Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	1.0	Basic Plane and Coordinate Geometry
Lesson Code	1.4	Circles
Time Limit		30 minutes

TARGET

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this module, the students will have been able to:

1. identify the equation of a circle in standard and general form given its center and radius; and
2. find the radius and center of a circle given its standard and general equation.

HOOK

Time Allocation: 4 minutes
Actual Time Allocation: _____ minutes

We have already discussed distance and midpoint formulas in our previous module.

Try to recall these!

1. Given two points in a coordinate plane, how can we determine the distance between these points?
2. A line segment is drawn in a coordinate plane. How can we determine the segment's midpoint?

To find the equation of a circle, we will apply these formulas to determine the center and the length of the radius of a circle that are needed to find the equation of a circle. There are two commonly used forms of equations for a circle. In this lesson, we will take a few minutes to explore and understand both forms and be able to convert from one to the other.

IGNITE

Time Allocation: 15 minutes
Actual Time Allocation: _____ minutes

Let us start exploring our topic!

Definition of a Circle

A **circle** is the set of all points in a plane that lie a fixed distance from a fixed point. The fixed distance is the **radius**, and the fixed point is the **center** of the circle.

Let us begin exploring the circle's equation using *Figure 1*. We have drawn the circle in a coordinate plane. As we can observe, the center of the circle is at (h, k) and its radius is r. The coordinates of any point on the circle will be denoted as (x, y).

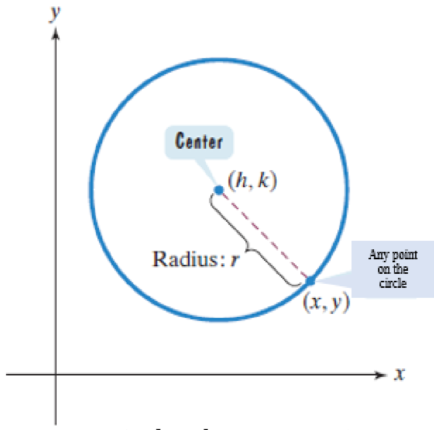


Figure 1 *A circle whose center is at* (h, k) *and has a radius* r.
Retrieved from *fl01000126.schoolwires.net*

Think about this!

Take a look at the circle in **Figure 1**, what does its geometric definition convey about the point (x, y)?

We have discussed already in our previous lesson the distance formula for d.

$$\boldsymbol{d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

We can determine the length of the raidus r by using the coordinates of the point on the circle (x, y) and the center (h, k). Replacing d with r in the distance formula, we now have

$$\boldsymbol{r = \sqrt{(x - h)^2 + (y - k)^2}}$$

Note: The distance from the center (h, k) to any point on the circle (x, y) is equal to $\boldsymbol{r}$.

EQUATION OF A CIRCLE IN STANDARD FORM

For all circles whose center is at the origin, its equation in standard form is $\boldsymbol{x^2 + y^2 = r^2}$ where $\boldsymbol{r}$ is the radius of the circle.

***Example 1*. Finding Equation of a Circle in Standard Form**

Determine the equation of the circle centered at the origin with radius of 3 units. Express answer in standard form and show the graph of the circle.

Solution:

Since the circle is centered at the origin and the given radius is 3 units, so $r = 3$. Substituting all the given values on the equation of the circle in standard form, we have

$$\boldsymbol{x^2 + y^2 = 3^2}$$

Then, simplify

$$\boldsymbol{x^2 + y^2 = 9}$$

Thus, the equation of the circle in standard form is $\boldsymbol{x^2 + y^2 = 9}$ and its graph is shown in *Figure 2.*

