KUTAISI
INTERNATIONAL
UNIVERSITY

Numerical Linear Algebra

Ramaz Botchorishvili

Kutaisi International University

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Matrix, Norm, Condition Number

- Recap of Previous Lecture
- Computational project 1
- Convergence of matrix sequences
- Matrix Series
- Triangular systems of linear equations
- Condition number
- Difficulties with theoretical linear algebra
- Q & A

Recap of Previous Lecture

- Matrix types examples
- Digital Image and Matrix, Visualisation
- Matrix Operations
- Some Useful Properties
- Matrix Norm
- Q & A

Topics

1. Operations with matrices
2. Matrix and vector norms
3. Modeling, refining and developing models
4. Applying methods to practical problems

Topics

1. Operations with matrices
2. Matrix and vector norms
3. Modeling, refining and developing models
4. Applying methods to practical problems

Connections with PLOs

- PLO7: an ability to search for, process and analyse information from a variety of sources and to communicate in a professional way orally and in written form)
- PLO5: an ability to design mathematical models in a broad range of intellectual domain
- PLO4: an ability to identify, formulate, abstract and solve mathematical problems applying analytical, symbolic and computatioanl methods together with computing facilities
- PLO3: understanding of limitations of mathematical methods and the constraints on their applicability

Assessment

- Homework is worth of 10 points subject to scaling and/or dropping out some of the points
- The problem/sub-problem is assigned 0 point if
  - same set of student defined parameters are used by two or more students
  - answer cannot be replicated
  - solution of sub-problem is submitted without explanation/proof
  - code fails: does not produce correct results on new tests

Problem 3.1

Blending digital images (1 point)

1. take two digital images
2. blend images for different values of parameters
3. you can use opencv library for creating video from set of blended images
4. Using arithmetic operations on matrices only is allowed for creating blended images.
5. Test: your code should work properly on digital images provided by TA.

Problem 3.2

Modeling with k-means clustering (1 different model= 1 point, MGMT students are eligible up to 4 points, MATH and CS students are eligible up to 2 points)

1. Find applications of k-means clustering
2. Formulate the problem. Discuss vector or matrix norm suitable for solving the problem.
3. Notice: 0 points is assigned to a model problem provided twice by students. Make sure, your model is different from models provided by others.

Problem 3.3

Implementation of k-means clustering for model problem (2 points)

1. Formulate problem including input data and suitable matrix or vector norm(s).
2. Solve the problem using k-means clustering.
3. Perform numerical experiments on how selection of different norms and starting centers or number of clusters affect the solution, number of iterations, convergence or divergence
4. Test: your code should work properly on data provided by TA.
5. Notice: 0 points is assigned to a model problem provided twice by students. Make sure, your model is different from models provided by others.

Problem 3.4

Ranking students for fellowship applications (2 points)

1. Students transcript is available for 4 semesters. In the fifth semester those who do not receive funding, can apply for scholarship.
2. students transcripts are given as 4×5 matrix where each line correspond to grades for one semester.
3. Which matrix norm is best for ranking the students ? explain your approach and justify your choice
4. Generate some data, test your approach and demonstrate
5. Test: your code should work properly on data provided by TA.

Problem 3.5

Partition of 225 freshmen students in 15 groups of 15 students in each (2 points)

- For each student 3-vector is available. These vectors contain NAEC examination scores in Georgian language, English language and mathematics.
- Requirement is that each group should have approximately similar strength from the viewpoint of NAEC examination scores.
- Compare groups using matrix norm. Which norm is better for the task? Explain.
- Describe your approach and algorithm. Give reasons for justifying your choice.
- Generate synthetic data, run code, visualize output
- investigate (experimentally) how solution depends on vector and matrix norms used in solution algorithm
- Are you happy with obtained solution? Explain why.
- Test: your code should work properly on data provided by TA.

