KUTAISI INTERNATIONAL UNIVERSITY

Numerical Linear Algebra

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LU factorization, pivoting and stability of Gaussian elimination

- Recap of Previous Lecture
- Examples of LU factorization
- Existence of LU factorization
- Gaussian elimination(GE) and LU factorization
- Pivoting
- Stability of GE
- Q & A

Recap of Previous Lecture

- Perturbations in right hand side and coefficients
- Round-of errors
- How to avoid cancellation and recursion errors
- Computational template for numerical linear algebra
- Thomas algorithm

LU Decomposition Examples 1

Example 8.1

General LU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & \dots & 0 \\ l_{21} & l_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

LU Decomposition Examples 1

Example 8.1 General LU decomposition

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Example 8.2 Doolittle LU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & u_{nn} \end{pmatrix}$$

LU Decomposition Examples 1

Example 8.1 General LU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ .. & .. & ... & .. \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & ... & 0 \\ l_{21} & l_{22} & ... & 0 \\ .. & .. & ... & .. \\ l_{n1} & l_{n2} & ... & l_{nn} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & ... & u_{1n} \\ 0 & u_{22} & ... & u_{2n} \\ .. & .. & ... & .. \\ 0 & 0 & ... & u_{nn} \end{pmatrix}$$

Example 8.2 Doolittle LU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ .. & .. & ... & .. \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & ... & 0 \\ l_{21} & 1 & ... & 0 \\ .. & .. & ... & .. \\ l_{n1} & l_{n2} & ... & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & ... & u_{1n} \\ 0 & u_{22} & ... & u_{2n} \\ .. & .. & ... & .. \\ 0 & 0 & ... & u_{nn} \end{pmatrix}$$

Example 8.3 Crout LU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ .. & .. & ... & .. \\ a_{n1} & a_{n2} & ... & a_{nn} \end{pmatrix} = \begin{pmatrix} l_{11} & 0 & ... & 0 \\ l_{21} & l_{22} & ... & 0 \\ .. & .. & ... & .. \\ l_{n1} & l_{n2} & ... & l_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & ... & u_{1n} \\ 0 & 1 & ... & u_{2n} \\ .. & .. & ... & .. \\ 0 & 0 & ... & 1 \end{pmatrix}$$

LU Decomposition Examples 2

Example 8.4 General LDU decomposition

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & d_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & \dots & u_{1n} \\ 0 & 1 & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

LU Decomposition Examples 2

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Example 8.5

- Doolittle decompoistion $A = LDU \Rightarrow A = (LD)U$

LU Decomposition Examples 2

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- Doolittle decompoistion $A = LDU \Rightarrow A = (LD)U$
- Crout decompoistion $A = LDU \Rightarrow A = L(DU)$

LU Decomposition Examples 2

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General LDU decomposition

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Example 8.5

- Doolittle decompoistion $A = LDU \Rightarrow A = (LD)U$
- Crout decompoistion $A = LDU \Rightarrow A = L(DU)$
- Exchanging rows or columns may be needed for implementation

LU Decomposition Examples 3

Example 8.6

- $B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, det(B) = 0, det(B_1) = 1 \neq 0$

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- $D = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}, det(D) = 0, det(D_1) = 0$

LU Decomposition Examples 3

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- $D = L_\beta U_\beta = \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 2 - \beta \end{pmatrix}$

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Example 8.8

- $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, det(C) = -1, det(C_1) = 0$: no LU factorization exists

LU Decomposition Examples 3

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- $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, det(C) = -1, det(C_1) = 0$: no LU factorization exists
- $\begin{pmatrix} 1 & 0 \\ l_{21} & 1 \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: no solution exists

Existence and uniqueness of LU factorization 1

Theorem 8.9

Unique LU factorization(Doolittle) of $A \in \mathbb{R}^{n \times n}$ *exists iff*

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Proof.

