Linear Independance

Linear Dependance: set of vectors is linearly dependant if one of the vectors can be written as a linear combination of others.

→ A collection of vectors $a_1, a_2, \dots, a_k$ is called linearly dependant if there are coefficients $\beta_1, \dots, \beta_K$, not all zero such that $\beta_1 a_1 + \beta_2 a_2 + \dots + \beta_K a_K = 0$ — ①

★ If one vector can be expressed as combination of others, then entire set is linearly dependant

Linearly Independant: If the only solution to ① is $\beta_1 = \beta_2 = \dots = \beta_K = 0$.

Examples: set consisting of single non-zero vector is linearly independant, while a set with single zero vector is linearly dependant.

$$a_1 = \begin{bmatrix} 0.2 \\ -7.0 \\ 8.6 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -0.1 \\ 2.0 \\ -1.0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0.0 \\ -1.0 \\ 2.2 \end{bmatrix}$$

$a_1 + 2a_2 + 3a_3 = 0$

∴ Linearly dependant.

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad a_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \quad a_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

To check if they are linearly independant we assume $\beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 = 0$.

This leads to equations: $\beta_1 - \beta_3 = 0$; $-\beta_2 + \beta_3 = 0$; $\beta_2 + \beta_3 = 0$.

From the above equations we get: $\beta_1 = \beta_2 = \beta_3 = 0$.

Uniqueness of coefficients: x = linear combination of linearly indepant vectors $a_1, a_2 \dots a_K$.

Then the coefficients of $\beta_1 \dots \beta_2$ are all <u>UNIQUE</u>

★ If there are 2 sets of coefficients (β_i and γ_i) such that:

$$\boxed{x = \beta_i a_1 + \dots + \beta_K a_K = \gamma_1 a_1 + \dots + \gamma_K a_K \text{ then } \beta_i = \gamma_i \text{ for all } i.}$$

Supersets & subsets: Adding vectors to a dependant set will keep it dependant, while any subset of an independant set ~~is~~ is still independant.

Basis:

Independance - Dimension Inequality: If you have n-dimensional vectors $a_1, \dots, a_K$ that are linearly independant, then <u>$K \leq n$</u>.

→ Any collection of $n+1$ or more n-dimensional vectors, they are linearly dependant.

Ex: collection of 3 2-vectors must be linearly dependant.

Basis: collection of n linearly independant n-dimensional vectors (max no. of linearly independant vector in that space.

★ Any n-dimensional vector b can be expressed as a linear combination of the basis vecs.

$$b = \alpha_1 a_1 + \dots + \alpha_n a_n$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are coefficients.

- The coefficients of expansion b are unique, which means that there is only one way to express b as a linear combination of the basis vectors.

For a vector to form a basis 2 conditions must be satisfied:

1. Linear independance: Lets take an example $a_1 = \begin{pmatrix} 1.2 \\ -2.6 \end{pmatrix}$ $a_2 = \begin{pmatrix} -0.3 \\ -3.7 \end{pmatrix}$

To check if they are linearly independant, we need to see if a non-trivial combo exists for $\alpha_1 a_1 + \alpha_2 a_2 = 0$. On substituting, we get 2 equations:

$$\left.\begin{aligned} 1.2\alpha_1 - 0.3\alpha_2 &= 0 \\ -2.6\alpha_1 - 3.7\alpha_2 &= 0 \end{aligned}\right\}$$ This will only be true if $\alpha_1 = \alpha_2 = 0$, confirming that a_1 & a_2 are linearly independant.

2. Spanning the space: a_1 & a_2 belong to the R^2 (2-dimensional plane)

If a set of 2 linearly independant vectors in R^2 exists, it spans the entire space, meaning: any vector $b \in R^2$ can be written as linear combo of a_1 & a_2.

For example, $b = (1, 1)$ can be written as $b = 0.6513 a_1 - 0.7280 a_2$.

Orthonormal Vectors:

Orthogonal Vectors: mutually perpendicular vectors. $(a_i \cdot a_j = 0)$ $i \neq j$.

- vectors are normalized if $\|a_i\| = 1$ for $i = 1, 2, \ldots, K$

Orthonormal Vectors: vectors which are orthogonal and normalized have length 1.

This can be expressed using inner products as: $\boxed{a_i^T a_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}}$

- orthonormal vectors are linearly independant.

- any vector x can be written as a linear combination of orthonormal vectors $a_1 \ldots a_K$ can have its coefficients found very easily.

suppose $x = B_1 a_1 + B_2 a_2 + \ldots + B_K a_K$

To find coefficients of B_i, take inner product of both sides with a_i

$$a_i \cdot x = a_i \cdot (B_1 a_1 + B_2 a_2 + \ldots B_K a_K) = B_i.$$

$$\boxed{a_i \cdot x = B_i}$$

Orthonormal expansion formula:

$$\boxed{x = (a_1 \cdot x) a_1 + (a_2 \cdot x) a_2 + \ldots + (a_K \cdot x) a_K}$$

Orthonormal Basis: set of orthonormal vectors that span the entire space. In R^n, a set of orthonormal vectors form a basis.

