Course: Calculus 1 - CS

Calculus: Early Transcendentals - James Stewart, Daniel Clegg, Saleem Watson (**Reader**) – Section 2.7

CALCULUS

EARLY TRANSCENDENTALS

NINTH EDITON

Metric Version

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71. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 + 1}{2x^2 + x + 1} - 1.5 \right| < 0.05$$

72. For the limit

$$\lim_{x \to \infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = -3$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

73. For the limit

$$\lim_{x \to -\infty} \frac{1 - 3x}{\sqrt{x^2 + 1}} = 3$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

74. For the limit

$$\lim_{x \to \infty} \sqrt{x \ln x} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

75. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

76. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?

(b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x \to \infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

77. Use Definition 8 to prove that $\displaystyle\lim_{x \to -\infty} \frac{1}{x} = 0$.

78. Prove, using Definition 9, that $\displaystyle\lim_{x \to \infty} x^3 = \infty$.

79. Use Definition 9 to prove that $\displaystyle\lim_{x \to \infty} e^x = \infty$.

80. Formulate a precise definition of

$$\lim_{x \to -\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x \to -\infty} (1 + x^3) = -\infty$$

81. (a) Prove that

$$\lim_{x \to \infty} f(x) = \lim_{t \to 0^+} f(1/t)$$

and

$$\lim_{x \to -\infty} f(x) = \lim_{t \to 0^-} f(1/t)$$

assuming that these limits exist.

(b) Use part (a) and Exercise 65 to find

$$\lim_{x \to 0^+} x \sin \frac{1}{x}$$

2.7 Derivatives and Rates of Change

Now that we have defined limits and have learned techniques for computing them, we revisit the problems of finding tangent lines and velocities from Section 2.1. The special type of limit that occurs in both of these problems is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the natural or social sciences or engineering.

Tangents

If a curve C has equation $y = f(x)$ and we want to find the tangent line to C at the point $P(a, f(a))$, then we consider (as we did in Section 2.1) a nearby point $Q(x, f(x))$, where $x \neq a$, and compute the slope of the secant line PQ:

$$m_{PQ} = \frac{f(x) - f(a)}{x - a}$$

Then we let Q approach P along the curve C by letting x approach a. If m_{PQ} approaches a number m, then we define the *tangent line* ℓ to be the line through P with slope m. (This amounts to saying that the tangent line is the limiting position of the secant line PQ as Q approaches P. See Figure 1.)

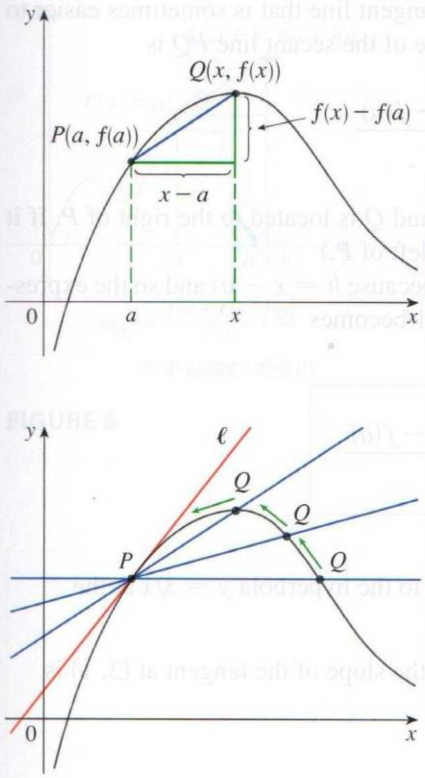


FIGURE 1

1 Definition The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

In our first example we confirm the guess we made in Example 2.1.1.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION Here we have $a = 1$ and $f(x) = x^2$, so the slope is

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x \to 1} (x + 1) = 1 + 1 = 2$$

Using the point-slope form of the equation of a line, we find that an equation of the tangent line at $(1, 1)$ is

Point-slope form for a line through the point (x_1, y_1) with slope m:
$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

We sometimes refer to the slope of the tangent line to a curve at a point as the **slope of the curve** at the point. The idea is that if we zoom in far enough toward the point, the curve looks almost like a straight line. Figure 2 illustrates this procedure for the curve $y = x^2$ in Example 1. The more we zoom in, the more the parabola looks like a line. In other words, the curve becomes almost indistinguishable from its tangent line.

