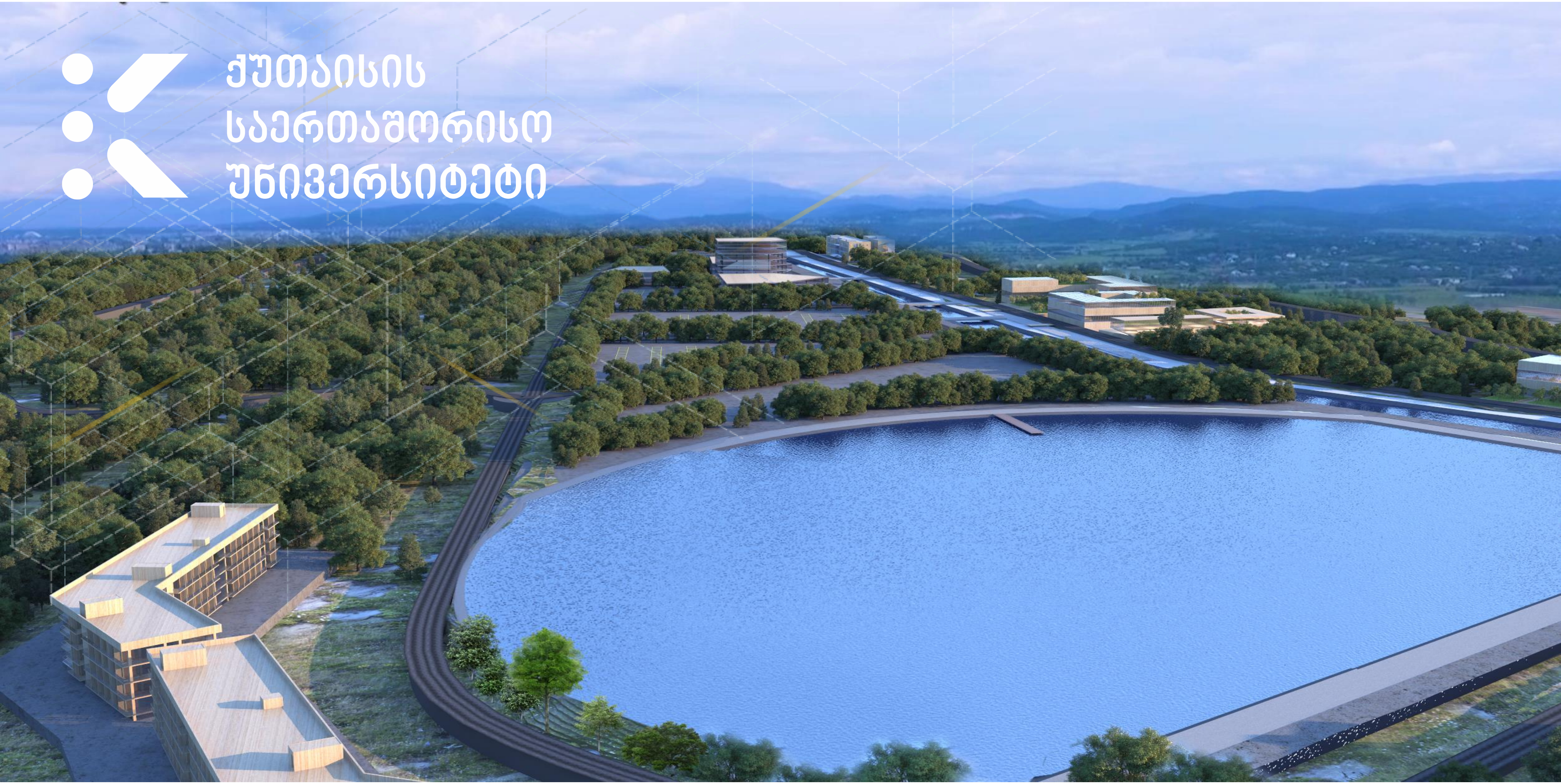
ქუთაისის საერთაშორისო უნივერსიტეტი



2.2 The Limit of a Function

The Limit of a Function (1 of 1)

To find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Finding Limits Numerically and Graphically

Finding Limits Numerically and Graphically (1 of 6)

Let's investigate the behavior of the function *f* defined by $f(x) = (x-1)/(x^2-1)$ for values of *x* near 1.

