# DOING PHYSICS WITH PYTHON

# COMPUTATIONAL OPTICS

# GAUSSIAN BEAMS

# (PARAXIAL REGIME)

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## DOWNLOAD DIRECTORIES FOR PYTHON CODE

**Google drive**

**GitHub**

**emGB01.py**

The Code **emGB01.py** is used to model the behaviour of a Gaussian beam propagating in the +Z direction with spherical wavefronts in the paraxial region. The beam is specified by its wavelength (visible part of the electromagnetic spectrum), power and the value of its waist.

# INTRODUCTION

Lasers are widely used in many fields today. Having a mathematical description of laser light is essential. The light emitted from a laser is composed of a narrow band of wavelengths, such that we can assume the light is monochromatic. The light emitted from the laser is usually collimated as it propagates in a straight line as a narrow beam. As a starting point, we will assume the beam to be unpolarized and the light intensity to have a Gaussian profile in any plane that is normal to the direction of propagation. This mode is referred to as the **$TEM_{00}$** mode and it describes the output of most lasers.

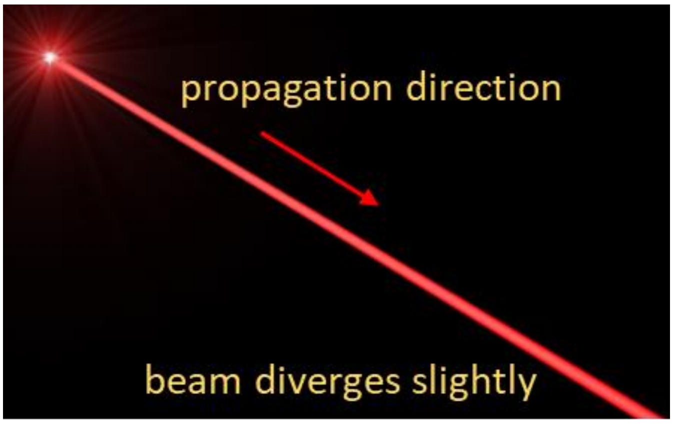


Fig. 1. Red light emitted from a HeNe laser. The light is seen due to scattering. The light propagates as a collimated beam in a straight line with slight divergence in its cross-section.

## MATHEMATICS OF A GAUSSIAN BEAM

The light emitted from a laser can be modelled by solving the [3D] scalar wave equation for the electric field $E(x,y,z,t)$

(1) $$\nabla^2 E(x,y,z,t) - \frac{1}{c^2}\frac{\partial^2 E(x,y,z,t)}{\partial t^2} = 0$$

We will assume that a monochromatic beam with a circular cross-section propagates in free spaces and that the electric field can be expressed as a product of its spatial part and time dependent part

(2)

$$E(x,y,z,t) = E_S(x,y,z)\, e^{-i\omega t} \qquad c = f\,\lambda = \omega / k \quad \omega = 2\pi\, f \quad k = 2\pi / \lambda$$

Substitution of equation 2 into equation 1 gives the **Helmholtz equation**

(3) $$\nabla^2 E_S(x,y,z) + k^2\, E_S(x,y,z) = 0$$

We will only consider the solution for a wave propagating in the Z direction where

(4) $$E_S(x,y,z) = E_{XY}(x,y,z)\, e^{ikz}$$

The term $e^{ikz}$ accounts for the wave oscillation along the propagation direction.

Substitution of equation 4 into equation 3 gives

(5) $$\frac{\partial^2 E_{XY}}{\partial x^2}+\frac{\partial^2 E_{XY}}{\partial y^2}+\frac{\partial^2 E_{XY}}{\partial z^2}+2ik\frac{\partial E_{XY}}{\partial x}=0$$

We will assume that there is a slow decrease in the amplitude of the wave as the wave propagates in the Z direction and only consider the wave far from the origin but close to the Z axis such that $x \ll z$ and $y \ll z$. Thus, we can say that $E_{XY}(x,y,z)$ varies slowly with $\mathbf{z}$, and thus we can neglect the term $\frac{\partial^2 E_{XY}}{\partial z^2}$. Therefore, the wave equation can be expressed as

(6) $$\frac{\partial^2 E_{XY}}{\partial x^2}+\frac{\partial^2 E_{XY}}{\partial y^2}+2ik\frac{\partial E_{XY}}{\partial x}=0$$

Without going into all the mathematical details, a solution to paraxial wave equation is

$$E(x,y,z,t) = E_0\left(\frac{w_0}{w}\right)\exp\left(-\frac{x^2+y^2}{w^2}\right) \exp\left(i(kz-\omega t)\right)\exp\left(ik\left(\frac{x^2+y^2}{2R}\right)\right)\exp(-i\phi) \tag{7}$$

Equation 7 us known as the **paraxial wave equation**.

$$E(x,y,z,t) = \overbrace{E_0\left(\frac{w_0}{w}\right)\underbrace{\exp\left(-\frac{x^2+y^2}{w^2}\right)}_{\text{Gaussian profile}}}^{\text{amplitude terms}} \overbrace{\underbrace{\exp\left(i(kz-\omega t)\right)}_{\text{unidirectional wave term}}\underbrace{\exp\left(ik\left(\frac{x^2+y^2}{2R}\right)\right)}_{\text{curvature of wavefront}}\exp(-i\phi)}^{\text{phase terms}}$$