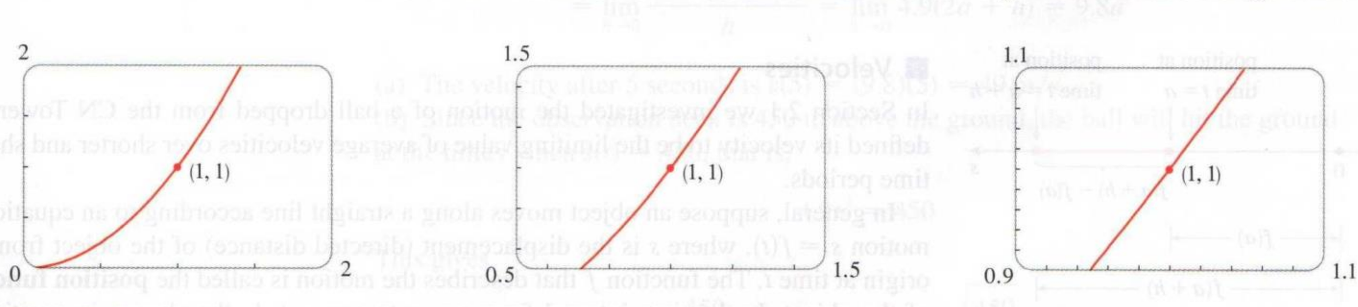


FIGURE 2
Zooming in toward the point $(1, 1)$ on the parabola $y = x^2$

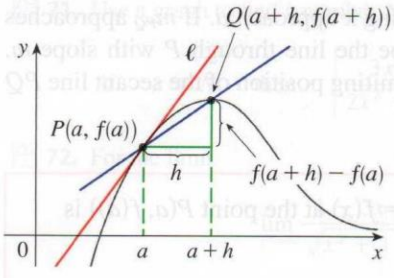


FIGURE 3

There is another expression for the slope of a tangent line that is sometimes easier to use. If $h = x - a$, then $x = a + h$ and so the slope of the secant line PQ is

$$m_{PQ} = \frac{f(a + h) - f(a)}{h}$$

(See Figure 3 where the case $h > 0$ is illustrated and Q is located to the right of P. If it happened that $h < 0$, however, Q would be to the left of P.)

Notice that as x approaches a, h approaches 0 (because $h = x - a$) and so the expression for the slope of the tangent line in Definition 1 becomes

$$\boxed{m = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}} \tag{2}$$

EXAMPLE 2 Find an equation of the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.

SOLUTION Let $f(x) = 3/x$. Then, by Equation 2, the slope of the tangent at $(3, 1)$ is

$$m = \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{3}{3 + h} - 1}{h} = \lim_{h \to 0} \frac{\dfrac{3 - (3 + h)}{3 + h}}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h(3 + h)} = \lim_{h \to 0} -\frac{1}{3 + h} = -\frac{1}{3}$$

Therefore an equation of the tangent at the point $(3, 1)$ is

$$y - 1 = -\tfrac{1}{3}(x - 3)$$

which simplifies to

$$x + 3y - 6 = 0$$

The hyperbola and its tangent are shown in Figure 4. ■

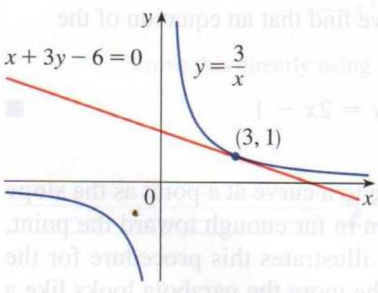


FIGURE 4

■ Velocities

In Section 2.1 we investigated the motion of a ball dropped from the CN Tower and defined its velocity to be the limiting value of average velocities over shorter and shorter time periods.

In general, suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. The function f that describes the motion is called the **position function** of the object. In the time interval from $t = a$ to $t = a + h$, the change in position is $f(a + h) - f(a)$. (See Figure 5.)

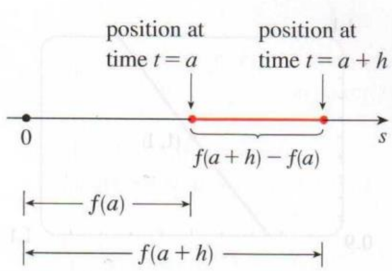
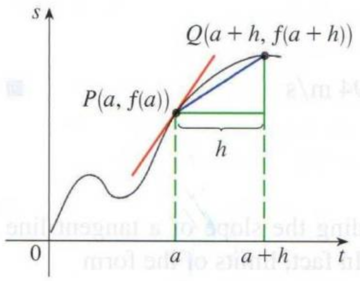


FIGURE 5



$$m_{PQ} = \frac{f(a+h) - f(a)}{h}$$
$= \text{average velocity}$

FIGURE 6

The average velocity over this time interval is

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}} = \frac{f(a+h) - f(a)}{h}$$

which is the same as the slope of the secant line PQ in Figure 6.

Now suppose we compute the average velocities over shorter and shorter time intervals $[a, a + h]$. In other words, we let h approach 0. As in the example of the falling ball, we define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ to be the limit of these average velocities.

3 Definition The **instantaneous velocity** of an object with position function $f(t)$ at time $t = a$ is

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

This means that the velocity at time $t = a$ is equal to the slope of the tangent line at P (compare Equation 2 and the expression in Definition 3).

Now that we know how to compute limits, let's reconsider the problem of the falling ball from Example 2.1.3.

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower, 450 m above the ground.
(a) What is the velocity of the ball after 5 seconds?
(b) How fast is the ball traveling when it hits the ground?

Recall from Section 2.1: The distance (in meters) fallen after t seconds is $4.9t^2$.