The following table gives values of *f*(*x*) for values of *x* close to 1 but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
$\downarrow$ 1	$\downarrow$ 0.5	$\downarrow$ 1	$\downarrow$ 0.5

Finding Limits Numerically and Graphically (2 of 6)

From the table and the graph of *f* shown in Figure 1 we see that the closer *x* is to 1 (on either side of 1), the closer *f*(*x*) is to 0.5.

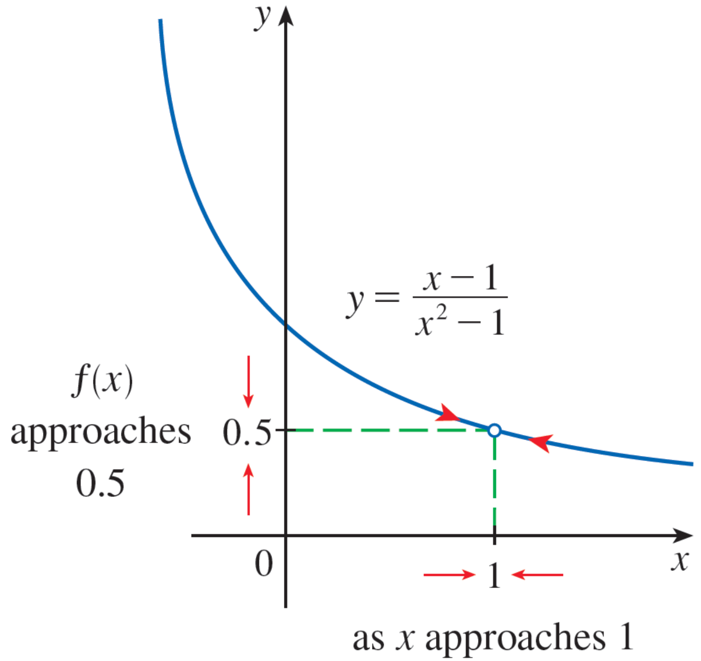


Figure 1

Finding Limits Numerically and Graphically (3 of 6)

In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x sufficiently close to 1.

We express this by saying "the limit of the function $f(x) = (x - 1) / (x^2 - 1)$ as x approaches 1 is equal to 0.5."

The notation for this is

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

Finding Limits Numerically and Graphically (4 of 6)

In general, we use the following notation.

1 Intuitive Definition of a Limit Suppose $f(x)$ is defined when x is near the number a. (This means chat f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

This says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$.

Finding Limits Numerically and Graphically (5 of 6)

An alternative notation for

$$\lim_{x \to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

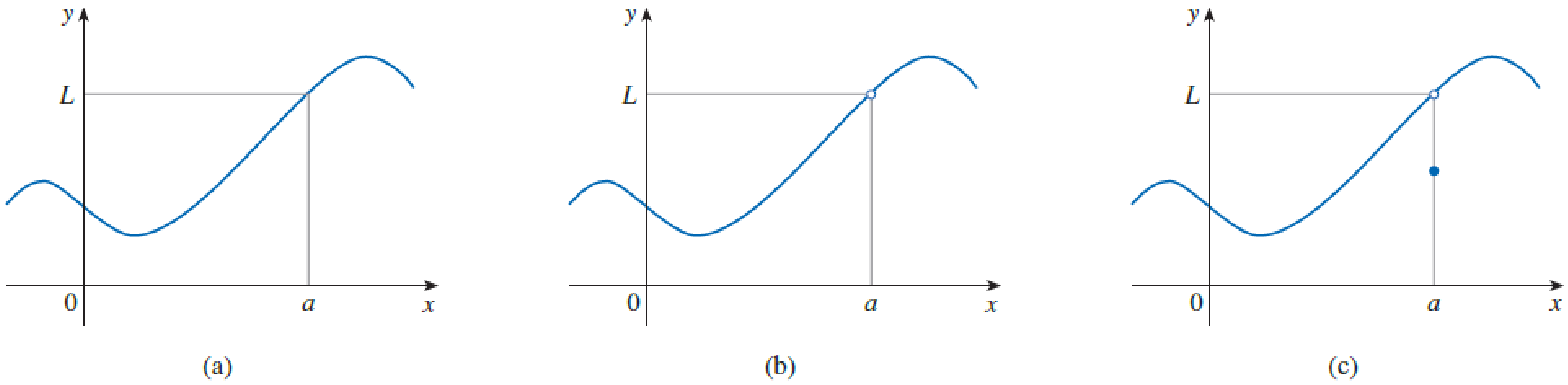
which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but x not equal to a" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near* a.

