Course: **Calculus 1 - CS**

Calculus: Early Transcendentals - James Stewart, Daniel Clegg, Saleem Watson (**Reader**) – Section 2.8

CALCULUS

EARLY TRANSCENDENTALS

NINTH EDITION

Metric Version

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2. Howard Eves, *An Introduction to the History of Mathematics*, 6th ed. (New York: Saunders, 1990), pp. 391, 395.

3. Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York: Oxford University Press, 1972), pp. 344, 346.

4. Uta Merzbach and Carl Boyer, *A History of Mathematics*, 3rd ed. (Hoboken, NJ: Wiley, 2011), pp. 323, 356.

2.8 The Derivative as a Function

The Derivative Function

In the preceding section we considered the derivative of a function f at a fixed number a:

$$\boxed{1} \qquad f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

Here we change our point of view and let the number a vary. If we replace a in Equation 1 by a variable x, we obtain

$$\boxed{2} \qquad \boxed{f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}}$$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can regard f' as a new function, called the **derivative of** f and defined by Equation 2. We know that the value of f' at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$.

The function f' is called the derivative of f because it has been "derived" from f by the limiting operation in Equation 2. The domain of f' is the set $\{x \mid f'(x) \text{ exists}\}$ and may be smaller than the domain of f.

EXAMPLE 1 The graph of a function f is given in Figure 1. Use it to sketch the graph of the derivative f'.

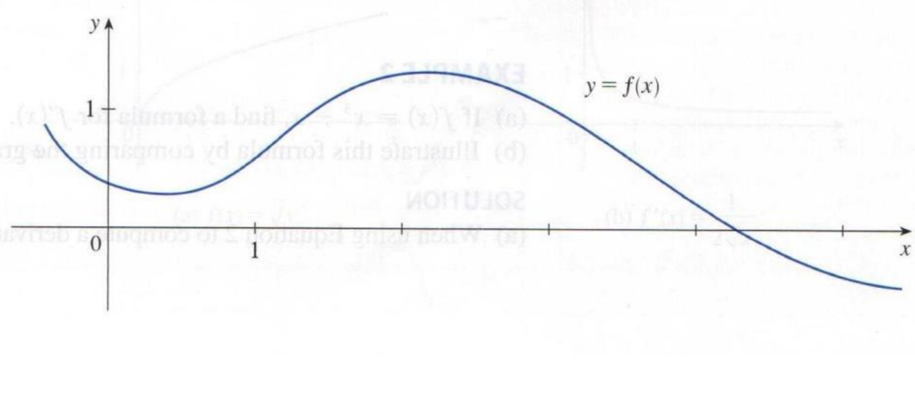


FIGURE 1

SOLUTION We can estimate the value of the derivative at any value of x by drawing the tangent at the point $(x, f(x))$ and estimating its slope. For instance, for $x = 3$ we draw a tangent at P in Figure 2 and estimate its slope to be about $-\frac{2}{3}$. (We have drawn a triangle to help estimate the slope.) Thus $f'(3) \approx -\frac{2}{3} \approx -0.67$ and this allows us to plot the point $P'(3, -0.67)$ on the graph of f' directly beneath P. (The slope of the graph of f becomes the y-value on the graph of f'.)

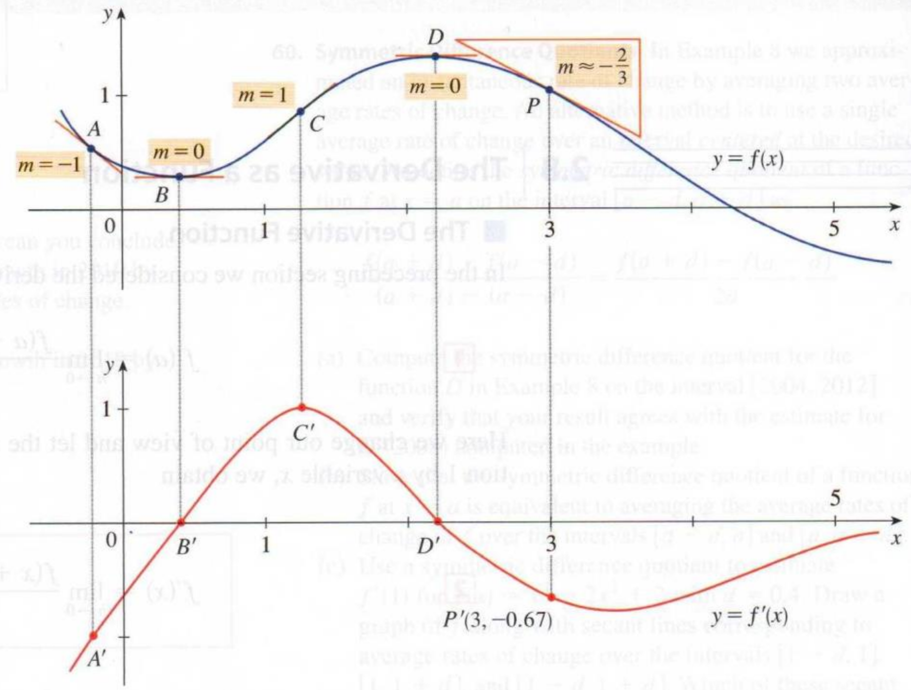


FIGURE 2

The slope of the tangent drawn at A appears to be about -1, so we plot the point A' with a y-value of -1 on the graph of f' (directly beneath A). The tangents at B and D are horizontal, so the derivative is 0 there and the graph of f' crosses the x-axis (where $y = 0$) at the points B' and D', directly beneath B and D. Between B and D, the graph of f is steepest at C and the tangent line there appears to have slope 1, so the largest value of $f'(x)$ between B' and D' is 1 (at C').

Notice that between B and D the tangents have positive slope, so $f'(x)$ is positive there. (The graph of f' is above the x-axis.) But to the right of D the tangents have negative slope, so $f'(x)$ is negative there. (The graph of f' is below the x-axis.) ■

EXAMPLE 2

(a) If $f(x) = x^3 - x$, find a formula for $f'(x)$.

(b) Illustrate this formula by comparing the graphs of f and f'.

SOLUTION

(a) When using Equation 2 to compute a derivative, we must remember that the variable is h and that x is temporarily regarded as a constant during the calculation of the limit.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{[(x+h)^3 - (x+h)] - [x^3 - x]}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 1) = 3x^2 - 1$$

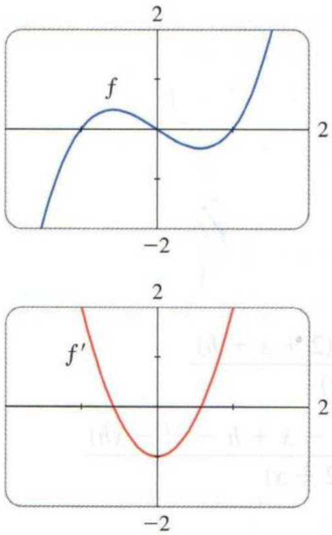


FIGURE 3

(b) We use a calculator to graph f and f' in Figure 3. Notice that $f'(x) = 0$ when f has horizontal tangents and $f'(x)$ is positive when the tangents have positive slope. So these graphs serve as a check on our work in part (a). ■

EXAMPLE 3 If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

SOLUTION

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \qquad \text{(Rationalize the numerator.)}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

We see that $f'(x)$ exists if $x > 0$, so the domain of f' is $(0, \infty)$. This is slightly smaller than the domain of f, which is $[0, \infty)$. ■

Let's check to see that the result of Example 3 is reasonable by looking at the graphs of f and f' in Figure 4. When x is close to 0, $\sqrt{x}$ is also close to 0, so $f'(x) = 1/(2\sqrt{x})$ is very large and this corresponds to the steep tangent lines near $(0, 0)$ in Figure 4(a) and the large values of $f'(x)$ just to the right of 0 in Figure 4(b). When x is large, $f'(x)$ is very small and this corresponds to the flatter tangent lines at the far right of the graph of f and the horizontal asymptote of the graph of f'.

