DOING PHYSICS WITH PYTHON

COMPUTATIONAL OPTICS

RAYLEIGH-SOMMERFELD 1

DIFFRACTION INTEGRAL:

BEAM PROPAGATION

UNIFORMLY ILLUMINATED CIRCULAR APERTURE

Ian Cooper

Please email me any corrections, comments, suggestions or additions: **matlabvisualphysics@gmail.com**

DOWNLOAD DIRECTORIES FOR PYTHON CODE

Google drive

GitHub

emRS02.py Irradiance in XY planes

emRS02Z.py Irradiance along the +Z axis

INTRODUCTION

The **Rayleigh-Sommerfeld diffraction integral of the first kind** is used to calculate the intensity from a circular aperture that is uniformly illuminated by monochromatic light of wavelength λ.

The geometry of the aperture and observation spaces is shown in figure 1, and figure 2 shows an outline of how to the RS1 diffraction integral is computed in Python. Figure 3 shows a [2D] and [3D] view of the uniform aperture intensity.

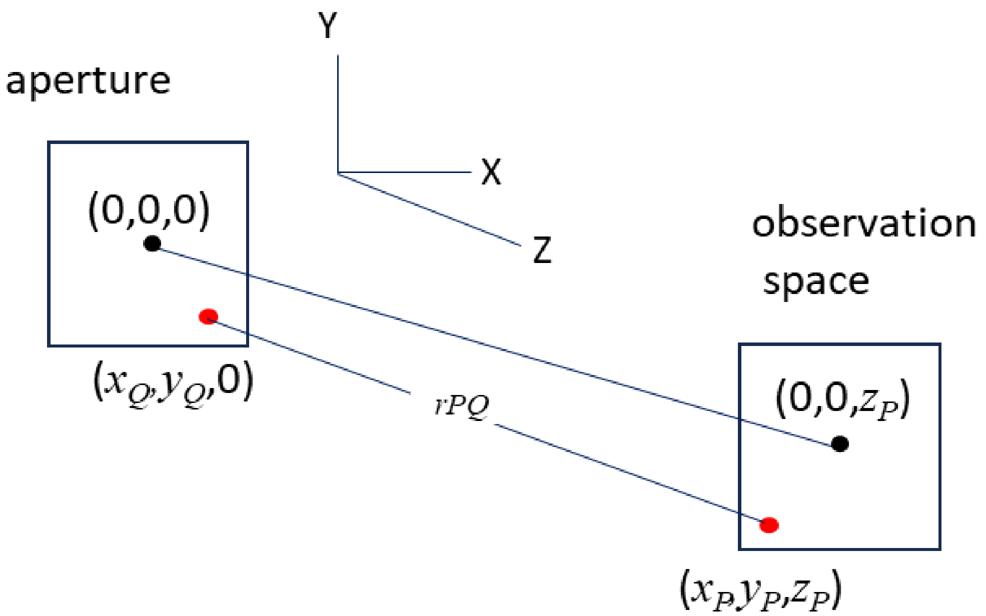


Fig. 1. Geometry of the aperture and observation spaces.

Rayleigh-Sommerfeld 1 diffraction integral

$$E_P = \frac{1}{2\pi}\int_{a_x}^{b_x}\int_{a_y}^{b_y} E_Q \frac{e^{j\,k\,r_{PQ}}}{r_{PQ}^{\;3}}\, z_p \,(j\,k\,r_{PQ} - 1)\; dx\, dy$$

numerical integration:
[2D] Simpson's 1/3 rule

$$E_P(x_P, y_P, z_P)$$

$$= \left(\frac{h_x\, h_y}{9}\right) z_P \sum_{m=1}^{n_Q} \sum_{n=1}^{n_Q} \left(\left(\frac{e^{j\,k\,r_{PQmn}}}{r_{PQmn}^{\;3}} \right) \left(j\,k\,r_{PQmn} - 1\right) \left(E_{Qmn}\, S_{mn}\right) \right)$$

$$h_x = \frac{b_x - a_x}{n_Q - 1}$$

$$h_y = \frac{b_y - a_y}{n_Q - 1}$$

matrix **MP** $n_Q \times n_Q$

matrix **EQ** $n_Q \times n_Q$

matrix **S** $n_Q \times n_Q$

Fig. 2. Matrices used in computed the diffraction integral.

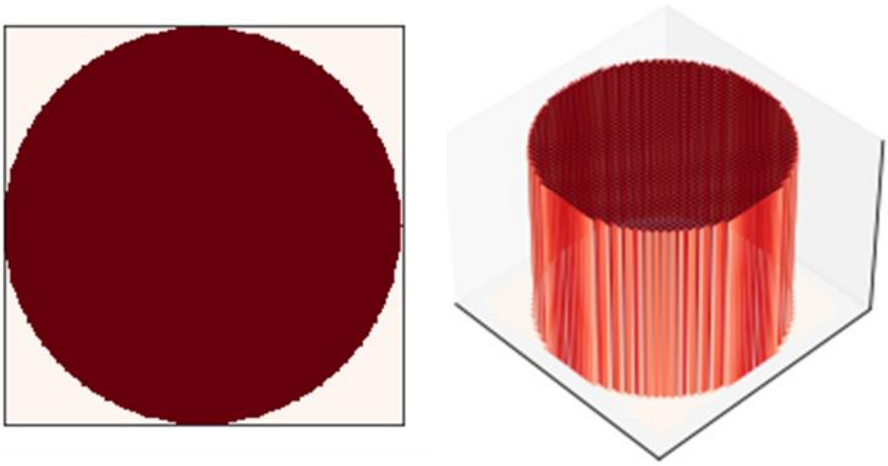


Fig. 3. [2D] and [3D] views of the uniform aperture intensity.

emRS102.py

Both the Fraunhoffer and Fresnel diffraction patterns in the observation space can be computed easily by evaluating the RS1 diffraction integral.

The **Fraunhofer diffraction** pattern for the circular aperture is circularly symmetric and consists of a bright central circle surrounded by series of bright rings of rapidly decreasing strength between a series of dark rings (figure 4). The bright and dark rings are not evenly spaced. The bright central region is known as the **Airy disk**.

