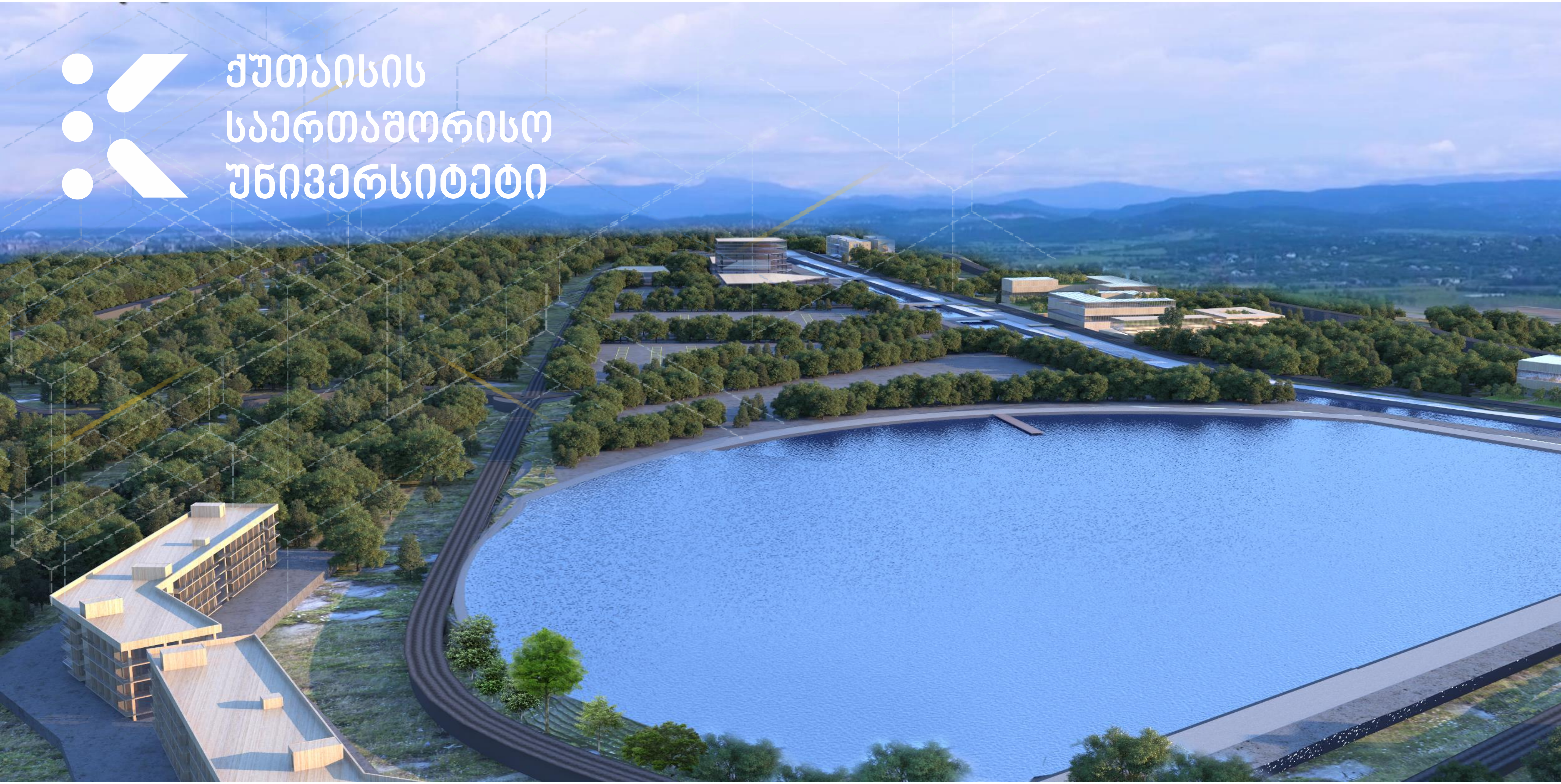
ქუთაისის საერთაშორისო უნივერსიტეტი



2.5 Continuity

Continuity of a Function

Continuity of a Function (1 of 8)

The limit of a function as x approaches a can often be found simply by calculating the value of the function at a. Functions with this property are called *continuous at a*.

