Rebase-and-Read Sequence

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1 Definition

Consider the following simple set of rules, starting with a number n:

1. Sum together the digits of n, when read in base 10. Call this sum b:

$$(n)_{10} = \sum d_i 10^i, \qquad b = \sum d_i$$

2. Write n in base b:

$$(n)_b = \sum c_i b^i$$

3. Now create a new number n' by using the digits c_i to create a base 10 number:

$$(n')_{10} = \sum c_i 10^i$$

We can represent this set of rules by a map F, so that $F(n) = n'$. There is a problem when the sum of the base 10 digits of n is equal to 1, since 1 is not a well defined base. However, if we define $F(n) := n$ whenever this occurs, then we get a well-defined map on the positive natural numbers $F : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$.

As an example, consider $n = 14$. If we sum together its digits we get 5, and 14 read in base 5 is $(14)_5 = 24$ (since $14 = 2 \times 5^1 + 4 \times 5^0$). We then read this number in base 10 and thus $F(14) = 24 = 2 \times 10^1 + 4 \times 10^0$. As another example, take $n = 845$. In this case, we have $845 = 2 \times 17^2 + 15 \times 17^1 + 12 \times 17^0$. We then read this in base 10 (i.e. we replace 17 by 10) which gives $2 \times 10^2 + 15 \times 10^1 + 12 \times 10^0 = 362$ and thus $F(845) = 362$.

We can then consider iterating F. For example, starting with $n = 13$ we get the following,

$$F(13) = 31, \quad F(31) = 133, \quad F(133) = 250, \quad F(250) = 505, \quad F(505) = 505, ... \tag{1}$$

We reached a fixpoint because the sum of the digits of 505 is 10. We can define $L(n)$ to be the number of terms in the sequence obtained by iterating F before we get a repeated digit (since this indicates either a fixed point or a cycle). For example, $L(13) = 5$. We get two interesting sequences given by F and L.

2 Examples

The early behaviour is quite simple. Firstly, $F(1) = 1$, by definition. Next, for all $n = 2, ..., 9$ we have $F(n) = 10$ because any number m read in base m is 10. The later behaviour is more interesting. For example, iterating F starting with $n = 11$ gives a sequence of length 9 before reaching a fixed point:

11, 1011, 1101110, 240213420, 7125580, 127810, 19216, 3547, 1063, ...

At $n = 15$, we encounter out first cycle:

15, 23, 43, 61, 115, 223, 436, 277, 115,...

Thus, if a sequence ever reaches the number 115, 223, 436 or 277 it will enter a cycle of length four.

For $1 \leq n \leq 100$, $L(n)$ is largest for $n = 96$, which gives rise to a sequence of length 15 under interation of F before reaching a fixed point:

96, 66, 56, 51, 123, 323, 503, 767, 287, 175, 106, 211, 3103, 12022, 50023, ...

It seems hard to predict when $L(n)$ will be big or small. One can choose very large n and $L(n)$ remains relatively small, for example $L(45708275980273450271) = 5$. It does however seem that the maximum value of L grows without bound as n increases.

3 Results

The first 100 terms in the sequence defined by F are:

