Homework 1

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1. We know that the eigenvalues of a matrix are the solutions to the characteristic equation of the matrix. So a matrix would have no real eigenvalues if the characteristic polynomial has no solutions.
$$\begin{aligned} \det(A - \lambda I) = 0 &\implies \\ (2-\lambda)(1-\lambda) + a = 0 &\implies \\ \lambda^2 - 3\lambda + 2 + a = 0 & \end{aligned}$$
we now know that
$$\begin{aligned} (-3)^2 - 8 - 4a < 0 &\implies \\ 4a > 1 &\implies \\ a > \frac{1}{4} & \end{aligned}$$

2. (a)
$$\begin{cases} 3x - 4y = -7 \\ -6x + 8y = 14 \end{cases} \implies \begin{cases} 3x - 4y = -7 \\ 0x + 0y = 0 \end{cases} \implies \begin{cases} x = \frac{4t-7}{3} \\ y = t \end{cases}$$
   (b)
$$\begin{cases} -x + 2y - 4z = 8 \\ 3y + 8z = -4 \\ -7x + y + 2z = 1 \end{cases} \implies \begin{cases} -x + 2y - 4z = 8 \\ y + \frac{8}{3}z = -\frac{4}{3} \\ -13y - 26z = -55 \end{cases} \implies$$
$$\begin{cases} -x - \frac{28}{3}z = \frac{32}{3} \\ y + \frac{8}{3}z = -\frac{4}{3} \\ z = -\frac{217}{26} \end{cases} \implies \begin{cases} -x = \frac{32}{3} - \frac{28}{3}\frac{217}{26} \\ y = -\frac{4}{3} + \frac{8}{3}\frac{217}{26} \\ z = -\frac{217}{26} \end{cases} \implies \begin{cases} x = \frac{874}{13} \\ y = \frac{272}{13} \\ z = -\frac{217}{26} \end{cases}$$

3. Let α be the smallest angle and β be the largest.
$$\begin{cases} \alpha = \frac{1}{2}\beta + 10^\circ \\ 180^\circ - \alpha - \beta = \alpha + 12^\circ \end{cases} \implies \begin{cases} \alpha - \frac{1}{2}\beta = 10^\circ \\ 2\alpha + \beta = 168^\circ \end{cases} \implies \begin{cases} \alpha = 57^\circ \\ \beta = 94^\circ \end{cases}$$

4.
   - $$\left\| \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \right\|_1 = |-5| + |4| + |5| = 14$$
   - $$\left\| \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \right\|_2 = \left((-5)^2 + 4^2 + 5^2\right)^{\frac{1}{2}} = \sqrt{66}$$

- $$\left\| \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \right\|_3 = \left((-5)^3 + 4^3 + 5^3\right)^{\frac{1}{3}} = \sqrt[3]{64} = 4$$

- $$\left\| \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \right\|_\infty = \max\left\{|-5|, |4|, |5|\right\} = 5$$

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- $$\left\| \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \right\|_A = \sqrt{A \begin{pmatrix} -5 \\ 4 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} -5 & 4 & 5 \end{pmatrix}} = \sqrt{\begin{pmatrix} -6 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} -5 & 4 & 5 \end{pmatrix}} = \sqrt{107}$$

5. We need to show all three properties of vector norms

property 1

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{k} x_i \right| \geq 0 \impliedby \left| \sum_{i=1}^{k} x_i \right| \geq 0$$

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{k} x_i \right| = 0 \iff x_i = 0 \ \forall i$$

property 2

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{k} \alpha x_i \right| = \sum_{k=1}^{n} \left| \alpha \cdot \sum_{i=1}^{k} x_i \right| = \sum_{k=1}^{n} |\alpha| \cdot \left| \sum_{i=1}^{k} x_i \right| = |\alpha| \cdot \sum_{k=1}^{n} \left| \sum_{i=1}^{k} x_i \right|$$

property 3

$$\sum_{k=1}^{n} \left| \sum_{i=1}^{k} (x_i + y_i) \right| = \sum_{k=1}^{n} \left| \sum_{i=1}^{k} x_i + \sum_{i=1}^{k} y_i \right|$$
$$\leq$$
$$\sum_{k=1}^{n} \left(\left| \sum_{i=1}^{k} x_i \right| + \left| \sum_{i=1}^{k} y_i \right| \right) = \sum_{k=1}^{n} \left| \sum_{i=1}^{k} x_i \right| + \sum_{k=1}^{n} \left| \sum_{i=1}^{k} y_i \right|$$

6. The problem can be rewriten as:

$$5|x_1| + |x_2| = 1 \implies \begin{cases} x_1 = \pm \frac{1-|x_2|}{5} \\ x_2 = \pm(1 - 5|x_2|) \end{cases}$$

7. $\begin{pmatrix} 0 & 9 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 9 & 0 \end{pmatrix} = \begin{pmatrix} 81 & 0 \\ 0 & 1 \end{pmatrix}$ has two eigenvalue, $\lambda = 1, 81$ so the spectral radius would be $\rho(A) = 81$.

- $$\|A\|_1 = \max\{1, 9\} = 9$$

- $$\|A\|_2 = \sqrt{\rho(A^T A)} = 9$$

- $$\|A\|_\infty = 9$$

8. They *really* are norms, because they are induced by their respective vector norms.

9. (a) First, we find the inverse of the matrix.

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$$

now we can calculate the condition number.

$$K(A) = \|A\|_1 \|A^{-1}\|_1 = 3 \cdot 1 = 3$$

(b) First, we find the inverse of the matrix

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 8 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & -12 \end{pmatrix} = \begin{pmatrix} -1/3 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & -1/2 \end{pmatrix}$$

now we can calculate the condition number.

$$K(A) = \|A\|_\infty \|A^{-1}\|_\infty = 4 \cdot \frac{1}{2} = 2$$