Numerical Linear Algebra

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Matrix, Norm, Condition Number

- Recap of Previous Lecture
- Matrix
- Digital Image and Matrix, Visualisation
- Matrix Operations
- Some Useful Properties
- Matrix Norm
- Q & A

Recap of Previous Lecture

- Course Overview
- Computational project 1
- Calculation errors
- Difficulties with theoretical linear algebra
- RGB colors and vectors
- Vector norms
- K-means clustering
- Q & A

Matrix, 1

Definition 2.1

Square matrix: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, A = (a_{ij})_{n \times n}, A \in \mathbb{R}^{n \times n}$

Matrix, 1

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Definition 2.2

Rectangular matrix: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, A = (a_{ij})_{m \times n}, A \in \mathbb{R}^{m \times n}$

Matrix, 2

Definition 2.3

Rectangular matrix: $A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, A = (a_{ij})_{m\times n}, A \in \mathbb{R}^{m\times n}$

Matrix, 2

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Definition 2.4

Rectangular block matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, a_{ij} \in \mathbb{R}^{k \times l}, A = (a_{ij})_{m \times n}, A \in \mathbb{R}^{m \times n}$$

Matrix, 3

Definition 2.5

Diagonal matrix: $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, A = diag(a_{ii})_{n\times n}, A \in \mathbb{R}^{n\times n}$

Definition 2.6

Diagonal part of matrix A: $diag(A)$

Matrix, 3

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Diagonal matrix: $A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, A = diag(a_{ii})_{n \times n}, A \in \mathbb{R}^{n \times n}$

Definition 2.6

Diagonal part of matrix A: $diag(A)$

Definition 2.7

Block diagonal matrix:

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, a_{ij} \in \mathbb{R}^{m \times m}, A = diag(a_{ii}), A \in \mathbb{R}^{mn \times mn}$$

Matrix(triangular), 4

Definition 2.8

Upper triangular matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, A = (a_{ij})_{n \times n}, , a_{ij} = 0 \text{ if } i > j, A \in \mathbb{R}^{n \times n}$$

Matrix(triangular), 4

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Upper triangular matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ .. & .. & \dots & .. \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}, A = (a_{ij})_{n\times n}, , a_{ij} = 0 \text{ if } i > j, A \in \mathbb{R}^{n\times n}$$

Definition 2.9

Lower triangular matrix:

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ .. & .. & \dots & .. \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, A = (a_{ij})_{m\times n}, a_{ij} = 0 \text{ if } i < j, A \in \mathbb{R}^{n\times n}$$

Matrix(Almost Triangular), 5

Definition 2.10

Upper Hessenberg matrix: $a_{ij} = 0$ if $i > j + 1, A = (a_{ij})_{n\times n}, A \in \mathbb{R}^{n\times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & \dots & a_{2n} \\ 0 & a_{32} & \dots & \dots & a_{3n} \\ .. & .. & \dots & \dots & .. \\ 0 & 0 & \dots & a_{nn-1} & a_{nn} \end{pmatrix}$$

Matrix(Almost Triangular), 5

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Definition 2.11

Lower Hessenberg matrix: $A = (a_{ij})_{m\times n}, a_{ij} = 0$ if $i < j - 1, A \in \mathbb{R}^{n\times n}$

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \dots & 0 \\ a_{21} & a_{22} & a_{23} & \dots & 0 \\ .. & .. & \dots & .. & a_{n-1n} \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{pmatrix}$$

Matrix, 6

Definition 2.12

Band matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & . & . & a_{1q+1} & 0 & . & . & 0 \\ a_{21} & a_{22} & a_{23} & & & & . & 0 & & . \\ a_{31} & a_{32} & a_{33} & & & & & . & 0 & . \\ . & & & . & & & & & . & 0 \\ . & & & & . & & & & & . \\ a_{p+1,1} & & & & & . & & & & \\ 0 & . & & & & & . & & & \\ 0 & 0 & . & & & & & . & & \\ 0 & & 0 & . & & & & & . & \\ 0 & . & . & 0 & . & & & & & a_{nn} \end{pmatrix}$$

Upper band width - q, lower band width -p, $p, q < n$, $A \in \mathbb{R}^{n \times n}$

Matrix, 7

Examples and Questions

- what is upper bandwidth of tridiagonal matrix?

