A Standard Atmosphere

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Figure 1: Solution to Task 1, graph of altitude in kilometres against Temperature in Kelvin. The tempereature reduces rapidly in the tropopause, and has an increase in the stratosphere, and then decreases in the mesosphere.

Abstract

In this paper, I will discuss my solutions to the British Physics Olympiad (BPhO) Computational Challenge. I visualized the International Standard Atmosphere model (ISA), and deduced mathematical models connecting altitude with lapse rate, temperature, dew temperature and boiling temperature. In addition, I produced a graphical user interface which allows users to input the parameters and compare different parameters by a live updating visualisation approach. I also compared atmosphere of the Earth and the other planets in the solar system.

Level	Base h (km)	Lapse rate L (K/km)
Troposphere	0.0	-6.5
Tropopause	11.0	0.0
Stratosphere	20.0	1.0
Stratosphere	32.0	2.8
Stratopause	47.0	0.0
Mesosphere	51.0	-2.8
Mesosphere	71.0	-2.0

Table 1: Data from the International Standard Atmosphere model. Includes the base altitude and lapse rate for each level.

Introduction

Modern airways are planned precisely, from the speed and direction of the aircraft to the altitude of the aircraft, to minimize the probability of tragic occurring. Therefore, it is significant for aero industries to understand and model the atmosphere. International Standard Atmosphere (ISA) is a model used for the standardisation of aircraft instruments. It was established with ta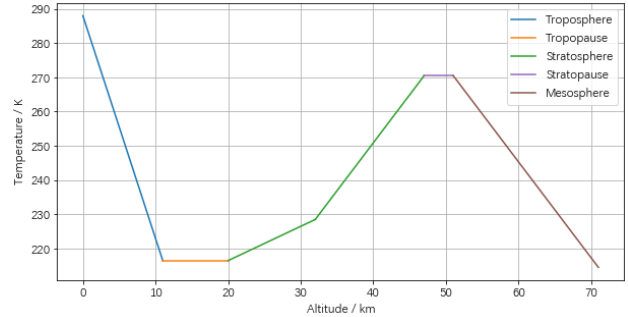bles of values over a range of altitudes, to provide a common reference for temperature and pressure. Flying in ISA-plus temperatures will have a negative impact on aircraft performance.

In the first task, I computed the change in temperature and plotted the graph on altitude in kilometres against temperature in kelvin. In the second task, I deduced a model for calculating pressure for dry air considering two circumstances, isothermal layers, and constant lapse rate. In the third task, I deduced an iterative model to calculate lapse rate, temperature, pressure, dew temperature and boiling temperature with respect to altitude. Deduction of all the mathematical models is supported by the BPhO challenge details presentation [French, 2019].

I used the programming language Python with Numpy, Pandas data analysis libraries to manipulate and compute the data, and Matplotlib library for visualization and I used Tkinter library for the Graphical User Interface.

1. Temperature vs Altitude

Plot temperature vs altitude for the International Standard Atmosphere (ISA) model.
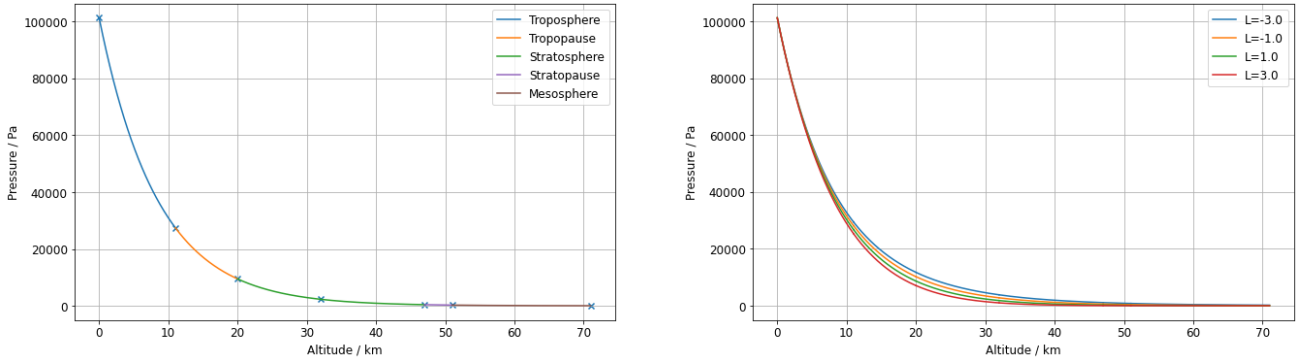
Figure 2 – Solution to Task 2. Left, graph of pressure against altitude at isothermal layer ($L = 0K/km$). Right, pressure against altitude at constant lapse rate $L = [-3.0, -1.0, 1.0, 3.0]\ K/km$. The graph shows that lower lapse rate reduces the rate of decrease of pressure.

