

DOING PHYSICS WITH PYTHON

QUANTUM MECHANICS

TRANSVERSE NORMAL MODES OF VIBRATION FOR STANDING WAVES ON A ROD

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DOWNLOAD DIRECTORY FOR PYTHON SCRIPTS

qm042.py (fixed – fixed) **qm042A** (fixed – free)

qm042B.py (free – free)

Solution of the wave equation for the transverse vibration of a rod (string) by finding the eigenvalues and eigenvectors (eigenfunctions).

Code also includes a Cell for the animation of the vibrating rod.

[GitHub](#)

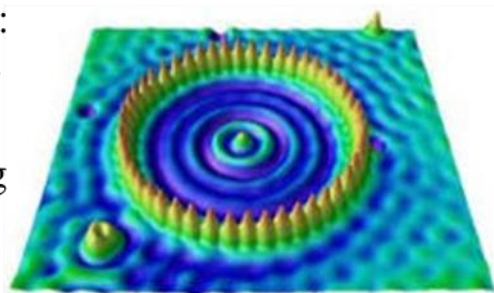
[Google Drive](#)

[VIEW ANIMATION OF VIBRATING ROD](#)

INTRODUCTION

For a particle such as an electron when it is confined in a region of atomic dimensions, the electron is best described by a set of wavefunctions (eigenfunctions). The characteristics of the wavefunctions are very similar to the of transverse vibrational modes of a rod (string) with fixed boundary conditions at the end of the rod. So, before we look at bound states for particles such as electrons, it is best to review the transverse vibrations of a rod and the necessary mathematics to solve the wave equation for the system of the rod plus boundary conditions.

Fig. 1 Tunnelling currents of electrons: iron atoms on a copper surface (ring – quantum corral). Surface electrons trapped inside the corral form standing waves.



Model parameters;

ω angular frequency [rad.s⁻¹]

f frequency [Hz]

T period [s]

λ wavelength [m]

k propagation constant [m⁻¹]

v speed of the wave along the rod [m.s⁻¹]

$y(x)$ transverse displacement [m]

$Y(t, x)$ transverse displacement of the rod at time t
and location x along the rod

$$\omega = 2\pi f = 2\pi / T \quad v = \omega / k \quad k = \omega / v = 2\pi / \lambda$$

The [1D] wave equation for the transverse waves on a rod is

$$\frac{\partial^2 Y(t, x)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 Y(t, x)}{\partial t^2}$$

For the normal modes of vibration:

$$Y(t, x) = y_n(x) \cos(\omega_n t) \quad n = 1, 2, 3, \dots$$

and the time independent wave equation (**Helmholtz equation**) can be expressed as

$$\frac{d^2 y(x)}{dx^2} = -k^2 y(x)$$

This is simply an eigenvalue problem, where $y(x)$ is the **eigenfunction** and $-k^2$ is the corresponding **eigenvalue**.

The second derivative of the wavefunction $y(x)$ can be approximated by the finite difference approximation

$$(1) \quad \frac{d^2 y(x)}{dx^2} \approx \frac{y(x - \Delta x) - 2y(x) + y(x + \Delta x)}{\Delta x^2}$$

The X domain is divided into $N+2$ evenly spaced grid points where $\Delta x = x(2) - x(1)$. The ends of the rod are $x(1) = 0$ and $x(N+2) = L$ where L is the length of the rod. The normal modes of vibration are

determined by the solutions of equation 1 that satisfy the imposed boundary conditions.

For the interior grid points, equation 1 can be expressed in matrix form

$$(2) \quad \left(\frac{1}{\Delta x^2} \right) \mathbf{A} y = -k^2 y$$

where \mathbf{A} is a matrix of dimensions $N \times N$ and y is a column vector with elements $y(0), y(1), \dots, y(N-1)$. The elements $\mathbf{A}(0, 0)$ and $\mathbf{A}(N-1, N-1)$ are determined by the boundary conditions.

Note: the first element of an array is referenced by a zero and not by one. For example, an array y with 5 elements is

$$y[0] \ y[1] \ y[2] \ y[3] \ y[4]$$

For the rod **fixed** (y displacement is zero) at both ends when $N = 5$

$$(3) \quad \frac{1}{\Delta x^2} \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_{-1} = 0 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 = 0 \end{pmatrix} = -k^2 \begin{pmatrix} y_{-1} = 0 \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 = 0 \end{pmatrix}$$

since from equations 1, 2 and 3

At the boundary $x = 0$

$$\frac{1}{\Delta x^2} (y_{-1} - 2y_0 + y_1) = \frac{1}{\Delta x^2} (-2y_0 + 1y_1) = -k^2 y_0$$

At an interior point

$$\frac{1}{\Delta x^2}(\textcolor{red}{1}y_1 - \textcolor{red}{2}y_2 + \textcolor{red}{1}y_3) = -k^2 y_2$$