We will see that the mathematical definition of continuity corresponds closely with the meaning of the word *continuity* in everyday language. (A continuous process is one that takes place without interruption.)

1 Definition A function f is **continuous at a number *a* if**

$$\lim_{x \to a} f(x) = f(a)$$

Continuity of a Function (2 of 8)

Notice that Definition 1 implicitly requires three things if *f* is continuous at *a*:

1. $f(a)$ is defined (that is, *a* is in the domain of *f*)

2. $\lim_{x \to a} f(x)$ exists

3. $\lim_{x \to a} f(x) = f(a)$

The definition says that *f* is continuous at *a* if $f(x)$ approaches $f(a)$ as *x* approaches *a*. Thus a continuous function *f* has the property that a small change in *x* produces only a small change in $f(x)$.

Continuity of a Function (3 of 8)

In fact, the change in $f(x)$ can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near a (in other words, f is defined on an open interval containing a, except perhaps at a), we say that f is **discontinuous at *a*** (or f has a **discontinuity** at a) if f is not continuous at a.

Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person’s height. But discontinuities do occur in such situations as electric currents.

Continuity of a Function (4 of 8)

Geometrically, you can think of a function that is continuous at every number in an interval as a function whose graph has no break in it: the graph can be drawn without removing your pen from the paper.

Example 1

Figure 2 shows the graph of a function *f*. At which numbers is *f* discontinuous? Why?

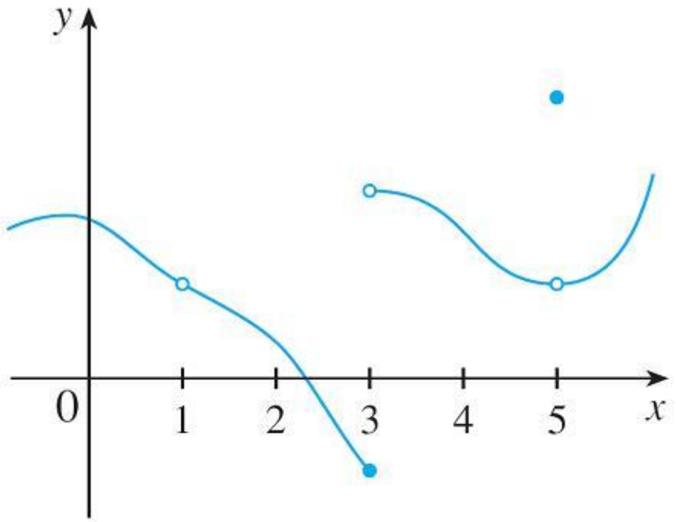


Figure 2

Solution:

It looks as if there is a discontinuity when $a = 1$ because the graph has a break there. The official reason that *f* is discontinuous at 1 is that *f* (1) is not defined.

Example 1 – Solution

The graph also has a break when $a = 3$, but the reason for the discontinuity is different. Here, $f(3)$ is defined, but $\lim_{x \to 3} f(x)$ does not exist (because the left and right limits are different). So f is discontinuous at 3.

What about $a = 5$? Here, $f(5)$ is defined and $\lim_{x \to 5} f(x)$ exists (because the left and right limits are the same).

But

$$\lim_{x \to 5} f(x) \neq f(5)$$

So f is discontinuous at 5.

Example 2

Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(c) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$

(b) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$

Solution:

(a) Notice that $f(2)$ is not defined, so f is discontinuous at 2. Later we'll see why f is continuous at all other numbers.

Example 2 – Solution (1 of 2)

(b) Here $f(2) = 1$ is defined and

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3 \text{ exists.}$$

But

$$\lim_{x\to 2} f(x) \neq f(2)$$

so f is not continuous at 2.

