Course: **Calculus 1 - CS**

Calculus: Early Transcendentals - James Stewart, Daniel Clegg, Saleem Watson (**Reader**) – Section 2.1; 2.2; 2.3; 2.5

CALCULUS

EARLY TRANSCENDENTALS

NINTH EDITION

Metric Version

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We know that when an object is dropped from a height it falls faster and faster. Galileo discovered that the distance the object has fallen is proportional to the square of the time elapsed. Calculus enables us to calculate the precise speed of the object at any time. In Exercise 2.7.11 you are asked to determine the speed at which a cliff diver plunges into the ocean.

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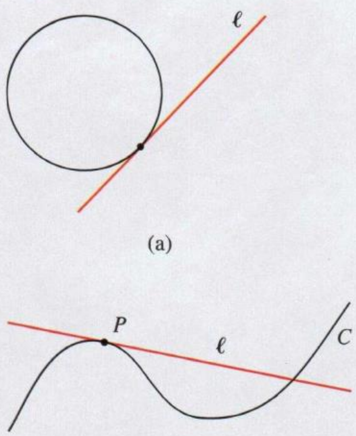
2 Limits and Derivatives

IN *A PREVIEW OF CALCULUS* (immediately preceding Chapter 1) we saw how the idea of a limit underlies the various branches of calculus. It is therefore appropriate to begin our study of calculus by investigating limits and their properties. The special type of limit that is used to find tangents and velocities gives rise to the central idea in differential calculus, the derivative.

2.1 The Tangent and Velocity Problems

In this section we see how limits arise when we attempt to find the tangent to a curve or the velocity of an object.



(a)

(b)

FIGURE 1

The Tangent Problem

The word *tangent* is derived from the Latin word *tangens*, which means "touching." We can think of a tangent to a curve as a line that touches the curve and follows the same direction as the curve at the point of contact. How can this idea be made precise?

For a circle we could simply follow Euclid and say that a tangent is a line ℓ that intersects the circle once and only once, as in Figure 1(a). For more complicated curves this definition is inadequate. Figure 1(b) shows a line ℓ that appears to be a tangent to the curve C at point P, but it intersects C twice.

To be specific, let's look at the problem of trying to find a tangent line ℓ to the parabola $y = x^2$ in the following example.

EXAMPLE 1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $P(1, 1)$.

SOLUTION We will be able to find an equation of the tangent line ℓ as soon as we know its slope m. The difficulty is that we know only one point, P, on ℓ, whereas we need two points to compute the slope. But observe that we can compute an approximation to m by choosing a nearby point $Q(x, x^2)$ on the parabola (as in Figure 2) and computing the slope m_{PQ} of the secant line PQ. (A **secant line**, from the Latin word *secans*, meaning cutting, is a line that cuts [intersects] a curve more than once.)

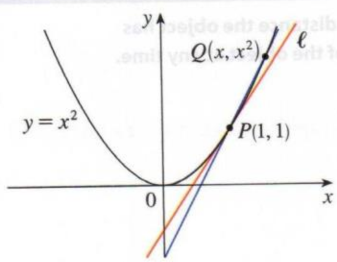


FIGURE 2

We choose $x \neq 1$ so that $Q \neq P$. Then

$$m_{PQ} = \frac{x^2 - 1}{x - 1}$$

For instance, for the point $Q(1.5, 2.25)$ we have

$$m_{PQ} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$$

x	m_{PQ}
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

x	m_{PQ}
0	1
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999

The tables in the margin show the values of m_{PQ} for several values of x close to 1. The closer Q is to P, the closer x is to 1 and, it appears from the tables, the closer m_{PQ} is to 2. This suggests that the slope of the tangent line ℓ should be $m = 2$.

We say that the slope of the tangent line is the *limit* of the slopes of the secant lines, and we express this symbolically by writing

$$\lim_{Q \to P} m_{PQ} = m \qquad \text{and} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$

Assuming that the slope of the tangent line is indeed 2, we use the point-slope form of the equation of a line [$y - y_1 = m(x - x_1)$, see Appendix B] to write the equation of the tangent line through $(1, 1)$ as

$$y - 1 = 2(x - 1) \qquad \text{or} \qquad y = 2x - 1$$ ■

Figure 3 illustrates the limiting process that occurs in Example 1. As Q approaches P along the parabola, the corresponding secant lines rotate about P and approach the tangent line ℓ.

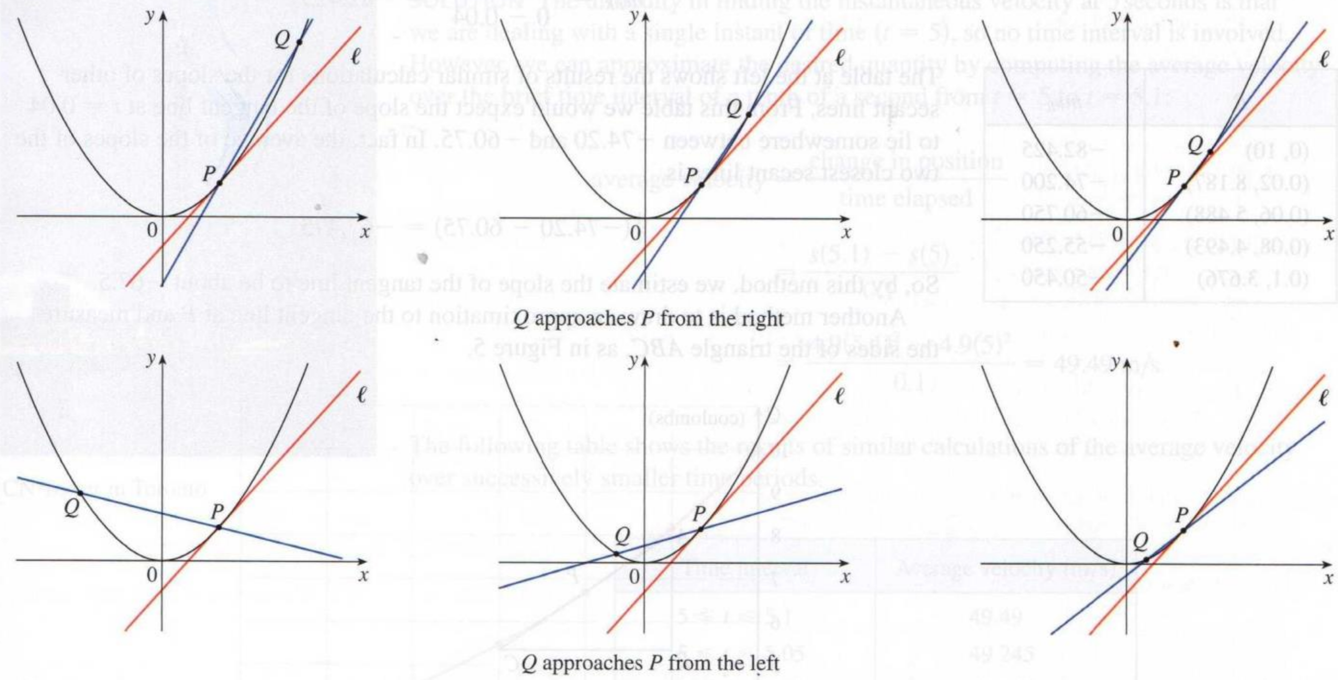


FIGURE 3

Many functions that occur in the sciences are not described by explicit equations; they are defined by experimental data. The next example shows how to estimate the slope of the tangent line to the graph of such a function.

t	Q
0	10
0.02	8.187
0.04	6.703
0.06	5.488
0.08	4.493
0.1	3.676

EXAMPLE 2 A pulse laser operates by storing charge on a capacitor and releasing it suddenly when the laser is fired. The data in the table describe the charge Q remaining on the capacitor (measured in coulombs) at time t (measured in seconds after the laser is fired). Use the data to draw the graph of this function and estimate the slope of the tangent line at the point where $t = 0.04$. (*Note:* The slope of the tangent line represents the electric current flowing from the capacitor to the laser [measured in amperes].)

SOLUTION In Figure 4 we plot the given data and use these points to sketch a curve that approximates the graph of the function.

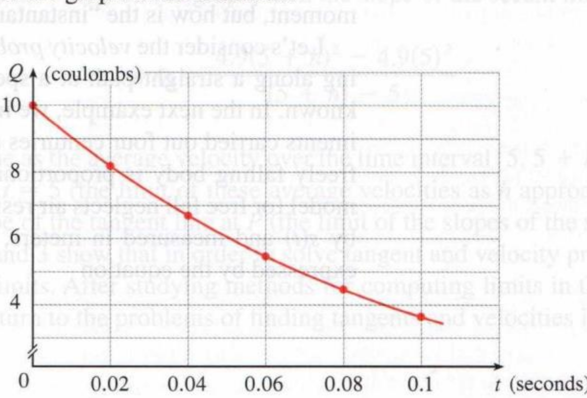


FIGURE 4

Given the points $P(0.04, 6.703)$ and $R(0, 10)$ on the graph, we find that the slope of the secant line PR is

$$m_{PR} = \frac{10 - 6.703}{0 - 0.04} = -82.425$$

R	m_{PR}
(0, 10)	−82.425
(0.02, 8.187)	−74.200
(0.06, 5.488)	−60.750
(0.08, 4.493)	−55.250
(0.1, 3.676)	−50.450

The table at the left shows the results of similar calculations for the slopes of other secant lines. From this table we would expect the slope of the tangent line at $t = 0.04$ to lie somewhere between -74.20 and -60.75. In fact, the average of the slopes of the two closest secant lines is

$$\tfrac{1}{2}(-74.20 - 60.75) = -67.475$$

So, by this method, we estimate the slope of the tangent line to be about -67.5.

Another method is to draw an approximation to the tangent line at P and measure the sides of the triangle ABC, as in Figure 5.

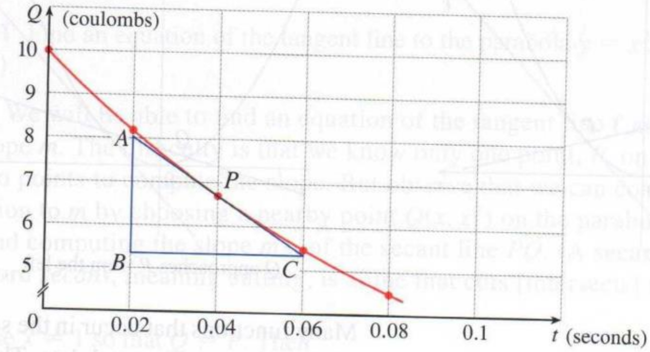


FIGURE 5

The physical meaning of the answer in Example 2 is that the electric current flowing from the capacitor to the laser after 0.04 seconds is about −65 amperes.

This gives an estimate of the slope of the tangent line as

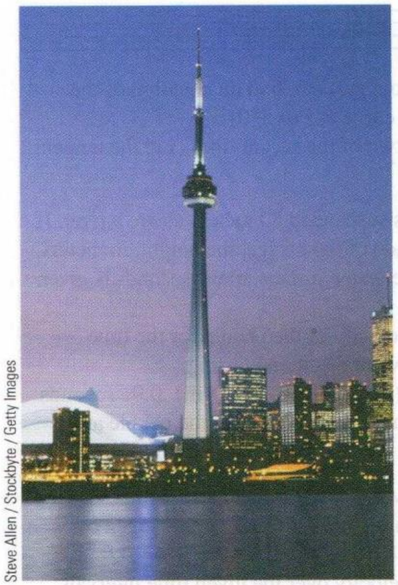
$$-\frac{|AB|}{|BC|} \approx -\frac{8.0 - 5.4}{0.06 - 0.02} = -65.0$$ ■

■ The Velocity Problem

If you watch the speedometer of a car as you drive in city traffic, you see that the speed doesn't stay the same for very long; that is, the velocity of the car is not constant. We assume from watching the speedometer that the car has a definite velocity at each moment, but how is the "instantaneous" velocity defined?

Let's consider the *velocity problem:* Find the instantaneous velocity of an object moving along a straight path at a specific time if the position of the object at any time is known. In the next example, we investigate the velocity of a falling ball. Through experiments carried out four centuries ago, Galileo discovered that the distance fallen by any freely falling body is proportional to the square of the time it has been falling. (This model for free fall neglects air resistance.) If the distance fallen after t seconds is denoted by $s(t)$ and measured in meters, then (at the earth's surface) Galileo's observation is expressed by the equation

$$s(t) = 4.9t^2$$



Steve Allen / Stockbyte / Getty Images

CN Tower in Toronto

EXAMPLE 3 Suppose that a ball is dropped from the upper observation deck of the CN Tower in Toronto, 450 m above the ground. Find the velocity of the ball after 5 seconds.

SOLUTION The difficulty in finding the instantaneous velocity at 5 seconds is that we are dealing with a single instant of time ($t = 5$), so no time interval is involved. However, we can approximate the desired quantity by computing the average velocity over the brief time interval of a tenth of a second from $t = 5$ to $t = 5.1$:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

$$= \frac{s(5.1) - s(5)}{0.1}$$

$$= \frac{4.9(5.1)^2 - 4.9(5)^2}{0.1} = 49.49 \text{ m/s}$$

The following table shows the results of similar calculations of the average velocity over successively smaller time periods.

Time interval	Average velocity (m/s)
$5 \leqslant t \leqslant 5.1$	49.49
$5 \leqslant t \leqslant 5.05$	49.245
$5 \leqslant t \leqslant 5.01$	49.049
$5 \leqslant t \leqslant 5.001$	49.0049

It appears that as we shorten the time period, the average velocity is becoming closer to 49 m/s. The **instantaneous velocity** when $t = 5$ is defined to be the *limiting value* of these average velocities over shorter and shorter time periods that start at $t = 5$. Thus it appears that the (instantaneous) velocity after 5 seconds is 49 m/s. ■

You may have the feeling that the calculations used in solving this problem are very similar to those used earlier in this section to find tangents. In fact, there is a close connection between the tangent problem and the velocity problem. If we draw the graph of the distance function of the ball (as in Figure 6) and we consider the points $P(5, 4.9(5)^2)$ and $Q(5 + h, 4.9(5 + h)^2)$ on the graph, then the slope of the secant line PQ is

$$m_{PQ} = \frac{4.9(5 + h)^2 - 4.9(5)^2}{(5 + h) - 5}$$

which is the same as the average velocity over the time interval $[5, 5 + h]$. Therefore the velocity at time $t = 5$ (the limit of these average velocities as h approaches 0) must be equal to the slope of the tangent line at P (the limit of the slopes of the secant lines).

Examples 1 and 3 show that in order to solve tangent and velocity problems we must be able to find limits. After studying methods for computing limits in the next five sections, we will return to the problems of finding tangents and velocities in Section 2.7.

