

Name: _____

Student No.: _____

Group A

For each of the following problems, find the correct answer (tick as appropriate!). No justifications are required. Each problem has exactly one correct solution, which is worth 1 mark. Incorrect solutions (including no answer, multiple answers, or unreadable answers) will be assigned 0 marks; there are no penalties.

1. Which of the following ODE's has distinct solutions $y_1, y_2: (0, 2) \rightarrow \mathbb{R}$ satisfying $y_1(1) = y_2(1)$?

☐ $y' = \sqrt{t}|y|$ ☒ $y'' = yy'$ ☒ $y' = t \ln y$ ☐ $y' = y \ln t$ ☐ $yy' = 0$

2. The ODE $3x dx - (y - 3x^2/y) dy$ has the integrating factor

☐ 0 ☐ x ☐ y ☐ x^2 ☒ y^2

3. For the solution $y(t)$ of the IVP $y' = y^4 - 1$, $y(2021) = 0$ the limit $\lim_{t \rightarrow +\infty} y(t)$ equals

☐ $-\infty$ ☒ -1 ☐ 0 ☐ 1 ☐ $+\infty$

4. For the solution $y(t)$ of the IVP $y' = e^{t-2y}$, $y(0) = 0$ the value $y(1)$ is contained in

☐ $[0, \frac{1}{2}]$ ☒ $[\frac{1}{2}, 1]$ ☐ $[1, \frac{3}{2}]$ ☐ $[\frac{3}{2}, 2]$ ☐ $[2, \frac{5}{2}]$

5. For the solution $y: [0, \infty) \rightarrow \mathbb{R}$ of the IVP $(t+1)(y'+1) = 2y$, $y(0) = 0$ the value $y(1)$ is equal to

☒ -2 ☐ -4 ☐ -6 ☐ -8 ☐ -10

6. The power series $z + z^2 + z^4 + z^8 + z^{16} + \dots$ has radius of convergence

☐ 0 ☐ $\frac{1}{2}$ ☒ 1 ☐ 2 ☐ ∞

7. The smallest integer a such that $f_a(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^a}$ is differentiable on \mathbb{R} is equal to

☐ 1 ☐ 2 ☒ 3 ☐ 4 ☐ 5

8. For which choice of $f_n(x)$ does the function sequence (f_n) converge uniformly on \mathbb{R} ?

☐ $f_n(x) = e^{-nx^2}$ ☒ $f_n(x) = x/(1+nx^2)$ ☐ $f_n(x) = 1/(1+nx^2)$
☒ $f_n(x) = x/(1+n^2)$ ☐ $f_n(x) = e^{-n^2x}$

9. If $y(t)$ solves $y' = 1 + y/t - y^2/t^2$ then $z = 1/(y(t) - t)$ solves

☐ $z' = z/t - 1/t^2$ ☐ $z' = tz + t^2$ ☐ $z' = tz - t^2$ ☐ $z' = t^2z + t$
☒ $z' = z/t + 1/t^2$

10. The sequence $\phi_0, \phi_1, \phi_2, \dots$ of Picard-Lindelöf iterates for the IVP $y' = 2y + 2$, $y(0) = 2$ has $\phi_2(t)$ equal to

☐ $2t + 6t^2$ ☐ $2t + 5t^2$ ☒ $2 + 6t + 6t^2$ ☐ $2t + 4t^2$ ☐ $1 + 4t + 4t^2$

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11. $y'' - 3y' + 2y = 2 + te^t$ has a particular solution $y_p(t)$ of the form

☐ $c_0 + c_1e^t + c_2te^t$

☒ $c_0 + c_1te^t + c_2t^2e^t$

☐ $c_0 + c_1t^2e^t$

☐ $c_0t + c_1te^t + c_2t^2e^t$

☐ $c_0t + c_1t^2e^t$

12. Maximal solutions of $y' = y^3 + 1$ satisfying $y(0) = 0$ are defined on an interval of the form

☒ (a, b)

☐ $[a, b]$

☐ $(a, +\infty)$

☒ $(-\infty, b)$

☐ $(-\infty, +\infty)$

with $a, b \in \mathbb{R}$.