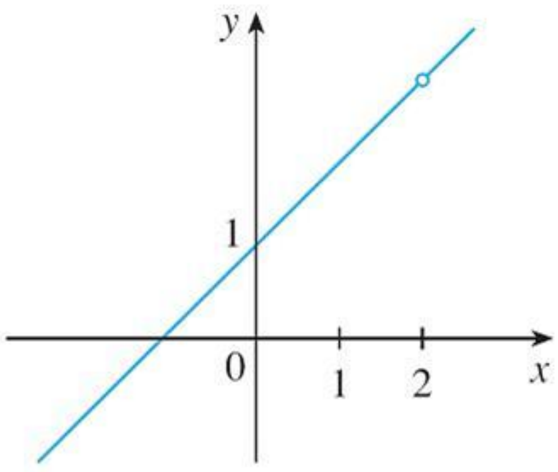
(c) Here $f(0) = 1$ is defined but

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2}$$

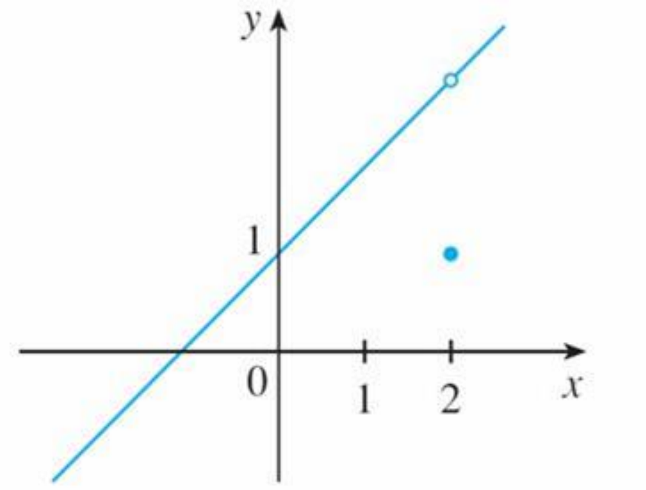
does not exist. So f is discontinuous at 0.

Continuity of a Function (5 of 8)

Figure 3 shows the graphs of the functions in Example 2.



(a) A removable discontinuity

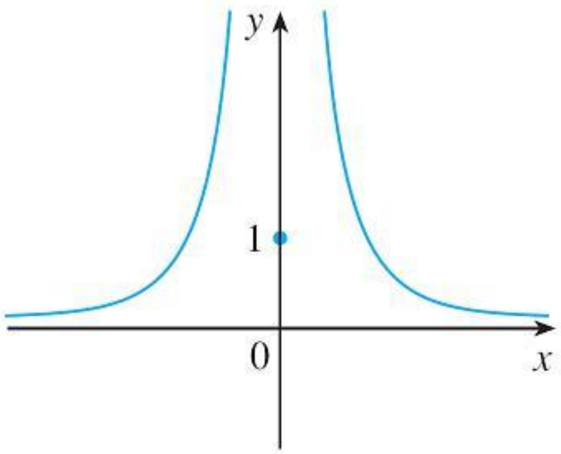


(b) A removable discontinuity

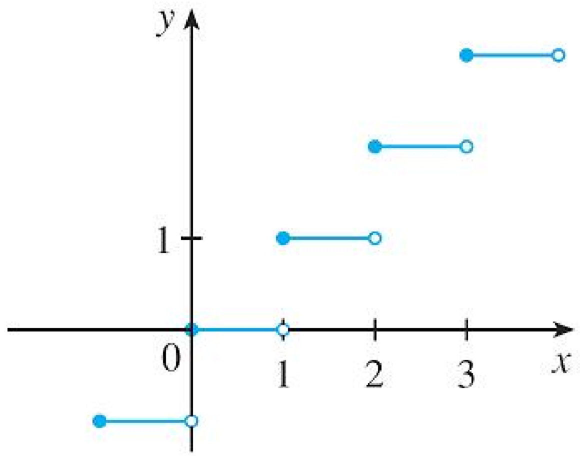
Graphs of the functions in Example 2

Figure 3

Continuity of a Function (6 of 8)



(c) An infinite discontinuity



(d) Jump discontinuities

Graphs of the functions in Example 2

Figure 3

Continuity of a Function (7 of 8)

In each case the graph can’t be drawn without lifting the pen from the paper because a hole or break or jump occurs in the graph.

The kind of discontinuity illustrated in parts (a) and (b) is called **removable** because we could remove the discontinuity by redefining f at just the single number 2. [If we redefine f to be 3 at $x = 2$, then f is equivalent to the function $g(x) = x + 1$, which is continuous.]

The discontinuity in part (c) is called an **infinite discontinuity**. The discontinuities in part (d) are called **jump discontinuities** because the function “jumps” from one value to another.

