Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	2.0	*Transformations on the Coordinate Plane*
Lesson Code	2.4.1	*Rotation 1*
Time Limit		30 minutes

TARGET

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this module, the students will have been able to
1. Define the transformation rotation about a fixed point through an angle
2. Demonstrate rotation about the origin using graphing tools and software and other manipulative devices
3. Illustrate rotation about the origin through an angle (90^o, 180^o, 270^o, 360^o) on the coordinate plane

HOOK

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

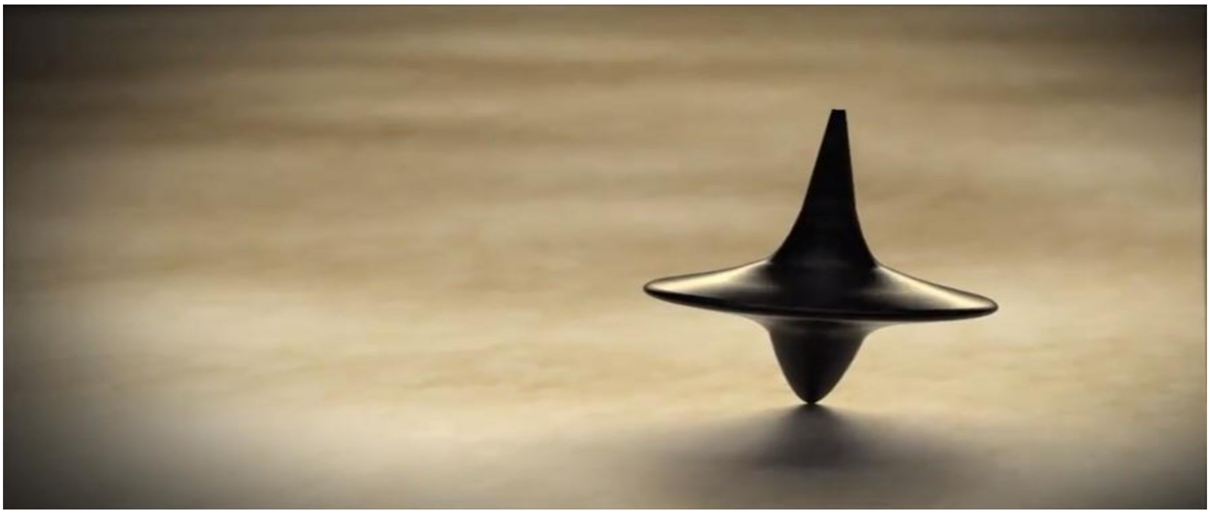


Figure 1. Screenshot of the Inception Movie ending scene. From the article *Christopher Nolan Explains the Spinning Top in Inception* by H. Horton, 2015, https://www.dazeddigital.com/artsandculture/article/24949/1/christopher-nolan-explains-the-spinning-top-in-inception.

If you watched the movie Inception, you are probably familiar with the picture shown above which was the ending scene of the said movie. An object rotating about an axis or a point is something that you can observe everywhere. You can think of a spinning top, a carousel, a ferris wheel, a windmill, a ceiling fan, and the movement of the stars in the night sky among others.

In this module, you will learn rotation as a type of geometric transformation on the coordinate plane. You will investigate, using a graphing tool, what happens to a figure when it is rotated about the origin at a given angle. What will be the coordinates of its image? Is the shape and size of the figure altered after the transformation?

Rotation is another type of rigid transformation where a set of points is turned at an angle about a fixed point. (Recall: Rigid transformation refers to transformations that do not change the shape or size of a figure that is being transformed.)

The fixed point is called the *center of rotation*. The angle of rotation can either be in *counterclockwise* or *clockwise* direction. We will use θ to denote the angle of rotation. When $\theta > 0$, it means the direction of rotation is counterclockwise while when $\theta < 0$, it indicates the clockwise direction.

Throughout this lesson, we will use the notation R_θ for this type of transformation, where θ is the angle of rotation and the origin is the center of rotation. (NOTE: Unless otherwise stated, we will use the origin as the center of rotation.)

The figure formed under rotation is called the *image* of the transformation while the figure to be rotated is called the *preimage*.

Figure 2 shows a polygon and its image under rotation about a fixed point through an angle θ in a counterclockwise direction. Note that all the points of the polygon are rotated about the same fixed point through the same angle. However, it will be enough to use the vertices of the polygon as we perform the transformation.

