DOING PHYSICS WITH PYTHON

COMPUTATIONAL OPTICS

RAY (GEOMETRIC) OPTICS

MATRIX METHODS IN PARAXIAL OPTICS

CONCAVE SPHERICAL MIRROR

Ian Cooper

Please email me any corrections, comments, suggestions or additions: **matlabvisualphysics@gmail.com**

DOWNLOAD DIRECTORIES FOR PYTHON CODE

[Google drive]

[GitHub]

S002.py

This Python Code is used to verify that Fermat's principle leads to the laws of reflection and refraction.

S002A.py

By specifying the start of the incident ray and its point of contact with the spherical mirror, you can use the reflection matrix to determine the direction of the reflected ray.

Inputs and setup:

Circular arc with |R| = 10: Grid 12x12

Incident ray A(0,yA) to P(xP,yP): inputs yA, yP

Reflected ray B(0,yB)

Calculations:

yP → xP

angle of radius vector OP → phi

slope (angle) of incident ray AP → aI

reflection matrix → slope (angle) of reflected aR

aR, yP → yB

aI, AR, phi → angle of incidence thetaI

angle of reflection thetaR

Output:

Display Console

Figure Window

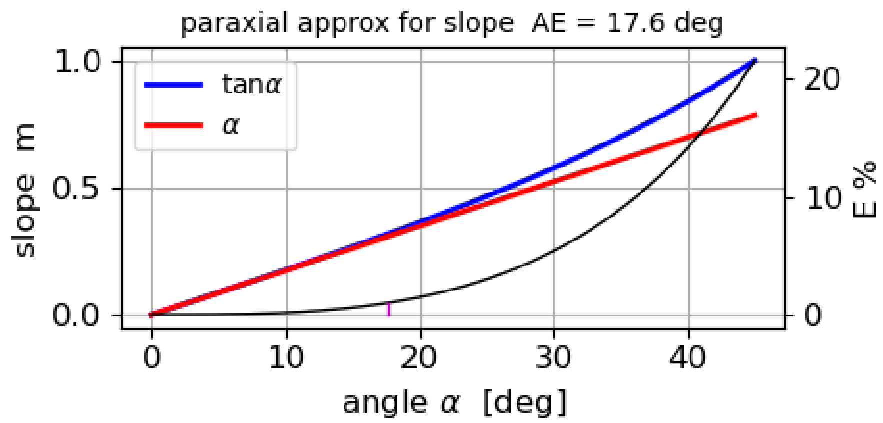
INTRODUCTION

We will consider a treatment of image formation that uses matrices to describe changes in the height and an angle as it makes its way through successive reflections and refractions in an optical system. The matrix approach is valid only in the paraxial regime.

PARAXIAL APPROXIMATION

Consider a simple translation of a ray in a homogeneous medium (uniform refractive index $n = 1$). Let the ray start at the point A(0, y_0) and end at the point B(L, y_1). The elevation angle of the ray is $\alpha_0 = \alpha_1$. The α angle is measured w.r.t. the X axis. Note: angles are measured in **radians** for all calculations, but for display purposes angles are often given in degrees.

Figure 1 shows plots of the slope of the ray, $m = \tan\alpha$ and the paraxial approximation $m = \alpha$. The twin axis of figure 1 gives the percentage error E for the different between the two slopes. The percentage error is less than 1% for angles less than 17.6° (Python variable AE). In summary, the paraxial approximation is most accurate for angles less than about 18°, with errors increasing significantly as the angle increases beyond this value.



