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Numerical Linear Algebra

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Kutaisi International University

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Direct and iterative methods for linear systems

- Recap of Previous Lecture
- Convergence of Richardson's iterations
- Optimal parameter parameter of Richardson's iterations
- Sufficient condition $P > 0.5A$
- Preconditioning matrices
- Quadratic functional and linear systems
- Q & A

Recap of Previous Lecture

- Sherman-Morrison formula
- Cholesky factorization of symmetric system
- Diagonally dominant system
- Necessary and sufficient conditions for convergence
- Classic iterative methods
- Convergence theorems of J,GS

Richardson method 1

- Richardson's method

$$P\frac{x^{(k+1)} - x^{(k)}}{\tau_k} + Ax^{(k)} = b, \quad k = 0, 1, 2, ...$$

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- $\frac{1}{\tau_k} Px^{(k+1)} = (\frac{1}{\tau_k} P - A)x^{(k)} + b, \quad k = 0, 1, 2, ...$

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- Pre-conditioner - $P \in \mathbb{R}^{n\times n}$

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- Numeric parameter - τ_k
- Stationary iteration method if $B_{R,k} \equiv B_R$
- Converges if $\|B_{R,k}\| < 1$
- Converges iff $\rho(B_{R,k}) < 1$

Richardson method 2

Example 10.1

Jacobi method = preconditioner is diagonal part of A

Richardson method 2

Example 10.1

Jacobi method = preconditioner is diagonal part of A

$$\begin{cases} P\frac{x^{(k+1)}-x^{(k)}}{\tau_k} + Ax^{(k)} = b \\ \tau_k = 1, P = D \end{cases} \Rightarrow \begin{cases} D(x^{(k+1)} - x^{(k)}) + Ax^{(k)} = b \\ x^{(k+1)} = (I - D^{-1}A)x^{(k)} + D^{-1}b \\ B_R = (I - D^{-1}A) = B_J \end{cases}$$

Richardson method 2

Example 10.1

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Example 10.2

Gaus-Seidel method = preconditioner is lower triangular part of A

Richardson method 2

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Example 10.2

Gaus-Seidel method = preconditioner is lower triangular part of A

$$\begin{cases} P\frac{x^{(k+1)}-x^{(k)}}{\tau_k} + Ax^{(k)} = b \\ \tau_k = 1, P = D + L \end{cases} \Rightarrow$$

$$\begin{cases} A = L + D + U \\ (D + L)(x^{(k+1)} - x^{(k)}) + Ax^{(k)} = b \\ x^{(k+1)} = -(D + L)^{-1}Ux^{(k)} + (D + L)^{-1}b \end{cases}$$

Richardson method 3

Theorem 10.3

The stationary Richarson method converges iff

$$\frac{2\mathfrak{Re}(\lambda)}{\tau|\lambda|^2} > 1$$

for all λ - eigenvalues of $P^{-1}A$

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- $P\frac{x^{(k+1)}-x^{(k)}}{\tau} + Ax^{(k)} = b$
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- $\rho(B_R) < 1 \Rightarrow |1-\tau\lambda| < 1$

$\square$

Richardson method 4

Proof.

- $P\frac{x^{(k+1)}-x^{(k)}}{\tau} + Ax^{(k)} = b$
- $x^{(k+1)} = B_R x^{(k)} + f,\ B_R = I - \tau P^{-1}A$
- Convergence $\equiv \rho(B_R) < 1$
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-
$$|1-\tau\lambda|^2 = (1-\tau\mathfrak{Re}(\lambda))^2 + (\tau\mathfrak{Im}(\lambda))^2 =$$
$$1 - 2\tau\mathfrak{Re}(\lambda) + (\tau\mathfrak{Re}(\lambda))^2 + (\tau\mathfrak{Im}(\lambda))^2 = 1 - 2\tau\mathfrak{Re}(\lambda) + \tau^2|\lambda|^2$$

Richardson method 4

Proof.