Finding Limits Numerically and Graphically (6 of 6)

Figure 2 shows the graphs of three functions. Note that in part (b), $f(a)$ is not defined and in part (c), $f(a) \neq L$.

But in each case, regardless of what happens at a, it is true that $\lim_{x \to a} f(x) = L$.



$\lim_{x \to a} f(x) = L$ in all three cases

Figure 2

Example 2

Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

Solution:

The function $f(x) = \sin x / x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}, \sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places.

Example 2 – Solution

From the table below and the graph in Figure 4,

Year	sin *x*/*x*
±1.0	0.84147098
±0.5	0.95885108
±0.4	0.97354586
±0.3	0.98506736
±0.2	0.99334665
±0.1	0.99833417
±0.05	0.99958339
±0.01	0.99998333
±0.005	0.99999583
±0.001	0.99999983

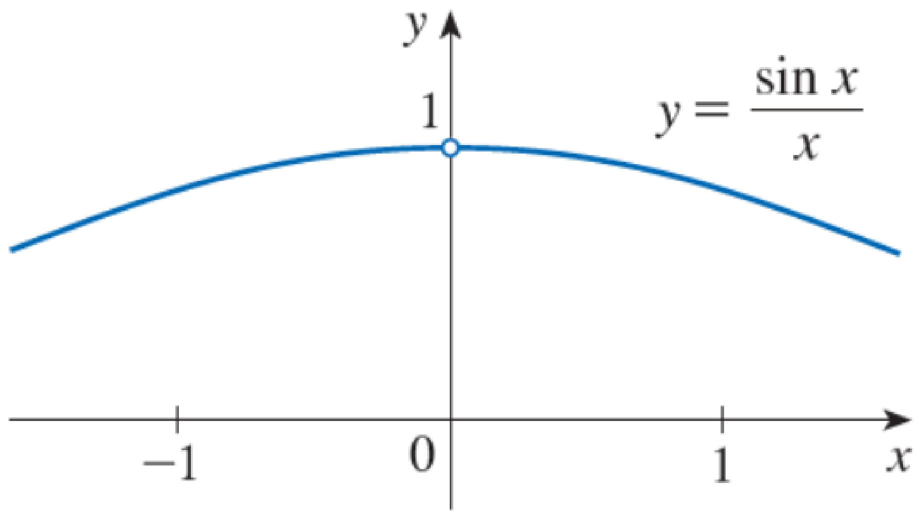


Figure 4

we guess that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

One-Sided Limits

One-Sided Limits (1 of 5)

The Heaviside function *H* is defined by

$$H(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

As *t* approaches 0 from the left, *H*(*t*) approaches 0. As *t* approaches 0 from the right, *H*(*t*) approaches 1.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0 \qquad \text{and} \qquad \lim_{t \to 0^+} H(t) = 1$$

and we call these *one-sided limits*.

One-Sided Limits (2 of 5)

The notation $t \to 0^-$ indicates that we consider only values of *t* that are less than 0.

