Central Exercise 4

October 2023

True or False

(a) The eigenvalues of an upper triangular matrix T are its diagonal entries.

(b) The eigenvalues of a real symmetric matrix are real.

(c) A matrix is nonsingular if and only if all its eigenvalues are nonzero.

(d) The eigenvalues of an orthogonal matrix are all equal to 1.

(e) An orthogonal matrix is not necessarily invertible.

(f) Two similar matrices have the same eigenvalues.

(g) The product of two upper (lower) triangular matrices does not need to be an upper (lower) triangular matrix.

(h) The length of a vector is preserved by an orthogonal multiplication.

(i) If $\|A\|_2 = 1$, then A must be orthogonal.

(j) The product of two orthogonal (unitary) matrices is an orthogonal (unitary) matrix.

(k) A symmetric matrix must be well-conditioned.

Condition Numbers

1. How are $\text{Cond}_2(A)$ and $\text{Cond}_2(A^{-1})$ related?
2. Show that $\text{Cond}(cA) = \text{Cond}(A)$ for any given norm.

Orthogonal Matrices

Show that if A is a scalar multiple of an orthogonal matrix $\text{Cond}_2(A) = 1$.

Properties of Matrix Norms

Let $\|x\| : \mathbb{R}^{n\times n} \to \mathbb{R}$, defined by $\|A\| = \max \left\|Ax\right\|_{\|x\|=1}$, be the matrix-norm induced by the vector-norm $\|\cdot\| : \mathbb{R}^{n\times n} \to \mathbb{R}$. Show:

(a) The compatibility, i.e., $\|Ay\| \leq \|A\| \cdot \|y\|$ for all $A \in \mathbb{R}^{n\times n}$ and $y \in \mathbb{R}^n$.

(b) The sub-multiplicativity, i.e., $\|AB\| \leq \|A\| \cdot \|B\|$ for all $A, B \in \mathbb{R}^{n\times n}$.

Matrix Products

Prove that for a given norm, $\text{Cond}(AB) < \text{Cond}(A) \cdot \text{Cond}(B)$.