KUTAISI INTERNATIONAL UNIVERSITY

Numerical Linear Algebra

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Direct and iterative methods for linear systems

- Recap of Previous Lecture
- Sherman-Morrison formula
- Cholesky factorization of symmetric system
- Diagonally dominant system
- Necessary and sufficient conditions for convergence
- Classic iterative methods
- Convergence theorems of J,GS
- Q & A

Recap of Previous Lecture

- Examples of LU factorization
- Existence of LU factorization
- Gaussian elimination(GE) and LU factorization
- Pivoting
- Stability of GE

Shermann-Morrison formula

Problem 9.1

Suppose A is square matrix and A^{-1} is known.

Shermann-Morrison formula

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- Compute inverse of $A + B$ by means of using A^{-1}, B is square matrix.

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Theorem 9.2

(Shermann-Morrison formula)

- $A \in \mathbb{R}^{n \times n}, u, v \in \mathbb{R}^n$

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- $(A + uv^T)^{-1} = A^{-1} - \alpha(A^{-1} uv^T A^{-1})$

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Generalization: Sherman-Morrison-Woodbury

Shermann-Morrison formula

Theorem 9.3

- $A \in \mathbb{R}^{n \times n}, u, v \in \mathbb{R}^n$
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Proof.

Shermann-Morrison formula

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Proof.

$$(A + uv^T)(A^{-1} - \alpha(A^{-1} uv^T A^{-1})) =$$

Shermann-Morrison formula

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Proof.

$$(A + uv^T)(A^{-1} - \alpha(A^{-1}uv^T A^{-1})) =$$
$$= AA^{-1} + uv^T A^{-1} - \alpha AA^{-1}uv^T A^{-1} - \alpha uv^T A^{-1}uv^T A^{-1} =$$

Shermann-Morrison formula

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Proof.

$$\begin{aligned} (A + uv^T)(A^{-1} - \alpha(A^{-1}uv^T A^{-1})) &= \\ = AA^{-1} + uv^T A^{-1} - \alpha AA^{-1}uv^T A^{-1} - \alpha uv^T A^{-1}uv^T A^{-1} &= \\ = I + uv^T A^{-1} - \alpha uv^T A^{-1} - \alpha u\lambda v^T A^{-1} &= \end{aligned}$$

Shermann-Morrison formula

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- $A \in \mathbb{R}^{n \times n}, u, v \in \mathbb{R}^n$
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Proof.

$$
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= I + uv^TA^{-1} - \alpha uv^TA^{-1} - \alpha u\lambda v^TA^{-1} = \\
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= I + uv^T A^{-1} - 1 \cdot uv^T A^{-1} =
\end{aligned}$$

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= I
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Sherman-Morrison-Woodbury formula

Theorem 9.4

(Sherman-Morrison-Woodbury formula)

- $A \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times k}, C \in \mathbb{R}^{k\times k}, V \in \mathbb{R}^{k\times n},$

Sherman-Morrison-Woodbury formula

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Proof.

Direct proof similar to Sherman-Morrison formula $\square$

Special matrix: diagonal dominance

Definition 9.5

- Strictly row diagonally dominant matrix
  $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, i = 1, 2, ..., n$

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Proposition 9.6

Suppose $A \in \mathbb{R}^{n\times n}$ is column diagonally dominant. Then Gaussian elimination algorithm coincides with Gaussian elimination algorithm with partial pivoting

Cholesky decomposition

Problem 9.7

- $A \in \mathbb{R}^{n\times n}, A = A^T, A \geq 0$

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- $A \in \mathbb{R}^{n\times n}, A = A^T, A \geq 0$
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Unique Cholesky decomposition exists

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Proof.

- Similar to general LU decomposition theorem

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- Similar to general LU decomposition theorem
- Apply mathematical induction

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- Similar to general LU decomposition theorem
- Apply mathematical induction
- Apply decomposition of main sub matrices

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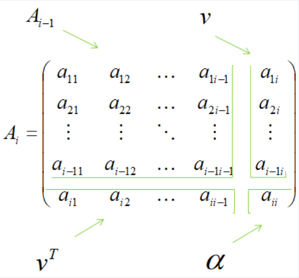

Convergence of iterative methods 1

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Suppose $\{x_k\}_{k=0}^{\infty}$ converges to x

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Convergence of iterative method

Suppose $\{x_k\}_{k=0}^{\infty}$ converges to x

- Does x satisfy any equation?

