Heap Sort and Insertion Sort

towards data structures

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- you know the (unoptimized) translation process
    - C0 compiler

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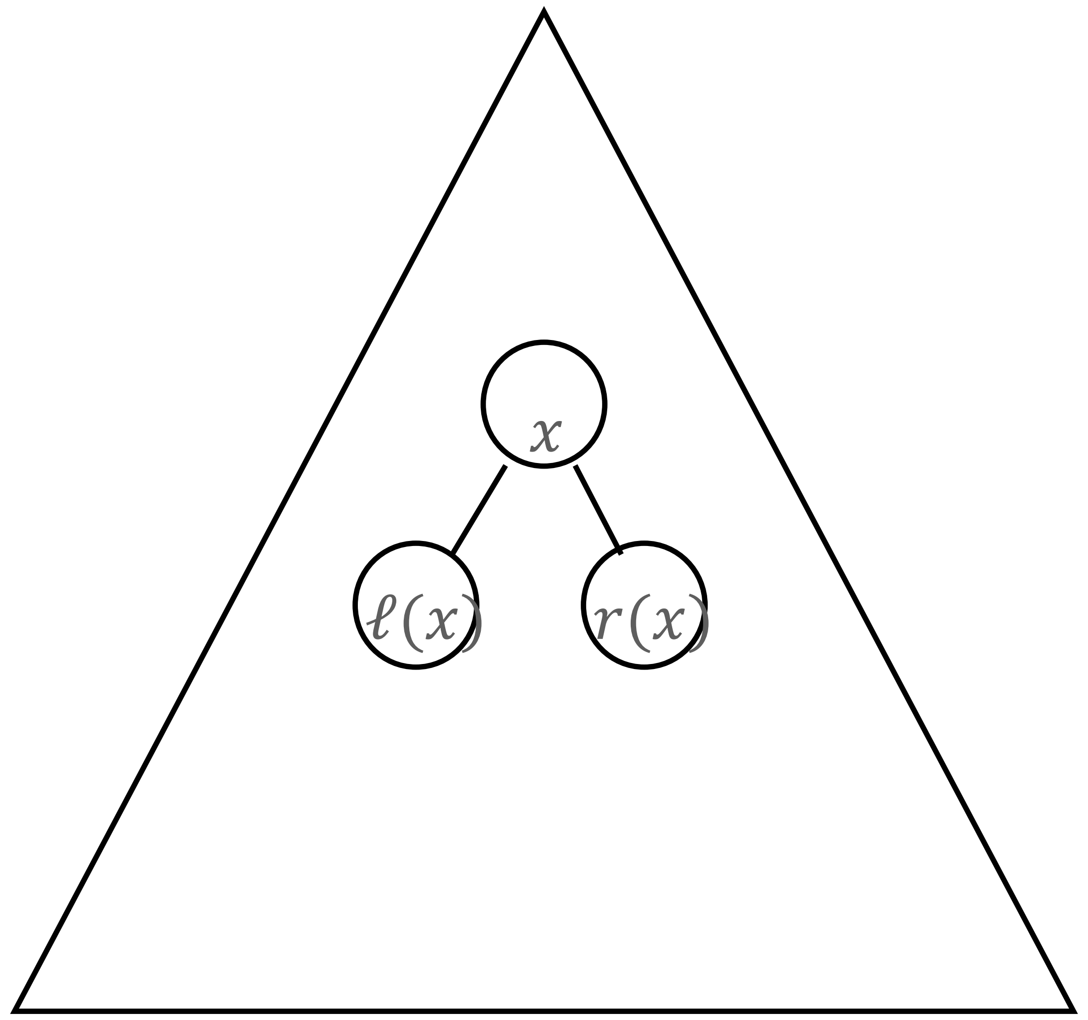
2 more sorting algorithms with O(n log n) comparisons

- so far
    - O(1) ISA instructions per comparison
    - 2 arrays of length n + …
- heap sort
    - O(1) ISA instructions/comparison
    - 1 array; only swaps in place
- insertion sort
    - O(n) ISA instructions/comparison with arrays
    - future: data structures; search trees…

storing the elements of a binary tree in an array.



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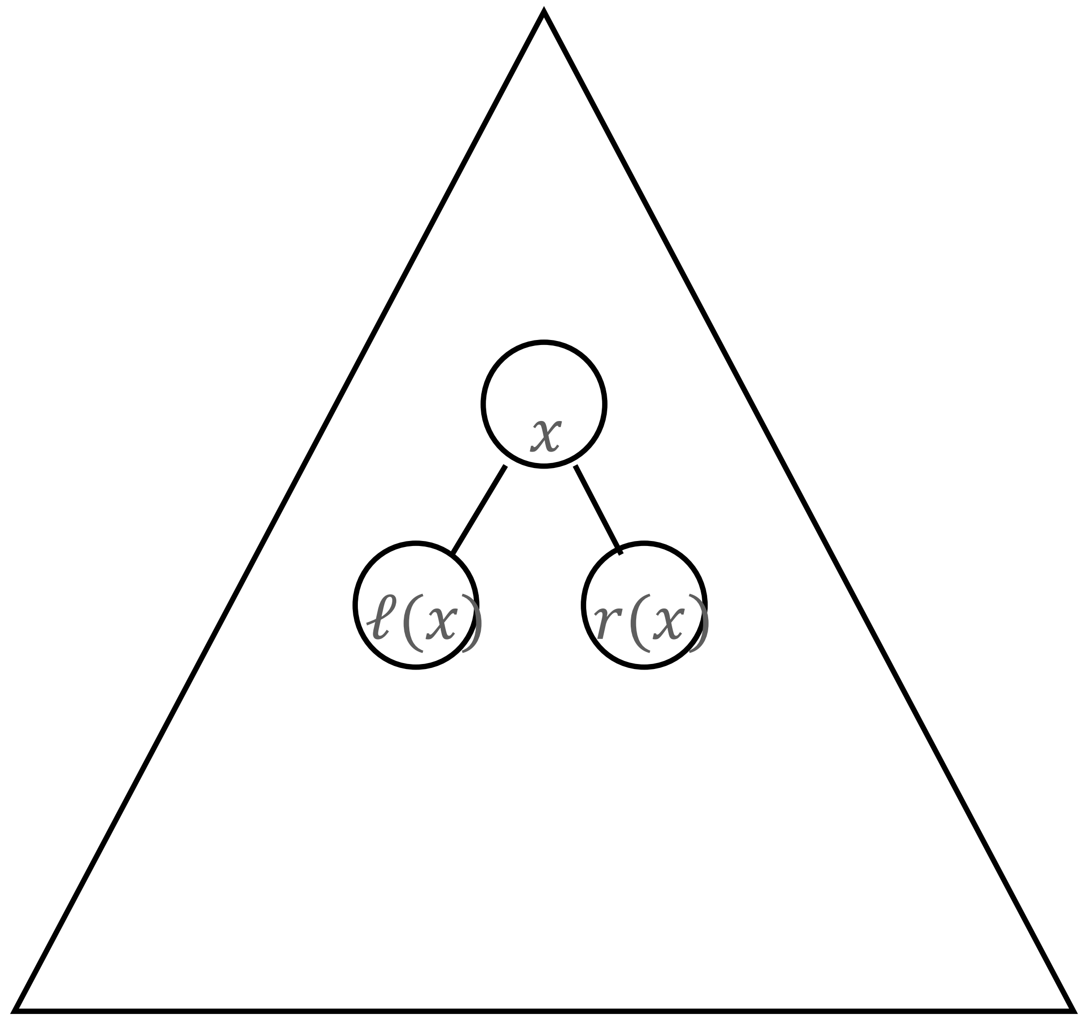

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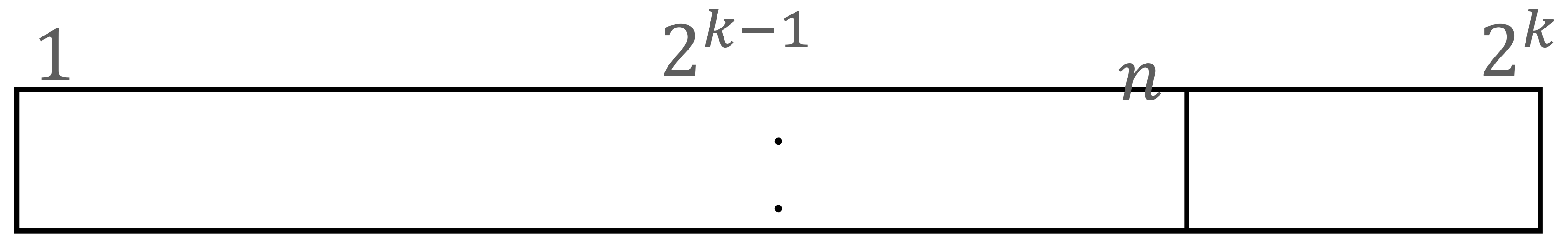
implement in array A of length $2^k = 2^{\lceil \log n \rceil}$