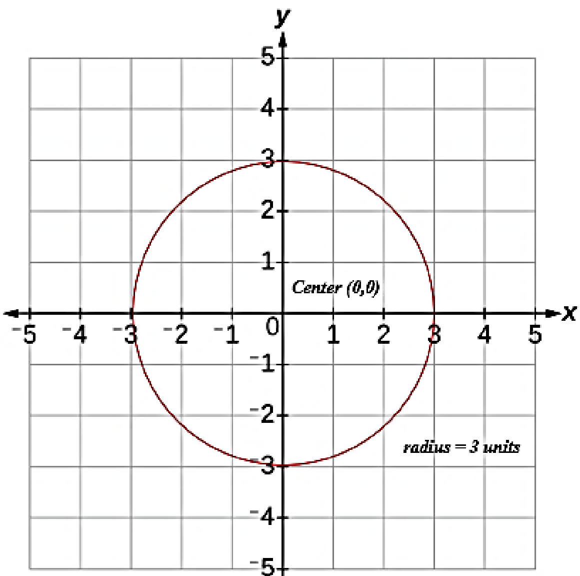


Figure 2

Now, let us take the next example whose circle's center is not at the origin.

The **equation of a circle in standard form** centered at (h, k) and with radius r is

$$\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$$

***Example 2.* Finding the Equation of a Circle in Standard Form**

Determine the equation of the circle in standard form whose center is at $(-3, 1)$and radius of $\sqrt{11}$ units. Show the graph of the circle.

Solution:

The center of the circle is at $(-3, 1)$, this means that the value of $h = -3$ and $k = 1$. The given radius is $\sqrt{11}$ units, so $r = \sqrt{11}$. Substituting all the given values on the standard form of the equation of the circle, we have

$$(x - (-3))^2 + (y - 1)^2 = \left(\sqrt{11}\right)^2$$

Then, simplify

$$(x + 3)^2 + (y - 1)^2 = 11$$

Thus, the equation of the circle in standard form is $\boldsymbol{(x + 3)^2 + (y - 1)^2 = 11.}$ The graph is shown in ***Figure 3***.

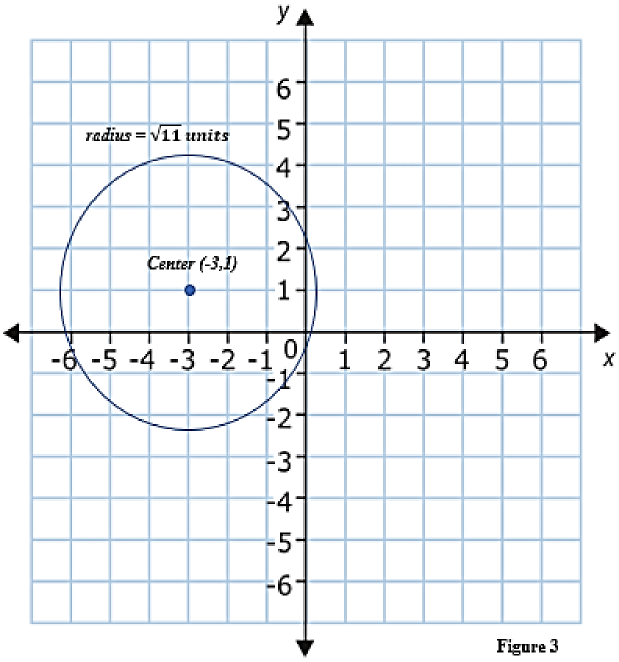


Figure 3

***Example 3.* Graphing a Circle Given its Equation in Standard Form**

a. Determine the center and radius of the circle with equation

$$\boldsymbol{(x - 4)^2 + (y + 3)^2 = 25}$$

b. Graph the circle.

Solution

a. First, let us determine the center of the circle, (h, k) and its radius, r, from its equation. Let us identify the values for (h, k) and r.

The standard form of the equation of a circle is $(x - \boldsymbol{h})^2 + (y - \boldsymbol{k})^2 = \boldsymbol{r^2}$.

