Theme 2: Continuity (continuation). Limits at infinity. Horizontal asymptotes. Derivatives and the rate of change. The derivative as a function. Sections 2.5 (part 2), 2.6, 2.7 and 2.8

Definitions, methods, formulas, theorems:

Definitions:

1) Removable, jump and infinite discontinuity.

2) **Continuity on an interval.** A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

3) **Intuitive Definition of a Limit at $+\infty$.** Let f be a function defined on some interval $(a, +\infty)$. Then $\lim\limits_{x\to\infty} = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

4) **Intuitive Definition of a Limit at $-\infty$.** Let f be a function defined on some interval $(-\infty, a)$. Then $\lim\limits_{x\to-\infty} = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

4) **Definition.** The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L.$$

1. **Definition (Tangent line).** The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$, is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}.$$

Another expression of the slope:

$$m = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}.$$

2. **Definition.** (Average velocity):

$$\text{Average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}.$$

3. **Definition.** (instantaneous velocity)
The velocity (or instantaneous velocity) $v(a)$ at time $t = a$ to be the limit of these average velocities:

$$v(a) = \lim_{h\to 0} \frac{s(a+h) - s(a)}{h}.$$

4. **Definition.** (Speed)

Speed: $|v(a)| = |s'(a)|$.

5. **Definition.** (Derivative as a function)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

6. **Definition.** (Differential operators):

Notation:

$$f'(x) = y' = \frac{dy}{dx} = \frac{dy}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x).$$

7. **Definition.** (Differentiability)

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or $(-\infty, a)$, $(-\infty, \infty)$ if it is differentiable at every number in the interval.

8. **Definition.** (Higher Derivatives)
Higher Derivatives: $f''(x) = (f'(x))'$; $f^{(n)}(x) = f^{(n-1)}(x)$.

Theorems without proofs.

1) **Direct Substitution Property:** If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

2) **Theorem.** If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:
a) $f + g$;
b) $f - g$;
c) cf;
d) fg;
e) $\frac{f}{g}$ provided that $g(a) \neq 0$.

3) **Theorem.**
(a) Any polynomial is continuous everywhere; that is, it is continuous on $(-\infty, \infty)$;
(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

4) **Theorem.** The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- rational functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

5) **Theorem.** If f is continuous at b and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$, i.e.,

$$\lim_{x \to a} f(g(x)) = f\Big(\lim_{x \to a} g(x)\Big).$$

6) **Theorem.** If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by the formula $f \circ g(x) = f(g(x))$ is continuous at a.

7) **Theorem.** If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0.$$

Theorems with proofs.

The Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ sad and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in $[a, b]$ such that $f(c) = N$.

1. **Theorem**. If f is differentiable at a, then f is continuous at a.

Proof. Since the limit exists:

$$f'(a) \lim_{x\to a} \frac{f(x) - f(a)}{x - a},$$

we have

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}(x-a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \lim_{x\to a}(x-a) = f'(a)\cdot \lim_{x\to a}(x-a) = f'(a)\cdot 0 = 0.$$

Problems of theoretical type; puzzles

Problems about transformations and combinations of functions;

Assessment: puzzles; online puzzles.

Resources:

Textbook: Stewart's Calculus; 9-the edition: Sections 2.5 (remaining part), 2.6, 27, 2.8

8. Total hours needed:

- Lecture 2
- Central Exercise 1
- TTF 1
- Office hour 1+1.