Likewise, $t \to 0^+$ indicates that we consider only values of *t* that are greater than 0.

One-Sided Limits (3 of 5)

2 Intuitive Definition of One-Sided Limits We write

$$\lim_{x \to a^-} f(x) = L$$

and say the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than* a.

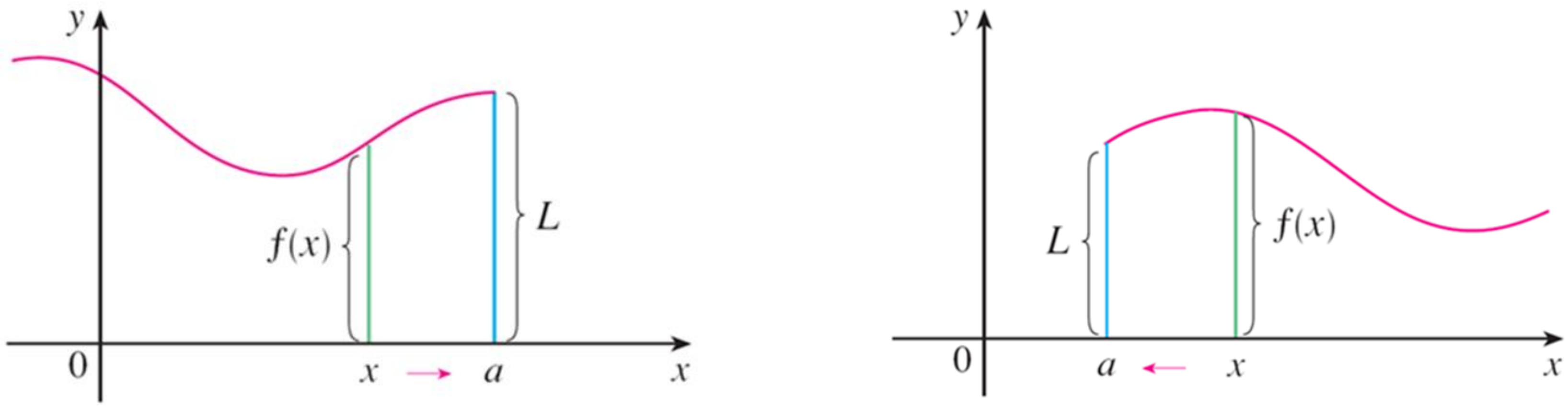
We write

$$\lim_{x \to a^+} f(x) = L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a *from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than* a.

One-Sided Limits (4 of 5)

For instance, the notation $x \to 5^-$ means that we consider only $x < 5$, and $x \to 5^+$ means that we consider only $x > 5$. Definition 2 is illustrated in Figure 6.



(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

Figure 6

One-Sided Limits (5 of 5)

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than (or greater than) a. By comparing these definitions, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

Example 4

The graph of a function *g* is shown in Figure 7. Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$ (d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

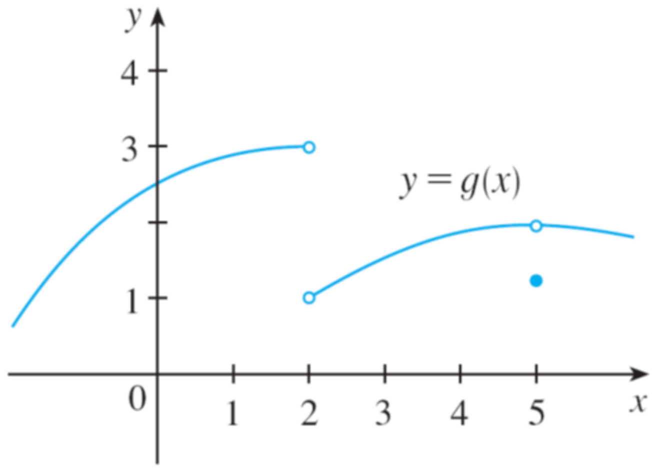


Figure 7

Example 4 – Solution (1 of 2)

Looking at the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right.

