

# DOING PHYSICS WITH PYTHON

## QUANTUM MECHANICS

### CENTRAL FORCES

### ANGULAR MOMENTUM and SPHERICAL HARMONICS

**Ian Cooper**

[matlabvisualphysics@gmail.com](mailto:matlabvisualphysics@gmail.com)

Please email any corrections, suggestions, improvements, etc

### DOWNLOAD DIRECTORY FOR PYTHON SCRIPTS

#### **qm061.py**

Polar plots of the azimuthal wavefunction and its probability density for different values of  $l$  and  $m_l$ .

#### **qm061A.py**

Angular dependent wavefunctions – spherical harmonics: Plots for  $Y_{l,m_l}(\theta,\phi)$  and  $Y_{l,m_l}^*(\theta,\phi) Y_{l,m_l}(\theta,\phi)$ .

#### **qm061C.py**

[3D] surface plots of the spherical harmonics.

#### **qm061S.py**

Spherical harmonics plotted on a unit sphere.

[GitHub](#)

[Google Drive](#)

## INTRODUCTION

It is often better to use the **spherical coordinate system** rather than the Carestian coordinate system in modelling quantum systems. For example, an electron is attracted to the nucleus by the Coulomb force between opposite charges. This is known as a **central force**, one that is directed toward a fixed point.

The spherical coordinates about an Origin O are  $(r, \theta, \phi)$

$r$  radius line

$\theta$  polar angle [rad]

$\phi$  azimuthal angle [rad]

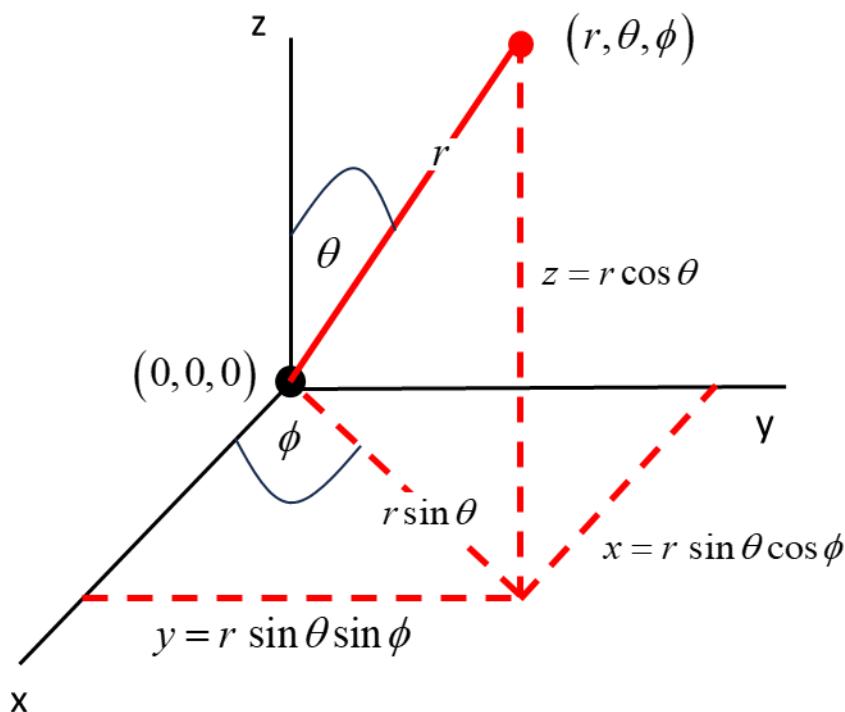


Fig. 1. Spherical coordinate system.

Just like energy for a bound particle is quantized, the angular momentum  $\vec{L}$  is also quantized. However, unlike energy, the angular momentum is a vector which has three components. For physically accepted wavefunctions for the angular momentum, not all angular momentum components can be known simultaneously. This is because the uncertainty principle applies to angular momentum just as it does to the momentum. The **uncertainty principle for angular momentum** is

It is impossible to specify simultaneously any two components of angular momentum – if one component of the angular momentum  $\vec{L}$  is sharp, then the remaining two components must be fuzzy.

Thus, we can have  $\vec{L}$  and  $L_z$  sharp (i.e. quantized) and the wavefunctions for  $\vec{L}$  and  $L_z$  are the **spherical harmonics**. The time-independent state wavefunctions are given by

$$\psi(\vec{r}) = \psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi)$$

The time-independent Schrödinger Equation for the eigenvalues  $E$  and eigenfunction  $\psi(\vec{r})$

$$\hat{H} \psi(\vec{r}) = E \psi(\vec{r})$$

where the Hamiltonian operator is

$$\hat{H} = \frac{-\hbar^2}{2m_\mu} \nabla^2 + U(\vec{r})$$

where  $\nabla^2$  is the Laplacian operator in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}$$

We can rewrite the Schrodinger equation as three separate equations:

- Radial equation  $R(r)$
- Polar equation  $\Theta(\theta)$
- Azimuthal equation  $\Phi(\phi)$

and solve each of these equations independently.

