DOING PHYSICS WITH PYTHON

NONLINEAR [1D] DYNAMICAL SYSTEMS

FIXED POINTS, STABILITY ANALYSIS, BIFURCATIONS

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Google drive

GitHub

cs100.py cs101.py cs103.py cs104.py cs10.5.py cs106.py

INTRODUCTION

To review many aspects of the behaviour of nonlinear systems, we will consider a number of examples of the solutions for nonlinear ordinary differential equation of the form

$$\dot{x} = f(x) \qquad \dot{x} \equiv dx / dt$$

The system will be in equilibrium at a fixed-point x_e where

$$\dot{x} = 0 \quad f(x_e) = 0$$

When $x = x_e$, $f(x_e) = 0$ then x_e often called a **steady state solution**.

To analyse the stability, consider a small perturbation $e(t)$ from an equilibrium position

$$x(t) = x_e(t) + e(t)$$

From a Taylor expansion, it can be shown that

$$e(t) = e(0)\, e^{f'(x_e)t}$$

If $f'(x_e) > 0$ then $e(t)$ grows exponentially and if $f'(x_e) < 0$, then $e(t)$ decays exponentially to zero.

Thus, the stability of a fixed point is determined from the function $f'(x_e)$ $\left(f'(x) \equiv df\,/\,dx\right)$

Stable fixed point $\quad f'(x_e) < 0 \quad$ where $x \to x_e$

Marginally stable fixed point $\quad f'(x_e) = 0$

where $x \to x_e$ or $x \to \pm\infty$

Unstable fixed point $\quad f'(x_e) > 0 \quad$ where $x \to \pm\infty$

The ODEs are solved using the Python function **odeint.** To reproduce the following plots, you need to change simulation parameters and comment/uncomment parts of the code.

Bifurcation means a structural change in the orbit of a system when a parameter is changed. The point where the bifurcation occurs is known as the **bifurcation point**. The orbit and the fixed point may change dramatically at bifurcation points as the character of an attractor or a repeller are altered. A graph of the parameter values versus the fixed points of the system is known as a **bifurcation diagram**.

The [1D] nonlinear system's ODE can be expressed as

$$\dot{x}(t) = f\left(x(t), r\right)$$

and the fixed points of the system are

$$f\left(x_e(t), r\right) = 0$$

where r is the bifurcation parameter. So, the fixed points and their stability depends upon the bifurcation parameter.

Using a number of examples, three important bifurcations, namely the **saddle node**, **pitchfork**, and **transcritical** bifurcations are discussed. for [1D] systems.

Example 1 SADDLE NODE BIFURCATION cs100.py

$\dot{x}(t) = r + x(t)^2$ r is an adjustable constant

$$f(x) = r + x^2 \quad f'(x) = 2x$$

$$\dot{x} = 0 \;\Rightarrow\; x_e = 0 \text{ and } x_e = \pm\sqrt{-r}$$

Thus, there are three possible fixed points;

$r > 0$ no fixed points

$r = 0$ one fixed point $x_e = 0$

$r < 0$ two fixed points $x_e = -\sqrt{-r}$ $x_e = +\sqrt{-r}$

The system's behaviour can be considered in terms of the **velocity vector field**. The system vector field is represented by a vector for the velocity at each position x. The arrow for the velocity vector at point x is to the right (+X direction) if $\dot{x} > 0$ and to the left (-X direction) if $\dot{x} < 0$. So, the flow is to the right when $\dot{x} > 0$ and to the left when $\dot{x} < 0$. At the points where $\dot{x} = 0$, there are no flows and such points are called **fixed points**.

$r > 0$ there are no fixed-points

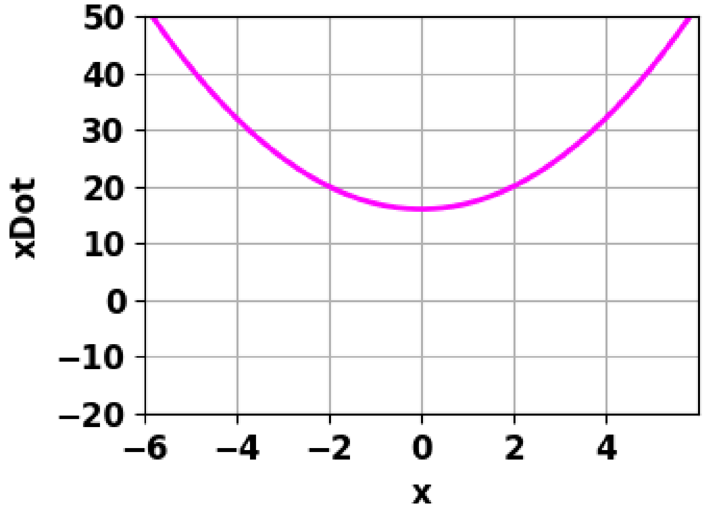


Fig. 1.1 If $r > 0$ then there are no fixed points

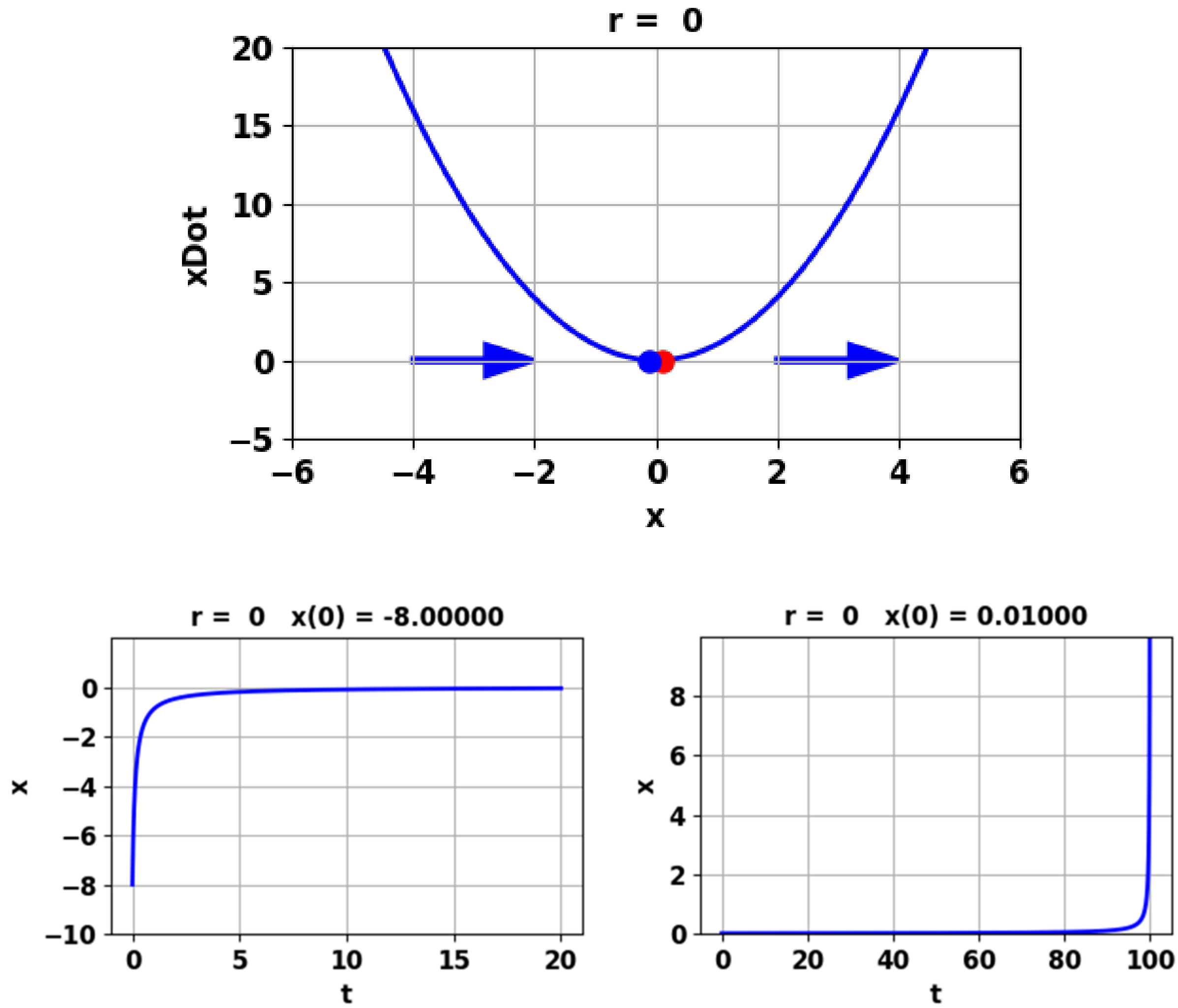
$r = 0$

$$r = 0 \quad \dot{x} = x^2 \quad x_e = 0 \quad f\,'(x_e = 0) = 0$$

$$x(0) = 0 \quad \dot{x}(t) = 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) < 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to 0$$

$$x(0) > 0 \quad \dot{x}(t) > 0 \quad \Rightarrow t \to \infty \quad x \to +\infty$$



Fig, 1.2 Fixed point: $r = 0,\ x_e = 0$.