Problem 3.6

Detect liquid level in a glass vessel (3 points)

- Simplified case: digital camera and glass vessel are at constant locations, e.g. in some laboratory. Vessel is filled in at various levels and with various liquids of predefined list. Detect from video digital images:
  - if vessel is empty, full, or filled in x percent, $0 < x < 100$
  - which liquid it is
- Use matrix norms for detecting liquid in a glass vessel from the predefined list of liquids. Consider three different liquids, for example, water, green tea, and coffee. How do you solve problem of possible color variations in each liquid? Which norms do you apply? Why?
- Describe your approach and algorithm in written. Give reasons for justifying your choice.
- Generate synthetic data, run code, visualize output
- Are you happy with obtained solution? Explain why.
- Test: your code should work properly on data provided by TA.

Image Blending Example

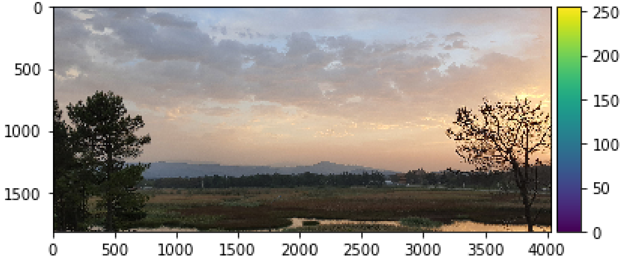


Image Blending Example

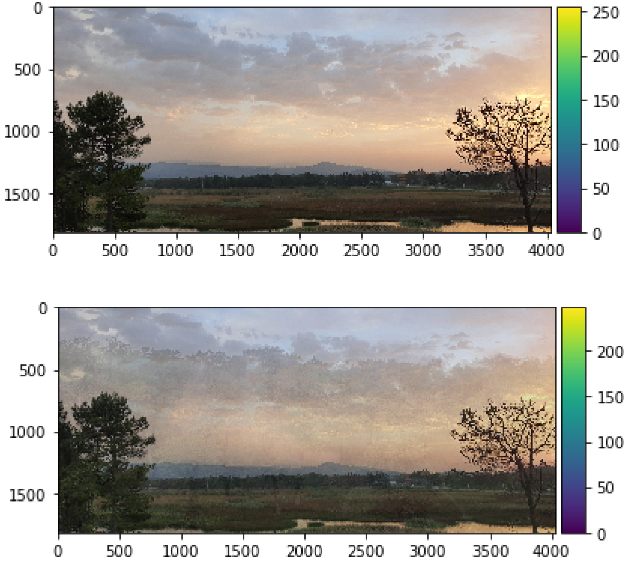
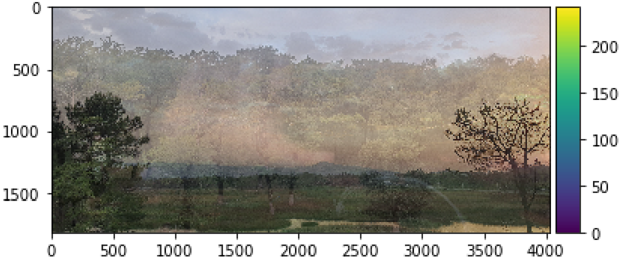


Figure:

Image Blending Example



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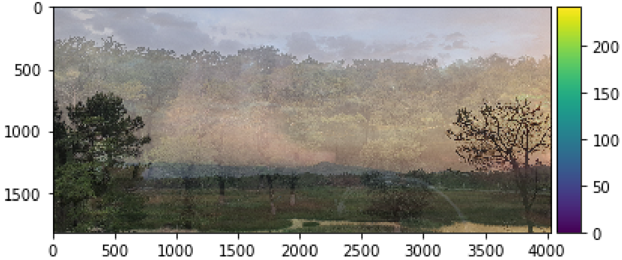