Therefore

(a) $\lim_{x \to 2^-} g(x) = 3$ and (b) $\lim_{x \to 2^+} g(x) = 1$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \to 2} g(x)$ does not exist.

Example 4 – Solution (2 of 2)

The graph also shows that

(d) $\lim_{x \to 5^-} g(x) = 2$ and (e) $\lim_{x \to 5^+} g(x) = 2$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$.

How Can a Limit Fail to Exist?

How Can a Limit Fail to Exist? (1 of 1)

We have seen that a limit fails to exist at a number *a* if the left- and right-hand limits are not equal (as in Example 4). The next example illustrate additional ways that a limit can fail to exist.

Example 5

Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$.

Solution:

Notice that the function $f(x) = \sin(\pi / x)$ is undefined at 0. Evaluating the function for some small values of *x*, we get

$$f(1) = \sin \pi = 0 \qquad f\left(\frac{1}{2}\right) = \sin 2\pi = 0$$

$$f\left(\frac{1}{3}\right) = \sin 3\pi = 0 \qquad f\left(\frac{1}{4}\right) = \sin 4\pi = 0$$

$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$.

Example 5 – Solution (1 of 2)

On the basis of this information we might be tempted to guess that the limit is 0, but this time our guess is wrong.

Note that although $f(1/n) = \sin n\pi = 0$ for any integer *n*, it is also true that *f*(*x*) = 1 for infinitely many values of *x* (such as 2/5 or 2/101) that approach 0. You can see this from the graph of *f* shown in Figure 8.

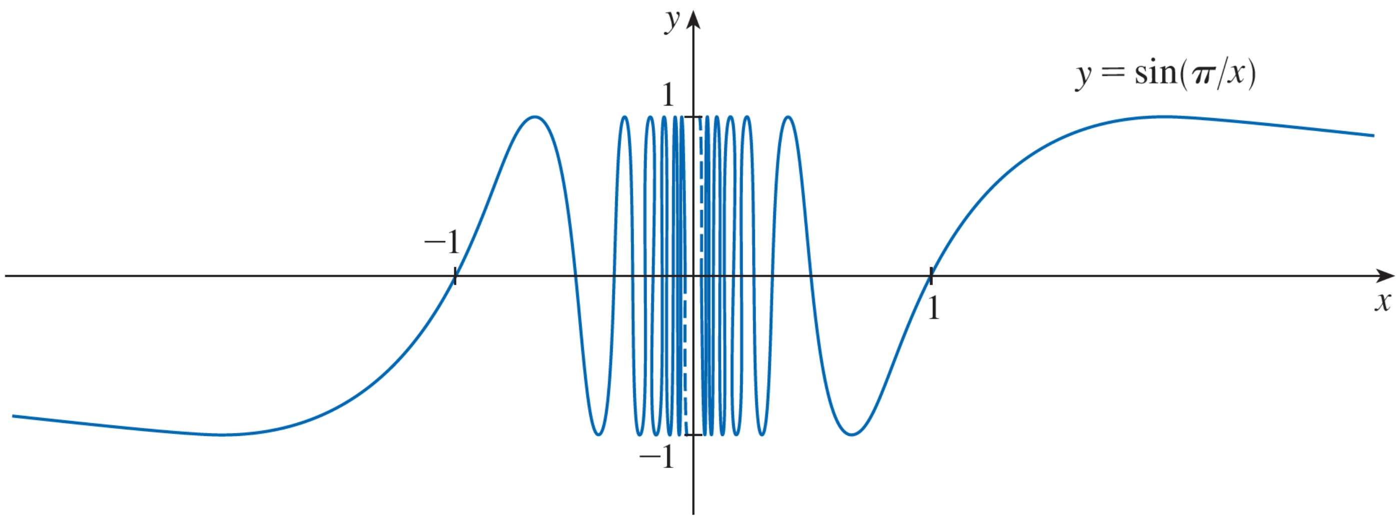


Figure 8

Example 5 – Solution (2 of 2)