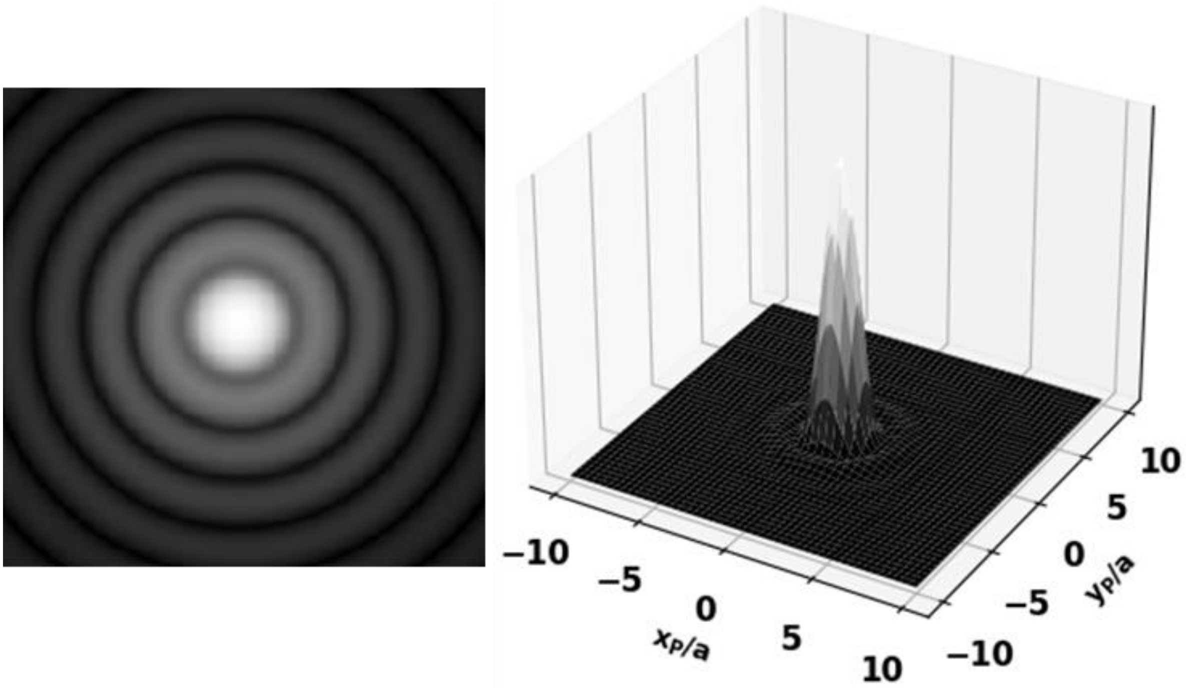


Fig. 4. Fraunhoffer diffraction pattern of a circular aperture of radius a. **emRS102.py**

The image shown in figure 4 is like a black and white time exposure photograph of the diffraction pattern that would be observed on a screen for a uniformly illuminated circular aperture. The bright centre spot corresponds to the zeroth order of diffraction and is known as the Airy disk and it extends to the first dark ring.

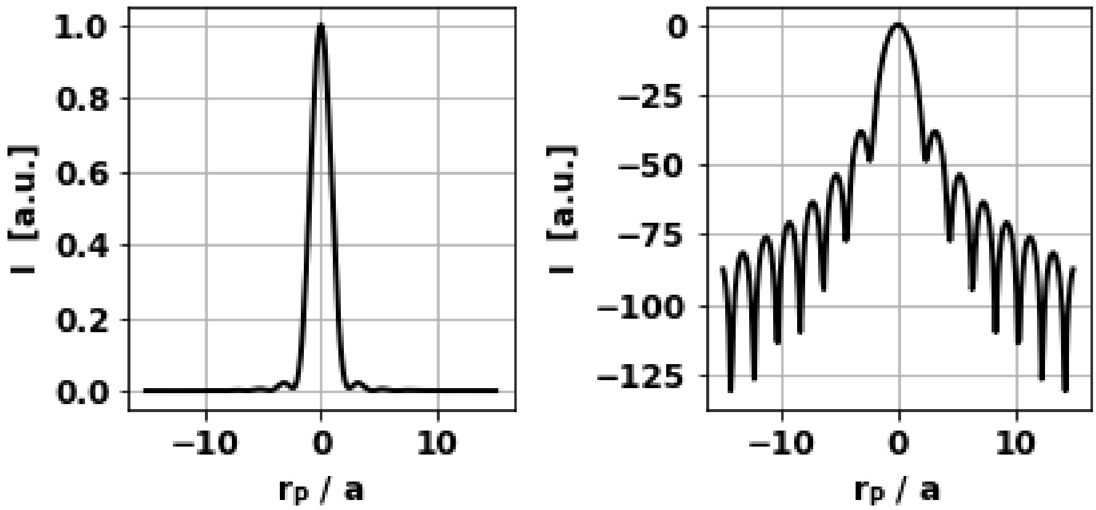


Fig. 4. The irradiance patterns in a radial direction (linear scale and decibel scale for the irradiance). **emRS102.py**

The default input parameters are

Wavelength [m] WL = 632.8e-9

Aperture radius [m] a = 4e-4

Grid NQ = 100 NP = 100

In the far-field or Fraunhofer region, the irradiance is given by equation 1

(1) $$I = I_o \left(\frac{J_1(v_P)}{v_P} \right)^2$$ **Fraunhofer diffraction only**

where J_1 is the Bessel function of the first kind and v_p is the radial optical coordinate and is a scaled perpendicular distance from the optical axis (equation 2).

