random routing

Valiant 1981

one of the great classics

the historic idea of ‚random routing‘

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- for enough cars p
  - hang car p with destination $\pi(p)$...
  - on random train with destination $\rho(p)$
  - then from station $\rho(p)$
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Valiant 1981:

- hypercube networks
  - 2^n nodes, diameter n
- route for each node p a packet to $\pi(p)$
  - π permutation of nodes
- with random routing for a constant c
  - run time $> nc$ extremely unlikely

n dimensional hypercubes

indexing of bits strings: $u = u[1:n]$

hypercubes: graphs $H_n = (V_n, E_n)$

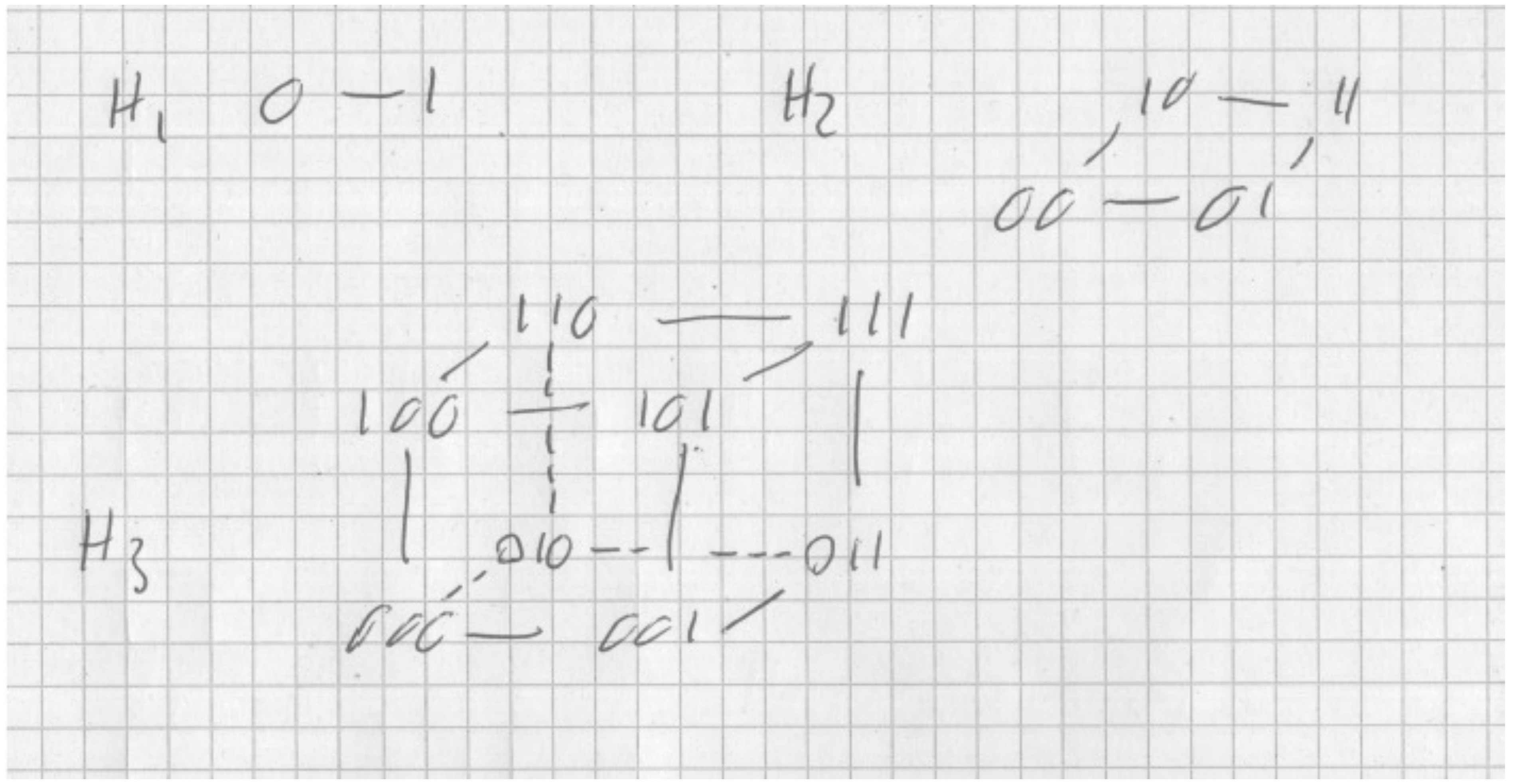
- nodes

$$V_n = \mathbb{B}^n$$

- edges

$$\{u,v\} \in E_n \leftrightarrow \#\{i \mid u_i \neq v_i\} = 1$$

nodes connected by an edge differ in exactly one position.



