power series the transformation $(a_0, a_1, a_2, \dots) \mapsto (a_0, a_0 + a_1, a_0 + a_1 + a_2, \dots)$, i.e., taking the partial sums of the sequence.

Example (cont'd)

Putting both computations together, we get

$$\sum_{n=1}^{\infty} s_n x^n = \frac{x+x^2}{(1-x)^4}.$$

This tells us that the so-called *generating function* of the sequence (s_n) is the rational function $\frac{x+x^2}{(1-x)^4}$. A closed formula for s_n may then be obtained by expanding $\frac{x+x^2}{(1-x)^4}$ into a power series and comparing coefficients. This can be done using partial fractions or, if you happen to know the power series expansion $(1-x)^{-s} = \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} x^n$, $|x| < 1$, $s \in \mathbb{N}$, quickly as follows:

$$\begin{aligned} \implies \sum_{n=0}^{\infty} s_n x^n &= \frac{x+x^2}{(1-x)^4} = (x+x^2) \sum_{n=0}^{\infty} \binom{n+3}{3} x^n \\ &= \sum_{n=0}^{\infty} \left(\binom{n+2}{3} + \binom{n+1}{3} \right) x^n, \end{aligned}$$

and hence $\quad s_n = \binom{n+2}{3} + \binom{n+1}{3} = \frac{(2n+1)(n+1)n}{6}.$

Example (geometry of the complex geometric series)

The geometric series evaluation

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}$$

is valid for all complex numbers z with $|z| < 1$. This follows from $1 + z + \cdots + z^n = \frac{1-z^{n+1}}{1-z}$ and $z^{n+1} \to 0$ for $n \to \infty$.

For example, since $\left|\frac{\mathrm{i}}{2}\right| = \frac{1}{2}$, $\left|\frac{1+\mathrm{i}}{2}\right| = \frac{1}{2}\sqrt{2}$, we have

$$\sum_{n=0}^{\infty} \left(\frac{\mathrm{i}}{2}\right)^n = \frac{1}{1-\mathrm{i}/2} = \frac{2}{2-\mathrm{i}} = \frac{4}{5} + \frac{2}{5}\,\mathrm{i},$$

$$\sum_{n=0}^{\infty} \left(\frac{1+\mathrm{i}}{2}\right)^n = \frac{1}{1-(1+\mathrm{i})/2} = \frac{2}{1-\mathrm{i}} = 1 + \mathrm{i}.$$

Since complex numbers are just vectors in the plane (which also can be multiplied), these limits have nice geometric illustrations; cf. next slide.

For the snakes' shapes note that multiplication by $\mathrm{i}/2$ amounts to a 90° rotation and a scaling by 0.5, and similarly for $(1+\mathrm{i})/2$.

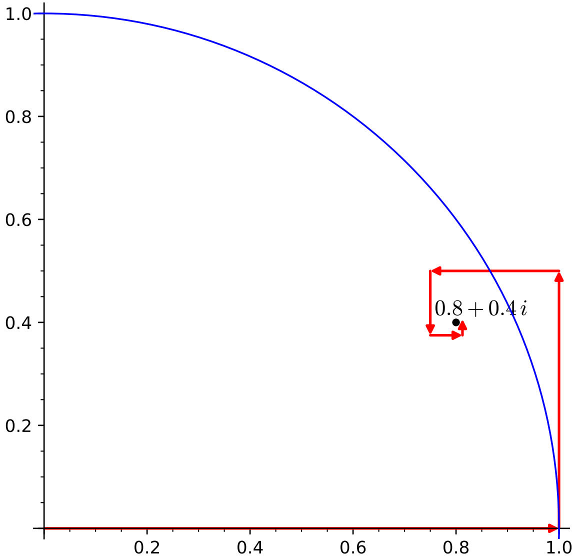


Figure: Illustration of $\displaystyle\sum_{k=0}^{5}\left(\frac{\mathrm{i}}{2}\right)^k \approx \frac{4}{5}+\frac{2}{5}\,\mathrm{i}$

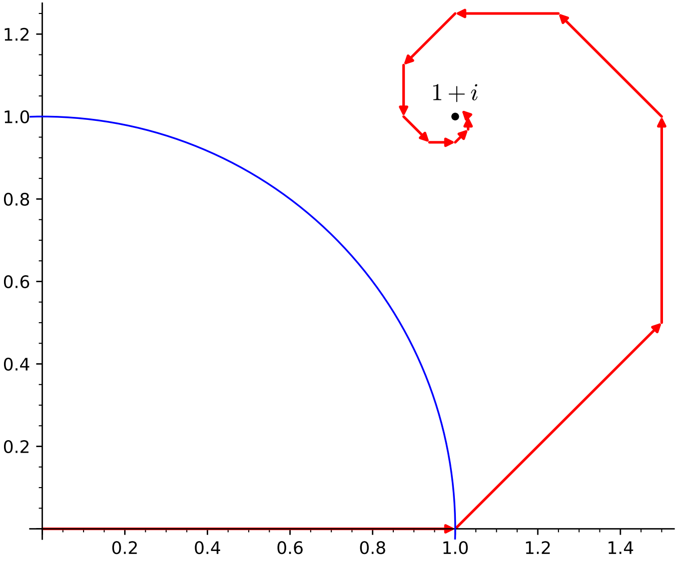


Figure: Illustration of $\sum_{k=0}^{11}\left(\frac{1+\mathrm{i}}{2}\right)^k \approx 1+\mathrm{i}$

Complex vs Real Differentiability

Since complex numbers $z=(x,y)=x+y\mathrm{i}$ are just points in the plane, a function $f\colon D\to\mathbb{C}$, $D\subseteq\mathbb{C}$, corresponds to a pair of real 2-variable functions $u,v\colon D\to\mathbb{R}$ via

$$f(z)=f(x,y)=u(x,y)+\mathrm{i}\,v(x,y),\quad\text{resp.,}\quad u=\operatorname{Re}f,\ v=\operatorname{Im}f.$$

Complex differentiabililty of f in $z=(x,y)\in D$ (which must be an inner point of D), i.e., the existence of the limit

$$f'(z)=\lim_{\substack{h\to 0\\ h\in\mathbb{C}}}\frac{f(z+h)-f(z)}{h}=\lim_{(h_1,h_2)\to(0,0)}\frac{f(x+h_1,y+h_2)-f(x,y)}{h_1+h_2\mathrm{i}},$$

has a nice characterization in terms of real (total) differentiability of u, v in (x,y) and certain conditions on the partial derivatives u_x,u_y,v_x,v_y in (x,y). For this note that it is equivalent to the existence of $c\in\mathbb{C}$ such that $\lim_{h\to 0}\bigl(f(z+h)-f(z)-ch\bigr)/h=0$. (If applicable, we have $f'(z)=c$.) Real differentiability of f in (x,y) in turn means the existence of $a_{11},a_{12},a_{21},a_{22}\in\mathbb{R}$ such that $\lim_{h\to 0}\bigl(u(x+h_1,y+h_2)-u(x,y)-a_{11}h_1-a_{12}h_2\bigr)/|h|=0$ and $\lim_{h\to 0}\bigl(v(x+h_1,y+h_2)-v(x,y)-a_{21}h_1-a_{22}h_2\bigr)/|h|=0$.

Theorem

f is complex differentiable in $z = (x, y)$ iff f is real differentiable in (x, y) (i.e., u, v are differentiable in (x, y)) and

$$u_x(x, y) = v_y(x, y), \quad u_y(x, y) = -v_x(x, y).$$

In particular, f is complex differentiable per se (i.e., in every point of D, which requires D to be open) iff f is real differentiable and u, v satisfy the so-called *Cauchy-Riemann PDE's* $u_x = v_y \wedge u_y = -v_x$.

Proof.

If f is complex differentiable in z with $f'(z) = a + b\mathrm{i}$ then

$$\begin{aligned} f(z+h) - f(z) &= f'(z)h + \mathrm{o}(h) = (a + b\mathrm{i})(h_1 + h_2\mathrm{i}) + \mathrm{o}(h) \\ &= ah_1 - bh_2 + (ah_2 + bh_1)\mathrm{i} + \mathrm{o}(h), \end{aligned}$$

where $g(h) = \mathrm{o}(h)$ means $g(h)/|h| \to 0$ for $h \to 0$.
Extracting real and imaginary part we obtain

$$\begin{aligned} u(x + h_1, y + h_2) - u(x, y) &= ah_1 - bh_2 + \mathrm{o}(h), \\ v(x + h_1, y + h_2) - v(x, y) &= bh_1 + ah_2 + \mathrm{o}(h), \end{aligned}$$

Proof cont'd.

... which says that u, v are differentiable in (x, y) with $u_x(x,y) = a$, $u_y(x,y) = -b$, $v_x(x,y) = b$, $v_y(x,y) = a$; in particular we have $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$.

Conversely, if u, v satisfy the conditions of the theorem, it is equally easy to see that f is complex differentiable in z with $f'(z) = u_x(x,y) + \mathrm{i}\, v_x(x,y)$. □

Note

The Cauchy-Riemann PDE's say that the Jacobi matrix $\mathbf{J}_f(x,y)$ has the special form

$$\mathbf{J}_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix},$$

i.e., it is at every point (x, y) a scaled rotation matrix.

