Guidelines for solutions of problems. Sections 2.5, 2.6

Name and section: ____________________

Instructor's name: ____________________

1. Find the slope of the tangent line to the graph of the function $y = x - x^2$ at the point $(1, 0)$;

(i) using the definition of derivative given by the formula:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a};$$

(ii) using the formula:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

(b) Find an equation of the tangent line in part (a).

Solution. (a) (i) The slope of the tangent line is:

$$m = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x - x^2 - (1 - 1^2)}{x - 1}$$
$$= \lim_{x \to 1} \frac{x - x^2}{x - 1} = \lim_{x \to 1} \frac{x(1-x)}{x - 1} \lim_{x \to 1}(-x) = -1.$$

(ii)

$$m = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h) - (1+h)^2 - 0}{h} = \lim_{h \to 0} \frac{(1+h) - (1+h)^2}{h}$$
$$\lim_{h \to 0} \frac{h^2 - h}{h} = \lim_{h \to 0}(h - 1) = -1.$$

(b) Equation of the line passing through (1,0) with the slope $m = -1$ is:
$y - 0 = (-1)(x - 1) = -x + 1$. Thus equation is

$$y = -x + 1.$$

Answer: $y = -x + 1$.

2. Find an equation of the tangent line to the graph of the function $y = 4x - 3x^2$ at the point $(2, -4)$.

Solution.
First of all find the slope of the tangent line:

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{4x - 3x^2 - (-4)}{x - 2} =$$

$$= \lim_{x\to 2} \frac{-3x^2+4x+4}{x-2} = \lim_{x\to 2} \frac{(-3)(x-2)(x+2/3)}{x-2} = \lim_{x\to 2})((-3)(x+2/3)) = -8.$$

Thus, equation of the tangent line is:

$y-(-4) = (-8)(x-2) \Rightarrow y = -8x+12.$

Answer: $y = -8x + 12$.

3. If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$

   Solution. velocity at t sec is given by the formula $v(t) = y'(t)$. In our case, we have $v(2) = y'(2) = \lim_{t\to 2} \frac{y(t)-y(2)}{t-2} = \lim_{t\to 2} \frac{40t-16t^2-y(2)}{t-2} = \lim_{t\to 2} \frac{40t-16t^2-16}{t-2} = \lim_{t\to 2} \frac{-16(t-2)t-1/2)}{t-2} = -16\lim_{t\to 2}(t-1/2) = -24.$

   Answer: -24 ft/s

4. The displacement (in feet) of a particle moving in a straight line is given by $s = \frac{1}{2}t^2 - 6t + 23$, where t is measured in seconds.

   (a) Find the average velocity over each time interval:

   $[4,8]$; $[6,8]$;

   (b) Find the instantaneous velocity when $t = 8$;

   Solution. (a) average velocity over the time interval $[4,8]$ is equal to

$$\frac{s(8)-s(4)}{8-4} = \frac{7-7}{8-4} = 0;$$

   average velocity over the time interval $[6,8]$ is

$$\frac{s(8)-s(6)}{8-6} = \frac{7-13}{8-6} = \frac{-6}{2} = -3.$$

   (b) instantaneous velocity when $t = 8$ is equal to

$$s'(8) = \lim_{t\to 8} \frac{s(t)-s(8)}{t-8} = \lim_{t\to 8} \frac{\frac{1}{2}t^2-6t+23-7}{t-8} = \lim_{t\to 8} \frac{\frac{1}{2}(t-4)(t-8)}{t-8} = \lim_{t\to 8} \frac{1}{2}(t-4) = \frac{1}{2}\lim_{t\to 8}(t-4) = \frac{1}{2}\cdot 8 = 4.$$

5. If the tangent line to $y = f(x)$ at $(4,3)$ passes through the point $(0,2)$, find $f(4)$ and $f'(4)$.

   Solution. It is clear that $f(4) = 3$ because the point (4.3) lies on the graph of $y = f(x)$. Further, since the tangent line passes through at $(4,3)$ and $(0,2)$, we have that the slope of this line would be