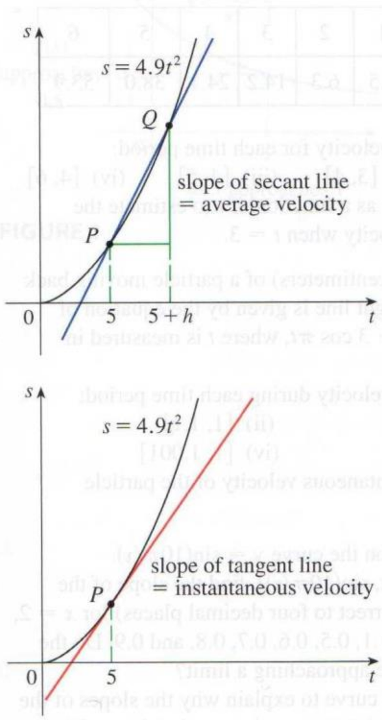


FIGURE 6

2.1 Exercises

1. A tank holds 1000 liters of water, which drains from the bottom of the tank in half an hour. The values in the table show the volume V of water remaining in the tank (in liters) after t minutes.

t (min)	5	10	15	20	25	30
V (L)	694	444	250	111	28	0

(a) If P is the point (15, 250) on the graph of V, find the slopes of the secant lines PQ when Q is the point on the graph with $t = 5, 10, 20, 25$, and 30.

(b) Estimate the slope of the tangent line at P by averaging the slopes of two secant lines.

(c) Use a graph of V to estimate the slope of the tangent line at P. (This slope represents the rate at which the water is flowing from the tank after 15 minutes.)

2. A student bought a smartwatch that tracks the number of steps she walks throughout the day. The table shows the number of steps recorded t minutes after 3:00 PM on the first day she wore the watch.

t (min)	0	10	20	30	40
Steps	3438	4559	5622	6536	7398

(a) Find the slopes of the secant lines corresponding to the given intervals of t. What do these slopes represent?
(i) [0, 40] (ii) [10, 20] (iii) [20, 30]

(b) Estimate the student's walking pace, in steps per minute, at 3:20 PM by averaging the slopes of two secant lines.

3. The point $P(2, -1)$ lies on the curve $y = 1/(1 - x)$.

(a) If Q is the point $(x, 1/(1 - x))$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 1.5 (ii) 1.9 (iii) 1.99 (iv) 1.999
(v) 2.5 (vi) 2.1 (vii) 2.01 (viii) 2.001

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(2, -1)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(2, -1)$.

4. The point $P(0.5, 0)$ lies on the curve $y = \cos \pi x$.

(a) If Q is the point $(x, \cos \pi x)$, find the slope of the secant line PQ (correct to six decimal places) for the following values of x:
(i) 0 (ii) 0.4 (iii) 0.49
(iv) 0.499 (v) 1 (vi) 0.6
(vii) 0.51 (viii) 0.501

(b) Using the results of part (a), guess the value of the slope of the tangent line to the curve at $P(0.5, 0)$.

(c) Using the slope from part (b), find an equation of the tangent line to the curve at $P(0.5, 0)$.

(d) Sketch the curve, two of the secant lines, and the tangent line.

5. The deck of a bridge is suspended 80 meters above a river. If a pebble falls off the side of the bridge, the height, in meters, of the pebble above the water surface after t seconds is given by $y = 80 - 4.9t^2$.

(a) Find the average velocity of the pebble for the time period beginning when $t = 4$ and lasting
(i) 0.1 seconds (ii) 0.05 seconds (iii) 0.01 seconds

(b) Estimate the instantaneous velocity of the pebble after 4 seconds.

6. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height in meters t seconds later is given by $y = 10t - 1.86t^2$.

(a) Find the average velocity over the given time intervals:
(i) [1, 2] (ii) [1, 1.5] (iii) [1, 1.1]
(iv) [1, 1.01] (v) [1, 1.001]

(b) Estimate the instantaneous velocity when $t = 1$.

7. The table shows the position of a motorcyclist after accelerating from rest.

t (seconds)	0	1	2	3	4	5	6
s (meters)	0	1.5	6.3	14.2	24.1	38.0	53.9

(a) Find the average velocity for each time period:
(i) [2, 4] (ii) [3, 4] (iii) [4, 5] (iv) [4, 6]

(b) Use the graph of s as a function of t to estimate the instantaneous velocity when $t = 3$.

8. The displacement (in centimeters) of a particle moving back and forth along a straight line is given by the equation of motion $s = 2 \sin \pi t + 3 \cos \pi t$, where t is measured in seconds.

(a) Find the average velocity during each time period:
(i) [1, 2] (ii) [1, 1.1]
(iii) [1, 1.01] (iv) [1, 1.001]

(b) Estimate the instantaneous velocity of the particle when $t = 1$.

9. The point $P(1, 0)$ lies on the curve $y = \sin(10\pi/x)$.

(a) If Q is the point $(x, \sin(10\pi/x))$, find the slope of the secant line PQ (correct to four decimal places) for $x = 2$, 1.5, 1.4, 1.3, 1.2, 1.1, 0.5, 0.6, 0.7, 0.8, and 0.9. Do the slopes appear to be approaching a limit?

(b) Use a graph of the curve to explain why the slopes of the secant lines in part (a) are not close to the slope of the tangent line at P.

(c) By choosing appropriate secant lines, estimate the slope of the tangent line at P.

2.2 The Limit of a Function

Having seen in the preceding section how limits arise when we want to find the tangent to a curve or the velocity of an object, we now turn our attention to limits in general and numerical and graphical methods for computing them.

Finding Limits Numerically and Graphically

Let's investigate the behavior of the function f defined by $f(x) = (x - 1)/(x^2 - 1)$ for values of x near 1. The following table gives values of $f(x)$ for values of x close to 1 but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975

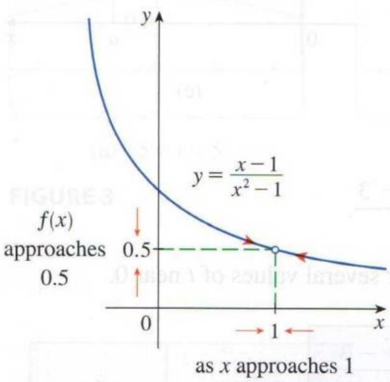


FIGURE 1

From the table and the graph of f shown in Figure 1 we see that the closer x is to 1 (on either side of 1), the closer $f(x)$ is to 0.5. In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x sufficiently close to 1. We express this by saying "the limit of the function $f(x) = (x - 1)/(x^2 - 1)$ as x approaches 1 is equal to 0.5." The notation for this is

$$\lim_{x \to 1} \frac{x - 1}{x^2 - 1} = 0.5$$

In general, we use the following notation.

> **1 Intuitive Definition of a Limit** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write
>
> $$\lim_{x \to a} f(x) = L$$
>
> and say "the limit of $f(x)$, as x approaches a, equals L"
>
> if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of $f(x)$ approach L as x approaches a. In other words, the values of $f(x)$ tend to get closer and closer to the number L as x gets closer and closer to the number a (from either side of a) but $x \neq a$. (A more precise definition will be given in Section 2.4.)

An alternative notation for

$$\lim_{x\to a} f(x) = L$$

is

$$f(x) \to L \quad \text{as} \quad x \to a$$

which is usually read "$f(x)$ approaches L as x approaches a."

Notice the phrase "but x not equal to a" in the definition of limit. This means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined when $x = a$. The only thing that matters is how f is defined *near a*.

Figure 2 shows the graphs of three functions. Note that in part (b), $f(a)$ is not defined and in part (c), $f(a) \neq L$. But in each case, regardless of what happens at a, it is true that $\lim_{x\to a} f(x) = L$.

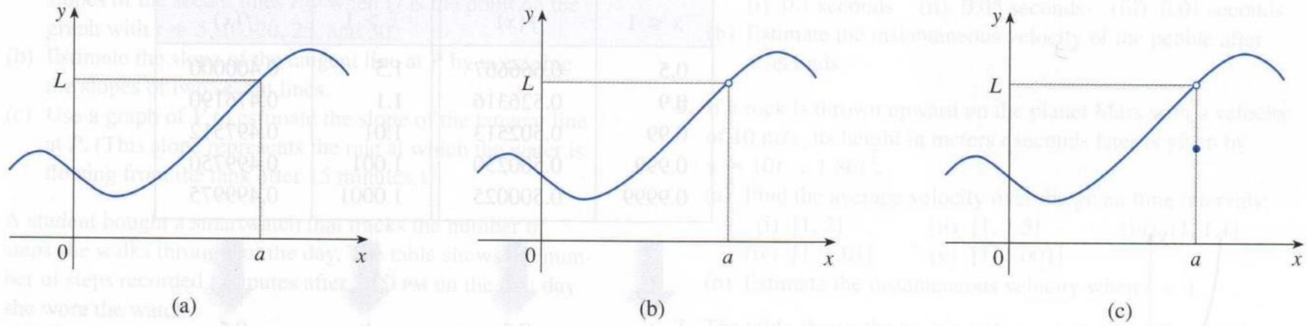


FIGURE 2 $\lim_{x\to a} f(x) = L$ in all three cases

EXAMPLE 1 Estimate the value of $\displaystyle\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2}$.

SOLUTION The table lists values of the function for several values of t near 0.

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 1.0	0.162277 . . .
± 0.5	0.165525 . . .
± 0.1	0.166620 . . .
± 0.05	0.166655 . . .
± 0.01	0.166666 . . .

As t approaches 0, the values of the function seem to approach 0.1666666 . . . and so we guess that

$$\lim_{t\to 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$$ ■

t	$\dfrac{\sqrt{t^2+9}-3}{t^2}$
± 0.001	0.166667
± 0.0001	0.166670
± 0.00001	0.167000
± 0.000001	0.000000

In Example 1 what would have happened if we had taken even smaller values of t? The table in the margin shows the results from one calculator; you can see that something strange seems to be happening.

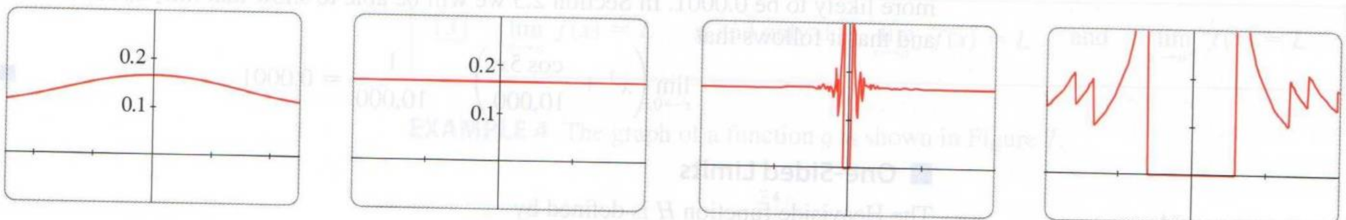
If you try these calculations on your own calculator you might get different values, but eventually you will get the value 0 if you make t sufficiently small. Does this mean that the answer is really 0 instead of $\frac{1}{6}$? No, the value of the limit is $\frac{1}{6}$, as we will show in the next section. The problem is that the calculator gave false values because $\sqrt{t^2 + 9}$ is very close to 3 when t is small. (In fact, when t is sufficiently small, a calculator's value for $\sqrt{t^2 + 9}$ is 3.000 . . . to as many digits as the calculator is capable of carrying.)

www.StewartCalculus.com
For a further explanation of why calculators sometimes give false values, click on *Lies My Calculator and Computer Told Me*. In particular, see the section called *The Perils of Subtraction*.

Something similar happens when we try to graph the function

$$f(t) = \frac{\sqrt{t^2 + 9} - 3}{t^2}$$

of Example 1 on a graphing calculator or computer. Parts (a) and (b) of Figure 3 show quite accurate graphs of f, and if we trace along the curve, we can estimate easily that the limit is about $\frac{1}{6}$. But if we zoom in too much, as in parts (c) and (d), then we get inaccurate graphs, again due to rounding errors within the calculations.



(a) $-5 \leqslant t \leqslant 5$ (b) $-0.1 \leqslant t \leqslant 0.1$ (c) $-10^{-6} \leqslant t \leqslant 10^{-6}$ (d) $-10^{-7} \leqslant t \leqslant 10^{-7}$

FIGURE 3

EXAMPLE 2 Guess the value of $\lim_{x \to 0} \frac{\sin x}{x}$.

SOLUTION The function $f(x) = (\sin x)/x$ is not defined when $x = 0$. Using a calculator (and remembering that, if $x \in \mathbb{R}$, $\sin x$ means the sine of the angle whose *radian* measure is x), we construct a table of values correct to eight decimal places. From the table at the left and the graph in Figure 4 we guess that

x	$\frac{\sin x}{x}$
± 1.0	0.84147098
± 0.5	0.95885108
± 0.4	0.97354586
± 0.3	0.98506736
± 0.2	0.99334665
± 0.1	0.99833417
± 0.05	0.99958339
± 0.01	0.99998333
± 0.005	0.99999583
± 0.001	0.99999983

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

This guess is in fact correct, as will be proved in Chapter 3 using a geometric argument. ■

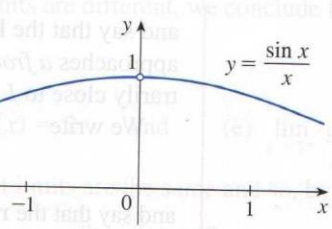


FIGURE 4

EXAMPLE 3 Find $\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10{,}000}\right)$.

SOLUTION As before, we construct a table of values. From the first table it appears that the limit might be zero.

x	$x^3+\frac{\cos 5x}{10{,}000}$
1	1.000028
0.5	0.124920
0.1	0.001088
0.05	0.000222
0.01	0.000101

x	$x^3+\frac{\cos 5x}{10{,}000}$
0.005	0.00010009
0.001	0.00010000

But if we persevere with smaller values of x, the second table suggests that the limit is more likely to be 0.0001. In Section 2.5 we will be able to show that $\lim_{x\to 0}\cos 5x = 1$ and that it follows that

$$\lim_{x\to 0}\left(x^3+\frac{\cos 5x}{10{,}000}\right)=\frac{1}{10{,}000}=0.0001$$

One-Sided Limits

The Heaviside function H is defined by

$$H(t)=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } t\geqslant 0\end{cases}$$

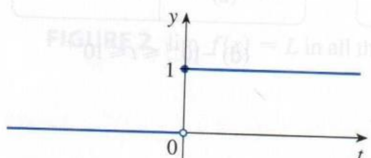


FIGURE 5
The Heaviside function

(This function is named after the electrical engineer Oliver Heaviside [1850–1925] and can be used to describe an electric current that is switched on at time $t=0$.) Its graph is shown in Figure 5.