Proof for sufficient condition, p1

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- $i = 1, A_1 = a_{11}, det(A_1) \neq 0 \Rightarrow A_1 = L_1 U_1 = 1 \cdot a_{11}$

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- Assume $A_{i-1} = L_{i-1} U_{i-1}$
- Denote $A_i = \begin{pmatrix} A_{i-1} & c_i \\ d_i & a_{ii} \end{pmatrix}, c_i = \begin{pmatrix} a_{1i} \\ .. \\ a_{i-1i} \end{pmatrix}, d_i = \begin{pmatrix} a_{i1} & .. & a_{ii-1} \end{pmatrix}$

$$L_i = \begin{pmatrix} L_{i-1} & 0 \\ l_i & 1 \end{pmatrix}, l_i = \begin{pmatrix} l_{i1} & .. & l_{ii-1} \end{pmatrix}, U_i = \begin{pmatrix} U_{i-1} & u_i \\ 0 & u_{ii} \end{pmatrix}, u_i = \begin{pmatrix} u_{1i} \\ .. \\ u_{i-1i} \end{pmatrix}$$

Proof.

Proof for sufficient condition, p2

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- Assume $A_{i-1} = L_{i-1} U_{i-1}$

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- Formula to prove: $A_i = L_iU_i$

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- Formula to prove: $A_i = L_iU_i$
- $\begin{pmatrix} L_{i-1} & 0 \\ l_i & 1 \end{pmatrix} \begin{pmatrix} U_{i-1} & u_i \\ 0 & u_{ii} \end{pmatrix} = \begin{pmatrix} A_{i-1} & c_i \\ d_i & a_{ii} \end{pmatrix} \Rightarrow \begin{cases} L_{i-1}U_{i-1} = A_{i-1} \\ L_{i-1}u_i = c_i \\ l_iU_{i-1} = d_i \\ l_i \cdot u_i + u_{ii} = a_{ii} \end{cases}$

Proof.

Proof for sufficient condition, p2

- Assume $A_{i-1} = L_{i-1}U_{i-1}$
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- $det(L_{i-1}) \neq 0, det(U_{i-1}) \neq 0 \Rightarrow$ proof is complete

$\square$

Existence and uniqueness of LU factorization 3

Theorem 8.10

Unique LU factorization(Doolittle) of $A \in \mathbb{R}^{n \times n}$ exists iff

- $det(A_i) \neq 0, i = 1, 2, .., n-1$
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Proof for necessary condition, p3:

Existence and uniqueness of LU factorization 3

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Proof for necessary condition, p3:

If unique LU factorization exists then first $n-1$ main submatrices are not degenerated

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If unique LU factorization exists then first $n-1$ main submatrices are not degenerated

- proof by contradiction

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- Assume unique LU factorization exists and for some $k < n$ main submatrix $A_k, 1 \leq k < n$ is degenerated and $A_{k-1}, 2 \leq k < n$ is not degenerated

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If unique LU factorization exists then first $n-1$ main submatrices are not degenerated

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- Assume unique LU factorization exists and for some $k < n$ main submatrix $A_k, 1 \leq k < n$ is degenerated and $A_{k-1}, 2 \leq k < n$ is not degenerated
- $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & u_{nn} \end{pmatrix} \Rightarrow det(A) = \prod_{i=1}^{n} u_{ii}$

Existence and uniqueness of LU factorization 4

Proof.

Proof for necessary condition, p4:

Existence and uniqueness of LU factorization 4

Proof.

Proof for necessary condition, p4:

- $A = \begin{pmatrix} 1 & 0 & ... & 0 \\ l_{21} & 1 & ... & 0 \\ .. & .. & ... & .. \\ l_{n1} & l_{n2} & ... & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & ... & u_{1n} \\ 0 & u_{22} & ... & u_{2n} \\ .. & .. & ... & .. \\ 0 & 0 & ... & u_{nn} \end{pmatrix} \Rightarrow det(A) = \prod_{i=1}^{n} u_{ii}$

Existence and uniqueness of LU factorization 4

Proof.

Proof for necessary condition, p4:

- $A = \begin{pmatrix} 1 & 0 & ... & 0 \\ l_{21} & 1 & ... & 0 \\ .. & .. & ... & .. \\ l_{n1} & l_{n2} & ... & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & ... & u_{1n} \\ 0 & u_{22} & ... & u_{2n} \\ .. & .. & ... & .. \\ 0 & 0 & ... & u_{nn} \end{pmatrix} \Rightarrow det(A) = \prod_{i=1}^{n} u_{ii}$
- $det(A_{n-1}) = 0, det(A_{n-2}) \neq 0 \Rightarrow u_{n-1n-1} = 0$

Existence and uniqueness of LU factorization 4

Proof.