At the boundary $x = L$

$$\frac{1}{\Delta x^2}(y_3 - 2y_4 + y_5) = \frac{1}{\Delta x^2}(\textcolor{red}{1}y_3 - \textcolor{red}{2}y_4) = -k^2 y_4$$

For the rod **free** at both ends (slope is zero: $y_{-1} = y_0$) when $N = 5$

$$(3) \quad \frac{1}{\Delta x^2} \begin{pmatrix} \textcolor{blue}{-1} & \textcolor{blue}{1} & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & \textcolor{blue}{1} & \textcolor{blue}{-1} \end{pmatrix} \begin{pmatrix} \textcolor{blue}{y_{-1} = y_0} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \textcolor{blue}{y_5 = y_4} \end{pmatrix} = -k^2 \begin{pmatrix} \textcolor{blue}{y_{-1} = y_0} \\ y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ \textcolor{blue}{y_5 = y_4} \end{pmatrix}$$

since from equations 1, 2 and 3

At the boundary $x = 0$

$$\frac{1}{\Delta x^2}(y_{-1} - 2y_0 + y_1) = \frac{1}{\Delta x^2}(\textcolor{blue}{-1}y_0 + \textcolor{blue}{1}y_1) = -k^2 y_0$$

At the boundary $x = L$

$$\frac{1}{\Delta x^2}(y_3 - 2y_4 + y_5) = \frac{1}{\Delta x^2}(\textcolor{blue}{1}y_3 - \textcolor{blue}{1}y_4) = -k^2 y_4$$

Increasing the number of grid points N improves the accuracy of the simulation but at the expense of the execution time.

$$(N = 99 \quad t_{exec} = 1 \text{ s}, \quad N = 599 \quad t_{exec} = 8 \text{ s})$$

The **eigenvalues** are given by the variable ev where $ev = -k^2$. The **eigenfunction** (**eigenvector**) is given by the variable ef . The eigenfunction returns the values for the displacements

$$y_0, y_1, \dots, y_{N-1}$$

thus, it does not include the value of the displacements at the end of the rod where $x = 0$ and $x = L$.

The Python Code **qm042.py** is used to simulation the normal modes of vibration of the rod (string). The matrix equation for the Helmholtz equation is expressed in the form

$$\mathbf{A} y = \left(-4 \pi^2 \Delta x^2 / \lambda^2 \right) y$$

where the eigenvalues are $\left(-4 \pi^2 \Delta x^2 / \lambda^2 \right)$. Computing the eigenvalues, thus gives us the wavelengths for the normal modes of vibration.

The input section specifies the properties of the string and M is the number of eigenvalues and eigenfunctions that are returned.

#%% INPUTS

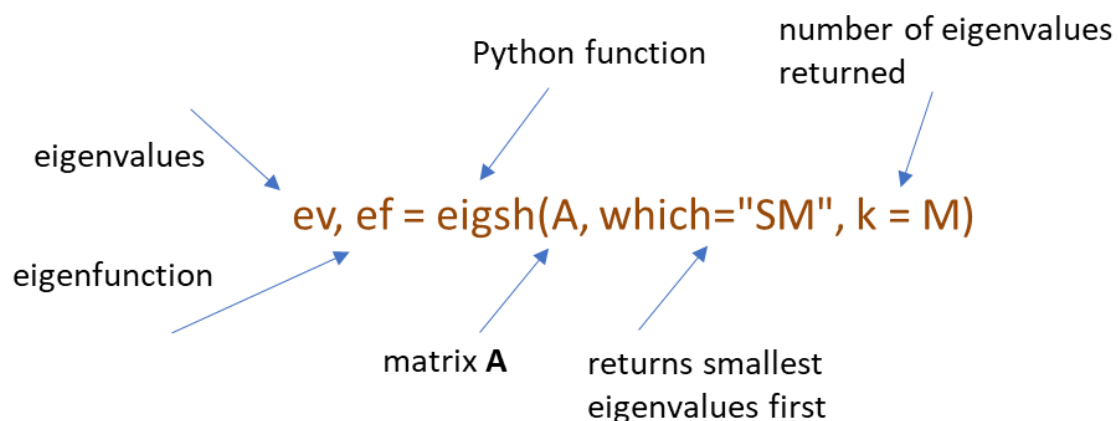
```

N = 999      # grid points
L = 1        # length of guitar string 0.640 [m]
T = 100      # string tension [N]
mu = 5.0e-3  # linear density of string [kg/m]
M = 6        # Number of modes

```

The matrix **A** is constructed and the eigenvalues and eigenfunctions are computed by the Code

```
### COMPUTATIONS
x = linspace(0,L,N)    # x domain [m]
dx = x[2] - x[1]       # dx [m]
# Second Derivative matrix
off = np.ones(N-1)
A = -2*np.eye(N) + np.diag(off,1) + np.diag(off,-1)
# Eigenvalues and eigenfunctions (eigenvectors)
ev, ef = eigsh(A, which="SM", k = M)
ef = ef / np.amax(ef)
```



Normal mode calculations

```
### Normal modes
wL = np.sqrt(-4*np.pi**2*dx**2/ev) # wavelengths [m]
v = sqrt(T/mu)                       # velocity [m/s]
f = v/wL                             # frequency [Hz]
```

`wL[0]` gives the wavelength for mode $n = 6$ and `wL[5]` gives the wavelength for mode $n = 1$.

