expected run time of quicksort

run time = number of comparisons

product of probability spaces

$A, B \subseteq S$ independent iff $p(A \cap B) = p(A) \cdot p(B)$

Lemma 1. *$W_1 \times W_2$ is a probability space*

single coin flip or single throw of dice:
too simple for example

let's consider *two* experiments

$$W_1 = (S_1, p_1), W_2 = (S_2, p_2)$$

set of outcomes for the pair of events

$$S = S_1 \times S_2 = \{(a, b) | a \in S_1, b \in S_2\}$$

$$p(a, b) = p_1(a) \cdot p_2(b)$$

$$W = W_1 \times W_2 = (S, p)$$

random variables from different independent experiments

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

For $i \in \{1,2\}$ let

$$W_i = (S_i, p_i)$$

be probability spaces and let

$$X_i : S_i \to \mathbb{R}$$

be random variables in these spaces.

sum if we perform independent experiments

$$X : S_1 \times S_2 \to \mathbb{R} \quad , \quad X(a,b) = X_1(a) + X_2(b)$$

Lemma 9.

$$E(X) = E(X_1) + E(X_2)$$

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

probability of B given A

$$A, B \subseteq S, p(A) \neq 0$$

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$

two experiments:

- throw coin
- if 0 throw coin *c*, otherwise dice *d*
- expected total number of points

it might be

$$E(c) + 1/2 \cdot E(c) + 1/2 \cdot E(d) = 1/2 + 1/4 + 7/4$$

sanity check 1

first experiment: $W_S = (S, p)$

second experiments: $W_i = (R_i, p_i)$ $\quad i \in S$

probability space: $W_Q = (Q, q)$

$$Q = \bigcup_{i \in S} \{i\} \times R_i$$

$$a \in R_i \to q(i, a) = p(i) \cdot p_i(a)$$

Lemma 10. *$W_Q = (Q, q)$ is a probability space* *first experiment:* $W_S = (S, p)$

second experiments: $W_i = (R_i, p_i)$ $i \in S$

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random variables on these spaces

$$X_0 : S \rightarrow \mathbb{R} \quad , \quad X_i : R_i \rightarrow \mathbb{R}$$

$$X : Q \rightarrow \mathbb{R}$$

$$X(i, r) = X_0(i) + X_i(r)$$

Lemma 12.

$$E(X) = E(X_0) + \Sigma_{i \in S} p(i) \cdot E(X_i)$$

two experiments:

- throw coin
- if 0 throw coin *c*, otherwise dice *d*
- expected total number of points

$$E(c) + 1/2 \cdot E(c) + 1/2 \cdot E(d) = 1/2 + 1/4 + 7/4$$

Quicksort

here: the algorithm does the random experiments

input: $(a(1), \dots, a(n))$ or set $A = \{a(1), \dots, a(n)\}$

we assume here: a(i) mutually distinct

random experiment: choose ‚splitter' $s \in \{a(1), \dots, a(n)\}$

all n splitters equally likely

$$A_< = \{a \in A | a < s\}$$

$$A_> = \{a \in A | a > s\}$$

$$sort(A) = sort(A_<) \circ s \circ sort(A_>)$$

$$W_1 = (S_1, p_1), W_2 = (S_2, p_2)$$

$$p(a,b) = p_1(a) \cdot p_2(b)$$

$$W = W_1 \times W_2 = (S, p)$$

Lemma 1. *$W_1 \times W_2$ is a probability space*

For $i \in \{1,2\}$ let

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$$X : S_1 \times S_2 \to \mathbb{R} \quad , \quad X(a,b) = X_1(a) + X_2(b)$$

Lemma 9.

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$$s \in \{a(1), \dots, a(n)\}$$

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$$E(X) = E(X_0) + \Sigma_{i \in S}\, p(i) \cdot E(X_i)$$

Define by induction on n probability spaces for sorting n distinct numbers

input: $(a(1),\ldots,a(n))$ or set $A=\{a(1),\ldots,a(n)\}$

$$QS_n=(Q_n,q_n)$$

random experiment: choose ‚splitter' $s\in\{a(1),\ldots,a(n)\}$

all n splitters equally likely

a(i) mutually distinct

$$A_<=\{a\in A|a<s\}$$

$$A_>=\{a\in A|a>s\}$$

$$sort(A)=sort(A_<)\circ s\circ sort(A_>)$$

Define by induction on n probability spaces for sorting n distinct numbers

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$$QS_n = (Q_n, q_n) \qquad RS_{n,i} = QS_{i-1} \times QS_{n-i}$$

and random variables

$$t_n : Q_n \to \mathbb{R}$$

$$x \in Q_n \ \to \ t_n(x) = \text{number of comparisons in run } x$$

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0 or 1 elements: no randomness
no comparisons

Define by induction on n probability spaces for sorting n distinct numbers

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$n \geq 2$: $S_n = \{1, \dots n\}$ $i \in S_n$ $i = \#A_< + 1$

rank of splitter s: number of a(i)≤ s.