(2) $$v_P = \frac{2\pi}{\lambda} a \sin\theta \qquad \sin\theta = \frac{\sqrt{{x_P}^2 + {y_P}^2}}{\sqrt{{x_P}^2 + {y_P}^2 + {z_P}^2}}$$

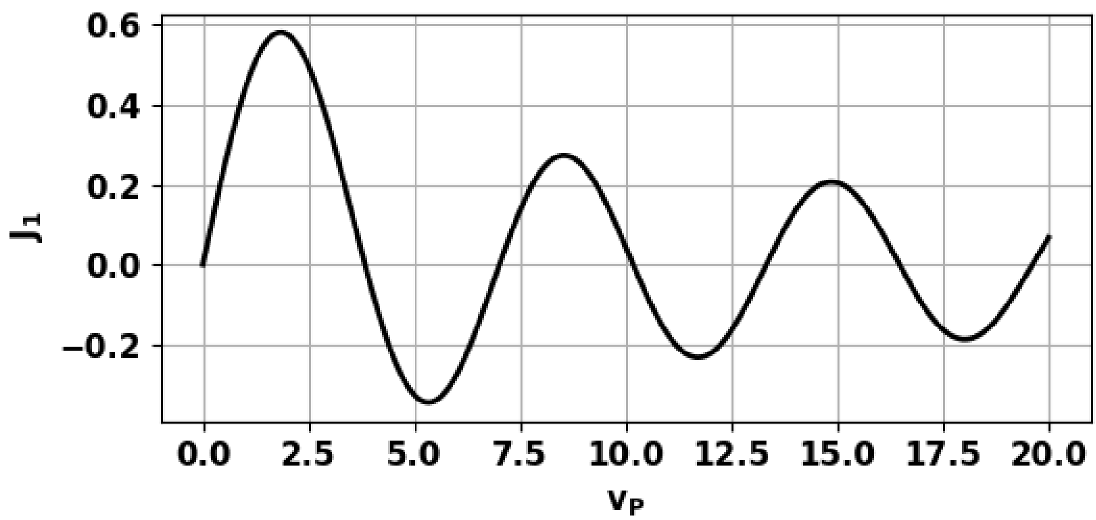


Fig. 5. Bessel function of the first kind using a Python function.
emRS102.py

```
from scipy.special import
j1   num = 9999
v = linspace(0,20,num)
J1 = j1(v)
```

Fraunhoffer to Fresnel diffraction

The Rayleigh-Sommerfeld diffraction integral of the first kind (figure 2) is valid right up to the aperture for the calculation of the electric field at an observation point P. The transition from Fraunhofer diffraction to Fresnel diffraction can be expressed in terms of the Rayleigh distance. The **Rayleigh distance** in optics is the axial distance from a radiating aperture to a point an observation point P at which the path difference between the axial ray and an edge ray is λ / 4. A good approximation of the Rayleigh distance RL is

$$RL = \frac{4a^2}{\lambda}$$

where a is the radius of the aperture. The Rayleigh distance is approximately the value of z_P where the first minimum in the irradiance is greater than zero.

$z_P < RL$ Fresnel diffraction

$z_P > RL$ Fraunhofer diffraction.

For the default parameters

$$a = 4\times10^{-4}\ \mathrm{m} \quad \lambda = 632.8\times10^{-9}\ \mathrm{m} \quad RL = 1.011\ \mathrm{m}$$

Irradiance I_z along the optical axis (+Z axis)

The Python Code **emRS1Z.py** is used to calculate the irradiance as a function of the distance from the centre of the aperture along the +Z axis. The Rayleigh-Sommerfeld diffraction integral can calculate electric field up to the aperture. Figure 6 shows the variation of the irradiance along the optical axis for RL = 1.01 m.

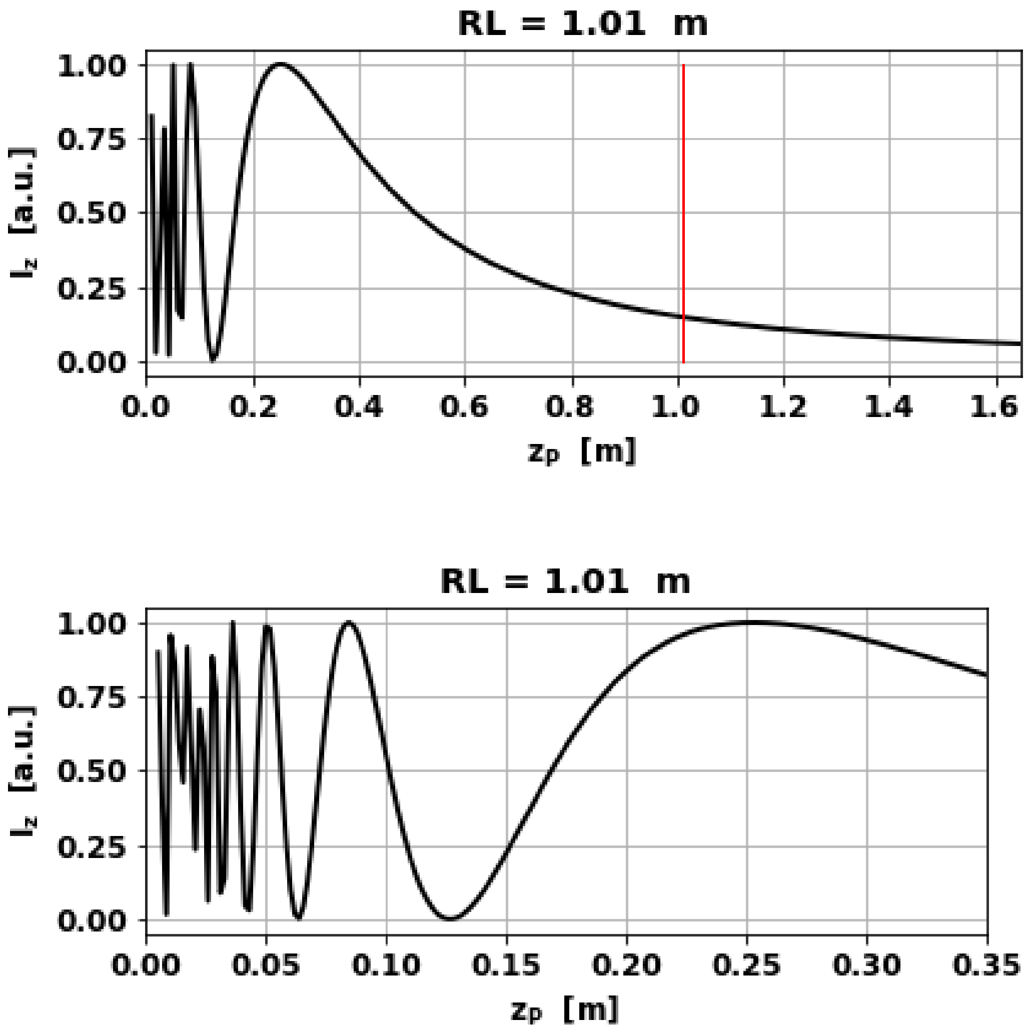


Fig. 6. Irradiance along the optical axis $RL = 1.011\ \mathrm{m}$. In the Fresnel region near the aperture, the irradiance oscillates due to the constructive and destruction effects of the light coming from different regions of the aperture. **emRS102Z.py**

