Union-Find with Balancing and Path Compresion

run time analysis

the most famous run time analysis there is

review: algorithm

- make-set

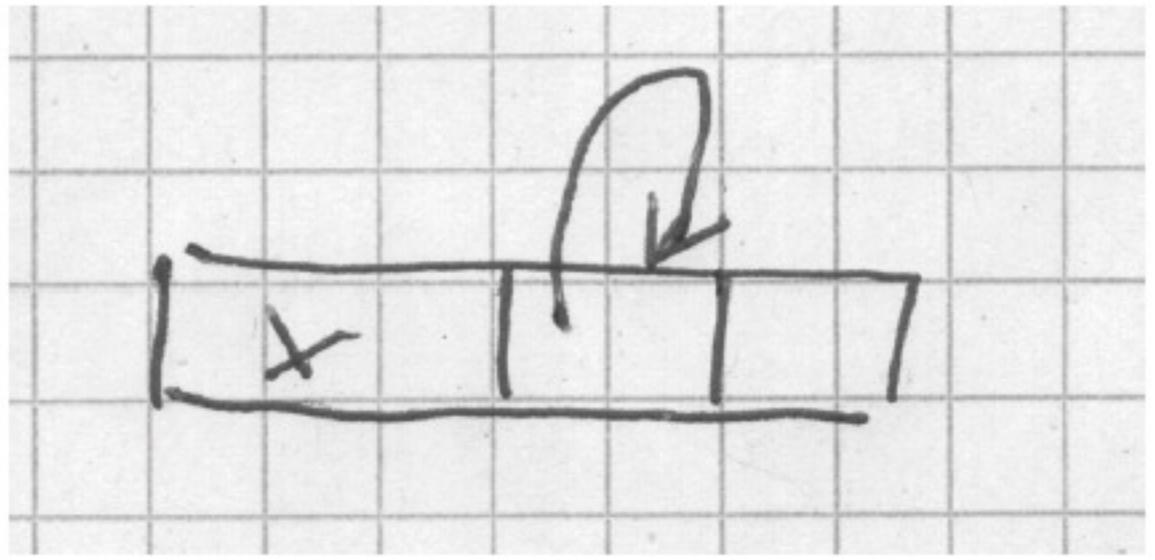


Figure 1: a record which points to itself is a root/representative

```
make-set(x):
TEA[x].e = x;
p(x):=x; r(x):=0
```

review: algorithm

- make-set

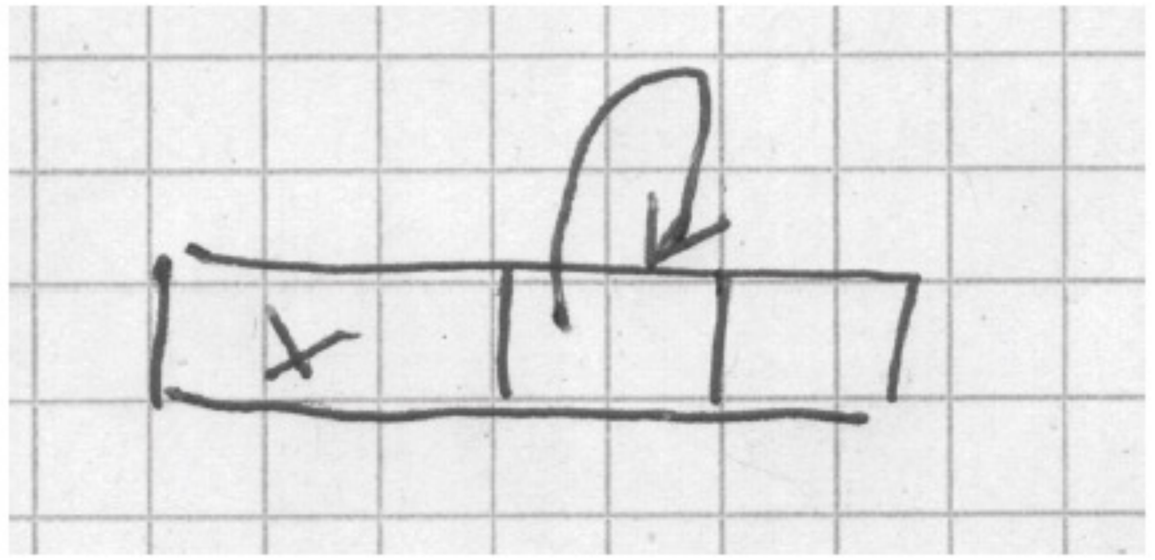


Figure 1: a record which points to itself is a root/representative

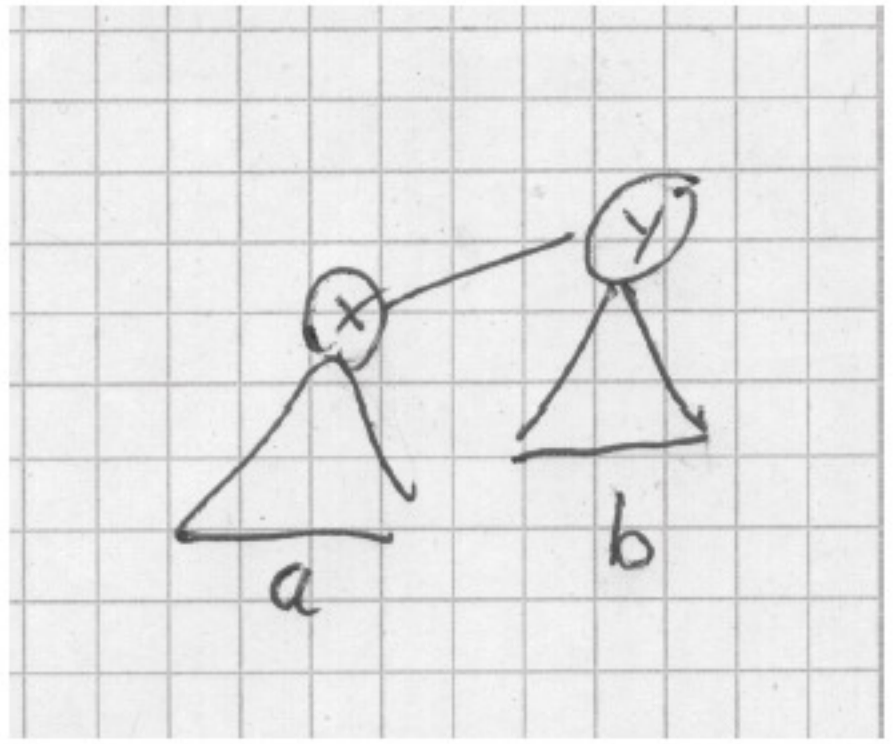
```
make-set(x):
TEA[x].e = x;
p(x):=x; r(x):=0
```

x = new TE;

- union and link with balancing

```
union(x,y): link(find(x), find(y))
```

```
link(x,y):
if r(x)<r(y) {p(x):=y} /*make y predecessor of x*/;
if r(x)>r(y) {p(y):=x} /*make x predecessor of y*/;
if r(x)=r(y) {p(x):=y; r(y)=r(y)+1} /*increase rank of y*/
```



review: algorithm

```
find(x):if x != p(x)
{p(x):= find(p(x)} /*recursive call with side effect*/
return p(x)
```

find with path compression

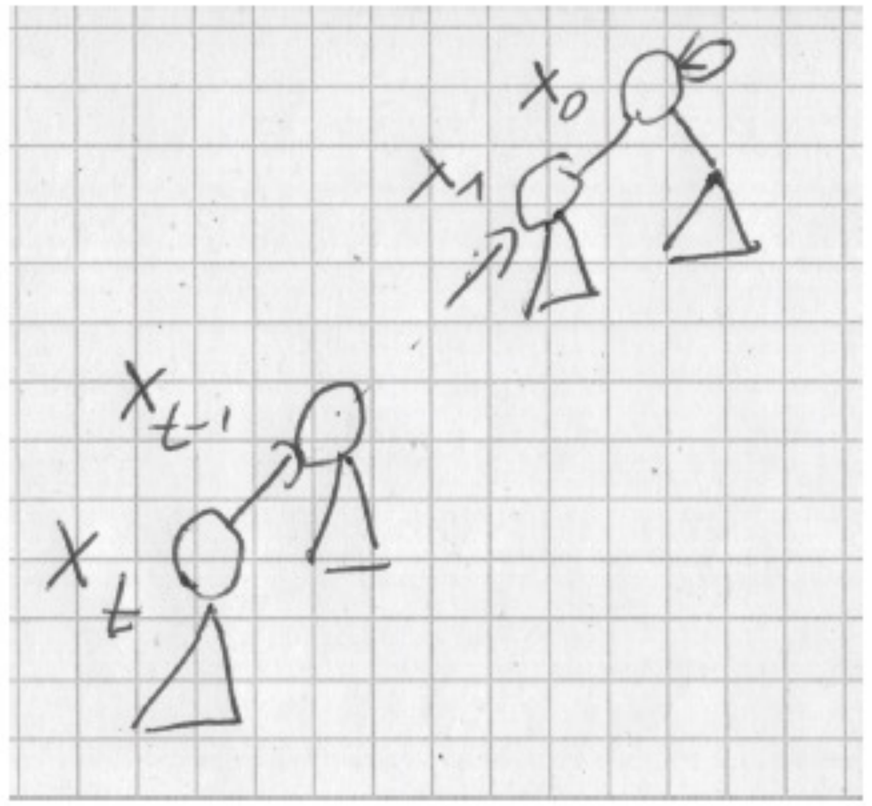


Figure 3: parent chasing from $x = x_t$ touches elements $x_{t-1}, \dots, x_0$

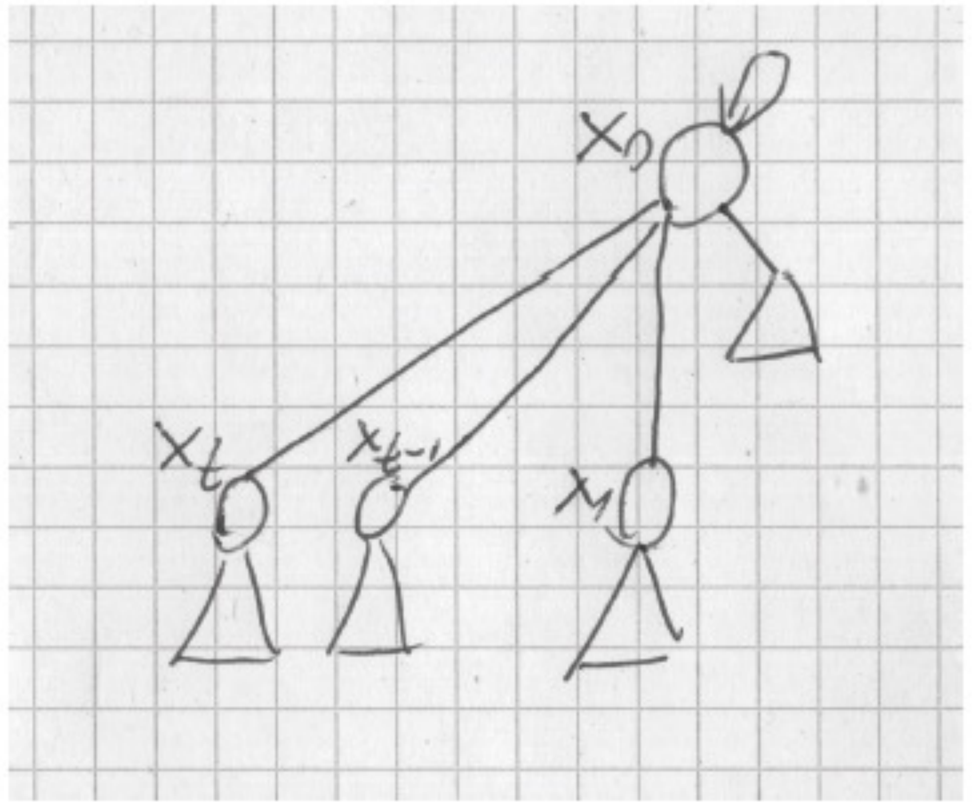


Figure 4: after path compression all nodes $x_t, \dots, x_1$ are sons of the root x_0

review: Ackermann’a function

Iterating i times function f:

$$
\begin{aligned}
f^{(0)}(j) &= j \\
f^{(i+1)}(j) &= f(f^{(i)}(j))
\end{aligned}
$$

$$
A_k(j) = \begin{cases} j+1 & k=0 \\ A_{k-1}^{(j+1)}(j) & \end{cases}
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$$A_0(j) = j+1 = S(j)$$

Lemma 1.

$$A_1(j) = 2j+1$$

Lemma 2.

$$A_2(j) = 2^{j+1}(j+1) - 1$$

Lemma 3.

$$A_k(j) < A_k(j+1)$$

Proof. exercise

review: Ackermann'a function

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def: 'inverse Ackermann function'

$$\alpha(n) = \min\{k \mid A_k(1) \geq n\}$$

$$\alpha(n) = \begin{cases} 0 & 0 \leq n \leq 2 \\ 1 & n = 3 \\ 2 & 4 \leq n \leq 7 \\ 3 & 8 \leq n \leq 2047 \\ 4 & 2048 \leq n \leq A_4(1) \end{cases}$$

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theorem:

run time of union-find algorithm with path compression: $O(n + m \cdot \alpha(n))$

Lemma 4.