$QS_n = (Q_n, q_n)$ $RS_{n,i} = QS_{i-1} \times QS_{n-i}$

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$Q_n = \bigcup_{i \in S_n} \{i\} \times (Q_{i-1} \times Q_{n-i})$

$q_n(i,(a,b)) = (1/n) \cdot q_{n,i}(a,b)$

$Q_2 = \{1\} \times Q_0 \times Q_1 \cup \{2\} \times Q_1 \times Q_0 =$

$= \{(1, \perp, \perp), (2, \perp, \perp)\}$

$q_2(1, \perp, \perp) = \frac{1}{2} q_{2,1}(\perp, \perp) = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2} = q_2(2, \perp, \perp)$

Define by induction on n probability spaces for sorting n distinct numbers

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$$q_n(i,(a,b)) = (1/n) \cdot q_{n,i}(a,b)$$

$Q_3 = \{1\} \times Q_0 \times Q_2 \cup \{2\} \times Q_1 \times Q_1 \cup \{3\} \times Q_2 \times Q_0 =$

$= \{(1,(\perp,(1,\perp,\perp)),\ (1,(\perp,(2,\perp,\perp)),\ (2,\perp,\perp),$

$\frac{1}{3} \cdot 1 \cdot \frac{1}{2} = \frac{1}{6} \qquad 1/6 \qquad 1/3$

$(3,(1,\perp,\perp),\perp),\ (3,(2,\perp,\perp),\perp)\}$

$1/6 \qquad 1/6$

Define by induction on n probability spaces for sorting n distinct numbers

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Lemma: For all i and n: QS_n and $RS_{n,i}$ are probability spaces

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proof: induction on n.
n = 0 or 1: trivial

$Q_n = \bigcup_{i \in S_n} \{i\} \times (Q_{i-1} \times Q_{n-i})$

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Define by induction on n probability spaces for sorting n distinct numbers

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$n \in \{0,1\}$: $Q_n = \{\bot\}$ $q_n(\bot) = 1$ $t_n(\bot) = 0$

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- n = 0 or 1: trivial
- assume true for j<n:

Define by induction on n probability spaces for sorting n distinct numbers

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$t_n(i,(a,b)) = n - 1 + t_{i-1}(a) + t_{n-i}(b)$

number of comparisons, random variable on Q_n
defined by induction on n

$q_n(i,(a,b)) = (1/n) \cdot q_{n,i}(a,b)$

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$q_n(i,(a,b)) = (1/n) \cdot q_{n,i}(a,b)$

Define by induction on n probability spaces for sorting n distinct numbers

input: $(a(1), \dots, a(n))$ or set $A = \{a(1), \dots, a(n)\}$

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and random variables

random experiment: choose ‚splitter‘ $s \in \{a(1), \dots, a(n)\}$

all n splitters equally likely

$t_n : Q_n \to \mathbb{R}$

$A_< = \{a \in A | a < s\}$

a(i) mutually distinct

$x \in Q_n \;\to\; t_n(x) = \text{number of comparisons in run } x$

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$n \in \{0,1\}$: $\quad Q_n = \{\perp\} \quad q_n(\perp) = 1 \quad t_n(\perp) = 0$

0 or 1 elements: no randomness
no comparisons

$n \geq 2$: $\quad S_n = \{1, \dots n\} \quad i \in S_n \quad i = \#A_< + 1 \quad r_n(i) = 1/n \quad W = (S_n, r_n)$ is probability space.