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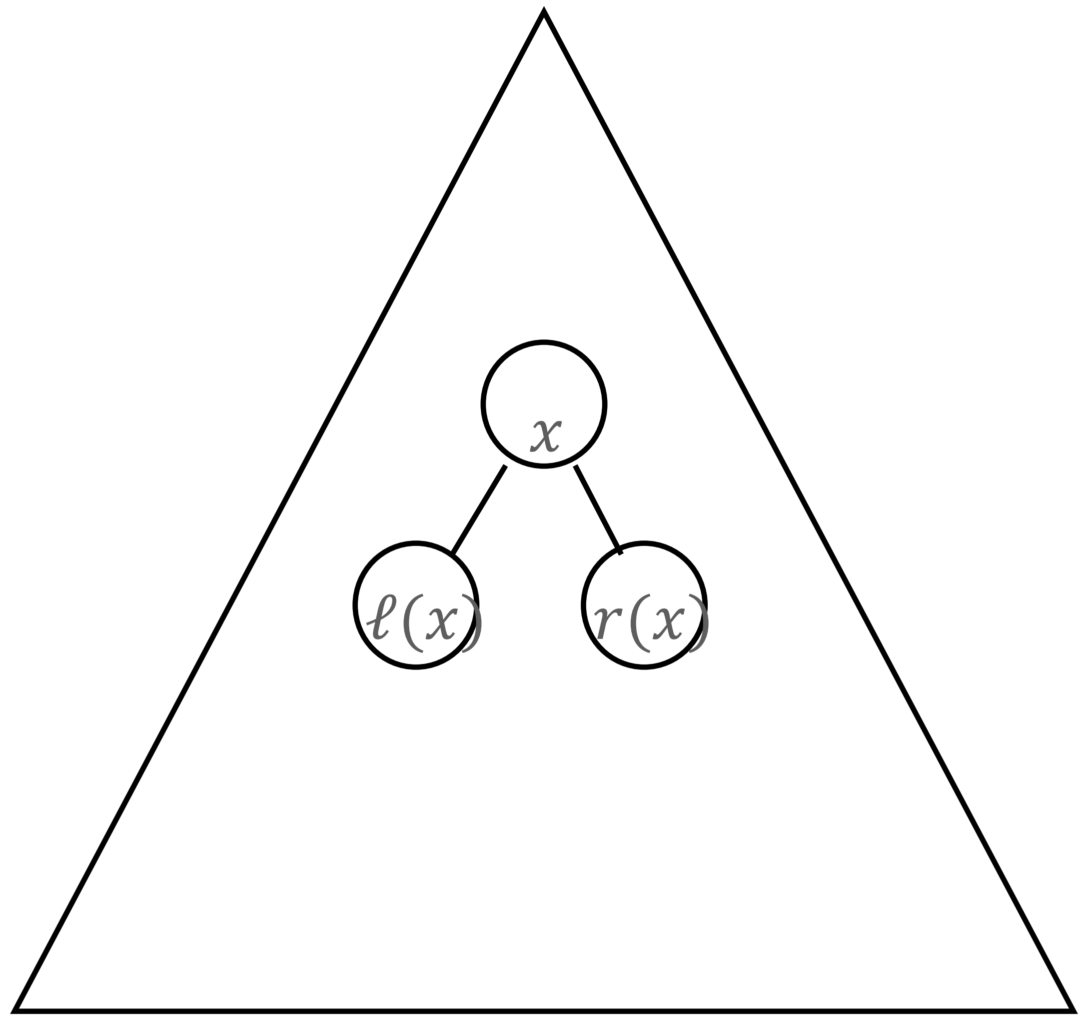

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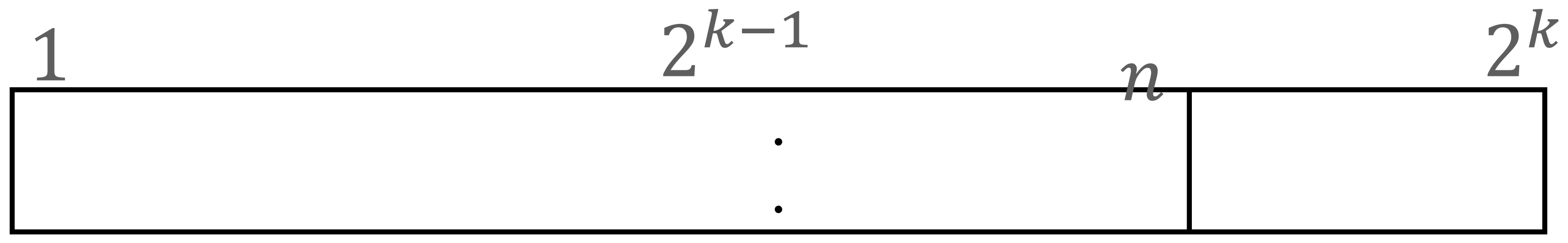

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\begin{aligned}
root &= 1 \\
\ell(x) &= 2x \quad \text{left son} \\
r(x) &= 2x+1 \quad \text{right son} \\
p(x) &= \cancel{\lfloor 2/x \rfloor} \quad \text{parent} \qquad \lfloor x/2 \rfloor
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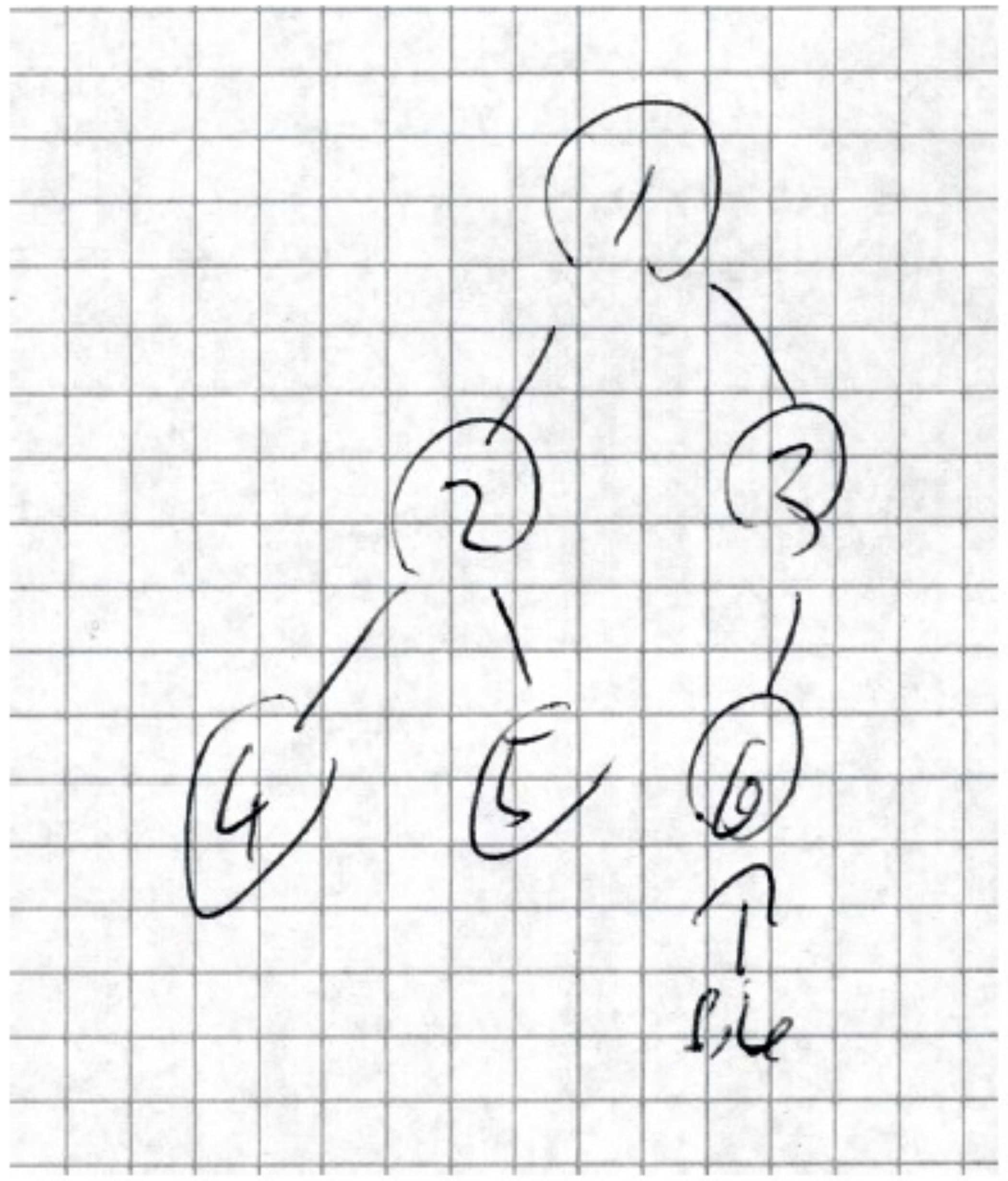