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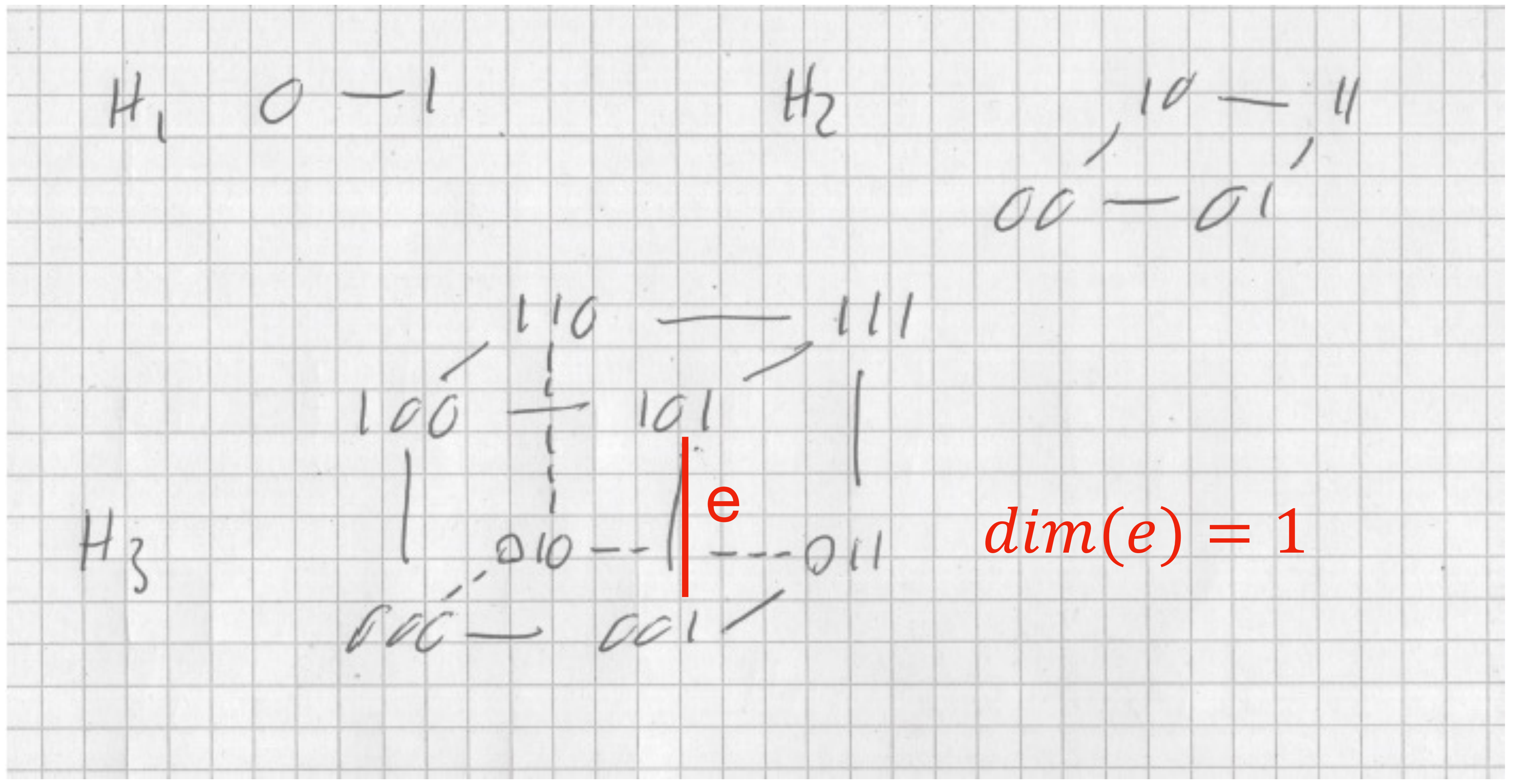
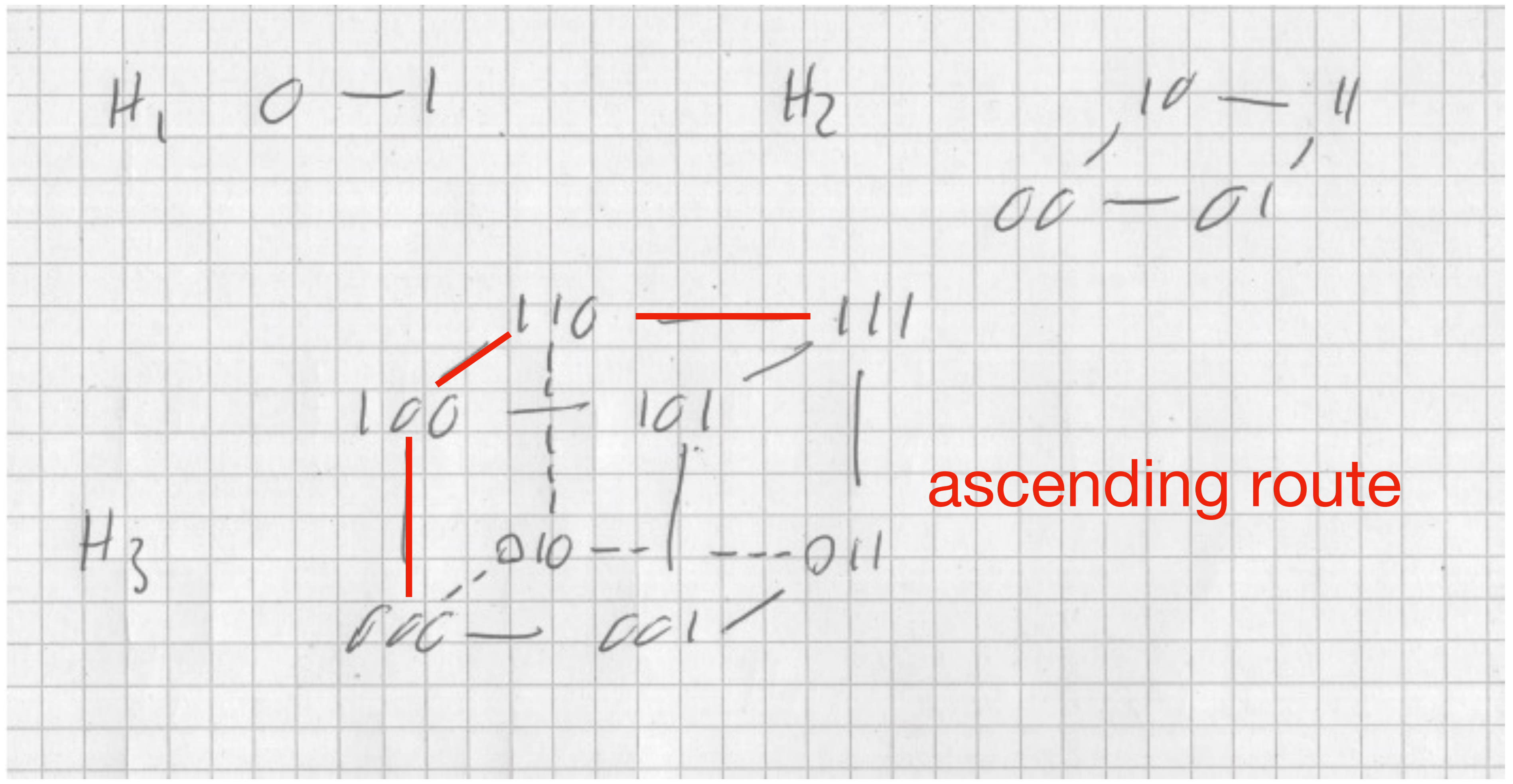