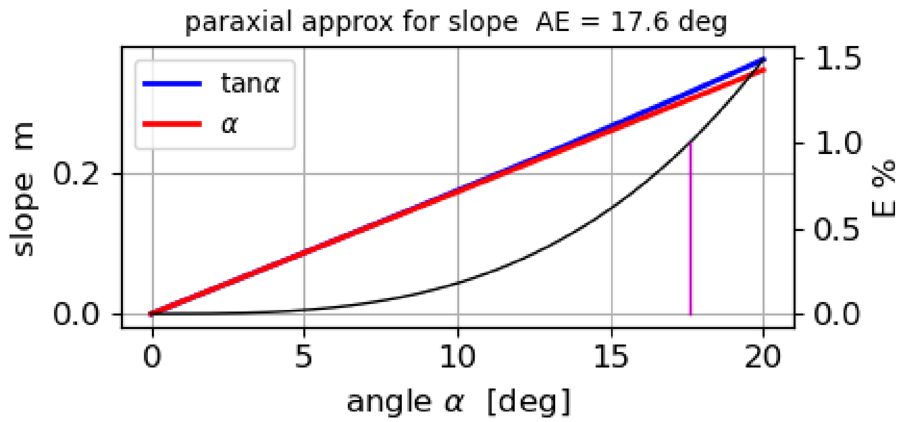


Fig. 1. Plots of the slope of the ray, $m = \tan\alpha$ and the paraxial approximation $m = \alpha$. The percentage error E is less than 1% for angles less than 17.6°. **S002.py**

ABCD MATRICES

The translation of a ray through a distance L is given by

$$\alpha_1 = \alpha_0 \quad y_1 = y_0 + L\ \tan\alpha_0$$

where the slope of the line is $m = \tan\alpha_0$. In the **paraxial approximation, the angle is assumed to be small** such that $\tan\alpha_0 \cong \alpha_0$. Therefore, using the paraxial approximation the translation can be written in matrix form

$$\begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_0 \\ \alpha_0 \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

where the $\mathbf{M}$ is the 2x2 **translation matrix**.

In optical systems, general linear transformations can be represented by 2x2 matrices. The linear transformation matrices are often called **ABCD** matrices since they are of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

REFLECTION FROM A CONCAVE SPHERICAL MIRROR

Consider a reflection at a concave spherical surface (figure 1). Sign convection: all angles pointing up are positive and all angles pointing down are negative. R is the radius of curvature and is the distance from the mirror's surface to the centre of the sphere. For a concave spherical mirror, the radius R is **negative**.

A property of a concave spherical mirror is that it can be used to focus light to the **focal point** where parallel rays of light converge after being reflected off the mirror. The **focal length** f of the mirror is

$$f = \frac{R}{2}$$

An ABCD matrix can be used to find the path of a reflected wave as shown below. The geometry of the optical system is shown in figure 1.

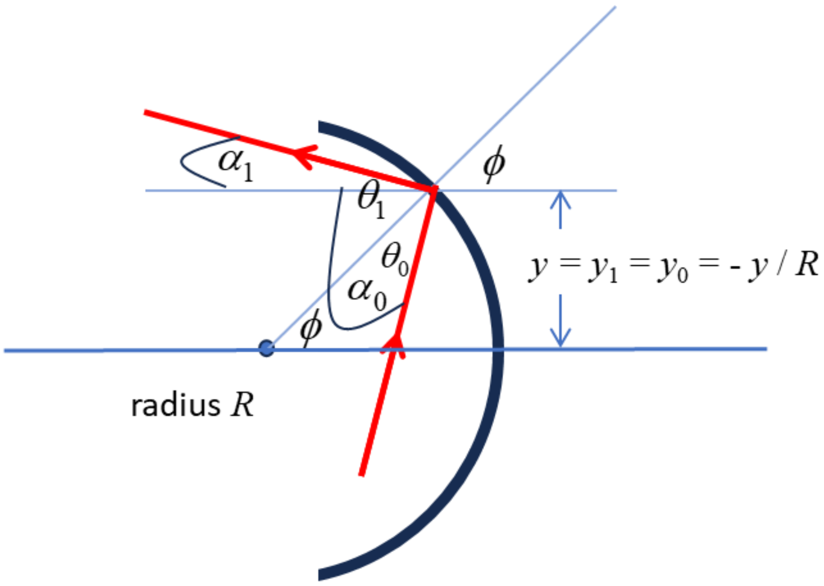


Fig. 1. Reflection of a ray at a spherical surface.
Incident ray: θ_I angle of incidence, α_I angle w.r.t X axis
Reflected ray: θ_R angle of reflection, α_R angle w.r.t X axis
Radius vector: ϕ angle w.r.t X axis

From the geometry shown in figure 1

$$\alpha_I = \theta_I + \phi = \theta_I + (-y / R)$$
$$\alpha_R = \theta_R + \phi = \theta_R - (-y / R)$$
$$R < 0$$