SOLUTION Since two different velocities are requested, it's efficient to start by finding the velocity at a general time $t = a$. Using the equation of motion $s = f(t) = 4.9t^2$, we have

$$v(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{4.9(a+h)^2 - 4.9a^2}{h}$$

$$= \lim_{h \to 0} \frac{4.9(a^2 + 2ah + h^2 - a^2)}{h} = \lim_{h \to 0} \frac{4.9(2ah + h^2)}{h}$$

$$= \lim_{h \to 0} \frac{4.9h(2a + h)}{h} = \lim_{h \to 0} 4.9(2a + h) = 9.8a$$

(a) The velocity after 5 seconds is $v(5) = (9.8)(5) = 49$ m/s.
(b) Since the observation deck is 450 m above the ground, the ball will hit the ground at the time t when $s(t) = 450$, that is,

$$4.9t^2 = 450$$

This gives

$$t^2 = \frac{450}{4.9} \qquad \text{and} \qquad t = \sqrt{\frac{450}{4.9}} \approx 9.6 \text{ s}$$

The velocity of the ball as it hits the ground is therefore

$$v\left(\sqrt{\frac{450}{4.9}}\right) = 9.8\sqrt{\frac{450}{4.9}} \approx 94 \text{ m/s}$$ ■

■ Derivatives

We have seen that the same type of limit arises in finding the slope of a tangent line (Equation 2) or the velocity of an object (Definition 3). In fact, limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

4 Definition The **derivative of a function f at a number a**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

$f'(a)$ is read "f prime of a."

If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative, as we saw in finding tangent lines (see Definition 1), is

5
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

EXAMPLE 4 Use Definition 4 to find the derivative of the function $f(x) = x^2 - 8x + 9$ at the numbers (a) 2 and (b) a.

Definitions 4 and 5 are equivalent, so we can use either one to compute the derivative. In practice, Definition 4 often leads to simpler computations.

SOLUTION

(a) From Definition 4 we have

$$\begin{aligned} f'(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \to 0} \frac{(2+h)^2 - 8(2+h) + 9 - (-3)}{h} \\ &= \lim_{h \to 0} \frac{4 + 4h + h^2 - 16 - 8h + 9 + 3}{h} \\ &= \lim_{h \to 0} \frac{h^2 - 4h}{h} = \lim_{h \to 0} \frac{h(h-4)}{h} = \lim_{h \to 0} (h - 4) = -4 \end{aligned}$$

(b)
$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h\to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$
$$= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h}$$
$$= \lim_{h\to 0} \frac{2ah + h^2 - 8h}{h} = \lim_{h\to 0} (2a + h - 8) = 2a - 8$$

As a check on our work in part (a), notice that if we let $a = 2$, then $f'(2) = 2(2) - 8 = -4$. ■

EXAMPLE 5 Use Equation 5 to find the derivative of the function $f(x) = 1/\sqrt{x}$ at the number a $(a > 0)$.

SOLUTION From Equation 5 we get

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$
$$= \lim_{x\to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} = \lim_{x\to a} \frac{\dfrac{1}{\sqrt{x}} - \dfrac{1}{\sqrt{a}}}{x - a} \cdot \frac{\sqrt{x}\sqrt{a}}{\sqrt{x}\sqrt{a}}$$
$$= \lim_{x\to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}\,(x - a)} = \lim_{x\to a} \frac{\sqrt{a} - \sqrt{x}}{\sqrt{ax}\,(x - a)} \cdot \frac{\sqrt{a} + \sqrt{x}}{\sqrt{a} + \sqrt{x}}$$
$$= \lim_{x\to a} \frac{-(x - a)}{\sqrt{ax}\,(x - a)(\sqrt{a} + \sqrt{x})} = \lim_{x\to a} \frac{-1}{\sqrt{ax}(\sqrt{a} + \sqrt{x})}$$
$$= \frac{-1}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{-1}{a \cdot 2\sqrt{a}} = -\frac{1}{2a^{3/2}}$$

You can verify that using Definition 4 gives the same result. ■

We defined the tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ to be the line that passes through P and has slope m given by Equation 1 or 2. Since, by Definition 4 (and Equation 5), this is the same as the derivative $f'(a)$, we can now say the following.

> The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

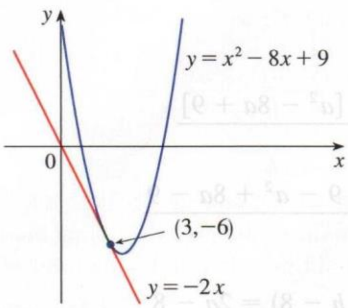


FIGURE 7

EXAMPLE 6 Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

SOLUTION From Example 4(b) we know that the derivative of $f(x) = x^2 - 8x + 9$ at the number a is $f'(a) = 2a - 8$. Therefore the slope of the tangent line at $(3, -6)$ is $f'(3) = 2(3) - 8 = -2$. Thus an equation of the tangent line, shown in Figure 7, is

$$y - (-6) = (-2)(x - 3) \qquad \text{or} \qquad y = -2x$$

Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

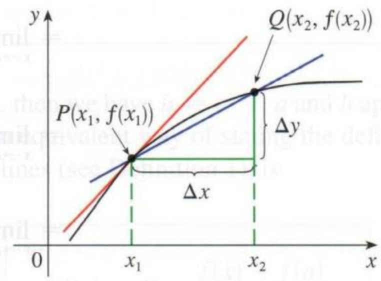
and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 8.



average rate of change $= m_{PQ}$
instantaneous rate of change = slope of tangent at P

FIGURE 8

By analogy with velocity, we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the **(instantaneous) rate of change of y with respect to x** at $x = x_1$, which (as in the case of velocity) is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

6 $$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

We recognize this limit as being the derivative $f'(x_1)$.