- dimension of edge: the position where end points differ

$$e = \{u,v\} \in E_n \wedge u_i \neq v_i \rightarrow dim(e) = i$$

routes

- route
$$R = (v_0, v_1, \ldots v_k) \text{ path in } H_n$$
- route R is *ascending* if dimensions are traversed in increasing order
$$\forall i \in [1, k-1].\ dim(\{v_{i-1}, v_i\}) < dim(\{v_i, v_{i+1}\})$$



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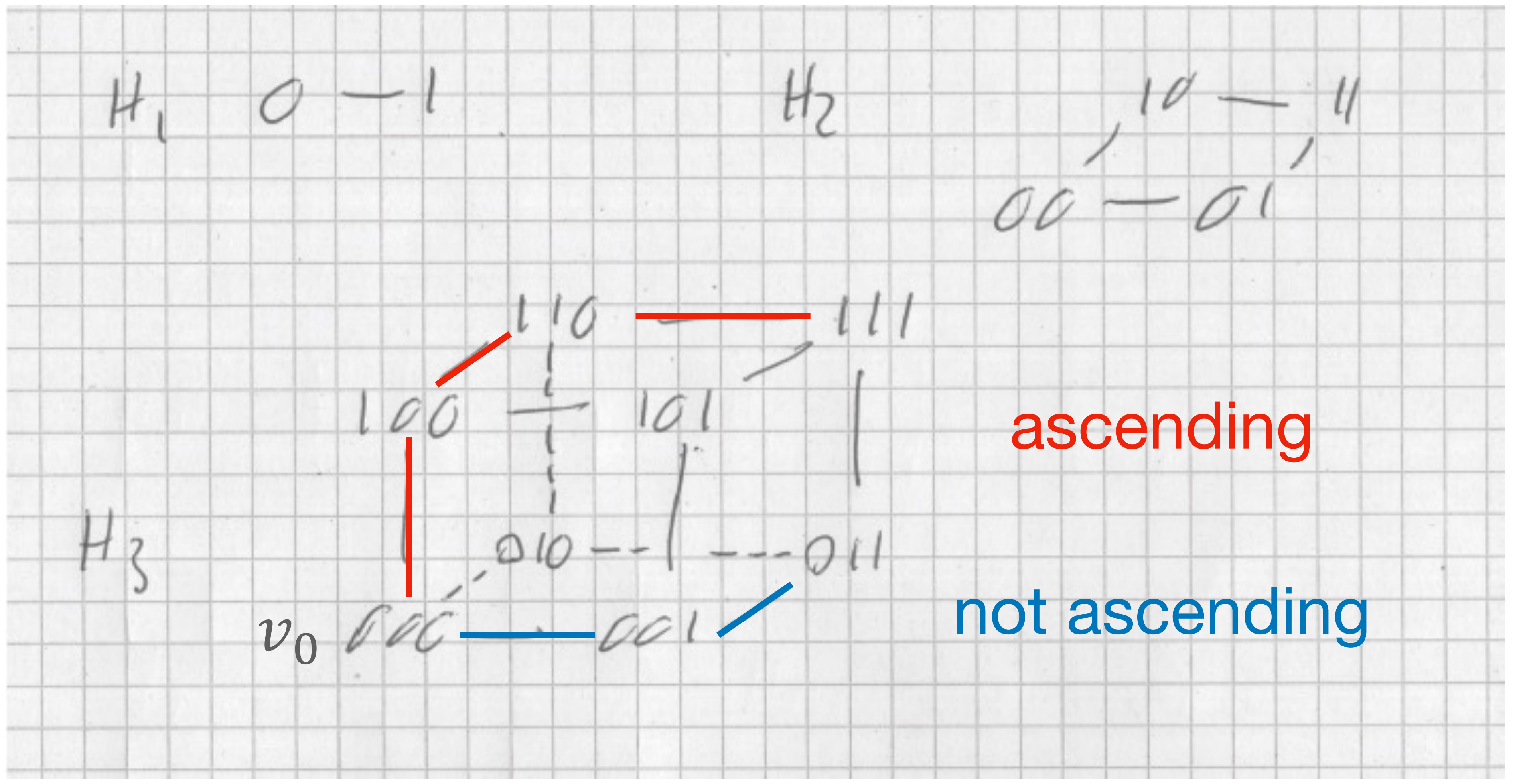

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- routes $R = (v_0, v_1, \ldots v_k)$ and $Q = (u_0, u_1, \ldots u_m)$ *collide*, if they share at least one edge

