graph algorithms 2

minimum spanning tree and shortest paths

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$\{u, v\} \in T' \rightarrow u, v \in V' \vee u, v \in V''$

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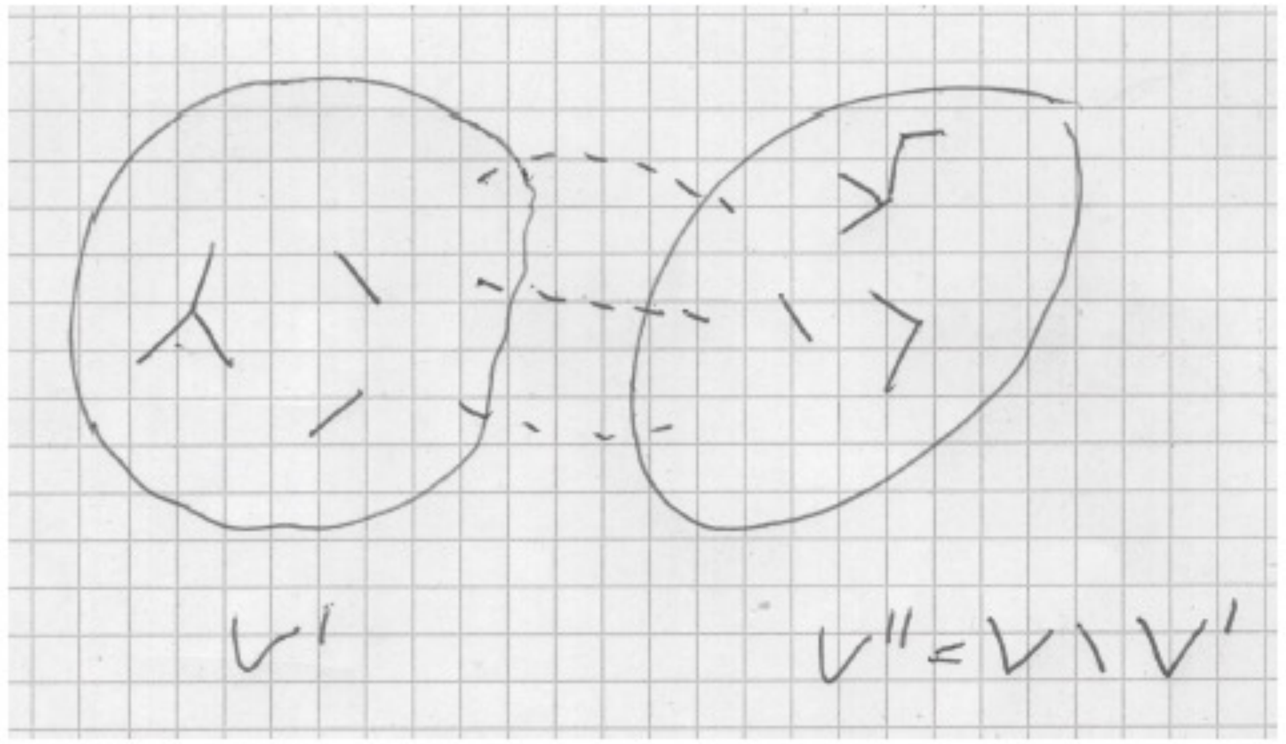

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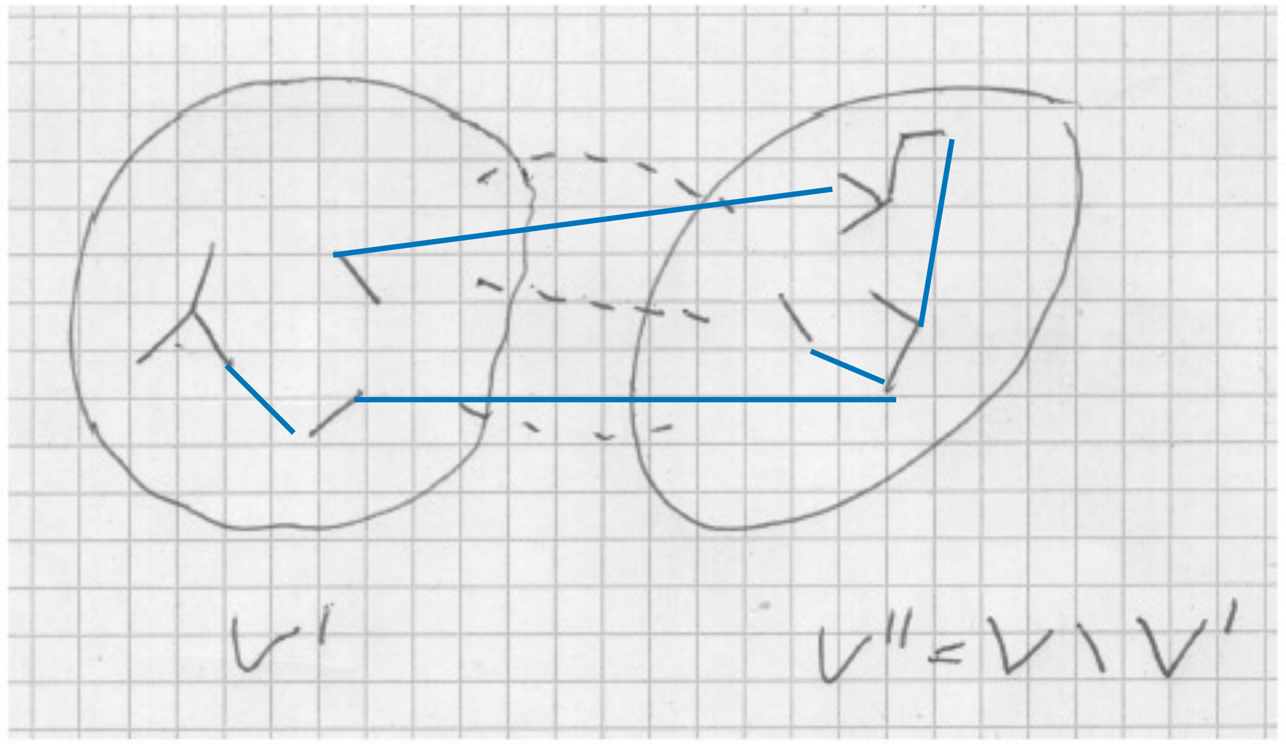

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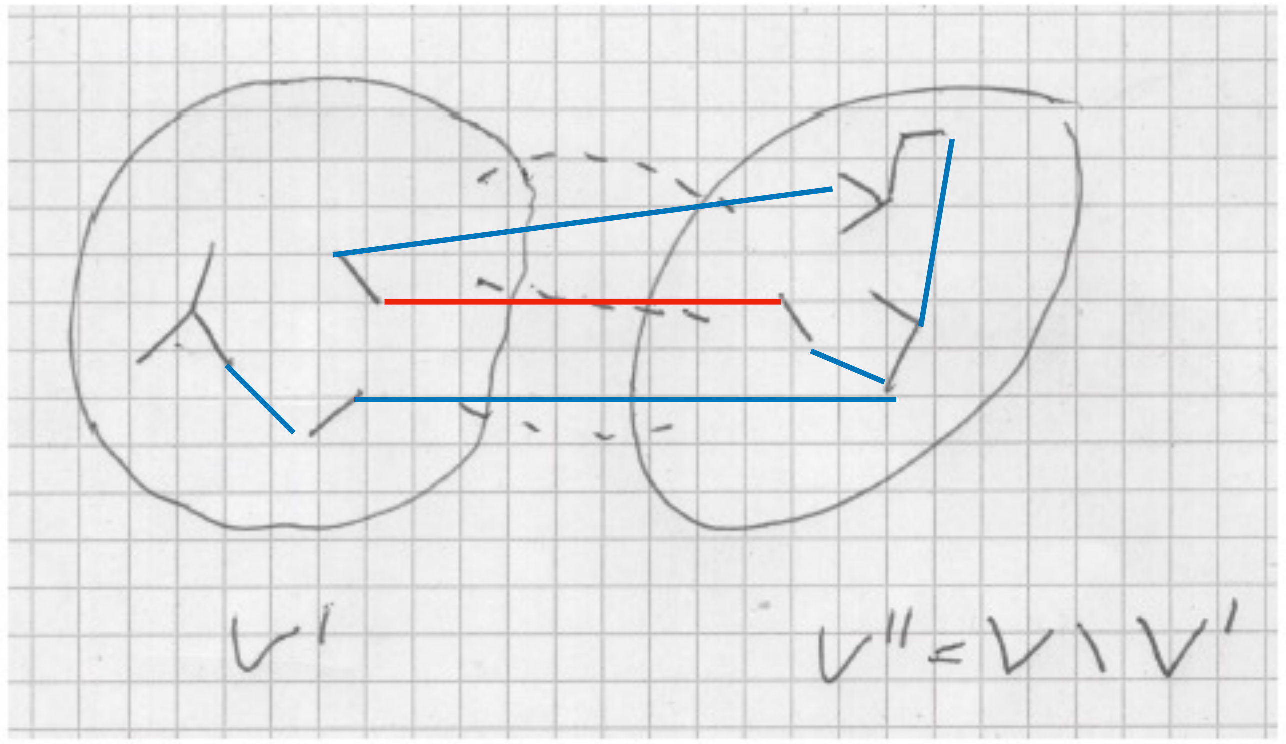

