3 Sorting Algorithms

Sorting

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output: sequence $(a(\pi(1)), \dots, a(\pi(n)))$ with $a(\pi(1)) \le \cdots \le a(\pi(n))$

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stop at

$$n - x = 1 \quad , \quad x = n - 1$$

$$M(n) = n - 1$$

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$$\begin{aligned} S(n) &= M(n) + S(n-1) \\ &= n - 1 + S(n-1) \\ &= n - 1 + n - 2 + S(n-2) \\ &\dots \\ &= \sum_{i=n-x}^{n-1} i + S(n-x) \end{aligned}$$

$n - x = 1$

$$S(n) = \sum_{i=1}^{n-1} i = (n-1) \cdot n/2$$

Merge Sort

merge 2 sorted lists of length n/2

input: sorted sequences

$(a(1), \dots, a(n/2))$

$(b(1)), \dots, b(n/2))$

with

$a(1) \leq \cdots \leq a(n/2)$

$b(1) \leq \cdots \leq b(n/2)$

output: merged sequence $(c(1), \dots, c(n))$ sorted $c(1) \leq \cdots \leq c(n))$

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input: sorted sequences

$(a(1), \ldots, a(n))$

$(b(1)), \ldots, b(m))$

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$a(1) \leq \cdots \leq a(n)$

$b(1) \leq \cdots \leq b(nm)$

output: merged sequence $(c(1), \ldots, c(n+m))$ sorted

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$$merge((a(1),\ldots,a(n)),(b(1),\ldots b(m)))$$
$$= \begin{cases} a(1) \circ merge((a(2),\ldots,a(n)),(b(1),\ldots b(m))) & a(1) \leq b(1) \\ b(1) \circ merge((a(1),\ldots,a(n)),(b(2),\ldots b(m))) & a(1) > b(1) \end{cases}$$

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$$sort((a(1), \dots, a(n)) = merge(sort(a(1) \dots a(n/2)), sort(a(n/2+1), \dots, a(n)))$$

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difference equations (and difference inequalities)

Let

$$n = 2^k \quad k \in \mathbb{N}$$

be a power of two and assume

$$\boxed{\begin{array}{rcl} f(1) & = & a \\ f(n) & = & 2 \cdot f(n/2) + b \cdot n \end{array}}$$
$$f(n) = ?$$

Idea: expand definition until you see something

$$\begin{array}{rcl} f(n) & = & 2 \cdot f(n/2) + b \cdot n \\ & = & 2 \cdot (2 \cdot f(n/4) + b \cdot n/2) + b \cdot n \\ & = & 2^2 \cdot f(n/2^2) + 2 \cdot b \cdot n \\ & = & \ldots \\ & = & 2^x \cdot f(n/2^x) + x \cdot b \cdot n \end{array}$$

Stop recursion at

$$n/2^x = 1 \quad , \quad x = k = \log n$$

Conjecture

$$\boxed{\begin{array}{rcl} f(n) & = & n \cdot f(1) + b \cdot n \cdot \log(n) \\ & = & b \cdot n \cdot \log n + a \cdot n \end{array}}$$

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$$\boxed{S(n) < n \cdot \log n}$$

can we sort asymptotically faster with comparisons?