Blue **dot** is a stable fixed point (negative slope)

Red **dot** is an unstable fixed point (positive slope).

$r < 0$

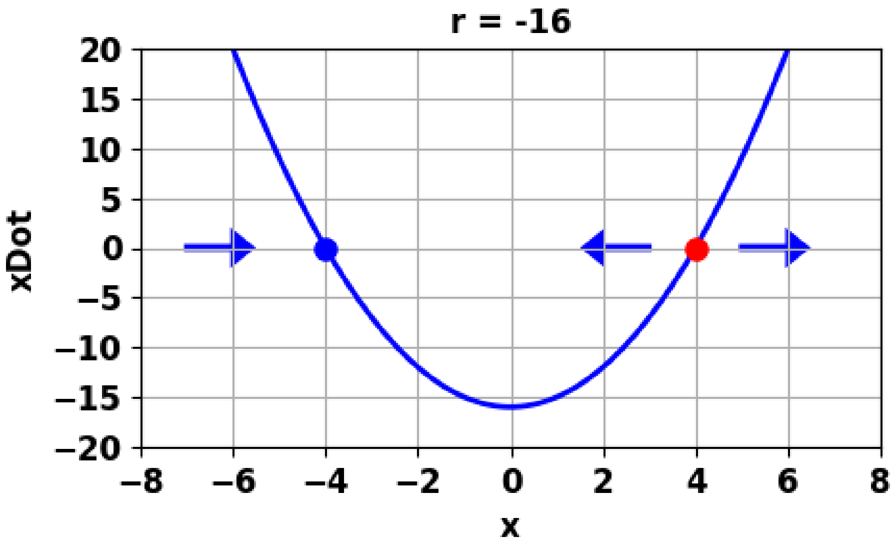
There are two fixed points

$$\dot{x} = r + x^2 \qquad f(x) = r + x^2 \qquad f'(x) = 2x$$

$$x_e = -\sqrt{-r} \qquad f'(x_e) < 0 \quad \Rightarrow \quad \text{stable}$$

$$x_e = +\sqrt{-r} \qquad f'(x_e) > 0 \quad \Rightarrow \quad \text{unstable}$$

Let r = -16 then the two fixed points are x_e = -4 (stable) and x_e = +4 (unstable).



This is a very simple system but its dynamics is highly interesting. The bifurcation in the dynamics occurred at $r = 0$ (bifurcation point), since the vector fields for $r < 0$ and $r > 0$ qualitatively different.

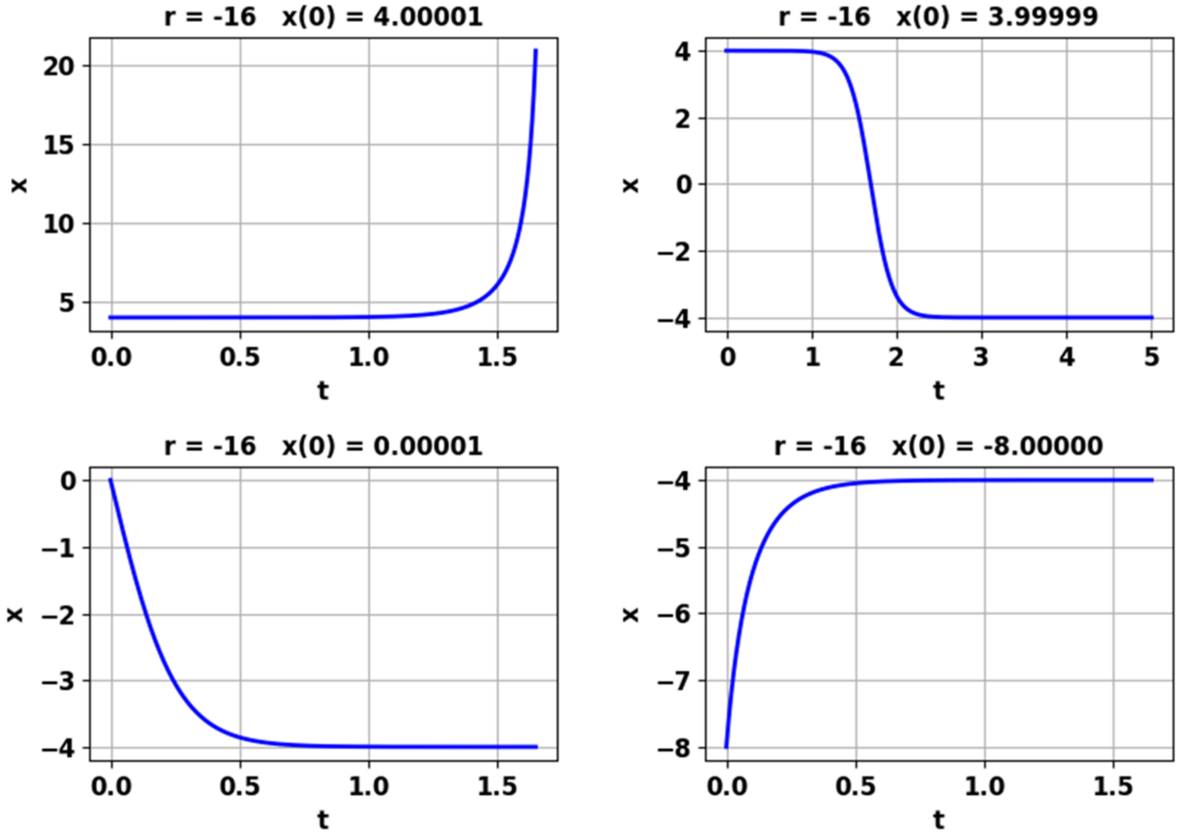


Fig. 1.3 Stable fixed point x_e = -4 (blue dot, negative slope)

Unstable fixed point x_e = +4 (red dot, positive slope)

$$x(0) > 4 \quad t \to \infty \quad \Rightarrow \quad x(t) \to \infty$$

$$x(0) < 4 \quad t \to \infty \quad \Rightarrow \quad x(t) \to -4$$

Figure 1.4 shows the **bifurcation diagram** for the fixed points x_e as a function of the **bifurcation parameter *r***.

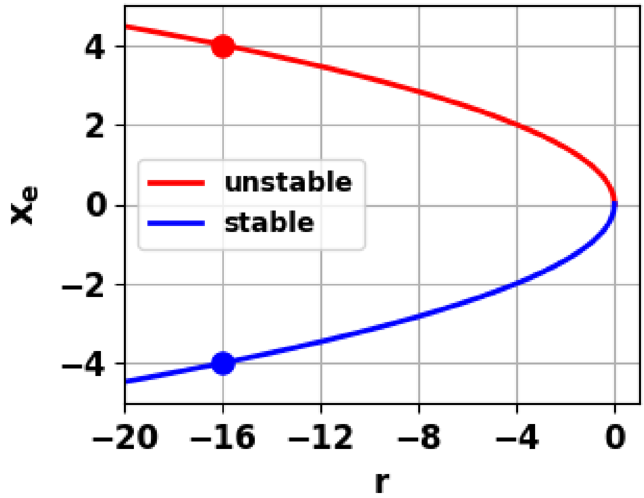


Fig. 1.4 Saddle node bifurcation diagram. The two fixed points for $r < 0$ merge as r goes to zero.

This is an example of a **subcritical saddle node bifurcation** since the fixed points exist for values of the parameter below the bifurcation point $r = 0$.

If we were to consider the system $\dot{x} = r - x^2$ than this would be an example of a **supercritical saddle node bifurcation**, since the equilibrium points exist for values of above the bifurcation point $r = 0 \left(r > 0 \Rightarrow x_e = \pm\sqrt{r}\right)$.

Example 2 Transcritical bifurcation cs101.py

The **transcritical bifurcation** is one type of bifurcation in which the stability characteristics of the fixed points are changed for varying values of the parameters.