order of nodes in array

1. top to bottom
2. left to right

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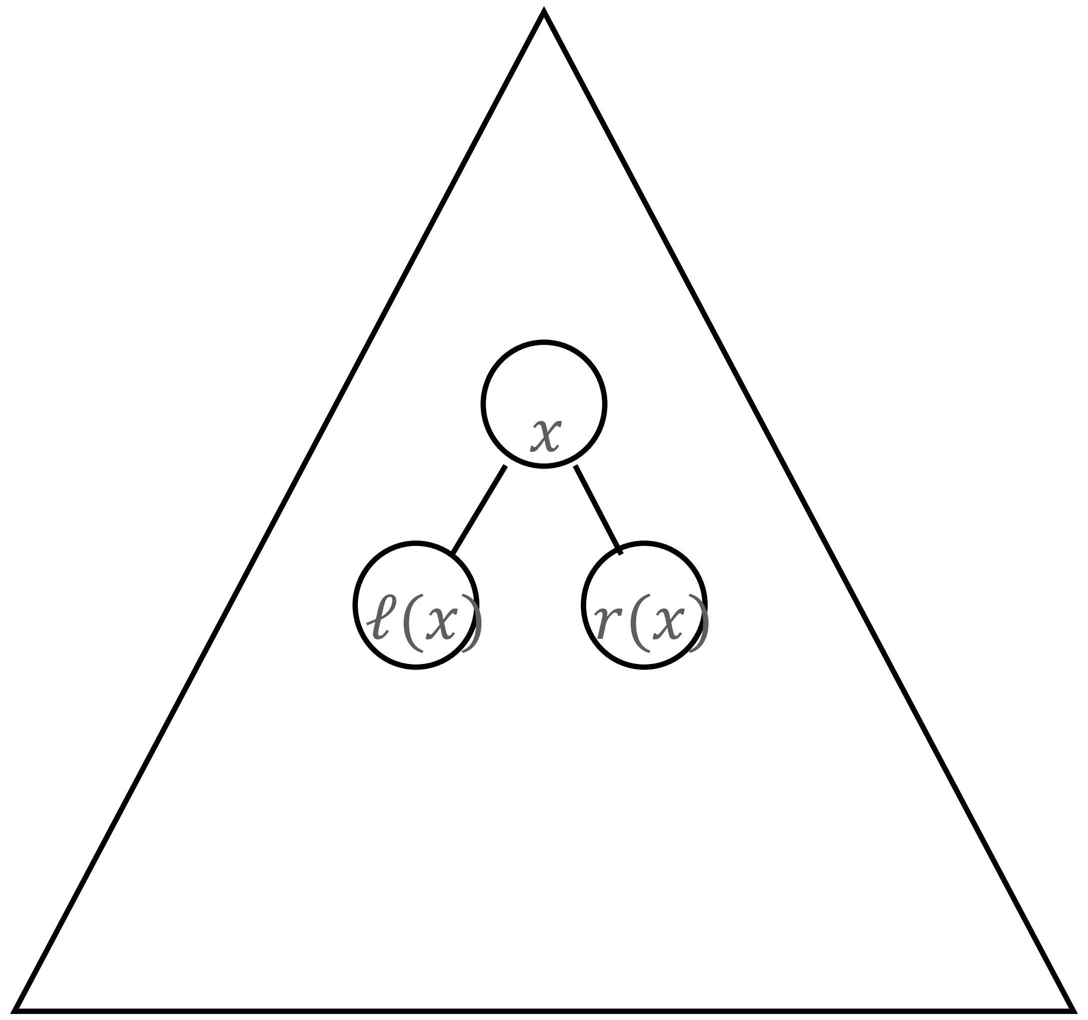

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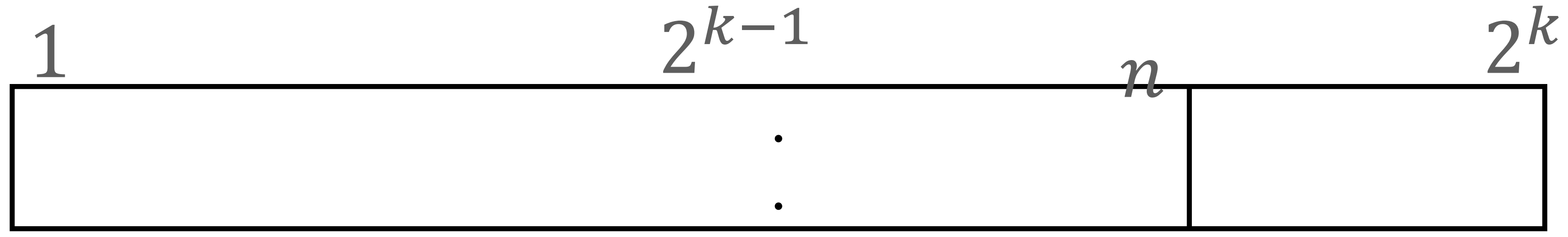

complete binary tree
except right end of bottom layer

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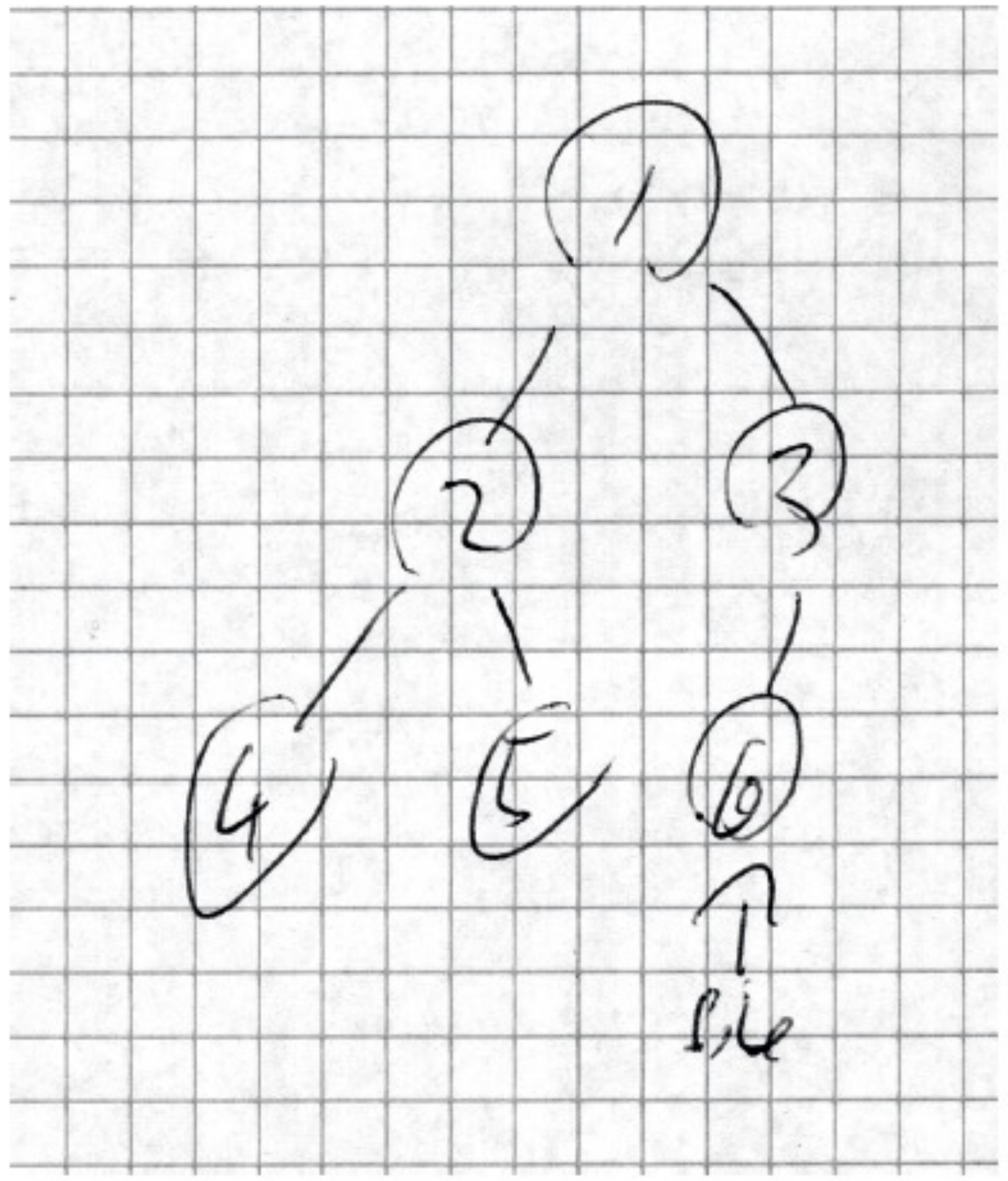