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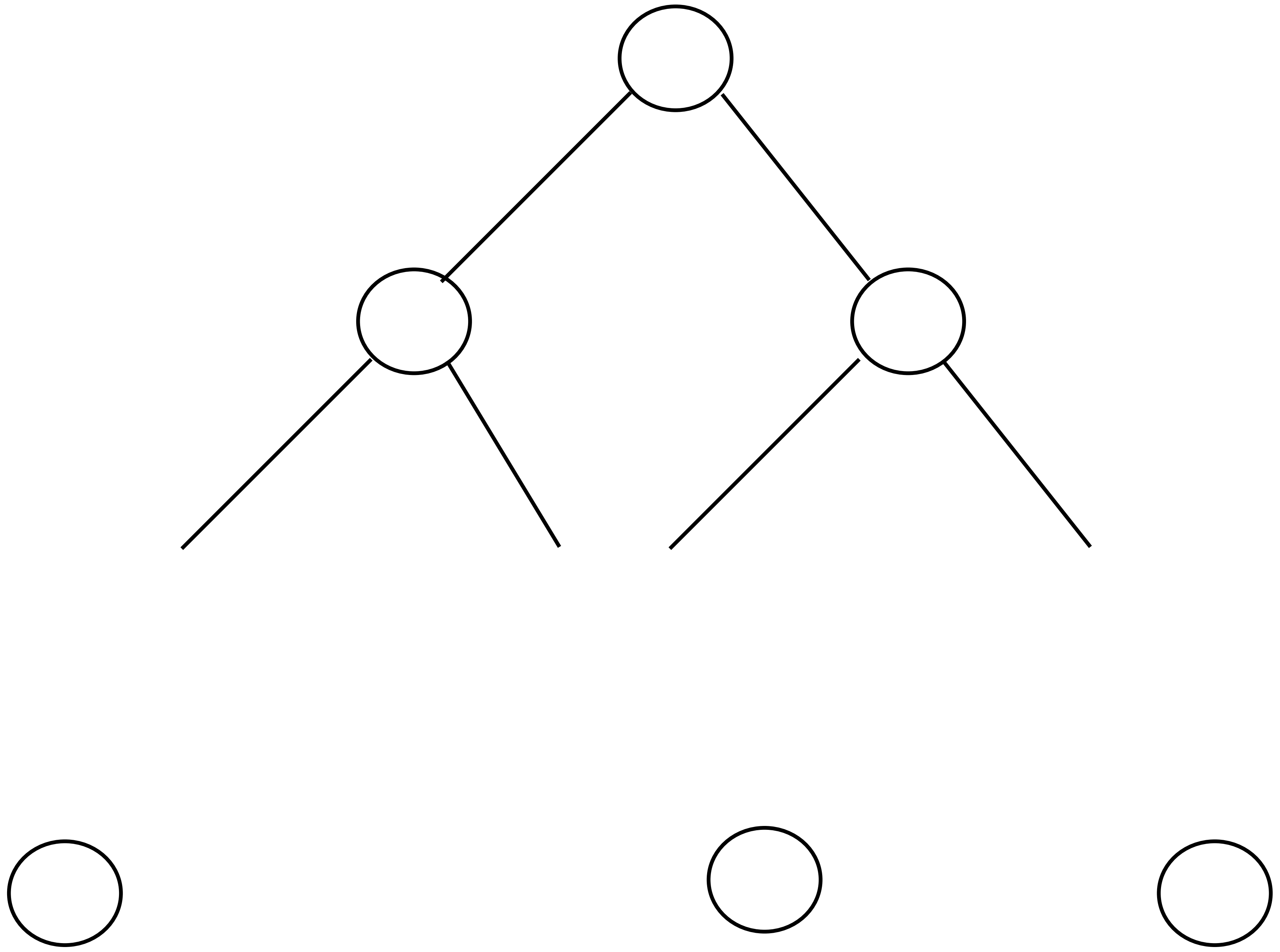

