

Random Numbers

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Abstract—This manual provides a simple introduction to the generation of random numbers

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Compile and execute the following C program

```
codes/exrand.c
codes/coeffs.h
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
codes/uni_cdf_plot.py
```

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^x p_U(u) du \quad (1.2)$$

There are three cases:

- a) $x < 0$:

$$p_X(x) = 0, \text{ and hence } F_U(x) = 0.$$

- b) $0 \leq x \leq 1$:

$$F_U(x) = \int_0^x du = x \quad (1.3)$$

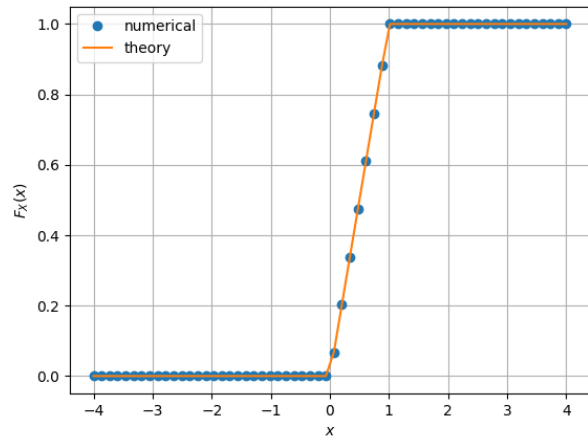


Fig. 1.2: The CDF of U

- c) $x > 1$:

$$F_U(x) = \int_0^1 du = 1$$

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.4)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.5)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.6)$$

Write a C program to find the mean and variance of U .

Solution: The C program can be found here

```
codes/uni_mean_var.c
```

The computed mean is 0.500137 and the variance is 0.083251.

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution:

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.8)$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \quad (1.9)$$

$$= \int_0^1 x dx = \frac{1}{2} \quad (1.10)$$

This verifies the empirical mean of 0.500137

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dF_U(x) \quad (1.11)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \quad (1.12)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \quad (1.13)$$

$$\text{var}[U] = E[U^2] - (E[U])^2 \quad (1.14)$$

$$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.15)$$

This verifies the empirical variance of 0.083251

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: The required samples are generated by the C file in Question 1.1

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2

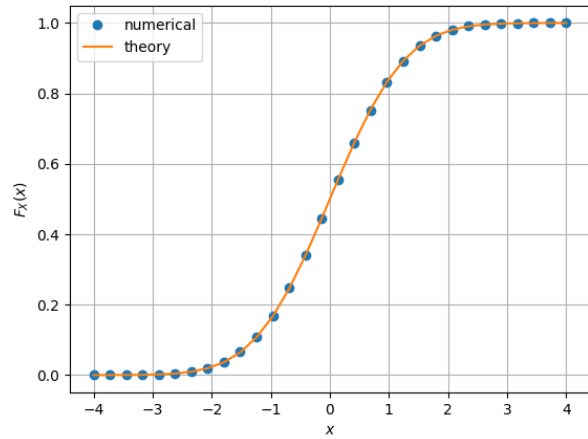


Fig. 2.2: The CDF of X

The code can be found at

codes/gau_cdf_plot.py

The properties of CDF of X are:

- a) The CDF is monotonically increasing
- b) $\lim_{x \rightarrow -\infty} F_X(x) = 0$
- c) $\lim_{x \rightarrow \infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code from

codes/pdf_gau_plot.py

The properties of pdf of X are:

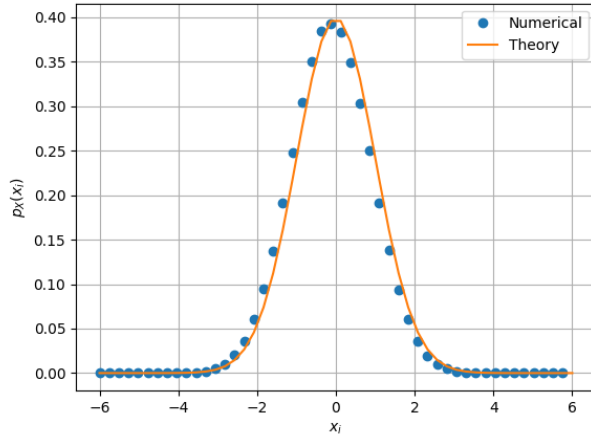
- a) $p_X(x) = p_X(-x)$
- b) $\int_{-\infty}^{\infty} p_X(x) dx = 1$

2.4 Find the mean and variance of X by writing a C program.

Solution: The code can be found at

codes/gau_mean_var.c

The computed mean is -0.000417 and the computed variance is 0.999902.

Fig. 2.3: The PDF of X

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

Solution: The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.4)$$

$$= 0 \text{ (Since the integrand is odd)} \quad (2.5)$$

$$\text{var}[X] = E[X^2] - (E[X])^2 \quad (2.6)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.7)$$

$$= x \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx - \int_{-\infty}^{\infty} \int x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.8)$$

$$= -xe^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} \quad (2.9)$$

$$(2.10)$$

Substituting limits we get $E[x^2] = 1$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

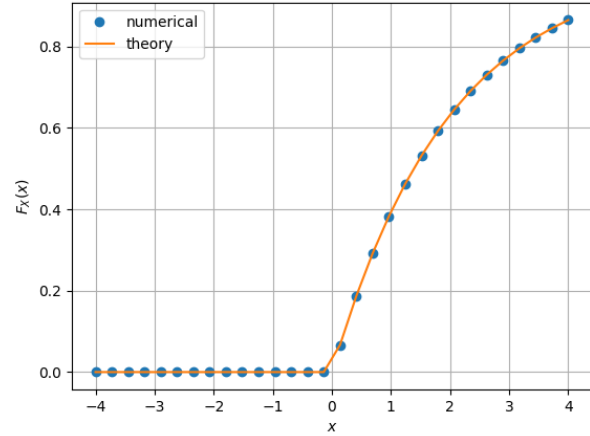
$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: The codes can be found at

codes/v_samp.c
codes/cdf_v_plot.py

and the CDF is plotted in Figure 3.1.

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution: The function

$$V = f(U) = -2 \ln(1 - U) \quad (3.2)$$

is monotonically increasing in $[0, 1]$. Hence, it is invertible and the inverse function is given by

$$U = f^{-1}(V) = 1 - \exp\left(-\frac{V}{2}\right) \quad (3.3)$$

$$F_V(x) = \Pr(V < x) \quad (3.4)$$

$$= \Pr(f(U) < x) \quad (3.5)$$

$$= \Pr\left(U < f^{-1}(x)\right) \quad (3.6)$$

$$= \Pr\left(U < 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.8)$$

$$\Rightarrow F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \quad (3.9)$$