Guidelines for solutions of problems. Sections 2.1-2.3, 2.5, 2.6

Name and section: ____________________

Instructor's name: ____________________

1. For the function f whose graph is given, state the value of each quantity, if it exists. If it does not exist, explain why.

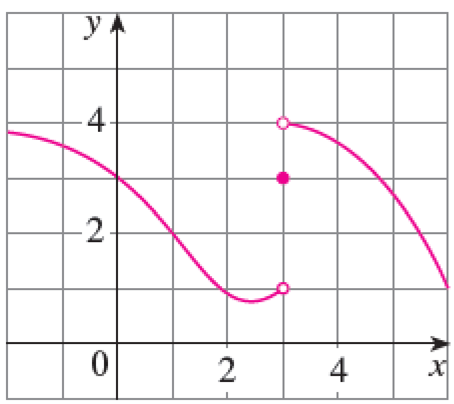
   (a) $\lim_{x\to 1} f(x)$;

   (b) $\lim_{x\to 3-} f(x)$;

   (c) $\lim_{x\to 3-} f(x)$;

   (d) $\lim_{x\to 3} f(x)$;

   (e) $f(3)$;



   Solution:

   (a) $\lim_{x\to 1} f(x) = 2$;

   (b) $\lim_{x\to 3-} f(x) = 1$;

   (c) $\lim_{x\to 3-} f(x) = 4$;

   (d) $\lim_{x\to 3} f(x)$ does not exist;

   (e) $f(3) = 3$;

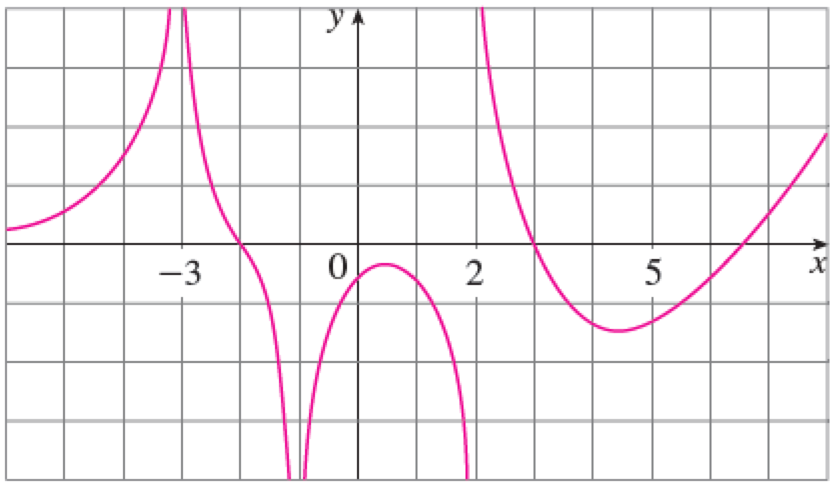
2. For the function A whose graph is shown, state the following.

   (a) $\lim_{x\to -3} A(x)$;

   (b) $\lim_{x\to 2-} A(x)$;

   (c) $\lim_{x\to 2+} A(x)$;

   (d) $\lim_{x\to -1} A(x)$



Solution

(a) $\lim_{x\to -3} A(x) = \infty$;

(b) $\lim_{x\to 2-} A(x) = -\infty$;

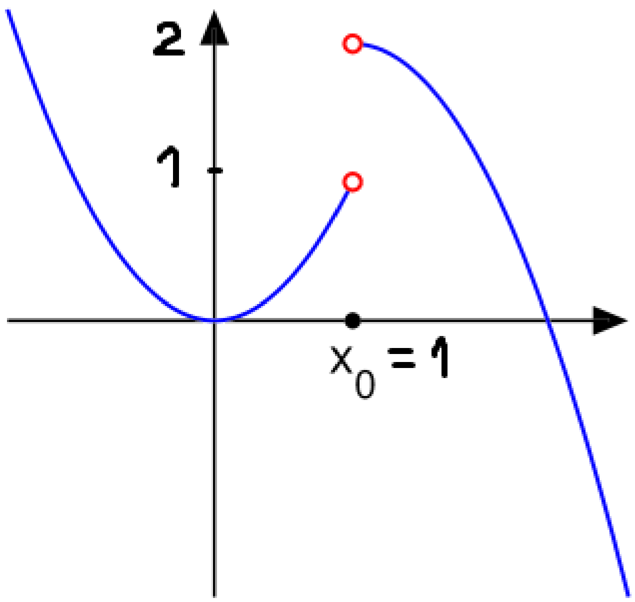
(c) $\lim_{x\to 2+} A(x) = \infty$;

(d) $\lim_{x\to -1} A(x) = -\infty$.

