Multivariable Calculus I

MATH 202

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Assignment 5

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Question 1

Compute the volume of a right cylindrical cone where the base radius is R and the height of the cone is h, using cylindrical coordinates.
The Jacobian determinant for cylindrical coordinates has already been computed in class, it is r.
The triple integral for the volume of the right cylindrical cone $\mathcal{C}$ is

$$V = \iiint_{\mathcal{C}} 1\, dV.$$

Converting to cylindrical coordinates

$$(x, y, z) \to (r\cos\theta, r\sin\theta, z)$$

yields

$$V = \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz\, d\theta.$$

The bounds for the integral in r are obtained by considering the slant of the boundary of the cone from $(0, 0, 0)$ to a point on the upper edge which would be $\frac{R}{h}z$. Evaluating the integral

$$V = \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz\, d\theta = 2\pi \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz = 2\pi \int_0^h \left[\frac{r^2}{2}\right]_0^{\frac{R}{h}z} dz = \pi \frac{R^2}{h^2} \int_0^h z^2\, dz = \frac{\pi\frac{R^2}{h^2}}{3}\left[z^3\right]_0^h$$

$$V = \frac{\pi h R^2}{3}.$$

Question 2

A spherical cap is the piece of a sphere (ball) sliced off by a plane. Suppose that a sphere has radius 10, and the height of the spherical cap is 6. Determine the volume of the spherical cap.
It can be seen that the volume of half a sphere is

$$V_{\text{half sphere}} = V_{\text{cap}} + \frac{1}{2}V_{\text{cylinder}} + \frac{1}{2}V_{\text{napkin ring}}.$$

Rearranging we get

$$V_{\text{cap}} = V_{sphere} - \frac{1}{2}V_{\text{cylinder}} - \frac{1}{2}V_{\text{napkin ring}}.$$

$$V_{\text{half sphere}} = \frac{2}{3}\pi R^3 = \frac{2000\pi}{3},$$

$$\frac{1}{2}V_{\text{cylinder}} = \frac{1}{2}\pi r^2 h = \pi(\sqrt{10^2 - 4^2}^2)(6) = \frac{1}{2}\pi(\sqrt{84}^2)(8) = 336\pi,$$

$$V_{\text{napkin ring}} = \frac{\pi h^3}{3} = \frac{128\pi}{3}.$$

Putting all of these together

$$V_{\text{cap}} = \frac{2000\pi}{3} - 336\pi - \frac{128\pi}{3} = 288\pi.$$

Question 3

A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent (That is, both X, Y are random variables). The joint probability density function of X and Y can be modelled by

$$f(x,y) = \begin{cases} ke^{-\frac{x+2y}{40}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of k and the probability that a client spends less than 60 minutes at the agent's office.

Solution.

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ke^{-\frac{x+2y}{40}}\,dy\,dx = k\int_{-\infty}^{\infty} e^{\frac{-x}{40}}\int_{-\infty}^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

Since the probability density function is non-zero only for non-zero values of x and y we can further simplify the double integral

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

$$u = \frac{y}{20},\ 20\,du = dy,\ y = 0 \Rightarrow u = 0,\ y = \infty \Rightarrow u = \infty$$

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} 20e^{-u}\,du\,dx = k\int_0^{\infty} e^{\frac{-x}{40}}\left[20e^{-u}\right]_0^{\infty}\,dx = -20k\int_0^{\infty} e^{\frac{-x}{40}}\,dx$$

$$v = \frac{x}{40},\ 40\,dv = dx,\ x = 0 \Rightarrow v = 0,\ x = \infty \Rightarrow v = \infty$$

$$-20k\int_0^{\infty} 40e^{-v}\,dv = 800k\left[e^{-v}\right]_0^{\infty} = 800k$$

We want the total probability to be 1, so we have

$$k = \frac{1}{800}.$$

We want to find $\mathcal{P}(X+Y<60)$, which can be done by evaluating the following integral

$$\mathcal{P}(X+Y<60) = \frac{1}{800}\int_{X=0}^{60}\int_{Y=0}^{60-x} e^{-\frac{x+2y}{40}}\,dy\,dx.$$

Using the same substitutions as the previous part we get

$$\frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\int_0^{60-x} e^{\frac{-y}{20}}\,dy\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left[-20e^{-u}\right]_0^{\frac{60-x}{20}}\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left(20 - 20e^{\frac{x-60}{20}}\right)\,dx$$