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- nodes: comparisons

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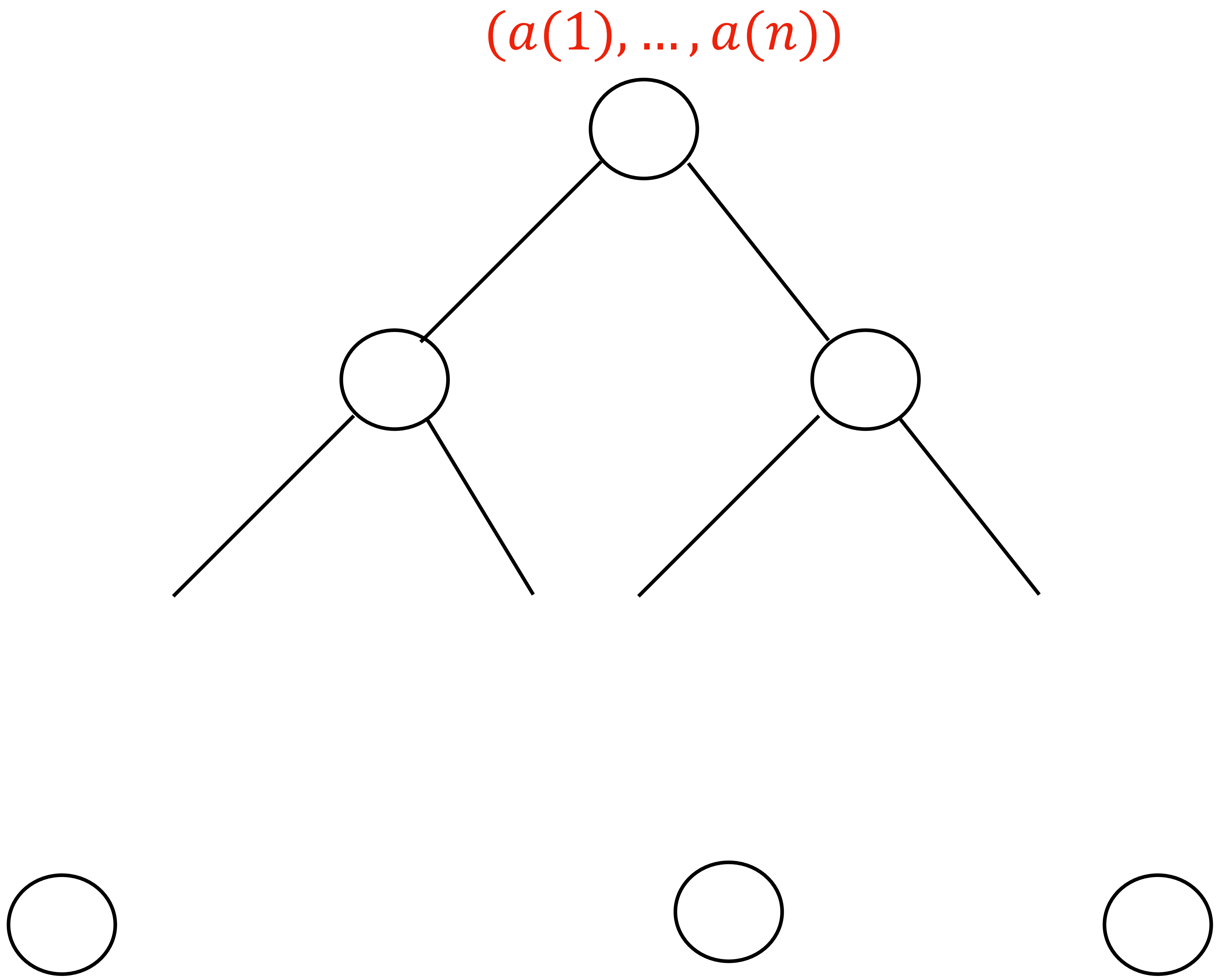

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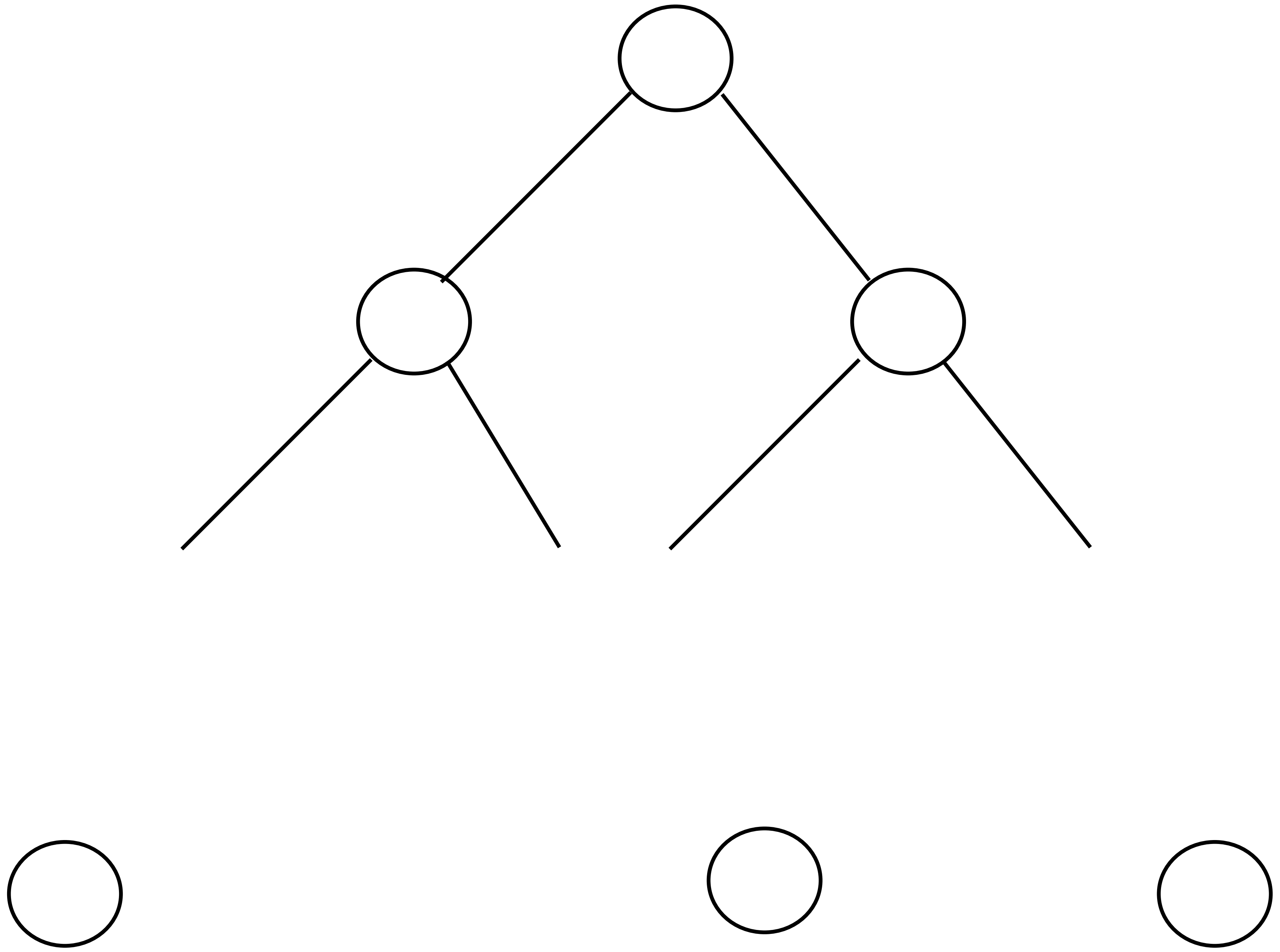

