Exercises below are your homework; they will be discussed during exercise classes. Problems marked with a (*) are more challenging.

Week 5

1. Prove that (1 p)
   a) if sets A and B are countable then $A \times B$ is countable;
   b) $\mathbb{N}_0^k$ is countable for any $k \in \mathbb{N}$;
   c) the set of all subsets of natural numbers in not countable;
   d) the set of all real numbers is not countable.

2. Show that following are primitive recursive (1 p)
   a) Functions $\min(x, y)$ and $\max(x, y)$.
   b) Function $|x - y|$.

3. Show that for a primitive recursive function f, the functions of *bounded sum* and *bounded product* of f, respectively given by

$$\mathrm{bsum}_f(x_0, \dots, x_{k-1}, y) := \sum_{i=0}^{y} f(x_0, \dots, x_{k-1}, i)$$

and

$$\mathrm{bprod}_f(x_0, \dots, x_{k-1}, y) := \prod_{i=0}^{y} f(x_0, \dots, x_{k-1}, i)$$

are also primitive recursive.

4. In the lecture, up to notation, we gave following examples of primitive recursive functions
   - addition:

$$\mathrm{add}(x, 0) = x,$$
$$\mathrm{add}(x, y + 1) = s(\mathrm{add}(x, y));$$

   - multiplication:

$$\mathrm{mult}(x, 0) = 0,$$
$$\mathrm{mult}(x, y + 1) = \mathrm{add}(x, \mathrm{mult}(x, y)).$$

However, the definition of primitive recursion operation takes as arguments a k-ary function[1] g, a $k + 2$-ary function h, and returns a $k + 1$-ary function f defined as follows

$$f(x_0, \dots, x_{k-1}, 0) = g(x_0, \dots, x_{k-1}),$$
$$f(x_0, \dots, x_{k-1}, y + 1) = h(x_0, \dots, x_{k-1}, y, f(x_0, \dots, x_{k-1}, y)).$$

The definition of multiplication and addition does not fit the format of this definition. Why? Provide the definition of add and mult so that it fits this format. (1 p)

[1] By k-ary function g we mean a function $g\colon \mathbb{N}_0^k \to \mathbb{N}_0$.