Today in Cryptography (5830)

Public-key encryption
The RSA permutation
PKCS#1 RSA encryption

<u>References:</u>
RSA discussed in many textbooks. See Katz & Lindell Sec. 8.1, 8.2
PKCS#1 encryption defined in PKCS#1 v1.5 standard.

Client

TLS handshake for RSA transport

Server

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods →

Pick random Ns

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

Check CERT using CA public verification key

← CERT = (pk of bank, signature over it)

Pick random PMS
C <- E(pk,PMS)

C →

PMS <- D(sk,C)

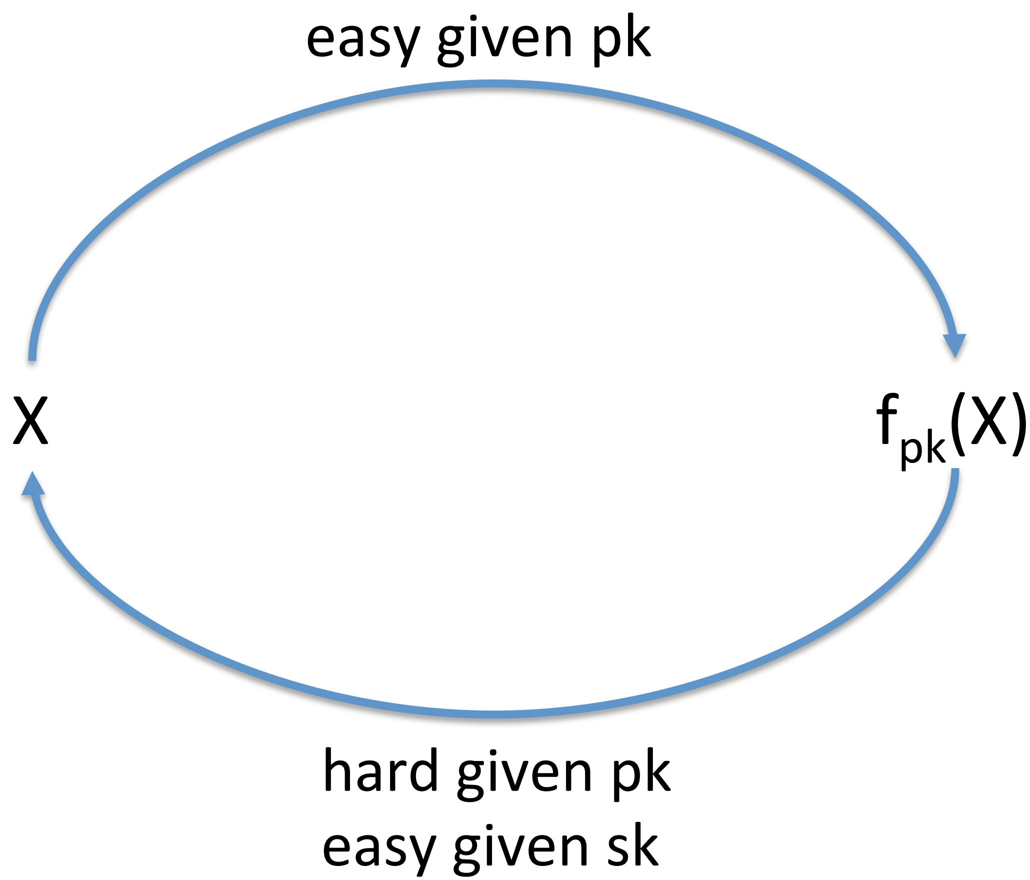
ChangeCipherSpec,
{ Finished, PRF(MS, "Client finished" || H(transcript)) } →

