Guidelines for solutions of problems. Sections 2.3

Name and section: ______________________________

Instructor's name: ______________________________

1. Evaluate the limit and justify each step.

   (a) $\lim\limits_{t\to -2} \frac{t^4-2}{2t^2-3t+2}$;

   (b) $\lim\limits_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}}$.

   Solution.

   (a) $\lim\limits_{t\to -2} \frac{t^4-2}{2t^2-3t+2} = \frac{\lim\limits_{t\to -2}(t^4-2)}{\lim\limits_{t\to -2}(2t^2-3t+2)} = \frac{14}{16} = \frac{7}{8}$;

   (b) $\lim\limits_{x\to 2} \sqrt{\frac{2x^2+1}{3x-2}} = \sqrt{\lim\limits_{x\to 2} \frac{2x^2+1}{3x-2}} = \sqrt{\frac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(3x-2)}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$.

2. Evaluate the limit

   (a) $\lim\limits_{t\to 5} \frac{x^2-6x+5}{x-5}$;

   (b) $\lim\limits_{h\to 0} \frac{(-5+h)^2-25}{h}$.

   Solution.

   (a) $\lim\limits_{t\to 5} \frac{x^2-6x+5x}{x-5} = \lim\limits_{t\to 5} \frac{(x-1)(x-5)}{x-5} = \lim\limits_{t\to 5}(x-1) = 4.$;

   (b) $\lim\limits_{h\to 0} \frac{(-5+h)^2-25}{h} = \lim\limits_{h\to 0} \frac{(-5+h-5)(-5+h+5)}{h} = \lim\limits_{h\to 0} \frac{(h-10)h}{h} == \lim\limits_{h\to 0}(h-10) = -10$;

3. Find the limit, if it exists. If the limit does not exist, explain why.

   $\lim\limits_{x\to -6} \frac{2x+12}{|x+6|}$;

   Solution. Observe that if $x > -6$, then $|x-6| = x-6$; if $x < -6$, then $|x-6| = 6-x$. That is why we have he limit does not exist because one-sided limits are different:

   $$\lim_{x\to -6+} \frac{2x+12}{|x+6|} = \lim_{x\to -6+} \frac{2x+12}{x+6} = \lim_{x\to -6+} \frac{2(x+6)}{x+6} = \lim_{x\to -6+} 2 = 2;$$

   $$\lim_{x\to -6-} \frac{2x+12}{|x+6|} = \lim_{x\to -6-} \frac{2x+12}{x+6} = \lim_{x\to -6+} \frac{2(x+6)}{6-x} = \lim_{x\to -6+} (-2) = -2.$$

4. If $\lim\limits_{x\to 1} \frac{f(x)-8}{x-1} = 10$, find $\lim\limits_{x\to 1} f(x)$.

   Solution.

   It is clear that $\lim\limits_{x\to 1} f(x) = 8$ otherwise $\lim\limits_{x\to 1} \frac{f(x)-8}{x-1} = \pm\infty$.

EXAMPLE 5 Evaluate $\lim\limits_{h \to 0} \dfrac{(3+h)^2 - 9}{h}$.

SOLUTION If we define

$$F(h) = \frac{(3+h)^2 - 9}{h}$$

then, as in Example 3, we can't compute $\lim_{h \to 0} F(h)$ by letting $h = 0$ because $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h}$$

$$= \frac{h(6+h)}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \to 0} \frac{(3+h)^2 - 9}{h} = \lim_{h \to 0} (6 + h) = 6$$

EXAMPLE 6 Find $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2 + 9} - 3}{t^2}$.

SOLUTION We can't apply the Quotient Law immediately because the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} = \lim_{t \to 0} \frac{\sqrt{t^2+9} - 3}{t^2} \cdot \frac{\sqrt{t^2+9} + 3}{\sqrt{t^2+9} + 3}$$

$$= \lim_{t \to 0} \frac{(t^2 + 9) - 9}{t^2\left(\sqrt{t^2+9} + 3\right)}$$

$$= \lim_{t \to 0} \frac{t^2}{t^2\left(\sqrt{t^2+9} + 3\right)}$$

$$= \lim_{t \to 0} \frac{1}{\sqrt{t^2+9} + 3}$$

$$= \frac{1}{\sqrt{\lim_{t \to 0} (t^2 + 9)} + 3}$$

(Here we use several properties of limits: 5, 1, 7, 8, 10.)

$$= \frac{1}{3+3} = \frac{1}{6}$$

This calculation confirms the guess that we made in Example 2.2.1.