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Matrix, 7

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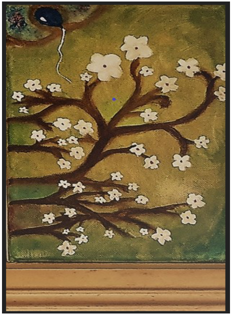
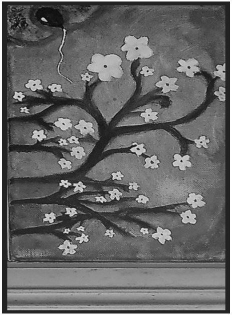
- what is lower bandwidth of upper triangular matrix?
- what is upper bandwidth of lower Hessenberg matrix?
- what is lower bandwidth of lower Hessenberg matrix?
- what is bandwidth of pentadiagonal matrix?

Matrices and Digital Images, 8





```
RGB matrix =
[[[135 108  37]
  [125  98  27]
  [119  92  21]
  [126  98  25]]

 [[134 107  36]
  [147 118  48]
  [151 123  50]
  [143 115  42]]]
```

```
grayscale matrix =
 [[108  98  92  98]
 [107 119 123 115]]
```

Matrix Operations, 1

Definition 2.13

Addition: $A + B = (a_{ij} + b_{ij})_{m \times n}, A, B, A + B \in \mathbb{R}^{m \times n}$

Matrix Operations, 1

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Definition 2.14

Product: $AB = (\sum_{k=1}^{n} a_{ik} b_{kj})_{m\times l}, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}, AB \in \mathbb{R}^{m\times l}$

Matrix Operations, 1

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Example 2.15

Alternative ways writing matrix products:

Matrix Operations, 1

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Example 2.15

Alternative ways writing matrix products:

- $AB = \begin{pmatrix} Ab_1 & Ab_2 & ... & Ab_l \end{pmatrix}, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}, b_i \in \mathbb{R}^n, i = 1, 2, .., l$

Matrix Operations, 1

Definition 2.13

Addition: $A + B = (a_{ij} + b_{ij})_{m\times n}, A, B, A + B \in \mathbb{R}^{m\times n}$

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Alternative ways writing matrix products:

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- $AB = \begin{pmatrix} a_1 B \\ a_2 B \\ \dots \\ a_m B \end{pmatrix}, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}, a_i \in \mathbb{R}^n, i = 1, 2, .., l$

Matrix Operations, 2

Definition 2.16

Inner product: $a^T b = \sum_{i=1}^{n} a_i b_i, a, b \in \mathbb{R}^n$

Matrix Operations, 2

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Inner product: $a^T b = \sum_{i=1}^{n} a_i b_i,\, a, b \in \mathbb{R}^n$

Definition 2.17

Outer product: $ab^T = (b_1 a, b_2 a, ..., b_n a),\, a \in \mathbb{R}^m,\, b \in \mathbb{R}^n,\, ab^T \in \mathbb{R}^{m \times n}$

Matrix Operations, 2

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Inner product: $a^T b = \sum_{i=1}^{n} a_i b_i$, $a, b \in \mathbb{R}^n$

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Outer product: $ab^T = (b_1 a, b_2 a, ..., b_n a)$, $a \in \mathbb{R}^m$, $b \in \mathbb{R}^n$, $ab^T \in \mathbb{R}^{m \times n}$

Definition 2.18

Determinant, Laplace expansion:

- $det(A) = (-1)^{i+1} a_{i1} det(A_{i1}) + (-1)^{i+2} a_{i1} det(A_{i2}) + ... + (-1)^{i+n} a_{in} det(A_{in})$
- $A \in \mathbb{R}^{n \times n}$, $A_{ij} \in \mathbb{R}^{n-1 \times n-1}$
- submatrix A_{ij} is obtained by eliminating i-th row and j-th column from matrix A

Matrix Operations, 3

Definition 2.19

Inverse matrix: $AB = BA = I; A, B, I \in \mathbb{R}^{n\times n}$, B is inverse of A, $B = A^{-1}$

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Definition 2.20

Characteristic polynomial $p_n(\lambda)$ of a matrix:

$$p_n(\lambda) = det(A - \lambda I), A, I \in \mathbb{R}^{n \times n}$$

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Definition 2.21

Spectral radius of a matrix

$$\rho(A) = \max_{i=1,2,..,n} |\lambda_i|, \quad Ax_i = \lambda_i x, x \neq 0, A \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n$$

Matrix Operations, 3

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Definition 2.22

Similarity transformation:

$$B^{-1}AB;\ A, B \in \mathbb{R}^{n\times n},\ A \text{ and } B^{-1}AB \text{ are similar matrices}$$

Matrix properties, 1

Theorem 2.23

Some determinant properties, $A, B \in \mathbb{R}^{n\times n}$

Matrix properties, 1

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Some determinant properties, $A, B \in \mathbb{R}^{n\times n}$

1. $det(\alpha A) = \alpha^n det(A)$

Matrix properties, 1

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2. $det(A) = det(A^T)$

Matrix properties, 1

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Matrix properties, 1

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4. $det(A) = 0$ *if its two rows or two columns are co-linear*