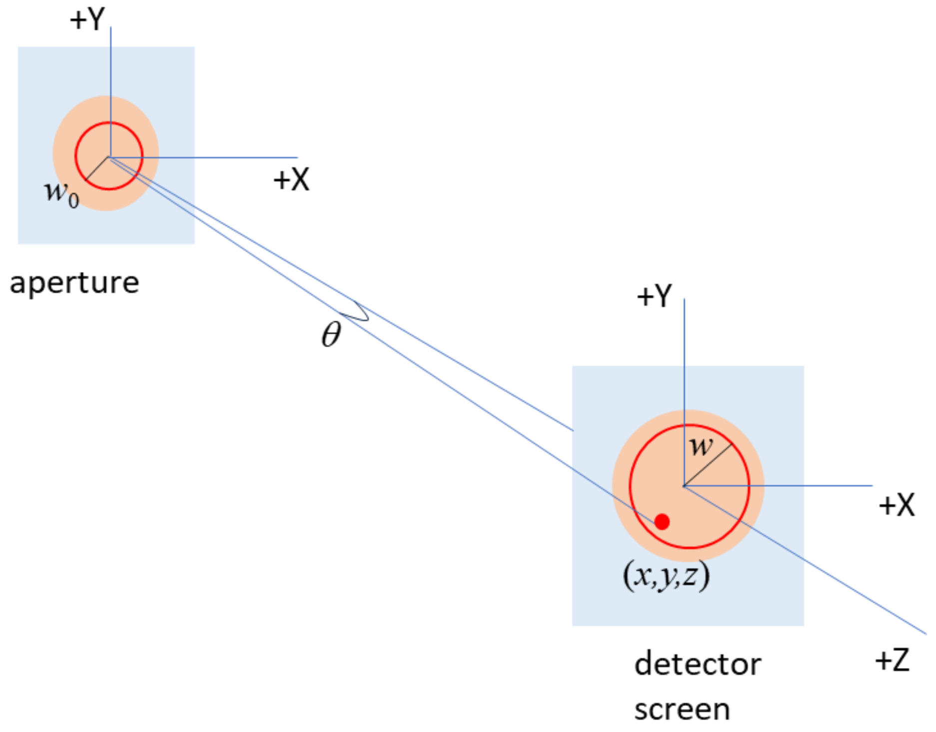


Fig. 1 Geometry for propagation of Gaussian bean.

$E_0$ is the **amplitude** of the wave

$w(z)$ **beam spot** and $w_0$ is the **beam waist**

$w(z)$ is the radius of the beam at position $z$ at which the amplitude has decreased by the factor $1/e$ from its axial value or where the irradiance is $1/e^2$ of its axial value. At position $z$ along the axis, the beam spot $w(z)$ is given by

(8)

$$w(z) = w_0 \sqrt{1 + \left(\frac{\lambda\, z}{\pi\, w_0}\right)^2} = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \qquad w(0) = w_0$$

where $z_R$ is the **Rayleigh range**

(9) $$z_R = \pi\, w_0^{\;2} / \lambda$$

The Raleigh range is an indicator of the divergence of the beam.

(10) $$R(z) = z + z_R^{\;2} / z = z\left[1 + \left(\frac{\pi\, w_0}{\lambda\, z}\right)^2\right]$$

$R(z)$ is the **radius of curvature of the wavefront** (surface that contains all points of the wave that have the same phase) at position $z$. As the beam propagates in the +Z direction, the width of the beam expands, and this is measured by the beam spot as it increases with $z$. For the expanding beam, its wavefront must have a spherical shape because a wave always propagates in a direction perpendicular to its wavefront.

$\phi(z)$ **Gouy phase**.

(11) $$\phi(z)=\tan^{-1}\left(\frac{z}{z_R}\right)$$

The Gouy phase slightly shifts the phase of the wavefront of the wave as a whole.

$\theta$ **divergence angle of the beam**

(12) $$\theta=\frac{\lambda}{\pi\, w_0} \qquad z>>z_R$$

When $z<z_R$ the ray description of the propagation of the light breaks down as the beam spot slowly increases. When $z\gg z_R$, the beam spot expands linearly as

(13)

$$z\gg z_R \quad\Rightarrow\quad w(z)=z\left(w_0\;/\;z_R\right)=z\;\theta \qquad \theta=w_0\;/\;z_R \qquad {w_0}^2=\frac{z_R\,\lambda}{\pi}$$

where $\theta$ is the limiting value of the divergence. Figure 2 shows the divergence of the beam for $w(z)=z\,\theta$.

Diffraction causes to spread as it propagates, and therefore it is impossible to have a perfectly collimated beam.

The paraxial approximation is only valid when $w_0 \gg \lambda$ and is not accurate for strongly diverging beams.

**ENERGY: Irradiance (Intensity) $S(r, z)$ and power $P(z)$**

The **irradiance** (intensity) $S$ is the time average flow of energy per unit time per unit area [W.m$^{-2}$]. The irradiance $S$ can be calculated from the electric field

(14) $$S = \left(\frac{c\,\varepsilon_0}{2}\right)|E|^2$$

The symbol $S$ is used for the irradiance in the Code and notes and not the more commonly used letter $I$. $c$ is the speed of light and $\varepsilon_0$ is the permittivity of free space.

For our propagating Gaussian beam, the electric field $E$ is given by equation 7, so the irradiance $S$ is

(15A) $$S(x, y, z) = E_0^{\,2}\left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 \exp\left[-2\left(\frac{x^2+y^2}{w^2}\right)\right]$$

or

(15B)

$$S(r,\, z) = E_0^{\,2}\left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 \exp\left[-2\left(\frac{r^2}{w^2}\right)\right] \qquad r = \sqrt{x^2+y^2}$$

The maximum irradiance occurs at the location $(r = 0,\ z = 0)$ where

(15C) $$S_{\max} = \left(\frac{c\,\varepsilon_0}{2}\right) E_0{}^2$$

The variation of the intensity along the Z axis $(r = 0)$ is given by

(15D)

$$S_z(z) = S(0,\, z) = E_0{}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 = E_0{}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{1}{1 + z^2 / z_R{}^2}\right)$$

The power $P$ transmitted through a circular disk of radius $r_P$ in an XY plane at position $z_P$ is

