Part 1: Theory

1. Question 1 is answered in Figure 1 shown below.

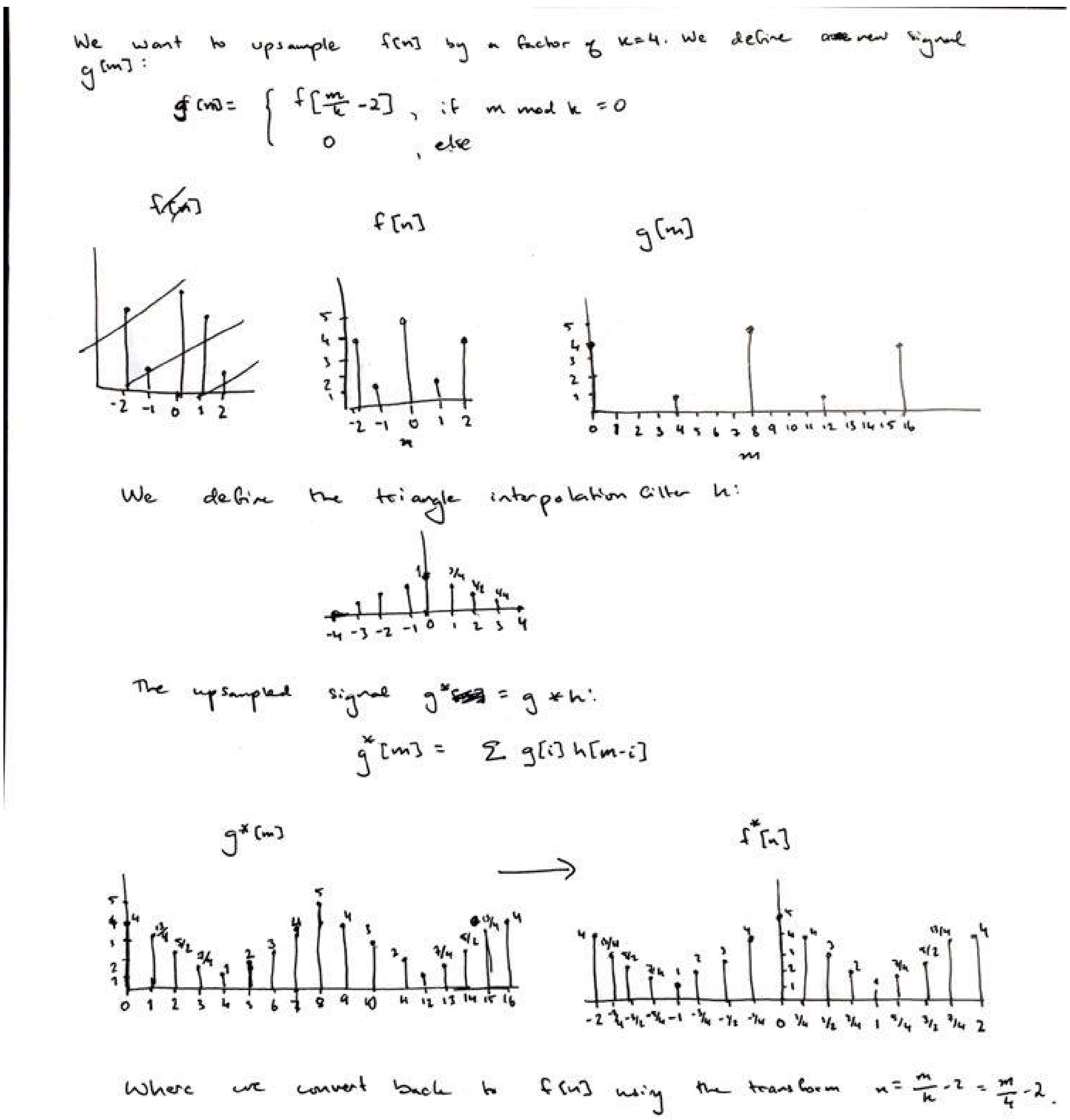

Figure 1: Question 1, upsampling a discrete signal.

