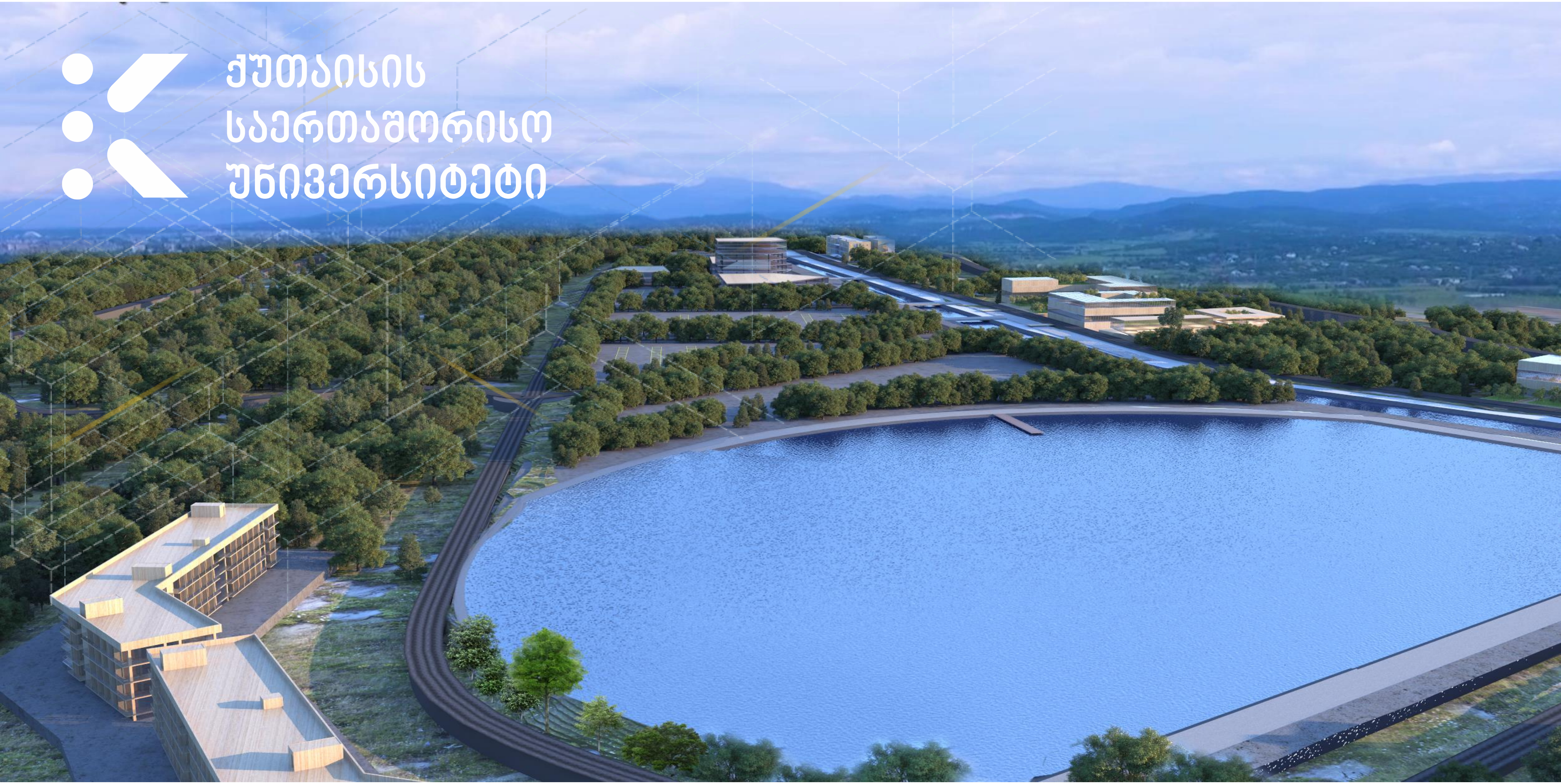
ქუთაისის საერთაშორისო უნივერსიტეტი



2. Limits and Derivatives

2.8 The Derivative as a Function

The Derivative Function

The Derivative Function (1 of 4)

We have considered the derivative of a function *f* at a fixed number *a*:

$$1 \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here we change our point of view and let the number *a* vary. If we replace *a* in Equation 1 by a variable *x*, we obtain

$$2 \quad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The Derivative Function (2 of 4)

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of *f*** and defined by Equation 2.