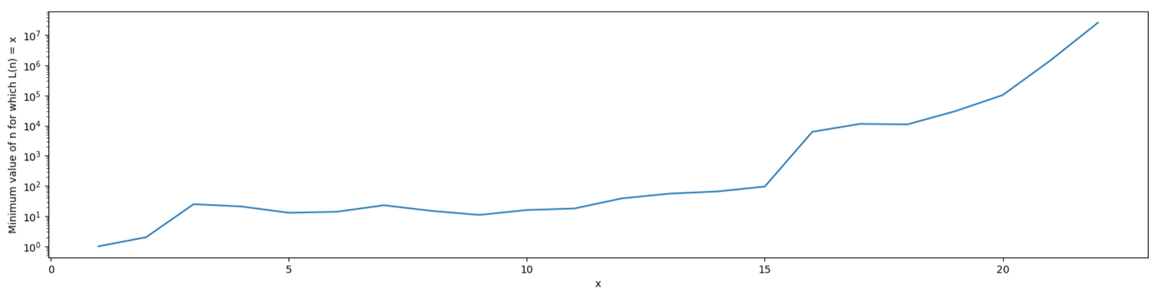
1, 10, 10, 10, 10, 10, 10, 10, 10, 10, 1011, 110, 31, 24, 23, 22, 21, 20, 19, 10100, 210, 112, 43, 40, 34, 32, 30, 28, 27, 1010, 133, 112, 53, 46, 43, 40, 37, 35, 33, 220, 131, 110, 61, 54, 50, 46, 43, 40, 40, 200, 123, 103, 65, 60, 55, 51, 49, 46, 43, 140, 115, 76, 70, 64, 60, 56, 52, 52, 49, 130, 107, 80, 73, 68, 63, 61, 57, 53, 55, 120, 100, 82, 76, 70, 67, 62, 62, 58, 54, 110, 91, 84, 79, 73, 71, 66, 61, 63, 59, 100, ...

The first 100 terms in the sequence defined by L are:

1, 2, 2, 2, 2, 2, 2, 2, 2, 1, 9, 9, 5, 6, 8, 10, 5, 11, 1, 10, 4, 9, 7, 5, 3, 10, 10, 1, 11, 9, 4, 9, 11, 2, 7, 5, 1, 8, 12, 4, 10, 9, 6, 10, 10, 1, 7, 5, 5, 9, 12, 5, 10, 9, 1, 13, 6, 2, 7, 8, 5, 7, 5, 1, 9, 14, 6, 6, 6, 4, 7, 9, 1, 7, 6, 6, 7, 11, 2, 8, 2, 1, 7, 5, 7, 8, 8, 3, 10, 9, 1, 6, 3, 2, 8, 15, 6, 6, 8, 1, ...

Some basic results observed for n up to 10^8 are the following:

- Every sequence obtained by iterating F either ends in a fixed point or one of the length four cycles discussed in section 2. For $1 \leq n \leq 10^8$, 615093 values of n lead to one of these cycles.
- The maximum value of $L(n)$ reached is 22, which is attained for $n = 26243999$, 29001119, 41311151, 42151103, 42200351, 76159999.
- The growth of $L(n)$ can be observed in the following graph:



3.1 Code

We leave here the code for the functions F and L, written in Python:

```
def list_from_number(num, base):
    digits = []
    while num > 0:
        digits.append(num % base)
        num = num // base
    digits.reverse()
    return digits

def number_from_list(lst, base):
    return sum([lst[len(lst) - i - 1] * base**i
                for i in range(0, len(lst))])


def F(n):
    b = sum(list_from_number(n, 10))

    if n <= 0:
        print("n must be a positive natural number")
    elif b == 1:
        return n
    else:
        n_base_b = list_from_number(n, b)
        n_prime = number_from_list(n_base_b, 10)
        return n_prime

def L(n, cutoff):
    seq = [n]
    found_cycle = False

    i = 0
    while i < cutoff:
        i += 1
```

```
        n_prime = F(seq[i-1])
        seq.append(n_prime)
        if seq[i] in seq[0:i-1]:
            found_cycle = True
            p = [x for x, n in enumerate(seq) if n == seq[i]]
            cycle_length = p[1] - p[0]
            seq.pop(i)
            break

    if found_cycle:
        return [len(seq), seq, cycle_length]
    else:
        return([len(seq), seq])
        print(cutoff, " iterations carried out without a cycle")
```

4 Open Questions

- Are there any other cycles than the cycles of length four discussed previously? If so, is four the smallest cycle length?
- Is $L(n)$ finite for every n? This can be framed as a Collatz-style conjecture: does every n reach a fixed point or a cycle under iteration of F?
- Assuming the previous is true, is $L(n)$ unbounded as n increases?