2

Stochastic resonance: its definition, history, and debates

Stochastic resonance (SR), being an interdisciplinary and evolving subject, has seen many debates. Indeed, the term SR itself has been difficult to comprehensively define to everyone's satisfaction. In this chapter we look at the problem of defining stochastic resonance, as well as exploring its history. Given that the bulk of this book is focused on suprathreshold stochastic resonance (SSR), we give particular emphasis to forms of stochastic resonance where *thresholding* of random signals occurs. An important example where thresholding occurs is in the generation of action potentials by spiking neurons. In addition, we outline and comment on some of the confusions and controversies surrounding stochastic resonance and what can be achieved by exploiting the effect. This chapter is intentionally qualitative. Illustrative examples of stochastic resonance in threshold systems are given, but fuller mathematical and numerical details are left for subsequent chapters.

2.1 Introducing stochastic resonance

Stochastic resonance, although a term originally used in a very specific context, is now broadly applied to describe any phenomenon where the presence of internal noise or external input noise in a nonlinear system provides a better system response to a certain input signal than in the absence of noise. The key term here is *nonlinear*. Stochastic resonance cannot occur in a linear system – linear in this sense means that the output of the system is a linear transformation of the input of the system. A wide variety of performance measures have been used – we shall discuss some of these later.

The term *stochastic resonance* was first used in the context of noise enhanced signal processing in 1980 by Roberto Benzi, at the NATO International School of Climatology. Since then it has been used – according to the ISI Web of Knowledge database – in around 2000 publications, over a period of a quarter of a century. The frequency of publication, by year, of these papers is shown in Fig. 2.1. This figure

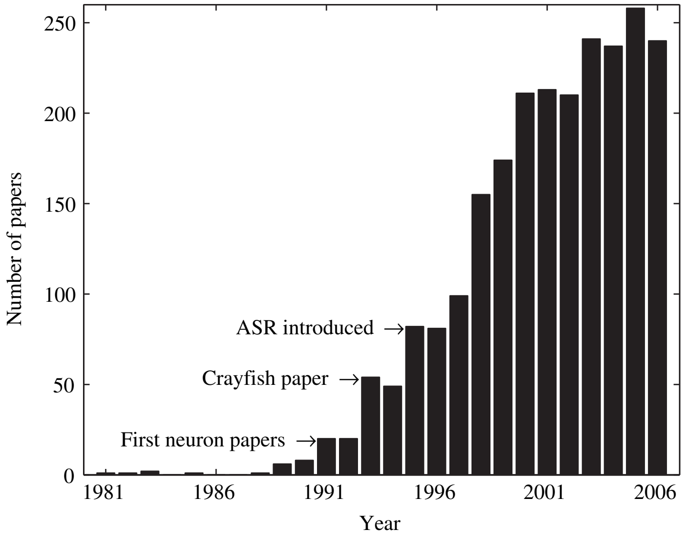


Fig. 2.1. Frequency of stochastic resonance papers by year – between 1980 and 2006 – according to the ISI Web of Knowledge database. There are several epochs in which large increases in the frequency of SR papers occurred. The first of these is between 1989 and 1992, when the most significant events were the first papers examining SR in neural models (Bulsara *et al.* 1991, Bulsara and Moss 1991, Longtin *et al.* 1991), and the description of SR by linear response theory (Dykman *et al.* 1990a). The second epoch is between about 1993 and 1996, when the most significant events were the observation of SR in physiological experiments on neurons (Douglass *et al.* 1993, Levin and Miller 1996), the popularization of array enhanced SR (Lindner *et al.* 1995, Lindner *et al.* 1996), and of aperiodic stochastic resonance (ASR) (Collins *et al.* 1995a). Around 1997, a steady increase in SR papers occurred, as investigations of SR in neurons and ASR became widespread.

illustrates how the use of the term *stochastic resonance* expanded rapidly in the 1990s, and is continuing to expand in the 2000s.

The 'resonance' part of 'stochastic resonance' was originally used because the signature feature of SR is that a plot of output signal-to-noise ratio (SNR) has a single maximum for some nonzero input noise intensity. Such a plot, as shown in Fig. 2.2, has a similar appearance to frequency dependent systems that have a maximum SNR, or output response, for some *resonant frequency*. However, in the case of SR, the resonance is 'noise induced', rather than at a particular frequency – see Section 2.3 for further discussion.

SR has been the subject of many reviews, including full technical journal articles (Jung 1993, Moss *et al.* 1994, Dykman *et al.* 1995, Gammaitoni *et al.* 1998, Wiesenfeld and Jaramillo 1998, Luchinsky *et al.* 1999a, Luchinsky *et al.* 1999b, Anishchenko *et al.* 1999, Hänggi 2002, Harmer *et al.* 2002, Wellens *et al.* 2004,

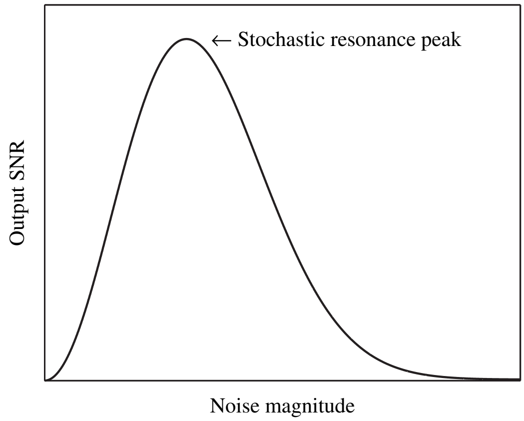


Fig. 2.2. Typical curve of output SNR vs. input noise magnitude, for systems capable of stochastic resonance. For small and large noise values, the output SNR is very small, while some intermediate nonzero noise value provides the maximum output SNR.

Shatokhin *et al.* 2004, Moss *et al.* 2004), editorial works (Bulsara *et al.* 1993, Astumian and Moss 1998, Petracchi *et al.* 2000, Abbott 2001, Gingl 2002), book chapters (Wiesenfeld 1993a), and magazine articles (Wiesenfeld 1993b, Moss and Wiesenfeld 1995, Bulsara and Gammaitoni 1996).

Some of the most influential workshops and conferences on SR include those held in San Diego, USA, in 1992 (published as a special issue of the journal *Il Nuovo Cimento*) (Moss *et al.* 1992) and 1997 (Kadtke and Bulsara 1997), Elba Island, Italy, in 1994 (published as a special issue of *Journal of Statistical Physics*) (Bulsara *et al.* 1994), and Ambleside, UK, in 1999 (Broomhead *et al.* 2000). Various other special issues of journals have been devoted to SR, such as the September 1998 issue of *Chaos*, the September 2000 issue of *Chaos Solitons and Fractals*, and the September 2002 issue of *Fluctuation and Noise Letters*.

There have been articles and letters on SR published in the prestigious journal, *Nature* (Douglass *et al.* 1993, Moss *et al.* 1993, Wiesenfeld and Moss 1995, Moss and Pei 1995, Bezrukov and Voydanoy 1995, Collins *et al.* 1995b, Noest 1995, Collins *et al.* 1995c, Levin and Miller 1996, Bezrukov and Voydanoy 1997b, Astumian *et al.* 1997, Dykman and McClintock 1998, Collins 1999, Russell *et al.* 1999, Moss and Milton 2003, Bulsara 2005, Badzey and Mohanty 2005). A book has been written exclusively on SR (Andò and Graziani 2000), as well as a large section of another book (Anischenko *et al.* 2002), and sections on SR written in popular science books (von Baeyer 2003, Kosko 2006). It has also started to appear in more general textbooks (Gerstner and Kistler 2002).

SR has been widely observed throughout nature – it has been quantified in such diverse systems as climate models (Benzi *et al.* 1982), electronic circuits (Fauve and Heslot 1983, Anishchenko *et al.* 1994), differential equations (Benzi *et al.* 1985, Hu *et al.* 1990), ring lasers (McNamara *et al.* 1988), semiconductor lasers (Iannelli *et al.* 1994), neural models (Bulsara *et al.* 1991, Longtin *et al.* 1991), physiological neural populations (Douglass *et al.* 1993), chemical reactions (Leonard and Reichl 1994), ion channels (Bezrukov and Voydanoy 1995), SQUIDs (Superconducting Quantum Interference Devices) (Hibbs *et al.* 1995), the behaviour of feeding paddlefish (Russell *et al.* 1999, Freund *et al.* 2002), ecological models (Blarer and Doebeli 1999), cell biology (Paulsson and Ehrenberg 2000, Paulsson *et al.* 2000), financial models (Mao *et al.* 2002), psychophysics (Ward *et al.* 2002), carbon-nanotube transistors (Lee *et al.* 2003, Lee *et al.* 2006), nanomechanical oscillators (Badzey and Mohanty 2005, Bulsara 2005), and even social systems (Wallace *et al.* 1997).

The first highly successful application inspired by SR involves the use of electrically generated subthreshold stimuli in biomedical prosthetics to improve human balance control and somatosensation (Priplata *et al.* 2004, Priplata *et al.* 2003, Collins *et al.* 2003, Moss and Milton 2003, Harry *et al.* 2005). This work led to James J. Collins winning a prestigious MacArthur Fellowship in October 2003 (Harry *et al.* 2005).

Prior to this, Collins was also an author on a correspondence to *Nature* (Suki *et al.* 1998) on a noise-enhanced application, described as analogous to SR. The *Nature* paper used a model to verify the experimental results of Lefevre *et al.* (1996). Random noise was introduced into the operation of mechanical life-support ventilators, in order to more closely replicate natural breathing. It was found that added noise enhanced artificial ventilation in several ways. See Brewster *et al.* (2005) for a review.

2.2 Questions concerning stochastic resonance

There are a number of misconceptions and controversies about stochastic resonance. The following list of questions attempts to encapsulate the main points of contention:

(i) What is the definition of stochastic resonance?
(ii) Is stochastic resonance exploited by the nervous system and brain as part of the neural code?
(iii) Does stochastic resonance occur only if a signal's power is weak compared with the power of the noise in a system?
(iv) Can stochastic resonance lead to a signal-to-noise ratio gain?
(v) Was stochastic resonance known prior to the first use in 1981 of the term 'stochastic resonance'?
(vi) How and when is stochastic resonance different from a signal processing technique called *dithering*?

Although question (ii) is quite clearly the interesting scientific question, and seemingly the motivation behind much SR research, it would appear the other questions in the above list have sometimes provided a diversion. The problem is that reaching a consensus on the answers to questions (ii)–(vi) really depends on an agreed answer to question (i).

The broadest possible definition of stochastic resonance is that it occurs when randomness may have some positive role in a signal processing context. Given this definition, *we believe* that the answers to these questions are: (ii) yes, although it is difficult to prove, the brain – or at least, some parts of the nervous system – would almost certainly not function as it does if it were completely deterministic; (iii) no, randomness can have a positive role even if it is only a small amount of randomness; (iv) yes, in the information-theoretic sense, random noise in a system can lead to a less noisy output signal, provided that the system is nonlinear; (v) yes, randomness has been known to have a positive role in many circumstances for decades, if not centuries; and (vi) stochastic resonance occurs when dithering is used – dithering can be described as the exploitation of SR.

On the other hand, if the definition of stochastic resonance is restricted to its original narrow context, then the answers to questions (ii)–(vi) change to: (ii) maybe – this is yet to be conclusively answered, (iii) yes, (iv) no, (v) no, and (vi) dithering is quite different from SR, as is comprehensively explained in Gammaitoni (1995a).

This discussion is intended to illustrate that the debate on the topics listed above can depend crucially on what we mean by *stochastic resonance*. In the remainder of this chapter we discuss some of these issues in more detail in order to bring some clarity to the debate and to illustrate the pitfalls and controversies for readers new to the stochastic resonance field, or who have held only a peripheral interest in the area.

2.3 Defining stochastic resonance

SR is often described as a counter-intuitive phenomenon. This is largely due to its historical background; in the first decade and a half since the coining of the term in 1980, virtually all research into SR considered only systems driven by a combination of a periodic single-frequency input signal and broadband noise. In such systems, a natural measure of system performance is the output signal-to-noise ratio (SNR), or, more precisely, often the ratio of the output power spectral density (PSD) at the input frequency to the output noise floor PSD measured with the signal present. The noise floor is measured with the signal present, rather than absent, as the output noise may change if the signal is not present. This is because the signal and output noise are generally not additive in a nonlinear system, or, in other words, the output noise is signal dependent.

In linear systems driven by periodic input signals – in fact, any linear system – it is well known that the output SNR is maximized in the absence of noise. When systems are analyzed in terms of SNR, it is the norm to assume that noise is a problem, usually with good reason. This means that observations of the presence of noise in a system providing the maximum output SNR are often seen to be highly counter-intuitive – see Kosko (2006, p. 149) for further discussion.