1. *ranks are nondecreasing along parent edges*

$$r(x) \leq r(p(x))$$

2. *they are strictly along edges increasing except at roots*

$$r(x) < r(p(x)) \leftrightarrow x \neq p(x)$$

3. *ranks are strictly increasing along a path to a root*

4. *ranks of parents increase in time*

$$r(p(x)) \leq r'(p'(x))$$

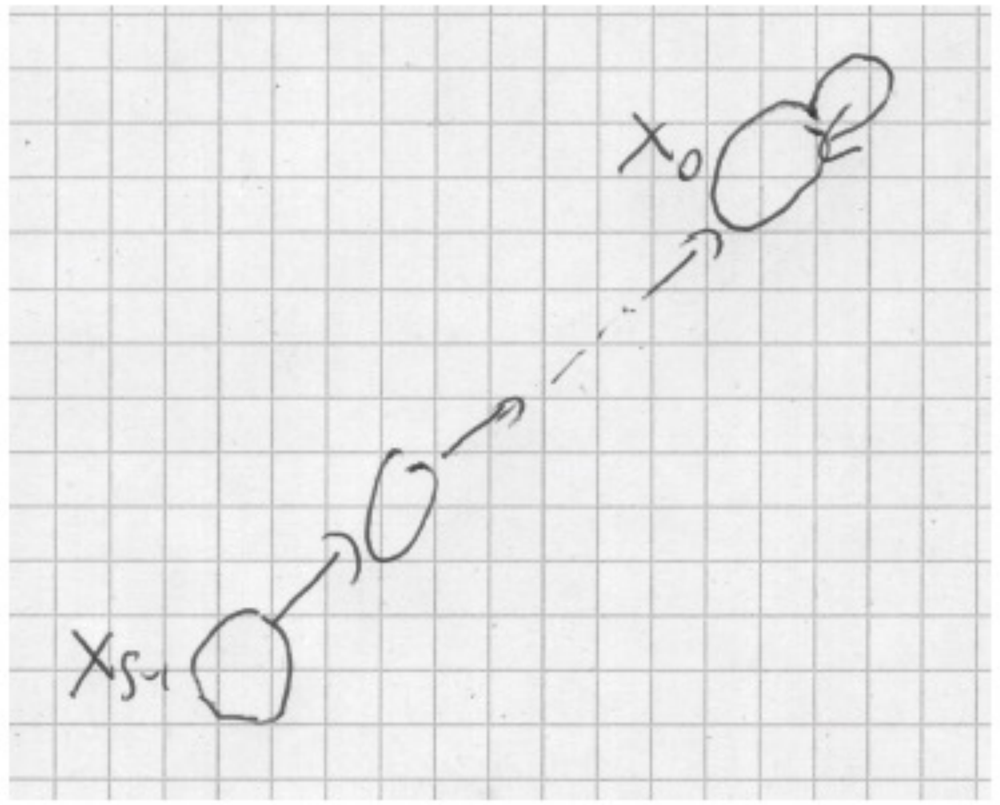

Figure 5: On a path to the root we have $r(x_{s-1}) < \ldots < r(x_0)$

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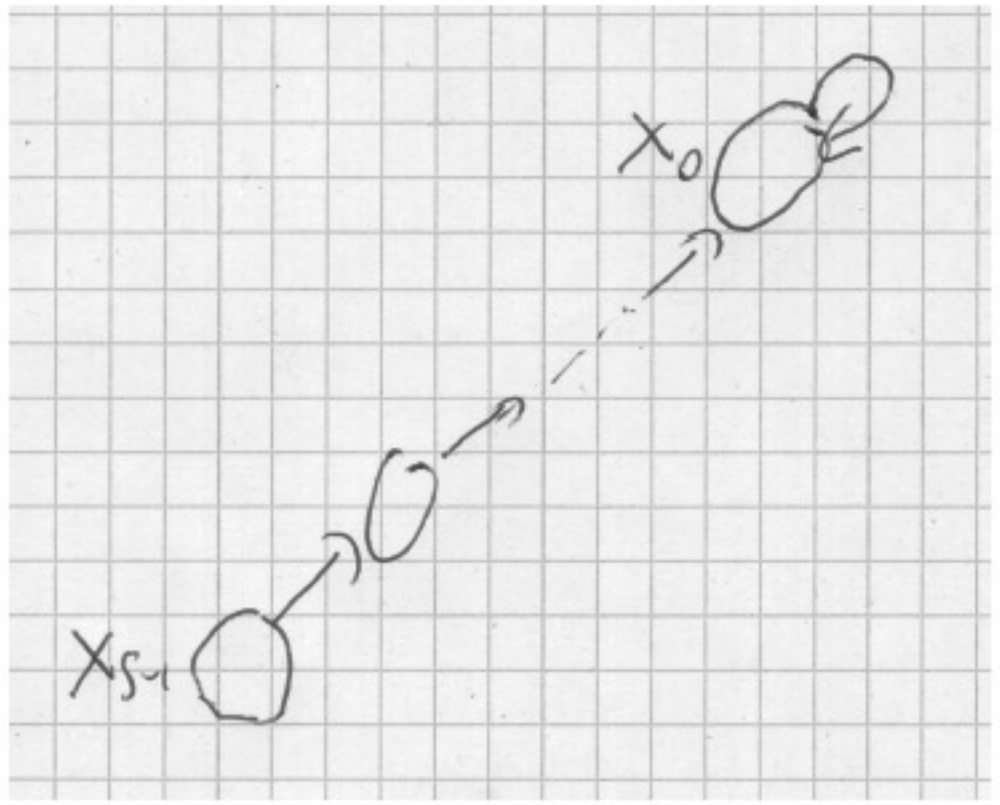

Figure 5: On a path to the root we have $r(x_{s-1}) < \ldots < r(x_0)$

Proof.

- Link and find operations do not change ranks except for the root

$$p(x) \neq x \rightarrow r(x) = r'(x)$$

properties of rank

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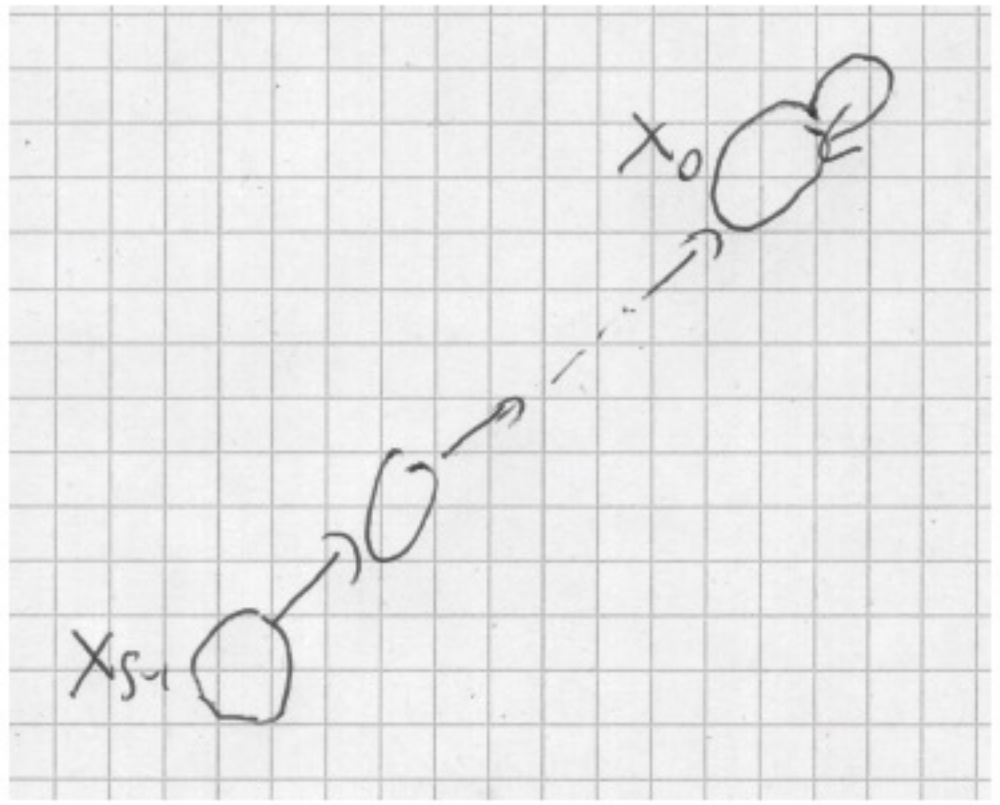

Figure 5: On a path to the root we have $r(x_{s-1}) < \ldots < r(x_0)$

- Now prove 1 and 2 by induction on link and find operations. Details: exercise.
- 3 follows from 1 and 2.
- 4 follows from 3 and the alpgorithms for link and path compressions. Details: exercise.

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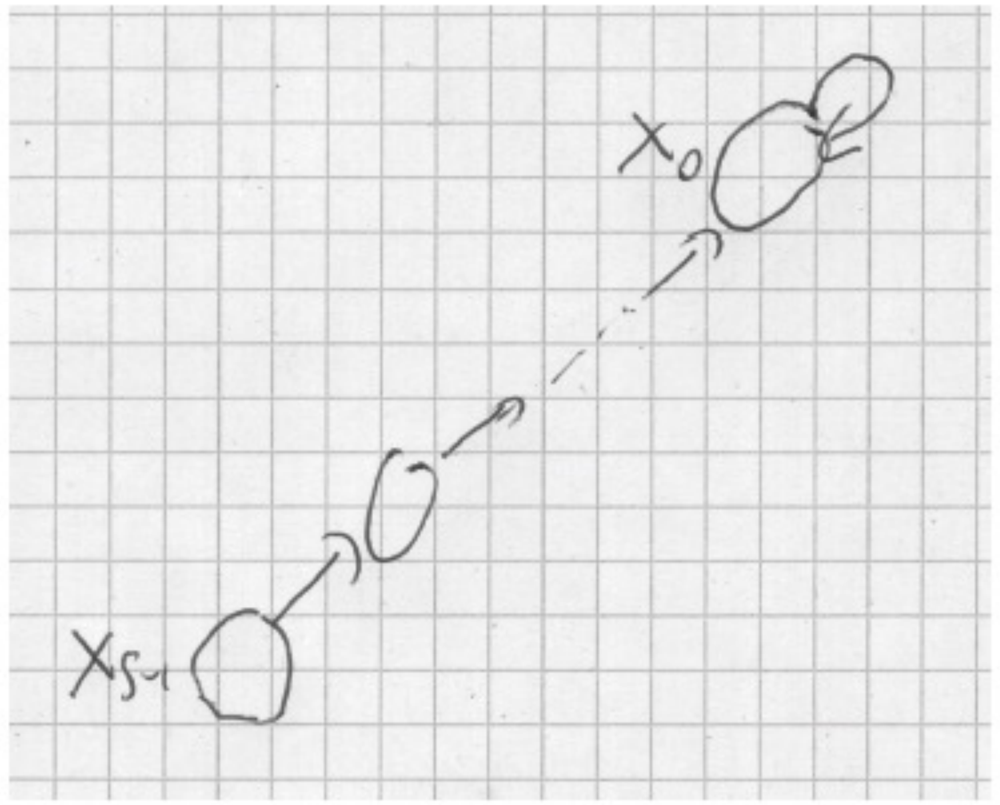

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Lemma 5.

$$r(x) \leq n-1$$

Proof. trivial induction on link operations: increase rank at most by 1. At most $n-1$ such operations possible. □

counting make-set, find and link operations

Lemma 6. *Let*

- m'*: number of make-set, find and union operations*
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union(x,y): /*1 operation*/
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□

Then

$$m' \cdot \alpha(m') = O(m \cdot \alpha(m))$$

Thus it sufices to estimate time for m make-set, find and link operations

partial definition of potential function

- $\phi_q(x)$: potential at node x after q operations
- Φ_q: potential after q operations

$$\Phi_q = \sum_x \phi_q(x)$$

$$\Phi_0 = 0$$

- invariant:

$$\Phi_q \geq 0$$

partial definition of potential function:

for roots or nodes with rank 0:

$$x = p(x) \lor r(x) = 0 \rightarrow \phi(x) = \alpha(n) \cdot r(x)$$

level

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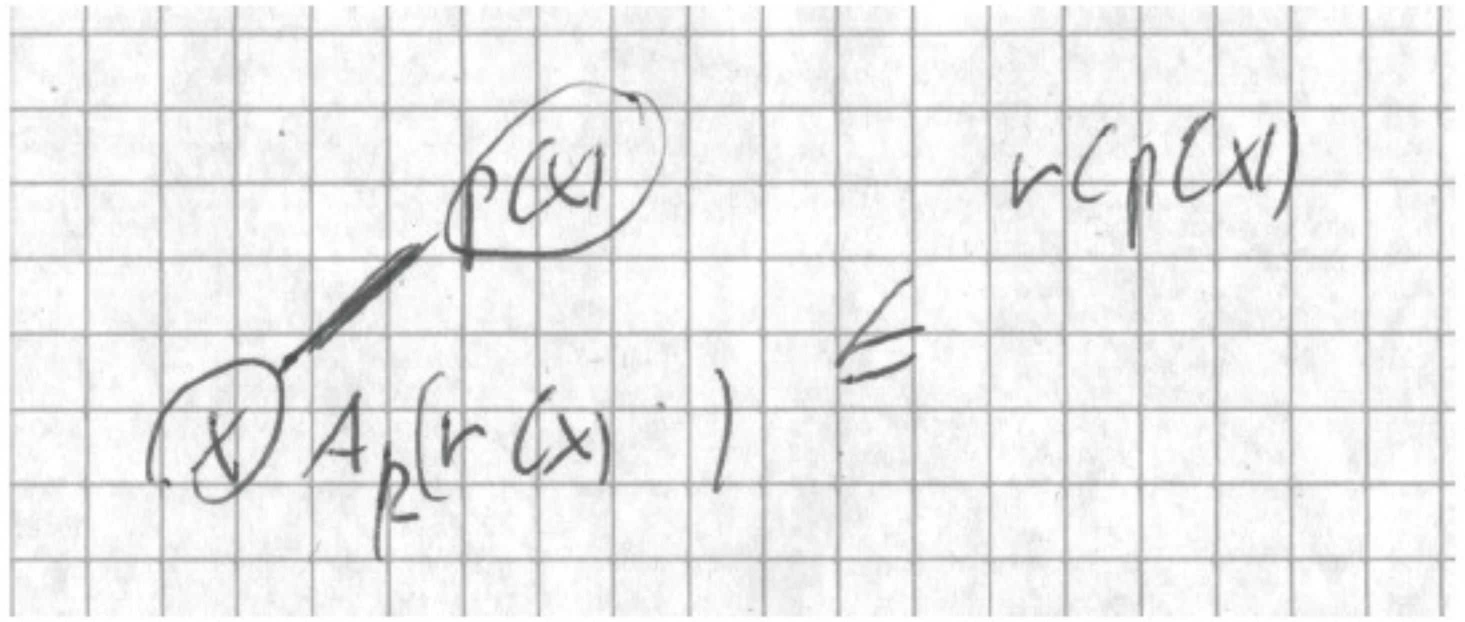
auxiliary function: level:

