Introduction to Optimization Homework (2)

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Problem 2.1:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)$$
$$f'(x_0) = \frac{f(x_0) - f(x_0 + h)}{h} + \frac{h^2}{2} f''(x_0) + O(h^2)$$

Then we apply extraplotaion to get a higher order approximation.

$$\begin{aligned} f'(x_0) &= \frac{4\left(2\frac{f(x_0)-f\left(x_0+\frac{h}{2}\right)}{h} + \frac{h^2}{8} f''(x_0) + O(h^2)\right) - \left(\frac{f(x_0)-f(x_0+h)}{h} + \frac{h^2}{2} f''(x_0) + O(h^2)\right)}{4-1} \\ &= \frac{8\frac{f(x_0)-f\left(x_0+\frac{h}{2}\right)}{h} + \frac{h^2}{2} f''(x_0) - \frac{f(x_0)-f(x_0+h)}{h} - \frac{h^2}{2} f''(x_0) + O(h^2)}{3} \\ &= \frac{7f(x_0) - 8f\left(x_0 + \frac{h}{2}\right) + f(x_0 + h)}{3} + O(h^2) \end{aligned}$$

Problem 2.2:

(a)

$$3f(x_0 - h) + f(x_0 + 3h) = 4f(x_0) + 6h^2 f''(x_0) + 4h^3 f'''(x_0) + O(h^3)$$
$$f''(x_0) = \frac{3f(x_0 - h) + f(x_0 + 3h) - 4f(x_0)}{6h^2} + \frac{2}{3} h f'''(x_0) + O(h)$$

(b)

$$\begin{aligned} f''(x_0) &= \frac{2^1\left(\frac{3f\left(x_0-\frac{h}{2}\right)+f\left(x_0+\frac{3}{2}h\right)-4f(x_0)}{\frac{3}{2}h^2} + O(h^2)\right) - \left(\frac{3f(x_0-h)+f(x_0+3h)-4f(x_0)}{6h^2} + \frac{2}{3}hf'''(x_0) + O(h)\right)}{1} \\ &= \frac{24f\left(x_0 - \frac{h}{2}\right) + 8f\left(x_0 + \frac{3}{2}h\right) - 3f(x_0 - h) - f(x_0 + 3h) - 28f(x_0)}{6h^2} + O(h^2) \end{aligned}$$

(c) with $h = 0.1$ the absolute error is $2.90187 \cdot 10^{-3}$ and with $h = 0.01$ it's $2.917 \cdot 10^{-5}$ which is about two times smaller

(d)

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h=0.500 0.13565659 -0.06405054
h=0.475 0.12329346 -0.05855471
h=0.450 0.11139753 -0.05319656
h=0.425 0.09999343 -0.04799719
h=0.400 0.08910493 -0.04297703
h=0.375 0.07875482 -0.03815570
h=0.350 0.06896488 -0.03355200
h=0.325 0.05975579 -0.02918381
h=0.300 0.05114710 -0.02506805
h=0.275 0.04315715 -0.02122063
h=0.250 0.03580302 -0.01765639
h=0.225 0.02910049 -0.01438903
h=0.200 0.02306395 -0.01143112
h=0.175 0.01770644 -0.00879399
h=0.150 0.01303953 -0.00648774
h=0.125 0.00907332 -0.00452119
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The first column is the absolute error of the first method (the one without extrapolation) and the second is the second column (the one with extrapolation).

Problem 2.3:

(a) $\frac{\partial f}{\partial x} \approx \frac{f(x+h,y)-f(x-h,y)}{2h}$

$$= \frac{\cancel{f(x,y)}+h\frac{\partial f}{\partial x}+\cancel{\frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}}+O(h^3)-\cancel{f(x,y)}+h\frac{\partial f}{\partial x}-\cancel{\frac{h^2}{2}\frac{\partial^2 f}{\partial x^2}}+O(h^3)}{2h}$$

$$= \frac{\partial f}{\partial x} + O(h^2)$$

(b) $\frac{\partial^2 f}{\partial y^2} \approx \frac{f(x,y+h)-2f(x,y)+f(x,y-h)}{h^2}$

$$= \frac{\cancel{f(x,y)}+\cancel{h\frac{\partial f}{\partial y}}+\frac{h^2}{2}\frac{\partial^2 f}{\partial y^2}+\cancel{\frac{h^3}{6}\frac{\partial^3 f}{\partial y^3}}+O(h^4)-\cancel{2f(x,y)}+\cancel{f(x,y)}-\cancel{h\frac{\partial f}{\partial y}}+\frac{h^2}{2}\frac{\partial^2 f}{\partial y^2}-\cancel{\frac{h^3}{6}\frac{\partial^3 f}{\partial y^3}}+O(h^4)}{h^2}$$

$$= \frac{\partial^2 f}{\partial y^2} + O(h^2)$$