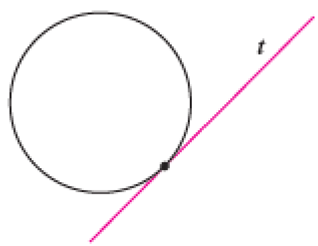
Theme 1: The tangent velocity problem. Limit of a Function. One-sides limits

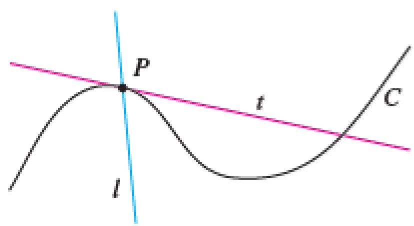
Definitions, methods, formulas, theoremss:

Definitions:

1. A tangent to a curve is a line that touches the curve. In other words, a tangent line should have the same direction as the curve at the point of contact.

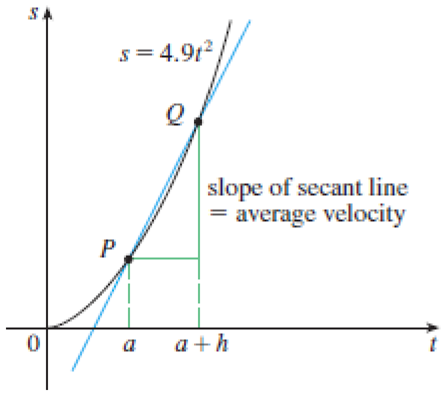


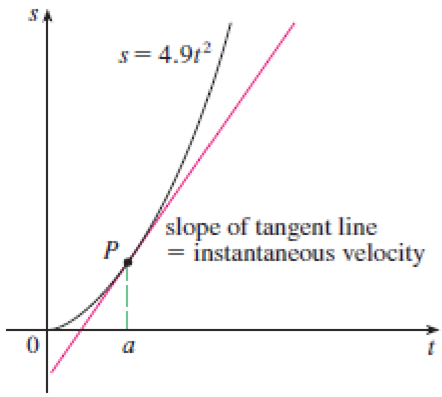
(a)



2) A secant line, from the Latin word secans, meaning cutting, is a line that cuts (intersects) a curve more than once

3) The instantaneous velocity when $t = a$ is defined to be the limiting value of these average velocities over shorter and shorter time periods that start at $t = a$. Let us see this concept for the function $s(t) = 4.9t^2$.





4) **Intuitive Definition of a Limit.** Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a (except possibly at a itself). Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of f is L as x approaches a, equals L" if we can make the values of f arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

5) **Definition of One-Sided Limits:** We write

$$\lim_{x \to a-} f(x) = L$$

and say the left-hand limit of f as x approaches a [or the limit of f as x approaches a from the left] is equal to L if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a with x less than a.

We write

$$\lim_{x \to a+} f(x) = L$$

and say the left-hand limit of f as x approaches a [or the limit of f as x approaches a from the right] is equal to L if we can make the values of f arbitrarily close to L by taking x to be sufficiently close to a with x greater than a.

6) **Intuitive Definition of an Infinite Limit.** Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

Intuitive Definition of an Infinite Limit. Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative (as large as we please) by taking x sufficiently close to a, but not equal to a.

7) Definition The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty; \quad \lim_{x \to a-} f(x) = \infty \quad \lim_{x \to a+} f(x) = \infty;$$

$$\lim_{x \to a} f(x) = -\infty; \quad \lim_{x \to a-} f(x) = -\infty \quad \lim_{x \to a+} f(x) = -\infty.$$

Sections 2.3; 2.5 (Partially), Calculation of limits using limit laws. The concept of continuity

Theorems without proof.