### Radial wavefunction $R(r)$

$$\frac{-\hbar^2}{2m} \left[ \frac{d^2 R(r)}{dr^2} + \frac{2}{r} \frac{dR(r)}{dr} \right] + \frac{l(l+1)\hbar^2}{2mr^2} R(r) + U(r)R(r) = E R(r)$$

The radial wave equation determines the radial part of the wavefunction  $\psi(r, \theta, \phi)$  and the allowed particle energies  $E$ . This radial wave equation is valid for any central force. The radial wavefunction  $R(r)$  for the state  $(n, l)$  is normalized

$$\int_0^\infty |R_{nl}(r)|^2 r^2 dr = 1$$

where  $n$  is the principal quantum number and  $l$  is the orbital quantum number.

### Azimuthal equation $\Phi(\phi)$

$$\frac{1}{\Phi} \frac{d^2\Phi}{d\phi^2} = -m_l^2 \quad m_l \text{ is the separation constant}$$

The solution for the azimuthal angle is

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m_l \phi) \quad \text{normalized}$$

where  $0 \leq \phi \leq 2\pi$  ( $-\pi \leq \phi \leq \pi$ ).

A complete rotation through  $2\pi$  rad must leave the quantum system in the same state, thus, for physically acceptable solution,  $m_l$  can only have values

$$m_l = 0, \pm 1, \pm 2, \pm 3, \dots$$

$m_l$  is known as the **magnetic quantum number**.

The probability density is

$$\Phi^*(\phi) \Phi(\phi) = \frac{1}{2\pi} \quad \text{independent of } \phi$$

and the probability of finding the azimuthal orientation of the molecule in the region from 0 to  $2\pi$  is

$$\text{probability}(0 \leq \phi \leq 2\pi) = \int_0^{2\pi} \frac{1}{2\pi} d\phi = 1$$

So, we never can know the orientation of the molecule, we only know that there is an equal probability of finding the azimuthal angle  $\phi$  when measured.

The azimuthal wavefunction can be expressed in terms of its real and imaginary parts

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m_l \phi) = \frac{1}{\sqrt{2\pi}} [\cos(m_l \phi) + i \sin(m_l \phi)]$$

We can visualize the azimuthal wavefunctions using polar plots (figure 2)