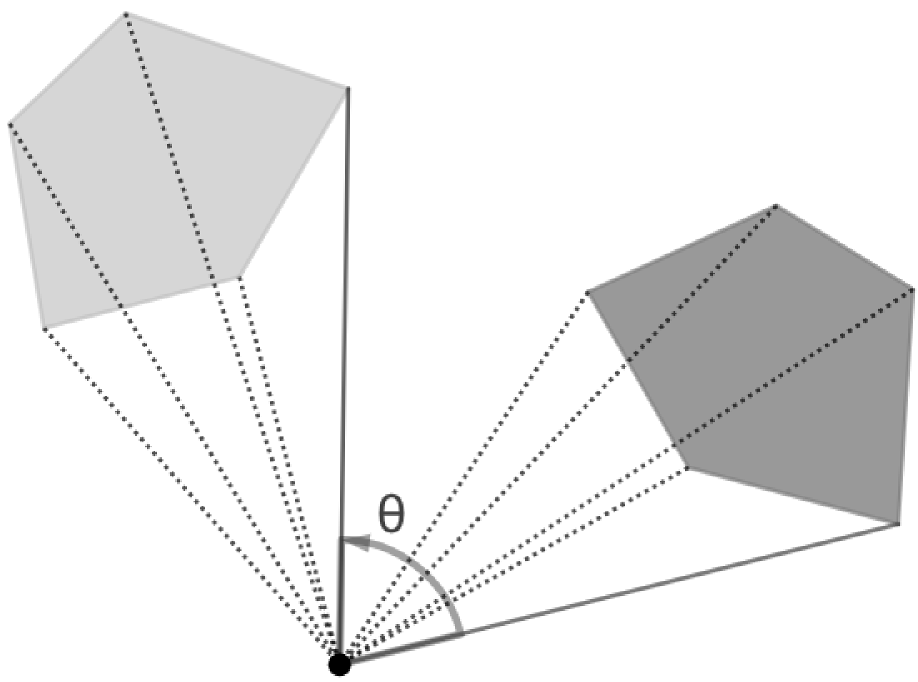


Figure 2. Rotation of a polygon about a fixed point through an angle θ in a counterclockwise direction

Hands-On Activity:

Objective: To investigate the transformation of a geometric figure under rotation about the origin through given angles 90^o, 180^o, and 270^o in a counterclockwise direction.

Graphing Tool/Software: GeoGebra

1. Using the graphing software, mark the origin as point O. This will be the *center of rotation*.

2. Plot the coordinates of triangle ABC with vertices $A(2,9)$, $B(-3,4)$, and $C(3,-1)$ and use the *Polygon Tool* to form the triangle.

3. Rotate triangle ABC about the origin 90^o counterclockwise, using the *Transformation Tool*: *Rotate around a point*. Select the polygon to rotate, then the center of rotation, then input the angle. Label the image as triangle $A'B'C'$.

4. Write down the coordinates of the vertices of triangle $A'B'C'$ below.
   R_{90^o}: $A(2,9) \rightarrow A'$ (___, ___)
   R_{90^o}: $B(-3,4) \rightarrow B'$(___, ___)
   R_{90^o}: $C(3,-1) \rightarrow C'$(___, ___)

   (The symbol "→" is read as "is mapped to".)

5. Now, rotate triangle ABC about the origin 180^o counterclockwise, using again the *Transformation Tool*: *Rotate around a point*. Do the same procedure and label the new image as triangle $A''B''C''$.

6. Write down the coordinates of the vertices of triangle $A''B''C''$ below.
   R_{180^o}: $A(2,9) \rightarrow A''$ (___, ___)
   R_{180^o}: $B(-3,4) \rightarrow B''$(___, ___)
   R_{180^o}: $C(3,-1) \rightarrow C''$(___, ___)

7. Lastly, rotate triangle ABC about the origin 270^o counterclockwise, using the same transformation tool. Do the same procedure and label the new image as triangle $A'''B'''C'''$.

8. Write down the coordinates of the vertices of triangle $A'''B'''C'''$ below.
   R_{270^o}: $A(2,9) \rightarrow A'''$ (___, ___)
   R_{270^o}: $B(-3,4) \rightarrow B'''$ (___, ___)
   R_{270^o}: $C(3,-1) \rightarrow C'''$ (___, ___)

9. What will be the image of the triangle if the angle of rotation is 360^o?

10. What is the result when the direction of the angle of rotation is in clockwise direction instead?

11. Summarize your observations by completing the statements below.
    a. A rotation about the origin through an angle 90^o **counterclockwise** maps the point (x, y) to its image (___, ___).
    b. A rotation about the origin through an angle 180^o **counterclockwise** maps the point (x, y) to its image (___, ___).
    c. A rotation about the origin through an angle 270^o **counterclockwise** maps the point (x, y) to its image (___, ___).

d. A rotation about the origin through an angle ________ maps the point (x, y) to itself.
e. A rotation of a point about the origin 90^o counterclockwise is equivalent to a rotation of the point about the origin ________ clockwise.
f. A rotation of a point about the origin 270^o counterclockwise is equivalent to a rotation of the point about the origin ________ clockwise.

Discussion/Processing:

Rotation of a point about the origin through angles 90^o, 180^o, 270^o, and 360^o

Based on the hands-on activity, we observed that when a point (x, y) is rotated counterclockwise about the origin through an angle of 90^o, the image will be $(-y, x)$.

For example, the point $P(-1,4)$ will have the image $P'(-4,-1)$ under this transformation. See Figure 3.

