Homework — Algorithms and Data Structures

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Worksheet 3

1. (a) (S, p) would be a probability space if $\sum_{s \in S} p(s) = 1$.

   Proof. by induction on n:

   Base case: $n = 1$

   $$p(s_1) = p_1(s_1), \sum_{s_1 \in S_1} p_1(s_1) = 1 \implies \sum_{s \in S} p(s) = 1$$

   Induction step:

   $$\begin{aligned} \sum_{s_n \in S_n} p(S_1, \ldots, S_{n-1}, s_n) &= \sum_{s_n \in S_n} p(S_1, \ldots, S_{n-1}) p_n(s_n) \\ &= p(S_1, \ldots, S_{n-1}) \cdot \sum_{s_n \in S_n} p_n(s_n) \\ &= p(S_1, \ldots, S_{n-1}) \cdot 1 \\ &= 1 \end{aligned}$$

   $\square$

   (b) Let $Y_i = \bigtimes_{j \neq i} S_j$. Then $e_i(A_i) = A_i \times Y_i$ since the order doesn't really matter.

   $$\begin{aligned} p(e_i(A_i)) &= \sum_{(s_1, s_2, \ldots, s_n) \in A_i \times Y_i} p_1(s_1) \cdot p_2(s_2) \cdot \cdots \cdot p_n(s_n) \\ &= \sum_{s_i \in A_i} \sum_{(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in Y_i} p_1(s_1) \cdot \cdots \cdot p_{i-1}(s_{i-1}) \cdot p_i(s_i) \cdot p_{i+1}(s_{i+1}) \cdot \cdots \cdot p_n(s_n) \\ &= \sum_{s_i \in A_i} p_i(s_i) \cdot \left(\sum_{(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n) \in Y_i} p_1(s_1) \cdot \cdots \cdot p_{i-1}(s_{i-1}) \cdot p_{i+1}(s_{i+1}) \cdot \cdots \cdot p_n(s_n) \right) \\ &= \sum_{s_i \in A_i} p_i(s_i) \cdot 1 \\ &= p_i(A_i) \end{aligned}$$

2. The expected value of a single item is $3 \cdot \frac{49}{50} - 80 \cdot \frac{1}{50} = \frac{147-80}{50} = \frac{67}{50}$ which is positive, meaning the company would make money in the long run.

3. Let the induction hypothesis be that

   $$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$

   Proof. by induction on n

Base case. $n = 1$

$$E\left(\sum_{i=1}^{1} X_i\right) = E(X_1) = \sum_{i=1}^{1} E(X_i)$$

Induction step. $n \to n+1$

$$\begin{aligned}\sum_{i=1}^{n} E(X_i) + E(X_{n+1}) &= \sum_{i=1}^{n+1} E(X_i)\\ &= E\left(\sum_{i=1}^{n+1} X_i\right)\end{aligned}$$

□

4. Let's build from ground up:

- $$Q_0 = \{\perp\}, \quad q_0(\perp) = 1$$
- $$Q_1 = \{\perp\}, \quad q_1(\perp) = 1$$
- $$Q_2 = \{(1, \perp, \perp), (2, \perp, \perp)\}, \quad q_2(i, a, b) = \frac{1}{2}$$
- $$Q_3 = \{(1, \perp, (1, \perp, \perp)), (1, \perp, (2, \perp, \perp)), (2, \perp, \perp), (3, (1, \perp, \perp), \perp), (3, (2, \perp, \perp), \perp)\}$$
  $$q_3(i, a, b) = \frac{1}{3} \cdot q_{i-1}(a) \cdot q_{3-i}(b) = \begin{cases} \frac{1}{6} & \text{if } (i, a, b) = (1, \perp, (1, \perp, \perp)) \\ \frac{1}{6} & \text{if } (i, a, b) = (1, \perp, (2, \perp, \perp)) \\ \frac{1}{3} & \text{if } (i, a, b) = (2, \perp, \perp) \\ \frac{1}{6} & \text{if } (i, a, b) = (3, (1, \perp, \perp), \perp) \\ \frac{1}{6} & \text{if } (i, a, b) = (3, (2, \perp, \perp), \perp) \end{cases}$$