However, although not all noise can be described as a random process – it can, for example, be constant or deterministic (even chaotic) – SR research has tended to focus on the stochastic case. The most common assumption is that the noise is Gaussian distributed, and white – that is, constant in power across all frequencies.

When it is noted that there are many examples of systems or algorithms where randomness is of benefit, SR does not seem quite so counter-intuitive. Such examples include:

- Brownian ratchets (Doering 1995) – mechanical applications of this idea include self-winding (batteryless) wristwatches (Paradiso and Starner 2005);
- dithering in signal processing and analogue-to-digital conversion (Schuchman 1964, Gray and Stockham 1993, Dunay *et al.* 1998, Wannamaker *et al.* 2000b);
- Parrondo's games – the random combination of losing games to produce a winning game (Harmer and Abbott 1999);
- noise-induced linearization (Yu and Lewis 1989, Dykman *et al.* 1994), noise-induced stabilization (Toral *et al.* 1999, Basak 2001), noise-induced synchronization (Neiman 1994), noise-enhanced phase coherence (Neiman *et al.* 1998), and noise-induced order (Matsumoto and Tsuda 1985);
- the use of mixed (probabilistic) optimal strategies in game theory (von Neumann and Morgenstern 1944);
- random switching to control electromagnetic compatibility performance (Allison and Abbott 2000);
- random search optimization techniques, including genetic algorithms (Gershenfeld 1999) and simulated annealing (Kirkpatrick *et al.* 1983);
- random noise radars – that is, radars which transmit random noise waveforms in order to provide immunity from jamming, detection, and interference (Narayanan and Kumru 2005);
- techniques involving the use of Brownian motion for solving nonstochastic partial differential equations, such as the Dirichlet problem (Øksendal 1998);
- stochastic iterative decoding in error control coding applications (Gaudet and Rapley 2003);
- estimation of linear functionals (Plaskota 1996a, 1996b).

Further discussion and other examples appear in Harmer and Abbott (2001), Abbott (2001), Zozor and Amblard (2005), and Kosko (2006).

SR was initially considered to be restricted to the case of periodic input signals. However, now the literature reveals that it is widely used as an all-encompassing term, whether the input signal is a periodic sine-wave, a periodic broadband signal, or aperiodic, and whether stationary, nonstationary (Fakir 1998a), or cyclostationary (Amblard and Zozor 1999). An appropriate measure of output performance depends on the task at hand, and the form of input signal. For example, for periodic signals and broadband noise, SNR is often used, while for random aperiodic signals, SR is usually measured by mutual information or correlation.

Is SR a bona fide *resonance?*

The definition of SR, and the word 'resonance' itself, have both been objects of debate. In particular, 'resonance' is usually thought of in the sense of a resonant frequency, rather than an optimal noise intensity. For the most conventional kinds of SR, 'resonance' was more or less resolved as being appropriate after a new way – using residence time distributions – of looking at SR found that the effect was a *bona fide* kind of resonance (Gammaitoni *et al.* 1995a), although there was some debate about this too (Choi *et al.* 1998, Giacomelli *et al.* 1999, Marchesoni *et al.* 2000). According to Gammaitoni,

> Even if the word resonance has been questioned since the very beginning, it has been recently demonstrated . . . that, for a diffusion process in a double-well system, the meaning of *resonance* as the matching of two characteristic frequencies (or physical time scales) is indeed appropriate for such a phenomenon. (Gammaitoni 1995a)

Stochastic resonance, static systems, and dithering

Although Gammaitoni's work appeared to end the debate for *bistable systems* – see Section 2.4 for discussion about SR in bistable systems – it opened up a new question: Is the term *stochastic resonance* appropriate for systems consisting of simple static threshold[1] nonlinearities? There are at least two reasons why this has been debated.

First, noise-enhanced behaviour in static threshold nonlinearities also occurs in a signal processing technique known as *dithering*. Dithering involves deliberately adding a random – or pseudo-random – signal to another signal, prior to its digitization or quantization (Schuchman 1964). It is most often associated with audio

[1] The term 'static threshold' often appears in relation to SR research. It is used to differentiate between *dynamical* systems – such as the bistable potential wells traditionally used in studies of SR – and nondynamical, or *static*, systems (Gingl *et al.* 1995a, Gingl *et al.* 1995b). A system is called static when nonlinear deformation – SR cannot occur in a linear system – of an input signal is not governed by time evolving differential equations, but by simple rules that produce an output signal based on the instantaneous value of the input signal.

or image processing, where the effect of the added noise signal, called the *dither signal*, is to randomize the error signal introduced by quantization. This randomization, although increasing the total power of the noise at the output, reduces undesirable harmonic distortion effects introduced by quantization.

Given that dithering is a way of improving a system using the presence of noise, the question is how to distinguish it from SR? That is, if the system being studied resembles dithering, should noise-enhanced behaviour be called dithering, rather than SR? Since dithering existed for decades prior to the first studies of SR, some SR traditionalists prefer to classify dithering as a completely separate form of noise-enhanced signal processing. However, this requires a quite restrictive definition of SR, where the system must be dynamical, and the presence of noise enables a matching of two time-scales. Such a definition has been superseded in the broader literature. As discussed below, the contemporary definition of SR is such that dithering can be described as a technique that *exploits* SR, and the two terms are not mutually exclusive.

As for determining whether an experiment or application should be labelled as a form of dithering that gives rise to SR, we suggest that the most natural distinction is in the motivation. If we are interested in studying noise-enhanced behaviour in a model or experiment, and how performance varies with noise intensity – for example, examining whether SR occurs – then there is no compulsion to call this dithering. Indeed, in most cases such as this, the signal processing goal is different from that when dither is used in image and audio processing, and therefore different performance measures are appropriate. On the other hand, as implied by Wannamaker *et al.* (2000a), if we are interested in deliberately introducing a random signal prior to digitization, with the sole aim of modifying the effects of quantization noise, then this could naturally be called dithering, while the fact that the presence of noise achieves the aim is the occurrence of SR. Further discussion of this problem is presented in Section 3.4 of Chapter 3.

Secondly, the initial questions about whether noise-enhanced behaviour in static threshold systems should be called *stochastic resonance* relate to whether a *bona fide* resonance occurs, and were concisely expressed by Gammaitoni, who states that:

> the use of the term resonance is questionable and the notion of noise induced threshold crossings is more appropriate

and that:

> this frequency matching condition, instead, does not apply to the threshold systems we consider here. (Gammaitoni 1995a)

Although these points are certainly fair, given common definitions of the word 'resonance', we take the point of view that such questions of semantic nomenclature are no longer relevant and argue this point in the following.

Stochastic resonance: noise-enhanced signal processing

The term *stochastic resonance* is now used so frequently in the much wider sense of being the occurrence of any kind of noise-enhanced signal processing that this common usage has, by 'weight of numbers', led to a re-definition. Although many authors still define SR only in its original narrow context, where a resonance effect can be considered to be *bona fide*, in line with the evolution of languages, words or phrases often end up with a different meaning from their original roots.

We emphasize here the fact that SR occurs only in the context of *signal* enhancement, as this is the feature that sets it apart from many of the list of randomness-enhanced phenomena above, which could all be described as benefiting in some way from noise, and yet cannot all be defined in terms of an enhanced signal. Furthermore, SR is usually[2] understood to occur in systems where there are both well-defined *input* and *output* signals, and the optimal output signal, according to some measure, occurs for some nonzero level and type of noise.

Indeed, Bart Kosko in his popular science book, *Noise*, succinctly defines SR as meaning 'noise benefit' (Kosko 2006). Implied in this definition – see Kosko (2006, pp. 148–149) – is the caveat that a *signal* should be involved, meaning that SR is 'noise benefit in a signal processing system', or alternatively 'noise-enhanced signal processing'. Put another way, SR occurs when the output signal from a system provides a better representation of the input signal than it would in the complete absence of noise.

With this definition in mind, we now provide a broad history of SR research.

2.4 A brief history of stochastic resonance

The early years: 1980–1992

The term *stochastic resonance* was first used[3] in the open literature – at least, in the context of a noise-optimized system – by Benzi, Sutera, and Vulpiani, in 1981, as a name for the mechanism they suggested is behind the periodic behaviour of the

[2] An early paper on SR examined a system in which SR is said to occur that did not have any input signal (Gang *et al.* 1993).

[3] A search in the Inspec database for the phrase 'stochastic resonance' returns a number of published papers prior to 1981, commencing in 1973 (Frisch *et al.* 1973). However, these all use the term 'stochastic resonance' in the context of 'stochastic wave parametric resonance', 'stochastic magnetic resonance', or other stochastic systems, where the term 'resonance' has nothing to do with noise benefits.

Earth's ice ages (Benzi *et al.* 1981, Benzi *et al.* 1982, Benzi *et al.* 1983). The term was apparently also mentioned one year earlier by Benzi (Abbott 2001) in discussions at a workshop on climatic variations and variability (Berger 1980). Climate records show that the period for the Earth's climate switching between ice ages and warmer periods is around 100 000 years. This also happens to be the period of the eccentricity of the Earth's orbit. However, current theories suggest that the eccentricity is not enough to cause such dramatic changes in climate. Benzi *et al.* (1981) – and, independently, Nicolis (1981), Nicolis (1982) – suggested that it is the combination of stochastic perturbations in the Earth's climate, along with the changing eccentricity, which is behind the ice age cycle. Benzi *et al.* (1981) gave this mechanism the name *stochastic resonance*. In its early years, the term was defined only in the very specific context of a bistable system driven by a combination of a periodic force and random noise. Note that Benzi *et al.* (1981) considered the Earth's orbital eccentricity to be a periodic driving signal, and the stochastic perturbations as the random noise. Interestingly, as pointed out in Hohn (2001), this theory for explaining the ice ages is still a subject of debate, even though SR is now well established as a general phenomenon in a huge variety of other systems.

Further mathematical investigations of SR in the following few years included the demonstration of SR in a simple two-state model (Eckmann and Thomas 1982), the quantification of SR in the Landau–Ginzberg equation (Benzi *et al.* 1985) – which is a partial differential equation – and the observation of SR in a system undergoing a Hopf bifurcation (Coullet *et al.* 1985).

Experimental observations of SR in physical systems also came quickly. In 1983, SR was reported in a Schmitt trigger electronic circuit (Fauve and Heslot 1983). Three years later, SR was observed in a bidirectional ring laser, where the deliberate addition of noise was shown to lead to an improved output SNR (McNamara *et al.* 1988, Vemuri and Roy 1989). Both the Schmitt trigger circuit and the bidirectional ring laser are bistable systems; it was originally thought that bistability is a necessary condition for SR (McNamara and Wiesenfeld 1989).

The ring laser paper brought about a large increase in interest in SR – approximately 50 published journal papers from 1989 to 1992 – with a number of theoretical treatments being published in the next few years, for example Gammaitoni *et al.* (1989a), Debnath *et al.* (1989), McNamara and Wiesenfeld (1989), Gammaitoni *et al.* (1989b), and Jung and Hänggi (1991). In particular, Jung and Hänggi (1991) provide a highly cited theoretical study that extends the results of McNamara and Wiesenfeld (1989) to show that SR can lead to peaks in the output PSD at harmonics of the driving frequency. This period also included the important realization that SR, in the limit of relatively small signal and relatively strong noise, can be described using linear response theory (Dykman *et al.* 1990a);

see Dykman *et al.* (1995) for a review. This insight was to lead directly to the realization that neither bistability nor a threshold was necessary for SR to occur, and it could, for example, occur in monostable systems (Stocks *et al.* 1993).

Incidentally, it has been suggested that for bistable systems the mechanism of SR has been known about for over 75 years. According to Dykman *et al.* (1995) and Luchinsky *et al.* (1998), the work of prolific Nobel Prize-winning chemist, Peter Debye, on the dielectric properties of polar molecules in a solid (Debye 1929), effectively shows SR behaviour. This fact is not entirely clear from a reading of Debye (1929), as a formula for SNR is not derived, and there is no comment made that some measure is optimized by a nonzero noise intensity; nevertheless, the relevant page is 105, and further explanation can be found in Dykman *et al.* (1995, pp. 669–671) and in Luchinsky *et al.* (1998, pp. 930–933). See also Luchinsky *et al.* (1999a). Dykman *et al.* (1995) also comment that the work of Snoek in the same field in the 1940s forms part of the 'prehistory of stochastic resonance'. More recently, Kalmykov *et al.* (2004) pointed out that another work of Debye can be related to SR, stating that

> it is possible to generalize the Debye–Fröhlich model of relaxation over a potential barrier . . . and so to estimate the effect of anomalous relaxation on the stochastic resonance effect.

