Introductory Mathematical Analysis

MATH 302

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Assignment 3

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Question 1

Let $\{f_n : n \geq 1\}$ be a sequence in $\mathcal{C}([0,1],\mathbb{R})$. Define a function F_n by

$$F_n(x) = \int_0^x \sin(f_n(t))\,dt, \quad x \in [0,1].$$

Use the Arzela-Ascoli theorem to prove that $\{F_n : n \geq 1\}$ has a uniformly convergent subsequence.

Proof.

$$\mathbb{Q} \cap [0,1] \subset [0,1]$$
$$\mathbb{Q} \cap [0,1] \subset \mathbb{Q}$$

Since $\mathbb{Q}$ is countable and dense then so is $\mathbb{Q} \cap [0,1]$. So, $[0,1]$ has a countable dense subset. $\mathbb{R}$ is a complete metric space.
Pick $\delta = \varepsilon$ then whenever $|x-y| < \delta$

$$|F_n(x) - F_n(y)| \leq \left| \int_0^x \sin\left(f_n(t)\right)\,dt - \int_0^y \sin\left(f_n(t)\right)\,dt \right| = \left| \int_x^y \sin\left(f_n(t)\right)\,dt \right|$$
$$\left| \int_x^y \sin\left(f_n(t)\right)\,dt \right| \leq \int_x^y \left|\sin\left(f_n(t)\right)\right|\,dt \leq \int_x^y |1|\,dt = |y-x| < \varepsilon.$$

Thus, $F_n(x)$, $x \in [0,1]$ is equicontinuous.

$$|F_n(x)| = \left| \int_0^x \sin\left(f_n(t)\right)\,dt \right| = \int_0^x \left|\sin\left(f_n(t)\right)\right|\,dt \leq \int_0^x \left|\sin\left(f_n(t)\right)\right|\,dt \leq 1.$$

Since $F_n(x)$ is bounded we have that the closure of $\{F_n(x)\}$ is compact.
Therefore, by the general *Arzela-Ascoli* theorem, $\{F_n(x) : n \geq 1\}$ has a convergent subsequence. Furthermore, since [0,1] is compact, this convergence is uniform. □

Question 2

(a) Prove that every open subset of $\mathbb{R}$ (with respect to the standard topology $\tau_{\mathbb{R}}$) is a countable union of disjoint open intervals.

Proof. A special property of $\mathbb{R}$ is that all intervals in $\mathbb{R}$ happen to also be a connected set. We can write any subset $E \subset \mathbb{R}$ as

$$E = E_1 \cup E_2 \cdots \cup E_n,$$

where each E_i $(1 \leq i \leq n)$ is a connected set. Furthermore, each of these connected sets are disjoint, for if they were not, we could simply make a new connected set that is the union of the non-disjoint connected sets. Due to the density of $\mathbb{Q} \in \mathbb{R}$ each of these connected sets (which are also open intervals) has a rational number in them. Since $\mathbb{Q}$ is a countable set, we have that the number of number of disjoint open intervals is countable. □

(b) Explain why this statement cannot be true for $\mathbb{R}^n$ for any $n \geq 2$.

Question 3

Let f be an integrable function on the unit circle (so f is 2π-periodic).

(a) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbb{R}$. Prove that $\hat{f}(n) = 0$ for all odd integers n.

Proof.

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\left[\int_0^{\pi} f(x)e^{-inx}\,dx + \int_{\pi}^{2\pi} f(x)e^{-inx}\,dx\right]$$

Let

$$I_1 = \int_0^{\pi} f(x)e^{-inx}\,dx,$$

$$I_2 = \int_{\pi}^{2\pi} f(x)e^{-inx}\,dx,$$

$$u = x - \pi,\ du = dx,\ x = \pi \Rightarrow 0,\ x = 2\pi \Rightarrow \pi.$$

then

$$I_2 = \int_0^{\pi} f(u+\pi)e^{-in(u+\pi)}\,du = e^{-in\pi}\int_0^{\pi} f(u)e^{-inu}\,du = (-1)^n I_1.$$

Thus,

$$\hat{f}(n) = I_1 + (-1)^n I_1 = \begin{cases} 2I_1, & \text{if } x \text{ is even} \\ 0, & \text{if } x \text{ is odd} \end{cases}.$$

□

(b) Compute the Fourier series of a trigonometric polynomial of the form

$$a_0 + a_1\cos(x) + \cdots + a_k\cos(kx). \tag{1}$$

Solution. There are no $\sin(nx)$ terms in (1), so $b_n = 0$ for all n. It is then easy to see that the Fourier series of (1) is

$$a_0 + \sum_{n=1}^{k} a_n\cos(nx).$$

Question 4

Consider the sequence

$$f_n(x) = \frac{\pi n + \sin(nx)}{2n + \cos(n^2 x)}, \quad x \in [0,1].$$

(a) Prove that (f_n) converges uniformly on $[0,1]$.

