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Problem 1. Compute (derive math formulas) the Fourier transform

a. $f(x) = x \cdot e^{-a|x|}$

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$tf(t)$	$i\frac{d}{d\omega}\widehat{f}(\omega)$	Derivative in frequency.	(10)

$$F\left[x \cdot e^{-a|x|}\right](\omega) = i \cdot \frac{d}{d\omega}\left(F\left[e^{-a|x|}\right](\omega)\right)$$

$e^{-\alpha\|t\|}$	$\dfrac{2\alpha}{\alpha^2 + \omega^2}$

$$i \cdot \left(F\left[e^{-a|x|}\right](\omega)\right)' = i \cdot \frac{1}{\sqrt{2 \cdot \pi}} \cdot \frac{d}{d\omega}\left(\frac{2 \cdot a}{a^2 + \omega^2}\right) = -\frac{4 \cdot a \cdot \omega \cdot i}{\sqrt{2 \cdot \pi}(\omega^2 + a^2)^2}$$

$$\frac{d}{d\omega}\left(\frac{2 \cdot a}{a^2 + \omega^2}\right) = \frac{(2 \cdot a)' \cdot (a^2 + \omega^2) - (2 \cdot a) \cdot (a^2 + \omega^2)'}{(a^2 + \omega^2)^2} = -\frac{4a\omega}{(a^2 + \omega^2)^2}$$

Answer:

$$-\frac{4 \cdot a \cdot \omega \cdot i}{\sqrt{2 \cdot \pi}(\omega^2 + a^2)^2}$$

b. $f(x) = e^{-a^2x^2}\cos(b \cdot x)$

$$\cos(b \cdot x) = \frac{e^{ix} + e^{-ix}}{2}\ (euler\ formula)$$

$$f(x) = e^{-a^2x^2} \cdot \frac{e^{ibx} + e^{-ibx}}{2} = \frac{1}{2}\left(e^{-a^2x^2} \cdot e^{ibx} + e^{-a^2x^2} \cdot e^{-ibx}\right)$$

$$F[f_1(x) + f_2(x)](\omega) = F[f_1(x)](\omega) + F[f_1(x)](\omega)$$

$$F[f(x)](\omega) = F\left[e^{-a^2x^2}\cos(b \cdot x)\right](\omega) = F\left[\frac{1}{2}\left(e^{-a^2x^2} \cdot e^{ibx} + e^{-a^2x^2} \cdot e^{-ibx}\right)\right](\omega) =$$

$$= \frac{1}{2} \cdot \left(F\left[e^{-a^2x^2} \cdot e^{ibx}\right](\omega) + F\left[e^{-a^2x^2} \cdot e^{-ibx}\right](\omega)\right)$$

So we need to find Fourier transform from 2 signals: $f_1 = e^{-a^2x^2} \cdot e^{ibx}$; $f_2 = e^{-a^2x^2} \cdot e^{-ibx}$

1) $F\left[e^{-a^2x^2} \cdot e^{ibx}\right](\omega)$

$e^{i\omega_0 t} f(t)$	$\widehat{f}(\omega - \omega_0)$	Modulation property.	(12)

$$F\left[e^{-a^2x^2} \cdot e^{ibx}\right](\omega) = F\left[e^{-a^2x^2}\right](\omega - b)$$

$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$

Replace the variables: $a^2 = \frac{1}{2\sigma^2}$; $\sigma = \sqrt{\frac{1}{2a^2}} = \frac{1}{\sqrt{2}\cdot a}$

$$F\left[e^{-a^2x^2}\right](\omega - b) = F\left[e^{-\frac{x^2}{2\sigma^2}}\right](\omega - b) = \sigma \cdot \sqrt{2 \cdot \pi} \cdot e^{\frac{-\sigma^2\cdot(\omega-b)^2}{2}} =$$

$$= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{\sqrt{2}\cdot a} \cdot \sqrt{2\cdot\pi} \cdot \exp\left(\frac{-\left(\frac{1}{\sqrt{2}\cdot a}\right)^2 \cdot (\omega - b)^2}{2}\right)\right) = \frac{1}{\sqrt{2}a} \cdot \exp\left(\frac{-\frac{1}{2a^2}\cdot(\omega-b)^2}{2}\right) =$$

$$\frac{1}{\sqrt{2}a}\exp\left(-\frac{(\omega-b)^2}{4a^2}\right) = \frac{1}{\sqrt{2}a}\exp\left(-\left(\frac{\omega-b}{2a}\right)^2\right)$$