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$T(n) = E(t_n)$: $T(n) = n - 1 + (1/n) \cdot \Sigma_{i=1}^{n}(T(i-1) + T(n-i))$ **back to the sixties!!!**

Lemma: $T(n) \leq 2n \cdot \ln(n)$

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$$
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T(n) &< n + \frac{1}{n} \cdot \sum_{i=1}^{n}(T(i-1) + T(n-i)) \\
&= n + \frac{1}{n} \cdot (\sum_{i=0}^{n-1} T(i) + \sum_{i=0}^{n-1} T(i)) \\
&= n + \frac{2}{n} \cdot (\sum_{i=0}^{n-1} T(i)) \\
&= n + \frac{2}{n} \cdot (\sum_{i=1}^{n-1} T(i)) \quad (T(0) = T(1) = 0) \\
&\leq n + \frac{2}{n} \cdot (\sum_{i=2}^{n-1} 2i \cdot \ln(i)) \quad \text{(induction hypothesis)} \\
&\leq n + \frac{2}{n} \cdot (\int_2^n 2x \cdot \ln(x) dx) \quad \text{(area under the curve)}
\end{aligned}
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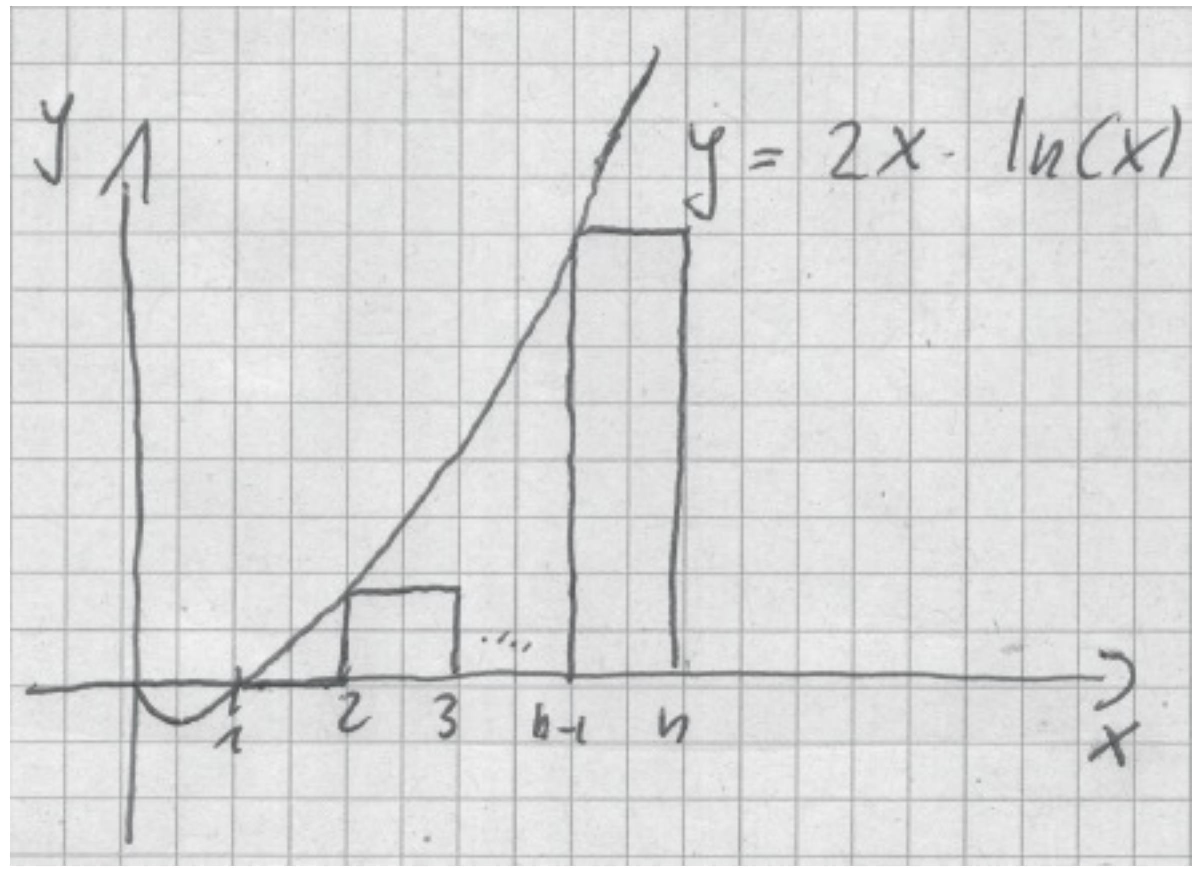

