Hiring Problem and Goat Problem

more about probability

1 Combinatorics

Def: set of permutations

$$P_n = \{\pi \mid \pi : [1:n] \to [1:n] \quad \text{bijective}\}$$

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Recursion

$$\begin{aligned} \#P_1 &= 1 \\ \#P_n &= n \cdot \#P_{n-1} \end{aligned}$$

$$\#P_n = \prod_{i=1}^{n} i = n!$$

Def: n choose i

$$\binom{n}{i} = \#\{A \mid A \subseteq [1:n]\ ,\ \#A = i\}$$

number of subsets of $[1:n]$ with i elements

$$\binom{n}{i} = \frac{n \cdot (n-1) \ldots \cdot (n-i+1)}{i!} = \frac{n!}{(n-i)! \cdot i!}$$

can pick each subset in i! orders

2 Hiring Algorithm 1

Input: $(a(1),\ldots,a(n))$, where a is permutation.

$a(i)$: quality of applicant i

Hiring strategy: always hire best applicant seen so far

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max = 0; i=1
while i <=n
{if a(i)> max {hire i ;max = a(i)};
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3 Hiring Problem 1

Input a is random, all permutations equally likely.

probability space

$$\begin{aligned} W &= (S,p) \\ S &= P_n \\ p(a) &= 1/n! \quad \text{for all } a \end{aligned}$$

$X(a)$ = number of applicants hired for permutation a

$$1 \leq X(a) \leq n$$

Problem:

$$E(X(a)) = ?$$

4 Indicator Variables

$W = (S, p)$ probability space

$A \subseteq S$ event

Indicator variable of A

$$X_A(a) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}$$

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Proof.

$$\begin{aligned} E(X_A) &= \sum_{a \in S} X_A(a) \cdot p(a) \\ &= \sum_{a \in A} X_A(a) \cdot p(a) + \sum_{a \notin A} X_A(a) \cdot p(a) \\ &= \sum_{a \in A} 1 \cdot p(a) + \sum_{a \notin A} 0 \cdot p(a) \\ &= p(A) \end{aligned}$$

‚exciting…‘

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Event: candidate i is hired

$$H(i) = \{a \mid a(i) = \max\{a(1), \ldots a(i)\}\}$$

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Let

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$$\begin{aligned}
H(A,i) &= \{a \mid \{a(1),\dots,a(i)\} = A\ ,\ a(i) = \max\{a(1),\dots a(i)\}\}\\
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\#H(i) &= \sum_{A} \#H(A,i)\\
&= \binom{n}{i} \#H(A,i)\\
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$$\boxed{\begin{aligned} E(X) &= \sum_{i=1}^{n} \frac{1}{i} \\ &< \int_{1}^{n+1} \frac{1}{x} dx \\ &= ln(n+1) \end{aligned}}$$

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- choose $j\in[1:i]$; random, equal probabilities. Output of function *random*(i).
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b=a, max = 0; i=n
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- $(b(j_n),\ldots,b(j_1))\in P_n$
- all such permutations are equally likely *(requires proof)*.
- then analysis of algorithm 1 with random inputs works.

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Probability space once vetor $b \in [1:n]^i$ is reached

$$\begin{aligned} W_b &= (S_b, p_b) \\ S_b &= \bigcup_{j=1}^{i} \{b(j)\} \times S_{b-b(j)} \\ p_b(b(j), c) &= (1/i) \cdot p_{b-b(j)}(c) \end{aligned}$$

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End of recursion:

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Outcome $b' \in S_b$:

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all outcomes equally likely:

Lemma 2. *For all $i \in [1:n]$, for all $b \in [1:n]^i$ with pairwise distinct lements and for all $b' \in S_b$*

$$p_b(b') = \frac{1}{i!}$$

Proof. induction on i □

6 Why I am paranoid about probability: the goat problem

Game show:

- three doors
- gold behind 1 door, goats behind 2 other doors
- you guess door with gold with $p = 1/3$
- show master opens a door without gold (if you guessed right he chooses door with $q = 1/2$)
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$D = \{w, a, b\}$ doors; gold behind w (like 'win')
Construct probability space; two experiments

1. your guess $g = random(D)$
2. door opened by show master

$$s = \begin{cases} random(\{a,b\}) & g = w \\ a & g = b \\ b & g = a \end{cases}$$