We know that one interpretation of the derivative $f'(a)$ is as the slope of the tangent line to the curve $y = f(x)$ when $x = a$. We now have a second interpretation:

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

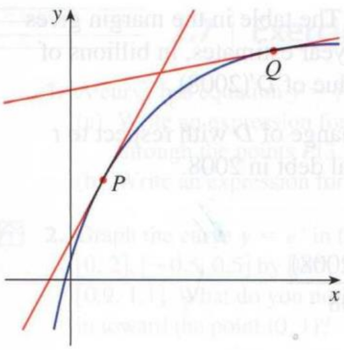


FIGURE 9
The y-values are changing rapidly at P and slowly at Q.

The connection with the first interpretation is that if we sketch the curve $y = f(x)$, then the instantaneous rate of change is the slope of the tangent to this curve at the point where $x = a$. This means that when the derivative is large (and therefore the curve is steep, as at the point P in Figure 9), the y-values change rapidly. When the derivative is small, the curve is relatively flat (as at point Q) and the y-values change slowly.

In particular, if $s = f(t)$ is the position function of a particle that moves along a straight line, then $f'(a)$ is the rate of change of the displacement s with respect to the time t. In other words, *$f'(a)$ is the velocity of the particle at time $t = a$*. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(a)|$.

In the next example we discuss the meaning of the derivative of a function that is defined verbally.

EXAMPLE 7 A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x meters of this fabric is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) In practical terms, what does it mean to say that $f'(1000) = 9$?
(c) Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

SOLUTION
(a) The derivative $f'(x)$ is the instantaneous rate of change of C with respect to x; that is, $f'(x)$ means the rate of change of the production cost with respect to the number of meters produced. (Economists call this rate of change the *marginal cost*. This idea is discussed in more detail in Sections 3.7 and 4.7.)

Because

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x}$$

the units for $f'(x)$ are the same as the units for the difference quotient $\Delta C/\Delta x$. Since ΔC is measured in dollars and Δx in meters, it follows that the units for $f'(x)$ are dollars per meter.

(b) The statement that $f'(1000) = 9$ means that, after 1000 meters of fabric have been manufactured, the rate at which the production cost is increasing is \$9/meter. (When $x = 1000$, C is increasing 9 times as fast as x.)

Since $\Delta x = 1$ is small compared with $x = 1000$, we could use the approximation

Here we are assuming that the cost function is well behaved; in other words, $C(x)$ doesn't oscillate rapidly near $x = 1000$.

$$f'(1000) \approx \frac{\Delta C}{\Delta x} = \frac{\Delta C}{1} = \Delta C$$

and say that the cost of manufacturing the 1000th meter (or the 1001st) is about \$9.

(c) The rate at which the production cost is increasing (per meter) is probably lower when $x = 500$ than when $x = 50$ (the cost of making the 500th meter is less than the cost of the 50th meter) because of economies of scale. (The manufacturer makes more efficient use of the fixed costs of production.) So

$$f'(50) > f'(500)$$

But, as production expands, the resulting large-scale operation might become inefficient and there might be overtime costs. Thus it is possible that the rate of increase of costs will eventually start to rise. So it may happen that

$$f'(5000) > f'(500)$$ ■

In the following example we estimate the rate of change of the national debt with respect to time. Here the function is defined not by a formula but by a table of values.

t	$D(t)$
2000	5662.2
2004	7596.1
2008	10,699.8
2012	16,432.7
2016	19,976.8

Source: US Dept. of the Treasury

EXAMPLE 8 Let $D(t)$ be the US national debt at time t. The table in the margin gives approximate values of this function by providing end of year estimates, in billions of dollars, from 2000 to 2016. Interpret and estimate the value of $D'(2008)$.

SOLUTION The derivative $D'(2008)$ means the rate of change of D with respect to t when $t = 2008$, that is, the rate of increase of the national debt in 2008.

According to Equation 5,

$$D'(2008) = \lim_{t \to 2008} \frac{D(t) - D(2008)}{t - 2008}$$

One way we can estimate this value is to compare average rates of change over different time intervals by computing difference quotients, as compiled in the following table.

t	Time interval	Average rate of change $= \dfrac{D(t) - D(2008)}{t - 2008}$
2000	[2000, 2008]	629.7
2004	[2004, 2008]	775.93
2012	[2008, 2012]	1433.23
2016	[2008, 2016]	1159.63

A Note On Units
The units for the average rate of change $\Delta D/\Delta t$ are the units for ΔD divided by the units for Δt, namely billions of dollars per year. The instantaneous rate of change is the limit of the average rates of change, so it is measured in the same units: billions of dollars per year.