$(a(\pi(1)), \ldots, a(\pi(n)))$

can represent any such algorithm as binary tree

- nodes: comparisons
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- at leaves: sorted sequence
- one leaf for each permutation π

depth of tree = length of longest path = worst case number of comparisons

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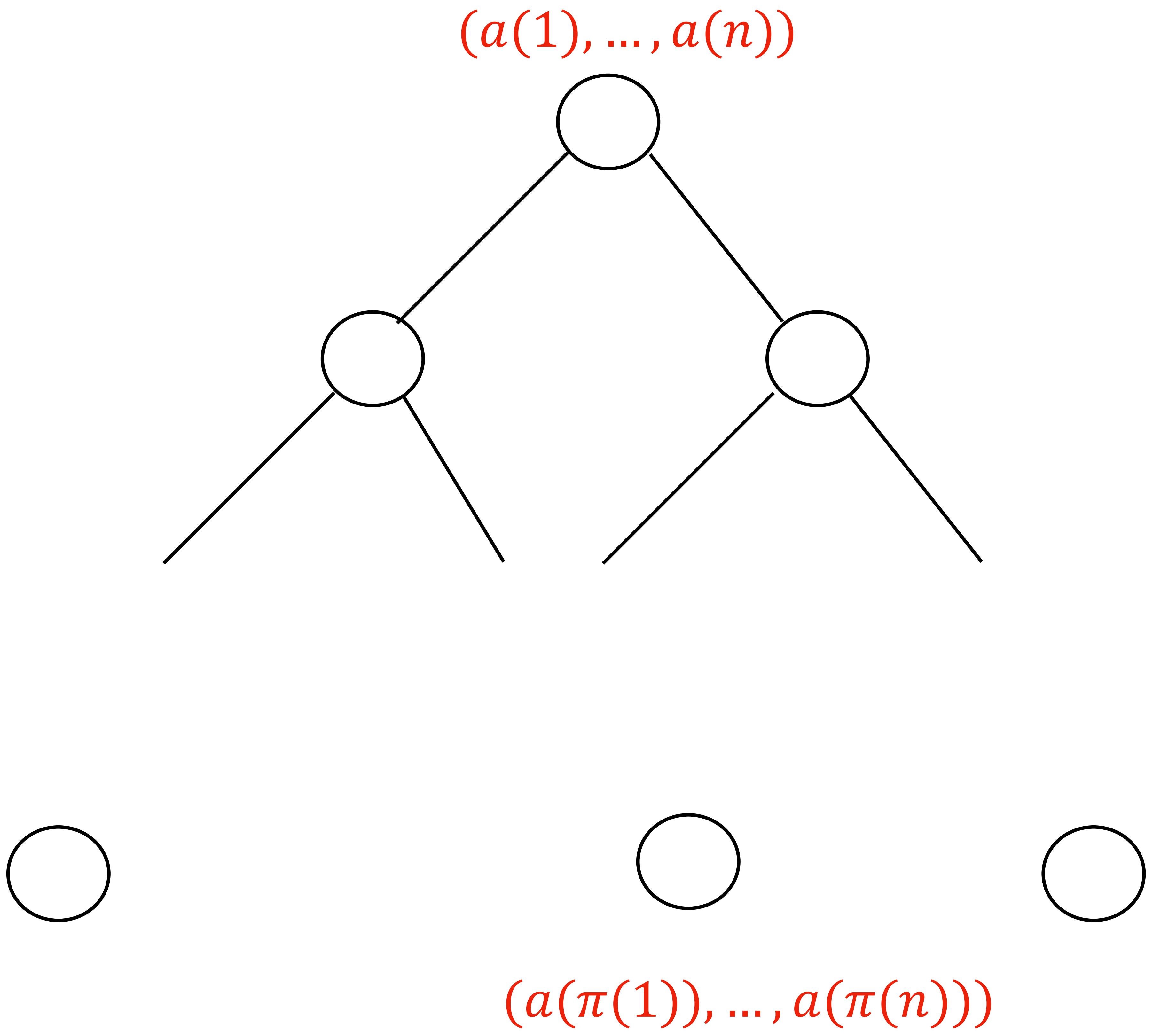