$\dot{x}(t) = r\,x(t) - x(t)^2$ r is an adjustable constant

$$f(x) = r\,x - x^2 \quad f'(x) = r - 2x$$

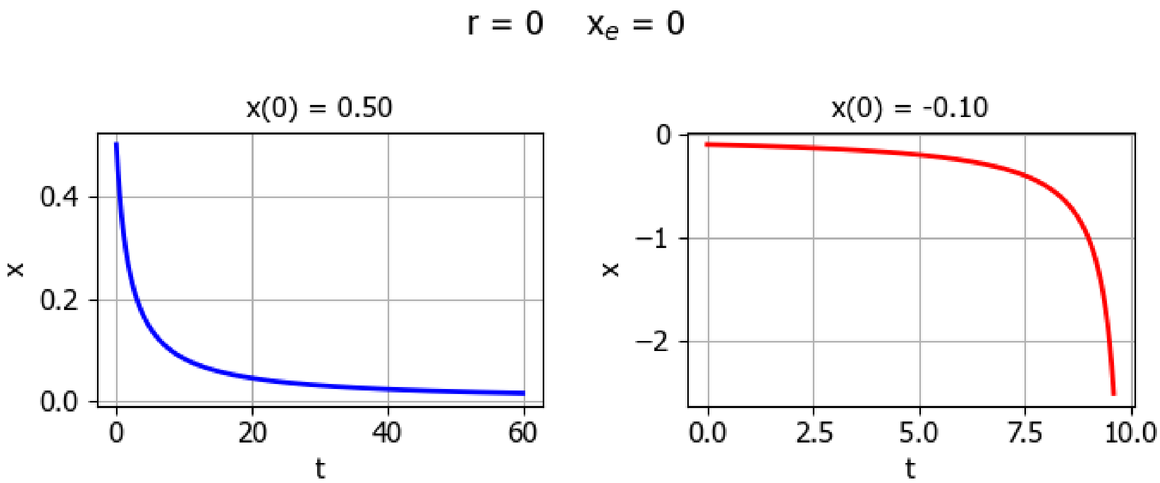
***r* = 0**

$$\dot{x} = -x^2 \Rightarrow x_e = 0 \quad f(x) = -x^2 \quad f'(x_e) = -2x_e = 0 \Rightarrow$$

System has only **one** equilibrium point at $x_e = 0$ and its stability is inconclusive from $f'(x_e) = 0$.

$$x < 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \; x \to -\infty$$
$$x > 0 \Rightarrow \dot{x} < 0 \Rightarrow t \to \infty \; x \to x_e = 0$$

The equilibrium points $x_e = 0$ is an unstable saddle point.



For $r \neq 0$, there are two distinct equilibrium, $x_e = 0$ and $x_e = r$.

***r* < 0**

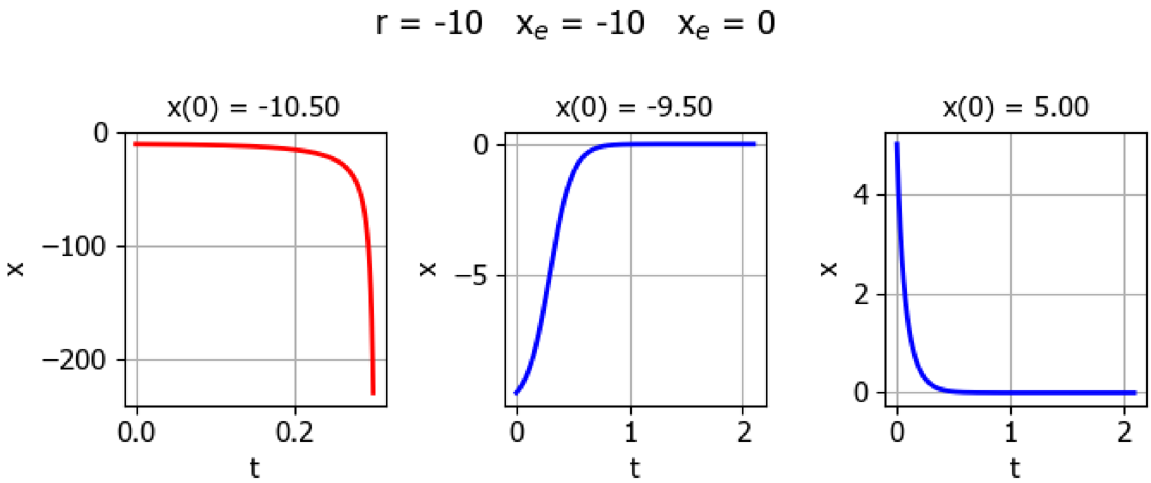
$$\dot{x}\big|_{x_e} = r\,x_e - x_e^{\,2} = 0 \quad f'(x_e) = r - 2x_e$$

$$x_e = 0 \quad f'(x_e) = f'(0) = r < 0 \;\Rightarrow$$

equilibrium point at the Origin $x_e = 0$ is **stable** (**sink**).

$$x_e = r \quad f'(x_e) = f'(r) = -r > 0 \;\Rightarrow$$

equilibrium point $x_e = r$ is **unstable** (**source**).



$r > 0$

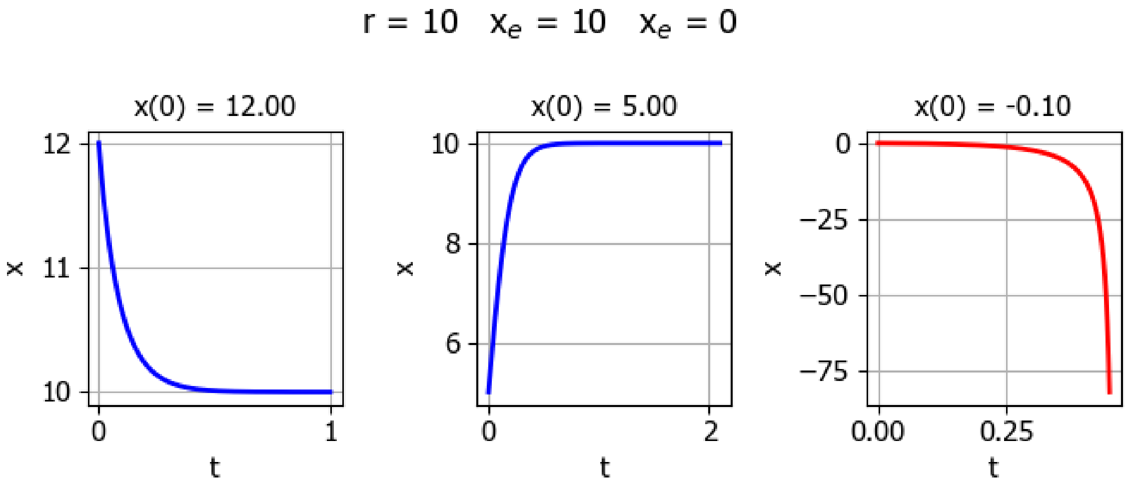
$$\dot{x}\big|_{x_e} = r\,x_e - x_e^{\,2} = 0 \qquad f'(x_e) = r - 2x_e$$

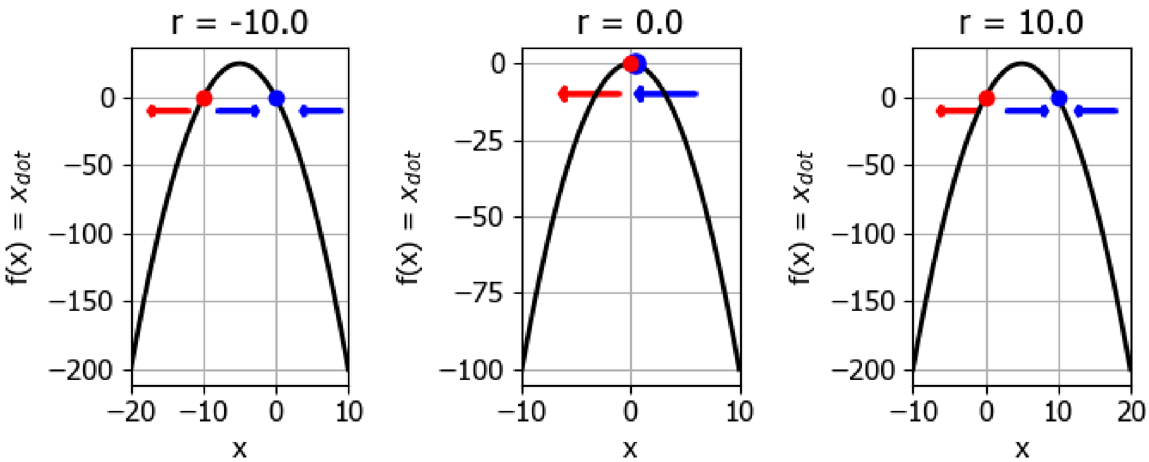
$$x_e = 0 \quad f'(x_e) = f'(0) = r > 0 \;\Rightarrow$$

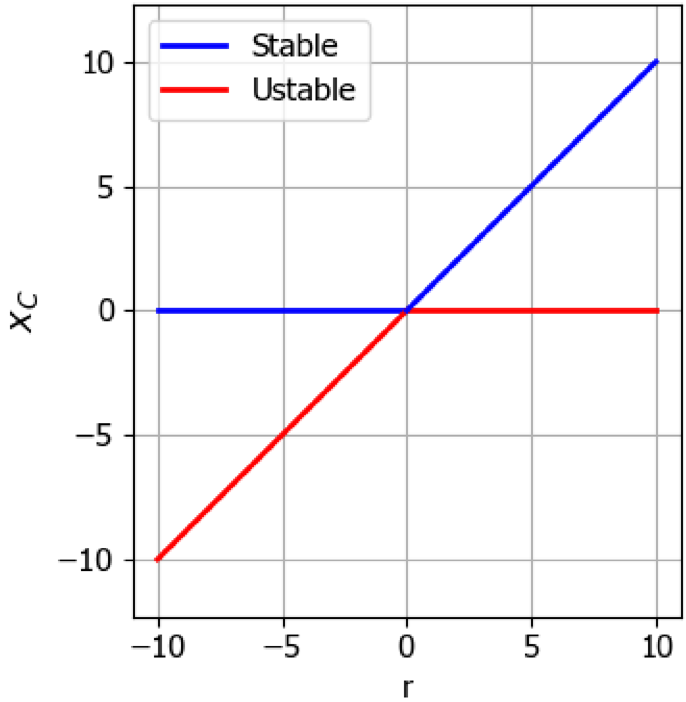
equilibrium point at the Origin $x_e = 0$ is **unstable** (**source**).

$$x_e = r \quad f'(x_e) = f'(r) = -r < 0 \;\Rightarrow$$

equilibrium point $x_e = r$ is **stable** (**sink**).