Principal Branch of the Complex Logarithm

The equation

$$w = \mathrm{e}^z = \mathrm{e}^{x+\mathrm{i}y} = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y} = \mathrm{e}^x \cos y + \mathrm{i}\,\mathrm{e}^x \sin y$$

is solvable for each nonzero $w \in \mathbb{C}$, and the solutions are

$$z_k = \ln|w| + \mathrm{i}(\arg w + 2k\pi), \quad k \in \mathbb{Z},$$

where $\arg w \in (-\pi, \pi]$ is the angle of w in polar coordinates. (To see this, write $w = r\mathrm{e}^{\mathrm{i}\phi}$ and compare with $\mathrm{e}^x \mathrm{e}^{\mathrm{i}y}$.)

Considering the "principal" solution z_0 as a function of w and swapping notation, we obtain the *principal branch of the complex logarithm*

$$\ln z = \ln|z| + \mathrm{i}\arg z = \ln\sqrt{x^2+y^2} + \mathrm{i}\arctan(y/x), \quad x = \operatorname{Re} z > 0.$$

By definition, the logarithm satisfies $\mathrm{e}^{\ln z} = z$ in its domain, which can be extended to the "slotted" plane $\mathbb{C} \setminus \{(x,0); x \le 0\}$ (with a different expression for the imaginary part) without affecting the truth of the following theorem.

Theorem

$f(z) = \ln z$ is complex differentiable with $f'(z) = 1/z$.

Proof.

We assume $\operatorname{Re} z > 0$, so that $v(x,y) = \arctan(y/x)$ can be used.

$$\begin{aligned}
f(x,y) &= \left(\ln\sqrt{x^2+y^2}, \arctan(y/x)\right),\\
u_x &= \frac{\mathrm{d}}{\mathrm{d}x}\ln\sqrt{x^2+y^2} = \frac{x}{x^2+y^2},\\
u_y &= \frac{\mathrm{d}}{\mathrm{d}y}\ln\sqrt{x^2+y^2} = \frac{y}{x^2+y^2},\\
v_x &= \frac{\mathrm{d}}{\mathrm{d}x}\arctan(y/x) = \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2},\\
v_y &= \frac{\mathrm{d}}{\mathrm{d}y}\arctan(y/x) = \frac{1/x^2}{1+(y/x)^2} = \frac{x}{x^2+y^2}.
\end{aligned}$$

Evidently the Cauchy-Riemann PDE's are satisfied, and hence f is complex differentiable with

$$f'(z) = u_x(x,y) + \mathrm{i}\, v_x(x,y) = \frac{x - y\mathrm{i}}{x^2+y^2} = \frac{\overline{z}}{|z|^2} = 1/z. \qquad \square$$

For $|z-1|<1$ we have

$$\begin{aligned}\ln z &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(z-1)^n \\ &= z-1-\frac{(z-1)^2}{2}+\frac{(z-1)^3}{3}-\frac{(z-1)^4}{4}\pm\cdots\end{aligned}$$

This can be seen as follows. The power series has radius of convergence 1, and hence for $|z-1|<1$ may be differentiated term-wise to yield

$$1-(z-1)+(z-1)^2-(z-1)^3\pm\ldots=\frac{1}{1+z-1}=\frac{1}{z},$$

the same derivative as $\ln z$.
$\Longrightarrow$ $\ln z$ differs from the power series by an additive constant. The constant must be zero, since both $\ln z$ and the power series vanish at $z=1$.

The following example plays an important role in the theory of Fourier series.

Example

The power series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence $R = 1$ and hence defines an analytic (holomorphic) function $f(z)$ on the open unit disk $\mathrm{B}_1(0) = \{z \in \mathbb{C}; |z| < 1\}$, whose derivative can be obtained by termwise differentiation:

$$f'(z) = \sum_{n=1}^{\infty} \frac{n\,z^{n-1}}{n} = \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1-z}.$$

Together with $f(0) = 0$ it follows that

$$f(z) = -\ln(1-z) = -\ln|1-z| - \mathrm{i}\arg(1-z) \quad \text{with}$$

$$\ln|1-z| = \ln\sqrt{(1-x)^2 + y^2} = \ln\sqrt{1 - 2x + |z|^2},$$

$$\arg(1-z) = \arg(1 - x - \mathrm{i}y) = -\arctan\left(\frac{y}{1-x}\right),$$

where we have written $z = x + \mathrm{i}y$; cf. the preceding discussion (or Calculus III) for principal branch of the complex logarithm.

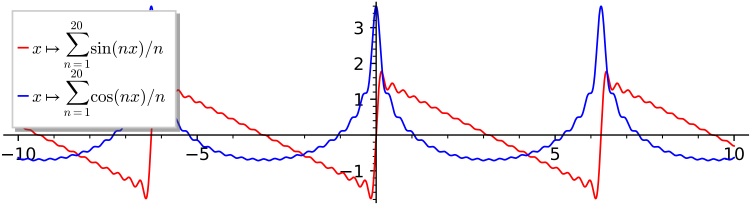
Example (cont'd)

Question: What happens on the boundary $\mathbb{S}^1 = \{z \in \mathbb{C}; |z| = 1\}$?

A point $z \in \mathbb{S}^1$ has the form $z = \mathrm{e}^{\mathrm{i}x}$ with $x \in [0, 2\pi)$ (with a different meaning of x !), and we are asking for the convergence of the series

$$f(\mathrm{e}^{\mathrm{i}x}) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{\mathrm{i}nx}}{n} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} + \mathrm{i} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}.$$

From Calculus I we know already that the series diverges for $z = 1$ ($x = 0$), because $f(1) = \sum_{n=1}^{\infty} \frac{1^n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is just the harmonic series, and converges for $z = -1$ ($x = \pi$), because $f(-1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \pm \cdots\right) = -\ln(2)$ (alternating harmonic series).



Example (cont'd)

From the plot it appears that:

- The series of cosines converges for $0 < x < 2\pi$ and represents a continuous (and maybe differentiable) function on $(0, 2\pi)$.
- The series of sines converges for all $x \in \mathbb{R}$ and represents the 2π-periodic function h defined by

$$h(x) = \begin{cases} (\pi - x)/2 & \text{for } 0 < x < 2\pi, \\ 0 & \text{for } x = 0, \end{cases}$$

  and 2π-periodic extension to $\mathbb{R}$.

  Since $h(0+) = \pi/2$, $h(0-) = h(2\pi-) = -\pi/2$, the function h has discontinuities at $x \in 2\pi\mathbb{Z}$. The value at any discontinuity x satisfies $h(x) = \frac{h(x+)+h(x-)}{2}$.

Example (cont'd)

As key step towards the proof of these assertions we now show that $f(z)=\sum_{n=1}^{\infty}\frac{z^n}{n}$ converges for all $z\in\mathbb{C}$ with $|z|\leq 1$ except for $z=1$, and the convergence is uniform on every subset D of $\overline{\mathrm{B}_1(0)}\setminus\{1\}$ that excludes a (small) circle around $z=1$.

For this we use a technique called "Abel summation" or "partial summation" (a discrete analogue of integration by parts). Setting $s_n(z)=\sum_{k=1}^{n}z^n$, we have for $m,n\in\mathbb{N}$ with $m<n$

$$\begin{aligned}\sum_{k=m}^{n}\frac{z^k}{k}&=\sum_{k=m}^{n}\frac{s_k(z)-s_{k-1}(z)}{k}\\&=-\frac{s_{m-1}(z)}{m}+\sum_{k=m}^{n-1}s_k(z)\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{s_n(z)}{n}\end{aligned}$$

Now suppose that $s_n(z)$ is uniformly bounded on D, i.e., there exists $M>0$ such that $|s_n(z)|\leq M$ for all $z\in D$ and all $n\in\mathbb{N}$. Then we obtain the estimate

$$\left|\sum_{k=m}^{n}\frac{z^k}{k}\right|\leq\frac{M}{m}+M\sum_{k=m}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)+\frac{M}{n}=\frac{2M}{m}.$$

Example (cont'd)

Hence, given $\epsilon > 0$, we have for the partial sums $f_n(z) = \sum_{k=1}^{n} z^k/k$ of our series the estimate

$$d_\infty(f_m, f_n) = \max\left\{\left|\sum_{k=m+1}^{n} \frac{z^k}{k}\right|; z \in D\right\} < \epsilon \quad \text{if } m, n > N_\epsilon = \lceil 2M/\epsilon \rceil.$$

This shows that the series satisfies the Cauchy-Criterion for uniform convergence on D and hence that it converges uniformly on D; cf. subsequent lecture for more details.

It yet remains to derive the bound M. This is easy, however, since

$$s_n(z) = \sum_{k=1}^{n} z^n = \frac{z^{n+1} - z}{z - 1}.$$

Hence, setting $D = D_r = \{z \in \mathbb{C}; |z| \leq 1, |z - 1| \geq r\}$ for $r > 0$, we have have $|s_n(z)| \leq 2/r$ for $z \in D_r$ and $n \in \mathbb{N}$, so that we can take $M = 2/r$.

