Guidelines for solutions of problems. Sections 2.5, 2.6

Name and section: ________________________________

Instructor's name: ________________________________

1. Explain why the function is discontinuous at the given number a. Sketch the graph of the function

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & if\ x \neq 1, \\ 1, & if\ \ x = 1, \end{cases} \qquad a = 1.$$

Solution. The function f is continuous everywhere when $x \neq 1$. First observe that $f(1) = 1$. We must check at $a = 1$. We have

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-x}{x^2-1} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$.

Thus $\lim_{x\to 1} f(x) \neq f(1)$. Hence, f is discontinuous at $a = 1$.

2. How would you "remove the discontinuity" of f ? In other words, how would you define $f(2)$ in order to make f continuous at 2 if

$$f(x) = \frac{x^2 - x - 2}{x - 2}?$$

Answer. The function f is continuous everywhere when $x \neq 2$. We should define f at 2 so that it would be also continuous ar 2. Let us calculate the limit at 2:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+1) = 3.$$

Thus, if $f(2) = 3$, then f is continuous because in this case

$$\lim_{x\to 2} f(x) = f(2).$$

3. Explain why the given function is discontinuous at a given number a?

(a)

$$f(x) = \begin{cases} \frac{1}{x+2}, & if\ x \neq -2; \\ 1, & if\ x = -2, \end{cases} \qquad a = -2.$$

(b)

$$f(x) = \begin{cases} x-1, & if\ x \leq 2; \\ e^x, & if\ x > 2, \end{cases} \qquad a = 2.$$

(c)

$$f(x) = \begin{cases} \frac{x^2-5x+6}{x^2-9} & if\ x \neq 3; \\ 1/2, & if\ x = 3, \end{cases} \qquad a = 3.$$

Solution

(a) We see that $\lim_{x \to -2+} f(x) = \infty$; $\lim_{x \to -2-} f(x) = -\infty$. That is why this function is neither right nor left continuous at $a = -2$. In fact, $a = -2$ is an infinite discontinuity point of f.

(b) $\lim_{x \to 2+} f(x) = e^2$; $\lim_{x \to 2-} f(x) = 1$. Since these one-sided limits exist but are different, f has jump discontinuity at $a = 2$.

(c) $\lim_{x \to 3} \frac{x^2-5x+6}{x^2-9} = \lim_{x \to 3} \frac{(x-3)(x-2)}{(x-3)(x+3)} = \lim_{x \to 3} \frac{x-2}{x+3} = \frac{1}{6} \neq \frac{1}{2}$. That is why we have removable discontinuity, because if we define $f(3) = \frac{1}{6}$, then f would be continuous at $a = 3$.

4. Find the numbers at which f is discontinuous. At which of these numbers f is continuous from the right? From the left? or neither?

$$f(x) = \begin{cases} 2^x & if\ x \leq 1; \\ 3-x, & if\ 1 < x \leq 4; \\ \sqrt{x}, & x > 4. \end{cases}$$

Solution The interesting for us points are $a = 1$; $a = 4$. Observe that $f(1) = 2^1 = 2$; $f(4) = 3-4 = -1$. At other points f is continuous.

Let us take the point $a = 1$. Then

$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} 2^x = 2 = f(1)$. Thus it is continuous from the left at $a = 1$;

$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} 3-x = 2 = f(1)$. Thus it is continuous from the right at $a = 1$. Hence f is continuous at $a = 1$.

Let us now take $a = 4$. Then

$\lim_{x \to 4-} f(x) = \lim_{x \to 4-} 3-x = -1 = f(4)$. Thus it is continuous from the left at $a = 4$;

$\lim_{x \to 4+} f(x) = \lim_{x \to 4+} \sqrt{4} = 2 \neq f(4)$. Thus it is not continuous from the right at $a = 4$. Hence f is not continuous at $a = 4$.

5. Evaluate the limit and justify each step by indicating the appropriate properties of limits. Find horizontal asymptotes of appropriate functions:

(a)

$$\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3};$$

(b)

$$\lim_{x\to\infty} \frac{1-x^2}{x^3-x-1};$$

(c)

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}};$$

(d)

$$\lim_{x\to\infty} (\ln (2+x) - \ln (1+x)).$$

Answer. Divide numerator and denominator by x^2. Then

(a) Divide numerator and denominator by x^2. Then

$$\lim_{x\to\infty} \frac{2x^2-7}{5x^2+x-3} = \lim_{x\to\infty} \frac{2-\frac{7}{x^2}}{5+\frac{1}{x}-\frac{3}{x^2}} = \frac{2}{5}.$$

Thus, $y = \frac{2}{5}$ is a horizontal Asymptote for the function $f(x) = \frac{2x^2-7}{5x^2+x-3}$.

(b) Divide numerator and denominator by x^3. Then

$$\lim_{x\to\infty} \frac{1-x^2}{x^3-x-1} = \lim_{x\to\infty} \frac{\frac{1}{x^3}-\frac{1}{x}}{1-\frac{1}{x^2}-\frac{1}{x^3}} = 0.$$

Thus, $y = 0$ (x axis) is a horizontal asymptote for the function $f(x) = \frac{1-x^2}{x^3-x-1}$.

(c) Divide numerator and denominator by e^{3x}. Then

$$\lim_{x\to\infty} \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \frac{1-e^{-6x}}{1+e^{-6x}} = 1.$$

Thus, $y = 1$ (x axis) is a horizontal asymptote for the function $f(x) = \frac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}}$.

(d) By using elementary properties of logarithm (recall that $\ln x = \log_e x$) we have

$$\lim_{x\to\infty} (\ln (2+x) - \ln (1+x)) = \lim_{x\to\infty} \ln \frac{2+x}{1+x} = \ln \Big(\lim_{x\to\infty} \frac{2+x}{1+x} \Big) = \ln\ 1 = 0.$$

Hence, $y = 0$ is a horizontal asymptote of the function $y = \ln (2+x) - \ln (1+x)$.

6. Find the horizontal and vertical asymptotes of each curve.

   $y = \frac{2x^2+1}{3x^2+2x-1}$.

   Answer.

   (a) Vertical asymptote:

$$3x^2 + 2x - 1 = 0 \Rightarrow x_1 = \frac{1}{3};\ \ x_2 = -1.$$

   Hence, we have 2 vertical asymptotes: $x = \frac{1}{3}$ and $x = -1$.

   To find horizontal asymptote we calculate the limits:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+1}{3x^2+2x-1} = \frac{2}{3};$$

   similarly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x^2+1}{3x^2+2x-1} = \frac{2}{3}.$$

   Thus, this curve has one horizontal asymptote $y = \frac{2}{3}$.

7. Find the limits of $f(x)$ as $x \to -\infty$ and $x \to \infty$ if

$$f(x) = 2x^3 - x^4.$$

   Answer.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} (2x^3 - x^4) = \lim_{x\to-\infty} x^4(\frac{2}{x} - 1) = -\infty;$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} (2x^3 - x^4) = \lim_{x\to\infty} x^4(\frac{2}{x} - 1) = -\infty.$$