Matrix properties, 1

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Notation equivalnce: $det(A) \equiv |A|$

Matrix properties, 2

Theorem 2.24

Some properties of invertible matrices, $A, B \in \mathbb{R}^{n\times n}$

Matrix properties, 2

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5. $(A^T)^{-1} = (A^{-1})^T$

Matrix properties, 2

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6. *All eigenvalues of A are nonzero*
7. *Columns and rows of A are linearly independent*

Matrix properties, 3

Theorem 2.25

Some properties of invertible matrices, $A, B \in \mathbb{R}^{n\times n}$

Matrix properties, 3

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Some properties of invertible matrices, $A, B \in \mathbb{R}^{n\times n}$

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3. *The diagonal entries of a lower(upper) triangular matrix are its eigenvalues*
4. *The inverse of a lower(upper) triangular matrix is a lower(upper) triangular matrix*

Matrix properties, 4

Theorem 2.26

Spectral decomposition of symmetric matrices

$$A = O\Lambda O^T, \Lambda = O^T A O,\ A = A^T, A \in \mathbb{R}^{n\times n}$$

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1. $\Lambda = diag(\lambda_1, ..., \lambda_n)$, $\lambda_i \in \mathbb{R}, i = 1, 2, .., n$, *are eigenvalues of A*

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3. $O^T O = I$; $O, O^T, I \in \mathbb{R}^{n\times n}$, O *is orthogonal matrix*

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Similar eigenvalue decomposition holds true for Hermitian matrices where orthogonal matrix O is substituted by unitary matrix U

Matrix norms, 1

Definition 2.27

$\| \cdot \| : \mathbb{R}^{m \times n} \to \mathbb{R}$:

Matrix norms, 1

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$\| \cdot \| : \mathbb{R}^{m\times n} \to \mathbb{R}$:

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Matrix norms, 1

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3. $\|A + B\| \leq \|A\| + \|B\|$(triangle inequality or subadditivity)

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Holds true for all $A, B \in \mathbb{R}^{m\times n}$ and for all scalars α

Matrix norms, 1

Definition 2.27

$\| \cdot \| : \mathbb{R}^{m\times n} \to \mathbb{R}$:

1. $\|A\| \geq 0$, $\|A\| = 0$ if and only if $A = 0$ (positivity)
2. $\|\alpha A\| = |\alpha| \|A\|$ (homogenity)
3. $\|A + B\| \leq \|A\| + \|B\|$(triangle inequality or subadditivity)

Holds true for all $A, B \in \mathbb{R}^{m\times n}$ and for all scalars α

Example 2.28

- $\|A\|_1 = \max_{\{1\leq j\leq n\}} \sum_{i=1}^{m} |a_{ij}|$ (one-norm or max. column sum norm)

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- $\|A\|_{max} = max_{\{1\leq i\leq m, 1\leq j\leq n\}}|a_{ij}|$ (maximum norm)

Matrix norms, 2

Example 2.29

- l_{pq}-norm: $\|A\|_{pq} = (\sum_{j=1}^{n}(\sum_{i=1}^{m}|a_{ij}|^p)^{\frac{q}{p}})^{\frac{1}{q}}$, $A \in \mathbb{R}^{m\times n}$
- Nuclear norm:

$$\|A\|_* = \sum_{k=1}^{min(m,n)} \sigma_k(A)$$

$\sigma_k(A), k = 1, ..., min(m,n)$ -singular values of $A, A \in \mathbb{R}^{m\times n}$

- Ky-Fan K-norm:

$$\|A\|_{Ky-Fan,K} = \sum_{k=1}^{K} \sigma_k(A), \quad K \leq min(m,n), A \in \mathbb{R}^{m\times n}$$

- Schatten p-norm:

$$\|A\|_{S,p} = (\sum_{k=1}^{min(m,n)} \sigma_k(A)^p)^{\frac{1}{p}}, \quad p \geq 1, A \in \mathbb{R}^{m\times n}$$

Matrix norms, 3

Definition 2.30

Equivalence of matrix norms

Matrix norms, 3

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Equivalence of matrix norms

1. $C_m\|A\|_\alpha \leq \|A\|_\beta \leq C_M\|A\|_\alpha$
2. C_m, C_M finite positive constants
3. holds true for all matrices $A \in \mathbb{R}^{m\times n}$

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- $\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty$

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- $\frac{1}{\sqrt{n}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$

Matrix norms, 4

Definition 2.33

Sub-multiplicative matrix norm:

$$\|AB\| \leq \|A\|\|B\|, A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times l}$$