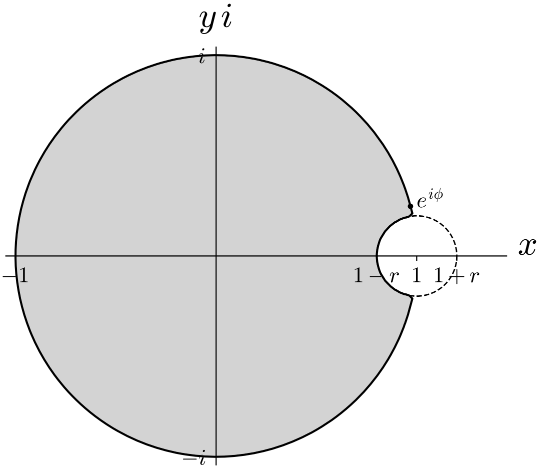


Figure: The region D_r

You may think of a map of China with Hangzhou at $1-r$ and Shanghai at $e^{i\phi}$. (Question: Where is Ningbo?)

Example (cont'd)

Now the continuity theorem gives that $f(z) = \sum_{n=1}^{\infty} z^n/n$ defines a continuous function on $\overline{\mathrm{B}_1(0)} \setminus \{1\}$: To prove continuity at a particular point z_0, let $r = \frac{1}{2}|z_0 - 1|$ and use the uniform convergence on D_r.

In particular the series $\sum_{n=1}^{\infty} \cos(nx)/n$ and $\sum_{n=1}^{\infty} \sin(nx)/n$ converge for every $x \in (0, 2\pi)$ (and the second series trivially converges also for $x = 0$).

Knowing that f is continuous in $z = \mathrm{e}^{\mathrm{i}\phi} \in \mathrm{S}^1 \setminus \{1\}$, we can compute $f(\mathrm{e}^{\mathrm{i}\phi})$ from the explicit representation of f in $\mathrm{B}_1(0)$ as the limit

$$
\begin{aligned}
f(\mathrm{e}^{\mathrm{i}\phi}) &= \lim_{r\uparrow 1} f(r\mathrm{e}^{\mathrm{i}\phi}) \\
&= \lim_{r\uparrow 1} \left[-\ln\sqrt{1 - 2r\cos\phi + r^2} + \mathrm{i}\arctan\left(\frac{r\sin\phi}{1 - r\cos\phi}\right)\right] \\
&= -\ln\sqrt{2(1-\cos\phi)} + \mathrm{i}\arctan\left(\frac{\sin\phi}{1-\cos\phi}\right) \\
&= -\ln\left(2\sin\frac{\phi}{2}\right) + \mathrm{i}\,\frac{\pi - \phi}{2}, \quad \dots
\end{aligned}
$$

Example (cont'd)

… where we have used

$$1-\cos\phi = 1-\left(\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}\right) = 2\sin^2\frac{\phi}{2},$$

$$\frac{\sin\phi}{1-\cos\phi} = \frac{2\sin\frac{\phi}{2}\cos\frac{\phi}{2}}{2\sin^2\frac{\phi}{2}} = \cot\frac{\phi}{2} = \tan\frac{\pi-\phi}{2}.$$

As a corollary we have the trigonometric series evaluations

$$\sum_{n=1}^{\infty}\frac{\cos(nx)}{n} = -\ln\left(2\sin\frac{x}{2}\right), \quad \sum_{n=1}^{\infty}\frac{\sin(nx)}{n} = \frac{\pi-x}{2} \quad (0<x<2\pi).$$

In particular, setting $x=\pi/2$ (or $z=\mathrm{e}^{\mathrm{i}x}=\mathrm{i}$) this gives

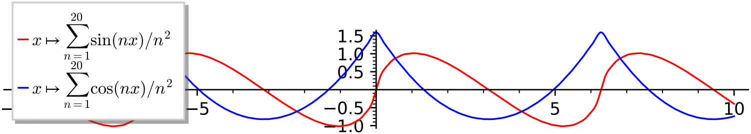
$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}\pm\cdots=\ln 2, \quad 1-\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{7}\pm\cdots=\pi/4.$$

In the next subsection the criterion we have used for proving the uniform convergence of $\sum_{n=1}^{\infty} z^n/n$ is stated in more generality. It is known as "Dirichlet's test for uniform convergence".

Example

We compute the related series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$.

Since $\left|\cos(nx)/n^2\right| \leq 1/n^2$ and $\sum_{n=1}^{\infty} 1/n^2$ converges, this series converges uniformly (and absolutely) on $\mathbb{R}$ and represents a continuous, 2π-periodic function $g\colon \mathbb{R} \to \mathbb{R}$.



The series of derivatives is

$$\sum_{n=1}^{\infty} \frac{-\sin(nx)n}{n^2} = -\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = -\operatorname{Im}\left(\sum_{n=1}^{\infty} \frac{(e^{ix})^n}{n}\right).$$

From the preceding example we know that the series of derivatives converges uniformly on every interval of the form $[\delta, 2\pi - \delta]$ with $\delta > 0$ (and δ sufficiently small).

Example (cont'd)

$\Longrightarrow$ The differentiation theorem can be applied and gives

$$g'(x) = -\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{x-\pi}{2} \quad \text{for } 0 < x < 2\pi.$$

$$\Longrightarrow g(x) = \frac{(x-\pi)^2}{4} + C \quad \text{for } 0 \leq x \leq 2\pi,$$

where C is some constant. (Note that $g(0) = g(2\pi) = \pi^2/4 + C$, so that the 2π-periodic extension to $\mathbb{R}$ will be automatically continuous.)

The constant can be determined by evaluating the integral $\int_0^{2\pi} g(x)\,\mathrm{d}x$ in two ways:

1. Applying the integration theorem to the series defining g, we obtain

$$\int_0^{2\pi} g(x)\,\mathrm{d}x = \int_0^{2\pi} \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_0^{2\pi} \frac{\cos(nx)}{n^2}\,\mathrm{d}x$$
$$= \sum_{n=1}^{\infty} \left[\frac{\sin(nx)}{n^3}\right]_0^{2\pi} = \sum_{n=1}^{\infty} \frac{\sin(2n\pi) - \sin(0)}{n^3} = 0.$$

Example (cont'd)

2. Using the expression $g(x) = (x-\pi)^2/4 + C$, we obtain

$$\int_0^{2\pi} g(x)\,\mathrm{d}x = \left[\frac{(x-\pi)^3}{12} + Cx\right]_0^{2\pi} = \frac{\pi^3}{12} + 2\pi C - \frac{(-\pi)^3}{12} = \frac{\pi^3}{6} + 2\pi C.$$

Since this is equal to zero, we conclude $C = -\pi^2/12$, and finally

$$g(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \quad \text{for } 0 \leq x \leq 2\pi.$$

As a by-product, setting $x = 0$, resp., $x = \pi$, we obtain from this the series evaluations

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}.$$

Exercise

a) Assuming that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, show that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$ without resorting to the evaluation of $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$ on the previous slide.

Hint: Add the two series.

b) Show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$.

Exercise

Determine the two series

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^3} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^4} \quad \text{for } x \in \mathbb{R},$$

and use the results to evaluate in turn $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$ and $\displaystyle\sum_{n=1}^{\infty} 1/n^4$.

Exercise

Riemann's Zeta function is defined for complex arguments $s = \sigma + \mathrm{i}t$ with $\sigma = \mathrm{Re}(s) > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{where} \quad n^s = \mathrm{e}^{\ln(n)s}.$$

1. Show that the above series converges uniformly on every closed half plane $H_\delta = \{s \in \mathbb{C}; \mathrm{Re}(s) \geq 1 + \delta, \delta > 0$, and conclude from this that ζ is continuous.
2. Using a variant of the Differentiation Theorem, show in a similar fashion that ζ is complex differentiable (in fact infinitely often) and give a series representation for $\zeta'(s)$.
3. Using properties of the prime factorization of integers, show
$$\zeta(s) = \frac{1}{(1-2^{-s})(1-3^{-s})(1-5^{-s})(1-7^{-s})(1-11^{-s})\cdots}.$$
4. Show that $(2^{1-s}-1)\zeta(s)$ has a series representation of the form $\sum_{n=1}^{\infty} a_n n^{-s}$, which converges for $\mathrm{Re}(s) > 0$ and uniformly for $\mathrm{Re}(s) \geq \delta > 0$. Conclude that $\zeta(s)$ is holomorphic in $\{s \in \mathbb{C}; \mathrm{Re}(s) > 0, s \neq 1\}$ and satisfies $\lim_{s\to 1}(s-1)\zeta(s) = 1$.

Further Tests for Uniform Convergence

Theorem

Let (f_n) be a monotonically decreasing sequence of real-valued functions on D (i.e., $f_1(x) \geq f_2(x) \geq f_3(x) \geq \cdots$ for $x \in D$) and (g_n) a sequence of complex-valued functions on D. The function series $\sum_{n=1}^{\infty} f_n g_n$ converges uniformly on D if one of the following two criteria is satisfied:

1. DIRICHLET*'s test for uniform convergence*
   (f_n) converges to 0 uniformly on D and the series $\sum_{n=1}^{\infty} g_n$ (i.e., its partial sums $G_n = g_1 + \cdots + g_n$) is uniformly bounded on D.
2. ABEL*'s test for uniform convergence*
   (f_n) is uniformly bounded on D and the series $\sum_{n=1}^{\infty} g_n$ converges uniformly on D.

The domain D in the theorem is completely arbitrary (i.e., any set).

Dirichlet's test in particular includes the case where f_n is constant, i.e., (f_n) is ordinary sequence of real numbers satisfying $f_n \downarrow 0$.