ChangeCipherSpec,
← { Finished, PRF(MS, "Server finished" || H(transcript')) }

Bracket notation means contents encrypted

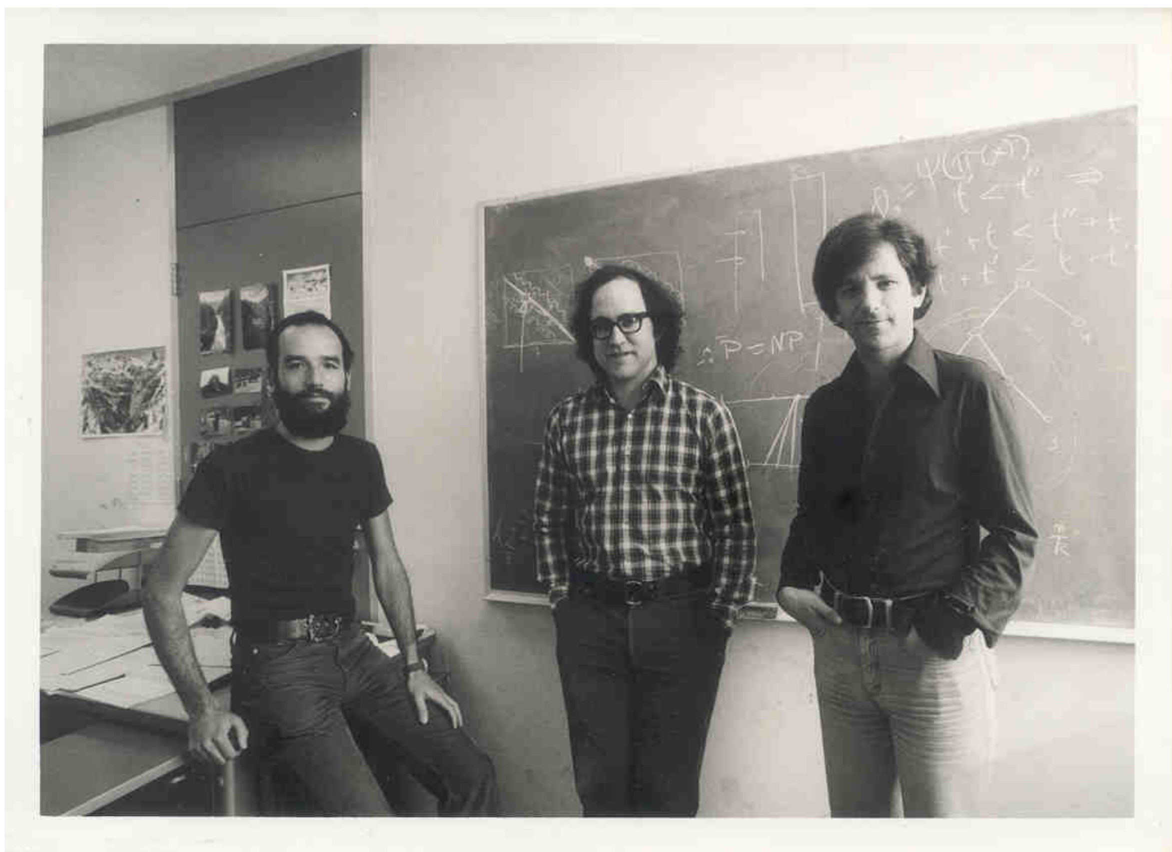
MS <- PRF(PMS, "master secret" || Nc || Ns)

Trapdoor functions



The RSA trapdoor function

- Rivest, Shamir, Adleman 1978
- Garnered them a Turing award



RSA math

p and q be large prime numbers
N = pq
N is called the modulus

p = 7, q = 13, gives	N = 91
p = 17, q = 53, gives	N = 901

RSA math

p and q be large prime numbers

$N = pq$

N is called the modulus

$\mathbf{Z}_N = \{0,1,2,3,\ldots, N-1\}$

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \text{ and } i < N\}$

$\gcd(X,Y) = 1$ if greatest common divisor of X,Y is 1

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

N = 13 $\quad\quad$ $\mathbf{Z}_{13}^* = \{ 1,2,3,4,5,6,7,8,9,10,11,12 \}$

N = 15 $\quad\quad$ $\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$

The size of a set S is denoted by |S|

Def. $\phi(N) = |\mathbf{Z}_N^*|$ (This is Euler's totient function)