2. Consider a signal:

$$x(t) = sin^2(2F_a\omega t) + \frac{1}{7}cos(F_b\omega t) + \frac{1}{5}sin(5F_c\omega t)$$

We want to determine a sufficient sampling rate to discretize this signal without aliasing. Applying the Nyquist-Shannon sampling theorem, we know that the sampling rate, f_s, is sufficient if it is greater than $2B$, where B is the band limit of the signal. The band limit of a signal is the largest nonzero frequency component of that signal, in other words the largest bound on the support of the signal's Fourier representation. As our signal $x(t)$ is a sum of sin and cosine terms, we can apply the Nyquist-Shannon theorem to determine f_s. Note:

$$\begin{aligned} 2sin^2(t) &= 1 - cos(2t) \\ \therefore \quad sin^2(2F_a\omega t) &= \frac{1}{2} - \frac{1}{2}cos(4F_a\omega t) \\ \text{and,} \quad x(t) &= \frac{1}{2} - \frac{1}{2}cos(4F_a\omega t) + \frac{1}{7}cos(F_b\omega t) + \frac{1}{5}sin(5F_c\omega t) \end{aligned}$$

As we have rewritten $x(t)$ as a sum of sin and cosine terms, along with a constant (which has a frequency of 0), we can directly analyze the frequencies making up the signal. Since

$$\begin{aligned} \mathcal{F}\{cos(\omega_0 t)\} &= \pi\Big[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\Big] \\ \mathcal{F}\{sin(\omega_0 t)\} &= \frac{\pi}{j}\Big[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\Big] \\ \mathcal{F}\{1\} &= \delta(\omega) \end{aligned}$$

The bandlimit of this signal is simply the largest frequency in the sin and cosine terms, thus the sampling frequency $f_s > 2\max\big\{4F_a, F_b, 5F_c\big\}$.

3. We define the Second Moment Matrix as:

$$M = \sum_x \sum_y w(x,y) \begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}$$

For an individual pixel (x, y), we define the matrix N:

$$N = \begin{bmatrix} I_x^2 & I_xI_y \\ I_xI_y & I_y^2 \end{bmatrix}$$

   (a) We determine the eigenvalues of the matrix N by solving the characteristic equation $det|N - \lambda I| = 0$:

$$\begin{aligned} det|N - \lambda I| &= 0 \\ (I_x^2 - \lambda)(I_y^2 - \lambda) - I_x^2 I_y^2 &= 0 \\ I_x^2 I_y^2 - \lambda I_x^2 - \lambda I_y^2 + \lambda^2 - I_x^2 I_y^2 &= 0 \\ \lambda^2 &= \lambda I_x^2 + \lambda I_y^2 \\ \therefore \lambda = 0 \quad \text{or} \quad \lambda &= I_x^2 + I_y^2 \end{aligned}$$

(b) The matrix N is clearly positive semi-definite, that is $N \succcurlyeq 0$, as all of its eigenvalues $\lambda_{N,i} \geq 0$. To show that M is a positive semi-definite matrix, we need to demonstrate that

$$\begin{aligned} \mathbf{a}^T M \mathbf{a} &\geq 0, \quad \forall \mathbf{a} \in \mathbb{R}^n \\ \mathbf{a}^T M \mathbf{a} &= \mathbf{a}^T \left(\sum_x \sum_y w(x,y) N \right) \mathbf{a} \\ &= \sum_x \sum_y w(x,y) \mathbf{a}^T N \mathbf{a} \end{aligned}$$

We note that $\mathbf{a}^T N \mathbf{a} \geq 0$ as $N \succcurlyeq 0$. Additionally, the window function $w(x,y) \geq 0$, and thus $\mathbf{a}^T M \mathbf{a} \geq 0,\ \forall \mathbf{a} \in \mathbb{R}^n$.

Part 2: Seam Carving

Here we simply compare the use of seam carving to resize images with content awareness versus simple cropping or scaling. In Figures 2, 3, 4, 5, 6 we see the results of the various methods (in order of *Original, Seam Carving, Cropping, and Scaling* applied to different images and resizing scenarios. As can be clearly seen in Figure 2, where the resizing only applies to the width of the image, we note that scaling performs poorly as it does not preserve the aspect ratio of the content, instead squeezing it along the width and stretching along the height. Cropping preserves the aspect ratio of the content, performs poorly as it removes part of the relevant content, the person, as the process has no content awareness. Seam carving performs well in preserving content and preserves the aspect ratio, however there are some artifacts such as errant clouds or visible jagged lines.