Proof for necessary condition, p4:

- $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & u_{nn} \end{pmatrix} \Rightarrow det(A) = \prod_{i=1}^{n} u_{ii}$
- $det(A_{n-1}) = 0, det(A_{n-2}) \neq 0 \Rightarrow u_{n-1n-1} = 0$
- $\begin{pmatrix} L_{n-1} & 0 \\ l_n & 1 \end{pmatrix} \begin{pmatrix} U_{n-1} & u_n \\ 0 & u_{nn} \end{pmatrix} = \begin{pmatrix} A_{n-1} & c_n \\ d_n & a_{nn} \end{pmatrix} \Rightarrow \begin{cases} L_{n-1} U_{n-1} = A_{n-1} \\ L_{n-1} u_n = A_{n-1} c_n \\ l_n U_{n-1} = d_n \\ l_n \cdot u_n + u_{nn} = a_{nn} \end{cases}$

Existence and uniqueness of LU factorization 4

Proof.

Proof for necessary condition, p4:

- $A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ l_{21} & 1 & \dots & 0 \\ .. & .. & \dots & .. \\ l_{n1} & l_{n2} & \dots & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & u_{nn} \end{pmatrix} \Rightarrow det(A) = \prod_{i=1}^{n} u_{ii}$
- $det(A_{n-1}) = 0, det(A_{n-2}) \neq 0 \Rightarrow u_{n-1n-1} = 0$
- $\begin{pmatrix} L_{n-1} & 0 \\ l_n & 1 \end{pmatrix} \begin{pmatrix} U_{n-1} & u_n \\ 0 & u_{nn} \end{pmatrix} = \begin{pmatrix} A_{n-1} & c_n \\ d_n & a_{nn} \end{pmatrix} \Rightarrow \begin{cases} L_{n-1}U_{n-1} = A_{n-1} \\ L_{n-1}u_n = A_{n-1}c_n \\ l_nU_{n-1} = d_n \\ l_n \cdot u_n + u_{nn} = a_{nn} \end{cases}$
- $det(U_{n-1}) = 0 \Rightarrow l_nU_{n-1} = d_n$ has no unique solution $\Rightarrow$contradiction, proof is complete

□

LU factorization from Gaussian elimination 1

Example 8.11

- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

LU factorization from Gaussian elimination 1

Example 8.11

- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ xy_1 + y_2 \\ y_3 \end{pmatrix}$$

LU factorization from Gaussian elimination 1

Example 8.11

- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ xy_1 + y_2 \\ y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ xy_1 + y_3 \end{pmatrix}$$

LU factorization from Gaussian elimination 1

Example 8.11

- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ xy_1 + y_2 \\ y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ xy_1 + y_3 \end{pmatrix}$$
- $$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ xy_1 + y_3 \\ y_4 \end{pmatrix}$$

LU factorization from Gaussian elimination 2

Example 8.12

- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_k & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_k \end{pmatrix} = \begin{pmatrix} y_1 \\ x_2 y_1 + y_2 \\ x_3 y_1 + y_3 \\ .. \\ x_k y_1 + y_k \end{pmatrix}$$

LU factorization from Gaussian elimination 2

Example 8.12

- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_k & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_k \end{pmatrix} = \begin{pmatrix} y_1 \\ x_2 y_1 + y_2 \\ x_3 y_1 + y_3 \\ .. \\ x_k y_1 + y_k \end{pmatrix}$$
- $$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ .. \\ y_k \end{pmatrix} = \begin{pmatrix} y_1 \\ 0 \\ 0 \\ .. \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_2 = -\frac{y_2}{y_1} \\ x_3 = -\frac{y_3}{y_1} \\ .. \\ x_k = -\frac{y_k}{y_1} \end{cases} \Rightarrow \begin{cases} \text{Gaussian elimination} \\ \text{multipliers} = x_i, i = 2, ..., n \end{cases}$$

LU factorization from Gaussian elimination 3

- $A \equiv A^{(0)}$

LU factorization from Gaussian elimination 3

- $A \equiv A^{(0)}$
- $A^{(1)} = M_1 A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & .. & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & a_{nn} \end{pmatrix}$

