

MULTIVARIABLE CALCULUS I

MATH 202

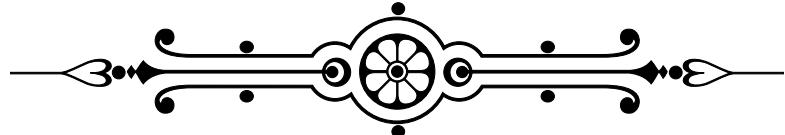
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Assignment 5

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Question 1

Compute the volume of a right cylindrical cone where the base radius is R and the height of the cone is h , using cylindrical coordinates.

Question 2

A spherical cap is the piece of a sphere (ball) sliced off by a plane. Suppose that a sphere has radius 10, and the height of the spherical cap is 6. Determine the volume of the spherical cap.

Question 3

A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent (That is, both X, Y are random variables). The joint probability density function of X and Y can be modelled by

$$f(x, y) = \begin{cases} ke^{-\frac{x+2y}{40}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of k and the probability that a client spends less than 60 minutes at the agent's office.

Solution.

$$\iint_{\mathbb{R}^2} f(x, y) dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} ke^{-\frac{x+2y}{40}} dy dx = k \int_{-\infty}^{\infty} e^{\frac{-x}{40}} \int_{-\infty}^{\infty} e^{\frac{-y}{20}} dy dx$$

Since the probability density function is non-zero only for non-zero values of x and y we can further simplify the double integral

$$\begin{aligned} & k \int_0^{\infty} e^{\frac{-x}{40}} \int_0^{\infty} e^{\frac{-y}{20}} dy dx \\ & u = \frac{y}{20}, 20 du = dy, y = 0 \Rightarrow u = 0, y = \infty \Rightarrow u = \infty \\ & k \int_0^{\infty} e^{\frac{-x}{40}} \int_0^{\infty} 20e^{-u} du dx = k \int_0^{\infty} e^{\frac{-x}{40}} [20e^{-u}]_0^{\infty} dx = -20k \int_0^{\infty} e^{\frac{-x}{40}} dx \\ & v = \frac{x}{40}, 40 dv = dx, x = 0 \Rightarrow v = 0, x = \infty \Rightarrow v = \infty \\ & -20k \int_0^{\infty} 40e^{-v} dv = 800k [e^{-v}]_0^{\infty} = 800k \end{aligned}$$

We want the total probability to be 1, so we have

$$k = \frac{1}{800}.$$

We want to find $\mathcal{P}(X + Y < 60)$, that is evaluate the integral

$$\frac{1}{800} \int_{X=0}^{60} \int_{Y=0}^{60-x} e^{-\frac{x+2y}{40}} dy dx.$$

Using the same substitutions as the previous part we get

$$\frac{1}{800} \int_0^{60} e^{\frac{-x}{40}} \int_0^{60-x} e^{\frac{-y}{20}} dy dx = \frac{1}{800} \int_0^{60} e^{\frac{-x}{40}} [20e^{-u}]_0^{\frac{60-x}{20}} dx = \frac{1}{800} \int_0^{60} e^{\frac{-x}{40}} (20e^{\frac{x-60}{20}} - 20) dx$$

Factoring and multiplying through

$$\frac{1}{40} \int_0^{60} e^{-3} e^{\frac{x}{40}} - e^{\frac{-x}{40}} dx$$

Question 4

(Worth 20 points). In this question, we derive some formulas for volumes of higher dimensional balls. Let $V_n(R)$ denote the volume of a n -ball (in \mathbb{R}^{n+1}) of radius R .

A 0-ball is just an interval. This allows us to compute the volume of a 2-ball, defined by the inequality

$$x^2 + y^2 + z^2 \leq R^2,$$

using cylindrical coordinates, we deduce:

$$V_2(R) = \int_0^{2\pi} \int_0^R \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r dh dr d\theta = \int_0^{2\pi} \int_0^R 2\sqrt{R^2 - r^2} dr d\theta.$$

(a) Verify that $V_2(R) = \frac{4\pi R^3}{3}$, by finishing with the iterated integral above.

(b) Using the same idea, deduce that

$$V_3(R) = \int_0^{2\pi} \int_0^R \pi(R^2 - r^2)r dr d\theta.$$

(c) Generalize this to obtain the formula

$$V_{n+2}(R) = \int_0^{2\pi} \int_0^R V_n(\sqrt{R^2 - r^2}) dr d\theta.$$

(d) Using the recursion in part (c), give formulas for $V_4(R)$, $V_5(R)$, $V_6(R)$.

(e) Evaluate the limit

$$\lim_{n \rightarrow \infty} \frac{V_n(R)}{R^{n+1}}.$$