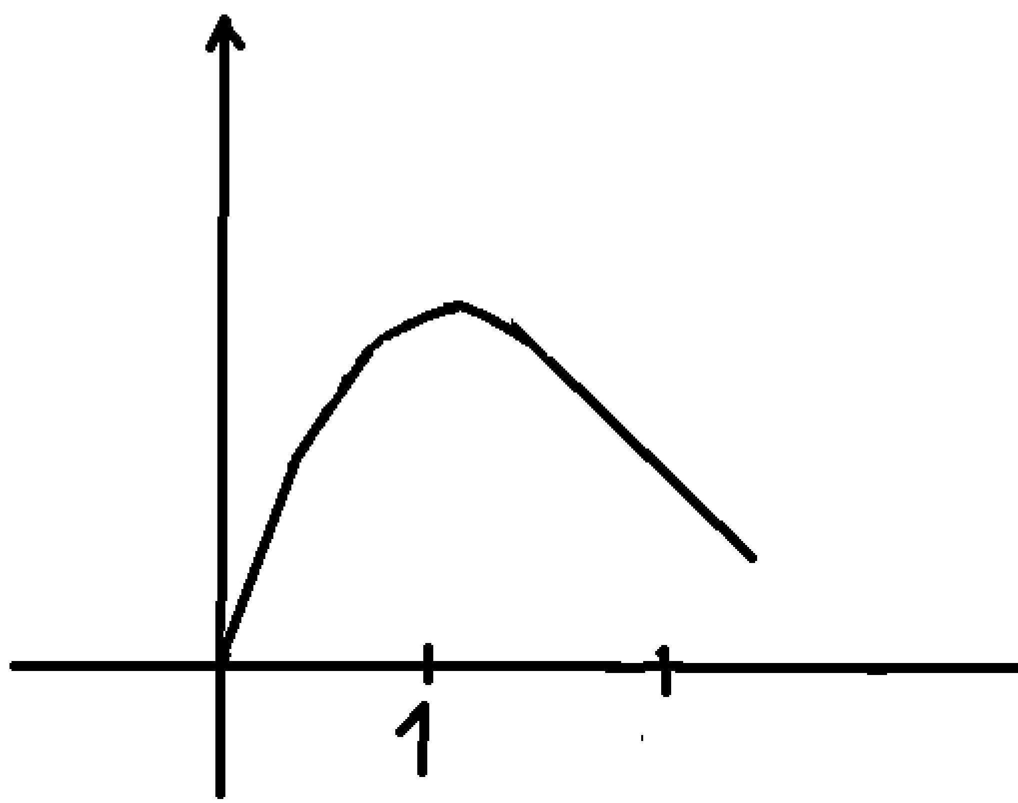
$$m = \frac{2-3}{0-4} = \frac{1}{4}.$$

   Hence,

$$f'(4) = m = \frac{1}{4}.$$

6. Sketch the graph of a function f for which $f(0) = 0$, $f'(0) = 3$, $f'(1) = 0$, $f'(2) = -1$.

   Solution. By assumption $f'(0) = 3$, $f'(1) = 0$, $f'(2) = -1$. This means that slopes of tangent lines at $(0, f(0))$, $(1, f(1))$ and $(2, f(2))$ are equal to 3, 0 and -1 respectively. The graph is given, for example, by the curve:



7. The limit represents the derivative of some function f at some number a. State such an f:
   $\lim\limits_{h\to 0} \frac{\sqrt{9+h}-3}{h}$.

   Solution. This limit is the derivative of the function $f(x) = \sqrt{x}$ at $t = 9$.

8. A particle moves along a straight line with equation of motion $s = f(t)$, where s is measured in meters and t in seconds. Find the velocity and the speed when $t = 4$ if $f(t) = 80t - 6t^2$.

   Solution. Recall that velocity of a particle at $= a$ is $f'(a)$; speed is $|f'(a)|$.

$f'(4) = \lim\limits_{t\to 4} \frac{f(t)-f(4)}{t-4} = \lim\limits_{t\to 4} \frac{(80t-6t^2)-224}{t-4} = \lim\limits_{t\to 4} \frac{(3t-28)t-4}{t-4} = \lim\limits_{t\to 4}(3t-28) = -16.$

Thus, velocity is $f'(4) = 16$; speed is $|f'(4)| = 12$.