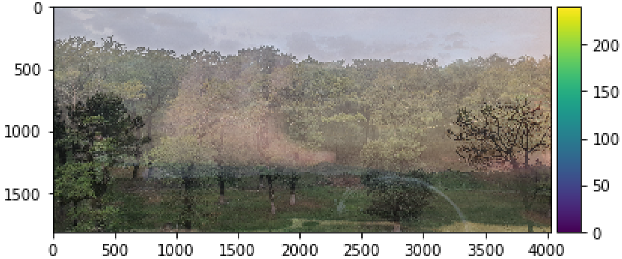

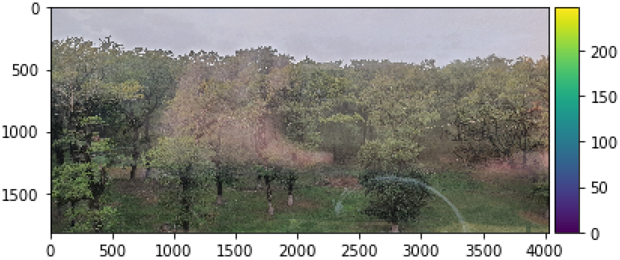
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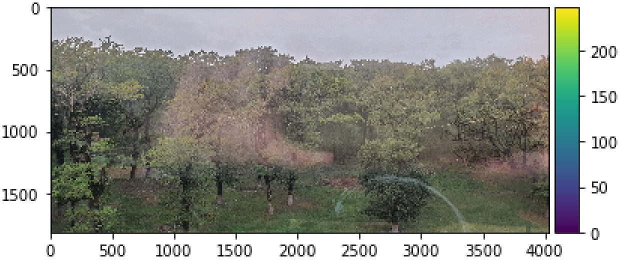
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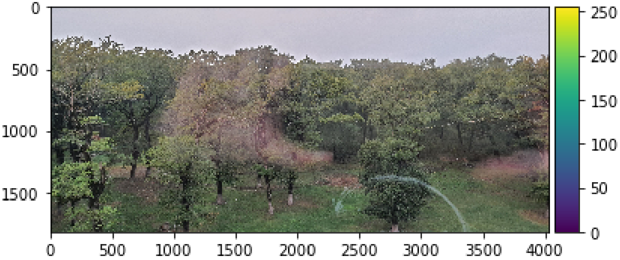
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Matrix series, 1

- sequence of matrices $\{A_k\}_1^\infty$
- how can you define convergence $lim_{k\to\infty} A_k = B$?

Matrix series, 2

- sequence of matrices $\{A_k\}_1^{\infty}$
- how can you define convergence $lim_{k\to\infty} A_k = B$?
- Is entry-wise convergence sufficient?
- Can you define convergence of matrix sequences using matrix norm?

Matrix series, 3

- Sequence of matrices $\{A_k\}_1^\infty$
- How can you define convergence $lim_{k\to\infty} A_k = B$?
- Is entry-wise convergence sufficient?
- Can you define convergence of matrix sequences using matrix norm?
- Would approach $\|A_n - B\| < \epsilon$ for all $n > n(\epsilon)$ work?
- Is entry-wise convergence equivalent to convergence by matrix norm?

Matrix series, 4

- Sequence of matrices $\{A_k\}_1^\infty$
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- Set $A_k = A^k$, A is square matrix. For which A this sequence converges to zero matrix?

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Theorem 3.7

if $\rho(A) < 1, A \in \mathbb{R}^{n\times n}$ then $lim_{k\to\infty} A^k$ converges to the matrix with all zero entries

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Theorem 3.8

if $\rho(A) < 1, A \in \mathbb{R}^{n\times n}$ then $lim_{k\to\infty}A^k$ converges to the matrix with all zero entries

Matrix sequences, 2

- Sequence of matrices $\{A_k\}_1^\infty$
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- Can you define convergence of matrix sequences using matrix norm?
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- Set $A_k = A^k$, A is square matrix. For which A this sequence converges to zero matrix?