(16) $$P(z) = \int_0^{r_P} S(z, r)\,(2\pi\, r)\,dr$$

$$S(r, z) = E_0{}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 \exp\left[-2\left(\frac{r^2}{w^2}\right)\right] \qquad r = \sqrt{x^2 + y^2}$$

$$P(z) = \int_0^{r_P} E_0{}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 \exp\left[-2\left(\frac{r^2}{w^2}\right)\right](2\pi\, r)\,dr$$

$$P(z) = E_0{}^2\,(\pi c\,\varepsilon_0)\left(\frac{w_0}{w}\right)^2 \int_0^{r_P} r \exp\left(\frac{-2r^2}{w^2}\right) dr$$

$$P(z) = E_0{}^2\,(\pi c\,\varepsilon_0)\left(\frac{w_0}{w}\right)^2 \left(-\frac{w^2}{4}\right)\left[\exp\left(\frac{-2r^2}{w^2}\right)\right]_0^{r_P}$$

$$P(z) = E_0^{\ 2}\left(\frac{\pi c\,\varepsilon_0\,w_0^{\ 2}}{4}\right)\left(1 - \exp\left(\frac{-2\,r_P^{\ 2}}{w^2}\right)\right)$$

(17) $$P(z) = P_0\left(1 - \exp\left(\frac{-2\,r_P^{\ 2}}{w^2}\right)\right)$$

When $r_P \to \infty$ then $P(z) \to P_0$

(18) $$P_0 = E_0^{\ 2}\left(\frac{\pi c\,\varepsilon_0\,w_0^{\ 2}}{4}\right)$$

where $P_0$ is the total power transmitted by the beam. If we know the total power $P_0$ transmitted by the beam, then we can calculate the electric field amplitude $E_0$ and the maximum irradiance of the beam $S_0$

(19) $$E_0 = \sqrt{\frac{4\,P_0}{\pi\,c\,\varepsilon_0\,w_0^{\ 2}}}$$

(20) $$S_0 = \frac{2\,P_0}{\pi\,w_0^{\ 2}}$$

## SIMULATIONS

The Python Code **emGB01.py** can be used for visualization of a Gaussian beam (TEM00 mode) propagating in the +Z direction.

The width of the laser beam is specified by the size of the beam spot as given by equation (8) and the beam spot as a function of $z$ is shown figure 2.

(8)

$$w(z) = w_0 \sqrt{1 + \left(\frac{\lambda z}{\pi w_0}\right)^2} = w_0 \sqrt{1 + \left(\frac{z}{z_R}\right)^2} \qquad w(0) = w_0$$

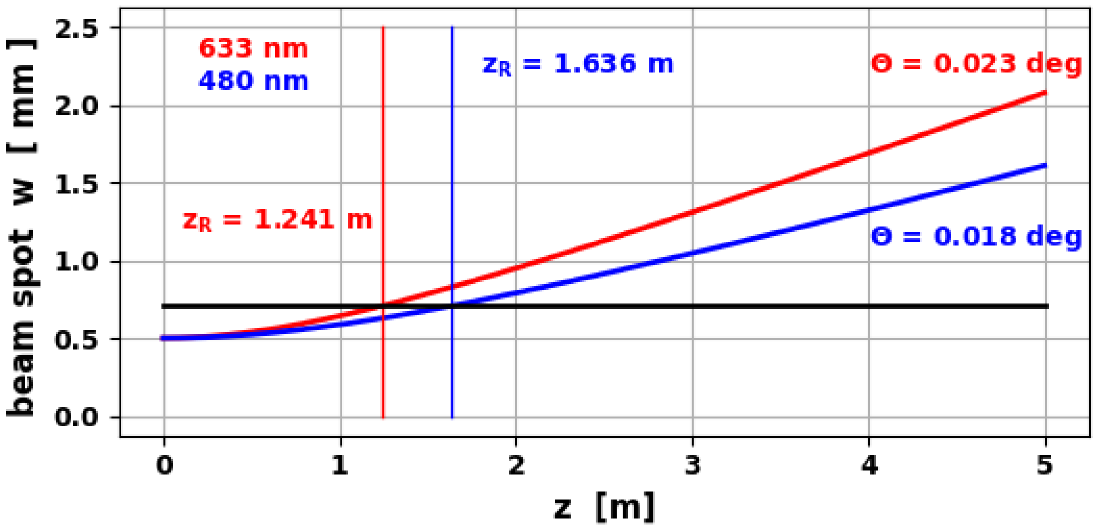


Fig. 2. Beam spot $w$ as a function of position $z$. The Rayleigh range $z_R$ is the $z$ value when $w = \sqrt{2}\, w_0$ as shown by the horizontal black where $w$ = 0.707 mm and the vertical lines $z_R$ = 1.636 m and $z_R$ = 1.241 m.

The longer the wavelength of the light, then the more rapidly the beam diverges

$$\theta = \frac{\lambda}{\pi w_0} \qquad z >> z_R$$

When $z$ = 0, we have $w(z=0) = w_0$ where $w_0$ is the beam waist. This the smallest value that $w(z)$ can have. Thus, $z$ = 0 is a special point in the propagation of the beam. For $z$ > 0 the beam spot increases. Also, in the negative Z direction the beam spot would also increase. When the beam is focused by a lens there is always a minimum width of the beam at the focal point such that the beam spot is equal to the value of the waist $w(z) = w_0$. It is important to note that the Raleigh range is an indicator of the divergence of the beam. When $z = z_R$ $\left(z / z_R = 1\right)$ we have $w(z = z_R) = \sqrt{2}\, w_0$ . This is a turning point in the propagation of the light as the beam spot makes the transition from being nearly constant to increasing linearly as shown in figure 2. The Rayleigh range is inversely proportional to the wavelength $\left(z_R \propto 1/\lambda\right)$, hence the longer the wavelength the smaller the value its value as shown in figure 2.

Note: the smaller the value of $z_R$ the quicker the beam will expand in a linear manner and the smaller the value of the waist, the larger the divergence angle of the beam. Because the divergence is inversely proportional to the spot size, for a given wavelength $\lambda$, a

Gaussian beam that is focused to a small spot diverges rapidly as it propagates away from the focus. Conversely, to minimize the divergence of a laser beam in the far field (and increase its peak intensity at large distances) it must have a large cross-section at the waist $w_0$ and thus a large diameter where it is launched, since $w(z) \geq w_0$. This relationship between beam width and divergence is a fundamental characteristic of diffraction, and of the Fourier transform which describes Fraunhofer diffraction. Since the Gaussian beam model uses the paraxial approximation, it fails when wavefronts are tilted by more than about 30° from the axis of the beam. From the above expression for divergence, this means the Gaussian beam model is only accurate for beams with waists where $w_0 >\sim 6\lambda / \pi^2$.