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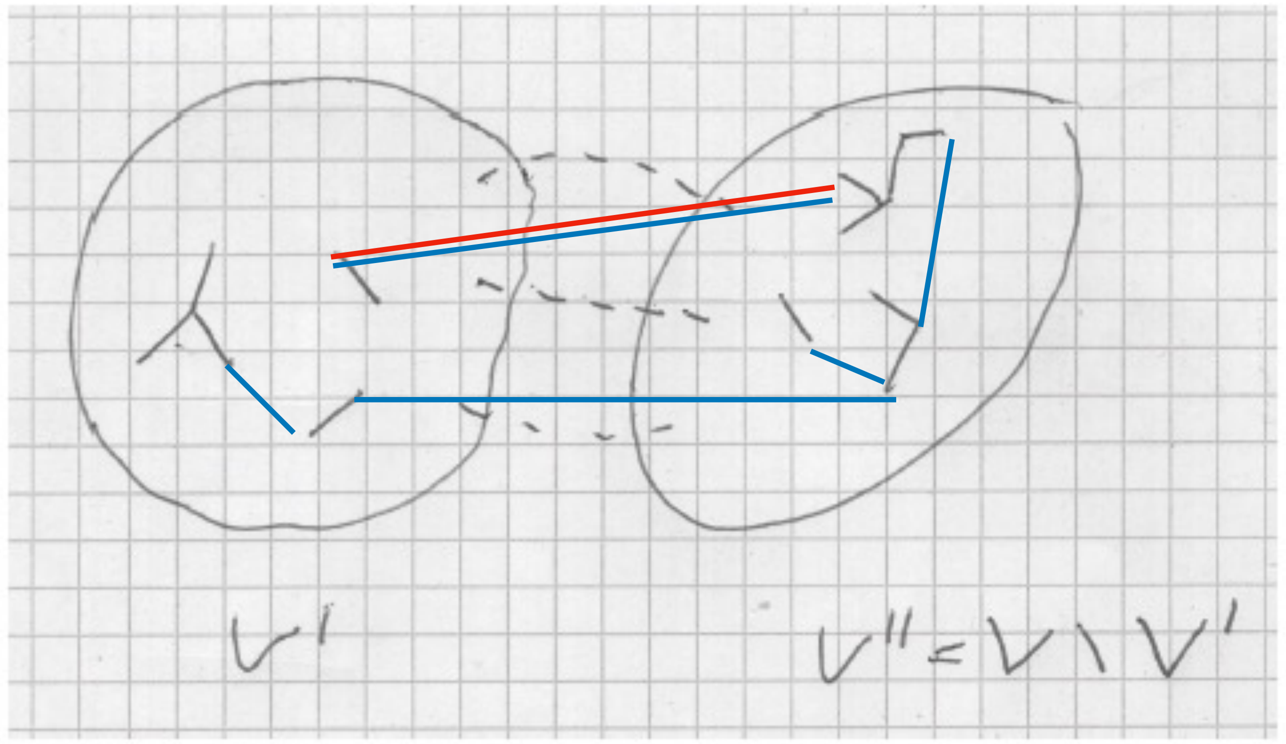

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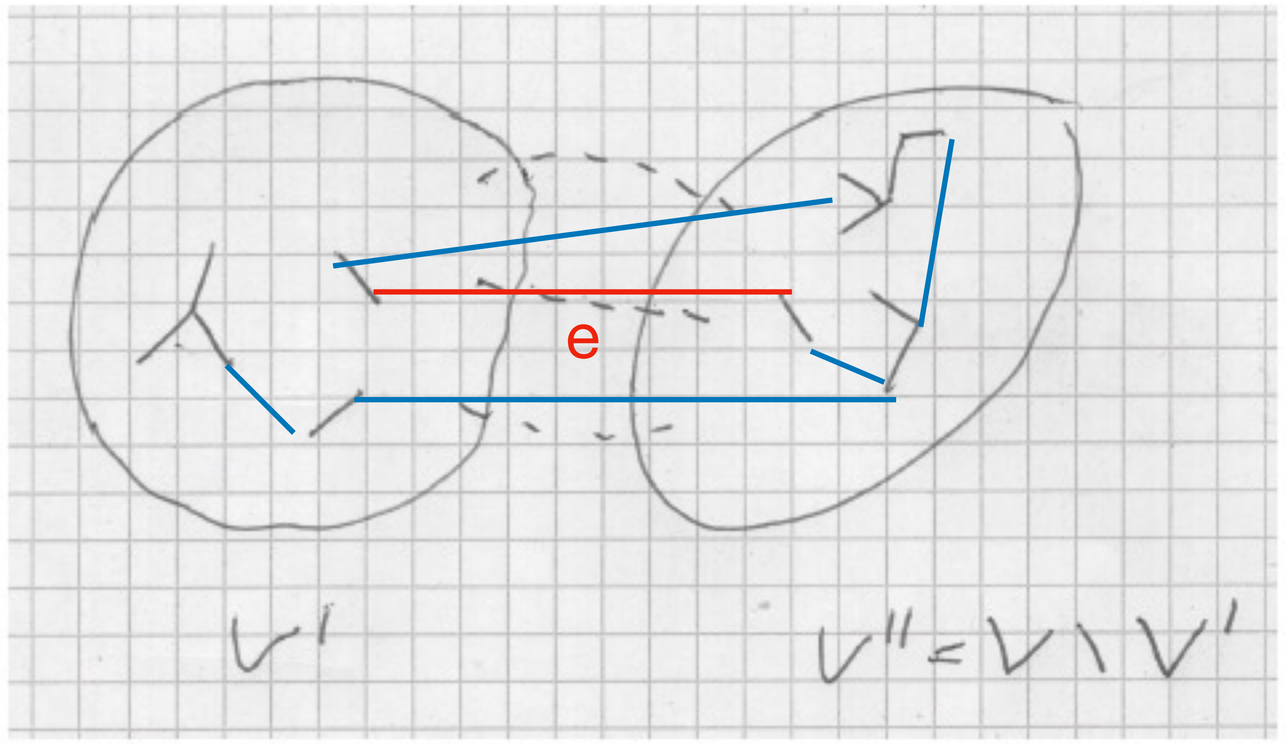