$$col(R, Q) \leftrightarrow \exists i, j.\ \{v_i, v_{i+1}\} = \{u_j, u_{j+1}\}$$

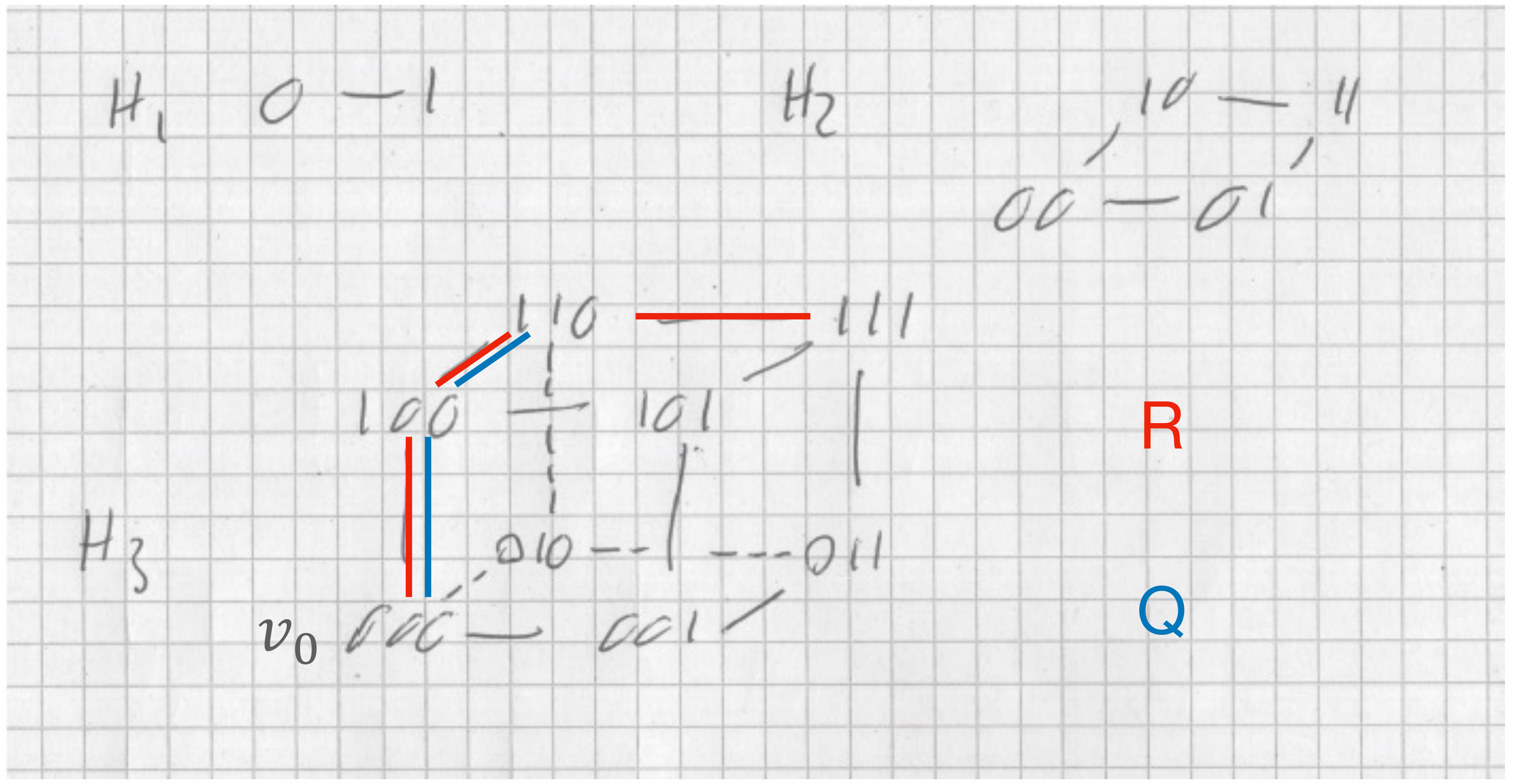
Their *collision dimension* is the dimension of the first edge on R which they share (if both are ascending, it is also the first edge on Q.

such

$$cd(R, Q) = \min i \,|\, \exists j.\ \{v_i, v_{i+1}\} = \{u_j, u_{j+1}\}$$

Their *collision edge* is the edge on R in the collision dimension

$$ce(R, Q) = \{v_{cd(R,Q)}, v_{cd(r,Q)+1}\}$$



- R, Q colliding
- collision dimension 1
- collision edge $\{000{,}100\}$

- $des(R) = 111$
- $des(Q) = 110$

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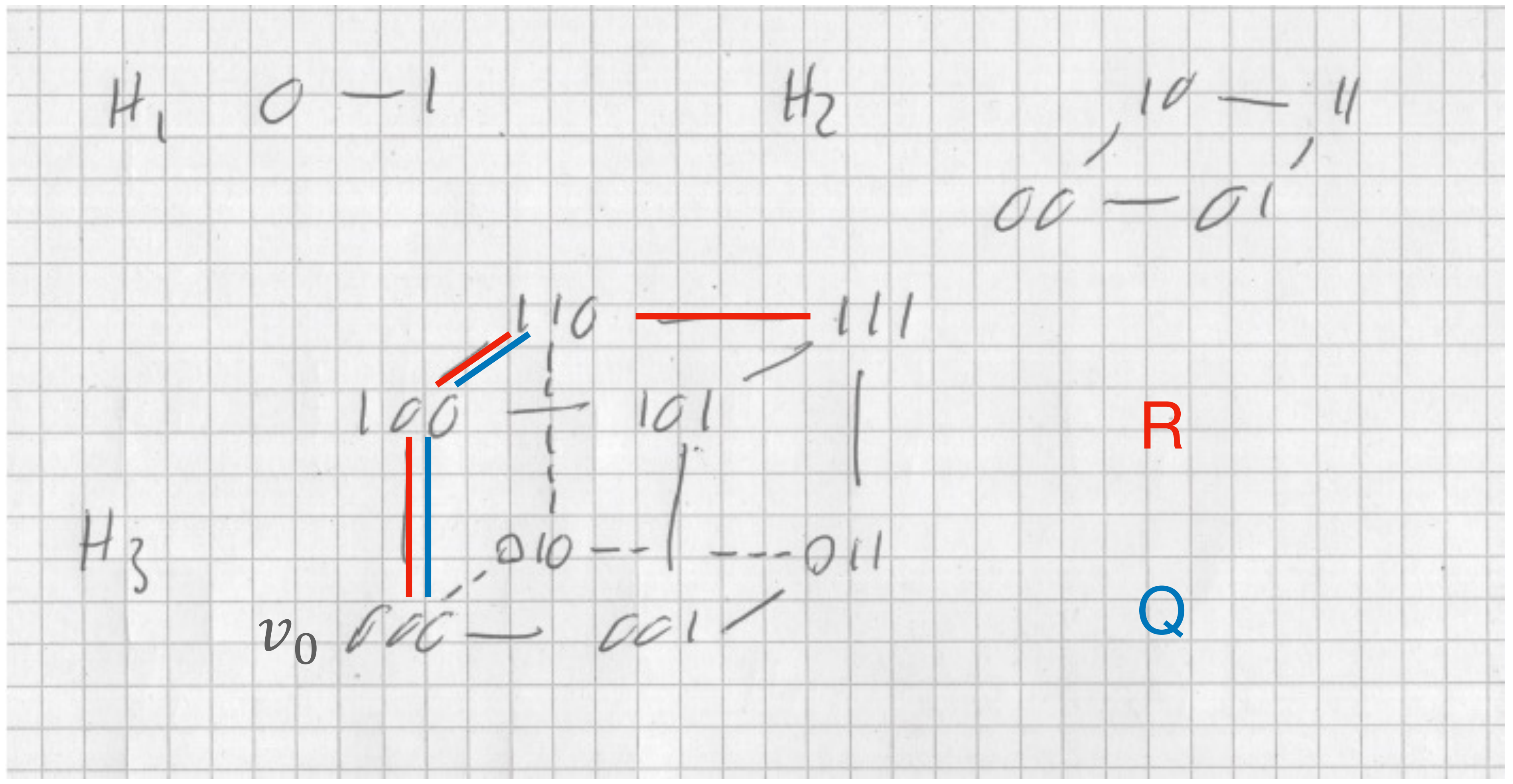
exercise

