Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	2.0	Transformations on the Coordinate Plane
Lesson Code	2.4.2	Rotation 2
Time Limit		30 minutes

TARGET

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this module, the students will have been able to

1. Illustrate rotation about a fixed point (not the origin) through an angle (90^o,180^o,270^o) on the coordinate plane
2. Determine the image of a geometric figure under rotation about a fixed point (not the origin) through an angle (90^o,180^o,270^o)

HOOK

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

In the previous module, we learned how to determine the coordinates of a point under rotation about the origin through given angles 90^o, 180^o, and 270^o. We recall the following information from the previous module:

$$R_{90^o}: (x, y) \rightarrow (-y, x)$$
$$R_{180^o}: (x, y) \rightarrow (-x, -y)$$
$$R_{270^o}: (x, y) \rightarrow (y, -x)$$

We also learned that rotation preserves the size and the shape of the object being transformed. In the case of a point, this means that the distance of the point from the center of rotation is fixed regardless of the angle it is being rotated. In the case of polygons, this implies that the lengths of its sides do not change. What about the angles? Does this also imply that the measure of the angles of the polygon remain the same under rotation? The answer is yes.

Just like translation, reflection, and glide reflection, the transformation rotation produces *congruent* figures. Two line segments are congruent when they have equal measures. Likewise, two polygons are said to be congruent when their corresponding sides and angles are congruent.

Now that we know this property of rotation, we will now explore what happens when the center of rotation is a point on the Cartesian plane other than the origin. How do we get the image of a geometric figure that is rotated about a fixed point through a given angle?

IGNITE

Time Allocation: 12 minutes
Actual Time Allocation: _____ minutes

Consider an arbitrary point $A(x, y)$ on the coordinate plane.

Let us rotate this point about the origin through a given angle θ. Graphically, its image A' will be on the circle with the origin as its center. See figure 1.

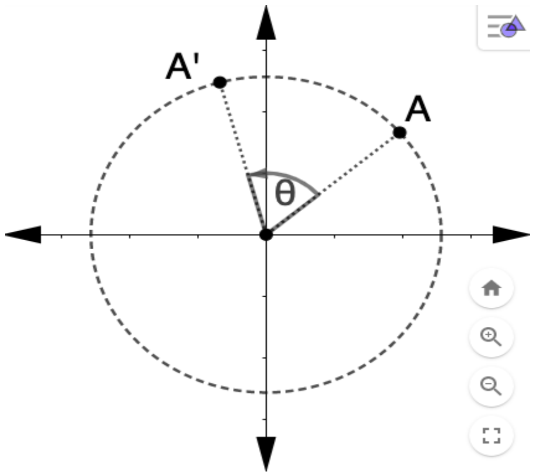


Figure 1. Point A and its image A' under rotation about the origin through angle θ

Now consider point A rotated about a fixed point P (not on the origin) through a given angle. Consequently, its image A' will be on the circle with center P. Refer to Figure 2.

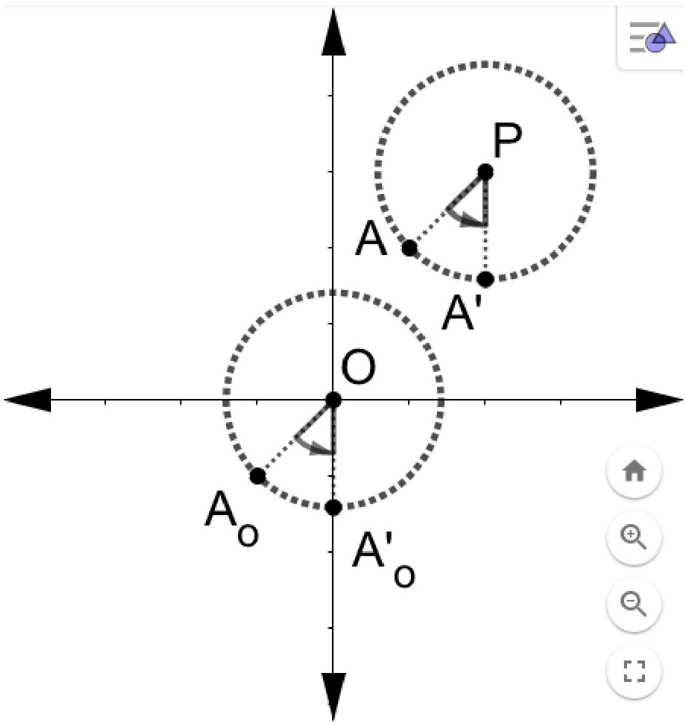


Figure 2. Point A and its image A' when rotated about a fixed point P with its counterpart A_o and A'_o when center of rotation is translated to the origin

Notice that we can look at the circle with center P as a result of transformation of the circle whose center is at the origin under translation, or vice versa. In this way, we can treat rotation about a fixed point (not on the origin) as a combination of two transformations – translation and rotation.