There is no single number that $H(t)$ approaches as t approaches 0, so $\lim_{t\to 0}H(t)$ does not exist. However, as t approaches 0 from the left, $H(t)$ approaches 0. As t approaches 0 from the right, $H(t)$ approaches 1. We indicate this situation symbolically by writing

$$\lim_{t\to 0^-}H(t)=0 \qquad \text{and} \qquad \lim_{t\to 0^+}H(t)=1$$

and we call these *one-sided limits*. The notation $t\to 0^-$ indicates that we consider only values of t that are less than 0. Likewise, $t\to 0^+$ indicates that we consider only values of t that are greater than 0.

2 Intuitive Definition of One-Sided Limits We write

$$\lim_{x\to a^-}f(x)=L$$

and say that the **left-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches *a from the left*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *less than a*.

We write

$$\lim_{x\to a^+}f(x)=L$$

and say that the **right-hand limit** of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches *a from the right*] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a with x *greater than a*.

For instance, the notation $x \to 5^-$ means that we consider only $x < 5$, and $x \to 5^+$ means that we consider only $x > 5$. Definition 2 is illustrated in Figure 6.

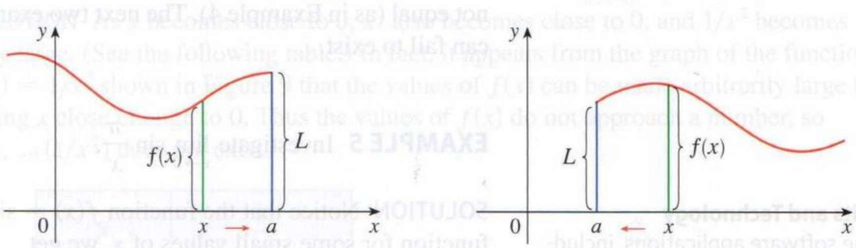


FIGURE 6

(a) $\lim_{x \to a^-} f(x) = L$

(b) $\lim_{x \to a^+} f(x) = L$

Notice that Definition 2 differs from Definition 1 only in that we require x to be less than (or greater than) a. By comparing these definitions, we see that the following is true.

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 4 The graph of a function g is shown in Figure 7.

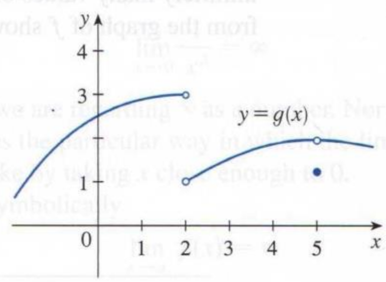


FIGURE 7

Use the graph to state the values (if they exist) of the following:

(a) $\lim_{x \to 2^-} g(x)$ (b) $\lim_{x \to 2^+} g(x)$ (c) $\lim_{x \to 2} g(x)$

(d) $\lim_{x \to 5^-} g(x)$ (e) $\lim_{x \to 5^+} g(x)$ (f) $\lim_{x \to 5} g(x)$

SOLUTION Looking at the graph we see that the values of $g(x)$ approach 3 as x approaches 2 from the left, but they approach 1 as x approaches 2 from the right. Therefore

$$\text{(a)} \lim_{x \to 2^-} g(x) = 3 \quad \text{and} \quad \text{(b)} \lim_{x \to 2^+} g(x) = 1$$

(c) Since the left and right limits are different, we conclude from (3) that $\lim_{x \to 2} g(x)$ does not exist.

The graph also shows that

$$\text{(d)} \lim_{x \to 5^-} g(x) = 2 \quad \text{and} \quad \text{(e)} \lim_{x \to 5^+} g(x) = 2$$

(f) This time the left and right limits are the same and so, by (3), we have

$$\lim_{x \to 5} g(x) = 2$$

Despite this fact, notice that $g(5) \neq 2$. ■

How Can a Limit Fail to Exist?

We have seen that a limit fails to exist at a number a if the left- and right-hand limits are not equal (as in Example 4). The next two examples illustrate additional ways that a limit can fail to exist.

EXAMPLE 5 Investigate $\lim_{x \to 0} \sin \dfrac{\pi}{x}$.

SOLUTION Notice that the function $f(x) = \sin(\pi/x)$ is undefined at 0. Evaluating the function for some small values of x, we get

$$f(1) = \sin \pi = 0 \qquad f(\tfrac{1}{2}) = \sin 2\pi = 0$$

$$f(\tfrac{1}{3}) = \sin 3\pi = 0 \qquad f(\tfrac{1}{4}) = \sin 4\pi = 0$$

$$f(0.1) = \sin 10\pi = 0 \qquad f(0.01) = \sin 100\pi = 0$$

Similarly, $f(0.001) = f(0.0001) = 0$. On the basis of this information we might be tempted to guess that the limit is 0, but this time our guess is wrong. Note that although $f(1/n) = \sin n\pi = 0$ for any integer n, it is also true that $f(x) = 1$ for infinitely many values of x (such as $2/5$ or $2/101$) that approach 0. You can see this from the graph of f shown in Figure 8.

Limits and Technology
Some software applications, including computer algebra systems (CAS), can compute limits. In order to avoid the types of pitfalls demonstrated in Examples 1, 3, and 5, such applications don't find limits by numerical experimentation. Instead, they use more sophisticated techniques such as computing infinite series. You are encouraged to use one of these resources to compute the limits in the examples of this section and check your answers to the exercises in this chapter.

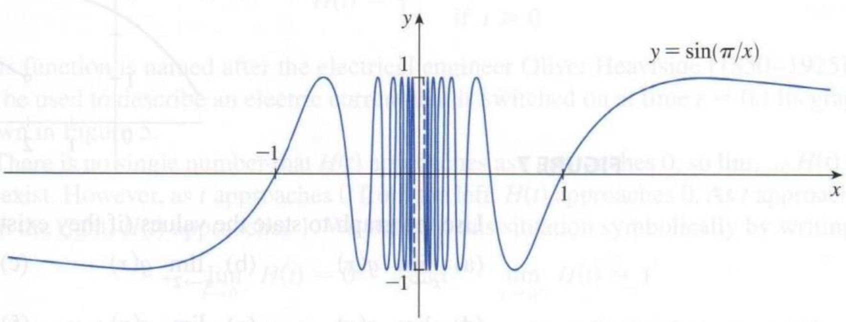


FIGURE 8

The dashed lines near the y-axis indicate that the values of $\sin(\pi/x)$ oscillate between 1 and -1 infinitely often as x approaches 0.

Since the values of $f(x)$ do not approach a fixed number as x approaches 0,

$$\lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

■

Examples 3 and 5 illustrate some of the pitfalls in guessing the value of a limit. It is easy to guess the wrong value if we use inappropriate values of x, but it is difficult to know when to stop calculating values. And, as the discussion after Example 1 shows, sometimes calculators and computers give the wrong values. In the next section, however, we will develop foolproof methods for calculating limits.

Another way a limit at a number a can fail to exist is when the function values grow arbitrarily large (in absolute value) as x approaches a.

EXAMPLE 6 Find $\lim_{x \to 0} \frac{1}{x^2}$ if it exists.

SOLUTION As x becomes close to 0, x^2 also becomes close to 0, and $1/x^2$ becomes very large. (See the following table.) In fact, it appears from the graph of the function $f(x) = 1/x^2$ shown in Figure 9 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0. Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} (1/x^2)$ does not exist.

x	$\frac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

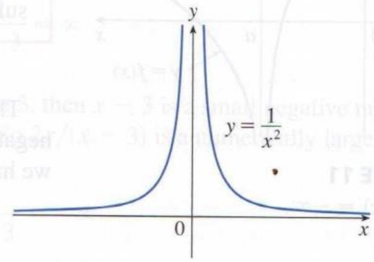


FIGURE 9

Infinite Limits; Vertical Asymptotes

To indicate the kind of behavior exhibited in Example 6, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

This does not mean that we are regarding ∞ as a number. Nor does it mean that the limit exists. It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or "increase without bound") as x becomes closer and closer to a.

4 Intuitive Definition of an Infinite Limit Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Another notation for $\lim_{x \to a} f(x) = \infty$ is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Again, the symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as

"the limit of $f(x)$, as x approaches a, is infinity"

or "$f(x)$ becomes infinite as x approaches a"

or "$f(x)$ increases without bound as x approaches a"

This definition is illustrated graphically in Figure 10.

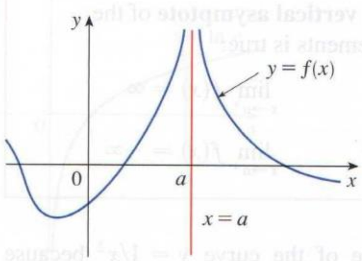


FIGURE 10
$\lim_{x \to a} f(x) = \infty$

When we say a number is "large negative," we mean that it is negative but its magnitude (absolute value) is large.

A similar sort of limit, for functions that become large negative as x gets close to a, is defined in Definition 5 and is illustrated in Figure 11.

5 Definition Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

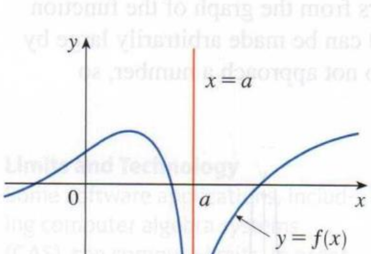


FIGURE 11
$\lim_{x \to a} f(x) = -\infty$

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as "the limit of $f(x)$, as x approaches a, is negative infinity" or "$f(x)$ decreases without bound as x approaches a." As an example we have

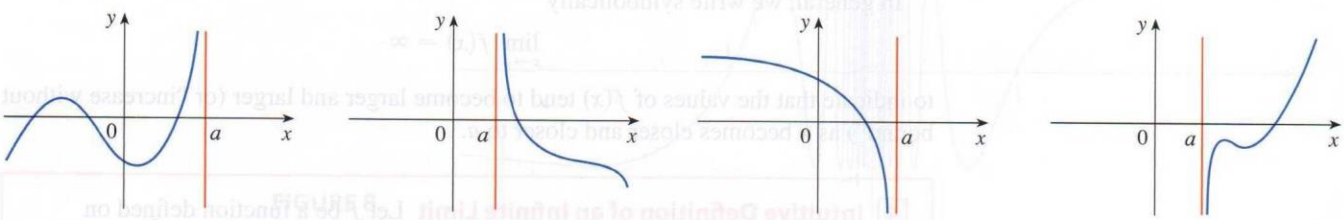
$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$

Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

remembering that $x \to a^-$ means that we consider only values of x that are less than a, and similarly $x \to a^+$ means that we consider only $x > a$. Illustrations of these four cases are given in Figure 12.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 12

6 Definition The vertical line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

For instance, the y-axis is a vertical asymptote of the curve $y = 1/x^2$ because $\lim_{x \to 0} (1/x^2) = \infty$. In Figure 12 the line $x = a$ is a vertical asymptote in each of the four cases shown. In general, knowledge of vertical asymptotes is very useful in sketching graphs.

EXAMPLE 7 Does the curve $y = \dfrac{2x}{x-3}$ have a vertical asymptote?

SOLUTION There is a potential vertical asymptote where the denominator is 0, that is, at $x = 3$, so we investigate the one-sided limits there.

If x is close to 3 but larger than 3, then the denominator $x - 3$ is a small positive number and $2x$ is close to 6. So the quotient $2x/(x-3)$ is a large *positive* number. [For instance, if $x = 3.01$ then $2x/(x-3) = 6.02/0.01 = 602$.] Thus, intuitively, we see that

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

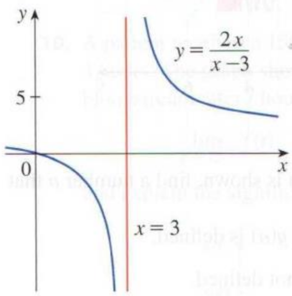


FIGURE 13

Likewise, if x is close to 3 but smaller than 3, then $x - 3$ is a small negative number but $2x$ is still a positive number (close to 6). So $2x/(x-3)$ is a numerically large *negative* number. Thus

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

The graph of the curve $y = 2x/(x-3)$ is given in Figure 13. According to Definition 6, the line $x = 3$ is a vertical asymptote. ■

NOTE Neither of the limits in Examples 6 and 7 exist, but in Example 6 we can write $\lim_{x \to 0} (1/x^2) = \infty$ because $f(x) \to \infty$ as x approaches 0 from either the left or the right. In Example 7, $f(x) \to \infty$ as x approaches 3 from the right but $f(x) \to -\infty$ as x approaches 3 from the left, so we simply say that $\lim_{x \to 3} f(x)$ does not exist.

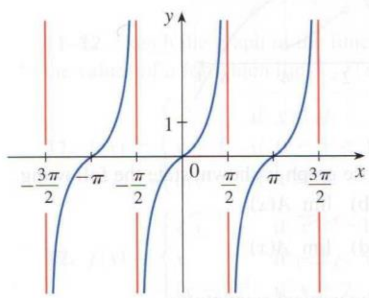


FIGURE 14
$y = \tan x$

EXAMPLE 8 Find the vertical asymptotes of $f(x) = \tan x$.

SOLUTION Because

$$\tan x = \frac{\sin x}{\cos x}$$

there are potential vertical asymptotes where $\cos x = 0$. In fact, since $\cos x \to 0^+$ as $x \to (\pi/2)^-$ and $\cos x \to 0^-$ as $x \to (\pi/2)^+$, whereas $\sin x$ is positive (near 1) when x is near $\pi/2$, we have

$$\lim_{x \to (\pi/2)^-} \tan x = \infty \qquad \text{and} \qquad \lim_{x \to (\pi/2)^+} \tan x = -\infty$$

This shows that the line $x = \pi/2$ is a vertical asymptote. Similar reasoning shows that the lines $x = \pi/2 + n\pi$, where n is an integer, are all vertical asymptotes of $f(x) = \tan x$. The graph in Figure 14 confirms this. ■

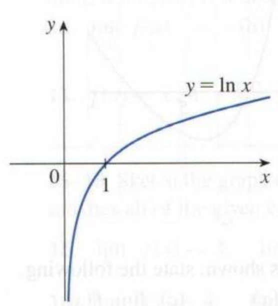


FIGURE 15
The y-axis is a vertical asymptote of the natural logarithmic function.

Another example of a function whose graph has a vertical asymptote is the natural logarithmic function $y = \ln x$. From Figure 15 we see that

$$\boxed{\lim_{x \to 0^+} \ln x = -\infty}$$

and so the line $x = 0$ (the y-axis) is a vertical asymptote. In fact, the same is true for $y = \log_b x$ provided that $b > 1$. (See Figures 1.5.11 and 1.5.12.)

2.2 Exercises

1. Explain in your own words what is meant by the equation

$$\lim_{x \to 2} f(x) = 5$$

Is it possible for this statement to be true and yet $f(2) = 3$? Explain.