- standard (partial) description of route R: Bit j of $des(R) \in \mathbb{B}^n$ is 1, if an edge of R has dimension j.

$$des(R) \in \mathbb{B}^n\ ,\ des(R)_j = 1 \leftrightarrow \exists i.\ dim\{v_{i-1}, v_i\} = j$$

An ascending route R is determined by start point v_0 and $des(R)$.



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packet routing

- one packet in each node u
- $\pi : \mathbb{B}^n \to \mathbb{B}^n$ permutation.
- goal: for all u move packet from u to $\pi(u)$.
- packets can be stored in nodes
- in each step each edge can only be passed by 1 packet.
- if several packets in a node try to traverse the same edge, the order cannot be predicted.

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random routing

two rounds:

- round 1: each node u determines random $des(R(u)) \in \mathbb{B}^n$. Packet from u is sent to random destination

$$\rho(u) = u \oplus des(R(u))$$

exercise: This is intuitive „thing"

via ascending route.

- round 2: packets from u are sent from $\rho(u)$ to $\pi(u)$ via ascending routes.

probability space

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$$d = des(R(0^n)) \circ \ldots \circ des(R(1^n)) \in \mathbb{B}^{n \cdot 2^n}$$

chosen independently and with probability $1/2$. Probability space $W = (S, p)$

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- traversal of different edges performed in parallel.
- fix any strategy to schedule conflicting packets on edges.
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probability of long runs

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long run times (in round 1) are unlikely:

Lemma 2. *For any constant $c > 0$:*

$$p(\{d \mid T(d) \geq n + c\}) \leq 2^{-(c-5n)/2}$$

routes with many collision routes

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- The routes $R_d(v_i)$ for these packets collide with route $R_d(u)$
- the event that this situation occurs:

$$E = \{d \mid \exists u, v_1, \ldots v_c. \, \forall j. \; col(R_d(u), R_d(v_j))\}$$

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idea:

- if a route $R_d(u)$ with c collision routes $R_d(v_j)$ exists, then this can be used to find a compressed encoding $d' \in \mathbb{B}^x$ of d with $x < n \cdot 2^n$.
- as each element in E is determined by such an encoding, the number of such encodings bouds $\#E$:

$$\#E \leq 2^x$$

compressing encodings along a route $R_d(u)$ with c collisions

compressed desciptions d' have 4 parts

$$d' = A \circ B \circ C \circ D$$

- the route from u

$$A = u \circ des(u)$$

We have

$$|A| = 2n$$

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- For each dimension $i > 1$ let $c(i)$ be the number of collision routes colliding with $R_d(u)$ in dimension i. With

$$c^*(i) = \begin{cases} c(i) & c(i) \text{ even} \\ c(i) - 1 & c(i) \text{ odd} \end{cases}$$

  For each odd $c(i)$ we disregard route $R_d(v_j)$ colliding in dimension i. This removes at most n routes and all $c^*(i)$ are even.

  c(1) at most 1.

$$c - n \leq c^* = \sum_{i=2}^{n} c^*(i) \leq c$$

  We code the sequence of the $c^*(i)/2$ in *unary* in the collision vector

$$B = 1^{c^*(2)/2}01^{c^*(3)/2}0\ldots01^{c^*(n)}$$

  Then

$$|B| < n + c^*/2$$

- we order nodes v_j with collision routes in the order of the collision dimension with $R_d(u)$. In this order the collision dimension or route $R_d(v_j)$

$$cd(j) = cd(R_d(u), R_d(v_j)) = \overset{\max}{\cancel{\min}}\{i \,|c^*(i) \neq 0\ ,\ \sum_{k<i} c^*(k) < j\}$$

We code the collision routes in this order with descriptions

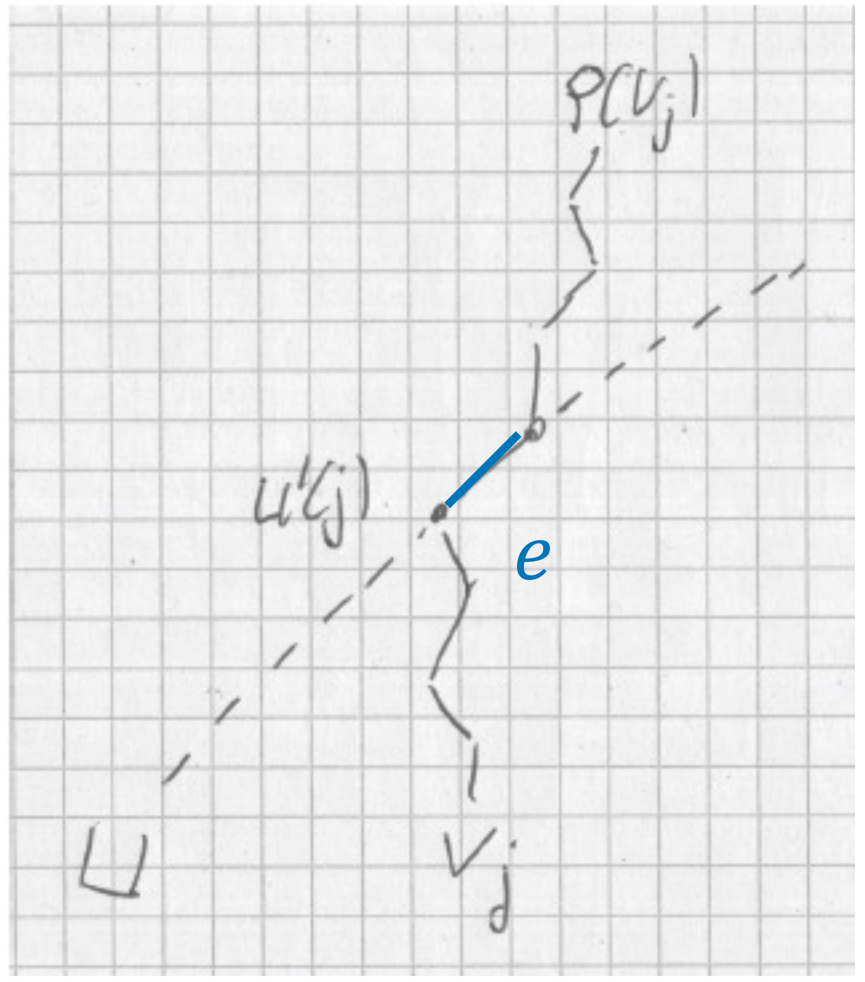
$$\underline{C = des'(v_1) \circ \ldots \circ des'(v_{c^*})}$$

