A&DS Worksheet 1 solutions

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1. (1) Given the definition

$$f(n) = O(g(n)) \leftrightarrow \exists n_0 \in \mathbb{N}, c > 0, f(n) \leq c \cdot g(n)\ \forall n \geq n_0$$

we can clearly see that $c \geq 2$ and $n_0 = 1$ satisfies the inequality

$$\forall n \geq n_0, \sqrt{n} + n \leq c \cdot n.$$

(2) Given the definition

$$f(n) = o(g(n)) \leftrightarrow \forall c > 0\ \exists n_0 \in \mathbb{N},\ f(n) \leq c \cdot g(n)\ \forall n \geq n_0$$

we can write n_0 in terms of c

$$
\begin{aligned}
n^i &\leq c \cdot n^j & \Longleftrightarrow \\
1 &\leq c \cdot n^{j-i} & \Longleftrightarrow \\
c^{-1} &\leq n^{j-i} & \Longleftrightarrow \\
\sqrt[j-i]{c^{-1}} = c^{\frac{1}{i-j}} &\leq n & \Longrightarrow \quad \boxed{n_0 = \left\lceil c^{\frac{1}{i-j}} \right\rceil}
\end{aligned}
$$

Now we need to prove that $n^i \leq c \cdot n^j$ for all $n \geq \left\lceil c^{\frac{1}{i-j}} \right\rceil$ given that $i < j$. We can do this by induction on n.

Base case: $n = \left\lceil c^{\frac{1}{i-j}} \right\rceil$

$$
\begin{aligned}
\left\lceil c^{\frac{1}{i-j}} \right\rceil^i &\leq c \cdot \left\lceil c^{\frac{1}{i-j}} \right\rceil^j & \Longrightarrow \\
\left\lceil c^{\frac{1}{i-j}} \right\rceil^{i-j} &\leq c & \Longrightarrow \\
\left[\left(\frac{1}{c} \right)^{\frac{1}{j-i}} \right]^{j-i} &\geq \frac{1}{c} & \Longrightarrow \quad \left\lceil \sqrt[j-i]{\frac{1}{c}} \right\rceil \geq \sqrt[j-i]{\frac{1}{c}}
\end{aligned}
$$

holds true only if $i - j$ is < 0, which it is since $i < j$.

Induction step:

$$
\begin{aligned}
n^i &\leq c \cdot n^j & \Longrightarrow \\
n^{i-j} &\leq c & \Longrightarrow \quad \frac{1}{n^{j-i}} \leq c
\end{aligned}
$$

Since $j - i > 0$, it's obvious that $n^{j-i} < (n+1)^{j-i}$.

$$
\begin{aligned}
\frac{1}{(n+1)^{j-i}} \leq \frac{1}{n^{j-i}} &\leq c & \Longrightarrow \\
\frac{1}{(n+1)^{j-i}} &\leq c & \Longrightarrow \\
(n+1)^{i-j} &\leq c & \Longrightarrow \quad (n+1)^i \leq c \cdot (n+1)^j
\end{aligned}
$$

1

(3) The fact that if $h(n) = o(g(n))$ then $h(n) = O(g(n))$ also holds true, by definition, we can simply say that

$$f(n) + h(n) = O(\max\{O(g(n)), o(g(n))\}) \qquad \text{(where } f(n) = O(g(n)))$$
$$= O(g(n))$$

(4) In the equation $(\log x)^k = o(x)$ we can substitute x with e^x to get

$$(x \log e)^k = o(e^x) \quad \implies$$
$$x^k \cdot \underbrace{(\log e)^k}_{\text{constant}} = o(e^x)$$

and by definition $f(n) = o(g(n))$, we can write

$$\forall c > 0 \; \exists n_0 \in \mathbb{N}, (\log x)^k \leq c \cdot x, \; \forall n \geq n_0$$

notice that we can replace c with $a \cdot t$ in the main condition where $t > 0$ is some constant ($t = (\log e)^k$ in this case).

$$\forall c > 0 \; \exists n_0 \in \mathbb{N}, (\log x)^k \leq a \cdot t \cdot x, \; \forall n \geq n_0$$

and the statement still holds true.

