Numerical Analysis Homework (week 7)

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Problem 7.1:

List the Chebyshev interpolation nodes $x_1, \ldots, x_n$ in the interval $[-1, 1]$, $n = 6$ and find the upper bound for $|(x - x_1) \cdot \cdots \cdot (x - x_n)|$ on this interval.

Solution

The formula for the Chebyshev nodes on the interval $[-1, 1]$ is as follows

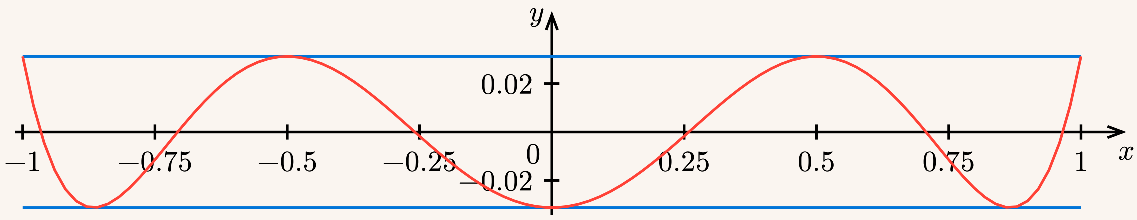
$$x_i = \cos\left(\frac{2k-1}{2n}\pi\right), i = 1, 2, \ldots, n$$

and for $n = 6$ we get

$$x_1 = \cos\left(\frac{1}{12}\pi\right) = \frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}, \qquad x_2 = \cos\left(\frac{3}{12}\pi\right) = \frac{1}{2}\sqrt{2}$$

$$x_3 = \cos\left(\frac{5}{12}\pi\right) = -\frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}, \qquad x_4 = \cos\left(\frac{7}{12}\pi\right) = -\frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{2}$$

$$x_5 = \cos\left(\frac{9}{12}\pi\right) = -\frac{1}{2}\sqrt{2}, \qquad x_6 = \cos\left(\frac{11}{12}\pi\right) = -\frac{1}{4}\sqrt{6} - \frac{1}{4}\sqrt{2}$$



and the bound for $|(x - x_1) \cdot \cdots \cdot (x - x_n)|$ is

$$\left|\left(\frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}\right)\left(\frac{1}{2}\sqrt{2}\right)\left(-\frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}\right)\left(-\frac{1}{4}\sqrt{6} + \frac{1}{4}\sqrt{2}\right)\left(-\frac{1}{2}\sqrt{2}\right)\left(-\frac{1}{4}\sqrt{6} - \frac{1}{4}\sqrt{2}\right)\right|$$

$$= \left|-\left(\frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}\right)^2\left(\frac{1}{2}\sqrt{2}\right)^2\left(-\frac{1}{4}\sqrt{2} + \frac{1}{4}\sqrt{6}\right)^2\right|$$

$$= \left(\frac{1}{4}\sqrt{3} + \frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{4}\sqrt{3}\right)$$

$$= \left(\frac{1}{4} - \frac{3}{16}\right)\left(\frac{1}{2}\right) = \frac{1}{32} = 0.03125$$

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Problem 7.2:

Let $T_n(x)$ denote the degree n Chebyshev polynomial. Find a formula for $T_n(0)$.

Solution

We know that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ with $T_0(x) = 1$ and $T_1(x) = x$. For x = 0 it's just $T_{n+1}(0) = -T_{n-1}(0)$ with $T_0(0) = 1$ and $T_1(0) = 0$ so it's clear that for every odd n $T_n(0) = 0$ and for every even $n = 2k$ we get $T_n(0) = (-1)^k$. ■

Problem 7.3:

Determine the following values

(a) $T_{999}(-1)$

(b) $T_{1000}(-1)$

(c) $T_{999}(0)$

(d) $T_{1000}(0)$

Solution

(a) $T_{999}(-1) = -1$

(b) $T_{1000}(-1) = 1$

(c) $T_{999}(0) = 0$

(d) $T_{1000}(0) = 1$

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Problem 7.4:

Determine the Pade approximations with $k = l = 3$ for $f(x) = \sin x$. Compare the results at $x_i = 0.1i$, for $i = 0, 1, \ldots, 5$, with the exact results of the sixth Maclaurin polynomial.

Solution

We know that the Maclaurin expansion of $f(x) = \sin x$ is as follows

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

From this, we can write

$$\begin{aligned} a_0 &= 0 & a_1 &= 1 \\ a_2 &= 0 & a_3 &= -1/3! \\ a_4 &= 0 & a_5 &= 1/5! \\ a_6 &= 0 \end{aligned}$$

Now we need to solve the following system of equations

$$\sum_{i=0}^{k} a_i q_{k-i} = p_k, \quad k = 0, 1, 2, 3$$

and

$$\sum_{i=4}^{k} a_i q_{k-i} = 0, \quad k = 4, 5, 6$$

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