FRAUNHOFFER DIFFRACTION

Figure 7 shows the irradiance in the XY plane at z_P = 1.10 m. There is excellent agreement between the RS1 solution (**black** line) and the predictions of the Fraunhoffer equation (equation 1) (**red** line). A summary of the simulation parameters is displayed in the Console Window.

```
nQ = 241   nP = 241
aperture radius = 0.400  mm
Wavelength wL = 632.8  nm
Observation space: max(rP/a) = 8
Observation space: zP = 1.100 m
Rayleigh length RL = 1.011  m
First dark ring XDark1/a = 2.667
First dark ring percentage power enclosed  PDark1/a = 87
Fraunhoffer diffraction: zero irradiance at rP/a
  2.67    4.87    7.07
Fraunhoffer diffraction: relative irradiance of the maxima
 0.0046   0.0213   1.0000   0.0213   0.0046
```

Python Code using find_peaks

```
# Find xP for first dark ring in IXY and xP for minima
q = find_peaks(-IdB)
xmin = xP[q[0]]/a
xDark1 = xmin[xmin>0][0]
PDark1 = Pr[r/a>xDark1][0]
xDark = xmin[xmin>0]

# Relative intensity of maxima
q = find_peaks(Ix)
peaks = Ix[q[0]]/max(Ix)
```

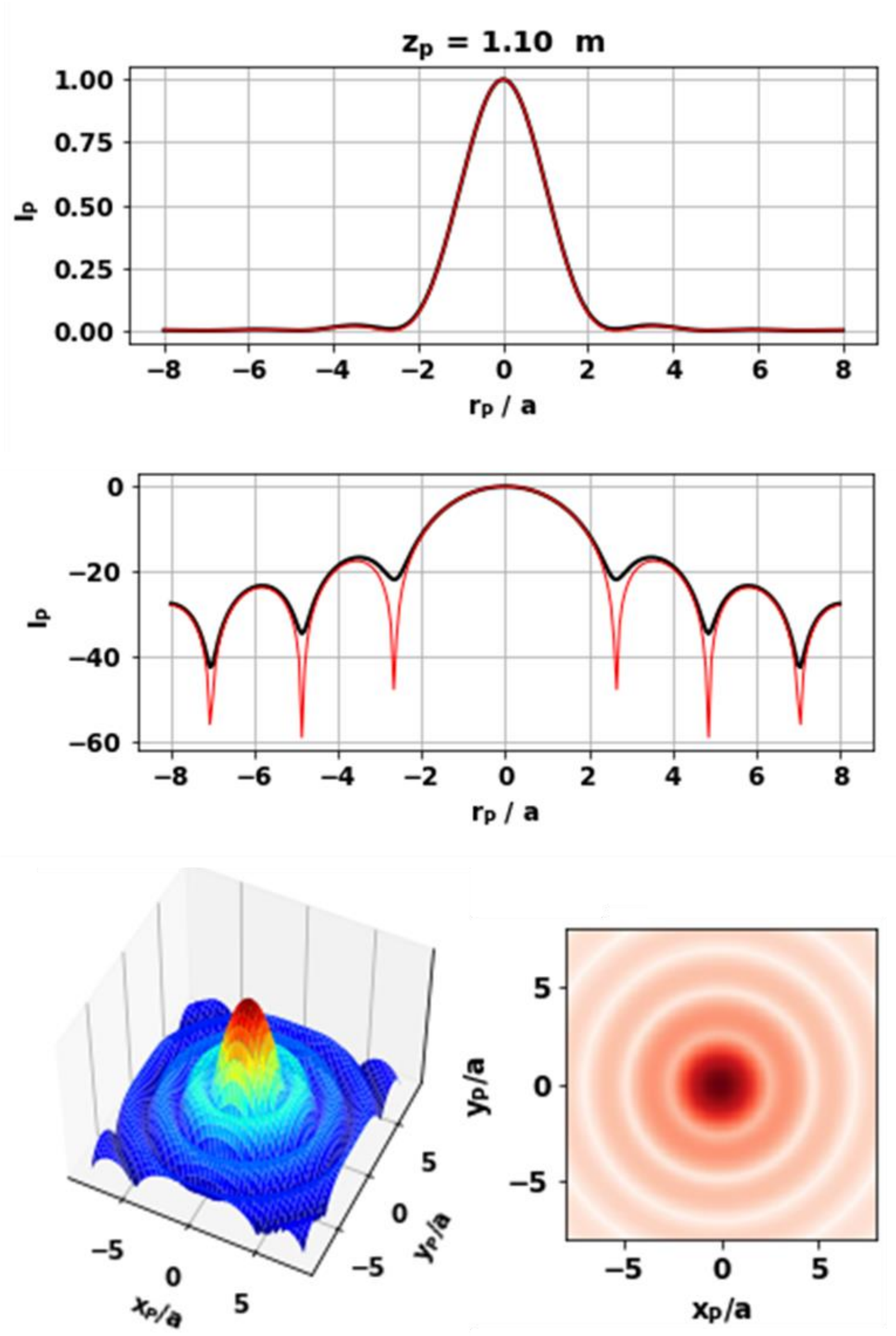


Fig. 7. Fraunhoffer diffraction for z_P = 1.10 m. **Black** line: RS! and **red** line (Fraunhoffer). Excellent agreement between RS1 computation and the Fraunhoffer equation (equation 1). **emRS102.py**

Fraunhoffer : Energy enclosed within the dark rings of the diffraction pattern

It is possible to calculate the energy enclosed within a ring of a specified radius on the observation screen by numerically integrating the irradiance with ever increasing radius.

```
# Power enclosed with a circle [a.u.]
r  = xP[indexXY:nP]
Ir  = Ix[indexXY:nP]
Pr = zeros(len(r))
for c in range(len(r)):
   if c > 1:
      Pr[c] = simps(r[0:c]*Ir[0:c],r[0:c])
Pr = 100*Pr/max(Pr)
```

Figure 8 shows the power as a total of maximum power enclosed within circles of increasing radius. About 84% of the energy from the aperture to the observation screen is enclosed within the Airy disk as indicated in figure 8.