(5) By the lemma of addition

$$O(n^2) + O(n \cdot (\log n)^k) = O(n^2) \quad \implies$$
$$n \cdot (\log n)^k = O(n^2) \quad \implies$$
$$(\log n)^k = O(n)$$

which we proved in the previous ecxercise.[1]

2. The growth rate (I.E. the derivative) of e^x is e^x itself, meaning that the growth is exponential. However, the derivative of any polynomial x^k is one degree lower $k \cdot x^{k-1}$ than the polynomial itself. Exponential growth is strictly faster than polynomial growth.

This can be shown by L'Hôpital's rule.

3. For $n = 2^k, k \in \mathbb{N}$

(1) First, the conjecture for a closed form solution. Let $k = 3$

$$f(2^k) = f(2^{k-1}) + b \cdot \log(2^k) \qquad \Longrightarrow$$
$$f(2^k) = f(2^{k-2}) + b \cdot \log(2^{k-1}) + b \cdot \log(2^k) \qquad \Longrightarrow$$
$$f(2^k) = f(2^{k-3}) + b \cdot \log(2^{k-2}) + b \cdot \log(2^{k-1}) + b \cdot \log(2^k) \qquad \Longrightarrow$$
$$f(2^k) = f(1) + b \cdot ((\log(2^k) - (k-1)) + \cdots + (\log(2^k) - 2) + (\log(2^k) - 1) + \log(2^k)) \qquad \Longrightarrow$$
$$f(2^k) = a + b \cdot ((k - (k-1)) + \cdots + (k-2) + (k-1) + k) \qquad \Longrightarrow$$
$$f(2^k) = a + b \cdot (1 + 2 + \cdots + (k-1) + k) \qquad \Longrightarrow$$
$$f(2^k) = a + b \cdot \frac{k(k+1)}{2} \qquad \Longrightarrow$$
$$f(n) = a + b \cdot \frac{\log n(\log n + 1)}{2}$$

The proof (by induction on k)

Base case: $k = 1$

$$f(2^1) = a + b \cdot \frac{1(1+1)}{2}$$
$$= a + b$$
$$= f(2^1/2) + b \cdot \log(2^1)$$

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[1] This is not a good proof.

Induction step:

$$f(2^k) = a + b \cdot \frac{k(k+1)}{2}$$

$$a + b \cdot \frac{k(k+1)}{2} + b \cdot \log(2^{k+1}) = a + b \cdot \frac{k(k+1)}{2} + b \cdot (k+1)$$

$$= a + b \cdot \frac{k(k+1) + 2(k+1)}{2}$$

$$= a + b \cdot \frac{(k+1)(k+2)}{2}$$

$$= a + b \cdot \frac{(k+1)((k+1)+1)}{2}$$

$$= f(2^{k+1})$$

Now all that's left to show is that

$$f(n) = a + b \cdot \frac{\log n(\log n + 1)}{2} = O((\log n)^2).$$

and we can do that by the lemma given in the slides

$$a + b \cdot \frac{\log n(\log n + 1)}{2} = O(1) + O(1) \cdot \frac{O(\log n) \cdot (O(\log n) + O(1))}{O(1)}$$

$$= O(1) + O(\log n) \cdot O(\log n)$$

$$= O(\log n \cdot \log n)$$

$$= O((\log n)^2)$$

(2) First, the conjecture for a closed form solution. Let $k = 3$

$$f(2^k) = 2 \cdot f(2^{k-1}) + b \cdot 2^k \cdot \log(2^k)$$

$$= 2 \cdot \left(2 \cdot f(2^{k-2}) + b \cdot 2^{k-1} \cdot \log(2^{k-1})\right) + b \cdot 2^k \cdot \log(2^k)$$