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Product rule of differentiation

$$(uv)' = u'v + uv'$$

Integrate

$$\int u'v = uv - \int uv'$$

$$\begin{aligned}
u' &= 2x \\
v &= ln(x) \\
u &= x^2 \\
v' &= 1/x \\
\int u'v &= x^2 \ln(x) - \int x^2/x \\
&= x^2 \cdot \ln(x) - \frac{x^2}{2}
\end{aligned}$$

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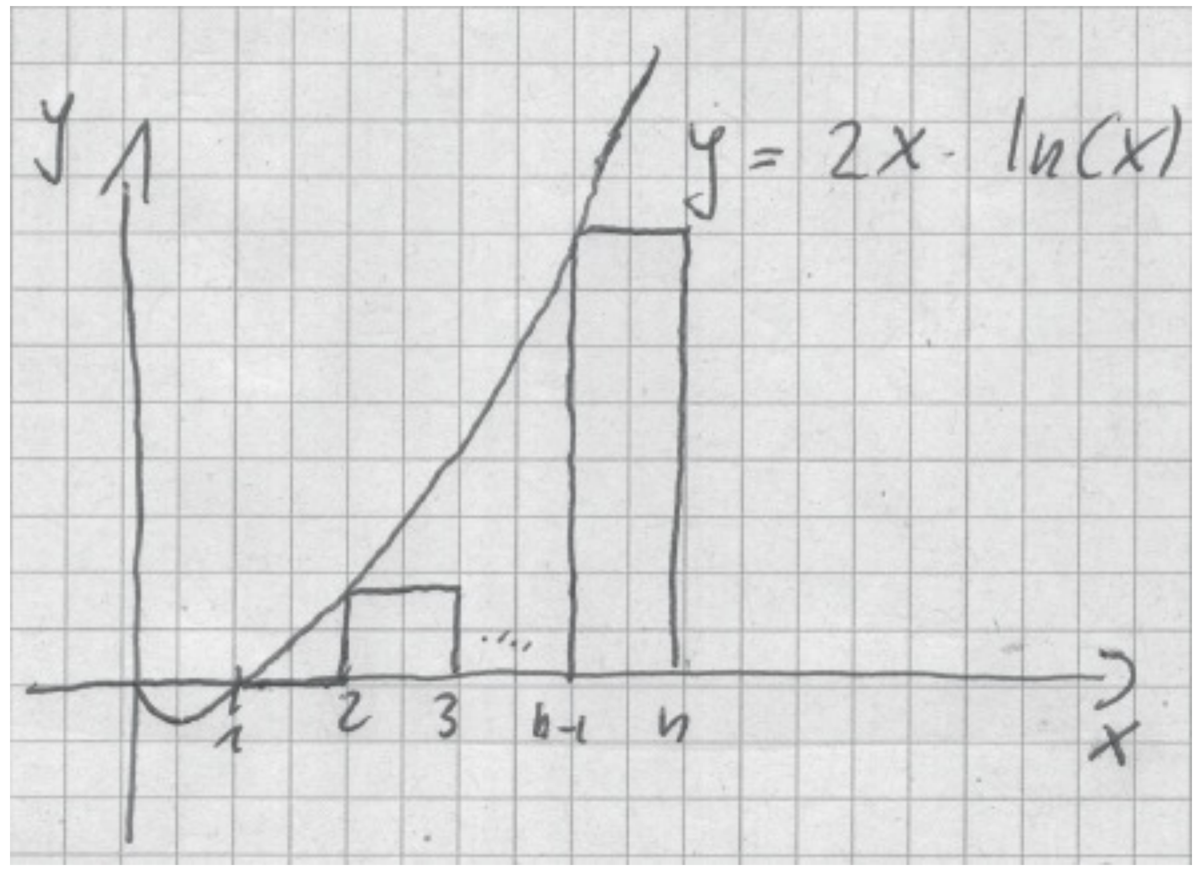

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&\leq n + \frac{2}{n} \cdot \left(\int_2^n 2x \cdot \ln(x) dx\right) \quad \text{(area under the curve)} \\
&= n + \frac{2}{n} \cdot [(n^2 \ln(n) - \frac{n^2}{2}) \quad \text{(product rule)} \\
&\quad -(2^2 \ln 2 - 2^2/2)] \\
&< 2n \cdot \ln(n) \quad (\ln 2 > 0.69 > 1/2)
\end{aligned}$$