Theorem 3.9

if $\rho(A) < 1, A \in \mathbb{R}^{n\times n}$ then $lim_{k\to\infty} A^k$ converges to the matrix with all zero entries

Theorem 3.10

For any square matrix A and $\delta > 0$ $\exists \|\cdot\| : \rho(A) < \|A\| < \rho(A) + \delta$

Matrix power series, 1

- Power series facts from calculus:
  - Infinite series

$$\sum_{k=0}^{\infty} a_k x^k, \quad \text{coefficients} \quad a_k \in \mathbb{R}$$

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$$S_l(x) = \sum_{k=0}^{l} a_k x^k$$

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  - Radius of convergence R:
$$\text{if } S_l(x) \text{ converges for} |x| < R$$

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Matrix power series, 1-c

- Cauchy sequences, facts from calculus:

Definition 3.11

- $x_1, x_2, x_3, ..., x_i \in \mathbb{R}$
- $\forall \epsilon > 0 \exists n(\epsilon)$ such that

$$|x_n - x_m| < \epsilon$$

  if $n, m > n(\epsilon)$

Matrix power series, 1-c

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- $\forall \epsilon > 0 \exists n(\epsilon)$ such that

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if $n, m > n(\epsilon)$

Theorem 3.12

- *Every convergent sequence is Cauchy sequence*
- *Cauchy sequence is bounded*
- *Cauchy sequence converges to some $x \in \mathbb{R}$*

Matrix power series, 2

Theorem 3.13

- *Power series*

$$\sum_{k=0}^{\infty} a_k x^k,$$

  - *converges absolutely*
  - *radius of convergence is R.*

Matrix power series, 2

Theorem 3.13

- *Power series*

$$\sum_{k=0}^{\infty} a_k x^k,$$

  - *converges absolutely*
  - *radius of convergence is R.*
- *Matrix power series*

$$\sum_{k=0}^{\infty} a_k A^k$$

  converges if
  - $\|A\| < R$ *or*
  - $\rho(A) < R$

Matrix power series, 3

Theorem 3.14

- *Suppose R is radius of convergence of absolutely convergent power series $\sum_{k=0}^{\infty} a_k x^k$,*
- *Matrix power series $\sum_{k=0}^{\infty} a_k A^k$, converges if $\|A\| < R$ or $\rho(A) < R$*

Matrix power series, 3

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- *Matrix power series $\sum_{k=0}^{\infty} a_k A^k$, converges if $\|A\| < R$ or $\rho(A) < R$*

Proof.

$$\|A\| < R :$$

$$\|S_{n+p} - S_n\| = \|\sum_{k=0}^{n+p} a_k A^k - \sum_{k=0}^{n} a_k A^k\| = \|\sum_{k=n+1}^{n+p} a_k A^k\| \leq$$

$$\leq \sum_{k=n+1}^{n+p} |a_k| \|A^k\| \leq \sum_{k=n+1}^{n+p} |a_k| \|A\|^k \leq \sum_{k=n+1}^{\infty} |a_k| \|A\|^k < \epsilon(n)$$

$$\exists \epsilon(n) : \lim_{n \to \infty} \epsilon(n) = 0$$

$\square$

Matrix power series, 4

Theorem 3.15

- *Suppose R is radius of convergence of absolutely convergent power series* $\sum_{k=0}^{\infty} a_k x^k$,
- *Matrix power series* $\sum_{k=0}^{\infty} a_k A^k$, *converges if* $\|A\| < R$ *or* $\rho(A) < R$

Matrix power series, 4

Theorem 3.15

- *Suppose R is radius of convergence of absolutely convergent power series $\sum_{k=0}^{\infty} a_k x^k$,*
- *Matrix power series $\sum_{k=0}^{\infty} a_k A^k$, converges if $\|A\| < R$ or $\rho(A) < R$*

Proof.

$$\rho(A) < R :$$

$$\exists \| \cdot \|, \delta > 0, 1 > \rho(A) + \delta > \|A\| :$$

$$\|S_{n+p} - S_n\| = \| \sum_{k=0}^{n+p} a_k A^k - \sum_{k=0}^{n} a_k A^k \| = \| \sum_{k=n+1}^{n+p} a_k A^k \| \leq$$

$$\leq \sum_{k=n+1}^{n+p} |a_k| \|A^k\| \leq \sum_{k=n+1}^{n+p} |a_k| \|A\|^k \leq \sum_{k=n+1}^{\infty} |a_k| \|A\|^k < \epsilon(n)$$

$$\exists \epsilon(n) : \lim_{n \to \infty} \epsilon(n) = 0$$

Matrix power series, 5

Example 3.16

- $$\frac{1}{(1-x)} = \sum_{k=0}^{\infty} x^k, \ R < 1$$

- $$(I-A)^{-1} = \sum_{k=0}^{\infty} A^k, \ \|A\| < R \text{ or } \rho(A) < 1$$

Matrix power series, 5

Example 3.16

- $$\frac{1}{(1-x)} = \sum_{k=0}^{\infty} x^k, \ R < 1$$
- $$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k, \ \|A\| < R \text{ or } \rho(A) < 1$$