2) *The same for* $F\left[e^{-a^2x^2} \cdot e^{i\cdot(-b)\cdot x}\right](\omega)$:

$$F\left[e^{-a^2x^2} \cdot e^{i\cdot(-b)\cdot x}\right](\omega) = F\left[e^{-a^2x^2}\right](\omega + b)) = \frac{1}{\sqrt{2}a}\exp\left(-\left(\frac{\omega+b}{2a}\right)^2\right)$$

Final result:

$$\frac{1}{2}\left(\frac{1}{\sqrt{2}a}\exp\left(-\left(\frac{\omega+b}{2a}\right)^2\right) + \frac{1}{\sqrt{2}a}\exp\left(-\left(\frac{\omega-b}{2a}\right)^2\right)\right) =$$

$$= \frac{1}{\sqrt{8}a}\left(\exp\left(-\left(\frac{\omega+b}{2a}\right)^2\right) + \exp\left(-\left(\frac{\omega-b}{2a}\right)^2\right)\right)$$

Answer:

$$\frac{1}{\sqrt{8}a}\left(\exp\left(-\left(\frac{\omega+b}{2a}\right)^2\right) + \exp\left(-\left(\frac{\omega-b}{2a}\right)^2\right)\right)$$

Problem 2. Compute (derive math formulas) the spectrum of the sine-filled sinc function

$$f(x) = \mathrm{sinc}(a \cdot x) \cdot \sin(b \cdot x)$$

$e^{i\omega_0 t} f(t)$	$\widehat{f}(\omega - \omega_0)$	Modulation property.	(12)

$$\cos(b \cdot x) = \frac{e^{ix} - e^{-ix}}{2i} \ (euler\ formula)$$

$$F[\mathrm{sinc}(a \cdot x) \cdot \sin(b \cdot x)](\omega) = 2i \cdot (F[\mathrm{sinc}(a \cdot x)](\omega - b) - F[\mathrm{sinc}(a \cdot x)](\omega + b))$$

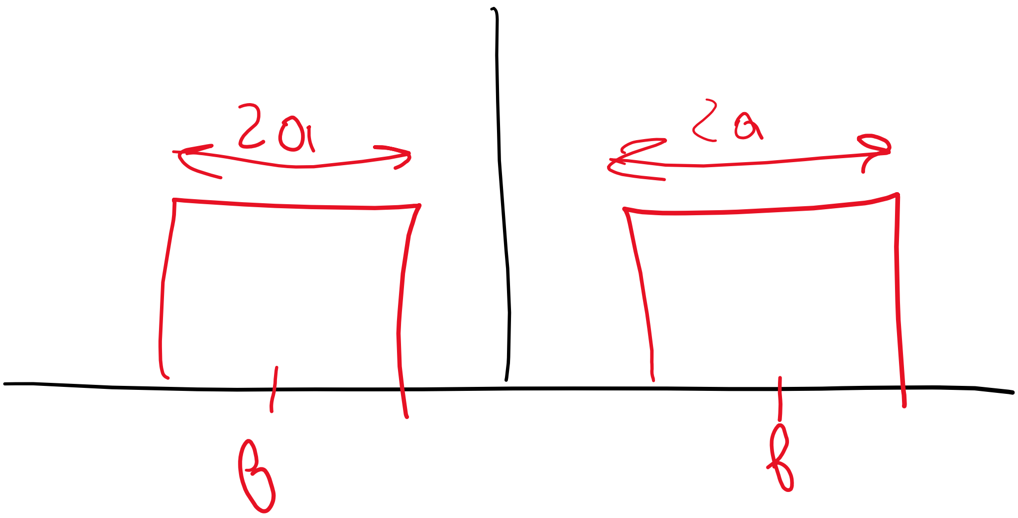
$\beta(t) = \begin{cases} 1, & \text{if } \lvert t \rvert < 1, \\ 0, & \text{if } \lvert t \rvert > 1 \end{cases}$	$2\,\mathrm{sinc}(\omega) = 2\frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi}\,\mathrm{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)