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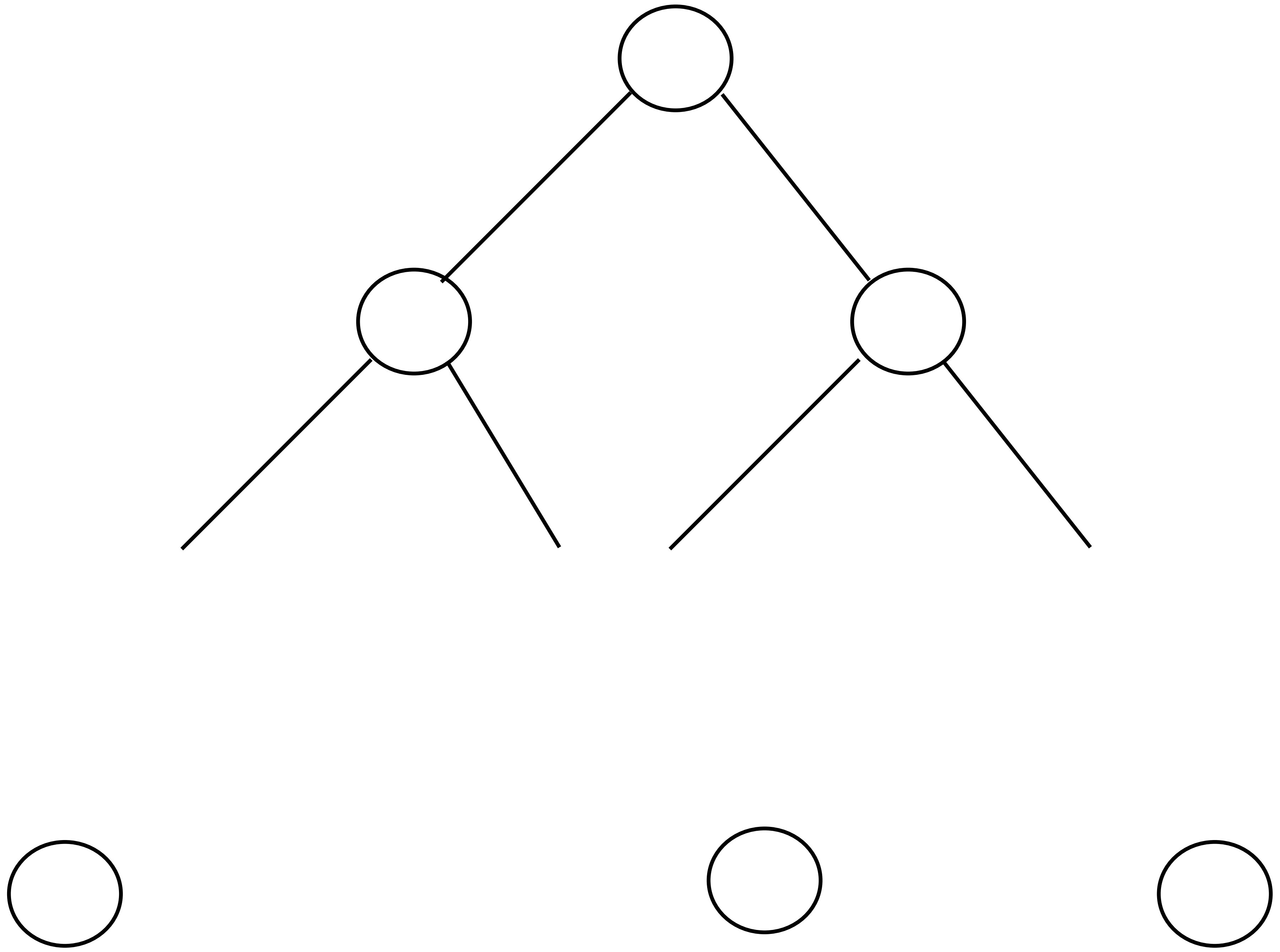

