Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	3.0	Analysis of Graphs: Polynomial and Rational Functions
Lesson Code	3.5	Asymptotes
Time Frame		30 minutes

After completing this module, you should be able to:

1. define and differentiate the different types of asymptotes: horizontal, vertical, and oblique; and
2. find the asymptote of a rational function.

Question 1 What happens to the function values $f(x)$ when the value of x approaches a zero of the denominator?
Question 2 What happens to the to the function values $f(x)$ when the value of x approaches a large positive or negative number?

Before we go through the lesson drill, study the following figures, definitions, and notations below:

10 MINUTES

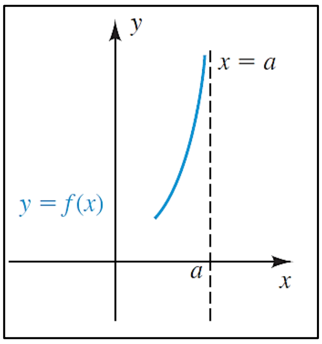


Figure 1

$f(x) \to \infty$ as $x \to a^-$

f(x) approaches positive infinity
(increases without bound)
as x approaches *a* from the left
(through values less than *a*)

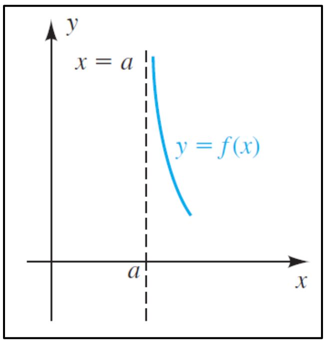


Figure 2

$f(x) \to \infty$ as $x \to a^+$

f(x) approaches positive infinity
(increases without bound)
as x approaches *a* from the right
(through values greater than *a*)

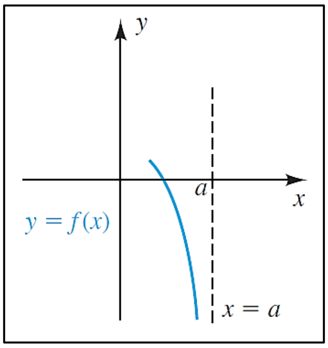


Figure 3

$f(x) \to -\infty$ as $x \to a^-$
f(x) approaches negative infinity
(decreases without bound)
as x approaches *a* from the left
(through values less than *a*)

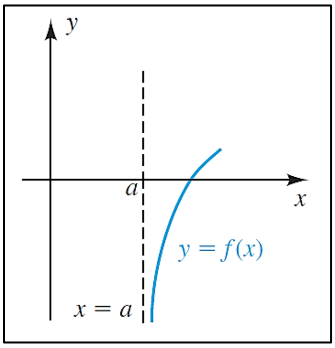


Figure 4

$f(x) \to -\infty$ as $x \to a^+$
f(x) approaches negative infinity
(decreases without bound)
as x approaches *a* from the right
(through values greater than *a*)

Vertical Asymptote *TIP (The Important Point)*

The line ***x*** **=** ***a*** is a vertical asymptote of a rational function if

$$\boxed{f(x) \to +\infty \quad or \quad f(x) \to -\infty}$$

as *x* approaches ***a*** from either the left or the right.

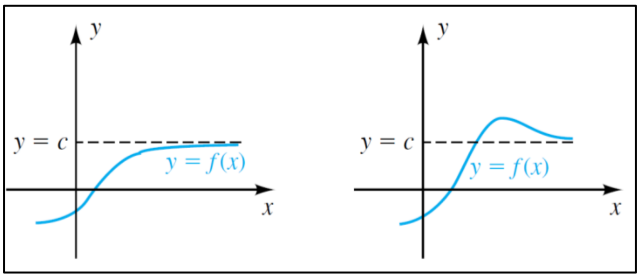


Figure 5

$f(x) \to c$ as $x \to \infty$
f(x) approaches *c*
as x approaches positive infinity
(increases without bound)