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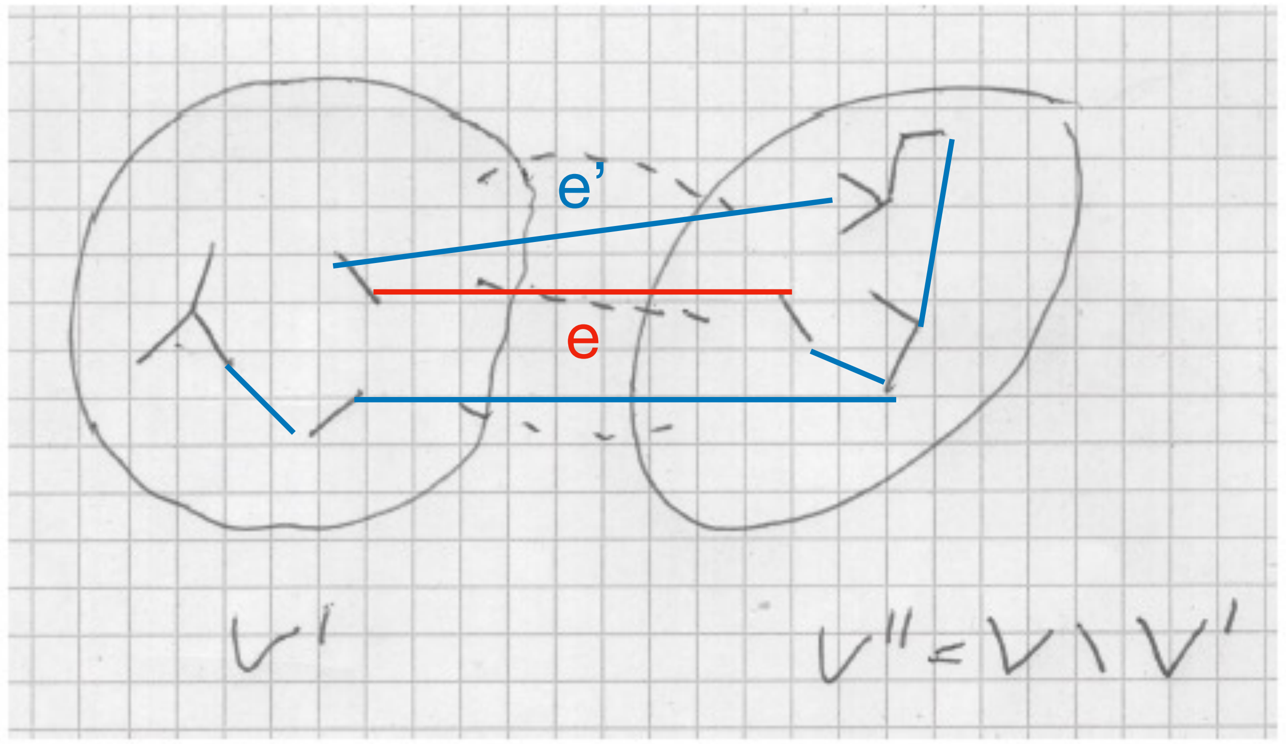

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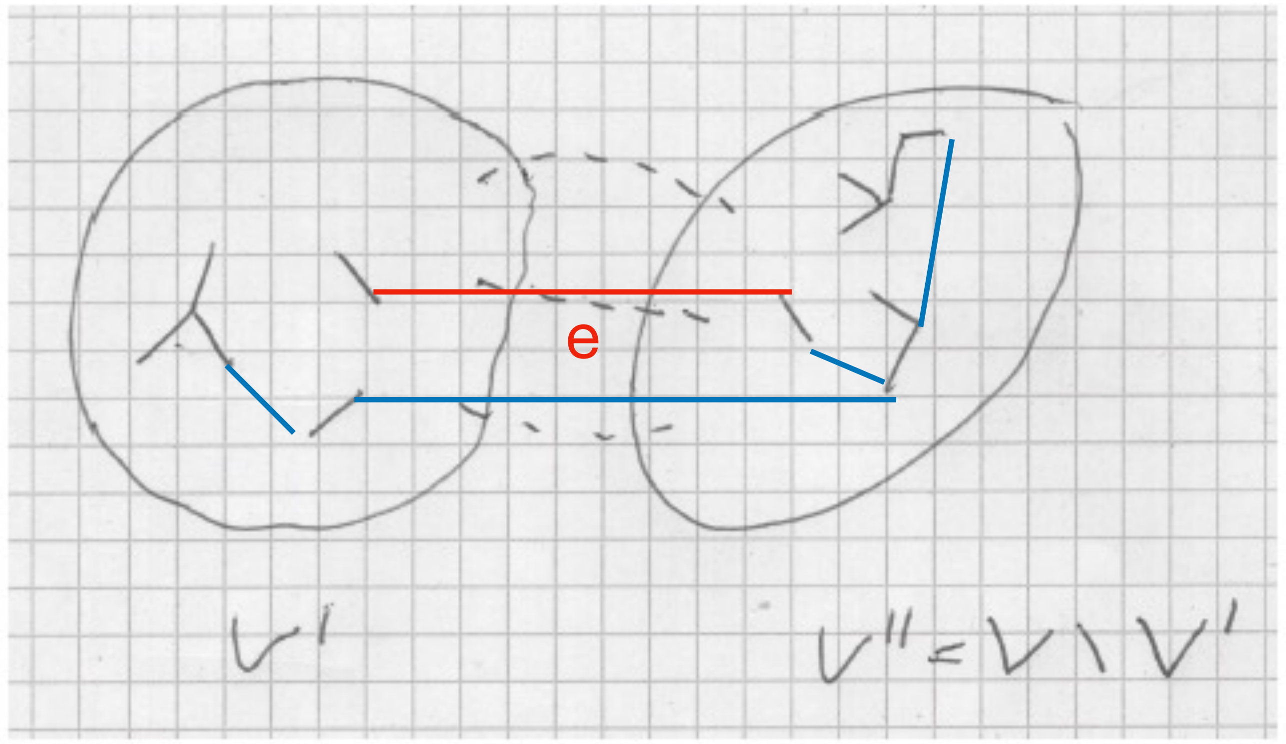

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Kruskal’s Algorithm

Algorithm 50 Kruskal-MST

Input: connected weighted undirected graph $G = (V, E, w)$
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for each vertex v ∈ V do
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- T contained in edges of minimum spanning tree
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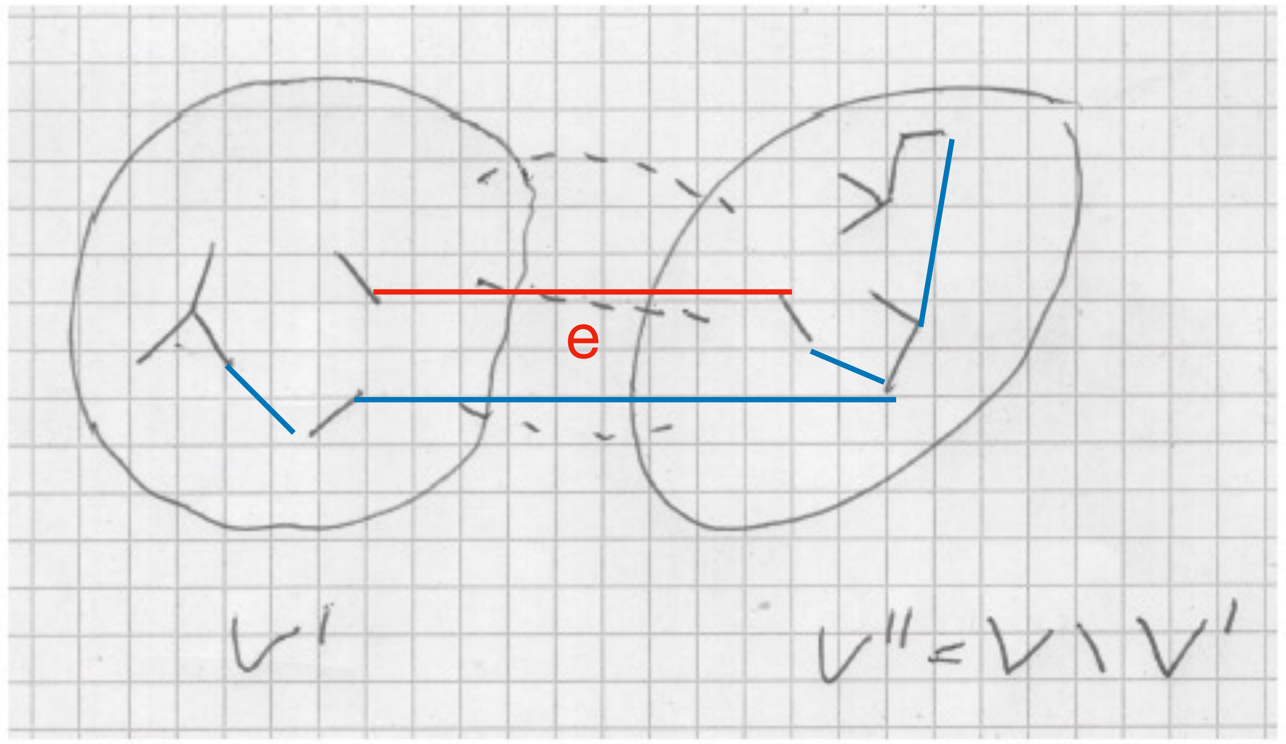

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greedy

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greedy algorithms:

