proofs of some of the lemmas are taken

1. (a) $3 + 1 = 4$ (Definition 2)
   (b) $x + S(y) = S(x + y)$ (Definition 3)
   (c) $1 + 4 = 5$ (Theorem)

   Proof.

$$\begin{aligned}
1 + 4 &= 1 + (3 + 1) && \text{(Definition of 4)} \\
&= 1 + ((2 + 1) + 1) && \text{(Definition of 3)} \\
&= 1 + (((1 + 1) + 1) + 1) && \text{(Definition of 2)} \\
&= (((1 + 1) + 1) + 1) + 1 && \text{(Lemma 4.)} \\
&= ((2 + 1) + 1) + 1 && \text{(Definition 2)} \\
&= (3 + 1) + 1 && \text{(Definition 2)} \\
&= 4 + 1 && \text{(Definition 2)} \\
&= 5 && \text{(Definition 2)}
\end{aligned}$$

   (d) $x + 1 = 1 + x$ (Lemma 3)

   Proof.
   For x = 0 we have

$$0 + 1 = 1 = 1 + 0$$

   .
   Assume $x + 1 = 1 + x$, we can show

$$\begin{aligned}
1 + S(x) &= S(1 + x) && \text{(Definition of addition)} \\
&= S(x + 1) && \text{(Induction hypothesis)} \\
&= S(x) + 1 && \text{(Definition of counting by adding 1)}
\end{aligned}$$

   (e) $x + y = y + x$ (Lemma 4)

   Proof.
   For $y = 0$ we have

$$\begin{aligned}
x + 0 &= x && \text{(Definition of addition)} \\
&= 0 + x && \text{(Lemma 2)}
\end{aligned}$$

   .
   Assume $x + y = y + x$, we can show

$$\begin{aligned}
x + S(y) &= S(x + y) && \text{(Definition of addition)} \\
&= S(y + x) && \text{(Induction hypothesis)} \\
&= y + S(x) && \text{(Definition 3)} \\
&= y + (x + 1) && \text{(Definition of counting by addoing 1)} \\
&= y + (1 + x) && \text{(Lemma 3)} \\
&= (y + 1) + x && \text{(Associativity of addition)} \\
&= S(y) + x && \text{(Definition of counting by addoing 1)}
\end{aligned}$$

(f) $x^{y+z} = x^y \cdot x^z$ (Lemma 7)

Proof.
For $z = 0$ we have

$$x^{y+0} = x^y = x^y \cdot 1 = x^y \cdot x^0$$

.
Assume $x^{y+z} = x^y \cdot x^z$, we can show

$$\begin{aligned}
x^{y+S(z)} &= x^{S(y+z)} && \text{(Definition of addition)}\\
&= x^{(y+z)} \cdot x && \text{(Definition of exponentiation)}\\
&= (x^y \cdot x^z) \cdot x && \text{(Induction hypothesis)}\\
&= x^y \cdot (x^z \cdot x) && \text{(Associativity of multiplication)}\\
x^{y+S(z)} &= x^y \cdot x^{S(z)} && \text{(Definition of exponentiation).}
\end{aligned}$$

(g) $(x + y) \cdot z = x \cdot z + y \cdot z$ (Lemma 5)

Proof.
For $z = 0$, we have

$$(x + y) \cdot 0 = 0 = x \cdot 0 + y \cdot 0 \qquad \text{(Definition of multiplication)}$$

.
Assume that $(x + y) \cdot z = x \cdot z + y \cdot z$, we can show

$$\begin{aligned}
(x + y) \cdot S(z) &= (x + y) \cdot z + (x + y) && \text{(Definition of multiplication)}\\
&= (x \cdot z + y \cdot z) + (x + y) && \text{(Induction hypothesis)}\\
&= x \cdot z + y \cdot z + x + y && \text{(Associativity of addition)}\\
&= x \cdot z + x + y \cdot z + y && \text{(Lemma 3)}\\
&= (x \cdot z + x) + (y \cdot z + y) && \text{(Associativity of addition)}\\
&= (x \cdot S(z)) + (y \cdot S(z)) && \text{(Definition of multiplication)}\\
&= x \cdot S(z) + y \cdot S(z) && \text{(Associativity of addition)}
\end{aligned}$$

(h) $x \cdot y = y \cdot x$ (Lemma 5)

Proof.
by induction on y. For $y = 0$, we have

$$x \cdot 0 = 0 = 0 \cdot x \qquad \text{(Definition 5)}$$

.
Assume that $x \cdot y = y \cdot x$, we can show

$$\begin{aligned}
x \cdot S(y) &= x \cdot y + x && \text{(Definition of multiplication)}\\
&= y \cdot x + x && \text{(Induction hypothesis)}\\
&= S(y) \cdot x && \text{(Definition of multiplication)}
\end{aligned}$$