Given the initial temperature T_0, we can use the data from *Table 1* to calculate the temperature at a certain altitude:

$$\Delta T_n = \Delta h_n \times L_n$$
$$T_n = T_{n-1} + \Delta T_n$$

Hence:

$$T_n = T_0 + \sum_0^n \Delta T$$

Where Δh_n is the difference in height between two adjacent levels, n and $n+1$. L_n is the lapse rate at n^{th} level. ΔT_n is the difference in temperature between two levels.

2. Pressure vs Altitude for dry air

Consider a $1m^2$ horizontal cross section parcel of air of density ρ at an altitude z, with vertical width dz. The atmospheric pressure change dP between altitudes z and $z + dz$ resulting from the removal of air parcel from the total weight of air above is:

$$dP = -\rho g dh$$

Where $g = 9.81 ms^{-2}$ is the gravitational constant.

Assume that the air column is an ideal gas. If n moles of gas occupy volume V at pressure P in Pascals and temperature T in Kelvin:

$$PV = nRT$$

Where $R = 8.134 Jmol^{-1}K^{-1}$ is the molar gas constant.

Assume the air column is comprised of dry air with molar mass:

$$M = 0.02896 kgmol^{-1}$$

The density of the air is the mass of n moles of dry air divided by the volume V of dry air:

$$\rho = \frac{nM}{V}$$

Hence:

$$\rho = \frac{MP}{RT}$$

Substitute into $dP = -\rho g dh$:

$$\frac{1}{P} dP = -\frac{Mg}{RT} dh$$
$$P_0 P \frac{1}{P} dP = z_0 z \frac{g}{T} dh$$
$$\ln\left(\frac{P}{P_0}\right) = -\frac{M}{R} \int_{z_0}^z \frac{g}{T} dz$$

Define temperature T in K to be a linear function of altitude h in m. The gradient is defined to be the lapse rate L in K/m.

$$T = T_0 - L(h - h_0)$$

2.1: Isothermal layer $L_0 = 0$

When $L = 0$, then $T = T0$:

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{RT_0} \int_{h_0}^h dh$$

$$P = P_0 e^{-\frac{Mg}{RT_0}(h-h_0)}$$

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2.2, Lapse rate constant, $L_0 \neq 0$

Consider $T = T0 - L(h - h0)$:

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{R}\int_{h_0}^{h}\frac{1}{T_0 - L(h - h_0)}dh$$

$$\ln\left(\frac{P}{P_0}\right) = -\frac{Mg}{(-L)R}\int_{h_0}^{h}\frac{-L}{T_0 + Lh_0 - Lh}dh$$

$$ln(\frac{P}{P_0}) = \frac{-L}{T_0 - L(h - h_0)}dh$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{Mg}{LR}[ln(T_0 + Lh_0 - Lh]_{h_0}^{h}$$

$$\ln\left(\frac{P}{P_0}\right) = \frac{Mg}{LR}ln\left(\frac{T_0 + Lh_0 - Lh}{T_0}\right)$$

$$\ln\left(\frac{P}{P_0}\right) = ln\left(\left(\frac{T_0 + Lh_0 - Lh}{T_0}\right)^{\frac{Mg}{LR}}\right)$$

$$P = P_0\left(1 - \frac{L(h - h_0)}{T_0}\right)^{\frac{Mg}{LR}}$$

Altitude h in $metres$, lapse rate L in K/m and temperature T in K.