Rotation of point A about a fixed point P through a given angle θ:

1. Locate A_o by translating point A by the same number of units required to translate P to the origin.
2. Get the image of A_o under rotation about the origin through the given angle θ. Call this A'_o.
3. We locate A' by translating A'_o back by the same number of units as it would take to translate the origin to point P.

Now let us apply the steps above in the following examples.

Example 1:

Find the image of the point $(-3,2)$ under rotation 90^o counterclockwise about the point $(1,1)$.

Solution:

1. We locate the counterpart of point $(-3,2)$ by translating it 1 unit to the left and 1 unit down. We get the point $(-4,1)$.
2. We get the image of $(-4,1)$ under rotation about the origin 90^o counterclockwise. We get the point $(-1,-4)$.
3. We translate $(-1,-4)$ 1 unit to the right and 1 unit up to get the image of $(-3,2)$. We obtain the point $(0,-3)$.

The image of $(-3,2)$ under 90^o counterclockwise about $(1,1)$ is $(0,-3)$. We verify this with the use of a graphing software. See Figure 2.5.3.

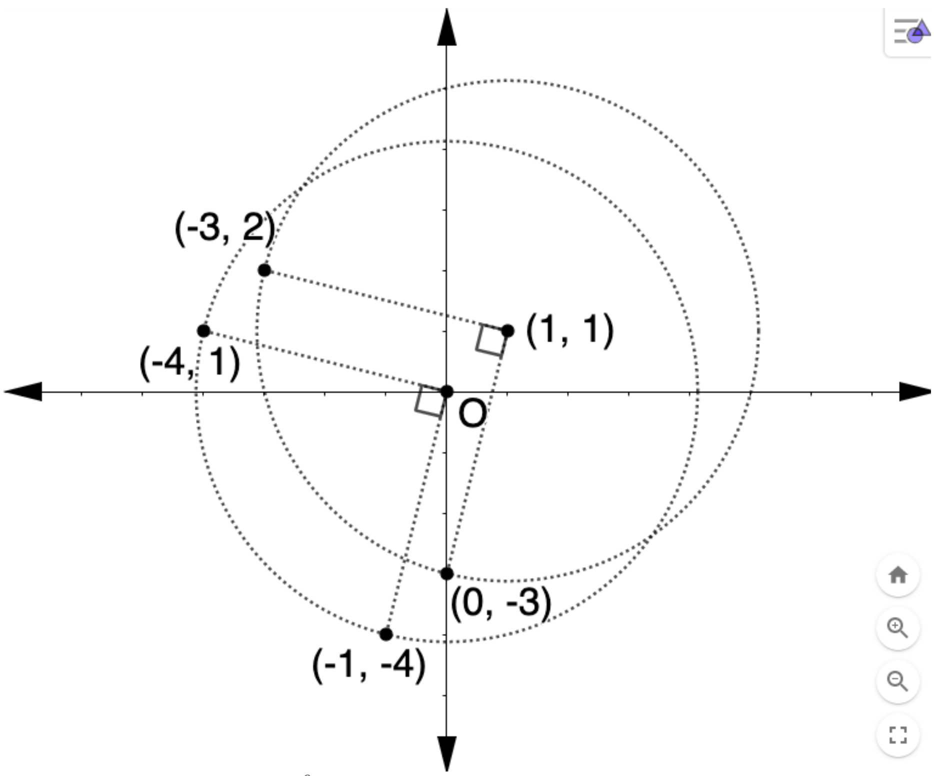


Figure 3. The point $(-3,2)$ and its image under 90^o counterclockwise rotation about $(1,1)$ with the corresponding points when the center of rotation is translated to the origin

In the absence of graphing software, we can manually locate the point using a protractor and a pen. Just position the protractor aligning the segment joining $(1,1)$ and $(-3,2)$ with its center at $(1,1)$. The image of the point lies on the ray that makes an angle of 90^o counterclockwise. We locate the exact point as the endpoint of the segment with the same measure as the segment joining $(1,1)$ and $(-3,2)$.

Example 2:
A triangle has vertices $A(5,1)$, $B(-1,-2)$, and $C(-2,3)$. Find the image of the triangle when rotated 180^o about point C. Label this image triangle $A'B'C'$. Plot the triangles on the Cartesian plane.

Solution:

Vertices of the triangle	Under $T_{2,-3}$	Under R_{180^o}	Under $T_{-2,3}$
$A(5,1)$ →	$A_o\ (7,-2)$ →	$A'_o\ (-7,2)$ →	$A'(-9,5)$
$B(-1,-2)$ →	$B_o\ (1,-5)$ →	$B'_o\ (-1,5)$ →	$B'(-3,8)$
$C(-2,3)$ →	$C_o\ (0,0)$ →	$C'_o\ (0,0)$ →	$C'(-2,3)$

The image of the triangle has vertices $(-9,5)$, $(-3,8)$, and $(-2,3)$. Observe that the image of the center of rotation $C(-2,3)$ under rotation with it as the center is itself. In general, under the transformation rotation, all the points in the coordinate plane are transformed except for the center of rotation, i.e. the fixed point.