For x with

$$x \neq p(x) \wedge r(x) \neq 0$$

define

$$\ell(x) = \max\{k \mid r(p(x)) \geq A_k(r(x))\}$$



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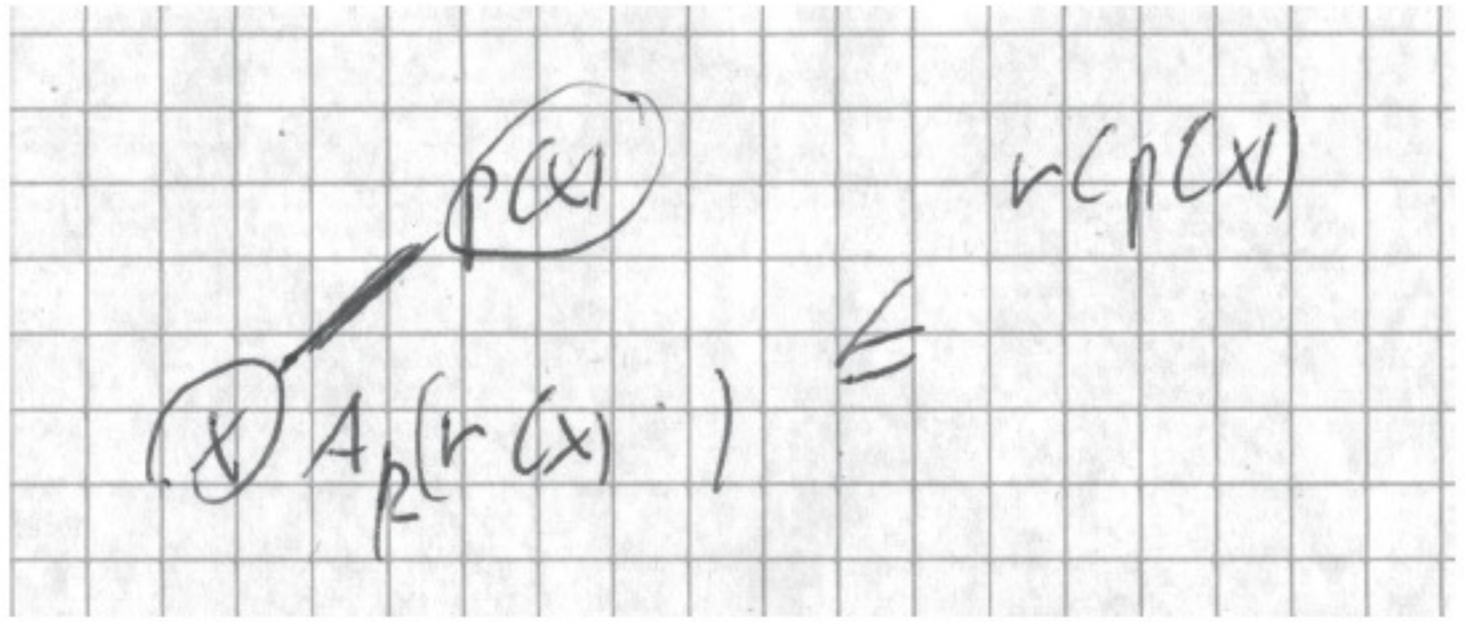

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eqn. 21.1 in [CLRS]:

Lemma 7.

$$0 \leq \ell(x) < \alpha(n)$$

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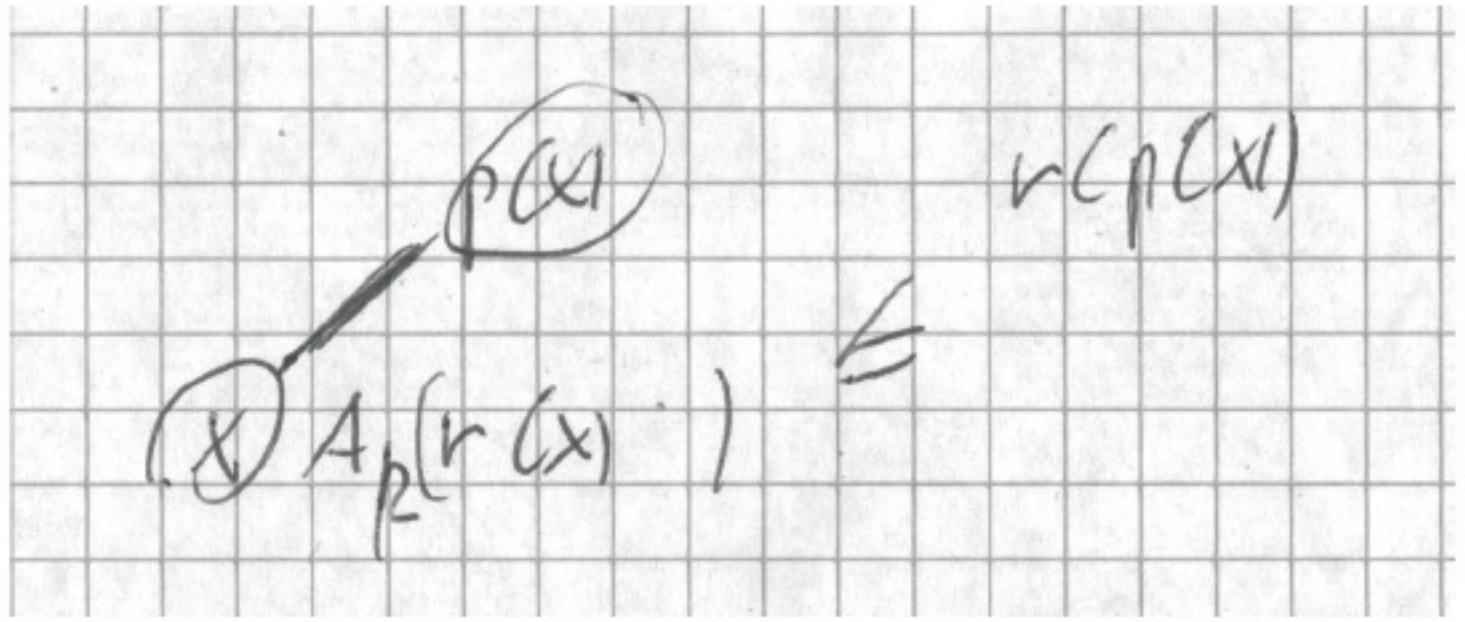

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$$\begin{aligned} A_0(r(x)) &= r(x) + 1 \quad \text{(def. of } A_0\text{)} \\ &\leq r(p(x)) \quad \text{(lemma 4)} \end{aligned}$$

$$\rightarrow \ell(x) \geq 0$$

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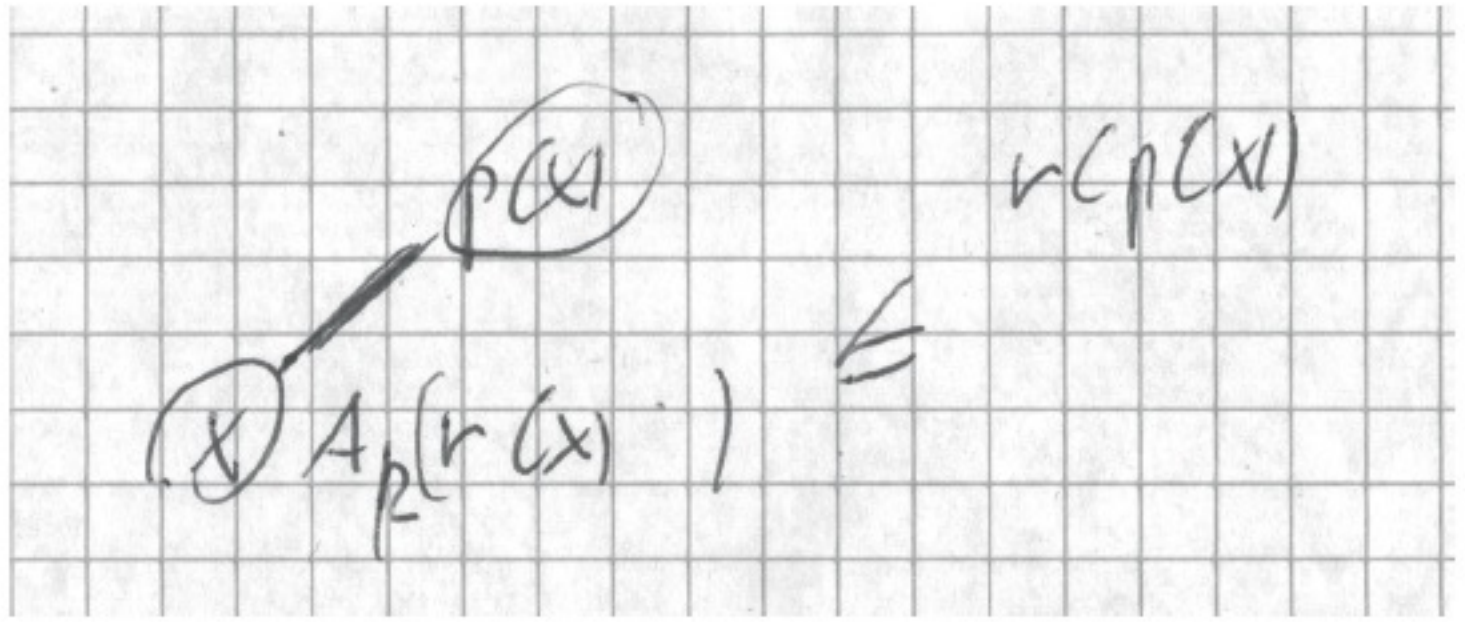

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$$\rightarrow \ell(x) \geq 0$$

$$\begin{aligned} r(p(x)) &< n \quad &\text{(lemma 5)} \\ &\leq A_{\alpha(n)}(1) \quad &\text{(def. of } \alpha(n)\text{)} \\ &\leq A_{\alpha(n)}(r(x)) \quad &\text{(lemma 3)} \end{aligned}$$

$$\rightarrow \ell(x) \leq \alpha(n)$$

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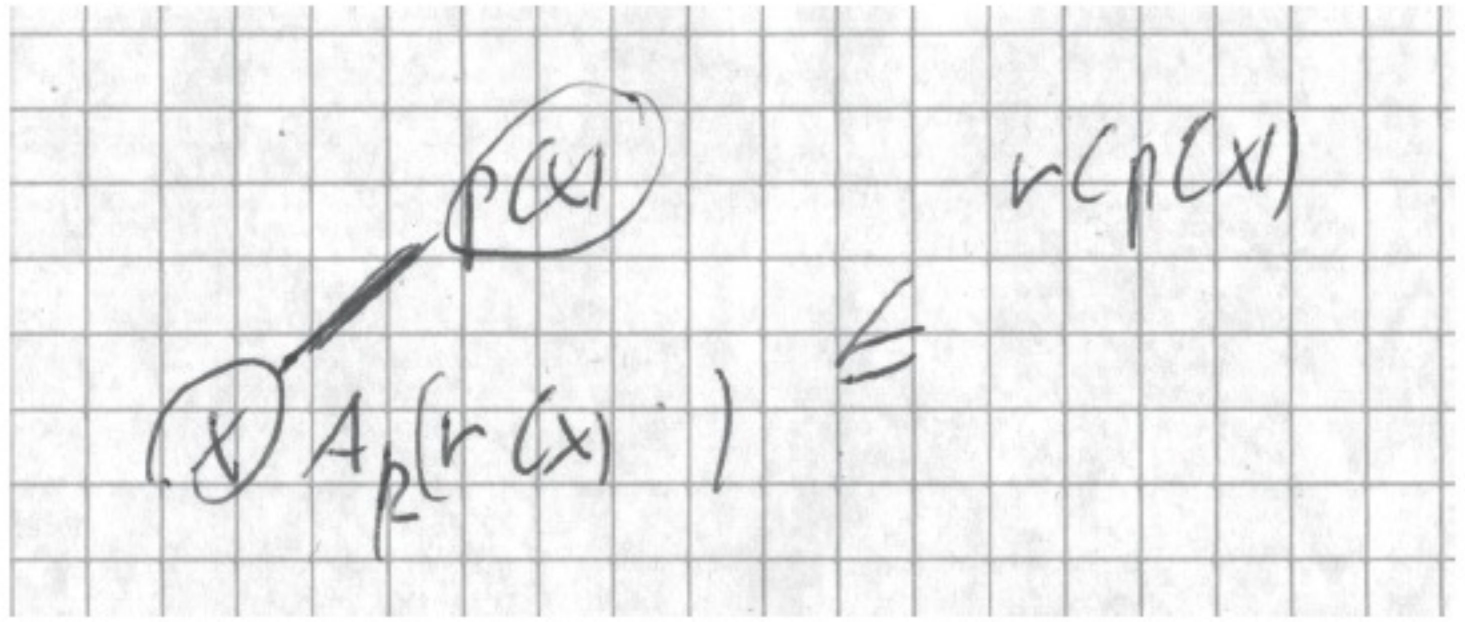

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$$\rightarrow \ell(x) \leq \alpha(n)$$

Lemma 8. *Ranks of predecessors and hence levels increase in time. For each operation*

$$r(p(x)) \leq r'(p'(x))$$

$$\cancel{\ell'(x) \leq \ell'(x)} \qquad \ell(x) \leq \ell'(x)$$

Proof. Exercise. Attention: parents change during find operations. $\square$

auxiliary function: iter:

$$i(x) = \max\{i \mid A^{(i)}_{\ell(x)}(r(x)) \leq r(p(x))\}$$

Figure 7: $i(x)$ is the largest k for which this inequality applies

iter

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Figure 7: $i(x)$ is the largest k for which this inequality applies

eqn. 21.2 in [CLRS]