3. Considering humidity

Any sensible description of non-arid climatology will need to consider the impact of variable amounts of water vapour contained within the air column. In the Troposphere at least, the presence of water vapour can have a dramatic influence upon thermodynamic variables such as temperature and pressure and is obviously a fundamental component of weather phenomena such as cloud and fog.

To model the effect of humidity upon temperature and pressure, we can modify our original single ideal gas assumption to consider a composite of dry air and water vapour. The molar masses of dry air and water vapour are, respectively:

$$M_d = 0.02896 kgmol^{-1}$$
$$M_v = 0.01802 kgmol^{-1}$$

The respective ideal gas equations are for n_d moles of dry air and n_v moles of water vapour:
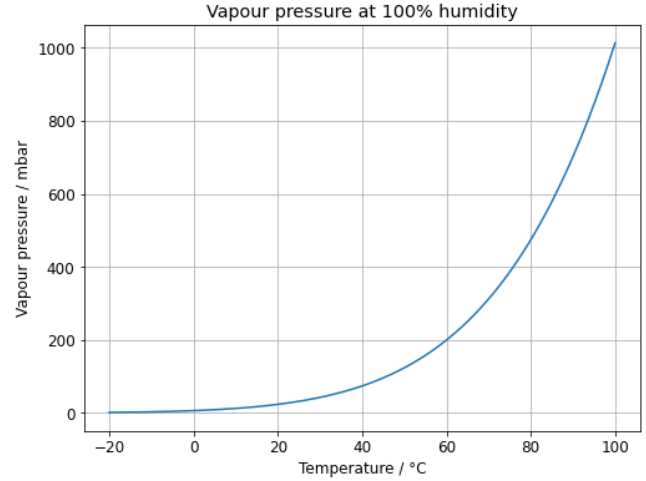

Figure 3 – Visualisation of vapour pressure at saturation in mbar against temperature in degree celcius. As temperature increases, the vapour pressure at saturation increases. The graph shows an exponential trend.

$$P = P_v + P_d$$
$$P_v = \frac{n_v RT}{V}, \qquad P_d = \frac{n_d RT}{V}$$

Where R is the Molar gas constant:
$$R = 8.314 Jmol^{-1}K^{-1}$$

3.1 Modelling by numerical method

Let us define relative humidity U as the ratio of water vapour pressure at saturation Es:
$$P_v = UE_s$$

$$E_s(T) = 6.1121e^{\left\{\left(18.678 - \frac{T_c}{234.5}\right)\left(\frac{T_c}{T_c + 257.14}\right)\right\}}$$

Where the saturation vapour pressure E_s is given in $mbar$ and temperature T_c in °C.

The overall density of the atmosphere is:

$$\rho = \frac{n_d M_d + n_v M_v}{V} = \frac{M_d}{V}\left(n_d + n_v\frac{M_v}{M_d}\right)$$

From the ideal gas equations:

$$n_v = \frac{V}{RT}P_v = \frac{V}{RT}UE_s$$

$$n_d = \frac{V}{RT}(P - P_v) = \frac{V}{RT}(P - UE_s)$$

Hence:

$$\rho = \frac{M_d}{RT}\left(P - U\left(1 - \frac{M_v}{M_d}\right)E_s(T)\right)$$

The pressure integral is now more complicated, but can be evaluated using a numerical method:

$$dP = -\rho g\, dh$$

$$\frac{dP}{dh} = -\frac{M_d g}{RT}\left(P - U\left(1 - \frac{M_v}{M_d}\right)E_s(T)\right)$$

A simple iterative numeric solution scheme might be to use a finite altitude change Δh, and start from a known temperature and pressure. E.g., 15°C, 1013.25 mbar.