LU factorization from Gaussian elimination 3

- $A \equiv A^{(0)}$
- $A^{(1)} = M_1 A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & .. & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & a_{nn} \end{pmatrix}$
- $A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & .. & a_{3n}^{(1)} \\ .. & .. & .. & .. & .. \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & .. & a_{nn}^{(1)} \end{pmatrix}, A^{(2)} = M_2 A^{(1)}$

LU factorization from Gaussian elimination 3

- $A \equiv A^{(0)}$
- $$A^{(1)} = M_1 A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ -\frac{a_{21}}{a_{11}} & 1 & 0 & .. & 0 \\ -\frac{a_{31}}{a_{11}} & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ -\frac{a_{n1}}{a_{11}} & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ a_{21} & a_{22} & a_{23} & .. & a_{2n} \\ a_{31} & a_{32} & a_{33} & .. & a_{3n} \\ .. & .. & .. & .. & .. \\ a_{n1} & a_{n2} & a_{n3} & .. & a_{nn} \end{pmatrix}$$
- $$A^{(1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & a_{2n}^{(1)} \\ 0 & a_{32}^{(1)} & a_{33}^{(1)} & .. & a_{3n}^{(1)} \\ .. & .. & .. & .. & .. \\ 0 & a_{n2}^{(1)} & a_{n3}^{(1)} & .. & a_{nn}^{(1)} \end{pmatrix}, A^{(2)} = M_2 A^{(1)}$$
- $$M_2 A^{(1)} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & -\frac{a_{32}^{(1)}}{a_{22}^{(1)}} & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & -\frac{a_{2n}^{(1)}}{a_{22}^{(1)}} & 0 & & 1 \end{pmatrix} \cdot A^{(2)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(2)} & .. & a_{2n}^{(2)} \\ 0 & 0 & a_{33}^{(2)} & .. & a_{3n}^{(2)} \\ .. & .. & .. & .. & .. \\ 0 & 0 & a_{n3}^{(2)} & .. & a_{nn}^{(2)} \end{pmatrix}$$

LU factorization from Gaussian elimination 4

- $$A^{(k-1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(2)} & .. & .. & a_{2n}^{(2)} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & a_{nk}^{(k-1)} & .. & .. & a_{nn}^{(k-1)} \end{pmatrix}$$

LU factorization from Gaussian elimination 4

- $$A^{(k-1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(2)} & .. & .. & a_{2n}^{(2)} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & a_{nk}^{(k-1)} & .. & .. & a_{nn}^{(k-1)} \end{pmatrix}$$
- $$A^{(k)} = M_k \cdot A^{(k-1)}$$

LU factorization from Gaussian elimination 4

- $A^{(k-1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(2)} & .. & .. & a_{2n}^{(2)} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & a_{nk}^{(k-1)} & .. & .. & a_{nn}^{(k-1)} \end{pmatrix}$
- $$A^{(k)} = M_k \cdot A^{(k-1)}$$
- Zero entries under main diagonal in the first k columns of $A^{(k)}$

LU factorization from Gaussian elimination 4

- $$A^{(k-1)} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & .. & .. & a_{1n} \\ 0 & a_{22}^{(1)} & a_{23}^{(2)} & .. & .. & a_{2n}^{(2)} \\ .. & .. & .. & .. & .. & .. \\ .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. \\ 0 & 0 & a_{nk}^{(k-1)} & .. & .. & a_{nn}^{(k-1)} \end{pmatrix}$$
- $$A^{(k)} = M_k \cdot A^{(k-1)}$$
- Zero entries under main diagonal in the first k columns of $A^{(k)}$
- $$M_k = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & -\frac{a_{k+1k}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & -\frac{a_{nk}^{(k-1)}}{(k-1)} & .. & .. & 1 \end{pmatrix}, \quad \begin{matrix} multipliers: \\ \begin{cases} m_{k+1,k} = -\frac{a_{k+1k}^{(k-1)}}{a_{kk}^{(k-1)}}, \\ .. \\ m_{n,k} = -\frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}} \end{cases} \end{matrix}$$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, \ldots, A^{(k)} = M_k \cdot A^{(k-1)}$$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, ..., A^{(k)} = M_k \cdot A^{(k-1)}$$

- $$A^{(n-1)} = M_{n-1} \cdot A^{(n-2)} \equiv U$$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, ..., A^{(k)} = M_k \cdot A^{(k-1)}$$