- $P\frac{x^{(k+1)}-x^{(k)}}{\tau}+Ax^{(k)}=b$
- $x^{(k+1)}=B_Rx^{(k)}+f,\ B_R=I-\tau P^{-1}A$
- Convergence $\equiv \rho(B_R)<1$
- y eigenvalue of $P^{-1}A \Rightarrow I-\tau P^{-1}Ay=(1-\tau\lambda)y$
- $\rho(B_R)<1 \Rightarrow |1-\tau\lambda|<1$
- $$|1-\tau\lambda|^2=(1-\tau\mathfrak{Re}(\lambda))^2+(\tau\mathfrak{Im}(\lambda))^2=$$ $$1-2\tau\mathfrak{Re}(\lambda)+(\tau\mathfrak{Re}(\lambda))^2+(\tau\mathfrak{Im}(\lambda))^2=1-2\tau\mathfrak{Re}(\lambda)+\tau^2|\lambda|^2$$
- $1-2\tau\mathfrak{Re}(\lambda)+\tau^2|\lambda|^2<1$

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- $\rho(B_R) < 1 \Rightarrow |1-\tau\lambda| < 1$
- $$|1-\tau\lambda|^2 = (1-\tau\mathfrak{Re}(\lambda))^2 + (\tau\mathfrak{Im}(\lambda))^2 =$$ $$1 - 2\tau\mathfrak{Re}(\lambda) + (\tau\mathfrak{Re}(\lambda))^2 + (\tau\mathfrak{Im}(\lambda))^2 = 1 - 2\tau\mathfrak{Re}(\lambda) + \tau^2|\lambda|^2$$
- $1 - 2\tau\mathfrak{Re}(\lambda) + \tau^2|\lambda|^2 < 1 \ \Rightarrow \frac{2\mathfrak{Re}(\lambda)}{\tau|\lambda|^2} > 1$

$\square$

Optimal Step(Parameter) for Stationary Richardson 1

Theorem 10.4

- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$

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- $\rho_{opt}=\min_\alpha\rho(B_{R,\alpha})=\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}$

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- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
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- $\lambda_i(P^{-1}A) \in \mathbb{R}$
- $\lambda_1(P^{-1}A) > \lambda_2(P^{-1}A) > ... > \lambda_n(P^{-1}A) > 0$
- $\Downarrow$
- $\alpha_{opt} = \frac{2}{\lambda_1+\lambda_n}$
- $\rho_{opt} = \min_\alpha \rho(B_{R,\alpha}) = \frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}$

Proof.

- $\lambda_i(B_{R,\alpha}) = 1 - \alpha\lambda_i(P^{-1}A), i = 1, 2, ..., n$

Optimal Step(Parameter) for Stationary Richardson 1

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- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
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Proof.

- $\lambda_i(B_{R,\alpha}) = 1 - \alpha\lambda_i(P^{-1}A), i = 1, 2, ..., n$
- $\rho(B_{R,\alpha}) < 1 \Rightarrow |1 - \alpha\lambda_i(P^{-1}A)| < 1, i = 1, 2, ..n$

$\square$

Optimal Step(Parameter) for Stationary Richardson 2

Proof.

- $\lambda_i(B_{R,\alpha}) = 1 - \alpha\lambda_i(P^{-1}A), i = 1, 2, ..., n$
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Proof.

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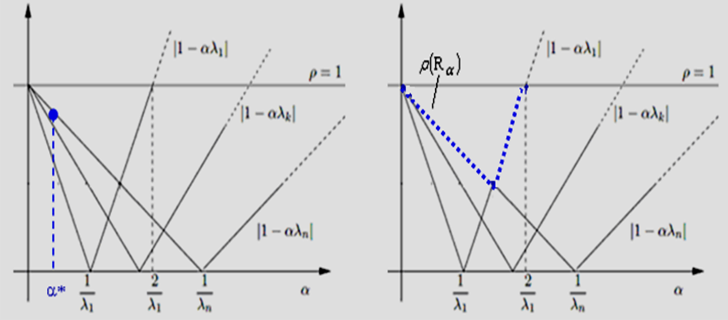

Figure: Eigenvalues of $B_{R,\alpha}$ are functions of α

Optimal Step(Parameter) for Stationary Richardson 3

Proof.