$\phi(13) = 12$

$\phi(15) = 8$

$\mathbf{Z}_{\phi(15)}^* = \mathbf{Z}_8^* = \{ 1,3,5,7 \}$

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

$\mathbf{Z}_N^*$ is a group under modular multiplication

Fact. For any a,N with N > 0, there exists unique q,r such that

$$a = Nq + r \qquad \text{and} \qquad 0 \le r < N$$

17 mod 15 = 2

105 mod 15 = 0

Def. $a \bmod N = r \in \mathbf{Z}_N$

Def. $a \equiv b \pmod N$ iff $(a \bmod N) = (b \bmod N)$

RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

$\mathbf{Z}_N^*$ is a group under modular multiplication

$$\mathbf{Z}_{15}^* = \{\, 1,2,4,7,8,11,13,14 \,\}$$

$2 \bullet 7 \equiv 14 \pmod{15}$

$4 \bullet 8 \equiv 2 \pmod{15}$

Closure: for any $a,b \in \mathbf{Z}_N^*$ $\quad a \bullet b \bmod N \in \mathbf{Z}_N^*$

Def. $\quad a^i \bmod N = \underbrace{a \bullet a \bullet a \bullet \ldots \bullet a}_{i \text{ times}} \bmod N$

Some needed algorithms

Algorithm	**Running time (n = log N)**
Modular multiplication ab mod N	$\mathcal{O}(n^2)$
Modular exponentation a^i mod N	$\mathcal{O}(n^3)$
Modular inverse a^{-1} mod N	$\mathcal{O}(n^2)$

Textbook exponentiation

Let G be a group.
How do we compute h^x for any $h \in G$?

```
Exp(h,x)
X' = h
For i = 2 to x do
    X' = X'*h
Return X'
```

Requires time O(|G|) in worst case.

```
SqrAndMulExp(h,x)
b_k,...,b_0 = x
f = 1
For i = k down to 0 do
    f = f^2
    If b_i = 1 then
        f = f*h
Return f
```

Requires time O(k) multiplies and squares in worst case.

```
SqrAndMulExp(h,x)
b_k,…,b_0 = x
f = 1
For i = k down to 0 do
    f = f^2
    If b_i = 1 then
        f = f*h
Return f
```

$$x = \sum_{b_i \neq 0} 2^i$$

$$h^x = h^{\sum_{b_i \neq 0} 2^i} = \prod_{b_i \neq 0} h^{2^i}$$

$$h^{11} = h^{8+2+1} = h^8 \bullet h^2 \bullet h$$

$b_3 = 1$ $\quad f_3 = 1 \bullet h$

$b_2 = 0$ $\quad f_2 = h^2$

$b_1 = 1$ $\quad f_1 = (h^2)^2 \bullet h$

$b_1 = 1$ $\quad f_0 = (h^4 \bullet h)^2 \bullet h \;=\; h^8 \bullet h^2 \bullet h$

Don’t implement this algorithm: side-channel attacks

RSA math

$\mathbf{Z}_N^* = \{ i \mid \gcd(i,N) = 1 \}$

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying $ed \bmod \phi(N) = 1$ then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$$\begin{aligned}(x^e)^d \bmod N &= x^{(ed \bmod \phi(N))} \bmod N \\ &= x^1 \bmod N \\ &= x \bmod N\end{aligned}$$

First equality is by Euler's Theorem

RSA math

$\mathbf{Z}_N^* = \{\, i \mid \gcd(i,N) = 1 \,\}$

Claim: Suppose $e,d \in \mathbf{Z}_{\phi(N)}^*$ satisfying $ed \bmod \phi(N) = 1$ then for any $x \in \mathbf{Z}_N^*$ we have that

$$(x^e)^d \bmod N = x$$

$\mathbf{Z}_{15}^* = \{ 1,2,4,7,8,11,13,14 \}$ $\mathbf{Z}_{\phi(15)}^* = \{ 1,3,5,7 \}$

$e = 3$, $d = 3$ gives $ed \bmod 8 = 1$

x	1	2	4	7	8	11	13	14
x^3 mod 15	1	8	4	13	2	11	7	14
y^3 mod 15	1	2	4	7	8	11	13	14

The RSA trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

$$\begin{tikzcd}[column sep=huge] X \arrow[r, bend left, "\text{easy given } N{,}e"] & f_{pk}(X) \arrow[l, bend left, "\substack{\text{hard given } N{,}e \\ \text{easy given } N{,}d}"] \end{tikzcd}$$

The RSA trapdoor permutation

$pk = (N,e)$ $sk = (N,d)$ with $ed \bmod \phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable N,e,d ?

