Math 286 Differential Equations (Prof. Honold) **Midterm Exam** 2020/05/14

Name: ____________________ Student No.: ____________________ All Groups

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. The (real or complex) solution space of $y - 2y' + t\,y^{(3)} = 0,\ \ t > 0,$ has dimension

   ☐ 1 ☐ 2 ☐ 3 ☐ 4 ☐ 5

2. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $y' = 2y,\ \ y(1) = 1$ has $\phi_2(t)$ equal to

   ☐ $-1 + 2t - 2t^2$ ☐ $1 - 2t + 2t^2$ ☐ $1 - t + t^2$ ☐ $-1 + t - t^2$
   ☐ $1 + t + t^2/2$

3. For the solution $y(t)$ of the IVP $y' = (y-1)\cos t,\ \ y(0) = 2$ the value $y(\pi/2)$ is equal to

   ☐ e ☐ $1 + \mathrm{e}$ ☐ $1 + 2\mathrm{e}$ ☐ $1 - \mathrm{e}$ ☐ $2 + \mathrm{e}$

4. Which of the following ODE's has distinct solutions $y_1, y_2 \colon [0,1) \to \mathbb{R}$ satisfying $y_1(0) = y_2(0)$?

   ☐ $y' = y^2$ ☐ $y' = y\sqrt{t}$ ☐ $y' = t\sqrt{y}$ ☐ $y' = ty$ ☐ $y' = |y|$

5. $\mathrm{e}^x(x+1)\,\mathrm{d}x + (y\mathrm{e}^y - x\mathrm{e}^x)\,\mathrm{d}y = 0$ has the integrating factor

   ☐ 0 ☐ 1 ☐ e^{-x} ☐ e^{-y} ☐ e^{-x-y}

6. For the solution $y(t)$ of the IVP $y' = y^3 - 4y,\ \ y(0) = 3$ the limit $\lim\limits_{t \to +\infty} y(t)$ is equal to

   ☐ 0 ☐ 2 ☐ -2 ☐ $+\infty$ ☐ $-\infty$

7. For the solution $y(t)$ of the IVP $y' = (\cos t)/y,\ \ y(0) = 1$ the value $y(\pi/2)$ is equal to

   ☐ 0 ☐ 1 ☐ $\frac{1}{2}$ ☐ $\sqrt{3}$ ☐ $\frac{1}{2}\sqrt{3}$

8. For which of the following ODE's does the set of solutions $\phi \colon \mathbb{R} \to \mathbb{R}$ <u>not</u> form a (linear) subspace of $\mathbb{R}^{\mathbb{R}}$?

   ☐ $y' = |t|\,y$ ☐ $yy' = 0$ ☐ $ty' = y$ ☐ $y' = t(y+1)$ ☐ $y'' = t^2(y' - y)$

9. For which choice of $f_n(x)$ does $\sum\limits_{n=1}^{\infty} f_n(x)$ converge uniformly on $[0, +\infty)$?

   ☐ $f_n(x) = \sin(x)/n$ ☐ $f_n(x) = \mathrm{e}^{-nx}/n$ ☐ $f_n(x) = x/n^4$
   ☐ $f_n(x) = 1/(n + x^2)$ ☐ $f_n(x) = 1/(n^2 + x)$

10. If $y = y(x)$ solves $y' = x/y$ then $z = y/x$ solves

    ☐ $z' = z$ ☐ $z' = (1 - z^2)/(xz)$ ☐ $z' = xz/(1 - z^2)$ ☐ $z' = 0$
    ☐ $z' = 1/z$

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11. For the solution $y\colon (0,+\infty) \to \mathbb{R}$ of the IVP $\quad t^2y'' - 2y = 0, \quad y(1) = y'(1) = 1 \quad$ the value $y(2)$ is equal to

- [ ] $\frac{2}{6}$
- [ ] $\frac{7}{6}$
- [ ] $\frac{12}{6}$
- [x] $\frac{17}{6}$
- [ ] $\frac{22}{6}$

12. Any solution $y(t)$ of $\quad y'' + 4y = 0 \quad$ satisfying $y(0) = 0$ also satisfies

- [ ] $y(\pi/4) = 0$
- [x] $y(\pi/2) = 0$
- [ ] $y'(0) = 0$
- [ ] $y'(0) = 1$
- [ ] $y'(0) = 2$

13. $y'' - 2y' + y = e^t - 2 \quad$ has a particular solution $y_p(t)$ of the form

- [ ] $c_0 + c_1 e^t$
- [ ] $c_0 + c_1 t e^t$
- [x] $c_0 + c_1 t^2 e^t$
- [ ] $(c_0 + c_1 t)e^t$
- [ ] $(c_0 + c_1 t^2)e^t$

14. The function $\quad f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^a} \cos\big((2k+1)x\big), \quad a \in \mathbb{Z}, \quad$ is continuous on $\mathbb{R}$ if and only if

- [ ] $a \geq 0$
- [ ] $a \geq 1$
- [x] $a \geq 2$
- [ ] $a \geq 3$
- [ ] $a \geq 4$

15. Maximal solutions of $\quad y' = y^2 - y + 1 \quad$ are defined on an interval of the form

- [x] (a,b)
- [ ] $[a,b]$
- [ ] $(a,+\infty)$
- [ ] $(-\infty,b)$
- [ ] $(-\infty,+\infty)$

with $a, b \in \mathbb{R}$.

