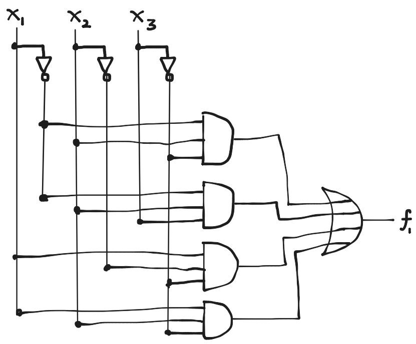
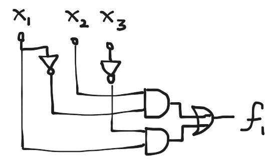
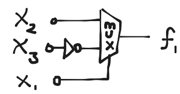
1.

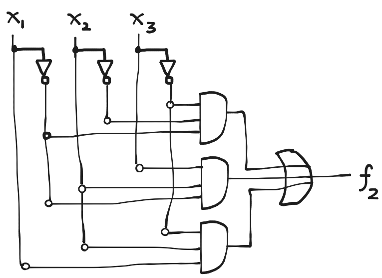


$\Longleftrightarrow$

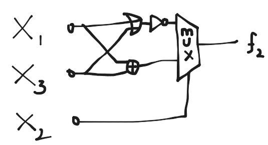


$\Longleftrightarrow$





$\Longleftrightarrow$



2. Given

$$y[i] = \begin{cases} 1 & i = \langle x \rangle \bmod 2^n,\ x[n-1] = 0 \\ 0 & i \neq \langle x \rangle \bmod 2^n \end{cases} \qquad 0 \leq i \leq 2^{n-1} - 1$$

$$y[i + 2^{n-1}] = \begin{cases} 1 & i = \langle x \rangle \bmod 2^{n-1},\ x[n-1] = 1 \\ 0 & i \neq \langle x \rangle \bmod 2^{n-1} \end{cases} \qquad 0 \leq i \leq 2^{n-1} - 1$$

we can say that

$$y[i] = \begin{cases} 1 & i = \langle x \rangle \bmod 2^{n-1} + x[n-1] \cdot 2^{n-1} \\ 0 & i \neq \langle x \rangle \bmod 2^{n-1} + x[n-1] \cdot 2^{n-1} \end{cases} \qquad 0 \leq i \leq 2^n - 1$$

$$\Downarrow$$

$$y[i] = \begin{cases} 1 & i = \langle x \rangle \\ 0 & i \neq \langle x \rangle \end{cases} \qquad 0 \leq i \leq 2^n - 1$$

$$\Downarrow$$

$$\langle y \rangle = 2^{\langle x \rangle} \iff y[\langle x \rangle] = 1$$

It is clear that this is a decoder.

3.

$$
\begin{aligned}
\langle x_0\rangle + \langle c_0\rangle &= \langle c_1 s_0\rangle \\
\langle x_1\rangle + \langle c_1\rangle &= \langle c_2 s_1\rangle \\
\langle x_1 0\rangle + \langle c_1 0\rangle &= \langle c_2 s_1 0\rangle \\
\langle x_1 0\rangle + \langle c_1 0\rangle + \langle s_0\rangle &= \langle c_2 s_1 s_0\rangle \\
\langle x_1 0\rangle + \langle c_1 s_0\rangle &= \langle c_2 s_1 s_0\rangle \\
\langle x_1 0\rangle + \langle x_0\rangle + \langle c_0\rangle &= \langle c_2 s_1 s_0\rangle \\
\langle x_1 x_0\rangle + \langle c_0\rangle &= \langle c_2 s_1 s_0\rangle \\
\langle x_1 x_0\rangle + 1 &= \langle c_2 s_1 s_0\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\langle c_2 s_1 s_0\rangle &= \langle x_1 x_0\rangle + 1 \\
\langle c_2\rangle \cdot 4 + \langle s_1\rangle \cdot 2 + \langle s_0\rangle &= \langle x_1\rangle \cdot 2 + \langle x_0\rangle + 1 \\
\langle s_1\rangle \cdot 2 + \langle s_0\rangle \bmod 4 &= \langle x_1\rangle \cdot 2 + \langle x_0\rangle + 1 \\
\langle s_1 s_0\rangle \bmod 4 &= \langle x_1 x_0\rangle + 1
\end{aligned}
$$

4.

$$\langle h^0.R\rangle = 0$$

$$
\begin{aligned}
\langle h^{t+1}.R\rangle &= \langle h^t.R\rangle + 1 \bmod 4 \\
&= (t \bmod 4) + 1 \bmod 4 \\
&= t + 1 \bmod 4
\end{aligned}
$$

$$\langle h^{t+1}.R\rangle = t + 1 \bmod 4 \iff h^{t+1}.R = \mathrm{bin}_2(t + 1 \bmod 4)$$