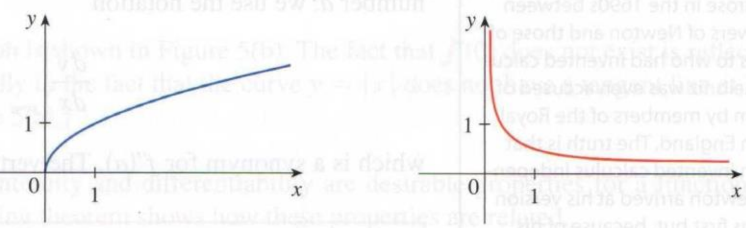


FIGURE 4 (a) $f(x) = \sqrt{x}$ (b) $f'(x) = \dfrac{1}{2\sqrt{x}}$

EXAMPLE 4 Find f' if $f(x) = \dfrac{1 - x}{2 + x}$.

SOLUTION

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{1 - (x + h)}{2 + (x + h)} - \dfrac{1 - x}{2 + x}}{h}$$

$$\frac{\dfrac{a}{b} - \dfrac{c}{d}}{e} = \frac{ad - bc}{bd} \cdot \frac{1}{e}$$

$$= \lim_{h \to 0} \frac{(1 - x - h)(2 + x) - (1 - x)(2 + x + h)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{(2 - x - 2h - x^2 - xh) - (2 - x + h - x^2 - xh)}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3h}{h(2 + x + h)(2 + x)}$$

$$= \lim_{h \to 0} \frac{-3}{(2 + x + h)(2 + x)} = -\frac{3}{(2 + x)^2}$$

Leibniz

Gottfried Wilhelm Leibniz was born in Leipzig in 1646 and studied law, theology, philosophy, and mathematics at the university there, graduating with a bachelor's degree at age 17. After earning his doctorate in law at age 20, Leibniz entered the diplomatic service and spent most of his life traveling to the capitals of Europe on political missions. In particular, he worked to avert a French military threat against Germany and attempted to reconcile the Catholic and Protestant churches.

His serious study of mathematics did not begin until 1672 while he was on a diplomatic mission in Paris. There he built a calculating machine and met scientists, like Huygens, who directed his attention to the latest developments in mathematics and science. Leibniz sought to develop a symbolic logic and system of notation that would simplify logical reasoning. In particular, the version of calculus that he published in 1684 established the notation and the rules for finding derivatives that we use today.

Unfortunately, a dreadful priority dispute arose in the 1690s between the followers of Newton and those of Leibniz as to who had invented calculus first. Leibniz was even accused of plagiarism by members of the Royal Society in England. The truth is that each man invented calculus independently. Newton arrived at his version of calculus first but, because of his fear of controversy, did not publish it immediately. So Leibniz's 1684 account of calculus was the first to be published.

Other Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called **differentiation operators** because they indicate the operation of **differentiation**, which is the process of calculating a derivative.

The symbol dy/dx, which was introduced by Leibniz, should not be regarded as a ratio (for the time being); it is simply a synonym for $f'(x)$. Nonetheless, it is a very useful and suggestive notation, especially when used in conjunction with increment notation. Referring to Equation 2.7.6, we can rewrite the definition of derivative in Leibniz notation in the form

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we want to indicate the value of a derivative dy/dx in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right|_{x=a} \qquad \text{or} \qquad \left.\frac{dy}{dx}\right]_{x=a}$$

which is a synonym for $f'(a)$. The vertical bar means "evaluate at."

3 Definition A function f is **differentiable at a** if $f'(a)$ exists. It is **differentiable on an open interval** (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

EXAMPLE 5 Where is the function $f(x) = |x|$ differentiable?

SOLUTION If $x > 0$, then $|x| = x$ and we can choose h small enough that $x + h > 0$ and hence $|x + h| = x + h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{(x + h) - x}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

and so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have $|x| = -x$ and h can be chosen small enough that $x + h < 0$ and so $|x + h| = -(x + h)$. Therefore, for $x < 0$,

$$f'(x) = \lim_{h \to 0} \frac{|x + h| - |x|}{h} = \lim_{h \to 0} \frac{-(x + h) - (-x)}{h}$$

$$= \lim_{h \to 0} \frac{-h}{h} = \lim_{h \to 0} (-1) = -1$$

and so f is differentiable for any $x < 0$.

For $x = 0$ we have to investigate

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \qquad \text{(if it exists)}$$

Let's compute the left and right limits separately:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

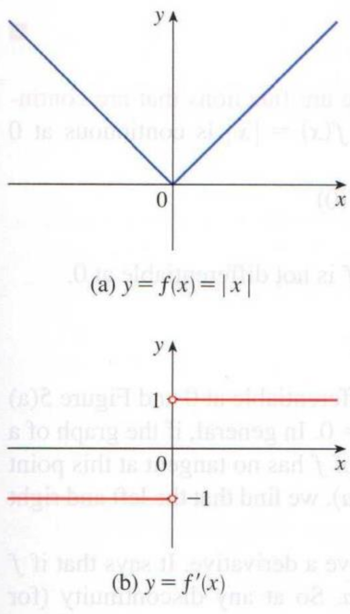
and $$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} (-1) = -1$$

Since these limits are different, $f'(0)$ does not exist. Thus f is differentiable at all x except 0.

A formula for f' is given by

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

and its graph is shown in Figure 5(b). The fact that $f'(0)$ does not exist is reflected geometrically in the fact that the curve $y = |x|$ does not have a tangent line at (0, 0). [See Figure 5(a).] ■



(a) $y = f(x) = |x|$

(b) $y = f'(x)$

FIGURE 5

Both continuity and differentiability are desirable properties for a function to have. The following theorem shows how these properties are related.

4 Theorem If f is differentiable at a, then f is continuous at a.

PROOF To prove that f is continuous at a, we have to show that $\lim_{x \to a} f(x) = f(a)$. We will do this by showing that the difference $f(x) - f(a)$ approaches 0.

The given information is that f is differentiable at a, that is,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

PS An important aspect of problem solving is trying to find a connection between the given and the unknown. See Step 2 (Think of a Plan) in Principles of Problem Solving following Chapter 1.

exists (see Equation 2.7.5). To connect the given and the unknown, we divide and multiply $f(x) - f(a)$ by $x - a$ (which we can do when $x \neq a$):

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a}(x - a)$$

Thus, using Limit Law 4, we can write

$$\begin{aligned}
\lim_{x \to a} [f(x) - f(a)] &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a}(x - a) \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \\
&= f'(a) \cdot 0 = 0
\end{aligned}$$

To use what we have just proved, we start with $f(x)$ and add and subtract $f(a)$:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} [f(a) + (f(x) - f(a))] \\
&= \lim_{x \to a} f(a) + \lim_{x \to a} [f(x) - f(a)] \\
&= f(a) + 0 = f(a)
\end{aligned}$$

Therefore f is continuous at a. ■

NOTE The converse of Theorem 4 is false; that is, there are functions that are continuous but not differentiable. For instance, the function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} |x| = 0 = f(0)$$

(See Example 2.3.7.) But in Example 5 we showed that f is not differentiable at 0.

How Can a Function Fail To Be Differentiable?

We saw that the function $y = |x|$ in Example 5 is not differentiable at 0 and Figure 5(a) shows that its graph changes direction abruptly when $x = 0$. In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]

Theorem 4 gives another way for a function not to have a derivative. It says that if f is not continuous at a, then f is not differentiable at a. So at any discontinuity (for instance, a jump discontinuity) f fails to be differentiable.

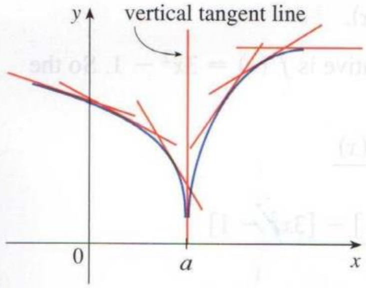


FIGURE 6

A third possibility is that the curve has a **vertical tangent line** when $x = a$; that is, f is continuous at a and

$$\lim_{x \to a} |f'(x)| = \infty$$

This means that the tangent lines become steeper and steeper as $x \to a$. Figure 6 shows one way that this can happen; Figure 7(c) shows another. Figure 7 illustrates the three possibilities that we have discussed.