- $$A^{(n-1)} = M_{n-1} \cdot A^{(n-2)} \equiv U$$

- $U = \begin{pmatrix} a_{11}^{(0)} & a_{11}^{(0)} & a_{13}^{(0)} & .. & .. & .. & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & .. & .. & a_{2n}^{(1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a_{n-1n-1}^{(n-2)} & a_{n-2n}^{(n-1)} \\ 0 & .. & .. & .. & .. & 0 & a_{nn}^{(n-1)} \end{pmatrix}$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, ..., A^{(k)} = M_k \cdot A^{(k-1)}$$

- $$A^{(n-1)} = M_{n-1} \cdot A^{(n-2)} \equiv U$$

- $U = \begin{pmatrix} a_{11}^{(0)} & a_{11}^{(0)} & a_{13}^{(0)} & .. & .. & .. & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & .. & .. & a_{2n}^{(1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a_{n-1n-1}^{(n-2)} & a_{n-2n}^{(n-1)} \\ 0 & .. & .. & .. & .. & 0 & a_{nn}^{(n-1)} \end{pmatrix}$
- $U = M_{n-1} \cdot A^{(n-2)} = ... = M_{n-1}M_{n-2}...M_1 \cdot A^{(0)}$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, ..., A^{(k)} = M_k \cdot A^{(k-1)}$$

- $$A^{(n-1)} = M_{n-1} \cdot A^{(n-2)} \equiv U$$

- $U = \begin{pmatrix} a_{11}^{(0)} & a_{11}^{(0)} & a_{13}^{(0)} & .. & .. & .. & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & .. & .. & a_{2n}^{(1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a_{n-1n-1}^{(n-2)} & a_{n-2n}^{(n-1)} \\ 0 & .. & .. & .. & .. & 0 & a_{nn}^{(n-1)} \end{pmatrix}$
- $U = M_{n-1} \cdot A^{(n-2)} = ... = M_{n-1} M_{n-2} ... M_1 \cdot A^{(0)}$
- $M_{n-1} M_{n-2} ... M_1 \cdot A = U \Rightarrow A = (M_{n-1} M_{n-2} ... M_1)^{-1} U$

LU factorization from Gaussian elimination 5

- $$A^{(0)} = A, A^{(1)} = M_1 \cdot A^{(0)}, ..., A^{(k)} = M_k \cdot A^{(k-1)}$$

- $$A^{(n-1)} = M_{n-1} \cdot A^{(n-2)} \equiv U$$

- $U = \begin{pmatrix} a_{11}^{(0)} & a_{11}^{(0)} & a_{13}^{(0)} & .. & .. & .. & a_{1n}^{(0)} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} & .. & .. & .. & a_{2n}^{(1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & a_{kk}^{(k-1)} & .. & .. & a_{kn}^{(k-1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & a_{n-1n-1}^{(n-2)} & a_{n-2n}^{(n-1)} \\ 0 & .. & .. & .. & .. & 0 & a_{nn}^{(n-1)} \end{pmatrix}$
- $U = M_{n-1} \cdot A^{(n-2)} = ... = M_{n-1} M_{n-2} ... M_1 \cdot A^{(0)}$
- $M_{n-1} M_{n-2} ... M_1 \cdot A = U \Rightarrow A = (M_{n-1} M_{n-2} ... M_1)^{-1} U$
- $(CB)^{-1} = B^{-1} C^{-1} \quad \Rightarrow \quad A = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1} U$

LU factorization from Gaussian elimination 6

- $$A = M_1^{-1} M_2^{-1} \dots M_{n-2}^{-1} M_{n-1}^{-1} U$$

LU factorization from Gaussian elimination 6

- $$A = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1} U$$

- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_k & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ -x_2 & 1 & 0 & .. & 0 \\ -x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ -x_k & 0 & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 6

