# Multivariable Calculus I

MATH 202

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# Assignment 5

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# Question 1

Compute the volume of a right cylindrical cone where the base radius is $R$ and the height of the cone is $h$, using cylindrical coordinates.
The Jacobian determinant for cylindrical coordinates has already been computed in class, it is $r$.
The triple integral for the volume of the right cylindrical cone $\mathcal{C}$ is

$$V = \iiint_{\mathcal{C}} 1\, dV.$$

Converting to cylindrical coordinates

$$(x, y, z) \to (r\cos\theta, r\sin\theta, z)$$

yields

$$V = \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz\, d\theta.$$

The bounds for the integral in $r$ are obtained by considering the slant of the boundary of the cone from $(0, 0, 0)$ to a point on the upper edge which would be $\frac{R}{h}z$. Evaluating the integral

$$V = \int_0^{2\pi} \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz\, d\theta = 2\pi \int_0^h \int_0^{\frac{R}{h}z} r\, dr\, dz = 2\pi \int_0^h \left[\frac{r^2}{2}\right]_0^{\frac{R}{h}z} dz = \pi\frac{R^2}{h^2} \int_0^h z^2\, dz = \frac{\pi\frac{R^2}{h^2}}{3}\left[z^3\right]_0^h$$

$$V = \frac{\pi h R^2}{3}.$$

# Question 2

A spherical cap is the piece of a sphere (ball) sliced off by a plane. Suppose that a sphere has radius 10, and the height of the spherical cap is 6. Determine the volume of the spherical cap.
It can be seen that the volume of half a sphere is

$$V_{\text{half sphere}} = V_{\text{cap}} + \frac{1}{2}V_{\text{cylinder}} + \frac{1}{2}V_{\text{napkin ring}}.$$

Rearranging we get

$$V_{\text{cap}} = V_{sphere} - \frac{1}{2}V_{\text{cylinder}} - \frac{1}{2}V_{\text{napkin ring}}.$$

$$V_{\text{half sphere}} = \frac{2}{3}\pi R^3 = \frac{2000\pi}{3},$$

$$\frac{1}{2}V_{\text{cylinder}} = \frac{1}{2}\pi r^2 h = \pi(\sqrt{10^2 - 4^2}^2)(6) = \frac{1}{2}\pi(\sqrt{84}^2)(8) = 336\pi,$$

$$V_{\text{napkin ring}} = \frac{\pi h^3}{3} = \frac{128\pi}{3}.$$

Putting all of these together

$$V_{\text{cap}} = \frac{2000\pi}{3} - 336\pi - \frac{128\pi}{3} = 288\pi.$$

# Question 3

A client spends $X$ minutes in an insurance agent's waiting room and $Y$ minutes meeting with the agent (That is, both $X, Y$ are random variables). The joint probability density function of $X$ and $Y$ can be modelled by

$$f(x,y) = \begin{cases} ke^{-\frac{x+2y}{40}} & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Determine the value of $k$ and the probability that a client spends less than 60 minutes at the agent's office.

*Solution.*

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ke^{-\frac{x+2y}{40}}\,dy\,dx = k\int_{-\infty}^{\infty} e^{\frac{-x}{40}}\int_{-\infty}^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

Since the probability density function is non-zero only for non-zero values of $x$ and $y$ we can further simplify the double integral

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} e^{\frac{-y}{20}}\,dy\,dx$$

$$u = \frac{y}{20},\ 20\,du = dy,\ y = 0 \Rightarrow u = 0,\ y = \infty \Rightarrow u = \infty$$

$$k\int_0^{\infty} e^{\frac{-x}{40}}\int_0^{\infty} 20e^{-u}\,du\,dx = k\int_0^{\infty} e^{\frac{-x}{40}}\left[20e^{-u}\right]_0^{\infty}\,dx = -20k\int_0^{\infty} e^{\frac{-x}{40}}\,dx$$

$$v = \frac{x}{40},\ 40\,dv = dx,\ x = 0 \Rightarrow v = 0,\ x = \infty \Rightarrow v = \infty$$

$$-20k\int_0^{\infty} 40e^{-v}\,dv = 800k\left[e^{-v}\right]_0^{\infty} = 800k$$

We want the total probability to be 1, so we have

$$k = \frac{1}{800}.$$

We want to find $\mathcal{P}(X+Y<60)$, which can be done by evaluating the following integral

$$\mathcal{P}(X+Y<60) = \frac{1}{800}\int_{X=0}^{60}\int_{Y=0}^{60-x} e^{-\frac{x+2y}{40}}\,dy\,dx.$$