- go locally for optimum
- hope globally for the best

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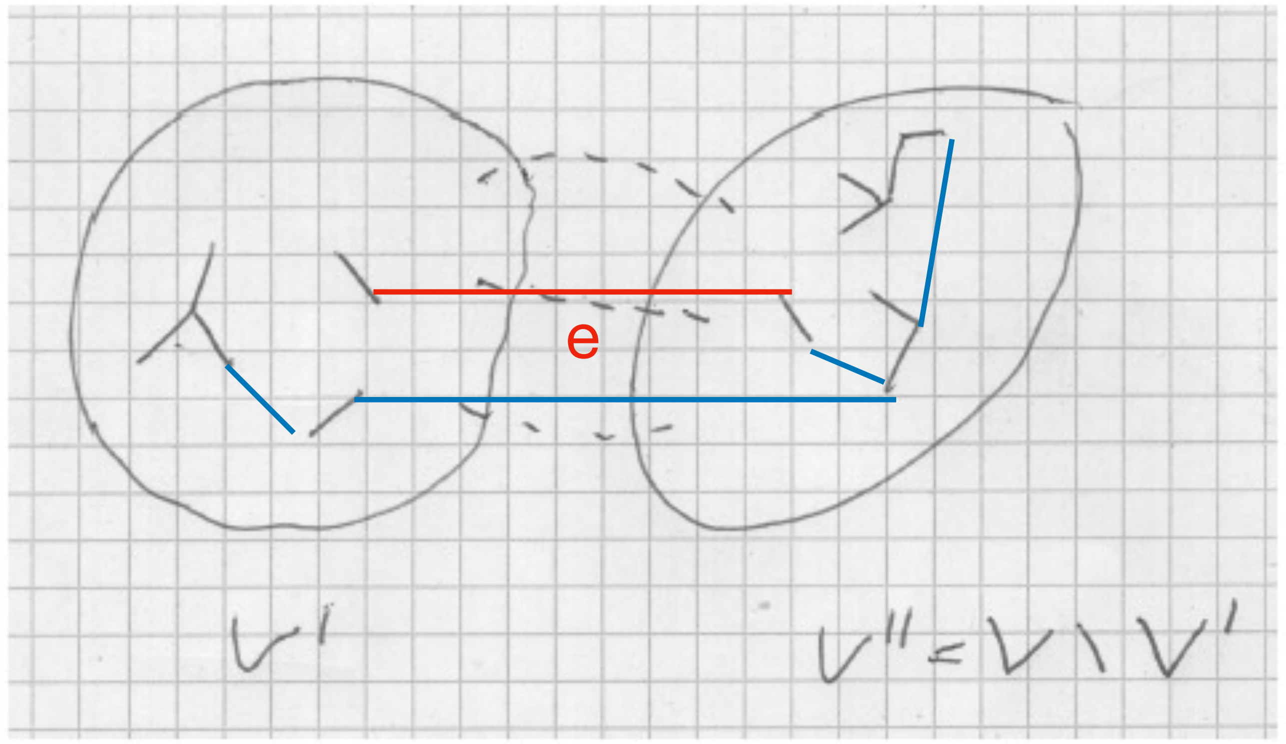

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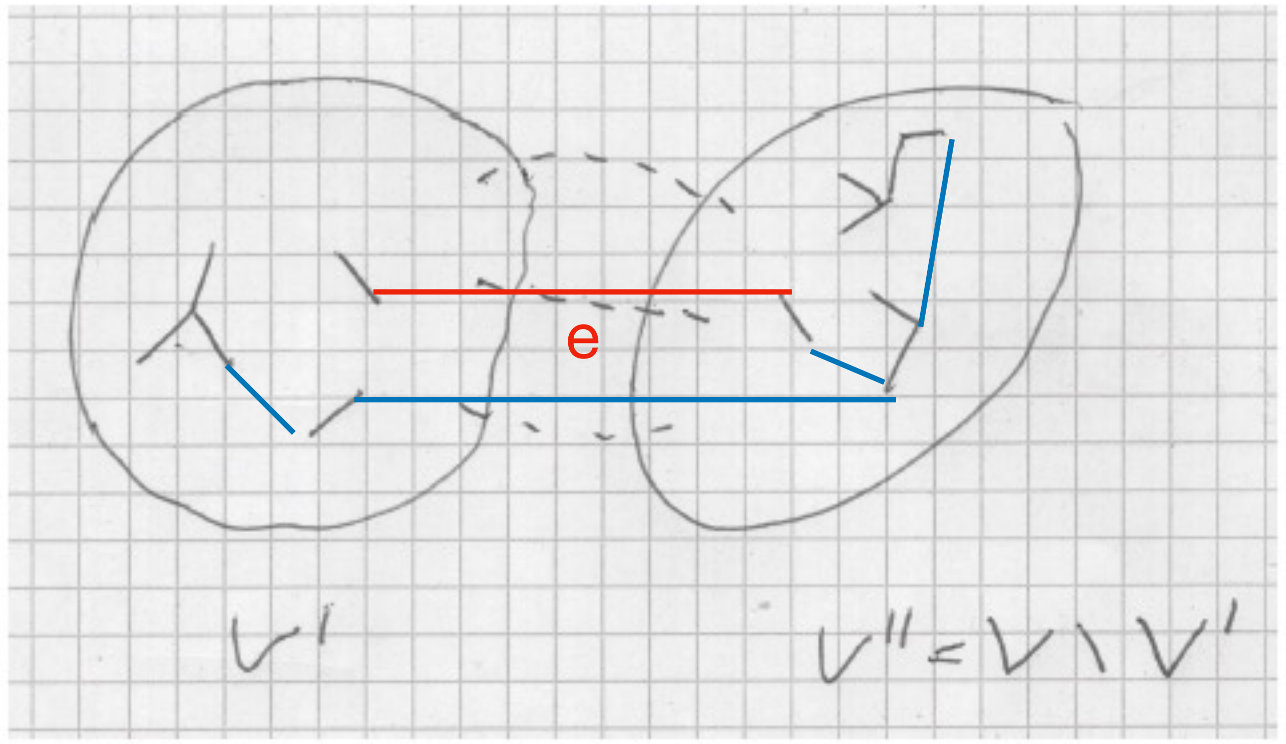

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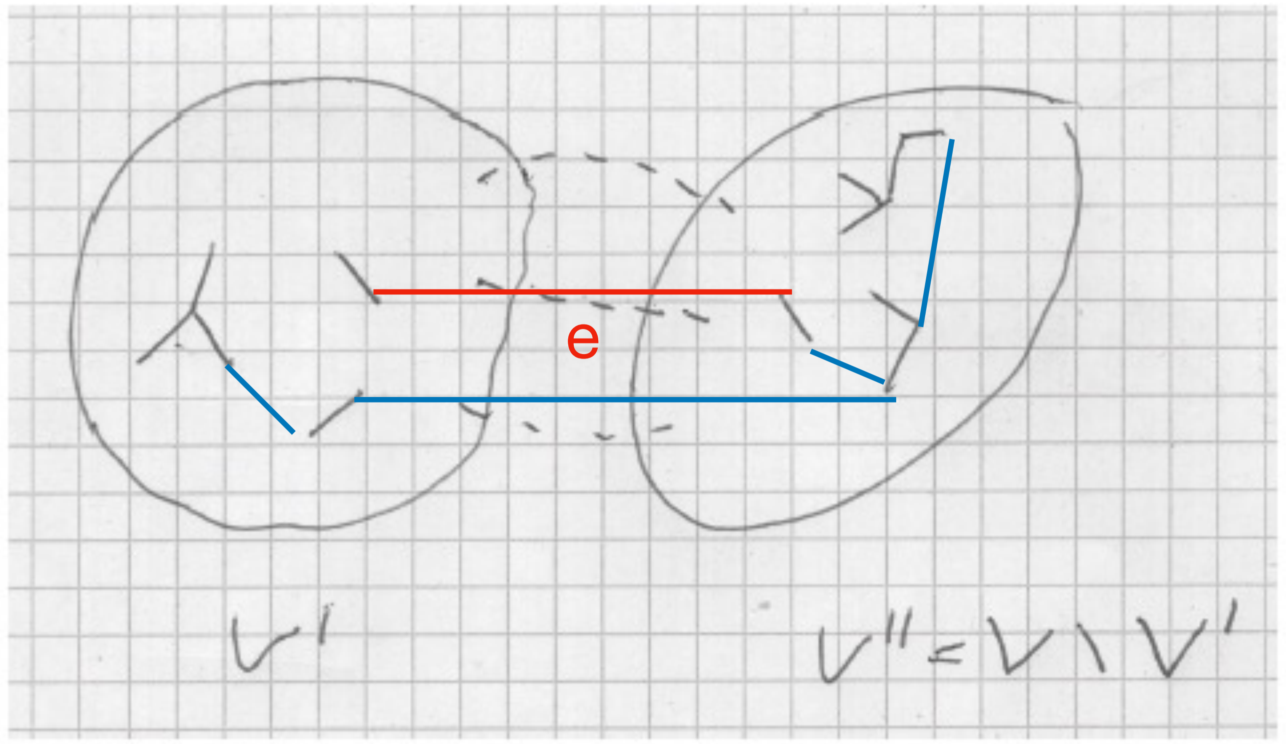

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length of paths

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$\Sigma_{i=1}^{\ell} w(p_{i-1}, p_i)$ length *of path*

sum of edge weights

single source shortest path

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algorithm

- input: weighted connected graph, start node $s \in V$
- output: for each node $\delta[v]$ =
  length of a shortest path from s to v

Dijkstra's Algorithm

idea:

- grow set of nodes S
- obtain upper bounds $d[x] \geq \delta(x)$ on length of shortest path
- $d[x]$ length of shortest path found so far, i.e in S or in S plus one further edge.