In symbols, we write $R_{90^o}: P(-1,4) \rightarrow P'(-4,-1)$.

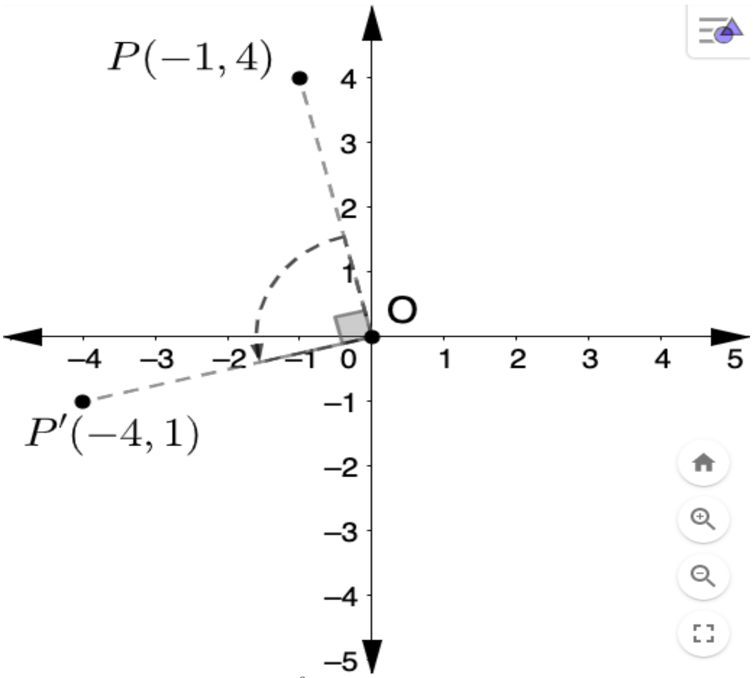


Figure 3. P and its image *P'* under a 90^o rotation about the origin in a counterclockwise direction

Likewise, a rotation of a point (x, y) about the origin through an angle 180^o counterclockwise will have the image $(-x, -y)$. We can think of this as rotating the point about the origin through a 90^o angle in a counterclockwise direction twice. We can represent this process in symbols as follows:

$$R_{180^o}: \ (x, y) \rightarrow_{R_{90^o}} (-y, x) \rightarrow_{R_{90^o}} (-x, -y)$$

As an example, see Figure 4. The figure shows the point $P(-1,4)$ and its image $P''(1,-4)$ after two 90^o counterclockwise rotations of P about the origin, which is the same as rotating it through an angle 180^o.

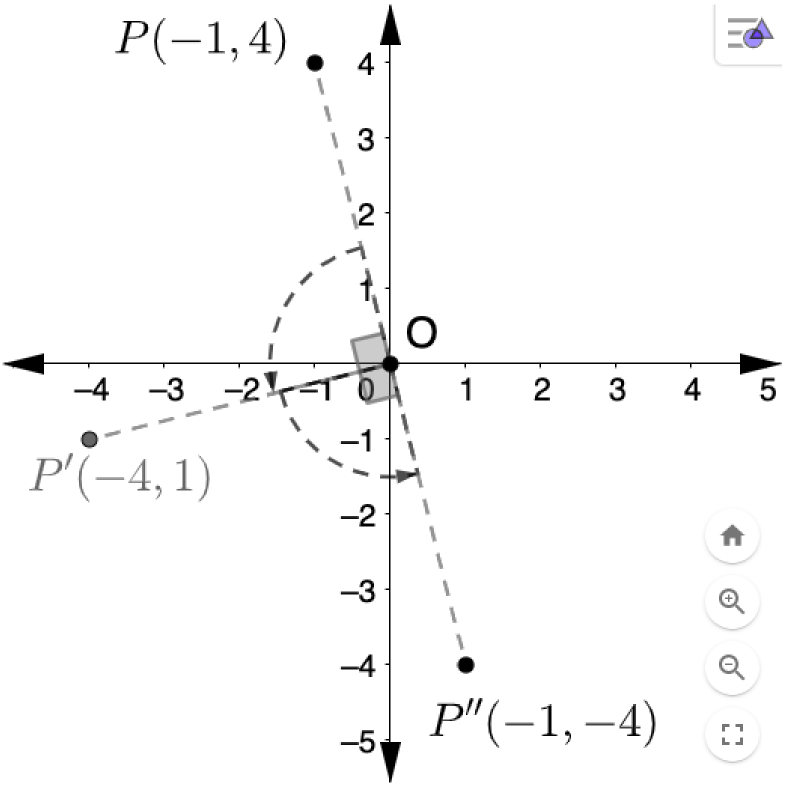


Figure 4. P and its image *P''* under two 90^o rotations about the origin in a counterclockwise direction

Using the same line of reasoning, we can deduce that a rotation of 270^o counterclockwise about the origin will map the point (x, y) to its image $(y, -x)$. We can represent this mapping as follows:

$$R_{270^o}: (x, y) \rightarrow_{R_{90^o}} (-y, x) \rightarrow_{R_{90^o}} (-x, -y) \rightarrow_{R_{90^o}} (y, -x)$$

For example, the image of $P(-1,4)$ under a 270^o counterclockwise rotation about the origin is the point $P'''(4,1)$. We will get the same result when we rotate P about the origin 90^o counterclockwise three times as shown in Figure 5.

