Course: Calculus 1 - CS

Calculus: Early Transcendentals - James Stewart, Daniel Clegg, Saleem Watson (**Reader**) – Section 2.6

CALCULUS

EARLY TRANSCENDENTALS

NINTH EDITON

Metric Version

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2.6 Limits at Infinity; Horizontal Asymptotes

In Sections 2.2 and 2.4 we investigated infinite limits and vertical asymptotes of a curve $y = f(x)$. There we let x approach a number and the result was that the values of y became arbitrarily large (positive or negative). In this section we let x become arbitrarily large (positive or negative) and see what happens to y.

Limits at Infinity and Horizontal Asymptotes

Let's begin by investigating the behavior of the function f defined by

$$f(x) = \frac{x^2 - 1}{x^2 + 1}$$

as x becomes large. The table at the left gives values of this function correct to six decimal places, and the graph of f has been drawn by a computer in Figure 1.

x	$f(x)$
0	−1
±1	0
±2	0.600000
±3	0.800000
±4	0.882353
±5	0.923077
±10	0.980198
±50	0.999200
±100	0.999800
±1000	0.999998

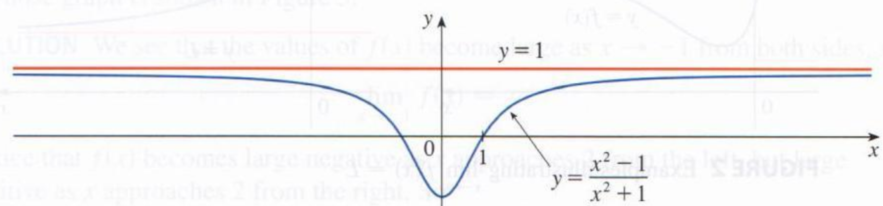

FIGURE 1

You can see that as x grows larger and larger, the values of $f(x)$ get closer and closer to 1. (The graph of f approaches the horizontal line $y = 1$ as we look to the right.) In fact, it seems that we can make the values of $f(x)$ as close as we like to 1 by taking x sufficiently large. This situation is expressed symbolically by writing

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

In general, we use the notation

$$\lim_{x \to \infty} f(x) = L$$

to indicate that the values of $f(x)$ approach L as x becomes larger and larger.

1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Another notation for $\lim_{x \to \infty} f(x) = L$ is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

The symbol ∞ does not represent a number. Nonetheless, the expression $\lim_{x\to\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches infinity, is L"

or "the limit of $f(x)$, as x becomes infinite, is L"

or "the limit of $f(x)$, as x increases without bound, is L"

The meaning of such phrases is given by Definition 1. A more precise definition, similar to the ε, δ definition of Section 2.4, is given at the end of this section.

Geometric illustrations of Definition 1 are shown in Figure 2. Notice that there are many ways for the graph of f to approach the line $y = L$ (which is called a *horizontal asymptote*) as we look to the far right of each graph.

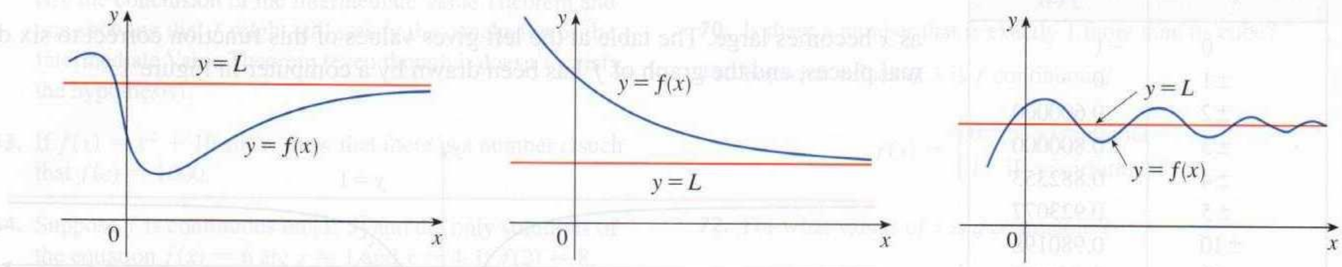

FIGURE 2 Examples illustrating $\lim_{x\to\infty} f(x) = L$

Referring back to Figure 1, we see that for numerically large negative values of x, the values of $f(x)$ are close to 1. By letting x decrease through negative values without bound, we can make $f(x)$ as close to 1 as we like. This is expressed by writing

$$\lim_{x\to-\infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

The general definition is as follows.

