A STUDY ON THE PARAMETERS OF BISTABLE STOCHASTIC RESONANCE SYSTEMS AND ADAPTIVE STOCHASTIC RESONANCE

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Abstract

Noise can improve the signal-to-noise ratio of many nonlinear dynamical systems. This "stochastic resonance" (SR) effect occurs in a wide range of physical and biological systems, and also occurs in engineering systems in signal processing, communications, and control. Most SR studies assume the signal and noise properties are known. To overcome this limitation, we give a type of adaptive stochastic resonance. This paper present a novel approach for investigating the relationships among the parameters of a bistable SR system, based on the characteristics of digital simulations. Using this approach, we can obtain the system's responses with varying parameters and these responses are consistent with experimental results. As a direct application, a system can be moved into the SR condition by conveniently changing its structural parameters, even when it is forced by an unknown periodic signal embedded in a white Gaussian noise with unknown statistical properties.

1 Stochastic Resonance- a Generic Model

The term stochastic resonance (SR) is given to a phenomenon that appears in nonlinear systems whereby the addition of noise to a system enhances its response to a periodic force [1-6]. The effect requires three basic ingredients: a form of threshold, a source of "noise," and a generally weak input source [3].

We consider an overdamped motion of a Brownian particle in a bistable potential in the presence of noise and periodic forcing [3]

$$\dot{x}(t) = -V'(x) + A_0\cos(\Omega t + \varphi) + \xi(t), \quad (1)$$

where A_0 is the signal amplitude and Ω is the modulation frequency. Here, we assume that the noise $\xi(t)$ is zero-mean, Gaussian white noise with an autocorrelation function given by $<\xi(t)\xi(0)> = 2D\delta(t)$. $V(x)$ denotes a symmetric bistable potential $V(x) = -\frac{a}{2}x^2 + \frac{b}{4}x^4$. The potential minimas are located at $\pm x_m$, with $x_m = \sqrt{a/b}$. The height of the potential barrier between the two minimas is $\Delta V = \frac{a^2}{4b}$.

In the absence of periodic forcing, $x(t)$ fluctuates around its local stable states. Noise-driven switching between the local equilibrium states occurs at the Kramers rate [7]

$$r_k = \frac{a}{\sqrt{2}\pi}\exp(-\frac{\Delta V}{D}). \quad (2)$$

When we apply a weak periodic forcing to the particle, noise-driven switching between the potential wells can become synchronized with the weak periodic forcing. This statistical synchronization takes place when the average waiting time $T_k(D) = 1/r_k$ between two noise-driven interwell transitions satisfies the time-scale matching condition, i.e.,

$$2T_k(D) = T_\Omega, \quad (3)$$

where T_Ω is the period of the periodic forcing [3].

1.1 The periodic response

For convenience, we choose the initial phase of the periodic driving $\varphi = 0$. When $t \to +\infty$, the memory of the initial conditions gets lost and the mean value $<x(t)>_{as}$ becomes a periodic function of time, i.e., $<x(t)>_{as} = <x(t + T_\Omega)>_{as}$. For small amplitudes, the response of the system to the periodic input signal can be written as [3]

$$<x(t)>_{as} = \bar{x}\cos(\Omega t - \bar{\phi}), \quad (4)$$

with amplitude $\bar{x}$ and a phase lag $\bar{\phi}$. Approximate expressions for $\bar{x}$ and $\bar{\phi}$ are

$$\bar{x}(D) = \frac{A_0 <x^2>_0}{D} \frac{2r_k}{\sqrt{4r_k{}^2 + \Omega^2}} \qquad (5)$$

and

$$\bar{\phi}(D) = \arctan(\frac{\Omega}{2r_k}), \qquad (6)$$

where $<x^2>_0$ is the D-dependent variance of the stationary unperturbed system ($A_0 = 0$). Eq. (5) has been shown to hold in leading order of the modulation amplitude $A_0 x_m / D$ for both discrete and continuous one-dimensional systems, i.e., $<x^2>_0 = x_m^2$.