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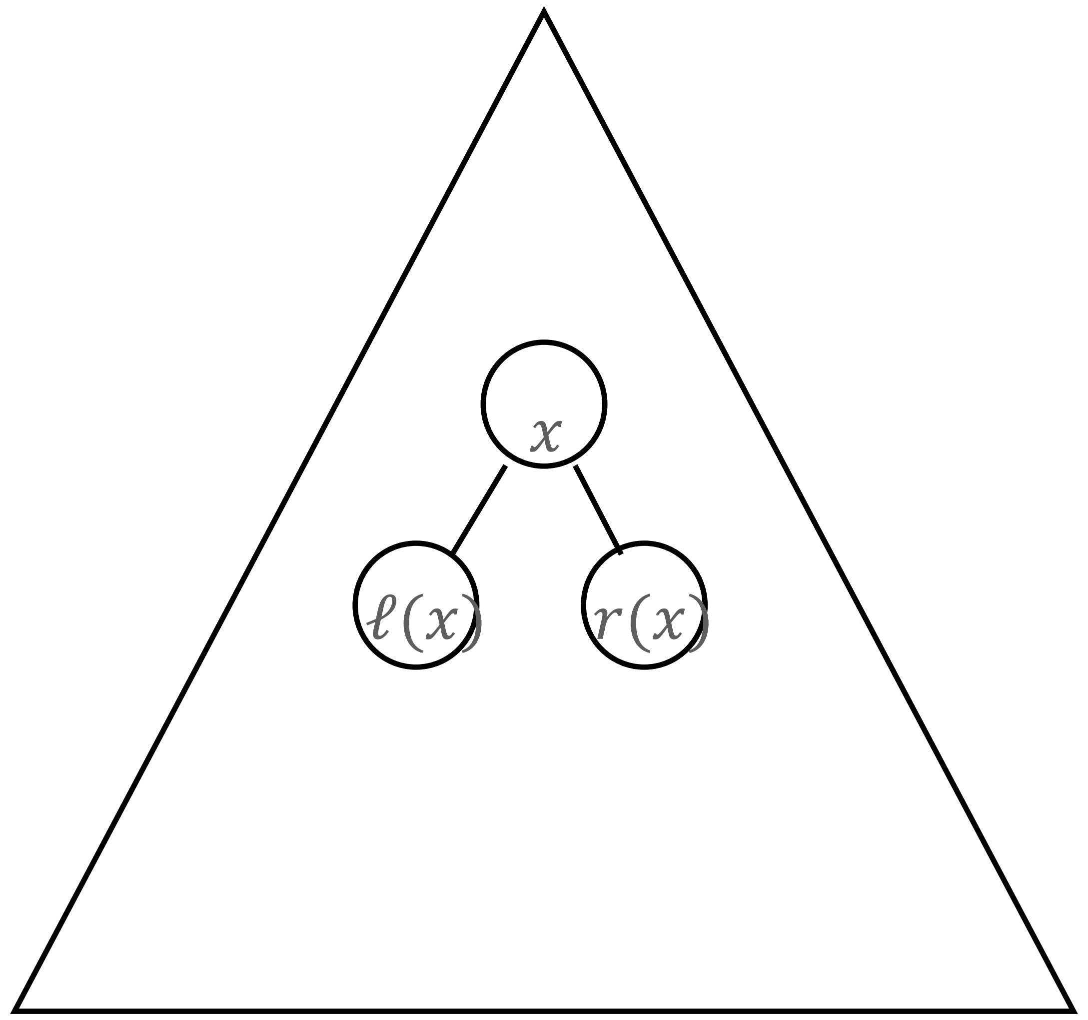

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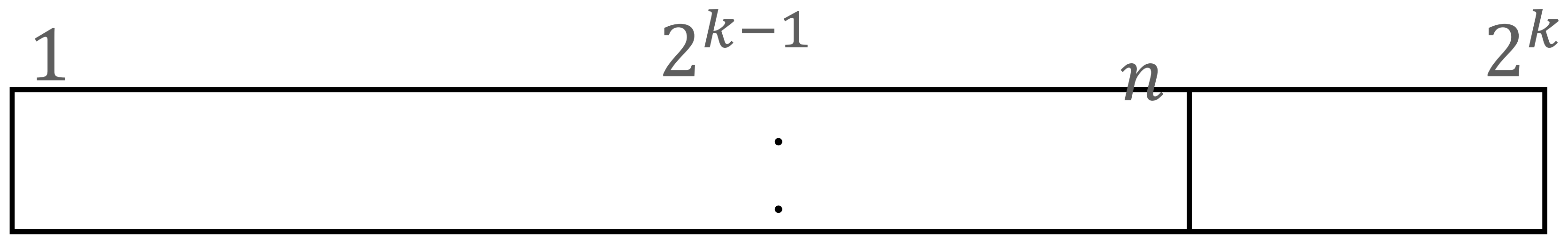

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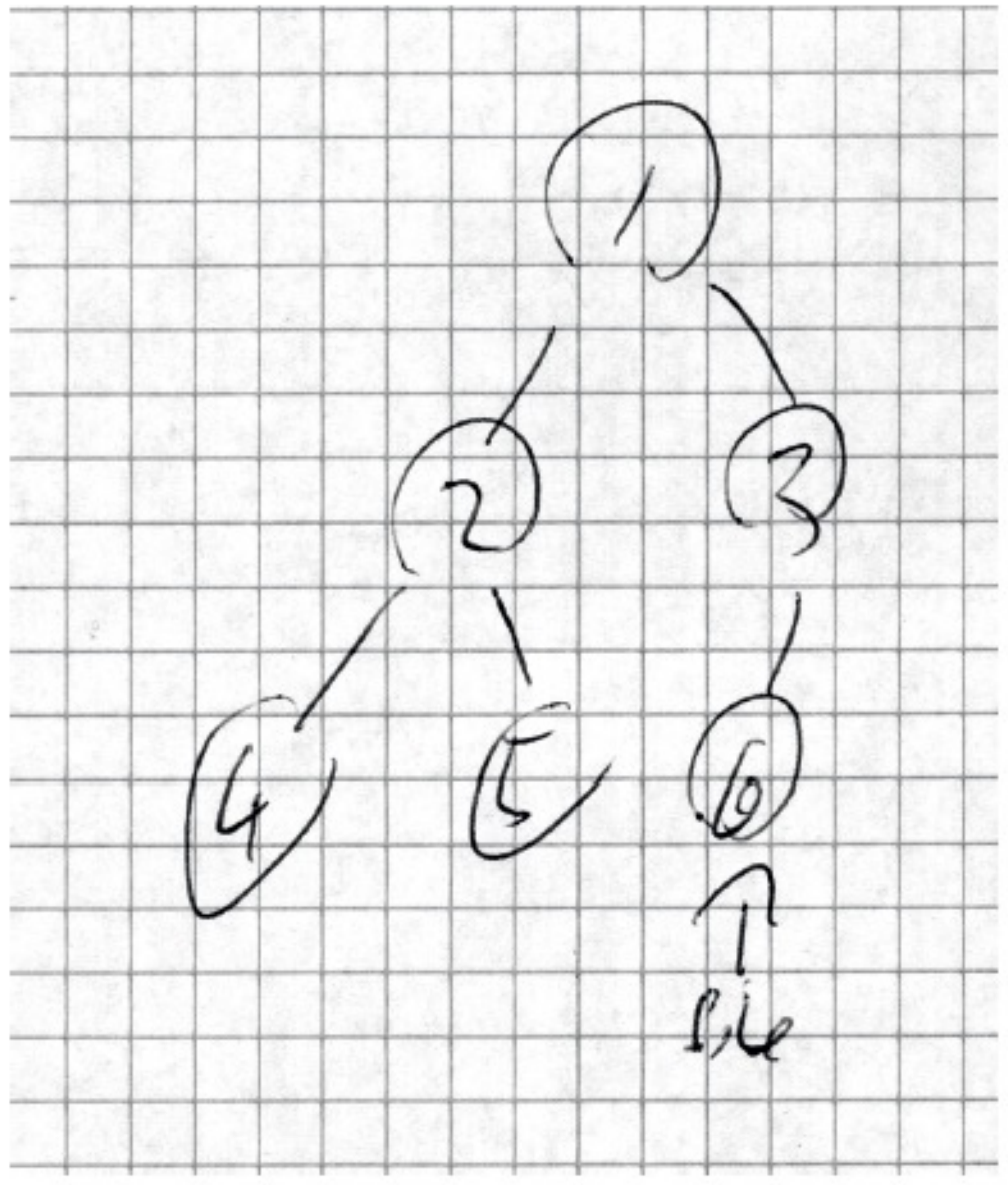
$\lfloor x/2 \rfloor$



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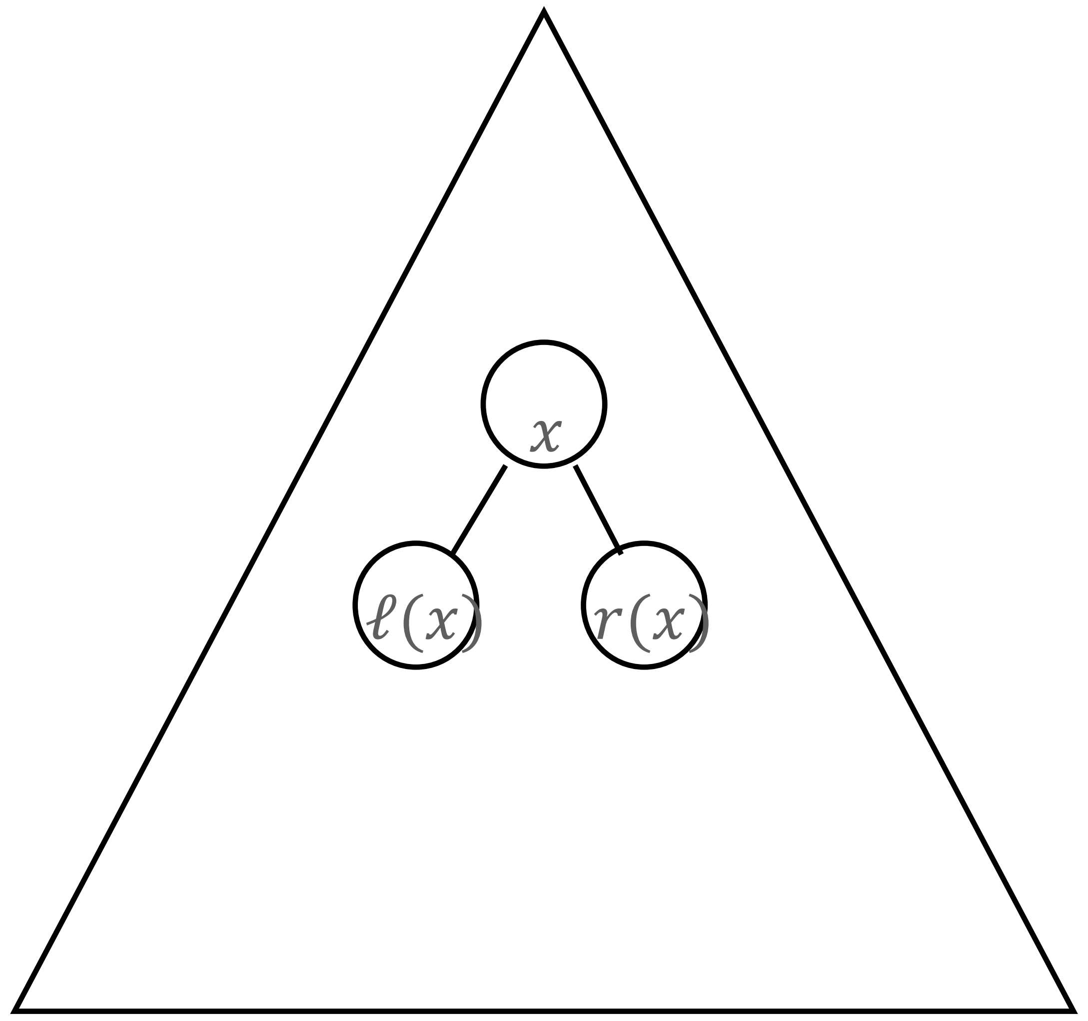

