Homework — Numerical Linear Algebra
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1. Sherman-Morrison-Woodbury formula

$$(A + UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$$

Where $A \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times k}, C \in \mathbb{R}^{k\times k}, V \in \mathbb{R}^{k\times n}$.

Proof.

$$\begin{aligned}
&(A + UCV)(A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}) \\
&= I - U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} + UCVA^{-1} - UCVA^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \\
&= I + UCVA^{-1} - U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} - UCVA^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \\
&= I + UCVA^{-1} - (U(C^{-1} + VA^{-1}U)^{-1} + UCVA^{-1}U(C^{-1} + VA^{-1}U)^{-1})VA^{-1} \\
&= I + UCVA^{-1} - (U + UCVA^{-1}U)(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \\
&= I + UCVA^{-1} - UC(C^{-1} + VA^{-1}U)(C^{-1} + VA^{-1}U)^{-1}VA^{-1} \\
&= I + UCVA^{-1} - UCVA^{-1} \\
&= I
\end{aligned}$$

(source: `https://en.wikipedia.org/wiki/Woodbury_matrix_identity#Direct_proof`) □

2. (a)

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} = HH^T$$

(b)

$$A = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 2 & -3 \\ 0 & -3 & 10 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} = HH^T$$

3. (a) If we try to decompose the matrix we get

$$\begin{aligned}
A = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} &= \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \\
&= \begin{pmatrix} a^2 & ab \\ ab & b^2 + c^2 \end{pmatrix} \\
&\Rightarrow \begin{cases} a^2 = 1 \\ ab = 2 \\ b^2 + c^2 = 2 \end{cases} \quad \Rightarrow \begin{cases} a = 1 \\ b = 2 \\ c^2 = -2. \end{cases}
\end{aligned}$$

But it is also clear that Cholesky decomposition would fail, since the matrix is not positive semi-definite.

(b) The matrix is not positive semi-definite, thus it won't have a Cholesky decomposition.

4.

$$A = \begin{pmatrix} 10 & 3 & 0 \\ 4 & 9 & 4 \\ 0 & 3 & 5 \end{pmatrix}, \quad Ax = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

(a) i.

$$x_1^{(k+1)} = \frac{b_1 - 3x_2^{(k)}}{10} \qquad x_2^{(k+1)} = \frac{b_2 - 4x_1^{(k)} - 4x_3^{(k)}}{9} \qquad x_3^{(k+1)} = \frac{b_3 - 3x_2^{(k)}}{5}$$

ii.

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x^{(k+1)} &= -\begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \left(\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} \right) x^{(k)} + \begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} b \\ &= -\begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} \begin{pmatrix} 0 & 3 & 0 \\ 4 & 0 & 4 \\ 0 & 3 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} b \\ &= -\begin{pmatrix} 0 & 3/10 & 0 \\ 4/9 & 0 & 4/9 \\ 0 & 3/5 & 0 \end{pmatrix} x^{(k)} + \begin{pmatrix} 1/10 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} b \end{aligned}$$

(b) i.

$$x_1^{(k+1)} = \frac{b_1 - 3x_2^{(k)}}{10}, \qquad x_2^{(k+1)} = \frac{b_2 - 4x_1^{(k+1)} - 4x_3^{(k)}}{9} \qquad x_3^{(k+1)} = \frac{b_3 - 3x_2^{(k+1)}}{5}$$

ii.

$$x^{(k+1)} = (D+L)^{-1}(b - Ux^{(k)})$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x^{(k+1)} &= \left(\begin{pmatrix} 10 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \right)^{-1} \left(b - \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} x^{(k)} \right) \\ &= \begin{pmatrix} 10 & 0 & 0 \\ 4 & 9 & 0 \\ 0 & 3 & 5 \end{pmatrix}^{-1} \left(b - \begin{pmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} x^{(k)} \right) \end{aligned}$$

(c) i. sufficient condition $\|A\|_1 = 15 \not< 1$ — not satisfied.

ii. necessary condition $\rho(A) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} \approx 2.531 \not< 1$ — not satisfied.

(d) We know that

$$\|e^{(k+1)}\| = \|B^{k+1}e^{(0)}\| \leq \|B^{k+1}\| \cdot \|e^{(0)}\| \leq \|B\|^{k+1} \cdot \|e^{(0)}\|$$

We can calculate $\|B\|$ to be $\|B\|_1 = 9/10$ for the Jacobi method and $\|B\|_1 = 1/5$ for the Gauss-Seidel method. We get that for Jacobi method we would need $\lceil \log_{9/10}(1/10) \rceil = \left\lceil \frac{\log_{10} 1 - 1}{\log_{10} 9 - 1} \right\rceil = \left\lceil \frac{1}{1 - \log_{10} 9} \right\rceil = 22$ iterations. For Gauss-Seidel method we would need $\lceil \log_{1/5}(1/10) \rceil = 2$. Allthough, from the previous sub-task we know that neither of the methods will converge.

5.

$$\begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix}, \quad \omega = 1.5, \quad u^{(0)} = v^{(0)} = w^{(0)} = 0$$

#1

$$u^{(1)} = \frac{1.5}{3}6 = 3$$
$$v^{(1)} = \frac{1.5}{3}(3-3) = 0$$
$$w^{(1)} = \frac{1.5}{3}(5-3) = 1$$

#2

$$u^{(2)} = \frac{1.5}{3}(6-1) + (1-1.5)3 = 1$$
$$v^{(2)} = \frac{1.5}{3}(3-1-1) = 0.5$$
$$w^{(2)} = \frac{1.5}{3}(5-1.5) + (1-1.5) = 1.25$$

Result:

$$\begin{pmatrix} u^{(2)} \\ v^{(2)} \\ w^{(2)} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \\ 1.25 \end{pmatrix}$$

6.

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 4 \end{pmatrix}, \quad P = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$