Continuity of a Function (8 of 8)

2 Definition A function *f* is **continuous from the right at a number** *a* if

$$\lim_{x \to a^+} f(x) = f(a)$$

and *f* is **continuous from the left at *a*** if

$$\lim_{x \to a^-} f(x) = f(a)$$

3 Definition A function *f* is **continuous on an interval** if it is continuous at every number in the interval. (If *f* is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right or continuous from the left*.)

Properties of Continuous Functions

Properties of Continuous Functions (1 of 8)

Instead of always using Definitions 1, 2, and 3 to verify the continuity of a function, it is often convenient to use the next theorem, which shows how to build up complicated continuous functions from simple ones.

4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$
2. $f - g$
3. cf
4. fg
5. $\dfrac{f}{g}$ if $g(a) \neq 0$

Properties of Continuous Functions (2 of 8)

It follows from Theorem 4 and Definition 3 that if f and g are continuous on an interval, then so are the functions $f + g$, $f - g$, cf, fg, and (if g is never 0) $\frac{f}{g}$.

The following theorem was stated as the Direct Substitution Property.

5 Theorem

(a) Any polynomial is continuous everywhere; that is, it is continuous on

$$\mathbb{R} = (-\infty, \infty).$$

(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

Properties of Continuous Functions (3 of 8)

As an illustration of Theorem 5, observe that the volume of a sphere varies continuously with its radius because the formula $V(r) = \frac{4}{3}\pi r^3$ shows that V is a polynomial function of r.

Likewise, if a ball is thrown vertically into the air with a velocity of 50 ft/s, then the height of the ball in feet t seconds later is given by the formula $h = 50t - 16t^2$.

Again this is a polynomial function, so the height is a continuous function of the elapsed time.

Example 5

Find $\lim\limits_{x \to -2} \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

Solution:

The function $f(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$

is rational, so by Theorem 5 it is continuous on its domain, which is $\left\{ x \mid x \neq \dfrac{5}{3} \right\}$.

Therefore

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \lim_{x \to -2} f(x) = f(-2)$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} = -\frac{1}{11}$$

Properties of Continuous Functions (4 of 8)

It turns out that most of the familiar functions are continuous at every number in their domains.

From the appearance of the graphs of the sine and cosine functions, we would certainly guess that they are continuous.

We know from the definitions of sin θ and cos θ that the coordinates of the point P in Figure 5 are (cos θ, sin θ). As $\theta \to 0$, we see that P approaches the point (1, 0) and so cos $\theta \to 1$ and sin $\theta \to 0$.

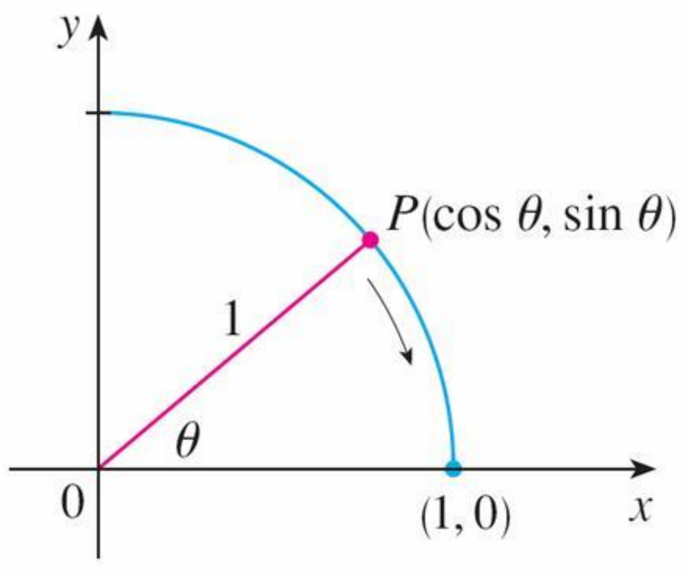


Figure 5

Properties of Continuous Functions (5 of 8)

Thus

$$6 \qquad \lim_{\theta \to 0} \cos\theta = 1 \qquad \lim_{\theta \to 0} \sin\theta = 0$$

Since cos 0 = 1 and sin 0 = 0, the equations in (6) assert that the cosine and sine functions are continuous at 0.