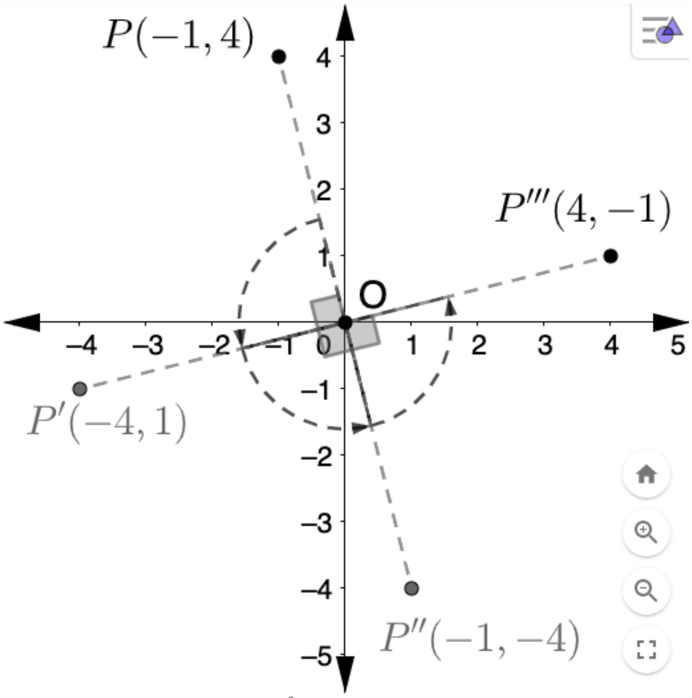


Figure 5. P and its image *P'''* under three *90°* rotations about the origin in a counterclockwise direction

It follows then that a rotation of 360^o of a point about the origin will lead us back to the point itself. The mapping will be as follows:

$$R_{360^o}: (x, y) \rightarrow_{R_{90^o}} (-y, x) \rightarrow_{R_{90^o}} (-x, -y) \rightarrow_{R_{90^o}} (y, -x) \rightarrow_{R_{90^o}} (x, y)$$

Counterclockwise versus Clockwise Rotation

It is important to take note of the direction of rotation as each may result in a different image. We can visualize that a 90^o *counterclockwise* rotation is equivalent to a 270^o *clockwise* rotation, in the same way that a 270^o *counterclockwise* rotation is equivalent to a 90^o *clockwise* rotation. Note that a rotation of 180^o will result in the same image regardless of the direction of rotation. The same can be said for a rotation of 360^o.

Table 1 summarizes the rules under the transformation rotation about the origin through given angles 90^o, 180^o, 270^o, and 360^o.

R_θ	The image of the point (x, y) given the transformation rotation R_θ
R_{90^o} or R_{-270^o}	$(-y, x)$
R_{180^o} or R_{-180^o}	$(-x, -y)$
R_{270^o} or R_{-90^o}	$(y, -x)$
R_{360^o} or R_{-360^o}	(x, y)

Table 1: Rules of the transformation rotation with the origin as center of rotation through an angle θ

Now let us try more examples of the transformation rotation about the origin.

Example 1:
Determine the image of a line segment with endpoints $(-3,2)$ and $(11,-8)$ under rotation about the origin, 90^o clockwise.

Solution:
It is possible to get the image of the line segment even without having to plot it on the Cartesian plane. Note that a 90^o rotation, clockwise is the same as a 270^o rotation, counterclockwise. Also, even though all the points on the line segment will be rotated (which is infinitely many!), it will be enough to find the image of its endpoints under the described transformation. Hence, we have

$$R_{-90^o} = R_{270^o}: (x, y) \to (y, -x)$$
$$R_{-90^o} : (-3,2) \to (2, 3)$$
$$R_{-90^o} : (11, -8) \to (-8, -11)$$

Therefore, the image of the line segment under rotation about the origin, 90^o clockwise is the line segment with endpoints $(2,3)$ and $(-8, -11)$. Figure 6 shows the line segment and its image.

