Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	2.0	*Transformations on the Coordinate Plane*
Lesson Code	2.5.2	*Scale Transformation (Dilation) 2*
Time Limit		30 minutes

TARGET

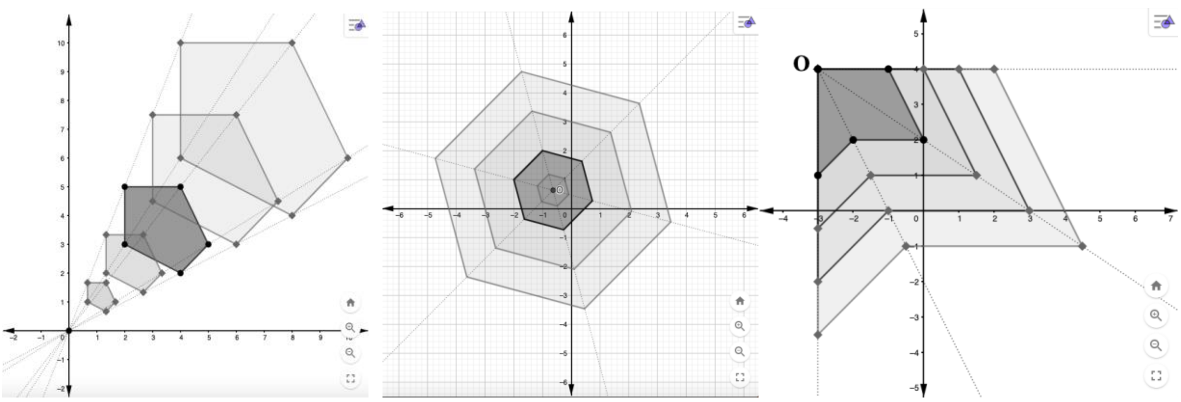
Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this module, the students will have been able to:

1. Illustrate scale transformations on the coordinate plane with center of dilation at a point other than the origin
2. Determine the image of a figure under dilation about a fixed point other than the origin

HOOK

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes



(1) (2) (3)

Figure 1. Different scale transformations of polygons with center of dilation (1) at the origin, (2) at the centroid, and (3) at a vertex in varying scale factors

In the previous module, we have observed through an activity what happens when a geometric figure is transformed under dilation given a scale factor and center of dilation at the origin. We also learned that a figure may be *enlarged* or *reduced* depending on the scale factor, k, and that the image of a point (x, y) under this dilation is (kx, ky).

Figure 1 shows the possible scenarios of scale transformations about a fixed point. The first picture shows a dilation about the origin at varying scale factors while the other two show dilations about the centroid (center point of the polygon) and about one vertex of the polygon. Notice that in all scenarios, the vertex and its corresponding images (at varying scale factors) lie on the same line containing the center of dilation. Why?

In this module we will deepen our discussion about scale transformations. Why does the figure enlarge (when $k > 1$), reduce (when $0 < k < 1$) or rotate (when $k < 0$)? What is the effect of this type of transformation on the perimeter and the area of a figure? How do we determine the image of a figure if the center of dilation is not at the origin?

Scale Transformation with the origin as the center of dilation

When an arbitrary point, say $A(x, y)$, is dilated by a scale factor k with the center of dilation at the origin, this means that the distance of point A from the center of dilation is multiplied by the scale factor k.

For instance, when point $A(2,3)$ is dilated by a scale factor $k = 3$, its image A' will be *three times* as far as point A is from the origin, which is the center of dilation. Hence, the resulting point will have coordinates $A'(6, 9)$. Refer to figure 2.

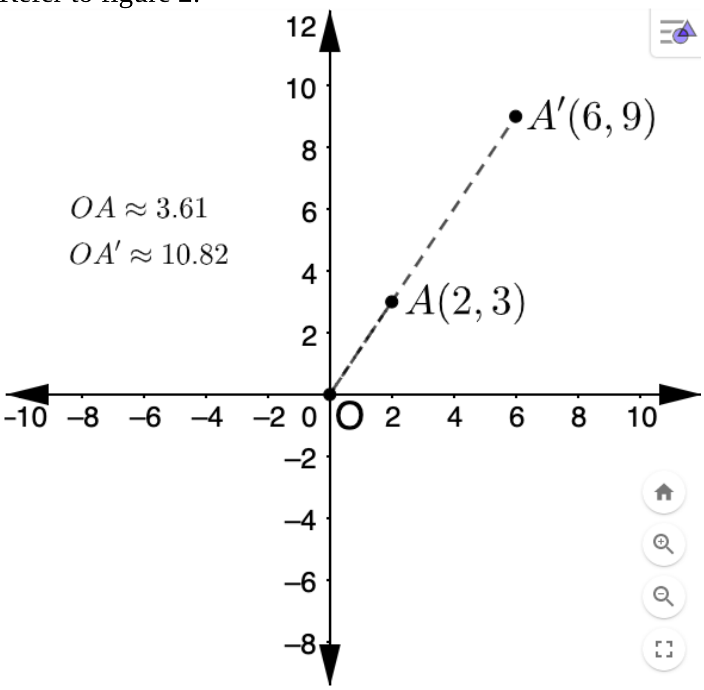


Figure 2. Point A and its image A' under scale transformation with center of dilation at the origin and scale factor, k= 3