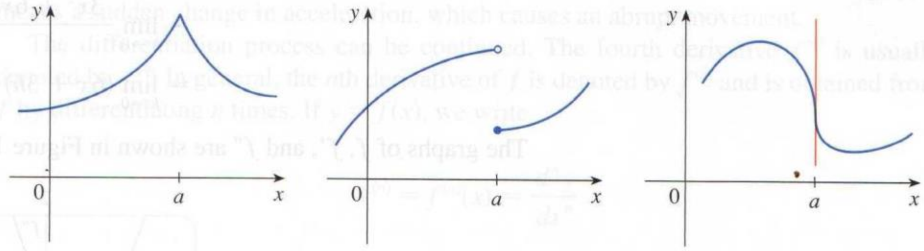


FIGURE 7
Three ways for f not to be differentiable at a

(a) A corner (b) A discontinuity (c) A vertical tangent

A graphing calculator or computer provides another way of looking at differentiability. If f is differentiable at a, then when we zoom in toward the point $(a, f(a))$ the graph straightens out and appears more and more like a line. (See Figure 8. We saw a specific example of this in Figure 2.7.2.) But no matter how much we zoom in toward a point like the ones in Figures 6 and 7(a), we can't eliminate the sharp point or corner (see Figure 9).

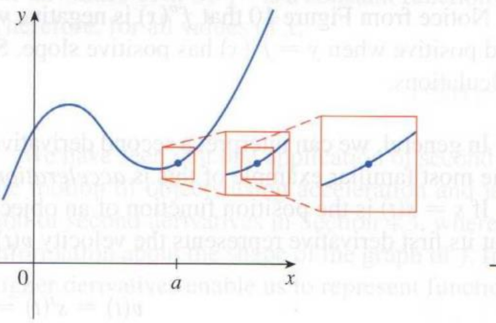


FIGURE 8
f is differentiable at a.

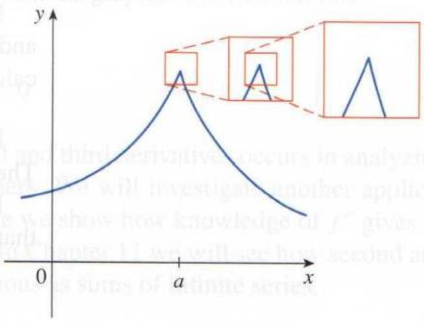


FIGURE 9
f is not differentiable at a.

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function, so f' may have a derivative of its own, denoted by $(f')' = f''$. This new function f'' is called the **second derivative** of f because it is the derivative of the derivative of f. Using Leibniz notation, we write the second derivative of $y = f(x)$ as

$$\underbrace{\frac{d}{dx}}_{\substack{\text{derivative}\\ \text{of}}}\underbrace{\left(\frac{dy}{dx}\right)}_{\substack{\text{first}\\ \text{derivative}}} = \underbrace{\frac{d^2y}{dx^2}}_{\substack{\text{second}\\ \text{derivative}}}$$

EXAMPLE 6 If $f(x) = x^3 - x$, find and interpret $f''(x)$.

SOLUTION In Example 2 we found that the first derivative is $f'(x) = 3x^2 - 1$. So the second derivative is

$$\begin{aligned} f''(x) = (f')'(x) &= \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} \\ &= \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h} \\ &= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - 3x^2 + 1}{h} \\ &= \lim_{h\to 0} (6x + 3h) = 6x \end{aligned}$$

The graphs of f, f', and f'' are shown in Figure 10.

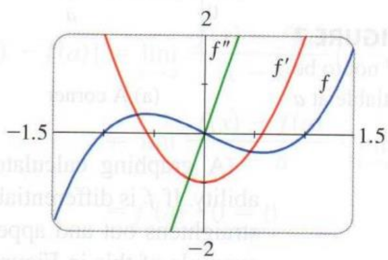


FIGURE 10

We can interpret $f''(x)$ as the slope of the curve $y = f'(x)$ at the point $(x, f'(x))$. In other words, it is the rate of change of the slope of the original curve $y = f(x)$.

Notice from Figure 10 that $f''(x)$ is negative when $y = f'(x)$ has negative slope and positive when $y = f'(x)$ has positive slope. So the graphs serve as a check on our calculations. ■

In general, we can interpret a second derivative as a rate of change of a rate of change. The most familiar example of this is *acceleration*, which we define as follows.

If $s = s(t)$ is the position function of an object that moves in a straight line, we know that its first derivative represents the velocity $v(t)$ of the object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to time is called the **acceleration** $a(t)$ of the object. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function:

$$a(t) = v'(t) = s''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

Acceleration is the change in velocity you feel when speeding up or slowing down in a car.

The **third derivative** f''' is the derivative of the second derivative: $f''' = (f'')'$. So $f'''(x)$ can be interpreted as the slope of the curve $y = f''(x)$ or as the rate of change of $f''(x)$. If $y = f(x)$, then alternative notations for the third derivative are

$$y''' = f'''(x) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$$

We can also interpret the third derivative physically in the case where the function is the position function $s = s(t)$ of an object that moves along a straight line. Because $s''' = (s'')' = a'$, the third derivative of the position function is the derivative of the acceleration function and is called the **jerk**:

$$j = \frac{da}{dt} = \frac{d^3s}{dt^3}$$

Thus the jerk j is the rate of change of acceleration. It is aptly named because a large jerk means a sudden change in acceleration, which causes an abrupt movement.

The differentiation process can be continued. The fourth derivative f'''' is usually denoted by $f^{(4)}$. In general, the nth derivative of f is denoted by $f^{(n)}$ and is obtained from f by differentiating n times. If $y = f(x)$, we write

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{dx^n}$$

EXAMPLE 7 If $f(x) = x^3 - x$, find $f'''(x)$ and $f^{(4)}(x)$.

SOLUTION In Example 6 we found that $f''(x) = 6x$. The graph of the second derivative has equation $y = 6x$ and so it is a straight line with slope 6. Since the derivative $f'''(x)$ is the slope of $f''(x)$, we have

$$f'''(x) = 6$$

for all values of x. So f''' is a constant function and its graph is a horizontal line. Therefore, for all values of x,

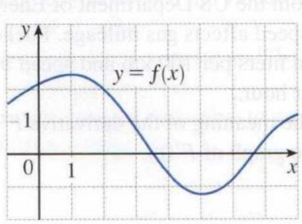
$$f^{(4)}(x) = 0$$ ■

We have seen that one application of second and third derivatives occurs in analyzing the motion of objects using acceleration and jerk. We will investigate another application of second derivatives in Section 4.3, where we show how knowledge of f'' gives us information about the shape of the graph of f. In Chapter 11 we will see how second and higher derivatives enable us to represent functions as sums of infinite series.

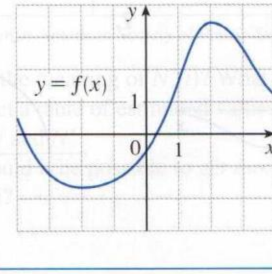
2.8 Exercises

1–2 Use the given graph to estimate the value of each derivative. Then sketch the graph of f'.

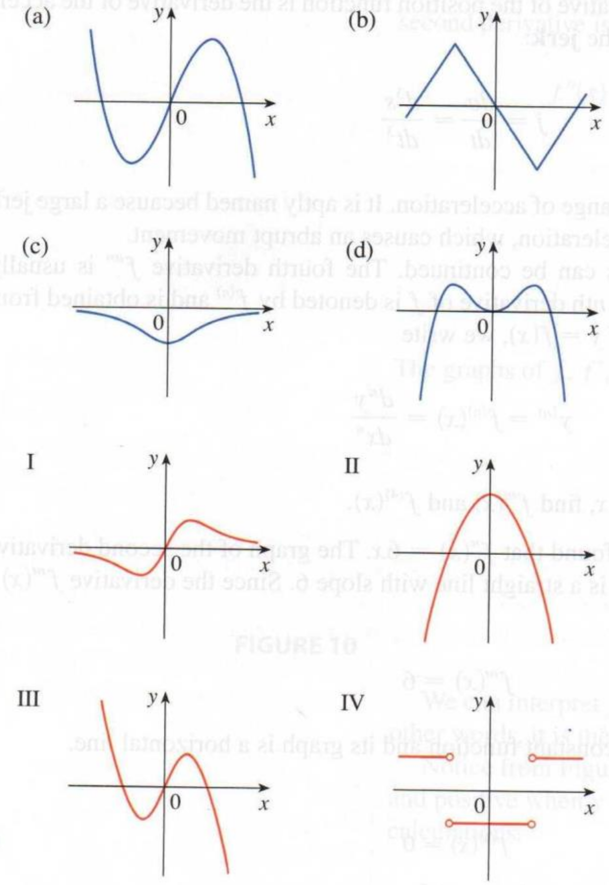
1. (a) $f'(0)$ (b) $f'(1)$ (c) $f'(2)$ (d) $f'(3)$
(e) $f'(4)$ (f) $f'(5)$ (g) $f'(6)$ (h) $f'(7)$



2. (a) $f'(-3)$ (b) $f'(-2)$ (c) $f'(-1)$
(d) $f'(0)$ (e) $f'(1)$ (f) $f'(2)$
(g) $f'(3)$

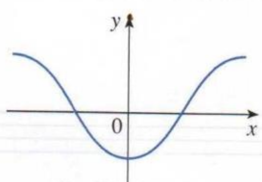


3. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for your choices.



