A&DS Worksheet 1 solutions

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1. (1) Given the definition

$$f(n) = O(g(n)) \leftrightarrow \exists n_0 \in \mathbb{N}, c > 0, f(n) \leq c \cdot g(n)\ \forall n \geq n_0$$

we can clearly see that $c \geq 2$ and $n_0 = 1$ satisfies the inequality

$$\forall n \geq n_0, \sqrt{n} + n \leq c \cdot n.$$

(2) Given the definition

$$f(n) = o(g(n)) \leftrightarrow \forall c > 0\ \exists n_0 \in \mathbb{N},\ f(n) \leq c \cdot g(n)\ \forall n \geq n_0$$

we can write n_0 in terms of c

$$\begin{aligned} n^i &\leq c \cdot n^j && \iff \\ 1 &\leq c \cdot n^{j-i} && \iff \\ c^{-1} &\leq n^{j-i} && \iff \\ \sqrt[j-i]{c^{-1}} = c^{\frac{1}{i-j}} &\leq n && \implies \quad \boxed{n_0 = \lceil c^{\frac{1}{i-j}} \rceil} \end{aligned}$$

Now we need to prove that $n^i \leq c \cdot n^j$ for all $n \geq \lceil c^{\frac{1}{i-j}} \rceil$ given that $i < j$. We can do this by induction on n.

Base case: $\mathbf{n} = \lceil \mathbf{c}^{\frac{1}{i-j}} \rceil$

$$\begin{aligned} \left\lceil c^{\frac{1}{i-j}} \right\rceil^i \leq c \cdot \left\lceil c^{\frac{1}{i-j}} \right\rceil^j & \implies \\ \left\lceil c^{\frac{1}{i-j}} \right\rceil^{i-j} \leq c & \implies \\ \left\lceil \left(\frac{1}{c}\right)^{\frac{1}{j-i}} \right\rceil^{j-i} \geq \frac{1}{c} & \implies \quad \left\lceil \sqrt[j-i]{\frac{1}{c}} \right\rceil \geq \sqrt[j-i]{\frac{1}{c}} \end{aligned}$$

holds true only if $i - j$ is < 0, which it is since $i < j$.

Induction step:

$$\begin{aligned} n^i \leq c \cdot n^j & \implies \\ n^{i-j} \leq c & \implies \quad \frac{1}{n^{j-i}} \leq c \end{aligned}$$

Since $j - i > 0$, it's obvious that $n^{j-i} < (n+1)^{j-i}$.

$$\begin{aligned} \frac{1}{(n+1)^{j-i}} \leq \frac{1}{n^{j-i}} \leq c & \implies \\ \frac{1}{(n+1)^{j-i}} \leq c & \implies \\ (n+1)^{i-j} \leq c & \implies \quad (n+1)^i \leq c \cdot (n+1)^j \end{aligned}$$

(3) The fact that if $h(n) = o(g(n))$ then $h(n) = O(g(n))$ also holds true, by definition, we can simply say that

$$\begin{aligned} f(n) + h(n) &= O(\max\{O(g(n)), o(g(n))\}) \qquad \text{(where } f(n) = O(g(n))\text{)} \\ &= O(g(n)) \end{aligned}$$

(4) In the equation $(\log x)^k = o(x)$ we can substitute x with e^x to get

$$\begin{aligned} (x \log e)^k &= o(e^x) \quad \Longrightarrow \\ x^k \cdot \underbrace{(\log e)^k}_{\text{constant}} &= o(e^x) \end{aligned}$$

and by definition $f(n) = o(g(n))$, we can write

$$\forall c > 0\ \exists n_0 \in \mathbb{N}, (\log x)^k \le c \cdot x,\ \forall n \ge n_0$$

notice that we can replace c with $a \cdot t$ in the main condition where $t > 0$ is some constant ($t = (\log e)^k$ in this case).

$$\forall c > 0\ \exists n_0 \in \mathbb{N}, (\log x)^k \le a \cdot t \cdot x,\ \forall n \ge n_0$$

and the statement still holds true.