Let us plot the two triangles on the Cartesian plane. See Figure 4 below.

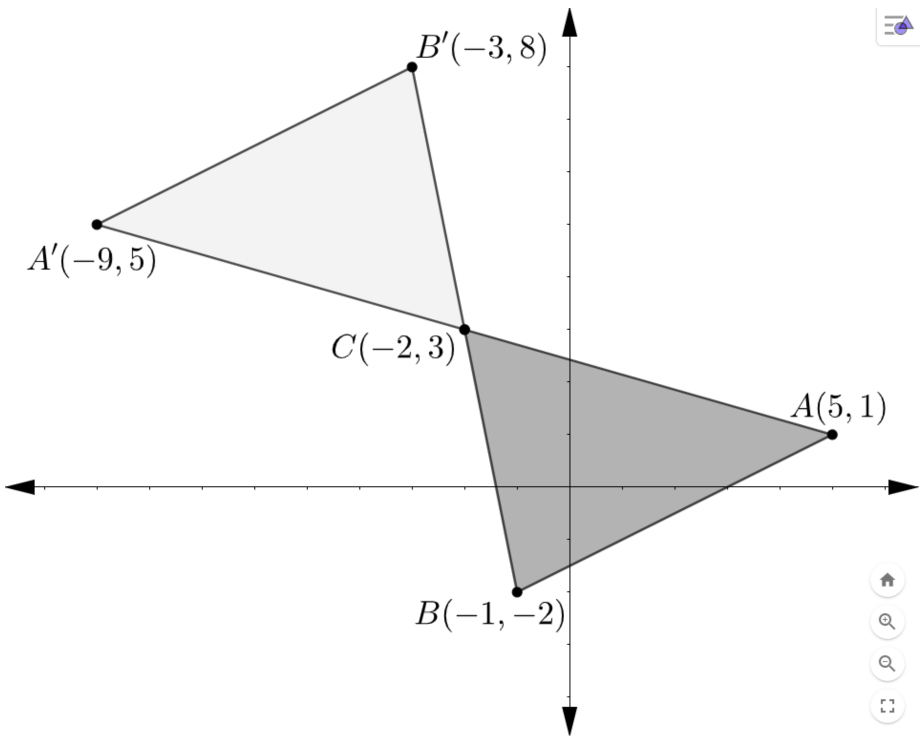


Figure 4. Triangle ABC and its image $A'B'C'$ under rotation about point C

Observe that each vertex and its corresponding image are equidistant from the fixed point, which is also the midpoint of the segment joining them. This is true for 180^o rotation. Other books use the term *half-turn* to refer to this rotation.

In addition, a half-turn may be expressed as a product of two reflections along two perpendicular lines. (See Extension of Module 2.4.1) In this case, we can treat 180^o rotation about point C as the image under reflection over the line $x = -2$ followed by reflection over the line $y = 3$. Is this process commutative? Why does this work?

Remark:
This module will only be limited to rotation about a fixed point through given angles 90^o, 180^o, and 270^o. Other angles would require introduction to other mathematical concepts (circular functions) which you will learn in Math 4.

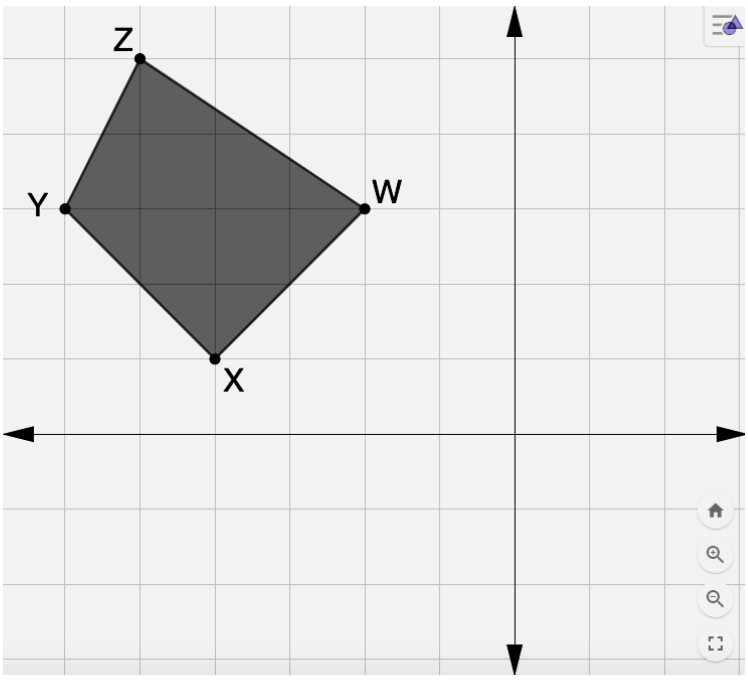
NAVIGATE

Time Allocation: 15 minutes
Actual Time Allocation: _____ minutes

Do as indicated.