> **2 Definition** Let f be a function defined on some interval $(-\infty, a)$. Then
>
> $$\lim_{x\to-\infty} f(x) = L$$
>
> means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.

Again, the symbol $-\infty$ does not represent a number, but the expression $\lim_{x\to-\infty} f(x) = L$ is often read as

"the limit of $f(x)$, as x approaches negative infinity, is L"

Definition 2 is illustrated in Figure 3. Notice that the graph approaches the line $y = L$ as we look to the far left of each graph.

> **3 Definition** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L$$

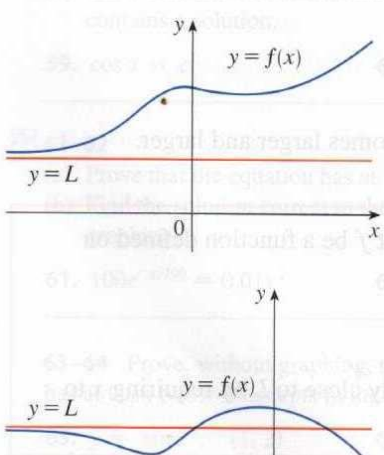

FIGURE 3
Examples illustrating $\lim_{x\to-\infty} f(x) = L$

For instance, the curve illustrated in Figure 1 has the line $y = 1$ as a horizontal asymptote because

$$\lim_{x \to \infty} \frac{x^2 - 1}{x^2 + 1} = 1$$

An example of a curve with two horizontal asymptotes is $y = \tan^{-1}x$. (See Figure 4.) In fact,

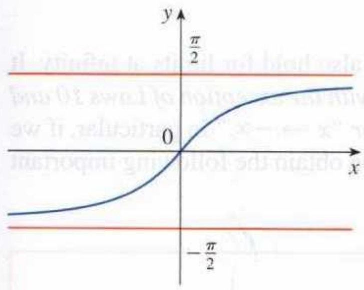

FIGURE 4
$y = \tan^{-1}x$

$$\boxed{4} \qquad \lim_{x \to -\infty} \tan^{-1}x = -\frac{\pi}{2} \qquad \lim_{x \to \infty} \tan^{-1}x = \frac{\pi}{2}$$

so both of the lines $y = -\pi/2$ and $y = \pi/2$ are horizontal asymptotes. (This follows from the fact that the lines $x = \pm\pi/2$ are vertical asymptotes of the graph of the tangent function.)

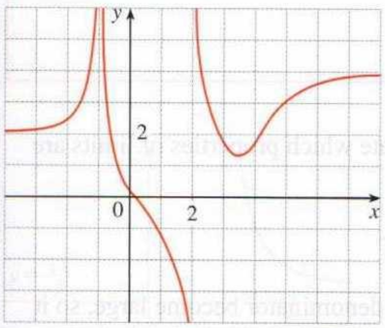

FIGURE 5

EXAMPLE 1 Find the infinite limits, limits at infinity, and asymptotes for the function f whose graph is shown in Figure 5.

SOLUTION We see that the values of $f(x)$ become large as $x \to -1$ from both sides, so

$$\lim_{x \to -1} f(x) = \infty$$

Notice that $f(x)$ becomes large negative as x approaches 2 from the left, but large positive as x approaches 2 from the right. So

$$\lim_{x \to 2^-} f(x) = -\infty \quad \text{and} \quad \lim_{x \to 2^+} f(x) = \infty$$

Thus both of the lines $x = -1$ and $x = 2$ are vertical asymptotes.

As x becomes large, it appears that $f(x)$ approaches 4. But as x decreases through negative values, $f(x)$ approaches 2. So

$$\lim_{x \to \infty} f(x) = 4 \quad \text{and} \quad \lim_{x \to -\infty} f(x) = 2$$

This means that both $y = 4$ and $y = 2$ are horizontal asymptotes. ■

EXAMPLE 2 Find $\displaystyle\lim_{x \to \infty} \frac{1}{x}$ and $\displaystyle\lim_{x \to -\infty} \frac{1}{x}$.