As r increases from -10 to 0 to 10, the two fixed points move towards each other, at $r = 0$, they merge and then for $r > 0$ they separate again with exchanged stabilities. The transcritical bifurcation point is $r = 0$.

This type of bifurcation diagram is known as a transcritical bifurcation. In this bifurcation, an exchange of stabilities has taken place between the two fixed points of the system.

Example 3 Supercritical pitchfork bifurcation cs103.py

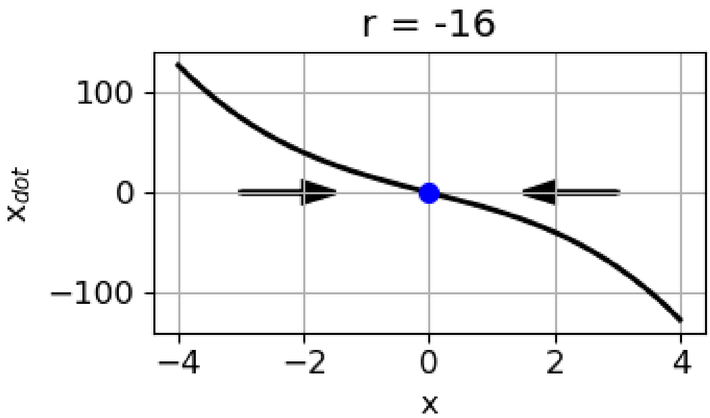
$\dot{x}(t) = r\,x(t) - x(t)^3$ r is an adjustable constant

$f(x,r) = r\,x - x^3 \quad f\,'(x,r) = r - 3x^2$

The system is invariant under the transformation

$$x \to -x \quad r(-x) - (-x)^3 = -\left(rx - x^3\right) = -\dot{x}$$

$r < 0$ one fixed point $x_e = 0$

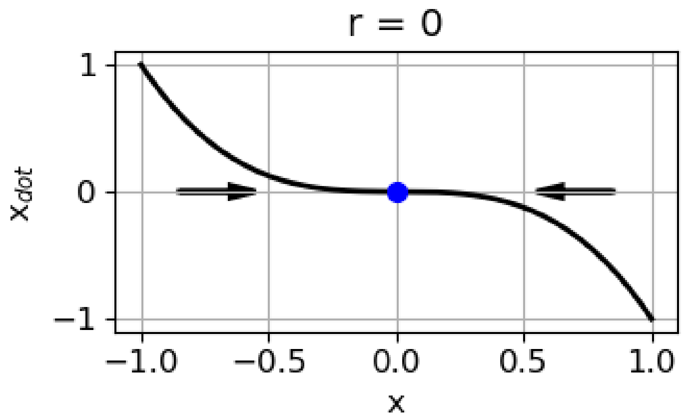


$$r < 0 \quad \dot{x} = 0 \Rightarrow x_e = 0 \quad f\,'(0) < 0 \quad \text{stable}$$

$$x(0) < 0 \quad t \to \infty \quad x(t) \to 0$$

$$x(0) > 0 \quad t \to \infty \quad x(t) \to 0$$

$r = 0$ one stable fixed point $x_e = 0$



$r = 0 \quad \dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = 0 \quad \text{stable}$

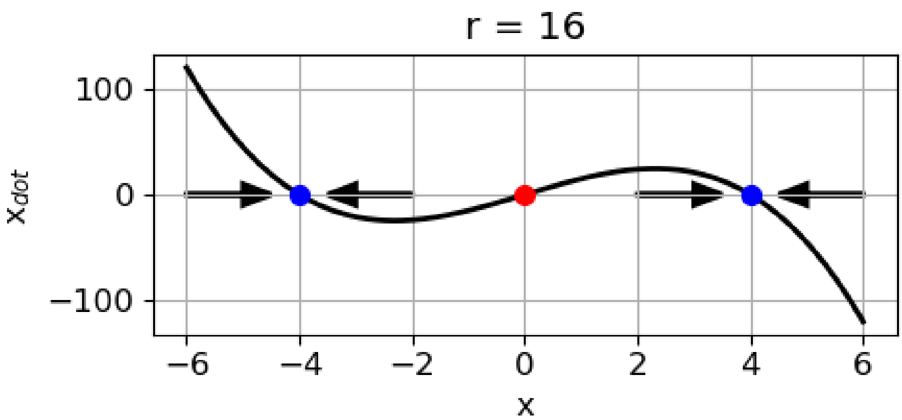
$x(0) < 0 \quad t \to \infty \quad x(t) \to 0$

$x(0) > 0 \quad t \to \infty \quad x(t) \to 0$

***r* > 0** three fixed points

$\dot{x} = 0 \quad x_e = 0 \quad f'(0) = r > 0 \quad \text{unstable}$

$\dot{x} = 0 \quad x_e = \pm\sqrt{r} \quad f'(\pm\sqrt{r}) = -2r < 0 \quad \text{stable}$

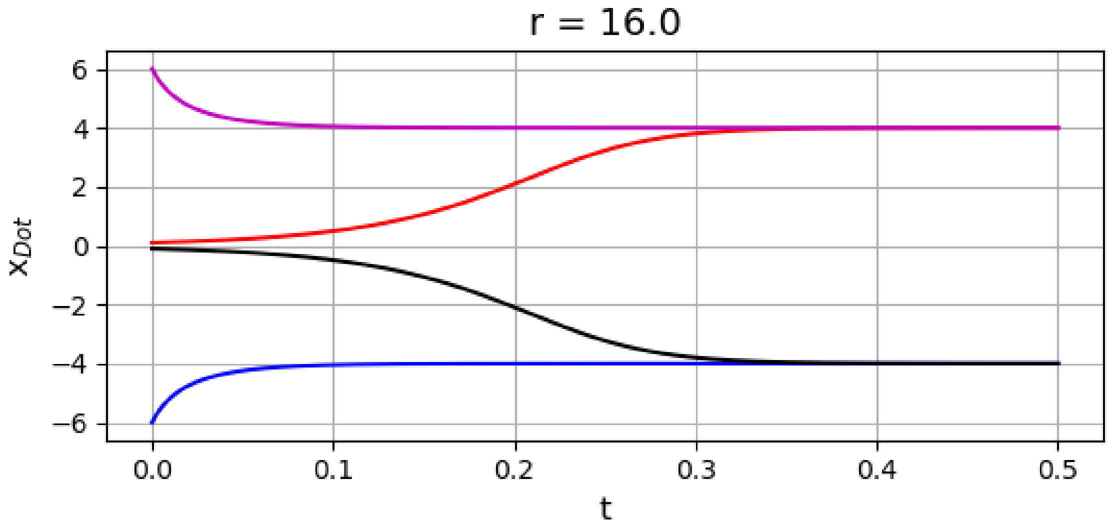


$x(0) < -4 \quad t \to \infty \quad x(t) \to -4$

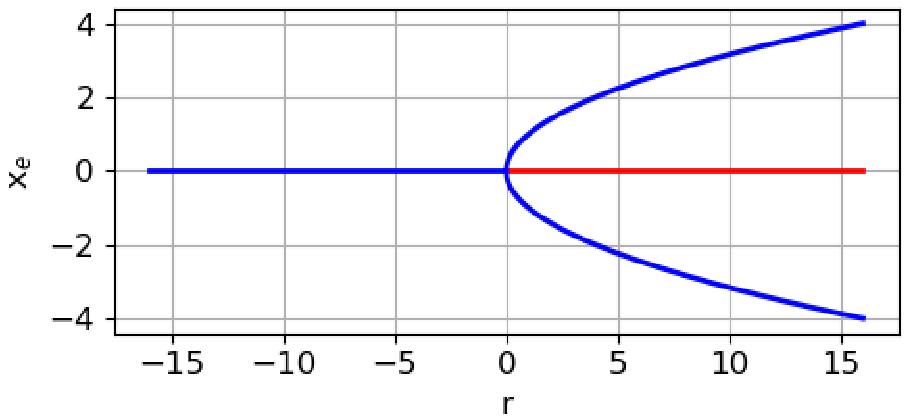
$-4 < x(0) < 0 \quad t \to \infty \quad x(t) \to -4$

$0 < x(0) < 4 \quad t \to \infty \quad x(t) \to +4$

$x(0) > 4 \quad t \to \infty \quad x(t) \to +4$



Time evolution of the flow for different initial conditions when $r = 16.0 > 0$.