$$F\left[\left(\frac{1}{\pi} \cdot \pi\right) \mathrm{sinc}(a \cdot x)\right](\omega) = \pi \cdot \beta(a \cdot \omega)$$

$$F[\mathrm{sinc}(a \cdot x) \cdot \sin(b \cdot x)](\omega) = \frac{2i}{\sqrt{2\pi}}\Big(\pi \cdot \beta\big(a \cdot (\omega - b)\big) - \pi \cdot \beta\big(a \cdot (\omega + b)\big)\Big) =$$

$$i\sqrt{2\pi}\Big(\beta\big(a \cdot (\omega - b)\big) - \beta\big(a \cdot (\omega + b)\big)\Big)$$

$$\beta(t) = rectangular\ pulse$$



Problem 3. Compute (derive math formulas) the autocorrelation of the triangular pulse

Problem 4. Compute (by hands) the convolution of the following signals:

a. h = [2 3 6 8]; x = [1 2 10 1]; b. h = [5 1 3 10]; x = [9 6 10 1];

Step 1.

		x				
		1	*2*	*10*	*1*	
h	*2*	2	4	20	2	
	3	3	6	30	3	
	6	6	12	60	6	
	8	8	16	80	8	$\leftarrow h_i \cdot x_j$

		x			
		9	*6*	*10*	*1*
h	*5*	45	30	50	5
	1	9	6	10	1
	3	27	18	30	3
	10	90	60	100	10

Step 2.

		x						
		1	*2*	*10*	*1*			
h	*2*	2	4	20	2	0	0	0
	3	0	3	6	30	3	0	0
	6	0	0	6	12	60	6	0
	8	0	0	0	8	16	80	8
		2	**7**	**32**	**52**	**79**	**86**	**8**

		x						
		9	*6*	*10*	*1*			
h	*5*	45	30	50	5	0	0	0
	1	0	9	6	10	1	0	0
	3	0	0	27	18	30	3	0
	10	0	0	0	90	60	100	10
		45	**39**	**83**	**123**	**91**	**103**	**10**

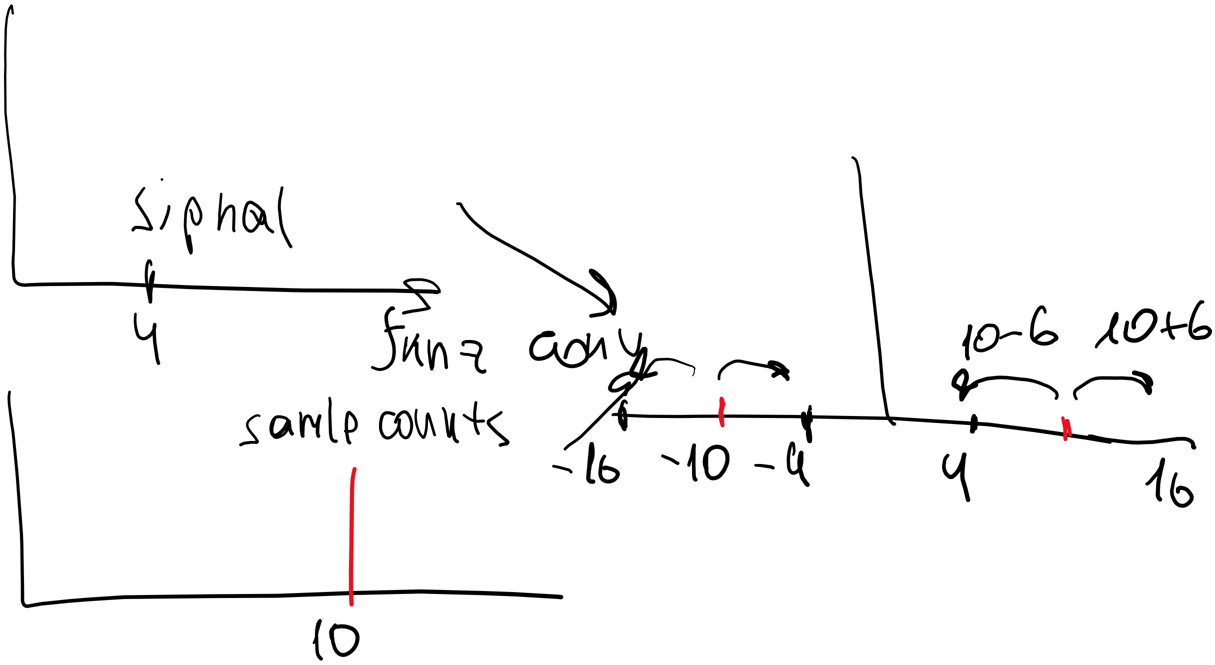
Step 3.