However, this does not imply conclusively that Debye knew of SR, only that his work has been generalized to show SR.

Expansion: 1993–1996

The first important milestone in the period from 1993 to 1996 was the initial investigation into SR in neural and excitable systems. Prior to this time SR was only observed in bistable systems. The second important milestone was the extension of SR from periodic to aperiodic driving signals. We shortly discuss both of these developments in more detail.

Further extensions that straddle the initial and expansion periods include analysis of different sorts of noise from the standard additive Gaussian white noise, including multiplicative noise (Dykman *et al.* 1992), coloured noise (Hänggi *et al.* 1993), $1/f$ noise (Kiss *et al.* 1993), and harmonic noise (Neiman and Schimansky-Geier 1994), as well as the observation that coupling together more than one SR-capable device can lead to increased output performance (Wiesenfeld 1991, Jung *et al.* 1992).

Stochastic resonance in excitable systems

An excitable system is one that has only one stable or rest state and a threshold above which an excited state can occur, but that is not stable. The excited state eventually decays to the rest state (Gammaitoni *et al.* 1998). Neurons are a significant example of an excitable system. The first papers on the observation of SR in neural models were published by Bulsara *et al.* (1991), Bulsara and Moss (1991), and Longtin *et al.* (1991), although an earlier paper by Yu and Lewis (1989) effectively demonstrates how noise in a neuron model linearizes the system response, a situation later described as SR. However, it was Longtin *et al.* (1991) that brought the initial attention of the broader scientific community to SR, after being featured in a *Nature* 'News and Views' article (Maddox 1991). However, research into SR in neurons and neural models only really took off in 1993, when a heavily cited *Nature* article reported the observation of SR in physiological experiments on crayfish mechanoreceptors[4] (Douglass *et al.* 1993). In the same year a heavily cited paper by Longtin (1993) on SR in neuron models also became widely known and since then many published papers examine stochastic resonance in neurons – whether in mathematical models of neurons, or in biological experiments on the sensory neurons of animals – triggering a large expansion in research into SR. Stochastic resonance in neurons is discussed further in Section 2.9. However, next we discuss the departure of SR research from its original context of periodic input signals – that is, aperiodic stochastic resonance.

Aperiodic stochastic resonance

This important extension of SR was first addressed in Hu *et al.* (1992), who examine a bistable system driven by an aperiodic input signal consisting of a sequence of binary pulses subject to noise. Extending SR to aperiodic signals is significant because, while some important signals in nature and electronic systems are periodic, very many signals are not.

Further discussion of aperiodic SR was not undertaken until 1995, when Collins *et al.* (1995a) popularized the term aperiodic stochastic resonance (ASR) in an investigation of an excitable system – a FitzHugh-Nagumo neuron model – subject to an aperiodic signal. Shortly afterwards, a letter to the journal *Nature* demonstrated the same behaviour in *arrays* of FitzHugh–Nagumo neuron models (Collins *et al.* 1995b). These two studies are amongst the most highly cited of all papers on SR. The *Nature* paper is not without criticism however, and a letter to *Nature* by Noest (1995), followed by a reply from Collins *et al.* (1995c), served only to enhance the exposure of this work to the field.

As an aside, in the abstract of Collins *et al.* (1996a), the term *aperiodic* appears to be equated with the term *broadband*. However, of course, a signal can be periodic

[4] Note, however, that it can be convincingly argued that these experiments do not prove that neurons utilize SR in any way. This is because both the signal and the noise were applied *externally* to the mechanoreceptors, and the fact that SR occurs only demonstrates that these cells are nonlinear. It remains an open question as to whether neurons make use of *internally* generated noise and SR effects.

and broadband, for example a periodically repeated radar chirp pulse, or a simple square wave. Equating broadband with aperiodic appears to be due to the early work in SR considering only single frequency periodic signals (Gammaitoni *et al.* 1998), usually a sine wave of the form $x(t) = A\cos(\omega t)$, where $\omega = 2\pi f$ is the frequency of the sine wave in radians per second. In general, an aperiodic signal can be considered to be broadband, but a broadband signal does not need to be aperiodic. The relevance to SR research is that the SNR measure used for single frequency signals is not applicable for either *broadband and periodic* signals or aperiodic signals.

Shortly after these initial two papers by Collins *et al.*, the same authors also demonstrated ASR in three other theoretical models: a bistable-well system, an integrate-and-fire-neuronal model, and the Hodgkin–Huxley neuronal model (Collins *et al.* 1996a); as well as in physiological experiments on rat mechanoreceptors (Collins *et al.* 1996b). In these papers, 'power-norm' measures – both the correlation between the input and output signals, and the normalized power-norm, or correlation coefficient – are used to characterize ASR, instead of SNR.

Almost simultaneously with Collins *et al.*, an alternative approach to measuring SR for aperiodic signals was proposed by Kiss (1996). By contrast, this work employs an SNR-like measure based on cross-spectral densities to show the existence of SR in simple threshold-based systems. The technique is demonstrated on a signal similar to that used by Hu *et al.* (1992), but can theoretically be applied to any broadband input. Although Collins *et al.* thought SNR measures to be inappropriate for aperiodic signals, it was soon demonstrated by Neiman *et al.* (1997) that the correlation measures of Collins *et al.* can be derived from the cross-spectral density, which forms the basis of the SNR measure proposed in Kiss (1996).

Furthermore, the SNR measure used by Kiss can be rewritten in terms of a correlation-like measure. This issue is examined in detail in McDonnell *et al.* (2004a). Kiss considers that one of the advantages of his method is that it is robust to phase shifts between the input and output signal, whereas the correlation-based measures of Collins *et al.* are not (L. B. Kish, personal communication, 2006).[5] However, this only applies to the first papers of Collins *et al.*, since in Collins *et al.* (1996a) – unlike Collins *et al.* (1995a) – the cross-correlation between the input and output signals is calculated as a function of time delay, which is the more conventional way of calculating cross-correlation. Such a measure is indeed robust to

[5] Note that Kiss and Kish are the same person. All pre-1999 papers spell his name using 'Kiss,' whereas later papers use the spelling 'Kish.' The pronunciation is the same in both cases – 'Kish.' This book will cite and refer to the spelling used in the corresponding publication.

phase shifts; the maximum cross-correlation simply occurs for a nonzero time lag. In fact, the cross-correlation is an ideal measurement of a time delay, and hence of phase shift.

Of these initial studies of ASR, it is the work by Collins *et al*. that gained the greatest exposure (Collins *et al.* 1995b, Collins *et al.* 1995c), with the result that many expositions on ASR have also used correlation-based measures. However, also of great influence is a paper presented before the work of Collins *et al*. at a 1994 workshop on stochastic resonance (Bulsara *et al.* 1994), which shows that SR can occur for an aperiodic input signal, and the mutual information measure. This paper was subsequently published in a journal (DeWeese and Bialek 1995), and is discussed in detail in Section 2.7

Subsequent to DeWeese and Bialek (1995), the next paper to use mutual information as a measure of ASR is a highly cited paper published in the journal *Nature*, which uses mutual information to experimentally show that SR occurs in the cercal sensory system of a cricket when noise is applied externally (Levin and Miller 1996). In the same year, two articles were published in the same issue of *Physical Review E* – those of Bulsara and Zador (1996) and Heneghan *et al.* (1996) – which theoretically examine the use of mutual information to measure SR for aperiodic signals. These papers paved the way for the use of information theory in SR research. The same year also saw a paper that applies other information-theoretic measures – dynamical entropies and Kullback entropy – to measure SR; however, unlike ASR, the system considered is driven by a periodic signal (Neiman *et al.* 1996).

Further extensions

Some further important extensions of SR in this period included the use of chaotic dynamics to act as the 'noise' source leading to SR (Anishchenko *et al.* 1993), analysis of quantum stochastic resonance (Lofstedt and Coppersmith 1994, Grifoni and Hänggi 1996), the first comprehensive studies of array enhanced stochastic resonance (Lindner *et al.* 1995, Lindner *et al.* 1996) – see Section 2.5 – and the observation of spatio-temporal stochastic resonance (Jung and Mayerkress 1995, Löcher *et al.* 1996, Marchesoni *et al.* 1996, Vilar and Rubi 1997).

Consolidation 1997–2007

Between 1997 and 2000 there was an acceleration in the rate of publication of papers either directly about SR, or listing SR as a keyword, and the number continues to grow. The main development in this period is that a large number of papers examine systems showing ASR for aperiodic input signals. For example some of the papers in 1997–1998 are Chialvo *et al.* (1997), Gailey *et al.* (1997), Eichwald and Walleczek (1997), Neiman *et al.* (1997), Vaudelle *et al.* (1998), Fakir (1998a), Fakir (1998b), and also Godivier and Chapeau-Blondeau (1998). Other developments in more recent years include:

- the first analysis of SR in discrete time rather than continuous time systems (Zozor and Amblard 1999, Zozor and Amblard 2001),
- approaches to controlling stochastic resonance (Mitaim and Kosko 1998, Gammaitoni *et al.* 1999, Löcher *et al.* 2000, Kosko and Mitaim 2001, Mitaim and Kosko 2004),
- comprehensive theoretical studies into the limits of when SR can occur, leading to the 'forbidden interval theorem', which states that SR will occur unless a simple condition relating the threshold value to the mean of the noise is violated (Kosko and Mitaim 2003, Mitaim and Kosko 2004, Patel and Kosko 2005, Lee *et al.* 2006),
- the observation of SR in carbon nanotube transistors (Lee *et al.* 2003, Lee *et al.* 2006),
- the observation of an SR-like effect called 'diversity-induced resonance' (Tessone *et al.* 2006) and
- a comprehensive signal-processing-based approach to understanding the detection capabilities of SR from a binary hypothesis testing perspective (Chen *et al.* 2007).

As discussed already, the popularization of SR as a phenomenon that is not restricted to periodic signals has seen its original definition expanded to encompass almost any system in which input and output signals can be defined, and in which noise can have some sort of beneficial role. This period of growth however appears to have slowed a little between 2001 and 2007. The general consensus appears to be that the most recent highly significant result in SR research is its expansion to aperiodic input signals. It could be said however, due to the number of papers investigating SR in neurons, that the most significant discovery on SR may be yet to come – see Section 2.9.

2.5 Paradigms of stochastic resonance

Stochastic resonance in bistable systems

As mentioned above, in the early years of the SR community, it was thought that SR effects were restricted to bistable systems (Fauve and Heslot 1983, Fox 1989, Dykman *et al.* 1990a, Gammaitoni *et al.* 1990, Dykman *et al.* 1990b, Zhou and Moss 1990, Jung and Hänggi 1991, Gammaitoni *et al.* 1991, Dykman *et al.* 1992). Here, following Harmer *et al.* (2002), a simple bistable system consisting of a periodically driven double-well potential is described.

A classic one-dimensional nonlinear system that exhibits stochastic resonance is the damped harmonic oscillator. Following Lanzara *et al.* (1997), this can be modelled with the Langevin equation of motion in the form

$$m\frac{d^2x(t)}{dt^2} + \gamma\frac{dx(t)}{dt} = -\frac{dU(x)}{dx} + \sqrt{D}\xi(t). \tag{2.1}$$

This equation describes the motion of a particle of mass m moving in the presence of friction, γ. The restoring force is expressed as the gradient of some bistable or multistable potential function $U(x)$. In addition, there is an additive stochastic force $\xi(t)$ with intensity D. Typically, this is white Gaussian noise with mean and autocorrelation given respectively by

$$\langle\xi(t)\rangle = 0 \quad \text{and} \quad \langle\xi(t)\xi(t')\rangle = \delta(t-t'). \tag{2.2}$$

This implies that $\xi(t)$ and $\xi(t')$ are statistically independent for $t \neq t'$. The angled brackets $\langle\cdot\rangle$ denote an ensemble average.

When $U(x)$ is a bistable potential, Eq. (2.1) may model several physical processes, ranging from the dynamics of a nonlinear elastic mechanical oscillator to the transient dynamics of a laser. A simple symmetric bistable potential has the form of a standard quartic

$$U(x) = -a\frac{x^2}{2} + b\frac{x^4}{4}. \tag{2.3}$$

If the system is heavily damped, the inertial $m\frac{d^2x(t)}{dt^2}$ term can be neglected. Rescaling the system in Eq. (2.1) with the damping term γ gives the stochastic overdamped Duffing equation

$$\frac{dx(t)}{dt} = -\frac{dU(x)}{dx} + \sqrt{D}\xi(t), \tag{2.4}$$

which is frequently used to model nonequilibrium critical phenomena. For $a > 0$ the potential is bistable as shown in Fig. 2.3. From simple algebra, there is an unstable state at $x = 0$ and two stable states at $x_s^{\pm} = \pm\sqrt{a/b}$, separated by a barrier of height $\Delta U = a^2/4b$ when the noise, $\xi(t)$, is zero. The position of the particle $x(t)$ is considered to be the output of the system and has a power spectral density (PSD) $S(\omega)$.