From this table we see that $D'(2008)$ lies somewhere between 775.93 and 1433.23 billion dollars per year. [Here we are making the reasonable assumption that the debt didn't fluctuate wildly between 2004 and 2012.] A good estimate for the rate of increase of the US national debt in 2008 would be the average of these two numbers, namely

$$D'(2008) \approx 1105 \text{ billion dollars per year}$$

Another method would be to plot the debt function and estimate the slope of the tangent line when $t = 2008$. ■

In Examples 3, 7, and 8 we saw three specific examples of rates of change: the velocity of an object is the rate of change of displacement with respect to time; marginal cost is the rate of change of production cost with respect to the number of items produced; the rate of change of the debt with respect to time is of interest in economics. Here is a small sample of other rates of change: In physics, the rate of change of work with respect to time is called *power*. Chemists who study a chemical reaction are interested in the rate of change in the concentration of a reactant with respect to time (called the *rate of reaction*). A biologist is interested in the rate of change of the population of a colony of bacteria with respect to time. In fact, the computation of rates of change is important in all of the natural sciences, in engineering, and even in the social sciences. Further examples will be given in Section 3.7.

All these rates of change are derivatives and can therefore be interpreted as slopes of tangents. This gives added significance to the solution of the tangent problem. Whenever we solve a problem involving tangent lines, we are not just solving a problem in geometry. We are also implicitly solving a great variety of problems involving rates of change in science and engineering.

2.7 Exercises

1. A curve has equation $y = f(x)$.
(a) Write an expression for the slope of the secant line through the points $P(3, f(3))$ and $Q(x, f(x))$.
(b) Write an expression for the slope of the tangent line at P.

2. Graph the curve $y = e^x$ in the viewing rectangles $[-1, 1]$ by $[0, 2]$, $[-0.5, 0.5]$ by $[0.5, 1.5]$, and $[-0.1, 0.1]$ by $[0.9, 1.1]$. What do you notice about the curve as you zoom in toward the point $(0, 1)$?

3. (a) Find the slope of the tangent line to the parabola $y = x^2 + 3x$ at the point $(-1, -2)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the parabola and the tangent line. As a check on your work, zoom in toward the point $(-1, -2)$ until the parabola and the tangent line are indistinguishable.

4. (a) Find the slope of the tangent line to the curve $y = x^3 + 1$ at the point $(1, 2)$
(i) using Definition 1 (ii) using Equation 2
(b) Find an equation of the tangent line in part (a).
(c) Graph the curve and the tangent line in successively smaller viewing rectangles centered at $(1, 2)$ until the curve and the line appear to coincide.

5–8 Find an equation of the tangent line to the curve at the given point.

5. $y = 2x^2 - 5x + 1, \quad (3, 4)$

6. $y = x^2 - 2x^3, \quad (1, -1)$

7. $y = \dfrac{x + 2}{x - 3}, \quad (2, -4)$

8. $y = \sqrt{1 - 3x}, \quad (-1, 2)$

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 5)$ and $(2, 3)$.
(c) Graph the curve and both tangents on a common screen.

10. (a) Find the slope of the tangent to the curve $y = 2\sqrt{x}$ at the point where $x = a$.
(b) Find equations of the tangent lines at the points $(1, 2)$ and $(9, 6)$.
(c) Graph the curve and both tangents on a common screen.

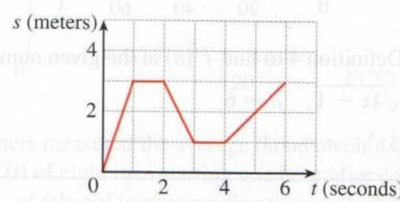
11. A cliff diver plunges from a height of 30 m above the water surface. The distance the diver falls in t seconds is given by the function $d(t) = 4.9t^2$ m.
(a) After how many seconds will the diver hit the water?
(b) With what velocity does the diver hit the water?

12. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in meters) after t seconds is given by $H = 10t - 1.86t^2$.
(a) Find the velocity of the rock after one second.
(b) Find the velocity of the rock when $t = a$.
(c) When will the rock hit the surface?
(d) With what velocity will the rock hit the surface?

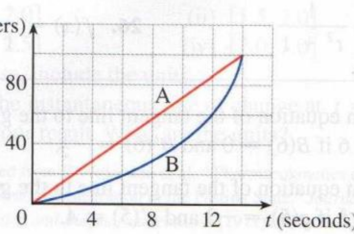
13. The displacement (in meters) of a particle moving in a straight line is given by the equation of motion $s = 1/t^2$, where t is measured in seconds. Find the velocity of the particle at times $t = a$, $t = 1$, $t = 2$ and $t = 3$.

14. The displacement (in meters) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.
(a) Find the average velocity over each time interval:
(i) $[4, 8]$ (ii) $[6, 8]$
(iii) $[8, 10]$ (iv) $[8, 12]$
(b) Find the instantaneous velocity when $t = 8$.
(c) Draw the graph of s as a function of t and draw the secant lines whose slopes are the average velocities in part (a). Then draw the tangent line whose slope is the instantaneous velocity in part (b).