The eigenfunction `ef[:,1]` gives the wavefunction as a function of x for the normal mode $n = 5$. Note the variable `ef` for the transverse displacement y does not include the end points $y(0) = 0$ and $y(L) = 0$.

SIMULATIONS

Boundary conditions fixed at both ends $y(0) = 0$ (node), $y(L) = 0$

(node) `qm042.py`

The strings of guitar are fixed at each end (nodes) and vibrate as standing waves when excited by finger movements.



Console output

$L = 1.0 \text{ m}$ $F = 100.0 \text{ N}$ $\mu = 5.0\text{e-}03 \text{ kg/m}$

$v = 141.42 \text{ m/s}$

n	λ [m]	f [Hz]	f_n/f_1
1	2.04	69.29	1.00
2	1.02	138.57	2.00
3	0.68	207.81	3.00
4	0.51	277.00	4.00
5	0.41	346.13	5.00
6	0.34	415.16	5.99

Graphical output

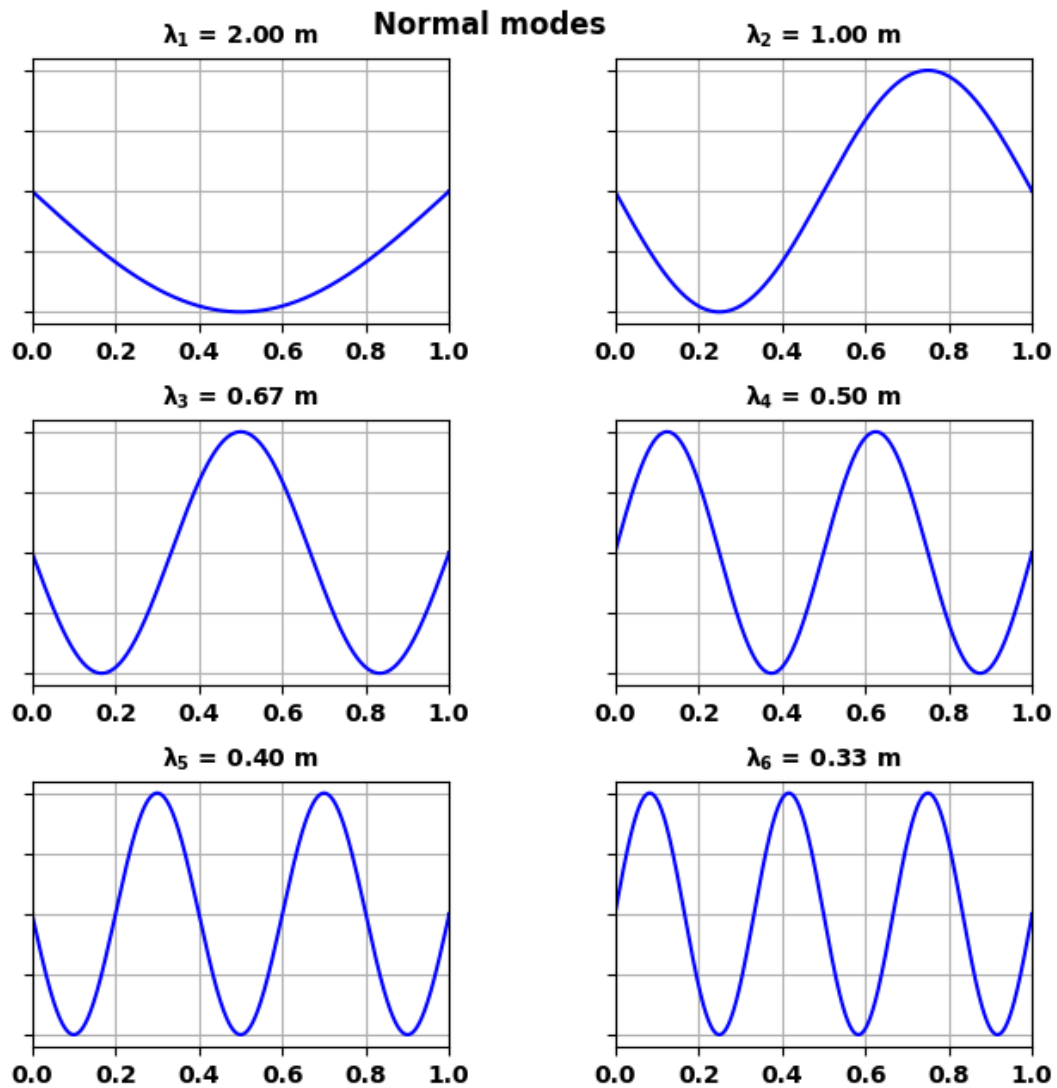