$$h \rightarrow h + \Delta h$$
$$T \rightarrow T - L\Delta h$$

$$\Delta P = -\frac{M_d g}{RT}\left(P - U\left(1 - \frac{M_v}{M_d}\right)E_s(T)\right)\Delta h$$

$$P \rightarrow P + \Delta P$$

3.2 Lapse rate

The partial pressure of water vapour is typically very small compared to dry air. Therefore, we can ignore it in pressure calculations. However, the lapse rate L is significantly affected by the presence of water vapour.

$$L = -\frac{dT}{dh}$$

$$L = g\frac{1 + \frac{r\Delta H_v}{R_{sd}T}}{c_{pd} + \frac{(\Delta H_v)^2 r}{R_{sw}T^2}}$$

Where:

$$r = \frac{\varepsilon U E_s}{P - U E_s}$$

$$\varepsilon = \frac{R_{sd}}{R_{sw}}$$

Specific latent heat of vapourisation of water:
$$\Delta H_v = 2{,}501{,}000 J kg^{-1}$$
Specific heat of dry air at constant pressure:
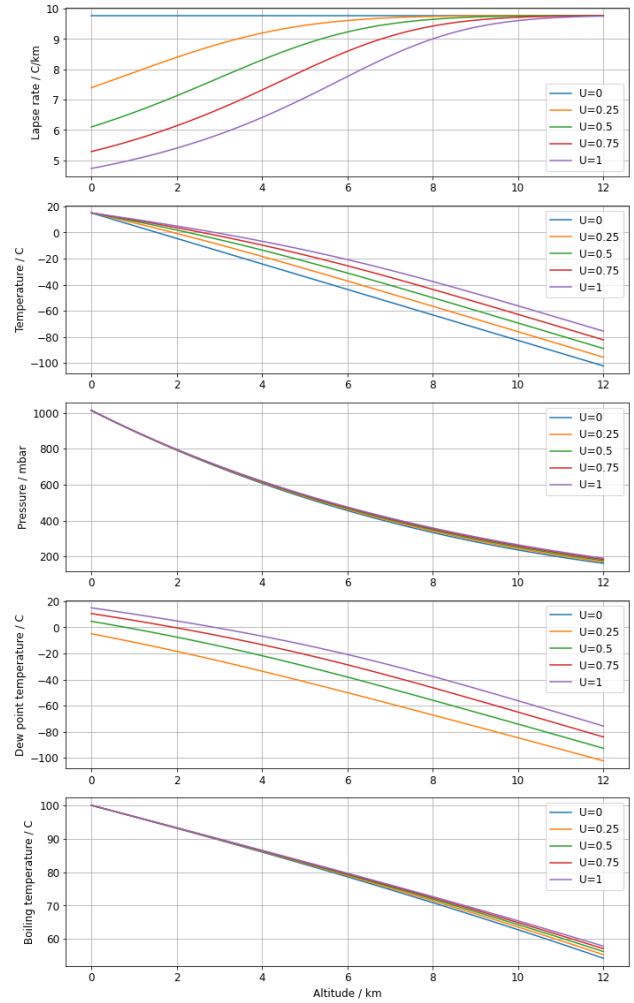$$c_{pd} = 1003.5 J kg^{-1}K^{-1}$$

Figure 4 – Solution to Task 3. U is the ratio of water vapour pressure at saturation. In order, graph of lapse rate, temperature, pressure, dew point and boiling point to altitude.

Specific gas constant for dry air:
$$R_{sd} = 287 J kg^{-1}K^{-1}$$
Specific gas constant for water vapour:
$$R_{sw} = 461.5 J kg^{-1}K^{-1}$$

3.3 Dew point

The dew point is the temperature to which a given parcel of air must be cooled at constant barometric pressure, for water vapour to condense into water.