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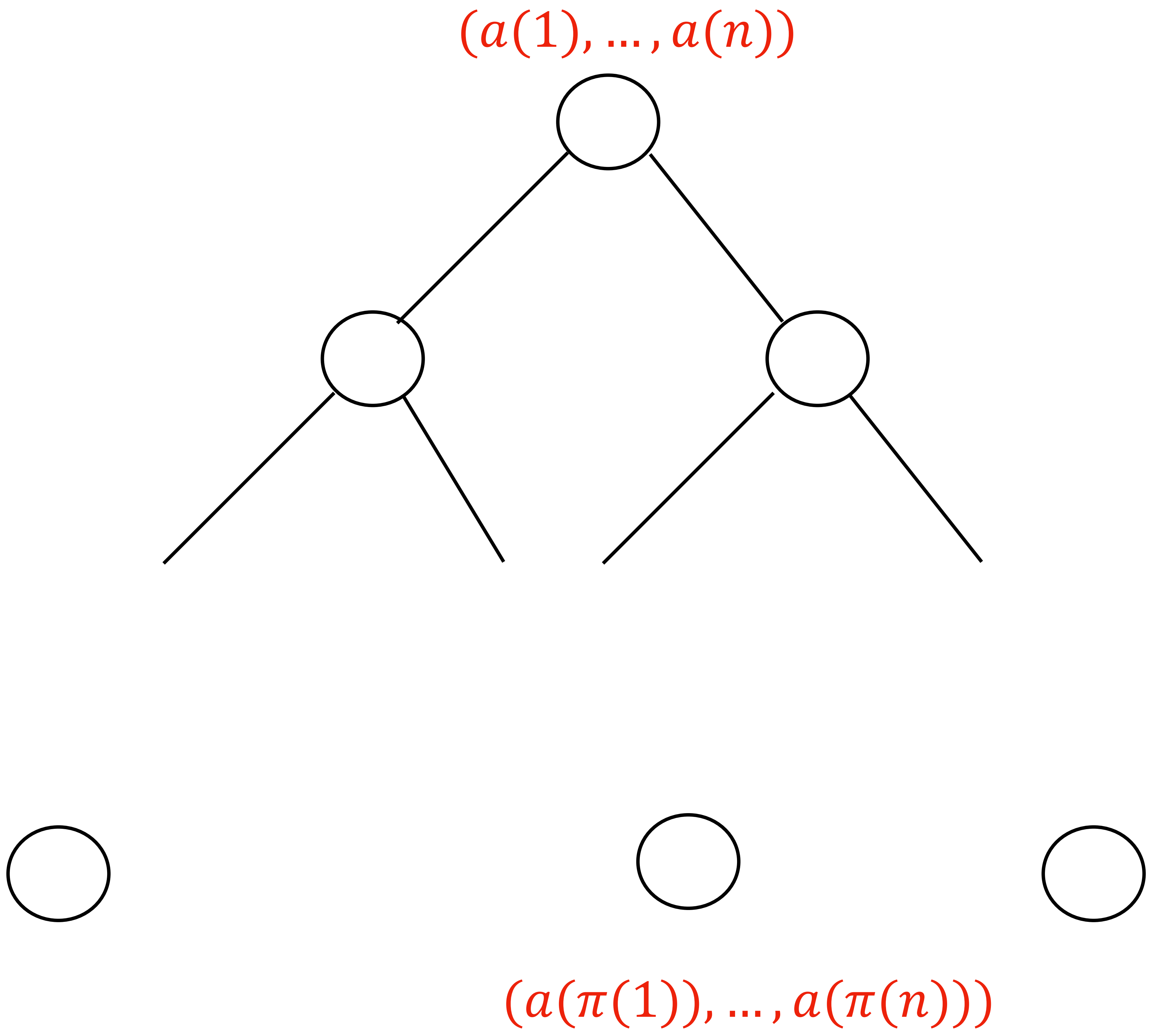

