Finals preparation

Dimitri Tabatadze

June 24, 2023

9.4

transformation from equation 4 to equation 7

$$\begin{aligned}
\frac{dP}{dt} &= kP\left(1-\frac{P}{M}\right) && \Longrightarrow \\
\int \frac{dP}{P\left(1-\frac{P}{M}\right)} &= \int k\ dt && \Longrightarrow \\
\int \frac{M}{P\left(M-P\right)} dP &= \int k\ dt && \Longrightarrow \\
\int \left(\frac{1}{P}+\frac{1}{M-P}\right) dP &= \int k\ dt && \Longrightarrow \\
\ln|P| - \ln|M-P| &= kt + C && \Longrightarrow \\
\ln\left|\frac{M-P}{P}\right| &= -kt - C && \Longrightarrow \\
\left|\frac{M-P}{P}\right| &= e^{-kt-C} && \Longrightarrow \\
\left|\frac{M-P}{P}\right| &= e^{-C}e^{-kt} && \Longrightarrow \\
A = \pm e^{-C},\ \frac{M-P}{P} &= Ae^{-kt} && \Longrightarrow \\
\frac{M}{P} &= 1 + Ae^{-kt} && \Longrightarrow \\
P &= \frac{M}{1+Ae^{-kt}}
\end{aligned}$$

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

EXAMPLE 2 If $f(x, y) = \sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

SOLUTION Using the Chain Rule for functions of one variable, we have

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = \cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = -\cos\left(\frac{x}{1+y}\right) \cdot \frac{x}{(1+y)^2}$$

The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

$u(x, t) = \sin(x - at)$ satisfies the wave equation.

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The linear function whose graph is this tangent plane, namely

3 $$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

4 $$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Recall that for a function of one variable, $y = f(x)$, if x changes from a to $a + \Delta x$, we defined the increment of y as

$$\Delta y = f(a + \Delta x) - f(a)$$

In Chapter 3 we showed that if f is differentiable at a, then

$$\boxed{5} \qquad \Delta y = f'(a)\,\Delta x + \varepsilon\,\Delta x \qquad \text{where } \varepsilon \to 0 \text{ as } \Delta x \to 0$$

Now consider a function of two variables, $z = f(x, y)$, and suppose x changes from a to $a + \Delta x$ and y changes from b to $b + \Delta y$. Then the corresponding **increment** of z is

$$\boxed{6} \qquad \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

Thus the increment Δz represents the change in the value of f when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. By analogy with (5) we define the differentiability of a function of two variables as follows.

7 Definition If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$

where ε_1 and ε_2 are functions of Δx and Δy such that ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

For a differentiable function of two variables, $z = f(x, y)$, we define the **differentials** dx and dy to be independent variables; that is, they can be given any values. Then the **differential** dz, also called the **total differential**, is defined by

$$\boxed{10} \qquad \boxed{dz = f_x(x, y)\,dx + f_y(x, y)\,dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy}$$

(Compare with Equation 9.) Sometimes the notation df is used in place of dz.

$$\begin{aligned}
z &= x^2y + 3xy^4,\ \ x = \sin 2t,\ \ y = \cos t \\
z &= (\sin 2t)^2 \cos t + 3(\sin 2t)(\cos t)^4 \\
z &= (2\sin t\cos t)(2\sin t\cos t)\cos t + 3(2\sin t\cos t)(\cos t)^4 \\
z &= 4(\sin t)^2(\cos t)^3 + 6(\sin t)(\cos t)^5 \\
\frac{dz}{dt} &= 8\sin t\cos^4 t - 12\sin^3 t\cos^2 t + 6\cos^6 t - 30\sin^2 t\cos^4 t
\end{aligned}$$

$$\begin{aligned}
z &= x^2y + 3xy^4,\ \ x = \sin 2t,\ \ y = \cos t \\
\frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\
\frac{dx}{dt} &= (2xy + 3y^4)(2\cos 2t) - (x^2 + 12xy^3)(\sin t) \\
\frac{dx}{dt} &= (4\sin t\cos^2 t + 3\cos^4 t)(2\cos^2 t - 2\sin^2 t) - 2\sin^3 t\cos^2 t - 24\sin^2 t\cos^4 t \\
\frac{dx}{dt} &= 8\sin t\cos^4 t + 6\cos^6 t - 10\sin^3 t\cos^2 t - 6\cos^4 t\sin^2 t - 24\sin^2 t\cos^4 t
\end{aligned}$$

2 Definition The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Strategy for Testing Series

1. **Test for Divergence** If you can see that $\lim_{n\to\infty} a_n$ may be different from 0, then apply the Test for Divergence.
2. ***p*-Series** If the series is of the form $\Sigma\, 1/n^p$, then it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leqslant 1$.
3. **Geometric Series** If the series has the form $\Sigma\, ar^{n-1}$ or $\Sigma\, ar^n$, then it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geqslant 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.
4. **Comparison Tests** If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered. In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series. Notice that most of the series in Exercises 11.4 have this form. (The value of p should be chosen as in Section 11.4 by keeping only the highest powers of n in the numerator and denominator.) The comparison tests apply only to series with positive terms, but if $\Sigma\, a_n$ has some negative terms, then we can apply a comparison test to $\Sigma\, |a_n|$ and test for absolute convergence.