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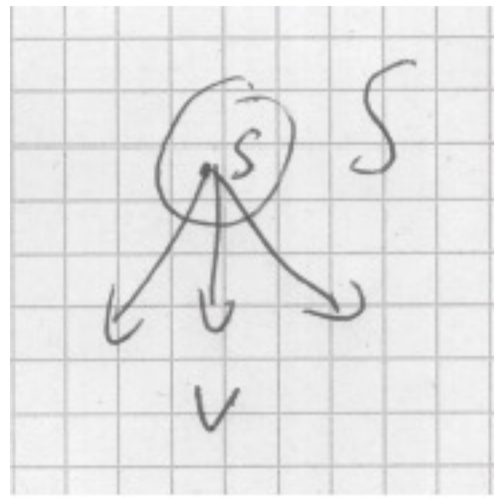

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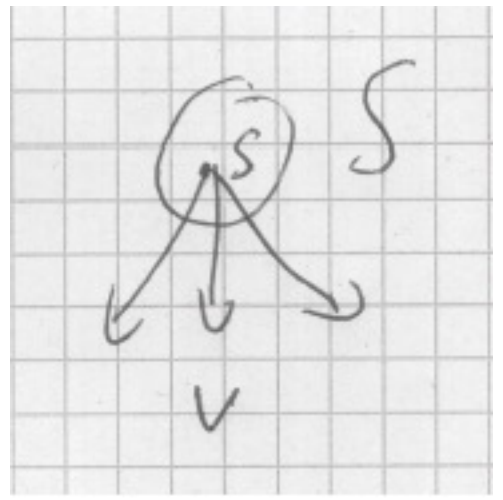

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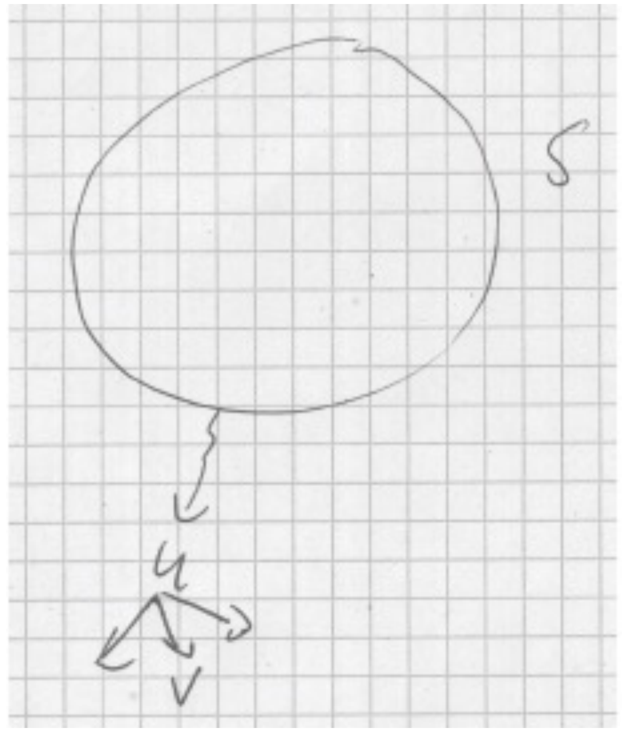

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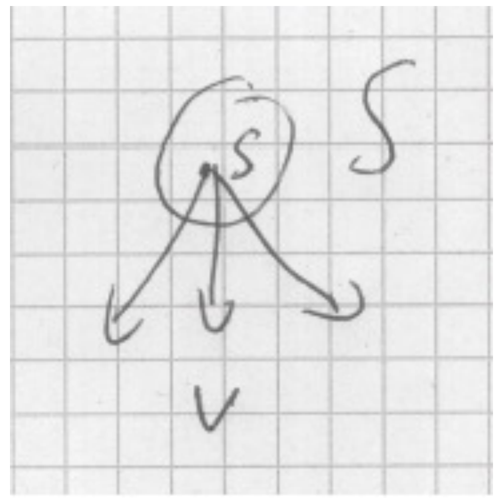

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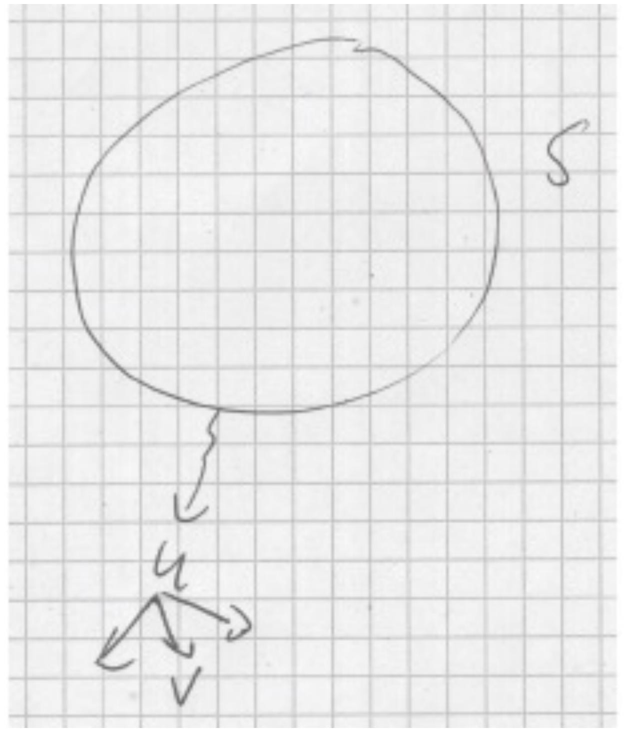

primed notation

- S^t and d^t: set S and distance bound d before iteration t of the while loop.
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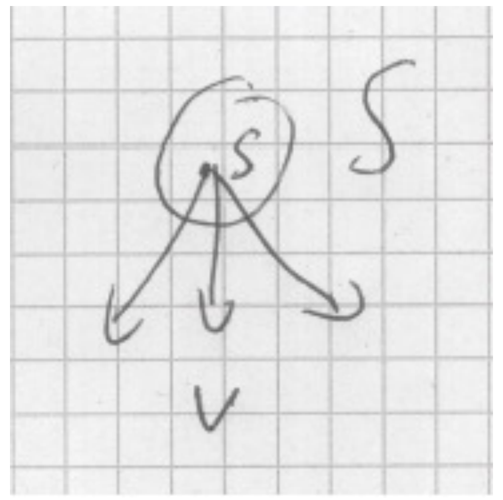

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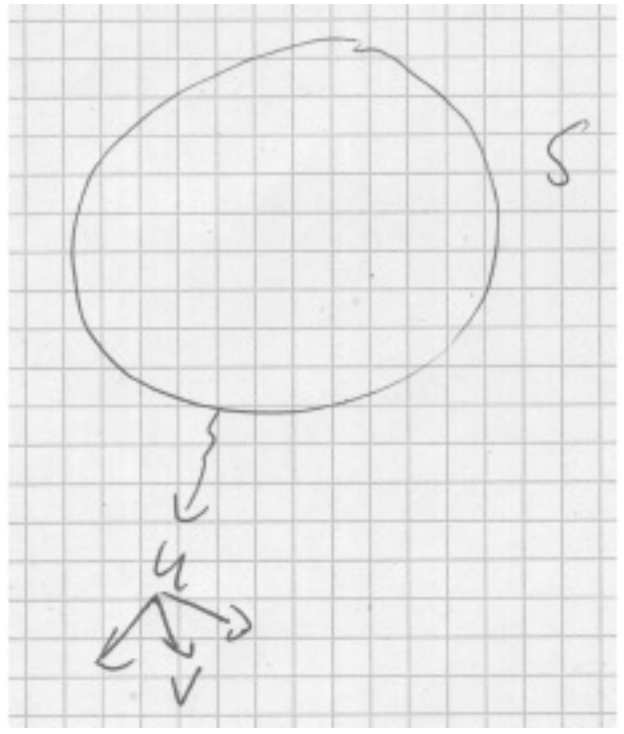