The August-Roche Magnus approximation defines the dew point to be defined by the following expression in terms of relative humidity U (with value 0-1) and ambient air temperature T.
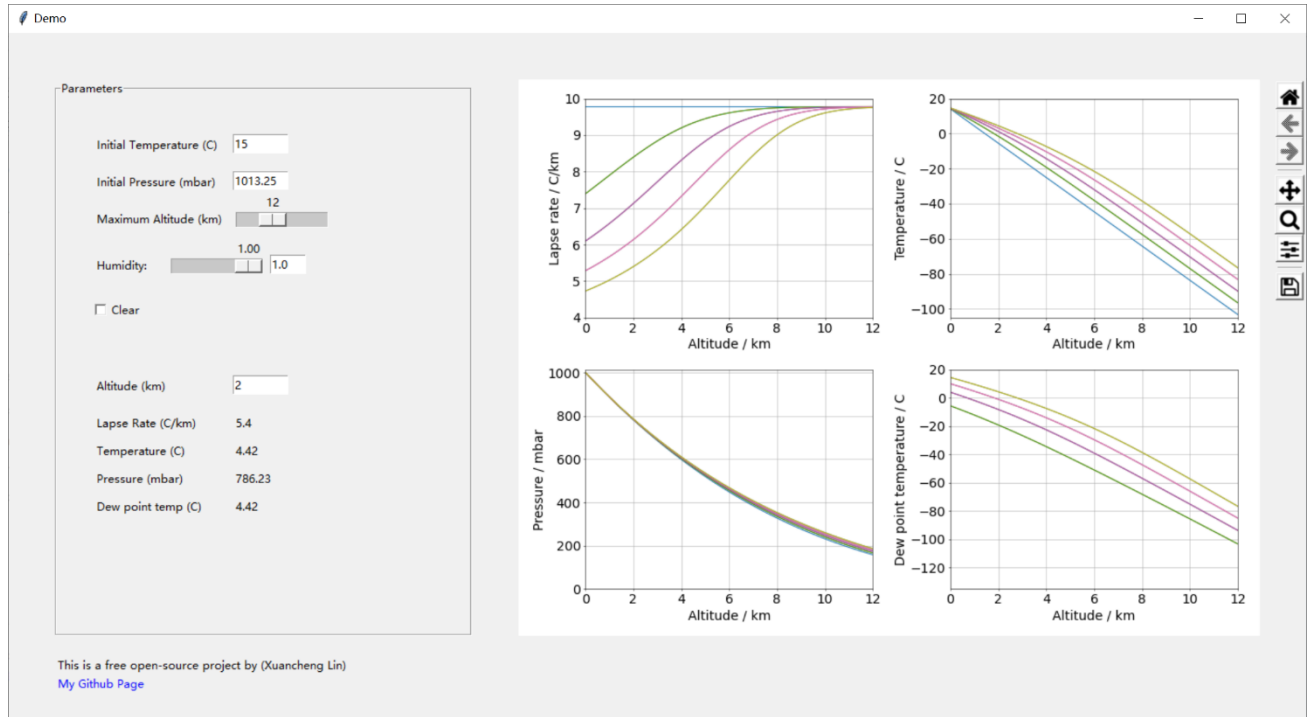
Figure 5 – Graphical User Interface based on the iterative model.

i.e., the air parcel being cooled will be colder than the ambient air.

$$T_d = \frac{b\left(lnU + \dfrac{aT}{b+T}\right)}{a - lnU - \dfrac{aT}{b+T}}$$

$$a = 17.625$$
$$b = 243.04$$

Where T_d is the dew temperature in °C, T is the temperature in °C, and U is the relative humidity. a and b are constants.

3.4 Boiling point

To boil water, it must undergo a phase transition from liquid to gas. This requires a certain amount of heat, the latent heat of vaporization, to break the inter-molecular bond inherent in the water.

$$\frac{dP}{dT} = \frac{\Delta H}{T \Delta V}$$

In a liquid to gas transition, I can assume the volume change is sufficiently large as to ignore the original fluid volume. If one assumes the resulting gas is ideal:

$$\Delta V = \frac{RT}{P}$$

Hence:

$$\frac{dP}{dT} = \frac{\Delta H}{RT^2} P$$

$$\int_{P_*}^{P} \frac{1}{P} dP = \frac{\Delta H}{R} \int_{T_*}^{T} \frac{1}{T^2} dT$$

$$ln\left(\frac{P}{P_*}\right) = -\frac{\Delta H}{R}\left(\frac{1}{T} - \frac{1}{T_*}\right)$$

$$P = P_* e^{-\frac{\Delta H}{R}\left(\frac{1}{T} - \frac{1}{T_*}\right)}$$

$$T_{boil} = \left(\frac{1}{T} - \frac{R}{\Delta H} ln(\frac{P}{P_*})\right)^{-1}$$

For every iteration, the parameters are updated, and the lapse rate, dew point and boiling point is recalculated.

