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Numerical Linear Algebra

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Vectors and matrices

- Eigenvalues and Eigenvectors
- Systems of Linear Equations

Definition 2.1

A **linear map (linear mapping, linear transformation** is a mapping) $T : V \to W$ between two vector spaces that preserves the operations of vector addition and scalar multiplication.

$$T(u+v) = T(u) + T(v)$$

$$T(cu) = cT(u)$$

where c is a scalar

- A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule T that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$
  - $\mathbb{R}^n$ is called the domain of T
  - $\mathbb{R}^m$ is called the codomain of T
- Let A be an $m \times n$ matrix. The **matrix transformation** associated to A is the transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$.
  - This transformation takes a vector x in $\mathbb{R}^n$ to the vector Ax in $\mathbb{R}^m$

Example 2.2

The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x, y) = (2x, y)$ is a linear map.

Example 2.3

Examples of $\mathbb{R}^2$ linear maps described by 2×2 matrices:

- rotation by 90° counterclockwise:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

- reflection through the x axis:

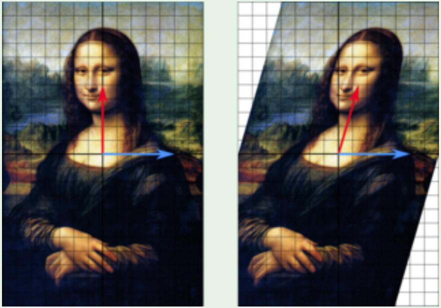
$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Definition 2.4

An **eigenvector or characteristic vector** of a linear transformation is a nonzero vector that changes at most by a scalar factor when that linear transformation is applied to it.
The corresponding eigenvalue, often denoted by λ *, is the factor by which the eigenvector is scaled.*

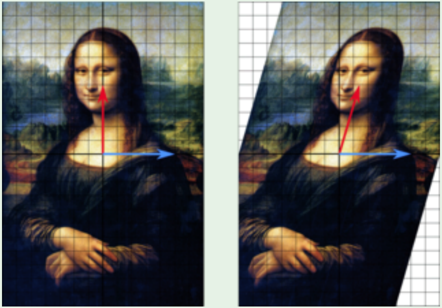
- Geometrically, an eigenvector, corresponding to a real nonzero eigenvalue, points in a direction in which it is stretched by the transformation and the eigenvalue is the factor by which it is stretched.
- If the eigenvalue is negative, the direction is reversed.
- In a multidimensional vector space, the eigenvector is not rotated.

Example 2.5



- Each point on the painting can be represented as a vector pointing from the center of the painting to that point.
- Points in the top half are moved to the right, and points in the bottom half are moved to the left, proportional to how far they are from the horizontal axis that goes through the middle of the painting.
- The vectors pointing to each point in the original image are therefore tilted right or left, and made longer or shorter by the transformation.

Example 2.6



- Points along the horizontal axis do not move at all when this transformation is applied.
- Therefore, any vector that points directly to the right or left with no vertical component is an eigenvector of this transformation, because the mapping does not change its direction.
- Moreover, these eigenvectors all have an eigenvalue equal to one, because the mapping does not change their length either.

Definition 2.7

Let A be an $n \times n$ matrix.

- An **eigenvector** of A is nonzero vector v in $\mathbb{R}^n$ such that $Av = \lambda v$, for some scalar λ
- An **eigenvalue** of A is a scalar λ such that the equation $Av = \lambda v$ has a nontrivial solution

If $Av = \lambda v$ for $v \neq 0$, we say that λ is the eigenvalue for v, and that v is an eigenvector for λ

Finding eigenvectors and eigenvalues

Definition 2.8

Let A be an $n \times n$ matrix.

- Find the eigenvalues λ of A by solving the equation $\det(\lambda I - A) = 0$.
- For each λ, find the basic eigenvectors $X \neq 0$ by finding the basic solutions to $(\lambda I - A)X = 0$

To verify, make sure that $AX = \lambda X$ for each λ and associated eigenvector X.

Example 2.9

Let $A = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}$. Find its eigenvalues and eigenvectors.

Solution:

Firts, find the eigenvalues of A by solving the equation

$$\det(\lambda I - A) = 0)$$

we have

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix} = 0$$

$$\begin{pmatrix} \lambda + 5 & -2 \\ 7 & \lambda - 4 \end{pmatrix} = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\lambda_1 = 2 \text{ and } \lambda_2 = -3$$

Example 2.9 - continued

Now, find the eigenvectors for each λ.
Let's start with $\lambda_1 = 2$.
We have

$$\left[2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}-5 & 2\\ -7 & 4\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}7 & -2\\ 7 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

The solution is any vector of the form $\begin{pmatrix}2\\ 7\end{pmatrix} t$, $t \in \mathbb{R}$

This gives the basic vector for $\lambda_1 = 2$ as $\begin{pmatrix}2\\ 7\end{pmatrix}$.

