

proofs of some of the lemmas are taken

1. (a) $3 + 1 = 4$ (Definition 2)
- (b) $x + S(y) = S(x + y)$ (Definition 3)
- (c) $1 + 4 = 5$ (Theorem)

Proof.

$$\begin{aligned}
 1 + 4 &= 1 + (3 + 1) && \text{(Definition of 4)} \\
 &= 1 + ((2 + 1) + 1) && \text{(Definition of 3)} \\
 &= 1 + (((1 + 1) + 1) + 1) && \text{(Definition of 2)} \\
 &= (((1 + 1) + 1) + 1) + 1 && \text{(Lemma 4.)} \\
 &= ((2 + 1) + 1) + 1 && \text{(Definition 2)} \\
 &= (3 + 1) + 1 && \text{(Definition 2)} \\
 &= 4 + 1 && \text{(Definition 2)} \\
 &= 5 && \text{(Definition 2)}
 \end{aligned}$$

- (d) $x + 1 = 1 + x$ (Lemma 3)

Proof.

For $x = 0$ we have

$$0 + 1 = 1 = 1 + 0$$

Assume $x + 1 = 1 + x$, we can show

$$\begin{aligned}
 1 + S(x) &= S(1 + x) && \text{(Definition of addition)} \\
 &= S(x + 1) && \text{(Induction hypothesis)} \\
 &= S(x) + 1 && \text{(Definition of counting by adding 1)}
 \end{aligned}$$

- (e) $x + y = y + x$ (Lemma 4)

Proof.

For $y = 0$ we have

$$\begin{aligned}
 x + 0 &= x && \text{(Definition of addition)} \\
 &= 0 + x && \text{(Lemma 2)}
 \end{aligned}$$

Assume $x + y = y + x$, we can show

$$\begin{aligned}
 x + S(y) &= S(x + y) && \text{(Definition of addition)} \\
 &= S(y + x) && \text{(Induction hypothesis)} \\
 &= y + S(x) && \text{(Definition 3)} \\
 &= y + (x + 1) && \text{(Definition of counting by adding 1)} \\
 &= y + (1 + x) && \text{(Lemma 3)} \\
 &= (y + 1) + x && \text{(Associativity of addition)} \\
 &= S(y) + x && \text{(Definition of counting by adding 1)}
 \end{aligned}$$

(f) $x^{y+z} = x^y \cdot x^z$ (Lemma 7)

Proof.

For $z = 0$ we have

$$x^{y+0} = x^y = x^y \cdot 1 = x^y \cdot x^0$$

.

Assume $x^{y+z} = x^y \cdot x^z$, we can show

$$\begin{aligned} x^{y+S(z)} &= x^{S(y+z)} && \text{(Definition of addition)} \\ &= x^{(y+z)} \cdot x && \text{(Definition of exponentiation)} \\ &= (x^y \cdot x^z) \cdot x && \text{(Induction hypothesis)} \\ &= x^y \cdot (x^z \cdot x) && \text{(Associativity of multiplication)} \\ x^{y+S(z)} &= x^y \cdot x^{S(z)} && \text{(Definition of exponentiation).} \end{aligned}$$

(g) $(x + y) \cdot z = x \cdot z + y \cdot z$ (Lemma 5)

Proof.

For $z = 0$, we have

$$(x + y) \cdot 0 = 0 = x \cdot 0 + y \cdot 0 \quad \text{(Definition of multiplication)}$$

.

Assume that $(x + y) \cdot z = x \cdot z + y \cdot z$, we can show

$$\begin{aligned} (x + y) \cdot S(z) &= (x + y) \cdot z + (x + y) && \text{(Definition of multiplication)} \\ &= (x \cdot z + y \cdot z) + (x + y) && \text{(Induction hypothesis)} \\ &= x \cdot z + y \cdot z + x + y && \text{(Associativity of addition)} \\ &= x \cdot z + x + y \cdot z + y && \text{(Lemma 3)} \\ &= (x \cdot z + x) + (y \cdot z + y) && \text{(Associativity of addition)} \\ &= (x \cdot S(z)) + (y \cdot S(z)) && \text{(Definition of multiplication)} \\ &= x \cdot S(z) + y \cdot S(z) && \text{(Associativity of addition)} \end{aligned}$$

(h) $x \cdot y = y \cdot x$ (Lemma 5)

Proof.

by induction on y . For $y = 0$, we have

$$x \cdot 0 = 0 = 0 \cdot x \quad \text{(Definition 5)}$$

.

Assume that $x \cdot y = y \cdot x$, we can show

$$\begin{aligned} x \cdot S(y) &= x \cdot y + x && \text{(Definition of multiplication)} \\ &= y \cdot x + x && \text{(Induction hypothesis)} \\ &= S(y) \cdot x && \text{(Definition of multiplication)} \end{aligned}$$

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ (Lemma 5)

Proof.