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Lemma 5)

Proof.
For $z = 0$, we have

$$(x \cdot y) \cdot 0 = 0 = x \cdot (y \cdot 0) = x \cdot 0$$

.
Assume that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, we can show

$$\begin{aligned}
(x \cdot y) \cdot S(z) &= (x \cdot y) \cdot z + (x \cdot y) && \text{(Definition of multiplication)} \\
&= x \cdot (y \cdot z) + (x \cdot y) && \text{(Induction hypothesis)} \\
&= x \cdot (y \cdot z + y) && \text{(Distributivity of multiplication)} \\
&= x \cdot (y \cdot S(z)) && \text{(Definition of multiplication)}
\end{aligned}$$

(j) $x \cdot 0 = 0$ (Definition 5)

2. - $(x \cdot y)^z = x^z \cdot y^z$.

Proof.
For $z = 0$, we have

$$(x \cdot y)^0 = 1 = 1 \cdot 1 = x^0 \cdot y^0$$

.
Assume that $(x \cdot y)^z = x^z \cdot y^z$, we ca show

$$\begin{aligned}
(x \cdot y)^{S(z)} &= (x \cdot y)^z \cdot (x \cdot y) && \text{(Definition of exponentiation)} \\
&= (x^z \cdot y^z) \cdot (x \cdot y) && \text{(Induction step)} \\
&= (x^z \cdot x) \cdot (y^z \cdot y) && \text{(Lemma 5)} \\
&= (x^{S(z)}) \cdot (y^{S(z)}) && \text{(Definition of exponentiation)}
\end{aligned}$$

- $(x^y)^z = x^{y \cdot z}$

Proof.
For $z = 0$ we have

$$(x^y)^0 = 1 = x^0 = x^{y \cdot 0}$$

.
Assume that $(x^y)^z = x^{y \cdot z}$, we can show

$$\begin{aligned}
(x^y)^{S(z)} &= (x^y)^z \cdot (x^y) && \text{(Definition of exponentiation)} \\
&= x^{y \cdot z} \cdot (x^y) && \text{(Induction hypothesis)} \\
&= x^{z \cdot y} \cdot (x^y) && \text{(Lemma 5)} \\
&= x^{z \cdot y + y} && \text{(Lemma 7)} \\
&= x^{S(z) \cdot y} && \text{(Definition of multiplication)}
\end{aligned}$$

3. - efghijklmn[7:3] = ghijk
   - abcdef[6:2] = abcd
   - lmno ∘ efgh = lmnoefgh
   - (abc ∘ defghi)[5:8] = ∅

4. - $A^2 = \{(a, a), (a, b), (a, c), \ldots, (d, b), (d, c), (d, d)\}$
   - $B^2 = \{(a, a), (a, b), (a, c), \ldots, (c, a), (c, b), (c, c)\}$

- $B^3 = \{(a,a,b),(a,a,b),\dots,(c,c,a),(c,c,b),(c,c,c)\}$

5.
- $A^\circ B = \{ad, ae, bd, be, cd, ce\}$
- $\#(A^\circ B) = 3 \cdot 2 = 6$
- $A^\circ A = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$
- $B^\circ B = \{ee, ed, de, dd\}$
- $\#(A^\circ A) = 9$
- $\#(B^\circ B) = 4$

6.
- $A \times A = \{(a,a),(a,b),(a,c),\dots,(d,b),(d,c),(d,d)\}$
- $\#(A \times A) = \#(A)^2 = 4^2 = 16$

7.
- $\langle 10100110 \rangle = 166$
- $\langle 11010 \rangle = 26$
- $bin_4(14) = 1100$
- $bin_6(47) = 101111$
- $bin_2 0(0) = 00000000000000000000$

8.

$54 = 110110$
$17 = 10001$

$$54 + 17 = 71 \iff \begin{array}{r} 110110 \\ +\quad 10001 \\ \hline 1000111 \end{array}$$

$$54 - 17 = 37 \iff \begin{array}{r} 110110 \\ -\quad 10001 \\ \hline 100101 \end{array}$$

$$54 \cdot 17 = 918 \iff \begin{array}{r} 110110 \\ \times \qquad 10001 \\ \hline 110110 \\ +\quad 110110 \\ \hline 1110010110 \end{array}$$

$$54 \div 17 = 3\ (3) \iff \begin{array}{r|l} 110110 & 10001 \\ \cline{2-2} -10001 & 11 \\ \cline{1-1} 10100 & \\ -10001 & \\ \cline{1-1} 11 & \end{array}$$

9. a) $\{1, 2, 8, 3, 5, 9, 10\}$

b) $[1] = \{1, 4, 6\}$
$[2] = \{2, 8\}$
$[3] = \{3\}$
$[4] = \{1, 4, 6\}$