Example 2.9. (continued)

Now, find the eigenvectors for each $\lambda_2 = -3$.
We have

$$\left[(-3)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}-5 & 2\\ -7 & 4\end{pmatrix}\right]\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}2 & -2\\ 7 & -7\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$$

The solution is any vector of the form $\begin{pmatrix}1\\ 1\end{pmatrix} s$, $s \in \mathbb{R}$

This gives the basic vector for $\lambda_2 = -3$ as $\begin{pmatrix}1\\ 1\end{pmatrix}$.

Example 2.10

Find the eigenvalues and eigenvectors of $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

Find, characteristic polynomial: $\det(\lambda I - A) = 0$

$$\det \begin{pmatrix} \lambda - 1 & 1 & 0 \\ 1 & \lambda - 2 & -1 \\ 0 & -1 & \lambda - 1 \end{pmatrix} = 0$$

$$(\lambda - 1)^2(\lambda - 2) - 2(\lambda - 1) = 0$$

$$(\lambda - 1)(\lambda^2 - 3\lambda) = 0$$

$$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 3$$

Example 2.10.(continued)

Find eigenvector for $\lambda_1 = 1$

We have

$$\begin{pmatrix} 1-1 & 1 & 0 \\ 1 & 1-2 & -1 \\ 0 & -1 & 1-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{cases} y = 0 \\ x - y - z = 0 \\ -y = 0 \end{cases}$$

The solution is any vector of form $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t, t \in \mathbb{R}$

This gives the basic vector for $\lambda_1 = 0$ as $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Example 2.10.(continued)

Find eigenvector for $\lambda_2 = 0$

We have

$$\begin{pmatrix} 0-1 & 1 & 0 \\ 1 & 0-2 & -1 \\ 0 & -1 & 0-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & -1 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{cases} -x + y = 0 \\ x - 2y - z = 0 \\ -y - z = 0 \end{cases}$$

The solution is any vector of form $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} s, s \in \mathbb{R}$

This gives the basic vector for $\lambda_2 = 0$ as $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

Example 2.10.(continued)

Find eigenvector for $\lambda_3 = 3$

We have

$$\begin{pmatrix} 3-1 & 1 & 0 \\ 1 & 3-2 & -1 \\ 0 & -1 & 3-1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\begin{cases} 2x + y = 0 \\ x + y - z = 0 \\ -y + 2z = 0 \end{cases}$$

The solution is any vector of form $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} r, r \in \mathbb{R}$

This gives the basic vector for $\lambda_3 = 3$ as $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

Systems of linear equations

Definition 2.11

A **linear equation** in variables $x_1, x_2, ..., x_n$ is an equation of the form

$$a_1x_1 + a_2x_2 + ... + a_nx_n = b$$

where $a_1, a_2, ..., a_n$ and b are called **constant term** of the equation.
A **system of linear equations (or linear system)** is a finite collection of linear equations in same variables.
For instance, a linear system of m equations in n variables $x_1, x_2, ..., x_n$ can be written as

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = b_2 \\ \quad ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n = b_m \end{cases}$$

Definition 2.12

- A **solution** of a linear system is a tuple $(s_1, s_2, ..., s_n)$ of numbers that makes each equation a true statement when the values $s_1, s_2, ..., s_n$ are substituted for $x_1, x_2, ..., x_n$ respectively.
- The set of all solutions of a linear system is called the **solution set** of the system
- A linear system is said to be consistent if it has at least one solution; and is said to be inconsistent if it has no solution.

Definition 2.13

We can write a linear system in the matrix form:

$$Ax = b$$

where A is a $m \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

x is a column vector with n entries, and b is a column vector with m entries.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \; b = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{pmatrix}$$

Example 2.14

Use Gaussian elimination to find the solution for the given system of equations:

$$\begin{cases} 3x + y - z = 1 \\ x - y + z = -3 \\ 2x + y + z = 0 \end{cases}$$

Solution:

Setup the augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{array}\right]$$

Perform row operations to reduce the matrix.