SOLUTION Observe that when x is large, $1/x$ is small. For instance,

$$\frac{1}{100} = 0.01 \qquad \frac{1}{10{,}000} = 0.0001 \qquad \frac{1}{1{,}000{,}000} = 0.000001$$

In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

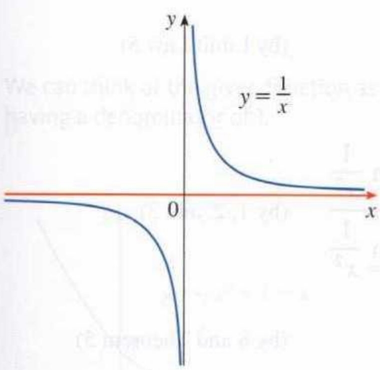

FIGURE 6
$\displaystyle\lim_{x \to \infty} \frac{1}{x} = 0, \ \lim_{x \to -\infty} \frac{1}{x} = 0$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 6.) ■

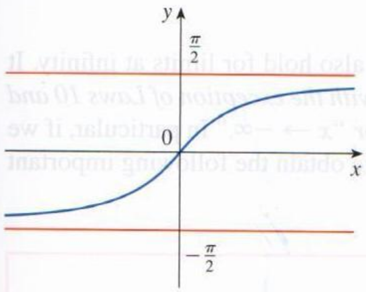

FIGURE 4
$y = \tan^{-1}x$

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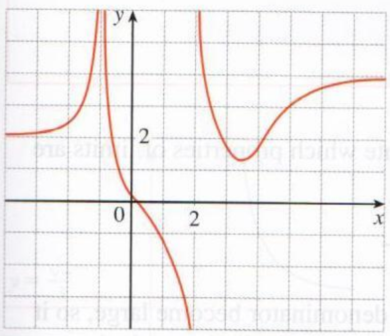

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In fact, by taking x large enough, we can make $1/x$ as close to 0 as we please. Therefore, according to Definition 1, we have

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Similar reasoning shows that when x is large negative, $1/x$ is small negative, so we also have

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

It follows that the line $y = 0$ (the x-axis) is a horizontal asymptote of the curve $y = 1/x$. (This is a hyperbola; see Figure 6.) ■

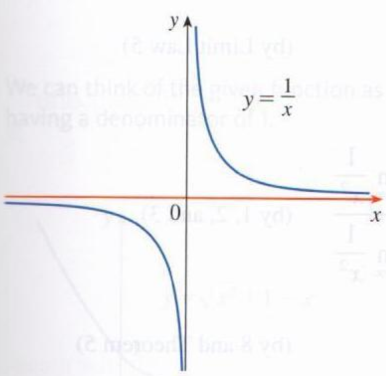


FIGURE 6
$\lim_{x \to \infty} \frac{1}{x} = 0, \quad \lim_{x \to -\infty} \frac{1}{x} = 0$

EXAMPLE 6 Evaluate $\lim_{x \to 2^+} \arctan\left(\dfrac{1}{x-2}\right)$.

SOLUTION If we let $t = 1/(x-2)$, we know that $t \to \infty$ as $x \to 2^+$. Therefore, by the second equation in (4), we have

$$\lim_{x \to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t \to \infty} \arctan t = \frac{\pi}{2}$$

The graph of the natural exponential function $y = e^x$ has the line $y = 0$ (the x-axis) as a horizontal asymptote. (The same is true of any exponential function with base $b > 1$.) In fact, from the graph in Figure 10 and the corresponding table of values, we see that

6 $$\boxed{\lim_{x \to -\infty} e^x = 0}$$

Notice that the values of e^x approach 0 very rapidly.

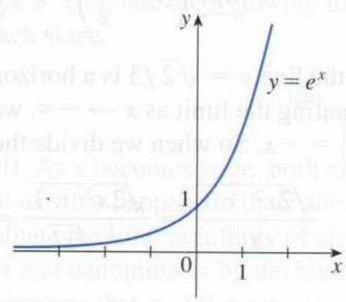


FIGURE 10

x	e^x
0	1.00000
−1	0.36788
−2	0.13534
−3	0.04979
−5	0.00674
−8	0.00034
−10	0.00005

EXAMPLE 7 Evaluate $\lim_{x \to 0^-} e^{1/x}$.

PS The problem-solving strategy for Examples 6 and 7 is *introducing something extra* (see Principles of Problem Solving following Chapter 1). Here, the something extra, the auxiliary aid, is the new variable t.