From the law of reflection $\theta = \theta_I = \theta_R$

Thus

$$\alpha_I = \theta + (-y / R) \to \theta = \alpha_I + y / R$$
$$\alpha_R = \alpha_I + y / R - (-y / R) = \alpha_I + 2y / R$$

Therefore, **reflection matrix M** for the spherical mirror is

$$\begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{pmatrix} = \begin{pmatrix} y_A \\ \alpha_A \end{pmatrix} \qquad \mathbf{M} = \begin{pmatrix} 1 & 0 \\ \frac{2}{R} & 1 \end{pmatrix} \quad R < 0$$

SIMULATIONS

Three horizontal incident rays $\left(\text{slope } \alpha_A = 0\right)$ and their reflections are shown in figure 2. The three rays converge to the focal point $f = \left|R/2\right| = 5$. For the rays close to the optical axis (y = 0) they almost converge to the focus, but as the distance increase of the rays from the optical axis increase, the focal length also increases. The reflected rays do not cross at the same point. Thus, the mirror does not have a well-defined focal point. This is called **spherical aberration** and results in a blurred image of an extended object.

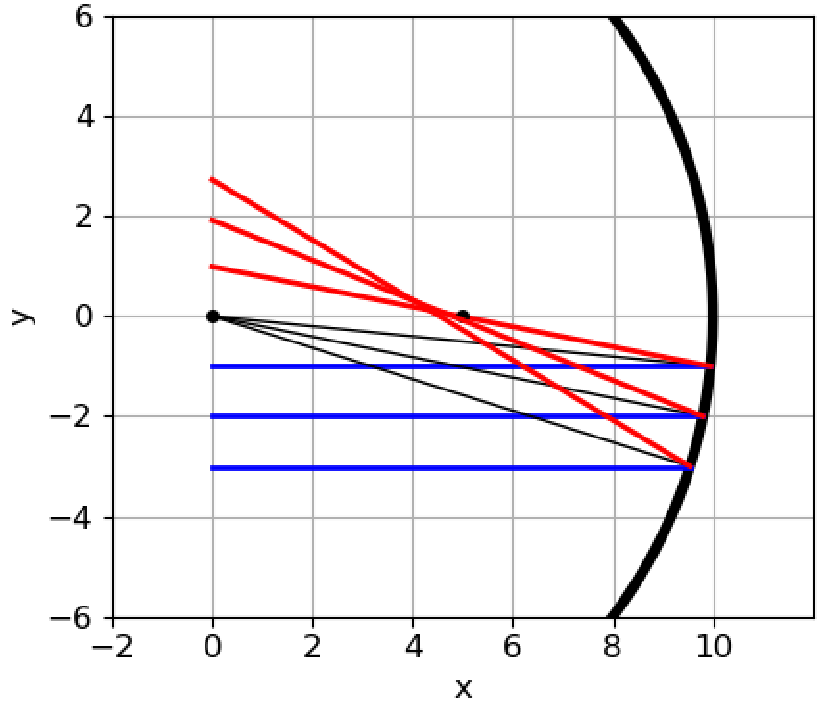


Fig. 2. Three incident rays that are parallel to the optical axis of the mirror follow the law of reflection. These rays are reflected so that they converge at a point, called the focal point. However, for a spherical mirror, the reflected rays do not cross at exactly the same point. This known as spherical aberration where the spherical mirror does not have a well-defined focal point. **S002A.py**

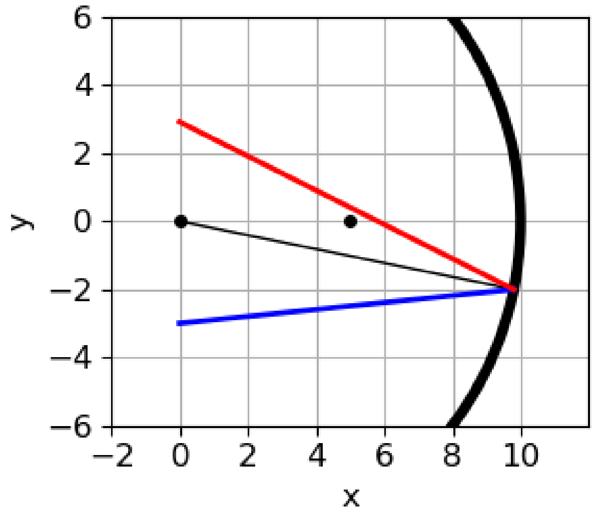
Table 1.