1.2 Quantifiers for SR

The most common quantifiers for SR are the spectral (power) amplification η, and the output signal-to-noise ratio (SNR). η is an alternative quantity of $\bar{x}(D)$: the integrated power p_1 stored in the delta-like spikes of $S(\omega)$ at $\pm\Omega$ is $p_1 = \pi\bar{x}^2(D)$. Analogously, the forcing signal carries a total power $p_A = \pi A_0^2$. Hence the spectral amplification is

$$\eta \equiv p_1/p_A = [\bar{x}(D)/A_0]^2. \qquad (7)$$

Instead of taking the spectral amplification, it sometimes can be more convenient to use the notion of signal-to-noise ratio (SNR). We adopt the following definition of the signal-to-noise ratio

$$SNR = 2[\lim_{\Delta\omega \to 0} \int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega)d\omega]/S_N(\Omega). \qquad (8)$$

For a symmetric bistable system, we obtain [3]

$$SNR = \pi(A_0 x_m / D)^2 r_k. \qquad (9)$$

2 Behavior of the System with Varying Parameters

Given a time sequence, we want to detect the weak signal using a SR system. The detection performance depends on the specific discrete computer simulation method. In this section we will explain how we can get the system's input-output responses with varying parameters for fixed discrete computer simulation.

2.1 Discrete time implementation

We use a discrete computer simulation with the stochastic version of Euler's method (the Euler-Maruyama scheme [5, 8])

$$x_{n+1} = x_n + \Delta T(ax_n - bx_n^3 + u_n), \qquad (10)$$

with initial condition $x_0 = x(0)$. Here $u_n = A_0\cos(\Omega\Delta Tn) + \sigma w_n$ is the input time sequence. The zero-mean white noise sequence $\{w_n\}$ has unit variance $\sigma_\omega^2{}' = 1$. The process's sampling period T_s can differ from the time step ΔT in (10). The subsampling rate in our simulation is $1 : 50$. We ignored all aliasing effects.

2.2 Relationships among parameters

By making the change of variables

$$\bar{x} = x\sqrt{b/a}, \bar{t} = at, \bar{A}_0 = A_0\sqrt{b/a^3}, \bar{\sigma} = \sigma\sqrt{b/a^3}, \qquad (11)$$

where σ is the standard deviation of the white noise, Eq. (1) takes on a dimensionless form

$$\frac{1}{a}d\bar{x}(\frac{1}{a}\bar{t})/d(\frac{1}{a}\bar{t}) = \bar{x}(\frac{1}{a}\bar{t}) - \bar{x}^3(\frac{1}{a}\bar{t}) + \bar{A}_0\cos(\frac{1}{a}\Omega\bar{t} + \varphi)$$

$$+ \bar{\xi}(\frac{1}{a}\bar{t}). \qquad (12)$$

Comparing this equation with

$$\dot{x}(t) = x(t) - x^3(t) + A_0\cos(\Omega t + \varphi) + \xi(t), \qquad (13)$$

and considering their discrete realization using the same simulation method, the following rules can be obtained:

Under a small amplitudes restriction, if we plot the spectral amplification η , or the ratio of the output SNR of the system to that at its input as a function of signal and noise properties, the pixels in these two potted maps (according to Eqs. (1) and (13)) have the one-to-one relationships

$$\Omega_1 = \frac{1}{a}\Omega_0, \qquad (14)$$

$$\sigma_1 = \sqrt{\frac{a}{b}}\sigma_0, \qquad (15)$$

where (Ω_1, σ_1) is a pixel in the map corresponding to Eq. (1), and (Ω_0, σ_0) is a pixel in the map corresponding to Eq. (13). The relationships between the optimal values of a and b will be discussed in the next section.

