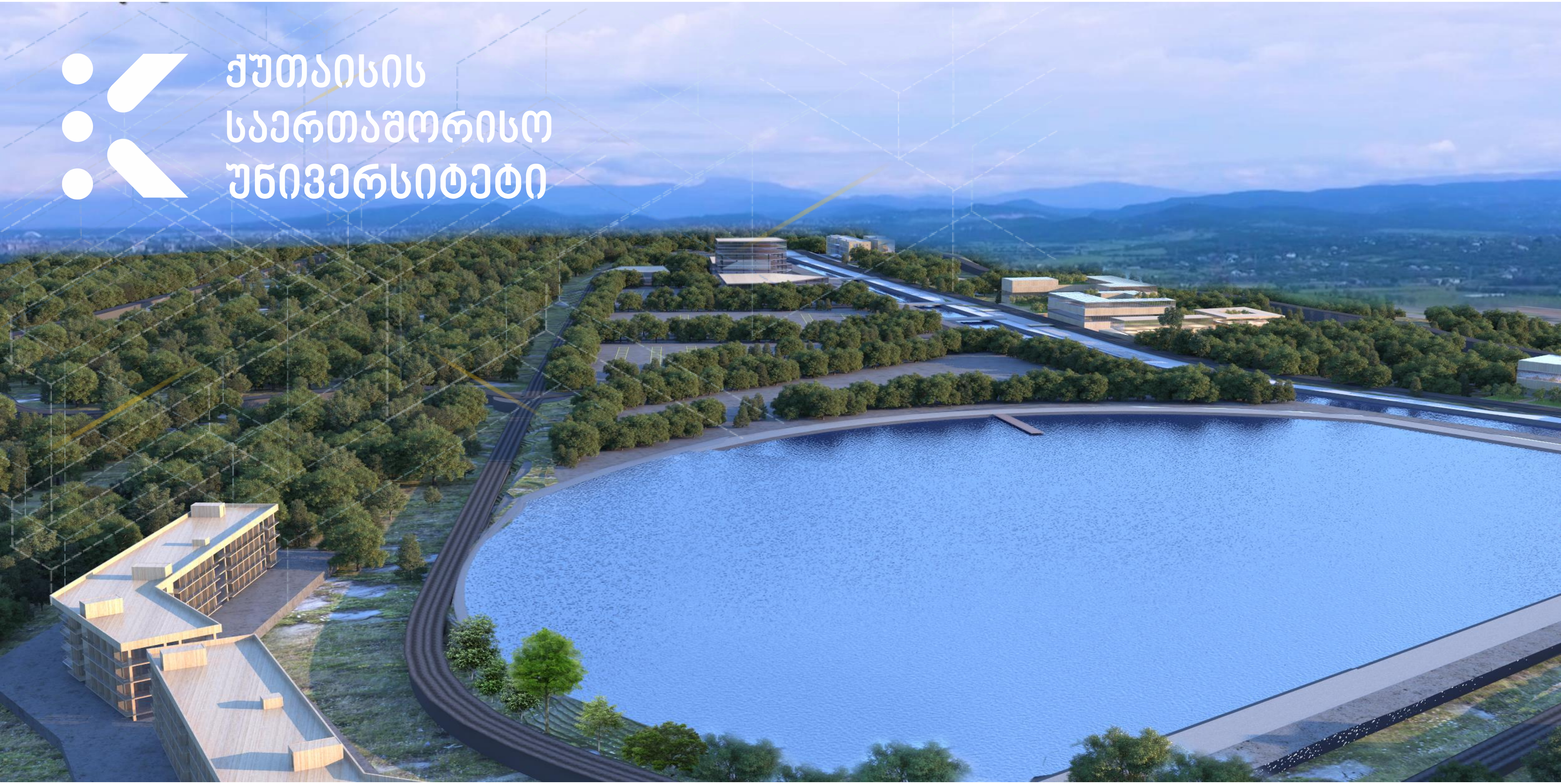

ქუთაისის
საერთაშორისო
უნივერსიტეტი

2.6 Limits at Infinity; Horizontal Asymptotes

Limits at Infinity; Horizontal Asymptotes (1 of 1)

In this section we let x become arbitrarily large (positive or negative) and see what happens to y.

Limits at Infinity and Horizontal Asymptotes

Limits at Infinity and Horizontal Asymptotes (1 of 10)

Let’s begin by investigating the behavior of the function *f* defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as *x* becomes large.