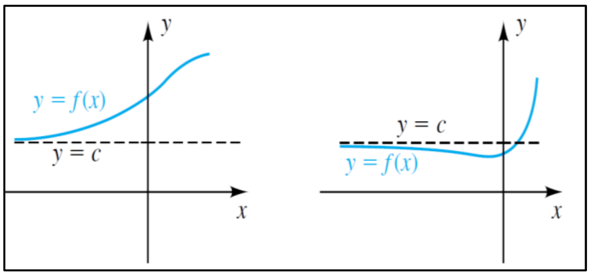


Figure 6

$f(x) \to c$ as $x \to -\infty$
f(x) approaches *c*
as x approaches negative infinity
(decreases without bound)

Horizontal Asymptote *TIP (The Important Point)*

The line ***y*** **=** ***c*** is a horizontal asymptote of a rational function if

$$\boxed{f(x) \to c}$$

as *x* approaches +∞ or as *x* approaches - ∞.

EXAMPLE 1

$$f(x) = \frac{1}{x}$$

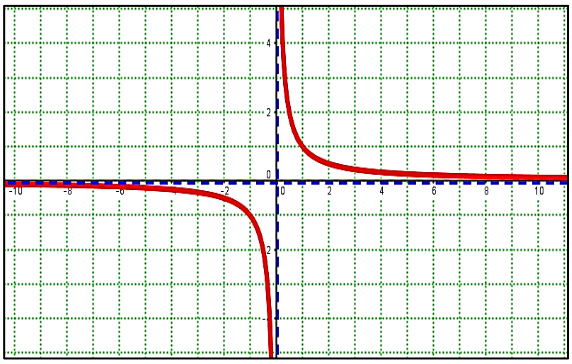


Figure 7. [X:Y=1:1]

(a) vertical asymptote: x = 0

(b) horizontal asymptote: x = 0

EXAMPLE 2

$$f(x) = \frac{x+1}{x-3}$$

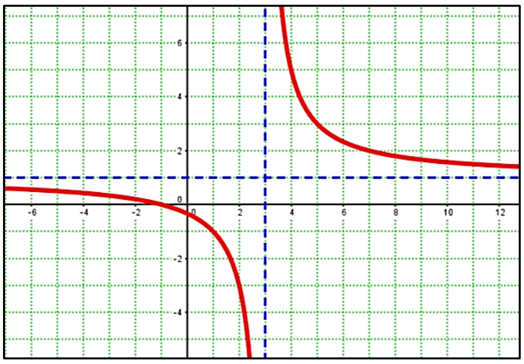


Figure 8. [X:Y=1:1]

(a) vertical asymptote: x = 3

(b) horizontal asymptote: x = 1

Are you confident to find the asymptotes without looking at the graph? If not, here's how to do it!

A. *Vertical Asymptote*

Vertical asymptote is expressed as a line $\boldsymbol{x = a}$.
The value/s of $\underline{\boldsymbol{a}}$ is the zero/s of the denominator.

B. *Horizontal Asymptote*

Horizontal asymptote is expressed as a line $\boldsymbol{y = c}$.
Let $f(x)$ be a rational function

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_k x^k + b_{k-1}x^{k-1} + \cdots + b_1 x + b_0}$$

where:
$a_n \neq 0$ and $b_k \neq 0$
a_n & b_k : *leading coefficients*
n & k : *degrees*

The value/s of $\underline{\boldsymbol{c}}$ can be identified from the following cases:

Case 1: If **n < k**, then HA: y = 0
Case 2: If **n = k**, then HA: $y = \frac{a_n}{b_k}$

Case 3: If **n > k**, then HA: **none**
No HA but has **oblique asymptote.**

C. *Oblique Asymptote*

Oblique asymptote is expressed as $\boldsymbol{y = Q(x)}$, where $Q(x)$ is a quotient when the function in the numerator is divided by the function in the denominator of the rational function.

We will use the two examples in the previous page.
Are you ready for the next level? Let's GO!!!

EXAMPLE 3

$$f(x) = \frac{1}{x}$$

A. vertical asymptote:
Equate denominator to zero.
$x = 0$

Thus, VA: $\boldsymbol{x = 0}$.