Fig.1. First six normal modes for a rod of length $L = 1.00$ m fixed at both ends at time $t = 0$ s. The mode number n gives the number of half-wavelengths that fit into the length of the rod.

$$\lambda_n = 2L / n = \lambda_1 / n \quad f_n = n f_1 \quad n = 1, 2, 3, \dots$$

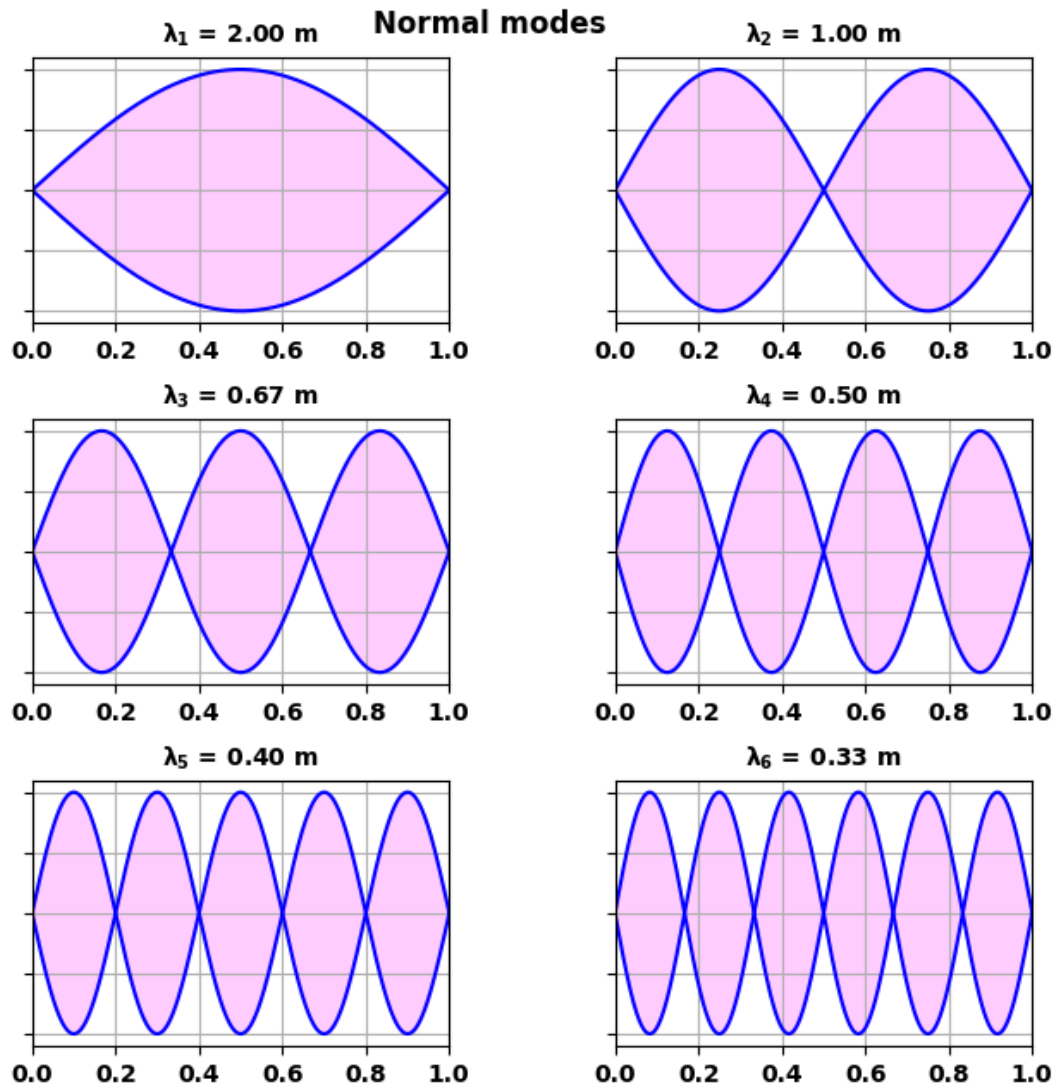


Fig.2. Standing wave patterns for the first six normal modes for a rod of length $L = 1.00$ m fixed at both ends. The mode number n gives the number of half-wavelengths that fit into the length of the rod.

$$\lambda_n = 2L / n = \lambda_1 / n \quad f_n = n f_1 \quad n = 1, 2, 3, \dots$$

The separation between adjacent nodes or adjacent antinodes is always $\lambda / 2$. The variable n corresponds to the harmonic number and for the rod fixed at both ends, all the harmonics can be excited.