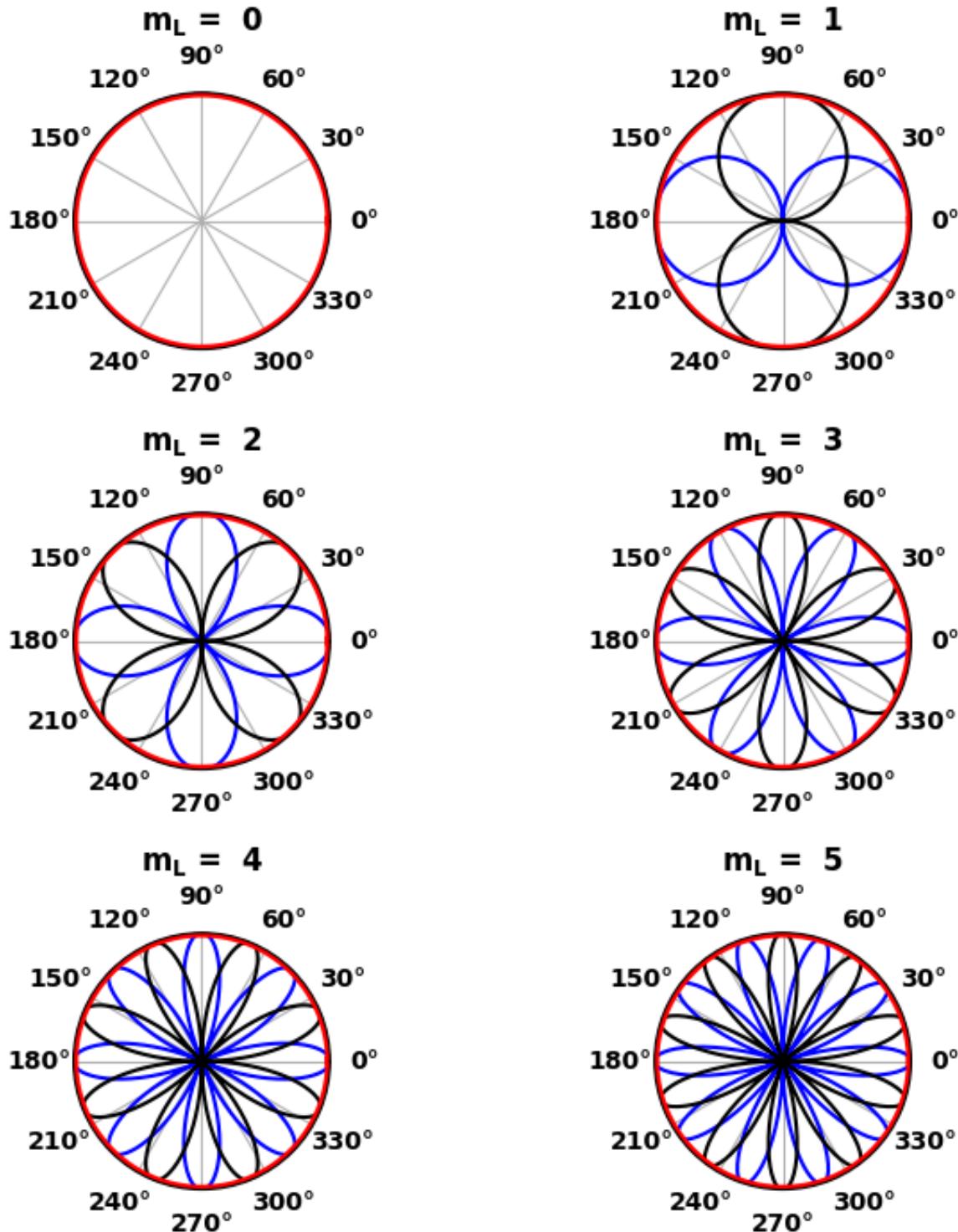


Fig. 2. Polar plots of the **real parts** (blue), **imaginary parts** (black) of the **azimuthal wavefunction** and the **probability density function** (red). Note: the probability density is independent of the azimuthal angle  $\phi$ .

**qm061.py**

**Polar equation**  $\Theta(\theta)$

$$\frac{d^2\Theta}{d\theta^2} + \frac{\cos\theta}{\sin\theta} \frac{d\Theta}{d\theta} + \left[ l(l+1) - \frac{m_l^2}{\sin^2\theta} \right] \Theta = 0$$

where  $l(l+1)$  is a separation constant.

The solutions  $\Phi_{l,m}(\theta)$  of this ODE are called the **associated Legendre functions** which are functions of sine and cosine terms. Solutions are only physically acceptable if  $l$  and  $m_l$  have the values

$$l \pm = 0, 1, 2, 3, \dots \quad m_l = 0, \pm 1, \pm 2, \pm 3, \dots, \pm l$$

$l$  is called the **orbital quantum number** and  $m$  the **magnetic quantum number**. (Note: chemists often use  $J$  not  $l$ ,  $J \equiv l$ )

The wavefunction for the angular dependence can be expressed in terms of **spherical harmonics** where

$$Y_{l,m_l}(\theta, \phi) = \Theta_{l,m_l}(\theta) \Phi_{ml}(\phi)$$

The separation constants  $l$  and  $m_l$  relate to the allowed values of the angular momentum and its  $z$  component.

$$|\vec{L}| = \sqrt{l(l+1)} \hbar \quad l = 0, 1, 2, 3, \dots$$

$$L_z = m_l \hbar \quad m_l = 0, \pm 1, \pm 2, \pm 3, \dots, \pm l$$

The solution of the polar equation in  $\Theta$  are known as **the associated Legendre functions**  $P_{l,m_l}(\cos \theta)$

$$\Theta(\theta) \equiv P_{l,m_l}(\cos \theta)$$

The full angular dependence of the central force wavefunction now becomes

$$Y_{l,m_l}(\theta, \phi) = P_{l,m_l}(\theta) \Phi_{ml}(\phi)$$

and for a normalized function

$$\int_0^{2\pi} \int_0^\pi Y_{l,m_l}^*(\theta, \phi) Y_{l,m_l}(\theta, \phi) \sin \theta d\theta d\phi = 1$$

Note: The probability density function  $probD$  is independent of the azimuthal angle since

$$e^{im_l\phi} e^{-im_l\phi} = e^0 = 1$$

$$probD(\theta) = Y_{l,m_l}^*(\theta, \phi) Y_{l,m_l}(\theta, \phi) = P_{l,m_l}^*(\theta) P_{l,m_l}(\theta)$$

The spherical harmonics can be computed in Python using the Code

```
import scipy.special
from scipy.special import sph_harm
Y = sph_harm(mL,L,phi,theta).real
```

From the spherical harmonics function we can get the associated Legendre function by setting  $\phi = 0$ .

**qm061A.py**      Note: for clarity  $l$  is used for  $L$  in the Code.

Input cell

```
#%% INPUTS >>>
L = 3      # ORBITAL QUANTUM NUMBER
mL = 0     # MAGNETIC QUANTUM NUMBER
N = 599    # number of grid points
phi = 0    # azimuthal angle [rad]
            # if phi = 0 --> Legendre function
```

