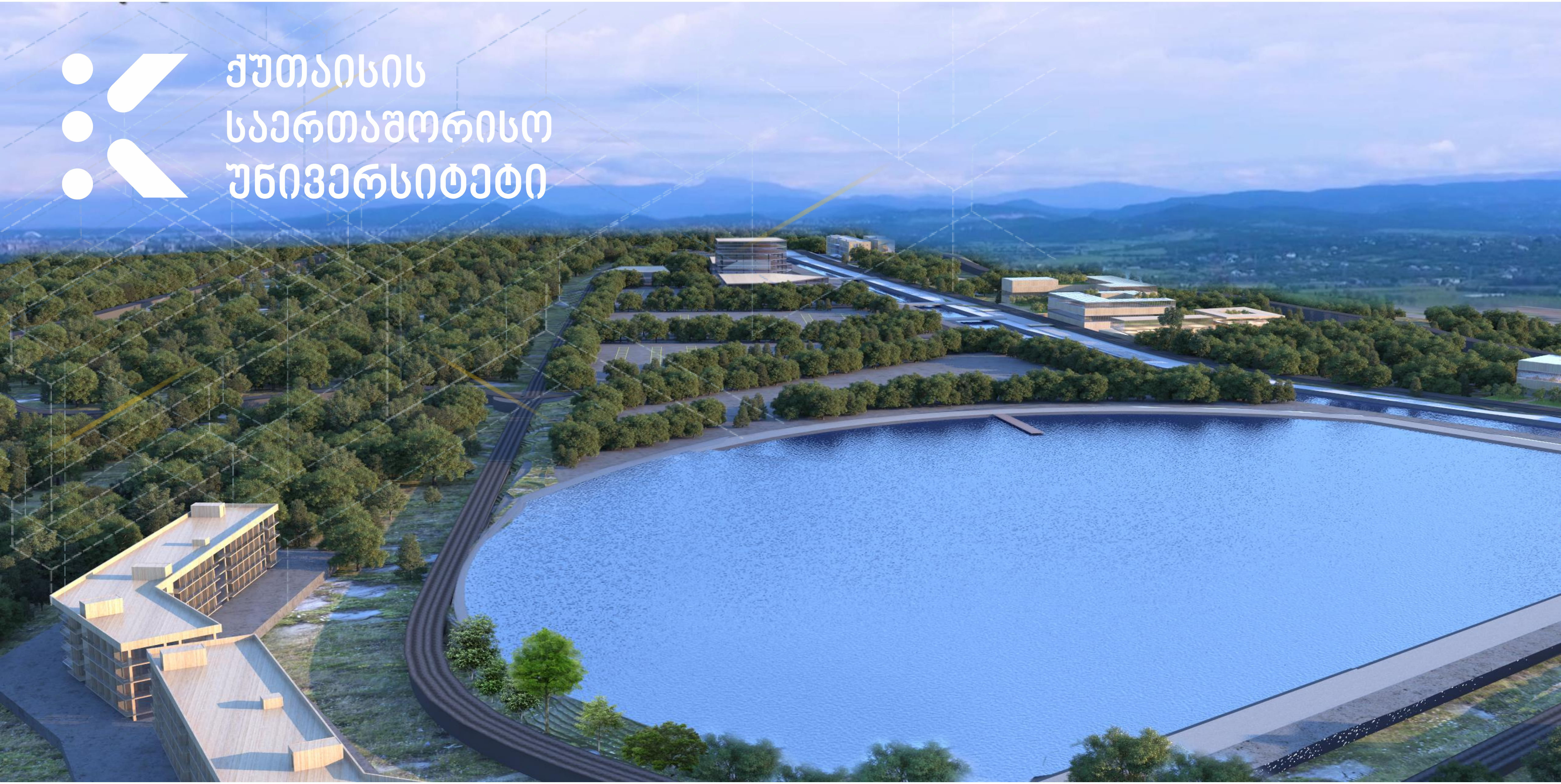

ქუთაისის
საერთაშორისო
უნივერსიტეტი

2. Limits and Derivatives

2.7 Derivatives and Rates of Change

Derivatives and Rates of Change (1 of 1)

This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

Tangents

Tangents (1 of 8)

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a.

