

# INTRODUCTORY MATHEMATICAL ANALYSIS

MATH 302

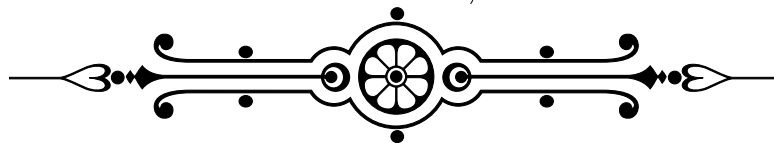
Dr. Stanley Yao Xiao

## Assignment 3

*Deepak Jassal*

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## Question 1

Let  $\{f_n : n \geq 1\}$  be a sequence in  $\mathcal{C}([0, 1], \mathbb{R})$ . Define a function  $F_n$  by

$$F_n(x) = \int_0^x \sin(f_n(t)) dt, \quad x \in [0, 1].$$

Use the Arzela-Ascoli theorem to prove that  $\{F_n : n \geq 1\}$  has a uniformly convergent subsequence.

*Proof.*

$$\mathbb{Q} \cap [0, 1] \subset [0, 1]$$

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Since  $\mathbb{Q}$  is countable and dense then so is  $\mathbb{Q} \cap [0, 1]$ . So,  $[0, 1]$  has a countable dense subset.  $\mathbb{R}$  is a complete metric space.  $\square$

## Question 2

- (a) Prove that every open subset of  $\mathbb{R}$  (with respect to the standard topology  $\tau_{\mathbb{R}}$ ) is a countable union of disjoint open intervals.
- (b) Explain why this statement cannot be true for  $\mathbb{R}^n$  for any  $n \geq 2$ .

## Question 3

Let  $f$  be an integrable function on the unit circle (so  $f$  is  $2\pi$ -periodic).

- (a) Suppose that  $f(\theta + \pi) = f(\theta)$  for all  $\theta \in \mathbb{R}$ . Prove that  $\hat{f}(n) = 0$  for all odd integers  $n$ .
- (b) Find an example of a metric space  $(X, d)$  and a subset  $S \subset X$  such that  $S = L(S)$ , where  $L(S)$  is the set of limit points of  $S$ .

## Question 4

Consider the sequence

$$f_n(x) = \frac{\pi n + \sin(nx)}{2n + \cos(n^2x)}, \quad x \in [0, 1].$$

- (a) Prove that  $(f_n)$  converges uniformly on  $[0, 1]$ .
- (b) Hence, or otherwise, evaluate the limit

$$\lim_{n \rightarrow \infty} \int_0^1 \frac{\pi n + \sin(nx)}{2n + \cos(n^2x)} dx.$$

## Question 5

Consider a second-order differential equation of the form

$$x''(t) + 5x(t) = F(t),$$

where

$$F(t) = a_0 + \sum_{n=1}^{\infty} b_n \cos(n\pi t) + \sum_{n=1}^{\infty} c_n \sin(n\pi t).$$

- (a) Solve this differential equation by considering a potential solution  $x(t)$  given as a Fourier series:

$$x(t) = A_0 + \sum_{n=1}^{\infty} B_n \cos(n\pi t) + \sum_{n=1}^{\infty} C_n \sin(n\pi t),$$

and obtain relations for  $A_0, B_n, C_n$  in terms of the coefficients  $a_0, b_n, c_n$  of  $F(t)$ .

- (b) Suppose

$$F(t) = \begin{cases} -1 & \text{if } -1 < t < 0 \\ 1 & \text{if } 0 < t < 1 \end{cases}.$$

Solve for  $x(t)$ .