5. **Alternating Series Test** If the series is of the form $\Sigma\, (-1)^{n-1} b_n$ or $\Sigma\, (-1)^n b_n$, then the Alternating Series Test is an obvious possibility. Note that if $\Sigma\, b_n$ converges, then the given series is absolutely convergent and therefore convergent.
6. **Ratio Test** Series that involve factorials or other products (including a constant raised to the nth power) are often conveniently tested using the Ratio Test. Bear in mind that $|a_{n+1}/a_n| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.
7. **Root Test** If a_n is of the form $(b_n)^n$, then the Root Test may be useful.
8. **Integral Test** If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

- Alternating series test (only formulation)

Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad (b_n > 0)$$

satisfies the conditions

$$\text{(i)}\quad b_{n+1} \leqslant b_n \qquad \text{for all } n$$

$$\text{(ii)}\quad \lim_{n\to\infty} b_n = 0$$

then the series is convergent.

- integral test (only formulation)

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent. In other words:

(i) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\displaystyle\int_1^{\infty} f(x)\,dx$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

- comparison tests (only formulations)

The Direct Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leqslant b_n$ for all n, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geqslant b_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \to \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

- ratio and root tests (only formulations)

The Ratio Test

(i) If $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$ or $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$, the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum a_n$.

The Root Test

(i) If $\displaystyle\lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\displaystyle\lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1$ or $\displaystyle\lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty$, then the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\displaystyle\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

- Theorems about term-by-term differentiation and integration of series (only formulations)

2 Theorem If the power series $\Sigma c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ and

(i) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$

(ii) $\int f(x)\,dx = C + c_0(x-a) + c_1\dfrac{(x-a)^2}{2} + c_2\dfrac{(x-a)^3}{3} + \cdots$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

- integration factor and solution of linear DE (with proof)

To solve the linear differential equation $y' + P(x)y = Q(x)$, multiply both sides by the **integrating factor** $I(x) = e^{\int P(x)\,dx}$ and then integrate both sides.

- divergence test for a number series (with proof)

7 Test for Divergence If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

PROOF Let $s_n = a_1 + a_2 + \cdots + a_n$. Then $a_n = s_n - s_{n-1}$. Since $\Sigma\ a_n$ is convergent, the sequence $\{s_n\}$ is convergent. Let $\lim_{n\to\infty} s_n = s$. Since $n - 1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$. Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0 \qquad \blacksquare$$

- the sum of geometric series (with proof)

Sum of a Geometric Series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

Each term is obtained from the preceding one by multiplying it by the **common ratio** r. (The series that arises from Zeno's paradox is the special case where $a = \frac{1}{2}$ and $r = \frac{1}{2}$.)

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \pm\infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges in this case.

If $r \neq 1$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

3 $$\boxed{s_n = \frac{a(1 - r^n)}{1 - r}}$$

4 The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \qquad |r| < 1$$

If $|r| \geqslant 1$, the geometric series is divergent.

- Taylor's inequality for n=1 (with proof)

9 Taylor's Inequality If $|f^{(n+1)}(x)| \leqslant M$ for $|x - a| \leqslant d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leqslant \frac{M}{(n+1)!}|x - a|^{n+1} \qquad \text{for } |x - a| \leqslant d$$

PROOF We first prove Taylor's Inequality for $n = 1$. Assume that $|f''(x)| \leqslant M$. In particular, we have $f''(x) \leqslant M$, so for $a \leqslant x \leqslant a + d$ we have

$$\int_a^x f''(t)\,dt \leqslant \int_a^x M\,dt$$

An antiderivative of f'' is f', so by Part 2 of the Fundamental Theorem of Calculus, we have

$$f'(x) - f'(a) \leqslant M(x - a) \qquad \text{or} \qquad f'(x) \leqslant f'(a) + M(x - a)$$

Thus

$$\int_a^x f'(t)\,dt \leqslant \int_a^x [f'(a) + M(t - a)]\,dt$$

$$f(x) - f(a) \leqslant f'(a)(x - a) + M\frac{(x - a)^2}{2}$$

$$f(x) - f(a) - f'(a)(x - a) \leqslant \frac{M}{2}(x - a)^2$$

But $R_1(x) = f(x) - T_1(x) = f(x) - f(a) - f'(a)(x - a)$. So

$$R_1(x) \leqslant \frac{M}{2}(x - a)^2$$

A similar argument, using $f''(x) \geqslant -M$, shows that

$$R_1(x) \geqslant -\frac{M}{2}(x - a)^2$$

So

$$|R_1(x)| \leqslant \frac{M}{2}|x - a|^2$$

Although we have assumed that $x > a$, similar calculations show that this inequality is also true for $x < a$.

This proves Taylor's Inequality for the case where $n = 1$. The result for any n is proved in a similar way by integrating $n + 1$ times. (See Exercise 95 for the case $n = 2$.) ■

- Theorem about directional derivative (with proof)

> **15 Theorem** Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}}f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

PROOF From Equation 9 or 14 and using Theorem 12.3.3, we have

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}|\cos\theta = |\nabla f|\cos\theta$$

where θ is the angle between ∇f and $\mathbf{u}$. The maximum value of $\cos\theta$ is 1 and this occurs when $\theta = 0$. Therefore the maximum value of $D_{\mathbf{u}}f$ is $|\nabla f|$ and it occurs when $\theta = 0$, that is, when $\mathbf{u}$ has the same direction as ∇f. ■

- Fermat's theorem for functions of two variables about critical points (with proof)

2 Theorem If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

PROOF Let $g(x) = f(x, b)$. If f has a local maximum (or minimum) at (a, b), then g has a local maximum (or minimum) at a, so $g'(a) = 0$ by Fermat's Theorem (see Theorem 4.1.4). But $g'(a) = f_x(a, b)$ (see Equation 14.3.1) and so $f_x(a, b) = 0$. Similarly, by applying Fermat's Theorem to the function $G(y) = f(a, y)$, we obtain $f_y(a, b) = 0$. ■

- McLaurin series of exponential and trigonometric functions (with proof)