2. Explain what it means to say that

$$\lim_{x \to 1^-} f(x) = 3 \quad \text{and} \quad \lim_{x \to 1^+} f(x) = 7$$

In this situation is it possible that $\lim_{x \to 1} f(x)$ exists? Explain.

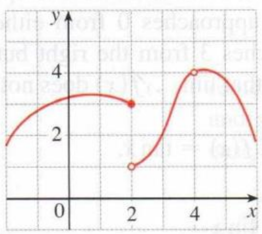
3. Explain the meaning of each of the following.

(a) $\lim_{x \to -3} f(x) = \infty$ (b) $\lim_{x \to 4^+} f(x) = -\infty$

4. Use the given graph of f to state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 2^-} f(x)$ (b) $\lim_{x \to 2^+} f(x)$ (c) $\lim_{x \to 2} f(x)$

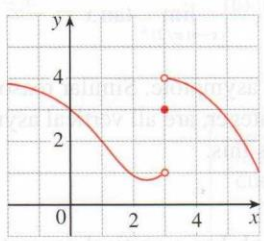
(d) $f(2)$ (e) $\lim_{x \to 4} f(x)$ (f) $f(4)$



5. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3^-} f(x)$ (c) $\lim_{x \to 3^+} f(x)$

(d) $\lim_{x \to 3} f(x)$ (e) $f(3)$



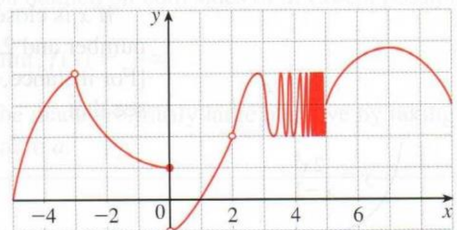
6. For the function h whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$



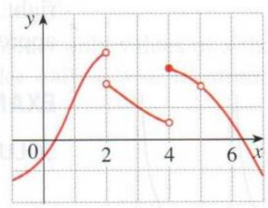
7. For the function g whose graph is shown, find a number a that satisfies the given description.

(a) $\lim_{x \to a} g(x)$ does not exist but $g(a)$ is defined.

(b) $\lim_{x \to a} g(x)$ exists but $g(a)$ is not defined.

(c) $\lim_{x \to a^-} g(x)$ and $\lim_{x \to a^+} g(x)$ both exist but $\lim_{x \to a} g(x)$ does not exist.

(d) $\lim_{x \to a^+} g(x) = g(a)$ but $\lim_{x \to a^-} g(x) \neq g(a)$.

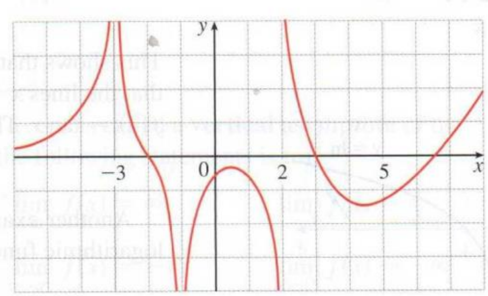


8. For the function A whose graph is shown, state the following.

(a) $\lim_{x \to -3} A(x)$ (b) $\lim_{x \to 2^-} A(x)$

(c) $\lim_{x \to 2^+} A(x)$ (d) $\lim_{x \to -1} A(x)$

(e) The equations of the vertical asymptotes

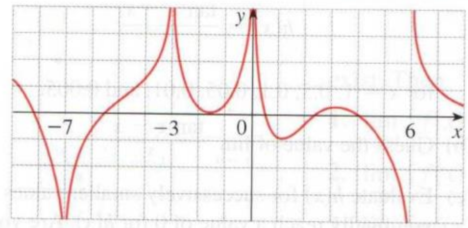


9. For the function f whose graph is shown, state the following.

(a) $\lim_{x \to -7} f(x)$ (b) $\lim_{x \to -3} f(x)$ (c) $\lim_{x \to 0} f(x)$

(d) $\lim_{x \to 6^-} f(x)$ (e) $\lim_{x \to 6^+} f(x)$

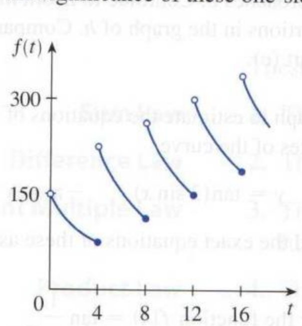
(f) The equations of the vertical asymptotes



10. A patient receives a 150-mg injection of a drug every 4 hours. The graph shows the amount $f(t)$ of the drug in the bloodstream after t hours. Find

$$\lim_{t\to12^-} f(t) \quad \text{and} \quad \lim_{t\to12^+} f(t)$$

and explain the significance of these one-sided limits.



11–12 Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$ exists.

11. $f(x) = \begin{cases} e^x & \text{if } x \leqslant 0 \\ x - 1 & \text{if } 0 < x < 1 \\ \ln x & \text{if } x \geqslant 1 \end{cases}$

12. $f(x) = \begin{cases} \sqrt[3]{x} & \text{if } x \leqslant -1 \\ x & \text{if } -1 < x \leqslant 2 \\ (x-1)^2 & \text{if } x > 2 \end{cases}$

13–14 Use the graph of the function f to state the value of each limit, if it exists. If it does not exist, explain why.

(a) $\lim_{x\to0^-} f(x)$ (b) $\lim_{x\to0^+} f(x)$ (c) $\lim_{x\to0} f(x)$

13. $f(x) = x\sqrt{1 + x^{-2}}$ **14.** $f(x) = \dfrac{e^{1/x} - 2}{e^{1/x} + 1}$

15–18 Sketch the graph of an example of a function f that satisfies all of the given conditions.

15. $\lim_{x\to1^-} f(x) = 3, \quad \lim_{x\to1^+} f(x) = 0, \quad f(1) = 2$

16. $\lim_{x\to0} f(x) = 4, \quad \lim_{x\to8^-} f(x) = 1, \quad \lim_{x\to8^+} f(x) = -3,$
$f(0) = 6, \quad f(8) = -1$

17. $\lim_{x\to-1^-} f(x) = 0, \quad \lim_{x\to-1^+} f(x) = 1, \quad \lim_{x\to2} f(x) = 3,$
$f(-1) = 2, \quad f(2) = 1$

18. $\lim_{x\to-3^-} f(x) = 3, \quad \lim_{x\to-3^+} f(x) = 2, \quad \lim_{x\to3^-} f(x) = -1,$
$\lim_{x\to3^+} f(x) = 2, \quad f(-3) = 2, \quad f(3) = 0$

19–22 Guess the value of the limit (if it exists) by evaluating the function at the given numbers (correct to six decimal places).

19. $\displaystyle\lim_{x\to3} \frac{x^2 - 3x}{x^2 - 9},$
$x = 3.1, 3.05, 3.01, 3.001, 3.0001,$
$2.9, 2.95, 2.99, 2.999, 2.9999$

20. $\displaystyle\lim_{x\to-3} \frac{x^2 - 3x}{x^2 - 9},$
$x = -2.5, -2.9, -2.95, -2.99, -2.999, -2.9999,$
$-3.5, -3.1, -3.05, -3.01, -3.001, -3.0001$

21. $\displaystyle\lim_{t\to0} \frac{e^{5t} - 1}{t}, \quad t = \pm0.5, \pm0.1, \pm0.01, \pm0.001, \pm0.0001$

22. $\displaystyle\lim_{h\to0} \frac{(2 + h)^5 - 32}{h},$
$h = \pm0.5, \pm0.1, \pm0.01, \pm0.001, \pm0.0001$

23–28 Use a table of values to estimate the value of the limit. If you have a graphing device, use it to confirm your result graphically.

23. $\displaystyle\lim_{x\to4} \frac{\ln x - \ln 4}{x - 4}$ **24.** $\displaystyle\lim_{p\to-1} \frac{1 + p^9}{1 + p^{15}}$

25. $\displaystyle\lim_{\theta\to0} \frac{\sin 3\theta}{\tan 2\theta}$ **26.** $\displaystyle\lim_{t\to0} \frac{5^t - 1}{t}$

27. $\displaystyle\lim_{x\to0^+} x^x$ **28.** $\displaystyle\lim_{x\to0^+} x^2 \ln x$

29–40 Determine the infinite limit.

29. $\displaystyle\lim_{x\to5^+} \frac{x + 1}{x - 5}$ **30.** $\displaystyle\lim_{x\to5^-} \frac{x + 1}{x - 5}$

31. $\displaystyle\lim_{x\to2} \frac{x^2}{(x - 2)^2}$ **32.** $\displaystyle\lim_{x\to3^-} \frac{\sqrt{x}}{(x - 3)^5}$

33. $\displaystyle\lim_{x\to1^+} \ln(\sqrt{x} - 1)$ **34.** $\displaystyle\lim_{x\to0^+} \ln(\sin x)$

35. $\displaystyle\lim_{x\to(\pi/2)^+} \frac{1}{x} \sec x$ **36.** $\displaystyle\lim_{x\to\pi^-} x \cot x$

37. $\displaystyle\lim_{x\to1} \frac{x^2 + 2x}{x^2 - 2x + 1}$ **38.** $\displaystyle\lim_{x\to3^-} \frac{x^2 + 4x}{x^2 - 2x - 3}$

39. $\displaystyle\lim_{x\to0} (\ln x^2 - x^{-2})$ **40.** $\displaystyle\lim_{x\to0^+} \left(\frac{1}{x} - \ln x\right)$

41. Find the vertical asymptote of the function

$$f(x) = \frac{x - 1}{2x + 4}$$

42. (a) Find the vertical asymptotes of the function

$$y = \frac{x^2 + 1}{3x - 2x^2}$$

(b) Confirm your answer to part (a) by graphing the function.

43. Determine $\lim_{x \to 1^-} \frac{1}{x^3 - 1}$ and $\lim_{x \to 1^+} \frac{1}{x^3 - 1}$

(a) by evaluating $f(x) = 1/(x^3 - 1)$ for values of x that approach 1 from the left and from the right,

(b) by reasoning as in Example 7, and

(c) from a graph of f.

44. (a) By graphing the function

$$f(x) = \frac{\cos 2x - \cos x}{x^2}$$

and zooming in toward the point where the graph crosses the y-axis, estimate the value of $\lim_{x \to 0} f(x)$.

(b) Check your answer in part (a) by evaluating $f(x)$ for values of x that approach 0.

45. (a) Estimate the value of the limit $\lim_{x \to 0} (1 + x)^{1/x}$ to five decimal places. Does this number look familiar?

(b) Illustrate part (a) by graphing the function $y = (1 + x)^{1/x}$.

46. (a) Graph the function $f(x) = e^x + \ln|x - 4|$ for $0 \leqslant x \leqslant 5$. Do you think the graph is an accurate representation of f?

(b) How would you get a graph that represents f better?

47. (a) Evaluate the function $f(x) = x^2 - (2^x/1000)$ for $x = 1, 0.8, 0.6, 0.4, 0.2, 0.1$, and 0.05, and guess the value of

$$\lim_{x \to 0} \left(x^2 - \frac{2^x}{1000} \right)$$

(b) Evaluate $f(x)$ for $x = 0.04, 0.02, 0.01, 0.005, 0.003$, and 0.001. Guess again.

48. (a) Evaluate the function

$$h(x) = \frac{\tan x - x}{x^3}$$

for $x = 1, 0.5, 0.1, 0.05, 0.01$, and 0.005.

(b) Guess the value of $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.

(c) Evaluate $h(x)$ for successively smaller values of x until you finally reach a value of 0 for $h(x)$. Are you still confident that your guess in part (b) is correct? Explain why you eventually obtained 0 values. (In Section 4.4 a method for evaluating this limit will be explained.)

(d) Graph the function h in the viewing rectangle $[-1, 1]$ by $[0, 1]$. Then zoom in toward the point where the graph crosses the y-axis to estimate the limit of $h(x)$ as x approaches 0. Continue to zoom in until you observe distortions in the graph of h. Compare with the results of part (c).

49. Use a graph to estimate the equations of all the vertical asymptotes of the curve

$$y = \tan(2 \sin x) \qquad -\pi \leqslant x \leqslant \pi$$

Then find the exact equations of these asymptotes.

50. Consider the function $f(x) = \tan \frac{1}{x}$.

(a) Show that $f(x) = 0$ for $x = \frac{1}{\pi}, \frac{1}{2\pi}, \frac{1}{3\pi}, \ldots$

(b) Show that $f(x) = 1$ for $x = \frac{4}{\pi}, \frac{4}{5\pi}, \frac{4}{9\pi}, \ldots$

(c) What can you conclude about $\lim_{x \to 0^+} \tan \frac{1}{x}$?

51. In the theory of relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light. What happens as $v \to c^-$?

2.3 Calculating Limits Using the Limit Laws

Properties of Limits

In Section 2.2 we used calculators and graphs to guess the values of limits, but we saw that such methods don't always lead to the correct answers. In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim_{x\to a} [f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a} [cf(x)] = c \lim_{x\to a} f(x)$
4. $\lim_{x\to a} [f(x)\, g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

These five laws can be stated verbally as follows:

Sum Law 1. The limit of a sum is the sum of the limits.

Difference Law 2. The limit of a difference is the difference of the limits.

Constant Multiple Law 3. The limit of a constant times a function is the constant times the limit of the function.

Product Law 4. The limit of a product is the product of the limits.

Quotient Law 5. The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

It is easy to believe that these properties are true. For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$. This gives us an intuitive basis for believing that Law 1 is true. In Section 2.4 we give a precise definition of a limit and use it to prove this law. The proofs of the remaining laws are given in Appendix F.

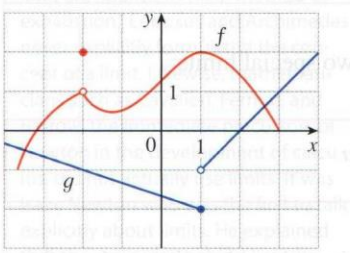


FIGURE 1

EXAMPLE 1 Use the Limit Laws and the graphs of f and g in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x\to -2} [f(x) + 5g(x)]$ (b) $\lim_{x\to 1} [f(x)g(x)]$ (c) $\lim_{x\to 2} \dfrac{f(x)}{g(x)}$

SOLUTION

(a) From the graphs of f and g we see that

$$\lim_{x\to -2} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to -2} g(x) = -1$$

Therefore we have

$$\begin{aligned}
\lim_{x\to -2} [f(x) + 5g(x)] &= \lim_{x\to -2} f(x) + \lim_{x\to -2} [5g(x)] && \text{(by Limit Law 1)}\\
&= \lim_{x\to -2} f(x) + 5 \lim_{x\to -2} g(x) && \text{(by Limit Law 3)}\\
&= 1 + 5(-1) = -4
\end{aligned}$$

(b) We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we can use Law 4 for the one-sided limits:

$$\lim_{x\to 1^-} [f(x)g(x)] = \lim_{x\to 1^-} f(x) \cdot \lim_{x\to 1^-} g(x) = 2 \cdot (-2) = -4$$

$$\lim_{x\to 1^+} [f(x)g(x)] = \lim_{x\to 1^+} f(x) \cdot \lim_{x\to 1^+} g(x) = 2 \cdot (-1) = -2$$

The left and right limits aren't equal, so $\lim_{x\to 1} [f(x)g(x)]$ does not exist.