9. Determine whether $f'(0)$ exists if

$$f(x) = \begin{cases} x\sin\frac{1}{x}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

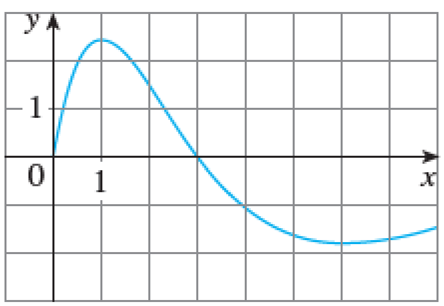
Solution. We have

$$\frac{f(x)-f(0)}{x-0} = \frac{f(x)-f(0)}{x} = \sin\frac{1}{x}.$$

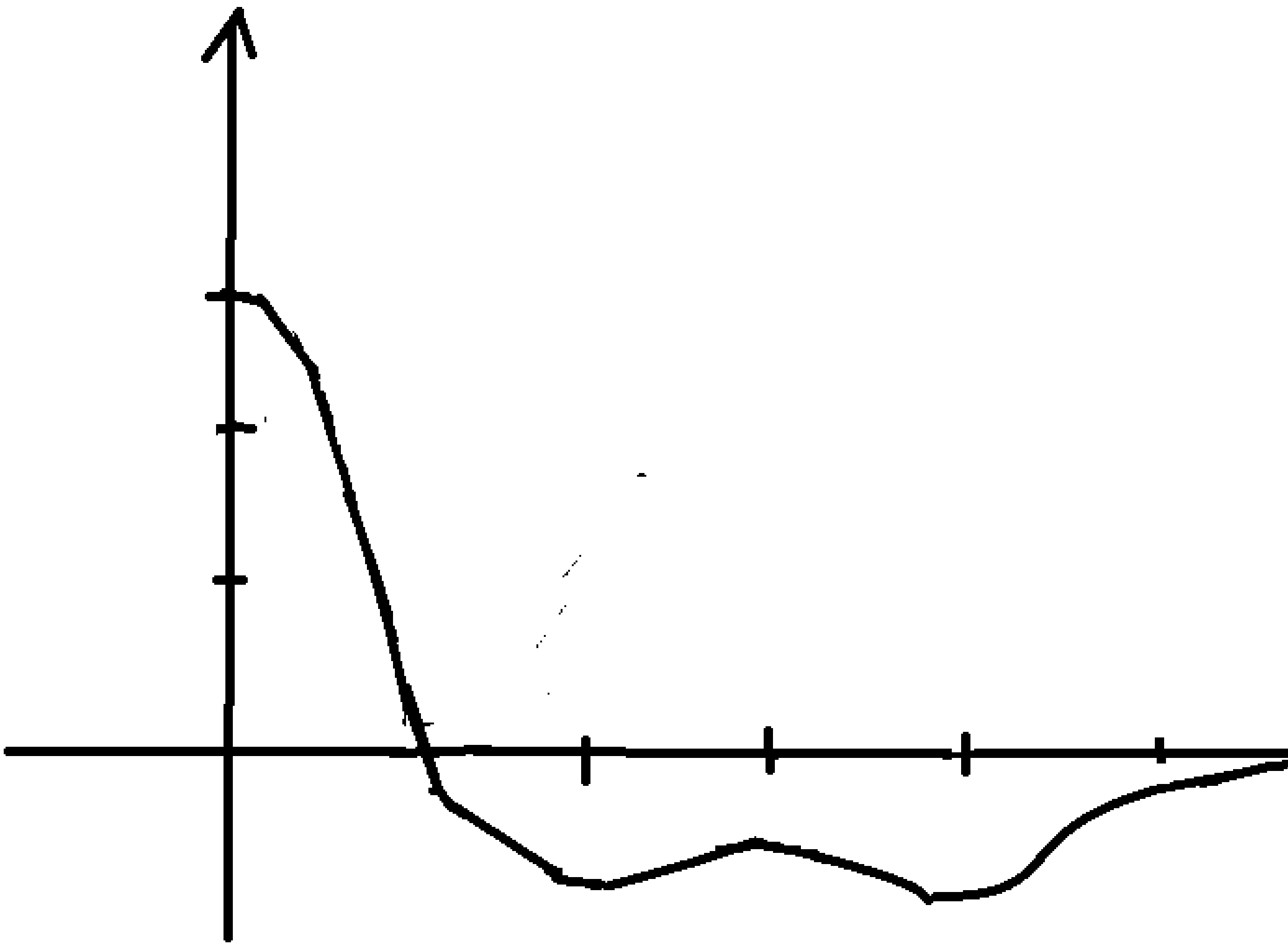
We know that the limit of $\frac{f(x)-f(0)}{x-0} = \sin\frac{1}{x}$ does not exist, because taking, for example, two different sequences $x_n = \frac{1}{\pi n}$ and $x_n = \frac{1}{\pi/2+2\pi n}$ approaching to 0, we see that along these sequences we get different limits:

$\lim\limits_{n\to\infty} \sin\frac{1}{x_n} = 0$ and $\lim\limits_{n\to\infty} \sin\frac{1}{x'_n} = 1.$

10. Use the given graph to estimate the value of each derivative. Then sketch the graph of f':
(a) $f'(0)$; (b) $f'(1)$; (c) $f'(2)$; (d) $f'(3)$; (e) $f'(4)$; (f) $f'(5)$; (g) $f'(6)$; (h) $f'(7)$;



Solution. (a) $f'(0) \approx 3$; (b) $f'(1) \approx 0$; (c) $f'(2) \approx -1$; (d) $f'(3) \approx -1/2$; (e) $f'(4) \approx -1$; (f) $f'(5) \approx -1/4$; (g) $f'(6) \approx 0$; (h) $f'(7) \approx 1/4$.



11. Find the derivative of the function $f(x) = \sqrt{9-x}$ using the definition of derivative. State the domain of the function and the domain of its derivative.

Solution. Using the definition of derivative at a point x we have

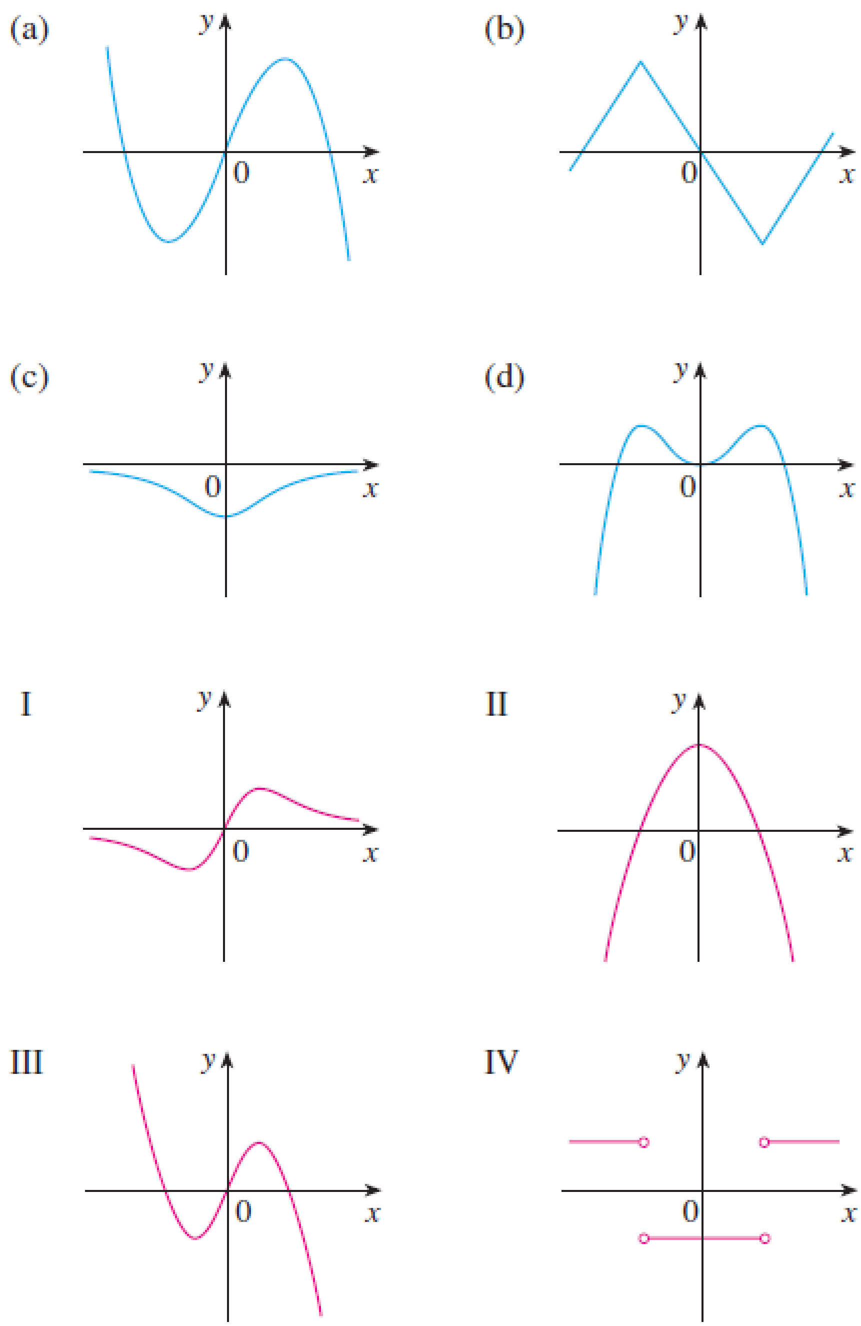
$$f'(x) = \lim_{h\to 0} \frac{\sqrt{9-(x+h)} - \sqrt{9-x}}{h} = \lim_{h\to 0} \frac{(\sqrt{9-(x+h)} - \sqrt{9-x})(\sqrt{9-(x+h)} + \sqrt{9-x})}{(\sqrt{9-(x+h)} + \sqrt{9-x})h}$$

