Numerical Analysis Homework (week 6)

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Problem 6.1:

Use Lagrange interpolation to find a polynomial that passes through the points:

(a) $\binom{0}{1}, \binom{2}{3}, \binom{3}{0}$

(b) $\binom{-1}{0}, \binom{2}{1}, \binom{3}{1}, \binom{5}{2}$

Solution

$$P(x) = \sum_{i=0}^{n} y_i \cdot l_{i(x)} \qquad l_{i(x)} = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

(a)

$$\begin{aligned} l_0(x) &= \frac{x-2}{0-2} * \frac{x-3}{0-3} = \frac{1}{6}(x-3)(x-2) \\ l_1(x) &= \frac{x-0}{2-0} * \frac{x-3}{2-3} = \frac{1}{2}x(3-x) \\ l_2(x) &= \frac{x-0}{3-0} * \frac{x-2}{3-2} = \frac{1}{3}x(x-2) \end{aligned}$$

so

$$\begin{aligned} P(x) &= 1 \cdot l_0(x) + 3 \cdot l_1(x) + 0 \cdot l_2(x) \\ &= \frac{1}{6}(x-2)(x-3) + \frac{3}{2}x(3-x) \\ &= (-4x-1)\left(\frac{1}{3}x - 1\right) \end{aligned}$$

(b)

$$\begin{aligned} l_0(x) &= \frac{x-2}{-1-2} \cdot \frac{x-3}{-1-3} \cdot \frac{x-5}{-1-5} = -\frac{1}{72}(x-5)(x-3)(x-2) \\ l_1(x) &= \frac{x+1}{2+1} \cdot \frac{x-3}{2-3} \cdot \frac{x-5}{2-5} = \frac{1}{9}(x+1)(x-5)(x-3) \\ l_2(x) &= \frac{x+1}{3+1} \cdot \frac{x-2}{3-2} \cdot \frac{x-5}{3-5} = -\frac{1}{8}(x+1)(x-5)(x-2) \\ l_3(x) &= \frac{x+1}{5+1} \cdot \frac{x-2}{5-2} \cdot \frac{x-3}{5-3} = \frac{1}{36}(x+1)(x-3)(x-2) \end{aligned}$$

so

$$\begin{aligned} P(x) &= 0 \cdot l_0(x) + 1 \cdot l_1(x) + 1 \cdot l_2(x) + 2 \cdot l_3(x) \\ &= \frac{1}{9}(x+1)(x-5)(x-3) - \frac{1}{8}(x+1)(x-5)(x-2) + \frac{1}{18}(x+1)(x-3)(x-2) \\ &= \frac{1}{24}x^3 - \frac{1}{4}x^2 + \frac{11}{24}x + \frac{3}{4} \end{aligned}$$

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Problem 6.2:

Use Newton's divided differences to find the interpolating polynomials of the points in Exercise 1.

Solution

(a)

$$\begin{aligned}
f[0] &= f(0) = 1\\
f[0,2] &= \frac{f(2)-f(0)}{2-0} = \frac{3-1}{2} = 1\\
f[0,2,3] &= \frac{f[2,3]-f[0,2]}{3-0}\\
&= \frac{\frac{f(3)-f(2)}{3-2}-1}{3-0}\\
&= \frac{\frac{0-3}{3-2}-1}{3-0} = -\frac{4}{3}
\end{aligned}$$

So the polynomial would look like

$$\begin{aligned}
P(x) &= f[0] + f[0,2](x-0) + f[0,2,3](x-0)(x-2)\\
&= 1 + (x-0) - \frac{4}{3}(x-0)(x-2)\\
&= -\frac{4}{3}x^2 + \frac{11}{3}x + 1
\end{aligned}$$

(b)

$$\begin{aligned}
f[-1] &= f(-1) = 0\\
f[-1,2] &= \frac{f(2)-f(-1)}{2+1} = \frac{1-0}{2+1} = \frac{1}{3}\\
f[-1,2,3] &= \frac{f[2,3]-f[-1,2]}{3+1}\\
&= \frac{\frac{f(3)-f(2)}{3-2}-\frac{1}{3}}{3+1}, = \frac{\frac{1-1}{3-2}-\frac{1}{3}}{3+1} = -\frac{1}{12}\\
f[-1,2,3,5] &= \frac{f[2,3,5]-f[-1,2,3]}{5+1}\\
&= \frac{\frac{f[3,5]-f[2,3]}{5-2}+\frac{1}{12}}{5+1}\\
&= \frac{\frac{\frac{f(5)-f(3)}{5-3}-\frac{f(3)-f(2)}{3-2}}{5-2}+\frac{1}{12}}{5+1}\\
&= \frac{\frac{\frac{2-1}{5-3}-\frac{1-1}{3-2}}{5-2}+\frac{1}{12}}{5+1} = \frac{\frac{\frac{1}{2}-0}{3}+\frac{1}{12}}{6} = \frac{\frac{1}{6}+\frac{1}{12}}{6} = \frac{1}{24}
\end{aligned}$$

Therefore the polynomial would be

$$\begin{aligned}
P(x) &= f[-1] + f[-1,2](x+1) + f[-1,2,3](x+1)(x-2) + f[-1,2,3,5](x+1)(x-2)(x-3)\\
&= \frac{1}{3}(x+1) - \frac{1}{12}(x+1)(x-2) + \frac{1}{24}(x+1)(x-2)(x-3)\\
&= \frac{1}{24}x^3 - \frac{1}{4}x^2 + \frac{11}{24}x + \frac{3}{4}
\end{aligned}$$

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Problem 6.3:

Find $P(0)$, where $P(x)$ is the degree 10 polynomial that is zero at $x = 1, \dots, 10$ and satisfies $P(12) = 44$.