4–11 Trace or copy the graph of the given function f. (Assume that the axes have equal scales.) Then use the method of Example 1 to sketch the graph of f' below it.

4.



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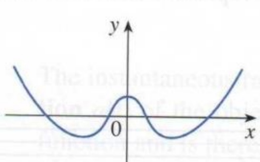

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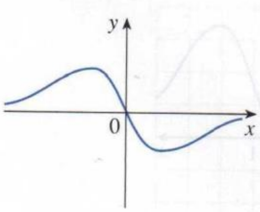

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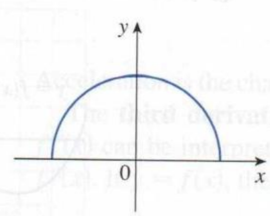

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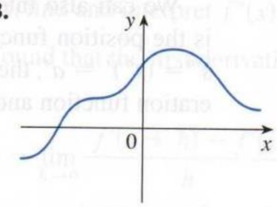
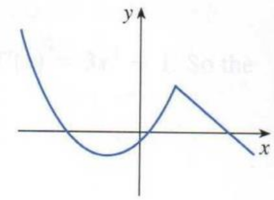


9.



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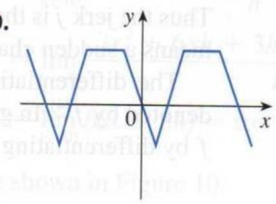

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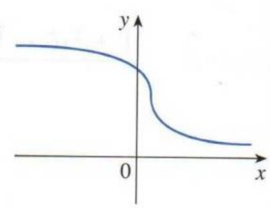
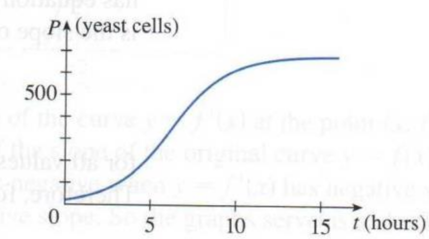


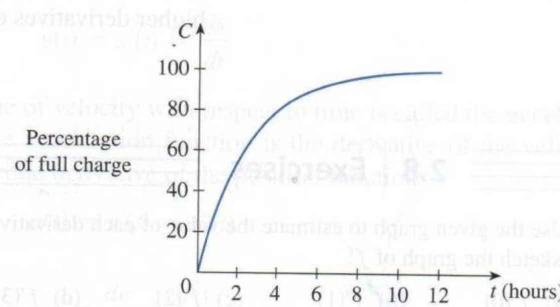
12. Shown is the graph of the population function $P(t)$ for yeast cells in a laboratory culture. Use the method of Example 1 to graph the derivative $P'(t)$. What does the graph of P' tell us about the yeast population?



13. A rechargeable battery is plugged into a charger. The graph shows $C(t)$, the percentage of full capacity that the battery reaches as a function of time t elapsed (in hours).

(a) What is the meaning of the derivative $C'(t)$?

(b) Sketch the graph of $C'(t)$. What does the graph tell you?

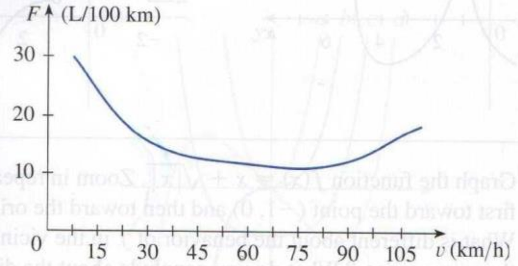


14. The graph (from the US Department of Energy) shows how driving speed affects gas mileage. Fuel economy F is measured in liters per 100 km and speed v is measured in kilometers per hour.

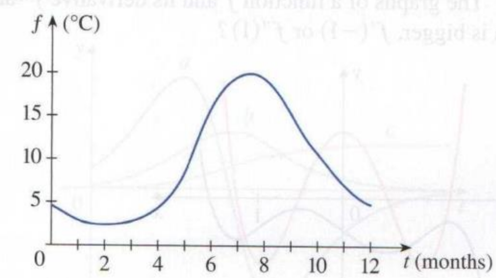
(a) What is the meaning of the derivative $F'(v)$?

(b) Sketch the graph of $F'(v)$.

(c) At what speed should you drive if you want to save on gas?



15. The graph shows how the average surface water temperature f of Lake Michigan varies over the course of a year (where t is measured in months with $t = 0$ corresponding to January 1). The average was calculated from data obtained over a 20-year period ending in 2011. Sketch the graph of the derivative function f'. When is $f'(t)$ largest?



16–18 Make a careful sketch of the graph of f and below it sketch the graph of f' in the same manner as in Exercises 4–11. Can you guess a formula for $f'(x)$ from its graph?

16. $f(x) = \sin x$ **17.** $f(x) = e^x$ **18.** $f(x) = \ln x$

19. Let $f(x) = x^2$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, and $f'(2)$ by zooming in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, and $f'(-2)$.
(c) Use the results from parts (a) and (b) to guess a formula for $f'(x)$.
(d) Use the definition of derivative to prove that your guess in part (c) is correct.

20. Let $f(x) = x^3$.
(a) Estimate the values of $f'(0)$, $f'(\frac{1}{2})$, $f'(1)$, $f'(2)$, and $f'(3)$ by zooming in on the graph of f.
(b) Use symmetry to deduce the values of $f'(-\frac{1}{2})$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.
(c) Use the values from parts (a) and (b) to graph f'.
(d) Guess a formula for $f'(x)$.
(e) Use the definition of derivative to prove that your guess in part (d) is correct.

21–32 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

21. $f(x) = 3x - 8$ **22.** $f(x) = mx + b$

23. $f(t) = 2.5t^2 + 6t$ **24.** $f(x) = 4 + 8x - 5x^2$

25. $A(p) = 4p^3 + 3p$ **26.** $F(t) = t^3 - 5t + 1$

27. $f(x) = \dfrac{1}{x^2 - 4}$ **28.** $F(v) = \dfrac{v}{v + 2}$

29. $g(u) = \dfrac{u + 1}{4u - 1}$ **30.** $f(x) = x^4$

31. $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **32.** $g(x) = \dfrac{1}{1 + \sqrt{x}}$

33. (a) Sketch the graph of $f(x) = 1 + \sqrt{x + 3}$ by starting with the graph of $y = \sqrt{x}$ and using the transformations of Section 1.3.
(b) Use the graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$. What are the domains of f and f'?
(d) Graph f' and compare with your sketch in part (b).

34. (a) If $f(x) = x + 1/x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

35. (a) If $f(x) = x^4 + 2x$, find $f'(x)$.
(b) Check to see that your answer to part (a) is reasonable by comparing the graphs of f and f'.

36. The table gives the number $N(t)$, measured in thousands, of minimally invasive cosmetic surgery procedures performed in the United States for various years t.

t	$N(t)$ (thousands)
2000	5,500
2002	4,897
2004	7,470
2006	9,138
2008	10,897
2010	11,561
2012	13,035
2014	13,945

Source: American Society of Plastic Surgeons

(a) What is the meaning of $N'(t)$? What are its units?
(b) Construct a table of estimated values for $N'(t)$.
(c) Graph N and N'.
(d) How would it be possible to get more accurate values for $N'(t)$?