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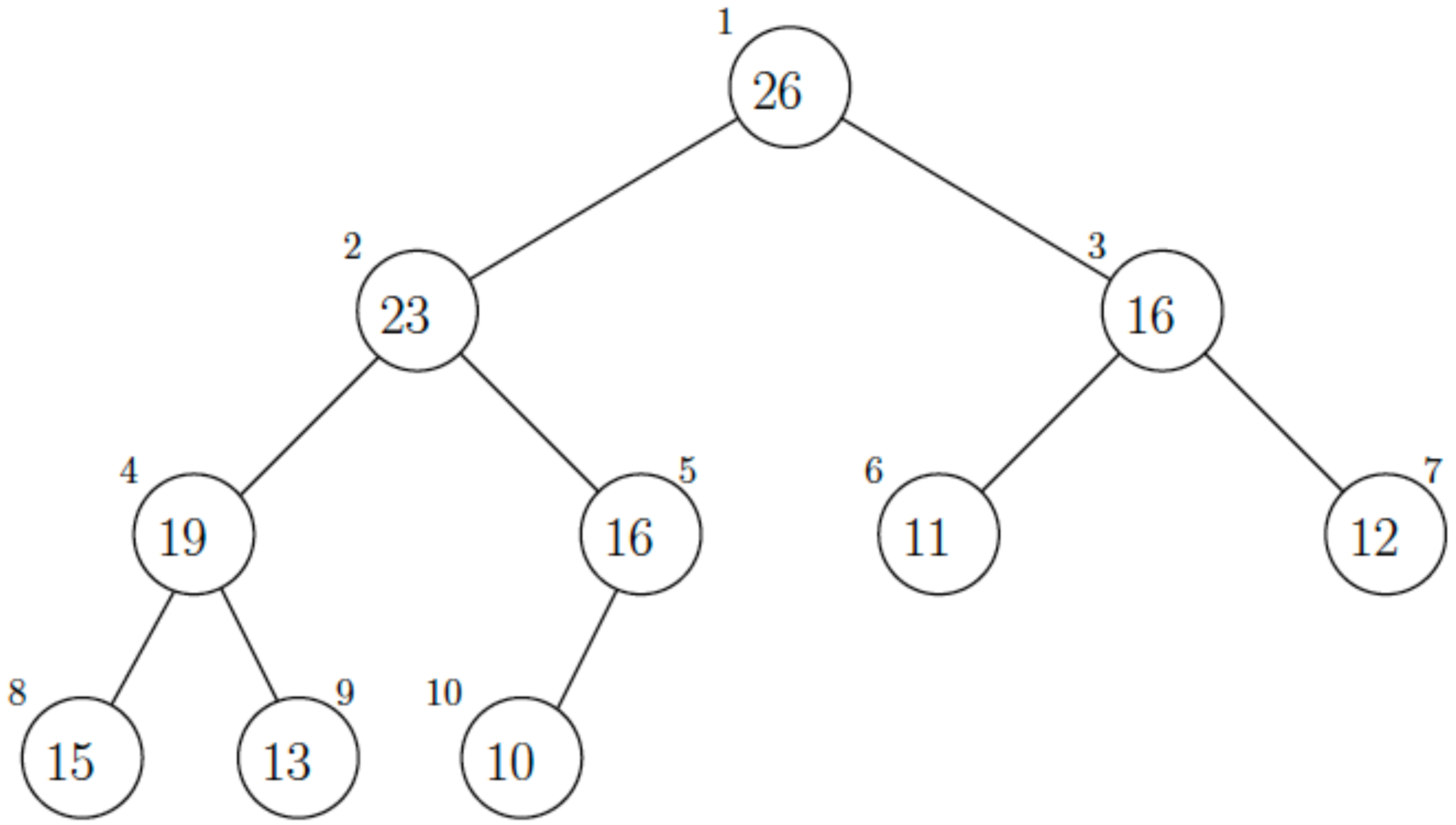
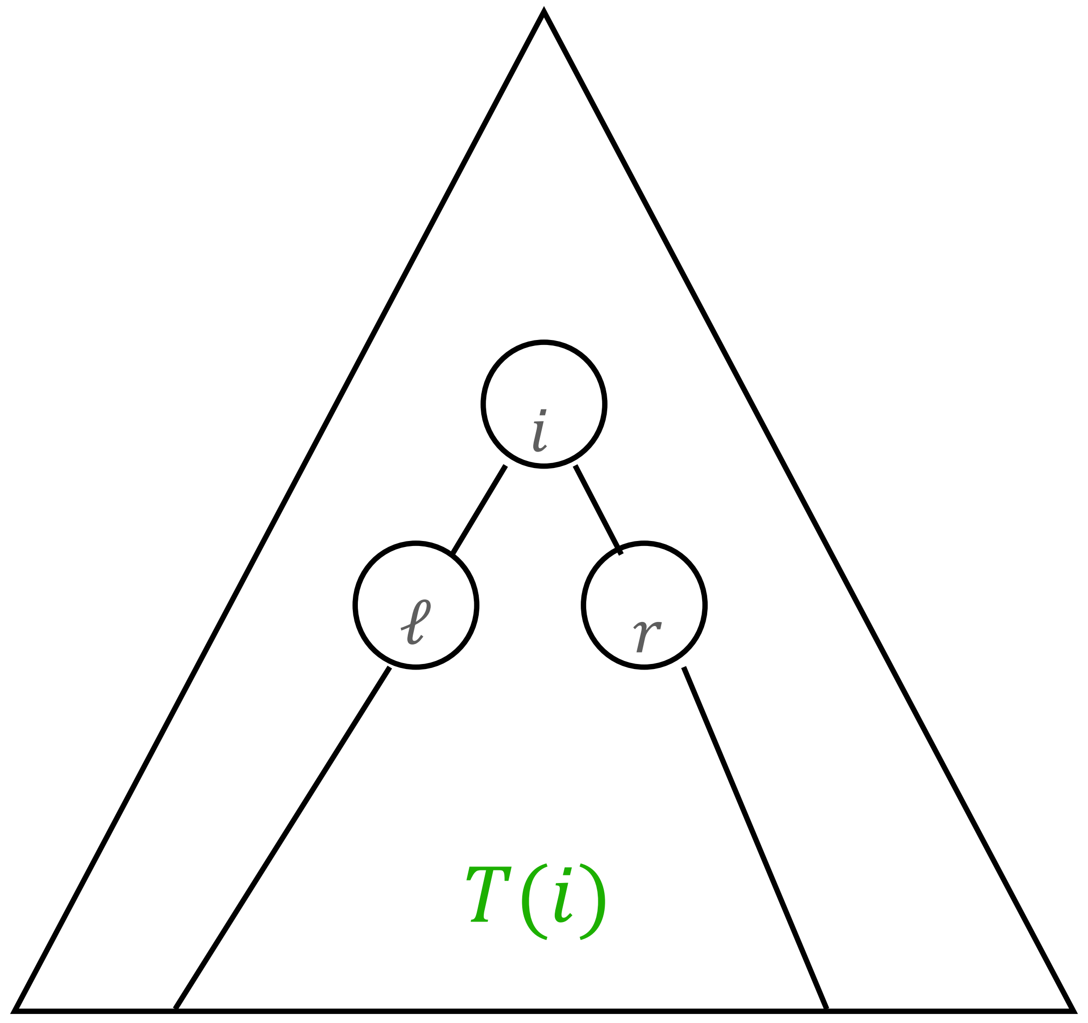


Figure 3.1: A heap. The large numbers in the circles are the elements stored in the heap. The small numbers next to the circle is the corresponding position in the array.