Supercritical pitchfork bifurcation

$r = 0$ one fixed point: $x_e = 0$ stable

$r < 0$ one fixed point: $x_e = 0$ stable

$r > 0$ three fixed points: $x_e = 0$ unstable

$$x_e = \pm\sqrt{r} \quad \text{stable}$$

The pitchfork bifurcations occur when one fixed point becomes three at the bifurcation point. Pitchfork bifurcations are usually associated with the physical phenomena called symmetry breaking. For the **supercritical pitchfork bifurcation**, the stability of the original fixed point changes from stable to unstable and a new pair of stable fixed points are created above and below the bifurcation point.

From the pitchfork-shape bifurcation diagram, the name ‘pitchfork’ becomes clear. But it is basically a pitchfork trifurcation of the system. The bifurcation for this vector field is called a supercritical pitchfork bifurcation, in which a stable equilibrium bifurcates into two stable equilibria.

Example 4 Subcritical pitchfork bifurcation cs104.py

The transformation $x \to -x$, gives the subcritical pitchfork bifurcation $\left(\ddot{x} = rx + x^3\right)$ as shown in the following example.

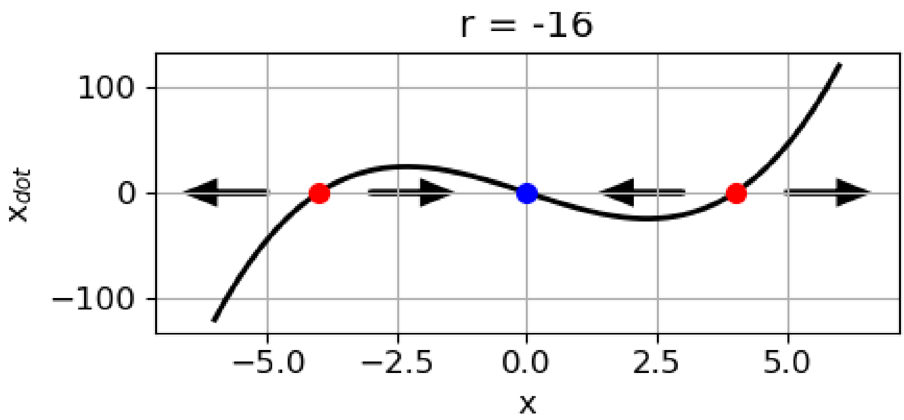
$\dot{x}(t) = r\,x(t) + x(t)^3$ $\quad r$ is an adjustable constant

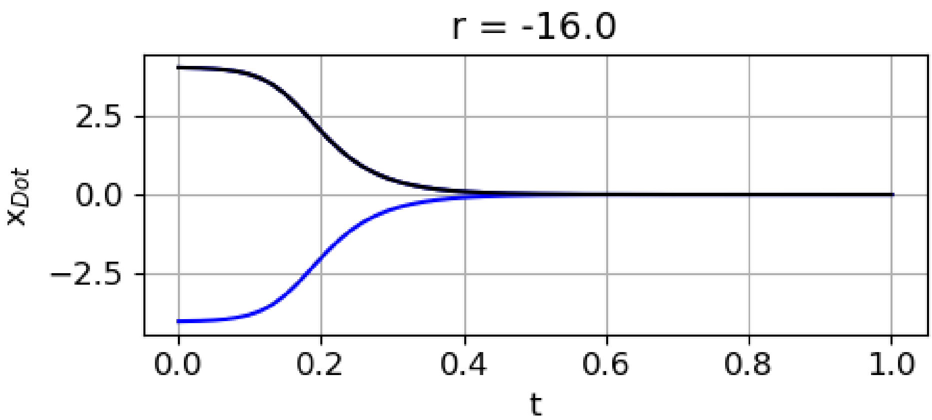
$$f(x) = r\,x + x^3 \qquad f\,'(x) = r + 3\,x^2$$

$r < 0$ three fixed points

$$\dot{x} = 0 \quad x_e = 0 \quad f\,'(0) = r < 0 \quad \textbf{stable}$$

$$r < 0 \quad \dot{x} = 0 \quad x_e = \pm\sqrt{-r} \quad f\,'(\pm\sqrt{-r}) = -2\,r > 0 \quad \textbf{unstable}$$





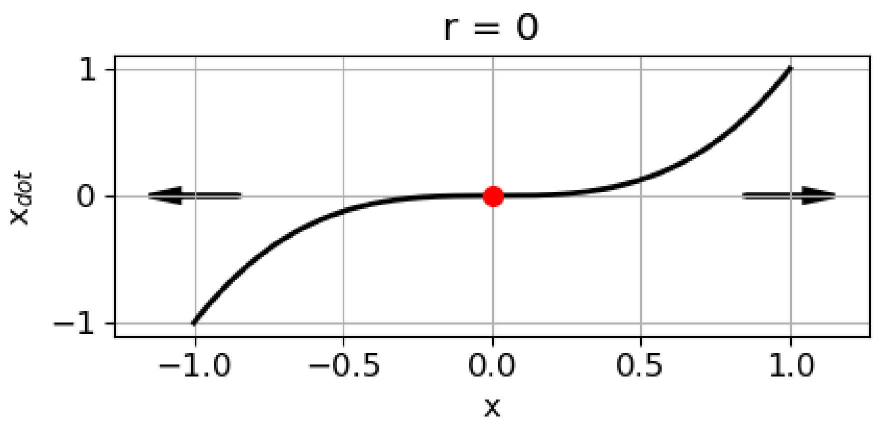
$$x(0) = 3.99 \text{ and } x(0) = -3.99$$

***r* = 0** one fixed point

$$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = 0 \quad \textbf{unstable}$$

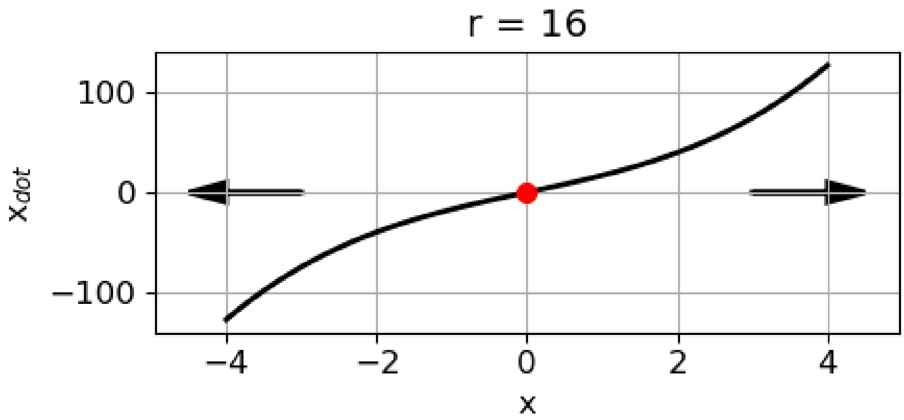
$$x(0) < 0 \quad \dot{x}(0) < 0 \quad t \to \infty \quad x(t) \to -\infty$$

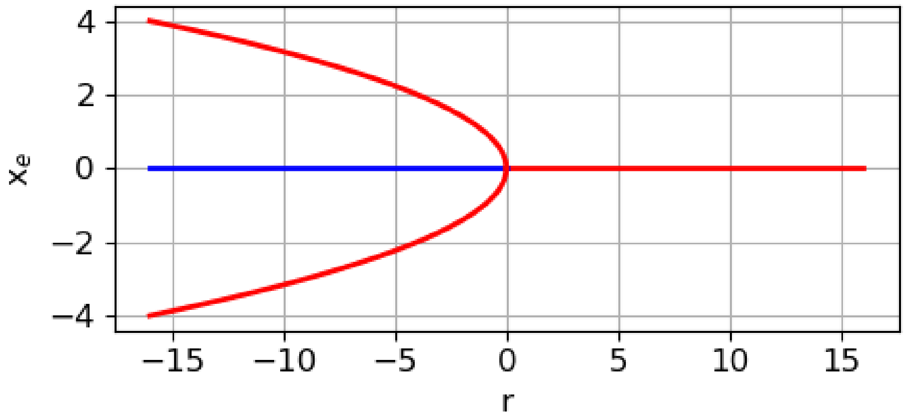
$$x(0) > 0 \quad \dot{x}(0) > 0 \quad t \to \infty \quad x(t) \to +\infty$$