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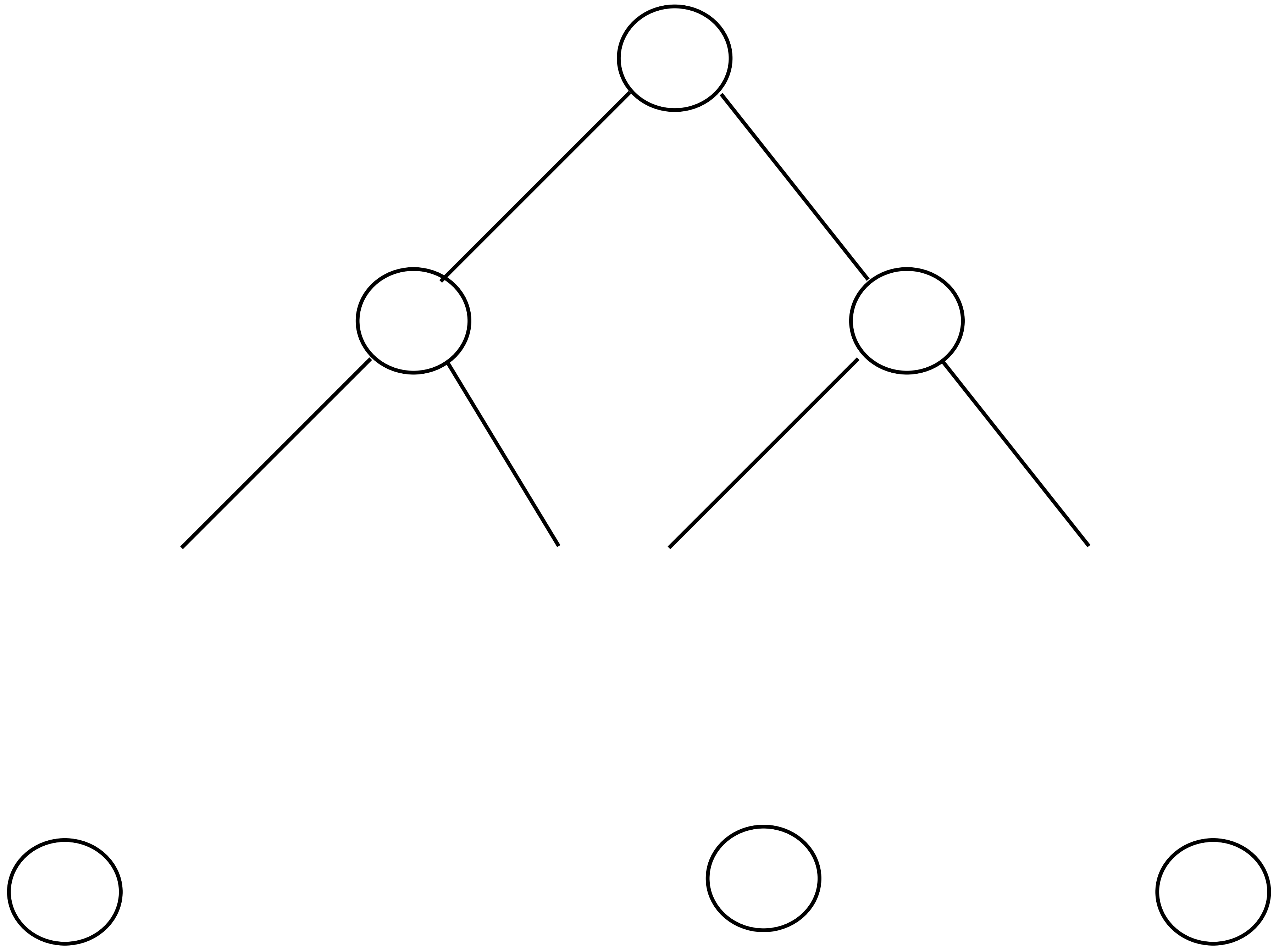