Cauchy's Test for Uniform Convergence

The mother of all such tests

Before proving the theorem, we state the analogue of Cauchy's convergence test for uniformly convergent function sequences/series, which was behind the scene also in Weierstrass' test.

Lemma

1. *A sequence (f_n) of (real-/complex-/vector-valued) functions on D converges uniformly iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_m(x) - f_n(x)| < \epsilon$ for all $m, n > N$ and all $x \in D$.*
2. *A series $\sum_{n=1}^{\infty} f_n$ of functions on D converges uniformly iff for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\left|\sum_{k=m}^{n} f_k(x)\right| < \epsilon$ for all $n \geq m > N$ and all $x \in D$.*

The proof is more or less the same as the earlier proof (discussed in Calculus III) that a sequence of real or complex numbers (or vectors) converges iff it is a Cauchy sequence, except that now all responses N must be uniform. We omit the details.

Proof of the theorem.

In both cases, given $\epsilon > 0$, we must find a response N such that $\left|\sum_{k=m}^{n} f_k(x)g_k(x)\right| < \epsilon$ for $n \geq m > N$. This can be done with the help of Abel summation:

$$\sum_{k=m}^{n} f_k(x)g_k(x) = \sum_{k=m}^{n} f_k(x)\left[G_k(x) - G_{k-1}(x)\right]$$

$$= -f_m(x)G_{m-1}(x) + f_n(x)G_n(x) + \sum_{k=m}^{n-1}\left[f_k(x) - f_{k+1}(x)\right]G_k(x)$$

Case 1 (Dirichlet): By assumption, there exists $M > 0$ such that $|G_k(x)| \leq M$ for all $k \in \mathbb{N}$ and $x \in D$, and since $f_n(x) \downarrow 0$ we must have $f_n(x) \geq 0$. This allows us to estimate as follows:

$$\left|\sum_{k=m}^{n} f_k(x)g_k(x)\right| \leq f_m(x)M + f_n(x)M + \sum_{k=m}^{n-1}\left[f_k(x) - f_{k+1}(x)\right]M$$

$$= 2M\, f_m(x). \qquad \text{(The sum "telescopes".)}$$

Since $f_n \downarrow 0$ uniformly, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $f_m(x) < \epsilon/(2M)$ for $m > N$ and $x \in D$, showing that $\sum_{n=1}^{\infty} f_n g_n$ satisfies the assumption of the Cauchy test for uniform convergence.

Proof cont'd.

Case 2 (Abel): Here the key observation is that in the expression obtained for $\sum_{k=m}^{n} f_k(x)g_k(x)$ by Abel summation the coefficient sum of the functions $G_k(x)$ is

$$f_n(x) - f_m(x) + f_m(x) - f_{m+1}(x) + \cdots + f_{n-1}(x) - f_n(x) = 0.$$

$\Longrightarrow$ We can add subtract $G(x) = \lim_{n\to\infty} G_n(x)$ from every summand without affecting the sum, i.e., we have (suppressing arguments)

$$\sum_{k=m}^{n} f_k g_k = -f_m(G_{m-1} - G) + f_n(G_n - G) + \sum_{k=m}^{n-1} (f_k - f_{k+1})(G_k - G).$$

Since $G_n \to G$ uniformly, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|G_n - G| < \epsilon$ for $n > N$. For $n \geq m > N + 1$ we then obtain

$$\left|\sum_{k=m}^{n} f_k g_k\right| \leq |f_m|\,\epsilon + |f_n|\,\epsilon + \sum_{k=m}^{n-1} (f_k - f_{k+1})\epsilon = (|f_m| + |f_n| + f_m - f_n)\,\epsilon.$$

If M is a uniform bound for f_n, i.e., $|f_n(x)| \leq M$ for all $n \in \mathbb{N}$ and $x \in D$, then $\left|\sum_{k=m}^{n} f_k g_k\right| \leq 4M\epsilon$ for $n \geq m > N + 1$, so that again the Cauchy test for uniform convergence can be applied. $\square$

Note

Uniform convergence of the series $\sum_{n=1}^{\infty} f_n g_n$ is not affected by deleting a finite number of summands from it. This can be helpful if some of the functions f_n or g_n are unbounded. For example, if g_1 is unbounded and all of $g_2, g_3, \dots$ are bounded then all partial sums $G_n = g_1 + \cdots + g_n$ are unbounded as well, and hence Dirichlet's test cannot be applied directly, but nevertheless it may be possible to obtain a uniform bound for $G_n' = g_2 + \cdots + g_n$ and apply it to the series $\sum_{n=2}^{\infty} f_n g_n$.

As discussed earlier, Dirichlet's test gives the uniform convergence of $\sum_{n=1}^{\infty} z^n/n$ on the regions D_r, $0 < r < 1$. Here $f_n(z) = 1/n$, $g_n(z) = z^n$, and the key observation is that $G_n(z) = z + z^2 + \cdots + z^n$ is uniformly bounded on D_r.

As an application of Abel's Test we prove the following

Theorem (Abel's Limit Theorem)

Suppose $\sum_{n=0}^{\infty} a_n(z-a)^n$ has radius of convergence $0 < R < \infty$. If the power series converges for a point $z_1 = a + R\,\mathrm{e}^{\mathrm{i}\phi}$ on the boundary of its disk of convergence, it converges uniformly on the line segment $[a, z_1] = \left\{a + r\,\mathrm{e}^{\mathrm{i}\phi}; 0 \leq r \leq R\right\}$, and hence represents a continuous function on $[a, z_1]$.

That the so-defined function f is continuous in z_1 means

$$\lim_{r\uparrow R} f(a+r\,\mathrm{e}^{\mathrm{i}\phi}) = \lim_{r\uparrow R}\sum_{n=0}^{\infty} a_n r^n \mathrm{e}^{\mathrm{i}n\phi} = \sum_{n=0}^{\infty} a_n R^n \mathrm{e}^{\mathrm{i}n\phi} = f(a+R\,\mathrm{e}^{\mathrm{i}\phi}),$$

and explains the name "limit theorem".

Proof.

Writing $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n$, we have

$$f(a+r\,\mathrm{e}^{\mathrm{i}\phi}) = \sum_{n=0}^{\infty} a_n r^n \mathrm{e}^{\mathrm{i}n\phi} = \sum_{n=0}^{\infty} \frac{r^n}{R^n}\, a_n R^n \mathrm{e}^{\mathrm{i}n\phi}.$$

Now define $f_n(r)=r^n/R^n=(r/R)^n$, $g_n(r)=a_nR^n\mathrm{e}^{\mathrm{i}n\phi}$ for $n\in\mathbb{N}_0$ and $r\in[0,R]$. Then $0\leq f_{n+1}(r)\leq f_n(r)\leq 1$ for all n, so that f_n has the properties required in Abel's test for uniform convergence.

The series $\sum_{n=0} g_n$ converges uniformly on $[0,R]$, since it converges by assumption and doesn't depend on r.

$\Longrightarrow$ Abel's test can be applied and yields the uniform convergence of $\sum_{n=0}^{\infty} f_ng_n$ on $[0,R]$, i.e., the uniform convergence of $\sum_{n=0}^{\infty} a_n(z-a)^n$ on $[a,z_1]$. $\square$

Example

The binomial series

$$\sum_{n=0}^{\infty}\binom{s}{n}z^n=\sum_{n=0}^{\infty}\frac{s(s-1)\cdots(s-n+1)}{1\cdot 2\cdots n}\,z^n,\quad s\notin\{0,1,2,\dots\},$$

has radius of convergence $R=1$ (by the ratio test) and represents the function $(1+z)^s=\mathrm{e}^{s\log(1+z)}$ in $\mathrm{B}_1(0)$; cf. Homework 4, Exercise H25. (Here, as usual, $\log$ denotes the principal branch of the complex logarithm. For $s=0,1,2,\dots$ the series terminates, and the identity reduces to the ordinary binomial theorem $(1+z)^s=\sum_{n=0}^{s}\binom{s}{n}z^n$.

Claim: For $s>-1$ the series $\sum_{n=0}^{\infty}\binom{s}{n}$ satisfies the assumptions of the alternating series test, and hence converges.

Proof: Since $s+1>0$ and

$$\binom{s}{n}=-\frac{n-s-1}{n}\binom{s}{n-1},$$

for large n the sequence $a_n=\binom{s}{n}$ will be alternating in sign and $|a_n|<|a_{n-1}|$.

Example (cont'd)

In order to show $\binom{s}{n} \to 0$ for $n \to \infty$, we rewrite the binomial coefficient as

$$\binom{s}{n} = \pm\frac{s}{n}\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n-1}\right)$$

For $s > 0$ we have $0 < 1 - s/k < 1$ except for $k = 1, 2, \dots, \lfloor s \rfloor$, and hence for $n > \lfloor s \rfloor$ the estimate $\left|\binom{s}{n}\right| \leq sP/n$, where $P = \prod_{k=1}^{\lfloor s \rfloor}(s/k - 1)$. This shows $\binom{s}{n} \to 0$ for $n \to \infty$.