establishing heap property



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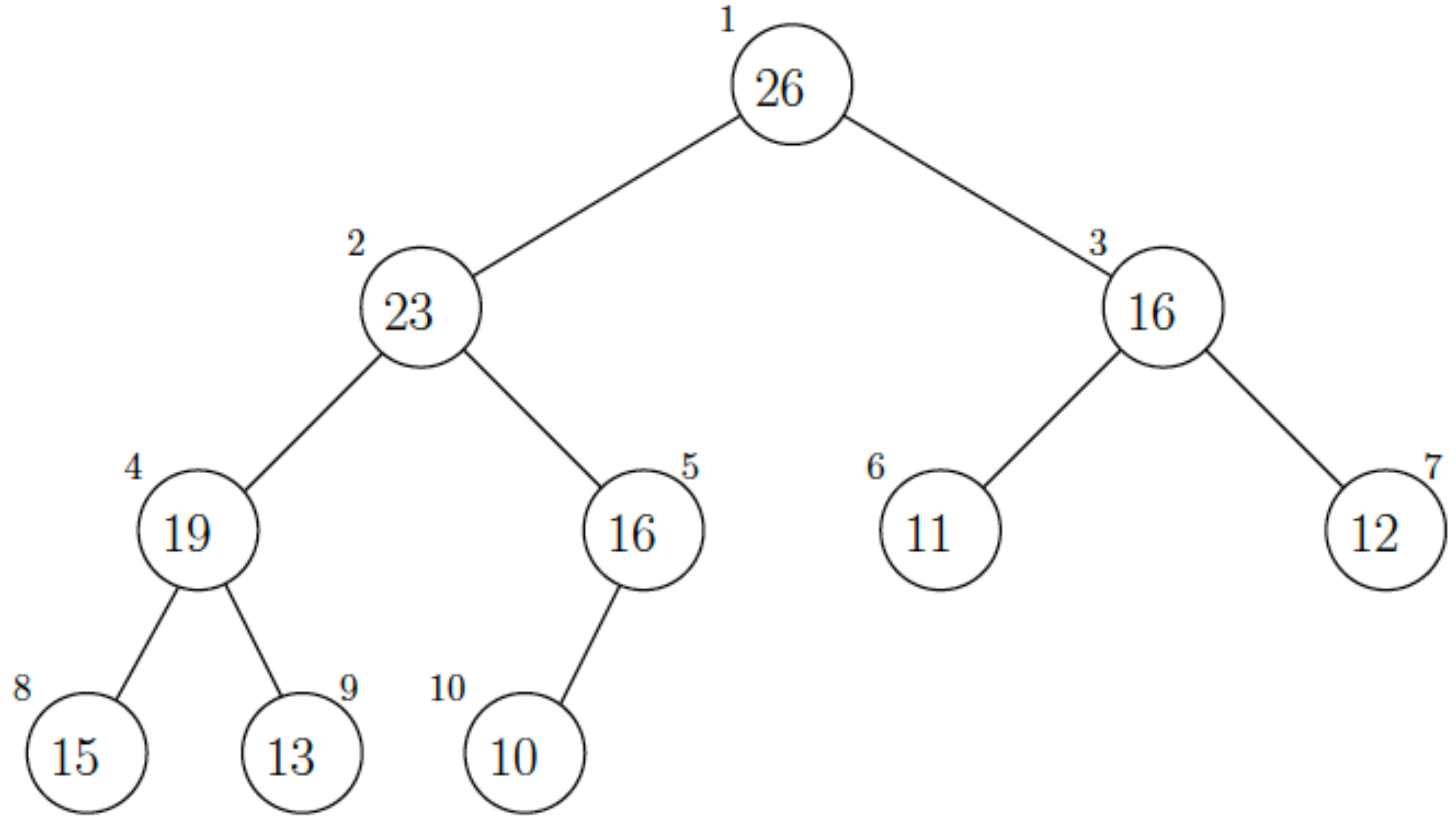

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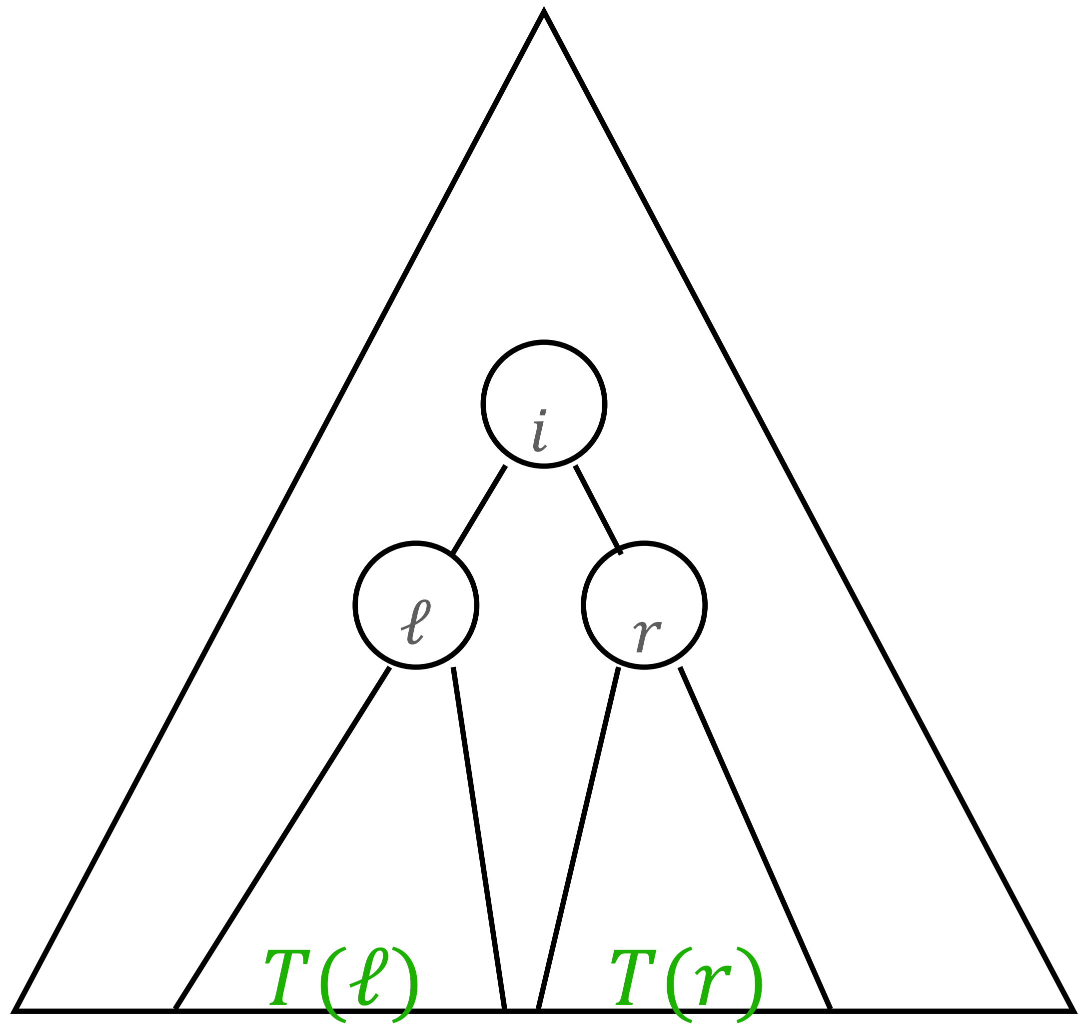

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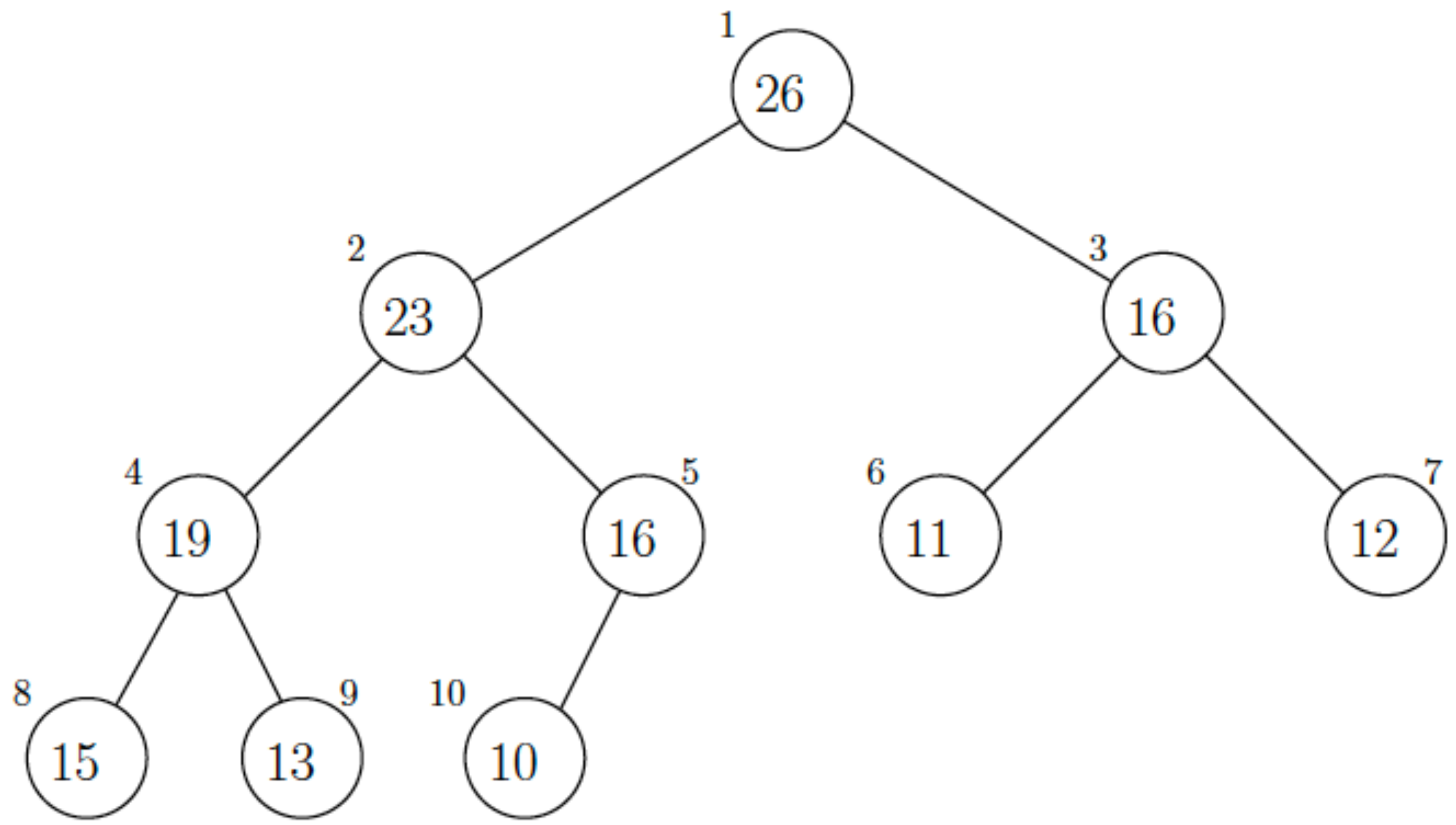