Lemma 9.

$$r(x) \ge 1 \;\rightarrow\; 1 \le i(x) \le r(x)$$

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$$\begin{aligned} A^{1}_{\ell(x)}(r(x)) &= A_{\ell(x)}(r(x) \quad \text{(def. of iteration)} \\ &\leq r(p(x)) \quad \text{(def. of } \ell) \end{aligned}$$

$$1 \leq i(x)$$

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$$1 \leq i(x)$$

$$\begin{aligned} r(p(x)) &< A_{\ell(x)+1}(r(x)) \quad \text{(def. of } \ell) \\ &= A^{(r(x)+1)}_{\ell(x)}(r(x)) \quad \text{(def. of } A_k(j)) \end{aligned}$$

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$$i \le r(x)$$

From $r(p(x)) \le r'(p'(x))$ follows

Lemma 10.

$$\ell'(x) = \ell(x) \rightarrow i(x) \le i'(x)$$

$$i'(x) < i(x) \rightarrow \ell(x) < \ell'(x)$$

potential function

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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$$0 \leq \phi(x) \leq \alpha(n) \cdot r(x)$$

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- otherwise

$$\begin{aligned} \phi(x) &= (\alpha(n) - \ell(x)) \cdot r(x) - i(x) \\ &\geq (\alpha(n) - (\alpha(n) - 1)) \cdot r(n) - r(x) \quad \text{(lemmas 7 and 9)} \\ &= 0 \end{aligned}$$

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the heavy weapon

def. of potential function

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Lemma 12. *Let $x \neq p(x)$, i.e. x is not a root and suppose a link or find operation is executed. Then*

- *the potential of x does not increase*

$$\phi'(x) \leq \phi(x)$$

- *if $r(x) \geq 1$ and if $\ell(x)$ or $i(x)$ change*

$$r(x) \geq 1 \ \wedge \ (\ell'(x) \neq \ell(x) \vee i'(x) \neq i(x))$$

then the potential $\phi(x)$ decreases

$$\phi'(x) \leq \phi(x) - 1$$

$$x \neq p(x) \rightarrow r(x) = r'(x)$$

$$r(x) = 0 \rightarrow \phi(x) = \phi'(x) = 0$$

the heavy weapon

def. of potential function

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$$\phi'(x) \leq \phi(x) - 1$$

$$x \neq p(x) \rightarrow r(x) = r'(x)$$

$$r(x) = 0 \rightarrow \phi(x) = \phi'(x) = 0$$

Assume $r(x) \geq 0$

- $$\ell(x) = \ell'(x) \;\rightarrow\; i(x) \leq i'(x) \quad \text{(lemma 10)}$$

$$i(x) = i'(x) \;\rightarrow \phi'(x) = \phi(x)$$

$$i'(x) \geq i(x) + 1 \rightarrow \phi'(x) \leq \phi(x) - 1$$

the heavy weapon

def. of potential function

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Assume $r(x) \geq 0$

•

$$\ell(x) = \ell'(x) \ \rightarrow \ i(x) \leq i'(x) \quad \text{(lemma 10)}$$

$$i(x) = i'(x) \ \rightarrow \phi'(x) = \phi(x)$$

$$i'(x) \geq i(x) + 1 \rightarrow \phi'(x) \leq \phi(x) - 1$$

the heavy weapon

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

Lemma 12. *Let $x \neq p(x)$, i.e. x is not a root and suppose a link or find operation is executed. Then*

- *the potential of x does not increase*

$$\phi'(x) \leq \phi(x)$$

- *if $r(x) \geq 1$ and if $\ell(x)$ or $i(x)$ change*

$$r(x) \geq 1 \ \wedge \ (\ell'(x) \neq \ell(x) \vee i'(x) \neq i(x))$$

then the potential $\phi(x)$ decreases

$$\phi'(x) \leq \phi(x) - 1$$

$$x \neq p(x) \to r(x) = r'(x)$$

$$r(x) = 0 \to \phi(x) = \phi'(x) = 0$$

Assume $r(x) \geq 0$

- $$\ell(x) = \ell'(x) \ \to \ i(x) \leq i'(x) \quad \text{(lemma 10)}$$

  $$i(x) = i'(x) \ \to \phi'(x) = \phi(x)$$

  $$i'(x) \geq i(x) + 1 \to \phi'(x) \leq \phi(x) - 1$$

- $$\ell'(x) \geq \ell(x) + 1$$

$$\begin{aligned} \phi(x) - \phi'(x) &= (\alpha(n) - \ell(x)) \cdot r(x) - i(x) - ((\alpha(n) - \ell'(x)) \cdot r(x) - i'(x)) \\ &= (\ell'(x) - \ell(x)) \cdot r(x) + i'(x) - i(x) \\ &\geq r(x) + i'(x) - i(x) \\ &\geq r(x) + 1 - r(x) \quad \text{(lemma 9)} \\ &= 1 \end{aligned}$$

amortized cost of operations: make-set

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

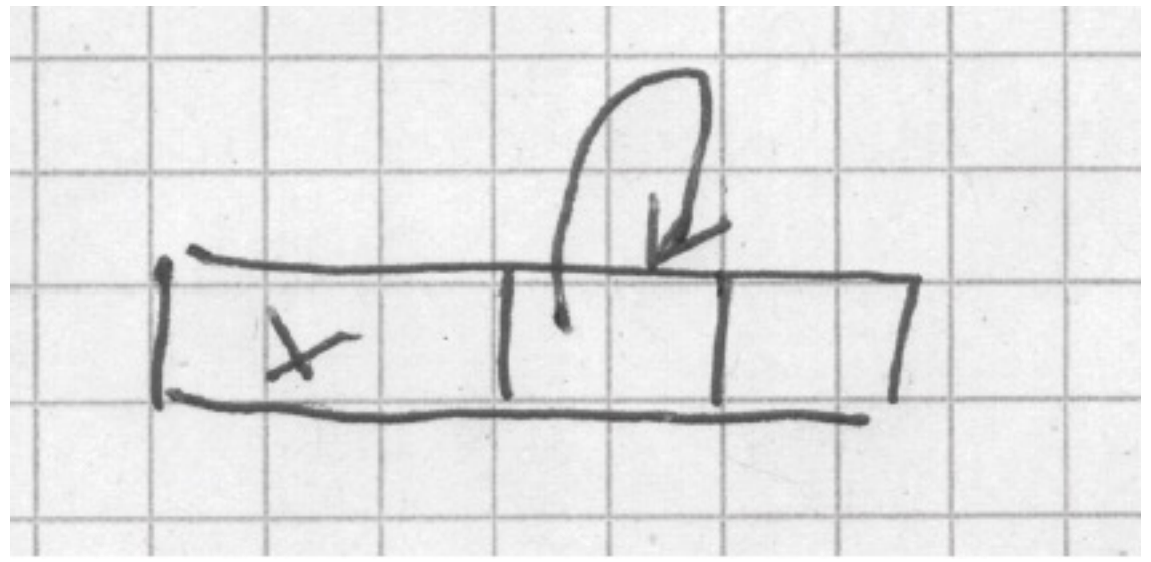
make-set:

Lemma 13.

$$op = make - set(x) \;\rightarrow\; \hat{c} = O(1)$$

Proof.

$$\phi'(x) = 0 \;,\; \Phi' = \Phi \;,\; \hat{c} = c + \Phi' - \Phi = c$$



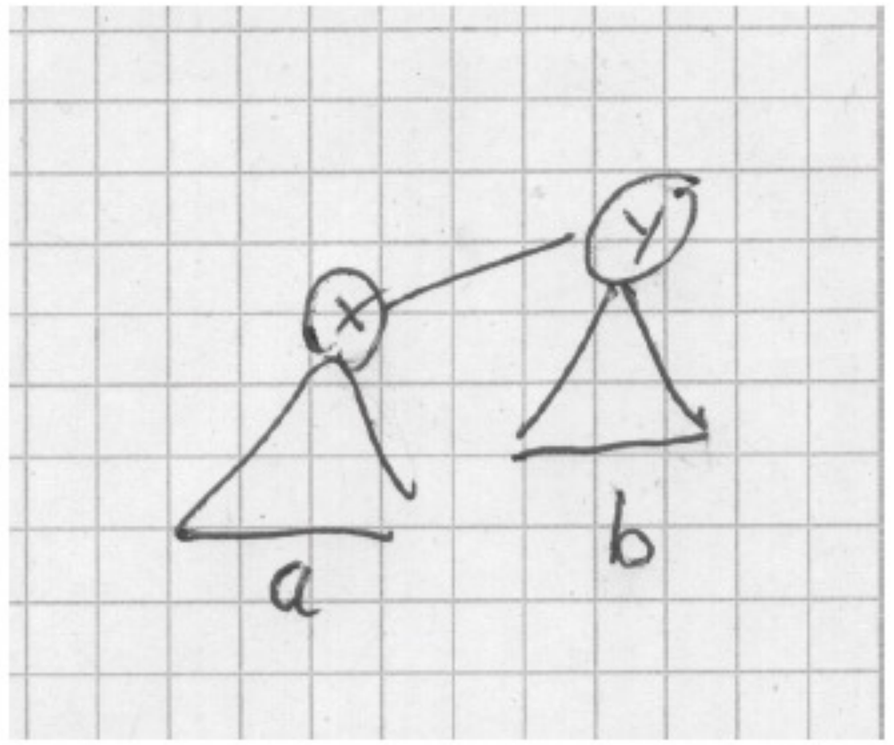
amortized cost of operations: link

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

Lemma 14.

$$op = link(x,y) \;\rightarrow\; \hat{c} = O(\alpha(n))$$



def. of potential function

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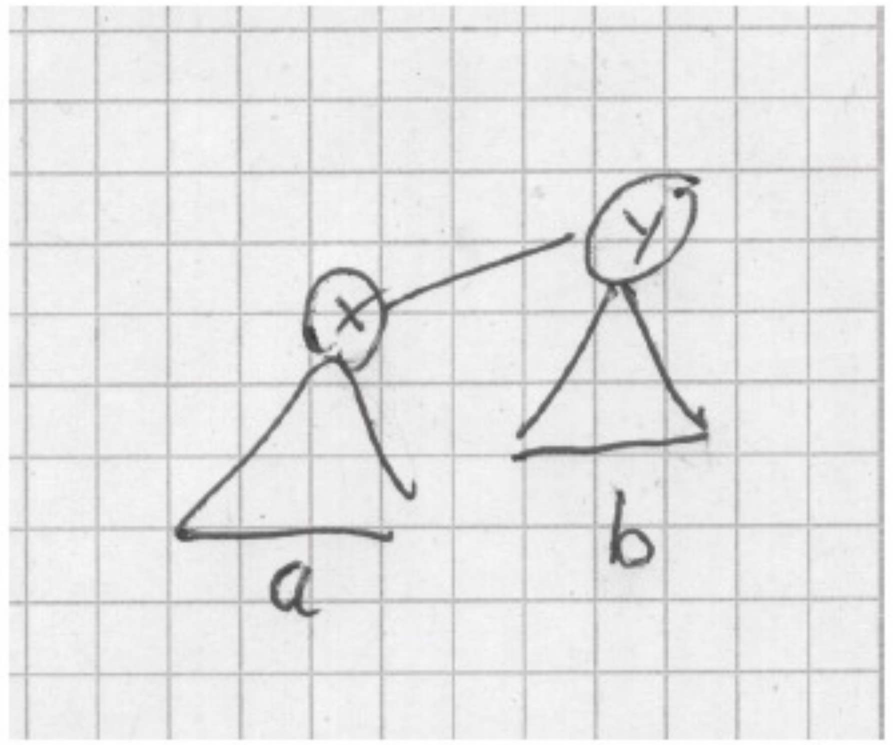

- nodes z which are no root. Potential not increasing.

$$z \notin \{x,y\} \ \rightarrow \phi'(z) \leq \phi(z) \quad \text{(lemma 12)}$$

amortized cost of operations: link

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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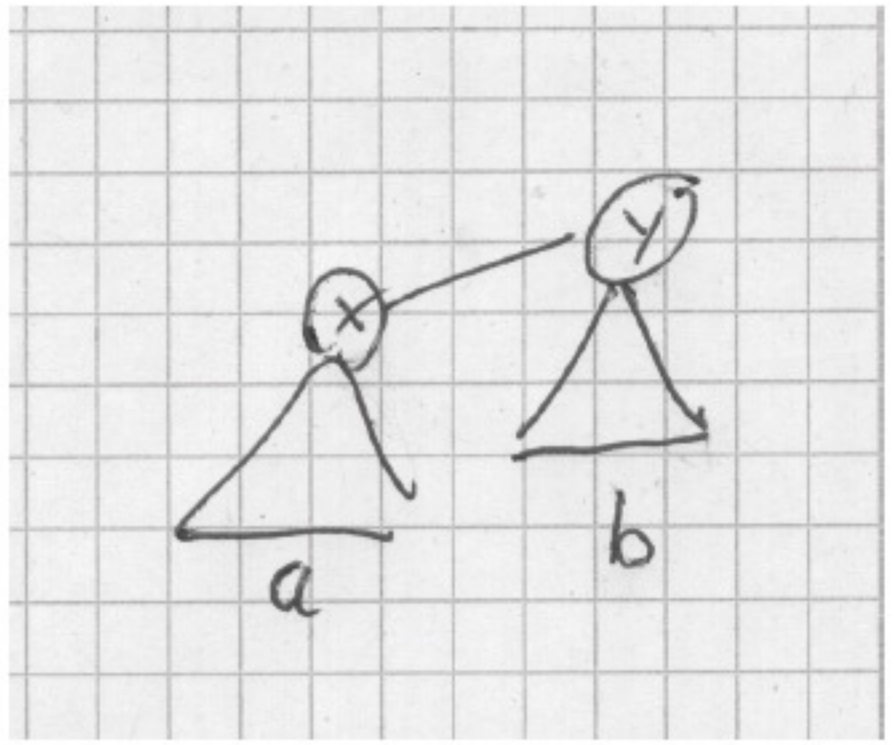