The shape of a Gaussian beam of a given wavelength $\lambda$ is governed solely by one parameter, the **beam waist** $w_0$. This is a measure of the beam size at the point of its focus ($z = 0$) and corresponds to the smallest value of the beam spot parameter. From value of the waist $w_0$, other parameters describing the beam geometry are determined, such as, the Rayleigh range $z_R$ and asymptotic beam divergence $\theta$.

## Radius of curvature of the wavefront $R(z)$

(10) $$R(z) = z + z_R^{\,2} / z$$

The phase term in equation (7)

$$\exp\left(i\,k\left(\frac{x^2+y^2}{2R}\right)\right) = \exp\left(i\,k\left(\frac{r^2}{2R}\right)\right) \qquad r^2 = x^2 + y^2$$

gives the curvature of the wavefront which is spherical with radius $R(z)$. Figure 3 shows the variation in the radius of curvature $R(z)$ as a function of $z$.

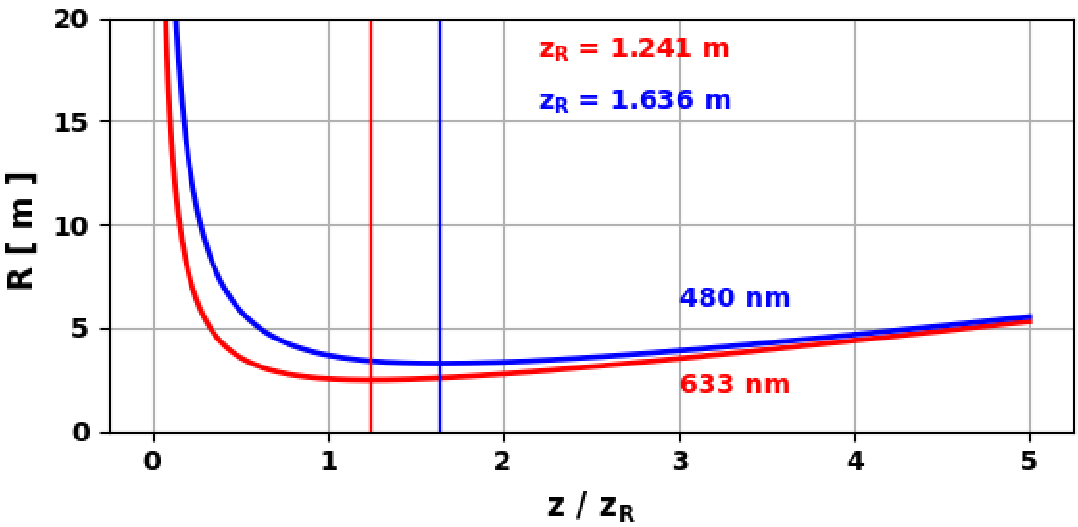


Fig.3. Radius of curvature of the wavefront as a function of $z$.
$\lambda = 633\ \text{nm}$ $\lambda = 480\ \text{nm}$

When $z$ = 0 we have $R(z=0) = \infty$, the wavefront is a plane wave and at this point all parts of the wave are moving in the same direction. When $z \gg z_R$, then $R(z) \approx z$. The wavefront is a nearly a spherical surface traveling away from $z$ = 0. This is the geometric optics limit. When we focus a beam of light, we expect the light rays to go in straight lines to and from the focal point ($z$ = 0). This is correct if $|z| \gg z_R$. For $|z| < z_R$, the wave aspect of the light applies. When $z = z_R$ then $R(z = zR) = 2\,z_R$ which is the minimum value of the radius of curvature and corresponds to the turning point between ray-optics and wave-optics.

## Axial irradiance

The variation of the intensity along the Z axis $(r = 0)$ is given by

(15D)

$$S_z(z) = S(0,\, z) = {E_0}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 = {E_0}^2 \left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{1}{1 + z^2 / {z_R}^2}\right)$$

Figure 4 shows a plot of the axial irradiance $S_z$ as a function of $z$.

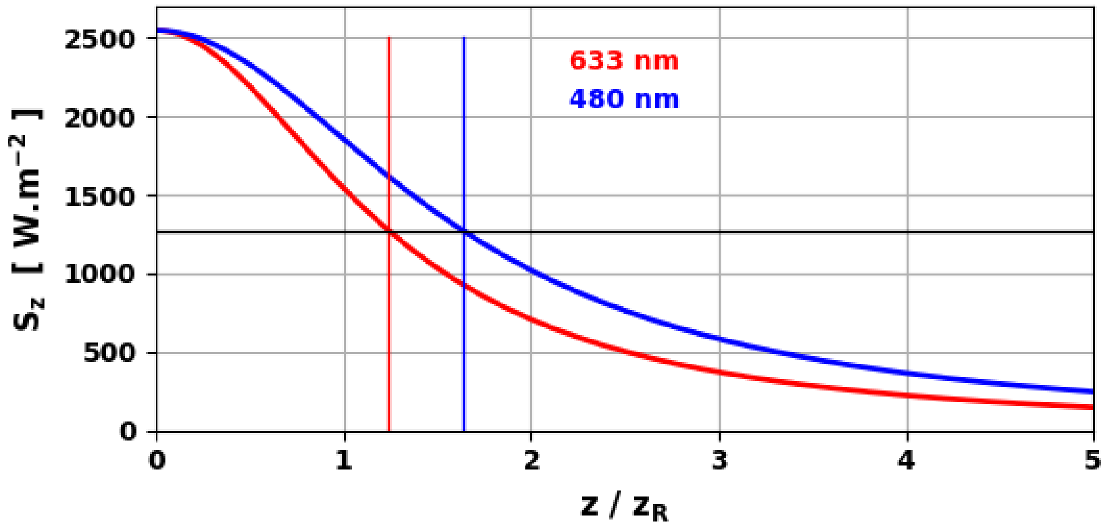


Fig. 4. Axial irradiance $S_z$ as a function of position $z$ along the axis of the beam. The maximum value of the irradiance from equation 15D is $S_{\max} = 2546\ \mathrm{W.m^{-2}}$. Note, at $z = z_R$ the on-axis irradiance is one-half the maximum irradiance at the waist $\left(z = 0\right)$ $S(z_R) = S_{\max} / 2$.

