# about groups

# 4 Groups

## 4.1 basic definitions

**group**

- $G = (S, \circ)$

$$S \text{ set },\ \circ : S \times S \to S$$

- identity

$$\exists e \in S.\ \forall a \in S. \quad e \circ a = a \circ e = e$$

- associativity

$$(a \circ b) \circ c = a \circ (b \circ c)$$

- inverse elements

$$\forall a \in S.\ \exists \text{ unique } b \in S. \quad a \circ b = b \circ a = e$$

  The inverse of $a$ is often denoted by $a^{-1}$.

- group is *abelian* if commutative law holds

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## 4.2 group for modular addition

$$(\mathbb{Z}_n, +_n) \ , \ \mathbb{Z}_n = [0 : n-1] \ , \ a +_n b = (a+b) \bmod n$$

- neutral element:

$$e = 0$$

- inverse of $a$

$$-a = n - a$$

$$(a + (n-a)) \bmod n = n \bmod n = 0$$

- abelian

### 4.3 group for modular multiplication

$$(Z_n^*, \cdot_n)\ ,\ a \cdot_n b = a \cdot b \bmod n$$

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- $\cdot_n : \mathbb{Z}_n^* \times \mathbb{Z}_n^* \to \mathbb{Z}_n^*$

$$a,b \in \mathbb{Z}_n^* \ \to gcd(a,n) = gcd(b.n) = 1$$

$$\begin{aligned}
gcd(ab,n) &= 1 \quad \text{(lemma 3)} \\
\exists x,y \in Z : 1\ abx + ny &= 1 \quad \text{(lemma 1)} \\
q &= \lfloor ab/n \rfloor \\
1 &= abx - qnx + qnx + ny \\
&= (ab - qn)x + n(qx + y) \\
&= (ab \bmod n) + n(qx + y) \\
gcd(ab \bmod n, n) &= 1 \quad \text{(lemma 1)}
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- uniqueness of inverse: later (lemma 23)

# 5 Euler's $\varphi$-function

$$\varphi(n) = |\mathbb{Z}_n^*| \quad \text{cardinality, number of elements)}$$

**Lemma 6.**

$$gcd(n,m) = 1 \rightarrow \varphi(nm) = \varphi(n) \cdot \varphi(m)$$

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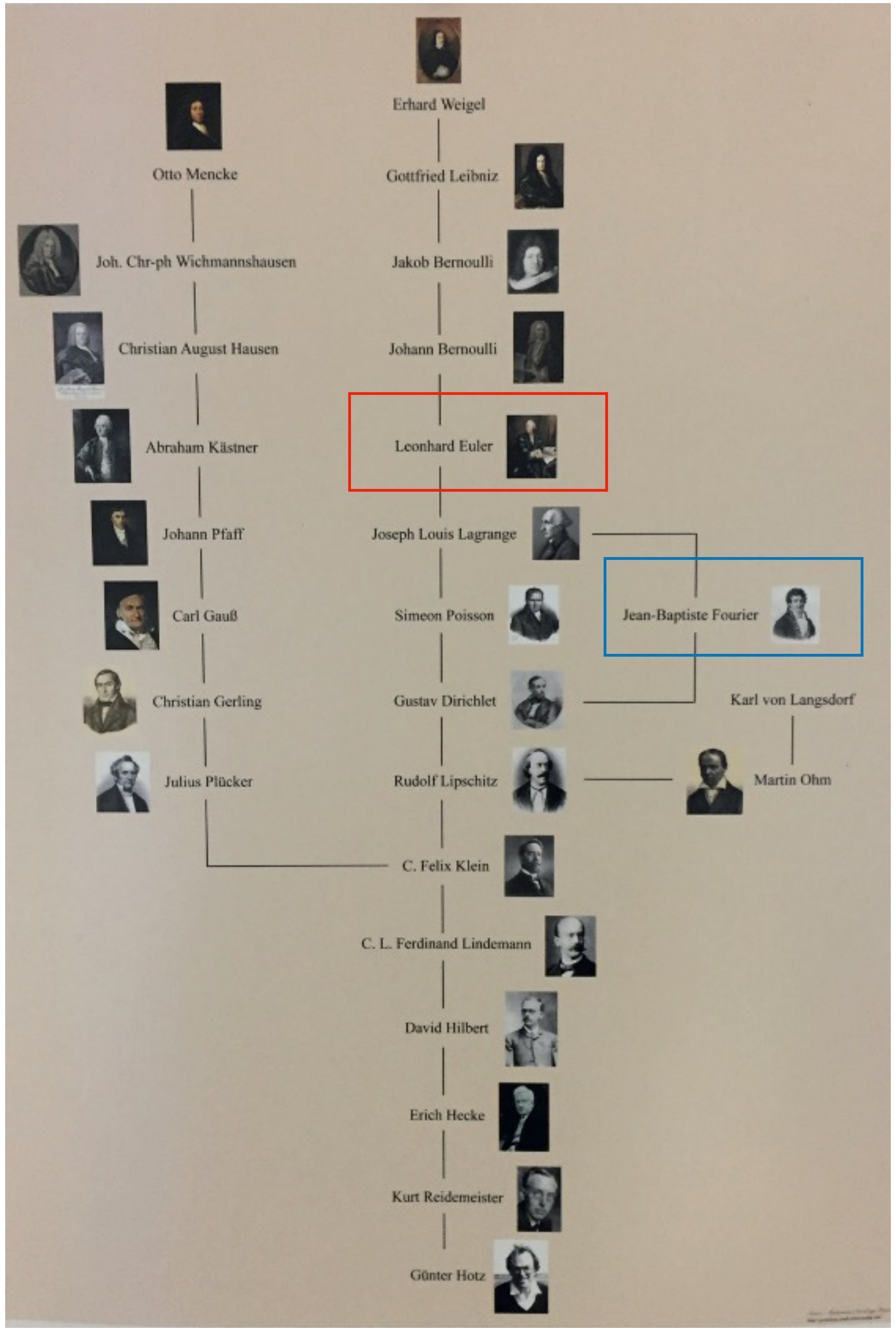