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properties

1. shortest discovered distances nonincreasing; stable for nodes $u \in S$

$$d'[u] \leq d[u] \ , \ u \in S \rightarrow d'[u] = d[u]$$

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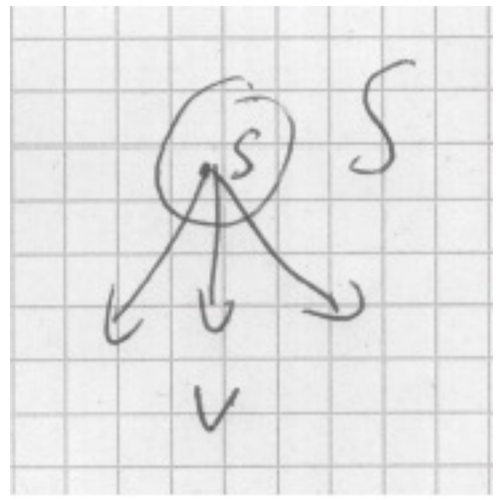

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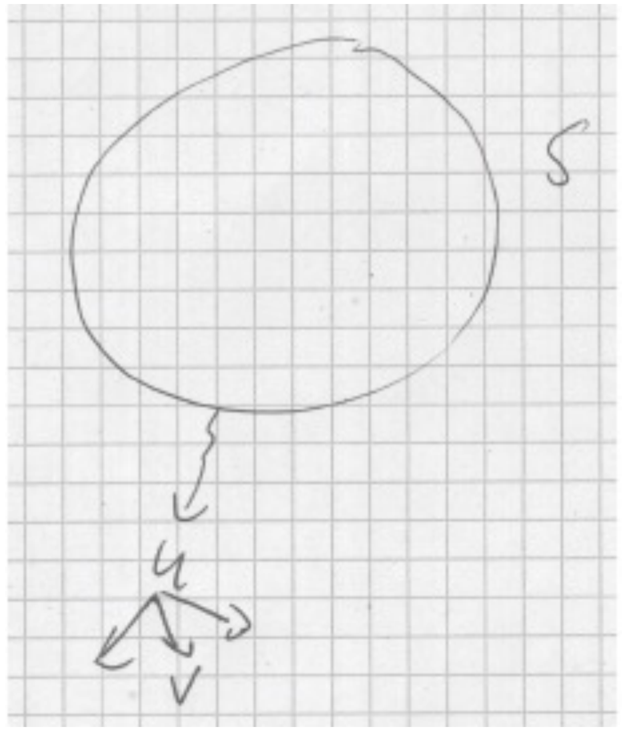

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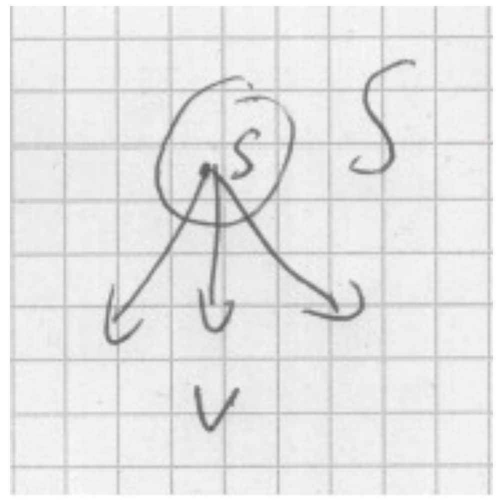

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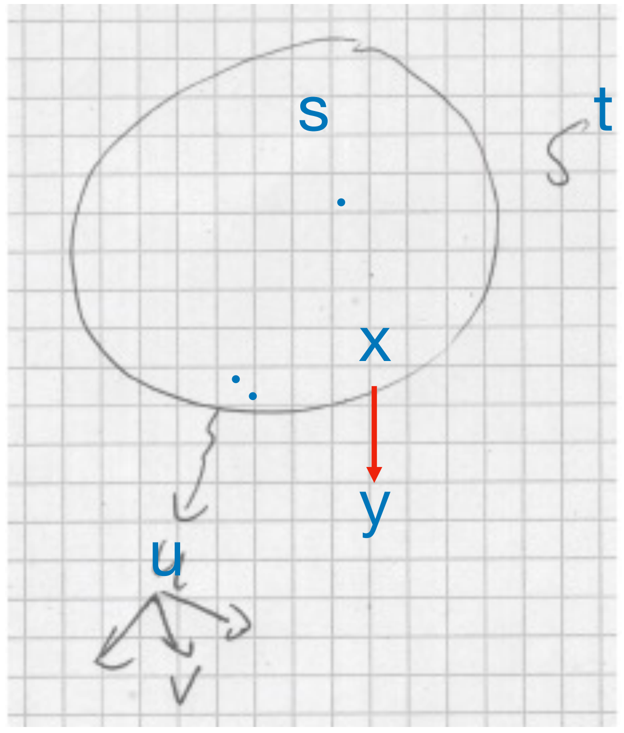

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$u = s$ trivial. For $u \neq s$ assume in some iteration of while loop:

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Consider *first* such t and u. Let P be shortest path from s to u.

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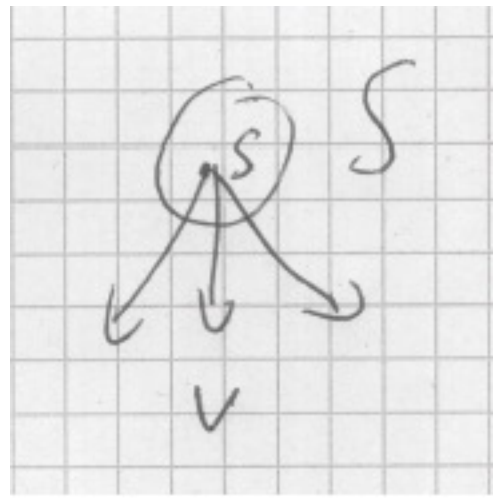

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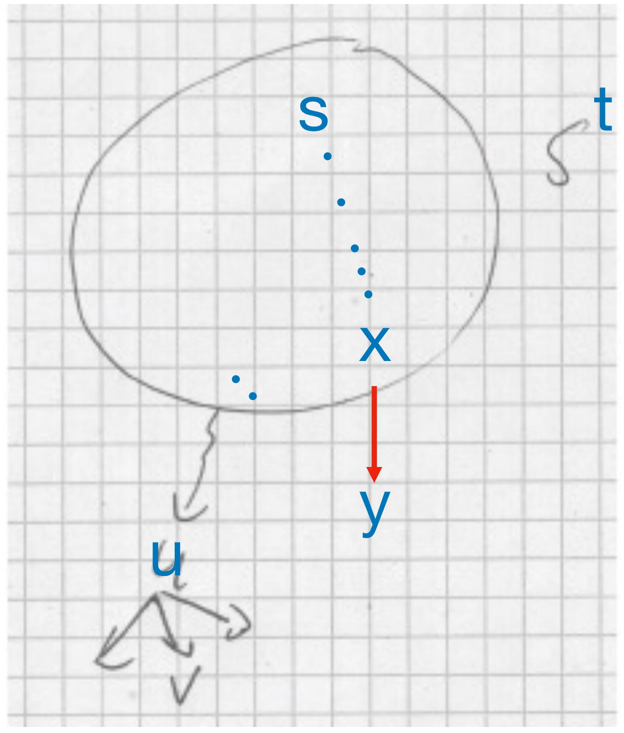

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$$x \in S^{r+1} \setminus S^r\ ,\ r < t$$

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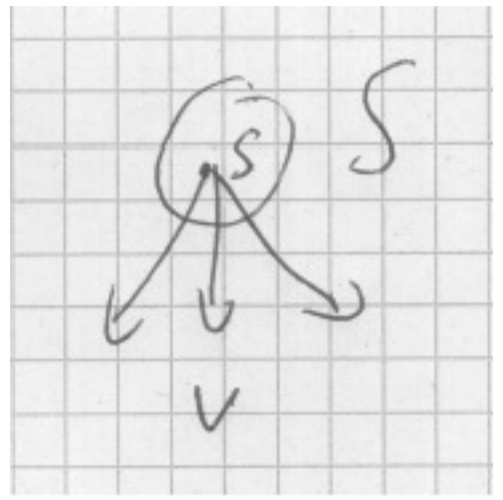