Time allowed: 45 min | CLOSED BOOK | ***Good luck!***

Notes

1. The ODE appeared in the three forms $y - 2y' + t\,y^{(n)} = 0$ with $n \in \{3,4,5\}$ and the correct answer is "n". Reason: On $(0,\infty)$ the ODE is equivalent to $y^{(n)} - \frac{2}{t}y' + \frac{1}{t}y = 0$, which is an explicit/monic (time-dependent) linear ODE of of order n and hence has a solution space of dimension n; cf. the lecture.

2. The initial values where $y(1) = 1$ (correct answer $1 - 2t + 2t^2$) and $y(1) = -1$ (correct answer $-1 + 2t - 2t^2$). The computation in the latter case is

$$
\begin{aligned}
&\phi_0(t) = -1,\\
&\phi_1(t) = -1 + \int_1^t 2\phi_0(s)\,\mathrm{d}s = -1 + (t-1)(-2) = -2t + 1,\\
&\phi_2(t) = -1 + \int_1^t 2\phi_1(s)\,\mathrm{d}s = -1 + \int_1^t -4s + 2\,\mathrm{d}s = -1 + \left[-2s^2 + 2s\right]_1^t = -1 - 2t^2 + 2t.
\end{aligned}
$$

3. The initial values where $y(0) = 2$ (correct answer $1 + \mathrm{e}$) and $y(0) = 3$ (correct answer $1 + 2\mathrm{e}$). The ODE is inhomogeneous linear with general solution $y(t) = 1 + c\,\mathrm{e}^{\sin t}$ (note that $y \equiv 1$ is a particular solution!). The initial conditions $y(0) = 2$ and $y(0) = 3$ give $c = 1$ and $c = 2$, respectively, from which the correct answers follow.

4. "$y' = t\sqrt{y}$" is the only candidate, since the other 4 ODE's satisfy the assumptions of the Existence and Uniqueness Theorem. In particular, $y' = y\sqrt{t}$ is 1st-order linear (the square root doesn't matter here, because the coefficient functions of a linear ODE need only be continuous), and $y' = |y|$ satisfies a Lipschitz condition with respect to y with Lipschitz constant $L = 1$.

   Of course this reasoning doesn't prove the existence of distinct solutions y_1, y_2 of $y' = t\sqrt{y}$ with $y_1(0) = y_2(0)$, but you can check that $y_1(t) \equiv 0$, $y_2(t) = \frac{1}{16}t^4$ provide an example.

5. You can check that $M\,\mathrm{d}x + N\,\mathrm{d}y = \mathrm{e}^{x-y}(x+1)\,\mathrm{d}x + (y - x\mathrm{e}^{x-y})\,\mathrm{d}y = 0$ satisfies the condition $M_y = -\mathrm{e}^{x-y}(x+1) = N_x$ and hence is exact on $\mathbb{R}^2$. Although $0\,\mathrm{d}x + 0\,\mathrm{d}y = 0$ is trivially exact, 0 is not an integrating factor, because integrating factors are required to be nonzero everywhere. The other 3 ODE's obtained by applying one of the remaining factors are not exact.

6. The statement of Q6 unfortunately contained an error.

   The phaseline can be used to answer such questions. The zeros of $f(y) = y^3 - 4y$ are 0, ± 2, and the interval containing $y_0 = 3$ determined by the zeros is $(2, +\infty)$. Since $f(y) > 0$ for $y \in (2, +\infty)$, solutions starting in $(2, +\infty)$ are monotonically increasing and satisfy $\lim_{t \to t_\infty} y(t) = +\infty$, where t_∞ is determined by

$$\int_{y_0}^{+\infty} \frac{\mathrm{d}u}{u^3 - 4u} = t_\infty - t_0.$$

   Since the improper integral is finite (because $\deg(u^3 - 4u) \geq 2$), we have $t_\infty \in \mathbb{R}$ and solutions blow up at finite time t_∞. (Evaluating the improper integral by means of partial fractions gives for $y_0 = 3$, $t_0 = 0$ the blow-up time as $t_\infty = \frac{1}{8}\ln\frac{9}{5} = \frac{1}{4}\ln 3 - \frac{1}{8}\ln 5 \approx 0.0735$.)

   Thus none of the offered answers can be correct. All students receive 1 mark for Q6.