Two examples (Bulsara and Gammaitoni 1996) of nonlinear dynamic systems are (i) the analogue Hopfield neuron, for which the potential is

$$U(x(t)) = \alpha x(t)^2 - \beta\ln(\cosh x(t)), \tag{2.5}$$

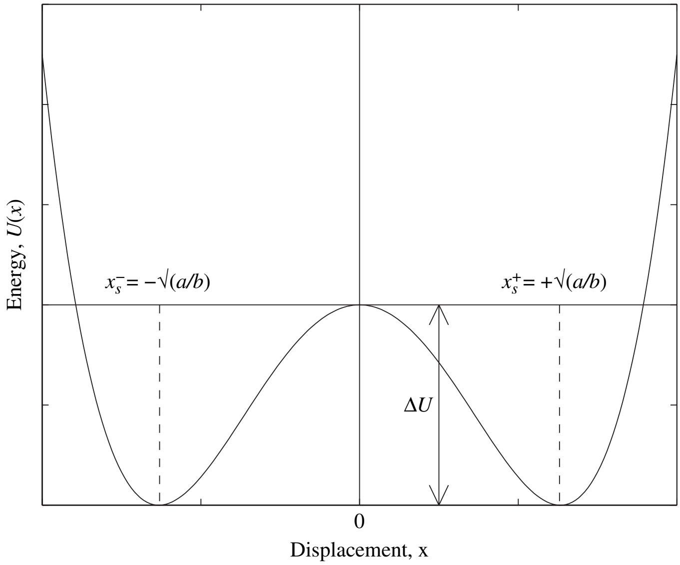


Fig. 2.3. The quartic bistable potential of Eq. (2.3), without any noise, with minima at $x_s^{\pm}$ and barrier height ΔU. Reprinted figure with permission after Harmer *et al.* (2002), © 2002 IEEE.

where the state point $x(t)$ denotes a cell membrane voltage; and (ii) the SQUID loop, for which the potential is

$$U(x(t)) = \alpha x(t)^2 - \beta \cos(2\pi x(t)), \tag{2.6}$$

where $x(t)$ denotes the magnetic field flux in the loop.

By itself, the bistable system is stationary as described by the motion of the particle. That is, if the particle is in one of the two wells, it will stay there indefinitely. By adding a periodic input signal, $A \sin(\omega_s t)$, to the bistable system, the dynamics are governed by the following equation

$$\frac{dx}{dt} = \left(-\frac{dU(x)}{dx} + A \sin(\omega_s t)\right) + \sqrt{D}\xi(t). \tag{2.7}$$

The bistable potential, which is now time dependent, becomes

$$\begin{aligned} U(x,t) &= U(x) - Ax\sin(\omega_s t) \\ &= -a\frac{x^2}{2} + b\frac{x^4}{4} - Ax\sin(\omega_s t), \end{aligned} \tag{2.8}$$

where A and ω_s are the amplitude and the frequency of the periodic signal respectively.

It is assumed that the signal amplitude is small enough that, in the absence of any noise, it is insufficient to force a particle to move from one well to another.

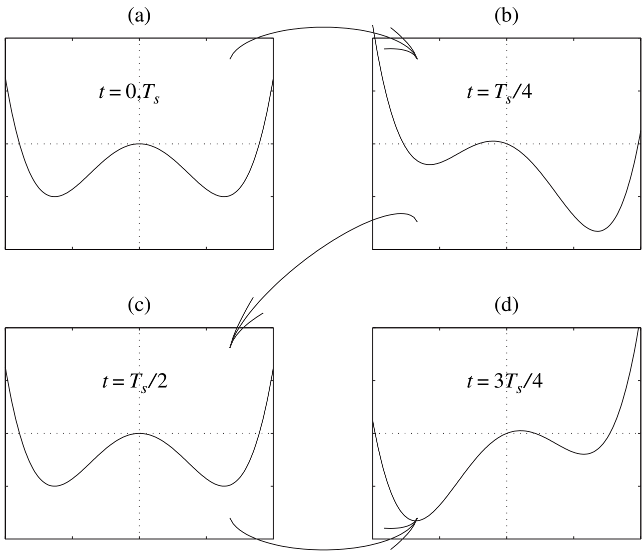


Fig. 2.4. Quartic bistable potential: fixed points. The periodic driving signal causes the double well potential to be tilted back and forth, antisymmetrically raising and lowering the left and right potential wells. Reprinted figure with permission after Harmer *et al.* (2002), © 2002 IEEE.

It is also assumed that the signal period is longer than some characteristic intrawell relaxation time for the system.

Due to the presence of the modulating signal, the double well potential $U(x, t)$ is periodically tilted back and forth with the same frequency, $\omega_s = 2\pi/T_s$. Relating this to the potentials in Fig. 2.4 the effect is to weakly tilt the potential barriers of the right and left sides respectively. In one period of the signal the potential is cycled through Fig. 2.4 (a)–(d). The maxima and minima of the signal correspond to when the potential barrier is at its lowest, which is shown in Fig. 2.4 (b) and (d).

While this tilting on its own is not enough to allow a particle to move from one well to another, the presence of the white noise process in Eq. (2.7) allows noise-induced transitions of the particle. For a certain range of values of the noise intensity, D, transitions will occur in synchronization with the periodic signal. One way of measuring how well the position of the particle represents the frequency of the input is to measure the PSD of the position, and determine the signal-to-noise ratio at ω_s. This will have a peak at a nonzero value of D, and hence SR occurs. The optimal value of D is the one that provides the best time-scale matching between ω_s and the residence times of the particle in each well (Gammaitoni *et al.* 1998, Wellens *et al.* 2004).

A natural simplification of the double-well problem is the discrete two-state system, in which the dynamical variable can take on only two discrete values in these systems. The Schmitt trigger is an example of such a discrete system. Another two-state system in common use is the simple threshold-based system, as we now consider.

Stochastic resonance in threshold systems

The prime objective of this book is to examine *stochastic quantization* of a signal.[6] We define stochastic quantization to mean:

The quantization of a signal by threshold values that are independent and identically distributed random variables.

The starting point of this work is a particular form of SR known as suprathreshold stochastic resonance (SSR). To set the context for SSR, it is necessary to briefly review SR in threshold-based systems – a story that began in 1994. In a decade, the phenomenon is now known to be so widespread that a recent paper has made the point that almost all threshold systems – and noise distributions – exhibit stochastic resonance (Kosko and Mitaim 2003, Kosko and Mitaim 2004).

A single threshold

The first paper to consider SR in a system consisting of a simple threshold, which when crossed by an input signal gives an output pulse, was published in April 1994 under the curious title of *Stochastic Resonance on a Circle* (Wiesenfeld *et al.* 1994). In this paper, the authors note that all previous treatments of SR are based on the classical motion of a particle confined in a monostable or multistable potential. This appears to not be strictly true, as in 1991 and 1993 Bulsara *et al.* (1991), Bulsara and Moss (1991), Longtin *et al.* (1991), Longtin (1993), Douglass *et al.* (1993), and Moss *et al.* (1993) all analyzed SR in neurons, which as excitable systems are not bistable systems in the classical sense. Additionally, although the work on SR in Schmitt trigger circuits (Fauve and Heslot 1983, Melnikov 1993) has been put into the bistable system category, a Schmitt trigger is closely related to the simple threshold crossing system – see the discussion on page 1219 of Luchinsky *et al.* (1999b).

Nevertheless, the paper by Wiesenfeld *et al.* (1994) does appear to be the first paper to show the presence of SR in a system consisting of a simple threshold and

[6] Note that this is a different usage of the term 'stochastic quantization' from that prevalent in areas of quantum physics, stochastic differential equations, and path-integrals such as the Parisi–Wu stochastic quantization method (Damgaard and Huffel 1988).

the sum of a subthreshold signal and noise. Unlike previous work on SR, this paper explores SR with:

> a different class of dynamical systems based not on bistability but rather an excitable dynamics. (Wiesenfeld *et al.* 1994)

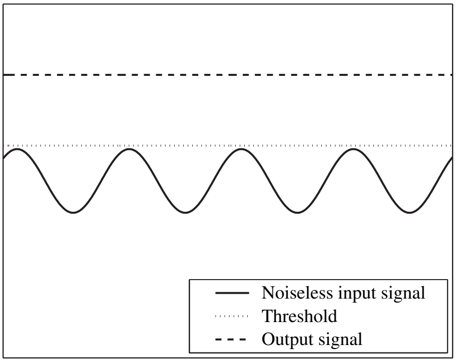
The system considered consists of a weak subthreshold periodic signal, a potential barrier, and zero-mean Gaussian white noise. The output is a 'spike', which is a short duration pulse, with a large amplitude and deterministic refractory time. Note that a short refractory time for the pulse is important; the refractory time must be much less than the correlation time of the noise, or threshold crossings will occur that cannot give an output pulse. This system is described as:

> a simple dynamical process based on a single potential well, for which SR can be observed. (Wiesenfeld *et al.* 1994)

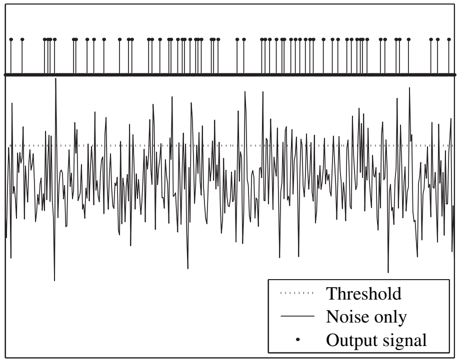
It is also observed to represent the process of action potential events in sensory neurons.

Figure 2.5 illustrates qualitatively why SR occurs in such a system. In each subfigure, the lower plot indicates the input signal's amplitude against increasing time, with the straight dotted line being the threshold value. The upper trace is the output signal plotted against the same time-scale as the input signal. This output signal is simply a narrow pulse, or 'spike', every time the input signal crosses the threshold with positive slope.

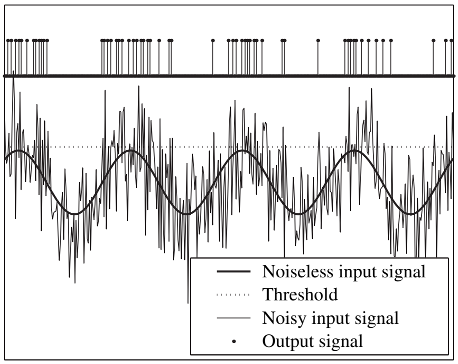
Figure 2.5(a) shows a noiseless subthreshold periodic input signal. Such a signal never induces a threshold crossing, and the output remains constant and spike-less. Clearly nothing at all can be said about the input signal by examining the output signal, except that the input is entirely subthreshold. Figure 2.5(b) indicates how in the absence of a signal, threshold crossings will occur randomly. The input in this plot is only bandlimited white Gaussian noise – that is, noise with a frequency spectrum that is flat within its bandwidth. The statistics of this case are required for obtaining output SNRs when a signal is present, although such formulas will only hold when the input signal is small compared to the noise. In Fig. 2.5(c), the lower trace shows the sum of a periodic input signal, and bandlimited Gaussian noise. Unlike Fig. 2.5(a), for this signal some threshold crossings do occur. The probability of an output pulse is dependent on the amplitude of the noiseless input signal. This is clear visually and intuitively; a threshold crossing is more likely to occur when the input signal is close to the threshold than when it is not. Compared with the absence of noise, some information about the original noiseless input signal is now available at the output, and removing the noise is counterproductive in this system. Note that the output pulse 'train' does not 'look' anything like the input signal, but in fact encodes the input by a form of stochastic frequency modulation.



(a) Subthreshold periodic input signal



(b) Noise only



(c) Subthreshold but noisy periodic input signal

Fig. 2.5. Qualitative illustration of SR occurring in a simple threshold-based system. In each subfigure, the lower trace indicates the input signal's amplitude against increasing time, with the dotted line being the threshold value. The upper trace is the output signal plotted against the same time-scale as the input signal. The output signal is a 'spike' – that is, a short duration pulse – every time the input crosses the threshold with positive slope, that is from subthreshold to suprathreshold but not *vice versa*. Figure 2.5(a) shows an entirely subthreshold periodic input signal. Such a signal will never cause an output spike. Figure 2.5(b) shows the case of threshold crossings occurring randomly due to bandlimited Gaussian noise. Figure 2.5(c) illustrates how the presence of noise enables some information about a subthreshold input signal to be present at the output. In the absence of noise, the output will always remain constant. When noise is added to the input, threshold crossings can occur, with a probability related to the amplitude of the input signal.