For $-1 < s < 0$ we have

$$\begin{aligned}\ln\left|\binom{s}{n}\right| &= \ln(-s) - \ln(n) + \sum_{k=1}^{n-1}\ln\left(1-\frac{s}{k}\right)\\ &\leq \ln(-s) - \ln(n) - \sum_{k=1}^{n-1}\frac{s}{k} = \ln(-s) - \ln(n) - s\sum_{k=1}^{n-1}\frac{1}{k} \to -\infty,\end{aligned}$$

since $\sum_{k=1}^{n-1} 1/k = \ln(n) + \mathrm{O}(1)$ and $-s < 1$. This shows $\binom{s}{n} \to 0$ for $n \to \infty$ also in this case and completes the proof of our claim.

Example (cont'd)

Now Abel's Limit Theorem gives that $z \mapsto \sum_{n=0}^{\infty} \binom{s}{n} z^n$ defines a continuous function on $(-1, 1]$ for $s > 1$ and hence represents $z \mapsto (1+z)^s$ also for $z = 1$.

$$\Longrightarrow \sum_{n=0}^{\infty} \binom{s}{n} = 2^s \quad \text{for } s > -1.$$

For $s = -1/2$ we have $\binom{-1/2}{n} = (-1)\,\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$ and obtain the series evaluation

$$1 - \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \mp \cdots = \frac{\sqrt{2}}{2}.$$

Similarly, one can show that for $s > 0$ the binomial series converges at $z = -1$ and hence represents $z \mapsto (1+z)^s$ also for $z = -1$.

$$\Longrightarrow \sum_{n=0}^{\infty} (-1)^n \binom{s}{n} = 0 \quad \text{for } s > 0.$$

The Multivariable Case

Usually uniform convergence is covered in textbooks as part of Calculus I or II, when the differential calculus of functions of several variables is not yet available. For this reason the main theorems about uniform convergence are usually stated for one-variable functions, as we have done.

But, of course, these theorems have multivariable generalizations, which are no less important. We consider only the case of the Differentiation Theorem.

Theorem (differentiation, multivariable case)

Suppose $f_k\colon D \to \mathbb{R}$, $D \subseteq \mathbb{R}^n$ ($k \in \mathbb{N}$) are C^1-functions, $(f_k)_{k\in\mathbb{N}}$ converges point-wise on D, and the n sequences of partial derivatives $(\partial f_k/\partial x_i)_{k\in\mathbb{N}}$, $1 \leq i \leq n$, converge uniformly on D. Then $f(\mathbf{x}) = \lim_{k\to\infty} f_k(\mathbf{x})$, $\mathbf{x} \in D$, is a C^1-function as well and satisfies

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \lim_{k\to\infty} \frac{\partial f_k}{\partial x_i}(\mathbf{x}) \quad \text{for } 1 \leq i \leq n \text{ and } \mathbf{x} \in D. \tag{$\star$}$$

Proof.

Recall from Calculus III that a function $g\colon D \to \mathbb{R}$ is C^1 iff the partial derivatives $\partial g/\partial x_i$ exist on D and are continuous as multivariable functions.

First consider a sequence of partial derivatives, say $(\partial f_k/\partial x_1)_{k\in\mathbb{N}}$. Since $x_1 \mapsto \frac{\partial f_k}{\partial x_1}(x_1, x_2, \dots, x_n)$ is the ordinary derivative of $x_1 \mapsto f_k(x_1, x_2, \dots, x_n)$ and since the uniform convergence of $(\partial f_k/\partial x_1)_{k\in\mathbb{N}}$ on D implies the uniform convergence of the one-variable functions $x_1 \mapsto \frac{\partial f_k}{\partial x_1}(x_1, x_2, \dots, x_n)$ on the set of all $x \in \mathbb{R}$ for which there exists an $(x_1, \dots, x_n) \in D$ such that $x = x_1$, we can apply the one-variable Differentation Theorem to conclude that $\partial f/\partial x_1$ exists on D and

$$\lim_{k\to\infty} \frac{\partial f_k}{\partial x_1}(x_1, \dots, x_n) = \frac{\partial f}{\partial x_1}(x_1, \dots, x_n) \quad \text{for } (x_1, \dots, x_n) \in D.$$

This shows that f is partially differentiable and satisfies $(\star)$. The proof is then finished by applying the Continuity Theorem (whose generalization to n-variable functions is straightforward) to the sequence $(\partial f_k/\partial x_1)_{k\in\mathbb{N}}$, say, which yields the continuity of $\frac{\partial f}{\partial x_1}$ as a multivariable function. $\square$

Note

Since differentiability is a local property, the conclusions of the Differentiation Theorem remain valid under the weaker assumption that every point $\mathbf{x} \in D$ has a neighborhood $D_{\mathbf{x}}$ on which the sequences $(\partial f_k/\partial x_i)_{k\in\mathbb{N}}$, $1 \leq i \leq n$, converge uniformly. This is also true for the Continuity Theorem (since continuity is a local property as well) and for the Integration Theorem (since the interval of integration is compact and locally uniform convergence on D implies uniform convergence on every compact subset of D by the Heine-Borel covering property; cf. Calculus III).

Application

Suppose $c_0, c_1, c_2, \dots$ is a sequence of real numbers which grows at most polynomially, i.e., there exists $d \in \mathbb{Z}^+$ such that $c_k = \mathrm{O}(k^d)$ for $k \to \infty$. We show that

$$f(x,y) = \sum_{k=0}^{\infty} c_k \mathrm{e}^{-ky} \cos(kx), \quad (x,y) \in \mathbb{R} \times \mathbb{R}^+,$$

solves Laplace's Equation $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$.

First let us assume that f is well-defined, is a C^2-function, and that partial differentiation can be done termwise.

$$\begin{aligned}
\implies \quad \frac{\partial f}{\partial x}(x,y) &= \sum\nolimits_{k=0}^{\infty} -kc_k \mathrm{e}^{-ky} \sin(kx), \\
\frac{\partial^2 f}{\partial x^2}(x,y) &= \sum\nolimits_{k=0}^{\infty} -k^2 c_k \mathrm{e}^{-ky} \cos(kx), \\
\frac{\partial f}{\partial y}(x,y) &= \sum\nolimits_{k=0}^{\infty} -kc_k \mathrm{e}^{-ky} \cos(kx), \\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \sum\nolimits_{k=0}^{\infty} k^2 c_k \mathrm{e}^{-ky} \cos(kx),
\end{aligned}$$

and clearly $\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0$.

Application cont'd

In order to justify the preceding computation, it suffices to show that all series involved (including that defining f) converge uniformly on some neighborhood of a given point (x_0, y_0) in the upper half-plane $\mathbb{R} \times \mathbb{R}^+$. (Be sure to check in detail how this and the Differentiation Theorem imply all assumptions made.)

We can take the neighborhoods as

$$H_\delta = \{(x, y) \in \mathbb{R}^2; y \geq \delta\}, \quad \delta > 0.$$

For the coefficients of the series representing $\partial^2 f/\partial x^2$ we have

$$\left|-k^2 c_k \mathrm{e}^{-ky} \cos(kx)\right| \leq k^2 |c_k| \mathrm{e}^{-ky} \leq M\, k^{d+2} \mathrm{e}^{-k\delta}$$

for $(x, y) \in H_\delta$ and k sufficiently large, where M is some constant.

The series $\sum_{k=0}^{\infty} k^{d+2} \mathrm{e}^{-k\delta}$ converges, because the rapid growth of $x \mapsto \mathrm{e}^x$ implies that $\mathrm{e}^{-k\delta} \leq k^{-d-4}$, and hence $k^{d+2}\mathrm{e}^{-k\delta} \leq 1/k^2$, for sufficiently large k.

$\Longrightarrow$ We can apply Weierstrass's Criterion to conclude the uniform convergence of $\sum_{k=0}^{\infty} -k^2 c_k \mathrm{e}^{-ky} \cos(kx)$ on H_δ.

The other series are treated similarly.

Uniform Convergence of Improper Parameter Integrals

So far we have considered uniform convergence of function sequences/series, the "discrete case" so-to-speak. The continuous analog of (function) series are improper (parameter) integrals, and accordingly it also makes sense to speak of uniform convergence of parameter integrals:

Definition

Suppose f is a real-valued function with domain $D \times [0, \infty)$ and such that $\int_0^R f(x,t)\,\mathrm{d}t$ is defined for every $R \in [0, \infty)$ and $x \in D$.

1. $\int_0^\infty f(x,t)\,\mathrm{d}t$ is said to converge point-wise on D if $\lim_{R\to\infty} \int_0^R f(x,t)\,\mathrm{d}t$ exists for every $x \in D$. If this is the case, $F(x) := \int_0^\infty f(x,t)\,\mathrm{d}t := \lim_{R\to\infty} \int_0^R f(x,t)\,\mathrm{d}t$ defines a real-valued function on D ("limit function").
2. $\int_0^\infty f(x,t)\,\mathrm{d}t$ is said to converge uniformly on D if it converges point-wise and for every $\epsilon > 0$ there exists a "uniform" response $R_0 \in [0, \infty)$ such that $\left|F(x) - \int_0^R f(x,t)\,\mathrm{d}t\right| = \left|\int_R^\infty f(x,t)\,\mathrm{d}t\right| < \epsilon$ for all $R > R_0$ and all $x \in D$.

