Multivariable Calculus I

MATH 202

Dr. Stanley Yao Xiao

Assignment 5

Deepak Jassal

Due Date:

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Question 1

Compute the volume of a right cylindrical cone where the base radius is R and the height of the cone is h, using cylindrical coordinates.

Question 2

A spherical cap is the piece of a sphere (ball) sliced off by a plane. Suppose that a sphere has radius 10, and the height of the spherical cap is 6. Determine the volume of the spherical cap.

Question 3

A client spends X minutes in an insurance agent's waiting room and Y minutes meeting with the agent (That is, both X, Y are random variables). The joint probability density function of X and Y can be modelled by

$$f(x,y) = \begin{cases} ke^{-\frac{x+2y}{40}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of k and the probability that a client spends less than 60 minutes at the agent's office.

Solution.

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ke^{-\frac{x+2y}{40}}\,dy\,dx = k\int_{-\infty}^{\infty} e^{\frac{-x}{40}}\int_{-\infty}^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

Since the probability density function is non-zero only for non-zero values of x and y we can further simplify the double integral

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

$$u = \frac{y}{20},\ 20\,du = dy,\ y = 0 \Rightarrow u = 0,\ y = \infty \Rightarrow u = \infty$$

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} 20e^{-u}\,du\,dx = k\int_0^{\infty} e^{\frac{-x}{40}}\left[20e^{-u}\right]_0^{\infty}\,dx = -20k\int_0^{\infty} e^{\frac{-x}{40}}\,dx$$

$$v = \frac{x}{40},\ 40\,dv = dx,\ x = 0 \Rightarrow v = 0,\ x = \infty \Rightarrow v = \infty$$

$$-20k\int_0^{\infty} 40e^{-v}\,dv = 800k\left[e^{-v}\right]_0^{\infty} = 800k$$

We want the total probability to be 1, so we have

$$k = \frac{1}{800}.$$

We want to find $\mathcal{P}(X+Y<60)$, that is evaluate the integral

$$\frac{1}{800}\int_{X=0}^{60}\int_{Y=0}^{60-x} e^{-\frac{x+2y}{40}}\,dy\,dx.$$

Using the same substitutions as the previous part we get

$$\frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\int_0^{60-x} e^{\frac{-y}{20}}\,dy\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left[20e^{-u}\right]_0^{\frac{60-x}{20}}\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left(20e^{\frac{x-60}{20}}-20\right)\,dx$$

Factoring and multiplying through

$$\frac{1}{40}\int_0^{60} e^{-3}e^{\frac{x}{40}} - e^{\frac{-x}{40}}\,dx$$

Question 4

(Worth 20 points). In this question, we derive some formulas for volumes of higher dimensional balls. Let $V_n(R)$ denote the volume of a n-ball (in $\mathbb{R}^{n+1}$) of radius R.

A 0-ball is just an interval. This allows us to compute the volume of a 2-ball, defined by the inequality

$$x^2+y^2+z^2\leq R^2,$$

using cylindrical coordinates, we deduce:

$$V_2(R)=\int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dh\,dr\,d\theta = \int_0^{2\pi}\int_0^R 2\sqrt{R^2-r^2}\,dr\,d\theta.$$

(a) Verify that $V_2(R)=\frac{4\pi R^3}{3}$, by finishing with the iterated integral above.

(b) Using the same idea, deduce that

$$V_3(R)=\int_0^{2\pi}\int_0^R \pi(R^2-r^2)r\,dr\,d\theta.$$

(c) Generalize this to obtain the formula

$$V_{n+2}(R)=\int_0^{2\pi}\int_0^R V_n(\sqrt{R^2-r^2})\,dr\,d\theta.$$

(d) Using the recursion in part (c), give formulas for $V_4(R), V_5(R), V_6(R)$.

(e) Evaluate the limit

$$\lim_{n\to\infty}\frac{V_n(R)}{R^{n+1}}.$$