- $$A = M_1^{-1} M_2^{-1} \ldots M_{n-2}^{-1} M_{n-1}^{-1} U$$
- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_k & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ -x_2 & 1 & 0 & .. & 0 \\ -x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ -x_k & 0 & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & & 1 \end{pmatrix}$$
- $$M_k = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & m_{k+1k}^{(k-1)} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & m_{nk}^{(k-1)} & .. & .. & 1 \end{pmatrix}, M_k^{-1} = \begin{pmatrix} 1 & 0 & 0 & .. & .. \\ 0 & 1 & 0 & .. & .. \\ .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. \\ .. & .. & .. & -m_{k+1k}^{(k-1)} & .. \\ .. & .. & .. & .. & .. \\ 0 & .. & .. & -m_{nk}^{(k-1)} & .. \end{pmatrix}$$

LU factorization from Gaussian elimination 7

$$M_k = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & -\frac{a_{k+1k}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & -\frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 7

$$M_k = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & -\frac{a_{k+1k}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & -\frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & 1 \end{pmatrix}$$

$$M_k^{-1} = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & \frac{a_{k+1k}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & \frac{a_{nk}^{(k-1)}}{a_{kk}^{(k-1)}} & .. & .. & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 8

- Product of lower triangular matrices is lower triangular matrix

LU factorization from Gaussian elimination 8

- Product of lower triangular matrices is lower triangular matrix
- Matrices $M_k^{-1}, k = 1, 2, ..., n-1$ are lower triangular matrices

LU factorization from Gaussian elimination 8

- Product of lower triangular matrices is lower triangular matrix
- Matrices $M_k^{-1}, k = 1, 2, ..., n-1$ are lower triangular matrices
- $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$ is lower triangular matrix

LU factorization from Gaussian elimination 8

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- Matrices $M_k^{-1}, k = 1, 2, ..., n-1$ are lower triangular matrices
- $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$ is lower triangular matrix
- $$A = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1} U$$

LU factorization from Gaussian elimination 8

- Product of lower triangular matrices is lower triangular matrix
- Matrices $M_k^{-1}, k = 1, 2, ..., n-1$ are lower triangular matrices
- $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$ is lower triangular matrix
- $$A = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1} U$$
- $$A = LU$$

LU factorization from Gaussian elimination 8

- Product of lower triangular matrices is lower triangular matrix
- Matrices $M_k^{-1}, k = 1, 2, ..., n-1$ are lower triangular matrices
- $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$ is lower triangular matrix
- $$A = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1} U$$
- $$A = LU$$
- Explicit expression for L?

LU factorization from Gaussian elimination 9

- Explicit expression for $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$?

LU factorization from Gaussian elimination 9

- Explicit expression for $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$?
- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & y_n & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & y_n & 0 & & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 9

- Explicit expression for $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$?
- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & y_n & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & y_n & 0 & & 1 \end{pmatrix}$$
- $$M_k^{-1} = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ 0 & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ .. & .. & .. & -m_{k+1k}^{(k-1)} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & -m_{nk}^{(k-1)} & .. & .. & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 9

- Explicit expression for $L = M_1^{-1} M_2^{-1} \ldots M_{n-2}^{-1} M_{n-1}^{-1}$
- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & y_n & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & y_n & 0 & & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 9

- Explicit expression for $L = M_1^{-1} M_2^{-1} ... M_{n-2}^{-1} M_{n-1}^{-1}$
- $$\begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & 0 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ 0 & 1 & 0 & .. & 0 \\ 0 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ 0 & y_n & 0 & & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & 0 \\ x_2 & 1 & 0 & .. & 0 \\ x_3 & y_3 & 1 & .. & 0 \\ .. & .. & .. & .. & .. \\ x_n & y_n & 0 & & 1 \end{pmatrix}$$
- $$L = \begin{pmatrix} 1 & 0 & 0 & .. & .. & .. & 0 \\ -m_{21}^{(1)} & 1 & 0 & .. & .. & .. & 0 \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & 1 & .. & .. & .. \\ -m_{k1}^{(1)} & .. & .. & -m_{k+1k}^{(k-1)} & .. & .. & .. \\ .. & .. & .. & .. & .. & .. & .. \\ -m_{n1}^{(1)} & .. & .. & -m_{nk}^{(k-1)} & .. & -m_{nn-1}^{(n-1)} & 1 \end{pmatrix}$$

LU factorization from Gaussian elimination 10

- Efficient storage scheme for LU factorization
  - store entries of U in upper triangular part of A including diagonal
  - store entries of L in lower triangular part of A excluding diagonal