Solution

We can take $P(x) = \prod_{i=1}^{10}(x - i)$ since it's zero at points $x = 1, \dots, 10$. However, it's $P(12) = (11!)$ at $x = 24$ so we just scale it by $\frac{44}{11!}$ to get $P(x) = \frac{44}{11!}\prod_{i=1}^{10}(x - i)$ which gives $P(0) = \frac{44}{11!}10! = \frac{44}{11} = 4$. ■

Problem 6.4:

Can a degree 3 polynomial intersect a degree 4 polynomial in exactly five points? Explain.

Solution

We can reformulate the qeustion as folows. Let $P(x)$ be a degree 3 polynomial and $Q(x)$ be a degree 4 polynomial. The intersection points of $P(x)$ and $Q(x)$ are the same as the roots of $P(x) - Q(x)$. For a polynomial to have n roots, it must be of order n. So we can ask if $P(x) - Q(x)$ is of order at least 5. Which it clearly isn't. ■

Problem 6.5:

Write down the degree 25 polynomial that passes through the points $\binom{1}{-1}, \binom{2}{-2}, \dots, \binom{25}{-25}$ and has constant term equal to 25.

Solution

If we take $P(x) = \prod_{i=1}^{25}(x - i)$ we know that $P(x) = 0$ for all $x \in [1 : 25]$. We also know that at $x = 0$, $P(x) = -25!$ so we can divide $P(x)$ by $-24!$ to make $P(0) = 25$. we can also subtract x so that $P(x) = -x \forall x \in [1 : 25]$. Finally we get

$$P(x) = \frac{\prod_{i=1}^{25}(x - i)}{-24!} - x.$$

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Problem 6.6:

Prove that the characteristic polynomials $l_i \in \mathbb{P}_n$ defined as $l_i(x) = \prod_{j=0,j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$ where $i = 0, \dots, n$ form a basis for $\mathbb{P}_n$.

Solution

It's clear that $l_i(x_k) = \begin{cases} 1 \text{ if } i=k \\ 0 \text{ otherwise} \end{cases}$ meaning that

$$a \cdot l_i(x_k) + b \cdot l_j(x_k) = \begin{cases} a & \text{if } k = i \\ b & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

and in general

$$\sum_{i=0}^{n} c_i \cdot l_i(x_k) = c_k$$

So we have a way of defining values of a degree n polynomial at $n+1$ points by specifying the corresponding c_i-s therefore $\{l_i : i = 0, 1, \ldots, n\}$ forms a basis of $\mathbb{P}_n$ ■

Problem 6.7:

Prove the recursive relation $f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}$, $n \geq 1$ for Newton's divided differences.

Solution

By the Newton's divided difference formula the interpolating polynomial for the given points $x_2, x_3, \ldots, x_{k-1}, x_1, x_k$ is

$$\begin{aligned} P_1(x) = {} & f[x_2] + f[x_2, x_3](x - x_2) + \cdots + f[x_2, x_3, \ldots, x_{k-1}, x_1](x - x_2) \cdots (x - x_{k-1}) \\ & + f[x_2, x_3, \ldots, x_{k-1}, x_1, x_k](x - x_2) \cdots (x - x_{k-1})(x - x_1) \end{aligned}$$

and the interpolating polynomial of points $x_2, x_3, \ldots, x_{k-1}, x_k, x_1$

$$\begin{aligned} P_2(x) = {} & f[x_2] + f[x_2, x_3](x - x_2) + \cdots + f[x_2, x_3, \ldots, x_{k-1}, x_k](x - x_2) \cdots (x - x_{k-1}) \\ & + f[x_2, x_3, \ldots, x_{k-1}, x_k, x_1](x - x_2) \cdots (x - x_{k-1})(x - x_k) \end{aligned}$$

By uniqueness, $P_1 = P_2$. Setting $P_1(x_k) = P_2(x_k)$ and canceling terms yields

$$\begin{aligned} & f[x_2, \ldots, x_{k-1}, x_1](x_k - x_2) \cdots (x_k - x_{k-1}) + f[x_2, \ldots, x_{k-1}, x_1, x_k](x_k - x_2) \cdots \\ & (x_k - x_{k-1})(x_k - x_1) = f[x_2, \ldots, x_{k-1}, x_k](x_k - x_2) \cdots (x_k - x_{k-1}) \end{aligned}$$

or

$$f[x_2, \ldots, x_{k-1}, x_1] + f[x_2, \ldots, x_{k-1}, x_1, x_k](x_k - x_1) = f[x_2, \ldots, x_k].$$

since $f[x_1, x_2, \ldots, x_k] = f[\sigma(x_1), \sigma(x_2), \ldots, \sigma(x_k)]$ for any permutation σ of x_i, the above equation can be rearanged to

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

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