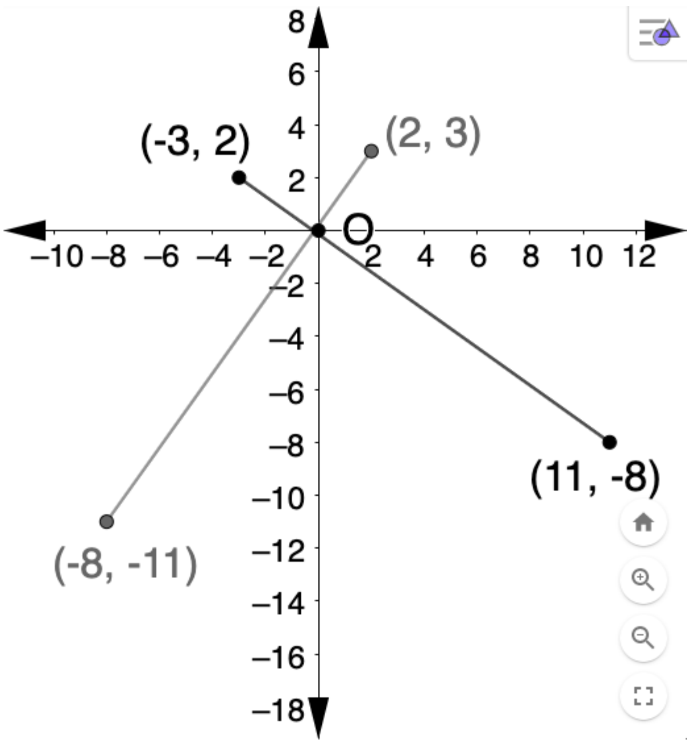


Figure 6. The line segment (black) with endpoints $(-3,2)$ and $(11, -8)$ and its image (gray) under 90^o clockwise rotation about the origin

CAUTION: Although the center of rotation is the origin, it is not contained on any of the two line segments (the preimage and the image). However, we can observe that the slopes of the two segments are negative reciprocals of each other, thus they are perpendicular.

Example 2:
A circle with equation $(x-2)^2+(y+2)^2=20$ is mapped to the circle with equation $(x+2)^2+(y-2)^2=20$ under the transformation rotation with the origin as the center of rotation. Identify the rule of transformation.

Solution:
We recall the standard form of the equation of a circle (center-radius form) $(x-h)^2+(y-k)^2=r^2$, where $(h,k)$is the center of the circle and r is the radius. Also, note that rotation is a type of rigid transformation, hence, the size and shape of the preimage is preserved (i.e. does not change) after the transformation. Therefore, we can expect that the radius will stay the same.

We get the following information from the given problem:

Preimage: $C(2,-2), r=2\sqrt{5}$
Image: $C(-2,2),\ r=2\sqrt{5}$

We are looking for θ such that R_θ will map $(2,-2)$ to its image $(-2,2)$. Note that $(x,y)\rightarrow(-x,-y)$ is the result of 180^o rotation. Therefore, $\theta=180^o$.

Hence, the rule of transformation is R_{180^o} or in words we say, rotation about the origin through an angle 180^o.