LU factorization from Gaussian elimination 10

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- $$\begin{pmatrix} a_{11}^{(0)} & a_{12}^{(0)} & a_{13}^{(0)} & .. & .. & .. & a_{1n}^{(0)} \\ -m_{21} & a_{22}^{(1)} & a_{23}^{(1)} & .. & .. & .. & a_{2n}^{(1)} \\ .. & .. & .. & .. & .. & .. & .. \\ .. & .. & .. & a_{kk}^{(k-1)} & .. & .. & .. \\ -m_{k+11} & .. & .. & -m_{k+1k} & .. & .. & a_{k+1n}^{(k)} \\ .. & .. & .. & .. & .. & .. & .. \\ -m_{n1} & .. & .. & -m_{nk} & .. & .. & a_{nn}^{(n-1)} \end{pmatrix}$$

LU factorization from Gaussian elimination 11

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   - Elimination step = update entries of submatrix $(k+1:n, k+1:n)$
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Difficulties with Gaussian elimination

Example 8.13

Forsythe, Moler (1967)

- Impact of Finite Digit Arithmetic(case: 3 digit arithmetic)

Difficulties with Gaussian elimination

Example 8.13

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$$A = \begin{pmatrix} 10^{-4} & 1 \\ 1 & 1 \end{pmatrix}$$

Difficulties with Gaussian elimination

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Difficulties with Gaussian elimination

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- In three digit arithmetic with rounding
$fl(1 - 10^{-4}) = fl(-9999) = fl(-0.9999 \cdot 10^4) = -10^4$ $L = \begin{pmatrix} 1 & 0 \\ 10^4 & 1 \end{pmatrix}$,
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Pivoting and Gaussian elimination 1

Example 8.14

Forsythe, Moler (1967)

- Impact of pivoting on finite precision arithmetic (3 digit example)

Pivoting and Gaussian elimination 1

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Pivoting and Gaussian elimination 1

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- $\boxed{fl(fl(L')fl(U')) = \begin{pmatrix} 1 & 1 \\ 10^{-4} & 1 \end{pmatrix} = A'}$

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CONCLUSIONS?

Pivoting and Gaussian elimination 2

Definition 8.15

P is permutation matrix if in each row and column exactly one entry is 1 and other entries are 0.

Pivoting and Gaussian elimination 2

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Example 8.16

- Unit vector $e_1, e_2, ..e_n$

Pivoting and Gaussian elimination 2

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Example 8.16

- Unit vector $e_1, e_2, ..e_n$
- $$P_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} e_2^T \\ e_1^T \end{pmatrix}, P_{12} P_{12}^T = I$$

Pivoting and Gaussian elimination 2

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- General permutation matrix

$$P = \begin{pmatrix} e_{\alpha_1}^T \\ e_{\alpha_2}^T \\ . \\ . \\ e_{\alpha_n}^T \end{pmatrix}, 1 \leq i, \alpha_i \leq n, PP^T = I, PA = \begin{pmatrix} \alpha_1 - \text{th row of } A \\ \alpha_2 - \text{th row of } A \\ . \\ . \\ \alpha_n - \text{th row of } A \end{pmatrix}$$

Pivoting and Gaussian elimination 3

- Premultiplication: $P_{left} A$

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,

Pivoting and Gaussian elimination 3

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,

$$P_{left}A = \begin{pmatrix} \alpha_1 - \text{th row of matrix } A \\ \alpha_2 - \text{th row of matrix } A \\ . \\ . \\ \alpha_n - \text{th row of matrix } A \end{pmatrix}$$

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$$AP_{right} = \left(\alpha_1 - \text{th column}, \alpha_2 - \text{th column}, ..., \alpha_n - \text{th column of } A\right)$$

Pivoting and Gaussian elimination 4

- Gaussian Elimination with Partial Pivoting (GEPP)

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$$
\begin{aligned}
A^{(1)} &= M_1 P_1 A \\
A^{(2)} &= M_2 P_2 A^{(1)} = M_2 P_2 M_1 P_1 A \\
&\vdots \\
A^{(n-1)} &= M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A
\end{aligned}
$$

Pivoting and Gaussian elimination 4

- Gaussian Elimination with Partial Pivoting (GEPP)

$$
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A^{(n-1)} &= M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A
\end{aligned}
$$