If p,q distinct primes and $N = pq$ then $\phi(N) = (p-1)(q-1)$

Why?

$$\begin{aligned}
\phi(N) &= |\{1,\ldots,N-1\}| - |\{ip : 1 \le i \le q-1\}| - |\{iq : 1 \le i \le p-1\}| \\
&= N-1 - (q-1) - (p-1) \\
&= pq - p - q + 1 \\
&= (p-1)(q-1)
\end{aligned}$$

The RSA trapdoor permutation

pk = (N,e) sk = (N,d) with ed mod $\phi(N) = 1$

$f_{N,e}(x) = x^e \bmod N$ $g_{N,d}(y) = y^d \bmod N$

But how do we find suitable N,e,d ?

If p,q distinct primes and N = pq then $\phi(N) = (p-1)(q-1)$

Given $\phi(N)$, choose $e \in \mathbf{Z}^*_{\phi(N)}$ and calculate

$$d = e^{-1} \bmod \phi(N)$$

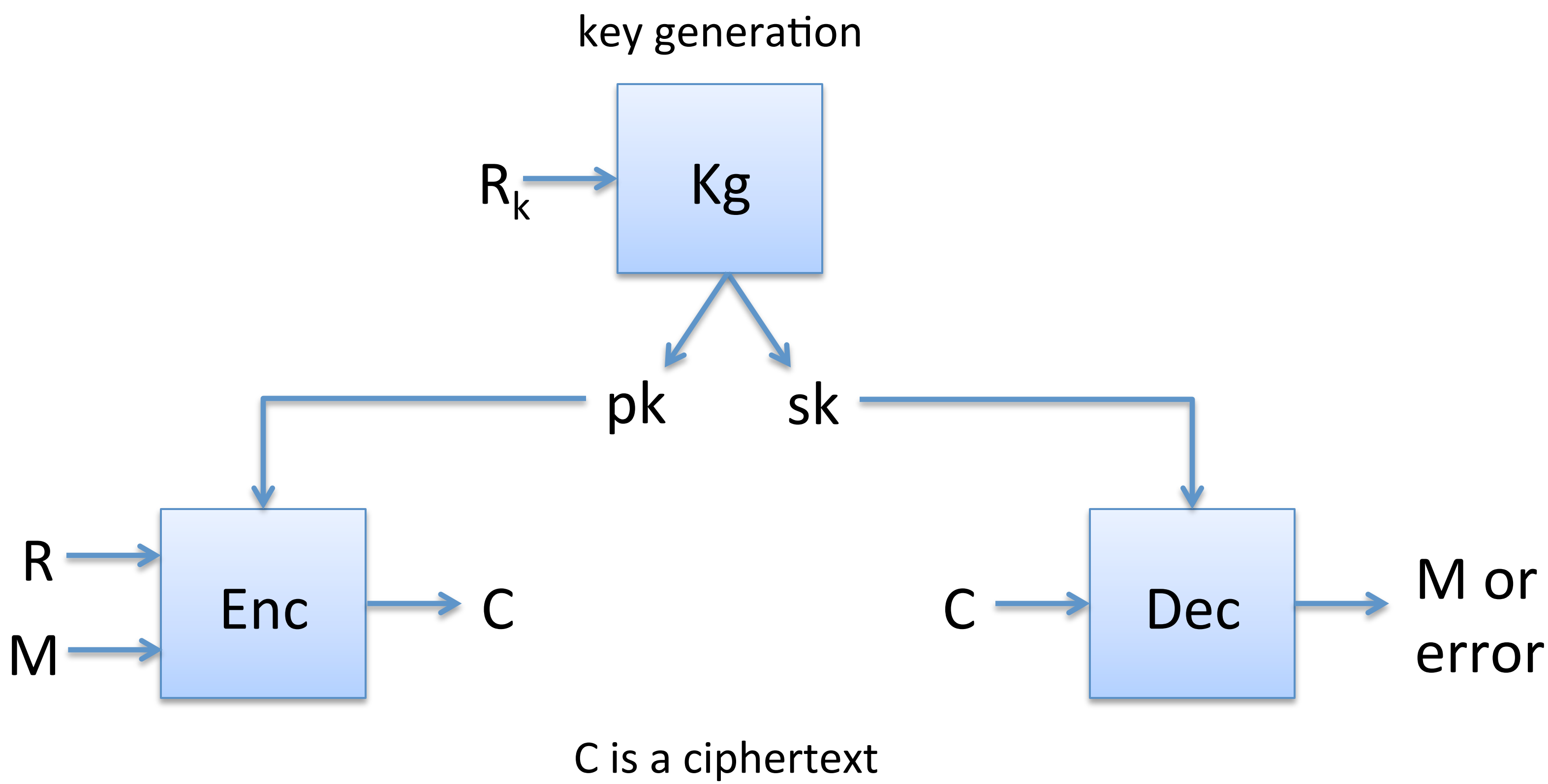
How to find suitable p,q prime?