We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

Example 1

The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

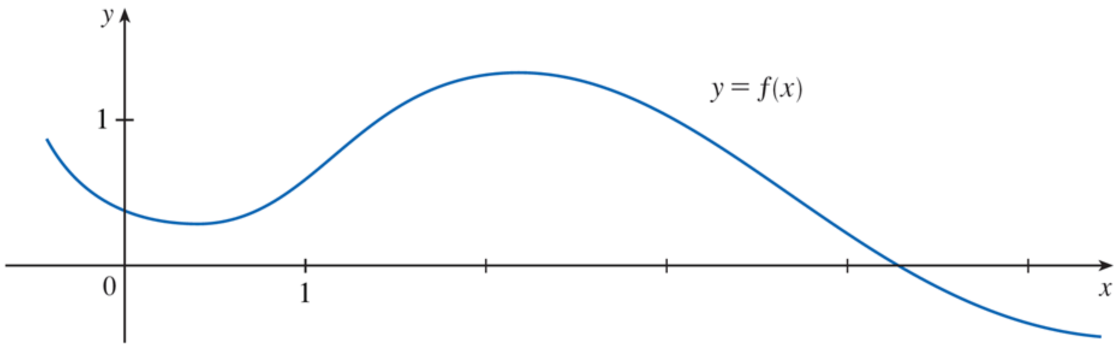


Figure 1

Example 1 – Solution (1 of 3)

We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 3$ we draw the tangent at P in Figure 2 and estimate its slope to be about $-\frac{2}{3}$.

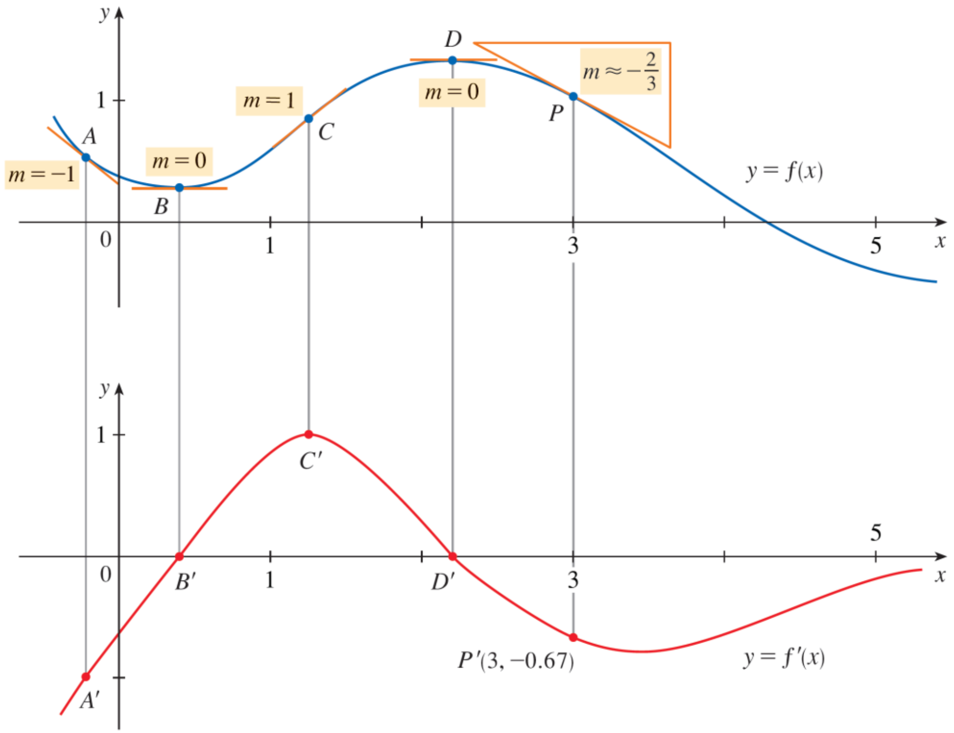


Figure 2

Example 1 – Solution (2 of 3)

This allows us to plot the point $P'(3, -0.67)$ on the graph of f' directly beneath *P*. (The slope of the graph of *f* becomes the *y*-value on the graph of f'.)

The slope of the tagent drawn at *A* appears to be about −1, so we plot the point A' with a *y*-value of −1 on the graph of f' (directly beneath *A*). The tangents at *B* and *D* are horizontal, so the derivative is 0 there and the graph of f' crosses the *x*-axis (where $y = 0$) at the points B' and D', directly beneath *B* and *D*.

Between *B* and *D*, the graph of *f* is steepest at *C* and the tangent line ther appears to have slope 1, so the largest value of $f'(x)$ between B' and D' is 1 (at C').