Claim 4.1. *$f_n(x)$ converges to $\dfrac{\pi}{2}$ uniformly.*

Proof.

$$\left|f_n(x)-\frac{\pi}{2}\right| = \left|\frac{\pi n+\sin(nx)}{2n+\cos(n^2x)}-\frac{\pi}{2}\right| \leq \left|\frac{\pi n+1}{2n-1}-\frac{\pi}{2}\right| = \left|\frac{2(\pi n+1)-\pi(2n-1)}{2(2n-1)}\right| = \left|\frac{2+\pi}{4n-2}\right|$$

Let $\varepsilon>0$, pick $N=\dfrac{\frac{2+\pi}{\varepsilon}+2}{4}$, then whenever $n>N$ we have

$$\left|\frac{2+\pi}{4n-2}\right| < \left|\frac{2+\pi}{4N-2}\right| = \left|\frac{2+\pi}{4\left(\frac{\frac{2+\pi}{\varepsilon}+2}{4}\right)-2}\right| = \left|\frac{2+\pi}{\frac{2+\pi}{\varepsilon}}\right| = \varepsilon.$$

□

(b) Hence, or otherwise, evaluate the limit

$$\lim_{n\to\infty}\int_0^1 \frac{\pi n+\sin(nx)}{2n+\cos(n^2x)}dx.$$

Solution. We know that $f_n(x)$ converges uniformly to $\frac{\pi}{2}$, due to this we can switch the order of integration and the limit.

$$\lim_{n\to\infty}\int_0^1 \frac{\pi n+\sin(nx)}{2n+\cos(n^2x)}\,dx = \int_0^1 \lim_{n\to\infty}\frac{\pi n+\sin(nx)}{2n+\cos(n^2x)}\,dx = \int_0^1 \frac{\pi}{2}\,dx = \left[\frac{\pi}{2}x\right]_0^1 = \frac{\pi}{2}.$$

Question 5

Consider a second-order differential equation of the form

$$x''(t)+5x(t)=F(t),$$

where

$$F(t)=a_0+\sum_{n=1}^{\infty}b_n\cos(n\pi t)+\sum_{n=1}^{\infty}c_n\sin(n\pi t).$$

(a) Solve this differential equation by considering a potential solution $x(t)$ given as a Fourier series:

$$x(t)=A_0+\sum_{n=1}^{\infty}B_n\cos(n\pi t)+\sum_{n=1}^{\infty}C_n\sin(n\pi t),$$

and obtain relations for A_0, B_n, C_n in terms of the coefficients a_0, b_n, c_n of $F(t)$.
Solution.

$$x''(t) = -\left[\sum_{n=1}^{\infty} B_n n^2\pi^2 \cos(n\pi t) + C_n n^2\pi^2 \sin(n\pi t)\right].$$

$$-\left[\sum_{n=1}^{\infty} B_n n^2\pi^2 \cos(n\pi t) + C_n n^2\pi^2 \sin(n\pi t)\right] + 5\left[A_0 + \sum_{n=1}^{\infty} B_n \cos(n\pi t) + \sum_{n=1}^{\infty} C_n \sin(n\pi t)\right] = F$$

$$a_0 + \sum_{n=1}^{\infty} b_n \cos(n\pi t) + \sum_{n=1}^{\infty} c_n \sin(n\pi t) = 5A_0 + \sum_{n=1}^{\infty}(5 - n^2\pi^2)\left[B_n \cos(n\pi t) + C_n \sin(n\pi t)\right].$$

It can be seen that

$$A_0 = \frac{a_0}{5}, \quad B_n = \frac{b_n}{5 - n^2\pi^2}, \quad C_n = \frac{c_n}{5 - n^2\pi^2}.$$

Then we have

$$x(t) = \frac{a_0}{5} + \sum_{n=1}^{\infty} \frac{b_n}{5 - n^2\pi^2} \cos(n\pi t) + \sum_{n=1}^{\infty} \frac{c_n}{5 - n^2\pi^2} \sin(n\pi t).$$

(b) Suppose

$$F(t) = \begin{cases} -1 & \text{if } -1 < t < 0 \\ 1 & \text{if } 0 < t < 1 \end{cases}.$$

so Solve for $x(t)$.
Solution. Period $T = 2$ so $L = 1$.

$$F(t) = a_0 + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi t}{L}\right) = a_0 + \sum_{n=1}^{\infty} b_n \cos(n\pi t) + \sum_{n=1}^{\infty} c_n \sin(n\pi t).$$

Since $F(-t) = -F(t)$ for all t, $F(t)$ is odd $a_0 = 0$ and $b_n = 0$ for all n.

$$c_n = \frac{2}{T}\int_{-1}^{1} F(t)\sin(n\pi t)\,dt = \int_{-1}^{0} F(t)\sin(n\pi t)\,dt + \int_{0}^{1} F(t)\sin(n\pi t)\,dt$$

$$c_n = -1\left[\frac{-\cos(n\pi t)}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos(n\pi t)}{n\pi}\right]_{0}^{1}$$

$$c_n = \frac{1}{n\pi}\left[1 - \cos(n\pi)\right] + \frac{1}{n\pi}\left[-\cos(n\pi) + 1\right]$$

$$c_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

$$c_n = \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

$$c_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$F(n) = \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{(n)\pi} \sin(n\pi t).$$

We know from part a that

$$C_n = \frac{c_n}{5 - n^2\pi^2} = \frac{\frac{2}{n\pi}\left[1-(-1)^n\right]}{5-n^2\pi^2} = \frac{2[1-(-1)^n]}{n\pi[5-n^2\pi^2]}.$$

Then,

$$x(t) = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi[5-n^2\pi^2]} \sin(n\pi t).$$

Let $n = 2k-1$,

$$x(t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi[5-(2k-1)^2\pi^2]} \sin((2k-1)\pi t).$$

$$x(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin((2k-1)\pi t)}{(2k-1)(5-(2k-1)^2\pi^2)}.$$