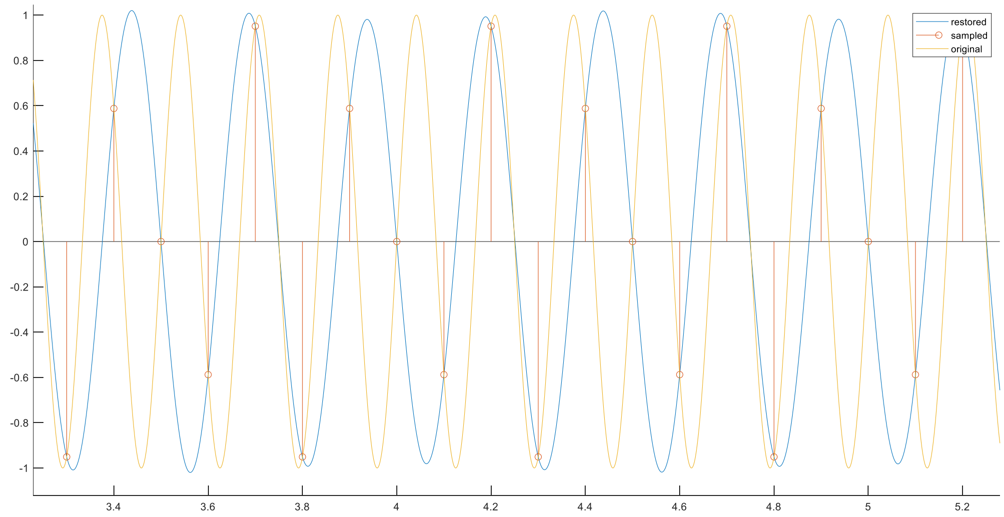
```
>> h = [2 3 6 8]; x = [1 2 10 1];
>> conv(x,h)

ans =

     2     7    32    52    79    86     8
```

```
>> h = [5 1 3 10]; x = [9 6 10 1];
>> conv(x,h)

ans =

    45    39    83   123    91   103    10
```

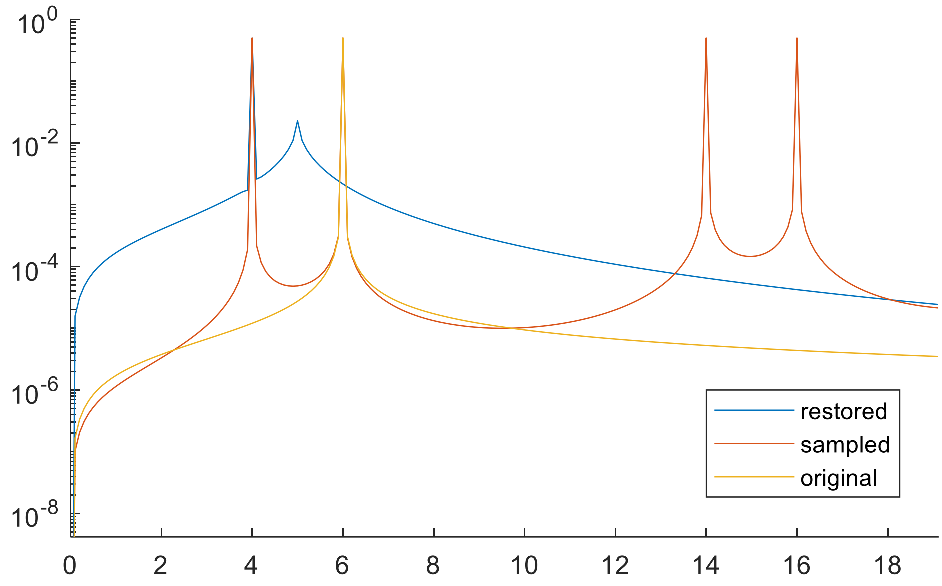
Problem 5. The sinusoidal signal with the frequency 6kHz ….

a. … is sampled with the frequency 10 kHz. Compute the apparent frequency after the signal reconstruction.