4. Application

In order to use the iterative model, I developed an application which allows users to input the hyperparameters, with simple tools which enable users to move, magnify and save the figures. The visual of the app can be found in *Figure 5*. It has

been created using Tkinter and Matplotlib library in Python.

The graphical user interface can update lively whenever users change any parameters. Users can decide whether to maintain the lines or clear the previous lines when parameters changes. Users can also look at data given the altitude to the nearest $1\,m$etre precision.

The app can be used for educational purposes, such as allow students to understand the atmosphere or being a demonstration for a competition like the BPhO Computational Challenge.

5. Comparison with other planets

Earth's atmosphere is composed of about 78% of nitrogen, 21% of oxygen, and 1% of other gases. In this section, I will be comparing the Earth's atmosphere against some other planets in the Solar system.

Due to lack of research ability and limited time. I selected Mars and Venus as my two sample planets.

Mars The Martian atmosphere is an extremely thin sheet of gas, principally carbon dioxide (95%), that extends from the surface of Mars to the edge of space. The Martian atmosphere is 1% of density of the Earth's atmosphere. [Glenn] proposed a model that separates the atmosphere into lower and upper atmosphere:

For $h < 7000$:
$$T = -31 - 0.000998h$$
$$P = 0.699e^{-0.00009h}$$
For $h > 7000$:
$$T = -23.4 - 0.00222h$$
$$P = 0.699e^{-0.00009h}$$

Density:
$$\rho = \frac{P}{0.1921(T + 273.1)}$$

Where height h in m, temperature T in ℃, pressure P in kPa, and density ρ in kg/m^3.

The visualization of the model is in *Figure 6*.

The initial temperature is cooler than in Figure 1,

Venus has a thick and toxic atmosphere filled with mostly carbon dioxide (96.5%). The Venusian atmosphere is $90\times$ denser than that on Earth. The atmosphere is perpetually shrouded by opaque clouds of sulfuric acid, making optical Earth-based and orbital observation of the surface impossible. Most of the information has been obtained by radar imaging. Aside from the surface layers, the atmosphere is in a state of vigorous circulation. The upper layer of troposphere exhibits a phenomenon of super-rotation, in which the atmosphere circles the planet in just four Earth days, much faster than the planet's sidereal day of 243 days, blowing at a speed of $100\,m/s$.

[Mulholland et al., 2019] provided the data of the atmosphere of Venus. *Figure 7* shows the visualization of the Venus atmosphere model.

Level	Base h (m)	Temperature (K)
Tropo-sphere	0 ~39220	699.000 ~386.534
Solar Heating Convection	39221 ~50000	386.527 ~314.200
Tropo-pause Convection Overshoot	50000 ~70000	314.200 ~180.000
Tropo-pause Breaking	50001 ~70477	314.194 ~228.686
Strato-sphere	70478 ~100000	228.683 ~173.350