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- old root x

$$\phi(x) = \alpha(n) \cdot r(x) \;, r'(x) = r(x)$$

case $r(x) = 0$:

$$r'(x) = 0 \;,\; \phi'(x) = \phi(x) = 0$$

amortized cost of operations: link

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n)\cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x))\cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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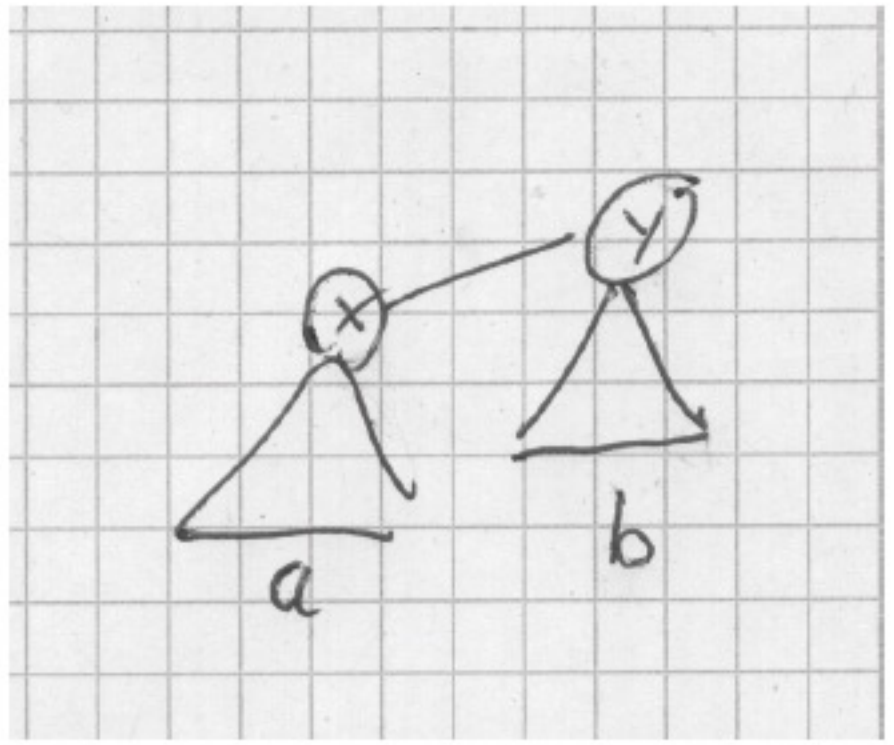

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$$\phi(x) = \alpha(n)\cdot r(x)\;, r'(x) = r(x)$$

case $r(x) = 0$:

$$r'(x) = 0\;,\; \phi'(x) = \phi(x) = 0$$

case $r(x) > 0$: potential falling.

$$\begin{aligned} \phi'(x) &= (\alpha(n) - \ell'(x))\cdot r'(x) - i'(x) \\ &= (\alpha(n) - \ell'(x))\cdot r(x) - i'(x) \\ &\leq \alpha(n)\cdot r(x) - 1 \quad \text{(lemmas 7 and 9)} \\ &= \phi(x) - 1 \end{aligned}$$

amortized cost of operations: link

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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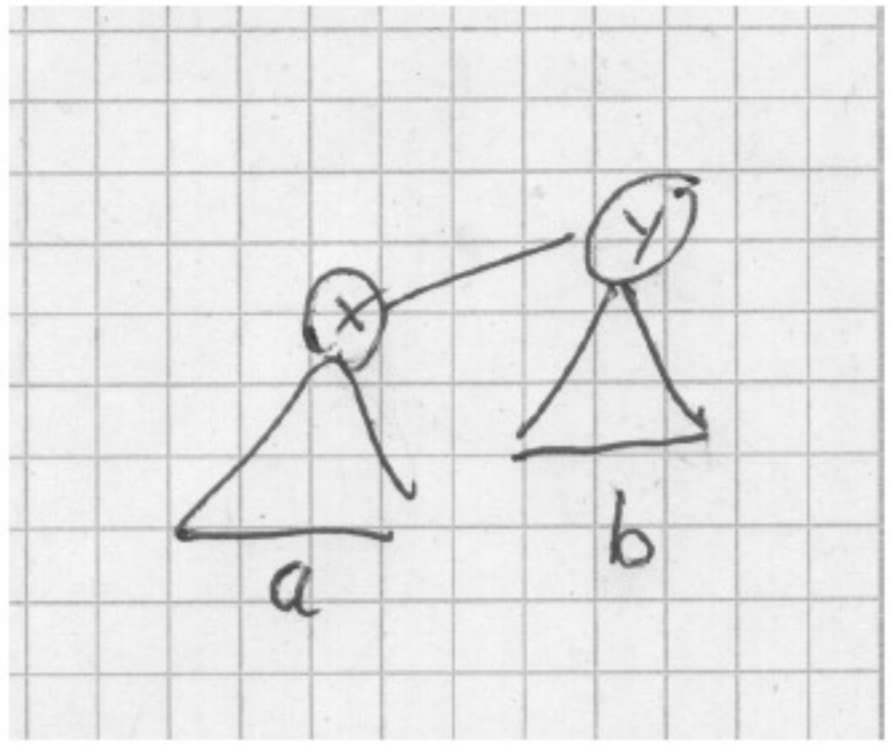

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- old and new root y:

$$\phi(y) = \alpha(n) \cdot r(y) \;,\; \phi'(y) = \alpha(n) \cdot r'(y)$$

$$r'(y) \in \{r(y), r(y)+1\}$$

$$\phi'(y) \in \{\alpha(n) \cdot r(y), \alpha(n) \cdot (r(y)+1)\}$$

$$\phi'(y) \leq \phi(y) + \alpha(n)$$

amortized cost of operations: link

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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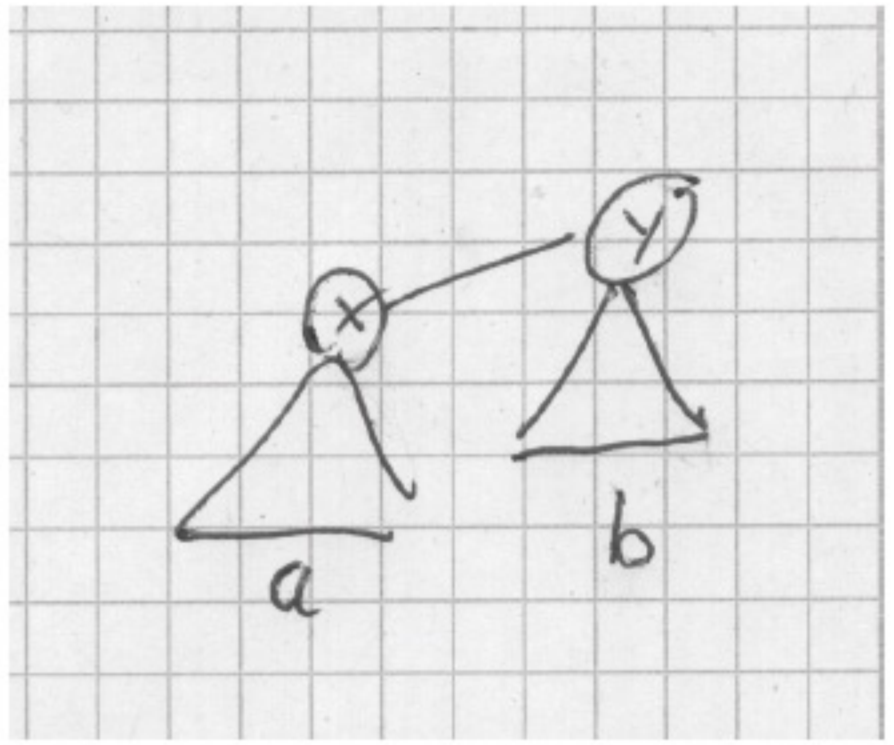

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- old and new root y:

$$\phi(y) = \alpha(n) \cdot r(y) \;, \; \phi'(y) = \alpha(n) \cdot r'(y)$$

$$r'(y) \in \{r(y), r(y)+1\}$$

$$\phi'(y) \in \{\alpha(n) \cdot r(y), \alpha(n) \cdot (r(y)+1)\}$$

$$\phi'(y) \le \phi(y) + \alpha(n)$$

$$\boxed{\hat{c} = c + \phi'(y) - \phi(y) = O(1) + O(\alpha(n))}$$

defining the unit of cost

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n)\cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x))\cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

We define first the unit of cost for c:

accessing (and processing) 1 node costs 1. Then the compression of a path with $\cancel{n}$ nodes has cost $c = s$

s

amortized cost of operations: find

defining the unit of cost

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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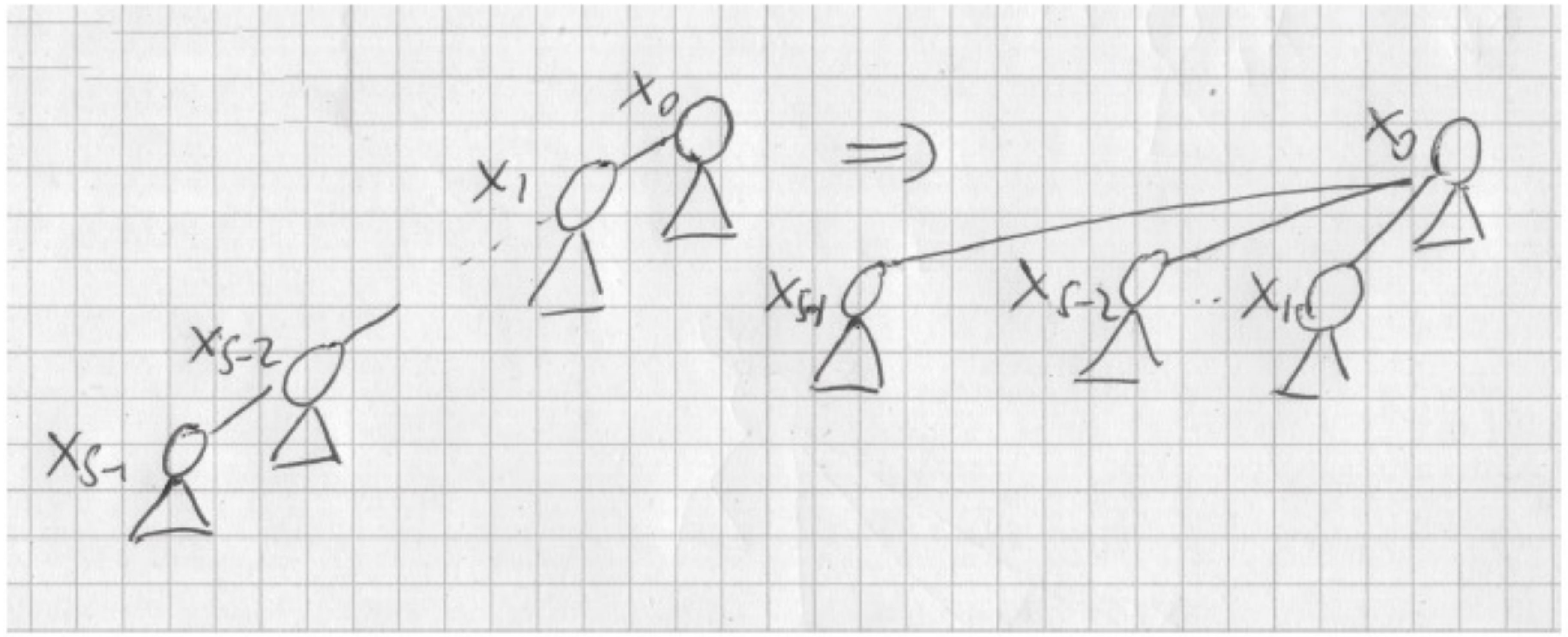

Figure 9: Compression of a find path with s nodes $x_{s-1}, \ldots, x_0$ processes exactly these nodes and has thus cost s

Lemma 15.

find

$$op = link\cancel{(x)} \to \hat{c} = O(\alpha(n))$$

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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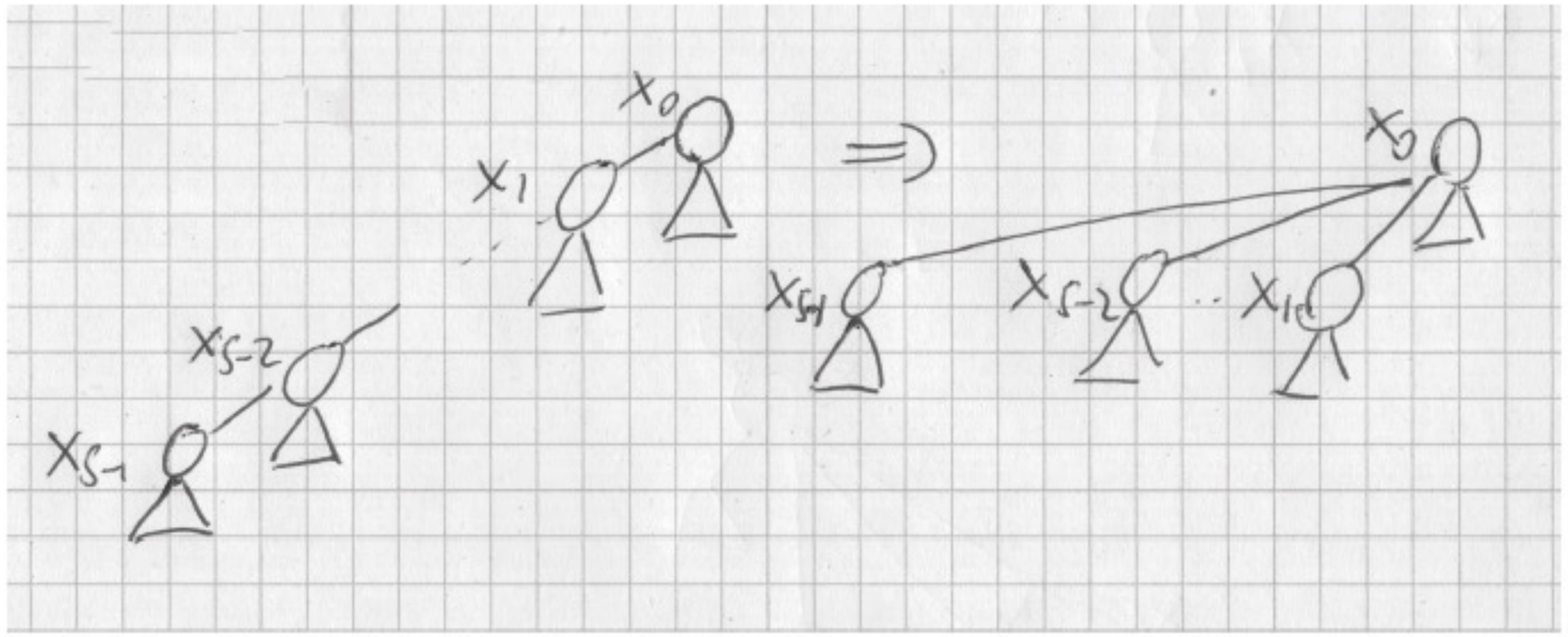