B. horizontal asymptote:
$n = 0$
$k = 1$
$\boldsymbol{n} < k$, **CASE 1**

Thus, HA: $\boldsymbol{y = 0}$.

EXAMPLE 4

$$f(x) = \frac{x+1}{x-3}$$

A. vertical asymptote:
Equate denominator to zero.
$x - 3 = 0$

Thus, VA: $\boldsymbol{x = 0}$.

B. horizontal asymptote:
$n = 1$
$k = 1$
$\boldsymbol{n = k}$, **CASE 2**
$a_n = 1$
$b_k = 1$

Thus, HA: $\boldsymbol{y = 1}$.

EXAMPLE 5

$$f(x) = \frac{x(x+1)(x-1)}{(x-1)(x+2)}$$

A. hole:
Note that there is a common factor $x - 1$. This common factor indicates the ***hole*** in the graph of $f(x)$. In this example, the hole is located at $(1, f(1))$. To find the value of $f(1)$, simplify $f(x)$ by cancelling the common factor first. Thus, $f(x)$ is discontinuous at $(1, 2/3)$.

B. vertical asymptote:
Cancel the common factors first.
Equate denominator to zero.
$x + 2 = 0$

Thus, VA: $\boldsymbol{x = -2}$.

A. horizontal asymptote:
$n = 2$
$k = 1$
$\boldsymbol{n} > k$, **CASE 3**

Thus, no HA but oblique asymptote.

To get the oblique asymptote of Example 5, use long division method:

$$\begin{array}{r|l} & x - 1 \quad r. -2 \\ \hline x+2 & x^2 + x \\ & \underline{x^2 + 2x} \\ & \quad -x \\ & \quad \underline{-x - 2} \\ & \qquad -2 \end{array}$$

The quotient is x – 1.

Thus, the **oblique asymptote** is the line $\boldsymbol{y = x - 1}$.

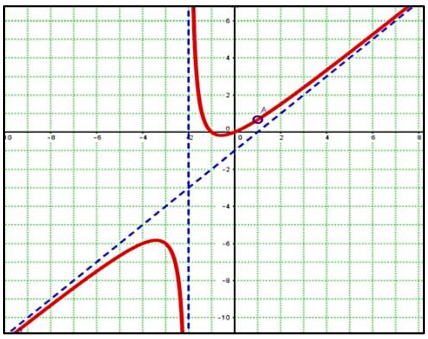
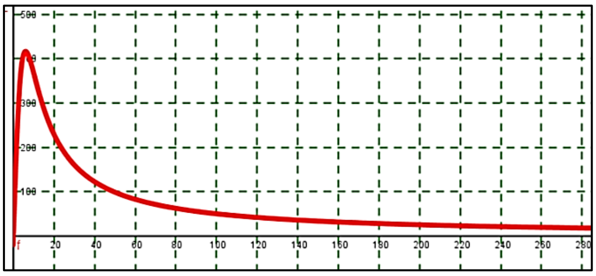


Figure 9. [X:Y=1:1]

Real-Life Application. *Population Density*

In a large city where the population density D (in people/mi2) is related to the distance x (in miles) from the center of the city by $= \frac{5000x}{x^2+36}$. The graph of D is shown in the graph below. What eventually happens to the density as you go farther from the center of the city?



Distance (x) vs. Density (D) graph

Solution.

The density approaches 0 as you go farther from the center of the city.

Now it's your turn to do the next problems on your own.

Find the asymptotes of the following rational functions.

