Chapter 3

The Symmetric Difference is Associative

Let X be a set.

Definition 4 *If* $A \subset X$ *then the* ***complement*** $A^c = X - A$.

Proposition 5 $A^{cc} = A$
$A \cup A^c = X$
$A \cap A^c = \phi$
$(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$
$A - B = A \cap B^c$

Definition 6 *The* ***symmetric difference*** *of two sets is* $A \Delta B = (A - B) \cup (B - A) = (A \cap B^c) \cup (B \cap A^c)$.

Proposition 7 $A \Delta B = (A \cup B) \cap (A^c \cup B^c)$

Proof.

$$\begin{aligned} A \Delta B &= (A \cap B^c) \cup (B \cap A^c) \\ &= (A \cup B) \cap (A \cup A^c) \cap (B^c \cup B) \cap (B^c \cup A^c) \\ &= (A \cup B) \cap X \cap X \cap (B^c \cup A^c) \\ &= (A \cup B) \cap (B^c \cup A^c) \end{aligned}$$

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Proposition 8 $(A \Delta B)^c = (A^c \cup B) \cap (A \cup B^c)$

Proof.

$$\begin{aligned} (A \Delta B)^c &= [(A \cap B^c) \cup (B \cap A^c)]^c \\ &= (A \cap B^c)^c \cap (A^c \cap B)^c \\ &= (A^c \cup B) \cap (A \cup B^c). \end{aligned}$$

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Proposition 9 $(A \Delta B) \Delta C = (A \cup B \cup C) \cap (A \cup B^c \cup C^c) \cap (A^c \cup B \cup C^c) \cap (A^c \cup B^c \cup C)$

Proof.

$$\begin{aligned} (A \Delta B) \Delta C &= \{(A \Delta B) \cup C\} \cap \{(A \Delta B)^c \cup C^c\} \\ &= \{[(A \cup B) \cap (A^c \cup B^c)] \cup C\} \cap \{[(A^c \cup B) \cap (A \cup B^c)] \cup C^c\} \\ &= \{(A \cup B \cup C) \cap (A^c \cup B^c \cup C)\} \cap \{(A^c \cup B \cup C^c) \cap (A \cup B^c \cup C^c)\} \end{aligned}$$

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Theorem 10 *The symmetric difference operator Δ is associative.*

Proof.

$$\begin{aligned}
(A\Delta B)\,\Delta C &= (A\cup B\cup C)\cap(A\cup B^c\cup C^c)\cap(A^c\cup B\cup C^c)\cap(A^c\cup B^c\cup C) \\
&= (B\cup C\cup A)\cap(B\cup C^c\cup A^c)\cap(B^c\cup C^c\cup A^c)\cap(B^c\cup C^c\cup A) \\
&= (B\Delta C)\,\Delta A \\
&= A\Delta\,(B\Delta C)
\end{aligned}$$

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