Introductory Mathematical Analysis

MATH 302

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Assignment 3

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Question 1

Let $\{f_n : n \geq 1\}$ be a sequence in $\mathcal{C}([0,1], \mathbb{R})$. Define a function F_n by

$$F_n(x) = \int_0^x \sin(f_n(t))dt, \quad x \in [0,1].$$

Use the Arzela-Ascoli theorem to prove that $\{F_n : n \geq 1\}$ has a uniformly convergent subsequence.

Proof.

$$\mathbb{Q} \cap [0,1] \subset [0,1]$$
$$\mathbb{Q} \cap [0,1] \subset \mathbb{Q}$$

Since $\mathbb{Q}$ is countable and dense then so is $\mathbb{Q} \cap [0,1]$. So, $[0,1]$ has a countable dense subset. $\mathbb{R}$ is a complete metric space. $\square$

Question 2

(a) Prove that every open subset of $\mathbb{R}$ (with respect to the standard topology $\tau_{\mathbb{R}}$) is a countable union of disjoint open intervals.

(b) Explain why this statement cannot be true for $\mathbb{R}^n$ for any $n \geq 2$.

Question 3

Let f be an integrable function on the unit circle (so f is 2π-periodic).

(a) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Prove that $\hat{f}(n) = 0$ for all odd integers n.

(b) Find an example of a metric space (X, d) and a subset $S \subset X$ such that $S = L(S)$, where $L(S)$ is the set of limit points of S.

Question 4

Consider the sequence

$$f_n(x) = \frac{\pi n + \sin(nx)}{2n + \cos(n^2 x)}, \quad x \in [0,1].$$

(a) Prove that (f_n) converges uniformly on $[0,1]$.

(b) Hence, or otherwise, evaluate the limit

$$\lim_{n\to\infty} \int_0^1 \frac{\pi n + \sin(nx)}{2n + \cos(n^2 x)} dx.$$

Question 5

Consider a second-order differential equation of the form

$$x''(t) + 5x(t) = F(t),$$

where

$$F(t) = a_0 + \sum_{n=1}^{\infty} b_n \cos(n\pi t) + \sum_{n=1}^{\infty} c_n \sin(n\pi t).$$

(a) Solve this differential equation by considering a potential solution $x(t)$ given as a Fourier series:

$$x(t) = A_0 + \sum_{n=1}^{\infty} B_n \cos(n\pi t) + \sum_{n=1}^{\infty} C_n \sin(n\pi t),$$

and obtain relations for A_0, B_n, C_n in terms of the coefficients a_0, b_n, c_n of $F(t)$.

(b) Suppose

$$F(t) = \begin{cases} -1 & \text{if } -1 < t < 0 \\ 1 & \text{if } 0 < t < 1 \end{cases}.$$

Solve for $x(t)$.