Using distance formula, we can verify the lengths of the segments:

$$OA = \sqrt{(0-2)^2 + (0-3)^2} = \sqrt{13} \approx 3.61$$

$$OA' = \sqrt{(0-6)^2 + (0-9)^2} = \sqrt{117} = 3\sqrt{13} \approx 10.81$$

We also know now that in general, under scale transformation with center of dilation at the origin, the point (x, y) is mapped to the point (kx, ky) where k is the scale factor. So, what happens when $k < 0$?

When $k < 0$, the scale factor is $|k|$ but because k is negative, the signs of the coordinates are inverted, thus results in a half-turn rotation about the origin. (Recall: $R_{180^o}: (x, y) \rightarrow (-x, -y)$.) Therefore, we can think of this case as a combination of transformation that involves rotation.

Figure 3 shows the images of a triangle under dilation about the origin using different scale factors $k = 0.5, 1, 2$, and -1. Observe that the preimage and its image lie on the same line containing the origin, regardless of the scale factor (even when $k < 0$).

Observe also that the triangle when $k = 1$ and when $k = -1$ are of the same size, but the latter is rotated through an angle of 180^o. Notice that the preimage and the images seem to radiate from the origin, which is the center of dilation.

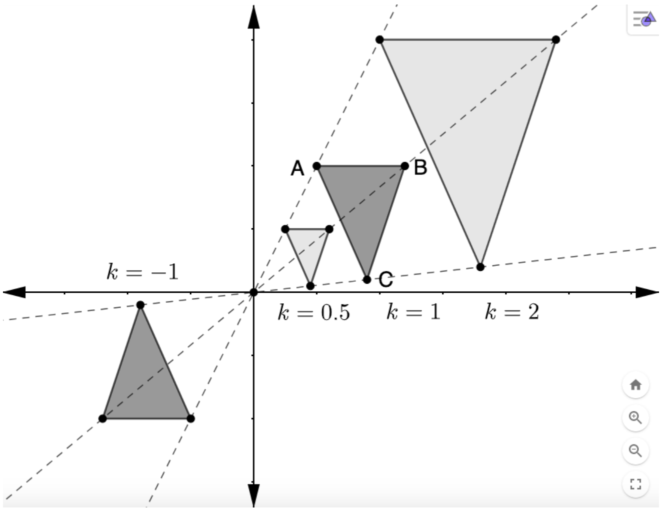


Figure 3. Triangle ABC and its images under dilation about the origin with scale factors k=0.5, 1, 2, and -1

Scale transformations do not preserve distance (as seen in the first example). This type of transformation produces *similar* figures. What does this mean? Let us take a look at the next example.

Example 1:
A trapezoid has vertices $M(3,3)$, $I(4,0)$, $N(-2,0)$, and $T(-1,3)$. Find its image under dilation about the origin with scale factor $k = 3/2$. Compare the perimeters and the areas of the preimage and its image.

Solution:

(1) The image of trapezoid $MINT$ under dilation about the origin with scale factor $k = 1.5$ is as follows:

Preimage	Image
$M(3,3)$	$M'(4.5, 4.5)$
$I(4,0)$	$I'(6, 0)$
$N(-2,0)$	$N'(-3,0)$
$T(-1,3)$	$T'(-1.5, 4.5)$

(2) We can compute the lengths of the sides of the trapezoid and of its image using the distance formula and determine the perimeter and the area.