15 MINUTES

1. $$f(x) = \frac{(x-3)}{x^2-4}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

***n*__ *k*, CASE ___**

2. $$f(x) = \frac{-x+3}{x-4}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

***n*__ *k*, CASE ___**

3.

$$f(x)=\frac{6(x+1)(x-2)}{(x+3)(x-1)(x-2)}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

$\boldsymbol{n}$__ $\boldsymbol{k}$, **CASE ___**

4.

$$f(x)=\frac{2(x+2)(x-1)\,x}{(x+3)(x-3)\,x}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

$\boldsymbol{n}$__ $\boldsymbol{k}$, **CASE ___**

5.

$$f(x)=\frac{x^2(x-3)}{(x^2+3x)}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

$\boldsymbol{n}$__ $\boldsymbol{k}$, **CASE ___**

6.

$$f(x)=\frac{x^3+x^2-2x}{x^2-x-2}$$

A. Hole:

B. vertical asymptote:

C. horizontal/oblique asymptote:

$n =$

$k =$

$\boldsymbol{n}$__ $\boldsymbol{k}$, **CASE ___**

___

7. vertical asymptotes: $x = -2, x = 0$
x-intercept: 2, $f(3) = 1$
What is the horizontal asymptote?

8. vertical asymptotes: $x = -1, x = 3$
x-intercept: -2, 1; $f(0) = 12$
What is the horizontal asymptote?

Answer to Hook Questions.
Q1. The values of f(x) approaches positive or negative infinity.
Q2. The values of f(x) approaches a value c.

KNOT

Before this lesson ends, Keep Note of these Outstanding Thoughts:

- Vertical asymptote is obtained by cancelling first the common factors of the rational function if exist then getting the zeros of the denominator.
- Horizontal asymptote is identified based on the degree of the polynomials in the numerator and denominator of the rational function.
- Oblique asymptote exists when a rational function has no horizontal asymptote. It is obtained by getting the quotient of the polynomials in the numerator and denominator of the rational function.

Reference:

[1] Swokowski, E., & Cole, J. (2008). *Algebra and Trigonometry with Analytic Geometry, 12th Edition.* Thomson Learning, Inc.
[2] International Geogebra Institute. (2020). *GeoGebra*. https://www.geogebra.org/
[3] Albarico, J.M. (2013). THINK Framework. (Based on Ramos, E.G. and N. Apolinario. (n.d.) *Science LINKS*. Rex Bookstore, Inc.)

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ANSWER KEY:

1. $$f(x) = \frac{(x-3)}{x^2-4}$$

A. hole: none

B. vertical asymptote:

$x^2 - 4 = 0$

$(x+2)(x-2) = 0$

$\boldsymbol{x = -2, x = 2}$

C. horizontal/oblique asymptote:

$n = 1$

$k = 2$

$\boldsymbol{n < k}$, **CASE 1**

HA: $\boldsymbol{y = 0}$

3. $$f(x) = \frac{6(x+1)(x-2)}{(x+3)(x-1)(x-2)}$$

A. hole: $\left(2, \frac{18}{5}\right)$

B. vertical asymptote:

$(x-2)$ is a common factor, cancel it

$(x+3)(x-1) = 0$

$\boldsymbol{x = -3, x = 1}$

C. horizontal/oblique asymptote:

$n = 1$

$k = 2$

$\boldsymbol{n < k}$, **CASE 1**

HA: $\boldsymbol{y = 0}$

5. $$f(x) = \frac{x^2(x-3)}{(x^2+3x)}$$

A. hole: (0,0)

B. vertical asymptote:

x is a common factor, cancel it

$(x+3) = 0$

$\boldsymbol{x = -3}$

C. horizontal/oblique asymptote:

$n = 2$

$k = 1$

$\boldsymbol{n > k}$, **CASE 3**

No HA but oblique asymptote is $\boldsymbol{y = x - 6}$.

7. vertical asymptotes: $x = -2$, $x = 0$
x-intercept: 2; $f(3) = 1$; hole at $x = -1$
What is the horizontal asymptote?

$$f(x) = \frac{a_n(x-2)(x+1)}{x(x+2)(x+1)}$$

$$3 = \frac{a_n(1-2)}{1(1+2)} = \frac{-a_n}{3}$$

$$a_n = -9$$

$$f(x) = \frac{-9(x-2)(x+1)}{x(x+2)(x+1)}$$

Thus, horizontal asymptote is $\boldsymbol{y = 0}$.