1. Give the image of the points below under rotation about $(3, -1)$ through 90^o counterclockwise.
   a. $(0,0)$
   b. $(3, -1)$
   c. $(\sqrt{2}, -3\sqrt{2})$

2. Trapezium $WXYZ$ is plotted on the Cartesian plane below. Rotate the figure 90^o clockwise about vertex X. Sketch the image below. (1 box=1 unit)



3. Find the coordinates of the image of the points below under a rotation of 180^o about the point $(0,2)$.
   a. $(4,1)$
   b. $(a, 2a)$
   c. $(a + 2, a - 2)$

4. A triangle with vertices $P(-4,3)$, $Q(-6,-3)$, and $R(-2,-2)$ is rotated about the point $(0,2)$ through an angle θ. Determine θ such that the image of P is $P'(-1,-2)$ and find the image of the other two vertices.

5. A circle C is defined by the equation $(x-2)^2 + (y+3)^2 = 1$. Consider the two geometric transformations described below:
   i. R: rotation about the point $(0,-2)$, 90^o counterclockwise
   ii. r : reflection over the line $x = 4$

   (Recall: Product/Composition of Transformations from Extension of Module 2.4.)

   (a) Give the equation of the new circle (in standard form) which is the image of C under the product of transformation $r \circ R$ (i.e. rotation followed by reflection). Express this product of transformation as a single transformation under translation.

   (b) Give the equation of the new circle which is the image of C under the product of transformation $R \circ r$ (i.e. reflection followed by rotation). Express this product of transformation as a single transformation under translation.

   (c) Is the combination of rotational transformation and reflection commutative?

KNOT

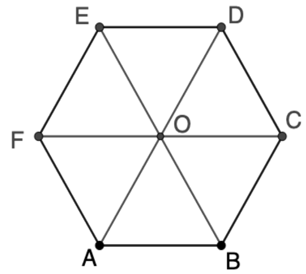
Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

In summary, we have the following points:

1. A rotation about a fixed point (not at the origin) through a given angle is a combination of rotation and translation.
2. To determine the coordinates of the image under rotation, with center of rotation at a fixed point other than the origin, through a given angle θ, we follow the steps below:
   a. Locate A_o by translating point A by the same number of units required to translate P to the origin.
   b. Get the image of A_o under rotation about the origin through the given angle θ. Call this A'_o.
   c. We locate A' by translating A'_o back by the same number of units as it would take to translate the origin to point P.
3. A half-turn may be treated as a product of two reflections over two perpendicular lines intersecting at the center of rotation.
4. The product of rotation and reflection can be expressed as a single transformation, i.e. translation.
5. A product of two or more transformations is not always commutative.

Extension/Optional Exercises:

1. Refer to the figure below. $ABCDEF$ is a regular hexagon with center O. (Lee, et.al, 2006)



Answer the following questions:

   a. What must be the angle of rotation such that ΔAOB is transformed to triangle ΔFOA under rotation about O?
   b. Determine the image of A under rotation 60^o clockwise about C.
   c. What is the image of rhombus $ODEF$ (i.e. a parallelogram with equal sides) when rotated 180^o clockwise about O?
   d. Describe using a single transformation under rotation to map trapezoid $ABCD$ to trapezoid $CDEF$.

2. Determine the center of rotation given the vertices $A(-3,2)$, $B(-7,1)$, and $C(-4,-1)$ and its image $A'(0,-1)$, $B'(1,-5)$, and $C'(3,-2)$ under rotation through any given angle. (Hint: Get the intersection of the perpendicular bisectors of the line segments connecting a vertex and its corresponding image.)

References:

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Answer Key:

Navigate:

1. (a) $(2,-4)$ (b) $(3,-1)$ (c) $(3\sqrt{2}+2,\sqrt{2}-4)$
3. (a) $(-4,3)$ (b) $(-a,-2a+4)$ (c) $(-a-2,-a+6)$
5. (a) $(x-7)^2+y^2=1$; $T_{5,3}$ (b) $(x-1)^2+(y-4)^2=1$; $T_{-1,7}$ (c) No

Knot:

Extension/Optional Exercises:

1. a. 60^o clockwise
   b. E
   c. rhombus $OABC$
   d. rotation of trapezoid $ABCD$ about O through an angle 120^o in counterclockwise direction