Survey of Algebra

MATH 320

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Assignment 6

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Question 1 [2 marks]

What is the order of the element $14 + \langle 8 \rangle$ in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$?

$$|\langle 8 \rangle| = 3,$$
$$|\mathbb{Z}_{24}/\langle 8 \rangle| = 8,$$
$$\langle 8 \rangle = \{0, 8, 16\}.$$

It can be seen that

$$8 + \langle 8 \rangle = 0 + \langle 8 \rangle,$$

that is in the factor group $\mathbb{Z}_{24}/\langle 8 \rangle$ $8 \equiv 0$. This tells us that $14 + \langle 8 \rangle = 6 + \langle 8 \rangle$, and that $|14 + \langle 8 \rangle| = |6 + \langle 8 \rangle|$. We know that for any two cosets in a factor group that

$$(a + H) + (b + H) = (a + b) + H$$

where H is the identity coset. If $a = b$ this can be generalized to

$$\underbrace{(a + H) + \cdots + (a + H)}_{n \text{ times}} = n(a) + H.$$

(A proper proof of this would be given through the use of the principle of Mathematical induction) Therefore to find the order of $6 + \langle 8 \rangle$ we need to find the smallest n such that $n6 + \langle 8 \rangle = \langle 8 \rangle$. We can see that if $n = 4$, then

$$4(6) + \langle 8 \rangle = 24 + \langle 8 \rangle = \langle 8 \rangle.$$

Now we need to see if 4 is the smallest number with this property. The only numbers that divide both 4 and 8 are 1,2,4. We can see that $n = 1$ does not work, and neither does $n = 2$. So,

$$|14 + \langle 8 \rangle| = 4.$$

Question 2 [2 marks]

Explain why the correspondence $x \mapsto 3x$ from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{10}$ is not a homomorphism.
Let $\varphi : \mathbb{Z}_{12} \mapsto \mathbb{Z}_{10}$ defined as above. Then if φ is a homomorphism we would have

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \forall a, b \in \mathbb{Z}_{12}.$$

Take $a = 6, b = 6$, then

$$\varphi(6 + 6) = \varphi(0) = 0$$
$$\varphi(6) + \varphi(6) = 6.$$

Thus, φ is not a homomorphism.

Question 3 [2 marks]

Let H be a normal subgroup of a finite group G, and let a belong to G. If the element aH has order 3 in the group G/H and $|H| = 10$, what are the possibilities for the order of a in G?
Let $|a| = n$. Then $3|n$. Let $h = a^3 \in H$, then $h^{10} = a^{3(10)} = e$. This shows that $n|30$. Putting these two together we see that the possible values of n are 1,3,6,15,30.

Question 4 [3 marks]

Prove that a factor group of a cyclic group is cyclic.

Proof. Let $G = \langle g \rangle G = \langle g \rangle$ be cyclic and $N \trianglelefteq G$. Then every element $n \in G/N$ is of the form $n = xN$, For some $x \in G$. Since G is cyclic we have $x = g^n$, for some $n \in \mathbb{Z}$. That is for all xN, we have $xN = g^k N = (gN)^K$ for some $k \in \mathbb{Z}$. Thus every coset in G/N is a power of gN, so $G/N = \langle g \rangle$, hence G/N is cyclic. $\square$

Question 5 [3 marks]

Let H and K be normal subgroups of a group G. Prove that HK is also a normal subgroup of G.

Proof. First we must show that HK is a subgroup of G. Let $a, b \in HK$, then for each of these elements in either in H, K or both (becuase $HK = \{hk : h \in H, k \in K\}$). So $ab \in HK$. We also have $a^{-1} \in HK$ becuase a is in either H, K or both, and each of those is a subgroup. So $HK \leq G$. We know that a subgroup is normal if and only if $xJx^{-1} \subset J$ for all $x \in G$. Let x be any element of the group G. Then since H and K are nomal in G we have

$$xHKx^{-1} = (xHx^{-1})(xKx^{-1}) = HK.$$

Therefore, $HK \trianglelefteq G$. $\square$

Question 6 [3 marks]

Let G be a group acting on a set X. Suppose that the stabilizer G_x of a certain point $x \in X$ is a proper normal subgroup of G. Prove that every element of G_x fixes every element $y \in G.x$.

Proof. An element $y \in G.x$ looks like $y = gx$ for some $g \in G$. Elements $h \in G_x$ have the unique property that $hx = x$ for all $h \in G_x$. For some arbitrary element $y \in G.x$ we have

$$\begin{aligned} h \cdot y &= h \cdot (y \cdot x) \\ &= (hg) \cdot x \\ &= (g^{-1}hg) \cdot x. \end{aligned}$$

Since $h \in G_x$ and G_x is a proper normal subgroup we have $g^{-1}hg \in G_x$. Since G_x is the stabilizer of x we have

$$(g^{-1}hg) \cdot x = x.$$

□

Question 7 [5 marks]

In what follows, you prove the third isomorphism theorem. Let M, N be normal sub-groups of a group G such that N is a subgroup of M.

(a) Show that N is a normal subgroup of M.

Proof. Since $N \trianglelefteq G$ we have $aN = Na \, \forall a \in G$. Since $M \leq G \Rightarrow M \subseteq G$, and $N \leq M$, then for all $m \in M$ we have $mN = Nm$. Therefore, $N \trianglelefteq M$. □

(b) Show that $(G/N)/(M/N) \cong G/M$.

Proof. The quotient groups $G/N, M/N, G/M$ are well defined as the group's are normal in the correct order. Define $\varphi : G/N \mapsto G/M$ by

$$\varphi(gN) = gM.$$

Claim 7.1. *φ is a surjective homomorphism.*

Proof of claim. For any $gM \in G/M$, we have $gN \in G/N$ with $\varphi(gN) = gM$.
Let $g_1N, g_2N \in G/N$ then

$$\varphi(g_1Ng_2N) = \varphi(g_1g_2N) = g_1g_2M$$

and

$$\varphi(g_1N)\varphi(g_2N) = g_1Mg_2M = g_1g_2M = \varphi(g_1Ng_2N).$$

□

The kernel of φ is

$$\ker \varphi = \{gN \in G/N : \varphi(gN) = gM = M\},$$

but if $gM = M$, then $g \in M$. So we can rewrite the kernel as

$$\ker \varphi = \{gN \in G/N : g \in M\}.$$

That is kernel consists of all gN such that $g \in M$, which is precisely M/N viewed as a subgroup of G/N.

$$(G/N)/(M/N) \cong G/M.$$

□