1. **Theorem.**

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = L.$$

2. Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x) \text{ exist.}$$

Then

(a)

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x);$$

(b)

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x);$$

(c)

$$\lim_{x\to a}(cf(x)) = c\lim_{x\to a} f(x);$$

(d)

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\lim_{x\to a} g(x);$$

(e)

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}, \text{provided that } \lim_{x\to a} g(x) \neq 0;$$

(f)

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n, \text{ where} n \text{is a real number;}$$

(g)

$$\lim_{x\to a} c = c;$$

(h)

$$\lim_{x\to a} x^n = a^n;$$

(i)

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a};$$

(i)

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}.$$

Definitions:

1) **Continuity at a single point.** A function f is continuous at a number a if

$$\lim_{x\to a} f(x) = f(a).$$

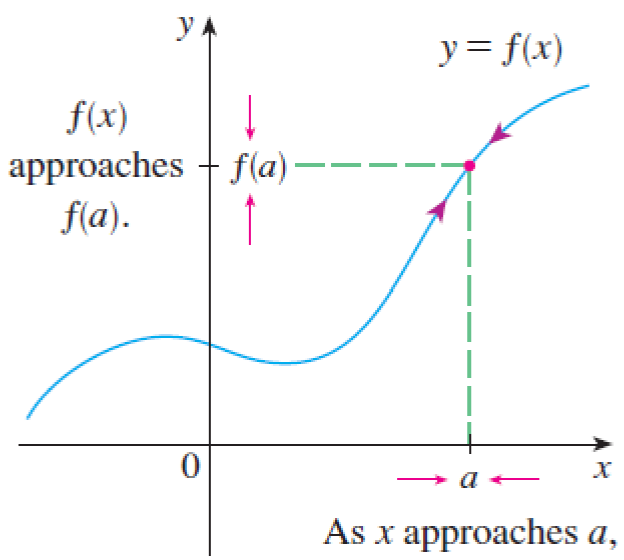
Notice that this definition implicitly requires three things if f is continuous at a:

a) $f(a)$ is defined (that is, a is in the domain of f);
b)

$$\lim_{x\to a} f(x)$$

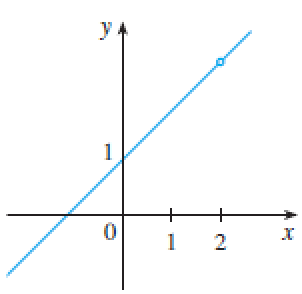
exists;
c)

$$\lim_{x\to a} f(x) = f(a).$$

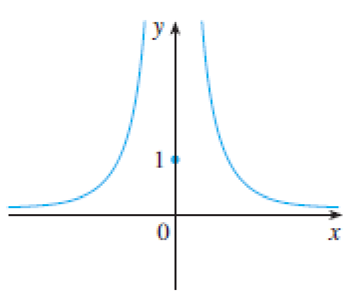


2) **Rough definition of continuity**:

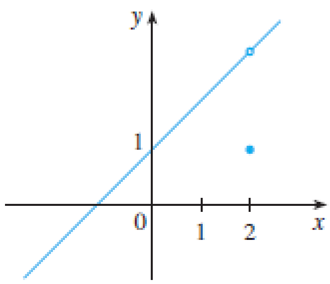
The function is discontinuous if the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. Consider the following figures:



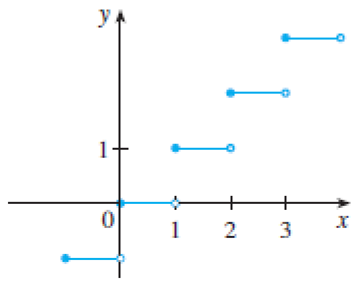
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$



(c) $f(x)=\begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x\neq 2 \\ 1 & \text{if } x=2 \end{cases}$



(d) $f(x)=[\![x]\!]$

In each case the graph can't be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph. The kind of discontinuity illustrated in parts (a) and (c) is called removable because we could remove the discontinuity by redefining f at just the single number 2. [The function $f(x)=x+1$ is continuous.] The discontinuity in part (b) is called an infinite discontinuity. The discontinuities in part (d) are called jump discontinuities because the function "jumps" from one value to another.

3) **One-sided continuity.** A function f is continuous from the right at a number a if

$$\lim_{x\to a+} f(x) = f(a).$$

A function f is continuous from the left at a number a if

$$\lim_{x\to a-} f(x) = f(a).$$

Formulas for:

$$\text{average velocity} = \frac{\text{change in position}}{\text{time elapsed}}$$

Theorems with proofs.

1) **Theorem.** Let $f(x)\leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x).$$

2) **The Squeeze Theorem.** Let $f(x)\leq g(x)\leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L.$$