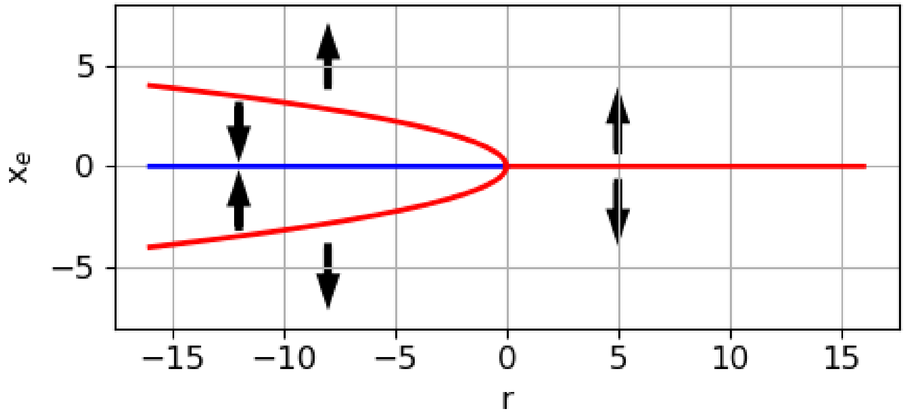
$r > 0$ one fixed point

$$\dot{x} = 0 \Rightarrow x_e = 0 \quad f'(0) = r > 0 \quad \textbf{unstable}$$





Bifurcation diagram: bifurcation parameter is r and the bifurcation point is $r = 0$.



Bifurcation diagram and flow along the line: the flow is aways directed towards a stable fixed point and away from an unstable fixed point.

$x_e = 0$ is unstable for $r \geq 0$ $\quad \leftarrow x_e \rightarrow$

$x_e = 0$ is stable for $r < 0$ $\quad \rightarrow x_e \leftarrow$

$x_e \neq 0$ is unstable for r < 0 $\quad \leftarrow x_e \rightarrow$

Example 5 $\dot{x}(t) = r\,x(t) + x(t)^3 - x(t)^5$ **cs105.py**

$\dot{x} = r\,x + x^3 - x^5$ r is an adjustable constant

$f(x) = r\,x + x^3 - x^5 \quad f\,'(x) = r + 3x^2 - 5x^4$

$$\dot{x} = 0 \quad \Rightarrow \quad x_e\left(r + {x_e}^2 - {x_e}^4\right) = 0$$

$$x_e = 0 \quad -{x_e}^4 + {x_e}^2 + r = 0$$

$$+z^2 - z - r = 0 \qquad z = {x_e}^2$$

$$z = \tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)$$

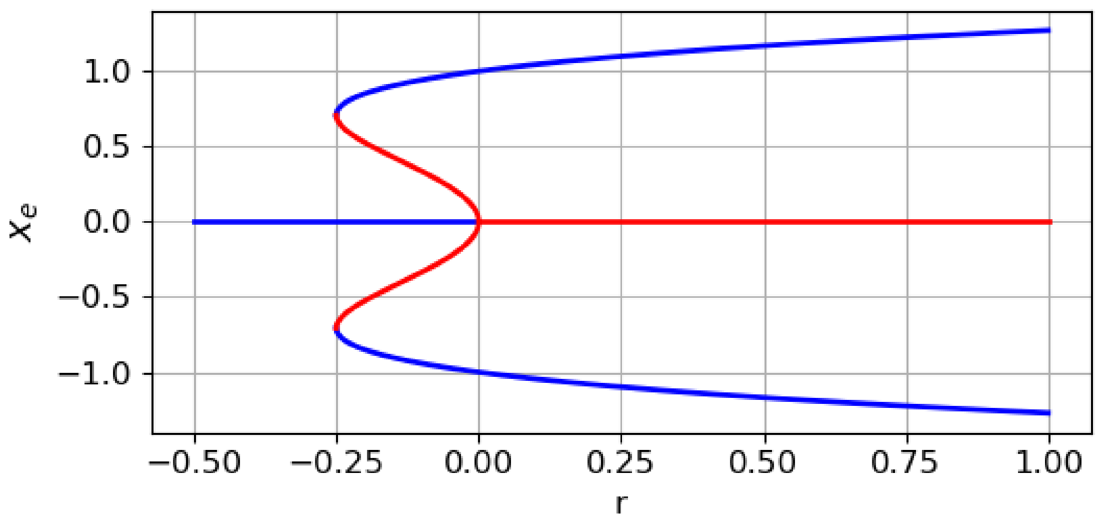
$$x_e = \pm\sqrt{\tfrac{1}{2}\left(1 \pm \sqrt{1+4r}\right)}$$

$$f\,'(x_e) = r + 3{x_e}^2 - 5{x_e}^4$$

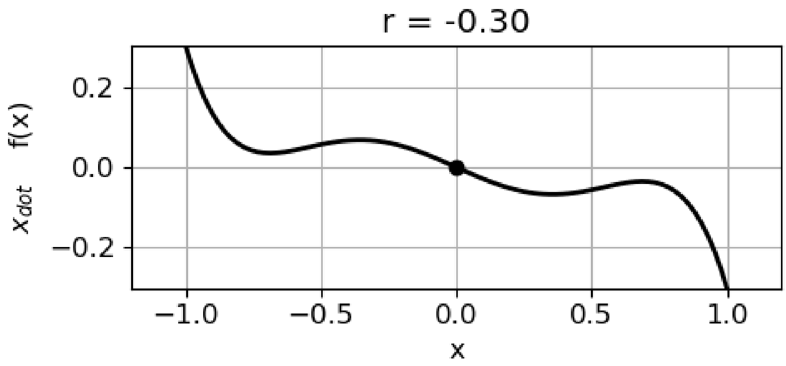
The bifurcation diagram shown below has in addition to a subcritical pitchfork bifurcation at the Origin, and two symmetric saddle node bifurcations that occur when $r = -1/4$. We can imagine what happens to the solution x(t) as r increases from negative values, assuming there is some noise in the system so that $x(t)$ fluctuates around a stable fixed point. For $r < -1/4$, the solution $x(t)$ fluctuates around the stable fixed point $x_e = 0$. As r increases into the range $-1/4 < r < 0$, the solution will remain close to the stable fixed point $x_e = 0$. However, a catastrophic event occurs as soon as $r > 0$. The fixed point $x_e = 0$ is lost and the solution will jump up or down to one of the fixed points. A similar catastrophe can happen as r decreases from positive values. In this case, the jump occurs as soon as $r < -1/4$

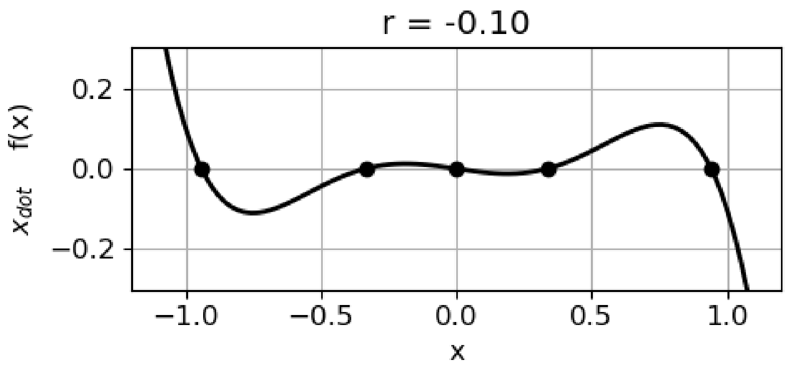
Since the behaviour of $x(t)$ is different depending on whether we increase or decrease r, we say that the system exhibits **hysteresis**.

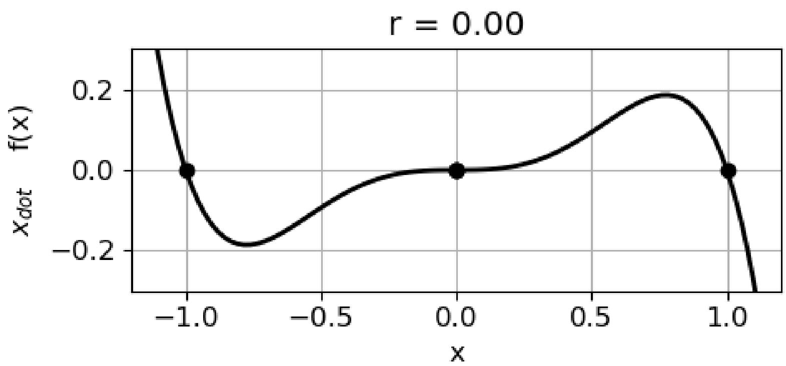
The existence of a subcritical pitchfork bifurcation can be very dangerous in engineering applications since a small change in the physical parameters of a problem can result in a large change in the equilibrium state. Physically, this can result in the collapse of a structure.