The given equation is $(x - 4)^2 + (y + 3)^2 = 25$.
We can rewrite this as $(x - \mathbf{4})^2 + (y - (-\mathbf{3}))^2 = \mathbf{5}^2$

Now, we determine that $h = 4$, $k = -3$ and $r = 5$. Thus, the circle is centered at $(4, -3)$ and has a radius of 5 units.

b. To graph the circle, we need to locate the center which is located at $(4, -3)$. Since we already know the radius which is 5 units, we can plot four points on the circle by counting five units to the left, to the right, up, and down from its center $(4, -3)$. By locating these points, *up* $(4, 2)$, *down* $(4, -8)$, *left* $(-1, -3)$ and *right* $(9, -3)$, we'll have the graph shown in **Figure 4** on the next page.

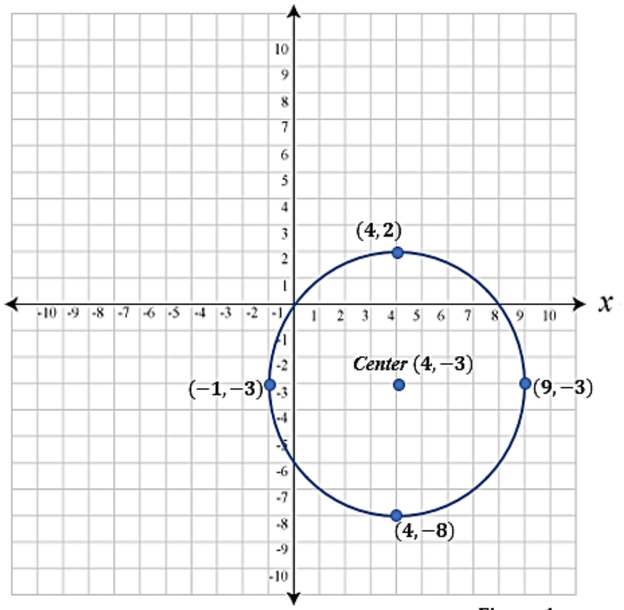


Figure 4

Take note!

In determining the coordinates of the center of a circle (h, k), always keep in mind that h and k are the numbers that *are next to the subtraction signs* in the equation of the circle. If the signs inside the parenthesis are plus signs, we need to change the signs into subtraction signs and make the numbers negative.

$$(x-4)^2 + (y+3)^2 = 25$$

$$(x-4)^2 + (y-(-3))^2 = 25$$

The number after the subtraction sign is 4, so $h = 4$

The number after the subtraction sign is -3, so $k = -3$

THE EQUATION OF A CIRCLE IN GENERAL FORM

The **equation of a circle in general form** is given by

$$x^2 + y^2 + Dx + Ey + F = 0$$

where D, E, and F are real numbers.

Do you remember on how to complete a square?

If we are going to convert the equation of a circle from general form to the standard form, we need to complete the square on both variables x and y. Let us explore how this is done in the next example.

***Example 4*. Converting the General Form of the Equation of the Circle to its Standard Form**

Convert $\boldsymbol{x^2 + y^2 + 6x - 4y - 23 = 0}$ to its standard form and graph the circle.

Solution:

We need to complete the square for both x and y by rearranging the terms to make the x – *terms* arranged together in descending order of powers, as well as the y – *terms*. Also, the constant term must be placed on the right side of the equation.

$$x^2 + y^2 + 6x - 4y - 23 = 0$$

We need to rewrite in anticipation of completing the square.

$$\left(x^2 + 6x\right) + \left(y^2 - 4y\right) = 23$$

Complete the square of x and y to make perfect square trinomials**.**

$$\left(x^2 + 6x + {\color{red}9}\right) + \left(y^2 - 4y + {\color{red}4}\right) = 23 + {\color{red}9} + {\color{red}4}$$

Remember!

The numbers we added on the left side of the equation must also be added on the right side.