15. (a) A particle starts by moving to the right along a horizontal line; the graph of its position function is shown in the figure. When is the particle moving to the right? Moving to the left? Standing still?
(b) Draw a graph of the velocity function.

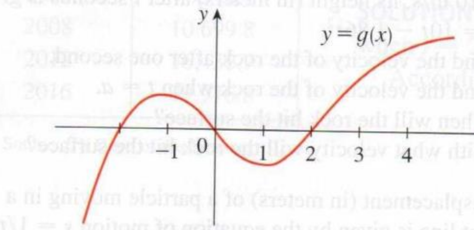


16. Shown are graphs of the position functions of two runners, A and B, who run a 100-meter race and finish in a tie.
(a) Describe and compare how the runners run the race.
(b) At what time is the distance between the runners the greatest?
(c) At what time do they have the same velocity?



17. For the function g whose graph is given, arrange the following numbers in increasing order and explain your reasoning:

$$0 \quad g'(-2) \quad g'(0) \quad g'(2) \quad g'(4)$$



18. The graph of a function f is shown.

(a) Find the average rate of change of f on the interval [20, 60].

(b) Identify an interval on which the average rate of change of f is 0.

(c) Compute

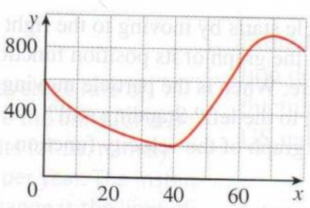
$$\frac{f(40) - f(10)}{40 - 10}$$

What does this value represent geometrically?

(d) Estimate the value of $f'(50)$.

(e) Is $f'(10) > f'(30)$?

(f) Is $f'(60) > \dfrac{f(80) - f(40)}{80 - 40}$? Explain.



19–20 Use Definition 4 to find $f'(a)$ at the given number a.

19. $f(x) = \sqrt{4x + 1}, \quad a = 6$

20. $f(x) = 5x^4, \quad a = -1$

21–22 Use Equation 5 to find $f'(a)$ at the given number a.

21. $f(x) = \dfrac{x^2}{x + 6}, \quad a = 3$

22. $f(x) = \dfrac{1}{\sqrt{2x + 2}}, \quad a = 1$

23–26 Find $f'(a)$.

23. $f(x) = 2x^2 - 5x + 3$

24. $f(t) = t^3 - 3t$

25. $f(t) = \dfrac{1}{t^2 + 1}$

26. $f(x) = \dfrac{x}{1 - 4x}$

27. Find an equation of the tangent line to the graph of $y = B(x)$ at $x = 6$ if $B(6) = 0$ and $B'(6) = -\frac{1}{2}$.

28. Find an equation of the tangent line to the graph of $y = g(x)$ at $x = 5$ if $g(5) = -3$ and $g'(5) = 4$.

29. If $f(x) = 3x^2 - x^3$, find $f'(1)$ and use it to find an equation of the tangent line to the curve $y = 3x^2 - x^3$ at the point (1, 2).

30. If $g(x) = x^4 - 2$, find $g'(1)$ and use it to find an equation of the tangent line to the curve $y = x^4 - 2$ at the point (1, −1).

31. (a) If $F(x) = 5x/(1 + x^2)$, find $F'(2)$ and use it to find an equation of the tangent line to the curve $y = 5x/(1 + x^2)$ at the point (2, 2).

(b) Illustrate part (a) by graphing the curve and the tangent line on the same screen.

32. (a) If $G(x) = 4x^2 - x^3$, find $G'(a)$ and use it to find equations of the tangent lines to the curve $y = 4x^2 - x^3$ at the points (2, 8) and (3, 9).

(b) Illustrate part (a) by graphing the curve and the tangent lines on the same screen.

33. If an equation of the tangent line to the curve $y = f(x)$ at the point where $a = 2$ is $y = 4x - 5$, find $f(2)$ and $f'(2)$.

34. If the tangent line to $y = f(x)$ at (4, 3) passes through the point (0, 2), find $f(4)$ and $f'(4)$.

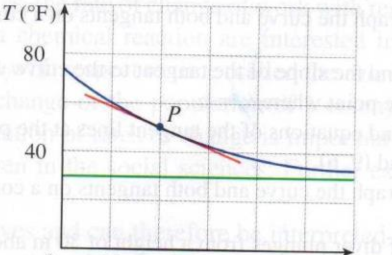
35–36 A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$.

35. $f(t) = 80t - 6t^2$

36. $f(t) = 10 + \dfrac{45}{t + 1}$

37. A warm can of soda is placed in a cold refrigerator. Sketch the graph of the temperature of the soda as a function of time. Is the initial rate of change of temperature greater or less than the rate of change after an hour?

38. A roast turkey is taken from an oven when its temperature has reached 85°C and is placed on a table in a room where the temperature is 24°C. The graph shows how the temperature of the turkey decreases and eventually approaches room temperature. By measuring the slope of the tangent, estimate the rate of change of the temperature after an hour.



39. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, and $f'(2) = -1$.

40. Sketch the graph of a function g for which $g(0) = g(2) = g(4) = 0$, $g'(1) = g'(3) = 0$, $g'(0) = g'(4) = 1$, $g'(2) = -1$, $\lim_{x\to\infty} g(x) = \infty$, and $\lim_{x\to-\infty} g(x) = -\infty$.

41. Sketch the graph of a function g that is continuous on its domain $(-5, 5)$ and where $g(0) = 1$, $g'(0) = 1$, $g'(-2) = 0$, $\lim_{x\to-5^+} g(x) = \infty$, and $\lim_{x\to5^-} g(x) = 3$.

42. Sketch the graph of a function f where the domain is $(-2, 2)$, $f'(0) = -2$, $\lim_{x\to2^-} f(x) = \infty$, f is continuous at all numbers in its domain except ±1, and f is odd.

43–48 Each limit represents the derivative of some function f at some number a. State such an f and a in each case.

43. $\displaystyle\lim_{h\to0} \frac{\sqrt{9+h} - 3}{h}$

44. $\displaystyle\lim_{h\to0} \frac{e^{-2+h} - e^{-2}}{h}$

45. $\displaystyle\lim_{x\to2} \frac{x^6 - 64}{x - 2}$

46. $\displaystyle\lim_{x\to1/4} \frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$

47. $\displaystyle\lim_{h\to0} \frac{\tan\left(\frac{\pi}{4} + h\right) - 1}{h}$

48. $\displaystyle\lim_{\theta\to\pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$

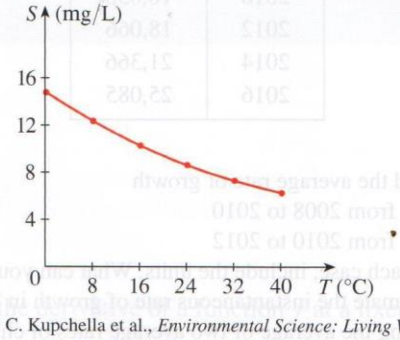
49. The cost (in dollars) of producing x units of a certain commodity is $C(x) = 5000 + 10x + 0.05x^2$.
(a) Find the average rate of change of C with respect to x when the production level is changed
(i) from $x = 100$ to $x = 105$
(ii) from $x = 100$ to $x = 101$
(b) Find the instantaneous rate of change of C with respect to x when $x = 100$. (This is called the *marginal cost*. Its significance will be explained in Section 3.7.)

50. Let $H(t)$ be the daily cost (in dollars) to heat an office building when the outside temperature is t degrees Celsius.
(a) What is the meaning of $H'(14)$? What are its units?
(b) Would you expect $H'(14)$ to be positive or negative? Explain.

51. The cost of producing x kilograms of gold from a new gold mine is $C = f(x)$ dollars.
(a) What is the meaning of the derivative $f'(x)$? What are its units?
(b) What does the statement $f'(22) = 17$ mean?
(c) Do you think the values of $f'(x)$ will increase or decrease in the short term? What about the long term? Explain.

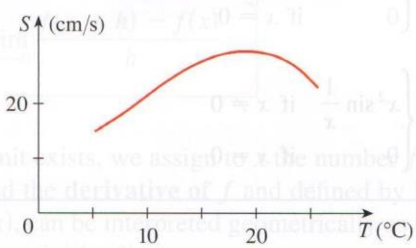
52. The quantity (in kilograms) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per kilogram is $Q = f(p)$.
(a) What is the meaning of the derivative $f'(8)$? What are its units?
(b) Is $f'(8)$ positive or negative? Explain.

53. The quantity of oxygen that can dissolve in water depends on the temperature of the water. (So thermal pollution influences the oxygen content of water.) The graph shows how oxygen solubility S varies as a function of the water temperature T.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the value of $S'(16)$ and interpret it.



Source: C. Kupchella et al., *Environmental Science: Living Within the System of Nature*, 2d ed. (Boston: Allyn and Bacon, 1989).

54. The graph shows the influence of the temperature T on the maximum sustainable swimming speed S of Coho salmon.
(a) What is the meaning of the derivative $S'(T)$? What are its units?
(b) Estimate the values of $S'(15)$ and $S'(25)$ and interpret them.



55. Researchers measured the average blood alcohol concentration $C(t)$ of eight men starting one hour after consumption of 30 mL of ethanol (corresponding to two alcoholic drinks).

t (hours)	1.0	1.5	2.0	2.5	3.0
$C(t)$ (g/dL)	0.033	0.024	0.018	0.012	0.007

(a) Find the average rate of change of C with respect to t over each time interval:
(i) [1.0, 2.0]
(ii) [1.5, 2.0]
(iii) [2.0, 2.5]
(iv) [2.0, 3.0]
In each case, include the units.
(b) Estimate the instantaneous rate of change at $t = 2$ and interpret your result. What are the units?