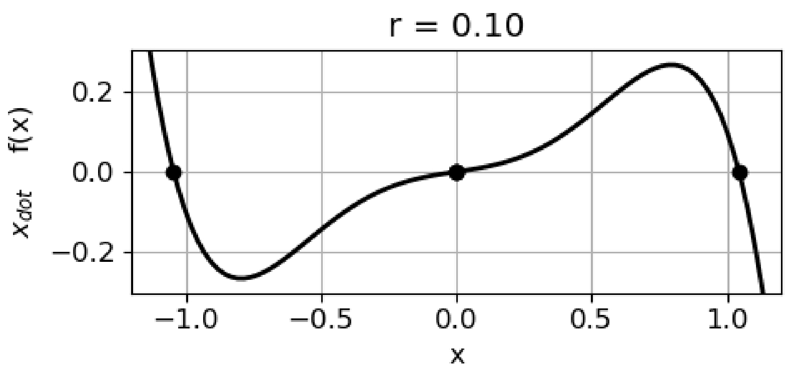


Subcritical pitchfork bifurcation at the Origin, and two symmetric saddle node bifurcations that occur when $r = -1/4$. In a local neighbourhood, the flow is always towards a stable fixed point and away from an unstable fixed point.

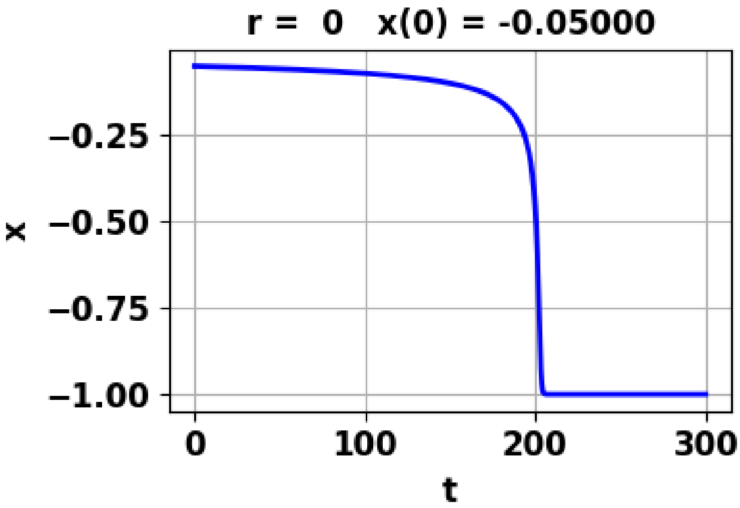


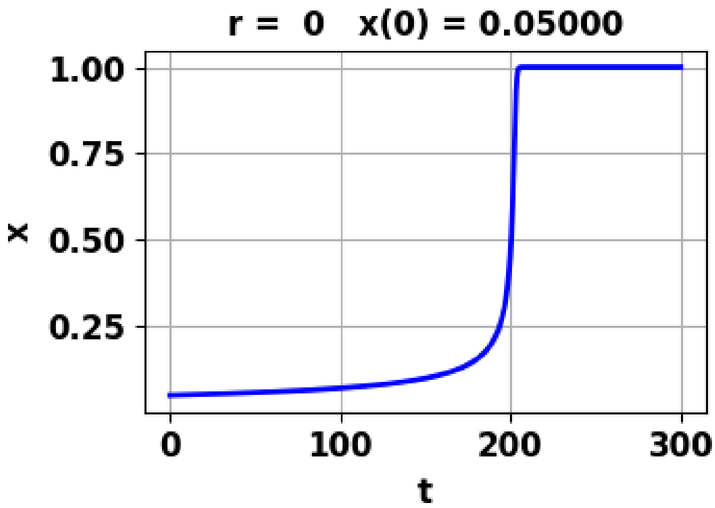




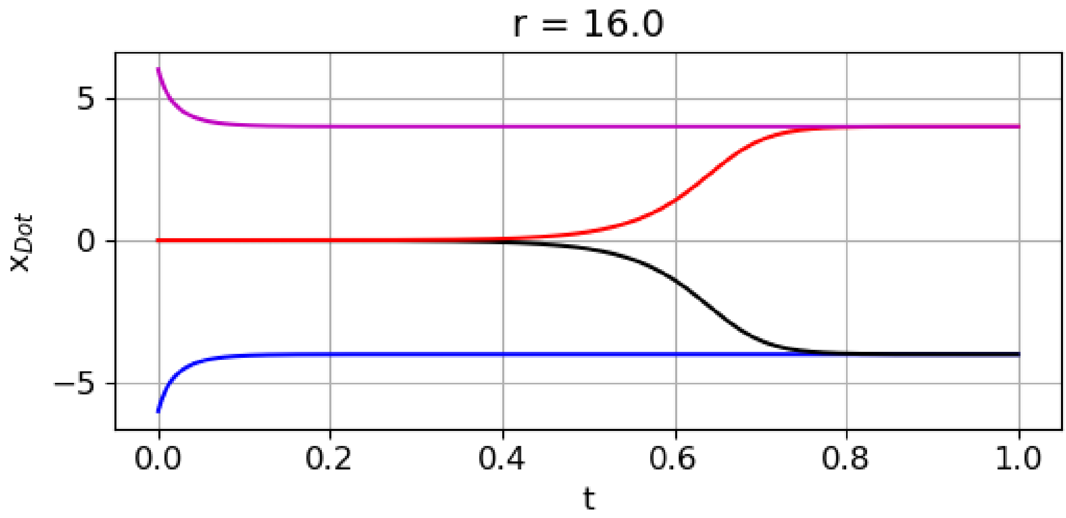


At a **fixed point**: negative slope → **stable**, positive slope → **unstable**





Slight differences in the initial conditions can lead to dramatic differences in the time evolution of the flow and steady state value for x.



$x(0) = + 6$ $x(0) = - 6$ $x(0) = + 0.00001$ $x(0) = - 0.00001$

You see that our system is extremely sensitive to the initial conditions. Although the system is deterministic, the system is not completely predictable for initial conditions near $x_e = 0$. In this instance, you cannot make useful predictions since unmeasurable differences in the initial conditions lead to dramatically different outcomes.

$\Rightarrow$ butterfly effect

The idea that a mathematical equation gave you the power to predict how a system will behave is ***dead – end of the Newtonian dream***.

Example 6 POPULATION GROWTH **cs106.py**

A simple mathematical model for the dynamics of a population is

$$\dot{N} = r\,N\left(1 - \frac{N}{k}\right)$$ **logistic equation**

where N is the population, $\dot{N}$ is the rate of change of the population (growth rate), r is a positive constant, and k is the carrying population (equilibrium population). For the logistic equation, $\dot{N} / N$ is linearly related to the population N.

The equilibrium points of the system are

$$N_e = 0 \quad N_e = k$$

To check the stability of the equilibrium, we need to consider the function

$$f\,' = df \,/\, dN\big|_{N_e} = r\left(1 - (2/k)N_e\right)$$

$N_e = 0 \quad f\,' = r > 0 \Rightarrow$ **unstable**

$N_e = k \quad f\,' = -r < 0 \Rightarrow$ **stable**

For small N, the growth rate equals r, and the population increases exponentially. For populations larger than the carrying capacity k, the growth rate becomes negative (death rate greater than birth rate).

The population N is always positive ($N > 0$), since it makes no sense to think about a negative population and if $N(0) = 0$ then there's nobody around to start reproducing, and so the population would be zero for all time ($N(t) = 0$). $N_e \sim 0$ is an unstable fixed point, so a small population will grow exponentially away from $N \sim 0$. $N = k$ is a stable fixed point, thus, if N is disturbed slightly from k, the disturbance will decay monotonically back to k.

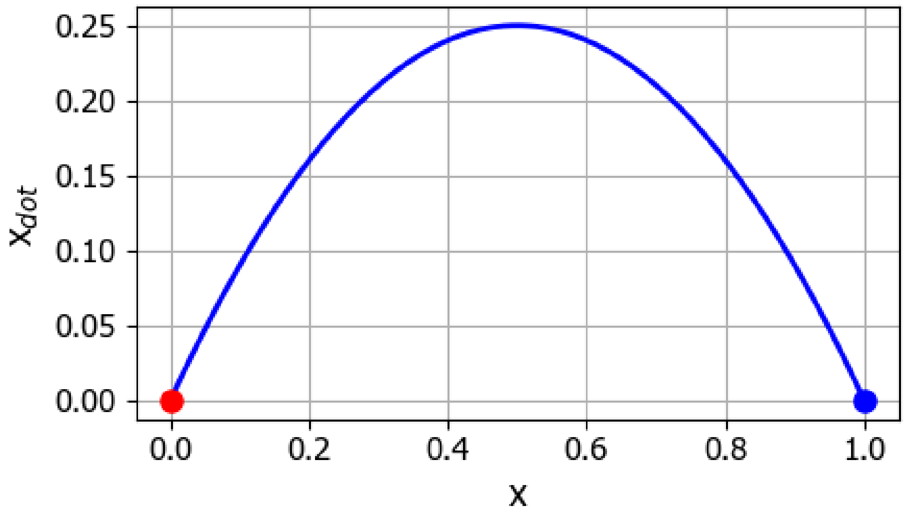


Fig. 6.1. Plot of the logical equation.