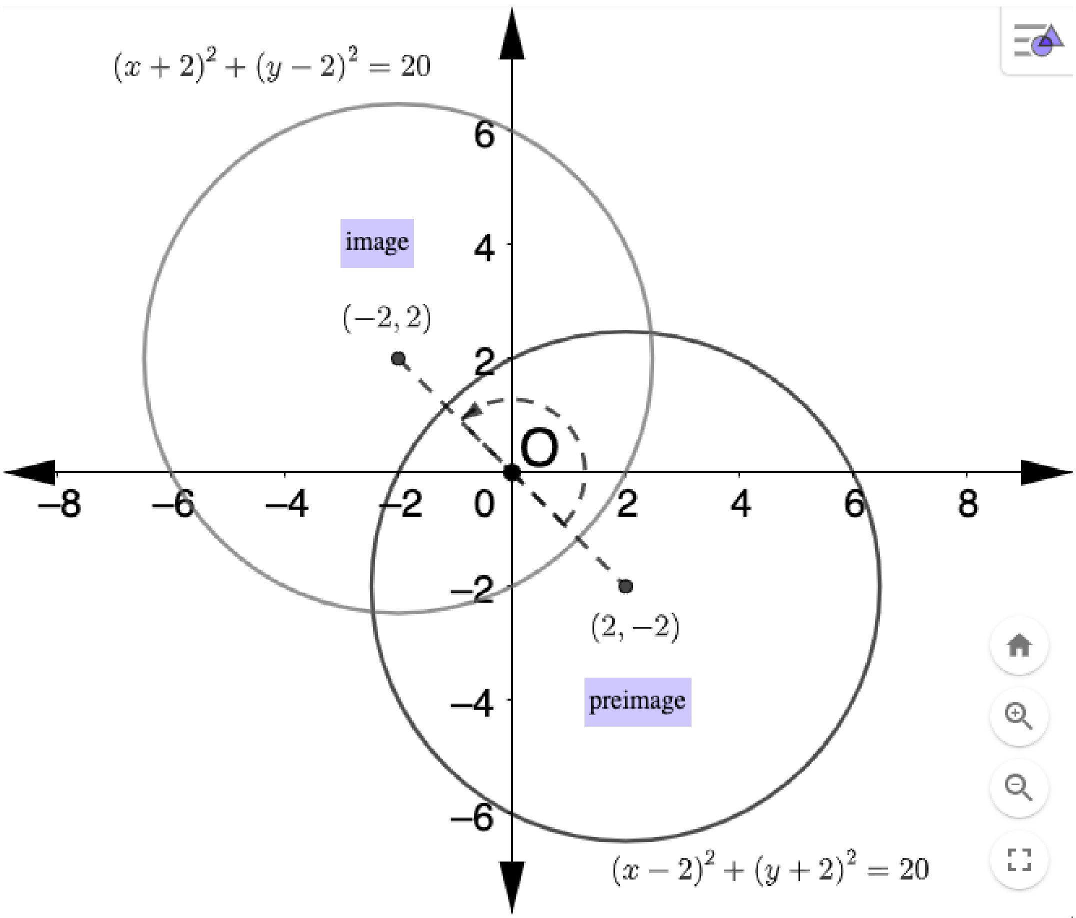


Figure 7. The circle and its image under the transformation rotation about the origin through 180^o

Example 3:
A square is transformed under rotation 90^o counterclockwise about the origin. If its image is the square bounded by the two axes and the lines $x = -6$ and $y = 6$, find the coordinates of the vertices of the original square.

Solution:
Firstly, we take note of the following given:

(1) the transformation is 90^o counterclockwise rotation about the origin
(2) The image is described as the square bounded by the lines $y = 0, x = 0, x = -6,$ and $y = 6$.

Since the figure is a square, the vertices of the image will lie on the intersection of any two of the boundary lines. (The trick is to identify which of the boundary lines are perpendicular to each other.) The vertices are $(0,0), (0,6), (-6,6),$ and $(-6,0)$.

Now we can have the following mapping of the vertices under $R_{90^o}: (x, y) \rightarrow (-y, x)$ and work backwards.

$$(x_1, y_1) \rightarrow (0,0) \quad \Rightarrow (x_1, y_1) = (0,0)$$
$$(x_2, y_2) \rightarrow (0,6) \quad \Rightarrow (x_2, y_2) = (6,0)$$
$$(x_3, y_3) \rightarrow (-6,6) \Rightarrow (x_3, y_3) = (6,6)$$
$$(x_4, y_4) \rightarrow (-6,0) \Rightarrow (x_4, y_4) = (0,6)$$

Therefore, the coordinates of the vertices of the original square are $(0,0), (6,0), (6,6),$ and $(0,6)$.

Alternatively, we can solve the problem by "undoing" the 90^o counterclockwise rotation. We can do this by rotating the image 90^o clockwise about the origin to get to the preimage.

Example 4:
A triangle is rotated 90^o counterclockwise about the origin and reflected over the line $x = 1$. Find the image of the triangle after the two transformations given its vertices $(-4,2), (-5,-2),$ and $(-1,-3)$.

Solution:
We can directly approach the problem by performing the rotation of the vertices first, and then by reflecting the resulting image over the given line of reflection. Since there are two transformations involved, it will help if we name these as follows:

A: transformation rotation about the origin, 90^o counterclockwise (i.e. R_{90^o})
B: transformation reflection over the line $x = 1$(i.e. $r_{x=1}$)

Below shows the result of the two transformations.

Preimage		Image under A		Image of A under B
$(-4,2)$	$\rightarrow_{R_{90^o}}$	$(-2,-4)$	$\rightarrow_{r_{x=}}$	$(4,-4)$
$(-5,-2)$	$\rightarrow_{R_{90^o}}$	$(2,-5)$	$\rightarrow_{r_{x=}}$	$(0,-5)$
$(-1,-3)$	$\rightarrow_{R_{90^o}}$	$(3,-1)$	$\rightarrow_{r_{x=}}$	$(-1,-1)$

Note that under B, the y-coordinates of the image are retained since the line of reflection is a vertical line. We can easily verify the results using a graphing software. See Figure 8.

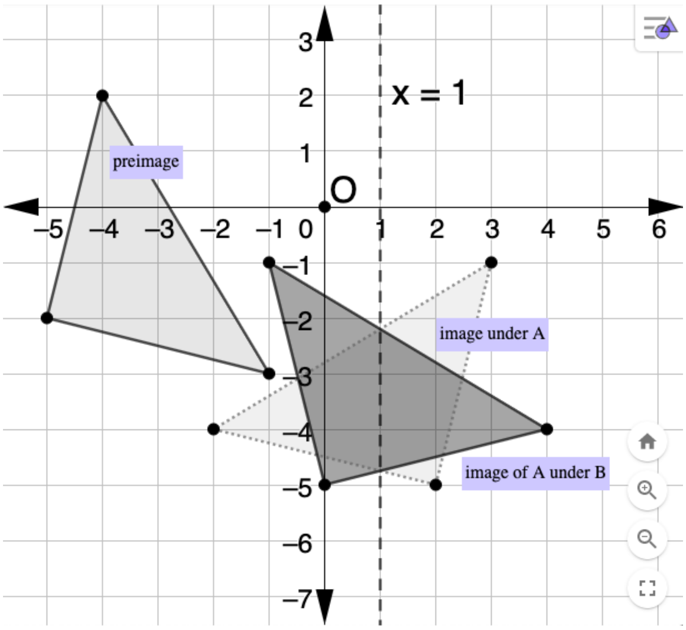


Figure 8. The triangle and its image after rotation followed by reflection

Now, think about this -- if suppose the order of transformations is reversed, will we get the same final image? That is, is rotation and reflection commutative? Verify this on your own.