Combined probability space (S, r)

$$S = \{(w,a), (w,b), (a,b), (b,a)\}$$

$$r(w,a) = r(w,b) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

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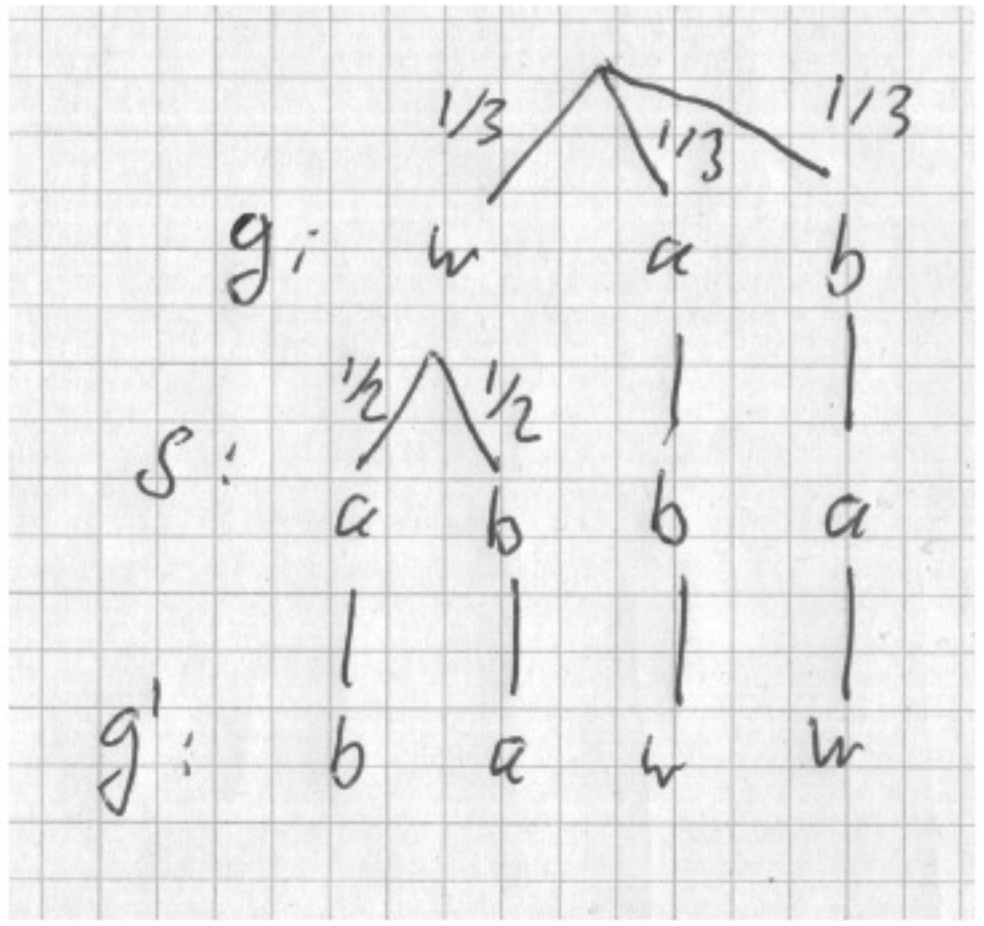

