Worksheet 7

1. For h to be a ring homomorphism, it needs to satisfy the two conditions:

$$h(a+b) = h(a) +' h(b)$$
$$h(a*b) = h(a) *' h(b)$$

Addition.

$$\begin{aligned} a, b \in S_n, (a +_n b)(i,j) &= a(i,j) + b(i,j) \\ h(a +_n b)(p,q)(i,j) &= (a +_n b)(i + (p-1)(n/2), j + (q-1)(n/2)) \\ &= a(i + (p-1)(n/2), j + (q-1)(n/2)) + \\ &\quad b(i + (p-1)(n/2), j + (q-1)(n/2)) \\ &= h(a)(p,q)(i,j) + h(b)(p,q)(i,j) \end{aligned}$$

where $\quad p, q \in \{1, 2\}, \quad i, j \in [1 : n/2]$.

Multiplication.

$$a, b \in S_n, (a +_n b)(i,j) = a(i,j) + b(i,j)$$
$$\begin{aligned} h(a *_n b)(p,q)(i,j) &= (a *_n b)(i + (p-1)(n/2), j + (q-1)(n/2)) \\ &= \sum_{t=1}^{n} a(i + (p-1)(n/2), t) * b(t, j + (q-1)(n/2)) \\ &= \sum_{k=1}^{2} \sum_{t=1}^{n/2} a(i + (p-1)(n/2), t + (k-1) \cdot (n/2)) * b(t + (k-1) \cdot (n/2), j + (q-1)(n/2)) \\ &= \sum_{k=1}^{2} \sum_{t=1}^{n/2} h(a)(p-1, (k-1))(i,t) * h(b)((k-1), q-1)(t,j) \\ &= \left(\sum_{k=1}^{2} h(a)(p-1, (k-1)) *_{n/2} h(b)((k-1), q-1) \right)(i,j) \\ &= \big(h(a) *_{n/2,2} h(b)\big)(p,q)(i,j) \end{aligned}$$

Multiplicative identity.

$$h(I_n) = \begin{pmatrix} I_{n/2} & 0_{n/2} \\ 0_{n/2} & I_{n/2} \end{pmatrix}$$
$$h(I_n)(p,q)(i,j) = I_n(i + (p-1)(n/2), j + (q-1)(n/2)) \implies h(I_n) = I_{n/2,2}$$

2. Since, for a ring homomorphism f

$$f(a+b) = f(a) +' f(b)$$

must hold, we can say that

$$\begin{aligned} f(x+0) &= f(x) +' f(0) \quad \implies \\ f(x) &= f(x) +' f(0) \quad \implies \\ f(0) &= 0' \end{aligned}$$

3. We can take $R = R' = \mathbb{Z}_{123}$ with $f(x) = 42 \cdot x$.
4. Let

$$t(n) = \sum_{i=0}^{k} \left\lfloor \frac{n}{2^i} \right\rfloor$$

where $k \in \mathbb{N}$ is the size of the binary counter.

5. First, for a decimal counter with k digits we set

- c_i: number of digits changed
- t_i: number of trailing 9s after op_i

and

- Φ_i: $\frac{\text{sum of the digits}}{9}$
- $\Phi_i > 0$: all trailing 9s become 0, but the digit just left of them gets incremented

$$\Phi_i = \Phi_{i-1} - t_{i-1} + \frac{1}{9}$$

- $\Phi_i = 0$: every digit could have just turned from 9 to 0.

$$t_{i-1} = \Phi_{i-1} = k, \quad \Phi_i = \Phi_{i-1} - t_{i-1} < \Phi_{i-1} - t_{i-1} + \frac{1}{9}$$

From this, we can infer that

$$\Phi_i - \Phi_{i-1} \leq \frac{1}{9} - t_{i-1}$$

It's clear that the cost of the i-th operation is

$$c_i = \begin{cases} t_{i-1} + 1 & \text{if } t_{i-1} < k \\ t_{i-1} & \text{if } t_{i-1} = k \end{cases}$$

giving us the upper bound

$$c_i \leq t_{i-1} + 1.$$

Now we just do some arithmetic

$$\begin{aligned} \hat{c}_i &= c_i + \Phi_i - \Phi_{i-1} \\ &\leq t_{i-1} + 1 + \frac{1}{9} - t_{i-1} \\ &= \frac{10}{9} \end{aligned}$$

$$\boxed{\sum_{i=1}^{n} c_i \leq \sum_{i=1}^{n} \frac{10}{9} + \underbrace{\Phi_0 - \Phi_n}_{\text{at most } k} \leq \frac{10}{9} n + k}$$