Using the same substitutions as the previous part we get

$$\frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\int_0^{60-x} e^{\frac{-y}{20}}\,dy\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left[-20e^{-u}\right]_0^{\frac{60-x}{20}}\,dx = \frac{1}{800}\int_0^{60} e^{\frac{-x}{40}}\left(20 - 20e^{\frac{x-60}{20}}\right)\,dx$$

Factoring and multiplying through

$$\frac{1}{40}\int_0^{60} e^{\frac{-x}{40}} - e^{-3}e^{\frac{x}{40}}\,dx = \frac{1}{40}\left[\int_0^{60} e^{\frac{-x}{40}} - e^{-3}\int_0^{60} e^{\frac{x}{40}}\right]$$

$$\frac{1}{40}\left[-40\left[e^{\frac{-x}{40}}\right]_0^{60}-40e^{-3}\left[e^{\frac{x}{40}}\right]_0^{60}\right]=\left[-\left[e^{\frac{-x}{40}}\right]_0^{60}-e^{-3}\left[e^{\frac{x}{40}}\right]_0^{60}\right]=-e^{\frac{3}{2}}+1-e^{\frac{3}{2}}+e^{-3}$$

$$\mathcal{P}(X+Y<60)=1-e^{-\frac{3}{2}}+e^{-3}.$$

# Question 4

(Worth 20 points). In this question, we derive some formulas for volumes of higher dimensional balls. Let $V_n(R)$ denote the volume of a $n$-ball (in $\mathbb{R}^{n+1}$) of radius $R$.

A 0-ball is just an interval. This allows us to compute the volume of a 2-ball, defined by the inequality

$$x^2+y^2+z^2\leq R^2,$$

using cylindrical coordinates, we deduce:

$$V_2(R)=\int_0^{2\pi}\int_0^R\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} r\,dh\,dr\,d\theta=\int_0^{2\pi}\int_0^R 2r\sqrt{R^2-r^2}\,dr\,d\theta.$$

(a) Verify that $V_2(R)=\frac{4\pi R^3}{3}$, by finishing with the iterated integral above.

$$V_2(R)=\int_0^{2\pi}\int_0^R 2\sqrt{R^2-r^2}\,dr\,d\theta.$$

Using the substitution

$$u=R^2-r^2,\ du=-2r\,dr,\ r=1\Rightarrow u=R^2,\ r=R\Rightarrow u=0$$

$$V_2(R)=\int_0^{2\pi}\int_{R^2}^0 -\sqrt{u}\,du\,d\theta$$

$$V_2(R)=\int_0^{2\pi}\int_0^{R^2} \sqrt{u}\,du\,d\theta$$

$$V_2(R)=2\pi\left[\frac{2u^{\frac{3}{2}}}{3}\right]_0^{r^2}=\frac{4\pi R^2}{3}.$$

(b) Using the same idea, deduce that

$$V_3(R)=\int_0^{2\pi}\int_0^R \pi(R^2-r^2)r\,dr\,d\theta.$$

In $\mathbb{R}^4$ a ball with radius $R$ is the set of points

$$x_1^2+x_2^2+x_3^2+x_4^2\leq R^2.$$

Converting to cylindrical coordinates in $\mathbb{R}^4$ we get $r^2 = x_1^2 + x_2^2$, which then gives

$$r^2 + x_3^2 + x_4^2 \leq R^2.$$

When $r$ is fixed the $(x_3, x_4)$ coordinates become a disk in $\mathbb{R}^2$, this disk has a radius of $\sqrt{R^2 - r^2}$ and area of $\pi(R^2 - r^2)$. The volume from the $(x_1, x_2)$ coordinates is given by $r\,dr\,d\theta$. All of this gives

$$V_3(R) = \int_0^{2\pi} \int_0^R \pi(R^2 - r^2) r\,dr\,d\theta.$$

(c) Generalize this to obtain the formula

$$V_{n+2}(R) = \int_0^{2\pi} \int_0^R V_n(\sqrt{R^2 - r^2}) r\,dr\,d\theta.$$

A ball in $\mathbb{R}^{n+2}$ with radius $R$ set of coordinates

$$x_1^2 + x_2^2 + \cdots + x_{n+2}^2 \leq R^2.$$

Converting to cylindrical coordinates we get $r^2 = x_1^2 + x_2^2$. Let $y = (x_3, \ldots, x_{n+2}) \in \mathbb{R}^n$. We then get

$$r^2 + \|y\|^2 \leq R^2.$$

Rearranging

$$\|y\|^2 \leq R^2 - r^2,$$

we see that $y$ is within an $n - ball$ with radius $\sqrt{R^2 - r^2}$, and the volume of this ball is $V_n(\sqrt{R^2 - r^2})$. The $(x_1, x_2)$ coordinates give $r\,dr\,d\theta$. This all yields

$$V_{n+2}(R) = \int_0^{2\pi} \int_0^R V_n(\sqrt{R^2 - r^2}) r\,dr\,d\theta.$$

(d) Using the recursion in part (c), give formulas for $V_4(R), V_5(R), V_6(R)$.
From the prvious part we know that for $V_{n+2}(R)$ the component $V_n(R)$ has a radius of $\sqrt{R^2-r^2}$ and that $V_n(R) \propto \left(\sqrt{R^2-r^2}\right)^{n+1}$, this will help simplfy the following integrals.