Choose random numbers and test primality

Summary

- Find 2 large primes p, q . Let $N = pq$
  - random integers + primality testing
- Choose e (usually 65,537)
  - Compute d using $\phi(N) = (p-1)(q-1)$
- $pk = (N,e)$ and $sk = (N,d)$
  - Often store p,q with sk to use Chinese Remainder Theorem

Public-key encryption



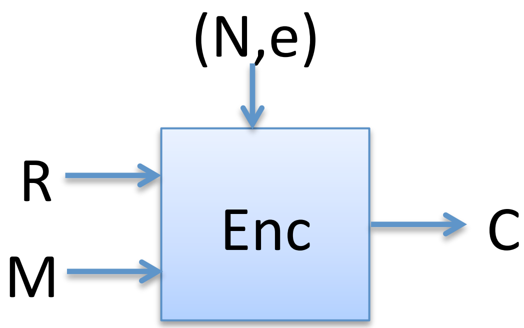
Correctness: D(sk , E(pk,M,R)) = M with probability 1 over randomness used

PKCS #1 RSA encryption

Kg outputs (N,e),(N,d) where $|N|_8 = n$

Let $B = \{0,1\}^8 / \{00\}$ be set of all bytes except 00

Want to encrypt messages of length $|M|_8 = m$

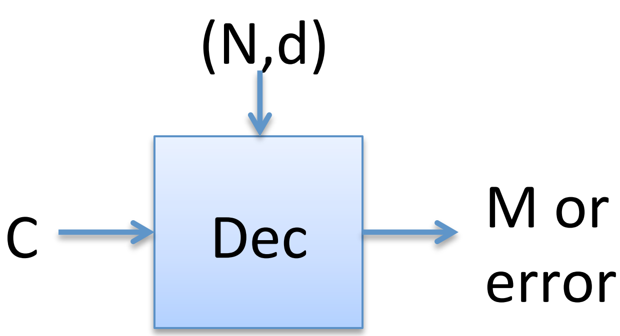


<u>Enc((N,e), M, R)</u>

pad = first n - m - 3 bytes from R that are in B

X = 00 || 02 || pad || 00 || M

Return X^e mod N



<u>Dec((N,d), C)</u>

$X = C^d$ mod N ; aa||bb||w = X

If (aa ≠ 00) or (bb ≠ 02) or (00 ∉ w)

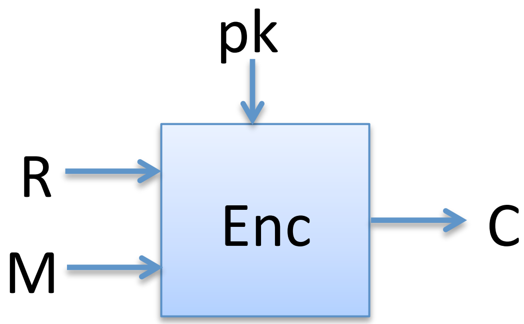
Return error

pad || 00 || M = w

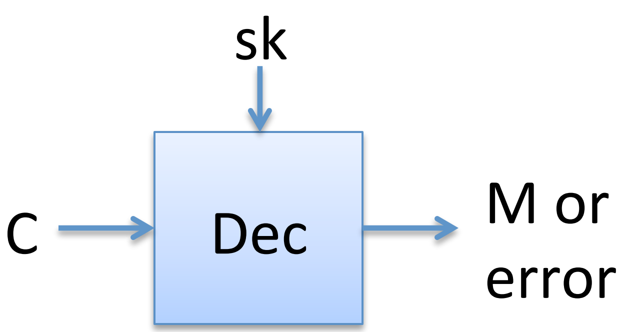
Return M

Hybrid encryption

Kg outputs (pk,sk)