The dashed lines near the *y*-axis indicate that the values of $\sin(\pi / x)$ oscillate between 1 and −1 infinitely often as *x* approaches 0. Since the values of *f*(*x*) do not approach a fixed number as *x* approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

Infinite Limits; Vertical Asymptotes

Infinite Limits; Vertical Asymptotes (1 of 8)

4 Intuitive Definition of an Infinite Limit Let *f* be a function defined on both sides of *a*, except possibly at *a* itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of *f*(*x*) can he made arbitrarily large (as large as we please) by taking *x* sufficiently close to *a*, but not equal to *a*.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Infinite Limits; Vertical Asymptotes (2 of 8)

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

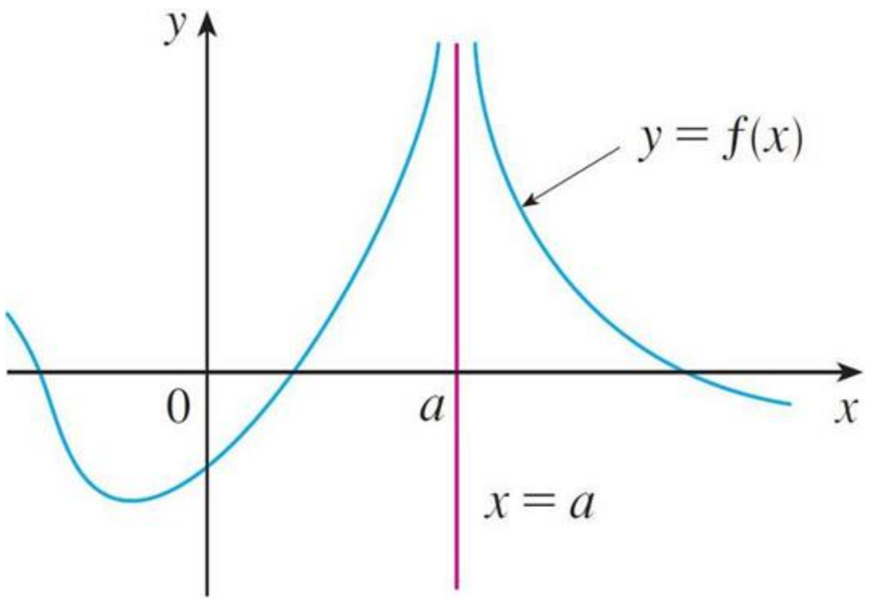
"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

Infinite Limits; Vertical Asymptotes (3 of 8)

This definition is illustrated graphically in Figure 10.

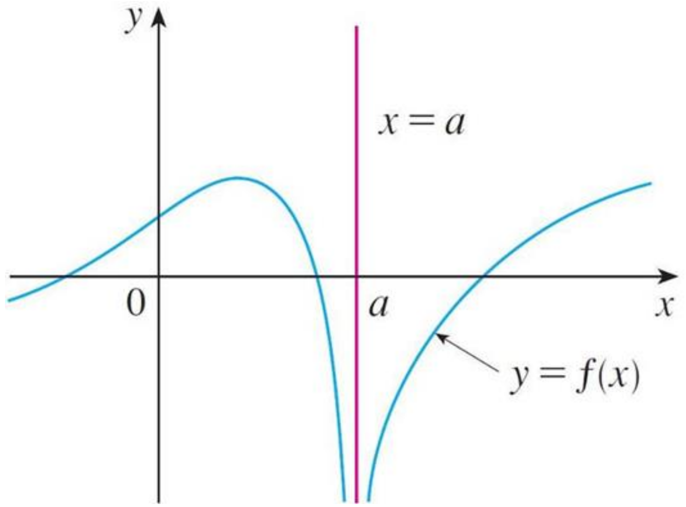


$$\lim_{x \to a} f(x) = \infty$$

Figure 10

Infinite Limits; Vertical Asymptotes (4 of 8)