Setup cell

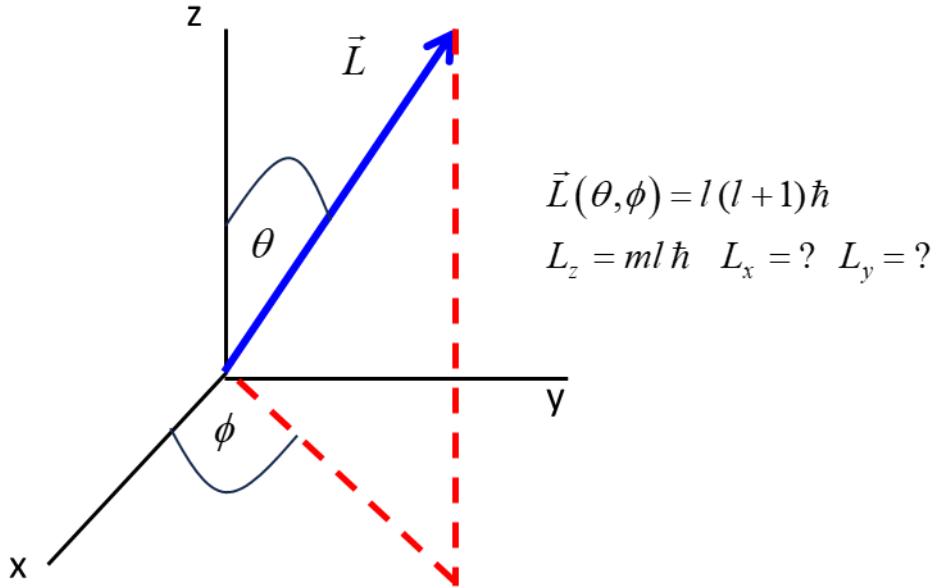
```
#%% SETUP
theta = linspace(0,pi,N)      # polar angle [ra]
Y = sph_harm(abs(mL),L,phi,theta).real  # spherical harmonics
probD = Y*Y                      # probability density
fn = 2*pi*Y*Y*sin(theta)        # check normalization
prob = simps(fn,theta)          # probability = 1 Or 0
print('check probability = 1 or 0: prob = %2.3f' % prob)
```

The variable **prob** gives the probability of finding the orientation of the angular momentum  $\hat{L}$  in space.

The variable **probD** is the probability density for the angular momentum  $\hat{L}$  having a polar angle  $\theta$

$$probD(\theta) = Y_{l,m_l}^*(\theta, \phi) Y_{l,m_l}(\theta, \phi) \quad \text{independent of } \phi$$

$$probD(\theta) = P_{l,m_l}^*(\theta) P_{l,m_l}(\theta)$$



$$Y_{l,m_l}^*(\theta, \phi) Y_{l,m_l}(\theta, \phi) = P_{l,m_l}^*(\theta) P_{l,m_l}(\theta)$$