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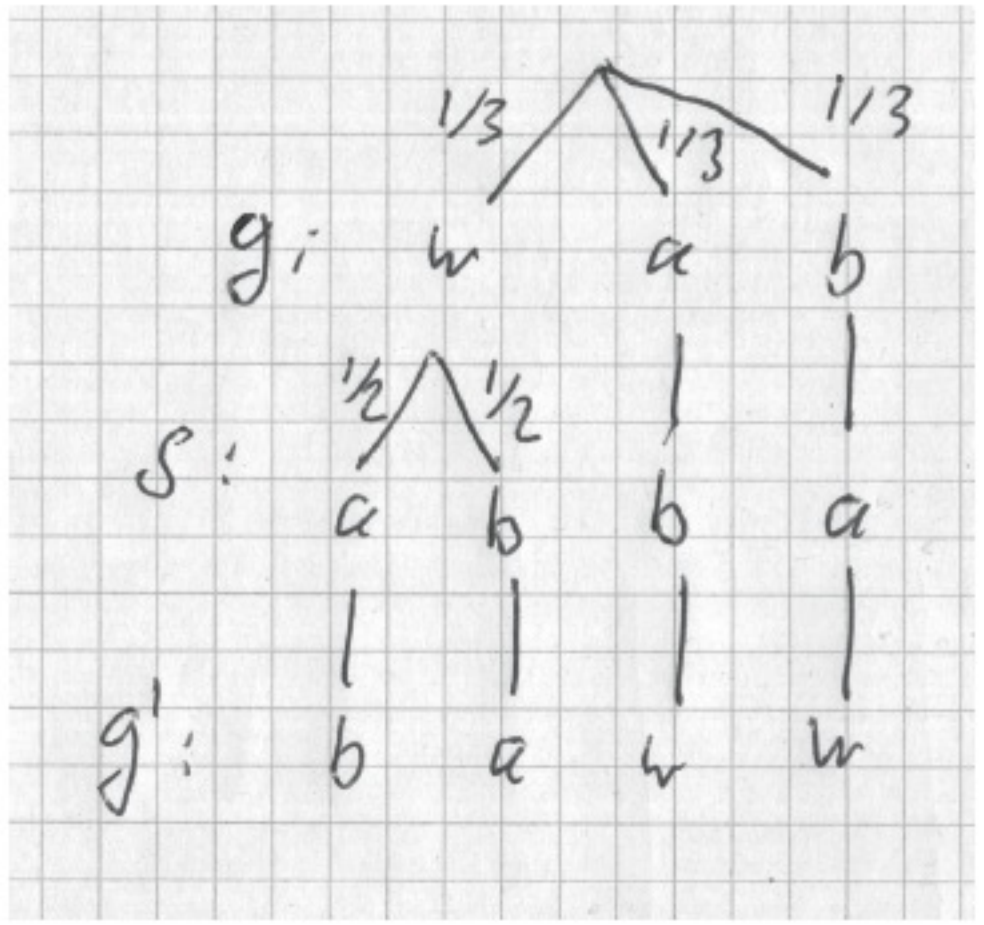

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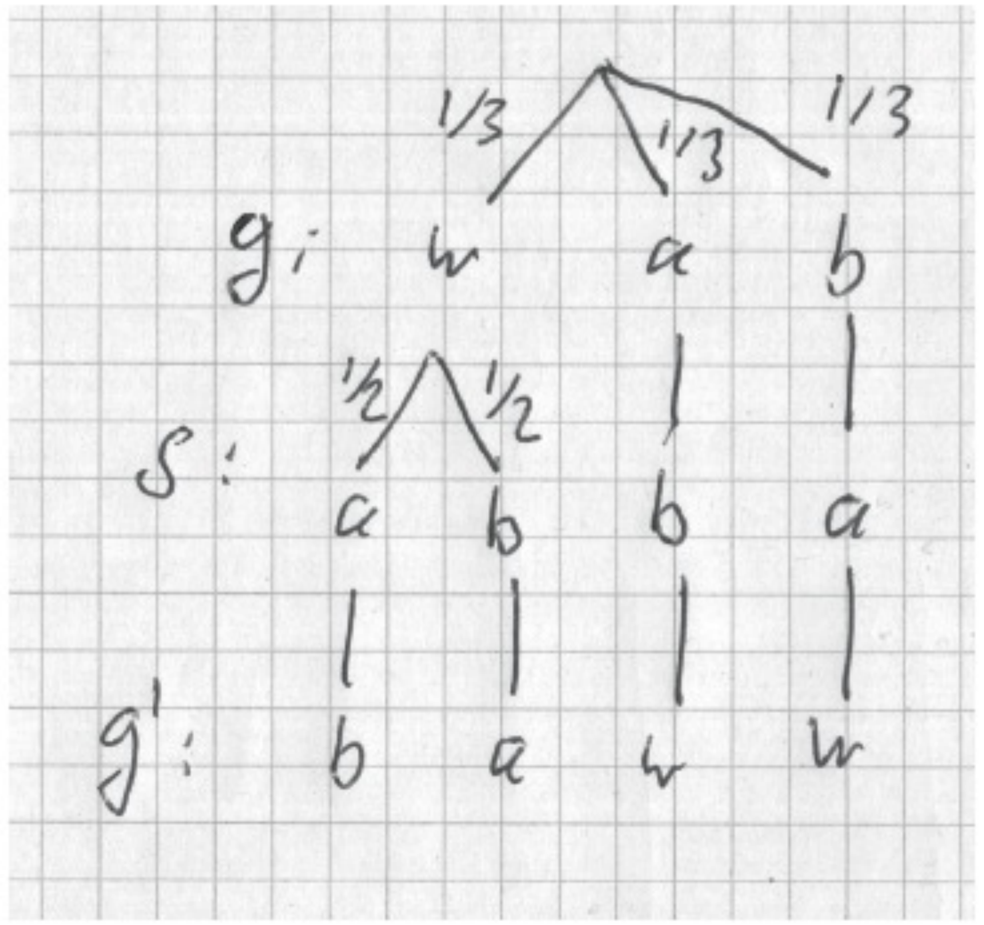