When $z \gg z_R$ $\left(1 + z^2 / z_R{}^2 \approx z^2 / z_R{}^2\right)$, then $S_z(z) \propto \dfrac{1}{z^2}$ and

$E(r = 0, z) \propto \dfrac{1}{z}$. Therefore, the amplitude of the wave decreases with increasing $z$ in the same manner as a spherical wave.

## Radial irradiance

The radial irradiance $S_r$ is the variation in the irradiance in the XY plane located at position *z*. and typical plots are shown in figure 5.

(15E)

$$S_r(r) = S(r,\, z) = E_0^{\;2}\left(\frac{c\,\varepsilon_0}{2}\right)\left(\frac{w_0}{w}\right)^2 \exp\left[-2\left(\frac{r^2}{w^2}\right)\right] \qquad z \text{ fixed}$$

For $z \ge z_R$ the on-axis $(r = 0)$ irradiance falls off to a good approximation according to the inverse square law $(S_r \propto 1/r)$

$$z = 5.00\ \text{m} \quad S_r(r = 0, \lambda = 633\ \text{nm})/4 = 147.8/4\ \text{W.m}^{-2} = 37\ \text{W.m}^{-2}$$

$$z = 10.00\ \text{m} \quad S_r(r = 0, \lambda = 633\ \text{nm}) = 39\ \text{W.m}^{-2}$$

$$z = 5.00\ \text{m} \quad S_r(r = 0, \lambda = 480\ \text{nm})/4 = 246.3/4\ \text{W.m}^{-2} = 62\ \text{W.m}^{-2}$$

$$z = 10.00\ \text{m} \quad S_r(r = 0, \lambda = 480\ \text{nm}) = 66\ \text{W.m}^{-2}$$

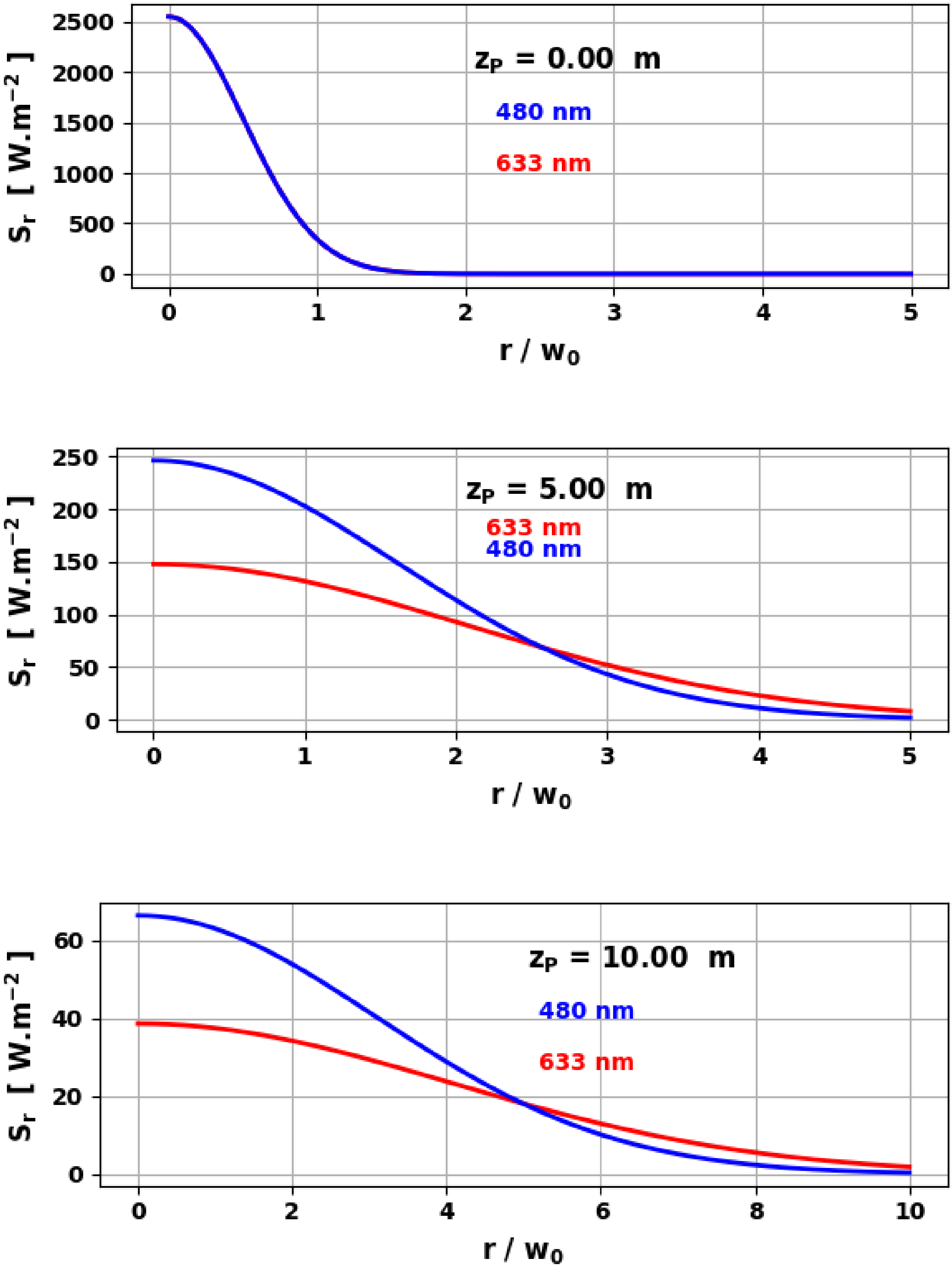


Fig. 5. The variation in the radial irradiance $S_r$ at three $z$ positions along the Z axis. The profile of each plot is a Gaussian function.