Example 3.17

- $$\mathrm{e}^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$
- $$\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

Triangular systems of linear equations 1

Example 3.18

$$\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \text{ augmented matrix: } \left(\begin{array}{cc|c} 2 & 1 & 6 \\ 0 & 4 & 4 \end{array}\right)$$

Triangular systems of linear equations 1

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$$\begin{aligned} 2x_1 + x_2 &= 6 \\ x_2 &= 4 \end{aligned}$$

Triangular systems of linear equations 1

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$$\begin{aligned} 2x_1 + x_2 &= 6 \\ x_2 &= 4 \end{aligned}$$

Back substitution:

$$\begin{aligned} x_2 &= 4 \\ 2x_1 + x_2 = 6 \;\Rightarrow\; 2x_1 + 4 = 6 \;\Rightarrow\; 2x_1 = 2 \;\Rightarrow\; x_1 &= 1 \end{aligned}$$

Triangular systems of linear equations 2

Example 3.19

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{pmatrix}, \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a_{22} & \dots & a_{2n} & b_2 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & a_{nn} & b_n \end{array}\right)$$

Triangular systems of linear equations 2

Example 3.19

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ .. \\ b_n \end{pmatrix}, \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ 0 & a_{22} & \dots & a_{2n} & b_2 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & a_{nn} & b_n \end{array}\right)$$

Back substitution:

$$a_{nn}x_n = b_n \Rightarrow x_n = b_n / a_{nn}$$

$$a_{n-1n-1}x_{n-1} + a_{n-1n}x_n = b_{n-1} \Rightarrow x_{n-1} = (-a_{n-1n}x_n + b_{n-1})/a_{n-1n-1}$$

$$\dots$$

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \Rightarrow x_1 = (-a_{12}x_2 \dots - a_{1n}x_n + b_1)/a_{11}$$

Triangular systems of linear equations 3

Example 3.20

$$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} =$$

$$\begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, \left(\begin{array}{ccccc|c} 1 & -1 & \dots & .. & -1 & -1 \\ 0 & 1 & \dots & .. & -1 & -1 \\ .. & .. & \dots & .. & .. & .. \\ 0 & 0 & \dots & 1 & -1 & -1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{array}\right)$$

Triangular systems of linear equations 3

Example 3.20

$$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, \left(\begin{array}{ccccc|c} 1 & -1 & \dots & .. & -1 & -1 \\ 0 & 1 & \dots & .. & -1 & -1 \\ .. & .. & \dots & .. & .. & .. \\ 0 & 0 & \dots & 1 & -1 & -1 \\ 0 & 0 & \dots & 0 & 1 & 1 \end{array}\right)$$

Back substitution:

$$x_n = 1$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 0$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 0$$

Triangular systems of linear equations 4

Example 3.21

Back substitution

$$x_n = 1$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 0$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 0$$

$$x_{n-3} - x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-3} = x_{n-2} + x_{n-1} + x_n - 1 = 2x_{n-2} = 0$$

$$\dots$$

$$x_{n-k} - x_{n-k+1} - \sum_{i=n-k+2}^{n} x_i = -1 \Rightarrow$$

$$\Rightarrow x_{n-k} = x_{n-k+1} + \sum_{i=n-k+2}^{n} x_i - 1 = 2x_{n-k+1} = 0$$

$$x_{n-k-1} - x_{n-k} - \sum_{i=n-k+1}^{n} x_i = -1 \Rightarrow$$

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$x_n = 1, x_{n-1} = 0, x_{n-2} = 0, \dots, x_1 = 0$