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- Can x be solution to $Ax = b$?

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Convergence of iterative methods 2

Convergence of iterative method

Suppose $\{x_k\}_{k=0}^{\infty}$ converges to x

- Does x satisfy any equation?

$$x = Bx + f$$

- Can x be solution to $Ax = b$?

Convergence of iterative methods 2

Convergence of iterative method

Suppose $\{x_k\}_{k=0}^{\infty}$ converges to x

- Does x satisfy any equation?

$$x = Bx + f$$

- Can x be solution to $Ax = b$?

Definition 9.9

Iterative method $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$ is compatible with $Ax = b$ if the latter is equivalent to $x = Bx + f$

Convergence of iterative methods 2

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Example 9.10

Richardson's method

$$P\frac{x^{(k+1)} - x^{(k)}}{\tau_k} + Ax^{(k)} = b$$

Convergence of iterative methods 3

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- Different initial guess defines different sequences $\{x_k\}_{k=0}^{\infty}$. Is it possible that for one initial guess method converges and for another does not?

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Definition 9.11

Iterative method $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$ is convergent if it converges to the same x for any initial guess.

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- Which parameters are important for convergence of ?

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Theorem 9.12

Sufficient condition of convergence: $\|B\| < 1$

Convergence of iterative methods 4

- $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$
  - $B \in \mathbb{R}^{n \times n}, x^{(k)} \in \mathbb{R}^n$
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Theorem 9.13

Sufficient condition of convergence: $\|B\| < 1$

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Theorem 9.13

Sufficient condition of convergence: $\|B\| < 1$

Proof.

- error - $e^{(k)} = x^{(k)} - x$

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Theorem 9.13

Sufficient condition of convergence: $\|B\| < 1$

Proof.

- error - $e^{(k)} = x^{(k)} - x$
- $x^{(k+1)} = Bx^{(k)} + f, x = Bx + f \Rightarrow x^{(k+1)} - x = B(x^{(k)} - x)$

Convergence of iterative methods 4

- $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$
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- $e^{(k+1)} = Be^{(k)}$

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- $x^{(k+1)} = Bx^{(k)} + f, x = Bx + f \Rightarrow x^{(k+1)} - x = B(x^{(k)} - x)$
- $e^{(k+1)} = Be^{(k)}$
- $e^{(k)} = Be^{(k-1)}, e^{(k+1)} = Be^{(k)} = BBe^{(k-1)} = B^2e^{(k-1)}$

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- $e^{(k)} = Be^{(k-1)}, e^{(k+1)} = Be^{(k)} = BBe^{(k-1)} = B^2 e^{(k-1)}$
- ...
- $e^{(k+1)} = B^{k+1} e^{(0)}$

Convergence of iterative methods 4

- $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$
  - $B \in \mathbb{R}^{n \times n}, x^{(k)} \in \mathbb{R}^n$
  - $x^{(0)}$ - initial guess
  - $x^{(1)}, x^{(2)}, ..., x^{(k)}, ..$ - infinite sequence

Theorem 9.13

Sufficient condition of convergence: $\|B\| < 1$

Proof.

- error - $e^{(k)} = x^{(k)} - x$
- $x^{(k+1)} = Bx^{(k)} + f, x = Bx + f \Rightarrow x^{(k+1)} - x = B(x^{(k)} - x)$
- $e^{(k+1)} = Be^{(k)}$
- $e^{(k)} = Be^{(k-1)}, e^{(k+1)} = Be^{(k)} = BBe^{(k-1)} = B^2 e^{(k-1)}$
- ...
- $e^{(k+1)} = B^{k+1} e^{(0)}$
- $\|e^{(k+1)}\| = \|B^{k+1} e^{(0)}\| \leq \|B^{k+1}\| \|e^{(0)}\| \leq \|B\|^{k+1} \|e^{(0)}\| \longrightarrow_{k \to \infty} 0$

Convergence of iterative methods 5

Theorem 9.14

$x^{(k+1)} = Bx^{(k)} + f,\ k = 0, 1, ..$ *converges iff* $\rho(B) < 1$

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- $\|e^{(k)}\| = \|\lambda^k y\| = |\lambda^k|\|y\| \longrightarrow_{k\to\infty} \infty$

