Guidelines for solutions of problems. Sections 2.1-2.3, 2.5, 2.6

Name and section: ____________________

Instructor's name: ____________________

1. Explain why the function is discontinuous at the given number a. Sketch the graph of the function

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & if\ x \neq 1, \\ 1, & if\ \ x = 1. \end{cases}$$

Solution. The function f is continuous everywhere when $x \neq 1$. We must check at $a = 1$. We have $\lim\limits_{x\to 1} f(x) = \lim\limits_{x\to 1} \frac{x^2-x}{x^2-1} = \lim\limits_{x\to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim\limits_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$.
Thus $\lim\limits_{x\to 1} f(x) \neq f(x)$. Hence, f is discontinuous at $a = 1$.

2. How would you "remove the discontinuity" of f ? In other words, how would you define $f(2)$ in order to make f continuous at 2 if

$$f(x) = \frac{x^2 - x - 2}{x - 2}?$$

Answer. The function f is continuous everywhere when $x \neq 2$. We should define f at 2 so that it would be also continuous ar 2. Let us calculate the limit at 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+1) = 3.$$

Thus, if $f(2) = 3$, then f is continuous because in this case

$$\lim_{x\to 2} f(x) = f(2).$$

3. Evaluate the limit and justify each step by indicating the appropriate properties of limits

(a)

$$\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3};$$

(b)

$$\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x - 1};$$

(c)

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}};$$

(d)

$$\lim_{x\to\infty} ln\ (2+x) - ln\ (1+x) = \lim_{x\to\infty} ln\ \frac{2+x}{1+x} = ln\Big(\lim_{x\to\infty} \frac{2+x}{1+x}\Big) = ln\ 1 = 0.$$

Answer. Divide numerator and denominator by x^2. Then

(a) Divide numerator and denominator by x^2. Then

$$\lim_{x\to\infty}\frac{2x^2-7}{5x^2+x-3}=\lim_{x\to\infty}\frac{2-\frac{7}{x^2}}{5+\frac{1}{x}-\frac{3}{x^2}}=\frac{2}{5}.$$

(b) Divide numerator and denominator by x^3. Then

$$\lim_{x\to\infty}\frac{1-x^2}{x^3-x-1}=\lim_{x\to\infty}\frac{\frac{1}{x^3}-\frac{1}{x}}{1-\frac{1}{x^2}-\frac{1}{x^3}}=0;$$

(c) Divide numerator and denominator by e^{3x}. Then

$$\lim_{x\to\infty}\frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}=\lim_{x\to\infty}\frac{1-e^{-6x}}{1+e^{-6x}}=1.$$

4. Find the horizontal and vertical asymptotes of each curve.

$y=\frac{2x^2+1}{3x^2+2x-1}$.

Answer.

(a) Vertical asymptote:

$$3x^2+2x-1=0\Rightarrow x_1=\frac{-1+\sqrt{13}}{3};\;\; x_1=\frac{-1-\sqrt{13}}{3}.$$

Hence, we have 2 vertical asymptotes: $x=\frac{-1+\sqrt{13}}{3}$ and $x=\frac{-1-\sqrt{13}}{3}$.

To find horizontal asymptote we calculate the limits:

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{2x^2+1}{3x^2+2x-1}=\frac{2}{3};$$

similarly,

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{2x^2+1}{3x^2+2x-1}=\frac{2}{3}.$$

Thus, this curve has 1 horizontal asymptote.

5. Find the limits of $f(x)$ as $x\to-\infty$ and $x\to\infty$ if

$$f(x)=2x^3-x^4.$$

Answer.

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}2x^3-x^4=\lim_{x\to-\infty}x^4(\frac{2}{x}-1)=-\infty;$$

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}2x^3-x^4=\lim_{x\to\infty}x^4(\frac{2}{x}-1)=-\infty.$$

6. Find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$ if

$$f(x) = 2x^3 - x^4.$$

Answer.

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} 2x^3 - x^4 = \lim_{x \to -\infty} x^4(\frac{2}{x} - 1) = -\infty;$$

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2x^3 - x^4 = \lim_{x \to \infty} x^4(\frac{2}{x} - 1) = -\infty.$$