37. The table gives the height as time passes of a typical pine tree grown for lumber at a managed site.

Tree age (years)	14	21	28	35	42	49
Height (meters)	12	16	19	22	24	25

Source: Arkansas Forestry Commission

If $H(t)$ is the height of the tree after t years, construct a table of estimated values for H' and sketch its graph.

38. Water temperature affects the growth rate of brook trout. The table shows the amount of weight gained by brook trout after 24 days in various water temperatures.

Temperature (°C)	15.5	17.7	20.0	22.4	24.4
Weight gained (g)	37.2	31.0	19.8	9.7	−9.8

If $W(x)$ is the weight gain at temperature x, construct a table of estimated values for W' and sketch its graph. What are the units for $W'(x)$?

Source: Adapted from J. Chadwick Jr., "Temperature Effects on Growth and Stress Physiology of Brook Trout: Implications for Climate Change Impacts on an Iconic Cold-Water Fish." *Masters Theses*. Paper 897. 2012. scholarworks.umass.edu/theses/897.

39. Let P represent the percentage of a city's electrical power that is produced by solar panels t years after January 1, 2020.

(a) What does dP/dt represent in this context?

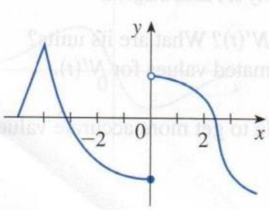
(b) Interpret the statement

$$\left.\frac{dP}{dt}\right|_{t=2} = 3.5$$

40. Suppose N is the number of people in the United States who travel by car to another state for a vacation in a year when the average price of gasoline is p dollars per liter. Do you expect dN/dp to be positive or negative? Explain.

41–44 The graph of f is given. State, with reasons, the numbers at which f is *not* differentiable.

41.



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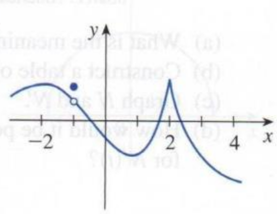

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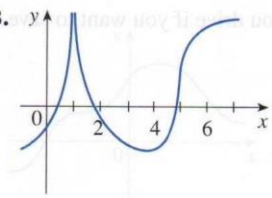

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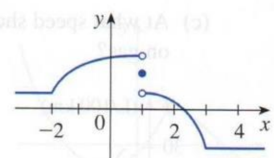
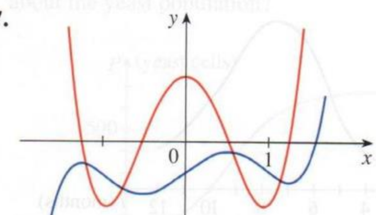


45. Graph the function $f(x) = x + \sqrt{|x|}$. Zoom in repeatedly, first toward the point $(-1, 0)$ and then toward the origin. What is different about the behavior of f in the vicinity of these two points? What do you conclude about the differentiability of f?

46. Zoom in toward the points $(1, 0)$, $(0, 1)$, and $(-1, 0)$ on the graph of the function $g(x) = (x^2 - 1)^{2/3}$. What do you notice? Account for what you see in terms of the differentiability of g.

47–48 The graphs of a function f and its derivative f' are shown. Which is bigger, $f'(-1)$ or $f''(1)$?

47.



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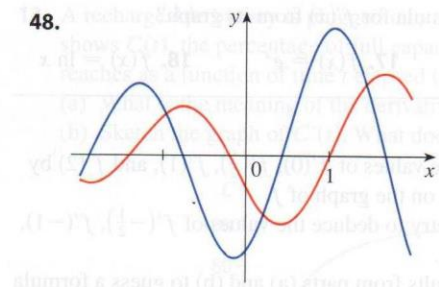
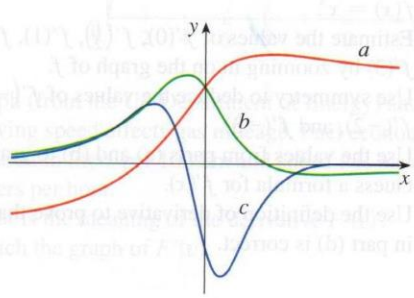
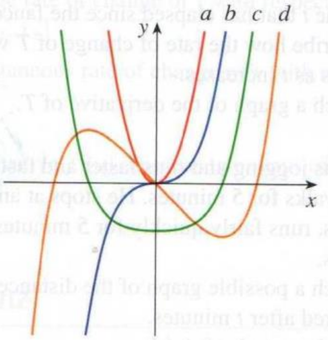


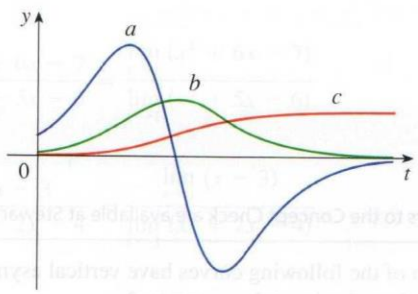
49. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



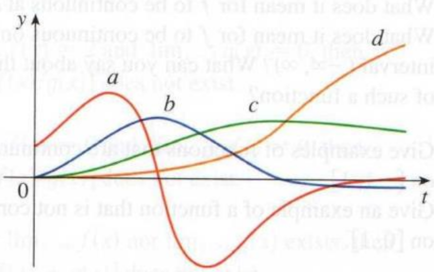
50. The figure shows graphs of f, f', f'', and f'''. Identify each curve, and explain your choices.



51. The figure shows the graphs of three functions. One is the position function of a car, one is the velocity of the car, and one is its acceleration. Identify each curve, and explain your choices.



52. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.

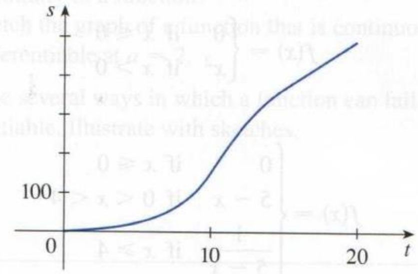


53–54 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

53. $f(x) = 3x^2 + 2x + 1$ **54.** $f(x) = x^3 - 3x$

55. If $f(x) = 2x^2 - x^3$, find $f'(x)$, $f''(x)$, $f'''(x)$, and $f^{(4)}(x)$. Graph f, f', f'', and f''' on a common screen. Are the graphs consistent with the geometric interpretations of these derivatives?

56. (a) The graph of a position function of a car is shown, where s is measured in meters and t in seconds. Use it to graph the velocity and acceleration of the car. What is the acceleration at $t = 10$ seconds?



(b) Use the acceleration curve from part (a) to estimate the jerk at $t = 10$ seconds. What are the units for jerk?

57. Let $f(x) = \sqrt[3]{x}$.

(a) If $a \neq 0$, use Equation 2.7.5 to find $f'(a)$.

(b) Show that $f'(0)$ does not exist.

(c) Show that $y = \sqrt[3]{x}$ has a vertical tangent line at $(0, 0)$. (Recall the shape of the graph of f. See Figure 1.2.13.)

58. (a) If $g(x) = x^{2/3}$, show that $g'(0)$ does not exist.

(b) If $a \neq 0$, find $g'(a)$.

(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0, 0)$.

(d) Illustrate part (c) by graphing $y = x^{2/3}$.

59. Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

60. Where is the greatest integer function $f(x) = [\![x]\!]$ not differentiable? Find a formula for f' and sketch its graph.

61. (a) Sketch the graph of the function $f(x) = x|x|$.

(b) For what values of x is f differentiable?

(c) Find a formula for f'.

62. (a) Sketch the graph of the function $g(x) = x + |x|$.

(b) For what values of x is g differentiable?

(c) Find a formula for g'.

63. Derivatives of Even and Odd Functions Recall that a function f is called *even* if $f(-x) = f(x)$ for all x in its domain and *odd* if $f(-x) = -f(x)$ for all such x. Prove each of the following.

(a) The derivative of an even function is an odd function.

(b) The derivative of an odd function is an even function.

64–65 Left- and Right-Hand Derivatives The *left-hand* and *right-hand derivatives* of f at a are defined by

$$f'_-(a) = \lim_{h \to 0^-} \frac{f(a + h) - f(a)}{h}$$

and

$$f'_+(a) = \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h}$$

if these limits exist. Then $f'(a)$ exists if and only if these one-sided derivatives exist and are equal.