SOLUTION If we let $t = 1/x$, we know that $t \to -\infty$ as $x \to 0^-$. Therefore, by (6),

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0$$

(See Exercise 81.)

EXAMPLE 8 Evaluate $\lim_{x \to \infty} \sin x$.

SOLUTION As x increases, the values of $\sin x$ oscillate between 1 and -1 infinitely often and so they don't approach any definite number. Thus $\lim_{x \to \infty} \sin x$ does not exist.

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x \to -\infty} f(x) = \infty \qquad \lim_{x \to \infty} f(x) = -\infty \qquad \lim_{x \to -\infty} f(x) = -\infty$$

EXAMPLE 9 Find $\lim_{x\to\infty} x^3$ and $\lim_{x\to-\infty} x^3$.

SOLUTION When x becomes large, x^3 also becomes large. For instance,

$$10^3 = 1000 \qquad 100^3 = 1{,}000{,}000 \qquad 1000^3 = 1{,}000{,}000{,}000$$

In fact, we can make x^3 as big as we like by requiring x to be large enough. Therefore we can write

$$\lim_{x\to\infty} x^3 = \infty$$

Similarly, when x is large negative, so is x^3. Thus

$$\lim_{x\to-\infty} x^3 = -\infty$$

These limit statements can also be seen from the graph of $y = x^3$ in Figure 11. ■

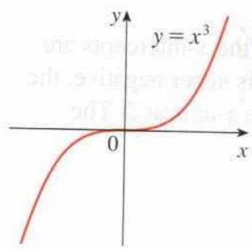


FIGURE 11
$\lim_{x\to\infty} x^3 = \infty$, $\lim_{x\to-\infty} x^3 = -\infty$

Looking at Figure 10 we see that

$$\lim_{x\to\infty} e^x = \infty$$

but, as Figure 12 demonstrates, $y = e^x$ becomes large as $x \to \infty$ at a much faster rate than $y = x^3$.

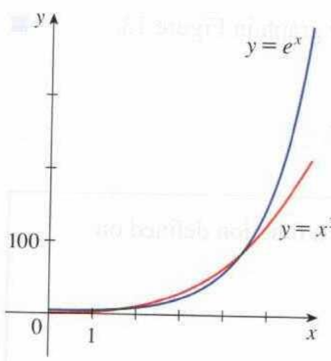


FIGURE 12
e^x is much larger than x^3 when x is large.

EXAMPLE 10 Find $\lim_{x\to\infty} (x^2 - x)$.

SOLUTION Limit Law 2 says that the limit of a difference is the difference of the limits, provided that these limits exist. We cannot use Law 2 here because

$$\lim_{x\to\infty} x^2 = \infty \quad \text{and} \quad \lim_{x\to\infty} x = \infty$$

In general, the Limit Laws can't be applied to infinite limits because ∞ is not a number ($\infty - \infty$ can't be defined). However, we *can* write

$$\lim_{x\to\infty} (x^2 - x) = \lim_{x\to\infty} x(x - 1) = \infty$$

because both x and $x - 1$ become arbitrarily large and so their product does too. ■

EXAMPLE 11 Find $\lim_{x\to\infty} \dfrac{x^2 + x}{3 - x}$.

SOLUTION As in Example 3, we divide the numerator and denominator by the highest power of x in the denominator, which is simply x:

$$\lim_{x\to\infty} \frac{x^2 + x}{3 - x} = \lim_{x\to\infty} \frac{x + 1}{\dfrac{3}{x} - 1} = -\infty$$

because $x + 1 \to \infty$ and $3/x - 1 \to 0 - 1 = -1$ as $x \to \infty$. ■

The next example shows that by using infinite limits at infinity, together with intercepts, we can get a rough idea of the graph of a polynomial without having to plot a large number of points.

EXAMPLE 12 Sketch the graph of $y = (x-2)^4(x+1)^3(x-1)$ by finding its intercepts and its limits as $x \to \infty$ and as $x \to -\infty$.

SOLUTION The y-intercept is $f(0) = (-2)^4(1)^3(-1) = -16$ and the x-intercepts are found by setting $y = 0$: $x = 2, -1, 1$. Notice that since $(x-2)^4$ is never negative, the function doesn't change sign at 2; thus the graph doesn't cross the x-axis at 2. The graph crosses the axis at -1 and 1.