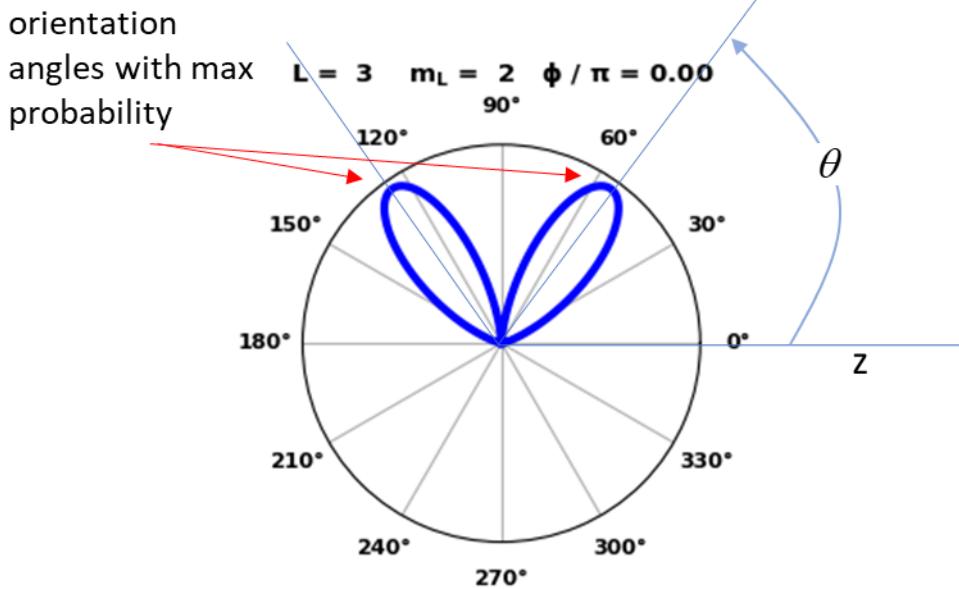
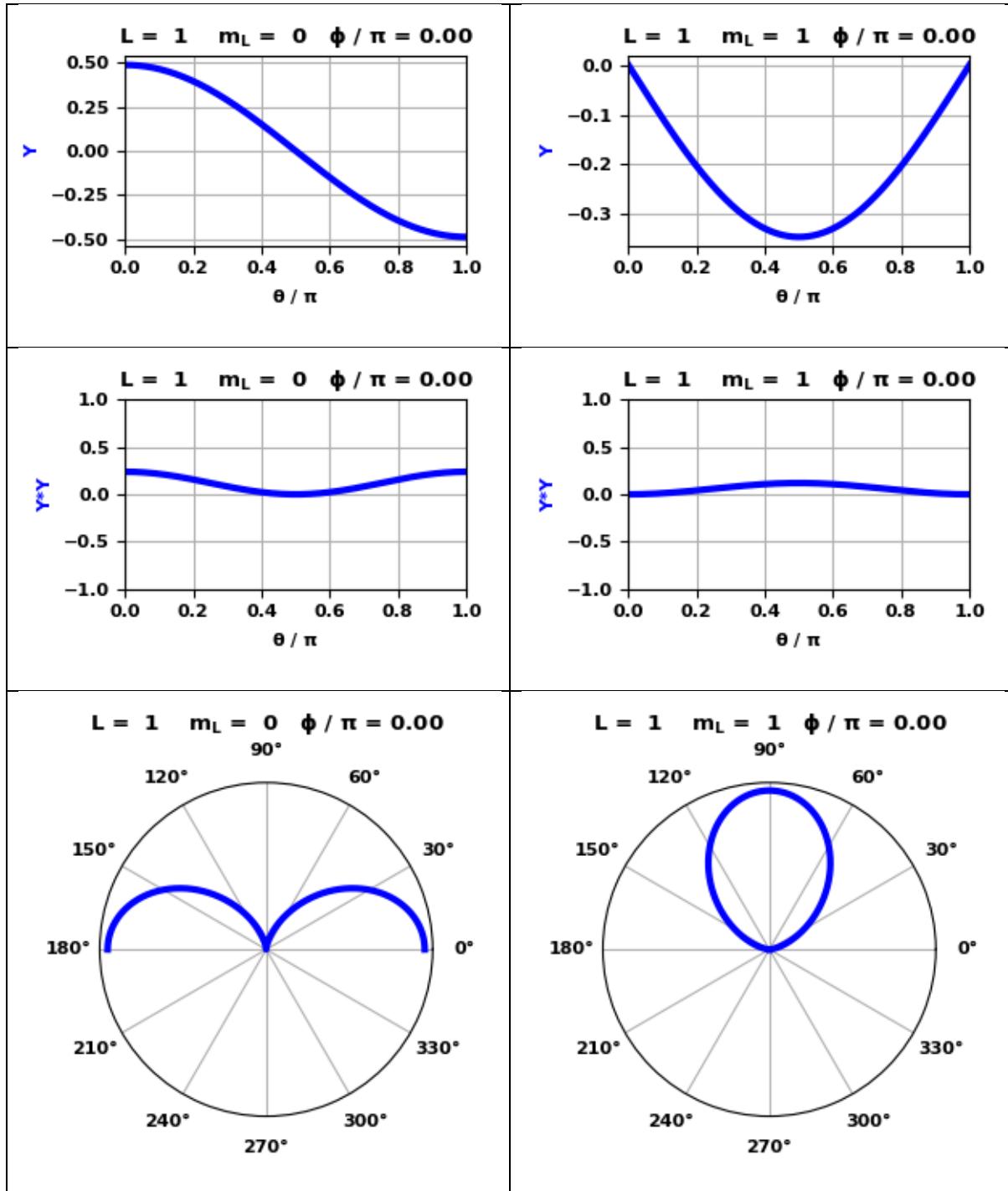
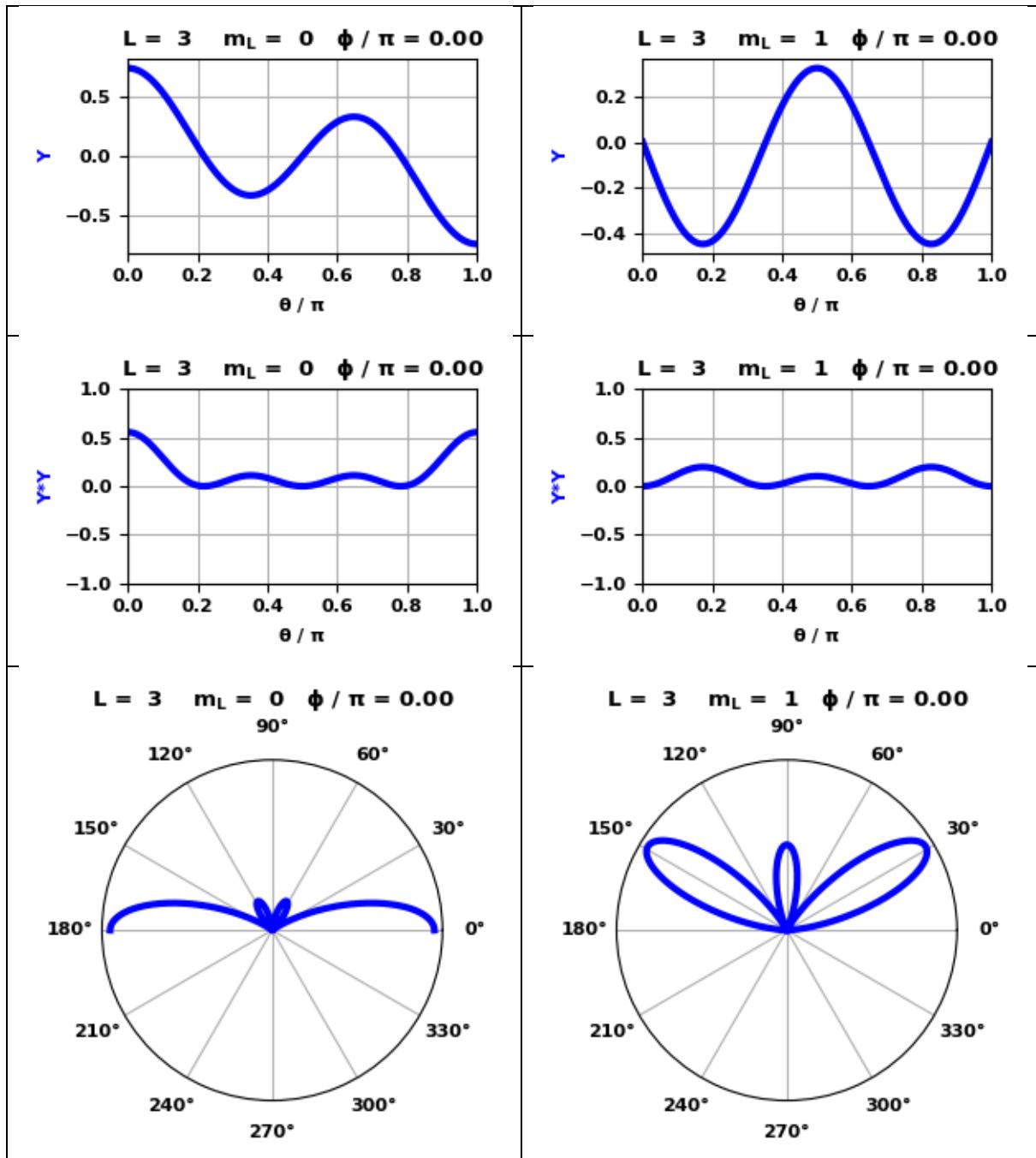


