Number Theory

MATH 480

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Assignment 4

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Due Date:

November 20th, 2025

Question 1 [2 marks]

Find all the primitive roots modulo 27.
$27 = 3^3$. $\varphi(3) = 2$, by *lemma 2.8.13* the number of integers less than 3 of order 2 do not exceed $\varphi(2) = 1$. Also, by *theorem 2.8.9* we know that 3 has a primitive root. Since $2^1 \not\equiv 1 \mod 3$ and $2^2 \equiv 1 \mod 3$ we have 2 as a primitive root of 3. By *theorem 2.8.15* we know that either 2 or $2 + 3 = 5$ is a primitive root(s) for 9. $\varphi(9) = 6$, so we need to check to see if the order of 2 or 5 are 6 in modulo 9.

$$2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16 \equiv 7,\ 2^5 = 32 \equiv 5 \mod 9,\ 2^6 \equiv 5 \times 2 \equiv 1 \mod 9,$$

and

$$5^1 = 5,\ 5^2 = 25 \equiv 7,\ 5^3 \equiv 7 \times 5 = 35 \equiv 8 \mod 9,\ 5^4 \equiv 8 \times 5 \equiv 4,$$
$$5^5 \equiv 4 \times 5 \equiv 2 \mod 9, 5^6 \equiv 2 \times 5 \equiv 1 \mod 9.$$

So, both 2 and 5 are primitive roots of 9. From *theorem 2.8.16* we know that both 2 and 5 are primitive roots of 27. The other primitive roots of 27 are of the form 2^a where a is any integer mod 27 such that $\gcd(a, \phi(27)) = \gcd(a, 18) = 1$. These integers are 1,5,7,11,13,17.

$$\begin{aligned}
2^1 &\equiv 2 \\
2^5 &\equiv 32 \equiv 5 \\
2^7 &\equiv 2^5 \cdot 2^2 \equiv 5 \cdot 4 = 20 \\
2^{11} &= 2^9 \cdot 2^2 \equiv (-1) \cdot 4 \equiv 23 \\
2^{13} &= 2^9 \cdot 2^4 \equiv (-1) \cdot 16 \equiv 11 \\
2^{17} &= 2^9 \cdot 2^8 \equiv (-1) \cdot 13 \equiv 14
\end{aligned}$$

So the primitive roots of 27 are 2,5,20,23,11,14.

Question 2 [2 marks]

Evaluate $\left(\frac{105}{1009}\right)$.

$$\left(\frac{105}{1009}\right) = \left(\frac{3}{1009}\right)\left(\frac{5}{1009}\right)\left(\frac{7}{1009}\right)$$

By *theorem 3.2.1* we have

$$\left(\frac{105}{1009}\right) = \left(\frac{1009}{3}\right)\left(\frac{1009}{5}\right)\left(\frac{1009}{7}\right)$$

$$\left(\frac{105}{1009}\right) = \left(\frac{1}{3}\right)\left(\frac{4}{5}\right)\left(\frac{1}{7}\right)$$

$$\left(\frac{105}{1009}\right) = \left(\frac{1}{3}\right)\left(\frac{2}{5}\right)\left(\frac{2}{5}\right)\left(\frac{1}{7}\right)$$
$$\left(\frac{105}{1009}\right) = \left(\frac{-1}{3}\right)\left(\frac{-1}{3}\right)(-1)\,(-1)\left(\frac{-1}{7}\right)\left(\frac{-1}{7}\right)$$
$$\left(\frac{105}{1009}\right) = (-1)\,(-1)\,(-1)\,(-1)\,(-1)\,(-1) = 1$$

Question 3 [3 marks]

Let m be a positive integer with a primitive root. Suppose that $(a, m) = 1$. Prove that then the congruence $x^n \equiv a \mod m$ has exactly $(n, \phi(m))$ solutions if and only if $a^{\frac{\phi(m)}{(\phi(m),n)}} \equiv 1 \mod m$.

Question 4 [3 marks]

Let p be an odd prime number, and suppose that $h \geq 2$. Denote by g a primitive root modulo p^h.

(a) List all the solutions of the congruence $x^p \equiv 1 \pmod{p^h}$ using the primitive root g modulo p^h.

(b) List all the solutions of the congruence $x^{2p} \equiv 1 \pmod{p^h}$ using the primitive root g modulo p^h.

Question 5 [2 marks]

Let n be a positive integer with a primitive root. Using this primitive root, prove that the product of all positive integers less than n and relatively prime to n is congruent to -1 modulo n.

Question 6 [2 marks]

Let $p_1, p_2, \ldots, p_r$ be distinct prime numbers. Show that there exists an integer g such that g is a primitive root modulo p_i for all $1 \leq i \leq r$.

Question 7 [2 marks]

(a) Let a be an integer with $a \geq 2$, and suppose that $q \in \mathbb{N}$. What is the smallest positive integer d satisfying the property that $a^d \equiv 1 \pmod{a^q - 1}$? Deduce that q divides $\varphi(a^q - 1)$.

(b) Let q be a prime number. By considering the prime factorisation of the integer $N = a^q - 1$, show that either N is divisible by q, or else N is divisible by a prime number p with $p \equiv 1 \pmod{q}$.

Question 8 [3 marks]

Let q be a prime number. Prove that there are infinitely many prime numbers p with $p \equiv 1 \pmod{q}$.

Question 9 [3 marks]

Let $p \geq 5$ be an odd prime, show that

$$\left(\frac{3}{p}\right) = \begin{cases} 1 & \text{if } \ p \equiv \pm 1 \mod 12 \\ -1 & \text{if } \ p \equiv \pm 5 \mod 12. \end{cases}$$

Question 10 [6 marks]

Let $n > 1$ be an odd integer. Write $n = p_1^{e_1} \cdots p_k^{e_k}$. Let a be an integer. We define the **Jacobi symbol** $\left(\frac{a}{n}\right)$ as follows:

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p_1}\right)^{e_1} \cdots \left(\frac{a}{p_k}\right)^{e_k}.$$

Prove the following properties:

(a) If $a \equiv b \mod n$, then $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$.

(b) If a and b are integers, then $\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right)$.

(c) If $x^2 \equiv a \mod n$ has a solution, then $\left(\frac{a}{n}\right) = 1$. Provide an example that shows that the converse of this statement isn't always true.

(d) If m, n are relatively prime odd integers, then

$$\left(\frac{m}{n}\right)\left(\frac{n}{m}\right) = (-1)^{\frac{m-1}{2}}(-1)^{\frac{n-1}{2}}.$$