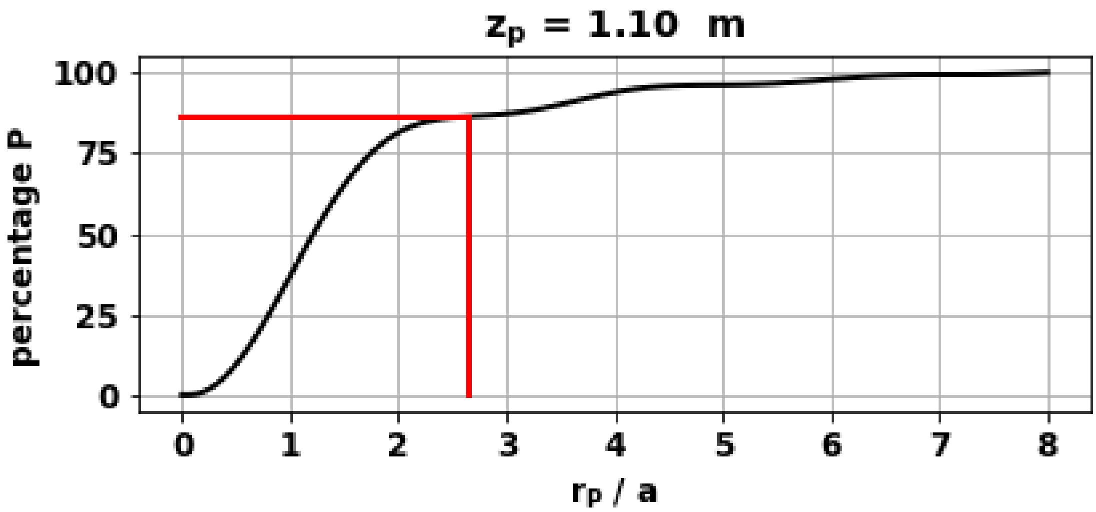


Fig.8. Percentage power enclosed within rings of increasing radius on the observation screen in the far field for a uniformly illuminated circular aperture. About 84% of the total power is enclosed within the first dark ring. **emRS102.py**

Divergence of the beam and wavelength

The wavelength can be changed in the Python Code **emRS102.py** to investigate the dependence of the irradiance pattern in an XY plane with wavelength (figure 9).