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Optimal Step(Parameter) for Stationary Richardson 3

Proof.

- $\lambda_i(B_{R,\alpha}) = 1 - \alpha\lambda_i(P^{-1}A), i = 1, 2, ..., n$
- $\rho(B_{R,\alpha}) < 1 \Rightarrow |1 - \alpha\lambda_i(P^{-1}A)| < 1, i = 1, 2, ..n$
- $\rho(B_{R,\alpha}) = \max_{1 \le i \le n, \alpha} |1 - \alpha\lambda_i(P^{-1}A)|$

Optimal Step(Parameter) for Stationary Richardson 3

Proof.

- $\lambda_i(B_{R,\alpha}) = 1 - \alpha\lambda_i(P^{-1}A), i = 1, 2, ..., n$
- $\rho(B_{R,\alpha}) < 1 \Rightarrow |1 - \alpha\lambda_i(P^{-1}A)| < 1, i = 1, 2, ..n$
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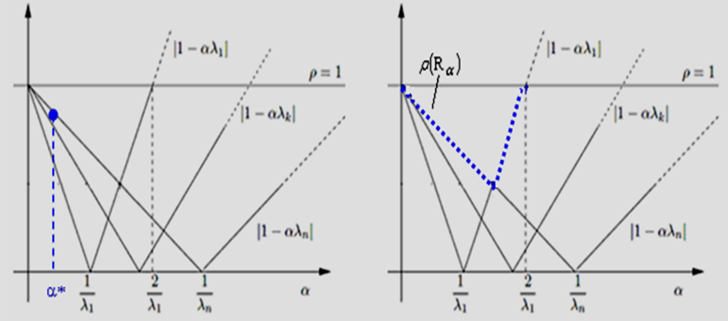

Figure: $\rho(B_{R,\alpha})$ is function of α

Optimal Step(Parameter) for Stationary Richardson 4

Proof.

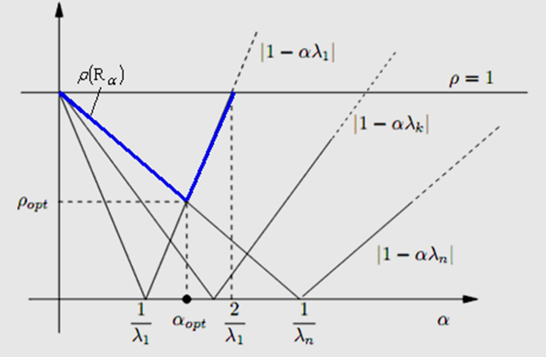

Figure: ρ_{opt} and $\rho(B_{R,\alpha})$

Optimal Step(Parameter) for Stationary Richardson 4

Proof.

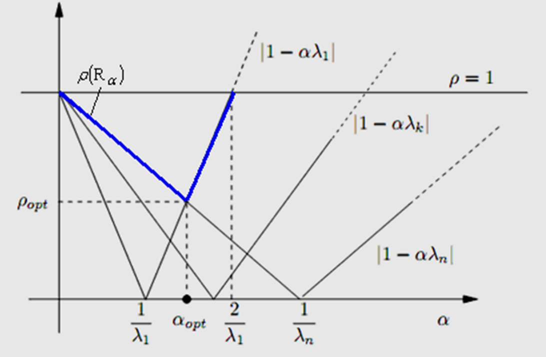

Figure: ρ_{opt} and $\rho(B_{R,\alpha})$

- $\alpha_{opt} = arg\min \rho(B_{R,\alpha})$

Optimal Step(Parameter) for Stationary Richardson 4

Proof.