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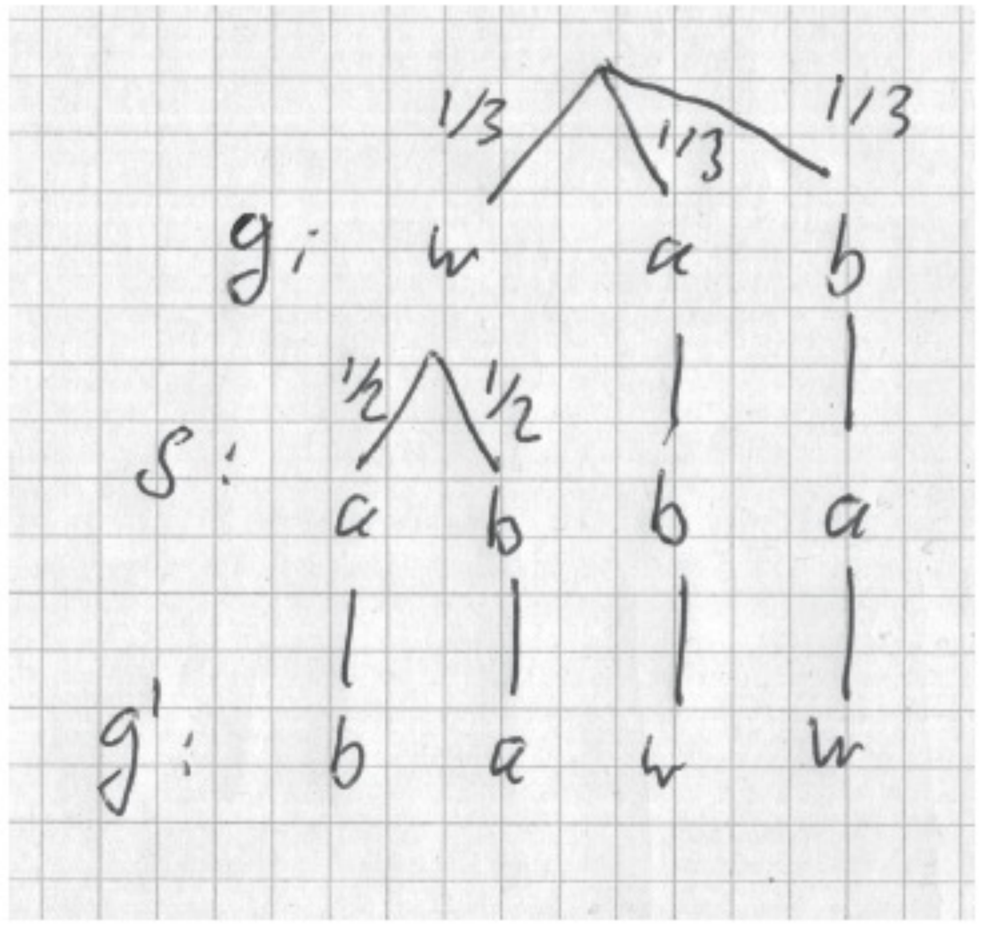

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You win if you first guessed wrong; you loose if you first guessed right. New winning event

$$\boxed{W' = \{(a,b),(b,a)\} \quad , \quad r(W') = 2/3}$$