ray	y_A	θ_I deg	θ_R deg	f
1	-1.00	5.739	5.720	5.05
2	-2.00	11.537	11.381	5.202
3	-3.00	17.458	16.920	5.461

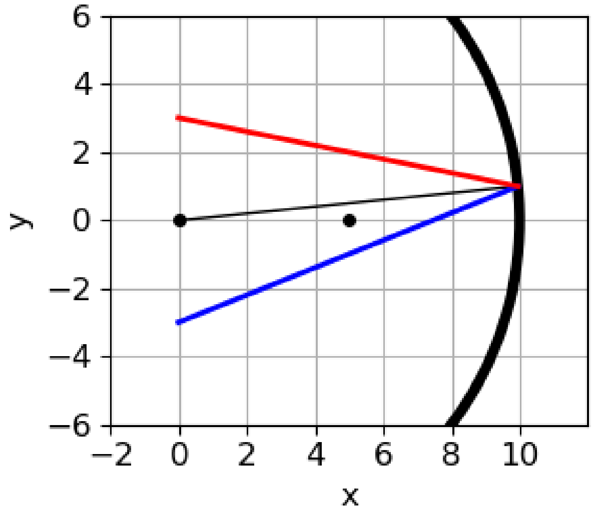
As the distance from the optical axis of the horizontal incident rays increases, the results of the paraxial approximation become less accurate.

Prediction using the ABCD reflection matrix

We can use the Python Code **S002A.py** to predict the direction of the reflected ray from our spherical mirror. The results of running different simulations are shown in figure 3.



A(xA,yA) = (0.00, -3.00)
P(xP,yP) = (9.80, -2.00)
B(xB,yB) = (0.00, 2.92)
incidence slope angle   alpha0, aI = 5.848 deg
reflection slope angle alpha1, aR = 28.766 deg
angle of incidence   thetaI = 17.385 deg
angle of reflection thetaR = 17.229 deg



A(xA,yA) = (0.00, -3.00)

P(xP,yP) = (9.95, 1.00)

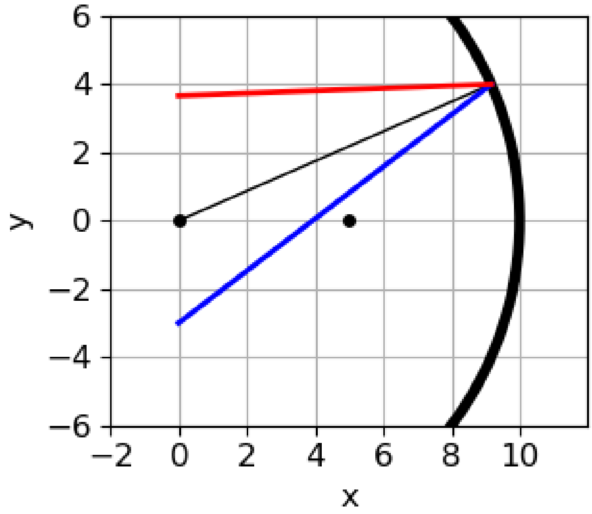
B(xB,yB) = (0.00, 3.01)

incidence slope angle   alpha0, aI = 23.034 deg

reflection slope angle alpha1, aR = 11.575 deg

angle of incidence   thetaI = 17.295 deg

angle of reflection thetaR = 17.314 deg



A(xA,yA) = (0.00, -3.00)

P(xP,yP) = (9.17, 4.00)

B(xB,yB) = (0.00, 3.67)

incidence slope angle   alpha0, aI = 43.760 deg

reflection slope angle alpha1, aR = -2.076 deg

angle of incidence   thetaI = 20.182 deg

angle of reflection thetaR = 21.502 deg

Fig. 3.   Ray diagrams showing the incident and reflected rays from a spherical mirror.

The accuracy of the prediction from the paraxial approximation decreases with increasing slope of the incident ray.