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in an $n \times m$-table as shown in table 2.

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For $k \in \mathbb{Z}_m$ collect elements in column $k$ into the set

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as elements in $C'(k)$ are mutually distinct

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$$\begin{aligned} 1 &< gcd(m\ell+k \text{ mod } n, n) \\ &= gcd(m\ell+k, n) \quad \text{(lemma 4 )} \\ &\leq gcd(m\ell+k, nm) \\ m\ell+k &\notin \mathbb{Z}_{mn}^* \end{aligned}$$

This excludes $n-\varphi(n)$ elements in each of the remaining $\varphi(m)$ columns.

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- for $\ell m + k \bmod n \in \mathbb{Z}_n^*$

$$\begin{aligned} 1 &= gcd(m\ell + k \bmod n, n) \\ &= gcd(m\ell + k, n) \quad \text{(lemma 4 )} \\ 1 &= gcd(m\ell + k, m) \quad \text{(shown above for } k \in \mathbb{Z}_m^*) \\ 1 &= gcd(m\ell + k, mn) \quad \text{(lemma 3)} \\ m\ell + k &\in \mathbb{Z}_{mn}^* \end{aligned}$$

This identifies $\varphi(n)$ elements in each of the remaining $\varphi(m)$ columns as elements of $\mathbb{Z}_{mn}^*$.

# 5 Euler's $\varphi$-function

**powers of primes:**

**Lemma 7.**
$$n = p^k \ , \ p \ prime \quad \rightarrow \quad \varphi(n) = n(1 - \frac{1}{p})$$

*Proof.*
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**general case:**

**Lemma 9.**

$$\varphi(n) = n \cdot \prod_{p|n, p \ prime} (1 - \frac{1}{p})$$

*Proof.* Let the prime factorization of $n$ be

$$n = p_1^{k_1} \ldots p_s^{k_s}$$

Then

$$\begin{aligned} \varphi(n) &= \prod_i \varphi(p^{k_i}) \quad \text{(lemma 6} \\ &= \prod_i n \cdot (1 - \frac{1}{p_i}) \quad \text{(lemma 7)} \\ &= n \cdot \prod_{p|n, p \text{ prime}} (1 - \frac{1}{p}) \end{aligned}$$