Preimage	**Image**
$MI = \sqrt{(3-4)^2 + (3-0)^2} = \sqrt{10}$	$M'I' = \sqrt{(4.5-6)^2 + (4.5-0)^2} = \frac{3\sqrt{10}}{2}$
$IN = \sqrt{(4+2)^2 + (0-0)^2} = \sqrt{36} = 6$	$I'N' = \sqrt{(6+3)^2 + (0-0)^2} = 9$

$NT = \sqrt{(-2+1)^2 + (0-3)^2} = \sqrt{10}$	$N'T' = \sqrt{(-3+1.5)^2 + (0-4.5)^2} = \frac{3\sqrt{10}}{2}$
$TM = \sqrt{(-1-3)^2 + (3-3)^2} = \sqrt{16} = 4$	$T'M' = \sqrt{(-1.5-4.5)^2 + (4.5-4.5)^2} = 6$

Remark:
Observe that the lengths of the corresponding sides of the preimage and the image are proportional, with ratio equal to 1.5.

$$\frac{M'I'}{MI} = \frac{I'N'}{IN} = \frac{N'T'}{NT} = \frac{T'M'}{TM} = 3/2$$

Since we already know the lengths of the sides of the two trapezoids, we can now compute for the perimeter and the area. (Recall: $A_{trapezoid} = \frac{1}{2}(b_1 + b_2)(h)$)

Perimeter	Area
$P_{MINT} = \sqrt{10} + \sqrt{10} + 4 + 6$ $= 10 + 2\sqrt{10}$ units	$A_{MINT} = \frac{1}{2}(10)(3) = 15$sq. units
$P_{M'I'N'T'} = \frac{3\sqrt{10}}{2} + \frac{3\sqrt{10}}{2} + 9 + 6$ $= 15 + 3\sqrt{10}$ units	$A_{M'I'N'T'} = \frac{1}{2}(15)(4.5) = \frac{135}{4}$ sq. units

The perimeter of the image is 3/2 times larger than that of the preimage.

The area of the image is 9/4 times larger than that of the preimage.

Figure 4 shows the trapezoid and its image when plotted on the Cartesian plane.

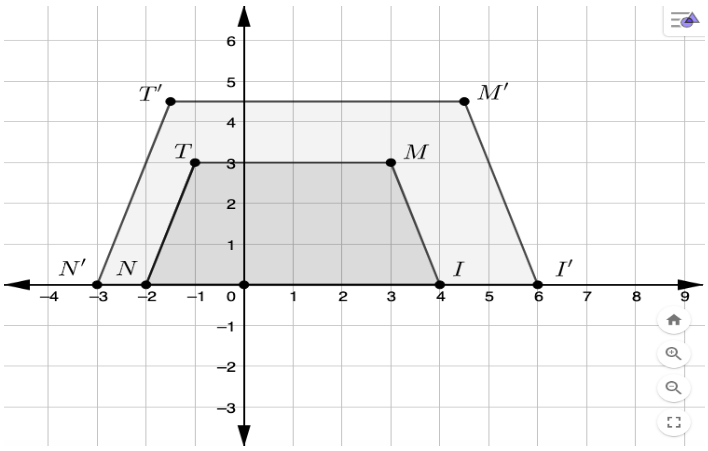


Figure 4. Trapezoid MINT and its image under dilation about the origin with scale factors $k = 1.5$

Scale Transformation given a fixed point and a scale factor

Just like in the transformation rotation, the image of the center of dilation under scale transformation is itself, regardless of the scale factor.

How do we determine the coordinates of the image under dilation when the center of dilation is not at the origin?

This can be done by considering the horizontal distance (x-coordinate) and the vertical distance (y-coordinate) of the preimage from the center of dilation.

For instance, let us assign point $A(1,2)$ as the center of dilation and suppose we want to determine the image of $P(-3,-1)$ under dilation with scale factor $k = 2$.

The image of P' must be located such that its horizontal and vertical distance is twice that of point P from point A. Note that point P is 4 units to the left of point A and 3 units down. Hence, the image of P must be located 8 units to the left of point A and 6 units down, which makes $P'(-7,-4)$.

Alternatively, we can deal with it in the same way as we do it with rotation about a fixed point not on the origin. In other words, we can perform the following steps:

(1) Translate point $P(-3,-1)$ the same number of units as it would require point $A(1,2)$ to be translated to the origin. Let's name this point P_o. (i.e., $P_o(-4,-3)$.)
(2) We dilate $P_o(-4,-3)$ by the scale factor $k = 2$ but the center of dilation is at the origin. Let's name this P'_o. (i.e., $P'_o\,(-8,-6)$.)
(3) We translate back $P'_o\,(-8,-6)$ by the same number of units as it would take the origin to point A. The resulting point is the image of point P under dilation about A given the scale factor $k = 2$. (i.e., $P'(-7,-4)$.)

Let us verify the result by computing for the following distance:

$$AP = \sqrt{(1+3)^2 + (2+1)^2} = \sqrt{25} = 5$$
$$AP' = \sqrt{(1+7)^2 + (2+4)^2} = \sqrt{100} = 10$$

Indeed, $AP' = 2(AP)$.