$$= \lim_{h\to 0} \frac{-h}{(\sqrt{9-(x+h)} + \sqrt{9-x})h} = -\lim_{h\to 0} \frac{1}{\sqrt{9-(x+h)} + \sqrt{9-x}} = -\frac{1}{2\sqrt{9-x}}.$$

Answer: $-\frac{1}{2\sqrt{9-x}}$.

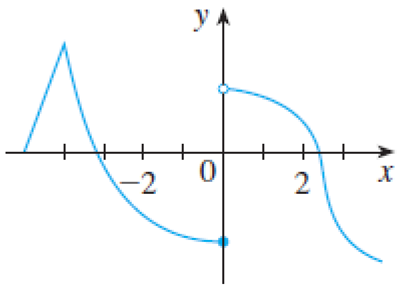
12. Match the graph of each function in (a)–(d) with the graph of its derivative in I–IV. Give reasons for

your choices.



Solution. We have the following correspondence: $(a) \to (ii)$; $(b) \to (iv)$; $(c) \to (i)$; $(d) \to (iii)$.

13. The graph of f is given. State, with reasons, the numbers at which f is not differentiable.



Solution. At the following points derivatives do not exist: $a = -4$ and $a = 0$.

In particular, at -4 the graph has a corner; at 0 the function is discontinuous (has jump discontinuity).

14. Let $g(x) = x^{2/3}$.

(a) show that $g'(0)$ does not exist;

(b) find $g'(a)$ and any $a \neq 0$;

(c) Show that $y = x^{2/3}$ has a vertical tangent line at $(0,0)$;

(d) Illustrate part (c) by graphing $y = x^{2/3}$.

Solution.

(a) $\frac{f(x)-f(0)}{x-0} = \frac{f(x)}{x} = x^{-1/3}$; since $\lim\limits_{x\to 0} x^{-1/3}$ does not exist; in particular, $\lim\limits_{x\to 0+} x^{-1/3} = \infty$; $\lim\limits_{x\to 0+} x^{-1/3} = -\infty$;

(b)

$$g'(a) = \lim_{x\to a} \frac{x^{2/3} - a^{2/3}}{x - a} = \lim_{x\to a} \frac{(x^{1/3} - a^{1/3})(x^{1/3} + a^{1/3})}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{2/3}a^{2/3} + a^{2/3})}$$

$$= \lim_{x\to a} \frac{x^{1/3} + a^{1/3}}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{2a^{1/3}}{3a^{2/3}} = \frac{2}{3}a^{-1/3};$$

(c) observe that $\lim\limits_{x\to 0-} g'(x) = -\infty$; $\lim\limits_{x\to 0+} g'(x) = \infty$;

(d)