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function heapify(A,i)

Input: A, i.

- i: interior node with sons $r = r(i)$, $ell = \ell(i)$
- subtrees $T(\ell), T(r)$ fulfill heap property

Output: afterwards $T(i)$ fulfills heap property.

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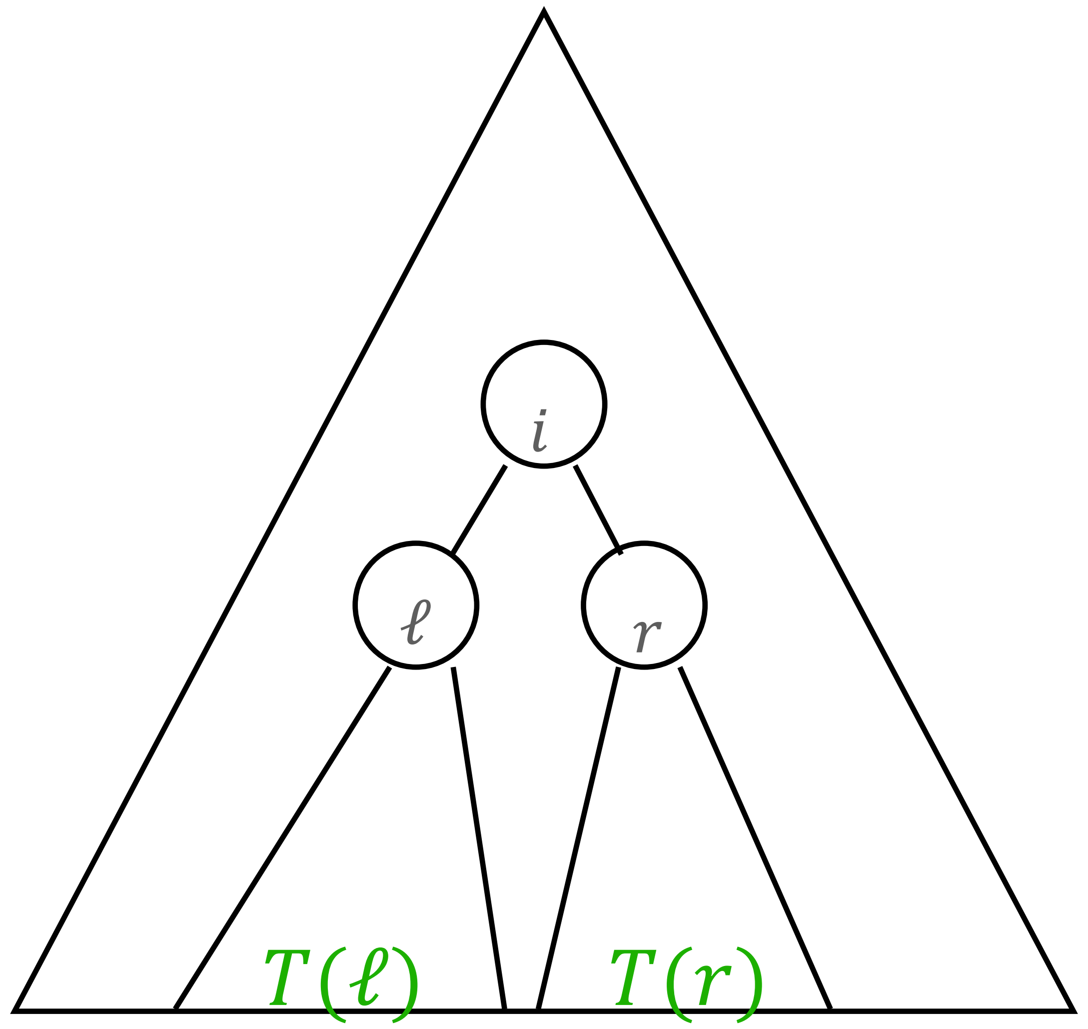

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h !=i: swap(A(i), A(h)) /* y=A(i);A(i)=A(h);A(h)=y*/
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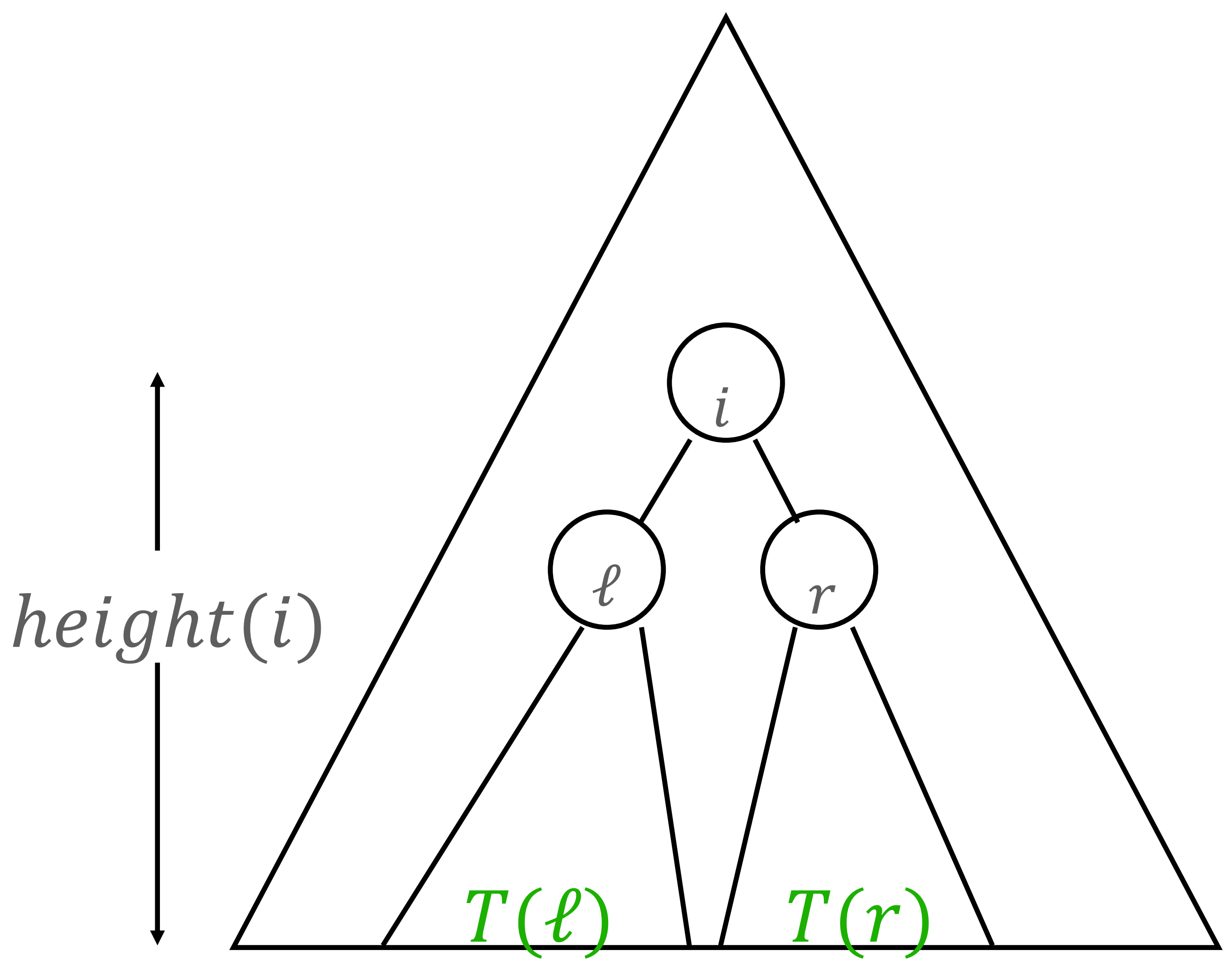