(c) The graphs show that

$$\lim_{x\to 2} f(x) \approx 1.4 \quad \text{and} \quad \lim_{x\to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5. The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number. ■

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law

6. $\lim_{x\to a} [f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ where n is a positive integer

A similar property, which you are asked to prove in Exercise 2.5.69, holds for roots:

Root Law

7. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim_{x\to a} f(x) > 0$.]

In applying these seven limit laws, we need to use two special limits:

8. $\lim_{x\to a} c = c$ 9. $\lim_{x\to a} x = a$

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$), but proofs based on the precise definition are requested in Exercises 2.4.23–24.

If we now put $f(x) = x$ in Law 6 and use Law 9, we get a useful special limit for power functions.

10. $\lim_{x\to a} x^n = a^n$ where n is a positive integer

Newton and Limits

Isaac Newton was born on Christmas Day in 1642, the year of Galileo's death. When he entered Cambridge University in 1661 Newton didn't know much mathematics, but he learned quickly by reading Euclid and Descartes and by attending the lectures of Isaac Barrow. Cambridge was closed because of the plague from 1665 to 1666, and Newton returned home to reflect on what he had learned. Those two years were amazingly productive for at that time he made four of his major discoveries: (1) his representation of functions as sums of infinite series, including the binomial theorem; (2) his work on differential and integral calculus; (3) his laws of motion and law of universal gravitation; and (4) his prism experiments on the nature of light and color. Because of a fear of controversy and criticism, he was reluctant to publish his discoveries and it wasn't until 1687, at the urging of the astronomer Halley, that Newton published *Principia Mathematica*. In this work, the greatest scientific treatise ever written, Newton set forth his version of calculus and used it to investigate mechanics, fluid dynamics, and wave motion, and to explain the motion of planets and comets.

The beginnings of calculus are found in the calculations of areas and volumes by ancient Greek scholars such as Eudoxus and Archimedes. Although aspects of the idea of a limit are implicit in their "method of exhaustion," Eudoxus and Archimedes never explicitly formulated the concept of a limit. Likewise, mathematicians such as Cavalieri, Fermat, and Barrow, the immediate precursors of Newton in the development of calculus, did not actually use limits. It was Isaac Newton who was the first to talk explicitly about limits. He explained that the main idea behind limits is that quantities "approach nearer than by any given difference." Newton stated that the limit was the basic concept in calculus, but it was left to later mathematicians like Cauchy to clarify his ideas about limits.

If we put $f(x) = x$ in Law 7 and use Law 9, we get a similar special limit for roots. (For square roots the proof is outlined in Exercise 2.4.37.)

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

EXAMPLE 2 Evaluate the following limits and justify each step.

(a) $\lim_{x \to 5} (2x^2 - 3x + 4)$ (b) $\lim_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

SOLUTION

(a)
$$\begin{aligned}
\lim_{x \to 5} (2x^2 - 3x + 4) &= \lim_{x \to 5} (2x^2) - \lim_{x \to 5} (3x) + \lim_{x \to 5} 4 && \text{(by Laws 2 and 1)} \\
&= 2 \lim_{x \to 5} x^2 - 3 \lim_{x \to 5} x + \lim_{x \to 5} 4 && \text{(by 3)} \\
&= 2(5^2) - 3(5) + 4 && \text{(by 10, 9, and 8)} \\
&= 39
\end{aligned}$$

(b) We start by using Law 5, but its use is fully justified only at the final stage when we see that the limits of the numerator and denominator exist and the limit of the denominator is not 0.

$$\begin{aligned}
\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)} && \text{(by Law 5)} \\
&= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 - \lim_{x \to -2} 1}{\lim_{x \to -2} 5 - 3 \lim_{x \to -2} x} && \text{(by 1, 2, and 3)} \\
&= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} && \text{(by 10, 9, and 8)} \\
&= -\frac{1}{11}
\end{aligned}$$

Evaluating Limits by Direct Substitution

In Example 2(a) we determined that $\lim_{x \to 5} f(x) = 39$, where $f(x) = 2x^2 - 3x + 4$. Notice that $f(5) = 39$; in other words, we would have gotten the correct result simply by substituting 5 for x. Similarly, direct substitution provides the correct answer in part (b). The functions in Example 2 are a polynomial and a rational function, respectively, and similar use of the Limit Laws proves that direct substitution always works for such functions (see Exercises 59 and 60). We state this fact as follows.

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions that have the Direct Substitution Property are called *continuous at a* and will be studied in Section 2.5. However, not all limits can be evaluated initially by direct substitution, as the following examples show.

EXAMPLE 3 Find $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$.

SOLUTION Let $f(x) = (x^2 - 1)/(x - 1)$. We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

Notice that in Example 3 we do not have an infinite limit even though the denominator approaches 0 as $x \to 1$. When both numerator and denominator approach 0, the limit may be infinite or it may be some finite value.

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor, $x - 1$, and then compute the limit by direct substitution as follows:

$$\lim_{x\to 1} \frac{x^2 - 1}{x - 1} = \lim_{x\to 1} \frac{(x - 1)(x + 1)}{x - 1}$$

$$= \lim_{x\to 1} (x + 1) = 1 + 1 = 2$$

The limit in this example arose in Example 2.1.1 in finding the tangent to the parabola $y = x^2$ at the point (1, 1).

NOTE In Example 3 we were able to compute the limit by replacing the given function $f(x) = (x^2 - 1)/(x - 1)$ by a simpler function, $g(x) = x + 1$, that has the same limit. This is valid because $f(x) = g(x)$ except when $x = 1$, and in computing a limit as x approaches 1 we don't consider what happens when x is actually *equal* to 1. In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\displaystyle\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$, provided the limits exist.

EXAMPLE 4 Find $\displaystyle\lim_{x\to 1} g(x)$ where

$$g(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \pi & \text{if } x = 1 \end{cases}$$

SOLUTION Here g is defined at $x = 1$ and $g(1) = \pi$, but the value of a limit as x *approaches* 1 does not depend on the value of the function *at* 1. Since $g(x) = x + 1$ for $x \neq 1$, we have

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} (x + 1) = 2$$

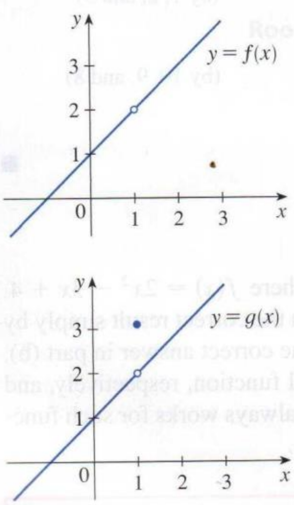


FIGURE 2
The graphs of the functions f (from Example 3) and g (from Example 4)

Note that the values of the functions in Examples 3 and 4 are identical except when $x = 1$ (see Figure 2) and so they have the same limit as x approaches 1.

EXAMPLE 5 Evaluate $\lim_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3+h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ because $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h}$$

$$= \frac{h(6+h)}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$

EXAMPLE 6 Find $\lim_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately because the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2+9} + 3\right)}$$

$$= \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9} + 3\right)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3}$$

$$= \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3}$$

(Here we use several properties of limits: 5, 1, 7, 8, 10.)

$$= \frac{1}{3 + 3} = \frac{1}{6}$$

This calculation confirms the guess that we made in Example 2.2.1.

Using One-Sided Limits

Some limits are best calculated by first finding the left- and right-hand limits. The following theorem is a reminder of what we discovered in Section 2.2. It says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

The result of Example 7 looks plausible from Figure 3.

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

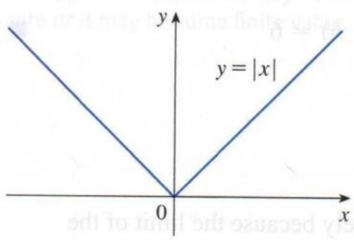


FIGURE 3

EXAMPLE 8 Prove that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

SOLUTION Using the facts that $|x| = x$ when $x > 0$ and $|x| = -x$ when $x < 0$, we have

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} (-1) = -1$$

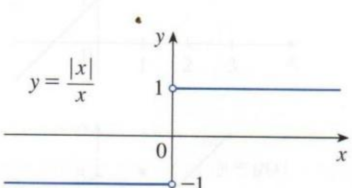


FIGURE 4

Since the right- and left-hand limits are different, it follows from Theorem 1 that $\lim_{x \to 0} |x|/x$ does not exist. The graph of the function $f(x) = |x|/x$ is shown in Figure 4 and supports the one-sided limits that we found.

EXAMPLE 9 If

$$f(x) = \begin{cases} \sqrt{x-4} & \text{if } x > 4 \\ 8 - 2x & \text{if } x < 4 \end{cases}$$

determine whether $\lim_{x \to 4} f(x)$ exists.

It is shown in Example 2.4.4 that $\lim_{x \to 0^+} \sqrt{x} = 0$.

SOLUTION Since $f(x) = \sqrt{x - 4}$ for $x > 4$, we have

$$\lim_{x \to 4^+} f(x) = \lim_{x \to 4^+} \sqrt{x - 4} = \sqrt{4 - 4} = 0$$

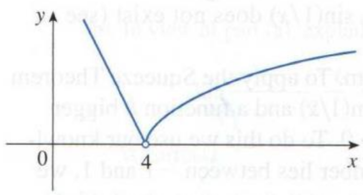


FIGURE 5

Since $f(x) = 8 - 2x$ for $x < 4$, we have

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} (8 - 2x) = 8 - 2 \cdot 4 = 0$$

The right- and left-hand limits are equal. Thus the limit exists and

$$\lim_{x \to 4} f(x) = 0$$

The graph of f is shown in Figure 5. ■

Other notations for $[\![x]\!]$ are $[x]$ and $\lfloor x \rfloor$. The greatest integer function is sometimes called the *floor function*.

EXAMPLE 10 The **greatest integer function** is defined by $[\![x]\!]$ = the largest integer that is less than or equal to x. (For instance, $[\![4]\!] = 4$, $[\![4.8]\!] = 4$, $[\![\pi]\!] = 3$, $[\![\sqrt{2}]\!] = 1$, $[\![-\frac{1}{2}]\!] = -1$.) Show that $\lim_{x \to 3} [\![x]\!]$ does not exist.

SOLUTION The graph of the greatest integer function is shown in Figure 6. Since $[\![x]\!] = 3$ for $3 \leqslant x < 4$, we have

$$\lim_{x \to 3^+} [\![x]\!] = \lim_{x \to 3^+} 3 = 3$$

Since $[\![x]\!] = 2$ for $2 \leqslant x < 3$, we have

$$\lim_{x \to 3^-} [\![x]\!] = \lim_{x \to 3^-} 2 = 2$$

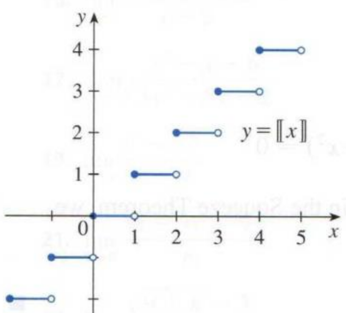


FIGURE 6
Greatest integer function

Because these one-sided limits are not equal, $\lim_{x \to 3} [\![x]\!]$ does not exist by Theorem 1. ■

■ The Squeeze Theorem

The following two theorems describe how the limits of functions are related when the values of one function are greater than (or equal to) those of another. Their proofs can be found in Appendix F.

> **2 Theorem** If $f(x) \leqslant g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then
>
> $$\lim_{x \to a} f(x) \leqslant \lim_{x \to a} g(x)$$

> **3 The Squeeze Theorem** If $f(x) \leqslant g(x) \leqslant h(x)$ when x is near a (except possibly at a) and
>
> $$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$
>
> then $$\lim_{x \to a} g(x) = L$$

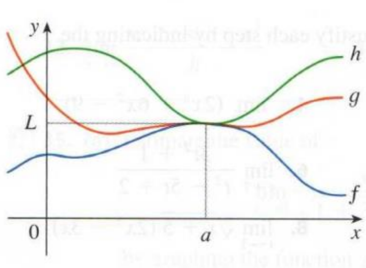


FIGURE 7

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7. It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

EXAMPLE 11 Show that $\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$.

SOLUTION First note that we **cannot** rewrite the limit as the product of the limits $\lim_{x \to 0} x^2$ and $\lim_{x \to 0} \sin(1/x)$ because $\lim_{x \to 0} \sin(1/x)$ does not exist (see Example 2.2.5).

We *can* find the limit by using the Squeeze Theorem. To apply the Squeeze Theorem we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0 as $x \to 0$. To do this we use our knowledge of the sine function. Because the sine of any number lies between -1 and 1, we can write

$$\boxed{4} \qquad -1 \leqslant \sin \frac{1}{x} \leqslant 1$$

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geqslant 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2, we get

$$-x^2 \leqslant x^2 \sin \frac{1}{x} \leqslant x^2$$

as illustrated by Figure 8. We know that

$$\lim_{x \to 0} x^2 = 0 \qquad \text{and} \qquad \lim_{x \to 0} (-x^2) = 0$$

Taking $f(x) = -x^2$, $g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin \frac{1}{x} = 0$$

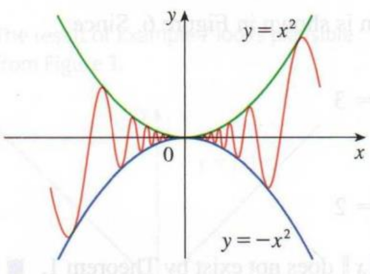


FIGURE 8
$y = x^2 \sin(1/x)$

2.3 Exercises

1. Given that

$$\lim_{x \to 2} f(x) = 4 \qquad \lim_{x \to 2} g(x) = -2 \qquad \lim_{x \to 2} h(x) = 0$$

find the limits that exist. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + 5g(x)]$ (b) $\lim_{x \to 2} [g(x)]^3$

(c) $\lim_{x \to 2} \sqrt{f(x)}$ (d) $\lim_{x \to 2} \dfrac{3f(x)}{g(x)}$

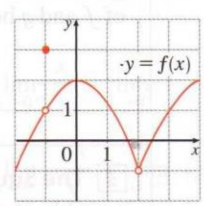
(e) $\lim_{x \to 2} \dfrac{g(x)}{h(x)}$ (f) $\lim_{x \to 2} \dfrac{g(x)h(x)}{f(x)}$

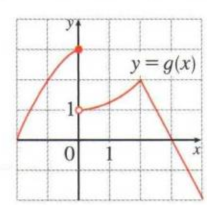
2. The graphs of f and g are given. Use them to evaluate each limit, if it exists. If the limit does not exist, explain why.