A similar sort of limit, for functions that become large negative as *x* gets close to *a*, is defined in Definition 5 and is illustrated in Figure 11.



$$\lim_{x \to a} f(x) = -\infty$$

Figure 11

Infinite Limits; Vertical Asymptotes (5 of 8)

5 Definition Let *f* be a function defined on both sides of *a*, except possibly at *a* itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking *x* sufficiently close to *a*, but not equal to *a*.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as *x* approaches *a*, is negative infinity" or "$f(x)$ decreases without bound as *x* approaches *a*." As an example we have

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Infinite Limits; Vertical Asymptotes (6 of 8)

Similar definitions can be given for the one-sided infinite limits

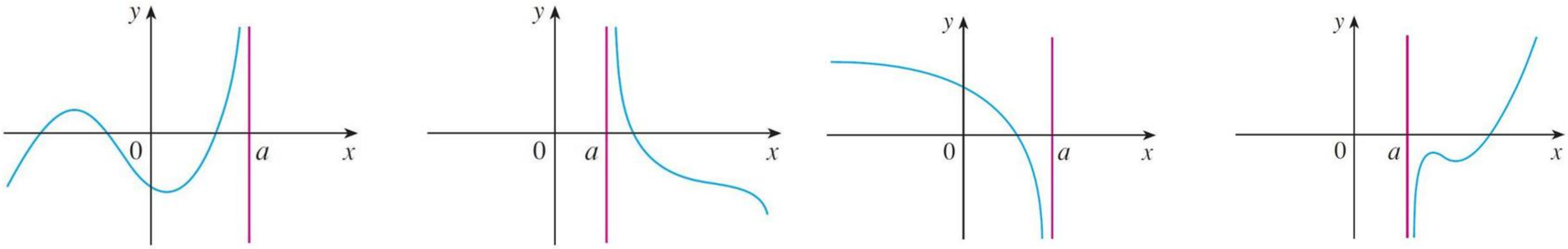
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that $x \to a^-$ means that we consider only values of x that are less than a, and similarly $x \to a^+$ means that we consider only $x > a$.

Infinite Limits; Vertical Asymptotes (7 of 8)

Illustrations of these four cases are given in Figure 12.



(a) $\lim_{x \to a^-} f(x) = \infty$

(b) $\lim_{x \to a^+} f(x) = \infty$

(c) $\lim_{x \to a^-} f(x) = -\infty$

(d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 12

Infinite Limits; Vertical Asymptotes (8 of 8)

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Example 8

Find the vertical asymptotes of $f(x) = \tan x$.

Solution:

Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$.

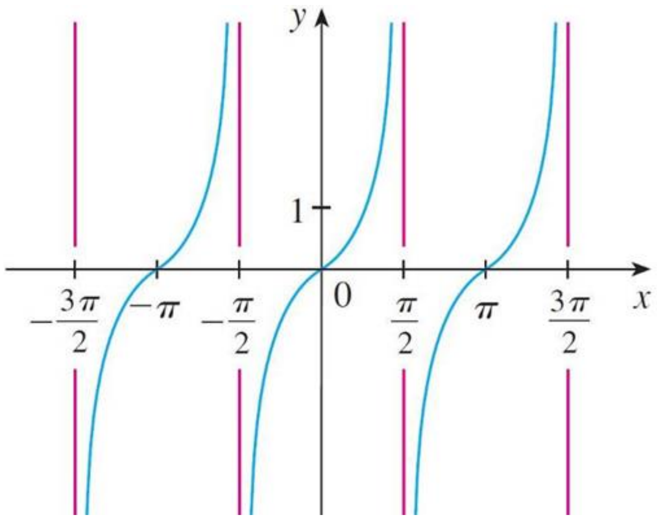
In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive (near 1) when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \quad \text{and} \quad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

Example 8 – Solution

This shows that the line $x = \pi / 2$ is a vertical asymptote. Similar reasoning shows that the lines $x = \pi / 2 + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$.

The graph in Figure 14 confirms this.



$y = \tan x$

Figure 14