Table 2 – Venusian Atmosphere Data. The table involves the range of altitude and temperature for each level. Unlike the ISA model, the data also involves additional layers such as solar heating convection and tropopause convection overshoot.
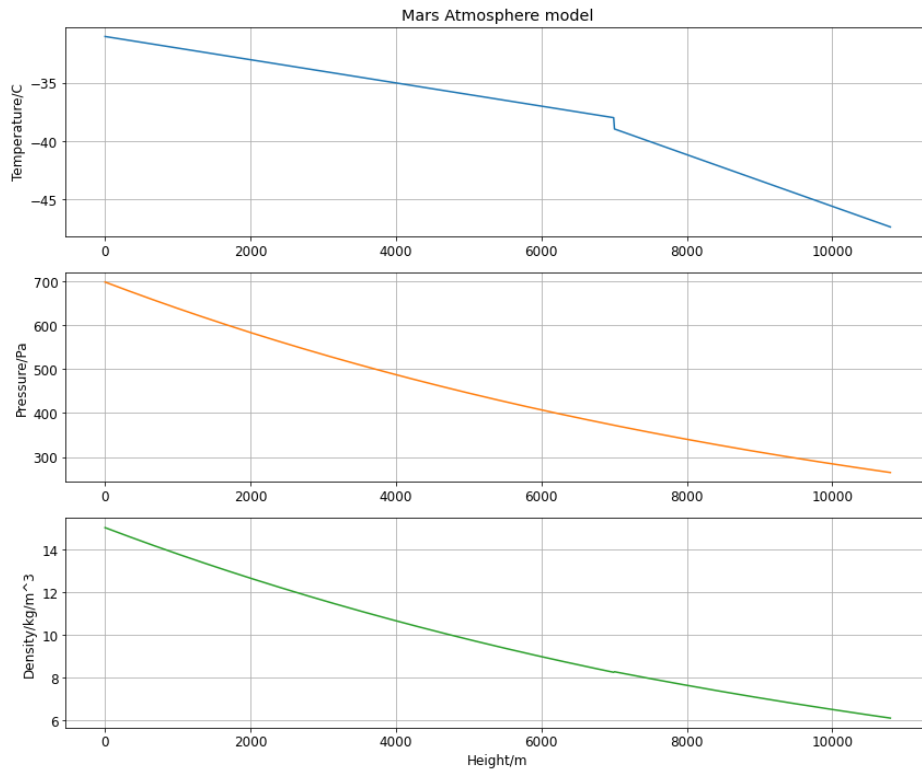
Figure 6 – Visualisation of Mars atmosphere model. (Top) Temperature in degree Celsius against altitude in metres. (Middle) Pressure in Pascals against altitude in metres. (Bottom) Density in kilograms per metre cube against altitude in metres. The scale height of the atmosphere of Mars is about 10.8km.
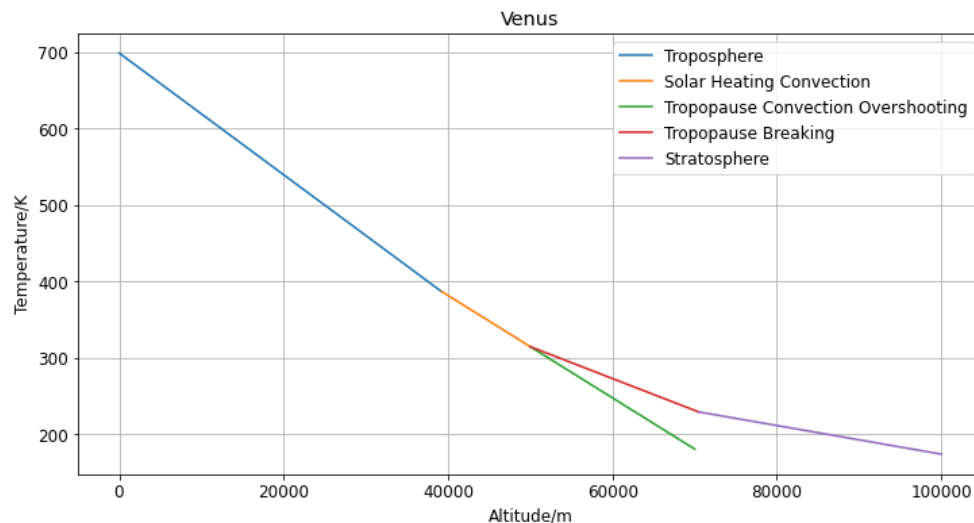

Figure 7 – Visualisation of Venus Temperature Altitude data. Temperature in Kelvin against altitude in metres. Compared to Figure 1, the initial temperature in troposphere is much higher and the gradient of the graph, lapse rate, is always negative.

References

[1] A French. British Physics Olympiad Computational Challenge 2022 – A Standard Atmosphere. https://www.bpho.org.uk/bpho/computational-challenge. 2022.

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