7. This is a separable equation and can be solved with the standard method: The general solution is $y(t) = \pm\sqrt{2\sin t + C}$, $C \in \mathbb{R}$, and $y(0) = 1$ gives $y(t) = \sqrt{2\sin t + 1}$.

8. The only candidates are $yy' = 0$ and $y' = t(y+1)$, because the other 3 ODE's are equivalent to homogeneous linear ODE's. The ODE $y' = t(y+1) = ty + t$ is inhomogeneous linear and hence its set of solutions doesn't form a subspace of $\mathbb{R}^{\mathbb{R}}$ (since, e.g., the all-zero function is not a solution). The solutions of $yy' = 0$ are precisely the constant functions $y(t) \equiv c$, $c \in \mathbb{R}$, which form a subspace of $\mathbb{R}^{\mathbb{R}}$. For this note that $y(t)y'(t) = 0$ is equivalent

to $y(t)^2 = c$, $c \geq 0$, i.e., to $y(t) = \pm\sqrt{c}$. Since solutions must be continuous, this forces $y(t)$ to be constant.

9. For $x \geq 0$ we have
$$\left|\frac{1}{n^2+x}\right| = \frac{1}{n^2+x} \leq \frac{1}{n^2} = M_n,$$
a bound that is independent of x. Since $\sum_{n=1}^{\infty} 1/n^2$ converges, the Weierstrass test gives that $\sum_{n=1}^{\infty} 1/(n^2+x)$ converges uniformly on $[0,\infty)$.

   The other 4 series don't converge uniformly on $[0,\infty)$. For the series $\sum_{n=1}^{\infty} x/n^4$ this can be seen as follows: The difference between the limit function and a partial sum $\sum_{k=1}^{n} x/k^4$ has the form $\sum_{k=n+1}^{\infty} x/k^4 = x\sum_{k=n+1}^{\infty} 1/k^4 = cx$ with $c > 0$. But an inequality $|cx| < \varepsilon$, $\varepsilon > 0$, is always violated by some $x \in [0,\infty)$. The remaining three series do not even converge point-wise on $[0,\infty)$ (look at $x = 0$ or $x = \pi/2$).

10. The ODE $y' = x/y = f(y/x)$ with $f(z) = 1/z$ is homogeneous. The substitution $z = y/x$ transforms it into the separable equation $z' = \big(f(z)-z)\big)/x = (1/z-z)/x = (1-z^2)/(xz)$.

11. The ODE is an Euler equation with $p_0 = 0$, $q_0 = -2$, indicial equation $r^2 + (p_0 - 1)r + q_0 = r^2 - r - 2 = 0$, exponents $r_1 = 2$, $r_2 = -1$, and general solution $y(t) = c_1 t^2 + c_2 t^{-1}$ on $(0,\infty)$. The initial conditions give $y(t) = \frac{2}{3}y^2 + \frac{1}{3}y^{-1}$.

12. The general real solution of $y'' + 4y = 0$ is $y(t) = a\cos(2t) + b\sin(2t)$ with $a, b \in \mathbb{R}$. Solutions with $y(0) = 0$ have $a = 0$ and hence satisfy $y(\pi/2) = b\sin\pi = 0$. The first answer is ruled out by setting $b = 1$ and the last three answers by setting $b = 0$.

13. A particular solution of $y'' - 2y' + y = -2$ is $y(t) \equiv -2$, and the correct „Ansatz" for obtaining a solution of $y'' - 2y' + y = \mathrm{e}^t$ is $y(t) = c_1 t^2 \mathrm{e}^t$ (because $\lambda = 1$ is a root of multiplicity 2 of the characteristic polynomial $X^2 - 2X + 1$). Superposition then gives a solution of $y'' - 2y' + y = \mathrm{e}^t - 2$ of the form $c_0 + c_1 t^2 \mathrm{e}^t$, viz. $y(t) = -2 + \frac{1}{2}t^2\mathrm{e}^t$.

14. The k-th summand of the series can be bounded in absolute value by $M_k = \frac{1}{(2k+1)^a}$. For $a \geq 2$ the series $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^a}$ converges (just like the series $\sum_{k=1}^{\infty} \frac{1}{k^a}$). The Weierstrass test then yields that the function series defining f converges uniformly on $\mathbb{R}$. By the Continuity Theorem, f is continuous. For $a = 1$ the series is essentially Fourier's cosine series (except for scale factors in the domain and codomain). Since Fourier's cosine series represents a discontinuous function (cf. lecture), the same is true of the series under consideration. (More precisely, f is discontinuous at $x = \pm\pi/2, \pm3\pi/2, \pm5\pi/2, \dots$)

15. Since $y^2 - y + 1 = 0$ has no real roots, the corresponding canonical form is $z' = z^2 + 1$, which is solved by $z(t) = \tan(t + C)$. Hence the domain of any maximal solution is a bounded open interval.