Exercises below are your homework; they will be discussed during exercise classes. Problems marked with a (*) are more challenging.

## Week 6

1. Show that Predicate $x \mid y$, that is a function $\mid : \mathbb{N}_0 \times \mathbb{N}_0 \to \{\text{true}, \text{ false}\}$, given by $x \mid y = \text{true}$ if and only if $x$ divides $y$ is primitive recursive.

2. In the Exercise for Week 5, you showed that a primitive recursive function $f$, the functions of *bounded sum* and *bounded product* of $f$, respectively given by

$$\text{bsum}_f(x_0, \ldots, x_{k-1}, y) = \sum_{i=0}^{y} f(x_0, \ldots, x_{k-1}, i), \; \text{bprod}_f(x_0, \ldots, x_{k-1}, y) = \prod_{i=0}^{y} f(x_0, \ldots, x_{k-1}, i)$$

are also primitive recursive.

Now, conclude (by proving) that primitive recursive predicates are closed under *bounded quantification.* That is, show that if $P$ on $\mathbb{N}_0^{k+1}$ is a primitive recursive predicate, then so are the predicates $\forall z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ and $\exists z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$.[1]

3. (*) In the lecture we sketched a proof that not every computable function is primitive recursive.
   a) Where does this proof fail for $\mu$-recursive functions?
   b) Where does this proof fail if we only consider total (defined everywhere) $\mu$-recursive functions?

Note that the set of $\mu$-recursive functions, as well as its proper subset of total $\mu$-recursive function are countably infinite.

4. What follows are the exercises from the lecture on Turing Machine. See the mentioned lecture for precise definitions.

Construct a Turing machine for

(a) "decrementing binary numbers." That is, given an input of a binary number $\text{bin}(n)$, it gives an output $\text{bin}(n-1)$;

(b) "concatenating tape inscriptions." That is two machines, one giving tape1 = tape1#tape2 and the other tape1 = tape2#tape1.

(c) "head and tail of tapes." That machines giving tape2 = head(tape1) and tape2 = tail(tape1).

[1] Here, $\forall z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ is an $\mathbb{N}_0^{k+1}$ predicate which is true for $(a_0, \ldots, a_{k-1}, a_k)$ if and only if $P(a_0, \ldots, a_{k-1}, a_k) = \text{true}$ for all $a_k \leq y$; and $\exists z \leq y(P(x_0, \ldots, x_{k-1}, z) = \text{true})$ is an $\mathbb{N}_0^{k+1}$ predicate which is true for $(a_0, \ldots, a_{k-1}, a_k)$ if and only if $P(a_0, \ldots, a_{k-1}, a_k) = \text{true}$ for some $a_k \leq y$.