13. For $A = \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$, the matrix e^{At} is equal to

☒ $\begin{pmatrix} e^{-t} & e^{2t} - e^{-t} \\ 0 & e^{2t} \end{pmatrix}$

☐ $\begin{pmatrix} e^{-t} & e^t \\ 0 & e^{2t} \end{pmatrix}$

☐ $\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{pmatrix}$

☐ $\begin{pmatrix} -e^t & e^t \\ 0 & 2e^t \end{pmatrix}$

☐ $\begin{pmatrix} e^{-t} & e^{-t} - e^{2t} \\ 0 & e^{2t} \end{pmatrix}$

14. The matrix norm of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ (subordinate to the Euclidean length on \mathbb{R}^2) is equal to

☐ 0

☐ 1

☐ $\sqrt{2}$

☐ 2

☒ $1 + \sqrt{2}$

15. A contraction $T: M \rightarrow M$, $M \subseteq \mathbb{R}^2$, has a fixed point if M is

☐ infinite

☐ connected

☐ open

☒ closed

☐ bounded

Time allowed: 50 min

CLOSED BOOK

Good luck!

Notes

1. The correct answer is $y'' = yy'$, since this is an explicit 2nd-order ODE and in addition to $y(1)$ we can also prescribe $y'(1)$ freely. Of course, one also needs to check that (maximal) solutions of such IVP's are defined on $(0, 2)$. This can be done, but you can also check that $y_1(t) \equiv -2$ and $y_2(t) = -2/t$ form a pair of solutions with the required property.

The implicit ODE $yy' = 0$ doesn't satisfy the assumptions of the EUT, but has only the constant solutions. The remaining three ODE's satisfy all assumptions of the EUT.

2. Multiplying the ODE with y^2 gives the ODE $3xy^2 dx - (y^3 - 3x^2y) dy = 0$, which is of the form $P dx + Q dy = 0$ with $P_y = 6xy = Q_x$, and hence exact. The other nonzero factors do not yield an ODE with $P_y = Q_x$. Although $0 dx + 0 dy = 0$ is trivially exact, 0 is not an integrating factor, because integrating factors are required to be nonzero everywhere.
3. The phaseline should be used to answer this question. We have $f(y) = y^4 - 1 = (y^2 - 1)(y^2 + 1)$ with roots $z_1 = -1, z_2 = 1$. Since $t_0 = 0 \in (z_1, z_2)$ and f is negative in (z_1, z_2) , we must have $\lim_{t \rightarrow +\infty} y(t) = z_1 = -1$; cf. lecture.
4. This is a separable ODE and can be solved with the standard method: $e^{2y} dy = e^t \implies \frac{1}{2} e^{2y} = e^t + C \implies y = \frac{1}{2} \ln(2e^t + 2C)$ with $C \in \mathbb{R}$. The initial value $y(0) = 0$ gives $y(t) = \frac{1}{2} \ln(2e^t - 1)$, $y(1) = \frac{1}{2} \ln(2e - 1)$. Since trivially $e < 2e - 1 < e^2$, we have $1 < \ln(2e - 1) < 2$, and hence $y(1) \in [\frac{1}{2}, 1]$.
5. The ODE has explicit form $y' = \frac{2y}{t+1} - 1$, which is 1st-order linear. The general solution of the associated homogeneous ODE $y' = \frac{2y}{t+1}$ is $y(t) = c(t+1)^2$, a particular solution of the inhomogeneous ODE is $y_p(t) = (t+1)^2 \int \frac{-1}{(t+1)^2} dt = t+1$, and hence the general solution of the inhomogeneous ODE is $y(t) = c(t+1)^2 + t+1$. The initial value $y(0) = 0$ gives $c = -1$, so that $y(t) = -(t+1)^2 + t+1 = -t(t+1)$ and $y(1) = -2, y(2) = -6$.
6. As mentioned in the lecture, the radius of convergence of any non-terminating power series with coefficients in $\{0, 1\}$ is equal to 1.
7. For checking the differentiability of $f_a(x)$ one has to look at the series of derivatives, which is