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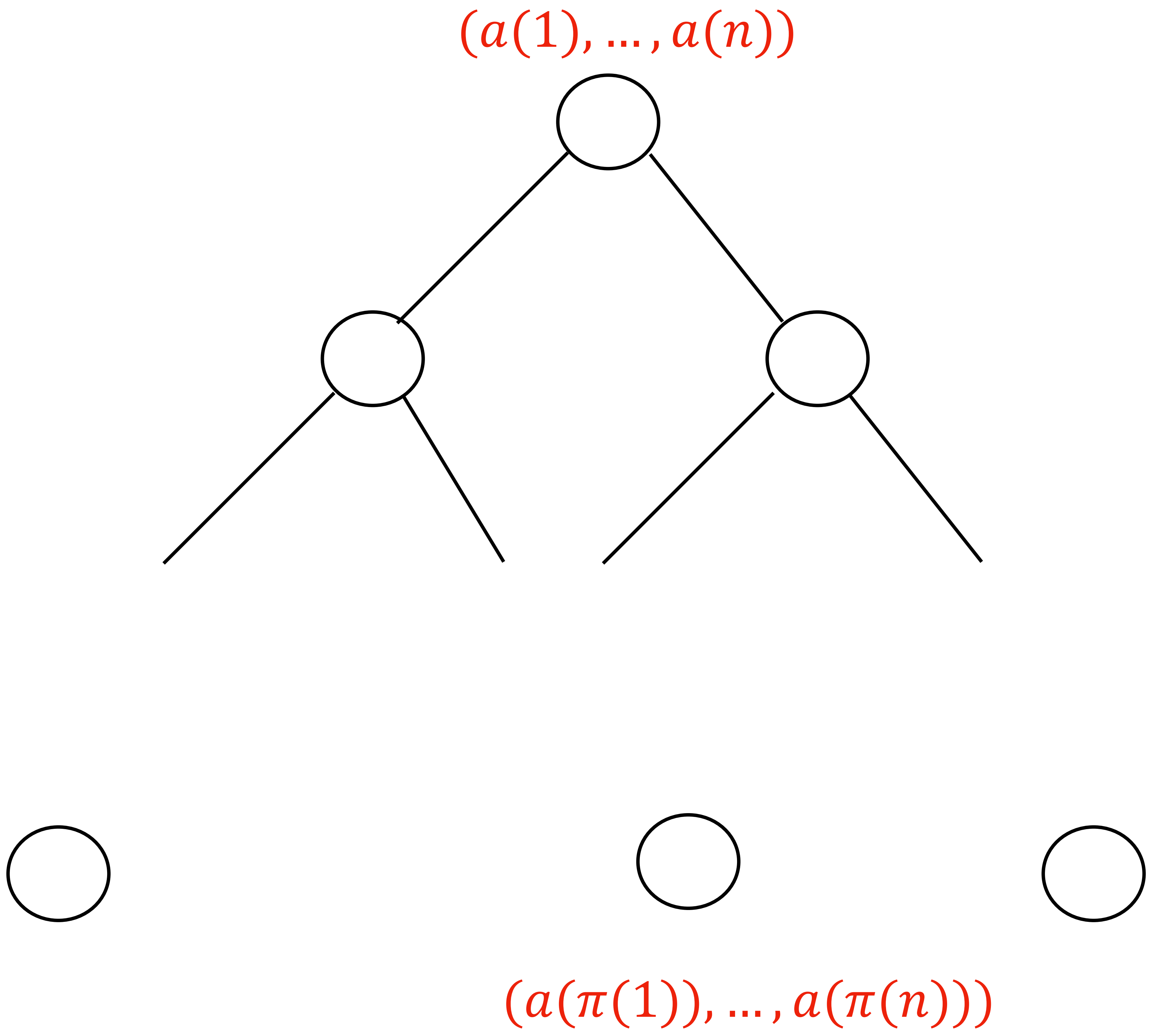

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based on ‚random events‘
whatever randomness is…

more in ‚theoretical computer science‘
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- you analyze as if the generated data was truly random
- the measured run time (usually) matches the analysis

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why does probabilistic modeling
of a completely deterministic -provably not random - algorithm
give the right results??

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Quicksort

here: the algorithm does the random experiments

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Quicksort. 1960, Tony Hoare

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2013

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$T(n)$ expected run time for sorting n elements

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this should have a proof!