Limits at Infinity and Horizontal Asymptotes (2 of 10)

The table gives values of this function correct to six decimal places, and the graph of *f* has been drawn by a computer in Figure 1.

x	$f(x)$
0	−1
± 1	0
± 2	0.600000
± 3	0.800000
± 4	0.882353
± 5	0.923077
± 10	0.980198
± 50	0.999200
± 100	0.999800
± 1000	0.999998

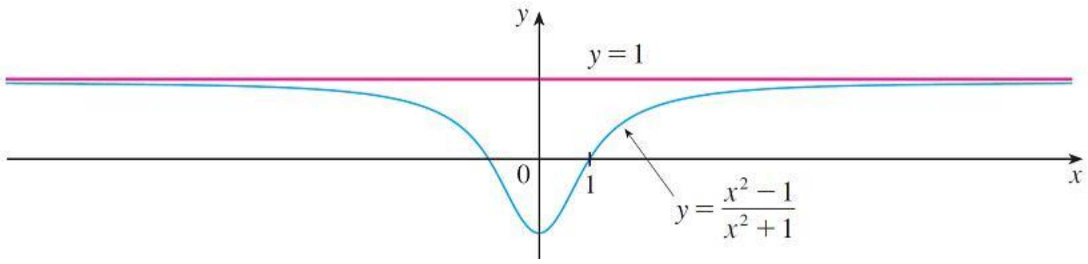


Figure 1

Limits at Infinity and Horizontal Asymptotes (3 of 10)

You can see that as x grows larger and larger, the values of $f(x)$ get closer and closer to 1. In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large.

This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity and Horizontal Asymptotes (4 of 10)

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

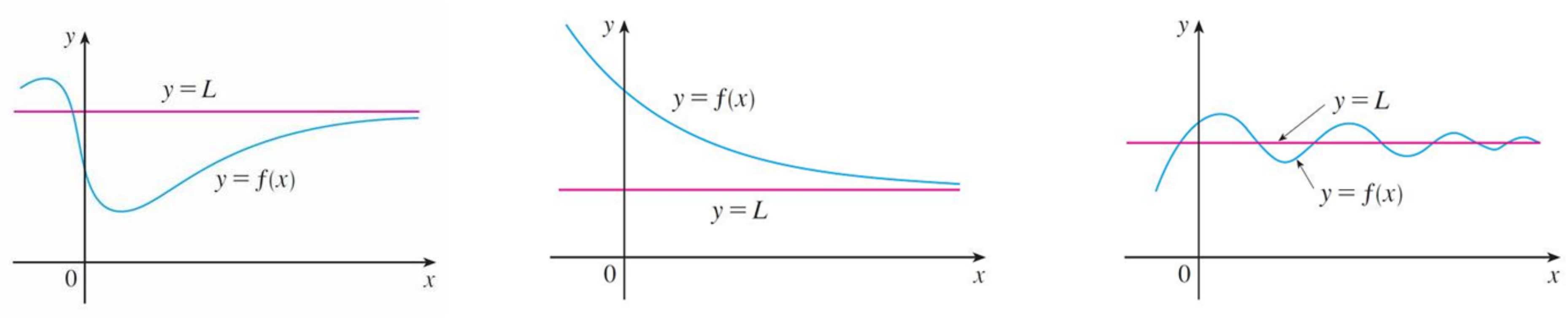
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Limits at Infinity and Horizontal Asymptotes (5 of 10)

Another notation for $\lim\limits_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Geometric illustrations of Definition 1 are shown in Figure 2.



Examples illustrating $\lim\limits_{x \to \infty} f(x) = L$

Figure 2

Limits at Infinity and Horizontal Asymptotes (6 of 10)

Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

Limits at Infinity and Horizontal Asymptotes (7 of 10)

Referring back to Figure 1, we see that for numerically large negative values of *x*, the values of *f*(*x*) are close to 1.