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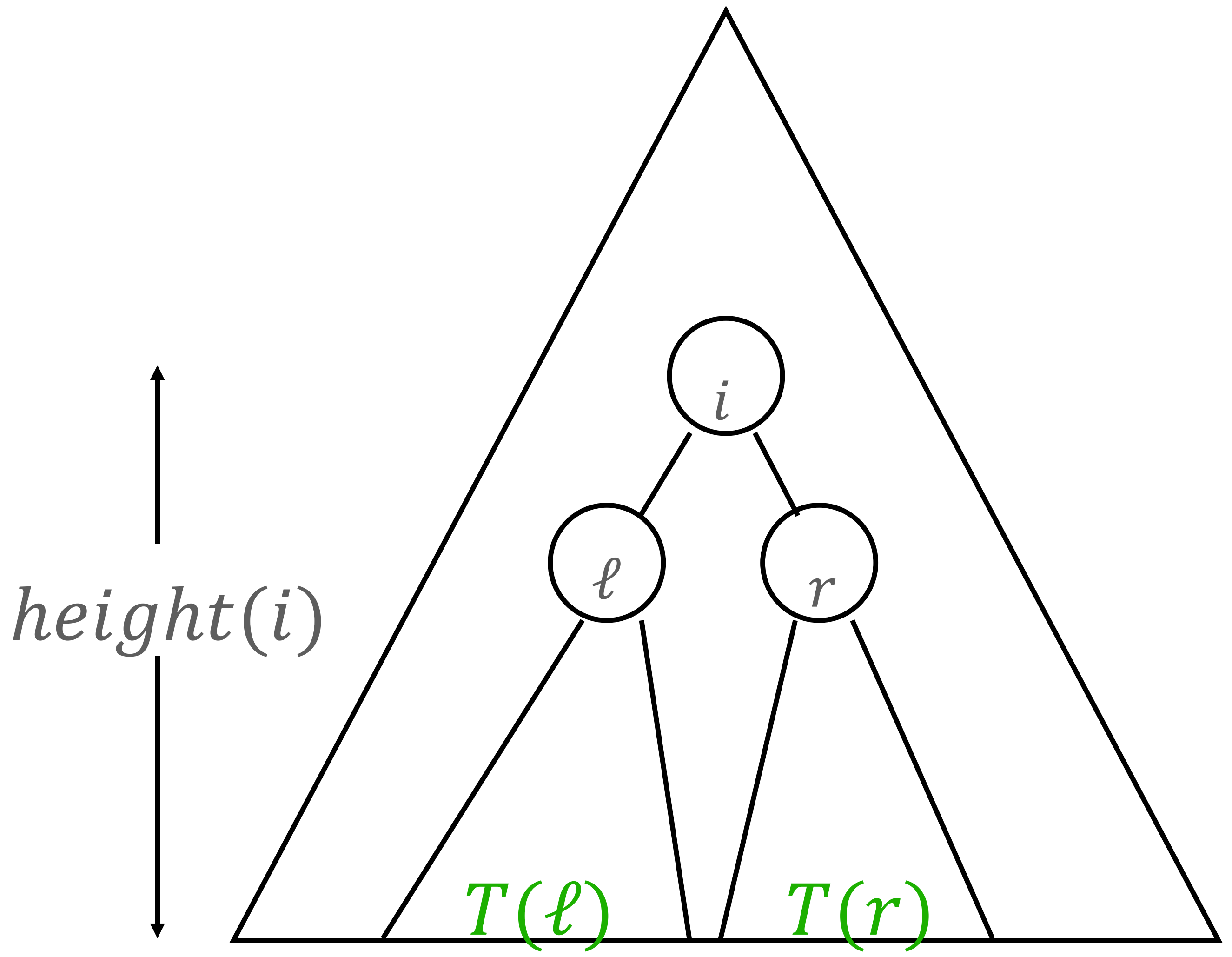
run time: O(height(i))

building a heap from scratch

Algorithm 11 Build-heap
Input: array $A[1..n]$
Output: afterwards, A satisfies the heap property
1: ~~*heap-size* := *n*~~
2: **for** $i = \lfloor n/2 \rfloor, \ldots, 1$ **do**
3: Heapify(A, i)

establishing heap



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$\lceil \frac{n}{2} \rceil$ nodes are leaves $\Rightarrow$ $x = 0 \xrightarrow{\text{at most}} \lceil \frac{n}{2} \rceil$ subtrees

$x = 1 \xrightarrow{\text{at most}} \lceil \frac{n}{4} \rceil$ subtrees

$\vdots$

runtime $t(n)$: $\left\lceil \frac{n}{2^{x+1}} \right\rceil$

For each height x at most $\cancel{n/2^x}$ subtrees of height x

height(n) = $\lceil \log n \rceil$

$$
\begin{aligned}
t(n) &= O\Big(\sum_{x=0}^{\lceil \log n \rceil} \frac{nx}{2^{x+1}}\Big) \\
&< n \cdot O\Big(\sum_{x=1}^{\lceil \log n \rceil} \frac{x}{2^x}\Big) \\
&< n \cdot O\Big(\sum_{x=1}^{2} \frac{x}{2^x} + \sum_{x=3}^{\infty} \frac{x}{2^x}\Big) \\
&< n \cdot \Big(O(1) + O\big(\sum_{x=3}^{\infty} (\tfrac{3}{4})^x\big)\Big) \\
&= n \cdot O(1) \quad \text{(convergent geometric series)}
\end{aligned}
$$

$\frac{x}{2^x} = \frac{2^x \cdot x}{4^x} < \frac{3^x}{4^x}$

heap sort

Algorithm 12 Heap sort

Input: array $A[1..n]$

Output: afterwards, A is sorted

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Build-heap(A)
for i := n, ..., 2 do
    Swap(A[1], A[i])
    heap-size := heap-size − 1
    ~~Heapify(A, 1)~~    /* A[i..n] IS SORTED AND CONTAINS THE n − i + 1
    LARGEST ELEMENTS */
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$heapify(A[1: heap - size], 1)$

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$$\boxed{T(n) = O(n\log n)}$$

binary locate

an easy variant of binary search

Algorithm 1 ~~Binary search~~ binary-locate(a,x)

Input: Sorted array $a[1..n]$, $a[1] < a[2] < \cdots < a[n]$, element x $\quad x \notin A = \{a[1], \ldots, a[n]\} \quad a[1] < x < a[n]$

Output: $\begin{cases} \cancel{m \quad \text{if there is an } 1 \le m \le n \text{ with } a[m] = x} \\ \cancel{-1 \quad \text{otherwise}} \end{cases}$ $\quad i$ with $\quad a[i] < x < a[i+1]$

$\ell := 0$; $r := n + 1$;
while $\ell + 1 < r$ **do** /* $0 \le \ell < r \le n+1$ AND $a[\ell] < x < a[r]$ */
  $m := \lfloor \frac{\ell + r}{2} \rfloor$;
  if $a[m] = x$ **then**
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return -1;

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input: $a[1:n]$ as above sorted, $x \notin A$

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insert-array(a,x):

if x < a[1] {output (x, a[1],..., a[n])}
if x > a[n] {output (a[1],..., a[n],x)}
i = binary-locate(a,x);
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insertion-sort

input $((a[1], \ldots a[n])$ pairwise different.
output: sorted sequence $b = (b[1], \ldots, b[n])$

```
b=(a[1]);
for i = 2 to n
{b = insert-array(a[i], b}
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O(n log n) comparisons, time $O(n^2)$

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shifting

time O(n) (with array)

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n times n-shifting

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insertion sort

Algorithm 1 ~~Binary search~~ binary-locate(a,x)

Input: Sorted array $a[1..n]$, $a[1] < a[2] < \cdots < a[n]$, element x $x \notin A = \{a[1], \ldots, a[n]\}$ $a[1] < x < a[n]$

Output: $\cancel{\begin{cases} m & \text{if there is an } 1 \le m \le n \text{ with } a[m] = x \\ -1 & \text{otherwise} \end{cases}}$ i with $a[i] < x < a[i+1]$

$\ell := 0$; $r := n + 1$;
while $\ell + 1 < r$ **do** /* $0 \le \ell < r \le n+1$ AND $a[\ell] < x < a[r]$ */
  $m := \lfloor \frac{\ell + r}{2} \rfloor$;
  ~~**if** $a[m] = x$ **then**~~
    ~~return m;~~
  if $a[m] < x$ **then**
    $\ell := m$
  else
    $r := m$;
return ~~-1;~~ ℓ;

$\boxed{\text{log n comparisons}}$

binary insert

input: $a[1:n]$ as above sorted, $x \notin A$

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insert-array(a,x):

if x < a[1] {output (x, a[1],..., a[n])}
if x > a[n] {output (a[1],..., a[n],x)}
i = binary-locate(a,x);
output (a[1],...,a[i],x,a[i+1],..., a[n])
```

time O(n) (with array)

insertion-sort

input $((a[1], \ldots a[n])$ pairwise different.
output: sorted sequence $b = (b[1], \ldots, b[n])$

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b=(a[1]);
for i = 2 to n
{b = insert-array(a[i], b}
```

O(n log n) comparisons, time $O(n^2)$

let's use balanced trees instead of arrays for this!