Problem 1.2.2.1:

prove that $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

Proof:

$$n = 1, \quad 1^3 = \frac{1^2(1+1)^2}{4} = \frac{1 \cdot 4}{4} \tag{1}$$

$$\begin{aligned}
\frac{(n-1)^2 n^2}{4} + n^3 = \frac{n^2(n+1)^2}{4} \quad &\Longrightarrow \\
\frac{(n-1)^2 n^2 + 4n^3}{4} = \frac{n^2(n+1)^2}{4} \quad &\Longrightarrow \\
(n-1)^2 + 4n = (n+1)^2 \quad &\Longrightarrow \\
n^2 - 2n + 1 + 4n = n^2 + 2n + 1 \quad &\Longrightarrow \\
n^2 + 2n + 1 = n^2 + 2n + 1 \quad &\Longrightarrow \\
\boxed{0 = 0} &
\end{aligned} \tag{2}$$

Problem 1.2.2.3:

prove that $n(n+1)(n+2)$ is divisible by 6

Proof:

$$n = 1, \quad 1 \cdot (1+1) \cdot (1+2) = 0 \ mod \ 6 \tag{3}$$

$$\begin{aligned}
n(n+1)(n+2) - 6k &= (n-1)(n)(n+1) \\
n(n+1)(n+2) - n(n-1)(n+1) &= 6k \\
n(n+1) \cdot ((n+2) - (n-1)) &= 6k \\
3n(n+1) &= 6k \\
n(n+1) &= 2k
\end{aligned} \tag{4}$$

Problem p:

rove that $n(n+1) = 2k$ where $k \in \mathbb{N}$

$$1(1+1) = 2 \tag{5}$$

$$\begin{aligned} n(n+1)+2k &= (n+1)((n+1)+1) \\ n(n+1)-(n+1)(n+2) &= -2k \\ (n+1)\cdot(n-(n+2)) &= -2k \\ (n+1)\cdot -2 &= -2k \\ n+1 &= k \end{aligned} \tag{6}$$

Problem 1.2.2.4:

prove $3^{2n}-1$ is divisible by 8

$$9^1 - 1 = 0 \; mod \; 8 \tag{7}$$

$$\begin{aligned} 9^n - 1 + 8k &= 9^{n+1} - 1 \\ 9^n + 8k &= 9 \cdot 9^n \\ 9^n - 9 \cdot 9^n &= -8k \\ -8 \cdot 9^n &= -8k \\ 9^n &= k \end{aligned} \tag{8}$$

$$adfva \begin{aligned} 2^k &> (k-1)^2 \\ 2 \cdot 2^k &> k^2 \end{aligned} \tag{9}$$

$$\begin{aligned} 2 \cdot (k-1)^2 &> k^2 \\ 2k^2 - 4k + 2 &> k^2 \\ 4k - 2 &< k^2 \end{aligned} \tag{10}$$

$$f : \mathbb{N} \to \mathbb{Z}, \; x \mapsto \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ \frac{x-1}{2} & \text{if } x \text{ is odd} \end{cases}$$