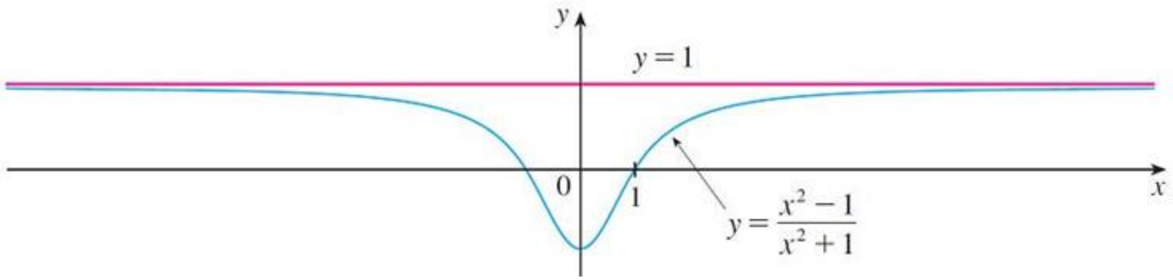


Figure 1

By letting *x* decrease through negative values without bound, we can make *f*(*x*) as close to 1 as we like.

This is expressed by writing

$$\lim_{x \to -\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

Limits at Infinity and Horizontal Asymptotes (8 of 10)

The general definition is as follows.

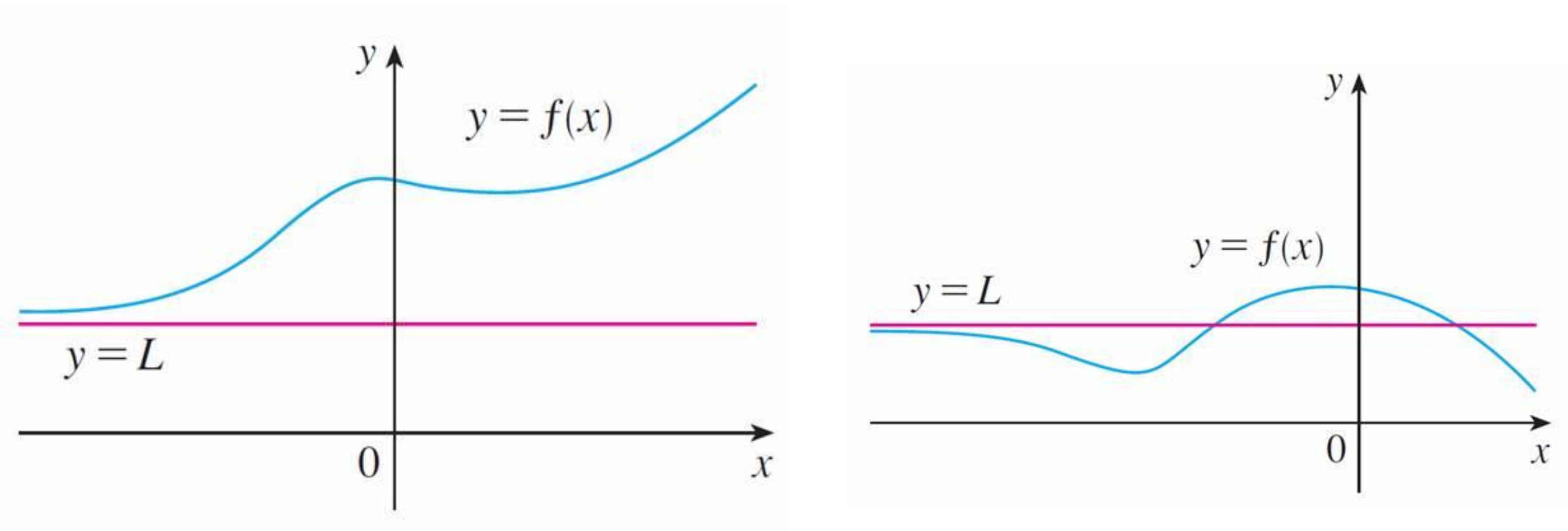
2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Limits at Infinity and Horizontal Asymptotes (9 of 10)

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.



Examples illustrating $\lim_{x \to -\infty} f(x) = L$

Figure 3

Limits at Infinity and Horizontal Asymptotes (10 of 10)

3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Example 2

Find $\lim_{x\to\infty} \frac{1}{x}$ and $\lim_{x\to-\infty} \frac{1}{x}$.

Solution:

Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \quad \frac{1}{10{,}000} = 0.0001 \quad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please.