Source: Adapted from P. Wilkinson et al., "Pharmacokinetics of Ethanol after Oral Administration in the Fasting State," *Journal of Pharmacokinetics and Biopharmaceutics* 5 (1977): 207–24.

56. The number N of locations of a popular coffeehouse chain is given in the table. (The numbers of locations as of October 1 are given.)

Year	N
2008	16,680
2010	16,858
2012	18,066
2014	21,366
2016	25,085

(a) Find the average rate of growth
(i) from 2008 to 2010
(ii) from 2010 to 2012
In each case, include the units. What can you conclude?
(b) Estimate the instantaneous rate of growth in 2010 by taking the average of two average rates of change. What are its units?
(c) Estimate the instantaneous rate of growth in 2010 by measuring the slope of a tangent.

57–58 Determine whether $f'(0)$ exists.

57. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

58. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

59. (a) Graph the function $f(x) = \sin x - \frac{1}{1000}\sin(1000x)$ in the viewing rectangle $[-2\pi, 2\pi]$ by $[-4, 4]$. What slope does the graph appear to have at the origin?
(b) Zoom in to the viewing window $[-0.4, 0.4]$ by $[-0.25, 0.25]$ and estimate the value of $f'(0)$. Does this agree with your answer from part (a)?
(c) Now zoom in to the viewing window $[-0.008, 0.008]$ by $[-0.005, 0.005]$. Do you wish to revise your estimate for $f'(0)$?

60. Symmetric Difference Quotients In Example 8 we approximated an instantaneous rate of change by averaging two average rates of change. An alternative method is to use a single average rate of change over an interval *centered* at the desired value. We define the *symmetric difference quotient* of a function f at $x = a$ on the interval $[a - d, a + d]$ as

$$\frac{f(a + d) - f(a - d)}{(a + d) - (a - d)} = \frac{f(a + d) - f(a - d)}{2d}$$

(a) Compute the symmetric difference quotient for the function D in Example 8 on the interval [2004, 2012] and verify that your result agrees with the estimate for $D'(2008)$ computed in the example.
(b) Show that the symmetric difference quotient of a function f at $x = a$ is equivalent to averaging the average rates of change of f over the intervals $[a - d, a]$ and $[a, a + d]$.
(c) Use a symmetric difference quotient to estimate $f'(1)$ for $f(x) = x^3 - 2x^2 + 2$ with $d = 0.4$. Draw a graph of f along with secant lines corresponding to average rates of change over the intervals $[1 - d, 1]$, $[1, 1 + d]$, and $[1 - d, 1 + d]$. Which of these secant lines appears to have slope closest to that of the tangent line at $x = 1$?

WRITING PROJECT | EARLY METHODS FOR FINDING TANGENTS

The first person to explicitly formulate the ideas of limits and derivatives was Sir Isaac Newton in the 1660s. But Newton acknowledged that "If I have seen further than other men, it is because I have stood on the shoulders of giants." Two of those giants were Pierre Fermat (1601–1665) and Newton's mentor at Cambridge, Isaac Barrow (1630–1677). Newton was familiar with the methods that these men used to find tangent lines, and their methods played a role in Newton's eventual formulation of calculus.

Learn about these methods by researching on the Internet or reading one of the references listed here. Write an essay comparing the methods of either Fermat or Barrow to modern methods. In particular, use the method of Section 2.7 to find an equation of the tangent line to the curve $y = x^3 + 2x$ at the point (1, 3) and show how either Fermat or Barrow would have solved the same problem. Although you used derivatives and they did not, point out similarities between the methods.

1. C. H. Edwards, *The Historical Development of the Calculus* (New York: Springer-Verlag, 1979), pp. 124, 132.

2. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.
3. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.
4. Uta Merzbach and Carl Boyer, *A History of Mathematics*, 3rd ed. (Hoboken, NJ: Wiley, 2011), pp. 323, 356.

2.8 The Derivative as a Function

The Derivative Function

In the preceding section we considered the derivative of a function f at a fixed number a:

$$\boxed{1} \qquad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

$$\boxed{2} \qquad \boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of** f and defined by Equation 2. We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

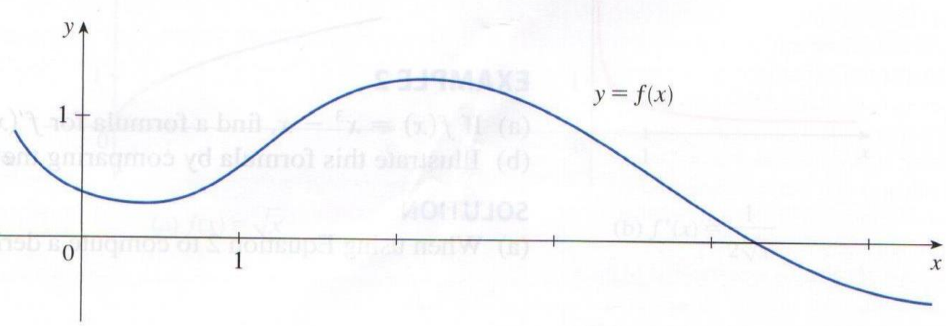


FIGURE 1