64. Find $f'_-(0)$ and $f'_+(0)$ for the given function f. Is f differentiable at 0?

(a) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x & \text{if } x > 0 \end{cases}$$

(b) $$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

65. Let

$$f(x) = \begin{cases} 0 & \text{if } x \leqslant 0 \\ 5 - x & \text{if } 0 < x < 4 \\ \dfrac{1}{5 - x} & \text{if } x \geqslant 4 \end{cases}$$

(a) Find $f'_-(4)$ and $f'_+(4)$.
(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

66. When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. In Example 1.1.4 we sketched a possible graph of T as a function of the time t that has elapsed since the faucet was turned on.
(a) Describe how the rate of change of T with respect to t varies as t increases.
(b) Sketch a graph of the derivative of T.

67. Nick starts jogging and runs faster and faster for 3 mintues, then he walks for 5 minutes. He stops at an intersection for 2 minutes, runs fairly quickly for 5 minutes, then walks for 4 minutes.
(a) Sketch a possible graph of the distance s Nick has covered after t minutes.
(b) Sketch a graph of ds/dt.

68. Let ℓ be the tangent line to the parabola $y = x^2$ at the point (1, 1). The *angle of inclination* of ℓ is the angle ϕ that ℓ makes with the positive direction of the x-axis. Calculate ϕ correct to the nearest degree.

2 REVIEW

CONCEPT CHECK

Answers to the Concept Check are available at StewartCalculus.com.

1. Explain what each of the following means and illustrate with a sketch.
(a) $\lim_{x \to a} f(x) = L$ (b) $\lim_{x \to a^+} f(x) = L$ (c) $\lim_{x \to a^-} f(x) = L$
(d) $\lim_{x \to a} f(x) = \infty$ (e) $\lim_{x \to \infty} f(x) = L$

2. Describe several ways in which a limit can fail to exist. Illustrate with sketches.

3. State the following Limit Laws.
(a) Sum Law
(b) Difference Law
(c) Constant Multiple Law
(d) Product Law
(e) Quotient Law
(f) Power Law
(g) Root Law

4. What does the Squeeze Theorem say?

5. (a) What does it mean to say that the line $x = a$ is a vertical asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.
(b) What does it mean to say that the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$? Draw curves to illustrate the various possibilities.

6. Which of the following curves have vertical asymptotes? Which have horizontal asymptotes?
(a) $y = x^4$ (b) $y = \sin x$ (c) $y = \tan x$
(d) $y = \tan^{-1} x$ (e) $y = e^x$ (f) $y = \ln x$
(g) $y = 1/x$ (h) $y = \sqrt{x}$

7. (a) What does it mean for f to be continuous at a?
(b) What does it mean for f to be continuous on the interval $(-\infty, \infty)$? What can you say about the graph of such a function?

8. (a) Give examples of functions that are continuous on $[-1, 1]$.
(b) Give an example of a function that is not continuous on $[0, 1]$.

9. What does the Intermediate Value Theorem say?

10. Write an expression for the slope of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$.

11. Suppose an object moves along a straight line with position $f(t)$ at time t. Write an expression for the instantaneous velocity of the object at time $t = a$. How can you interpret this velocity in terms of the graph of f?

12. If $y = f(x)$ and x changes from x_1 to x_2, write expressions for the following.
   (a) The average rate of change of y with respect to x over the interval $[x_1, x_2]$
   (b) The instantaneous rate of change of y with respect to x at $x = x_1$

13. Define the derivative $f'(a)$. Discuss two ways of interpreting this number.

14. Define the second derivative of f. If $f(t)$ is the position function of a particle, how can you interpret the second derivative?

15. (a) What does it mean for f to be differentiable at a?
   (b) What is the relation between the differentiability and continuity of a function?
   (c) Sketch the graph of a function that is continuous but not differentiable at $a = 2$.

16. Describe several ways in which a function can fail to be differentiable. Illustrate with sketches.

TRUE-FALSE QUIZ

Determine whether the statement is true or false. If it is true, explain why. If it is false, explain why or give an example that disproves the statement.

1. $\displaystyle\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to 4}\frac{2x}{x-4}-\lim_{x\to 4}\frac{8}{x-4}$

2. $\displaystyle\lim_{x\to 1}\frac{x^2+6x-7}{x^2+5x-6}=\frac{\lim_{x\to 1}(x^2+6x-7)}{\lim_{x\to 1}(x^2+5x-6)}$

3. $\displaystyle\lim_{x\to 1}\frac{x-3}{x^2+2x-4}=\frac{\lim_{x\to 1}(x-3)}{\lim_{x\to 1}(x^2+2x-4)}$

4. $\displaystyle\frac{x^2-9}{x-3}=x+3$

5. $\displaystyle\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}(x+3)$

6. If $\lim_{x\to 5} f(x) = 2$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5}[f(x)/g(x)]$ does not exist.

7. If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5}[f(x)/g(x)]$ does not exist.

8. If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a}[f(x) + g(x)]$ does not exist.

9. If $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a}[f(x) + g(x)]$ does not exist.

10. If p is a polynomial, then $\lim_{x\to b} p(x) = p(b)$.

11. If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0}[f(x) - g(x)] = 0$.

12. A function can have two different horizontal asymptotes.

13. If f has domain $[0, \infty)$ and has no horizontal asymptote, then $\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to\infty} f(x) = -\infty$.

14. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

15. If $f(1) > 0$ and $f(3) < 0$, then there exists a number c between 1 and 3 such that $f(c) = 0$.

16. If f is continuous at 5 and $f(5) = 2$ and $f(4) = 3$, then $\lim_{x\to 2} f(4x^2 - 11) = 2$.

17. If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there exists a number r such that $|r| < 1$ and $f(r) = \pi$.

18. Let f be a function such that $\lim_{x\to 0} f(x) = 6$. Then there exists a positive number δ such that if $0 < |x| < \delta$, then $|f(x) - 6| < 1$.

19. If $f(x) > 1$ for all x and $\lim_{x\to 0} f(x)$ exists, then $\lim_{x\to 0} f(x) > 1$.

20. If f is continuous at a, then f is differentiable at a.

21. If $f'(r)$ exists, then $\lim_{x\to r} f(x) = f(r)$.

22. $\displaystyle\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$

23. The equation $x^{10} - 10x^2 + 5 = 0$ has a solution in the interval $(0, 2)$.

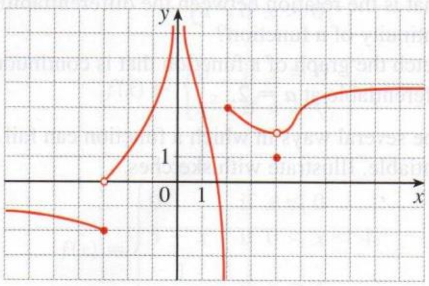
24. If f is continuous at a, so is $|f|$.

25. If $|f|$ is continuous at a, so is f.

26. If f is differentiable at a, so is $|f|$.

EXERCISES

1. The graph of f is given.



(a) Find each limit, or explain why it does not exist.
(i) $\lim_{x \to 2^+} f(x)$ (ii) $\lim_{x \to -3^+} f(x)$ (iii) $\lim_{x \to -3} f(x)$
(iv) $\lim_{x \to 4} f(x)$ (v) $\lim_{x \to 0} f(x)$ (vi) $\lim_{x \to 2^-} f(x)$
(vii) $\lim_{x \to \infty} f(x)$ (viii) $\lim_{x \to -\infty} f(x)$

(b) State the equations of the horizontal asymptotes.
(c) State the equations of the vertical asymptotes.
(d) At what numbers is f discontinuous? Explain.

2. Sketch the graph of a function f that satisfies all of the following conditions:
$\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to -3} f(x) = \infty$,
$\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = 2$,
f is continuous from the right at 3

3–20 Find the limit.