(5) By the lemma of addition

$$\begin{aligned} O(n^2) + O(n \cdot (\log n)^k) &= O(n^2) \quad \Longrightarrow \\ n \cdot (\log n)^k &= O(n^2) \quad \Longrightarrow \\ (\log n)^k &= O(n) \end{aligned}$$

which we proved in the previous ecxercise.[1]

2. The growth rate (I.E. the derivative) of e^x is e^x itself, meaning that the growth is exponential. However, the derivative of any polynomial x^k is one degree lower $k \cdot x^{k-1}$ than the polynomial itself. Exponential growth is strictly faster than polynomial growth.

   This can be shown by L'Hôpital's rule.

3. For $n = 2^k, k \in \mathbb{N}$

(1) First, the conjecture for a closed form solution. Let $k = 3$

$$\begin{aligned} f(2^k) &= f(2^{k-1}) + b \cdot \log(2^k) && \Longrightarrow \\ f(2^k) &= f(2^{k-2}) + b \cdot \log(2^{k-1}) + b \cdot \log(2^k) && \Longrightarrow \\ f(2^k) &= f(2^{k-3}) + b \cdot \log(2^{k-2}) + b \cdot \log(2^{k-1}) + b \cdot \log(2^k) && \Longrightarrow \\ f(2^k) &= f(1) + b \cdot ((\log(2^k) - (k-1)) + \cdots + (\log(2^k) - 2) + (\log(2^k) - 1) + \log(2^k)) && \Longrightarrow \\ f(2^k) &= a + b \cdot ((k - (k-1)) + \cdots + (k-2) + (k-1) + k) && \Longrightarrow \\ f(2^k) &= a + b \cdot (1 + 2 + \cdots + (k-1) + k) && \Longrightarrow \\ f(2^k) &= a + b \cdot \frac{k(k+1)}{2} && \Longrightarrow \\ f(n) &= a + b \cdot \frac{\log n(\log n + 1)}{2} \end{aligned}$$

The proof (by induction on k)

Base case: $k = 1$

$$\begin{aligned} f(2^1) &= a + b \cdot \frac{1(1+1)}{2} \\ &= a + b \\ &= f(2^1/2) + b \cdot \log(2^1) \end{aligned}$$

[1] This is not a good proof.

Induction step:

$$\begin{aligned}
f(2^k) &= a + b \cdot \frac{k(k+1)}{2} \\
a + b \cdot \frac{k(k+1)}{2} + b \cdot \log(2^{k+1}) &= a + b \cdot \frac{k(k+1)}{2} + b \cdot (k+1) \\
&= a + b \cdot \frac{k(k+1) + 2(k+1)}{2} \\
&= a + b \cdot \frac{(k+1)(k+2)}{2} \\
&= a + b \cdot \frac{(k+1)((k+1)+1)}{2} \\
&= f(2^{k+1})
\end{aligned}$$

Now all that's left to show is that

$$f(n) = a + b \cdot \frac{\log n(\log n + 1)}{2} = O((\log n)^2).$$

and we can do that by the lemma given in the slides

$$\begin{aligned}
a + b \cdot \frac{\log n(\log n + 1)}{2} &= O(1) + O(1) \cdot \frac{O(\log n) \cdot (O(\log n) + O(1))}{O(1)} \\
&= O(1) + O(\log n) \cdot O(\log n) \\
&= O(\log n \cdot \log n) \\
&= O((\log n)^2)
\end{aligned}$$

