Introduction to Optimization Mock Quiz (1)

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Problem 1.1:

a) We need to calculate $\nabla f(x)$ and show that it is 0 at $\binom{0}{0}$. We write

$$\nabla f(x) = \begin{pmatrix} 2x_1 \\ 3x_2^2 \end{pmatrix}$$

which gives us $\nabla f(0,0) = \binom{2 \cdot 0}{3 \cdot 0^2} = \binom{0}{0}$.

b) For f to be convex, H_f needs to be positive semi-definite everywhere. We can calculate the hessian:

$$H_f(x) = \begin{pmatrix} 2 & 0 \\ 0 & 6x_2 \end{pmatrix}$$

which has eigenvalues 2 and $6x_2$ which is clearly not bounded by zero from below, therefore f is not convex. If f were to be convex, a local minimizer would also be a global minimizer, but that is not the case.

c) For us to know that a point x is local minimizer $\nabla f(x) = 0$ should hold and $H_f(x)$ should be positive definite. We know that $\nabla f(0,0) = 0$ and we can easily calculate $H_f(0)$ to be $\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ which is positive semi-definite, not positive definite. This means that we have to show that there is no direction s and $\varepsilon > 0$ such that $f(0 + ts) < f(0) \forall t \in (0, \varepsilon)$. By looking at $f(x)$ it's easy to see that going in the direction $\binom{0}{-1}$ will lead to a decrease in the value of the function. We can check to get

$$f\left(0 + \begin{pmatrix} 0 \\ -t \end{pmatrix}\right) = f(0, -t) = -3t^3 < 0 = f(0) \forall t \in (0, \varepsilon).$$

d) For a function to be coercive $f(x)$ should tend to infinity as $\|x\|$ tends to infinity. We can show that that's not the case by picking x from the set $\{(u,v) : u^2 = -v^3\}$. This way, when $\|x\| \to \infty$, $f(x) = f(u,v) = u^2 + v^3 = 0$ stays 0, therefore f is not coercive.

Problem 1.2:

a)
- $x^1 = x^0 + \frac{1}{2}(-\nabla f(x^0)) = \binom{2}{2} - \frac{1}{2}\binom{2}{4} = \binom{1}{0}$
- $x^2 = x^1 + \frac{1}{2}(-\nabla f(x^1)) = \binom{1}{0} - \frac{1}{2}\binom{1}{0} = \binom{1/2}{0}$
- $x^3 = x^2 + \frac{1}{2}(-\nabla f(x^2)) = \binom{1/2}{0} - \frac{1}{2}\binom{1/2}{0} = \binom{1/4}{0}$

b) The global minimum of f is 0. The iterates seem to not have zig-zaged very much, however, it will take infinite iterations to converge to the solution.

Problem 1.3:

a)
- $\kappa(A) = \frac{4}{2} = 2$
- $\kappa(B) = \frac{40}{10} = 4$

b)
- A: $\sqrt{2}\left(\frac{2-1}{2+1}\right) = \frac{\sqrt{2}}{3}$
- B: $\sqrt{4}\left(\frac{4-1}{4+1}\right) = \frac{6}{5}$

c) Since both examples are diagonal matrices, using $D_{i,i} = H_f\left(x^k\right)^{-1}_{i,i}$ would give $D^{\frac{1}{2}} A D^{\frac{1}{2}} = I$ and also $D^{\frac{1}{2}} B D^{\frac{1}{2}} = I$ so

$$\sqrt{\kappa(I)}\left(\frac{\kappa(I) - 1}{\kappa(I) + 1}\right) = \sqrt{1}\left(\frac{1 - 1}{1 + 1}\right) = 0.$$

meaning it would converge in one iteration.