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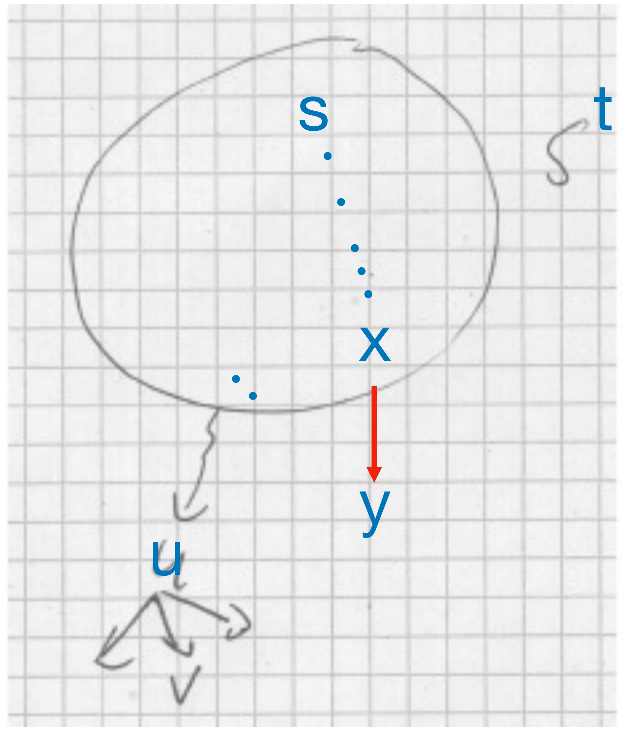

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$$\begin{aligned} d^t[y] &\leq d^r[y] && (\text{property 1}) \\ &\leq d^r[x] + w(x,y) && (\text{algorithm}) \\ &= \delta[x] + w(x,y) && (\text{above}) \\ &= \delta[y] && ((x,y) \text{ on shortest path } P) \\ &\leq d^t[y] && (\text{property 2}) \end{aligned}$$

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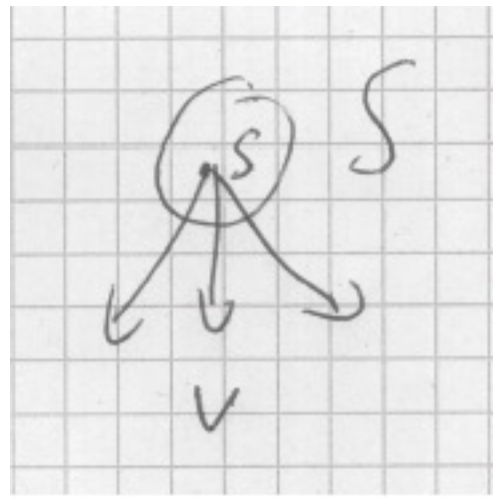

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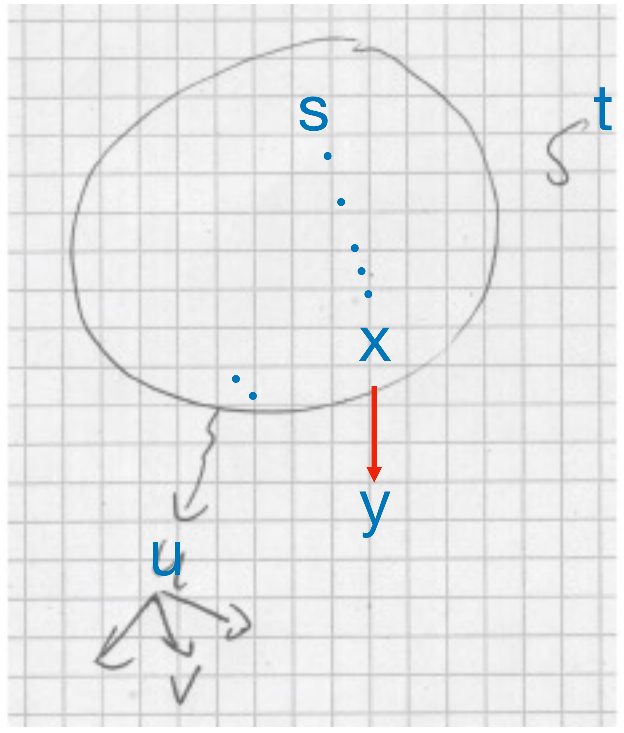

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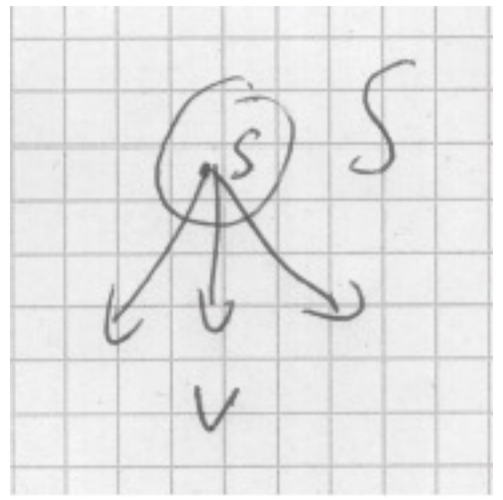

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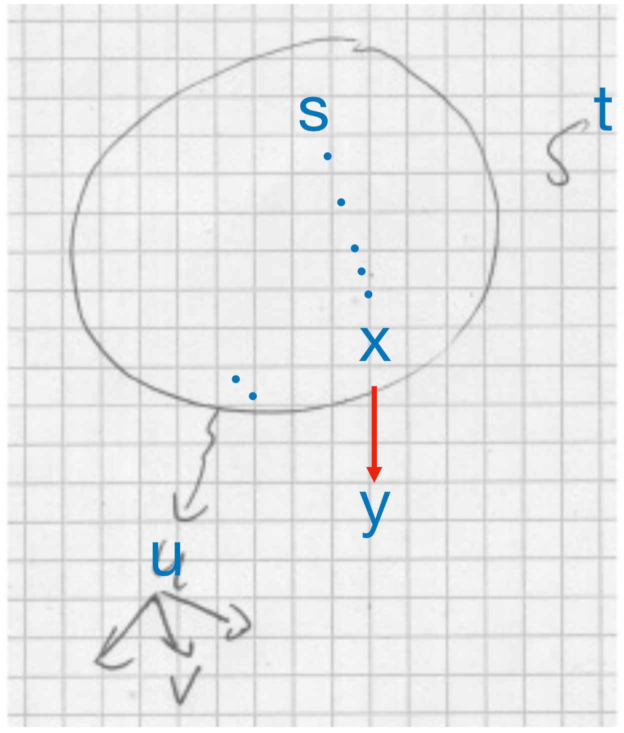

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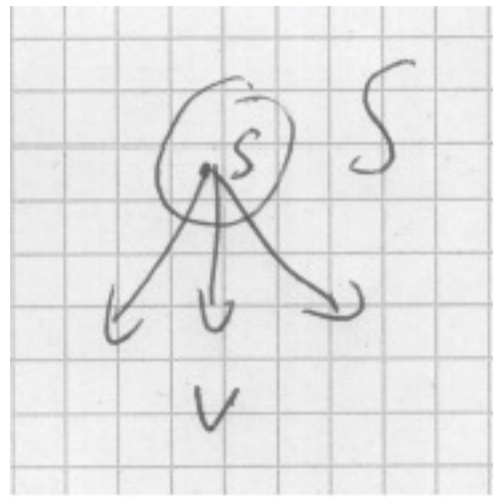

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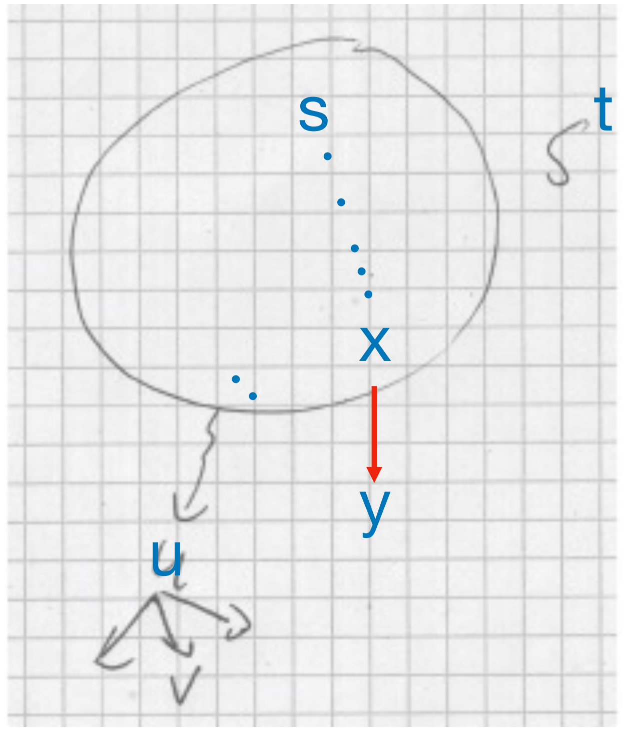