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{a-1}},$$

and prove that this series converges uniformly on \mathbb{R} (or on all intervals of the form $[-R, R]$, $R > 0$). For $a \geq 3$ we can apply the Weierstrass test with $M_n = 1/n^2$ and conclude that the series of derivatives converges uniformly on \mathbb{R} . The Differentiation Theorem then gives that f_3, f_4, f_5 are differentiable. For $a = 2$ we obtain the series $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$, which doesn't converge for $x = 0$ and represents the function $x \mapsto -\ln(2 \sin \frac{x}{2})$ on $(0, 2\pi)$; cf. the lecture. Although the divergence for $x = 0$ doesn't imply that f_2 is not differentiable at 0, the latter is nevertheless true and can be seen as follows: Since $\sum_{n=1}^{\infty} \frac{\cos(nx)}{n}$ converges uniformly on every interval $[\delta, 2\pi - \delta]$, $\delta > 0$, the function f_2 is differentiable in $(0, 2\pi)$ with derivative $f_2'(x) = -\ln(2 \sin \frac{x}{2})$. Since f_2 is continuous at 0 (by the Continuity Theorem), we can apply the Mean Value Theorem and obtain

$$\frac{f_2(x)}{x} = \frac{f_2(x) - f_2(0)}{x - 0} = f'(\xi) = -\ln\left(2 \sin \frac{\xi}{2}\right) \quad \text{with } \xi \in (0, x) \text{ for small } x > 0.$$

Since the right-hand side tends to $+\infty$ for $\xi \downarrow 0$, f_2 can't be differentiable at 0.

For $a = 1$ we have seen in the lecture that $f_1(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n}$ is discontinuous at 0, let alone differentiable.

8. The correct answer is $f_n(x) = x/(1 + nx^2)$, because we can use the inequality $1 + nx^2 \geq 2\sqrt{n}x$ to conclude that $|f_n(x)| \leq \frac{1}{2\sqrt{n}}$ for $x \neq 0$, which is independent of x . Trivially this holds also for $x = 0$, and hence $f_n(x) \rightarrow 0$ uniformly.

Independently, the other 4 answers can be excluded as follows: e^{-n^2x} doesn't converge at all for $x < 0$, the limit function f of both $1/(1 + nx^2)$ and e^{-nx^2} is discontinuous ($f(x) = 0$ for $x \neq 0$ but $f(0) = 1$), which would contradict the Continuity Theorem, and $f_n(x) = x/(1 + n^2)$ converges point-wise to 0 but satisfies $f_n(1 + n^2) = 1$, implying that for $\varepsilon = 1$ no uniform response can be found.

9. This is a Riccati equation with solution $y_1(t) = t$, and it is known that the substitution $z = 1/(y - y_1)$ transforms it into a 1st-order linear equation. (You are not supposed to know this, and in fact this knowledge doesn't help since all offered answers are 1st-order linear.)

$$z = \frac{1}{y-t} \Rightarrow z' = -\frac{y' - 1}{(y-t)^2} = -\frac{y/t - y^2/t^2}{(y-t)^2} = \frac{y}{t^2(y-t)} = \frac{zy}{t^2} = \frac{z(t + 1/z)}{t^2} = \frac{z}{t} + \frac{1}{t^2}$$