3. Sketch the graph of the function and use it to determine the values of a for which $\lim_{x\to a} f(x)$. Calculate $\lim_{x\to 1+} f(x)$; $\lim_{x\to 1-} f(x)$; $f(1)$ if $f(x)$ is given by

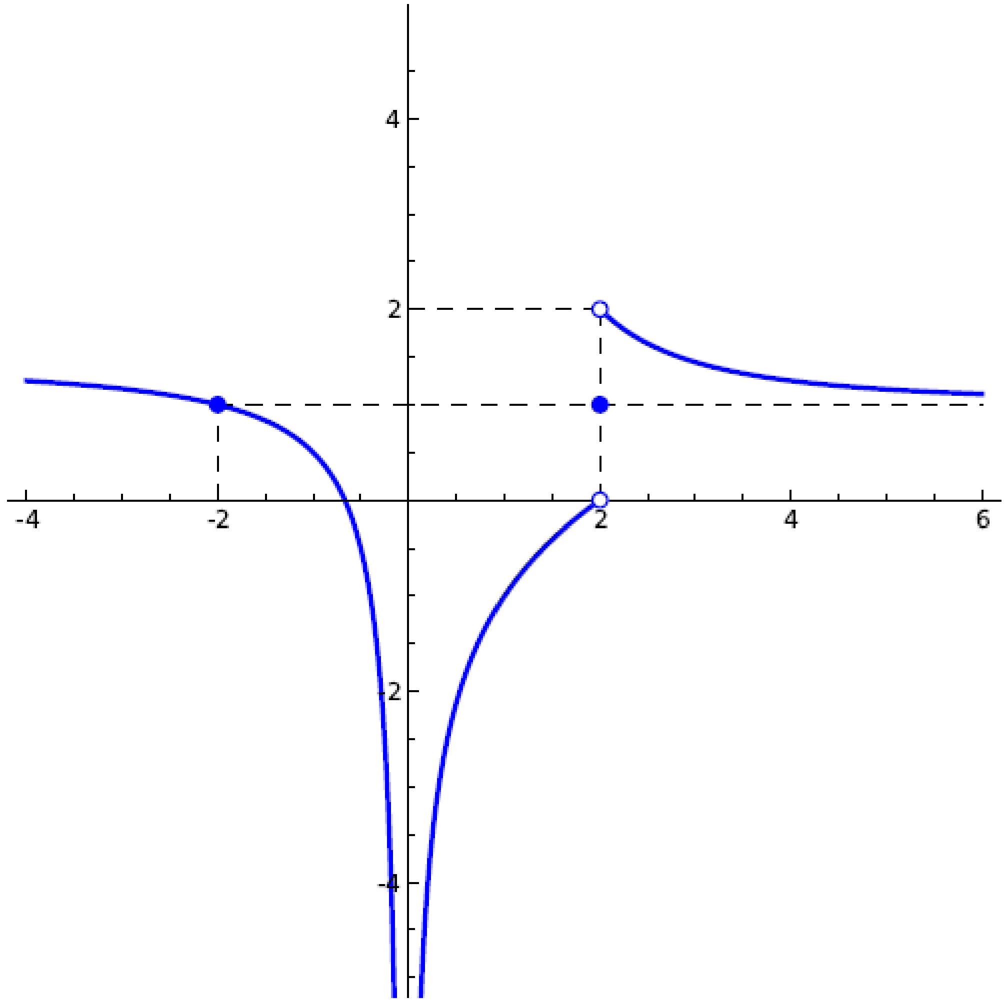
$$f(x) = \begin{cases} x^2, \ \ if \ \ x < 1, \\ -x^2 + 2x + 1, \ \ if \ \ x > 2, \\ 0, \ \ x = 1 \end{cases}$$

Solution. Observe that $-x^2 + 2x + 1 = -(x-1)^2 + 2$. That is why, the graph is given by