Figure 6 shows the radial irradiance $S_r$ for the wavelengths 633 nm and 480 nm.

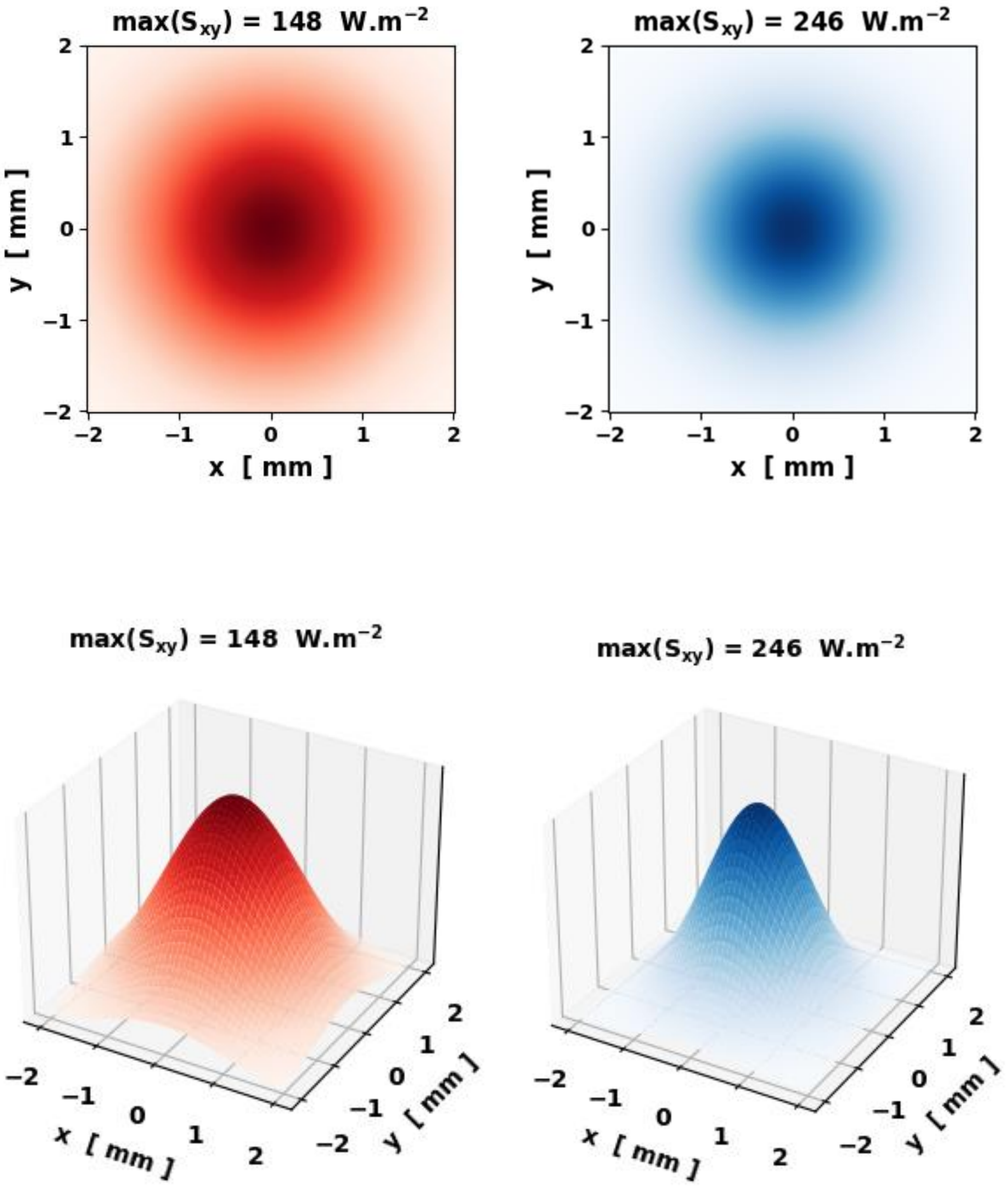


Fig. 6. Beam profile: Radial irradiance $S_r$ at $z_P$ = 5.00 m

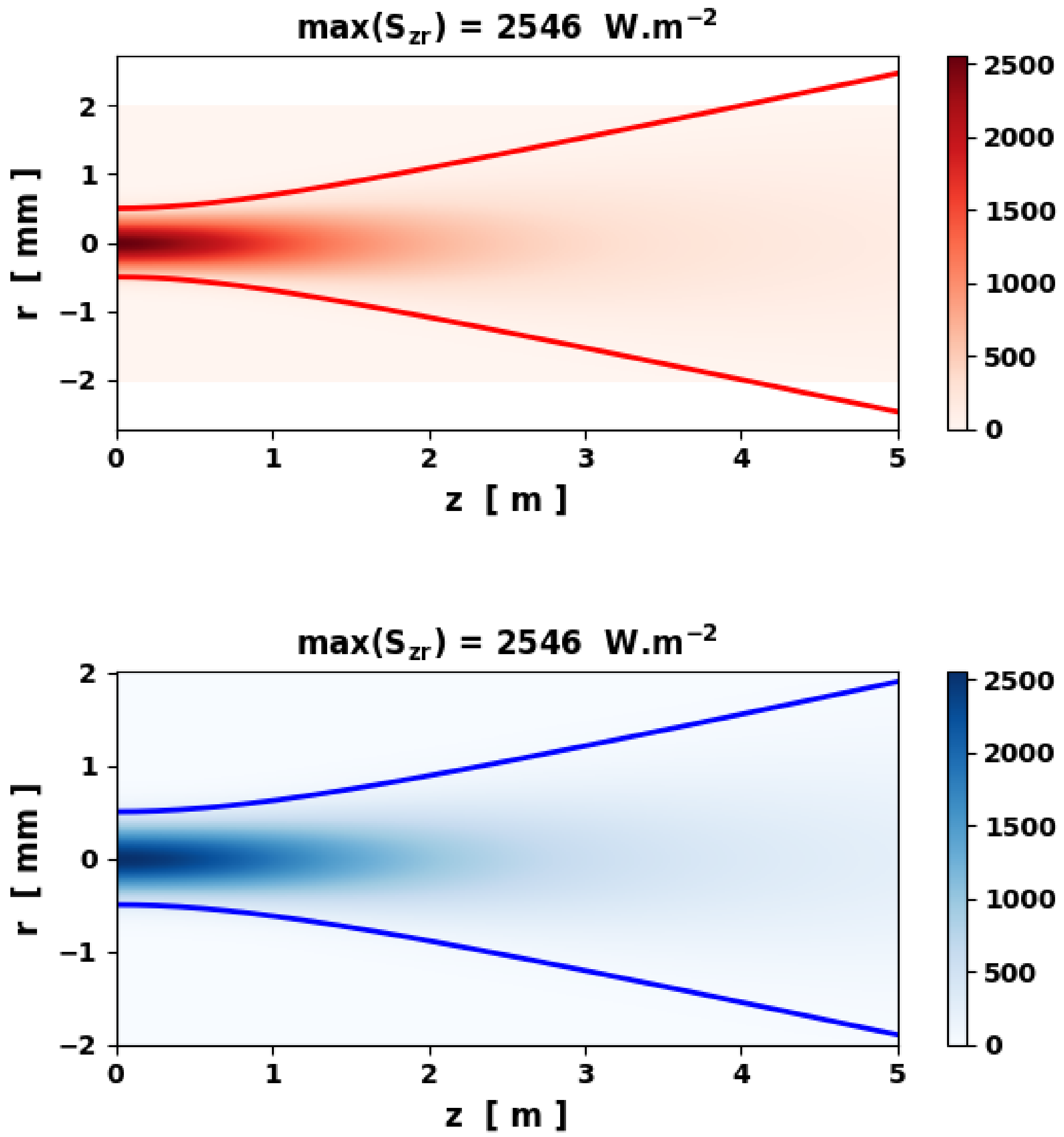


Fig. 7. Beam profile: Radial irradiance $S_r$ as a function $z$.

Figure 8 shows the beam in the XZ plane and how the beam diverges and how its intensity decreases with increasing $z$ distance from the waist at $z = 0$.

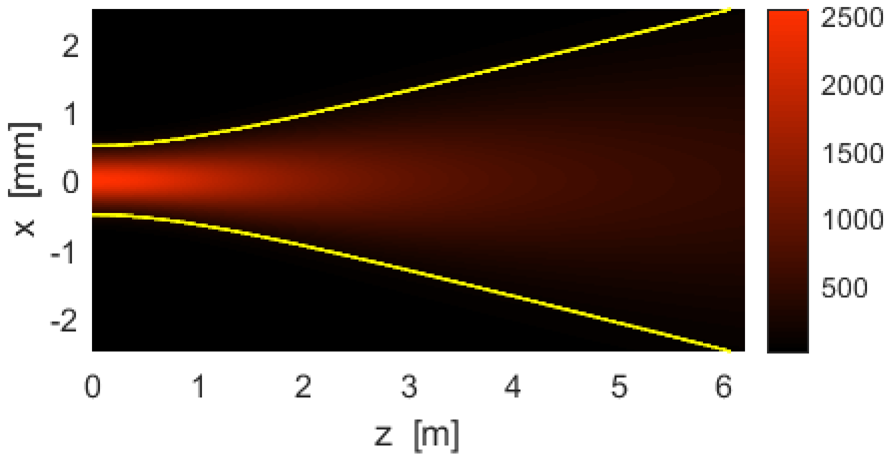


Fig. 8. Profile of the beam in the XZ plane. The yellow lines show the beam spot as shown in figure 2.