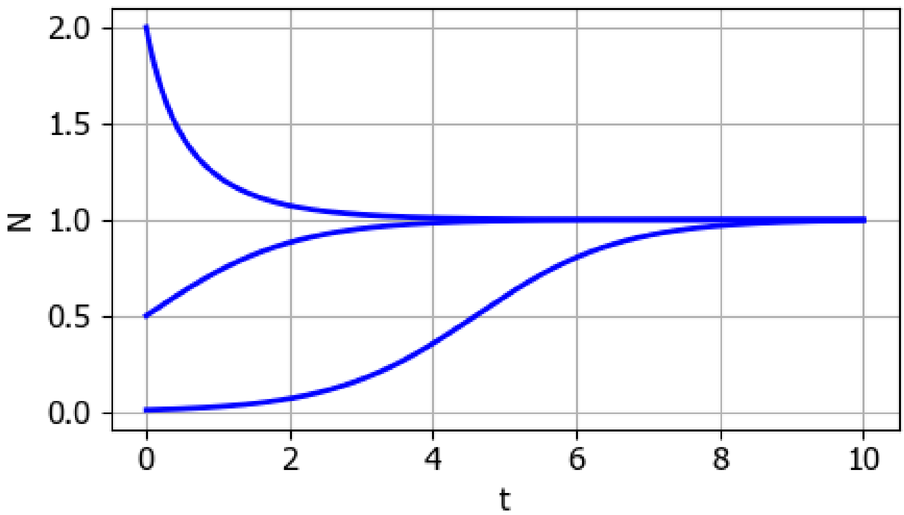


Fig. 6.2. Time evolution of the population for three initial conditions. The three populations converge to the population capacity, k.

The logistic equation has been tested in laboratory experiments in which colonies of bacteria, yeast, or other simple organisms were grown in conditions of constant climate, food supply, and absence of predators. These experiments often yielded growth curves with an impressive match to the logistic predictions. However, for organisms with more complicated lifecycles, the agreement between experiment and predictions was not so good.

Another way to explore [1D] dynamical systems is to plot the **slope field** for the system in the (t, x) plane (figure 6.3). The equation $\dot{x} = x(1-x)$ with $r = 1$ and $k = 1$, can be interpreted in a new way: for each point (t, x), the equation gives the slope dx / dt of the solution passing through that point. The slope field can be show using a quiver plot (figure 6.3) or a streamplot (figure 6.4). Then, finding a solution now becomes a problem of drawing a curve that is always tangent to the local slope.

Fixed points dominate the dynamics of first-order systems. In all our examples, all trajectories either approached a fixed point, or diverged to $\pm\infty$. These are the only things that can happen for a vector field on the real line [1D]. The reason is that trajectories are forced to increase or decrease monotonically, or remain constant. To put it more geometrically, the phase point never reverses. Hence the **impossibility of oscillations.**

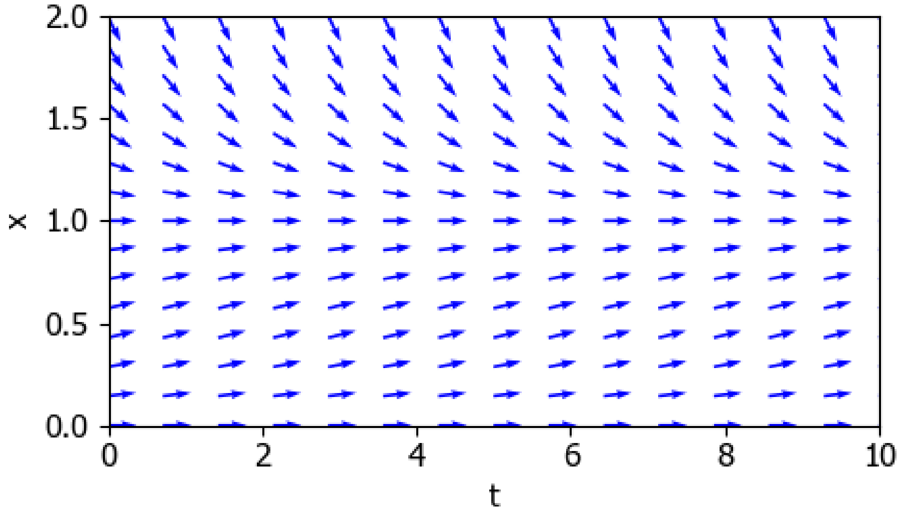


Fig. 6.3. Slope field quiver plot (normalized so that all arrows have unit length).

The slope is given by dY/dX for dY is the function for $\dot{x}$ and dX is set to 1.

```
N = 15; t = linspace(0,10,N); x = linspace(0,2,N)
f = x*(1-x)
T,X = np.meshgrid(t,x)
dX = np.ones([N,N])
F = X*(1-X)
dY = F/(np.sqrt(dX**2 + F**2))
dX = dX/(np.sqrt(dX**2 + F**2))
```

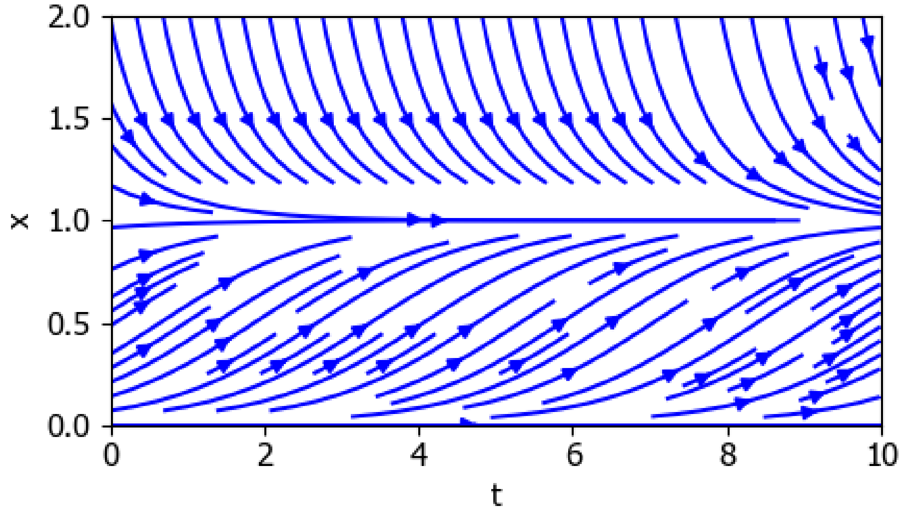


Fig. 6.4. Slope field streamplot.

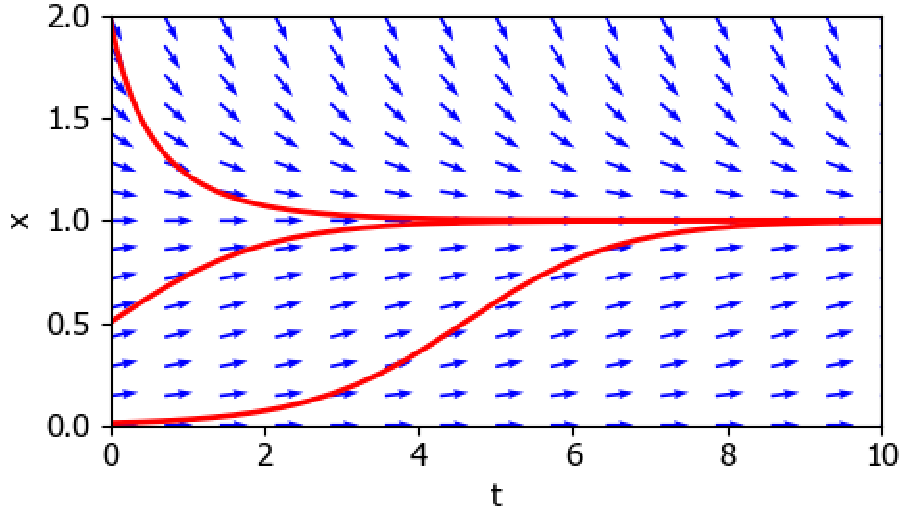


Fig. 6.5. Slope field and three trajectories with different initial conditions.