* If $a_1, \ldots, a_n$ form an orthogonal basis, then any vector 'x' in the space can be written as orthonormal expansion. *

Example: $a_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$ $a_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

are orthonormal basis for $\mathbb{R}^3$. To express the vector $x = (1, 2, 3)$ in terms of this basis, we calculate inner product of x with each basis vector.

$$q_1 \cdot x = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = -3 \; ; \quad q_2 \cdot x = \frac{3}{\sqrt{2}} \; ; \quad q_3 x = \frac{-1}{\sqrt{2}}.$$

Thus, x can be written as: $x = -3 \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \frac{3}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Gram-Schmidt Algorithm:

Takes collection of linearly independant vectors & converts them into orthormal collection.

- **Algorithm explained:** It performs 3 main tasks:

  ① Orthogonalization: For each vector a_i, subtract its projection onto previously computed orthonormal vectors. $(q_1, q_2, \ldots q_{i-1})$

$$\boxed{\tilde{q}_i = a_i - (q_1^T a_1) q_1 - (q_2^T a_2) q_2 - \ldots - (q_{i-1}^T a_i) q_{i-1}}$$

→ The scalar $(q_j^T a_i)$ gives how much of a_i lies in direction of q_j.

→ multiplying this scalar with q_j gives the projection of a_i on q_j

* Subtracting this projection removes component of a_i in direct q_j giving us a $\tilde{q}_i$ vector orthogonal to q_1, q_2 ~~an~~ $\ldots q_{i-1}$.

If $i=1$, then $\tilde{q}_1 = a_1$ because there are no previous vectors to subtract from.

~~Linear~~ ② Linear Dependance Test: Now the algo checks if the $\tilde{q}_i$ is a zero vector.

If $\tilde{q}_i = 0$, then a_i is a linearly dependant on previous $a_1, \ldots a_{i-1}$

If this is the case, the algo terminates. or else it goes to next step.

③ Normalization: The norm of $\tilde{q}_i$ is denoted as $\|\tilde{q}_i\|$

$$\boxed{q_i = \frac{\tilde{q}_i}{\|q_i\|}}$$ This makes sure vector is orthonormal.

Example: $a_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$; $a_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Step 1: Orthogonalize a_1: As there are no previous vectors: $\tilde{q}_1 = a_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

~~Normalize~~ Norm $\tilde{q}_1$: $\|\tilde{q}_1\| = \sqrt{3^2 + 1^2} = \sqrt{9+1} = \sqrt{10}$

Normalize $\tilde{q}_1$ to get q_1 : $q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|} = \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$

Step 2: We have to orthogonalize a_2 with respect to q_1

Projection of a_2 on q_1 : $\text{proj}_{q_1}(a_2) = (q_1^T a_2) q_1$

$$= \frac{3}{\sqrt{10}} \cdot 2 + \frac{1}{\sqrt{10}} \cdot 2 = \frac{8}{\sqrt{10}}$$

multiplying $q_1 = \frac{8}{\sqrt{10}} \begin{pmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix} = \frac{8}{10} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{matrix} 2.4 \\ 0.8 \end{matrix}$

To make it $\tilde{q}_2 = a_2 - (q_1^T a_2) q_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2.4 \\ 0.8 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 1.2 \end{pmatrix}$

~~Normalizing~~ Norm $\tilde{q}_2$: $\|\tilde{q}_2\| = \sqrt{(-0.4)^2 + (1.2)^2} = \sqrt{1.6}$

Normalize $\tilde{q}_2$ to get q_2 = $q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|} = \frac{1}{\sqrt{1.6}} \begin{pmatrix} -0.4 \\ 1.2 \end{pmatrix} = \begin{pmatrix} -0.4/\sqrt{1.6} \\ 1.2/\sqrt{1.6} \end{pmatrix}$

$\therefore$ Orthonormal vectors are : $q_1 = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right)$ $q_2 = \left(\frac{-0.4}{\sqrt{1.6}}, \frac{1.2}{\sqrt{1.6}} \right)$

Steps Involved:

Let's take example of 3 2-dim vectors a_1, a_2, a_3.

Step 1: orthonormalizing a_1:

→ $\tilde{q}_1 = a_1$

→ Norm $(q_1) = \|\tilde{q}_1\|$

→ Normalize $q_1 = \frac{\tilde{q}_1}{\|\tilde{q}_1\|}$

Step 2: orthogonalizing a_2:

→ $\tilde{q}_2 = a_2 - (q_1^T a_2) q_1$

→ Norm $(\tilde{q}_2) = \|\tilde{q}_2\|$

→ Normalize $q_2 = \frac{\tilde{q}_2}{\|\tilde{q}_2\|}$

Step 3: Orthogonalizing a_3:

→ $\tilde{q}_3 = a_3 - (q_1^T a_3) q_1 - (q_2^T a_3) q_2$

→ Normalize $\tilde{q}_3$ = $q_3 = \frac{\tilde{q}_3}{\|\tilde{q}_3\|}$

* If the algorithm completes without any early termination, vectors are LIn.

Applications:

1. Checking if vector is linear combination:
   - New vector (b) & want to check if its linear combo.
   - Apply Gram-Schmidt on $a_1, \ldots, a_k$.
   - If algo terminates early, then b is linear combo of original set.

2. Basis: If Gram-Schmidt runs to completion, the vectors from a basis.

Complexity: Total flops count = $\boxed{2kn^2}$

→ running time grows linearly with length of the vector.

→ and quadratically with the no. of vectors.

Special case : $k = n$

Then the complexity becomes $2n^3$