Fig. 3. The orientation of the angular momentum vector  $\vec{L}$  is specified by its polar angle  $\theta$  and azimuthal angle  $\phi$ . The orientation probability density depends only upon  $\theta$ . The magnitudes of the angular momentum and its z-component are quantized, but we have no knowledge of the x-component and the y-component. There is zero probability of the angular momentum being aligned along the Z-axis.

The following figures (figure 4) show plots for  $Y_{l,m_l}(\theta, \phi)$  and  $Y_{l,m_l}^*(\theta, \phi) Y_{l,m_l}(\theta, \phi)$ . When  $\phi = 0$  then you get plots of the Associated Legendre functions as

$$\phi = 0 \quad \Phi_{m_l}(0) = e^{-i m_l \phi} = 1 \quad Y_{l,m_l}(\theta, 0) = P_{l,m_l}(\theta)$$





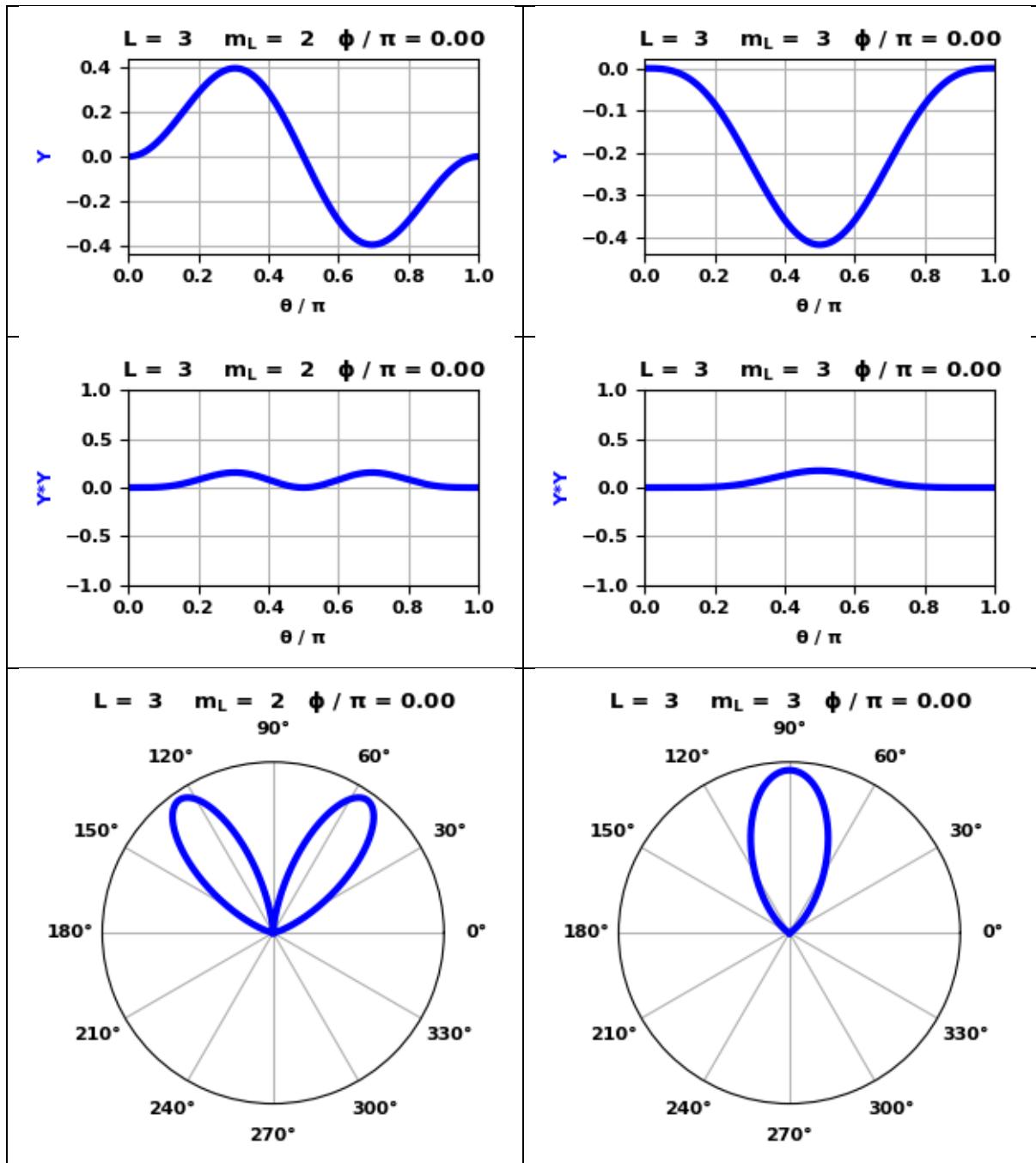
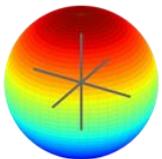