Then, factor the left side of the equation and simplify the right side. We'll have

$$(x + 3)^2 + (y - 2)^2 = 36$$

Then, the equation of the circle in the standard form is $(\boldsymbol{x} + \mathbf{3})^{\mathbf{3}} + (\boldsymbol{y} - \mathbf{2})^{\mathbf{2}} = \mathbf{36}$. From the equation, we can determine the center of the circle:

$$(x - (-3))^2 + (y - 2)^2 = 36$$

Therefore, the center (h, k) has coordinates $(-3, 2)$. To find radius r, we need to take the square root of the value on the right side of the equation. Since $r^2 = 36$, the radius is ***6 units***. (Disregard $r = -6$, since r is always **positive**.)

Take a look at the graph of $\boldsymbol{x^2 + y^2 + 6x - 4y - 23 = 0}$ below**.** (See figure 5)

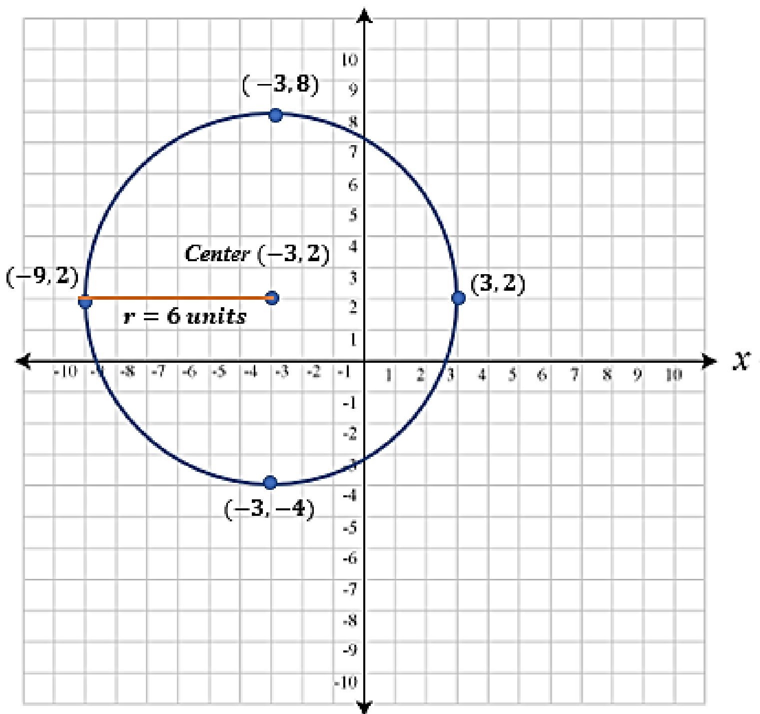


Figure 5

Here are some supplemental videos to watch:

a. Equation of a Circle

https://www.youtube.com/watch?v=Sl0VeTcL-s4

b. How to find the center and radius of a circle in standard form

https://www.youtube.com/watch?v=JbEfcJyOd2Y

NAVIGATE

Time Allocation: 5 minutes
Actual Time Allocation: ______ minutes

Check Your Understanding!
Directions: Answer the following problems.

A. A line segment whose endpoints lie on a circle and passes through the center is called the **diameter** of the. Based on the given figure, answer the following (see Figure 6).
1. Determine the coordinates of the center of the circle.
2. Find the radius of the circle.
3. Write the equation of the circle in standard form using your answers in parts (1) and (2).

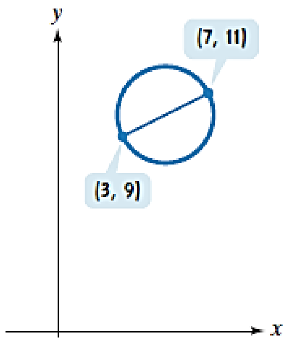


Figure 6
Retrieved from fl01000126.schoolwires.net

B. In Exercises 4 - 6, tell whether if each statement is TRUE or FALSE. If the statement is false, rewrite the statement to make it a true statement.

4. The standard form of the equation of the circle whose center is located at the origin with diameter of 12 units is $x^2 + y^2 = 36$.