The addition formulas for cosine and sine can then be used to deduce that these functions are continuous everywhere.

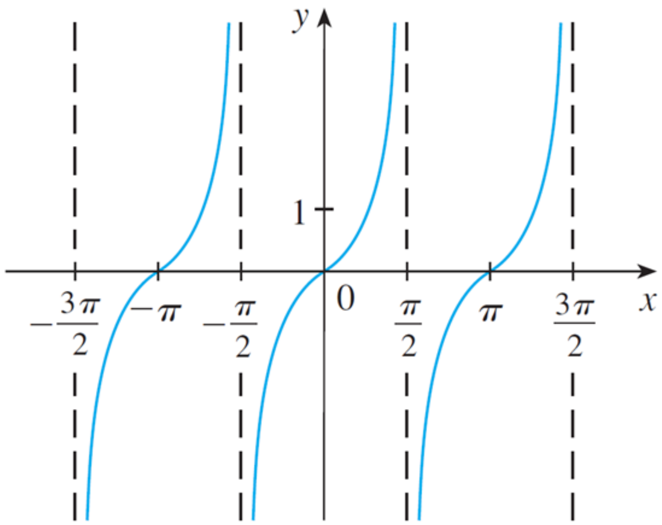
It follows from part 5 of Theorem 4 that

$$\tan x = \frac{\sin x}{\cos x}$$

is continuous except where cos x = 0.

Properties of Continuous Functions (6 of 8)

This happens when x is an odd integer multiple of $\frac{\pi}{2}$, so $y = \tan x$ has infinite discontinuities when $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2},$ and so on (see Figure 6).



$y = \tan x$

Figure 6

Properties of Continuous Functions (7 of 8)

7 Theorem The following types of functions are continuous at every number in their domains:

- Polynomials
- trigonometric functions
- exponential functions
- rational functions
- root functions
- inverse trigonometric functions
- logarithmic functions

Another way of combining continuous functions *f* and *g* to get a new continuous function is to form the composite function $f \circ g$. This fact is a consequence of the following theorem.

8 Theorem If *f* is continuous at *b* and $\lim_{x \to a} g(x) = b$, then $\lim_{x \to a} f(g(x)) = f(b)$.

In other words,

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Properties of Continuous Functions (8 of 8)

Intuitively, Theorem 8 is reasonable because if *x* is close to *a*, then *g*(*x*) is close to *b*, and since *f* is continuous at *b*, if *g*(*x*) is close to *b*, then *f*(*g*(*x*)) is close to *f*(*b*).

EXAMPLE 8 Evaluate $\lim_{x \to 1} \arcsin\left(\dfrac{1 - \sqrt{x}}{1 - x}\right)$.

SOLUTION Because arcsin is a continuous function, we can apply Theorem 8:

$$\begin{aligned}
\lim_{x \to 1} \arcsin\left(\frac{1 - \sqrt{x}}{1 - x}\right) &= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})}\right) \\
&= \arcsin\left(\lim_{x \to 1} \frac{1}{1 + \sqrt{x}}\right) \\
&= \arcsin \frac{1}{2} = \frac{\pi}{6}
\end{aligned}$$

9 Theorem If g is continuous at a and f is continuous at $g(a)$. then the composition function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a

The Intermediate Value Theorem

The Intermediate Value Theorem (1 of 7)

An important property of continuous functions is expressed by the following theorem, whose proof is found in more advanced books on calculus.

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem (2 of 7)

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrated by Figure 8.

Note that the value N can be taken on once [as in part (a)] or more than once [as in part (b)].