If the input signal is to be recovered, we must obtain an ensemble average to make sense of this, as is demonstrated shortly, and described in Sulcs *et al.* (2000).

Another way of understanding this effect is to realize that adding noise to a subthreshold periodic signal, and then thresholding the result, is equivalent to thresholding the noiseless signal with a time-varying stochastic threshold value. This is equivalent to having a signal that is thresholded at a random amplitude, where sometimes the random amplitude is greater than the signal's maximum amplitude. This means the periodic signal is thresholded at a random phase in its period, and sometimes not at all.

Returning to Wiesenfeld's work, his paper notes that such a threshold-based system can be realized by a Josephson junction biased in its zero voltage state. He then provides a general theoretical approach for what he calls 'generic threshold-plus-reinjection dynamics' and shows how it can lead to SR. The main theoretical result given is a derivation of the SNR at the output of the generic system, using the autocorrelation of the output. This formula for SNR is shown to have a maximum for a nonzero value of noise strength, which shows that SR exists in such a system.

Shortly after this initial paper, several other authors also analyzed SR in threshold systems subject to periodic input signals and additive noise. The first of these is an excellent paper by Jung (1994), which extends the theory of Wiesenfeld *et al.* (1994) by removing an assumption of Poisson statistics, and using the decades-old work of S. O. Rice, who analyzed the expected number of threshold crossings by either broadband noise alone, or the sum of a sinusoid and broadband noise (Rice 1944, 1945, 1948). Jung (1994) shows mathematically that the correlation coefficient between a subthreshold periodic input signal and its corresponding random output pulse train possesses a noise-induced maximum.

Jung (1994) also gives a convincing argument, which has been much used in SR research, for why such simple threshold crossing systems can be used to provide simple models of neurons, particularly in studies of neural networks, with large numbers of neurons. As noted by Jung (1994), such a simple two-state model of a neuron was first proposed over 50 years ago by McCulloch and Pitts (1943).

Another observation made is that SR occurs for subthreshold signals since, in the absence of noise, no threshold crossings can occur, whereas

> in the presence of noise, there will be noise-induced threshold crossing(s), but at preferred instants of time, that is, when the signal is larger. (Jung 1994)

In addition, the conclusion is made that suprathreshold signals will never show stochastic resonant behaviour. We shall see later in this chapter that this is not true, once some of the assumptions used by Jung are discarded. Jung (1994) was followed up shortly afterwards in two more papers on the same topic (Jung and Mayer-Kress 1995, Jung 1995).

Simultaneously with Jung, an alternative approach was taken by Gingl *et al.* (1995b) – aspects of which also appeared in Kiss (1996) and Gingl *et al.* (1995a) – who consider a threshold-based system to be a 'level crossing detector' (LCD), that is a system that detects whether or not an input signal has crossed a certain voltage level. Such a system is described as 'non-dynamical', to differentiate it from the bistable systems used in 'classical' – that is, pre 1994 – SR studies.

Like Jung (1994), this work also uses the established work of Rice. Formulas first given in Rice (1944) are applied to derive a formula for the SNR in the linear response limit at the output of an LCD, a formula that is quite similar to that of Wiesenfeld *et al.* (1994). The equation obtained is verified by simulation in Gingl *et al.* (1995b).

Other early papers on SR in threshold-based systems appeared in 1996 from Bulsara and Zador (1996) and Bulsara and Gammaitoni (1996), who use mutual information to measure SR for a subthreshold aperiodic input signal. Shortly after this, Chapeau-Blondeau and co-authors published several papers examining threshold-based SR from a wide variety of new angles (Chapeau-Blondeau 1996, Chapeau-Blondeau and Godivier 1996, Chapeau-Blondeau and Godivier 1997, Chapeau-Blondeau 1997b, Godivier and Chapeau-Blondeau 1997, Chapeau-Blondeau 1997a, Godivier *et al.* 1997).

Array enhanced stochastic resonance

The bulk of this book concerns a form of SR that occurs in arrays of parallel threshold devices. We briefly mention some of the history of what is sometimes known as array enhanced stochastic resonance (AESR). There have been many observations that coupling together more than one SR-capable device can lead to increased output performance. Perhaps the first to demonstrate this were Wiesenfeld (1991), Jung *et al.* (1992), and Bulsara and Schmera (1993). However, Lindner *et al.* (1995) were the first to use the term AESR, and show that a chain of coupled nonlinear oscillators can provide an enhanced SR effect when compared with a single oscillator. A very similar effect is discussed in Collins *et al.* (1995b), where ASR is studied in an array of FitzHugh–Nagumo neuron models. Each neuron is considered to receive the same subthreshold aperiodic input signal, but independent noise, and the overall output is the summed response from all neurons. This result is also discussed in Moss and Pei (1995).

Other early papers showing the effect of AESR include Bezrukov and Voydanoy (1995), Lindner *et al.* (1996), Pei *et al.* (1996), Gailey *et al.* (1997), Neiman *et al.* (1997), and Chialvo *et al.* (1997). An unpublished preprint gives a more detailed history (P. F. Góra, arXiv:cond-mat/0308620). The main point in these works is that the magnitude of SR effects can be enhanced by combining the outputs of more than one single SR-capable component.

Multiple thresholds and soft thresholds

The first paper to consider systems consisting of more than one static threshold is Gammaitoni (1995b), which considers SR in threshold-based systems to be equivalent to dithering. A comparison between dithering and stochastic resonance is given in Section 3.4 of Chapter 3. The second is Gailey *et al.* (1997), which analyzes an ensemble of N threshold elements, using the classical theory of nonlinear transformations of a Gaussian process to obtain cross correlations.

Another extension was the realization of the existence of a:

> class of non-dynamical and threshold-free systems that also exhibit stochastic resonance. (Bezrukov and Voydanoy 1997b, Bezrukov and Voydanoy 1997a)

This work demonstrates that a 'hard' threshold – that is, a threshold that divides its inputs into exactly two states, rather than a continuum of states – is not a necessary condition for SR to occur in nondynamical systems. Further development of this approach can be found in Bezrukov (1998).

Forbidden interval theorems

Although demonstrations of SR in threshold systems since the initial 1994 paper are many and varied, detailed theoretical proofs of the fact that SR effects should be expected in nearly all threshold systems were not published until much later (Kosko and Mitaim 2003). A number of theorems, collectively known as the *forbidden interval theorems*, give simple conditions predicting when SR will occur for random binary (Bernoulli distributed) signals, and static threshold systems (Kosko and Mitaim 2003, Kosko and Mitaim 2004). Specifically, the theorems prove that SR will occur (i) for all finite variance noise distributions, if and only if the mean of the noise is outside a 'forbidden interval', and (ii) all infinite variance *stable* noise distributions, if and only if the location parameter of the noise is outside the forbidden interval. The forbidden interval is the region $[\theta - A, \theta + A]$, where θ is the threshold value, and the binary signal takes values from $\{-A, A\}$.

More recently, similar forbidden interval theorems were published that partially extend their validity from the basic static threshold to spiking neuron models (Patel and Kosko 2005).

When the consequences of the forbidden interval theorem are combined with the observation that SR occurs for impulsive (infinite variance) noise, as pointed out by Kosko and Mitaim (2001), it is clear that SR is actually very robust. This fact highlights that, although published SR research usually includes an assumption of finite variance noise – usually Gaussian – this is not a necessary condition for SR to occur.

The validity of the forbidden interval theorem has been tested experimentally using carbon nanotube transistors (Lee *et al.* 2003, Lee *et al.* 2006), both with finite variance and with impulsive noise. This demonstration that SR effects are extremely robust is potentially important for future applications making use of novel transistor technologies, as it means that SR could be utilized even with little control over the distribution of SR-inducing noise signals.

To date, the forbidden interval theorem has not been extended beyond binary input signals. It is an open question whether conditions exist stating whether SR can or cannot occur when the input signal is a continuously valued random variable. This latter situation is the one studied in the remaining chapters of this book.

2.6 How should I measure thee? Let me count the ways . . .

The reason for the name of this section[7] is to draw attention to the fact that SR has been measured in many different ways. Examples include SNR (Benzi *et al.* 1981), spectral power amplification (Jung and Hänggi 1991, Rozenfeld and Schimansky-Geier 2000, Imkeller and Pavlyukevich 2001, Drozhdin 2001), correlation coefficient (Collins *et al.* 1995a), mutual information (Levin and Miller 1996), Kullback entropy (Neiman *et al.* 1996), channel capacity (Chapeau-Blondeau 1997b), Fisher information (Greenwood *et al.* 1999), ϕ–divergences (Inchiosa *et al.* 2000, Robinson *et al.* 2001), and mean square distortion (McDonnell *et al.* 2002a). Stochastic resonance has been analyzed in terms of residence time distributions – see Gammaitoni *et al.* (1998) for a review – as well as Receiver Operating Characteristic (ROC) curves (Robinson *et al.* 1998, Galdi *et al.* 1998, Zozor and Amblard 2002), which are based on probabilities of detecting a signal to be present, or falsely detecting a nonexisting signal (Urick 1967).

The key point is that the measure appropriate to a given task should be used. Unfortunately, due to historical reasons, some authors tend to employ the original measure used, SNR, in contexts where it is effectively meaningless. This section gives a brief history of the use of SNR in SR research, and a discussion of some of the criticisms of its use.

It was first thought that SR occurs only in bistable dynamical systems, generally driven by a periodic input signal, $A \sin(\omega_0 t + \phi)$, and broadband noise. Since the input to such systems is a simple sinusoid, the SNR at the output is a natural measure to use to determine how well the output signal can reflect the input periodicity, with the following definition most common

$$\mathrm{SNR} = \frac{P(\omega_0)}{S_N}. \tag{2.9}$$

In Eq. (2.9), $P(\omega_0)$ is the output power spectral density (PSD) at the frequency of the input signal, ω_0, and S_N is the PSD of the output background noise, as measured with a signal present. This definition assumes that the overall output PSD is the superposition of a constant noise background, corresponding to white noise, and a

[7] Apologies to Elizabeth Barrett Browning (1850).

delta-function spike at the input frequency. Stochastic resonance occurs when the SNR is maximized by a nonzero value of input noise intensity.

It is well known in electronic engineering that nonlinear devices cause output frequency distortion – that is, for a single frequency input, the output will consist of various harmonics of the input (Cogdell 1996). This means that basic circuit design requires the use of filters that remove unwanted output frequencies. For example, this *harmonic distortion* in audio amplifiers is very undesirable. On the other hand, high frequency oscillators make use of this effect by starting with a very stable low frequency oscillator and sending the generated signal through a chain of frequency multipliers. The final frequency is harmonically related to the low frequency source.

For more than one input frequency, the output of the nonlinear device will contain the input frequencies, as well as integer multiples of the sum and difference between all frequencies (Cogdell 1996). This effect of creating new frequencies is known as *intermodulation distortion*. In the field of optics this phenomenon can be used to generate lower frequency signals – for example, T-rays (that is, terahertz radiation) – from different optical frequencies by a method known as *optical rectification* (Mickan *et al.* 2000, Mickan and Zhang 2003).

A study of such higher harmonics generated by a nonlinear system exhibiting SR has been published (Bartussek *et al.* 1994), and the phenomenon is also discussed in subsequent works (Bulsara and Inchiosa 1996, Inchiosa and Bulsara 1998). However, much research into SR has only been interested in the output frequency component that corresponds to the fundamental frequency of the periodic input signal, in which case the output SNR is given by Eq. (2.9) and ignores all other output harmonics.

More recently, attempts have been made to overcome this, by defining the output SNR as a function of all frequencies present at the input, even if not in a narrow band around the fundamental (Kiss 1996, Gingl *et al.* 2001, Mingesz *et al.* 2005). However, while such formulations may have some uses (McDonnell *et al.* 2004a), there has been much discussion regarding the inadequacies of SNR as an appropriate measure for many signal processing tasks (DeWeese and Bialek 1995, Galdi *et al.* 1998).

One of the most important objections can be illustrated as follows. Consider a periodic, but broadband input signal, such as a regularly repeated radar chirp signal. The use of SNR as the ratio of the output power of the fundamental frequency to the background noise PSD is meaningless for signal recovery here, unless only the fundamental period is of interest. This output SNR measure only provides information about the period of the signal – the output SNR at that frequency – and nothing about the shape of the chirp in the time domain.