Triangular systems of linear equations 5

Example 3.22

What if we have small error in data, for example $b_n = 1 + \epsilon$

$$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1+\epsilon \end{pmatrix},$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & \dots & .. & -1 & -1 \\ 0 & 1 & \dots & .. & -1 & -1 \\ .. & .. & \dots & .. & .. & .. \\ 0 & 0 & \dots & 1 & -1 & -1 \\ 0 & 0 & \dots & 0 & 1 & 1+\epsilon \end{array}\right)$$

Triangular systems of linear equations 6

Example 3.23

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$

$$x_n = 1 + \epsilon$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 1 + \epsilon - 1 = \epsilon$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 2\epsilon$$

$$x_{n-3} - x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-3} = x_{n-2} + x_{n-1} + x_n - 1 = 2x_{n-2} = 2^2\epsilon$$

$$\ldots$$

$$x_{n-k} = x_{n-k+1} + \sum_{i=n-k+2}^{n} x_i - 1 = 2x_{n-k+1} = 2^{k-1}\epsilon$$

$$x_{n-k-1} - x_{n-k} - \sum_{i=n-k+1}^{n} x_i = -1 \Rightarrow$$

$$\Rightarrow x_{n-k-1} = x_{n-k} + \sum_{i=n-k+1}^{n} x_i - 1 = 2x_{n-k} = 2^k\epsilon$$

$$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, \ldots, x_1 = 2^{n-1}\epsilon$$

Triangular systems of linear equations 6

Example 3.24

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$
$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, ..., x_1 = 2^{n-1}\epsilon$

Triangular systems of linear equations 6

Example 3.24

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$

$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, ..., x_1 = 2^{n-1}\epsilon$

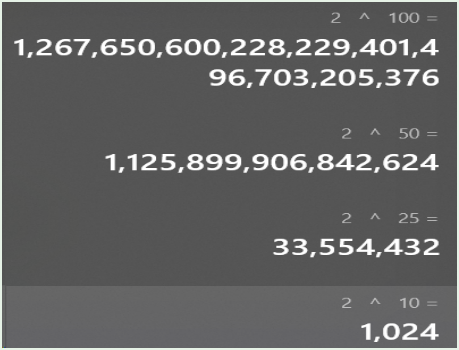

Figure: Error amplification factors 2^{n-1}

Triangular systems of linear equations 7

- $$\begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}$$

Triangular systems of linear equations 7

- $$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ .. \\ \tilde{x}_{n-1} \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1+\epsilon \end{pmatrix}, \tilde{x} = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ 1+\epsilon \end{pmatrix}$$

Triangular systems of linear equations 8, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

Triangular systems of linear equations 8, errors

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Definition 3.25

Let $x \in \mathbb{R}$ and let $\tilde{x} \in \mathbb{R}$ denote an approximation of it.

- Absolute error: $|x - \tilde{x}|$
- Relative error: $\frac{|x-\tilde{x}|}{|x|}$

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Definition 3.26

Let $x \in \mathbb{R}^n$ and let $\tilde{x} \in \mathbb{R}^n$ denote an approximation of it.

- Absolute error: $\|x - \tilde{x}\|$
- Relative error: $\frac{\|x-\tilde{x}\|}{\|x\|}$

Triangular systems of linear equations 9, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

$$x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}, \tilde{x} = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ 1+\epsilon \end{pmatrix}, \qquad \tilde{x} - x = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ \epsilon \end{pmatrix}, \|\tilde{x} - x\| = 2^{n-1}|\epsilon|$$

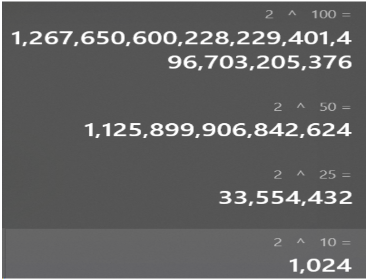

Figure: Error amplification factors 2^{n-1}

Triangular systems of linear equations 10, ill conditioned matrix

Example 3.27

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem
- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.
- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

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Theorem 3.28

The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Q & A