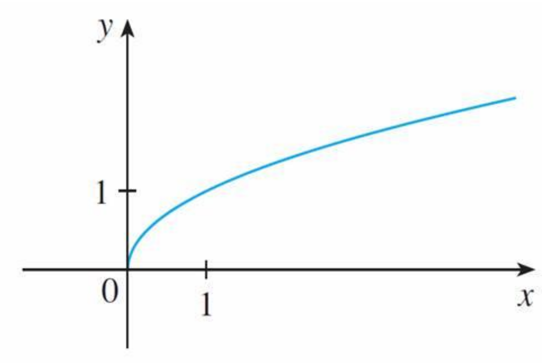
Example 1 – Solution (3 of 3)

Notice that between *B* and *D* the tangents have positive slope, so $f'(x)$ is positive there. (The graph of f' is above the x-axis.) But to the right of *D* the tangents have negative slope, so $f'(x)$ is negative there.

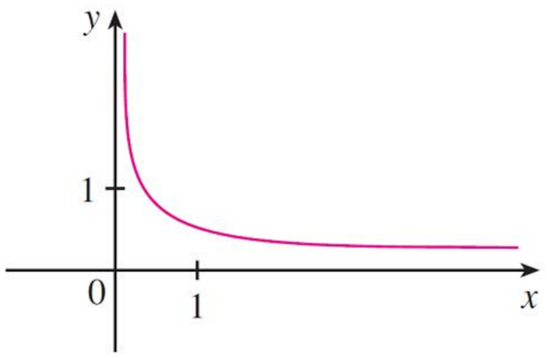
The Derivative Function (3 of 4)

When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = \frac{1}{(2\sqrt{x})}$ is very large and

this corresponds to the steep tangent lines near (0, 0) in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b).



(a) $f(x) = \sqrt{x}$



(b) $f'(x) = \frac{1}{2\sqrt{x}}$

Figure 4

The Derivative Function (4 of 4)

When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

Other Notations

Other Notations (1 of 4)

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and $\frac{d}{dx}$ are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

Other Notations (2 of 4)

The symbol $\frac{dy}{dx}$, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation.

We can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Other Notations (3 of 4)

If we want to indicate the value of a derivative $\frac{dy}{dx}$ in Leibniz notation at a specific number *a*, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$.

3 Definition A function *f* is **differentiable at *a*** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Example 5

Where is the function $f(x) = |x|$ differentiable?

Solution:

If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x+h| = x+h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h} = \lim_{h \to 0} \frac{(x+h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Example 5 – Solution (1 of 4)

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x+h| = -(x+h)$.

Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x+h| - |x|}{h}$$

$$= \lim_{h \to 0} \frac{-(x+h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0}(-1) = -1$$

and so f is differentiable for any $x < 0$.

Example 5 – Solution (2 of 4)

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \qquad (\text{if it exists})$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

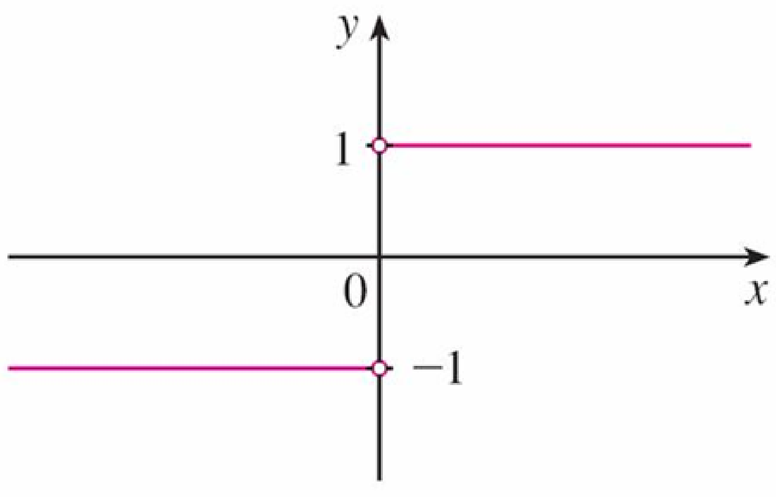
Example 5 – Solution (3 of 4)

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b).

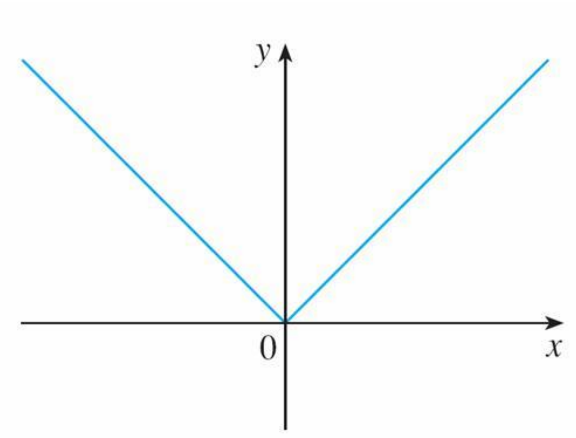


$y = f'(x)$

Figure 5(b)

Example 5 – Solution (4 of 4)

The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 5(a).]