(2) First, the conjecture for a closed form solution. Let $k = 3$

$$\begin{aligned}
f(2^k) &= 2 \cdot f(2^{k-1}) + b \cdot 2^k \cdot \log(2^k) \\
&= 2 \cdot \big(2 \cdot f(2^{k-2}) + b \cdot 2^{k-1} \cdot \log(2^{k-1})\big) + b \cdot 2^k \cdot \log(2^k) \\
&= 2 \cdot \big(2 \cdot \big(2 \cdot f(2^{k-3}) + b \cdot 2^{k-2} \cdot \log(2^{k-2})\big) + b \cdot 2^{k-1} \cdot \log(2^{k-1})\big) + b \cdot 2^k \cdot \log(2^k) \\
&= 2^3 \cdot a + 2^2 \cdot b \cdot 2^{k-2} \cdot \log(2^{k-2}) + 2^1 \cdot b \cdot 2^{k-1} \cdot \log(2^{k-1}) + 2^0 \cdot b \cdot 2^k \cdot \log(2^k) \\
&= 2^k \cdot a + 2^k \cdot b \left(\log(2^{k-2}) + \log(2^{k-1}) + \log(2^k)\right) \\
&= 2^k \cdot a + 2^k \cdot b \left(\log(2^k) - 2 + \log(2^k) - 1 + \log(2^k)\right) \\
&= 2^k \cdot a + 2^k \cdot b \left(k \cdot \log(2^k) - 2 - 1\right) \\
&= 2^k \cdot a + 2^k \cdot b \left(k^2 - \sum_{i=1}^{k-1} i\right) \\
&= 2^k \cdot \left(a + b\left(k^2 - \frac{k(k-1)}{2}\right)\right) \\
&= 2^k \cdot \left(a + b\left(\frac{2k^2 - k^2 + k)}{2}\right)\right) \\
&= 2^k \cdot \left(a + b\left(\frac{k(k+1)}{2}\right)\right) \\
f(n) &= n \cdot \left(a + \frac{b}{2} \cdot (\log n \cdot (\log n + 1))\right)
\end{aligned}$$

The proof by induction on k:

Base case: k = 1

$$\begin{aligned}
f(2^1) &= 2^1 \cdot \left(a + \frac{b}{2} \cdot 1(1+1)\right) \\
&= 2^1 \cdot (a + b) \\
&= 2 \cdot f(2^0) + b \cdot 2^1 \cdot 1
\end{aligned}$$

Induction step:

$$2 \cdot \underbrace{\left(2^k \cdot \left(a + b\left(\frac{k(k+1)}{2}\right)\right)\right)}_{\text{value for } k} + b \cdot 2^{k+1} \cdot \log(2^{k+1}) = 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2}\right)\right) + b \cdot 2^{k+1} \cdot (k+1)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2}\right) + b(k+1)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2} + (k+1)\right)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1) + 2(k+1)}{2}\right)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{(k+1)(k+2)}{2}\right)\right)$$

$$= f(2^{k+1})$$

now we with the lemma we show that

$$\begin{aligned} n \cdot \left(a + \frac{b}{2} \cdot (\log n \cdot (\log n + 1))\right) &= O(n) \cdot (O(1) + O(1) \cdot (O(\log n) \cdot (O(\log n) + O(1)))) \\ &= O(n) \cdot (O(1) + O(1) \cdot (O(\log n) \cdot (O(\log n)))) \\ &= O(n) \cdot \big(O(1) + O(1) \cdot O((\log n)^2)\big) \\ &= O(n) \cdot \big(O(1) + O((\log n)^2)\big) \\ &= O(n) \cdot O((\log n)^2) \\ &= O(n \cdot (\log n)^2) \end{aligned}$$