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As $u \notin S^t$ is chosen in pass t with minimal $d[u]$

$$d^t[u] \leq d^t[y] \quad \text{contradiction}$$

run time

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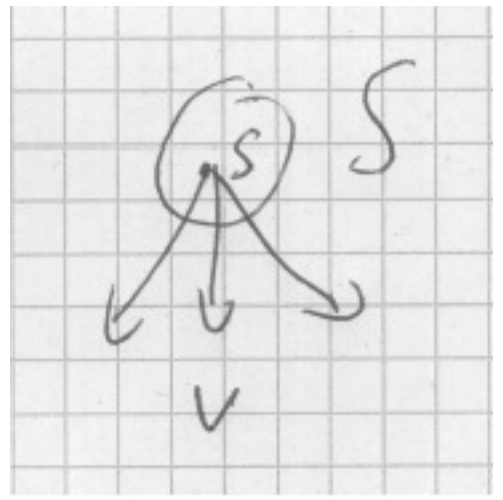

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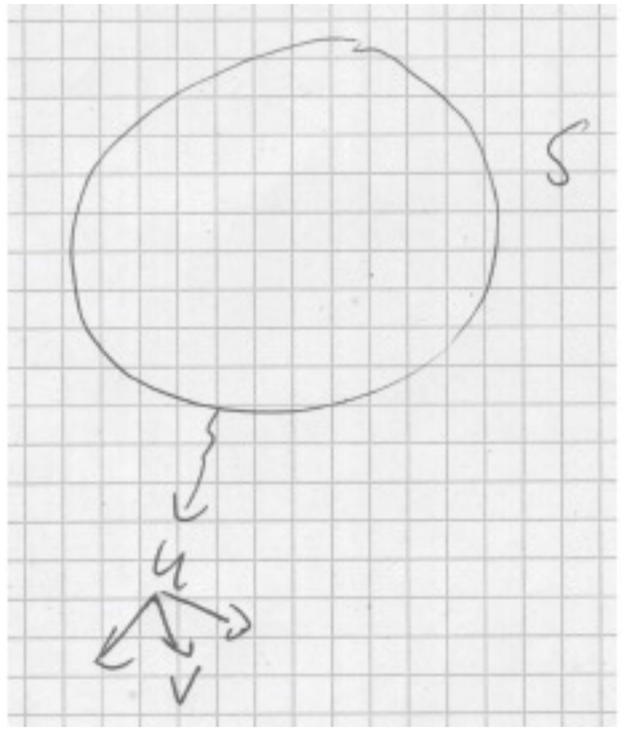

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data structure for $V \setminus S$

balanced search tree: AVL or 2/3

records

u	d[u]	

use d[u] as key;
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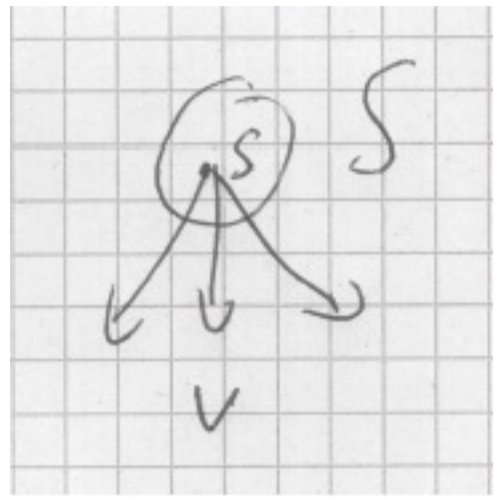

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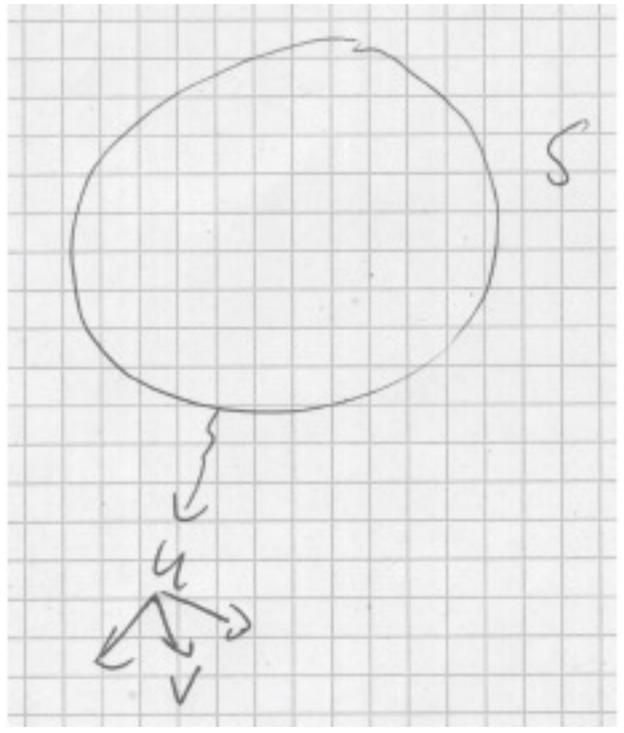

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at most |V| insert operations

iterations

at most |E| operations

find
- ~~delete~~ min (go over lef edges or sth.)
- delete ~~v~~ u
- ~~insert v with~~ d'[v]
update

run time

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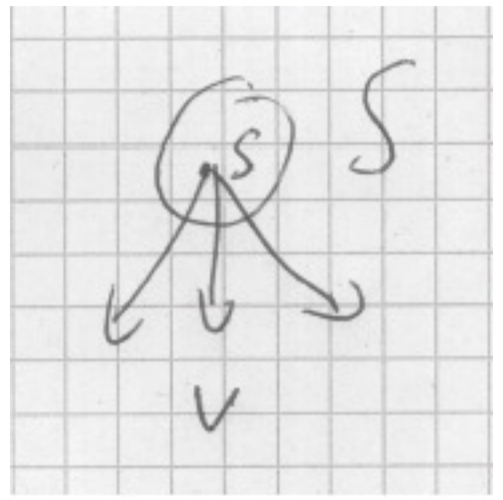

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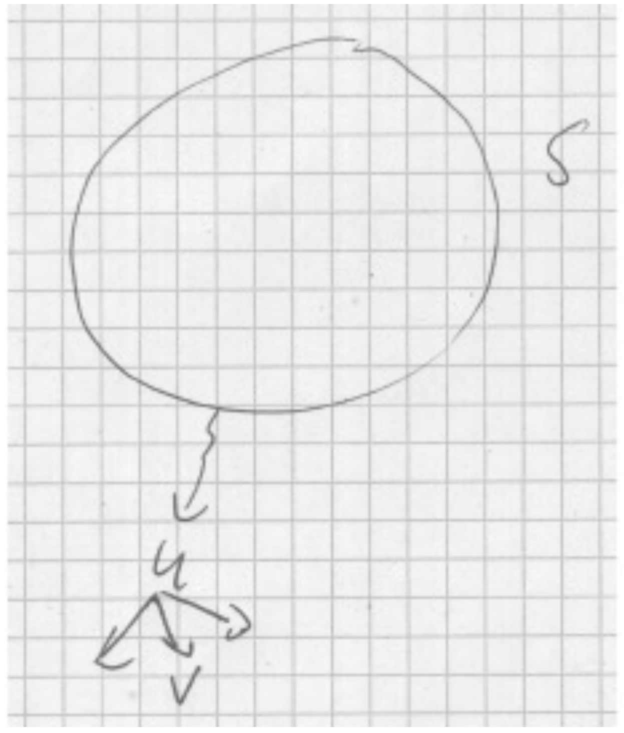

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This log V time for E operat.

iterations

at most |E| operations
- delete min
- delete v
- insert v with d'[v]

nodes of tree maintaining V\S at most V.

$\boxed{\text{run time = } O(|E|\log|V|)}$