The modes of vibration of the rod are identical to the standing wave patterns of the matter waves of an electron bound in a well with infinite depth.

Boundary conditions fixed at $y(0) = 0$ (node), and free at $y(L) = 0$ (antinode) **qm042A.py**

Console output

$L = 1.0 \text{ m}$ $F = 100.0 \text{ N}$ $\mu = 5.0\text{e-}03 \text{ kg/m}$

$v = 141.42 \text{ m/s}$

n	lambda [m]	f [Hz]	fn/f1
1	4.06	34.82	1.00
3	1.35	104.46	3.00
5	0.81	174.07	5.00
7	0.58	243.63	7.00
9	0.45	313.14	8.99
11	0.37	382.56	10.99

Graphical output

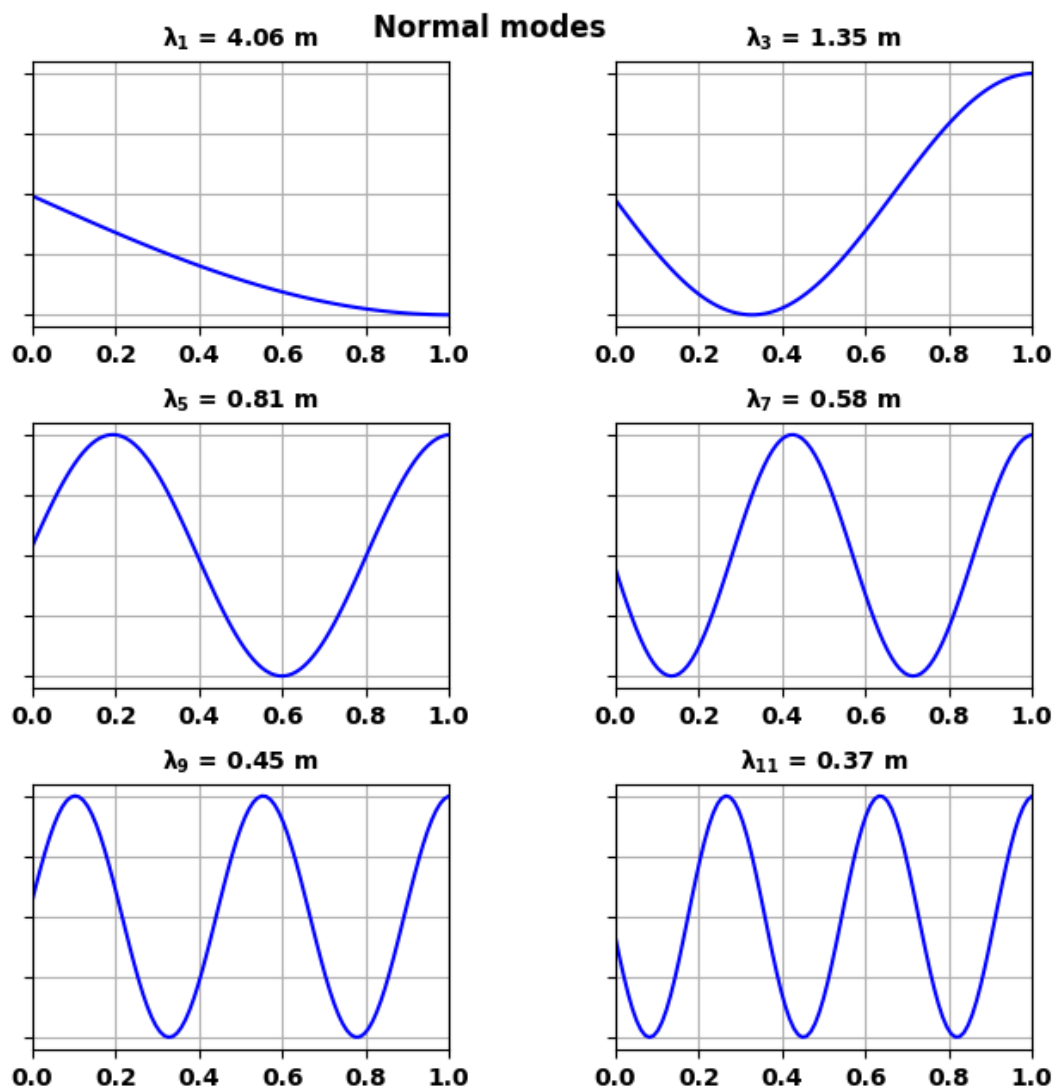


Fig. 3. Standing wave patterns at time $t = 0$ for the rod fixed at $x = 0$ (node) and free to vibrate at the end $x = L$ (antinode). The distance between adjacent nodes or adjacent antinodes is $\lambda / 2$. Only the odd harmonics can be excited where $n = 1, 3, 5, \dots$.