5. The graph of $(x+8)^2 + (y-6)^2 = 81$ is a circle with radius of 9 units and its center is at $(-8, 6)$.

6. The graph of $(x-4)^2 + (y+3)^2 = -100$ is a circle with radius of 10 units and its center is at $(4, -3)$.

Definition: A line that intersects the circle at exactly one point is called the **tangent line** of a circle. This line forms a right angle (is perpendicular) with the radius of the circle at the point of tangency.

C. Determine the equation in slope-intercept form of the line that is tangent to the circle whose equation in standard form is $\boldsymbol{(x-1)^2 + (y+2)^2 = 25}$ at the point $\boldsymbol{(5, -5)}$.

KNOT

Time Allocation: 5 minutes
Actual Time Allocation: _____ minutes

Let us summarize!

A ***circle*** is the set of all points in a plane that are of the same distance from a fixed point, called the ***center***. ***Radius*** is defined as the fixed distance from the center to any point on the circle.

For all circles whose center is at the origin, its equation in standard form is $\boldsymbol{x^2 + y^2 = r^2}$ where $\boldsymbol{r}$ is the radius of the circle.

The ***standard form of the equation of a circle*** whose center is at (h, k) and radius r is

$$\boldsymbol{(x-h)^2 + (y-k)^2 = r^2}$$

The ***general form of the equation of a circle*** is given by

$$\boldsymbol{x^2 + y^2 + Dx + Ey + F = 0}$$

where D, E, and F are real numbers.

Remember this!

You can change the general form of the equation of the circle to its standard form by completing the square for each of the variables, x and y with the following steps:

1. Group the x's and y's parts together and the constant must be placed on the other side of the right side of the equation.
2. Complete the square for each variable. Do not forget to also add the values you added in the *x's* and *y's* parts to the right side of the equation.
3. Factor each perfect square trinomial.

To change the standard form of the equation of the circle to its general form, take the following steps:

1. Expand the binomials (x's and y's parts).
2. Combine like terms and simplify.
3. Let the right side of the equation be equal to zero.

<u>SUMMATIVE TEST</u>

Do as indicated. Provide your complete solutions.

1. Determine the equation of the circle whose center is at (-3,2) and radius $2\sqrt{3}$ in standard form.

2. The center of the circle shown in Figure 7 is C. Determine the equation of circle C in general form.

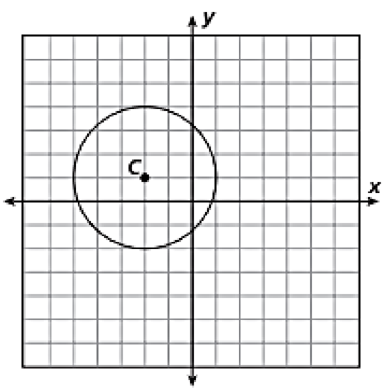
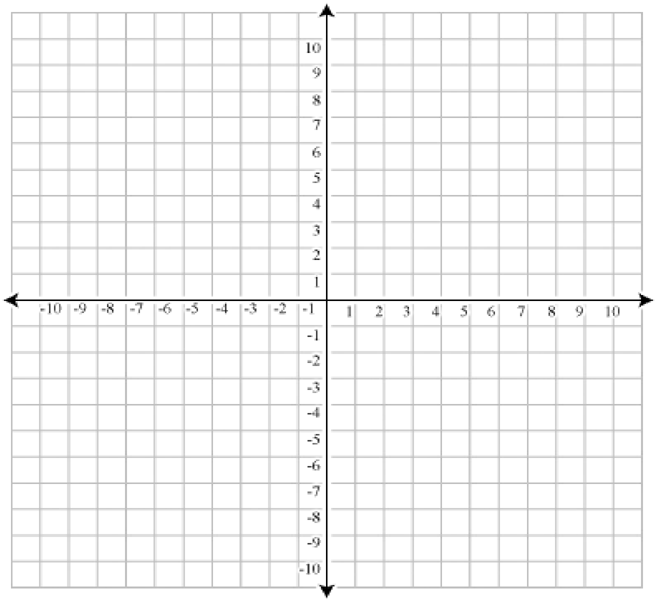