□

Convergence of iterative methods 6

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Convergence of iterative methods 6

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$$\rho(B)^k \sum_{i=1}^{n} |c_i|\|y_i\| \longrightarrow_{k\to\infty} 0$$

Convergence of iterative methods 7

Important theorem on norms and spectral radius

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Convergence of iterative methods 7

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Convergence of iterative methods 7

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Other proof of sufficient condition of convergence

- $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$
- $\rho(B) < 1 \Rightarrow \|B\|_* < 1$

Convergence of iterative methods 7

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Other proof of sufficient condition of convergence

- $x^{(k+1)} = Bx^{(k)} + f, k = 0, 1, ..$
- $\rho(B) < 1 \Rightarrow \|B\|_* < 1$
- Apply sufficient condition: $\|B\|_* < 1 \Rightarrow$ convergence

Convergence of iterative methods 8

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- $\rho(B_2) < \rho(B_1)$
- Which method converges faster?

Convergence of iterative methods 8

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- $\rho(B_2) < \rho(B_1)$
- Which method converges faster?

Answer:

$$\|e^{(k)}\| = \|\sum_{i=1}^{n} c_i \lambda_i^k y_i\| = \sum_{i=1}^{n} |c_i||\lambda^k|\|y_i\| \leq$$

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Iterative methods for linear systems of algebraic equations

Iterative methods for linear systems of algebraic equations: example - iterative refinement

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    4. $tol = \frac{\|x^{(k+1)} - x^{(k)}\|}{\|x^{(k)}\|}$
- Produces very accurate solution if $cond(A)$ not too large
- Convergence depends on $cond(A)$

Direct and Iterative methods

- Direct methods:
  - Finite number of arithmetic operations for finding exact solution

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- What is term "exact solution" in floating point arithmetic?

Jacobi Method

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$$A = L + D + U, A, L, D, U \in \mathbb{R}^{n \times n}, a_{ii} \neq 0,$$
$$l_{ij} = 0, \text{if } i < j; d_{ij} = 0, \text{if } i \neq j; u_{ij} = 0, \text{if } i > j;$$
$$x^{(k+1)} = -D^{-1}(L + U)x^{(k)} + D^{-1}b, k = 0, 1, ...$$
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4. Convergence: $\|D^{-1}(L + U)\| < 1, \rho(D^{-1}(L + U)) < 1$

Gaus-Seidel

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4. Convergence: $\|(D + L)^{-1}U\| < 1, \rho((D + L)^{-1}U) < 1$

JOR - Jacobi over relaxation

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$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}) + (1-\omega) x_i^{(k)}, i = 1, 2, ..., n$$

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$$B_{SOR} = (D + \omega L)^{-1}((1-\omega)D - \omega U), f_{SOR} = (D + \omega L)^{-1}f,$$
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$\|(D+\omega L)^{-1}((1-\omega)D-\omega U)\| < 1, \rho((D+\omega L)^{-1}((1-\omega)D-\omega U)) < 1$

Richardson's method

- Richardson's method

$$P\frac{x^{(k+1)} - x^{(k)}}{\tau_k} + Ax^{(k)} = b, \quad k = 0, 1, 2, ...$$

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- $\frac{1}{\tau_k}Px^{(k+1)} = (\frac{1}{\tau_k}P - A)x^{(k)} + b, \quad k = 0, 1, 2, \dots$

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- Pre-conditioner - $P \in \mathbb{R}^{n\times n}$
- Numeric parameter - τ_k
- Stationary iteration method if $B_{R,k} \equiv B_R$
- Converges if $\|B_{R,k}\| < 1$
- Converges iff $\rho(B_{R,k}) < 1$

Convergence Jacobi, GS, JOR 1

Theorem 9.17

A strictly row diagonally dominant $\Rightarrow$ Jacobi method converges

Convergence Jacobi, GS, JOR 1

Theorem 9.17

A strictly row diagonally dominant $\Rightarrow$ Jacobi method converges

Proof.