Factoring and multiplying through

$$\frac{1}{40}\int_0^{60} e^{\frac{-x}{40}} - e^{-3}e^{\frac{x}{40}}\,dx = \frac{1}{40}\left[\int_0^{60} e^{\frac{-x}{40}} - e^{-3}\int_0^{60} e^{\frac{x}{40}}\right]$$

$$\frac{1}{40}\left[-40\left[e^{\frac{-x}{40}}\right]_0^{60}-40e^{-3}\left[e^{\frac{x}{40}}\right]_0^{60}\right]=\left[-\left[e^{\frac{-x}{40}}\right]_0^{60}-e^{-3}\left[e^{\frac{x}{40}}\right]_0^{60}\right]=-e^{\frac{3}{2}}+1-e^{\frac{3}{2}}+e^{-3}$$

$$\mathcal{P}(X+Y<60)=1-e^{-\frac{3}{2}}+e^{-3}.$$

Question 4

(Worth 20 points). In this question, we derive some formulas for volumes of higher dimensional balls. Let $V_n(R)$ denote the volume of a n-ball (in $\mathbb{R}^{n+1}$) of radius R.

A 0-ball is just an interval. This allows us to compute the volume of a 2-ball, defined by the inequality

$$x^2+y^2+z^2\leq R^2,$$

using cylindrical coordinates, we deduce:

$$V_2(R)=\int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dh\,dr\,d\theta=\int_0^{2\pi}\int_0^R 2r\sqrt{R^2-r^2}\,dr\,d\theta.$$

(a) Verify that $V_2(R)=\frac{4\pi R^3}{3}$, by finishing with the iterated integral above.

$$V_2(R)=\int_0^{2\pi}\int_0^R 2\sqrt{R^2-r^2}\,dr\,d\theta.$$

Using the substitution

$$u=R^2-r^2,\ du=-2r\,dr,\ r=1\Rightarrow u=R^2,\ r=R\Rightarrow u=0$$

$$V_2(R)=\int_0^{2\pi}\int_{R^2}^0 -\sqrt{u}\,du\,d\theta$$

$$V_2(R)=\int_0^{2\pi}\int_0^{R^2} \sqrt{u}\,du\,d\theta$$

$$V_2(R)=2\pi\left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^{r^2}=\frac{4\pi R^2}{3}.$$

(b) Using the same idea, deduce that

$$V_3(R)=\int_0^{2\pi}\int_0^R \pi(R^2-r^2)r\,dr\,d\theta.$$

In $\mathbb{R}^4$ a ball with radius R is the set of points

$$x_1^2+x_2^2+x_3^2+x_4^2\leq R^2.$$

Converting to cylindrical coordinates in $\mathbb{R}^4$ we get $r^2 = x_1^2 + x_2^2$, which then gives

$$r^2 + x_3^2 + x_4^2 \leq R^2.$$

When r is fixed the (x_3, x_4) coordinates become a disk in $\mathbb{R}^2$, this disk has a radius of $\sqrt{R^2 - r^2}$ and area of $\pi(R^2 - r^2)$. The volume from the (x_1, x_2) coordinates is given by $r\,dr\,d\theta$. All of this gives

$$V_3(R) = \int_0^{2\pi} \int_0^R \pi(R^2 - r^2) r\,dr\,d\theta.$$

(c) Generalize this to obtain the formula

$$V_{n+2}(R) = \int_0^{2\pi} \int_0^R V_n(\sqrt{R^2 - r^2}) r\,dr\,d\theta.$$

A ball in $\mathbb{R}^{n+2}$ with radius R set of coordinates

$$x_1^2 + x_2^2 + \cdots + x_{n+2}^2 \leq R^2.$$

Converting to cylindrical coordinates we get $r^2 = x_1^2 + x_2^2$. Let $y = (x_3, \ldots, x_{n+2}) \in \mathbb{R}^n$. We then get

$$r^2 + \|y\|^2 \leq R^2.$$

Rearranging

$$\|y\|^2 \leq R^2 - r^2,$$

we see that y is within an $n - ball$ with radius $\sqrt{R^2 - r^2}$, and the volume of this ball is $V_n(\sqrt{R^2 - r^2})$. The (x_1, x_2) coordinates give $r\,dr\,d\theta$. This all yields

$$V_{n+2}(R) = \int_0^{2\pi} \int_0^R V_n(\sqrt{R^2 - r^2}) r\,dr\,d\theta.$$

(d) Using the recursion in part (c), give formulas for $V_4(R), V_5(R), V_6(R)$.
From the prvious part we know that for $V_{n+2}(R)$ the component $V_n(R)$ has a radius of $\sqrt{R^2 - r^2}$ and that $V_n(R) \propto \left(\sqrt{R^2 - r^2}\right)^{n+1}$, this will help simplfy the following integrals.