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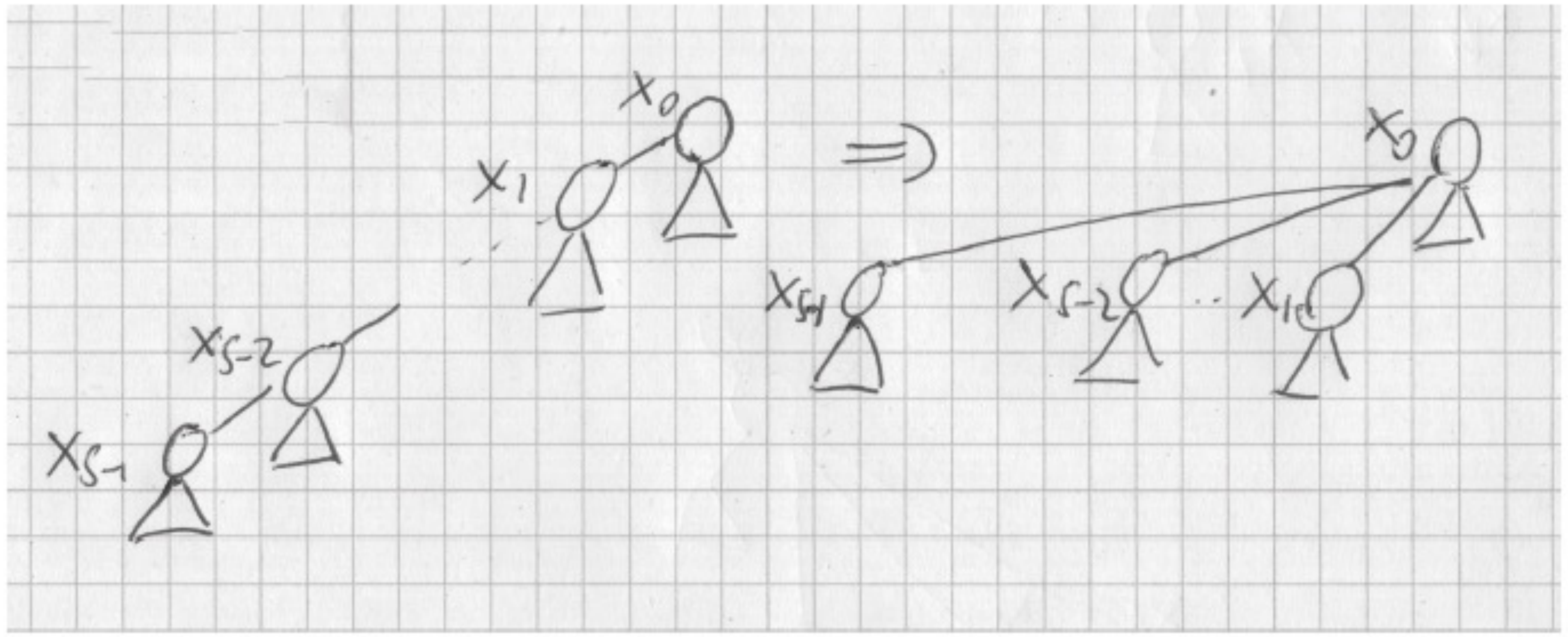

Figure 9: Compression of a find path with s nodes $x_{s-1}, \ldots, x_0$ processes exactly these nodes and has thus cost s

Lemma 15. $op = \overset{\text{find}}{\cancel{\mathit{link}}}(x) \rightarrow \hat{c} = O(\alpha(n))$

$$\hat{c} = s + \sum_{i=0}^{s-1} (\phi'(x_i) - \phi(x_i))$$

$$\phi'(x_0) = \phi(x_0) = \alpha(n) \cdot r(x_0)$$

Lemma 12 $\rightarrow$:

$$\phi'(x_i) \leq \phi(x_i) \text{ for } 1 \leq i \leq s-1$$

amortized cost of operations: find

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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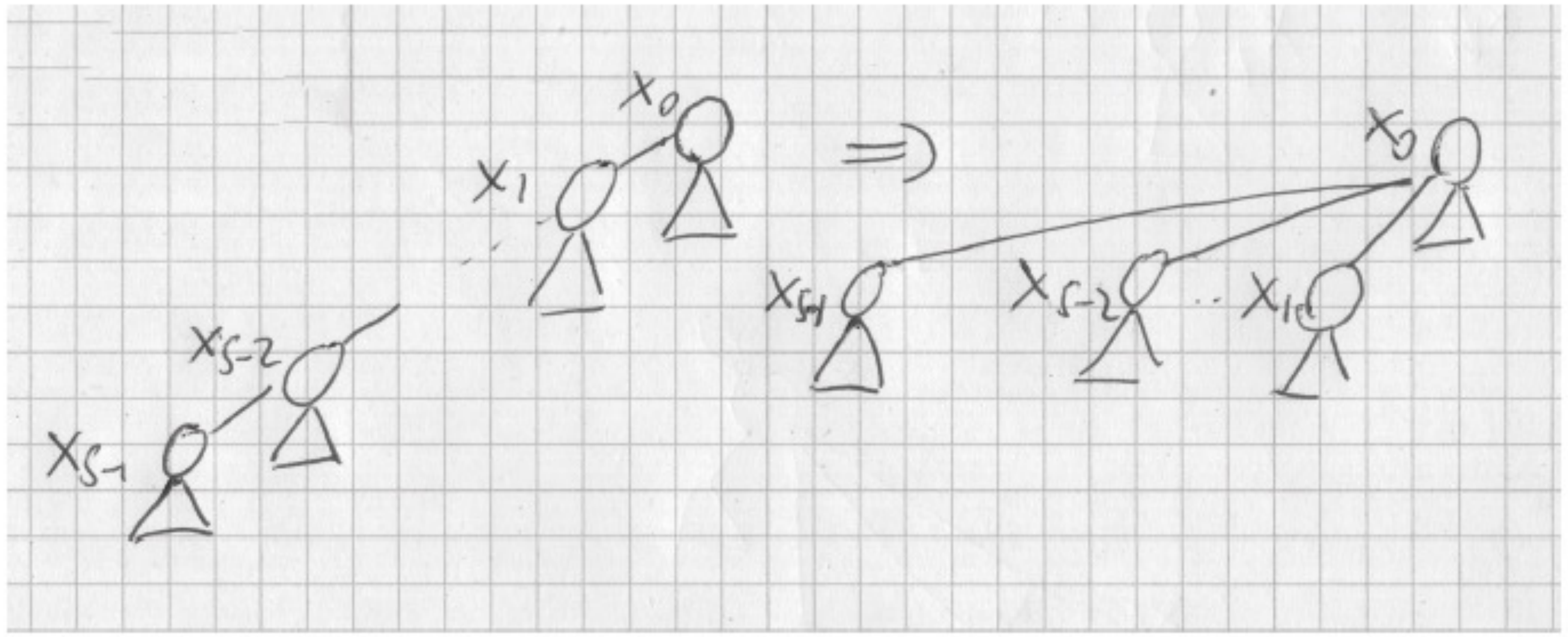

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Lemma 15.

find

$$op = \cancel{link}(x) \;\rightarrow\; \hat{c} = O(\alpha(n))$$

$$\hat{c} = s + \sum_{i=0}^{s-1} (\phi'(x_i) - \phi(x_i))$$

$$\phi'(x_0) = \phi(x_0) = \alpha(n) \cdot r(x_0)$$

Lemma 12 $\rightarrow$:

$$\phi'(x_i) \leq \phi(x_i) \text{ for } 1 \leq i \leq s-1$$

goal: define large enough set

$$E \subseteq \{x_0, \ldots, x_{s-1}\}$$

such that

$$\phi'(x) \leq \phi(x) - 1 \text{ for all } x \in E$$

Then

$$\hat{c} \leq s + \sum_{x \in E} (\phi'(x) - \phi(x)) = s - \#E$$

Done if

$$\#E \geq s - O(\alpha(n))$$

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

Lemma 15.

find

$$op = \cancel{link}(x) \;\rightarrow\; \hat{c} = O(\alpha(n))$$

$$E = \{x_i \mid r(x_i) > 0 \,,\; \exists j.\, 0 < j < i \wedge \ell(x_i) = \ell(x_j)\}$$

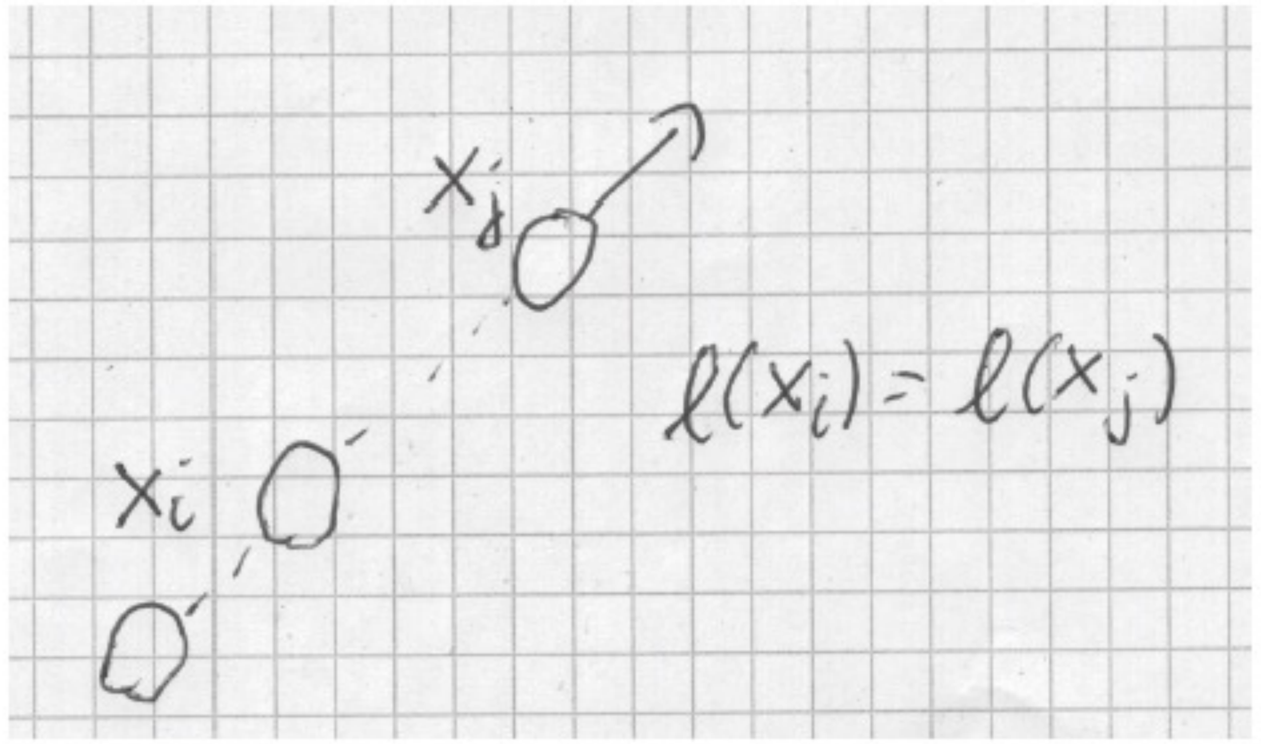

Figure 10: For $x_i \in E$ we need $r(x_i) > 0$ and some node x_j with the same level must be properly between x_i and the root

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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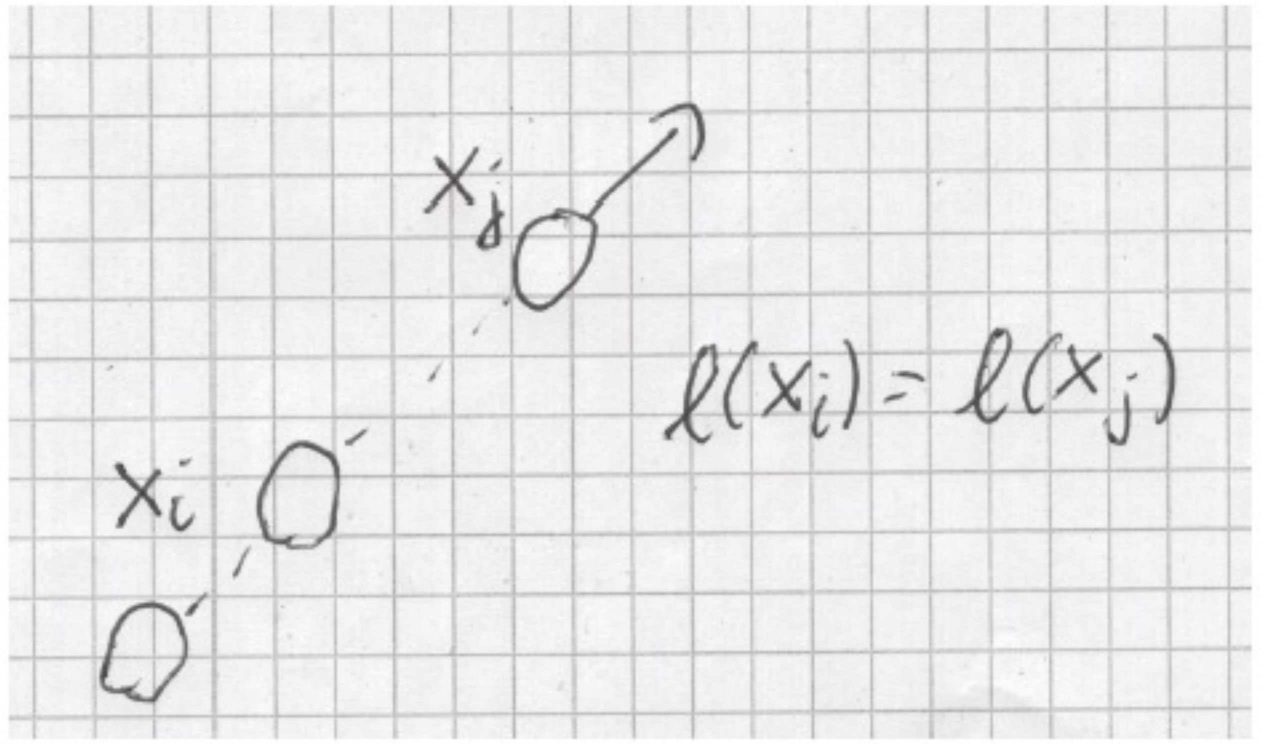

Figure 10: For $x_i \in E$ we need $r(x_i) > 0$ and some node x_j with the same level must be properly between x_i and the root

Exercise: construct an example where x_j is not the parent of x_i.

