performing random experiment 2 as a function of experiment 1

happens in probabilistic algorithms

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

probability of B given A

$$A, B \subseteq S, p(A) \neq 0$$

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$

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- throw coin
- if 0 throw coin *c*, otherwise dice *d*
- expected total number of points

it might be

$$E(c) + 1/2 \cdot E(c) + 1/2 \cdot E(d) = 1/2 + 1/4 + 7/4$$

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$$\begin{aligned} \sum_{(i,a) \in Q} q(i,a) &= \sum_{i \in S} \sum_{a \in R_i} p(i) \cdot p_i(a) \\ &= \sum_{i \in S} p(i) \cdot (\sum_{a \in R_i} p_i(a)) \\ &= \sum_{i \in S} p(i) \cdot 1 \\ &= 1 \end{aligned}$$

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Lemma 11. *For events $A \subseteq R_i$ the conditional probability that A occurs second given that i occurred first is*

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$$\begin{aligned}
(\{i\} \times R_i) \cap (\{i\} \times A) &= \{i\} \times A \\
q(\{i\} \times A \,|\{i\} \times R_i) &= \frac{q((\{i\} \times R_i) \cap (\{i\} \times A))}{q(\{i\} \times R_i)} \\
&= \frac{q(\{i\} \times A)}{q(\{i\} \times R_i)} \\
&= \frac{\Sigma_{a \in A} q(i,a)}{\Sigma_{r \in R_i} q(i,r)} \\
&= \frac{\Sigma_{a \in A} p(i) \cdot p_i(a)}{\Sigma_{r \in R_i} p(i) \cdot p_i(r)} \\
&= \frac{\Sigma_{a \in A} p_i(a)}{\Sigma_{r \in R_i} p_i(r)} \\
&= \frac{p_i(A)}{1} \\
&= p_i(A)
\end{aligned}$$

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random variables on these spaces

$$X_0 : S \to \mathbb{R} \quad , \quad X_i : R_i \to \mathbb{R}$$

$$X : Q \to \mathbb{R}$$

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$$E(X) = E(X_0) + \Sigma_{i \in S} p(i) \cdot E(X_i)$$

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Lemma 12.

$$E(X) = E(X_0) + \Sigma_{i \in S}\, p(i) \cdot E(X_i)$$

$$\begin{aligned}
E(X) &= \sum_{i\in S}\sum_{r\in R_i} q(i,r)\cdot X(i,r) \\
&= \sum_{i\in S}\sum_{r\in R_i} p(i)\cdot p_i(r)\cdot (X_0(i)+X_i(r)) \\
&= \sum_{i\in S} p(i)\cdot (\sum_{r\in R_i} p_i(r)\cdot (X_0(i)+X_i(r)) \\
&= \sum_{i\in S} p(i)\cdot X_0(i)\cdot (\sum_{r\in R_i} p_i(r)) + \sum_{i\in S} p(i)\cdot (\sum_{r\in R_i} p_i(r)\cdot X_i(r)) \\
&= \sum_{i\in S} p(i)\cdot X_0(i)\cdot 1 + \sum_{i\in S} p(i)\cdot E(X_i) \\
&= E(X_0) + \sum_{i\in S} p(i)\cdot E(X_i)
\end{aligned}$$

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