Tangents (2 of 8)

If m_{PQ} approaches a number m, then we define the *tangent line* ℓ to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

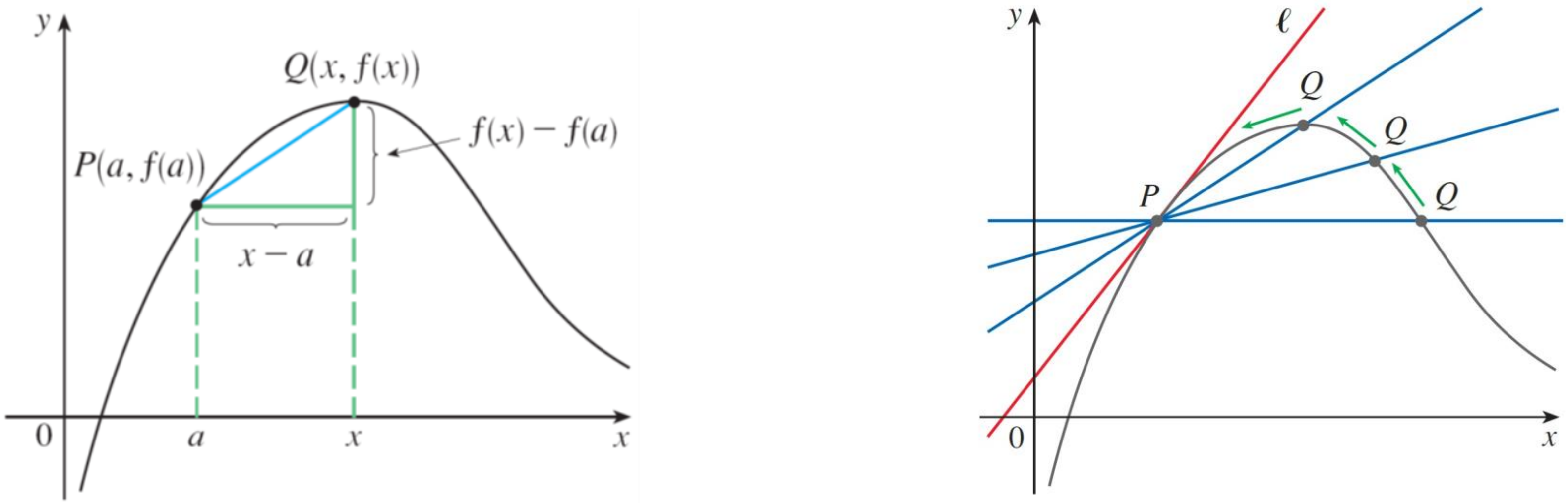


Figure 1

Tangents (3 of 8)

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Example 1

Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

Solution:

Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1}$$

Example 1 – Solution

$$
\begin{aligned}
&= \lim_{x \to 1} (x+1) \\
&= 1+1 \\
&= 2
\end{aligned}
$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at (1, 1) is

$$y - 1 = 2(x - 1) \quad \text{or} \quad y = 2x - 1$$

Velocities

Velocities (1 of 4)

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t.

The function f that describes the motion is called the **position function** of the object.

In the time interval from $t = a$ to $t = a + h$ the change in position is $f(a + h) - f(a)$.

Velocities (2 of 4)

See Figure 5.

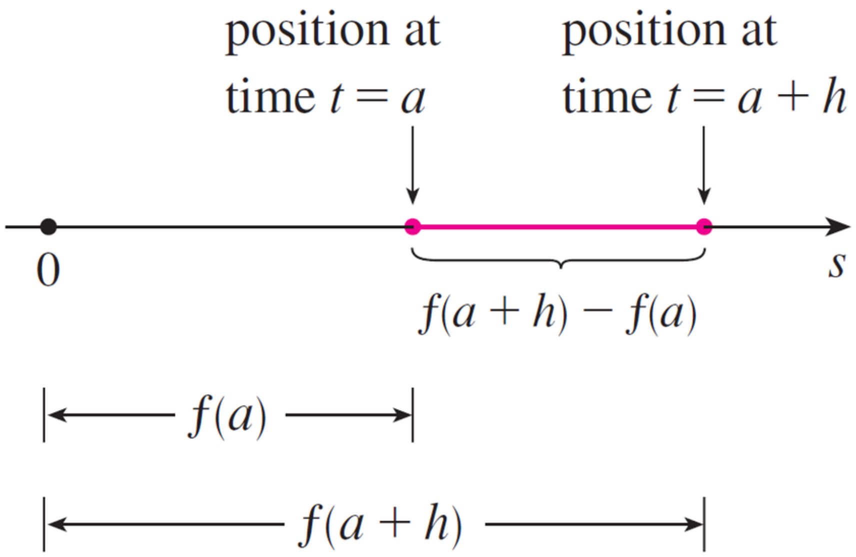


Figure 5

Velocities (3 of 4)

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line *PQ* in Figure 6.