Enc(pk, M, R)
K||R1||R2 = R
C1 = Enc(pk,K,R1)
C2 = Enc(K,M,R2)
Return (C1,C2)



Dec(sk, (C1,C2))
K = Dec(sk,C1)
M = Dec(K,C2)
Return M

TLS handshake for RSA transport

Bank customer

Bank

Pick random Nc

ClientHello, MaxVer, Nc, Ciphers/CompMethods →

Pick random Ns

← ServerHello, Ver, Ns, SessionID, Cipher/CompMethod

← CERT = (pk of bank, signature over it)

Check CERT using CA public verification key

Pick random PMS
C <- E(pk,PMS)

C →

PMS <- D(sk,C)

ChangeCipherSpec,
{ Finished, PRF(MS, “Client finished” || H(transcript)) } →

ChangeCipherSpec,
← { Finished, PRF(MS, “Server finished” || H(transcript’)) }

Bracket notation means contents encrypted

MS <- PRF(PS, “master secret” || Nc || Ns)

Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?

Inverting RSA : given N, e, y find x such that $x^e \equiv y \pmod{N}$

EASY because $f^{-1}(y) = y^d \bmod N$

Know d

EASY because $d = e^{-1} \bmod \varphi(N)$

Know $\varphi(N)$

EASY because $\varphi(N) = (p-1)(q-1)$

Know p, q

? Learning p,q from N is the factoring problem

Know N

We don't know if inverse is true, whether inverting RSA implies ability to factor

Factoring composites

- What is p,q for N = 901?

Factor(N):

```
for i = 2 , … ,  sqrt(N) do
     if N mod i = 0 then
          p = i
          q = N / p
          Return (p,q)
```

Woops… we can always factor

But not always efficiently:
Run time is sqrt(N)

$\mathcal{O}(\mathrm{sqrt}(N)) = \mathcal{O}(e^{0.5 \ln(N)})$

Factoring composites

Algorithm	**Time to factor N**
Naïve	$\mathcal{O}(e^{0.5 \ln(N)})$
Quadratic sieve (QS)	$\mathcal{O}(e^{c})$ $c = d (\ln N)^{1/2} (\ln \ln N)^{1/2}$
Number Field Sieve (NFS)	$\mathcal{O}(e^{c})$ $c = 1.92 (\ln N)^{1/3} (\ln \ln N)^{2/3}$

Factoring records

Challenge	**Year**	**Algorithm**	**Time**
RSA-400	1993	QS	830 MIPS years
RSA-478	1994	QS	5000 MIPS years
RSA-515	1999	NFS	8000 MIPS years
RSA-768	2009	NFS	~2.5 years
RSA-512	2015	NFS	$75 on EC2 / 4 hours

RSA-x is an RSA challenge modulus of size x bits

Security of RSA PKCS#1

- Passive adversary sees (N,e),C
- Attacker would like to invert C
- Possible attacks?
  - Pick |N| > 1024 and factoring will fail
  - Active attacks?

Bleichanbacher attack

I've just learned some information about C_1^d mod N

C_1 →

← padding error?

C_2 →

← padding error?

…

Dec((N,d), C)

X = C^d mod N ; aa||bb||w = X

If (aa ≠ 00) or (bb ≠ 02) or (00 ∉ w)

Return error

pad || 00 || M = w

Return M

We can take a target C and decrypt it using a sequence of chosen ciphertexts C_1, … , C_q where q ≈ 1 million

[Bardou et al. 2012] q = 9400 ciphertexts on average

Response to this attack

- Ad-hoc fix: Don't leak whether padding was wrong or not
  - This is harder than it looks (timing attacks, control-flow side channel attacks, etc.)
- Better:
  - use chosen-ciphertext secure encryption
  - OAEP is common choice

Summary

- RSA is example of trapdoor one-way function
  - Security conjectured. Relies on factoring being hard
- RSA security scales somewhat poorly with size of primes
- RSA PKCS#1 v1.5 is insecure due to padding oracle attacks. Don't use it in new systems.
  - Use OAEP instead