# 6 Subgroups and Lagrange's Theorem

**subgroups:**

Let $G = (S, \circ)$ be a group, $S' \subseteq S$

$$a \circ' b = a \circ b \text{ for all } a, b \in S'$$

Then $G'$ is a *subgroup* of $G$ iff

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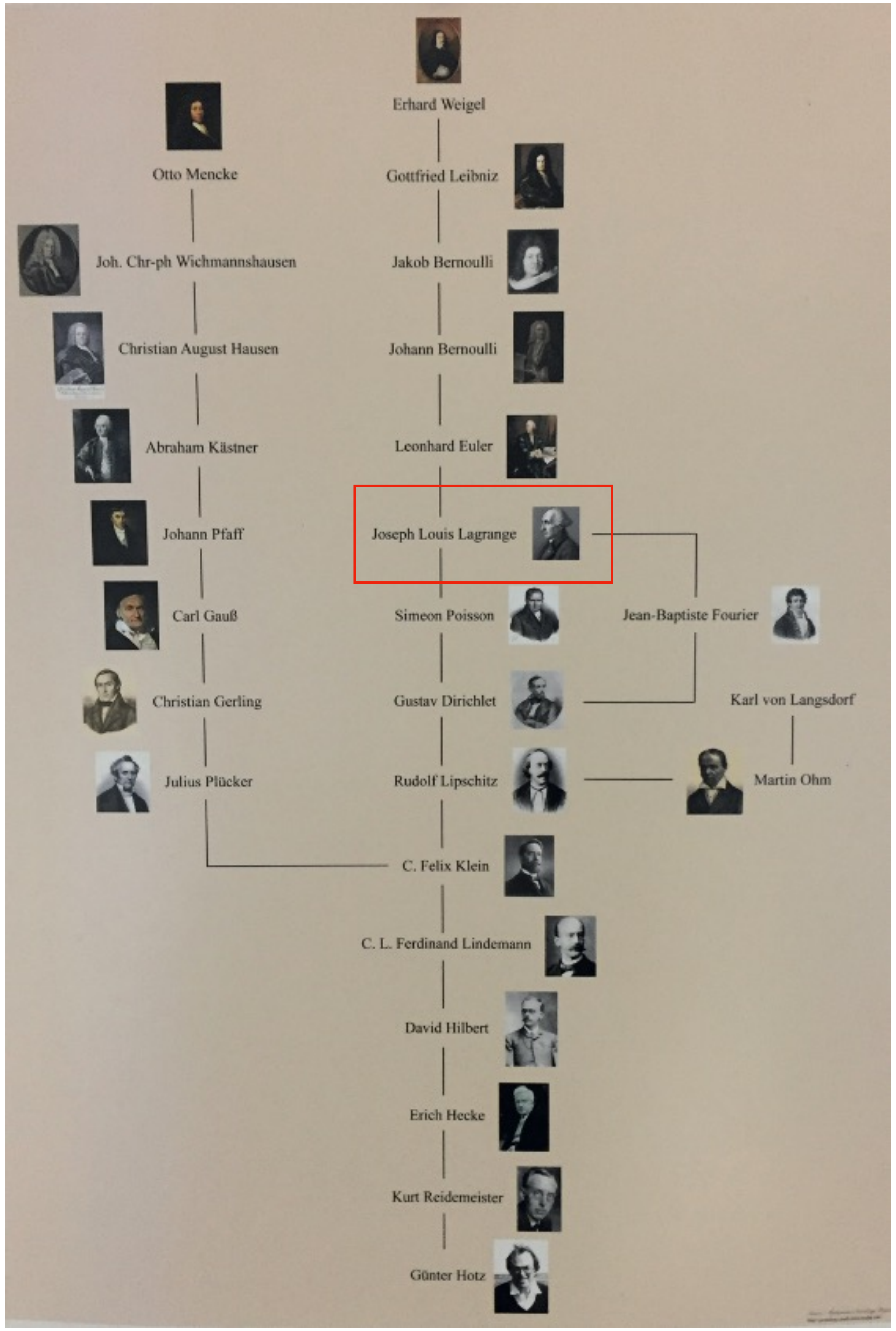
**examples:**

- let $\mathbb{Z}_{even} = \{2z \ : \ z \in \mathbb{Z}\}$ be the set of even integers. Then $(\mathbb{Z}_{even}, +)$ is subgroup of $(\mathbb{Z}, +)$.
- let $S' = \{0, 2, 4\}$. Then $(S', +_6)$ is subgroup of $(\mathbb{Z}_6, +_6)$.