Notes

- The definition is easily extended to improper integrals with other domains of integration, such as $[a, \infty)$, $(-\infty, b]$, $(-\infty, \infty)$, (a, b), etc. The subsequent discussion applies mutatis mutandis to all these cases.
- Uniform convergence makes sense for any limit involving a further parameter, e.g., under the assumptions of the definition it makes sense to define "$f(x, t) \to F(x)$ uniformly for $t \to \infty$" if $\lim_{t\to\infty} f(x, t) = F(x)$ for every $x \in D$ and in a proof of this responses $R_0 = R_0(\epsilon)$ can be found that do not depend on x.
- The theory we shall now develop overlaps with that of the Lebesgue integral, but is not contained in it, because it also applies to improper integrals that don't converge absolutely, e.g., $\int_0^\infty \sin(t)/t \,\mathrm{d}t$.

Our first goal is to prove analogues of the Continuity and Differentiation Theorems for proper parameter integrals with continuous integrands. These are special cases of theorems for the Lebesgue integral stated earlier in Calculus III. "Elementary" proofs that are independent of the Lebesgue theory will be given.

Lemma (continuity)

Suppose $I \subseteq \mathbb{R}$ is an interval and $f\colon I \times [a,b] \to \mathbb{R}$ is a continuous two-variable function. Then $F\colon I \to \mathbb{R}$, $x \mapsto \int_a^b f(x,t)\,\mathrm{d}t$ is continuous.

Proof.

Since all functions $t \mapsto f(x,t)$, $x \in I$, are continuous, existence of all (Riemann) integrals involved is trivial. For $x, x_0 \in I$ we have

$$F(x) - F(x_0) = \int_a^b [f(x,t) - f(x_0,t)]\,\mathrm{d}t,$$

and, given $\epsilon > 0$, need to find a response δ such $|f(x,t) - f(x_0,t)| < \epsilon$ for $x \in [x_0 - \delta, x_0 + \delta] \cap I$ and all $t \in [a,b]$. (For such x we then have $|F(x) - F(x_0)| \leq \epsilon(b-a)$, showing that F is continuous in x_0.)

Now we use the uniform continuity of f on compact rectangles $K = [\alpha,\beta] \times [a,b]$ with $[\alpha,\beta] \subseteq I$. If x_0 is an inner point of I, we can choose $\alpha < x < \beta$, and if x_0 is the left end point of I, say, we can choose $\alpha = x_0 < \beta$.

Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x_1,t_1) - f(x_2,t_2)| < \epsilon$ if $(x_1,t_1), (x_2,t_2) \in K$ and $|x_1 - x_2| < \delta$, $|t_1 - t_2| < \delta$. Specializing to $x_1 = x$, $x_2 = x_0$, $t_1 = t_2 = t$ shows that this δ can serve as the desired response. $\square$

Lemma (differentiation)

Suppose $I \subseteq \mathbb{R}$ is an interval, $f\colon I \times [a,b] \to \mathbb{R}$ is a continuous two-variable function, and the partial derivative $f_x = \frac{\partial f}{\partial x}\colon I \times [a,b] \to \mathbb{R}$ exists and is a continuous two-variable function as well. Then $F\colon I \to \mathbb{R}$, $x \mapsto \int_a^b f(x,t)\,\mathrm{d}t$ is differentiable with

$$F'(x) = \int_a^b f_x(x,t)\,\mathrm{d}t,$$

i.e., we can differentiate F under the integral sign.

Proof.

For $x_0 \in I$ and $h \neq 0$ such that $x_0 + h \in I$ we have

$$\frac{F(x_0+h)-F(x_0)}{h} - \int_a^b f_x(x_0,t)\,\mathrm{d}t = \int_a^b \left[\frac{f(x_0+h,t)-f(x_0,t)}{h} - f_x(x_0,t)\right]\mathrm{d}t,$$

$$\left|\frac{F(x_0+h)-F(x_0)}{h} - \int_a^b f_x(x_0,t)\,\mathrm{d}t\right| \leq \int_a^b \left|\frac{f(x_0+h,t)-f(x_0,t)}{h} - f_x(x_0,t)\right|\mathrm{d}t,$$

and all integrals involved exist because of the continuity assumptions on f and f_x.

Proof cont'd.

Using the Mean Value Theorem, the last integrand can also be expressed as

$$\left|\frac{f(x_0+h,t)-f(x_0,t)}{h}-f_x(x_0,t)\right| = \left|f_x(\xi_{h,t},t)-f_x(x_0,t)\right|$$

with $\xi_{h,t}$ between x_0 and x_0+h. (Considering x_0 as fixed, there is no dependence of $\xi_{h,t}$ on x_0.)

Now the proof can be finished as in the Continuity Lemma, this time using the uniform continuity of f_x on compact rectangles $K=[\alpha,\beta]\times[a,b]$: If $\delta=\delta(\epsilon)$ is such that $|f_x(x_1,t_1)-f_x(x_2,t_2)|<\epsilon$ for all (x_1,t_1), $(x_2,t_2)\in K$ with $|x_1-x_2|<\delta$, $|t_1-t_2|<\delta$, the previous estimates imply that

$$\left|\frac{F(x_0+h)-F(x_0)}{h}-\int_a^b f_x(x_0,t)\,\mathrm{d}t\right|\leq\epsilon(b-a)$$

for all nonzero $h\in(-\delta,\delta)$ such that $x_0+h\in I$.

$\Longrightarrow \lim\limits_{h\to0}\frac{F(x_0+h)-F(x_0)}{h}=\int_a^b f_x(x_0,t)\,\mathrm{d}t$, as desired. $\square$

Now we are ready to state and prove analogues of our Continuity and Differentiation Theorems for uniformly convergent improper parameter integrals.

Theorem (continuity)

Suppose $I \subseteq \mathbb{R}$ is an interval, $f\colon I \times [a,\infty) \to \mathbb{R}$ is a continuous two-variable function, and $\int_a^\infty f(x,t)\,\mathrm{d}t$ converges uniformly on I. Then $F\colon I \to \mathbb{R}$, $x \mapsto \int_a^\infty f(x,t)\,\mathrm{d}t$ is continuous.

Proof.

For $n \in \mathbb{N}$ define $F_n\colon I \to \mathbb{R}$ by $F_n(x) = \int_a^{a+n} f(x,t)\,\mathrm{d}t$. The functions F_n are continuous by the Continuity Lemma, and converge uniformly to $F(x) = \int_a^\infty f(x,t)\,\mathrm{d}t$, $x \in I$. (If R_0 is such that $\left|\int_R^\infty f(x,t)\,\mathrm{d}t\right| < \epsilon$ for all $R > R_0$ then $|F(x) - F_n(x)| = \int_{a+n}^\infty f(x,t)\,\mathrm{d}t < \epsilon$ for all $n > R_0 - a$, so that we can take $N = \lceil R_0 - a \rceil$ as corresponding response.)
$\Longrightarrow$ By the Continuity Theorem for function sequences, F is continuous. $\square$

Theorem (differentiation)

Suppose $I \subseteq \mathbb{R}$ is an interval, $f\colon I \times [a,\infty) \to \mathbb{R}$ is a continuous two-variable function, the partial derivative $f_x = \frac{\partial f}{\partial x}\colon I \times [a,\infty) \to \mathbb{R}$ exists and is a continuous two-variable function as well, $\int_a^\infty f(x,t)\,\mathrm{d}t$ converges point-wise on I, and $\int_a^\infty f_x(x,t)\,\mathrm{d}t$ converges uniformly on I. Then $F\colon I \to \mathbb{R}$, $x \mapsto \int_a^\infty f(x,t)\,\mathrm{d}t$ is differentiable with

$$F'(x) = \int_a^\infty f_x(x,t)\,\mathrm{d}t,$$

i.e., we can differentiate F under the integral sign.

Proof.

As before we set $F_n(x) = \int_a^{a+n} f(x,t)\,\mathrm{d}t$ for $x \in I$, which converges point-wise on I to F by assumption. The Differentiation Lemma gives that F_n is differentiable with $F_n'(x) = \int_a^{a+n} f_x(x,t)\,\mathrm{d}t$, and the uniform convergence of $\int_a^\infty f_x(x,t)\,\mathrm{d}t$ that (F_n') converges uniformly on I to $x \mapsto \int_a^\infty f_x(x,t)\,\mathrm{d}t$. Hence the Differentiation Theorem for function sequences can be used to finish the proof. □

In a way the deepest result used in the foregoing "elementary" proofs is that continuous functions on compact subsets of $\mathbb{R}^n$, here rectangles $K = [\alpha, \beta] \times [a, b] \subset \mathbb{R}^2$, are uniformly continuous. In the Calculus III lecture slides this was shown for the 1-dimensional case $K = [a, b]$ on two different occasions with two different proofs, one based on the Bolzano-Weierstrass Theorem and the other on the Heine-Borel covering property.

Exercise

Translate in detail one of the two proofs mentioned above into the present 2-dimensional setting.

Exercise

Suppose $U \subseteq \mathbb{R}^2$ is open and contains a compact segment $\{(0, y); a \leq y \leq b\}$ of the y-axis. Show that there exists $\delta > 0$ such that $[-\delta, \delta] \times [a, b] \subseteq U$.

A proof of the Continuity Lemma can also be based on this property.