4. Draw the graph of a function $y = f(x)$ such that:
   $\lim_{x\to -2} f(x) = 1;$
   $\lim_{x\to 0} f(x) = \infty;$
   $\lim_{x\to 2-} f(x) = 0;$
   $\lim_{x\to 2+} f(x) = 2;$
   $\lim_{x\to 2-} f(x) = 0;$
   $f(2) = 1;$
   $\lim_{x\to \infty} f(x) = 1.$

   Solution. There are many ways to draw this. Here is one possibility:



5. Determine the infinite limit.

   (a) $\lim\limits_{x\to 3+} \frac{x+2}{x-3}$;

   (b) $\lim\limits_{x\to 1} \frac{2-x}{(x-1)^2}$;

   (c) $\lim\limits_{x\to \pi/2-} x\csc x$;

   (d) $\lim\limits_{x\to 2+} \frac{x^2-2x-8}{x^2-5x+6}$.

   Solution.

   (a) $\lim\limits_{x\to 3+} \frac{x+2}{x-3} = \infty$ because when x approaches 3 from the right we have that $x-3>0$ ($x-3$ approaches to 0 from the right) and $x+2>0$.

   (b) $\lim\limits_{x\to 1} \frac{2-x}{(x-1)^2} = \infty$ because when x approaches to 1, $2-x>0$, $(x-1)^2>0$.

   (c) $\lim\limits_{x\to 2\pi-} x\csc x = -\infty$ because when $x \to 2\pi-$ we have that $\sin x$ approaches to 0 from the left ($\sin x < 0$).

(d) $\lim\limits_{x\to 2+}\frac{x^2-2x-8}{x^2-5x+6}=\lim\limits_{x\to 2+}\frac{(x+2)(x-4)}{(x-2)(x+3)}=-\infty$ because if x approaches to 2 from the right, then $x-2>0$, $x+3>0$, $x+2>0$, $x-4<0$.

6. Evaluate the limit and justify each step.

(a) $\lim\limits_{t\to -2}\frac{t^4-2}{2t^2-3t+2}$;

(b) $\lim\limits_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}}$.

Solution.

(a) $\lim\limits_{t\to -2}\frac{t^4-2}{2t^2-3t+2}=\frac{\lim\limits_{t\to -2}(t^4-2)}{\lim\limits_{t\to -2(2t^2-3t+2)}}$;

(b) $\lim\limits_{x\to 2}\sqrt{\frac{2x^2+1}{3x-2}}=\sqrt{\lim\limits_{x\to 2}\frac{2x^2+1}{3x-2}}=\sqrt{\frac{\lim\limits_{x\to 2}(2x^2+1)}{\lim\limits_{x\to 2}(3x-2)}}=\sqrt{9/4}=3/2.$

7. Evaluate the limit

(a) $\lim\limits_{t\to 5}\frac{x^2-6x+5x}{x-5}$;

(b) $\lim\limits_{h\to 0}\frac{(-5+h)^2-25}{h}$;

(c) $\lim\limits_{x\to -6}\frac{2x+12}{|x+6|}$;

Solution.

(a) $\lim\limits_{t\to 5}\frac{x^2-6x+5x}{x-5}=\lim\limits_{t\to 5}\frac{(x-1)(x-5)}{x-5}=\lim\limits_{t\to 5}(x-1)=4.$;

(b) $\lim\limits_{h\to 0}\frac{(-5+h)^2-25}{h}=\lim\limits_{h\to 0}\frac{(-5+h-5)(-5+h=5)}{h}=\lim\limits_{h\to 0}\frac{(h-10)h}{h}==\lim\limits_{h\to 0}(h-10)=-10$;

(c) $\lim\limits_{x\to -6}\frac{2x+12}{|x+6|}=$;

8. Find the limit, if it exists. If the limit does not exist, explain why.

$\lim\limits_{x\to -6}\frac{2x+12}{|x+6|}$;

Solution. The limit does not exist because one-sided limits are different:

$\lim\limits_{x\to -6+}\frac{2x+12}{|x+6|}=\lim\limits_{x\to -6+}\frac{2x+12}{x+6}=\lim\limits_{x\to -6+}\frac{2(x+6)}{x+6}\lim\limits_{x\to -6+}2=2$;;

$\lim\limits_{x\to -6-}\frac{2x+12}{|x+6|}=\lim\limits_{x\to -6-}\frac{2x+12}{-(x+6)}=\lim\limits_{x\to -6-}\frac{2(x+6)}{-(x+6)}\lim\limits_{x\to -6-}(-2)=-2.$

9. If $\lim\limits_{x\to 1}\frac{f(x)-8}{x-1}=10$, find $\lim\limits_{x\to 1}f(x)$.

Solution.

