Discrete Probability Theory Homwework Variant 2 (week 6)

Dimitri Tabatadze · Wednesday 10-04-2024

Problem 6.1:

You ask your neighbor to water a sickly plant while you are on vacation. Without water, it will die with probability 0.7; with water, it will die with probability 0.1. You are 95 percent certain that your neighbor will remember to water the plant. What is the probability that the plant will be alive when you return?

Solution

Let W be the event that the plant gets watered and D be the event that it dies, so $p(E) = 0.95$, $\frac{P(D \cap E)}{P(E)} = 0.1$, $\frac{P(D \cap E^c)}{P(E^c)} = 0.7$. We can calculate

$$P(D \cap E) = 0.1 \cdot 0.95 = 0.095$$

$$P(D \cap E^c) = 0.7 \cdot (1 - 0.95) = 0.035$$

and we know that $D \cap E$ and $D \cap E^c$ are independent, since E and E^c are independent. Thus, we can write

$$\begin{aligned} P(D) &= P(D \cap E \cup D \cap E^c) \\ &= P(D \cap E) + P(D \cap E^c) \\ &= 0.095 + 0.035 \\ &= 0.13 \end{aligned}$$

and the probability that the plant will be alive is $p(D^c) = 1 - p(D) = 0.87$ ■

Problem 6.2:

At all times, a pipe-smoking mathematician carries 2 matchboxes — 1 in his left-hand pocket and 1 in his right-hand pocket. Each time he needs a match, he is equally likely to take it from either pocket. Consider the moment when the mathematician first discoveres that one of his matchboxes is empty. If it is assumed that both of the matchboxes initially contained N matches, what is the probability that there are exactly k matches, $k = 0, 1, \ldots, N$, in the other box?

Solution

We can think about this problem as a sequence of N `0`s and N `1`s shuffled where `0` represents the one pocket and `1` the other and the order in which they appear is the order the mathematician took the matches out of corresponding pockets. Without loss of generality, we can assume that the first matchbox that runs out of matches is the one represented by `0`s. Since you'd need to try picking $N + 1$ matches to find out that they've run out, we can put $N + 1$ `0`s instead of N into the sequence (the $(N + 1)$-th `0` being the act of discovering that the matchbox is empty).

Now we can ask the question in such a way: what's the probability that the number of `1`s that follow the $(N + 1)$-th `0` is k? Or if you reverse the sequence: What's the probability that a sequence consisting of N `1`s and $(N + 1)$ `0`s starts with k ones? The solution is very clear and can be written in the following way

$$
\begin{aligned}
p_k &= \prod_{i=0}^{k} \frac{N+1-i}{2N+1-i} \\
&= \frac{\binom{2N-k}{N}}{\binom{2N+1}{N}} = \frac{(2N-k)!(N+1)!}{(2N+1)!(N-k)!}
\end{aligned}
$$

■

Problem 6.3:

Independent trials consisting of rolling a pair of fair dice are performed. What is the probability that an outcome of 4 appears before an outcome of 7 when the outcome of a roll is the sum of the dice?

Solution

The probability that 7 doesn't get rlooed in k rolls is equal to $\left(1 - \frac{6}{36}\right)^k = \left(\frac{5}{6}\right)^k$ and the probability that the first 7 gets rolled on the $(k+1)$-th roll is

$$
\left(\frac{5}{6}\right)^k \cdot \frac{6}{36} = \frac{5^k}{6^{k+1}}.
$$

Now the probability that a 4 gets rolled in the first k rolls with no 7 being rolled and teh $(k+1)$-th being a 7 is

$$
\begin{aligned}
\left(1 - \left(1 - \frac{3}{36-6}\right)^k\right) \cdot \frac{5^k}{6^{k+1}} &= \frac{5^k}{6^{k+1}} - \frac{5^k}{6^{k+1}} * \frac{9^k}{10^k} \\
&= \frac{1}{6} \cdot \left(\frac{5^k}{6^k} - \frac{3^k}{4^k}\right)
\end{aligned}
$$

Now since these events are independent, the probability of their union would be equal to the sum of their probabilities

$$
\begin{aligned}
\sum_{i=0}^{\infty} \frac{1}{6} \cdot \left(\frac{5^k}{6^k} - \frac{3^k}{4^k}\right) &= \frac{1}{6} \cdot \lim_{h\to\infty} \sum_{i=0}^{h} \left(\frac{5^k}{6^k} - \frac{3^k}{4^k}\right) \\
&= \frac{1}{6} \cdot \lim_{h\to\infty} \left(\frac{\left(\frac{5}{6}\right)^{h+1} - 1}{\frac{5}{6} - 1} - \frac{\left(\frac{3}{4}\right)^{h+1} - 1}{\frac{3}{4} - 1}\right) \\
&= \frac{1}{6} \cdot \lim_{h\to\infty} \left(6 - \frac{5^{h+1}}{6^h} - 4 + \frac{3^{h+1}}{4^h}\right) \\
&= 1 - \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}
$$

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Problem 6.4:

A coin that, when flipped, comes up with heads with probability p is flipped until either heads or tails has occured twice. Find the expected number of flips.

Solution

By pigeon hole principle, for $n > 2$ throws, one side of the coin will come up at least twice. This means that the maximum number of throws needed is the smallest $n > 2$ which is 3. It's also clear that only one throw is not gonna be enough, so we can look at the 4 cases that can happen:

- `heads`, `heads` — with probability p^2.
- `heads`, `tails` — with probability $p(1-p)$.
- `tails`, `heads` — with probability $p(1-p)$.
- `tails`, `tails` — with probability $(1-p)^2$.

Out of these, only two are desired with probabilities p^2 and $(1-p)^2$ and they are independent so their combined probability would be $p^2 + (1-p)^2$. Now we can calculate the expected value:

$$2 \cdot \underbrace{\left(p^2 + (1-p)^2\right)}_{\text{only two needed}} + 3 \cdot \underbrace{\left(1 - p^2 - (1-p)^2\right)}_{\text{three needed}} = -2p^2 + 2p + 2$$

■