Random Numbers(Note: sec 2.5 var part and some Q function no built-in)

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Compile and execute the following C program

```
codes/exrand.c
codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
codes/uni_cdf_plot.py
```

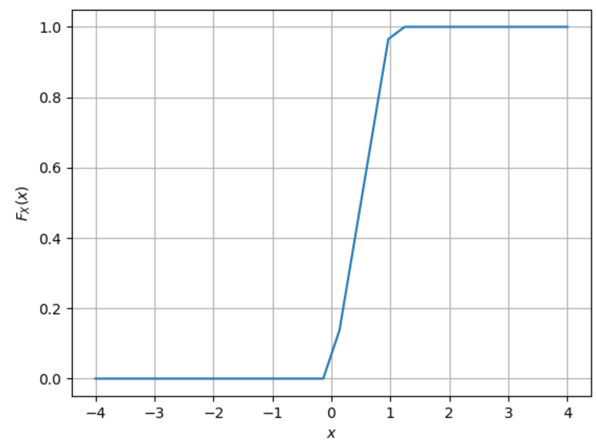

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

There are three cases:

a) $x < 0$:
$p_X(x) = 0$, and hence $F_U(x) = 0$.

b) $0 \le x \le 1$:

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x > 1$:

$$F_U(x) = \int_0^1 du = 1$$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.4}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: The C program can be found here

```
codes/uni_mean_var.c
```

The computed mean is 0.500137 and the variance is 0.083251.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution:

$$\begin{aligned} E[U] &= \int_{-\infty}^{\infty} x dF_U(x) && (1.8) \\ &= \int_{-\infty}^{\infty} x p_U(x) dx && (1.9) \\ &= \int_{0}^{1} x dx = \frac{1}{2} && (1.10) \end{aligned}$$

This verifies the empirical mean of 0.500137

$$\begin{aligned} E\left[U^2\right] &= \int_{-\infty}^{\infty} x^2 dF_U(x) && (1.11) \\ &= \int_{-\infty}^{\infty} x^2 p_U(x) dx && (1.12) \\ &= \int_{0}^{1} x^2 dx = \frac{1}{3} && (1.13) \end{aligned}$$

$$\begin{aligned} \text{var}[U] &= E\left[U^2\right] - (E[U])^2 && (1.14) \\ &= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} && (1.15) \end{aligned}$$

This verifies the empirical variance of 0.083251

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The required samples are generated by the C file in Question 1.1

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

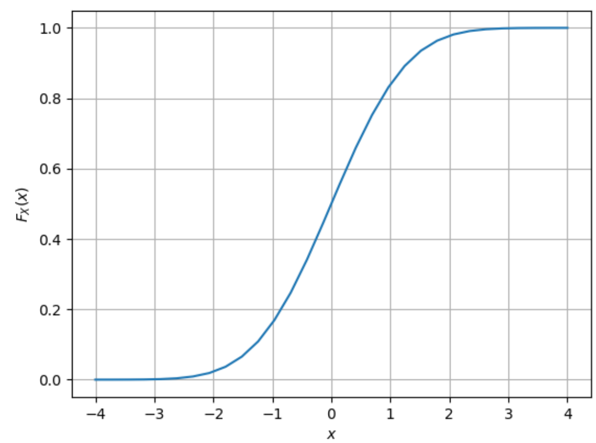

Fig. 2.2: The CDF of X

The code can be found at

```
codes/gau_cdf_plot.py
```

The properties of CDF of X are:

a) The CDF is monotonically increasing
b) $\lim_{x\to-\infty} F_X(x) = 0$
c) $\lim_{x\to\infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code from

```
codes/pdf_gau_plot.py
```

The properties of pdf of X are:

a) $p_X(x) = p_X(-x)$
b) $\int_{-\infty}^{\infty} p_X(x) dx = 1$

2.4 Find the mean and variance of X by writing a C program.

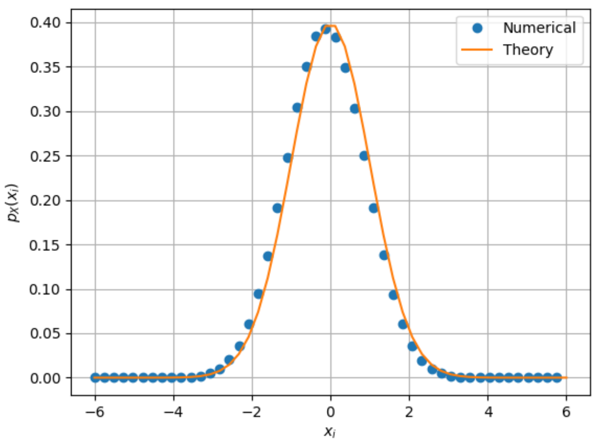

Fig. 2.3: The PDF of X

Solution: The code can be found at

```
codes/gau_mean_var.c
```

The computed mean is -0.000417 and the computed variance is 0.999902.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.4)$$

$$= 0 \text{ (Since the integrand is odd)} \quad (2.5)$$

$$\text{var}[X] = E\left[X^2\right] - (E[X])^2 \quad (2.6)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} \sqrt{2t} e^{-t} dt \quad (2.8)$$

$$= \frac{2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \quad (2.9)$$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right) = 1 \quad (2.10)$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function given as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1} e^{-x} dx \quad (2.11)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \quad (2.12)$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$.

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \quad (3.1)$$

and plot its CDF.
Solution: The codes can be found at

```
codes/v_samp.c
codes/cdf_v_plot.py
```

and the CDF is plotted in Figure 3.1.

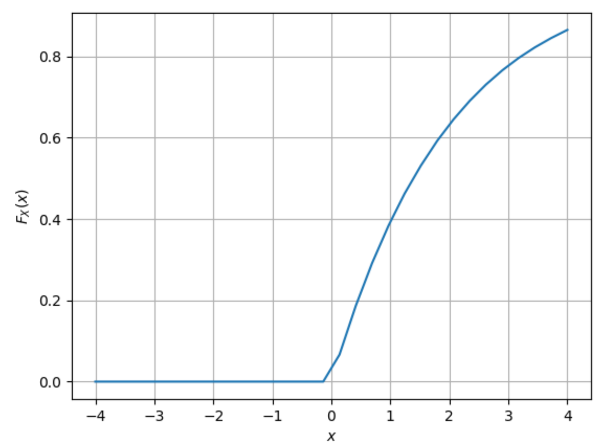

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution: The function

$$V = f(U) = -2 \ln (1 - U) \quad (3.2)$$

is monotonically increasing in $[0, 1]$. Hence, it is invertible and the inverse function is given by

$$U = f^{-1}(V) = 1 - \exp\left(-\frac{V}{2}\right) \quad (3.3)$$

$$
\begin{aligned}
F_V(v) &= \Pr\left(V < v\right) && (3.4)\\
&= \Pr\left(f(U) < v\right) && (3.5)\\
&= \Pr\left(U < f^{-1}(v)\right) && (3.6)\\
&= \Pr\left(U < 1 - \exp\left(-\frac{v}{2}\right)\right) && (3.7)\\
&= F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) && (3.8)\\
\implies F_V(v) &= \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} && (3.9)
\end{aligned}
$$