1. We can rewrite the system in matrix form:

$$\begin{cases} 2x_1 + x_2 = 3 \\ x_1 + 2x_2 + x_3 = -2 \\ 2x_2 + 3x_3 = 0 \end{cases} \implies \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}.$$

Now we do LU factorization on the matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix} = LU = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 1 & \alpha_2 & 0 \\ 0 & 2 & \alpha_3 \end{pmatrix} \begin{pmatrix} 1 & \beta_1 & 0 \\ 0 & 1 & \beta_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{aligned} 1 \cdot \alpha_1 = 2 \qquad & \implies \alpha_1 = 2 \\ \beta_1 \cdot \alpha_1 = \beta_1 \cdot 2 = 1 \qquad & \implies \beta_1 = 1/2 \\ \beta_1 + \alpha_2 = 1/2 + \alpha_2 = 2 \qquad & \implies \alpha_2 = 3/2 \\ \beta_2 \cdot \alpha_2 = \beta_2 \cdot 3/2 \qquad & \implies \beta_2 = 2/3 \\ 2\beta_2 + \alpha_3 = 4/3 + \alpha_3 = 3 \qquad & \implies \alpha_3 = 5/3 \end{aligned}$$

giving us

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 0 & 2 & 5/3 \end{pmatrix}, U = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, we do back-substiturion and forward-substiturion separately on U and L.

$$U\vec{t} = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \implies \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \qquad \text{(via back-substiturion)}$$

$$LU\vec{t} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 0 & 2 & 5/3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix} \implies \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -8/3 \\ 16/5 \end{pmatrix} \qquad \text{(via forward-substiturion)}$$

2. (a)

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{pmatrix} \xrightarrow{r_2 - 2r_1, r_3 - r_1} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{pmatrix} \xrightarrow{r_2 - 2r_1, r_3 + r_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 2 & 2 \end{pmatrix} \xrightarrow{r_3 - r_2 \cdot 1/2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ -1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1/2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 1/2 \end{pmatrix}$$

3. (a)

$$PA = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 2 \\ 1 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

(b)

$$PA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 & 2 \\ -1 & -1 & 1 & 5 \\ 2 & 2 & 3 & 7 \\ 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & 4 & 5 \\ 2 & 2 & 3 & 7 \\ -1 & -1 & 1 & 5 \end{pmatrix}$$

4.

$$\begin{pmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{pmatrix} \xrightarrow{r_2 - r_1, r_3 - r_2 \cdot 1/2} \begin{pmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, U = \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$$