where we omit t for each v_j the bit for the collision dimension $cd(j)$ (which we already know to be 1).

$$\underline{des'(v_j) = des(v_j)[1 : cd(j) - 1] \circ des(v_j)[cd(j) + 1 : n]}$$

Then for all j

$$\begin{aligned} |des'(v_j)| &= n - 1 \\ |C| &= c^* \cdot (n - 1) \end{aligned}$$



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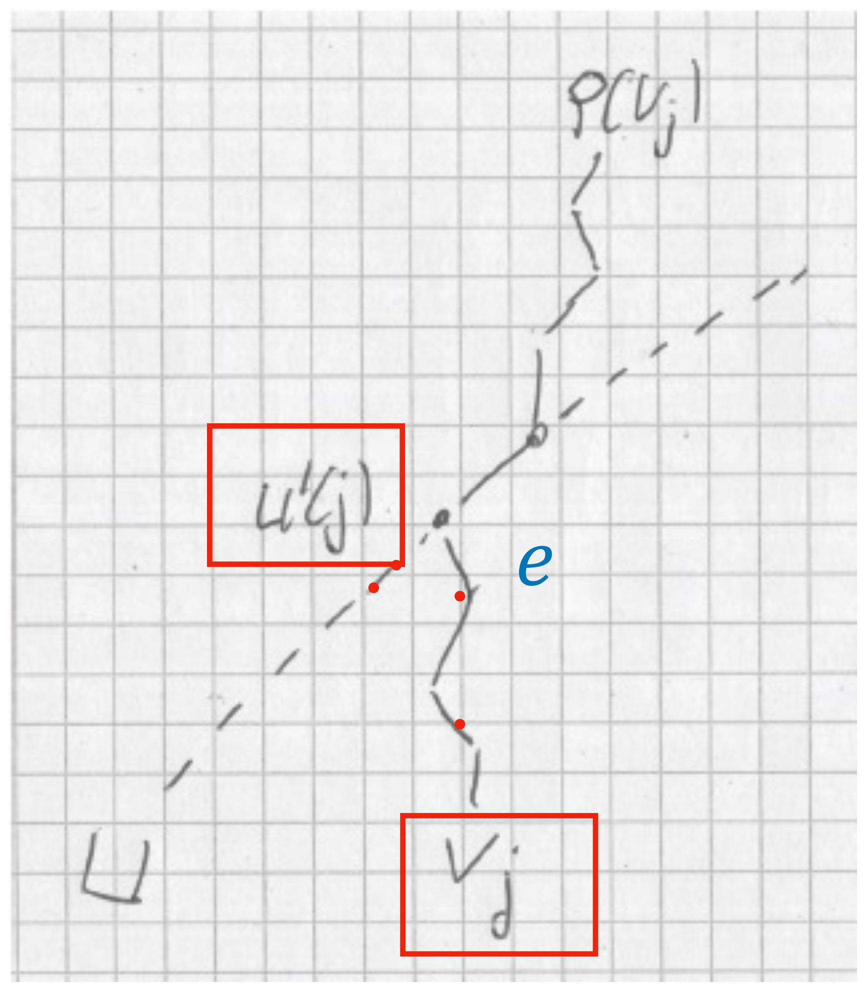

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With known collision dimensions $cd(j)$ the start points v_j themselves can be reconstructed: from u follow path $R_d(u)$ until the start u' of the collision edge

$$u'(j) = u \oplus des(u)[1 : cd(j) - 1] \circ 0^{n-cd(j)+1}$$

then follow route $r_d(v_j)$ backward in the dimensions before the collision dimension

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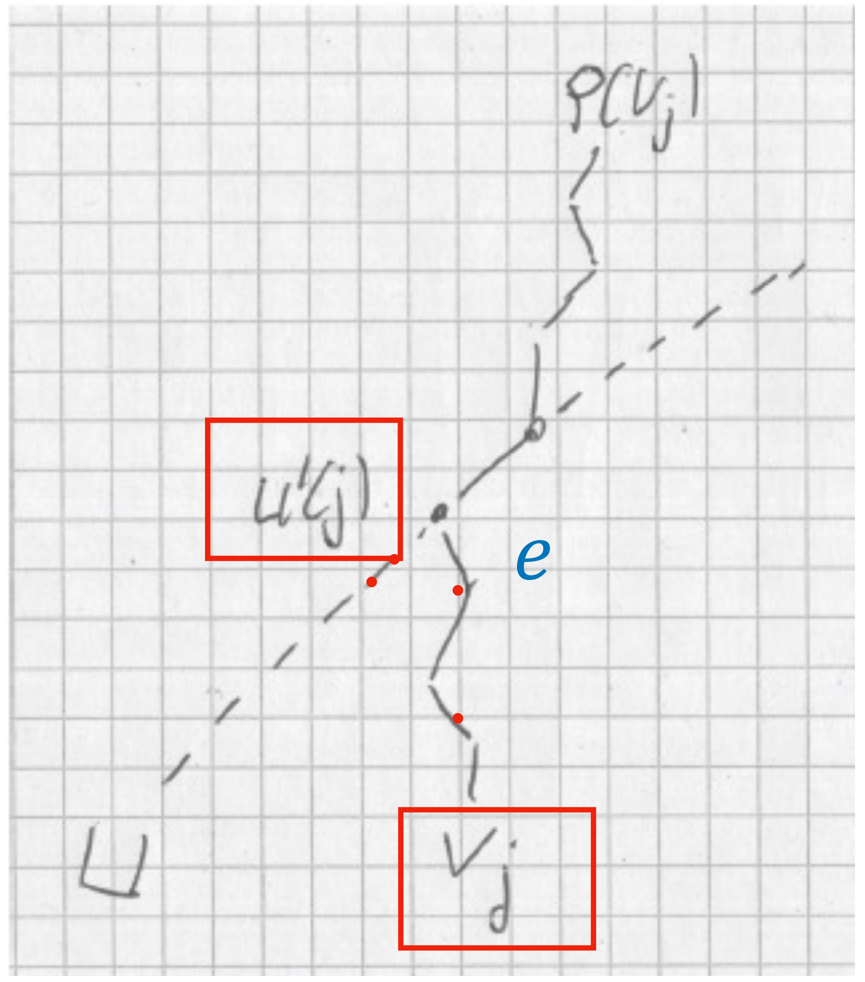