This inadequacy was recognized when researchers first turned their attention to ASR, which ushered in the widespread use of cross-correlation and information-theoretic measures, which can, in some sense, describe how well the shape of the output signal is related to the input signal. An excellent description of the issue is that:

> a nonlinear signal processor may output a signal that has infinite SNR but is useless because it has no correlation with the input signal. Such a system would be one which simply generates a sine wave at the signal frequency, totally ignoring its input. (Inchiosa and Bulsara 1995)

While many SR researchers realized that studying aperiodic input signals substantially increases the relevance of SR to applications such as studies of neural coding, some did not get past the need to move on to measures other than SNR in such circumstances. This has led to a somewhat strange debate about whether or not 'SNR gains' can be made to happen in a nonlinear system by the addition of noise. Next we discuss the main questions on this issue.

The SNR gain debate

In the last decade, a number of researchers have reported results claiming that it is possible to obtain an SNR gain in some nonlinear systems by the addition of noise (Kiss 1996, Loerincz *et al.* 1996, Vilar and Rubí 1996, Chapeau-Blondeau 1997a, Chapeau-Blondeau and Godivier 1997, Chapeau-Blondeau 1999, Gingl *et al.* 2000, Liu *et al.* 2001, Gingl *et al.* 2001, Makra *et al.* 2002, Casado-Pascual *et al.* 2003, Duan *et al.* 2006). There has been some criticism of these works, for example the comment on Liu *et al.* (2001) given in Khovanov and McClintock (2003), and in the SR community, such results have been seen as fairly controversial, for two reasons, as discussed in the remainder of this section.

Can SNR gains occur at all?

Initially it seemed that SNR gains contradicted proofs that SNR gains cannot occur. For example DeWeese and Bialek (1995) – see also Dykman *et al.* (1995) – show for stationary Gaussian noise and a signal that is small compared to the noise, that for nonlinear systems the gain, $G = \text{SNR}_{\text{out}}/\text{SNR}_{\text{in}}$, must be less than or equal to unity, and that no SNR gain can be induced by utilizing SR. This proof is based on the use of linear response theory, where, since the signal is small compared to the noise, both the signal and noise are transferred linearly to the output, and, as in a linear system, no SNR gain is possible. Much attention has been given to this fact, since most of the earlier studies on SR were kept to cases where the linear response limit applies, to ensure that the output is not subject to the above-mentioned harmonic distortion (Gingl *et al.* 2000).

Once this fact was established, researchers still hoping to be able to find systems in which SNR gains due to noise could occur turned their attention to situations not covered by the proof – that is, the case of a signal that is not small compared to the noise, or broadband signals or non-Gaussian noise.

For example, Kiss (1996) considers a broadband input signal, and, being broadband, the conventional SNR definition cannot be used. Instead, a new frequency dependent SNR measure is derived, a measure with which an SNR gain is shown to occur. Further examples are Chapeau-Blondeau and Godivier (1997) and Chapeau-Blondeau (1997a), which use the conventional SNR definition, but the large signal regime to show the existence of SNR gains. Furthermore, Chapeau-Blondeau (1999) also considers the case of non-Gaussian noise.

However, the interpretation in some of the papers on this topic can be a little fuzzy. For example, it is sometimes implied that an SNR gain is *due* to the addition of more noise to an already noisy signal.

Instead, the SNR gains reported are caused by an increase in input noise in order to find the optimal point on the system's SR curve. The side effect of this is a decrease in input SNR. This leads to an increase in SNR gain, simply due to the same mechanism that causes SR itself. This means that the gain is due to the characteristics of the system itself rather than the addition of noise.

Nevertheless, the main point emphasized is that the SNR gain can be greater than one, which does not occur for the linear response regime, and is a valid point. Our conclusion is that the answer to 'can SNR gains occur at all?' is 'yes, SNR gains can occur'. The more important question is whether such gains are meaningful.

Are SNR gains meaningful?

By looking outside the conditions of the proof that SNR gains cannot occur in the linear response limit, SNR gains can be found. However, the second reason that an emphasis on SNR gains due to SR are seen to be controversial is that the definitions of SNR used in cases where SNR gains occur are not always particularly meaningful. Taking the approach of looking outside the parameters of the proof assumes that SNR is still a useful measure outside these parameters. A strong argument against this assumption, and for the use of information theory, rather than SNRs, is given in DeWeese and Bialek (1995), as discussed in Section 2.7.

Useful discussions of this point, and discussions of signal detection theory in the context of SR are given in Inchiosa and Bulsara (1995), Galdi *et al.* (1998), Robinson *et al.* (1998), Petracchi (2000), Hänggi *et al.* (2000), Robinson *et al.* (2001) and Chen *et al.* (2007). For example, Hänggi *et al.* (2000) give a general investigation of SNR gains due to noise, are highly critical of the use of SNR in such systems, and indicate more appropriate measures to use, at least for signal detection or estimation problems. Our conclusion is that the answer to the title of this subsection is ‘probably not, for most tasks’.

Recall the quote above: Inchiosa and Bulsara (1995) recognize what is well-known to electronic engineers – that an SNR gain is not in itself a remarkable thing, and that SNR gains are routinely obtained by filtering – for example, the bandpass filter. The reason that more is made of such phenomena in the SR literature is that the reported SNR gains are said to be due to the *addition of noise* to an already noisy signal, rather than a deliberately designed filter. Another paper by the same authors also discusses this topic (Inchiosa and Bulsara 1996). As discussed above, the view that SNR gains occur *due to* SR can be misleading.

An associated problem is that of relating SNRs to information theory. For example, it has sometimes been stated that an SNR gain in a periodic system is analogous to an increase in information. We now investigate such a claim.

2.7 Stochastic resonance and information theory

The previous section highlighted that problems can occur if SNR measures are used in situations where they are not appropriate. Here, we indicate why this is the case. The crux of the matter is that SNR measures are, for example, appropriate when the goal is to decide if a signal is present or not – that is, the problem of *signal detection*. In the original context of SR, the detection of a weak periodic signal was certainly the goal, and a small SNR can make it very difficult to detect the signal. This is quite a different matter from other signal processing problems, such as estimation, compression, error-free transmission, and classification.

Rather than attempting to observe SR effects in a particular measure, it makes more sense to first define the signal processing objective of a system. This leads to an appropriate measure of performance quality, which can then be analyzed to see whether SR can occur.

One influential paper on SR that takes this approach is now discussed.

Signal detection vs. information transmission

The first paper to discuss stochastic resonance in the context of information theory was DeWeese and Bialek (1995). In this paper, it is considered that the signal processing objective of a neuron is to transmit as much information about its input as possible. The measure used is mutual information (Cover and Thomas 1991).

There are several reasons why this paper has been significant for SR researchers:

- The point is made that one potentially universal characteristic of neural coding is that the:

  SNR (is) of order unity over a broad bandwidth. (DeWeese and Bialek 1995)

Since this means that the environment in which sensory neural coding takes place appears to be very noisy, it is highly plausible that neural coding makes use of SR. This point is also made in Bialek *et al.* (1993) and DeWeese (1996).

- It is pointed out that measuring information transfer for single frequency sine waves by SNR is only really applicable in linear systems. Since SR cannot occur in linear systems, the use of SNR only really applies to the case of small input signals so that the output exhibits a linear response.
- A proof is given of the fact that, for small signals in Gaussian noise, it is impossible for the output SNR to be greater than the input SNR. This fact has led some researchers to search for – and find (see Section 2.6) – circumstances in which the proof does not apply and that SNR gains can occur. It could be argued that any such work that does not have a detection goal does not pay attention to the previous point above, and is possibly not really proving much.
- It is pointed out that sine waves do not carry information that increases with the time of observation:

  No information can be carried by the signal unless its entropy is an extensive quantity. In other words, if we choose to study a signal composed of a sine wave, the information carried by the signal will not grow linearly with the length of time we observe it, whether or not the noise is present. In addition to this, we would like to compare our results to the performance of real neurons in as natural conditions as possible, so we should use ensembles of broadband signals, not sine waves. (DeWeese and Bialek 1995)

- It is demonstrated for the case of a subthreshold signal in a single threshold system that the information transferred through such a system can be optimized by modifying the threshold setting. With the optimal value for the threshold, the mutual information is strictly decreasing for increasing noise, and SR does not occur.

  So it seems that if you adapt your coding strategy, you discover that *stochastic resonance* effects disappear ... More generally, we can view the addition of noise to improve information transmission as a strategy for overcoming the *incorrect* setting of the threshold. (DeWeese and Bialek 1995)

The conclusions drawn are that single-frequency periodic signals are not relevant for information transfer, and particularly not relevant for neurons. Indeed, it has been pointed out that rather than using SNR:

> It is the total information encoded about a signal that is the biologically relevant quantity to consider. (Levin and Miller 1996)

At the time, with a growing interest into SR in neurons, such a realization led researchers away from studying conventional single frequency SR in neurons, to more realistic broadband and aperiodic signals. For most authors, this naturally led to using measures other than SNR.

One exception to the above reasoning is that there are some neurons that act as binary switches; that is, they are essentially simple signal detectors asking the question: ‘is there a signal present at my input or not?’

Perhaps another important exception to these ideas is in the encoding of sound by the cochlear nerve. The mechanism by which audio signals are encoded has been likened to a biological Fourier transform; different spatial regions in the *organ of Corti* – that is, the part of the inner ear that contains sensory neurons – are sensitive to different frequencies of sound waves (Kandel *et al.* 1991).

The conclusion that SR in threshold systems is simply a way to overcome the incorrect threshold setting seems to have led many to think that making use of noise is a suboptimal means of designing a system. The contrasting viewpoint is that noise is ubiquitous; since it is virtually impossible to remove all noise completely from systems, design methods should consider the effects of SR, and that various design parameters, such as a threshold value, may in some circumstances need to be set in ways that make use of the inherent noise to obtain an optimal response. We discuss exactly this situation in Chapter 8.

Furthermore, the analysis in DeWeese and Bialek (1995) assumes that the input signal to a neuron is random with an ideal white spectrum. In other words, there is no time correlation in the signal. As is argued by illustration in Section 2.8, time correlation can lead to a noise-enhanced benefit in a single threshold system, even if the threshold is optimally set.

Information theory and SNR gains

The search for SNR gains due to SR in the case of periodic input signals naturally led some authors to look for an analogy to compare input performance to output performance for aperiodic stochastic resonance (ASR). As mentioned, Kiss (1996) defines a frequency dependent SNR measure based on cross-spectral densities, which he considers to be valid for such aperiodic input signals. This method is discussed further in McDonnell *et al*. (2004a).

An alternative approach for measuring ASR is mutual information. A special case of mutual information is known as *channel capacity* (Cover and Thomas 1991). Channel capacity is simply defined as the maximum possible mutual information through a ‘channel’ or system. It is usually defined in terms of the input probability distribution that provides the maximum mutual information, subject to certain constraints. For example, the input signal may be restricted to two states – that is, a binary signal – or to be a continuously valued random variable, but with a specified power.

The most widely known formula describing channel capacity is the Shannon–Hartley formula, which gives the channel capacity for the transmission of a power-limited and band-limited signal through an additive, signal-independent, Gaussian white noise channel. As mentioned in Berger and Gibson (1998), this formula is often misused in situations where it does not apply, including, one could argue, in the SR literature.

Channel capacity as a measure of SR is discussed in Chapeau-Blondeau (1997b), Godivier and Chapeau-Blondeau (1998), Goychuk and Hänggi (1999), Kish *et al.* (2001), Goychuk (2001), and Bowen and Mancini (2004), all of which show that the right level of noise can provide the maximum channel capacity. However, in general, this means only that the right level of input noise optimizes the channel; that is, SR occurs.

Of more interest to us here is a way of comparing the input signal to the output signal in a way analogous to SNR gains for periodic signals. For example, in Chapeau-Blondeau (1999) it is considered that comparing the channel capacity at the input and the output of a system for an aperiodic input signal is analogous to a comparison of the input and output SNRs for periodic input signals. Here we investigate the use of channel capacity in simple threshold-based systems where SR can occur, and show by use of a well-known theorem of information theory that such an analogy is a false one.

The data processing inequality

The data processing inequality (DPI) of information theory asserts that no more information can be obtained out of a set of data than is there to begin with. It states that given random variables X, Y, and Z that form a Markov chain in the order $X \rightarrow Y \rightarrow Z$, then the mutual information between X and Y is greater than or equal to the mutual information between X and Z (Cover and Thomas 1991). That is

$$I(X, Y) \geq I(X, Z). \tag{2.10}$$

In practice, this means that no signal processing on Y can increase the information that Y contains about X.

It should be noted that the terminology *Markov chain* used in connection with the DPI is somewhat more inclusive than that prevalent in applied probability. DPI usage requires only the basic Markov property that Z and X are conditionally independent given Y. By contrast the usage in applied probability requires also that X, Y, and Z range over the same set of values and that the distribution of Z given Y be the same as that of Y given X.