$y = f(x) = |x|$

Figure 5(a)

Other Notations (4 of 4)

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

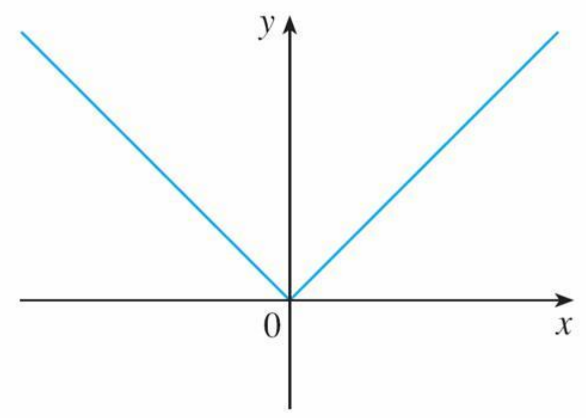
Note: The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable.

How Can a Function Fail to Be Differentiable?

How Can a Function Fail to Be Differentiable? (1 of 4)

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$.

In general, if the graph of a function *f* has a "corner" or "kink" in it, then the graph of *f* has no tangent at this point and *f* is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]



$y = f(x) = |x|$

Figure 5(a)

How Can a Function Fail to Be Differentiable? (2 of 4)

Theorem 4 gives another way for a function not to have a derivative. It says that if *f* is not continuous at *a*, then *f* is not differentiable at *a*. So at any discontinuity (for instance, a jump discontinuity) *f* fails to be differentiable.

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, *f* is continuous at *a* and

$$\lim_{x \to a} |f'(x)| = \infty$$

How Can a Function Fail to Be Differentiable? (3 of 4)

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another.

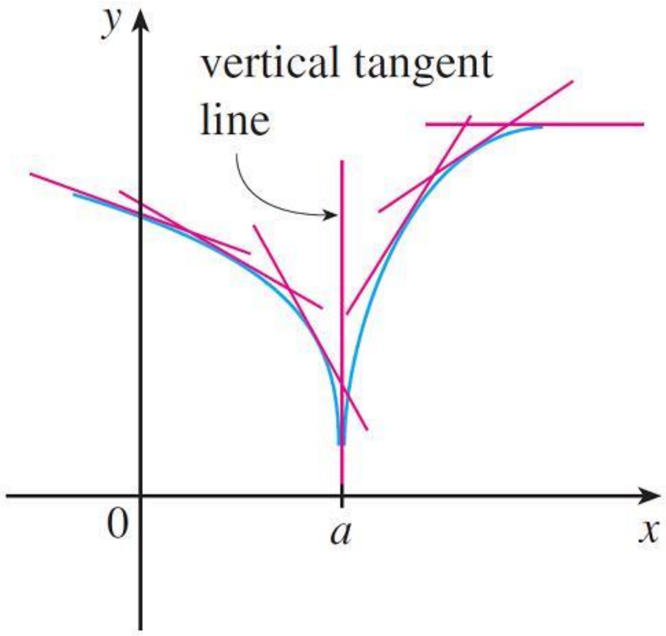
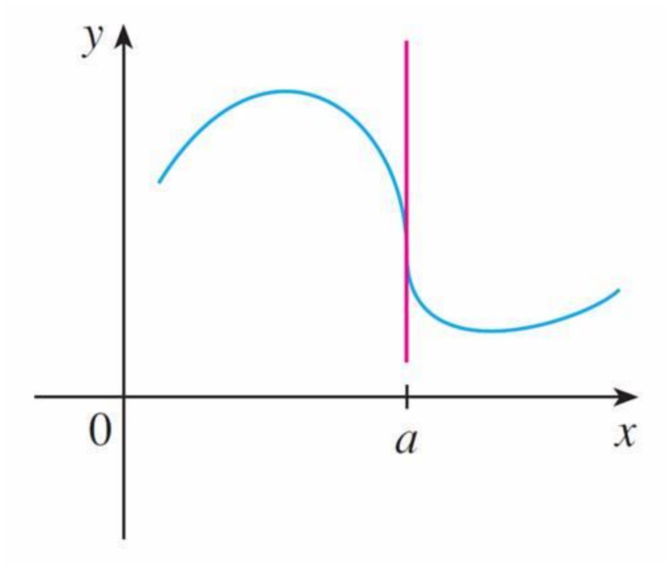


