KUTAISI INTERNATIONAL UNIVERSITY

Homework 3
Numerical Linear Algebra
Dimitri Tabatadze
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Part 1

1. To find the condition number, we first choose a norm. Let's pick $\|\cdot\|_1$ for simplicity sake. Let's also precompute the inverse of A:

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} 2 & -2 \\ -1.0001 & 1 \end{pmatrix} = \begin{pmatrix} 2/-0.0002 & 2/0.0002 \\ 1.0001/0.0002 & 1/0.0002 \end{pmatrix} = \begin{pmatrix} -10000 & 10000 \\ 5000.5 & 5000 \end{pmatrix}$$

$$K(A) = \|A\|_1\|A^{-1}\|_1 = 4 \cdot 15000.5 = 60002$$

2. First, let's find the inverse of the matrix $A = \begin{pmatrix} 8 & 5 & 2 \\ 21 & 19 & 16 \\ 39 & 48 & 53 \end{pmatrix}$.

$$\begin{aligned} \det(A) &= 8 \cdot \det\begin{pmatrix} 19 & 16 \\ 48 & 53 \end{pmatrix} - 5 \cdot \det\begin{pmatrix} 21 & 16 \\ 39 & 53 \end{pmatrix} + 2 \cdot \det\begin{pmatrix} 21 & 19 \\ 39 & 48 \end{pmatrix} \\ &= 8 \cdot (19 \cdot 53 - 16 \cdot 48) - 5 \cdot (21 \cdot 53 - 16 \cdot 39) + 2 \cdot (21 \cdot 48 - 19 \cdot 39) \\ &= 1912 - 2445 + 534 \\ &= 1 \end{aligned}$$

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = \begin{pmatrix} 239 & -169 & 42 \\ -489 & 346 & -86 \\ 267 & -189 & 47 \end{pmatrix}$$

Now, to find $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we multiply $\begin{pmatrix} 15 \\ 56 \\ 140 \end{pmatrix}$ by the inverse

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 15 \\ 56 \\ 140 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and now if we change the 15 to 14, we get

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = A^{-1}\begin{pmatrix} 14 \\ 56 \\ 140 \end{pmatrix} = \begin{pmatrix} -238 \\ 490 \\ -266 \end{pmatrix}.$$

It's clear that with relatively tiny perturbation, the error is very big, therefore this matrix can be considered to be ill-conditioned.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}$

$$Ax = b \implies x = A^{-1}b = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

now, if we perturbate b by $\Delta b = \begin{pmatrix} 0.0001 \\ 0.0001 \end{pmatrix}$ to get $\tilde{b} = b + \Delta b = \begin{pmatrix} 3.0001 \\ 7.0001 \end{pmatrix}$

$$\tilde{x} = A^{-1}\tilde{b} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3.0001 \\ 7.0001 \end{pmatrix} = \begin{pmatrix} 1.0001 \\ 1.0001 \end{pmatrix}$$

We can see that by tiny perturbation, the error is also very tiny, so the matrix can be considered well-conditioned.