Numerical Analysis Homework (5)

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Problem 5.1:

(a) By theorem 10.6[1] , if $\left|\frac{\partial g_{i(x)}}{\partial x_j}\right| \leq \frac{K}{n}$ for all $i, j = 1, 2$ where K is some constant in $(0, 1)$, then D contains a unique fixed point.

$$\nabla G(x_1, x_2) = \begin{pmatrix} \frac{x_1}{5} & \frac{x_2}{5} \\ \frac{x_2^2+1}{10} & \frac{x_1 x_2}{5} \end{pmatrix}$$

We can calculate the upper bounds of each component for points in D

$$\max_{x \in D} \frac{x_1}{5} = \frac{3}{10}$$

$$\max_{x \in D} \frac{x_2}{5} = \frac{3}{10}$$

$$\max_{x \in D} \frac{x_2^2 + 1}{10} = \frac{13}{40}$$

$$\max_{x \in D} \frac{x_1 x_2}{5} = \frac{9}{20}$$

So we get that $K = 2 \cdot \max\left\{\frac{3}{10}, \frac{3}{10}, \frac{13}{40}, \frac{9}{20}\right\} = \frac{9}{10} \in (0, 1)$.

(b) With the following code

```
fn main() {
    let (mut x1, mut x2) = (0.0, 0.0);
    let mut count = 0;
    loop {
        let (nx1, nx2) = ((
            x1*x1 + x2*x2 + 8.0) / 10.0,
            (x1*x2*x2 + x1 + 8.0) / 10.0);

        if (x1, x2) == (nx1, nx2) {
            println!("{x1}, {x2}  in {count} iterations");
            break;
        }
        (x1, x2) = (nx1, nx2);
        count += 1;
    }

}
```

The solution is $x = \binom{0}{0}$

(c) Yes, it takes 40 iterations and only 32 with Gauss-Seidel.

Problem 5.2:

First let's wrewrite the equations:

[1]Burden, page 633.

$$3x_1^2 - x_2^2 = 0 \implies x_1 = \frac{x_2}{\sqrt{3}}$$

$$3x_1 x_2^2 - x_1^3 - 1 = 0 \implies x_2 = \sqrt{\frac{x_1^2 - \frac{1}{x_1}}{3}}$$

```
fn main() {
    let (mut x1, mut x2) = (1.0f64, 1.0f64);
    let mut count = 0usize;

    loop {
        let nx1 = (x2 * x2 / 3.0).sqrt();
        let nx2 = ((x1 * x1 + 1.0 / x1) / 3.0).sqrt();

        if (x1 - nx1).abs().max((x2 - nx2).abs()) < 1e-10 {
            println!("{x1}, {x2}  in {count} iterations");
            break;
        }

        (x1, x2) = (nx1, nx2);
        count += 1;
    }
}
```

The solution is $x = \begin{pmatrix} 0.49999999994365524 \\ 0.8660254038294863 \end{pmatrix}$

Problem 5.3:

```rust
use std::f64::consts::PI;

fn a() {
    let (mut x1, mut x2) = (0.0, 0.0);

    for count in 0..10000 {
        let nx1 = x1 * x1 / 5.0 + x2 * x2 / 80.0 + 0.4;
        let nx2 = nx1 * x2 * x2 / 10.0 + nx1 * 0.4 + 1.6;

        if (x1, x2) == (nx1, nx2) {
            println!("{x1}, {x2}  in {count} iterations");
            break;
        }

        (x1, x2) = (nx1, nx2);
    }
}

fn b() {
    let (mut x1, mut x2, mut x3) = (2.1f64, 0.0f64, 0.1f64);

    for count in 0..10000 {
        let nx1 = (x2 * x3).cos() / 3.0 + 1.0 / 6.0;
        let nx2 = 0.5 + 625.0 / 2.0 * x2 * x2 - 2.0 * nx1 * nx1;
        let nx3 = (3.0 - 10.0 * PI) / 60.0 - (-nx1 * nx2).exp() / 20.0;

        if (x1, x2, x3) == (nx1, nx2, nx3)  {
            println!("{x1}, {x2}, {x3}  in {count} iterations");
            break;
        }

        (x1, x2, x3) = (nx1, nx2, nx3);
    }
}

fn main() {
    a(); b();
}
```

(a)

$$4x_1^2 - 20x_1 + \frac{1}{4}x_2^2 + 8 = 0 \implies x_1 = \frac{1}{5}x_1^2 + \frac{1}{80}x_2^2 + \frac{2}{5}$$

$$\frac{1}{2}x_1x_2^2 + 2x_1 - 5x_2 + 8 = 0 \implies x_2 = \frac{1}{10}x_1x_2^2 + \frac{2}{5}x_1 + \frac{8}{5}$$

The solution is $x = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix}$

(b)

$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0 \implies x_1 = \frac{1}{3}\cos(x_2x_3) + \frac{1}{6}$$

$$4x_1^2 - 625x_2^2 + 2x_2 - 1 = 0 \implies x_2 = \frac{1}{2} + \frac{625}{2}x_2^2 - 2x_1^2$$

$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0 \implies x_3 = \frac{3 - 10\pi}{60} - \frac{1}{20}e^{-x_1x_2}$$

The solution is $x = \begin{pmatrix} 0.5 \\ 0 \\ -0.5235987755982988 \end{pmatrix}$