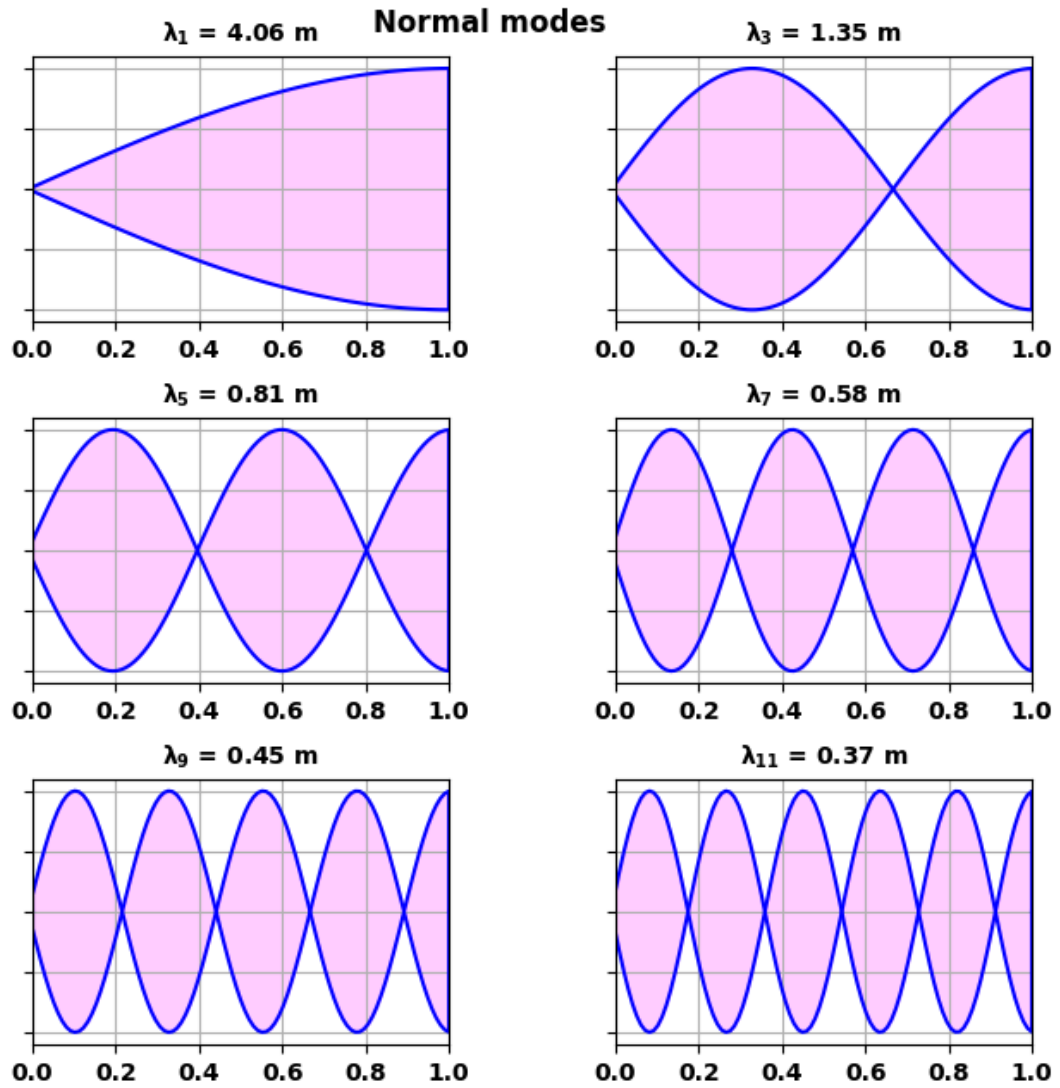
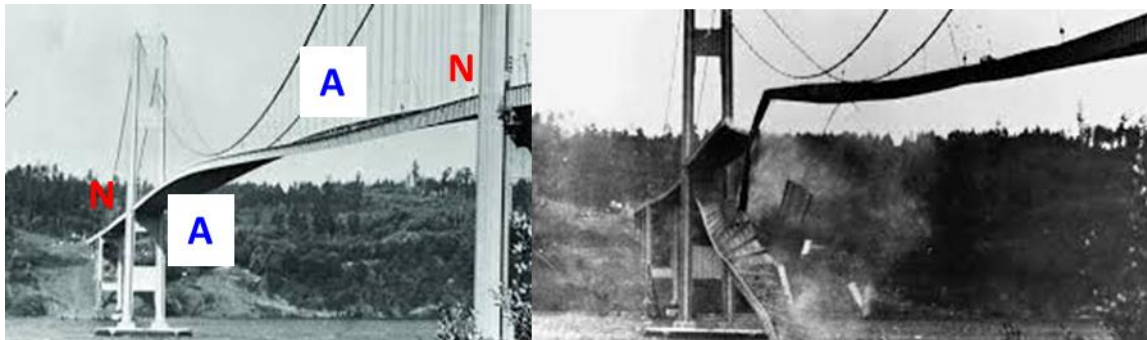