(a) $\lim_{x \to 2} [f(x) + g(x)]$ (b) $\lim_{x \to 0} [f(x) - g(x)]$

(c) $\lim_{x \to -1} [f(x)g(x)]$ (d) $\lim_{x \to 3} \dfrac{f(x)}{g(x)}$

(e) $\lim_{x \to 2} [x^2 f(x)]$ (f) $f(-1) + \lim_{x \to -1} g(x)$





3–9 Evaluate the limit and justify each step by indicating the appropriate Limit Law(s).

3. $\lim_{x \to 5} (4x^2 - 5x)$

4. $\lim_{x \to -3} (2x^3 + 6x^2 - 9)$

5. $\lim_{v \to 2} (v^2 + 2v)(2v^3 - 5)$

6. $\lim_{t \to 7} \dfrac{3t^2 + 1}{t^2 - 5t + 2}$

7. $\lim_{u \to -2} \sqrt{9 - u^3 + 2u^2}$

8. $\lim_{x \to 3} \sqrt[3]{x + 5}\,(2x^2 - 3x)$

9. $\lim_{t \to -1} \left(\dfrac{2t^5 - t^4}{5t^2 + 4} \right)^3$

10. (a) What is wrong with the following equation?

$$\frac{x^2 + x - 6}{x - 2} = x + 3$$

(b) In view of part (a), explain why the equation

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x\to 2} (x + 3)$$

is correct.

11–34 Evaluate the limit, if it exists.

11. $\displaystyle\lim_{x\to -2} (3x - 7)$

12. $\displaystyle\lim_{x\to 6} \left(8 - \tfrac{1}{2}x\right)$

13. $\displaystyle\lim_{t\to 4} \frac{t^2 - 2t - 8}{t - 4}$

14. $\displaystyle\lim_{x\to -3} \frac{x^2 + 3x}{x^2 - x - 12}$

15. $\displaystyle\lim_{x\to 2} \frac{x^2 + 5x + 4}{x - 2}$

16. $\displaystyle\lim_{x\to 4} \frac{x^2 + 3x}{x^2 - x - 12}$

17. $\displaystyle\lim_{x\to -2} \frac{x^2 - x - 6}{3x^2 + 5x - 2}$

18. $\displaystyle\lim_{x\to -5} \frac{2x^2 + 9x - 5}{x^2 - 25}$

19. $\displaystyle\lim_{t\to 3} \frac{t^3 - 27}{t^2 - 9}$

20. $\displaystyle\lim_{u\to -1} \frac{u + 1}{u^3 + 1}$

21. $\displaystyle\lim_{h\to 0} \frac{(h - 3)^2 - 9}{h}$

22. $\displaystyle\lim_{x\to 9} \frac{9 - x}{3 - \sqrt{x}}$

23. $\displaystyle\lim_{h\to 0} \frac{\sqrt{9 + h} - 3}{h}$

24. $\displaystyle\lim_{x\to 2} \frac{2 - x}{\sqrt{x + 2} - 2}$

25. $\displaystyle\lim_{x\to 3} \frac{\dfrac{1}{x} - \dfrac{1}{3}}{x - 3}$

26. $\displaystyle\lim_{h\to 0} \frac{(-2 + h)^{-1} + 2^{-1}}{h}$

27. $\displaystyle\lim_{t\to 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$

28. $\displaystyle\lim_{t\to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t}\right)$

29. $\displaystyle\lim_{x\to 16} \frac{4 - \sqrt{x}}{16x - x^2}$

30. $\displaystyle\lim_{x\to 2} \frac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$

31. $\displaystyle\lim_{t\to 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t}\right)$

32. $\displaystyle\lim_{x\to -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$

33. $\displaystyle\lim_{h\to 0} \frac{(x + h)^3 - x^3}{h}$

34. $\displaystyle\lim_{h\to 0} \frac{\dfrac{1}{(x + h)^2} - \dfrac{1}{x^2}}{h}$

35. (a) Estimate the value of

$$\lim_{x\to 0} \frac{x}{\sqrt{1 + 3x} - 1}$$

by graphing the function $f(x) = x/\left(\sqrt{1 + 3x} - 1\right)$.

(b) Make a table of values of $f(x)$ for x close to 0 and guess the value of the limit.

(c) Use the Limit Laws to prove that your guess is correct.

36. (a) Use a graph of

$$f(x) = \frac{\sqrt{3 + x} - \sqrt{3}}{x}$$

to estimate the value of $\lim_{x\to 0} f(x)$ to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Use the Limit Laws to find the exact value of the limit.

37. Use the Squeeze Theorem to show that

$$\lim_{x\to 0} x^2 \cos 20\pi x = 0$$

Illustrate by graphing the functions $f(x) = -x^2$, $g(x) = x^2 \cos 20\pi x$, and $h(x) = x^2$ on the same screen.

38. Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \sqrt{x^3 + x^2}\, \sin \frac{\pi}{x} = 0$$

Illustrate by graphing the functions f, g, and h (in the notation of the Squeeze Theorem) on the same screen.

39. If $4x - 9 \leqslant f(x) \leqslant x^2 - 4x + 7$ for $x \geqslant 0$, find $\lim_{x\to 4} f(x)$.

40. If $2x \leqslant g(x) \leqslant x^4 - x^2 + 2$ for all x, evaluate $\lim_{x\to 1} g(x)$.

41. Prove that $\displaystyle\lim_{x\to 0} x^4 \cos \frac{2}{x} = 0$.

42. Prove that $\displaystyle\lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

43–48 Find the limit, if it exists. If the limit does not exist, explain why.

43. $\displaystyle\lim_{x\to -4} \left(|x + 4| - 2x\right)$

44. $\displaystyle\lim_{x\to -4} \frac{|x + 4|}{2x + 8}$

45. $\displaystyle\lim_{x\to 0.5^-} \frac{2x - 1}{|2x^3 - x^2|}$

46. $\displaystyle\lim_{x\to -2} \frac{2 - |x|}{2 + x}$

47. $\displaystyle\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

48. $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

49. The Signum Function The *signum* (or sign) *function*, denoted by sgn, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\displaystyle\lim_{x\to 0^+} \operatorname{sgn} x$ (ii) $\displaystyle\lim_{x\to 0^-} \operatorname{sgn} x$

(iii) $\displaystyle\lim_{x\to 0} \operatorname{sgn} x$ (iv) $\displaystyle\lim_{x\to 0} |\operatorname{sgn} x|$

50. Let $g(x) = \operatorname{sgn}(\sin x)$.

(a) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x\to 0^+} g(x)$ (ii) $\lim_{x\to 0^-} g(x)$ (iii) $\lim_{x\to 0} g(x)$

(iv) $\lim_{x\to \pi^+} g(x)$ (v) $\lim_{x\to \pi^-} g(x)$ (vi) $\lim_{x\to \pi} g(x)$

(b) For which values of a does $\lim_{x\to a} g(x)$ not exist?

(c) Sketch a graph of g.

51. Let $g(x) = \dfrac{x^2 + x - 6}{|x - 2|}$.

(a) Find

(i) $\lim_{x\to 2^+} g(x)$ (ii) $\lim_{x\to 2^-} g(x)$

(b) Does $\lim_{x\to 2} g(x)$ exist?

(c) Sketch the graph of g.

52. Let

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ (x-2)^2 & \text{if } x \geqslant 1 \end{cases}$$

(a) Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

(b) Does $\lim_{x\to 1} f(x)$ exist?

(c) Sketch the graph of f.

53. Let

$$B(t) = \begin{cases} 4 - \frac{1}{2}t & \text{if } t < 2 \\ \sqrt{t + c} & \text{if } t \geqslant 2 \end{cases}$$

Find the value of c so that $\lim_{t\to 2} B(t)$ exists.

54. Let

$$g(x) = \begin{cases} x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2 - x^2 & \text{if } 1 < x \leqslant 2 \\ x - 3 & \text{if } x > 2 \end{cases}$$

(a) Evaluate each of the following, if it exists.

(i) $\lim_{x\to 1^-} g(x)$ (ii) $\lim_{x\to 1} g(x)$ (iii) $g(1)$

(iv) $\lim_{x\to 2^-} g(x)$ (v) $\lim_{x\to 2^+} g(x)$ (vi) $\lim_{x\to 2} g(x)$

(b) Sketch the graph of g.

55. (a) If the symbol $\llbracket\ \rrbracket$ denotes the greatest integer function defined in Example 10, evaluate

(i) $\lim_{x\to -2^+} \llbracket x \rrbracket$ (ii) $\lim_{x\to -2} \llbracket x \rrbracket$ (iii) $\lim_{x\to -2.4} \llbracket x \rrbracket$

(b) If n is an integer, evaluate

(i) $\lim_{x\to n^-} \llbracket x \rrbracket$ (ii) $\lim_{x\to n^+} \llbracket x \rrbracket$

(c) For what values of a does $\lim_{x\to a} \llbracket x \rrbracket$ exist?

56. Let $f(x) = \llbracket \cos x \rrbracket$, $-\pi \leqslant x \leqslant \pi$.

(a) Sketch the graph of f.

(b) Evaluate each limit, if it exists.

(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to (\pi/2)^-} f(x)$

(iii) $\lim_{x\to (\pi/2)^+} f(x)$ (iv) $\lim_{x\to \pi/2} f(x)$

(c) For what values of a does $\lim_{x\to a} f(x)$ exist?

57. If $f(x) = \llbracket x \rrbracket + \llbracket -x \rrbracket$, show that $\lim_{x\to 2} f(x)$ exists but is not equal to $f(2)$.

58. In the theory of relativity, the Lorentz contraction formula

$$L = L_0\sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v\to c^-} L$ and interpret the result. Why is a left-hand limit necessary?

59. If p is a polynomial, show that $\lim_{x\to a} p(x) = p(a)$.

60. If r is a rational function, use Exercise 59 to show that $\lim_{x\to a} r(x) = r(a)$ for every number a in the domain of r.

61. If $\lim_{x\to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x\to 1} f(x)$.

62. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 5$, find the following limits.

(a) $\lim_{x\to 0} f(x)$ (b) $\lim_{x\to 0} \dfrac{f(x)}{x}$

63. If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x\to 0} f(x) = 0$.

64. Show by means of an example that $\lim_{x\to a} [f(x) + g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

65. Show by means of an example that $\lim_{x\to a} [f(x)\,g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

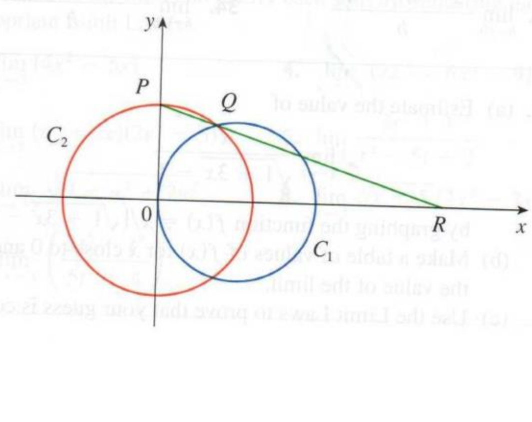
66. Evaluate $\lim_{x\to 2} \dfrac{\sqrt{6 - x} - 2}{\sqrt{3 - x} - 1}$.

67. Is there a number a such that

$$\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and the value of the limit.

68. The figure shows a fixed circle C_1 with equation $(x - 1)^2 + y^2 = 1$ and a shrinking circle C_2 with radius r and center the origin. P is the point $(0, r)$, Q is the upper point of intersection of the two circles, and R is the point of intersection of the line PQ and the x-axis. What happens to R as C_2 shrinks, that is, as $r \to 0^+$?



2.5 Continuity

Continuity of a Function

We noticed in Section 2.3 that the limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions having this property are called *continuous at a*. We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place without interruption.)

1 Definition A function f is **continuous at a number a** if

$$\lim_{x \to a} f(x) = f(a)$$

As illustrated in Figure 1, if f is continuous, then the points $(x, f(x))$ on the graph of f approach the point $(a, f(a))$ on the graph. So there is no gap in the curve.

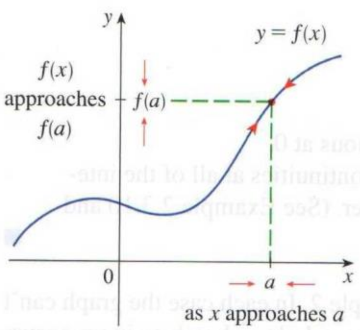


FIGURE 1

Notice that Definition 1 implicitly requires three things if f is continuous at a:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists
3. $\lim_{x \to a} f(x) = f(a)$

The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a. Thus a continuous function f has the property that a small change in x produces only a small change in $f(x)$. In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at a** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a moving vehicle varies continuously with time, as does a person's height. But discontinuities do occur in such situations as electric currents. [The Heaviside function, introduced in Section 2.2, is discontinuous at 0 because $\lim_{t \to 0} H(t)$ does not exist.]

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

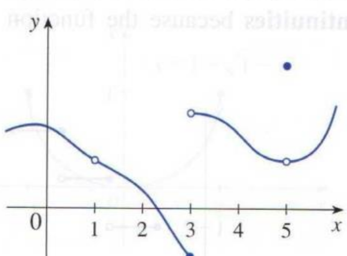


FIGURE 2

EXAMPLE 1 Figure 2 shows the graph of a function f. At which numbers is f discontinuous? Why?

SOLUTION It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that f is discontinuous at 1 is that $f(1)$ is not defined.

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same). But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5. ■

Now let's see how to detect discontinuities when a function is defined by a formula.