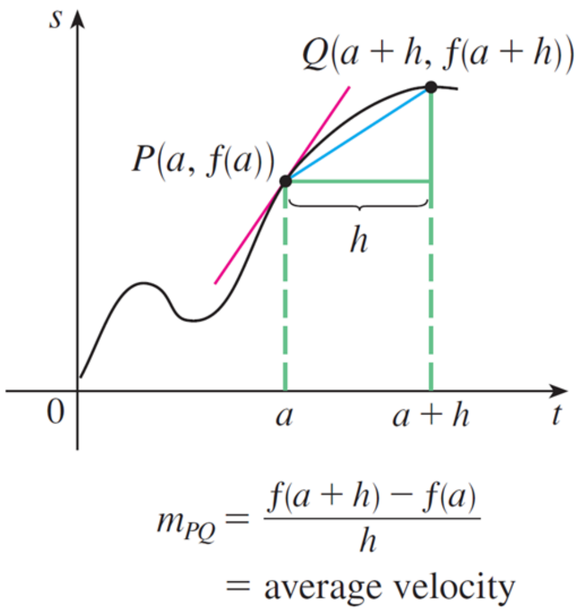


Figure 6

Velocities (4 of 4)

Now suppose we compute the average velocities over shorter and shorter time intervals [*a*, *a* + *h*].

In other words, we let *h* approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities:

3 Definition The **instantaneous velocity** of an object with position function $f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P.

Example 3

Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.

(a) What is the velocity of the ball after 5 seconds?

(b) How fast is the ball traveling when it hits the ground?

Solution:

Since two different velocities are requested, it’s efficient to start by finding the velocity at a general time $t = a$.

Example 3 – Solution (1 of 4)

Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$

$$= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a$$

Example 3 – Solution (2 of 4)

(a) The velocity after 5 seconds is $v(5) = (9.8)(5)$

$= 49$ m/s.

Example 3 – Solution (3 of 4)

(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t when $s(t) = 450$, that is,

$$4.9t^2 = 450$$

This gives

$$t^2 = \frac{450}{4.9} \quad \text{and} \quad t = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

Example 3 – Solution (4 of 4)

The velocity of the ball as it hits the ground is therefore

$$v\left(\sqrt{\frac{450}{4.9}}\right) = 9.8\sqrt{\frac{450}{4.9}}$$

$$\approx 94 \text{ m/s}$$

Derivatives

Derivatives (1 of 4)

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Definition 3).

In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics.

Since this type of limit occurs so widely, it is given a special name and notation.

Derivatives (2 of 4)

4 Definition The **derivative of a function *f* at a number *a***, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines, is

5
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example 4

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) *a.*

Solution:

(a) From Definition 4 we have

$$
\begin{aligned}
f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h} \\
&= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h}
\end{aligned}
$$

Example 4 – Solution (1 of 2)

$$= \lim_{h\to 0} \frac{h^2 - 4h}{h}$$

$$= \lim_{h\to 0} \frac{h(h-4)}{h}$$

$$= \lim_{h\to 0} (h-4) = -4$$

(b) $$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(a+h)^2 - 8(a+h) + 9\right] - \left[a^2 - 8a + 9\right]}{h}$$

$$= \lim_{x\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$

Example 4 – Solution (2 of 2)

$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h\to 0} (2a + h - 8)$$

$$= 2a - 8$$

As a check on our work in part (a), notice that if we let a = 2, then $f'(2) = 2(2) - 8 = -4$.

Derivatives (3 of 4)

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2.

Since, by Definition 4, this is the same as the derivative $f'(a)$, we can now say the following.

The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Derivatives (4 of 4)

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Rates of Change

Rates of Change (1 of 6)

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$.