Generic nonlinear noisy system

To illustrate the arguments we now present, consider a generic system where a signal, $s(t)$, is subject to independent additive random noise, $n(t)$, to form another random signal, $x(t) = s(t) + n(t)$. The signal $x(t)$ is then subjected to a nonlinear transformation, $T[\cdot]$, to give a final random signal, $y(t) = T[x(t)]$. A block diagram of such a system is shown in Fig. 2.6.

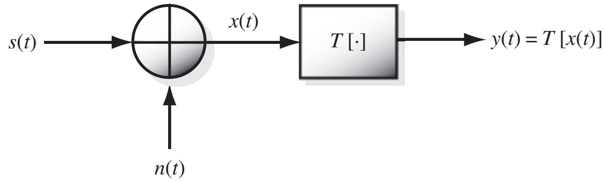


Fig. 2.6. Schematic diagram of a generic noisy nonlinear system. The input signal, $s(t)$, is subject to additive noise, $n(t)$. The sum of the signal and noise, $x(t)$, is subjected to the nonlinear transfer function, $T[\cdot]$, to give $y(t) = T[x(t)]$.

As noted above, many papers have demonstrated that SNR gains can occur due to SR. Such a system as that of Fig. 2.6 describes many of those reported to show SNR gains for both periodic or aperiodic input signals.

Channel capacity

Ignoring for now the question over whether SNR measures have much relevance in such cases, the observation of SNR gains can appear on the surface to contradict the DPI. The reason for this is that one could be led to believe that information can always be related to SNR by the Shannon–Hartley channel capacity formula[8]

$$C = 0.5 \log_2 (1 + \mathrm{SNR}) \quad \text{bits per sample.} \tag{2.11}$$

Clearly, when this formula applies, an increasing SNR leads to an increase in the maximum possible mutual information through a channel. Suppose that Eq. (2.11) does apply in Fig. 2.6 and that the SNR of $s(t)$ in $x(t)$ is SNR_1. This means that the maximum mutual information between $s(t)$ and $x(t)$ is $I(s, x) = 0.5 \log_2 (1 + \mathrm{SNR}_1)$ bits per sample.

Suppose also that the operation $T[\cdot]$ filters $x(t)$ to obtain $y(t)$, such that the filtering provides an output SNR for $s(t)$ in $y(t)$ of SNR_2. If the filtering provides an SNR gain, then $\mathrm{SNR}_2 > \mathrm{SNR}_1$. Consider the overall system that has input, $s(t)$, and output, $y(t)$. If Eq. (2.11) applies for this whole system, the mutual information between $s(t)$ and $y(t)$ is $I(s, y) = 0.5 \log_2 (1 + \mathrm{SNR}_2) > I(s, x)$. This is clearly a violation of the DPI, and an SNR gain either cannot occur in a system in which Eq. (2.11) applies, or Eq. (2.11) does not apply. If we believe that Eq. (2.11) always applies, then scepticism about the occurrence of SNR gains can be forgiven.

However, it is instead the validity of Eq. (2.11) that needs consideration. As mentioned, this formula is often widely misused (Berger and Gibson 1998), as it applies only for additive Gaussian white noise channels, where the signal is independent of the noise. Of particular relevance here is the *additive noise* part. No SNR gain such

[8] Some references instead refer to this formula as the 'Hartley–Shannon formula', the 'Shannon–Hartley–Tuller law', or simply as 'Shannon's channel capacity formula'. It is also variously known as a 'theory', 'law', 'equation', 'limit', or 'formula'.

as that from SNR_1 to SNR_2 can be achieved in such an additive noise channel. This means that Eq. (2.11) can never apply to the situation mentioned above between signals $s(t)$ and $y(t)$, since even if it applies between $s(t)$ and $x(t)$, the SNR gain required in the filtering operation rules it invalid. The conclusion of this reasoning is that there is no reason to be sceptical about SNR gains, except for cases where the Shannon–Hartley channel capacity formula is actually valid.

However, such a discussion does indicate that any analogy between SNR gains and mutual information is fraught with danger. For example, it is shown using simple examples in McDonnell *et al.* (2003b) and McDonnell *et al.* (2003c) that such an analogy is generally false. From these investigations, it is clear that although SNR gains may exist due to SR for periodic input signals, no information-theoretical analogy exists for random noisy aperiodic signals. The simplest illustration of this is to threshold a noisy binary pulse train at its mean. For uniform noise with a maximum value less than half the pulse amplitude, the mutual information between input and output remains constant, regardless of the input SNR.

Furthermore, since the DPI holds, the addition of more noise to a noisy signal cannot be of benefit as far as obtaining an input–output mutual information gain is concerned. Such a result does not rule out the fact that the addition of noise at the input to a channel can maximize the mutual information at the output; in other words, the effect of SR for aperiodic signals is perfectly valid. When this occurs, an optimal value of input noise means minimizing the information lost in the channel.

2.8 Is stochastic resonance restricted to subthreshold signals?

The common aspect of the cited works in the section on SR in threshold systems is that, for a single threshold, SR is shown to occur only for subthreshold input signals. It has been discussed several times that SR cannot occur in a threshold-based system for an optimally placed threshold. However, this is true only for certain situations. One such situation is where only the output SNR at the frequency of a periodic signal is measured. For example, consider the case of a threshold system where the output is a pulse whenever the input signal crosses the threshold with positive slope. In the absence of noise, placing the threshold at the mean of the input signal will cause output pulses to occur once per input period. In the presence of noise, the output signal will be noisy, since spurious pulses, or jitter in the timing of the desired pulse, will occur, and the absence of noise is desirable in this case. This situation is illustrated in Fig. 2.7.

Another such situation, as already discussed, is given in DeWeese and Bialek (1995). When the input signal is aperiodic, with no time correlation, and the measure used is mutual information, an optimally placed threshold also precludes SR from occurring.

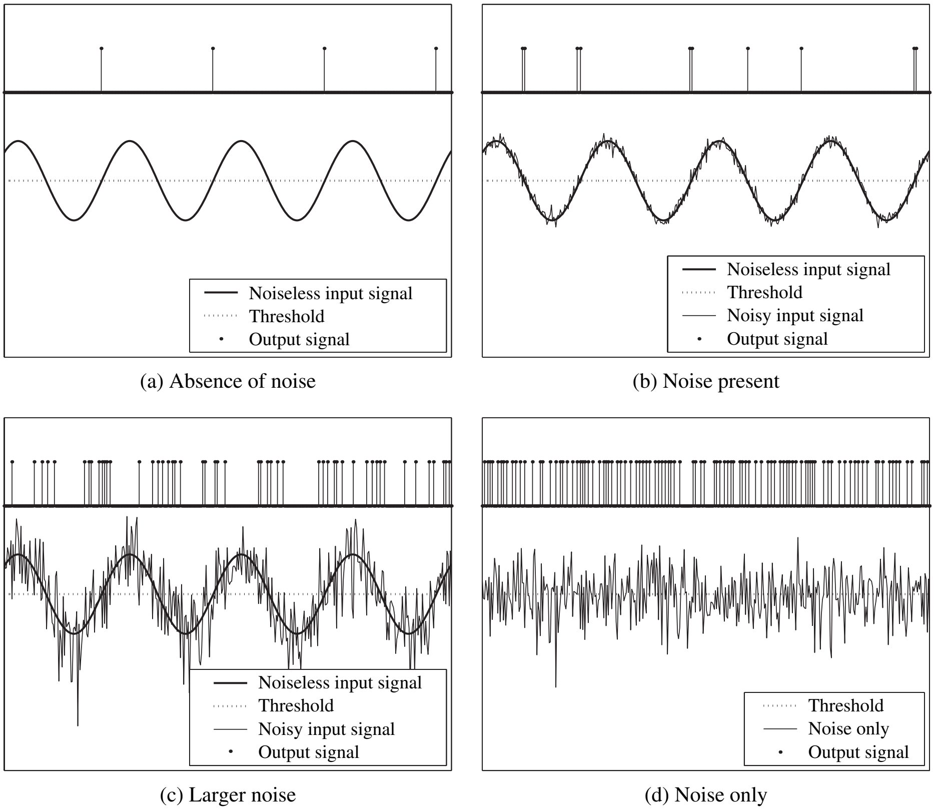


Fig. 2.7. Thresholding a periodic signal at its mean. This plot illustrates that for an optimally placed threshold, at least as far as determining the frequency of a sine wave, the optimal output occurs in the absence of noise. In each subfigure, the lower trace indicates the input signal's amplitude against increasing time, with the straight line being the threshold value. The upper plot is the output signal plotted against the same time-scale as the input signal. The output signal is a 'spike' – that is, a short duration pulse – every time the input crosses the threshold with positive slope – that is, from subthreshold to suprathreshold but not *vice versa*. Figure 2.7(a) shows that in the absence of noise, there is exactly one threshold crossing per period. Figure 2.7(b) shows that as soon as some small amount of additive noise is present, extra threshold crossings occur. This results in some noise being present in the output, as there is no longer exactly one output pulse per period. Figure 2.7(c) shows that as the noise becomes larger, many more output pulses occur, although with a higher frequency when the input signal is close to the threshold than when it is near its maximum or minimum. This indicates that averaging should recover the signal period. Figure 2.7(d) shows that, in the absence of a signal, the occurrence of output pulses is completely random.

However, one example that shows that SR can indeed occur for a signal that is not entirely subthreshold[9] is when averaging is allowed for a periodic – but not

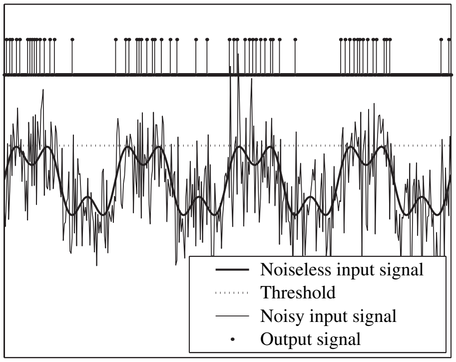
[9] In this text, the term 'suprathreshold' is not intended to mean that a signal is always entirely greater than a certain threshold. Instead, it refers to a signal that is allowed to have values that are both above and below a threshold. This includes the special case of entirely suprathreshold signals.

single frequency – signal. Figs 2.8 and 2.9 illustrate that the ensemble average of a noisy thresholded periodic signal – with an optimal threshold – can be better in a certain sense in the presence of noise than without it.

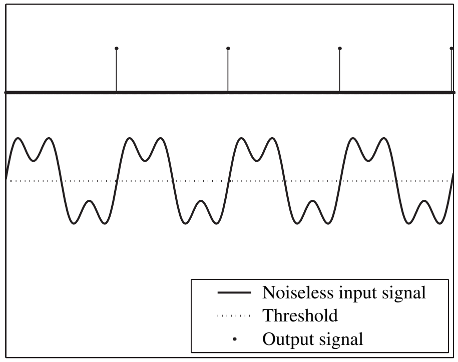
Figures 2.8(b) and 2.8(c) show a periodic but *not single frequency* signal being thresholded at its mean. Figure 2.8(a), which shows the same signal completely subthresholded, is provided for comparison. In Fig. 2.8(b) where noise is absent, the output is identical to that of the single frequency sine wave of Fig. 2.7(a), and no amount of averaging can differentiate between the two output signals. However, in Fig. 2.8(c), where noise is present, the shape of the periodic signal can be recovered upon averaging. This is illustrated in Figs 2.9(a) and 2.9(b), which show that the shape of both subthreshold and suprathreshold input signals can be recovered in the presence of noise by ensemble averaging the output, albeit with a certain amount of distortion.

The use of ensemble averaging to increase the SNR of a noisy signal is a well-known technique in areas such as sonar signal processing – see also Section 6.2 in Chapter 6. When N independently noisy realizations of the same signal are averaged, the resulting SNR is known to be increased by a factor of N. This is illustrated in Figs 2.9(c) and 2.9(d), where the input signals shown in Figs 2.5 and 2.8 respectively have been ensemble averaged for 1000 different noise realizations. The difference between this technique and the situation described in Figs 2.9(a) and 2.9(b) is that in one case the signal being averaged is continuously valued, whereas in the thresholded signal case, the signal being averaged is binary. This distinction is crucial. Different techniques must be used to properly analyze each case. Furthermore, in practice, ensemble averaging a continuously valued signal to increase its SNR is simply not practical, due to the difficulty of precisely storing an analogue quantity. Instead, such a signal is converted to a digital signal by quantization, prior to averaging, giving rise to a situation highly analogous to the scenario in Figs 2.9(a) and 2.9(b). The difference is that quantization is performed by a quantizer with a number of different threshold values, on each signal realization. In the case of Figs 2.9(a) and 2.9(b), a one-bit quantization is performed. The presence of noise allows ensemble averaging to improve the resulting output SNR.