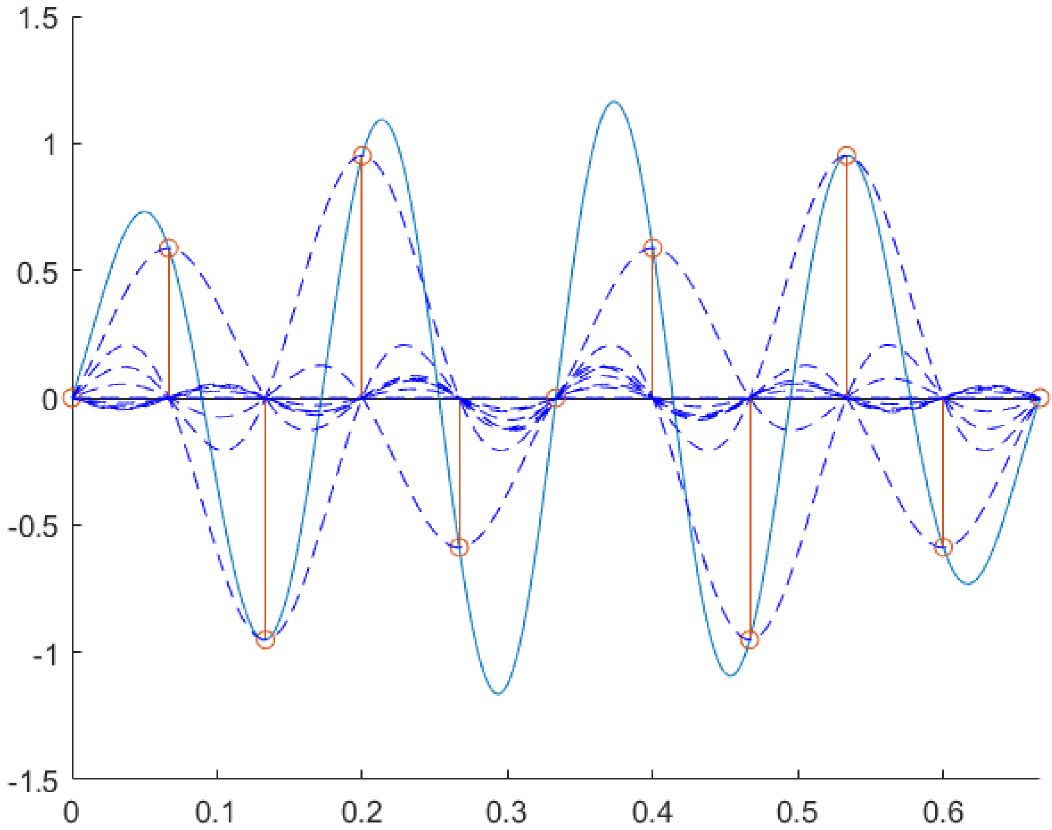
The signals in time domain



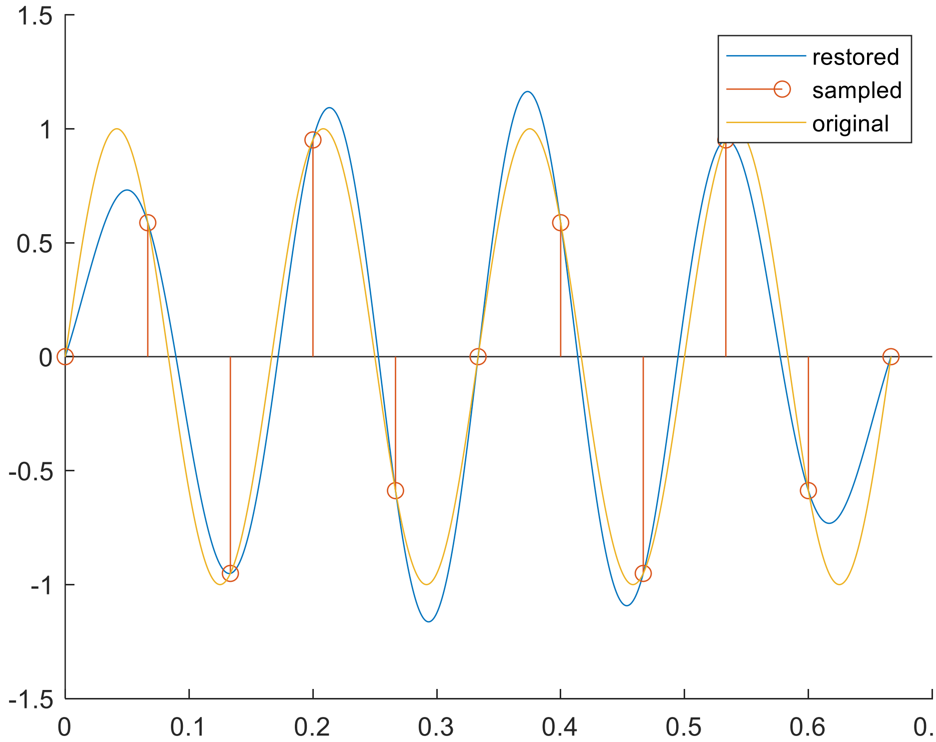
The signals in frequency domain

Answer: The reconstructed with Nyquist filter signal will have the frequency equal to 4kHz.

b. … is represented by 4 signal periods sampled with the sample clock frequency 15 kHz. Are these samples enough to reconstruct the initial signal correctly (without ANY error)? Plot the original and the reconstructed signals. Explain the difference.



The reconstruction of signal using Nyquist filter



The reconstructed signal

Answer: this quantity of samples is not enough to reconstruct the limited in time signal. With incrasing a number of periods error will decrease

MATLAB code for task №5:

```
close all
clear all
fss = 15e3;   % sampling frequency for modeling
fs = 15;      % sampling frequency
f_signal = 6;  % signal frequency
w =2*pi*f_signal;  % signal angular frequency
Num_of_periods = 4;  % number of periods of signal
t = Num_of_periods/f_signal;  % modeling time: 0..t
time_signal = 0:(1/fss):t;  % time counts for modeling
time_sampled = 0:(1/fs):t;  % time counts for
discretization
signal = sin(w*time_signal);  % raw signal
discrete = sin(w*time_sampled);  % sample counts

figure(1);plot(time_signal,signal);
hold on;
% plot(time_sampled,discrete,'ks')
stem(time_sampled,discrete,'ks--');