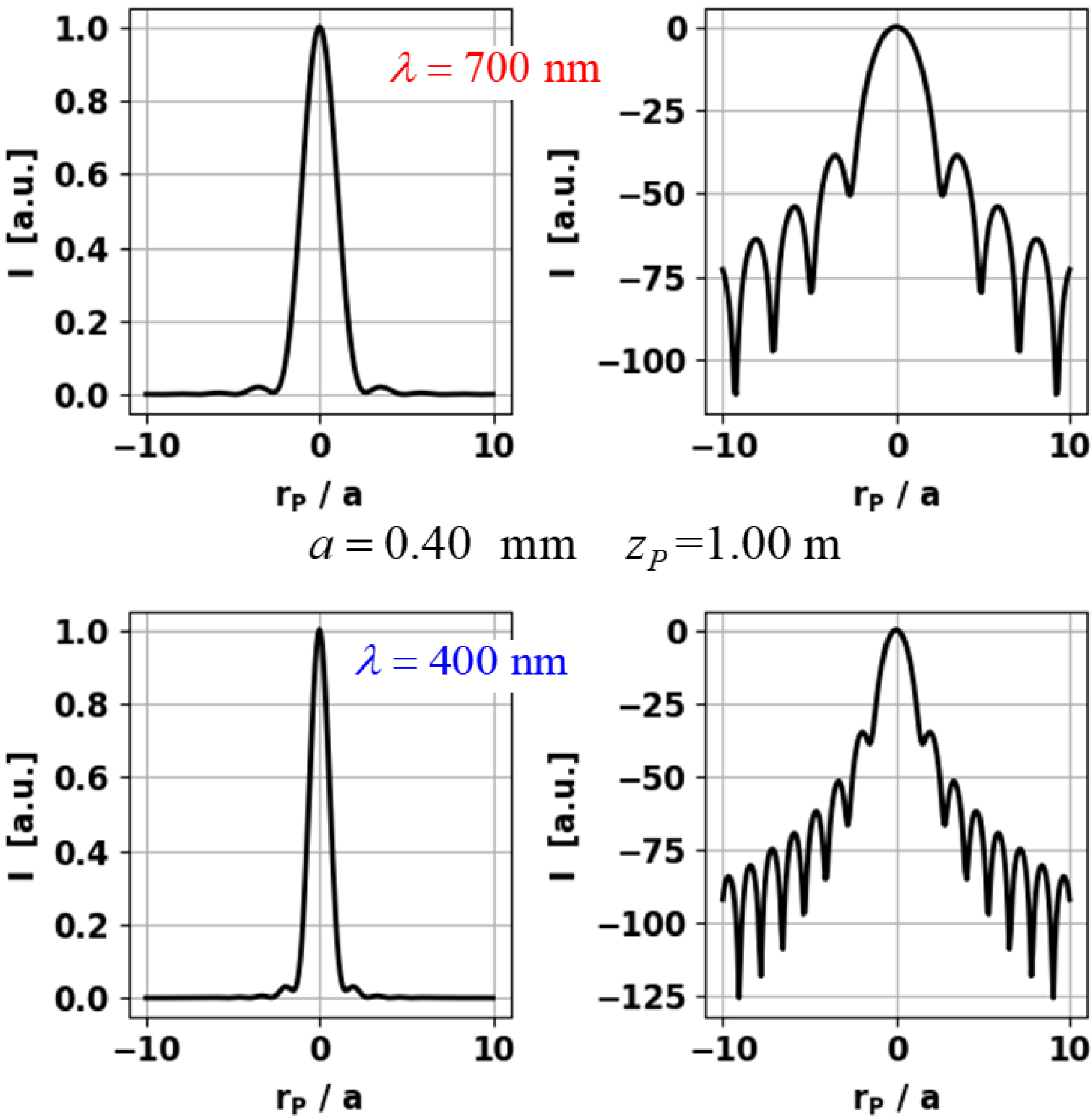
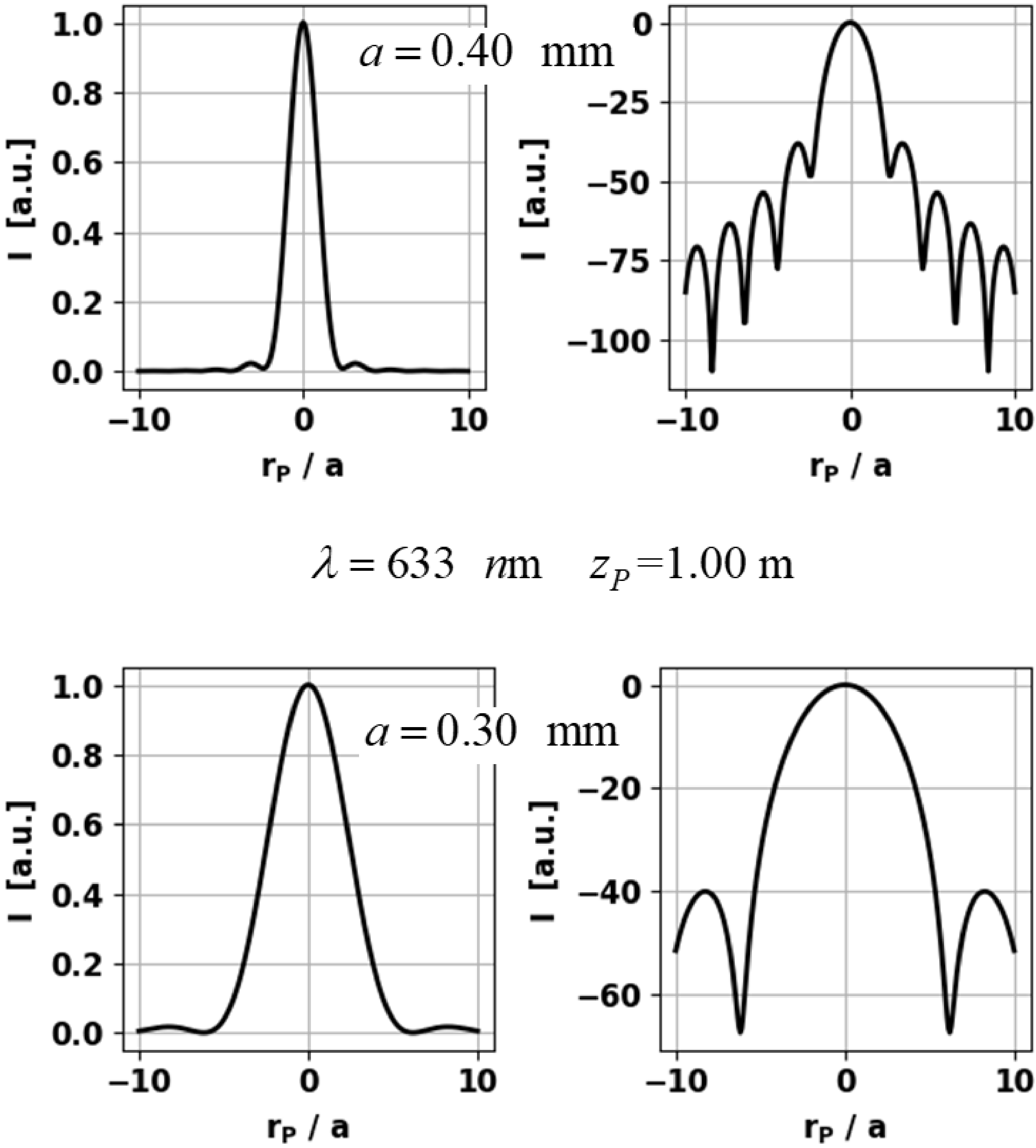


Fig. 9. The smaller the wavelength of the incident radiation, then the narrower the beam. For the blue light, the dark rings are much closer together than for the red light. **emRS102.py**

Divergence of the beam and aperture radius

The radius of the aperture can be changed in the Python Code **emRS102.py** to investigate the dependence of the irradiance pattern in an XY plane with aperture size (figure 10).