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\begin{aligned}
Q_4 = \{&(1, \perp, (1, \perp, (1, \perp, \perp))),\\
&(1, \perp, (1, \perp, (2, \perp, \perp))),\\
&(1, \perp, (2, \perp, \perp)),\\
&(1, \perp, (3, (1, \perp, \perp), \perp)),\\
&(1, \perp, (3, (2, \perp, \perp), \perp)),\\
&(2, \perp, (1, \perp, \perp)),\\
&(2, \perp, (2, \perp, \perp)),\\
&(3, (1, \perp, \perp), \perp),\\
&(3, (2, \perp, \perp), \perp),\\
&(4, (1, \perp, (1, \perp, \perp)), \perp),\\
&(4, (1, \perp, (2, \perp, \perp)), \perp),\\
&(4, (2, \perp, \perp), \perp),\\
&(4, (3, (1, \perp, \perp), \perp), \perp),\\
&(4, (3, (2, \perp, \perp), \perp), \perp)\}
\end{aligned}
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q_4(i, a, b) = \frac{1}{4} \cdot q_{i-1}(a) \cdot q_{4-i}(b) = \begin{cases} \frac{1}{24} & \text{if } i = 1, b \notin Q_2 \\ \frac{1}{12} & \text{if } i = 1, b \in Q_2 \\ \frac{1}{8} & \text{if } i \in \{2, 3\} \\ \frac{1}{12} & \text{if } i = 4, a \in Q_2 \\ \frac{1}{24} & \text{if } i = 4, a \notin Q_2 \end{cases}
$$

it's clear that if you sum up the probabilities, you'll get 1.

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\begin{aligned}
t_3(1, \perp, (1, \perp, \perp)) &= 2 + 0 + 1 + 0 + 0 = 3\\
t_3(1, \perp, (2, \perp, \perp)) &= 2 + 0 + 1 + 0 + 0 = 3\\
t_3(2, \perp, \perp) &= 2 + 0 + 0 = 2\\
t_3(3, (1, \perp, \perp), \perp) &= 2 + 1 + 0 + 0 + 0 = 3\\
t_3(3, (2, \perp, \perp), \perp) &= 2 + 1 + 0 + 0 + 0 = 3
\end{aligned}
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\begin{aligned}
t_4(1, \perp, (1, \perp, (1, \perp, \perp))) &= 3 + 0 + 2 + 0 + 1 + 0 + 0 = 6\\
t_4(1, \perp, (1, \perp, (2, \perp, \perp))) &= 3 + 0 + 2 + 0 + 1 + 0 + 0 = 6\\
t_4(1, \perp, (2, \perp, \perp)) &= 3 + 0 + 2 + 0 + 0 = 5\\
t_4(1, \perp, (3, (1, \perp, \perp), \perp)) &= 3 + 0 + 2 + 1 + 0 + 0 + 0 = 6\\
t_4(1, \perp, (3, (2, \perp, \perp), \perp)) &= 3 + 0 + 2 + 1 + 0 + 0 + 0 = 6\\
t_4(2, \perp, (1, \perp, \perp)) &= 3 + 0 + 1 + 0 + 0 = 4\\
t_4(2, \perp, (2, \perp, \perp)) &= 3 + 0 + 1 + 0 + 0 = 4\\
t_4(3, (1, \perp, \perp), \perp) &= 3 + 1 + 0 + 0 + 0 = 4\\
t_4(3, (2, \perp, \perp), \perp) &= 3 + 1 + 0 + 0 + 0 = 4\\
t_4(4, (1, \perp, (1, \perp, \perp)), \perp) &= 3 + 2 + 0 + 1 + 0 + 0 + 0 = 6\\
t_4(4, (1, \perp, (2, \perp, \perp)), \perp) &= 3 + 2 + 0 + 1 + 0 + 0 + 0 = 6\\
t_4(4, (2, \perp, \perp), \perp) &= 3 + 2 + 0 + 0 + 0 = 5\\
t_4(4, (3, (1, \perp, \perp), \perp), \perp) &= 3 + 2 + 1 + 0 + 0 + 0 + 0 = 6\\
t_4(4, (3, (2, \perp, \perp), \perp), \perp) &= 3 + 2 + 1 + 0 + 0 + 0 + 0 = 6
\end{aligned}
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I actually wrote a script to generate most the mathematics that's written above :)