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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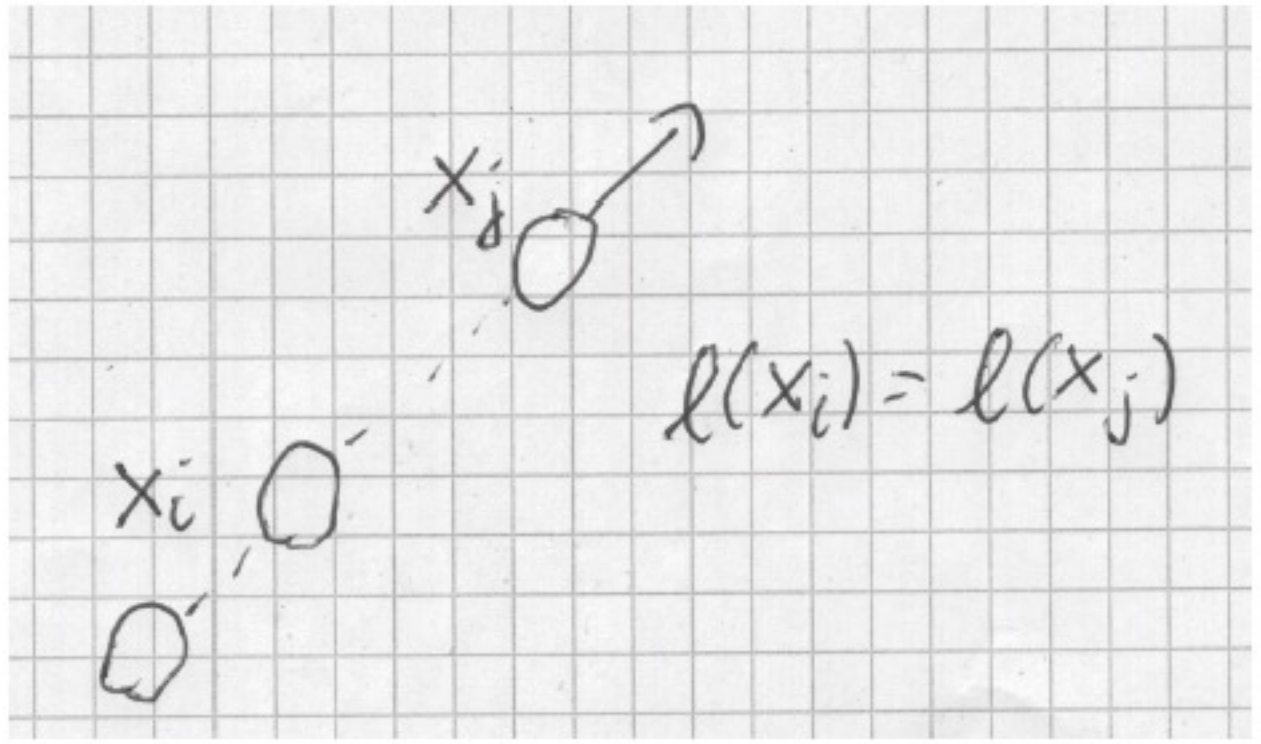

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Exercise: construct an example where x_j is not the parent of x_i.

Lemma 16.

$$\# E \geq s - \alpha(n) - 2$$

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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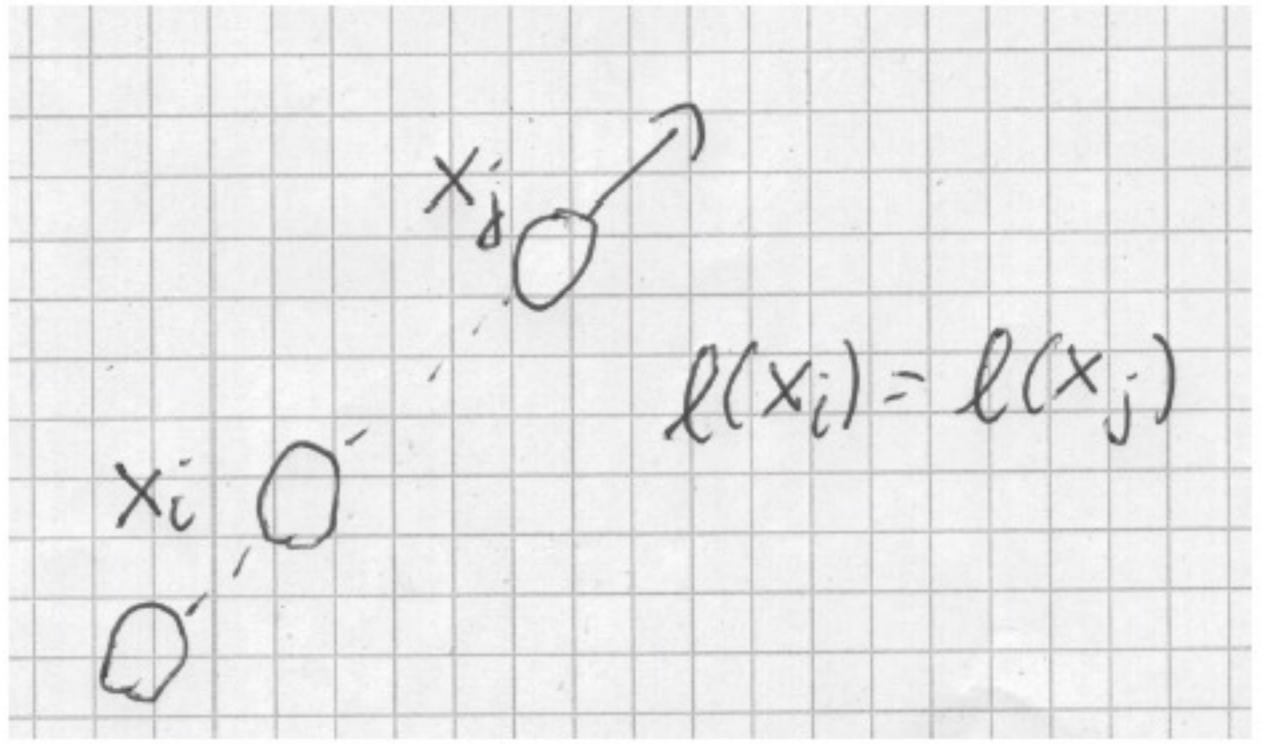

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Lemma 16.

$$\# \; E \geq s - \alpha(n) - 2$$

for $i \in [0 : s-1]$ node x_i not in E

- possibly if $r(x_i) = 0$, i.e. $x_i = x_{s-1}$ is a leaf
- if $x_i = x_0$, i.e. x_i is the root
- if x_i is last node on path with level $\ell(x)$

$$\forall j.\, 0 < j < i \rightarrow \ell(x_j) \neq \ell(x_i)$$

Lemma 7

$$0 \leq \ell(x) < \alpha(n)$$

excludes at most $\alpha(n)$ nodes

amortized cost of operations: find

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

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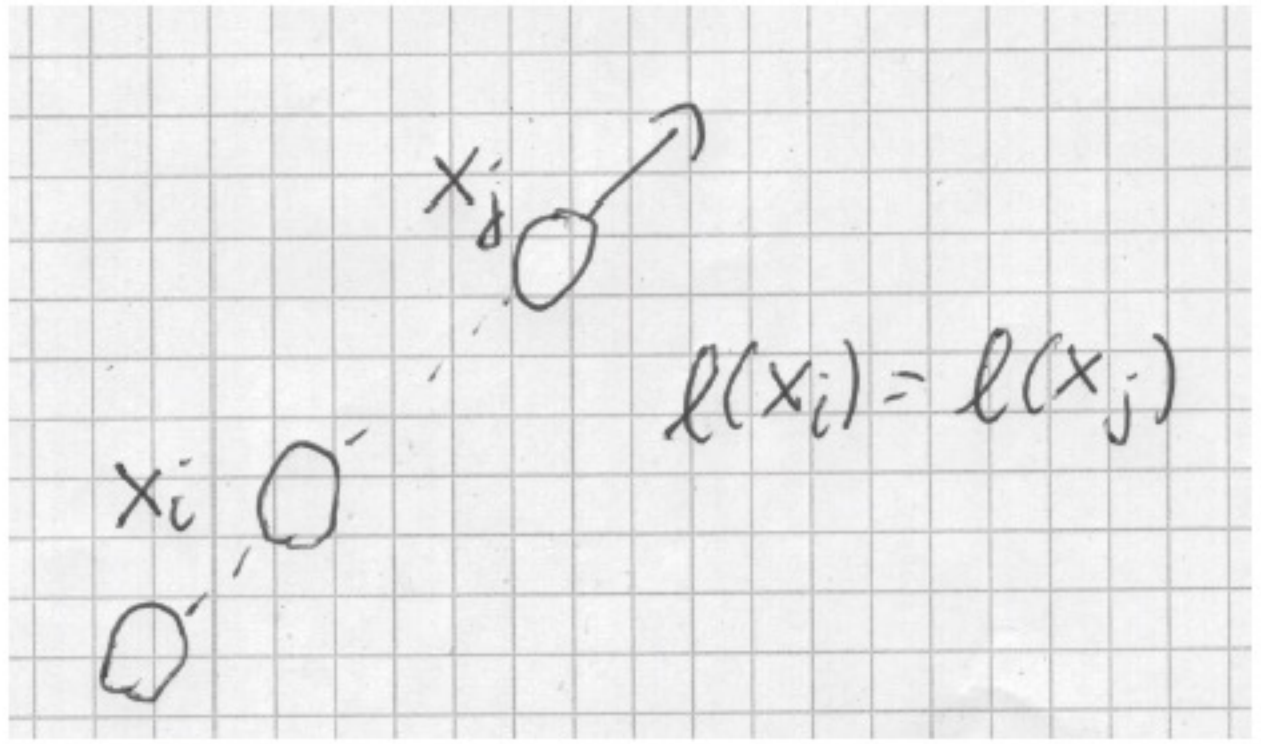

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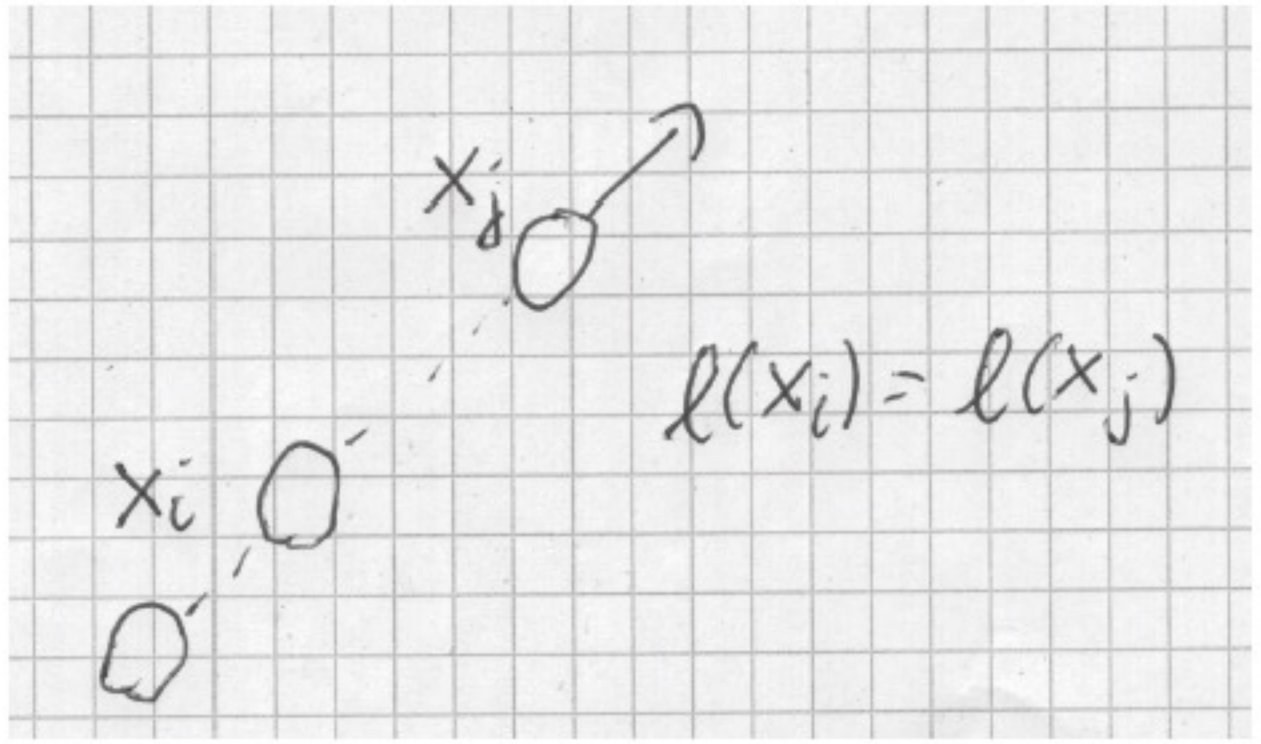