$$= 2 \cdot \left(2 \cdot \left(2 \cdot f(2^{k-3}) + b \cdot 2^{k-2} \cdot \log(2^{k-2})\right) + b \cdot 2^{k-1} \cdot \log(2^{k-1})\right) + b \cdot 2^k \cdot \log(2^k)$$

$$= 2^3 \cdot a + 2^2 \cdot b \cdot 2^{k-2} \cdot \log(2^{k-2}) + 2^1 \cdot b \cdot 2^{k-1} \cdot \log(2^{k-1}) + 2^0 \cdot b \cdot 2^k \cdot \log(2^k)$$

$$= 2^k \cdot a + 2^k \cdot b \left(\log(2^{k-2}) + \log(2^{k-1}) + \log(2^k)\right)$$

$$= 2^k \cdot a + 2^k \cdot b \left(\log(2^k) - 2 + \log(2^k) - 1 + \log(2^k)\right)$$

$$= 2^k \cdot a + 2^k \cdot b \left(k \cdot \log(2^k) - 2 - 1\right)$$

$$= 2^k \cdot a + 2^k \cdot b \left(k^2 - \sum_{i=1}^{k-1} i\right)$$

$$= 2^k \cdot \left(a + b \left(k^2 - \frac{k(k-1)}{2}\right)\right)$$

$$= 2^k \cdot \left(a + b \left(\frac{2k^2 - k^2 + k)}{2}\right)\right)$$

$$= 2^k \cdot \left(a + b \left(\frac{k(k+1)}{2}\right)\right)$$

$$f(n) = n \cdot \left(a + \frac{b}{2} \cdot (\log n \cdot (\log n + 1))\right)$$

The proof by induction on k:

Base case: k = 1

$$f(2^1) = 2^1 \cdot \left(a + \frac{b}{2} \cdot 1(1+1)\right)$$

$$= 2^1 \cdot (a + b)$$

$$= 2 \cdot f(2^0) + b \cdot 2^1 \cdot 1$$

3

Induction step:

$$2 \cdot \underbrace{\left(2^k \cdot \left(a + b\left(\frac{k(k+1)}{2}\right)\right)\right)}_{\text{value for } k} + b \cdot 2^{k+1} \cdot \log(2^{k+1}) = 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2}\right)\right) + b \cdot 2^{k+1} \cdot (k+1)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2}\right) + b(k+1)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1)}{2} + (k+1)\right)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{k(k+1) + 2(k+1)}{2}\right)\right)$$

$$= 2^{k+1} \cdot \left(a + b\left(\frac{(k+1)(k+2)}{2}\right)\right)$$

$$= f(2^{k+1})$$

now we with the lemma we show that

$$n \cdot \left(a + \frac{b}{2} \cdot (\log n \cdot (\log n + 1))\right) = O(n) \cdot (O(1) + O(1) \cdot (O(\log n) \cdot (O(\log n) + O(1))))$$

$$= O(n) \cdot (O(1) + O(1) \cdot (O(\log n) \cdot (O(\log n))))$$

$$= O(n) \cdot \left(O(1) + O(1) \cdot O((\log n)^2)\right)$$

$$= O(n) \cdot \left(O(1) + O((\log n)^2)\right)$$

$$= O(n) \cdot O((\log n)^2)$$

$$= O(n \cdot (\log n)^2)$$

(3) The conjecture. Let $k = 3$

$$f(2^k) = 7 \cdot f(2^{k-1}) + b \cdot 2^{2k}$$

$$= 7 \cdot \left(7 \cdot f(2^{k-2}) + b \cdot 2^{2k-2}\right) + b \cdot 2^{2k}$$

$$= 7 \cdot \left(7 \cdot \left(7 \cdot f(2^{k-3}) + b \cdot 2^{2(k-2)}\right) + b \cdot 2^{2(k-1)}\right) + b \cdot 2^{2k}$$

$$= 7^k \cdot f(2^{k-3}) + 7^{k-1} \cdot b \cdot 2^{2(k-2)} + 7^{k-2} \cdot b \cdot 2^{2(k-1)} + 7^{k-3} \cdot b \cdot 2^{2k}$$