2.3 System's input-output responses with varying parameters

On the basis of the above consideration, it is interesting to investigate the system's response when the parameters a and b vary. The results presented in this section concern a system forced by a sinusoid with frequency Ω_1 to which noise with known standard deviation σ_1 is added.
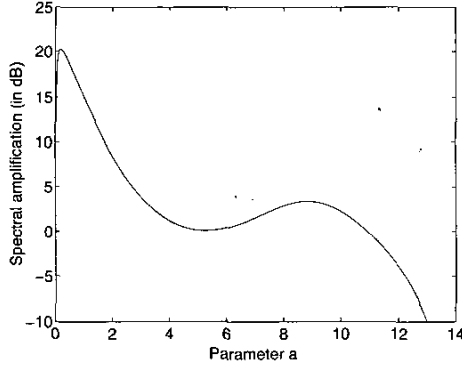
485

Figure 1: *The spectral amplification η in Eq. (20) as a function of parameter a.*

Figure 2: *The output SNR in Eq. (21) as a function of parameter a.*

We first show that the optimal value b can be determined when a is fixed. As above, Ω_1, σ_1 denotes the input signal and noise properties of the system with parameters a, b (for convenience, we use the notion of system (a, b)), and Ω_0, σ_0 denotes the input signal and noise properties of the system with parameters $a = b = 1$ (system $(1,1)$). The time-scale matching condition Eq. (3) can sometimes be used to estimate the optimal noise intensity for system $(1,1)$

$$\sigma_0 = (-\frac{1}{2\log(\sqrt{2}\Omega_0)})^{\frac{1}{2}}. \qquad (16)$$

However Eq. (16) cannot be used for any class of SR systems. In discrete simulations, the optimal noise intensity is different from σ_0 determined by (16). We can rewrite σ_0 as the form

$$\sigma_0 = f(\Omega_0) \approx M(-\frac{1}{2\log(\sqrt{2}\Omega_0)})^{\frac{1}{2}}. \qquad (17)$$

In a fixed discrete simulation, $f(\Omega_0)$ can be measured using experimental method. M is a constant.

Recall that a and Ω_1 are fixed. Eq.(14) gives

$$\Omega_0 = a\Omega_1. \qquad (18)$$

Then we put (18) into (17) to get the estimated optimal noise intensity for system $(1,1)$ when the input signal frequency is Ω_1 to system (a, b). Because the values of σ_0 and σ_1 are known, from (15) we can get the relationship between the optimal a and b

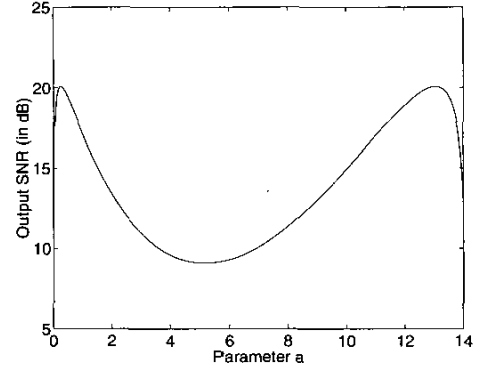$$\frac{a}{b} = (\frac{\sigma_1}{\sigma_0})^2. \qquad (19)$$

So in order to obtain the system's input-output responses with varying parameters, we just need to vary the value of a with b adjusting Eq. (19).

We put a reshaped form of (19) into (7) and (9) to get the corresponding responses. They vary with a.

$$\eta = (\frac{2\sqrt{2}a\exp(-\frac{a}{2\sigma_0^2})}{\sigma_0^2\sqrt{2a^2\exp^2(-\frac{a}{2\sigma_0^2}) + \pi^2\Omega_1^2}})^2, \qquad (20)$$

$$SNR = \frac{4a}{\sqrt{2}(\sigma_0\sigma_1)^2}\exp(-\frac{a}{2\sigma_0^2}). \qquad (21)$$

Here σ_0 is determined by (17) and (18). In (21), we omit A_0^2.

Figures 1, 2 show how the spectral amplification η and the output SNR for input signal frequency $\Omega_1 = 0.02\pi(rad/s)$ vary with parameter a. The parameter b has been selected as the optimal value using Eq. (19). The scale effect of coefficient M in (17) has been modified in the discrete simulations. From Figs. 1 and 2 we can see the tendencies of the input-output responses when a varies, and these tendencies coincide with the experiments results. There are similar results using the same approach for different input signal frequency Ω_1.