- Strictly row diagonally dominance $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}|, , i = 1, 2, ..., n$

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- $|a_{ii}| > \sum_{j=1, j\neq i}^{n} |a_{ij}| \Rightarrow \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1, i = 1, 2, ..., n$

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- $\sum_{j=1,j\neq i}^{n} \frac{|a_{ij}|}{|a_{ii}|} < 1, i = 1, 2, ..., n \Rightarrow \max_{1\leq i\leq n} \frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{ij}| < 1$

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- $\frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{ij}| = \frac{1}{|a_{ii}|} (\sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|)$

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- $1 > \max_{1\leq i\leq n} \frac{1}{|a_{ii}|}(\sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|) = \|D^{-1}(L+U)\|_\infty$

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- $\frac{1}{|a_{ii}|} \sum_{j=1,j\neq i}^{n} |a_{ij}| = \frac{1}{|a_{ii}|}(\sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|)$
- $1 > \max_{1\le i\le n} \frac{1}{|a_{ii}|}(\sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|) = \|D^{-1}(L+U)\|_\infty$
- $\|D^{-1}(L+U)\|_\infty = \|-D^{-1}(L+U)\|_\infty = \|B_J\|_\infty < 1$

$\square$

Convergence Jacobi, GS, JOR 2

Theorem 9.18

A strictly row diagonally dominant $\Rightarrow$ GS method converges

Convergence Jacobi, GS, JOR 2

Theorem 9.18

A strictly row diagonally dominant $\Rightarrow$ GS method converges

Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$

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Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
- $B_{GS} = -(D+L)^{-1}U$, if $\rho(B_{GS}) < 1 \Rightarrow$ convergence

Convergence Jacobi, GS, JOR 2

Theorem 9.18

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Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
- $B_{GS} = -(D + L)^{-1} U$, if $\rho(B_{GS}) < 1 \Rightarrow$ convergence
- $B_{GS} y = \lambda y \ \Rightarrow \ -(D + L)^{-1} U y = \lambda y$

Convergence Jacobi, GS, JOR 2

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A strictly row diagonally dominant $\Rightarrow$ GS method converges

Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
- $B_{GS} = -(D+L)^{-1}U$, if $\rho(B_{GS}) < 1 \Rightarrow$ convergence
- $B_{GS}y = \lambda y \;\Rightarrow\; -(D+L)^{-1}Uy = \lambda y$
- $-(D+L)^{-1}Uy = \lambda y \;\Rightarrow\; Uy = -\lambda(D+L)y$

Convergence Jacobi, GS, JOR 2

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Proof.

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- $-(D+L)^{-1}Uy = \lambda y \;\Rightarrow\; Uy = -\lambda(D+L)y$
- $Uy = -\lambda(D+L)y \;\Rightarrow\; -\lambda Dy = (\lambda L + U)y$

Convergence Jacobi, GS, JOR 2

Theorem 9.18

A strictly row diagonally dominant $\Rightarrow$ *GS method converges*

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- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
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- $-(D+L)^{-1}Uy = \lambda y \ \Rightarrow \ Uy = -\lambda(D+L)y$
- $Uy = -\lambda(D+L)y \ \Rightarrow \ -\lambda Dy = (\lambda L + U)y$
- Component-wise: $-\lambda a_{ii}y_i = \lambda \sum_{j=1}^{i-1} a_{ij}y_j + \sum_{j=i+1}^{n} a_{ij}y_j$

Convergence Jacobi, GS, JOR 2

Theorem 9.18

A strictly row diagonally dominant $\Rightarrow$ GS method converges

Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
- $B_{GS} = -(D+L)^{-1}U$, if $\rho(B_{GS}) < 1 \Rightarrow$ convergence
- $B_{GS}y = \lambda y \;\Rightarrow\; -(D+L)^{-1}Uy = \lambda y$
- $-(D+L)^{-1}Uy = \lambda y \;\Rightarrow\; Uy = -\lambda(D+L)y$
- $Uy = -\lambda(D+L)y \;\Rightarrow\; -\lambda Dy = (\lambda L + U)y$
- Component-wise: $-\lambda a_{ii} y_i = \lambda \sum_{j=1}^{i-1} a_{ij} y_j + \sum_{j=i+1}^{n} a_{ij} y_j$
- $|\lambda| \cdot |a_{ii}| \cdot |y_i| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}| \cdot |y_j| + \sum_{j=i+1}^{n} |a_{ij}| \cdot |y_j|$

Convergence Jacobi, GS, JOR 2

Theorem 9.18

A strictly row diagonally dominant $\Rightarrow$ GS method converges

Proof.