One important observation is that the content and composition of the image play an important role in selection of a resizing method. For Image 2, shown in Figures 3, 4, we note that there is a single point of focus onto the trees, the remaining clouds are largely sectioned off into uniform layers. Here, cropping does very well and scaling does not seem to alter the aspect ratio too negatively. For Image 3, shown in Figures 5, 6, seam carving performs well at altering the height as there are low energy seams along the dark clouds at the top. However seam carving removes part of the rock structure when resizing along the width. Here cropping once again performs well, as the content of the image is centered and cropping does not remove anything of focus.

Part 3: Corner Detection

2. We show the scatterplots of λ_1, λ_2 for both Image 1 and 2 in Figure 7.

3. Based on the scatter plots shown in Figure 7, a reasonable threshold for $min(\lambda_1, \lambda_2)$ is 0.5×10^7. Using this threshold, we determine a threshold for $R(x, y) = \det(\mathbf{M}(x, y)) - \alpha \mathrm{tr}(\mathbf{M}(x, y))$. This threshold is applied to locating corners in both images, which are displayed in Figure 8.

4. We now replicate the corner detection process, but use a smaller standard deviation for the Gaussian kernel, $\sigma = 2$. The eigenvalues are plotted once again and shown in Figure 9, from which it becomes clear that a different threshold should be chosen for corner detection. Selecting a threshold of 0.45×10^7, the corners detected are shown in Figure 10. We can clearly note that using a smaller σ results in sharper corners being detected, which is intuitive since a larger σ smooths the image and will make sharp corners (which have large gradients) be smoothed and less prominent.

Figure 2: Resizing Image 1 (968, 1428) -> (968, 957) with various methods

Figure 3: Resizing Image 2 (961, 1440) -> (961,1200) with various methods

Figure 4: Resizing Image 2 (961, 1440) -> (861,1200) with various methods

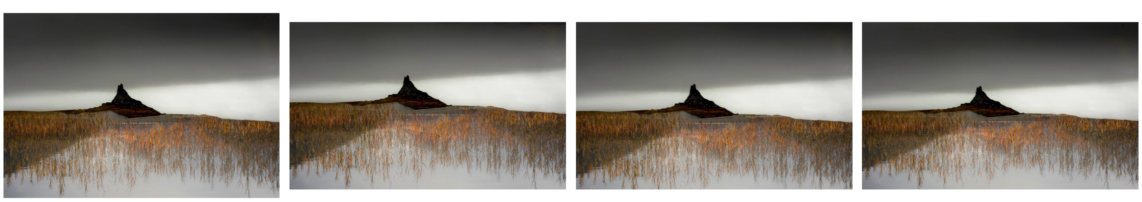

Figure 5: Resizing Image 3 (960, 1440) -> (870 , 1440) with various methods

(a) Original (b) Seam Carving (c) Cropping (d) Scaling

Figure 6: Resizing Image 3 (960, 1440) -> (870 , 1200) with various methods

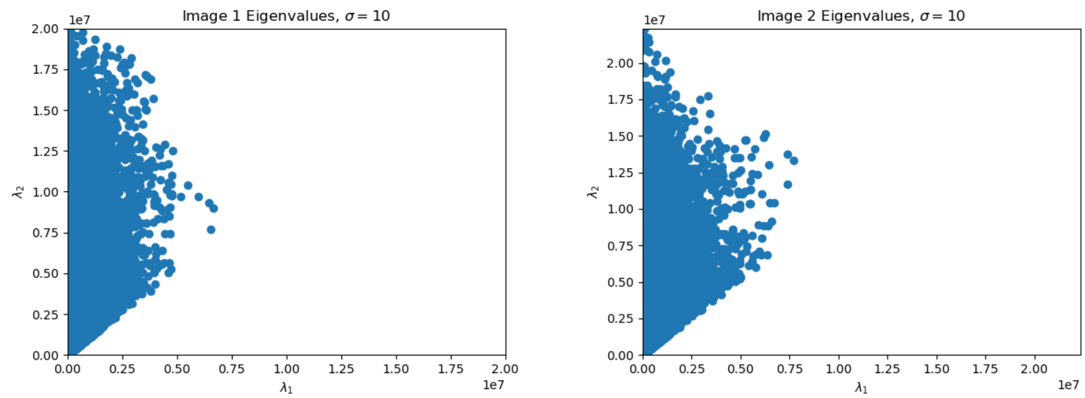

Figure 7: Eigenvalues for Image 1 and 2.