3 A SR System by Adaptively Selecting the Parameters

Given the prior information and an observation sequence we proceed as follows. We select a numerical method to simulate the SR system. Then we compute a crude estimate of the noise intensity by
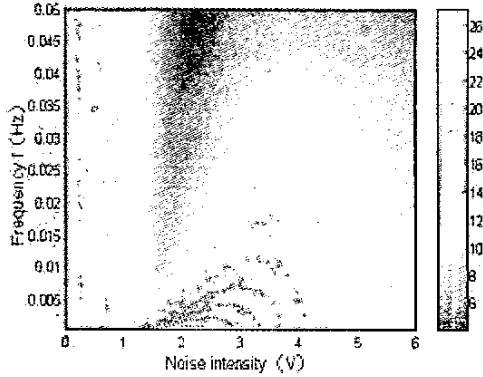
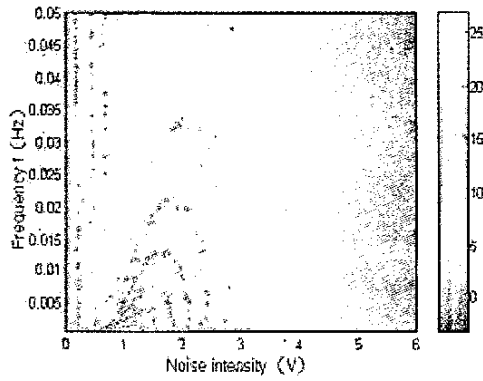**Figure 3:** *Output SNR as a function of signal frequency and noise intensity when $a = 1, b = 1$.*

Figure 5: *Output PSD through an adaptive SR system. $A = 0.2, \Omega = 0.02\pi, \sigma = 2$.*

Figure 4: *Output SNR as a function of signal frequency and noise intensity when $a = 1, b = 4$.*

Figure 6: *Output PSD through an adaptive SR system. $A = 0.2, \Omega = 0.02\pi, \sigma = 4$.*

measuring the variance of the observation sequence in those bands that do not contain signal energy.

Next, we pass the observation sequence through a set of resonators. The value of a in each resonator can be selected in the range where the system's response has good performance. In our simulations, a is selected in $(0, 15)$. If the signal frequency is known, b can be computed by (19). If the signal frequency is not known, b must be adjusted to fit different frequencies. Finally, a algorithm computes the frequency corresponding to the peak value for each spectrum, and choose the frequency with the maximum number of occurrences. It has been experimentally proved that this value is reached by the forcing frequency.
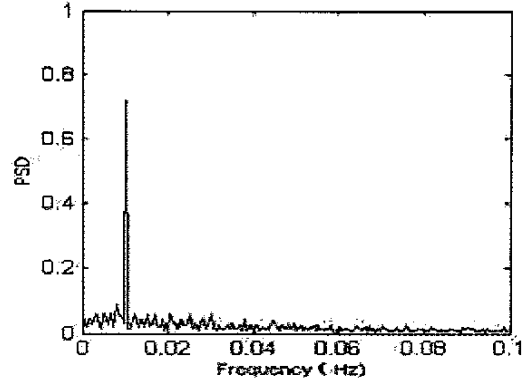
4 Experimental Results

Experimental results of our simulations for the relationships among parameters described in section 2.2 are presented in Figures 3 and 4. Figures 5 and 6 show some results corresponding to the selecting algorithm described in section 3.

Figures 3 and 4 compares the output SNR as a function of signal frequency and noise intensity through system $(1, 1)$ and system $(1, 4)$ respectively. Obviously they coincide with Eq. (19).

Figures 5 and 6 show two output PSDs with different signal and noise properties. We use $a = 0.1, b$ is determined by (19). A strong signal peak appears in the output PSD plots at the signal frequency.

5 Conclusion

In this paper, we studied the problem of investigating the relationships among the parameters of a discrete bistable SR system. We derived the system's responses with varying parameters. Using this approach, it is easy to design an adaptive SR system to detect a weak signal in the presence of heavy noise by conveniently changing its structural parameters.

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