- $|a_{ii}| - \sum_{j=1}^{i-1} |a_{ij}| > \sum_{j=i+1}^{n} |a_{ij}|, , i = 1, 2, ..., n$
- $B_{GS} = -(D + L)^{-1} U$, if $\rho(B_{GS}) < 1 \Rightarrow$ convergence
- $B_{GS} y = \lambda y \;\Rightarrow\; -(D + L)^{-1} U y = \lambda y$
- $-(D + L)^{-1} U y = \lambda y \;\Rightarrow\; U y = -\lambda (D + L) y$
- $U y = -\lambda (D + L) y \;\Rightarrow\; -\lambda D y = (\lambda L + U) y$
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- $|y_k| = \|y\|_\infty \Rightarrow |\lambda| \cdot |a_{kk}| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}| \frac{|y_j|}{|y_k|} + \sum_{j=i+1}^{n} |a_{ij}| \frac{|y_j|}{|y_k|}$

Convergence Jacobi, GS, JOR 2

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- $|\lambda| \cdot |a_{kk}| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|$

Convergence Jacobi, GS, JOR 2

Theorem 9.18

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- $|\lambda| \cdot |a_{kk}| \leq |\lambda| \sum_{j=1}^{i-1} |a_{ij}| + \sum_{j=i+1}^{n} |a_{ij}|$
- $|\lambda|(|a_{kk}| - \sum_{j=1}^{i-1} |a_{ij}|) \leq \sum_{j=i+1}^{n} |a_{ij}| \Rightarrow |\lambda| \leq \frac{\sum_{j=i+1}^{n} |a_{ij}|}{|a_{kk}| - \sum_{j=1}^{i-1} |a_{ij}|} < 1$

Convergence Jacobi, GS, JOR 3

Theorem 9.19

Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

Convergence Jacobi, GS, JOR 3

Theorem 9.19

Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

Proof.

- Jacobi method converges $\Rightarrow \rho(B_J) < 1 \Rightarrow B_J y = \lambda y, |\lambda| < 1$

Convergence Jacobi, GS, JOR 3

Theorem 9.19

Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

Proof.

- Jacobi method converges $\Rightarrow \rho(B_J) < 1 \Rightarrow B_J y = \lambda y, |\lambda| < 1$
- $B_{JOR} = \omega B_J + (1-\omega)I \Rightarrow B_{JOR} y = \omega B_J y + (1-\omega)y = \omega\lambda y + (1-\omega)y$

Convergence Jacobi, GS, JOR 3

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- $B_{JOR} = \omega B_J + (1-\omega)I \Rightarrow B_{JOR} y = \omega B_J y + (1-\omega)y = \omega\lambda y + (1-\omega)y$
- $B_{JOR} y = \mu y \Rightarrow \mu = \omega\lambda + (1-\omega)$

Convergence Jacobi, GS, JOR 3

Theorem 9.19

Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

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- $B_{JOR} = \omega B_J + (1-\omega)I \Rightarrow B_{JOR} y = \omega B_J y + (1-\omega)y = \omega\lambda y + (1-\omega)y$
- $B_{JOR} y = \mu y \Rightarrow \mu = \omega\lambda + (1-\omega)$
- $\lambda = re^{i\theta}, e^{i\theta} = \cos(\theta) + i\sin(\theta), r < 1$

Convergence Jacobi, GS, JOR 3

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Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

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- $B_{JOR} y = \mu y \Rightarrow \mu = \omega \lambda + (1-\omega)$
- $\lambda = re^{i\theta}, e^{i\theta} = \cos(\theta) + i \sin(\theta), r < 1$
- $\mu = \omega r(\cos(\theta) + i \sin(\theta)) + 1 - \omega = \omega r \cos(\theta) + 1 - \omega + i\omega r \sin(\theta)$

Convergence Jacobi, GS, JOR 3

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Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