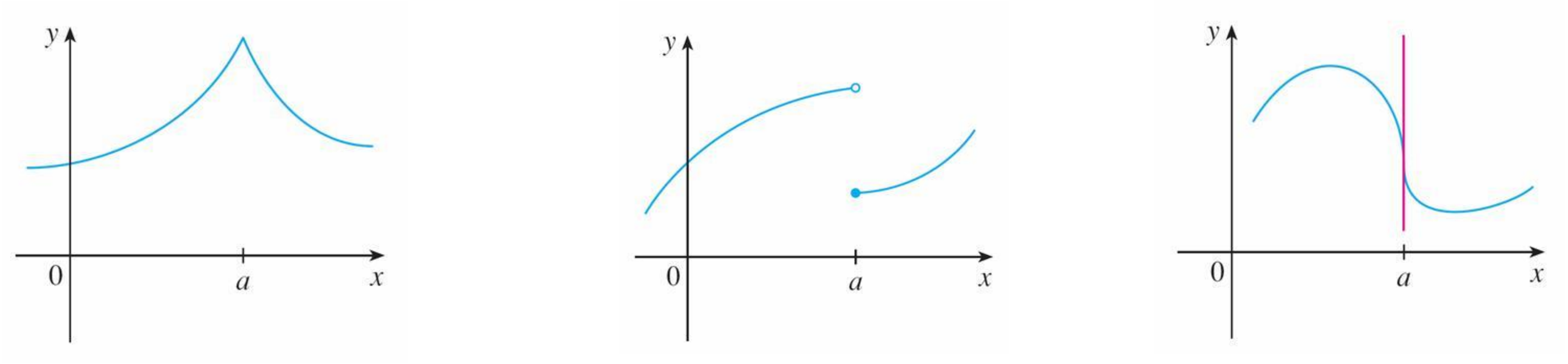
Figure 6



A vertical tangent

Figure 7(c)

Figure 7 illustrates the three possibilities that we have discussed.



(a) A corner (b) A discontinuity (c) A vertical tangent

Three ways for *f* not to be differentiable at *a*

Figure 7

Higher Derivatives

Higher Derivatives (1 of 7)

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f .

Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\underbrace{\frac{d}{dx}}_{\text{derivative of}} \underbrace{\left(\frac{dy}{dx}\right)}_{\text{first derivative}} = \underbrace{\frac{d^2y}{dx^2}}_{\text{second derivative}}$$

Example 6

If $f(x) = x^3 - x$, find and interpret $f''(x)$.

Solution:

The first derivative is $f'(x) = 3x^2 - 1$.

So the second derivative is

$$\begin{aligned} f''(x) &= (f')'(x) \\ &= \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h \to 0} \frac{\left[3(x+h)^2 - 1\right] - \left[3x^2 - 1\right]}{h} \end{aligned}$$

Example 6 – Solution (1 of 2)

$$= \lim_{h \to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h}$$

$$= \lim_{h \to 0} (6x + 3h)$$

$$= 6x$$

The graphs of f, f', and f'' are shown in Figure 10.

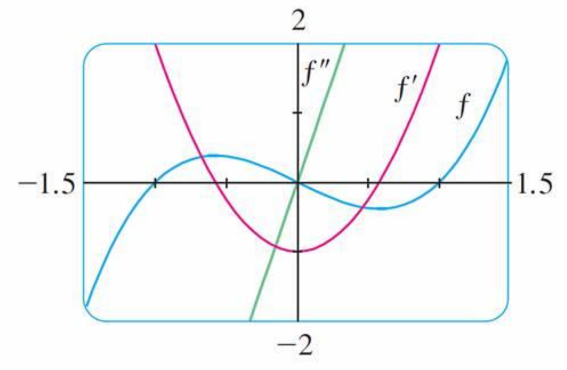


Figure 10

Example 6 – Solution (2 of 2)

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations.

Higher Derivatives (2 of 7)

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration,* which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

Higher Derivatives (3 of 7)

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Higher Derivatives (4 of 7)

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$.

If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

Higher Derivatives (5 of 7)

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line.

Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Higher Derivatives (6 of 7)

Thus the jerk j is the rate of change of acceleration.

It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement.

Higher Derivatives (7 of 7)

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$.

In general, the *n*th derivative of *f* is denoted by $f^{(n)}$ and is obtained from *f* by differentiating *n* times.

If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$