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Numerical Linear Algebra

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- Quiz Solutions

Example 7.1

Prove if statement below is true, else give counter example:

1. $cond(A^{-1}) = cond(\alpha A),\ a \in \mathbb{R}^{n\times n},\ \alpha \in \mathbb{R},\ \alpha \neq 0;$
2. $cond(A) = 1$ if and only if $A^T A = \alpha I,\ \alpha \in \mathbb{R}, \alpha \neq 0;$
3. $cond_2(A) = cond_2(A^T)$ iff $A \in \mathbb{R}^{n\times n}$;
4. $cond_\infty(A) = cond_1(A)$ iff A is symmetric.

Solution 7.1

1. $cond(A^{-1}) = cond(\alpha A),\ a \in \mathbb{R}^{n\times n},\ \alpha \in \mathbb{R},\ \alpha \neq 0;$

Solution:

$$cond(\alpha A) = \|\alpha A\| \cdot \|(\alpha A)^{-1}\| = |\alpha| \cdot \|A\| \cdot |\alpha^{-1}| \cdot \|A^{-1}\| =$$
$$= \|A\| \cdot \|A^{-1}\| = cond(A^{-1})$$

Solution 7.1

2. $cond(A) = 1$ if and only if $A^T A = \alpha I,\ \alpha \in \mathbb{R}, \alpha \neq 0$;

Solution:

Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$A^T A = A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

But $cond_F(A) = \sqrt{2}$.

Solution 7.1

3. $cond_2(A) = cond_2(A^T)$ iff $A \in \mathbb{R}^{n\times n}$;

Solution:

I. Prove $cond_2(A) = cond_2(A^T) \Rightarrow A \in \mathbb{R}^{n\times n}$

$$cond_2(A) = \|A\|_2\|A^{-1}\|_2$$

The inverse matrix exist only for square matrices. So, $A \in \mathbb{R}^{n\times n}$.

II. Prove $A \in \mathbb{R}^{n\times n} \Rightarrow cond_2(A) = cond_2(A^T)$.
Matrix and its transpose have the same set of eigenvalues. Thus,

$$cond_2(A) = \frac{\sigma_1}{\sigma_2} = cond_2(A^T)$$

Solution 7.1

4. $cond_{\infty}(A) = cond_1(A)$ iff A is symmetric.

Solution:

Consider $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

$A^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & 0 & -1/2 \\ -1/2 & 1 & -1/2 \end{pmatrix}$.

$$cond_{\infty}(A) = \|A\|_{\infty} \cdot \|A^{-1}\|_{\infty} = 3 \cdot 2 = 6$$

$$cond_1(A) = \|A\|_1 \cdot \|A^{-1}\|_1 = 3 \cdot 2 = 6$$

$cond_{\infty}(A) = cond_1(A)$ but A is not symmetric.

Example 7.2

Suppose

$$Ax = b,\ A = \begin{pmatrix} 0.99 & 1 & 1 \\ 1 & 0.9999 & 1 \\ 1 & 1 & 0.999999 \end{pmatrix},\ cond_1(A) \approx 59991, \|b\|_1 = \|A\|_1$$

What is acceptable absolute error in the right hand side and coefficients of linear system $Ax = b$ for obtaining solution with relative error not higher than 10^{-3}?

Example 7.3

Give example of 2×2 matrix such that $cond_2(A) = 10^9$.

Solution:

Use the fact that $cond_2(A) = \frac{\lambda_{max}}{\lambda_{min}}$, where A is a symmetric positive definite matrix.

We want $cond_2(A) = 10^9 \Rightarrow cond_2(A) = \frac{\lambda_{max}}{\lambda_{min}} = 10^9$.

If we take $A = \begin{pmatrix} 10^9 & 0 \\ 0 & 1 \end{pmatrix}$, we get that $\lambda_{max} = 10^9$ *and* $\lambda_{min} = 1$. Thus, $cond_2(A) = \frac{\lambda_{max}}{\lambda_{min}} = 10^9$

Example 7.4

Suppose $v_1 = (1,1,1,1), v_2 = (2,2,2,2), v_3 = (1.5, 0.5, 1.5, 0.5)$. Using different markers for different vectors draw points $(i, v_{1i}), (i, v_{2i}), (i, v_{3i}),\ i = 1,2,3,4$ on a plane. Give example of a vector norm formula such that:

1. $\|v_2 - v_1\| < \|v_3 - v_1\|$
2. $\|v_3 - v_1\| < \|v_2 - v_1\|$

Solution 7.4

1. $\|v_2 - v_1\| < \|v_3 - v_1\|$

Solution:

$(v_2 - v_1) = (1, 1, 1, 1)$
$(v_3 - v_1) = (0.5, -0.5, 0.5, -0.5)$

Check if $\|x\|_* = |x_1| + \sum_{i=2}^{n} |x_i - x_{i-1}|$ is a norm.

1. $\|x\|_* > 0$ when $x \neq 0$ and $\|x\|_* = 0$ iff $x = 0$;
2. $\|kx\|_* = |kx_1| + \sum_{i=2}^{n} |kx_i - kx_{i-1}| = |k||x_1| + |k| \sum_{i=2}^{n} |x_i - x_{i-1}| = |k| (= |x_1| + \sum_{i=2}^{n} |x_i - x_{i-1}|) = |k|\|x\|$ for any scalar k.
3. $\|x + y\|_* = |x_1 + y_1| + \sum_{i=2}^{n} |(x_i + y_i) - (x_{i-1} + y_{i-1})| \leq |x_1| + |y_1| + \sum_{i=2}^{n}(|x_i - x_{i-1}| + |y_i - y_{i-1}|)$ Thus, $\|x\|_*$ is a norm.

   Check if given inequality is satisfied.

   $\|v_2 - v_1\|_* = 1 + 0 + 0 + 0 = 1 < 0.5 + 1 + 1 + 1 = 3.5 = \|v_3 - v_1\|_*$

Solution 7.4

2. $\|v_3 - v_1\| < \|v_2 - v_1\|$

Solution:

$(v_2 - v_1) = (1, 1, 1, 1)$
$(v_3 - v_1) = (0.5, -0.5, 0.5, -0.5)$

Consider 2-norm: $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^2$

$\|v_3 - v_1\|_2 = 1/4 + 1/4 + 1/4 + 1/4 = 1 < 1 + 1 + 1 + 1 = 4 = \|v_2 - v_1\|_2$

Example 7.5

Consider $n \times n$ bi-diagonal matrix with ones on main diagonal and twos on upper diagonal. Show that this matrix is ill-conditioned for $n = 101$.

Example 7.6

Prove: $\|A\| = inf\{\lambda \in \mathbb{R} : \|Ax\| \leq \lambda\|x\|, x \in \mathbb{R}^n\},\ A \in \mathbb{R}^{n\times n}$