3. $\lim_{x \to 0} \cos(x^3 + 3x)$ **4.** $\lim_{x \to 3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

5. $\lim_{x \to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$ **6.** $\lim_{x \to 1^+} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

7. $\lim_{h \to 0} \dfrac{(h - 1)^3 + 1}{h}$ **8.** $\lim_{t \to 2} \dfrac{t^2 - 4}{t^3 - 8}$

9. $\lim_{r \to 9} \dfrac{\sqrt{r}}{(r - 9)^4}$ **10.** $\lim_{v \to 4^+} \dfrac{4 - v}{|4 - v|}$

11. $\lim_{r \to -1} \dfrac{r^2 - 3r - 4}{4r^2 + r - 3}$ **12.** $\lim_{t \to 5} \dfrac{3 - \sqrt{t + 4}}{t - 5}$

13. $\lim_{x \to \infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$ **14.** $\lim_{x \to -\infty} \dfrac{\sqrt{x^2 - 9}}{2x - 6}$

15. $\lim_{x \to \pi^-} \ln(\sin x)$ **16.** $\lim_{x \to -\infty} \dfrac{1 - 2x^2 - x^4}{5 + x - 3x^4}$

17. $\lim_{x \to \infty} \left(\sqrt{x^2 + 4x + 1} - x\right)$ **18.** $\lim_{x \to \infty} e^{x - x^2}$

19. $\lim_{x \to 0^+} \tan^{-1}(1/x)$ **20.** $\lim_{x \to 1} \left(\dfrac{1}{x - 1} + \dfrac{1}{x^2 - 3x + 2}\right)$

21–22 Use graphs to discover the asymptotes of the curve. Then prove what you have discovered.

21. $y = \dfrac{\cos^2 x}{x^2}$ **22.** $y = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

23. If $2x - 1 \leqslant f(x) \leqslant x^2$ for $0 < x < 3$, find $\lim_{x \to 1} f(x)$.

24. Prove that $\lim_{x \to 0} x^2 \cos(1/x^2) = 0$.

25–28 Prove the statement using the precise definition of a limit.

25. $\lim_{x \to 2} (14 - 5x) = 4$ **26.** $\lim_{x \to 0} \sqrt[3]{x} = 0$

27. $\lim_{x \to 2} (x^2 - 3x) = -2$ **28.** $\lim_{x \to 4^+} \dfrac{2}{\sqrt{x - 4}} = \infty$

29. Let

$$f(x) = \begin{cases} \sqrt{-x} & \text{if } x < 0 \\ 3 - x & \text{if } 0 \leqslant x < 3 \\ (x - 3)^2 & \text{if } x > 3 \end{cases}$$

(a) Evaluate each limit, if it exists.
(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 0^-} f(x)$ (iii) $\lim_{x \to 0} f(x)$
(iv) $\lim_{x \to 3^-} f(x)$ (v) $\lim_{x \to 3^+} f(x)$ (vi) $\lim_{x \to 3} f(x)$

(b) Where is f discontinuous?
(c) Sketch the graph of f.

30. Let

$$g(x) = \begin{cases} 2x - x^2 & \text{if } 0 \leqslant x \leqslant 2 \\ 2 - x & \text{if } 2 < x \leqslant 3 \\ x - 4 & \text{if } 3 < x < 4 \\ \pi & \text{if } x \geqslant 4 \end{cases}$$

(a) For each of the numbers 2, 3, and 4, determine whether g is continuous from the left, continuous from the right, or continuous at the number.
(b) Sketch the graph of g.

31–32 Show that the function is continuous on its domain. State the domain.

31. $h(x) = xe^{\sin x}$ **32.** $g(x) = \dfrac{\sqrt{x^2 - 9}}{x^2 - 2}$

33–34 Use the Intermediate Value Theorem to show that there is a solution of the equation in the given interval.

33. $x^5 - x^3 + 3x - 5 = 0$, $(1, 2)$

34. $\cos\sqrt{x} = e^x - 2$, $(0, 1)$

35. (a) Find the slope of the tangent line to the curve $y = 9 - 2x^2$ at the point $(2, 1)$.
(b) Find an equation of this tangent line.

36. Find equations of the tangent lines to the curve

$$y = \frac{2}{1 - 3x}$$

at the points with x-coordinates 0 and -1.

37. The displacement (in meters) of an object moving in a straight line is given by $s = 1 + 2t + \frac{1}{4}t^2$, where t is measured in seconds.
(a) Find the average velocity over each time period.
(i) $[1, 3]$ (ii) $[1, 2]$ (iii) $[1, 1.5]$ (iv) $[1, 1.1]$
(b) Find the instantaneous velocity when $t = 1$.

38. According to Boyle's Law, if the temperature of a confined gas is held fixed, then the product of the pressure P and the volume V is a constant. Suppose that, for a certain gas, $PV = 4000$, where P is measured in pascals and V is measured in liters.
(a) Find the average rate of change of P as V increases from 3 L to 4 L.
(b) Express V as a function of P and show that the instantaneous rate of change of V with respect to P is inversely proportional to the square of P.

39. (a) Use the definition of a derivative to find $f'(2)$, where $f(x) = x^3 - 2x$.
(b) Find an equation of the tangent line to the curve $y = x^3 - 2x$ at the point $(2, 4)$.
(c) Illustrate part (b) by graphing the curve and the tangent line on the same screen.

40. Find a function f and a number a such that

$$\lim_{h \to 0} \frac{(2 + h)^6 - 64}{h} = f'(a)$$

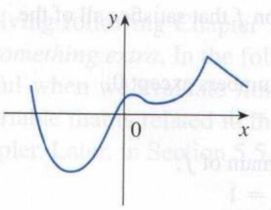
41. The total cost of repaying a student loan at an interest rate of r% per year is $C = f(r)$.
(a) What is the meaning of the derivative $f'(r)$? What are its units?
(b) What does the statement $f'(10) = 1200$ mean?
(c) Is $f'(r)$ always positive or does it change sign?

42–44 Trace or copy the graph of the function. Then sketch a graph of its derivative directly beneath.

42.

43.

44.



45–46 Find the derivative of f using the definition of a derivative. What is the domain of f'?

45. $f(x) = \dfrac{2}{x^2}$ **46.** $f(t) = \dfrac{1}{\sqrt{t + 1}}$

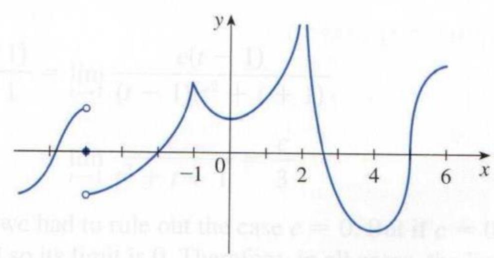
47. (a) If $f(x) = \sqrt{3 - 5x}$, use the definition of a derivative to find $f'(x)$.
(b) Find the domains of f and f'.
(c) Graph f and f' on a common screen. Compare the graphs to see whether your answer to part (a) is reasonable.

48. (a) Find the asymptotes of the graph of

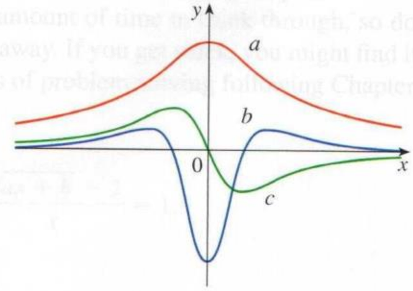
$$f(x) = \frac{4 - x}{3 + x}$$

and use them to sketch the graph.
(b) Use your graph from part (a) to sketch the graph of f'.
(c) Use the definition of a derivative to find $f'(x)$.
(d) Graph f' and compare with your sketch in part (b).

