DOING PHYSICS WITH PYTHON

[2D] NON-LINEAR DYNAMICAL SYSTEMS
BIFURCATIONS

Ian Cooper
matlabvisualphysics@gmail.com

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

INTRODUCTION

The evolution of a [2D] system can lead to growth, decay, equilibrium, oscillations, periodic motion, aperiodic motion, and chaos. Often the best way to understand the time evolution of a system is not through the mathematics, but through the visualisation of the orbits (trajectories) in phase space where the vector field can give a qualitative view. Local stability equilibrium at a point is characterised by small disturbances being damped out in time, whereas local instability, the disturbance grows with time. Stable fixed points (equilibrium points) are referred to as attractors or sinks. An unstable fixed point is referred to as a repeller or a source.

To analyse a dynamical system, it is important to determine the existence of equilibrium points. For example, the acrobats in the photograph are in a stable equilibrium position: if an acrobat tilts laterally, the long rod moves to causes the system to tilt in the opposite direction, returning to the equilibrium position. If the acrobat did not have the rod, that equilibrium position would be unstable: if an acrobat tilted sideways, then the acrobat would make them tilt further, moving the system away from the equilibrium position.

BIFURCATIONS

A bifurcation is the division of something into two branches or parts.

If the phase portrait changes its topological structure as a parameter (**bifurcation parameter** or **control parameter**) is varied, then a **bifurcation** has occurred. The bifurcation occurs at a specific parameter value, leading to a qualitative change in the system's dynamics. Examples include changes in the number or stability of fixed points, closed orbits, or saddle connections as the control parameter is varied. That is, the qualitative structure of the flow can change as control parameters are varied. In particular, fixed points can be created or destroyed, or their stability can change. These qualitative changes in the dynamics are called bifurcations, and the parameter values at which they occur are called **bifurcation points**. Bifurcations are important scientifically—they provide models of transitions and instabilities as some control parameter is varied.

In dynamical systems, a bifurcation is a sudden, qualitative change in a system's behaviour that occurs when a small change in a parameter value crosses a critical threshold. Bifurcation theory studies these phenomena, explaining how a small tweak in a system's parameters can lead to significant shifts in its stability or the number of its possible states, such as the appearance or disappearance of stable states or the onset of oscillations. Bifurcations help us understand how complex behaviours, such as stability, bistability (having two stable states), and oscillations, arise from simple rules. Bifurcation theory is applied in fields like physics, biology, and engineering to model and predict the behaviour of systems such as neuronal networks, gene regulation models, and mechanical systems.

Key bifurcation types include saddle-node bifurcations (creation/destruction of fixed points), Hopf bifurcations (birth of periodic orbits), and homoclinic bifurcations (collision of limit cycles with saddle points). These phenomena are identified by changes in eigenvalues at fixed points or by geometric changes in phase portraits, such as the tangency of nullclines. Changes in the number of intersections of nullclines (curves where one variable is constant, e.g., $dx/dt = 0$ or $dy/dt = 0$) can indicate fixed point bifurcations.

An important diagram in the study of dynamical systems is the **bifurcation diagram**: A plot that shows the stable states of a system in phase space (e.g., fixed points) as a function of a bifurcation parameter.

TYPES OF BIFURCATIONS IN [2D] DYNAMICAL SYSTEMS

SADDLE – NODE BIFURCATIONS

A saddle-node bifurcation in a dynamical system is a local bifurcation where two equilibrium points (fixed points) collide and then annihilate each other, or are created from nothing. This phenomenon is also known as a fold bifurcation, tangential bifurcation, or blue skies bifurcation (refers to the sudden creation of two fixed points from nothing). Two fixed points emerge or are destroyed when nullclines become tangent.

Collision and Annihilation: Two fixed points, one stable and one unstable, merge into a single semi-stable fixed point at the bifurcation point.

Creation of Fixed Points: As the bifurcation parameter changes, two new fixed points can appear at the bifurcation point.

TRANSCRITICAL BIFURCATIONS

Two fixed points cross, exchanging stability. A transcritical bifurcation is a phenomenon where two fixed points collide, exchange their stability, and then separate again, without being created or destroyed. This occurs as a bifurcation parameter in the system's equations is varied, leading to a qualitative change in the system's long-term behaviour. A stable fixed point becomes unstable, and an unstable fixed point becomes stable, as they cross a bifurcation point. The two fixed points exist for all values of the bifurcation parameter but merge and exchange their properties at the bifurcation point.

Transcritical bifurcations are fundamental to understanding how the behaviour of a dynamical system can change dramatically in response to small variations in its parameters. They are found in many scientific fields, including population dynamics, where they can represent transitions between population extinction and growth, or in epidemic models.

PITCHFORK BIFURCATIONS

A pitchfork bifurcation is a type of local bifurcation in a dynamical system where, as a control parameter is changed, a single equilibrium point becomes unstable and splits into three, or vice versa, due to a system's inherent symmetry. This phenomenon can be either supercritical, where a stable equilibrium gives way to two new stable branches and one unstable one, or subcritical, where one stable and two unstable equilibria collapse into a single stable one, with the bifurcating branches becoming unstable.

BIFURCATIONS OF PERIODIC ORBITS

Bifurcations of periodic orbits are qualitative changes in the behaviour of a dynamical system's periodic solutions as a parameter is varied, leading to the creation, annihilation, or modification of these orbits. Key examples include the Andronov-Hopf bifurcation (appearance of small-amplitude orbits), period-doubling bifurcations (creation of orbits with double the period), and saddle-node bifurcations (coalescence and disappearance of orbits). These events often mark transitions in system behavior, from stable oscillations to more complex patterns, including chaos.

HOPF BIFURCATIONS

This type of bifurcation leads to the onset of oscillations. Think of a stable equilibrium that starts to oscillate back and forth, like a pendulum that has been pushed. A pair of complex conjugate eigenvalues cross into the right half of the complex plane, becoming purely imaginary at a fixed point, leading to the birth of a stable closed orbit (periodic motion).

HOMOCLINIC BIFURCATIONS

A homoclinic bifurcation is a global dynamic event where a limit cycle (a periodic orbit) in a dynamical system evolves to become a homoclinic orbit, which is an orbit that returns to the same equilibrium point (usually a saddle point) both forward and backward in time. This transition occurs when the limit cycle "touches" or "collides" with the stable and unstable manifolds of the saddle point.

https://ocw.mit.edu/courses/12-006j-nonlinear-dynamics-chaos-fall-2022/mit12_006jf22_lec10-11.pdf

https://ocw.mit.edu/courses/12-006j-nonlinear-dynamics-chaos-fall-2022/pages/lecture-notes/