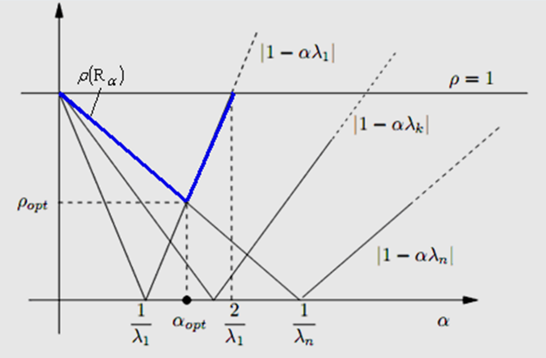

Figure: ρ_{opt} and $\rho(B_{R,\alpha})$

- $\alpha_{opt} = arg\min \rho(B_{R,\alpha})$
- $\alpha_{opt}\lambda_1(P^{-1}A) - 1 = 1 - \alpha_{opt}\lambda_n(P^{-1}A) \Rightarrow \alpha_{opt} = \frac{2}{\lambda_1+\lambda_n}$

Optimal Step(Parameter) for Stationary Richardson 4

Proof.

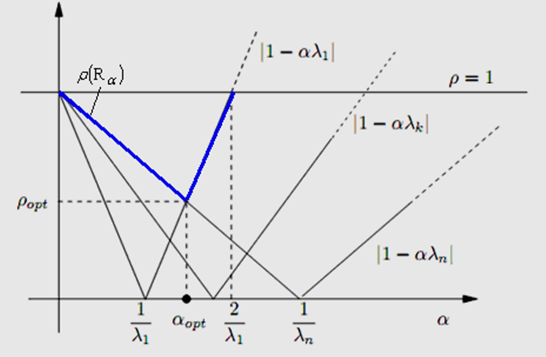

Figure: ρ_{opt} and $\rho(B_{R,\alpha})$

- $\alpha_{opt} = arg\min \rho(B_{R,\alpha})$
- $\alpha_{opt}\lambda_1(P^{-1}A) - 1 = 1 - \alpha_{opt}\lambda_n(P^{-1}A) \Rightarrow \alpha_{opt} = \frac{2}{\lambda_1+\lambda_n}$
- $\rho_{opt} = \min_\alpha \rho(B_{R,\alpha}) = 1 - \alpha_{opt}\lambda_n = 1 - \frac{2}{\lambda_1+\lambda_n}\lambda_n = \frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n}$

Sufficient condition for convergence $P > 0.5\alpha A$, 1

Theorem 10.5

- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$

Sufficient condition for convergence $P > 0.5\alpha A$, 1

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Sufficient condition for convergence $P > 0.5\alpha A$, 1

Theorem 10.5

- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
- $A \in \mathbb{R}^{n\times n}, A = A^T > 0$
- $\alpha > 0$

Sufficient condition for convergence $P > 0.5\alpha A$, 1

Theorem 10.5

- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
- $A \in \mathbb{R}^{n\times n}, A = A^T > 0$
- $\alpha > 0$
- $P > 0.5\alpha A$

Sufficient condition for convergence $P > 0.5\alpha A$, 1

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- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
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Sufficient condition for convergence $P > 0.5\alpha A$, 1

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- $P > 0.5\alpha A$
- $\Downarrow$
- *Errors decrease monotonically in A-norm*

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- $A \in \mathbb{R}^{n\times n}, A = A^T > 0$
- $\alpha > 0$
- $P > 0.5\alpha A$
- $\Downarrow$
- *Errors decrease monotonically in A-norm*
- *Richardson's method converges*

Sufficient condition for convergence $P > 0.5\alpha A$, 1

Theorem 10.5

- $P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b$
- $A \in \mathbb{R}^{n\times n}, A = A^T > 0$
- $\alpha > 0$
- $P > 0.5\alpha A$
- $\Downarrow$
- *Errors decrease monotonically in A-norm*
- *Richardson's method converges*

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- $A > 0 \equiv \exists\beta > 0, (Ay, y) > \beta(y, y), \ \forall y \neq 0, y \in \mathbb{R}^n$, $(.,.)$ - inner product

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Sufficient condition for convergence $P > 0.5\alpha A$, 2

Proof.