Let us work on a few more examples.

Example 2:
Circle C is given by the equation $(x-1)^2 + (y+1)^2 = 25$ is transformed under dilation about its center by a scale factor of 3. Find the equation of the new circle formed after the transformation.

Solution:
Since the circle is in center-radius form we know that its center is at $(1,-1)$ and its radius is 5.

Note that the center of the circle will serve as the center of dilation, hence, its image will be itself.

What will be the radius of the new circle? It will be the radius of the preimage multiplied by the scale factor, i.e. $k = 3$. Why?

We now have the new circle centered at $(1,-1)$ with radius equal to 15. Therefore, the equation of the new circle is given by $(x-1)^2 + (y+1)^2 = 225$.

Example 3:
A triangle with vertices $(0,0)$, $(-2,-3)$, and $(4,-5)$ is dilated about the point $(4,2)$ by the same dilation constant as the line segment with endpoints at $(3,1)$ and $(2,6)$ is dilated about the origin. If the image of the line segment has endpoints $(1.5,0.5)$ and $(1,3)$, respectively, find the image of the triangle under its described transformation. (Note: *Dilation constant* is another term for scale factor.)

Solution:

(1) We need to know the scale factor by which the line segment is dilated about the origin. In other words, we have:

$$D_k: (3,1) \rightarrow (1.5,0.5) \;\Rightarrow\; k = 1/2$$
$$D_k: (2,6) \rightarrow (1,3) \;\Rightarrow\; k = 1/2$$

(2) We now find the image of the triangle under dilation about the point $(4,2)$ by a scale factor of k = ½ .

Note that it takes four units to the left and two units down (i.e. $T_{-4,-2}$) to translate the center of dilation (4,2) to the origin. The plan is to translate the vertices of the triangle by this same number of units, then find their image under dilation about the origin, and then translate back the image by the same number of units as it would take to shift the origin to the center of dilation, (i.e. $T_{4,2}$). The table summarizes this process.

Vertices of the triangle	Image under translation, $T_{-4,-2}$	Image under dilation about the origin with $k = 1/2$	Image under translation, $T_{4,2}$
$(0,0)$	$(-4,-2)$	$(-2,-1)$	$(2,1)$
$(-2,-3)$	$(-6,-5)$	$(-3,-2.5)$	$(1,-0.5)$
$(4,-5)$	$(0,-7)$	$(0,-3.5)$	$(4,-1.5)$

NAVIGATE

Time Allocation: 10 minutes
Actual Time Allocation: _____ minutes

Do as indicated.

1. Tell whether the statement about scale transformation is true or false.
   a. A figure is enlarged when the scale factor k is greater than 1.
   b. Two congruent figures are formed when the scale factor k is equal to 1.
   c. The images of a given point under dilation at varying scale factors lie on the same line containing the center of dilation.
   d. The area of a rectangle is $1/k$ times the area of its image under dilation with scale factor k.
   e. If the image of a point under dilation with scale factor $0 < k < 1$ is itself, then the given point is the center of dilation.

2. Identify the scale factor given the following information about the preimage and the image under dilation about the origin.
    a. Point $A(-8,7)$ and its image $A'(2,-1.75)$
    b. The lengths of a line segment and its image are 10 units and 6 units, respectively
    c. The perimeters of a triangle and its image are $5\sqrt{5}$ inches and 25 inches
    d. The areas of a square and its image are $8\ cm^2$ and $4\ cm^2$, respectively

3. Determine the image of the following points under dilation given the indicated scale factor. Use point $(-2,4)$ as the center of dilation.
    a. $(4,-2), k = 5$
    b. $(1,1), k = 1$
    c. $(0,0), k = 0.4$
    d. $(-2,-2), k = -2$

4. Give the image of a square under dilation with center of dilation at $(0,-3)$ and scale factor $k = -3$. The vertices of the square are as follows: $(-2,0), (0,2), (2,0)$, and $(0,-2)$.

KNOT

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes

Some points to remember:

1. The image of the center of dilation under scale transformation is itself.
2. To determine the coordinates of the image of a point under dilation, with center of dilation not at the origin, given a scale factor, we can perform the following steps:
    a. Translate the pre-image by the same number of units as it would require to translate the center of dilation to the origin.
    b. Dilate the resulting point in (a) with the given scale factor but using the origin as the center of dilation.
    c. Translate back the resulting point in (b) by the same number of units it would require to translate the origin to the given center of dilation.

References:

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Answer Key:

1. (a) True (b) True (c) True (d) False (e) True
3. (a) $(28,-26)$ (b) $(1,1)$ (c) $(-1.2,2.4)$ (d) $(-2,16)$