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- we order the remaining

$$m = 2^n - c^* - 1$$

nodes u_i in lexicographic order and store their descriptors

$$D = des(u_1) \circ \ldots \circ des(u_m)$$

We have

$$|D| = n \cdot (2^n - c^* - 1)$$

estimating the length $|d| = |ABCD|$ of the compressed description

- estimating $|d|$:

$$
\begin{aligned}
|d| &= |A| + |B| + |C| + |D| \\
&\leq 2n + c^*/2 + n + c^* \cdot (n-1) + n \cdot (2^n - c^* - 1) \\
&= n \cdot 2^n - \frac{c^*}{2} + 2n \\
&\leq n \cdot 2^n - \frac{c-n}{2} + 2n \quad (c^* \geq c - n) \\
&= n \cdot 2^n + \frac{5n}{2} - \frac{c}{2}
\end{aligned}
$$

estimating the probability of long runs

- estimating $|d|$:

$$
\begin{aligned}
|d| &= |A|+|B|+|C|+|D| \\
&\leq 2n+c^*/2+n+c^*\cdot(n-1)+n\cdot(2^n-c^*-1) \\
&= n\cdot 2^n-\frac{c^*}{2}+2n \\
&\leq n\cdot 2^n-\frac{c-n}{2}+2n \quad (c^*\geq c-n) \\
&= n\cdot 2^n+\frac{5n}{2}-\frac{c}{2}
\end{aligned}
$$

- estimating $\#E$ and $p(E)$

$$
\begin{aligned}
\#E &\leq 2^{|d|} \\
&\leq 2^{n\cdot 2^n-\frac{c-5n}{2}} \\
p(E) &= 2^{-n\cdot 2^n}\cdot \#E \quad (\text{lemma } 1\ ) \\
&\leq 2^{-\frac{c-5n}{2}}
\end{aligned}
$$

estimating the probability of long runs

- estimating $|d|$:

$$\begin{aligned} |d| &= |A|+|B|+|C|+|D| \\ &\leq 2n + c^*/2 + n + c^* \cdot (n-1) + n \cdot (2^n - c^* - 1) \\ &= n \cdot 2^n - \frac{c^*}{2} + 2n \\ &\leq n \cdot 2^n - \frac{c-n}{2} + 2n \quad (c^* \geq c-n) \\ &= n \cdot 2^n + \frac{5n}{2} - \frac{c}{2} \end{aligned}$$

- estimating $\#E$ and $p(E)$

$$\begin{aligned} \#E &\leq 2^{|d|} \\ &\leq 2^{n \cdot 2^n - \frac{c-5n}{2}} \\ p(E) &= 2^{-n \cdot 2^n} \cdot \#E \quad \text{(lemma 1)} \\ &\leq 2^{-\frac{c-5n}{2}} \end{aligned}$$

- instantiating $c = 9$: 9n

$$\boxed{\begin{aligned} p(\{d \mid T(d) \geq 10n\}) &\leq p(E) \\ &\leq 2^{-2n} \\ &= \frac{1}{N^2} \quad (\text{number of nodes } N = 2^n) \end{aligned}}$$

round 2

The probability of runs longer than $n+c$ can be estimated exactly as in phase 1. The previous argument is adapted as follows.

- start points of routes u and v_j are replaced by final destinations $\pi(u)$ and $\pi(v_j)$ of packets on ascending routes from intermediate destination $\rho(v_j)$ to $\pi(v_j)$.
- as collision edges one takes the *last* edge (i.e. with the largest dimension) shared by colliding pathes.
- descriptions $des[1:n]$ of paths specify as before the dimensions, in which edges are traversed, but to completely specify the path, the endpoints (of the form $\pi(v)$) are specified.