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- $B_{JOR} = \omega B_J + (1-\omega)I \Rightarrow B_{JOR} y = \omega B_J y + (1-\omega)y = \omega\lambda y + (1-\omega)y$
- $B_{JOR} y = \mu y \Rightarrow \mu = \omega\lambda + (1-\omega)$
- $\lambda = re^{i\theta}, e^{i\theta} = \cos(\theta) + i\sin(\theta), r < 1$
- $\mu = \omega r(\cos(\theta) + i\sin(\theta)) + 1 - \omega = \omega r\cos(\theta) + 1 - \omega + i\omega r\sin(\theta)$
- $$|\mu|^2 = (\omega r\cos(\theta) + 1 - \omega)^2 + \omega^2 r^2 \sin^2(\theta) =$$
  $$= \omega^2 r^2 \cos^2(\theta) + 2\omega r(1-\omega)\cos(\theta) + (1-\omega)^2 + \omega^2 r^2 \sin^2(\theta)$$
  $$= \omega^2 r^2 + 2\omega r(1-\omega)\cos(\theta) + (1-\omega)^2 \leq (\omega \cdot r + (1-\omega)\cdot 1)^2 < 1$$

Convergence Jacobi, GS, JOR 3

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Jacobi method converges $\Rightarrow$ *JOR method converges for* $0 < \omega \leq 1$

Proof.

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- $B_{JOR} = \omega B_J + (1-\omega)I \Rightarrow B_{JOR}y = \omega B_J y + (1-\omega)y = \omega\lambda y + (1-\omega)y$
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- $\lambda = re^{i\theta}, e^{i\theta} = \cos(\theta) + i\sin(\theta), r < 1$
- $\mu = \omega r(\cos(\theta) + i\sin(\theta)) + 1 - \omega = \omega r\cos(\theta) + 1 - \omega + i\omega r\sin(\theta)$
-

$$|\mu|^2 = (\omega r\cos(\theta) + 1 - \omega)^2 + \omega^2 r^2 \sin^2(\theta) =$$
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$$= \omega^2 r^2 + 2\omega r(1-\omega)\cos(\theta) + (1-\omega)^2 \leq (\omega \cdot r + (1-\omega)\cdot 1)^2 < 1$$

Convergence Jacobi, GS, JOR 4

Comparing Jacobi and Gaus-Seidel: Stein-Rosenberg Theorem

Theorem 9.20

- $a_{ij} < 0, i \neq j, \quad i, j = 1, 2, ..., n$

Convergence Jacobi, GS, JOR 4

Comparing Jacobi and Gaus-Seidel: Stein-Rosenberg Theorem

Theorem 9.20

- $a_{ij} < 0, i \neq j, \quad i, j = 1, 2, ..., n$
- $a_{ii} > 0, i = 1, 2, ..n$

Convergence Jacobi, GS, JOR 4

Comparing Jacobi and Gaus-Seidel: Stein-Rosenberg Theorem

Theorem 9.20

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Only one of the following statements holds true

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- $a_{ij} < 0, i \neq j, \quad i, j = 1, 2, ..., n$
- $a_{ii} > 0, i = 1, 2, ..n$

Only one of the following statements holds true

1. $0 \leq \rho(G_{GS}) < \rho(B_J) < 1$

Convergence Jacobi, GS, JOR 4

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- $a_{ij} < 0, i \neq j, \quad i, j = 1, 2, ..., n$
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Only one of the following statements holds true

1. $0 \leq \rho(G_{GS}) < \rho(B_J) < 1$
2. $1 < \rho(B_J) < \rho(G_{GS})$

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Only one of the following statements holds true

1. $0 \leq \rho(G_{GS}) < \rho(B_J) < 1$
2. $1 < \rho(B_J) < \rho(G_{GS})$
3. $\rho(B_J) =< \rho(B_G SJ) = 0$

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2. $1 < \rho(B_J) < \rho(G_{GS})$
3. $\rho(B_J) =< \rho(B_G SJ) = 0$
4. $\rho(B_J) =< \rho(B_G SJ) = 1$

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3. $\rho(B_J) =< \rho(B_G SJ) = 0$
4. $\rho(B_J) =< \rho(B_G SJ) = 1$

Theorem 9.21

- *Tridiagonal system*
- *Positive definite*

Convergence Jacobi, GS, JOR 4

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1. $0 \leq \rho(G_{GS}) < \rho(B_J) < 1$
2. $1 < \rho(B_J) < \rho(G_{GS})$
3. $\rho(B_J) =< \rho(B_G SJ) = 0$
4. $\rho(B_J) =< \rho(B_G SJ) = 1$

Theorem 9.21

- *Tridiagonal system*
- *Positive definite*

$\Rightarrow \rho(B_{GS}) = \rho^2(B_J)$

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