Example 2 – Solution (1 of 2)

Therefore, according to Definition 1, we have

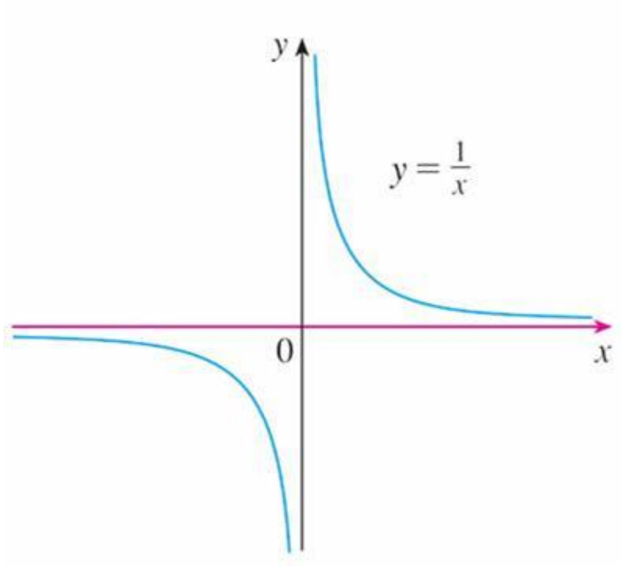
$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

Example 2 – Solution (2 of 2)

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 6.)



$$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$$

Figure 6

Evaluating Limits at Infinity

Evaluating Limits at Infinity (1 of 1)

5 Theorem If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0$$

Example 3

Evaluate the following limit and indicate which properties of limits are used at each stage.

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Solution:

As x becomes large, both numerator and denominator become large, so it isn't obvious what happens to their ratio. We need to do some preliminary algebra.

To evaluate the limit at infinity of any rational function, we first divide both the numerator and denominator by the highest power of x that occurs in the denominator. (We may assume that $x \neq 0$, since we are interested only in large values of x.)

Example 3 – Solution (1 of 3)

In this case the highest power of x in the denominator is x^2, so we have

$$\lim_{x\to\infty}\frac{3x^2-x-2}{5x^2+4x+1} = \lim_{x\to\infty}\frac{\dfrac{3x^2-x-2}{x^2}}{\dfrac{5x^2+4x+1}{x^2}}$$

$$= \lim_{x\to\infty}\frac{3-\dfrac{1}{x}-\dfrac{2}{x^2}}{5+\dfrac{4}{x}+\dfrac{1}{x^2}}$$

Example 3 – Solution (2 of 3)

$$= \frac{\lim_{x \to \infty} \left(3 - \frac{1}{x} - \frac{2}{x^2} \right)}{\lim_{x \to \infty} \left(5 + \frac{4}{x} + \frac{1}{x^2} \right)}$$

(by Limit Law 5)

$$= \frac{\lim_{x \to \infty} 3 - \lim_{x \to \infty} \frac{1}{x} - 2 \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 5 + 4 \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}}$$

(by 1, 2, and 3)

$$= \frac{3 - 0 - 0}{5 + 0 + 0}$$

(by 8 and Theorem 5)

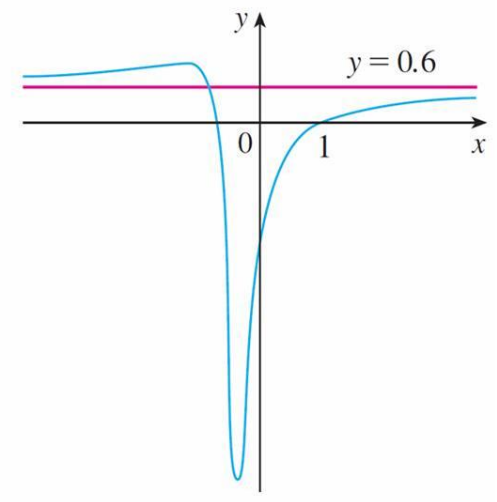
$$= \frac{3}{5}$$

Example 3 – Solution (3 of 3)

A similar calculation shows that the limit as $x \to -\infty$ is also $\frac{3}{5}$.

Figure 7 illustrates the results of these calculations by showing how the graph of the given rational function approaches the horizontal asymptote

$$y = \frac{3}{5} = 0.6.$$



$$y = \frac{3x^2 - x - 2}{5x^2 + 4x + 1}$$

Figure 7

Example 4

Find the horizontal asymptotes of the graph of the function

$$f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$$

Solution:

Dividing both numerator and denominator by x (which is the highest power of x in the denominator) and using the properties of limits, we have

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \lim_{x\to\infty} \frac{\dfrac{\sqrt{2x^2+1}}{x}}{\dfrac{3x-5}{x}}$$