If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

Rates of Change (2 of 6)

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of *y* with respect to *x*** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line *PQ* in Figure 8.

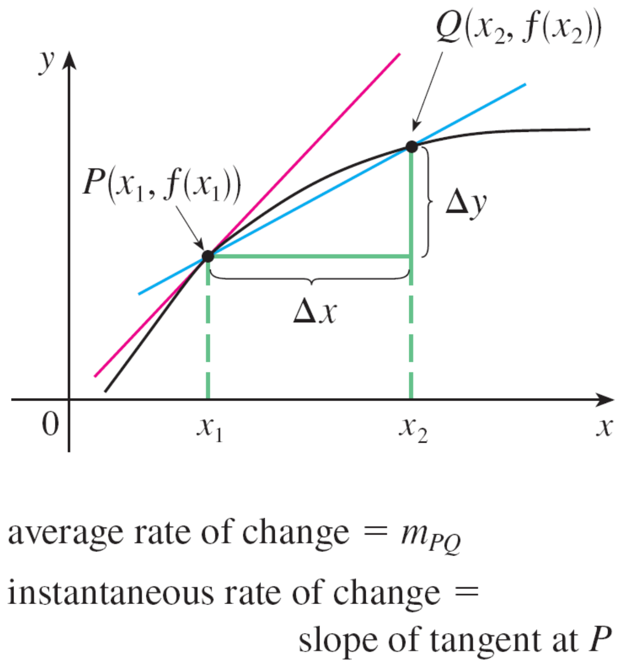


Figure 8

Rates of Change (3 of 6)

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0.

The limit of these average rates of change is called the (**instantaneous**) **rate of change of *y* with respect to *x*** at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\textbf{6} \quad \text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

Rates of Change (4 of 6)

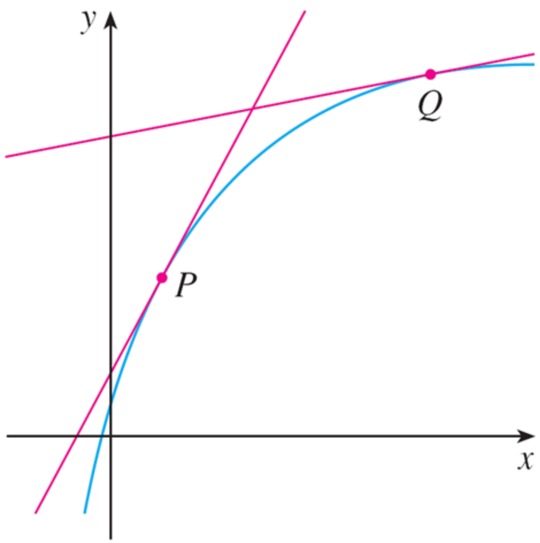
We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$.

Rates of Change (5 of 6)

This means that when the derivative is large (and therefore the curve is steep, as at the point *P* in Figure 9), the *y*-values change rapidly.



The *y*-values are changing rapidly at *P* and slowly at *Q*.

Figure 9

Rates of Change (6 of 6)

When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t.

In other words, $f'(a)$ *is the velocity of the particle at time* $t = a$.

The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

Example 7

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

(a) What is the meaning of the derivative $f'(x)$? What are its units?

(b) In practical terms, what does it mean to say that $f'(1000) = 9$?

(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Example 7(a) – Solution

The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of yards produced.

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta c}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\frac{\Delta C}{\Delta x}$.

Since ΔC is measured in dollars and Δx in yards, it follows that the units for $f'(x)$ are dollars per yard.

Example 7(b) – Solution

The statement that $f'(1000) = 9$ means that, after 1000 yards of fabric have been manufactured, the rate at which the production cost is increasing is \$9/yard. (When x = 1000, C is increasing 9 times as fast as x.)

Since Δx = 1 is small compared with x = 1000, we could use the approximation

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th yard (or the 1001st) is about \$9.

Example 7(c) – Solution (1 of 2)

The rate at which the production cost is increasing (per yard) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th yard is less than the cost of the 50th yard) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.)

So

$$f'(50) > f'(500)$$

Example 7(c) – Solution (2 of 2)

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs.

Thus it is possible that the rate of increase of costs will eventually start to rise.

So it may happen that

$$f'(5000) > f'(500)$$