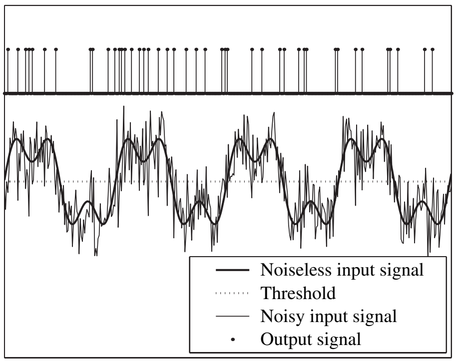
It seems that, at least in terms of efficiency, if the input SNR is very large, it might be worthwhile to perform the one-bit quantization N times, provided the noise is independent in each realization. This is in contrast with trying to obtain a high precision quantization of a very noisy signal – which is stored in a multi-bit binary number – and then averaging the result N times. Such a technique has indeed been performed in sonar signal processing, in a method known as DIgital MUltibeam Steering (DIMUS), employed in submarine sonar arrays (Rudnick 1960) – see Section 4.2 of Chapter 4.



(a) Subthreshold signal

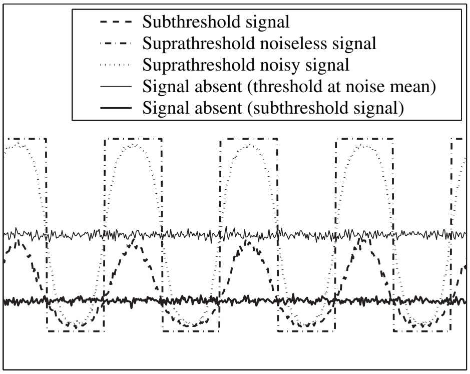


(b) Suprathreshold, noise absent

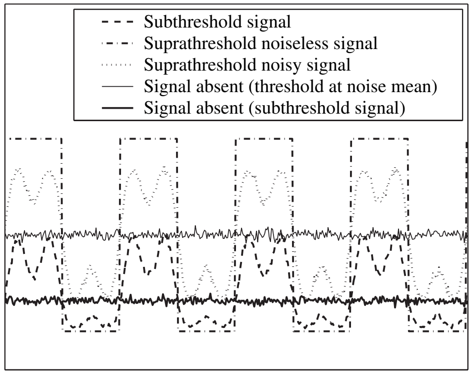


(c) Suprathreshold, noise present

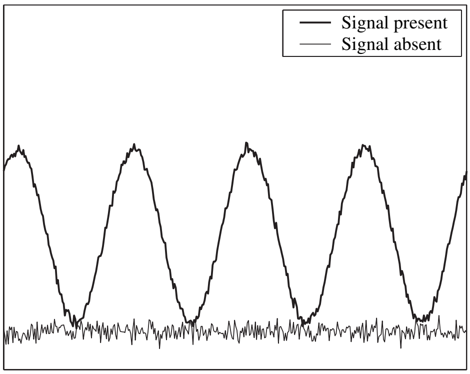
Fig. 2.8. Threshold SR for a periodic but not sinusoidal signal. This figure shows an input signal that, while periodic, is not a single frequency sine wave. Instead, the input is the sum of two sine waves of different frequencies. In each subfigure, the lower trace indicates the input signal's amplitude against increasing time, with the straight line being the threshold value. The upper plot is the output signal plotted against the same time-scale as the input signal. The output signal is a 'spike' – that is, a short duration pulse – every time the input crosses the threshold with positive slope – that is, from subthreshold to suprathreshold but not *vice versa*. Figure 2.8(a) shows the output signal for the case where the input signal is subthreshold, but additive noise causes threshold crossings. As with the single frequency input signal of Fig. 2.5, the noise causes output pulses to occur. The probability of a pulse is higher when the signal is closer to the threshold. Figure 2.8(b) illustrates how, in the absence of noise, thresholding this signal at its mean will provide exactly the same output as a single frequency sine wave. Figure 2.8(c) shows the output signal when the input is thresholded at its mean and additive noise is present. Unlike in the absence of noise, more than one pulse per period can occur. This means output noise is created, if it is only the input signal's period that is to be recovered. On the other hand, the presence of noise allows the shape of the input signal to be recovered by ensemble averaging – see Fig. 2.9.



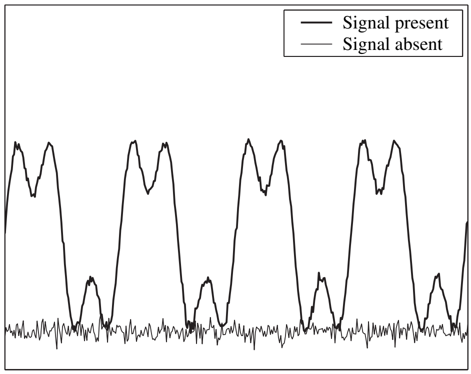
(a) Ensemble average of thresholded signal



(b) Ensemble average of thresholded signal



(c) Ensemble averaged signal



(d) Ensemble averaged signal

Fig. 2.9. Ensemble averages of thresholded and unthresholded signals. This figure shows the result of ensemble averaging 1000 realizations of the outputs shown in Figs 2.5, 2.7, and 2.8, as well as ensemble averages of the input signals. Figure 2.9(a) shows ensemble averages when the input is the single frequency sine wave of Figs 2.5 and 2.7. In the absence of noise, the ensemble average is a square wave. However, in the presence of noise, the ensemble average is clearly closer in shape to the original input signal, although also somewhat distorted. This is true for both subthreshold and suprathreshold input signals. The average of the noise is also shown for comparison. Figure 2.9(b) shows the ensemble averages for the periodic but not single frequency input signal of Fig. 2.8. The absence of noise gives the same signal as for the single frequency input, whereas again the presence of noise provides a signal that has a shape close to that of the input signal. It is well known that averaging a noisy signal N times reduces the SNR by a factor of N. This is illustrated for the unthresholded input signals in Figs 2.9(c) and 2.9(d). Clearly the ensemble averaging of the thresholded signals in the presence of noise provides a similar effect to this, although since distortion is introduced due to the discrete nature of the output, the output SNR is greater.

In light of the above discussion, it can be said that SR can occur for non-subthreshold signals; we simply need to clarify what it is that is being measured! This fact was perhaps overlooked due to an ingrained emphasis on measuring SR by the output SNR at the fundamental frequency of the input periodic signal. When this was starting to be questioned, the emphasis switched to aperiodic input signals. As discussed below, the ensemble averaging performed here cannot be carried out for aperiodic signals in the same way.

Suprathreshold stochastic resonance

The fact that ensemble averaging a thresholded periodic signal can provide a better response in the presence of noise rather than its absence leads, almost, but not quite, to the concept of suprathreshold stochastic resonance (SSR) (Stocks 2000c).

In the situations illustrated above, it is the periodicity of the input – regardless of the shape – that allows ensemble averages, taken over a period of time, to increase the output SNR. Since we know that the signal is periodic, provided that separate ensembles of the input are all mutually in phase, each signal segment that is averaged can be collected at any point in time. This allows independent amplitudes of noise to be added to each amplitude of the signal, since each amplitude of signal is periodically repeated, whereas the noise is not. This situation is described in Gammaitoni *et al.* (1998). In such a situation, if the input signal is aperiodic, ensemble averaging in this fashion would not work.

By contrast, we shall see in Chapter 4 that SSR refers to the *instantaneous* averaging of the outputs from an *array* of independently noisy threshold devices that all receive the same signal.

We note briefly that although it is normally considered that SR does not occur in a single device for suprathreshold signals, it has in fact been reported that an SR effect, known as *residual* stochastic resonance, can occur in a bistable system for a weakly suprathreshold signal (Gammaitoni *et al.* 1995b, Apostolico *et al.* 1997, Duan *et al.* 2004).

SSR has also received some criticism because of its apparent similarity to a situation where independently noisy versions of the same signal are each passed through linear amplifiers, and then averaged. However, this view misses a crucial point. The result of this procedure would remain an analogue signal. In the SSR model discussed in this book, the output signal is a quantized – that is, digital – signal. This fact should be considered to be equally important to the fact that the output signal is less noisy than the input, as digitizing a signal – that is, *compressing* it – provides the possibility of immunity from the effects of more noise in subsequent propagation and computation. This is the case whether we are considering artificial digital technology – where nearly all modern systems are digital – or biology, where the senses communicate analogue stimuli that they observe to the brain predominantly using 'all-or-nothing' electrical action potentials.

Further discussion of SSR is deferred until Chapters 4–9. We demonstrate in these chapters that SSR can be described as a form of nondeterministic quantization. As an aid to those unfamiliar with quantization in the signal processing context, the next chapter discusses quantization theory. Before then, however, we give a brief pointer to some of the most important results on SR in neurons and neural systems.

2.9 Does stochastic resonance occur *in vivo* in neural systems?

According to the ISI Web of Knowledge database, at the time of writing, about 20% of SR papers also contain a reference in the title, abstract, or keywords to the words *neuron* or *neural*.

As mentioned previously, the first papers investigating SR in neuron models appeared in 1991 (Bulsara *et al.* 1991, Bulsara and Moss 1991, Longtin *et al.* 1991) with such research accelerating – for example Longtin (1993), Chialvo and Apkarian (1993) and Longtin *et al.* (1994) – after the 1993 observation of SR in physiological experiments where external signal and noise were applied to crayfish mechanoreceptors (Douglass *et al.* 1993). A good history of this early work on SR in neurons is given in Hohn (2001), and a recent summary of progress in the field was published in Moss *et al.* (2004).

However, there are some other published works that indicate that the positive role of noise in neurons was noticed prior to 1991.

For example, in 1971 the first comprehensive analytical studies of the effects of noise on neuron firing demonstrated that noise 'smoothes' the firing response of neurons (Lecar and Nossal 1971a, Lecar and Nossal 1971b). Later, Horsthemke and Lefever (1980) discuss noise-induced transitions in neural models, and in particular Yu and Lewis (1989) advocate noise as being an important element in signal modulation by neurons.

Crucially, none of the above cited papers has been able to prove that neurons use SR in a natural setting – the evidence for neurons 'using SR' is only indirect. What has been observed is that neurons are nonlinear dynamical systems for which SR effects occur when signal and noise are both added externally. A direct observation of SR would require an application of an external signal, and measurements of *internal* neural noise.

If it can be established that SR plays an important role in the encoding and processing of information in the brain, and that it somehow provides part of the brain's superior performance to computers and artificial intelligence in some areas, then using this knowledge in engineering systems may revolutionize the way we design computers, sensors, and communications systems.

One of the most intriguing proposed applications inspired by SR, as first suggested by Morse and Evans (1996), is that of enhanced cochlear implant signal encoding. Various authors have since advocated the exploitation of SR in this area (Morse and Evans 1996, Morse and Roper 2000, Morse and Meyer 2000, Hohn and Burkitt 2001, Stocks *et al.* 2002, Chatterjee and Robert 2001, Rubinstein and Hong 2003, Behnam and Zeng 2003, Moore 2003, Chatterjee and Oba 2005, Morse *et al.* 2007). The basic idea is that several sources of substantial randomness are known to exist in healthy functioning inner ears (Hudspeth 1989, Lewis *et al.* 2000, Henry and Lewis 2001, Robles and Roggero 2001). Well-controlled noise in the output of cochlear implant electrical signals would therefore stimulate nerve fibres in a more natural way, and hopefully improve hearing in deaf cochlear-implant patients. The same principle of making the output of biomedical prosthetics more like biology – including any random aspects – has previously been applied in mechanical ventilators (Suki *et al.* 1998). The subject of cochlear implants is discussed further in Chapter 10.

2.10 Chapter summary

In this chapter a historical perspective is used to review SR, and the main sub-areas of SR important to this book. In particular, we define SR, as it is most widely understood, and discuss its occurrence in simple threshold-based systems.

Chapter 2 in a nutshell

This chapter includes the following highlights:

- A discussion of the evolution of the term 'stochastic resonance'.
- A historical review and elucidation of the major epochs in the history of SR research.
- A qualitative demonstration that SR can occur in a single threshold device, where the threshold is set to the signal mean. SR will not occur in the conventional SNR measure in this situation, but only in a measure of distortion, after ensemble averaging.
- A discussion of some of the confusing and controversial aspects of SR research, and a critique of the application of SNR and information-theoretic measures.

This concludes Chapter 2, which sets the historical context for this book. The next chapter presents some information-theoretic definitions required in the remainder of this book, and overviews signal quantization theory.