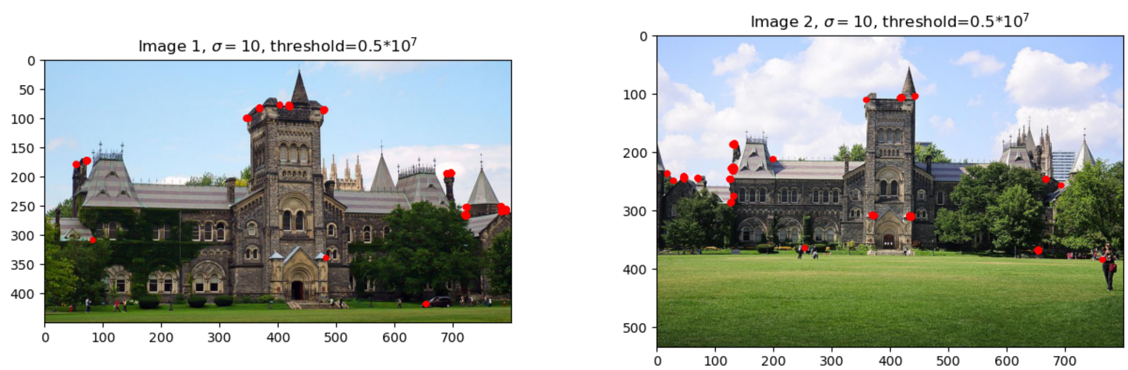

Figure 8: Corners detected for Images 1 and 2.

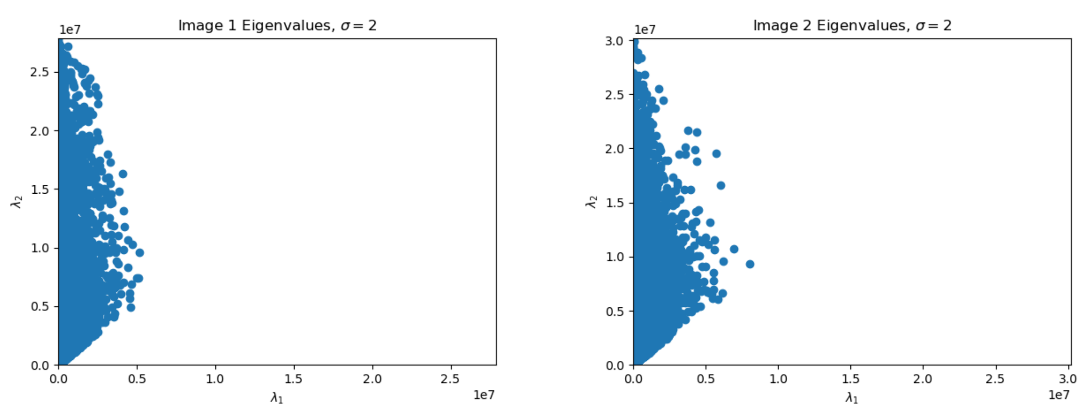

Figure 9: Eigenvalues for Image 1 and 2.

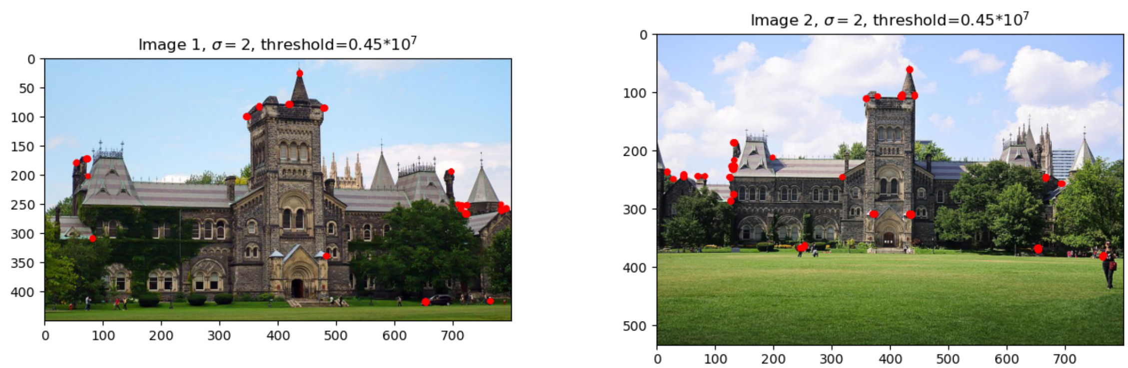

Figure 10: Eigenvalues for Image 1 and 2, using $\sigma = 2$.