$$V_4(R) = \int_0^{2\pi}\int_0^R \left(\frac{4\pi R^2}{3}\right)(\sqrt{R^2-r^2})r\,dr\,d\theta = 2\pi\int_0^R \left(\frac{4\pi}{3}\right)\left(R^2-r^2\right)^{\frac{3}{2}} r\,dr$$

$$u = R^2-r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_4(R) = \frac{4\pi^2}{3}\int_{R^2}^0 -u^{\frac{3}{2}}\,du = \frac{4\pi^2}{3}\int_0^{R^2} u^{\frac{3}{2}}\,du = \frac{8\pi^2}{15}\left[u^{\frac{5}{2}}\right]_0^{R^2} = \frac{8\pi^2}{15}R^5$$

$$V_4(R) = \frac{8\pi^2}{15}R^5.$$

$$V_3(R) = \int_0^{2\pi}\int_0^R \pi(R^2-r^2)r\,dr\,d\theta = 2\pi\int_0^R \pi(R^2-r^2)r\,dr$$

$$u = R^2-r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_3(R) = 2\pi^2\int_0^R R^2r - r^3\,dr = 2\pi^2\left[\frac{R^2r^2}{2} - \frac{r^4}{4}\right]_0^R = 2\pi^2\left(\frac{R^4}{2} - \frac{R^4}{4}\right) = \frac{\pi^2R^4}{2}.$$

$$V_5(R) = \int_0^{2\pi}\int_0^R V_3(\sqrt{R^2-r^2})r\,dr\,d\theta = \int_0^{2\pi}\int_0^R \left(\frac{\pi^2R^4}{2}\right)(\sqrt{R^2-r^2})r\,dr\,d\theta$$

$$V_5(R) = 2\pi^3\int_0^R (R^2-r^2)^2r\,dr = 2\pi^3\int_0^R R^4r-2R^2r^3+r^5\,dr = 2\pi^3\left[\frac{R^4r^2}{2} - \frac{2R^2r^4}{4} + \frac{r^6}{6}\right]_0^{R^2},$$

$$V_5(R) = \pi^3\left(\frac{R^6}{2} - \frac{R^6}{2} + \frac{R^6}{6}\right) = \frac{\pi^3}{6}R^6.$$

$$V_6(R) = \int_0^{2\pi}\int_0^R V_4(\sqrt{R^2-r^2})r\,dr\,d\theta = 2\pi\int_0^R \frac{8\pi^2}{15}R^5(\sqrt{R^2-r^2})r\,dr = \frac{8\pi^3}{15}\int_0^R \left(R^2-r^2\right)^{\frac{5}{2}} r\,dr.$$

$$u = R^2-r^2,\ du = -2r\,dr,\ r = 1 \Rightarrow u = R^2,\ r = R \Rightarrow u = 0$$

$$V_6(R) = \frac{8\pi^3}{15}\int_0^{R^2} u^{\frac{5}{2}}\,du = \frac{16\pi^3}{105}\left[u^{\frac{7}{2}}\right]_0^{R^2},$$

$$V_6(R) = \frac{16\pi^3}{105}R^7.$$

(e) Evaluate the limit
$$\lim_{n\to\infty}\frac{V_n(R)}{R^{n+1}}.$$
Since $V_n(R) \propto \left(\sqrt{R^2-r^2}\right)^{n+1}$ we have
$$\frac{V_n(R)}{R^{n+1}} = \alpha\frac{\left(\sqrt{R^2-r^2}\right)^{n+1}}{R^{n+1}},$$
for some $\alpha$ which is independant of $n$. So in the limit we obtain
$$\lim_{n\to\infty}\frac{V_n(R)}{R^{n+1}} = \lim_{n\to\infty}\alpha\frac{\left(\sqrt{R^2-r^2}\right)^{n+1}}{R^{n+1}} = \alpha\lim_{n\to\infty}\frac{\left(\sqrt{R^2-r^2}\right)^{n+1}}{R^{n+1}}.$$
And
$$\lim_{n\to\infty}\frac{\left(\sqrt{R^2-r^2}\right)^{n+1}}{R^{n+1}} = 0,$$
since the denominator grows faster than the numerator. So,
$$\lim_{n\to\infty}\frac{V_n(R)}{R^{n+1}} = 0.$$