We can use only three elementary operations:

1. Row swap;
2. Scalar multiplication;
3. Row sum.

Example 2.14 - continued

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 1 & -1 & 1 & -3 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 3 & 1 & -1 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-3R_1+R_2 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & 4 & -4 & 10 \\ 2 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & 4 & -4 & 10 \\ 0 & 3 & -1 & 6 \end{array}\right] \xrightarrow{R_2 \cdot \frac{1}{4} \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & -3 \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 3 & -1 & 6 \end{array}\right] \xrightarrow{R_2+R_1 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 3 & -1 & 6 \end{array}\right] \xrightarrow{-3R_2+R_3 \to R_3}$$

Example 2.14 - continued

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 2 & -\frac{3}{2} \end{array}\right] \xrightarrow{R_3\cdot\frac{1}{2}\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & -1 & \frac{5}{2} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right] \xrightarrow{R_3+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{7}{4} \\ 0 & 0 & 1 & -\frac{3}{4} \end{array}\right]$$

Therefore, the solution is $(x, y, z) = (-\frac{1}{2}, \frac{7}{4}, -\frac{3}{4})$

Vector norms

Definition 2.15

A general vector norm, denoted by $\|x\|$, is a nonnegative norm defined such that

1. $\|\mathrm{x}\| > 0$ when $\mathrm{x} \neq 0$ and $\|\mathrm{x}\| = 0$ iff $\mathrm{x} = 0$;
2. $\|k\mathrm{x}\| = |k|\|\mathrm{x}\|$ for any scalar k;
3. $\|\mathrm{x{+}y}\| \leq \|\mathrm{x}\| + |\mathrm{y}\|$

The vector norm $\|\mathrm{x}\|_p$ for $p = 1, 2, \ldots$ is defined as

$$\|\mathrm{x}\|_p = \left(\sum_i |x_i|^p\right)^{1/p}$$

The special case $\|\mathrm{x}\|_\infty$ is defined as

$$\|\mathrm{x}\|_\infty = \max_i |x_i|$$

Example 2.16

Find the $\|.\|_1$, $\|.\|_2$, $\|.\|_3$, $\|.\|_\infty$, $\|.\|_A$ norms for the vector $v = (2, 0, -1)^T$ if $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Solution:

$$\|v\|_1 = \|(2, 0, -1)^T\|_1 = |2| + |0| + |-1| = 3$$
$$\|v\|_2 = \|(2, 0, -1)^T\|_2 = (|2|^2 + |0|^2 + |-1|^2)^{1/2} = \sqrt{5}$$
$$\|v\|_3 = \|(2, 0, -1)^T\|_3 = (|2|^3 + |0|^3 + |-1|^3)^{1/3} = 9^{1/3}$$
$$\|v\|_\infty = \|(2, 0, -1)^T\|_\infty = \max_{1 \le i \le 3} |v_i| = \max(|2|, |0|, |-1|) = 2$$
$$\|v\|_A = \|(2, 0, -1)^T\|_A = (Av, v)^{1/2} =$$
$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} (2, 0, -1)^T, (2, 0, -1) \right) = ((4, 1, -2), (2, 0, -1)) =$$
$$4 \cdot 2 + 1 \cdot 0 + (-2) \cdot (-1) = 10$$

Matrix norms

Given a matrix A. A matrix norm $\|A\|$ is a nonnegative number associated with A having the properties:

1. $\|A\| > 0$ when $A \neq 0$ and $\|A\| = 0$ iff $A = 0$;
2. $\|kA\| = |k|\|A\|$ for any scalar k;
3. $\|A + B\| \leq \|A\| + \|B\|$

- 1-norm or maximum absolute column sum norm is defined as

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|$$

- ∞-norm or maximum absolute row sum norm is defined by

$$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

- 2-norm or spectral norm:

$$\|A\|_2 = \sqrt{\rho(A^T A)} \text{ i.e.} \|A\|_2 = \sqrt{\text{maximum eigenvalue of } A^T A}$$

- Frobenius norm:

$$\|A\|_F = \sqrt{\sum_{i,j=1}^{n} |a_{ij}|^2}$$

Example 2.17

Find the $\|.\|_1$, $\|.\|_\infty$, norms for the matrix $\begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Solution:

- $$\|A\|_1 = \left\| \begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\|_1 = \max_{j=1,2,3} \sum_{i=1}^{3} |a_{ij}| =$$
  $$= \max\left[(1+1+0), (3+0+1), (0+1+1)\right] = 4$$
- $$\|A\|_\infty = \left\| \begin{pmatrix} 1 & 3 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\|_\infty = \max_{i=1,2,3} \sum_{j=1}^{3} |a_{ij}| =$$
  $$= \max\left[(1+3+0), (1+0+1), (0+1+1)\right] = 4$$

Example 2.18

Find the $\|.\|_2$ norm for the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

Solution:

$$\|A\|_2 = \sqrt{\rho(A^T A)} = \sqrt{\rho\left(\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\right)} = \sqrt{\rho\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}}$$

Eigenvalues of $\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ are 9 and 1. So,

$$\sqrt{\rho\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}} = \sqrt{\max(9,1)} = \sqrt{9} = 3$$