% попытка восстановить сигнал с использованием синков
tt = (-t:(1/fss):t);  % time counts for filter (based on
modeling frequency)
Nyquist_filter = sinc(fs*tt);  % Nyquist filter in time
domain
% plot(tt,Nyquist_filter);
upsampled = upsample(discrete,fss/fs);  % upsample discrete
signal to the modeling frequency (new samples is equal to
zero)
upsampled(end+1-(fss/fs-1):end) = [];  % remove last unused
zeros: S000S000S[000] (S = sample)

restored = conv(upsampled,Nyquist_filter,'same');  %
restored signal: signal -> sample -> filter

ttt = linspace(0,t,length(restored));
figure(2);hold on;
plot(ttt,restored);stem(time_sampled,discrete);plot(time_si
gnal,signal);
legend('restored','sampled','original');
```

```
figure(4);hold on;

fft_restored = abs(fft(restored)/length(restored));
plot(linspace(0,fss,length(restored)),fft_restored)

fft_sampled =
abs(fft(upsampled)/length(upsampled)).*fss/fs; %energy loss
compensation
plot(linspace(0,fss,length(upsampled)),fft_sampled)

fft_signal = abs(fft(signal)/length(signal));
plot(linspace(0,fss,length(signal)),fft_signal)

legend('restored','sampled','original');xlim([0,
2*fs]);set(gca, 'YScale', 'log')

% return

[rId, cId] = find(upsampled);
figure(3);hold on;
plot(ttt,restored);stem(time_sampled,discrete);
for i = cId
    plot(tt+((i-1)*1/fss),upsampled(i)*Nyquist_filter,'b--
');
end
xlim([0 t]);
```