Criteria for Uniform Convergence

The tests for uniform convergence of function series have continuous analogues, which we now discuss. We state these in the original setting for a function $f\colon D \times [0,\infty) \to \mathbb{R}$ and tacitly assume that the Riemann integrals $\int_0^R f(x,t)\,\mathrm{d}t$, and hence also $\int_0^R |f(x,t)|\,\mathrm{d}t$, exist for all $R \in [0,\infty)$ and $x \in D$. (In particular this is the case if D is an interval in $\mathbb{R}$ and f is continuous as a two-variable function.) As usual, the Cauchy test is the basis for all others.

Cauchy test

$\int_0^\infty f(x,t)\,\mathrm{d}t$ converges uniformly on D iff for every $\epsilon > 0$ there exists $R_0 > 0$ such that $\left|\int_R^{R'} f(x,t)\,\mathrm{d}t\right| < \epsilon$ for all $R' > R > R_0$ and $x \in D$.

Proof.

We only prove "$\Longleftarrow$", which is more difficult. Define a sequence of functions $F_n\colon D \to \mathbb{R}$ by $F_n(x) = \int_0^n f(x,t)\,\mathrm{d}t$, $n = 0,1,2,\dots$. Under the given assumption F_n clearly satisfies the Cauchy test for uniform convergence of function sequences, and hence converges uniformly to a function $F\colon D \to \mathbb{R}$.

Proof cont'd.

Further, given $\epsilon > 0$, let R_0 be the corresponding response as stated in the Cauchy test. For $n > R > R_0$ we then have

$$\left| F_n(x) - \int_0^R f(x,t)\,\mathrm{d}t \right| = \left| \int_R^n f(x,t)\,\mathrm{d}t \right| < \epsilon \quad \text{for } x \in D.$$

Letting $n \to \infty$ we obtain $\left| F(x) - \int_0^R f(x,t)\,\mathrm{d}t \right| \leq \epsilon$ for all $R > R_0$ and $x \in D$, which shows that $\int_0^\infty f(x,t)\,\mathrm{d}t$ converges uniformly on D to $F(x)$. □

Weierstrass's Test

Suppose there exists a function $\Phi\colon [0,\infty) \to \mathbb{R}$ such that $|f(x,t)| \leq \Phi(t)$ for all $(x,t) \in D \times [0,\infty)$ and $\int_0^\infty \Phi(t)\,\mathrm{d}t$ converges in $\mathbb{R}$. Then $\int_0^\infty f(x,t)\,\mathrm{d}t$ converges uniformly and (absolutely) on D.

Since necessarily $\Phi \geq 0$, this is actually a special case of Lebesgue's Dominated Convergence Theorem, but it has a simple proof based on the Cauchy test: One needs only observe that $\left| \int_R^{R'} f(x,t)\,\mathrm{d}t \right| \leq \int_R^{R'} |f(x,t)|\,\mathrm{d}t \leq \int_R^{R'} \Phi(t)\,\mathrm{d}t$ independently of $x \in D$, and use the (reverse) Cauchy test for the ordinary improper integral $\int_0^\infty \Phi(t)\,\mathrm{d}t$.

Example

The Gamma function

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t,\quad x\in(0,\infty)$$

can be discussed in the present framework without recourse to Lebesgue integration theory. Since for $x<1$ the integral is improper on both ends, an appropriate definition is

$$\Gamma(x)=\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t:=\int_0^1 t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t+\int_1^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$$

reducing the discussion to the one-sided improper integrals

$$\Gamma_0(x)=\int_0^1 t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t=\lim_{r\downarrow 0}\int_r^1 t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t,$$
$$\Gamma_1(x)=\int_1^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t=\lim_{R\to\infty}\int_1^R t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t\,.$$

Provided we are allowed to differentiate k-times under the integral sign, the corresponding derivatives are

Example (cont'd)

$$\Gamma_0^{(k)}(x) = \int_0^1 (\ln t)^k t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t, \quad \Gamma_1^{(k)}(x) = \int_1^\infty (\ln t)^k t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t\,.$$

To justify the differentiations, it suffices to show uniform convergence of these integrals on intervals of the form $[a, b]$ with $0 < a < b$. This can be done with the Weierstrass test:

$$\begin{aligned}
\left|(\ln t)^k t^{x-1} \mathrm{e}^{-t}\right| &\leq |\ln t|^k\, t^{a-1}, && t \in (0,1],\ x \geq a,\\
\left|(\ln t)^k t^{x-1} \mathrm{e}^{-t}\right| &\leq (\ln t)^k t^{b-1} \mathrm{e}^{-t}, && t \in [1,\infty)\ x \leq b.
\end{aligned}$$

The integrals $\int_0^1 |\ln t|^k\, t^{a-1}\,\mathrm{d}t$, $\int_1^\infty (\ln t)^k t^{b-1} \mathrm{e}^{-t}\,\mathrm{d}t$ exist, as is easily shown using the growth behavior of $\log$, $\exp$. Thus the Weierstrass can be applied as claimed.

Putting things together, we obtain that Γ has derivatives of all orders, given by

$$\Gamma^{(k)}(x) = \Gamma_0^{(k)}(x) + \Gamma_1^{(k)}(x) = \int_0^\infty (\ln t)^k t^{x-1} \mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for } k = 1, 2, \ldots$$

Example (cont'd)

Recall that we had derived these results using the theory of the Lebesgue integral. In particular, by applying the Monotone Convergence Theorem to the sequence of functions $f_n \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(t) = \begin{cases} t^{x-1}\mathrm{e}^{-t} & \text{if } x \in \left[\frac{1}{n}, n\right], \\ 0 & \text{otherwise} \end{cases}$$

(considering x as fixed), we had established the existence of $\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t$ as a Lebesgue integral.

The argument used in an essential way the non-negativity of the integrand $t^{x-1}\mathrm{e}^{-t}$, which implies that (f_n) is monotonically increasing. The Monotone Convergence Theorem then gives

$$\int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\,\mathrm{d}t = \lim_{n\to\infty} \int_{1/n}^n t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t\,.$$

For general functions, however, the existence of such very special limits do not imply the existence of the corresponding “infinite” (improper Riemann or Lebesgue) integral.

Example (cont'd)

As an example consider

$$\int_{1/n}^{n} \frac{\ln x}{x}\,\mathrm{d}x = \left[\tfrac{1}{2}\ln(x)^2\right]_{1/n}^{n} = \ln(n)^2 - \ln(1/n)^2 = 0.$$

Thus we also have

$$\lim_{n\to\infty} \int_{1/n}^{n} \frac{\ln x}{x}\,\mathrm{d}x = 0,$$

but the improper integral

$$\int_0^{\infty} \frac{\ln x}{x}\,\mathrm{d}x = \int_0^{1} \frac{\ln x}{x}\,\mathrm{d}x + \int_1^{\infty} \frac{\ln x}{x}\,\mathrm{d}x = -\infty + \infty$$

doesn't exist.

Exercise

1. Show that in the definition of a two-sided improper integral over $(0,\infty)$ in terms of two one-sided improper integrals we can take any number $a>0$ instead of $a=1$ as intermediate point, i.e., the value of

$$\int_0^a f(x)\,\mathrm{d}x+\int_a^\infty f(x)\,\mathrm{d}x=\lim_{r\downarrow 0}\int_r^a f(x)\,\mathrm{d}x+\lim_{R\to\infty}\int_a^R f(x)\,\mathrm{d}x$$

doesn't depend on a.

2. In stark contrast with this, prove the following fact regarding the previous example. Given any numbers $0<r<R$ and $C\in\mathbb{R}$ show that there exists $a\in(0,r)$ and $b\in(R,\infty)$ such that

$$\int_a^b \frac{\ln x}{x}\,\mathrm{d}x=C.$$

In other words, by letting $a\downarrow 0$ and $b\to\infty$ in a certain "dependent" way, we can achieve that $\int_0^\infty \frac{\ln x}{x}\,\mathrm{d}x$ converges to any prescribed value.

Dirichlet's and Abel's Tests

Suppose that for every $x \in D$ the function $t \mapsto f(x,t)$ is continuously differentiable and monotonically decreasing, and $t \mapsto g(x,t)$ is continuous. Then $\int_0^\infty f(x,t)g(x,t)\,\mathrm{d}t$ converges uniformly on D under each of the following assumptions:

1. $f(x,t)$ converges uniformly to zero for $t \to \infty$, and there exists a "uniform bound" $M > 0$ such that $\left|\int_0^R g(x,t)\,\mathrm{d}t\right| \leq M$ for all $R \in [0,\infty)$ and $x \in D$.
2. There exists $M > 0$ such that $|f(x,t)| \leq M$ for all $t \in [0,\infty)$ and $x \in D$, and $\int_0^\infty g(x,t)\,\mathrm{d}t$ converges uniformly on D.

Proof.

The functions $t \mapsto f(x,t)$ and $t \mapsto g(x,t)$ satisfy the assumptions of the Second Mean Value Theorem for integrals; cf. subsequent slide. Hence, given $x \in D$ and $0 < R < R'$, there exists $\tau \in [R,R']$ such that

$$\int_R^{R'} f(x,t)g(x,t)\,\mathrm{d}t = f(x,R)\int_R^\tau g(x,t)\,\mathrm{d}t + f(x,R')\int_\tau^{R'} g(x,t)\,\mathrm{d}t\,.$$

Proof cont'd.