NAVIGATE

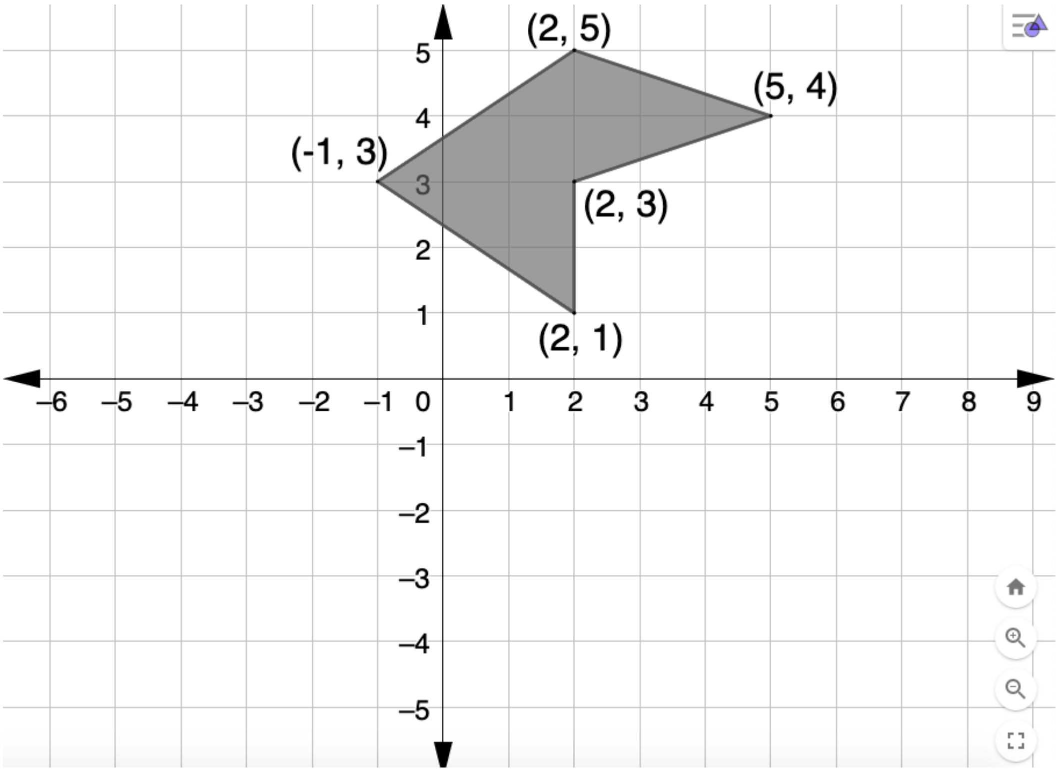
Time Allocation:	10 minutes
Actual Time Allocation:	_____ minutes

Do as indicated.

1. Give the image of the points below under transformation R_{90^o}.
   a. $(0,0)$
   b. $(1,-4)$
   c. $(-\frac{2}{3},\frac{1}{2})$
   d. $(-2\sqrt{2}\,,-\sqrt{5}\,)$
   e. $(\,2p+q,-p+q\,)$ where p,q are real numbers
   f. $(\,pq,\frac{p}{q})$ where p,q are real numbers and $q \neq 0$

2. The coordinates of the vertices of a parallelogram are $(0,1)$, $(4,0)$, $(3,-3)$, and $(-1,-2)$. Identify the coordinates of the vertices of its image under transformation R_{180^o}.

3. Sketch the image of the figure on the Cartesian plane below under transformation R_{270^o}. Write the coordinates of the vertices of the image.



4. Under rotation about the origin, the vertices of a trapezium are $(1,2), (-2,-1), (1,-3)$, and $(4,-2)$and its corresponding images are $(2,-1), (-1,2), (-3,-1)$, and (a,b), respectively. Determine the rule of transformation and find the values of a and b.

5. Identify the quadrant where the image of point (x,y) will be located given the following conditions.
   a. $x < 0, y < 0$, R_{-90^o}
   b. $x > 0, y > 0$, R_{180^o}
   c. $x > 0, y < 0$, R_{-270^o}

6. A circle given by the equation $x^2 + y^2 - 8x + 2y + 12 = 0$ is rotated about the origin at an angle180^o. Give the equation (in general form) of its image under the said transformation and determine the midpoint of the segment joining the centers of the two circles.