49. The graph of f is shown. State, with reasons, the numbers at which f is not differentiable.



50. The figure shows the graphs of f, f', and f''. Identify each curve, and explain your choices.



51. Sketch the graph of a function f that satisfies all of the following conditions:

The domain of f is all real numbers except 0,

$\lim_{x\to 0^-} f(x) = 1, \quad \lim_{x\to 0^+} f(x) = 0,$

$f'(x) > 0$ for all x in the domain of f,

$\lim_{x\to -\infty} f'(x) = 0, \quad \lim_{x\to \infty} f'(x) = 1$

52. Let $P(t)$ be the percentage of Americans under the age of 18 at time t. The table gives values of this function in census years from 1950 to 2010.

t	$P(t)$	t	$P(t)$
1950	31.1	1990	25.7
1960	35.7	2000	25.7
1970	34.0	2010	24.0
1980	28.0		

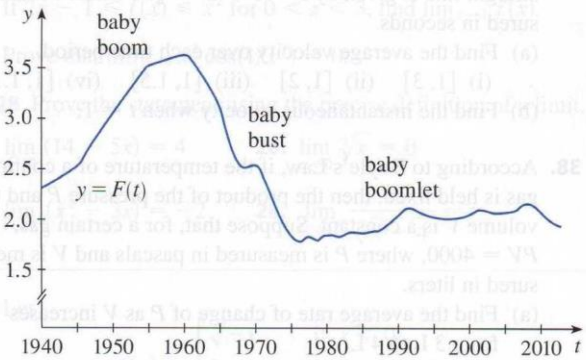
(a) What is the meaning of $P'(t)$? What are its units?
(b) Construct a table of estimated values for $P'(t)$.
(c) Graph P and P'.
(d) How would it be possible to get more accurate values for $P'(t)$?

53. Let $B(t)$ be the number of US \$20 bills in circulation at time t. The table gives values of this function from 1995 to 2015, as of December 31, in billions. Interpret and estimate the value of $B'(2010)$.

t	1995	2000	2005	2010	2015
$B(t)$	4.21	4.93	5.77	6.53	8.57

54. The *total fertility rate* at time t, denoted by $F(t)$, is an estimate of the average number of children born to each woman (assuming that current birth rates remain constant). The graph of the total fertility rate in the United States shows the fluctuations from 1940 to 2010.

(a) Estimate the values of $F'(1950)$, $F'(1965)$, and $F'(1987)$.
(b) What are the meanings of these derivatives?
(c) Can you suggest reasons for the values of these derivatives?



55. Suppose that $|f(x)| \leqslant g(x)$ for all x, where $\lim_{x\to a} g(x) = 0$. Find $\lim_{x\to a} f(x)$.

56. Let $f(x) = [\![x]\!] + [\![-x]\!]$.
(a) For what values of a does $\lim_{x\to a} f(x)$ exist?
(b) At what numbers is f discontinuous?

Problems Plus

In the Principles of Problem Solving following Chapter 1 we considered the problem-solving strategy of *introducing something extra*. In the following example we show how this principle is sometimes useful when we evaluate limits. The idea is to change the variable—to introduce a new variable that is related to the original variable—in such a way as to make the problem simpler. Later, in Section 5.5, we will make more extensive use of this general idea.

EXAMPLE Evaluate $\lim_{x \to 0} \dfrac{\sqrt[3]{1 + cx} - 1}{x}$, where c is a constant.

SOLUTION As it stands, this limit looks challenging. In Section 2.3 we evaluated limits in which both numerator and denominator approached 0. There our strategy was to perform some sort of algebraic manipulation that led to a simplifying cancellation, but here it's not clear what kind of algebra is necessary.

So we introduce a new variable t by the equation

$$t = \sqrt[3]{1 + cx}$$

We also need to express x in terms of t, so we solve this equation:

$$t^3 = 1 + cx \qquad x = \frac{t^3 - 1}{c} \quad (\text{if } c \neq 0)$$

Notice that $x \to 0$ is equivalent to $t \to 1$. This allows us to convert the given limit into one involving the variable t:

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt[3]{1 + cx} - 1}{x} &= \lim_{t \to 1} \frac{t - 1}{(t^3 - 1)/c} \\ &= \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} \end{aligned}$$

The change of variable allowed us to replace a relatively complicated limit by a simpler one of a type that we have seen before. Factoring the denominator as a difference of cubes, we get

$$\begin{aligned} \lim_{t \to 1} \frac{c(t - 1)}{t^3 - 1} &= \lim_{t \to 1} \frac{c(t - 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \to 1} \frac{c}{t^2 + t + 1} = \frac{c}{3} \end{aligned}$$

In making the change of variable we had to rule out the case $c = 0$. But if $c = 0$, the function is 0 for all nonzero x and so its limit is 0. Therefore, in all cases, the limit is $c/3$. ■

The following problems are meant to test and challenge your problem-solving skills. Some of them require a considerable amount of time to think through, so don't be discouraged if you can't solve them right away. If you get stuck, you might find it helpful to refer to the discussion of the principles of problem solving following Chapter 1.

Problems

1. Evaluate $\lim_{x \to 1} \dfrac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

2. Find numbers a and b such that $\lim_{x \to 0} \dfrac{\sqrt{ax + b} - 2}{x} = 1$.

3. Evaluate $\lim_{x \to 0} \dfrac{|2x - 1| - |2x + 1|}{x}$.

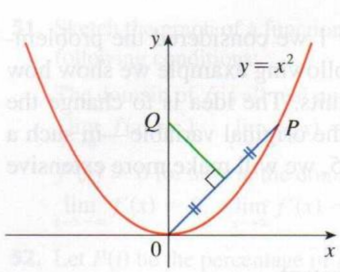


FIGURE FOR PROBLEM 4

4. The figure shows a point P on the parabola $y = x^2$ and the point Q where the perpendicular bisector of OP intersects the y-axis. As P approaches the origin along the parabola, what happens to Q? Does it have a limiting position? If so, find it.

5. Evaluate the following limits, if they exist, where $[\![x]\!]$ denotes the greatest integer function.

(a) $\displaystyle\lim_{x\to 0} \frac{[\![x]\!]}{x}$ (b) $\displaystyle\lim_{x\to 0} x[\![1/x]\!]$

6. Sketch the region in the plane defined by each of the following equations.

(a) $[\![x]\!]^2 + [\![y]\!]^2 = 1$ (b) $[\![x]\!]^2 - [\![y]\!]^2 = 3$

(c) $[\![x + y]\!]^2 = 1$ (d) $[\![x]\!] + [\![y]\!] = 1$

7. Let $f(x) = x/[\![x]\!]$.

(a) Find the domain and range of f. (b) Evaluate $\lim_{x\to\infty} f(x)$.

8. A **fixed point** of a function f is a number c in its domain such that $f(c) = c$. (The function doesn't move c; it stays fixed.)

(a) Sketch the graph of a continuous function with domain $[0, 1]$ whose range also lies in $[0, 1]$. Locate a fixed point of f.

(b) Try to draw the graph of a continuous function with domain $[0, 1]$ and range in $[0, 1]$ that does *not* have a fixed point. What is the obstacle?

(c) Use the Intermediate Value Theorem to prove that any continuous function with domain $[0, 1]$ and range in $[0, 1]$ must have a fixed point.

9. If $\lim_{x\to a}[f(x) + g(x)] = 2$ and $\lim_{x\to a}[f(x) - g(x)] = 1$, find $\lim_{x\to a}[f(x)\,g(x)]$.

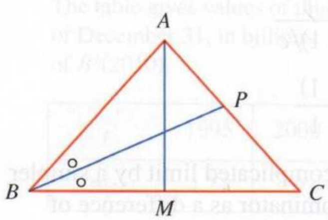


FIGURE FOR PROBLEM 10

10. (a) The figure shows an isosceles triangle ABC with $\angle B = \angle C$. The bisector of angle B intersects the side AC at the point P. Suppose that the base BC remains fixed but the altitude $|AM|$ of the triangle approaches 0, so A approaches the midpoint M of BC. What happens to P during this process? Does it have a limiting position? If so, find it.

(b) Try to sketch the path traced out by P during this process. Then find an equation of this curve and use this equation to sketch the curve.

11. (a) If we start from 0° latitude and proceed in a westerly direction, we can let $T(x)$ denote the temperature at the point x at any given time. Assuming that T is a continuous function of x, show that at any fixed time there are at least two diametrically opposite points on the equator that have exactly the same temperature.

(b) Does the result in part (a) hold for points lying on any circle on the earth's surface?

(c) Does the result in part (a) hold for barometric pressure and for altitude above sea level?

12. If f is a differentiable function and $g(x) = xf(x)$, use the definition of a derivative to show that $g'(x) = xf'(x) + f(x)$.

13. Suppose f is a function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose also that

$$\lim_{x\to 0} \frac{f(x)}{x} = 1$$

(a) Find $f(0)$. (b) Find $f'(0)$. (c) Find $f'(x)$.

14. Suppose f is a function with the property that $|f(x)| \le x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.