Case 1 (Dirichlet): We have

$$\left|\int_R^\tau g(x,t)\,\mathrm{d}t\right| = \left|\int_0^\tau g(x,t)\,\mathrm{d}t - \int_0^R g(x,t)\,\mathrm{d}t\right| \leq 2M,$$

and similarly $\left|\int_\tau^{R'} g(x,t)\,\mathrm{d}t\right| \leq 2M$. Since $f(x,t) \to 0$ uniformly for $t \to \infty$, there exists $R_0 \in [0,\infty)$ such that $|f(x,t)| < \epsilon/(4M)$ for all $t > R_0$ and $x \in D$. For $R' > R > R_0$ we then get

$$\left|\int_R^{R'} f(x,t)g(x,t)\,\mathrm{d}t\right| < \frac{\epsilon}{4M}\cdot 2M + \frac{\epsilon}{4M}\cdot 2M = \epsilon,$$

independently of $x \in D$. Applying the Cauchy test for uniform convergence finishes the proof.

Case 2 (Abel): Here we can apply the reverse Cauchy test to $\int_0^\infty g(x,t)\,\mathrm{d}t$. If R_0 denotes a response to $\epsilon/(2M)$ in this test and $R' > R > R_0$, we can upper-bound $\left|\int_R^\tau g(x,t)\,\mathrm{d}t\right|$ and $\left|\int_\tau^{R'} g(x,t)\,\mathrm{d}t\right|$ by $\epsilon/(2M)$, and then finish the proof in the same way. □

Exercise

The following facts are commonly called the First and Second Mean Value Theorem for (Riemann) integrals. Prove these facts.

1. Suppose $f\colon [a,b] \to \mathbb{R}$ is non-negative and integrable and $g\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $\tau \in [a,b]$ such that
$$\int_a^b f(t)g(t)\,\mathrm{d}t = g(\tau)\int_a^b f(t)\,\mathrm{d}t\,.$$
Hint: Let $m = \min\{g(t); t \in [a,b]\}$, $M = \max\{g(t); t \in [a,b]\}$. Then $m \leq g(t) \leq M$ for $t \in [a,b]$. Multiply these inequalities by $f(t)$ and integrate over $[a,b]$.

2. Suppose $f\colon [a,b] \to \mathbb{R}$ is continuously differentiable and monotonic and $g\colon [a,b] \to \mathbb{R}$ is continuous. Then there exists $\tau \in [a,b]$ such that
$$\int_a^b f(t)g(t)\,\mathrm{d}t = f(a)\int_a^\tau g(t) + f(b)\int_\tau^b g(t)\,\mathrm{d}t\,.$$
Hint: Use integration by parts with $G(t) = \int_a^t g(s)\,\mathrm{d}s$, and apply the First Mean Value Theorem to the resulting integral.

Example

We consider the function

$$F(x)=\int_0^\infty \frac{\sin t}{t}\,\mathrm{e}^{-xt}\,\mathrm{d}t,\quad x\in[0,\infty). \tag{FE}$$

For $x\geq\delta>0$ we can estimate the absolute values of the integrand $f(x,t)=\frac{\sin t}{t}\,\mathrm{e}^{-xt}$ and its partial derivative $f_x(x,t)=-\sin t\,\mathrm{e}^{-xt}$ by $\Phi(t)=\mathrm{e}^{-\delta t}$, and apply Weierstrass's test to show the uniform convergence of the integrals $\int_0^\infty \frac{\sin t}{t}\,\mathrm{e}^{-xt}\,\mathrm{d}t$ and $\int_0^\infty \sin t\,\mathrm{e}^{-xt}\,\mathrm{d}t$ on $[\delta,\infty)$ for any $\delta>0$. The Differentiation Theorem then implies that F is differentiable on $(0,\infty)$ with $F'(x)=-\int_0^\infty \sin t\,\mathrm{e}^{-xt}\,\mathrm{d}t$.

Using integration by parts on the expression for $F'(x)$ two times (either way), one finds that $F'(x)=-1/(1+x^2)$ and hence $F(x)=-\arctan x+C$ for $x>0$, where C is some constant. In fact $C=\pi/2$ and $F(x)=\pi/2-\arctan x=\operatorname{arccot} x$, as follows from $\lim_{x\to\infty}F(x)=0$. The latter can be proved by (with justification!) interchanging the order of limit and integral in (FE), but is also easy to see directly:

$$|F(x)|\leq\int_0^\infty\left|\frac{\sin t}{t}\,\mathrm{e}^{-xt}\right|\mathrm{d}t\leq\int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}t=\frac{1}{x}\quad\text{for } x>0.$$

Example (cont'd)

The actual motivation to consider the function $F(x)$ is that it leads to an evaluation of the famous *Dirichlet integral*

$$F(0) = \int_0^\infty \frac{\sin t}{t} \, \mathrm{d}t = \frac{\pi}{2}. \tag{D}$$

For this we need to show that F is continuous in $x = 0$, because then we can use $F(0) = \lim_{x \downarrow 0} F(x) = \lim_{x \downarrow 0} [\pi/2 - \arctan x] = \pi/2$. However, since (D) doesn't converge absolutely, it is impossible to argue with the Weierstrass test or Lebesgue's Bounded Convergence Theorem, making this step the most difficult in the evaluation of (D).

But with the Dirichlet test for uniform convergence at hand, it is easy to do: For $x \geq 0$, $t \geq 1$ let $f(x,t) = \mathrm{e}^{-xt}/t$, $g(x,t) = \sin t$. All assumptions of the test are satisfied (e.g., $f(x,t) \to 0$ uniformly for $t \to \infty$ follows from $0 < f(x,t) \leq 1/t$).
$\Longrightarrow \int_1^\infty \frac{\sin t}{t} \, \mathrm{e}^{-xt} \, \mathrm{d}t$ converges uniformly on $[0, \infty)$ and hence represents a continuous function $F_1(x)$ (by the Continuity Theorem).
But $F(x) = F_1(x) + \int_0^1 \frac{\sin t}{t} \, \mathrm{e}^{-xt} \, \mathrm{d}t$, and the latter is also continuous (by the Continuity Lemma).

Example

Similar reasoning can be used to evaluate the integral $\int_0^\infty \frac{\cos t}{t^2+1}\,\mathrm{d}t$.

This is the subject of an accompanying exercise (H18 of Homework 3). Here, as a preparation for the exercise, we show that $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t$$

is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ with derivative

$$F'(x) = -\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t, \quad x \neq 0.$$

Since $\left|\frac{\cos(xt)}{t^2+1}\right| \leq \frac{1}{t^2+1}$ and $\int_0^\infty \frac{\mathrm{d}t}{t^2+1} = [\arctan(t)]_0^\infty = \pi/2$ is finite, $\int_0^\infty \frac{\cos(xt)}{t^2+1}\,\mathrm{d}t$ converges uniformly on $\mathbb{R}$ (Weierstrass's test).
$\Longrightarrow$ F is continuous on $\mathbb{R}$ (Continuity Theorem).

Justifying the differentiation under the integral sign is more complicated, since the corresponding integrand $t \mapsto \frac{t\sin(xt)}{t^2+1}$ is not absolutely integrable.

Example (cont'd)

It suffices to show that $\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$ converges uniformly on $[\delta,\infty)$ for every $\delta>0$. Then the Differentiation Theorem gives $F'(x)=-\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$ for $x>0$ and, since F is even, this also holds for $x<0$.

For a proof we can apply Dirichlet's test with $f(x,t)=\frac{t}{t^2+1}$, $g(x,t)=\sin(xt)$. Since

$$\int_0^R \sin(xt)\,\mathrm{d}t=\left[-\frac{\cos(xt)}{x}\right]_0^R=\frac{1-\cos(xR)}{x}\leq\frac{2}{\delta},$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\frac{t}{t^2+1}=\frac{1-t^2}{(t^2+1)^2}<0\quad\text{for } t>1,$$

and of course $\lim_{t\to\infty}\frac{t}{t^2+1}=0$, the assumptions of Dirichlet's test are satisfied (strictly speaking, only for $\int_1^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$, but uniform convergence of $\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$ is equivalent to that of $\int_1^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$).

Example (cont'd)

The following, more direct proof using integration by parts is also instructive:

$$\begin{aligned}\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t &= \left[-\frac{\cos(xt)}{x}\,\frac{t}{t^2+1}\right]_0^\infty - \int_0^\infty -\frac{\cos(xt)}{x}\,\frac{1-t^2}{(t^2+1)^2}\,\mathrm{d}t\\ &= \frac{1}{x}\int_0^\infty \cos(xt)\,\frac{1-t^2}{(t^2+1)^2}\,\mathrm{d}t\,.\end{aligned}$$

Since $\frac{1-t^2}{(t^2+1)^2} = \mathrm{O}(t^{-2})$ for $t\to\infty$, the last integral converges uniformly for $x\in[0,\infty)$ (Weierstrass's test), and hence $\int_0^\infty \frac{t\sin(xt)}{t^2+1}\,\mathrm{d}t$ converges uniformly on each interval $[\delta,\infty)$, $\delta>0$.

The example nicely illustrates what can go wrong if you blindly interchange limits and integration without thinking about proper justification: From the formula for $F'(x)$ one is tempted to conclude that $F'(0) = -\int_0^\infty \frac{t\sin(0\,t)}{t^2+1}\,\mathrm{d}t = 0$, but this is wrong! When you solve Exercise H18 you will see that F is not differentiable at $x=0$.