- GEPP and LU decomposition

Pivoting and Gaussian elimination 4

- Gaussian Elimination with Partial Pivoting (GEPP)

$$A^{(1)} = M_1 P_1 A$$

$$A^{(2)} = M_2 P_2 A^{(1)} = M_2 P_2 M_1 P_1 A$$

$$..$$

$$A^{(n-1)} = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A$$

- GEPP and LU decomposition

$$\begin{aligned} A^{(n-1)} = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 A = \\ = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 (P_2^T ... P_{n-1}^T P_{n-1} ... P_2) P_1 A = \\ = M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 (P_2^T ... P_{n-1}^T) P_{n-1} ... P_2 P_1 A = \\ = M_{n-1} P_{n-1} \cdots M_2 (P_3^T ... P_{n-1}^T P_{n-1} ... P_3) P_2 M_1 (P_2^T ... P_{n-1}^T) PA = \\ = M_{n-1} P_{n-1} \cdots M_2 (P_3^T ... P_{n-1}^T (P_{n-1} ... P_3 P_2 M_1 P_2^T ... P_{n-1}^T) PA = \\ = M_{n-1} P_{n-1} \cdots M_2 (P_3^T ... P_{n-1}^T) M_1' PA = \\ = M_{n-1}' \cdots M_2' M_1' PA \end{aligned}$$

Pivoting and Gaussian elimination 5

- GEPP and LU decomposition

$$
\begin{aligned}
A^{(n-1)} &= M'_{n-1} \cdots M'_2 M'_1 PA \\
P &= P_{n-1} ... P_2 P_1 \\
M'_1 &= P_{n-1} ... P_3 P_2 M_1 P_2^T ... P_{n-1}^T \\
M'_2 &= P_{n-1} ... P_3 M_1 P_3^T ... P_{n-1}^T \\
M'_3 &= P_{n-1} ... P_4 M_1 P_4^T ... P_{n-1}^T \\
&.. \\
M'_{n-1} &= M_{n-1}
\end{aligned}
$$

Pivoting and Gaussian elimination 5

- GEPP and LU decomposition

$$A^{(n-1)} = M'_{n-1} \cdots M'_2 M'_1 PA$$
$$P = P_{n-1} ... P_2 P_1$$
$$M'_1 = P_{n-1} ... P_3 P_2 M_1 P_2^T ... P_{n-1}^T$$
$$M'_2 = P_{n-1} ... P_3 M_1 P_3^T ... P_{n-1}^T$$
$$M'_3 = P_{n-1} ... P_4 M_1 P_4^T ... P_{n-1}^T$$
$$..$$
$$M'_{n-1} = M_{n-1}$$

-

$$A^{(n-1)} = U$$
$$PA = LU$$
$$L = (M'_{n-1} \cdots M'_2 M'_1)^{-1}$$

Pivoting and Gaussian elimination 5

- Gaussian elimination with complete pivoting (GECP)

Pivoting and Gaussian elimination 5

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- $$A^{(k)} = M_k P_k A^{(k-1)} Q_k$$

Pivoting and Gaussian elimination 5

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- $$A^{(k)} = M_k P_k A^{(k-1)} Q_k$$
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Pivoting and Gaussian elimination 5

- Gaussian elimination with complete pivoting (GECP)
- $$A^{(k)} = M_k P_k A^{(k-1)} Q_k$$
- P_k - for row exchange
- Q_k - for column exchange
- M_k - for elimination step in column k

Pivoting and Gaussian elimination 5

- Gaussian elimination with complete pivoting (GECP)
- $$A^{(k)} = M_k P_k A^{(k-1)} Q_k$$
- P_k - for row exchange
- Q_k - for column exchange
- M_k - for elimination step in column k
- $$A^{(n-1)} = U$$
  $$PAQ = LU$$
  $$P = P_{n-1} ... P_2 P_1$$
  $$Q = Q_1 Q_2 ... Q_{n-1}$$

Stability of Gaussian elimination 1

Definition 8.17

- The **growth factor** ρ
- $$\rho = \frac{\max\{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}}{\alpha_0}$$
  $$\alpha_k = \max_{i,j} |a_{ij}^{(k)}|, k = 0, 1, .., n-1$$
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- The textbook, chapter 5, pp.108,109

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  - Scaling example: textbook, chapter 6, p.135

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Q & A