EXAMPLE 2 Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ (b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ (d) $f(x) = [\![x]\!]$

SOLUTION

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

(b) Here $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2}\frac{(x-2)(x+1)}{x - 2} = \lim_{x\to 2}(x + 1) = 3$$

exists. But

$$\lim_{x\to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

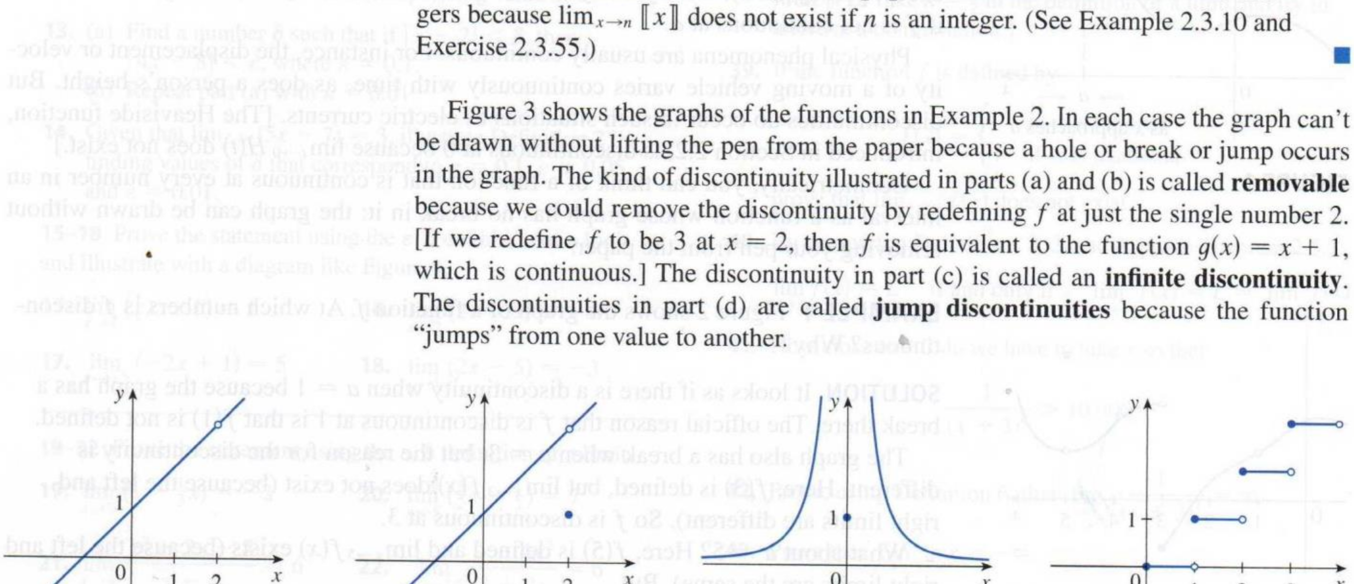
(c) Here $f(0) = 1$ is defined but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{1}{x^2}$$

does not exist. (See Example 2.2.6.) So f is discontinuous at 0.

(d) The greatest integer function $f(x) = [\![x]\!]$ has discontinuities at all of the integers because $\lim_{x\to n} [\![x]\!]$ does not exist if n is an integer. (See Example 2.3.10 and Exercise 2.3.55.) ■

Figure 3 shows the graphs of the functions in Example 2. In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (b) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [If we redefine f to be 3 at $x = 2$, then f is equivalent to the function $g(x) = x + 1$, which is continuous.] The discontinuity in part (c) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function "jumps" from one value to another.



(a) A removable discontinuity (b) A removable discontinuity (c) An infinite discontinuity (d) Jump discontinuities

FIGURE 3

Graphs of the functions in Example 2

2 Definition A function f is **continuous from the right at a number a** if

$$\lim_{x\to a^+} f(x) = f(a)$$

and f is **continuous from the left at a** if

$$\lim_{x\to a^-} f(x) = f(a)$$

EXAMPLE 3 At each integer n, the function $f(x) = [\![x]\!]$ [see Figure 3(d)] is continuous from the right but discontinuous from the left because

$$\lim_{x\to n^+} f(x) = \lim_{x\to n^+} [\![x]\!] = n = f(n)$$

but.
$$\lim_{x\to n^-} f(x) = \lim_{x\to n^-} [\![x]\!] = n - 1 \neq f(n)$$
■

3 Definition A function f is **continuous on an interval** if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right* or *continuous from the left.*)

EXAMPLE 4 Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

SOLUTION If $-1 < a < 1$, then using the Limit Laws from Section 2.3, we have

$$\begin{aligned}
\lim_{x\to a} f(x) &= \lim_{x\to a} \left(1 - \sqrt{1 - x^2}\right) \\
&= 1 - \lim_{x\to a} \sqrt{1 - x^2} && \text{(by Laws 2 and 8)} \\
&= 1 - \sqrt{\lim_{x\to a} (1 - x^2)} && \text{(by 7)} \\
&= 1 - \sqrt{1 - a^2} && \text{(by 2, 8, and 10)} \\
&= f(a)
\end{aligned}$$

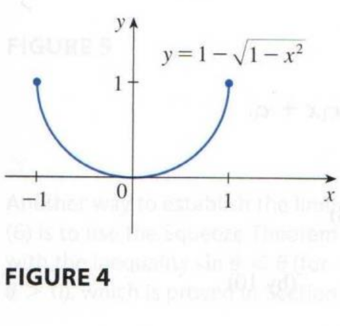


FIGURE 4

Thus, by Definition l, f is continuous at a if $-1 < a < 1$. Similar calculations show that

$$\lim_{x\to -1^+} f(x) = 1 = f(-1) \qquad \text{and} \qquad \lim_{x\to 1^-} f(x) = 1 = f(1)$$

so f is continuous from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, f is continuous on $[-1, 1]$.

The graph of f is sketched in Figure 4. It is the lower half of the circle

$$x^2 + (y - 1)^2 = 1$$
■

■ Properties of Continuous Functions

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function as we did in Example 4, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ 2. $f - g$ 3. cf
4. fg 5. $\dfrac{f}{g}$ if $g(a) \neq 0$

PROOF Each of the five parts of this theorem follows from the corresponding Limit Law in Section 2.3. For instance, we give the proof of part 1. Since f and g are continuous at a, we have

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a)$$

Therefore

$$\begin{aligned} \lim_{x \to a} (f + g)(x) &= \lim_{x \to a} [f(x) + g(x)] \\ &= \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \qquad \text{(by Law 1)} \\ &= f(a) + g(a) \\ &= (f + g)(a) \end{aligned}$$

This shows that $f + g$ is continuous at a. ■

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) f/g. The following theorem was stated in Section 2.3 as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

PROOF

(a) A polynomial is a function of the form

$$P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_n$ are constants. We know that

$$\lim_{x \to a} c_0 = c_0 \qquad \text{(by Law 8)}$$

and

$$\lim_{x \to a} x^m = a^m \qquad m = 1, 2, \ldots, n \qquad \text{(by 10)}$$

This equation is precisely the statement that the function $f(x) = x^m$ is a continuous function. Thus, by part 3 of Theorem 4, the function $g(x) = cx^m$ is continuous. Since P is a sum of functions of this form and a constant function, it follows from part 1 of Theorem 4 that P is continuous.

(b) A rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials. The domain of f is $D = \{x \in \mathbb{R} \mid Q(x) \neq 0\}$. We know from part (a) that P and Q are continuous everywhere. Thus, by part 5 of Theorem 4, f is continuous at every number in D. ■

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r. Likewise, if a ball is thrown vertically into the air with an initial velocity of 15 m/s, then the height of the ball in meters t seconds later is given by the formula $h = 15t - 4.9t^2$. Again this is a polynomial function, so the height is a continuous function of the elapsed time, as we might expect.

Knowledge of which functions are continuous enables us to evaluate some limits very quickly, as the following example shows. Compare it with Example 2.3.2(b).

EXAMPLE 5 Find $\displaystyle\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$.

SOLUTION The function

$$f(x) = \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

is rational, so by Theorem 5 it is continuous on its domain, which is $\{x \mid x \neq \frac{5}{3}\}$. Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$ ■

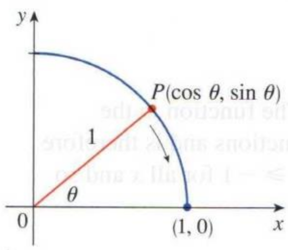


FIGURE 5

It turns out that most of the familiar functions are continuous at every number in their domains. For instance, Limit Law 11 in Section 2.3 is exactly the statement that root functions are continuous.

From the appearance of the graphs of the sine and cosine functions (Figure 1.2.19), we would certainly guess that they are continuous. We know from the definitions of $\sin\theta$ and $\cos\theta$ that the coordinates of the point P in Figure 5 are $(\cos\theta, \sin\theta)$. As $\theta \to 0$, we see that P approaches the point $(1, 0)$ and so $\cos\theta \to 1$ and $\sin\theta \to 0$. Thus

6 $$\lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

Another way to establish the limits in (6) is to use the Squeeze Theorem with the inequality $\sin\theta < \theta$ (for $\theta > 0$), which is proved in Section 3.3.

Since $\cos 0 = 1$ and $\sin 0 = 0$, the equations in (6) assert that the cosine and sine functions are continuous at 0. The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere (see Exercises 66 and 67).

It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where $\cos x = 0$. This happens when x is an odd integer multiple

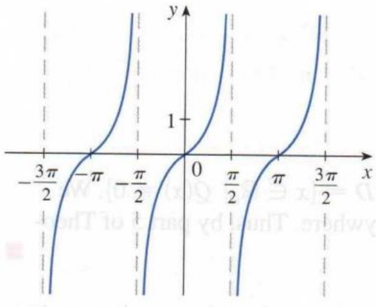


FIGURE 6
$y = \tan x$

of $\pi/2$, so $y = \tan x$ has infinite discontinuities when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, and so on (see Figure 6).

The inverse function of any continuous one-to-one function is also continuous. (This fact is proved in Appendix F, but our geometric intuition makes it seem plausible: the graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$. So if the graph of f has no break in it, neither does the graph of f^{-1}.) Thus the inverse trigonometric functions are continuous.

The inverse trigonometric functions are reviewed in Section 1.5.

In Section 1.4 we defined the exponential function $y = b^x$ so as to fill in the holes in the graph of $y = b^x$ where x is rational. In other words, the very definition of $y = b^x$ makes it a continuous function on $\mathbb{R}$. Therefore its inverse function $y = \log_b x$ is continuous on $(0, \infty)$.

7 Theorem The following types of functions are continuous at every number in their domains:

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

EXAMPLE 6 Where is the function $f(x) = \dfrac{\ln x + \tan^{-1}x}{x^2 - 1}$ continuous?

SOLUTION We know from Theorem 7 that the function $y = \ln x$ is continuous for $x > 0$ and $y = \tan^{-1}x$ is continuous on $\mathbb{R}$. Thus, by part 1 of Theorem 4, $y = \ln x + \tan^{-1}x$ is continuous on $(0, \infty)$. The denominator, $y = x^2 - 1$, is a polynomial, so it is continuous everywhere. Therefore, by part 5 of Theorem 4, f is continuous at all positive numbers x except where $x^2 - 1 = 0 \iff x = \pm 1$. So f is continuous on the intervals $(0, 1)$ and $(1, \infty)$. ■

EXAMPLE 7 Evaluate $\displaystyle\lim_{x \to \pi} \frac{\sin x}{2 + \cos x}$.

SOLUTION Theorem 7 tells us that $y = \sin x$ is continuous. The function in the denominator, $y = 2 + \cos x$, is the sum of two continuous functions and is therefore continuous. Notice that this function is never 0 because $\cos x \geqslant -1$ for all x and so $2 + \cos x > 0$ everywhere. Thus the ratio

$$f(x) = \frac{\sin x}{2 + \cos x}$$

is continuous everywhere. Hence, by the definition of a continuous function,

$$\lim_{x \to \pi} \frac{\sin x}{2 + \cos x} = \lim_{x \to \pi} f(x) = f(\pi) = \frac{\sin \pi}{2 + \cos \pi} = \frac{0}{2 - 1} = 0$$ ■

Another way of combining continuous functions f and g to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

This theorem says that a limit symbol can be moved through a function symbol if the function is continuous and the limit exists. In other words, the order of these two symbols can be reversed.

8 Theorem If f is continuous at b and $\displaystyle\lim_{x \to a} g(x) = b$, then $\displaystyle\lim_{x \to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Intuitively, Theorem 8 is reasonable because if x is close to a, then $g(x)$ is close to b, and since f is continuous at b, if $g(x)$ is close to b, then $f(g(x))$ is close to $f(b)$. A proof of Theorem 8 is given in Appendix F.

EXAMPLE 8 Evaluate $\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x \to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{1-x}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1}{1+\sqrt{x}}\right) \\
&= \arcsin\frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

Let's now apply Theorem 8 in the special case where $f(x) = \sqrt[n]{x}$, with n being a positive integer. Then

$$f(g(x)) = \sqrt[n]{g(x)}$$

and

$$f\left(\lim_{x \to a} g(x)\right) = \sqrt[n]{\lim_{x \to a} g(x)}$$

If we put these expressions into Theorem 8, we get

$$\lim_{x \to a} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x \to a} g(x)}$$

and so Limit Law 7 has now been proved. (We assume that the roots exist.)

9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

This theorem is often expressed informally by saying "a continuous function of a continuous function is a continuous function."

PROOF Since g is continuous at a, we have

$$\lim_{x \to a} g(x) = g(a)$$

Since f is continuous at $b = g(a)$, we can apply Theorem 8 to obtain

$$\lim_{x \to a} f(g(x)) = f(g(a))$$

which is precisely the statement that the function $h(x) = f(g(x))$ is continuous at a; that is, $f \circ g$ is continuous at a.

EXAMPLE 9 Where are the following functions continuous?

(a) $h(x) = \sin(x^2)$ (b) $F(x) = \ln(1 + \cos x)$

SOLUTION

(a) We have $h(x) = f(g(x))$, where

$$g(x) = x^2 \quad \text{and} \quad f(x) = \sin x$$

We know that g is continuous on $\mathbb{R}$ since it is a polynomial, and f is also continuous everywhere. Thus $h = f \circ g$ is continuous on $\mathbb{R}$ by Theorem 9.