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Unitarily invariant matrix norm:

$$\|UAV\| = \|A\|, A \in \mathbb{C}^{m\times n}, U \in \mathbb{C}^{m\times m}, V \in \mathbb{C}^{n\times n}$$

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Definition 2.35

Consistent matrix norm is a submultiplicative matrix norm which is defined for all $m, n \in \mathbb{N}$

Matrix norms, 5

Example 2.36

- Not all norms are submultiplicative

Matrix norms, 5

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Matrix norms, 5

Example 2.36

- Not all norms are submultiplicative
- Some books require sub-multiplicativity in norm definition : $\|AB\| \leq \|A\|\|B\|$
- The max-norm is not submultiplicative

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, AB = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\|A\|_{max} = max_{\{1 \leq i \leq m, 1 \leq j \leq n\}} |a_{ij}|,$$

$\|AB\|_{max} = 2, \|A\|_{max} = 1, \|B\|_{max} = 1, 2 > 1$, and therefore sub-multiplicative property does not hold true

Matrix norms, 6

Definition 2.37

Matrix norm β and vector norm α ar compatible if

$$\|Ax\|_\alpha \leq \|A\|_\beta \|x\|_\alpha \quad A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n$$

Matrix norms, 6

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Example 2.38

The one vector and matrix norms are compatible:

$$\|Ax\|_1 \leq \|A\|_1 \|x\|_1 = (\max_{\{1\leq j\leq n\}} \sum_{i=1}^{m} |a_{ij}|)(\sum_{i=1}^{n} |x_i|)$$

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Definition 2.39

Subordinate matrix norm:

$$\|Ax\|_\alpha \leq \|A\|_\beta \|x\|_\gamma, \quad A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n,$$

matrix norm $\|.\|_\beta$ is subordinate to vector norms $\|.\|_\alpha$ and $\|.\|_\gamma$

Matrix norms, 7

Definition 2.40

Matrix norm $\|.\|$ induced by vector norm $\|.\|_\alpha$:

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha} = \sup_{\|x\|_\alpha = 1} \|Ax\|_\alpha$$

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Example 2.41

- Frobenius norm is not induced by any vector norm
- Spectral matrix norm is induced by Euclidean norm

$$\|A\|_2 = \sqrt{\rho(A^T A)}, \quad \|x\|_2 = (x_1^2 + x_2^2 + ... + x_n^2)^{\frac{1}{2}}$$

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Definition 2.42

Operator norm:

$$\|A\|_{\alpha,\beta} = \sup_{x \neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\beta}$$

Matrix norms, 8

Theorem 2.43

Properties of Frobenius norm

$$\|A\|_F = (\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2)^{\frac{1}{2}}$$

- $\|A\|_F = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$,

Matrix norms, 8

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- $\|A\|_F = \sqrt{tr(A^*A)} = \sqrt{tr(AA^*)}$, $tr(B) = \sum_{i=1}^{n}|b_{ii}|$, $B \in \mathbb{C}^{n\times n}$
- $\|AB\|_F \leq \|B\|_F\|B\|_F$

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- $\sqrt{\rho(A^*A)} \leq \|A\|_F \leq \sqrt{n}\sqrt{\rho(A^*A)}$

Matrix norms, 9

Theorem 2.44

For any induced matrix norm

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha}$$

the following holds true:

Matrix norms, 9

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Matrix norms, 9

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the following holds true:

- $$\rho(A) \leq \|A\|, A \in \mathbb{C}^{n \times n}$$

- *(Gelfand's formula)*

  $$\rho(A) = \lim_{k \to \infty} (\|A^k\|)^{\frac{1}{k}}, A \in \mathbb{C}^{n \times n}$$

- $$\|A\| = \inf\{\lambda \in \mathbb{R} : \|Ax\| \leq \lambda \|x\|, x \in \mathbb{C}^n\}$$

Matrix series, 1

- sequence of matrices $\{A_k\}_1^\infty$
- how can you define convergence $lim_{k\to\infty} A_k = B$?

Matrix series, 2

- sequence of matrices $\{A_k\}_1^\infty$
- how can you define convergence $lim_{k\to\infty} A_k = B$?
- Is entry-wise convergence sufficient?
- Can you define convergence of matrix sequences using matrix norm?

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- Would approach $\|A_n - B\| < \epsilon$ for all $n > n(\epsilon)$ work?
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Matrix series, 4

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- Set $A_k = A^k$, A is square matrix. For which A this sequence converges to zero matrix?

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Theorem 2.45

if $\rho(A) < 1, A \in \mathbb{R}^{n\times n}$ then $lim_{k\to\infty} A^k$ converges to the matrix with all zero entries

Q & A