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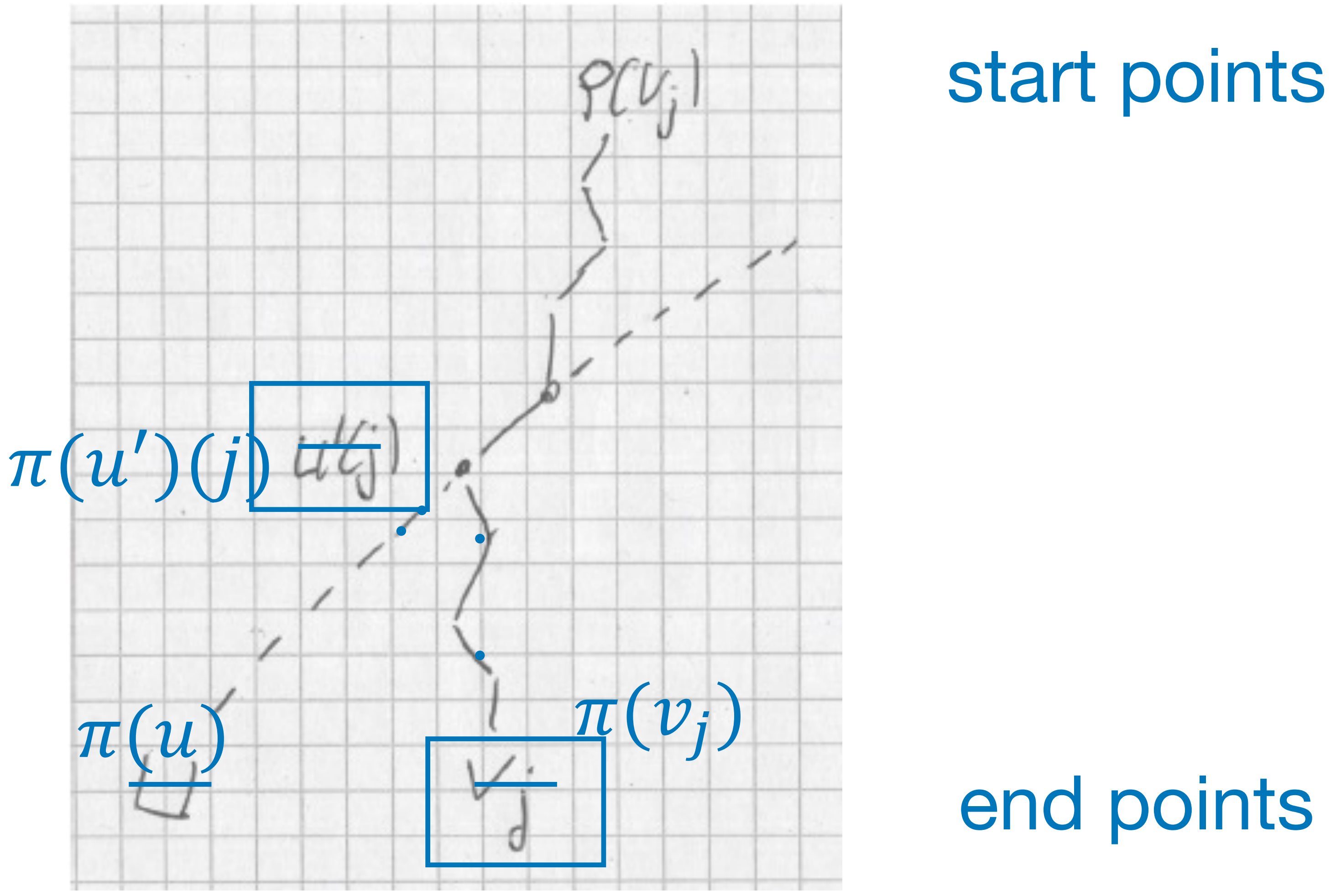
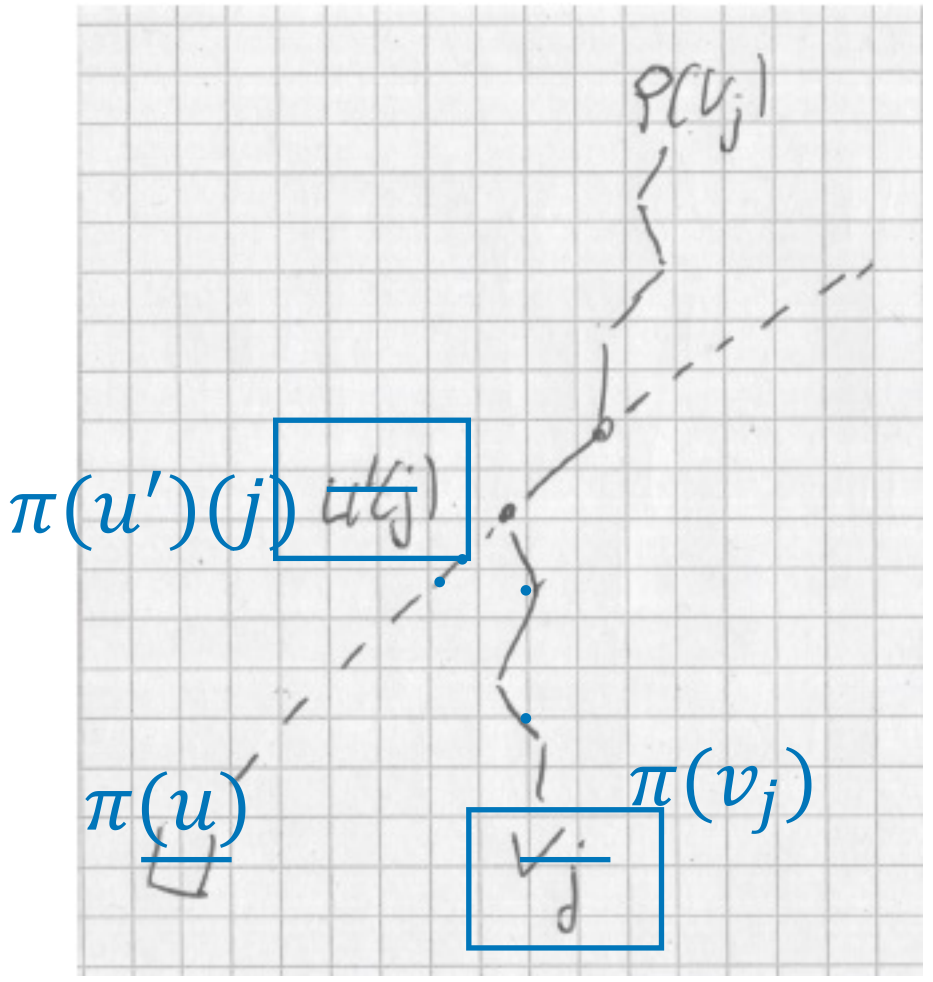


- reconstruction of an endpoint $\pi(v_j)$ from a compressed description $des'(\pi(v_j))$ and known collision dimendsion i is done with the *last* bits of the descriptions.

$$
\begin{aligned}
\pi(u)'(j) &= \pi(u) \oplus 0^i \circ des(\pi(u))[n-i+1:n] \\
\pi(v_j) &= \pi(u)'(j) \oplus 0^i \circ des'(\pi(v_j))[n-i+1:n]
\end{aligned}
$$

round 2



start points

end points

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\end{aligned}
$$

- this gives short descriptions d' of all paths in phase 2, in particular all intermediate destinations $\rho(v)$. From them one gets the routes and the sequence of random bits d generated in phase 1.