(b) We know from Theorem 7 that $f(x) = \ln x$ is continuous and $g(x) = 1 + \cos x$ is continuous (because both $y = 1$ and $y = \cos x$ are continuous). Therefore, by Theorem 9, $F(x) = f(g(x))$ is continuous wherever it is defined. The expression $\ln(1 + \cos x)$ is defined when $1 + \cos x > 0$, so it is undefined when $\cos x = -1$, and this happens when $x = \pm\pi, \pm 3\pi, \ldots$. Thus F has discontinuities when x is an odd multiple of π and is continuous on the intervals between these values (see Figure 7). ■

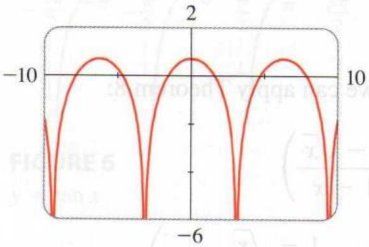


FIGURE 7
$y = \ln(1 + \cos x)$

■ The Intermediate Value Theorem

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

> **10 The Intermediate Value Theorem** Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8. Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

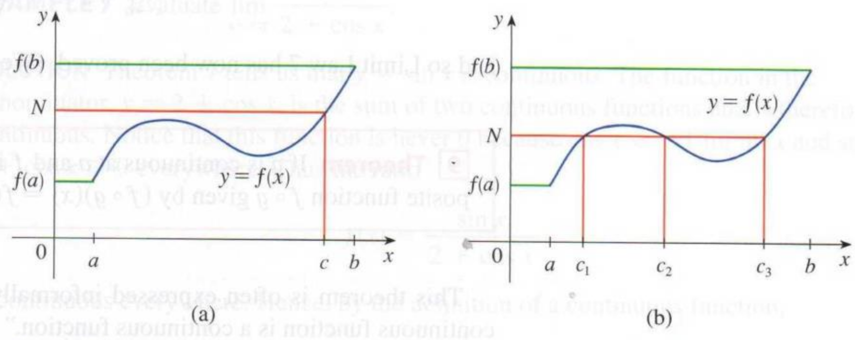


FIGURE 8

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true. In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

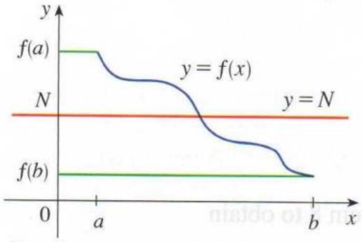


FIGURE 9

It is important that the function f in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions (see Exercise 52).

One use of the Intermediate Value Theorem is in locating solutions of equations as in the following example.

EXAMPLE 10 Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0$$

between 1 and 2.

SOLUTION Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. The function f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one solution c in the interval $(1, 2)$.

In fact, we can locate a solution more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a solution must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \quad \text{and} \quad f(1.23) = 0.056068 > 0$$

so a solution lies in the interval $(1.22, 1.23)$. ■

We can use a graphing calculator or computer to illustrate the use of the Intermediate Value Theorem in Example 10. Figure 10 shows the graph of f in the viewing rectangle $[-1, 3]$ by $[-3, 3]$ and you can see that the graph crosses the x-axis between 1 and 2. Figure 11 shows the result of zooming in to the viewing rectangle $[1.2, 1.3]$ by $[-0.2, 0.2]$.

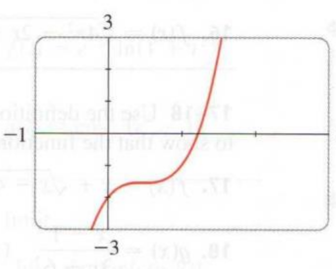


FIGURE 10

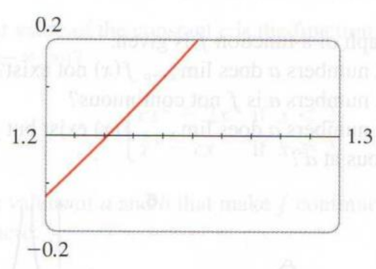


FIGURE 11

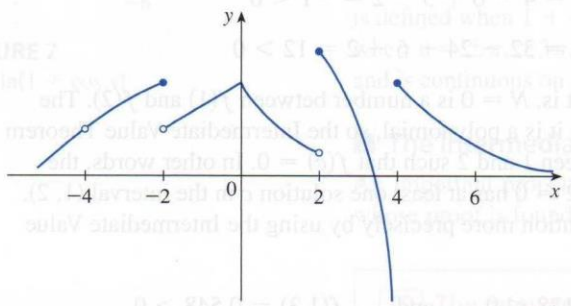
In fact, the Intermediate Value Theorem plays a role in the very way these graphing devices work. A computer calculates a finite number of points on the graph and turns on the pixels that contain these calculated points. It assumes that the function is continuous and takes on all the intermediate values between two consecutive points. The computer therefore "connects the dots" by turning on the intermediate pixels.

2.5 Exercises

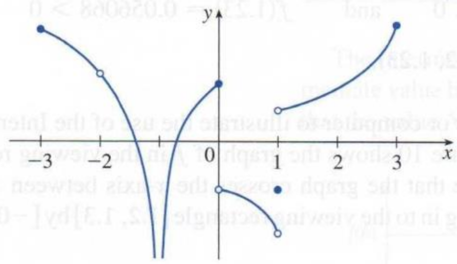
1. Write an equation that expresses the fact that a function f is continuous at the number 4.

2. If f is continuous on $(-\infty, \infty)$, what can you say about its graph?

3. (a) From the given graph of f, state the numbers at which f is discontinuous and explain why.
(b) For each of the numbers stated in part (a), determine whether f is continuous from the right, or from the left, or neither.

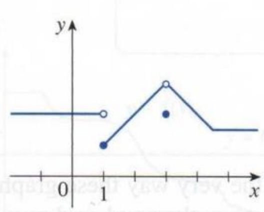


4. From the given graph of g, state the numbers at which g is discontinuous and explain why.

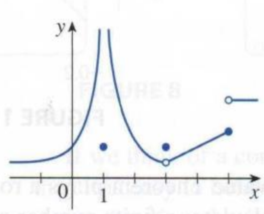


5–6 The graph of a function f is given.
(a) At what numbers a does $\lim_{x \to a} f(x)$ not exist?
(b) At what numbers a is f not continuous?
(c) At what numbers a does $\lim_{x \to a} f(x)$ exist but f is not continuous at a?

5.



6.



7–10 Sketch the graph of a function f that is defined on $\mathbb{R}$ and continuous except for the stated discontinuities.

7. Removable discontinuity at -2, infinite discontinuity at 2

8. Jump discontinuity at -3, removable discontinuity at 4

9. Discontinuities at 0 and 3, but continuous from the right at 0 and from the left at 3

10. Continuous only from the left at -1, not continuous from the left or right at 3

11. The toll T charged for driving on a certain stretch of a toll road is \$5 except during rush hours (between 7 AM and 10 AM and between 4 PM and 7 PM) when the toll is \$7.
(a) Sketch a graph of T as a function of the time t, measured in hours past midnight.
(b) Discuss the discontinuities of this function and their significance to someone who uses the road.

12. Explain why each function is continuous or discontinuous.
(a) The temperature at a specific location as a function of time
(b) The temperature at a specific time as a function of the distance due west from New York City
(c) The altitude above sea level as a function of the distance due west from New York City
(d) The cost of a taxi ride as a function of the distance traveled
(e) The current in the circuit for the lights in a room as a function of time

13–16 Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

13. $f(x) = 3x^2 + (x + 2)^5, \quad a = -1$

14. $g(t) = \dfrac{t^2 + 5t}{2t + 1}, \quad a = 2$

15. $p(v) = 2\sqrt{3v^2 + 1}, \quad a = 1$

16. $f(r) = \sqrt[3]{4r^2 - 2r + 7}, \quad a = -2$

17–18 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

17. $f(x) = x + \sqrt{x - 4}, \quad [4, \infty)$

18. $g(x) = \dfrac{x - 1}{3x + 6}, \quad (-\infty, -2)$

19–24 Explain why the function is discontinuous at the given number a. Sketch the graph of the function.

19. $f(x) = \dfrac{1}{x + 2} \qquad a = -2$

20. $f(x) = \begin{cases} \dfrac{1}{x + 2} & \text{if } x \neq -2 \\ 1 & \text{if } x = -2 \end{cases} \qquad a = -2$

21. $f(x) = \begin{cases} x + 3 & \text{if } x \leqslant -1 \\ 2^x & \text{if } x > -1 \end{cases} \qquad a = -1$

22. $f(x) = \begin{cases} \dfrac{x^2 - x}{x^2 - 1} & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} \qquad a = 1$

23. $f(x) = \begin{cases} \cos x & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 - x^2 & \text{if } x > 0 \end{cases} \qquad a = 0$

24. $f(x) = \begin{cases} \dfrac{2x^2 - 5x - 3}{x - 3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases} \qquad a = 3$

25–26

(a) Show that f has a removable discontinuity at $x = 3$.
(b) Redefine $f(3)$ so that f is continuous at $x = 3$ (and thus the discontinuity is "removed").

25. $f(x) = \dfrac{x - 3}{x^2 - 9}$ **26.** $f(x) = \dfrac{x^2 - 7x + 12}{x - 3}$

27–34 Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain. State the domain.

27. $f(x) = \dfrac{x^2}{\sqrt{x^4 + 2}}$ **28.** $g(v) = \dfrac{3v - 1}{v^2 + 2v - 15}$

29. $h(t) = \dfrac{\cos(t^2)}{1 - e^t}$

30. $B(u) = \sqrt{3u - 2} + \sqrt[3]{2u - 3}$

31. $L(v) = v \ln(1 - v^2)$ **32.** $f(t) = e^{-t^2} \ln(1 + t^2)$

33. $M(x) = \sqrt{1 + \dfrac{1}{x}}$ **34.** $g(t) = \cos^{-1}(e^t - 1)$

35–38 Use continuity to evaluate the limit.

35. $\lim\limits_{x \to 2} x\sqrt{20 - x^2}$ **36.** $\lim\limits_{\theta \to \pi/2} \sin(\tan(\cos \theta))$

37. $\lim\limits_{x \to 1} \ln\left(\dfrac{5 - x^2}{1 + x}\right)$ **38.** $\lim\limits_{x \to 4} 3^{\sqrt{x^2 - 2x - 4}}$

39–40 Locate the discontinuities of the function and illustrate by graphing.

39. $f(x) = \dfrac{1}{\sqrt{1 - \sin x}}$ **40.** $y = \arctan \dfrac{1}{x}$

41–42 Show that f is continuous on $(-\infty, \infty)$.

41. $f(x) = \begin{cases} 1 - x^2 & \text{if } x \leqslant 1 \\ \ln x & \text{if } x > 1 \end{cases}$

42. $f(x) = \begin{cases} \sin x & \text{if } x < \pi/4 \\ \cos x & \text{if } x \geqslant \pi/4 \end{cases}$

43–45 Find the numbers at which f is discontinuous. At which of these numbers is f continuous from the right, from the left, or neither? Sketch the graph of f.

43. $f(x) = \begin{cases} x^2 & \text{if } x < -1 \\ x & \text{if } -1 \leqslant x < 1 \\ 1/x & \text{if } x \geqslant 1 \end{cases}$

44. $f(x) = \begin{cases} 2^x & \text{if } x \leqslant 1 \\ 3 - x & \text{if } 1 < x \leqslant 4 \\ \sqrt{x} & \text{if } x > 4 \end{cases}$

45. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ e^x & \text{if } 0 \leqslant x \leqslant 1 \\ 2 - x & \text{if } x > 1 \end{cases}$

46. The gravitational force exerted by the planet Earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \dfrac{GMr}{R^3} & \text{if } r < R \\ \dfrac{GM}{r^2} & \text{if } r \geqslant R \end{cases}$$

where M is the mass of Earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r?

47. For what value of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geqslant 2 \end{cases}$$

48. Find the values of a and b that make f continuous everywhere.

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x < 2 \\ ax^2 - bx + 3 & \text{if } 2 \leqslant x < 3 \\ 2x - a + b & \text{if } x \geqslant 3 \end{cases}$$

49. Suppose f and g are continuous functions such that $g(2) = 6$ and $\lim_{x \to 2} [3f(x) + f(x)g(x)] = 36$. Find $f(2)$.

50. Let $f(x) = 1/x$ and $g(x) = 1/x^2$.
(a) Find $(f \circ g)(x)$.
(b) Is $f \circ g$ continuous everywhere? Explain.

51. Which of the following functions f has a removable discontinuity at a? If the discontinuity is removable, find a function g that agrees with f for $x \neq a$ and is continuous at a.

(a) $f(x) = \dfrac{x^4 - 1}{x - 1}$, $a = 1$

(b) $f(x) = \dfrac{x^3 - x^2 - 2x}{x - 2}$, $a = 2$

(c) $f(x) = [\![\sin x]\!]$, $a = \pi$

52. Suppose that a function f is continuous on $[0, 1]$ except at 0.25 and that $f(0) = 1$ and $f(1) = 3$. Let $N = 2$. Sketch two possible graphs of f, one showing that f might not satisfy the conclusion of the Intermediate Value Theorem and one showing that f might still satisfy the conclusion of the Intermediate Value Theorem (even though it doesn't satisfy the hypothesis).

53. If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

54. Suppose f is continuous on $[1, 5]$ and the only solutions of the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

55–58 Use the Intermediate Value Theorem to show that there is a solution of the given equation in the specified interval.

55. $-x^3 + 4x + 1 = 0$, $(-1, 0)$

56. $\ln x = x - \sqrt{x}$, $(2, 3)$

57. $e^x = 3 - 2x$, $(0, 1)$

58. $\sin x = x^2 - x$, $(1, 2)$

59–60

(a) Prove that the equation has at least one real solution.

(b) Use a calculator to find an interval of length 0.01 that contains a solution.

59. $\cos x = x^3$

60. $\ln x = 3 - 2x$

61–62

(a) Prove that the equation has at least one real solution.

(b) Find the solution correct to three decimal places, by graphing.

61. $100e^{-x/100} = 0.01x^2$

62. $\arctan x = 1 - x$

63–64 Prove, without graphing, that the graph of the function has at least two x-intercepts in the specified interval.

63. $y = \sin x^3$, $(1, 2)$

64. $y = x^2 - 3 + 1/x$, $(0, 2)$

65. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

66. To prove that sine is continuous, we need to show that $\lim_{x \to a} \sin x = \sin a$ for every real number a. By Exercise 65 an equivalent statement is that

$$\lim_{h \to 0} \sin(a + h) = \sin a$$

Use (6) to show that this is true.

67. Prove that cosine is a continuous function.

68. (a) Prove Theorem 4, part 3.

(b) Prove Theorem 4, part 5.

69. Use Theorem 8 to prove Limit Laws 6 and 7 from Section 2.3.

70. Is there a number that is exactly 1 more than its cube?

71. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

72. For what values of x is g continuous?

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$$

73. Show that the function

$$f(x) = \begin{cases} x^4 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $(-\infty, \infty)$.

74. If a and b are positive numbers, prove that the equation

$$\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0$$

has at least one solution in the interval $(-1, 1)$.

75. A woman leaves her house at 7:00 AM and takes her usual path to the top of a mountain, arriving at 7:00 PM. The following morning, she starts at 7:00 AM at the top and takes the same path back, arriving at her home at 7:00 PM. Use the Intermediate Value Theorem to show that there is a point on the path that the woman will cross at exactly the same time of day on both days.

76. Absolute Value and Continuity

(a) Show that the absolute value function $F(x) = |x|$ is continuous everywhere.

(b) Prove that if f is a continuous function on an interval, then so is $|f|$.

(c) Is the converse of the statement in part (b) also true? In other words, if $|f|$ is continuous, does it follow that f is continuous? If so, prove it. If not, find a counterexample.