(3) The conjecture. Let $k = 3$

$$\begin{aligned} f(2^k) &= 7 \cdot f(2^{k-1}) + b \cdot 2^{2k} \\ &= 7 \cdot \big(7 \cdot f(2^{k-2}) + b \cdot 2^{2k-2}\big) + b \cdot 2^{2k} \\ &= 7 \cdot \left(7 \cdot \left(7 \cdot f(2^{k-3}) + b \cdot 2^{2(k-2)}\right) + b \cdot 2^{2(k-1)}\right) + b \cdot 2^{2k} \\ &= 7^k \cdot f(2^{k-3}) + 7^{k-1} \cdot b \cdot 2^{2(k-2)} + 7^{k-2} \cdot b \cdot 2^{2(k-1)} + 7^{k-3} \cdot b \cdot 2^{2k} \\ &= 7^k \cdot a + b \cdot 7^k 4^k \left(7^{-1} \cdot 2^{2(-2)} + 7^{-2} \cdot 2^{2(-1)} + 7^{-3} \cdot 2^{2(0)}\right) \\ &= 7^k \cdot a + b \cdot 7^k \left(\sum_{i=1}^{k} 7^{-i} \cdot 2^{2i}\right) \\ &= 7^k \cdot a + b \cdot 7^k \left(\sum_{i=1}^{k} \left(\frac{4}{7}\right)^i\right) \\ &= 7^k \cdot a + b \cdot 7^k \cdot \left(\frac{4}{7} \cdot \frac{\left(\frac{4}{7}\right)^k - 1}{\frac{4}{7} - 1}\right) \\ &= 7^k \cdot a + b \cdot 7^k \cdot \left(\frac{4}{-3} \cdot \left(\left(\frac{4}{7}\right)^k - 1\right)\right) \\ &= 7^k \cdot \left(a + \frac{4}{3} b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right)\right) \\ f(n) &= n^{\log(7)} \cdot a + 7^{\log(n)} \cdot \frac{4}{3} b \cdot \left(1 - \left(\frac{4}{7}\right)^{\log(n)}\right) \end{aligned}$$

We can prove this by induction on k.

Base case: k = 1

$$\begin{aligned}
f(2^1) &= 7^1 \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^1\right)\right) \\
&= 7a + \frac{7 \cdot 4}{3}b \cdot \left(1 - \frac{4}{7}\right) \\
&= 7a + \frac{7 \cdot 4}{3}b \cdot \frac{3}{7} \\
&= 7a + 4b \\
&= 7 \cdot f(2^1/2) + b \cdot (2^1)^2
\end{aligned}$$

Induction step:

$$\begin{aligned}
f(2^{k+1}) &= 7 \cdot \left(f(2^k)\right) + b \cdot \left(2^{k+1}\right)^2 \\
&= 7 \cdot \left(7^k \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right)\right)\right) + b \cdot \left(2^{k+1}\right)^2 \\
&= 7^{k+1} \cdot a + 7^{k+1} \cdot \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right) + b \cdot \left(2^{k+1}\right)^2 \\
&= 7^{k+1} \cdot a + b \cdot \left(7^{k+1} \cdot \frac{4}{3}\left(1 - \left(\frac{4}{7}\right)^k\right) + 4^{k+1}\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k\right) + 4^k\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3 \cdot 4^k}{7^{k+1}}\right)\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3}{7} \cdot \frac{4^k}{7^k}\right)\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3}{7} \cdot \left(\frac{4}{7}\right)^k\right)\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 + \left(\frac{4}{7}\right)^k \cdot \left(\frac{3}{7} - 1\right)\right)\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 + \left(\frac{4}{7}\right)^k \cdot \frac{-4}{7}\right)\right) \\
&= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^{k+1}\right)\right) \\
&= 7^{k+1} \cdot a + 7^{k+1} \cdot b \cdot \frac{4}{3}\left(1 - \left(\frac{4}{7}\right)^{k+1}\right) \\
&= 7^{k+1} \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^{k+1}\right)\right)
\end{aligned}$$

now we with the lemma we show that

$$\begin{aligned}
n^{\log 7} \cdot a + n^{\log 7} \cdot \frac{4}{3}b \cdot \left(1 - n^{\log \frac{4}{7}}\right) &= O(n^{\log 7}) \cdot O(1) + O(n^{\log 7}) \cdot O(1) \cdot O(1) \cdot \left(O(1) - O(n^{\log \frac{4}{7}})\right) \\
&= O(n^{\log 7}) + O(n^{\log 7}) \cdot O(n^{\log \frac{4}{7}}) \qquad \text{(since } \log \frac{4}{7} \text{ is a negative power)} \\
&= O(n^{\log 7}) + O(n^{\log 7}) \\
&= O(n^{\log 7})
\end{aligned}$$