7. Describe as a single transformation a rotation about the origin, 90^oclockwise followed by a rotation about the origin 180^ocounterclockwise.

8. A triangle bounded by the y-axis and the lines $y = -2$ and $5x + 3y = 9$ is mapped to its image using the following transformations:

   A: Rotation about the origin through an angle 180^o (i.e. R_{180^o})
   B: Translation 2 units to the left and 1 unit up (i.e. $T_{-2,1}$)
   C: Reflection over the x-axis (i.e. r_{x-axis})

   a. Identify the vertices of the triangle and determine its final image under transformation **A** followed by transformation **C**.

b. Determine the final image of the triangle under transformation **A** followed by transformation **B**. Then, reverse the order of transformation. Is the final image the same?
c. Can we write as a single transformation the one described in (a)? in (b)?

KNOT

Time Allocation: 3 minutes
Actual Time Allocation: _____ minutes

Key terms to remember:
1. rigid transformation
2. rotation
3. fixed point or center of rotation
4. angle of rotation
5. counterclockwise or clockwise

Important points to remember:
1. The transformation rotation does not change the shape and the size of the figure being transformed.
2. Under the transformation rotation, we consider the center of rotation, the angle of rotation, and the direction of rotation.
3. A positive angle of rotation implies counterclockwise direction while a negative angle of rotation implies clockwise direction.
4. The image of the center of rotation under rotation at any given angle is itself.
5. A point (x, y) under rotation at an angle 360^o is mapped to itself.
6. The rules of rotation of a point (x, y) under transformation R_θ is as follows:

R_θ	The image of the point (x, y) given the transformation rotation
R_{90^o} or R_{-270^o}	$(-y, x)$
R_{180^o} or R_{-180^o}	$(-x, -y)$
R_{270^o} or R_{-90^o}	$(y, -x)$
R_{360^o} or R_{-360^o}	(x, y)

Extension:

Composition or Product of Transformation
The *product of transformation*, AB, refers to a combination of two transformations done on a set of points, where the first transformation is B (the one on the right), followed by the transformation A (the one on the left). (Other reference materials use the notation $A \circ B$ to refer to the product of transformations as *composition* of transformations. That is, we perform transformation A to the image under transformation B.)

For instance, $T_{2,3}\, r_{x-axis}$ refers to a combination of reflection over the x-axis followed by a translation of 2 units to the right and 3 units up. (Alternatively, we can write $T_{2,3} \circ r_{x-axis}$.)

Some examples of combinations of transformations were provided in the Ignite and Navigate sections of this module.

Answer the following:

1. Every rotation of a point given the center of rotation through an angle can be described as a product of two reflections. Express the transformation rotation about the origin at 180^o as a product of two reflections.
2. Describe as a single transformation Navigate item 8a and 8b.
3. (Always, Sometimes, Never True) A combination of two different types of transformation (e.g. translation and reflection, or rotation and translation, etc), can be expressed as a single transformation.

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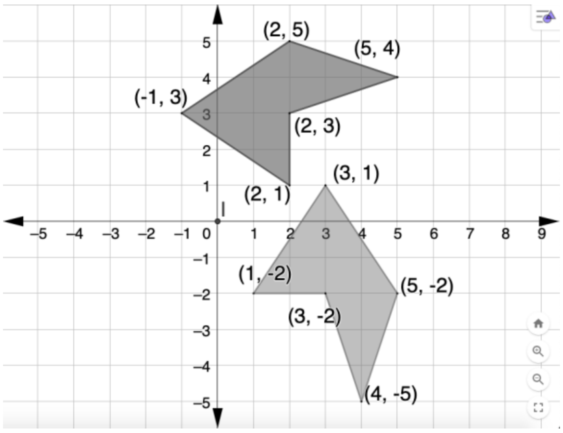
Answer Key:

Hand-on Activity:

4. $A'(-9,2)$; $B'(-4,-3)$; $C'(1,3)$
6. $A''(-2,-9)$; $B''(3,-4)$; $C''(-3,1)$
8. $A'''(9,-2)$; $B'''(4,3)$; $C'''(-1,-3)$
9. The image will be the triangle ABC itself.
10. $R_{90^o} = R_{-270^o}$ and $R_{270^o} = R_{-90^o}$
11. $a. (-y,x);\ b. (-x,-y);\ c. (y,-x);\ d. 360^o;\ e.\ 270^o;\ f.\ 90^o$

Navigate:

1. a. $(0,0)$; b. $(4,1)$; c. $(-\frac{1}{2},-\frac{2}{3})$; d. $(\sqrt{5},-2\sqrt{2})$; e. $(p-q, 2p+q)$; f. $(-\frac{p}{q}, pq)$
3.



5. a. QII ; b. QIII; c. QI

7. R_{90^o} or R_{-270^o}

Knot:

(Extension)

1. $R_{180^o} = r_{x-axis}r_{y-axis} = r_{y-axis}r_{x-axis}$
3. Sometimes true