- $\begin{cases} P\frac{x^{(k+1)}-x^{(k)}}{\alpha} + Ax^{(k)} = b \\ P\frac{x-x}{\alpha} + Ax = b \end{cases} \quad \Rightarrow P\frac{e^{(k+1)}-e^{(k)}}{\alpha} + Ae^{(k)} = 0$

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$$
\begin{aligned}
s_{k+1} = (Ae^{(k+1)}, e^{(k+1)}) = (Ae^{(k)} - \alpha Ay_k, e^{(k)} - \alpha y_k) = \\
(Ae^{(k)}, e^{(k)}) - \alpha(Ae^{(k)}, y_k) - \alpha(Ay_k, e^{(k)}) + \alpha^2(Ay_k, y_k) = \\
(Ae^{(k)}, e^{(k)}) - 2\alpha(Ae^{(k)}, y_k) + \alpha^2(Ay_k, y_k) = \\
s_k - 2\alpha(AA^{-1}Py^{(k)}, y_k) + \alpha^2(Ay_k, y_k) = \\
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Sufficient condition for convergence $P > 0.5\alpha A$, 3

Proof.

- $s_{(k+1)} = s_k - 2\alpha\left((P - 0.5\alpha A)y_k, y_k\right)$

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- $\begin{cases} \|e^{(k)}\|_2 = \|A^{-1}Py_k\|_2 \leq \|A^{-1}P\|_2 \cdot \|y_k\|_2 \\ \lim_{k\to\infty} \|y_k\|_2 = 0 \end{cases} \Rightarrow \lim_{k\to\infty} \|e^{(k)}\|_2 = 0$

$\square$

Convergence SOR, 1

Example 10.7

- $P\frac{x^{(k+1)}-x^{(k)}}{\omega} + Ax^{(k)} = b$

Convergence SOR, 1

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Precondtioning matrices

- Q: How to Build optimal preconditioning matrix?

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- Reduce spectral radius of $(I - \tau P^{-1}A)$?

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What are requirements to preconditioning matrix?

- Reduce spectral radius of $(I - \tau P^{-1}A)$?
- $P^{-1}A$ almost normal, eigenvalues clustered in small region

Precondtioning matrices

How to devise preconditioning matrix?

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- Diagonal preconditioners

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```
[n,m]=size(A);
if n ~= m, error('Only square matrices'); end
for k=1:n-1
    for i=k+1:n,
        if A(i,k) ~= 0
            if A(k,k) == 0, error('Null pivot element'); end
            A(i,k)=A(i,k)/A(k,k);
            for j=k+1:n
                if A(i,j) ~= 0
                    A(i,j)=A(i,j)-A(i,k)*A(k,j);
                end
            end
        end
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Figure: Quarteroni et al.

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Precondtioning matrices

How to devise preconditioning matrix?

- Diagonal preconditioners $p_{ii} = \sum_{j=1}^{n} |a_{ij}|$
- ILU factorization, for sparse matrices

```
[n,m]=size(A);
if n ~= m, error('Only square matrices'); end
for k=1:n-1
    for i=k+1:n,
        if A(i,k) ~= 0
            if A(k,k) == 0, error('Null pivot element'); end
            A(i,k)=A(i,k)/A(k,k);
            for j=k+1:n
                if A(i,j) ~= 0
                    A(i,j)=A(i,j)-A(i,k)*A(k,j);
                end
            end
        end
    end
end
```

Figure: Quarteroni et al.

- Von Neuman method $P^{-1} = p(A)$, $A = (I - CD^{-1})D$
  $A^{-1} = D^{-1}(I - CD^{-1})^{-1} = D^{-1}(I + CD^{-1} + (CD^{-1})^2 + ...)$

Q & A