$$V_4(R) = \int_0^{2\pi} \int_0^R \left(\frac{4\pi R^2}{3}\right)(\sqrt{R^2 - r^2})r\,dr\,d\theta = 2\pi \int_0^R \left(\frac{4\pi}{3}\right)\left(R^2 - r^2\right)^{\frac{3}{2}} r\,dr$$

$$u = R^2 - r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_4(R) = \frac{4\pi^2}{3} \int_{R^2}^0 -u^{\frac{3}{2}}\,du = \frac{4\pi^2}{3} \int_0^{R^2} u^{\frac{3}{2}}\,du = \frac{8\pi^2}{15}\left[u^{\frac{5}{2}}\right]_0^{R^2} = \frac{8\pi^2}{15} R^5$$

$$V_4(R) = \frac{8\pi^2}{15} R^5.$$

$$V_3(R) = \int_0^{2\pi} \int_0^R \pi(R^2 - r^2)r\,dr\,d\theta = 2\pi \int_0^R \pi(R^2 - r^2)r\,dr$$

$$u = R^2 - r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_3(R) = 2\pi^2 \int_0^R R^2 r - r^3\,dr = 2\pi^2 \left[\frac{R^2 r^2}{2} - \frac{r^4}{4}\right]_0^R = 2\pi^2\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\pi^2 R^4}{2}.$$

$$V_5(R) = \int_0^{2\pi} \int_0^R V_3(\sqrt{R^2 - r^2})r\,dr\,d\theta = \int_0^{2\pi} \int_0^R \left(\frac{\pi^2 R^4}{2}\right)(\sqrt{R^2 - r^2})r\,dr\,d\theta$$

$$V_5(R) = 2\pi^3 \int_0^R (R^2 - r^2)^2 r\,dr = 2\pi^3 \int_0^R R^4 r - 2R^2 r^3 + r^5\,dr = 2\pi^3\left[\frac{R^4 r^2}{2} - \frac{2R^2 r^4}{4} + \frac{r^6}{6}\right]_0^{R^2},$$

$$V_5(R) = \pi^3\left(\frac{R^6}{2} - \frac{R^6}{2} + \frac{R^6}{6}\right) = \frac{\pi^3}{6} R^6.$$

$$V_6(R) = \int_0^{2\pi} \int_0^R V_4(\sqrt{R^2 - r^2})r\,dr\,d\theta = 2\pi \int_0^R \frac{8\pi^2}{15} R^5(\sqrt{R^2 - r^2})r\,dr = \frac{8\pi^3}{15} \int_0^R \left(R^2 - r^2\right)^{\frac{5}{2}} r\,dr.$$

$$u = R^2 - r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_6(R) = \frac{8\pi^3}{15} \int_0^{R^2} u^{\frac{5}{2}}\,du = \frac{16\pi^3}{105}\left[u^{\frac{7}{2}}\right]_0^{R^2},$$

$$V_6(R) = \frac{16\pi^3}{105} R^7.$$

(e) Evaluate the limit

$$\lim_{n\to\infty} \frac{V_n(R)}{R^{n+1}}.$$

Since $V_n(R) \propto \left(\sqrt{R^2 - r^2}\right)^{n+1}$ we have

$$\frac{V_n(R)}{R^{n+1}} = \alpha \frac{\left(\sqrt{R^2 - r^2}\right)^{n+1}}{R^{n+1}},$$

for some α which is independant of n. So in the limit we obtain

$$\lim_{n\to\infty} \frac{V_n(R)}{R^{n+1}} = \lim_{n\to\infty} \alpha \frac{\left(\sqrt{R^2 - r^2}\right)^{n+1}}{R^{n+1}} = \alpha \lim_{n\to\infty} \frac{\left(\sqrt{R^2 - r^2}\right)^{n+1}}{R^{n+1}}.$$

And

$$\lim_{n\to\infty} \frac{\left(\sqrt{R^2 - r^2}\right)^{n+1}}{R^{n+1}} = 0,$$

since the denominator grows faster than the numerator. So,

$$\lim_{n\to\infty} \frac{V_n(R)}{R^{n+1}} = 0.$$