emRS101.py

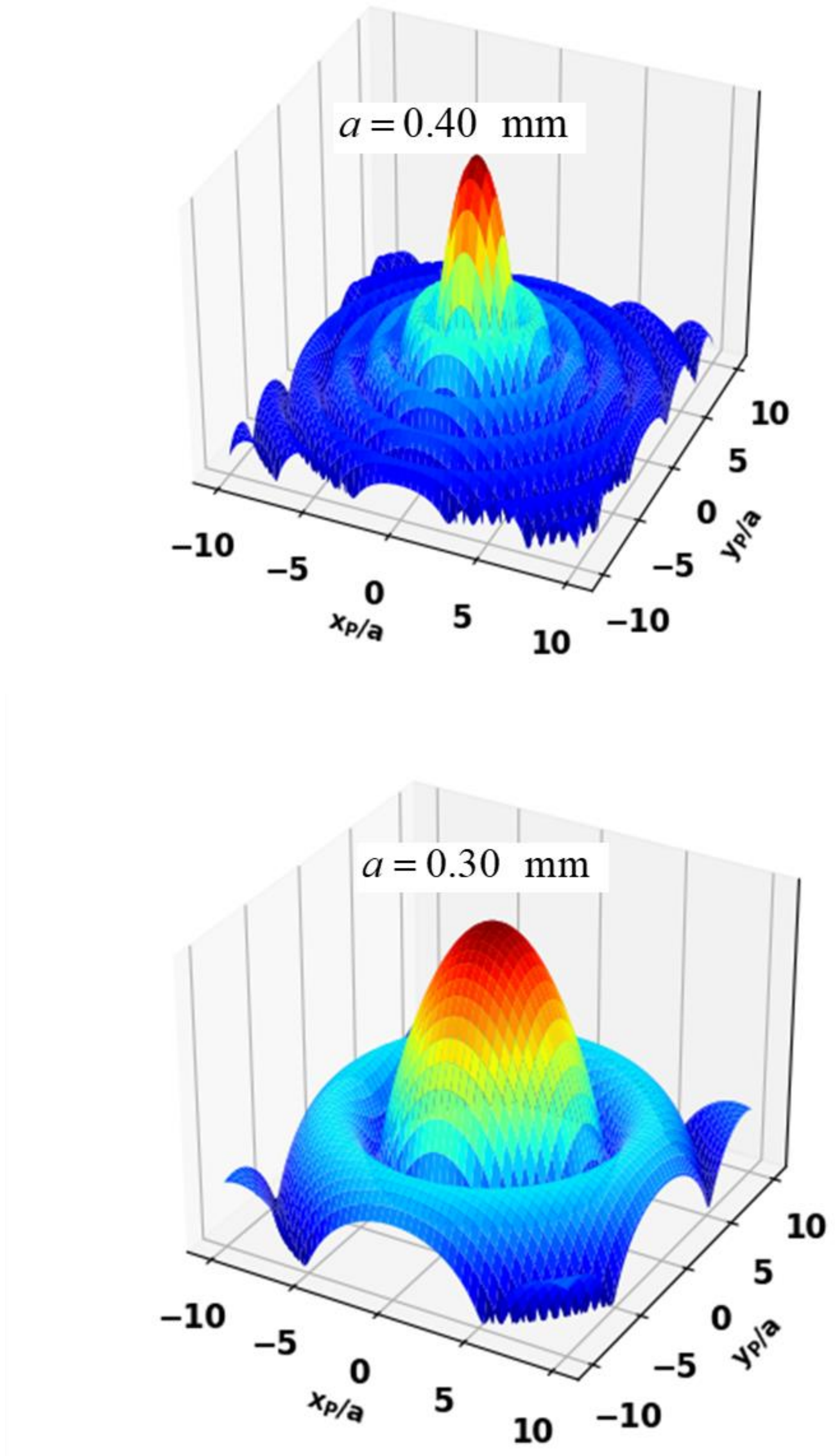


Fig. 10. The smaller the aperture radius, then the wider the beam.

For the larger radius aperture, the dark rings are much further apart.

emRS102.py

FRESNEL DIFFRACTION

When the observation point is within the Rayleigh length $\left(z_P < RL\right)$, equation 1 describing Fraunhoffer diffraction is no longer valid.

Figure 11 shows that the irradiance for the first dark ring to no longer zero.

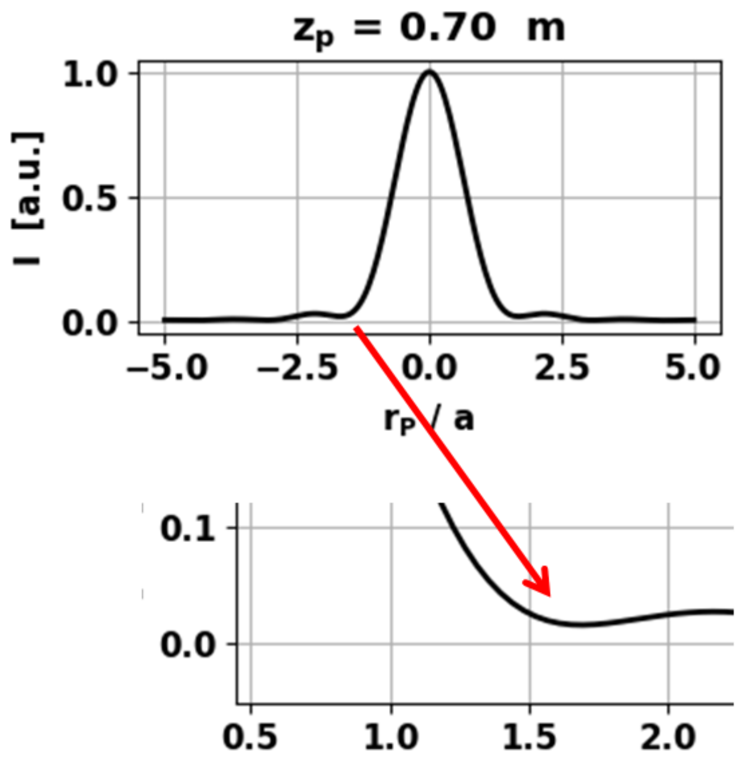


Fig. 11. The first minimum is greater than zero $\left(z_P = 0.70\ \text{m} < RL = 1.01\ \text{m}\right)$. **emRS102.py**