Example 4 – Solution (1 of 5)

$$= \lim_{x\to\infty} \frac{\sqrt{\dfrac{2x^2+1}{x^2}}}{\dfrac{3x-5}{x}}$$

$\left(\text{since } \sqrt{x^2} = x \text{ for } x > 0\right)$

$$= \frac{\lim\limits_{x\to\infty} \sqrt{2+\dfrac{1}{x^2}}}{\lim\limits_{x\to\infty}\left(3-\dfrac{5}{x}\right)}$$

$$= \frac{\sqrt{\lim\limits_{x\to\infty} 2 + \lim\limits_{x\to\infty}\dfrac{1}{x^2}}}{\lim\limits_{x\to\infty} 3 - 5\lim\limits_{x\to\infty}\dfrac{1}{x}}$$

$$= \frac{\sqrt{2+0}}{3-5\cdot 0}$$

Example 4 – Solution (2 of 5)

$$= \frac{\sqrt{2}}{3}$$

Therefore the line $y = \sqrt{2}/3$ is a horizontal asymptote of the graph of *f*.

In computing the limit as $x \to -\infty$, we must remember that for $x < 0$, we have $\sqrt{x^2} = |x| = -x$.

Example 4 – Solution (3 of 5)

So when we divide the numerator by x, for $x < 0$ we get

$$\frac{\sqrt{2x^2+1}}{x} = \frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}$$

$$= -\sqrt{\frac{2x^2+1}{x^2}}$$

$$= -\sqrt{2+\frac{1}{x^2}}$$

Example 4 – Solution (4 of 5)

Therefore

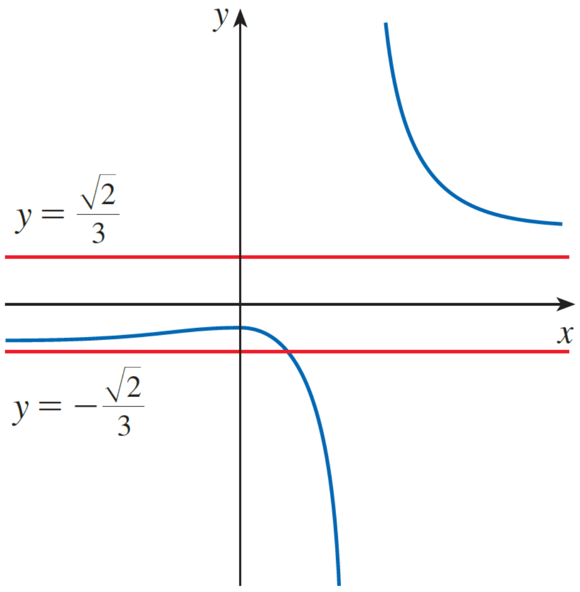
$$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{-\sqrt{2 + \frac{1}{x^2}}}{3 - \frac{5}{x}}$$

$$= \frac{-\sqrt{2 + \lim\limits_{x \to -\infty} \frac{1}{x^2}}}{3 - 5 \lim\limits_{x \to -\infty} \frac{1}{x}}$$

$$= -\frac{\sqrt{2}}{3}$$

Example 4 – Solution (5 of 5)

Thus the line $y = -\sqrt{2}/3$ is also a horizontal asymptote. See Figure 8.



$$y = \frac{\sqrt{2x^2+1}}{3x-5}$$

Figure 8

Infinite Limits at Infinity

Infinite Limits at Infinity (1 of 1)

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \quad \lim_{x \to \infty} f(x) = -\infty \quad \lim_{x \to -\infty} f(x) = -\infty$$

Example 9

Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

Solution:

When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by requiring x to be large enough.

Therefore we can write

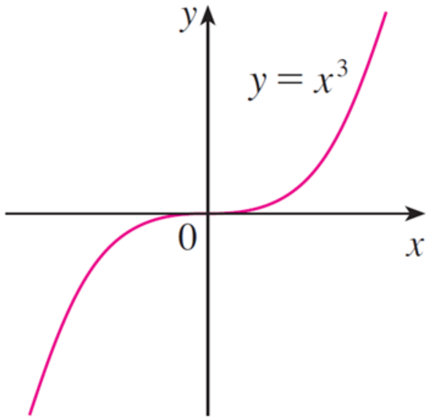
$$\lim_{x\to\infty} x^3 = \infty$$

Example 9 – Solution

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x \to -\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11.



$$\lim_{x \to \infty} x^3 = \infty, \quad \lim_{x \to -\infty} x^3 = -\infty$$

Figure 11