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$$x = x_i \;,\; y = x_j \;,\; k = \ell(x) = \ell(y)$$

$$\begin{aligned} r(p(x)) &\geq A_k^{(i(x))}(r(x)) && (\text{ def. of } i(x)) && (1) \\ r(p(y)) &\geq A_k(r(y)) && (\text{ def. of } \ell(x)) && (2) \\ r(y) &\geq r(p(x)) && (\text{ lemma 4}) && (3) \end{aligned}$$

def. of potential function

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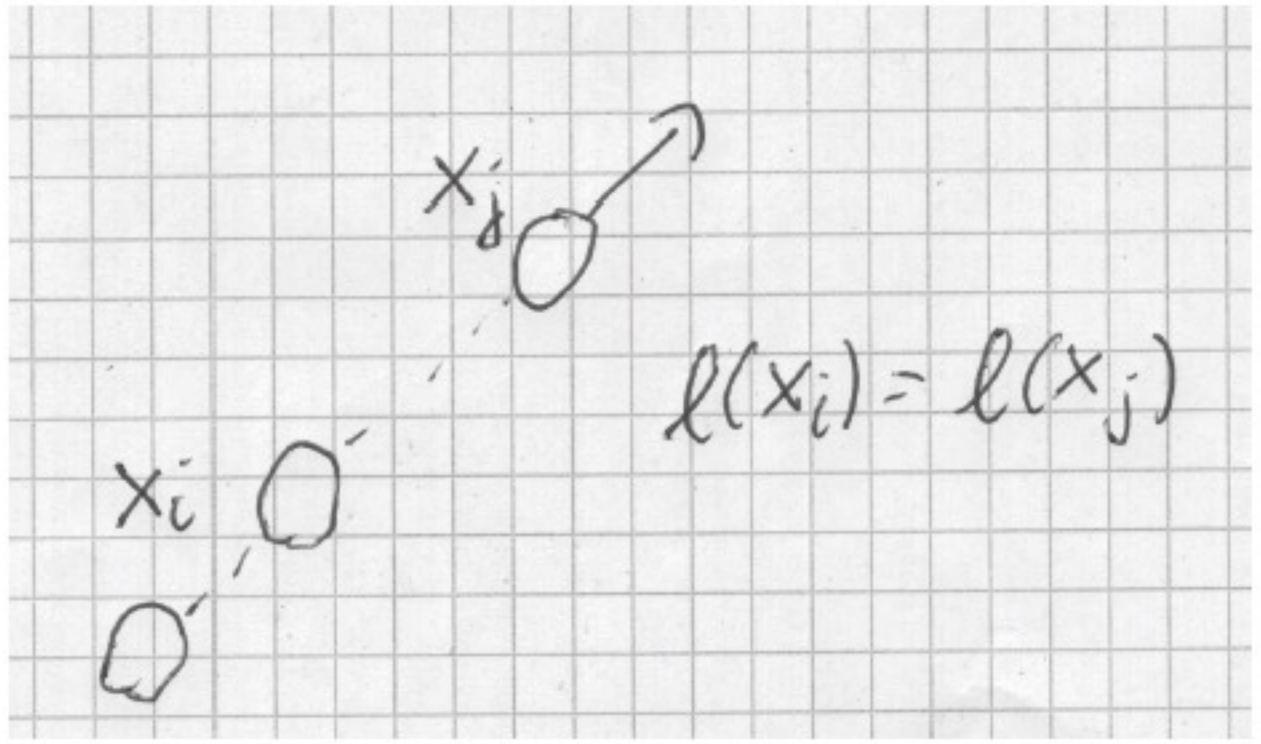

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$$x = x_i \ , \ y = x_j \ , \ k = \ell(x) = \ell(y)$$

$$\begin{aligned} r(p(x)) &\geq A_k^{(i(x))}(r(x)) && (\text{ def. of } i(x)) && (1) \\ r(p(y)) &\geq A_k(r(y)) && (\text{ def. of } \ell(x)) && (2) \\ r(y) &\geq r(p(x)) && (\text{ lemma } 4) && (3) \end{aligned}$$

$$\begin{aligned} r(p(y)) &\geq A_k(r(y)) && (\text{eqn. } 2) && (4) \\ &\geq A_k(r(p(x))) && (\text{lemma 3 and eqn. } 3) && (5) \\ &\geq A_k(A_k^{(i(x))}(r(x))) && (\text{lemma 3 and eqn. } 1) && (6) \\ &= A_k^{(i(x)+1)}(r(x)) && (\text{def. of iteration}) && (7) \end{aligned}$$

amortized cost of operations: find

def. of potential function

$$\phi(x) = \begin{cases} \alpha(n) \cdot r(n) & x = p(x) \vee r(x) = 0 \\ (\alpha(n) - \ell(x)) \cdot r(x) - i(x) & \text{otherwise} \end{cases}$$

Lemma 15.

find

$$op = \cancel{link}(x) \;\rightarrow\; \hat{c} = O(\alpha(n))$$

$$E = \{x_i \mid r(x_i) > 0 \;,\; \exists j.\, 0 < j < i \;\wedge\; \ell(x_i) = \ell(x_j)\}$$

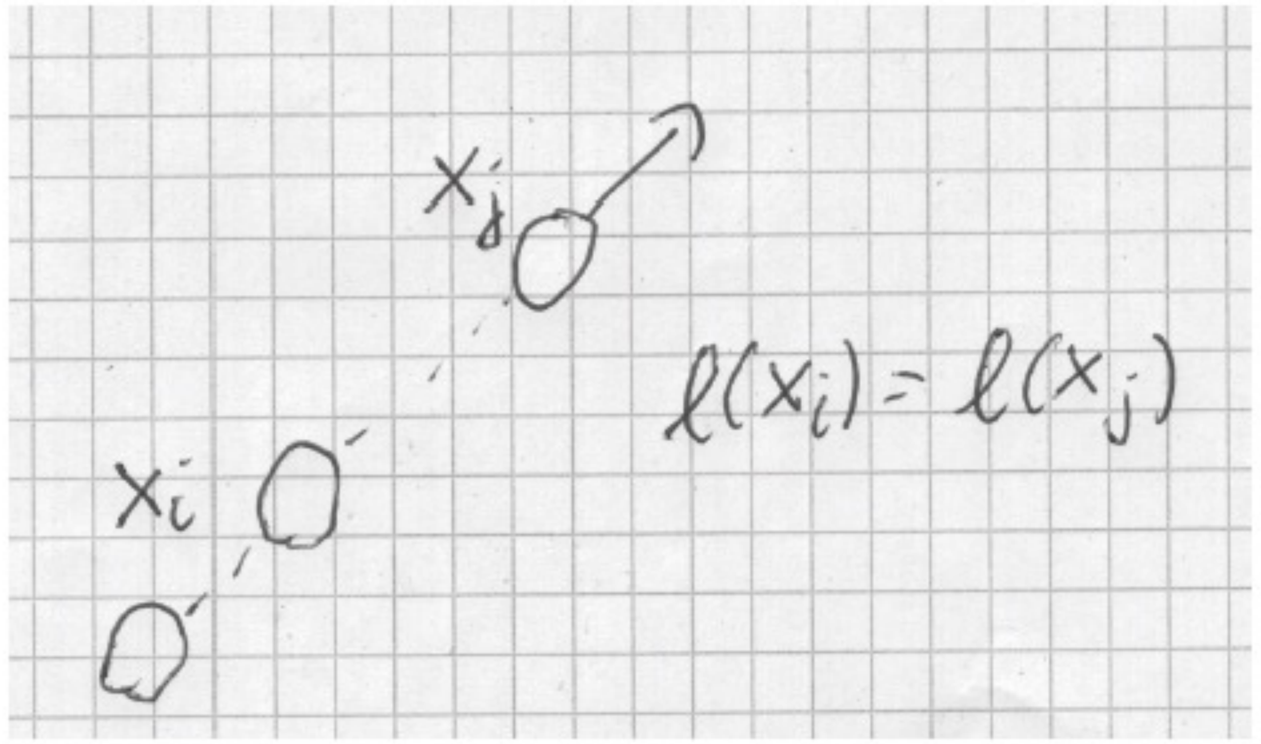

Figure 10: For $x_i \in E$ we need $r(x_i) > 0$ and some node x_j with the same level must be properly between x_i and the root

Lemma 16.

$$\# \; E \geq s - \alpha(n) - 2$$

Lemma 17.

$$x \in E \rightarrow \phi'(x) \leq \phi(x) - 1$$

$$x = x_i \;,\; y = x_j \;,\; k = \ell(x) = \ell(y)$$

$$\begin{aligned} r(p(x)) &\geq A_k^{(i(x))}(r(x)) \quad (\text{ def. of } i(x)) && (1) \\ r(p(y)) &\geq A_k(r(y)) \quad (\text{ def. of } \ell(x)) && (2) \\ r(y) &\geq r(p(x)) \quad (\text{ lemma 4}) && (3) \end{aligned}$$

$$\begin{aligned} r(p(y)) &\geq A_k(r(y)) \quad (\text{eqn. 2}) && (4) \\ &\geq A_k(r(p(x)) \quad (\text{lemma 3 and eqn. 3}) && (5) \\ &\geq A_k(A_k^{(i(x))}(r(x))) \quad (\text{lemma 3 and eqn. 1}) && (6) \\ &= A_k^{(i(x)+1)}(r(x)) \quad (\text{def. of iteration}) && (7) \end{aligned}$$

after path compression:

$$\begin{aligned} r'(p'(x)) &= r(x_0) \\ &= r'(p'(y)) \\ &\geq r(p(y)) \quad (\text{lemma 4 }) \\ &\geq A_k^{(i(x)+1)}(r(x)) \quad (\text{eqn. 7}) \\ &= A_k^{(i(x)+1)}(r'x)) \quad (\text{lemma 4 }) \end{aligned}$$

proof of..

Lemma 17.

$$x \in E \to \phi'(x) \leq \phi(x) - 1$$

$$x = x_i \;,\; y = x_j \;,\; k = \ell(x) = \ell(y)$$

$$\begin{aligned}
r(p(x)) &\geq A_k^{(i(x))}(r(x)) \quad (\text{ def. of } i(x)) && (1)\\
r(p(y)) &\geq A_k(r(y)) \quad (\text{ def. of } \ell(x)) && (2)\\
r(y) &\geq r(p(x)) \quad (\text{ lemma 4}) && (3)
\end{aligned}$$

$$\begin{aligned}
r(p(y)) &\geq A_k(r(y)) \quad (\text{eqn. 2}) && (4)\\
&\geq A_k(r(p(x)) \quad (\text{lemma 3 and eqn. 3}) && (5)\\
&\geq A_k(A_k^{(i(x))}(r(x))) \quad (\text{lemma 3 and eqn. 1}) && (6)\\
&= A_k^{(i(x)+1)}(r(x)) \quad (\text{def. of iteration}) && (7)
\end{aligned}$$

after path compression:

$$\begin{aligned}
r'(p'(x)) &= r(x_0)\\
&= r'(p'(y))\\
&\geq r(p(y)) \quad (\text{lemma 4 })\\
&\geq A_k^{(i(x)+1)}(r(x)) \quad (\text{eqn. 7})\\
&= A_k^{(i(x)+1)}(r'x)) \quad (\text{lemma 4 })
\end{aligned}$$ proof of..

- if $k = \ell'(x)$

$$r'(p'(x)) \geq A_{\ell'(x)}^{(i(x)+1)}(r'x))$$

$$i'(x) \geq i(x) + 1$$

- lemma 12

$$\ell'(x) \neq \ell(x) \;\vee\; i'(x) \neq i(x) \;\to\; \phi'(x) \leq \phi(x) - 1$$

Lemma 17.

$$x \in E \to \phi'(x) \leq \phi(x) - 1$$

$$x = x_i \ , \ y = x_j \ , \ k = \ell(x) = \ell(y)$$

$$\begin{aligned} r(p(x)) &\geq A_k^{(i(x))}(r(x)) \quad (\text{ def. of } i(x)) && (1) \\ r(p(y)) &\geq A_k(r(y)) \quad (\text{ def. of } \ell(x)) && (2) \\ r(y) &\geq r(p(x)) \quad (\text{ lemma 4}) && (3) \end{aligned}$$

$$\begin{aligned} r(p(y)) &\geq A_k(r(y)) \quad (\text{eqn. 2}) && (4) \\ &\geq A_k(r(p(x)) \quad (\text{lemma 3 and eqn. 3}) && (5) \\ &\geq A_k(A_k^{(i(x))}(r(x))) \quad (\text{lemma 3 and eqn. 1}) && (6) \\ &= A_k^{(i(x)+1)}(r(x)) \quad (\text{def. of iteration}) && (7) \end{aligned}$$

after path compression:

$$\begin{aligned} r'(p'(x)) &= r(x_0) \\ &= r'(p'(y)) \\ &\geq r(p(y)) \quad (\text{lemma 4 }) \\ &\geq A_k^{(i(x)+1)}(r(x)) \quad (\text{eqn. 7}) \\ &= A_k^{(i(x)+1)}(r'x)) \quad (\text{lemma 4 }) \end{aligned}$$

proof of..

- if $k = \ell'(x)$

$$r'(p'(x)) \geq A_{\ell'(x)}^{(i(x)+1)}(r'x))$$

$$i'(x) \geq i(x) + 1$$

- lemma 12

$$\ell'(x) \neq \ell(x) \ \vee \ i'(x) \neq i(x) \ \to \ \phi'(x) \leq \phi(x) - 1$$

$\boxed{\textbf{done!}}$