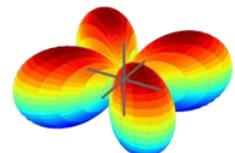
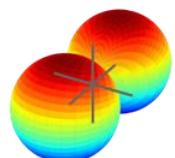
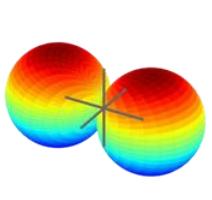
Fig. 4. Plots for  $Y_{l,m_l}(\theta, \phi)$  and  $|Y_{l,m_l}(\theta, \phi)|^2$ . In the polar format  $|Y(\theta, \phi)|^2$ , if you rotate the curve around the  $180^\circ - 0^\circ$  axis, you can visualize the [3D] pattern for the spherical harmonics.  
**qm061A.py**

The Code **qm061C.py** can be used to graph [3D] surface plots of the spherical harmonics  $Y_{l,m_l}(\theta, \phi)$  (figure 5).

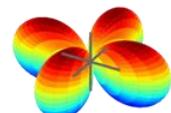
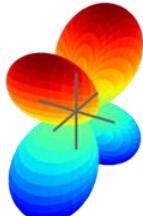
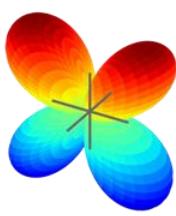
$l = 0 \ m_l = 0$



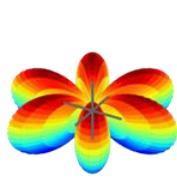
$l = 1 \ m_l = -1 \quad l = 1 \ m_l = 0 \quad l = 1 \ m_l = 1 \quad l = 2 \ m_l = -2$



$l = 2 \ m_l = -1 \quad l = 2 \ m_l = 0 \quad l = 2 \ m_l = 1 \quad l = 2 \ m_l = 2$



$l = 3 \ m_l = -3 \quad l = 3 \ m_l = -2 \quad l = 3 \ m_l = -1 \quad l = 3 \ m_l = 0$



$l = 3 \ m_l = 1 \quad l = 3 \ m_l = 2 \quad l = 3 \ m_l = 3 \quad l = 7 \ m_l = 6$

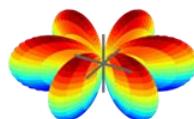


Fig. 5. Spherical harmonics. Note: that all the spherical harmonics are rotationally symmetric about the Z-axis. In the Python figure window you can [3D] rotate the plots. [qm061C.py](#)

## ANGULAR MOMENTUM: SPACE QUANTIZATION

The magnitude of the orbital angular momentum magnitude  $|\vec{L}|$  and its projection along the z-axis  $L_z$ , are both sharp and quantized according to the restrictions imposed by the orbital and magnetic quantum numbers, respectively. Together,  $l$  and  $m_l$  specify the orientation of the angular momentum vector  $|\vec{L}|$ . The direction of  $|\vec{L}|$  being quantized with respect to the Z-axis (arbitrary axis) is referred to as **space quantization**.