10. $\phi_0(t) = 2$; $\phi_1(t) = 2 + \int_0^t 2\phi_0(s) + 2 ds = 2 + \int_0^t 6 ds = 2 + 6t$; $\phi_2(t) = 2 + \int_0^t 2\phi_1(s) + 2 ds = 2 + \int_0^t 2(2 + 6s) + 2 ds = 2 + \int_0^t 6 + 12s ds = 2 + [6s + 6s^2]_0^t = 2 + 6t + 6t^2$.
11. A solution of $y'' - 3y' + 2y = 2$ is $y_1(t) = 1$. The correct „Ansatz“ for obtaining solution of $y'' - 3y' + 2y = te^t$ is $y_2(t) = (c_1t + c_2t^2)e^t$, giving after a short computation $y_2(t) = (-t - t^2/2)e^t$. Thus a particular solution of $y'' - 3y' + 2y = 2 + te^t$ is $y_p(t) = y_1(t) + y_2(t) = 1 - (t + t^2/2)e^t$, and the general (real) solution of $y'' - 3y' + 2y = 2 + te^t$ is

$$ae^t + be^{2t} + 1 - (t + t^2/2)e^t, \quad \text{with } a, b \in \mathbb{R}.$$

Hence none of the other forms offered can yield a solution.

12. The only equilibrium solution of this autonomous ODE is $y \equiv -1$. Integrating gives

$$\int_0^y \frac{d\eta}{\eta^3 + 1} = \int_0^t d\tau = t.$$

Since the integral $\int_0^{+\infty} \frac{d\eta}{\eta^3 + 1}$ is finite, the solution is only defined for $t < t_\infty = \int_0^{+\infty} \frac{d\eta}{\eta^3 + 1}$. Since $\lim_{y \downarrow -1} \int_0^y \frac{d\eta}{\eta^3 + 1} = -\lim_{y \downarrow -1} \int_y^0 \frac{d\eta}{\eta^3 + 1} = -\infty$, there is no further restriction on the domain of y . Hence y is defined on $(-\infty, t_\infty)$.

13. Substituting $t = 0$ should produce $e^{A0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. This excludes $\begin{pmatrix} e^{-t} & e^t \\ 0 & 2e^t \end{pmatrix}$ and $\begin{pmatrix} e^{-t} & e^t \\ 0 & 2e^t \end{pmatrix}$. For the remaining three matrices we check whether the 2nd column solves $\mathbf{y}' = \mathbf{A}\mathbf{y}$. (The 1st column of all matrices is a solution.)

$$\begin{aligned} \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}' &= \begin{pmatrix} 0 \\ 2e^{2t} \end{pmatrix}, & \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix} &= \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}; \\ \begin{pmatrix} e^{2t} - e^{-t} \\ e^{2t} \end{pmatrix}' &= \begin{pmatrix} 2e^{2t} + e^{-t} \\ 2e^{2t} \end{pmatrix}, & \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{2t} - e^{-t} \\ e^{2t} \end{pmatrix} &= \begin{pmatrix} 2e^{2t} + e^{-t} \\ 2e^{2t} \end{pmatrix}; \\ \begin{pmatrix} e^{-t} - e^{2t} \\ e^{2t} \end{pmatrix}' &= \begin{pmatrix} -e^{-t} - 2e^{2t} \\ 2e^{2t} \end{pmatrix}, & \begin{pmatrix} -1 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} - e^{2t} \\ e^{2t} \end{pmatrix} &= \begin{pmatrix} -e^{-t} + 4e^{2t} \\ 2e^{2t} \end{pmatrix}. \end{aligned}$$

Thus the correct answer is $\begin{pmatrix} e^{-t} & e^{2t} - e^{-t} \\ 0 & e^{2t} \end{pmatrix}$.

14. $\|\mathbf{A}\| = \max\{|\mathbf{A}\mathbf{x}|; |\mathbf{x}| = 1\}$ cannot be smaller than the length of a column of \mathbf{A} , which is the image of a standard unit vector under $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$. Since the first column of the given matrix has length $\sqrt{5} > 2$, the correct answer must be $1 + \sqrt{2}$.
15. The correct answer is “closed”, since for Banach's Fixed Point Theorem to hold M needs to be a complete metric space, and only closed subspaces of \mathbb{R}^2 have this property.