$$= 7^k \cdot a + b \cdot 7^k 4^k \left(7^{-1} \cdot 2^{2(-2)} + 7^{-2} \cdot 2^{2(-1)} + 7^{-3} \cdot 2^{2(0)}\right)$$

$$= 7^k \cdot a + b \cdot 7^k \left(\sum_{i=1}^{k} 7^{-i} \cdot 2^{2i}\right)$$

$$= 7^k \cdot a + b \cdot 7^k \left(\sum_{i=1}^{k} \left(\frac{4}{7}\right)^i\right)$$

$$= 7^k \cdot a + b \cdot 7^k \cdot \left(\frac{4}{7} \cdot \frac{\left(\frac{4}{7}\right)^k - 1}{\frac{4}{7} - 1}\right)$$

$$= 7^k \cdot a + b \cdot 7^k \cdot \left(\frac{4}{-3} \cdot \left(\left(\frac{4}{7}\right)^k - 1\right)\right)$$

$$= 7^k \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right)\right)$$

$$f(n) = n^{\log(7)} \cdot a + 7^{\log(n)} \cdot \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^{\log(n)}\right)$$

We can prove this by induction on k.

4

Base case: k = 1

$$f(2^1) = 7^1 \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^1\right)\right)$$

$$= 7a + \frac{7 \cdot 4}{3}b \cdot \left(1 - \frac{4}{7}\right)$$

$$= 7a + \frac{7 \cdot 4}{3}b \cdot \frac{3}{7}$$

$$= 7a + 4b$$

$$= 7 \cdot f(2^1/2) + b \cdot (2^1)^2$$

Induction step:

$$f(2^{k+1}) = 7 \cdot \left(f(2^k)\right) + b \cdot \left(2^{k+1}\right)^2$$

$$= 7 \cdot \left(7^k \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right)\right)\right) + b \cdot \left(2^{k+1}\right)^2$$

$$= 7^{k+1} \cdot a + 7^{k+1} \cdot \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^k\right) + b \cdot \left(2^{k+1}\right)^2$$

$$= 7^{k+1} \cdot a + b \cdot \left(7^{k+1} \cdot \frac{4}{3}\left(1 - \left(\frac{4}{7}\right)^k\right) + 4^{k+1}\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k\right) + 4^k\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3 \cdot 4^k}{7^{k+1}}\right)\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3}{7} \cdot \frac{4^k}{7^k}\right)\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^k + \frac{3}{7} \cdot \left(\frac{4}{7}\right)^k\right)\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 + \left(\frac{4}{7}\right)^k \cdot \left(\frac{3}{7} - 1\right)\right)\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 + \left(\frac{4}{7}\right)^k \cdot \frac{-4}{7}\right)\right)$$

$$= 7^{k+1} \cdot a + b \cdot 4 \cdot \left(\frac{7^{k+1}}{3}\left(1 - \left(\frac{4}{7}\right)^{k+1}\right)\right)$$

$$= 7^{k+1} \cdot a + 7^{k+1} \cdot b \cdot \frac{4}{3}\left(1 - \left(\frac{4}{7}\right)^{k+1}\right)$$

$$= 7^{k+1} \cdot \left(a + \frac{4}{3}b \cdot \left(1 - \left(\frac{4}{7}\right)^{k+1}\right)\right)$$

now we with the lemma we show that

$$n^{\log 7} \cdot a + n^{\log 7} \cdot \frac{4}{3}b \cdot \left(1 - n^{\log \frac{4}{7}}\right) = O(n^{\log 7}) \cdot O(1) + O(n^{\log 7}) \cdot O(1) \cdot O(1) \cdot \left(O(1) - O(n^{\log \frac{4}{7}})\right)$$

$$= O(n^{\log 7}) + O(n^{\log 7}) \cdot O(n^{\log \frac{4}{7}}) \quad \text{(since } \log \frac{4}{7} \text{ is a negative power)}$$

$$= O(n^{\log 7}) + O(n^{\log 7})$$

$$= O(n^{\log 7})$$