Fig. 4. Standing wave patterns for the rod fixed at $x = 0$ (node) and free to vibrate at the end $x = L$ (antinode). The distance between adjacent nodes or adjacent antinodes is $\lambda / 2$. Only the odd harmonics can be excited where $n = 1, 3, 5, \dots$.

Boundary conditions free at both ends $y(0)$ (antinode), $y(L)$ (antinode) `qm042B.py`



Tacoma Narrows Bridge

Longitudinal vibration: node **N** at each end of bridge (fixed-fixed boundary conditions). Transverse vibrations: antinode **A** at edges of bridge (free - free boundary conditions).

The boundary conditions are that the rod is free to vibrate at both ends. So, antinodes will exist at the ends of the rod, $x = 0$ and $x = L$.

Console output

$L = 1.0$ m $F = 100.0$ N $\mu = 5.0e-03$ kg/m

$v = 141.42$ m/s

n	lambda [m]	f [Hz]	fn/f1
1	2.00	70.59	1.00
2	1.00	141.18	2.00
3	0.67	211.78	3.00
4	0.50	282.37	4.00
5	0.40	352.95	5.00
6	0.33	423.54	6.00

Graphical output

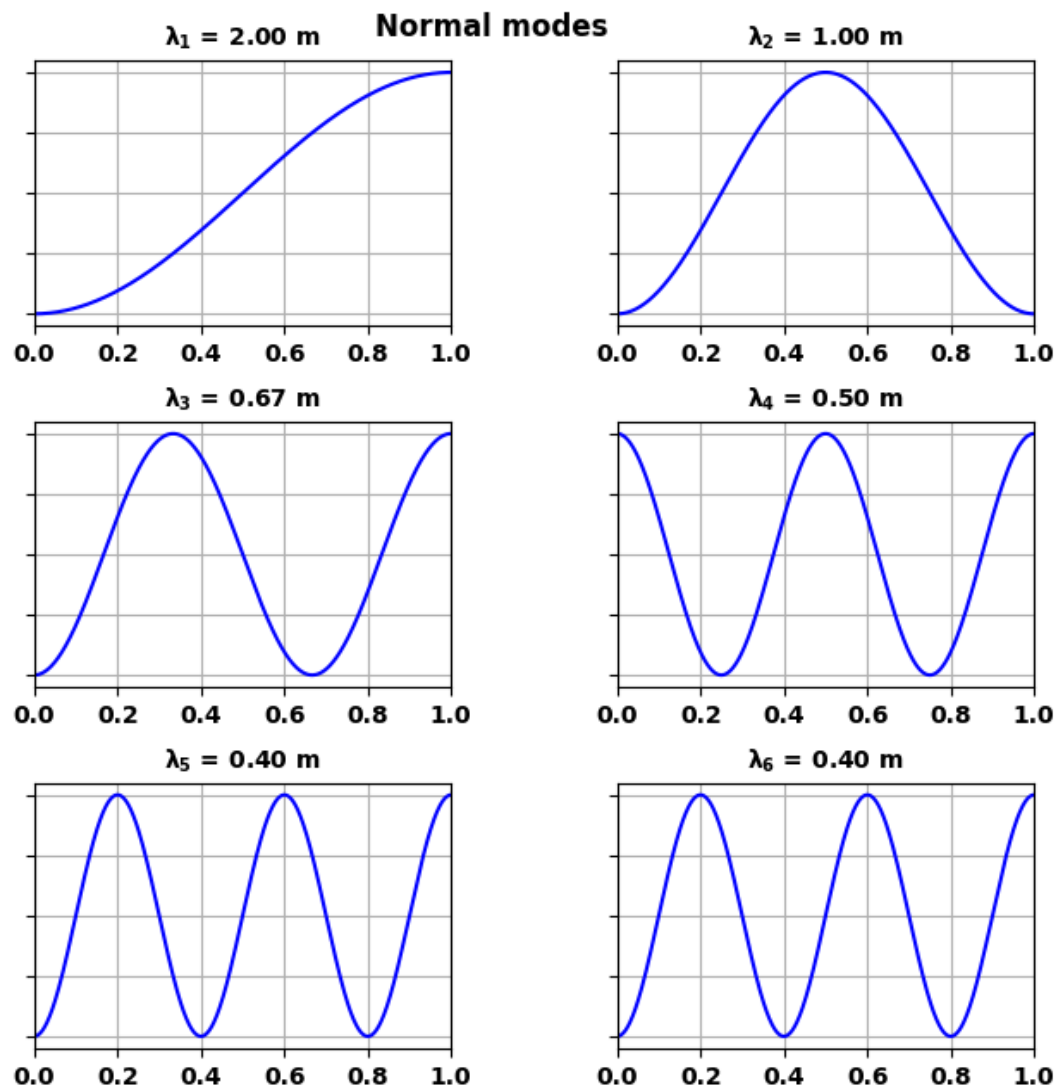


Fig. 5. Standing wave patterns with antinodes at both ends of the rod at time $t = 0$.

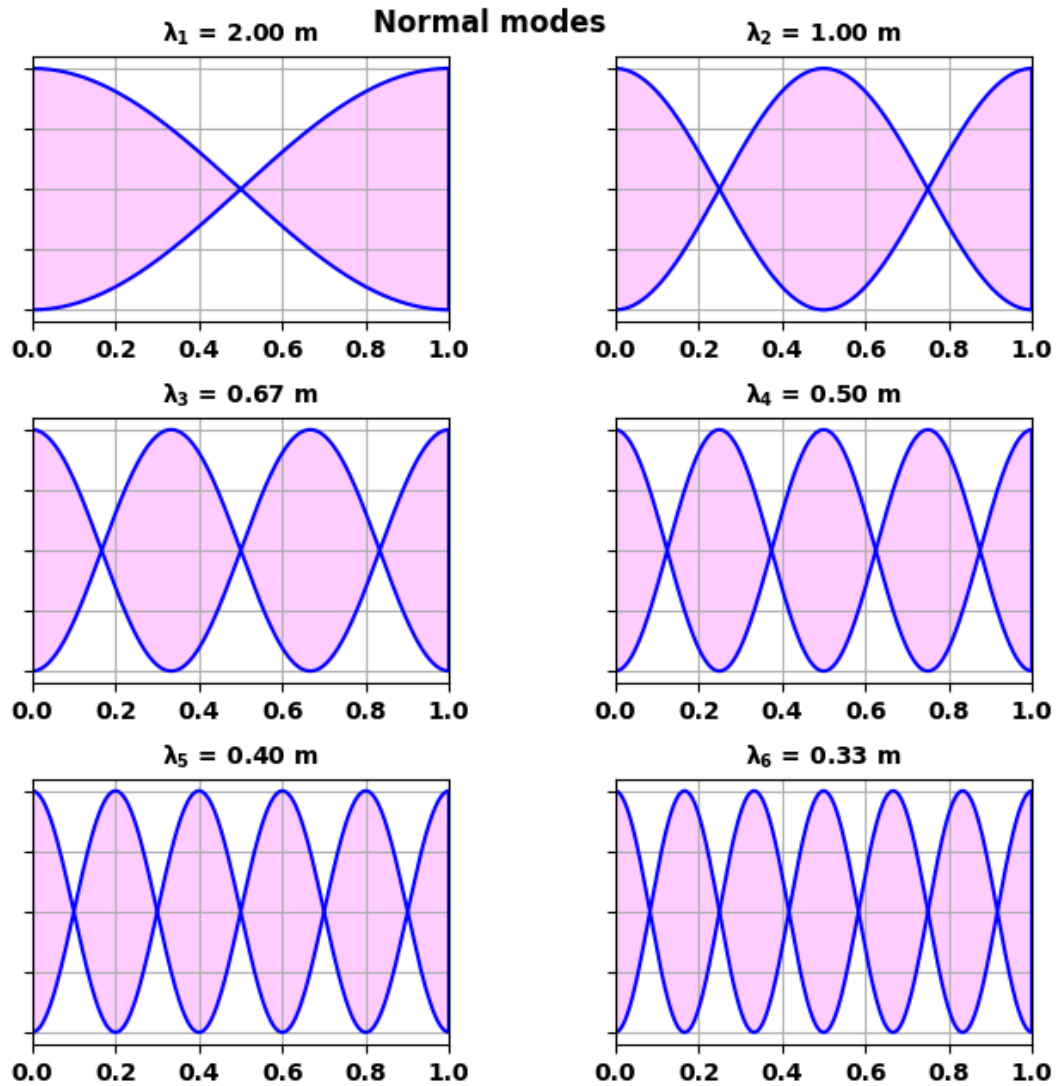


Fig. 6. Standing wave patterns for rod free at each end.

The normal mode wavelengths depend only on the length of the string. The vibrational frequency is dependent upon the wavelength (length), the rod tension and the linear density of the rod.