Figure 7 *Circle C*
Retrieved from *http://msastete.com/yahoo_site_admin1/assets/docs*

3. Find the equation of the circle in standard form with a diameter whose endpoints are $(-2, -1)$ and $(4,7)$.

4. Identify the center (h, k) and radius r of the circle $x^2 + y^2 - 16x + 6y + 24 = 0$

5. Convert the equation of a circle $\boldsymbol{x^2 + y^2 - 14x + 4y + 49 = 0}$ to its standard form. Then, graph the circle.



References:

Albarico, J.M. (2013). THINK Framework. (Based on Ramos, E.G. and N. Apolinario. (n.d.) *Science LINKS*. Rex Bookstore, Inc.)

Analyzemath Website (2020). *Distance Problems with Solutions.* Retrieved from https://www.analyzemath.com/math_problems/distance_problems.html

Cabrillo College (n.d). *Distance and Midpoint Formulas; Circles.* Modules Retrieved from https://cabrillo.instructure.com/courses/10269/modules/items/256611

Schoolwires Website (n.d), *Distance and Midpoint Formulas; Circles*, pp. 243-253. Retrieved from https://fl01000126.schoolwires.net/cms/lib/FL01000126/Centricity/Domain/261/Ch1_Section9.pdf

Prepared by: Frederick F. Faz
Position: Special Science Teacher (SST) II
Campus: PSHS - MRC

Reviewed by: Virginia A. Barlas
Position: Special Science Teacher (SST) IV
Campus: PSHS - WVC

ANSWER KEY

Try to recall these!

1. When two points are given, $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, the distance between P_1 and P_2 can be solved using the distance formula, $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

2. We determine the midpoint of a line segment drawn in a coordinate plane using the midpoint formula.

$$M = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

Check Your Understanding!

A. 1. $(5{,}10)$ 3. $(x-5)^2 + (y-10)^2 = 5$

B. In Exercises 4 - 6, determine whether each statement is true or false. If the statement is false, make the necessary change(s) to produce a true statement.

5. True

C. Determine the slope from the center to the point on the circle.

Center: $(1, -2)$ and Point $(5, -5)$

$$m = \frac{y_2-y_1}{x_2-x_1} \Longrightarrow m = \frac{-5-(-2)}{5-1} \Longrightarrow \boldsymbol{m = -\frac{3}{4}}$$

The slope of the line perpendicular to a given line is equal to $-\frac{1}{m}$.

So, the slope of the perpendicular line is $\frac{4}{3}$.

Using the point (5, -5) and m= 4/3. Substituting the values to the slope - point form of the equation of the line, we have

$$y - y_1 = m(x - x_1) \Longrightarrow y - (-5) = \frac{4}{3}(x - 5)$$

$$y + 5 = \frac{4}{3}(x - 5)$$

$3y + 15 = 4x - 20$ *Multiplying both sides by 3 to get rid of the denominator.*

$\boldsymbol{4x - 3y = 35}$ *Simplify*

The equation of the perpendicular line is $y = \frac{4}{3}\boldsymbol{x} - \boldsymbol{\frac{35}{3}}$.

Summative Test

1. $(x+3)^2 + (y-2)^2 = 12$
3. $(x-1)^2 + (y-3)^2 = 25$
5. $\boldsymbol{x^2 + y^2 - 14x + 4y + 49 = 0}$

Solution: By completing the square,

$$(x^2 - 14x + 49) + (y^2 + 4y + 4) = -49 + 49 + 4$$

$$(x-7)^2 + (y+2)^2 = 4$$

Therefore, the center is at $(7, -2)$ and $r = 2$

The graph is shown below.