### Power $P$ transmitted through a circular disk of radius $r_P$ in an XY plane at position $z_P$

(16A) $$P(z) = P_0\left(1 - \exp\left(\frac{-2\,{r_P}^2}{w^2}\right)\right)$$

Figure 9 shows the percentage power that would pass through a circular aperture of varying radius placed perpendicular to the beam at the beam waist.

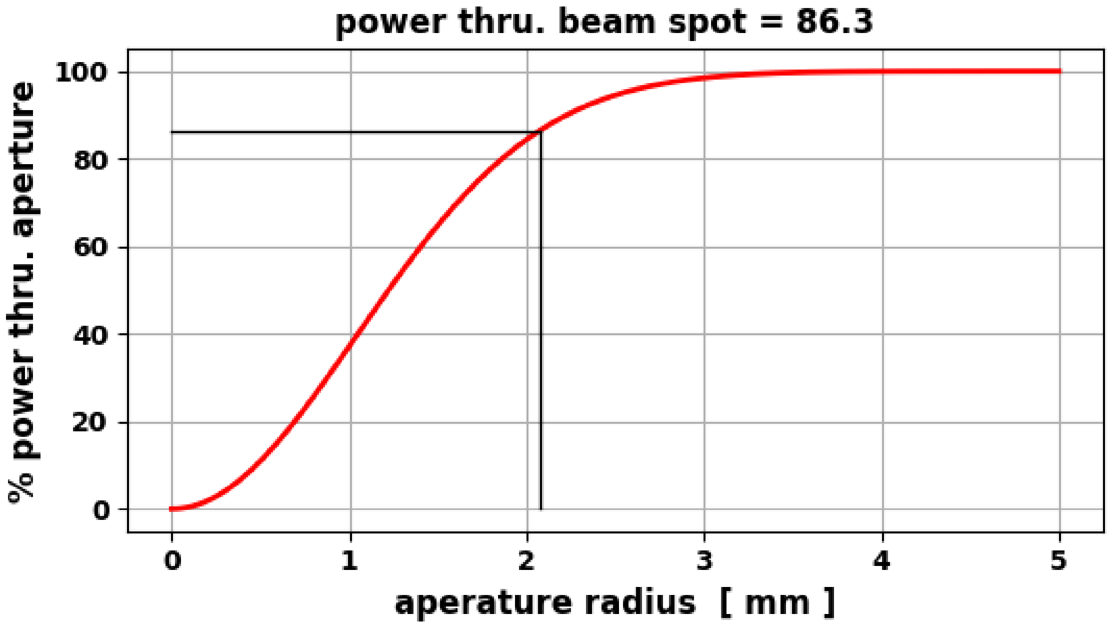


Fig. 9. The percentage power that would pass through a circular aperture of varying radius placed perpendicular to the beam at the beam waist. 86% of the light passes through the aperture when the radius of the aperture is equal to the beam waist.