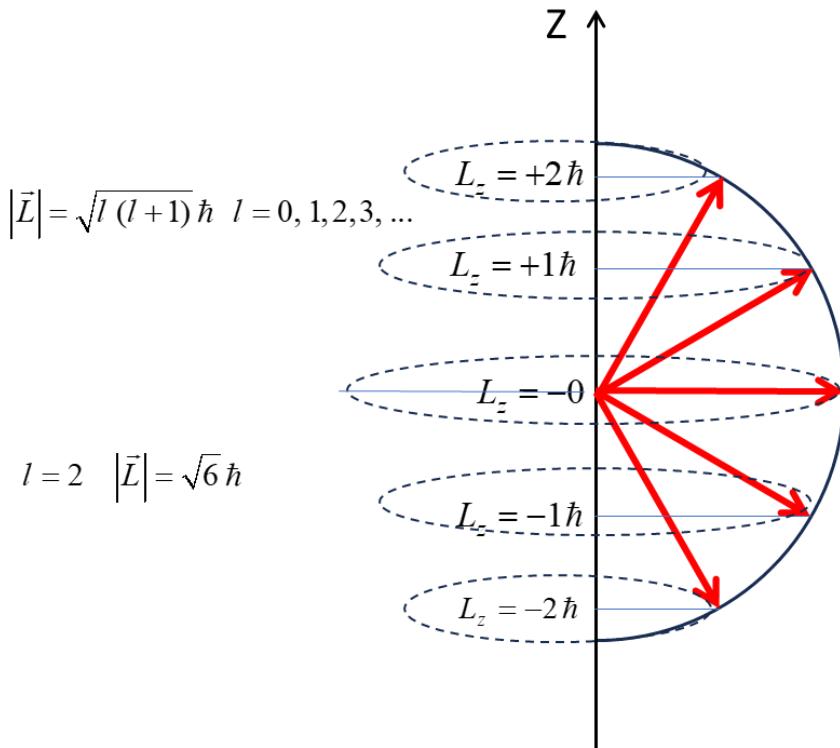


Fig. 6. Allowed projections of  $\vec{L}$  onto the Z-axis for  $l = 2$ . From a [3D] perspective,  $\vec{L}$  lies on the surface of a cone. The fuzzy character of  $L_x$  and  $L_y$  are shown by allowing  $\vec{L}$  to precess about the Z-axis, so that  $L_x$  and  $L_y$  change continually while  $L_z$  maintains the fixed value  $L_z = m_l \hbar$ .

## SPHERICAL HARMONICS (further notes)

### Reference

<https://irhum.github.io/blog/spherical-harmonics/index.html>

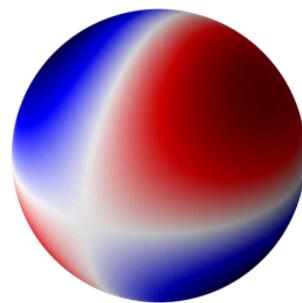
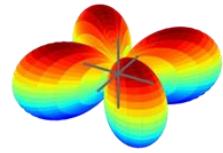
Spherical harmonics are powerful mathematical tools, allowing us to represent any function on a sphere. That is, spherical harmonics are basis functions defined on a spherical surface. A function  $f(x, y, z)$  defined on the unit sphere can be expressed as

$$f(x, y, z) = \sum_{l=-0}^{\infty} \sum_{m_l=-l}^l a_{l,m_l} Y_{lm_l}(x, y, z)$$

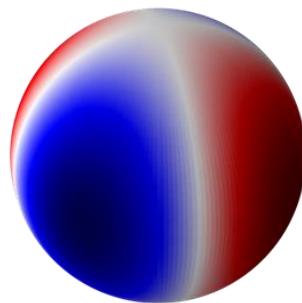
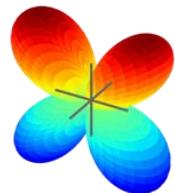
$$x^2 + y^2 + z^2 = 1$$

$l$  is the degree of the function and can be thought of as a “frequency”:  $l = 1$  gives 1 cycle,  $l = 2$  gives 2 cycles, . . . . as shown in figure 7.

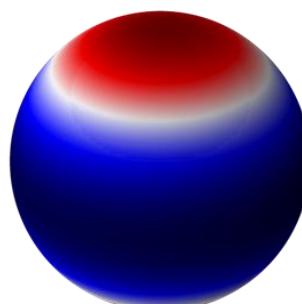
$l = 2 \quad m_l = -2$



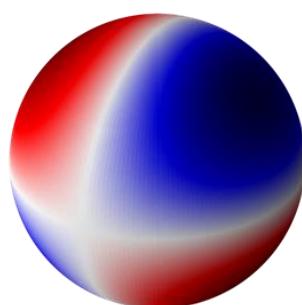
$l = 2 \quad m_l = -1$



$l = 2 \quad m_l = 0$



$l = 2 \quad m_l = 1$



$l = 2 \quad m_l = 2$

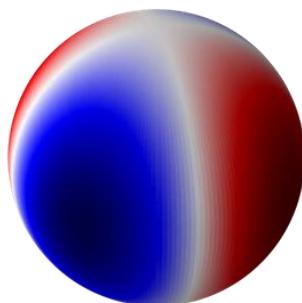
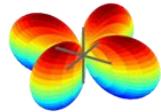


Fig. 7. Visualization of the spherical harmonics for  $l = 2$ .