For $z = 0$, we have

$$(x \cdot y) \cdot 0 = 0 = x \cdot (y \cdot 0) = x \cdot 0$$

.

Assume that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, we can show

$$\begin{aligned} (x \cdot y) \cdot S(z) &= (x \cdot y) \cdot z + (x \cdot y) && \text{(Definition of multiplication)} \\ &= x \cdot (y \cdot z) + (x \cdot y) && \text{(Induction hypothesis)} \\ &= x \cdot (y \cdot z + y) && \text{(Distributivity of multiplication)} \\ &= x \cdot (y \cdot S(z)) && \text{(Definition of multiplication)} \end{aligned}$$

(j) $x \cdot 0 = 0$ (Definition 5)

2. • $(x \cdot y)^z = x^z \cdot y^z$.

Proof.

For $z = 0$, we have

$$(x \cdot y)^0 = 1 = 1 \cdot 1 = x^0 \cdot y^0$$

.

Assume that $(x \cdot y)^z = x^z \cdot y^z$, we can show

$$\begin{aligned} (x \cdot y)^{S(z)} &= (x \cdot y)^z \cdot (x \cdot y) && \text{(Definition of exponentiation)} \\ &= (x^z \cdot y^z) \cdot (x \cdot y) && \text{(Induction step)} \\ &= (x^z \cdot x) \cdot (y^z \cdot y) && \text{(Lemma 5)} \\ &= (x^{S(z)}) \cdot (y^{S(z)}) && \text{(Definition of exponentiation)} \end{aligned}$$

- $(x^y)^z = x^{y \cdot z}$

Proof.

For $z = 0$ we have

$$(x^y)^0 = 1 = x^0 = x^{y \cdot 0}$$

.

Assume that $(x^y)^z = x^{y \cdot z}$, we can show

$$\begin{aligned} (x^y)^{S(z)} &= (x^y)^z \cdot (x^y) && \text{(Definition of exponentiation)} \\ &= x^{y \cdot z} \cdot (x^y) && \text{(Induction hypothesis)} \\ &= x^{z \cdot y} \cdot (x^y) && \text{(Lemma 5)} \\ &= x^{z \cdot y + y} && \text{(Lemma 7)} \\ &= x^{S(z) \cdot y} && \text{(Definition of multiplication)} \end{aligned}$$

3. • $efghijklmn[7:3] = ghijk$
 • $abcdef[6:2] = abcd$
 • $lmno \circ efgh = lmnoefgh$
 • $(abc \circ defghi)[5:8] = \emptyset$
4. • $A^2 = \{(a, a), (a, b), (a, c), \dots, (d, b), (d, c), (d, d)\}$
 • $B^2 = \{(a, a), (a, b), (a, c), \dots, (c, a), (c, b), (c, c)\}$

- $B^3 = \{(a, a, b), (a, a, b), \dots, (c, c, a), (c, c, b), (c, c, c)\}$
- 5.
- $A^\circ B = \{ad, ae, bd, be, cd, ce\}$
 - $\#(A^\circ B) = 3 \cdot 2 = 6$
 - $A^\circ A = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$
 - $B^\circ B = \{ee, ed, de, dd\}$
 - $\#(A^\circ A) = 9$
 - $\#(B^\circ B) = 4$
- 6.
- $A \times A = \{(a, a), (a, b), (a, c), \dots, (d, b), (d, c), (d, d)\}$
 - $\#(A \times A) = \#(A)^2 = 4^2 = 16$
- 7.
- $\langle 10100110 \rangle = 166$
 - $\langle 11010 \rangle = 26$
 - $\text{bin}_4(14) = 1100$
 - $\text{bin}_6(47) = 101111$
 - $\text{bin}_2(0) = 00000000000000000000$

8.

$$54 = 110110$$

$$17 = 10001$$

$$54 + 17 = 71 \quad \Longleftrightarrow \quad \begin{array}{r} 110110 \\ + 10001 \\ \hline 1000111 \end{array}$$

$$54 - 17 = 37 \quad \Longleftrightarrow \quad \begin{array}{r} 110110 \\ - 10001 \\ \hline 100101 \end{array}$$

$$54 \cdot 17 = 918 \quad \Longleftrightarrow \quad \begin{array}{r} 110110 \\ \times 10001 \\ \hline 110110 \\ + 110110 \\ \hline 1110010110 \end{array}$$

$$54 \div 17 = 3 \text{ (3)} \quad \Longleftrightarrow \quad \begin{array}{r|l} 110110 & 10001 \\ -10001 & 11 \\ \hline 10100 & \\ -10001 & \\ \hline 11 & \end{array}$$

9. a) $\{1, 2, 8, 3, 5, 9, 10\}$

b) $[1] = \{1, 4, 6\}$

$[2] = \{2, 8\}$

$[3] = \{3\}$

$[4] = \{1, 4, 6\}$