## Gouy phase $\phi(z)$

(11) $$\phi(z) = \tan^{-1}\left(\frac{z}{z_R}\right)$$

The Gouy phase slightly shifts the phase of the wavefront of the wave as a whole. For a focused beam, the dependence of the Gouy phase as a function of $z$ is displayed in figure 4. As $z \to \pm\infty$ the Gouy phase asymptotes to $\phi(z) = \pm\pi/2$. Thus, only a $\pi$ phase shift occurs from $z = -\infty$ to $z = +\infty$. This results in inversion of a signal that has passed through the waist $\left(w(z=0) = w_0\right)$, that corresponds to the inversion obtained with the ray-tracing approximation (Geometrical Optics).

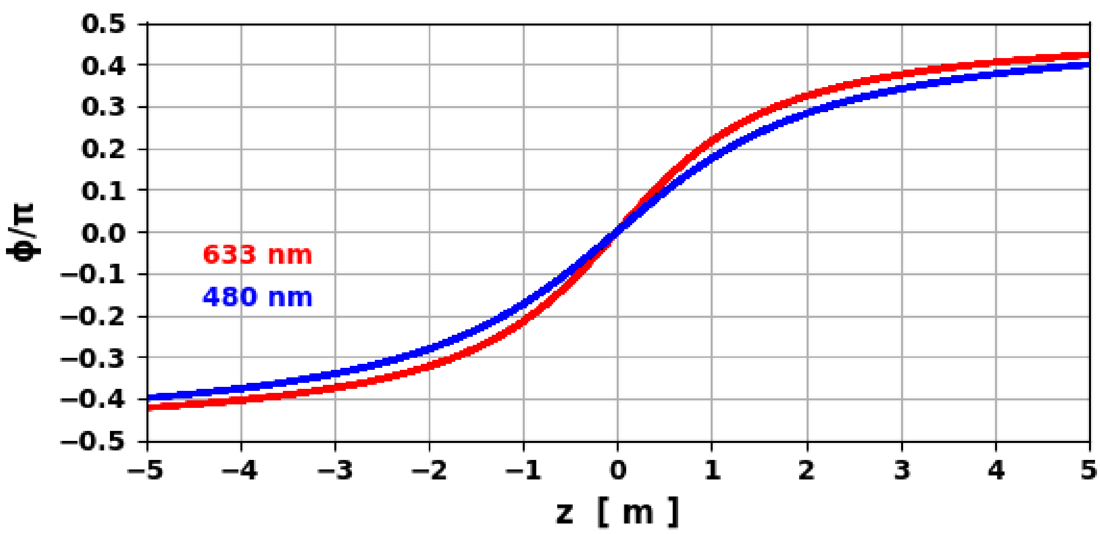


Fig.10A. Gouy phase plot for a focused Gaussian beam.

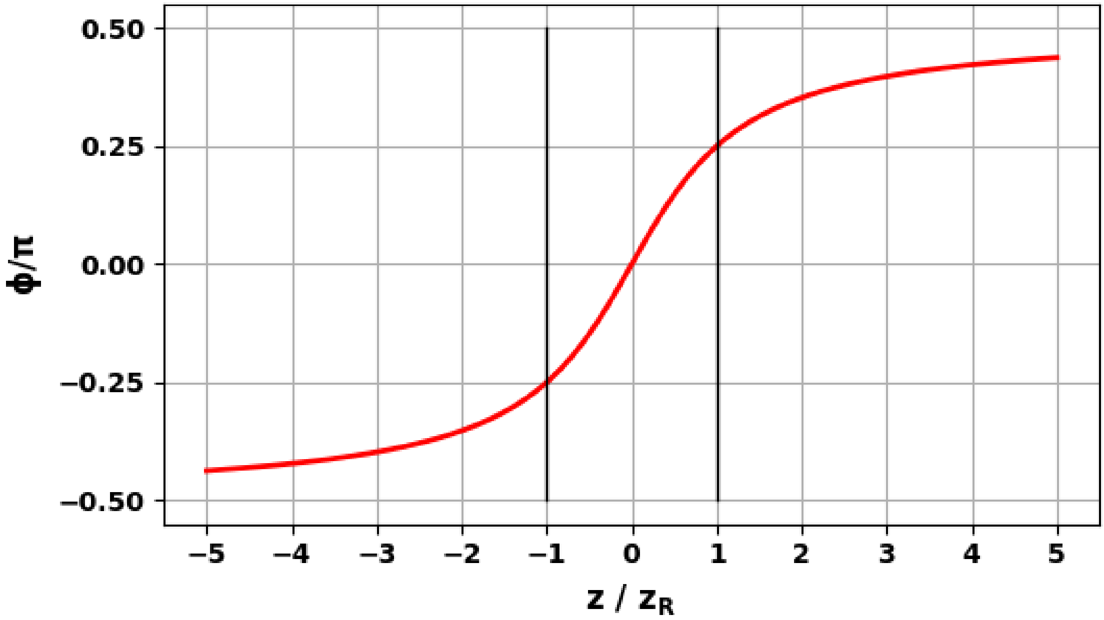


Fig.10B. Gouy phase plot for a focused Gaussian beam.

The most rapid change in phase occurs in the region from $z = -z_R$ to $z = +z_R$ and when $z = -z_R$, then $\phi(z = -z_R) = -\pi / 4$ and $z = +z_R$, then $\phi(z = +z_R) = +\pi / 4$. The Gouy phase shift along the beam remains within the range $\pm\pi / 2$ (for a fundamental Gaussian beam) and is not observable in most experiments. However, it is of theoretical importance and takes on a greater range for higher-order Gaussian modes.