It is clear that $\lim\limits_{x\to 1}f(x)=8$ otherwise $\lim\limits_{x\to 1}\frac{f(x)-8}{x-1}=\pm\infty$.

1. Explain why the function is discontinuous at the given number a. Sketch the graph of the function

$$f(x)=\begin{cases}\frac{x^2-x}{x^2-1} & if\ x\neq 1,\\ 1, & if\ \ x=1.\end{cases}$$

Solution. The function f is continuous everywhere when $x \neq 1$. We must check at $a = 1$. We have $\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$.
Thus $\lim_{x\to 1} f(x) \neq f(x)$. Hence, f is discontinuous at $a = 1$.

2. How would you "remove the discontinuity" of f ? In other words, how would you define $f(2)$ in order to make f continuous at 2 if

$$f(x) = \frac{x^2 - x - 2}{x - 2}?$$

Answer. The function f is continuous everywhere when $x \neq 2$. We should define f at 2 so that it would be also continuous ar 2. Let us calculate the limit at 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3.$$

Thus, if $f(2) = 3$, then f is continuous because in this case

$$\lim_{x\to 2} f(x) = f(2).$$

3. Evaluate the limit and justify each step by indicating the appropriate properties of limits

(a)

$$\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3};$$

(b)

$$\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x - 1};$$

(c)

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}};$$

(d)

$$\lim_{x\to\infty} ln\ (2+x) - ln\ (1+x) = \lim_{x\to\infty} ln\ \frac{2+x}{1+x} = ln\Big(\lim_{x\to\infty} \frac{2+x}{1+x}\Big) = ln\ 1 = 0.$$

Answer. Divide numerator and denominator by x^2. Then

(a) Divide numerator and denominator by x^2. Then

$$\lim_{x\to\infty} \frac{2x^2 - 7}{5x^2 + x - 3} = \lim_{x\to\infty} \frac{2 - \frac{7}{x^2}}{5 + \frac{1}{x} - \frac{3}{x^2}} = \frac{2}{5}.$$

(b) Divide numerator and denominator by x^3. Then

$$\lim_{x\to\infty} \frac{1 - x^2}{x^3 - x - 1} = \lim_{x\to\infty} \frac{\frac{1}{x^3} - \frac{1}{x}}{1 - \frac{1}{x^2} - \frac{1}{x^3}} = 0;$$

(c) Divide numerator and denominator by e^{3x}. Then

$$\lim_{x\to\infty} \frac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim_{x\to\infty} \frac{1 - e^{-6x}}{1 + e^{-6x}} = 1.$$

4. Find the horizontal and vertical asymptotes of each curve.

   $y = \frac{2x^2+1}{3x^2+2x-1}$.

   Answer.

   (a) Vertical asymptote:

$$3x^2 + 2x - 1 = 0 \Rightarrow x_1 = \frac{-1+\sqrt{13}}{3}; \;\; x_1 = \frac{-1-\sqrt{13}}{3}.$$

   Hence, we have 2 vertical asymptotes: $x = \frac{-1+\sqrt{13}}{3}$ and $x = \frac{-1-\sqrt{13}}{3}$.

   To find horizontal asymptote we calculate the limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+1}{3x^2+2x-1} = \frac{2}{3};$$

   similarly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2+1}{3x^2+2x-1} = \frac{2}{3}.$$

   Thus, this curve has 1 horizontal asymptote.

5. Find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$ if

$$f(x) = 2x^3 - x^4.$$

   Answer.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^3 - x^4 = \lim_{x\to-\infty} x^4(\frac{2}{x} - 1) = -\infty;$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x^3 - x^4 = \lim_{x\to\infty} x^4(\frac{2}{x} - 1) = -\infty.$$

6. Find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$ if

$$f(x) = 2x^3 - x^4.$$

   Answer.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2x^3 - x^4 = \lim_{x\to-\infty} x^4(\frac{2}{x} - 1) = -\infty;$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} 2x^3 - x^4 = \lim_{x\to\infty} x^4(\frac{2}{x} - 1) = -\infty.$$