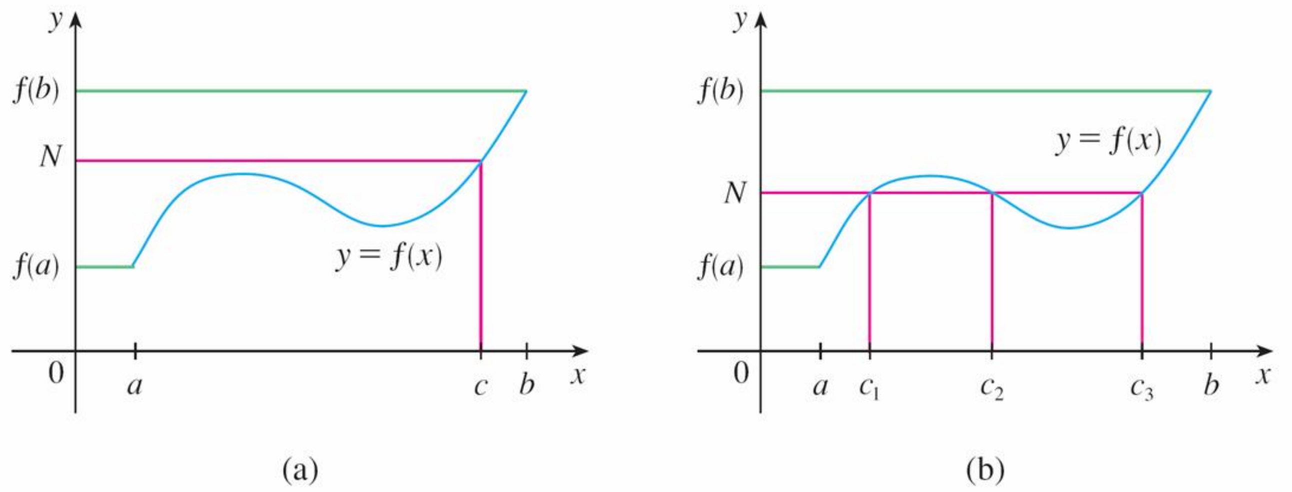


Figure 8

The Intermediate Value Theorem (3 of 7)

If we think of a continuous function as a function whose graph has no hole or break, then it is easy to believe that the Intermediate Value Theorem is true.

In geometric terms it says that if any horizontal line $y = N$ is given between $y = f(a)$ and $y = f(b)$ as in Figure 9, then the graph of f can't jump over the line. It must intersect $y = N$ somewhere.

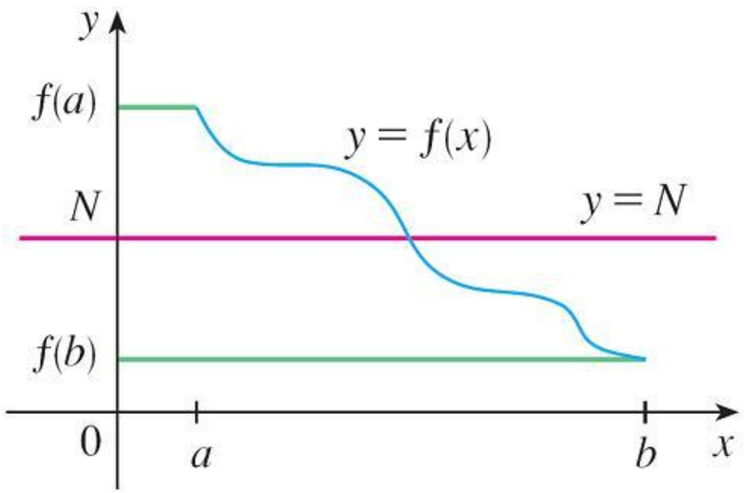


Figure 9

The Intermediate Value Theorem (4 of 7)

It is important that the function *f* in Theorem 10 be continuous. The Intermediate Value Theorem is not true in general for discontinuous functions.

Example 10

Show that there is a solution of the equation

$$4x^3 - 6x^2 + 3x - 2 = 0.$$

between 1 and 2.

Solution:

Let $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0.$

We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore we take $a = 1$, $b = 2$, and $N = 0$ in Theorem 10.

Example 10 – Solution (1 of 2)

We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0.$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0.$$

Thus $f(1) < 0 < f(2)$; that is, $N = 0$ is a number between $f(1)$ and $f(2)$. The function f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that $f(c) = 0$.

In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one solution c in the interval (1, 2).

Example 10 – Solution (2 of 2)

In fact, we can locate a solution more precisely by using the Intermediate Value Theorem again. Since

$$f(1.2) = -0.128 < 0 \quad \text{and} \quad f(1.3) = 0.548 > 0$$

a solution must lie between 1.2 and 1.3. A calculator gives, by trial and error,

$$f(1.22) = -0.007008 < 0 \text{ and } f(1.23) = 0.056068 > 0$$

so a solution lies in the interval (1.22, 1.23).