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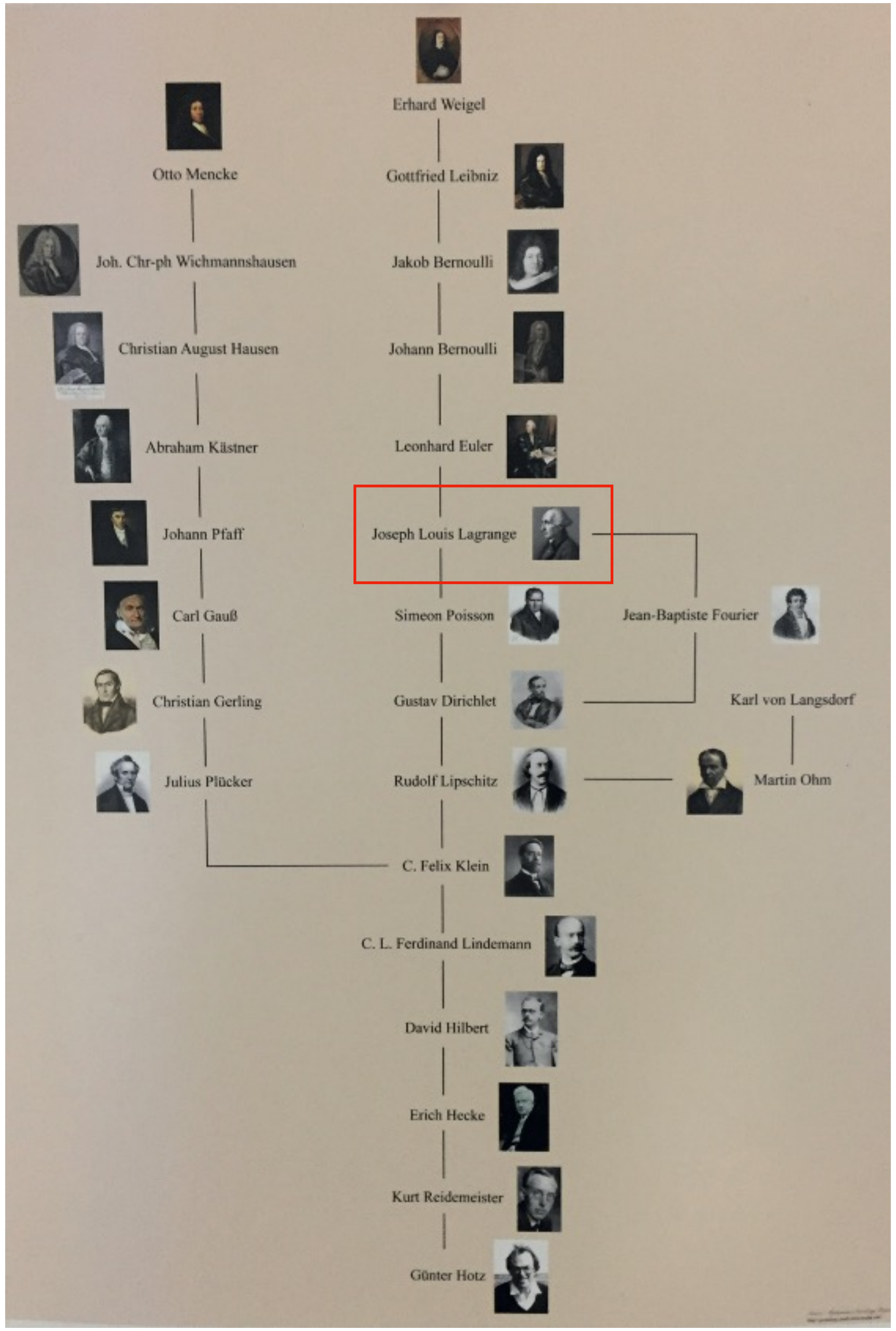

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$$\begin{aligned} h \circ r &= h' \circ r \quad || \circ r^{-1} \\ \to h &= h' \end{aligned}$$

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$$e \in H \;, r = er \in Hr$$

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- assume $h_1 r = h_2 s$ with $h_1, h_2 \in H$

$$\begin{aligned} h_1 r &= h_2 s \quad || h_1^{-1} \circ \\ r &= h_1^{-1} h_2 s \quad || \circ s^{-1} \\ rs^{-1} &= h_1^{-1} h_2 \in H \end{aligned}$$

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Exhaust $G$ by
$i = 1;\; P(1) = H;$

while $\exists s_i \in G \setminus H(i)$

$P = P \cup Hs_i; i = i + 1$

As $G$ is finite this terminates with some finite $i = n \leq |G|$.

$$G = \bigcup_{i=1}^{n} Hs_i$$

By lemma 11 the union is disjoint.

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