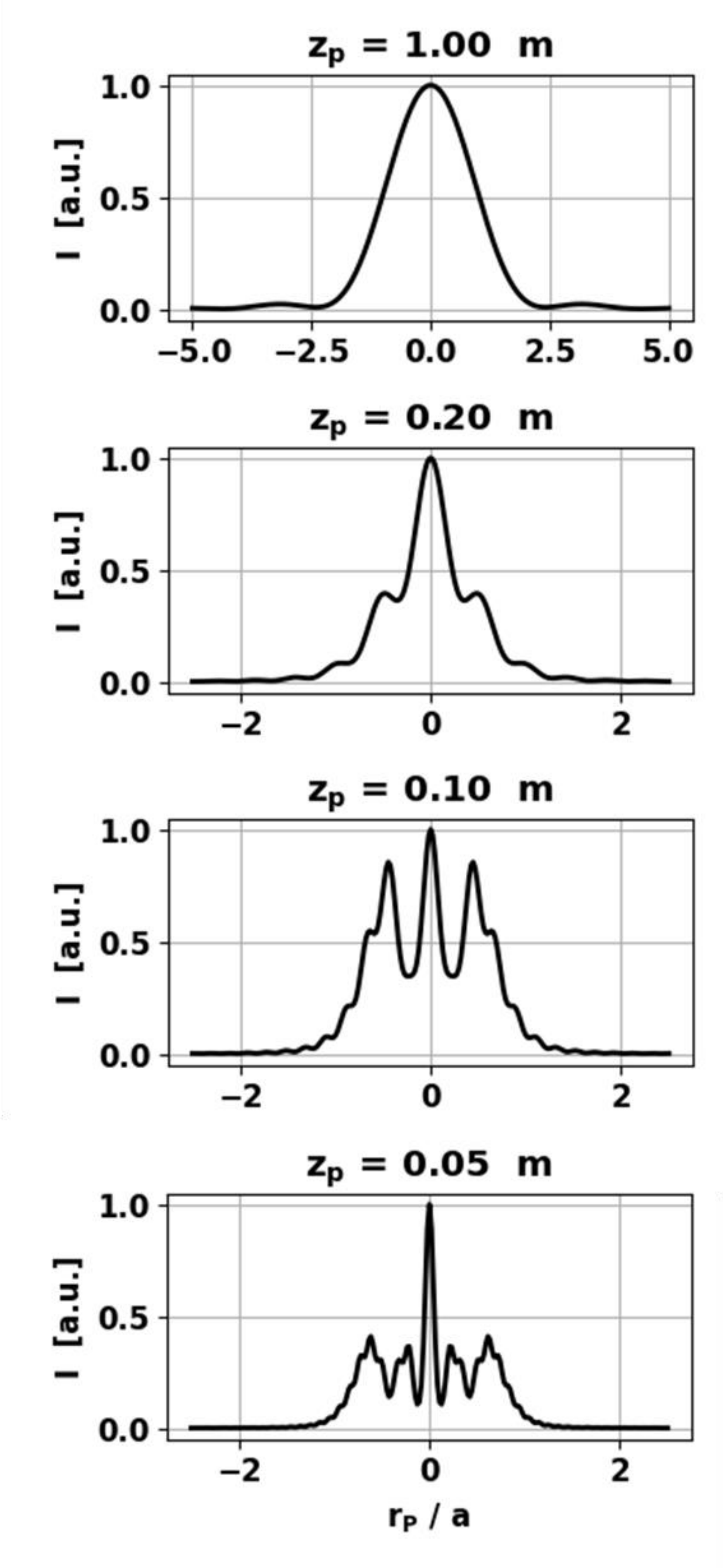


Fig. 12. Transition from Fraunhofer diffraction to Fresnel diffraction.
emRS102.py

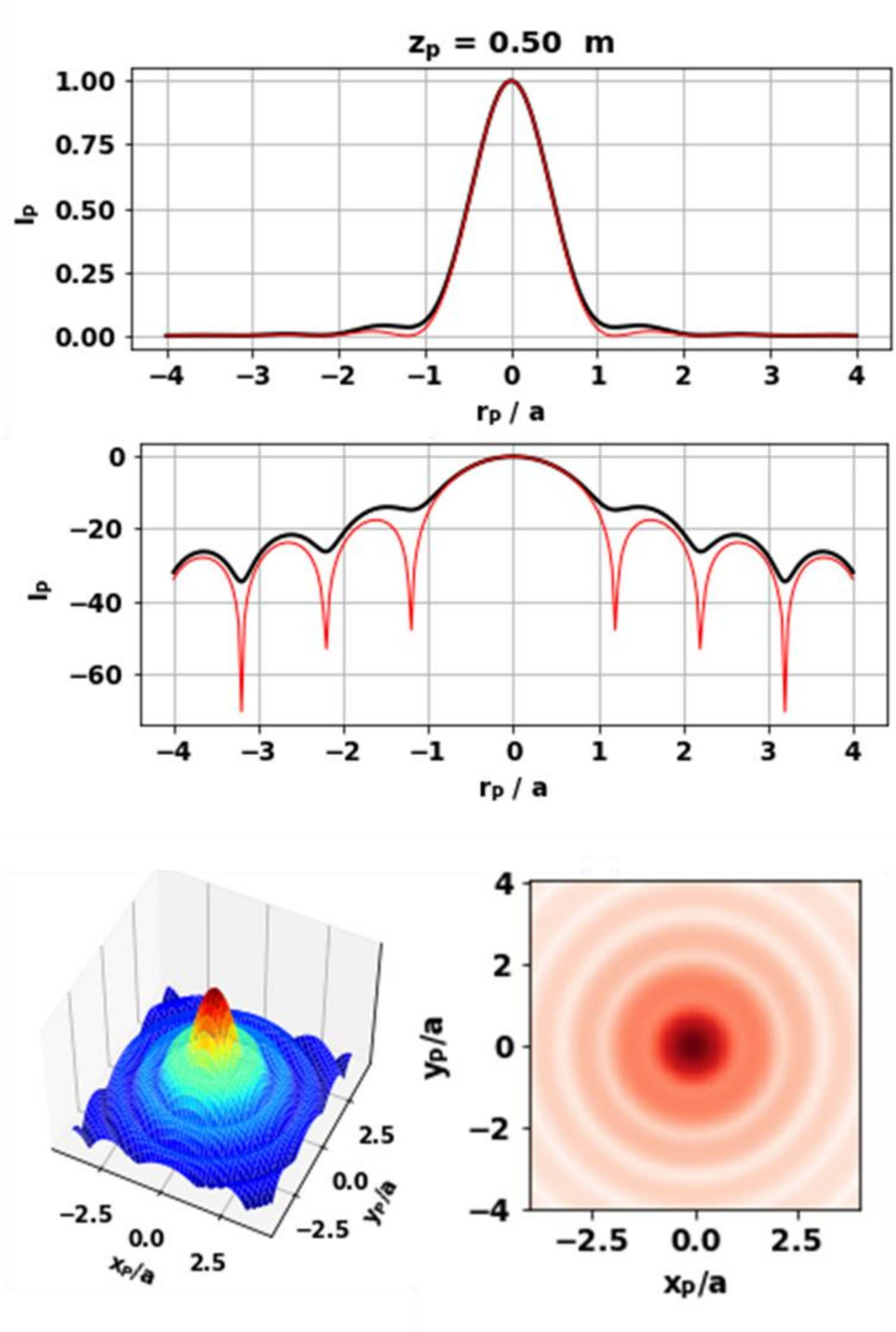


Fig. 13. Fresnel diffraction for $\left(z_P = 0.50\ \ \mathrm{m} < RL = 1.01\,\mathrm{m}\right)$.

RS1 (**black** line) Fraunhoffer (red line). The irradiance at the position of the first dark ring is greater than zero, the Fraunhoffer equation (equation 1) is not valid.   **emRS102.py**