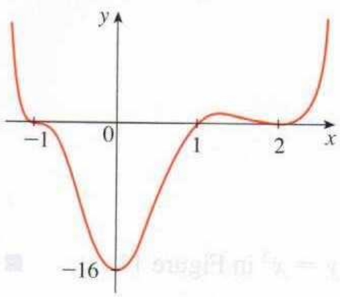


FIGURE 13
$y = (x-2)^4(x+1)^3(x-1)$

When x is large positive, all three factors are large, so

$$\lim_{x\to\infty} (x-2)^4(x+1)^3(x-1) = \infty$$

When x is large negative, the first factor is large positive and the second and third factors are both large negative, so

$$\lim_{x\to-\infty} (x-2)^4(x+1)^3(x-1) = \infty$$

Combining this information, we give a rough sketch of the graph in Figure 13. ■

Precise Definitions

Definition 1 can be stated precisely as follows.

> **7 Precise Definition of a Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x\to\infty} f(x) = L$$
>
> means that for every $\varepsilon > 0$ there is a corresponding number N such that
>
> $$\text{if} \quad x > N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In words, this says that the values of $f(x)$ can be made arbitrarily close to L (within a distance ε, where ε is any positive number) by requiring x to be sufficiently large (larger than N, where N depends on ε). Graphically, it says that by keeping x large enough (larger than some number N) we can make the graph of f lie between the given horizontal lines $y = L - \varepsilon$ and $y = L + \varepsilon$ as in Figure 14. This must be true no matter how small we choose ε.

FIGURE 14
$\lim_{x\to\infty} f(x) = L$

Figure 15 shows that if a smaller value of ε is chosen, then a larger value of N may be required.

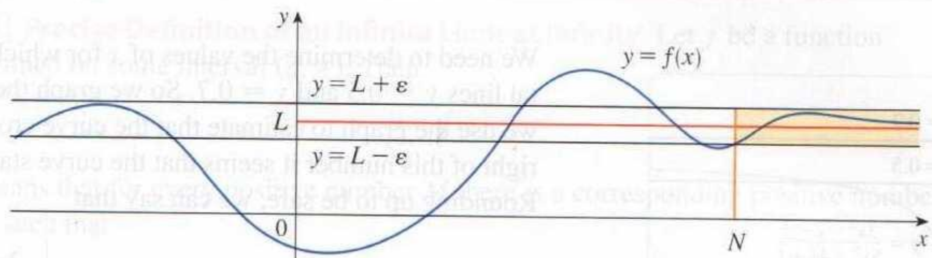


FIGURE 15
$\lim_{x\to\infty} f(x) = L$

Similarly, a precise version of Definition 2 is given by Definition 8, which is illustrated in Figure 16.

8 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to -\infty} f(x) = L$$

means that for every $\varepsilon > 0$ there is a corresponding number N such that

$$\text{if} \quad x < N \quad \text{then} \quad |f(x) - L| < \varepsilon$$

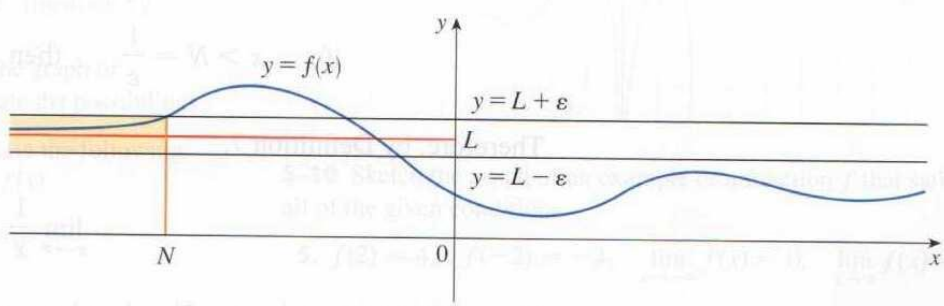


FIGURE 16
$\lim_{x\to -\infty} f(x) = L$

In Example 3 we calculated that

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

In the next example we use a calculator (or computer) to relate this statement to Definition 7 with $L = \frac{3}{5} = 0.6$ and $\varepsilon = 0.1$.

EXAMPLE 13 Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

SOLUTION We rewrite the given inequality as

$$0.5 < \frac{3x^2 - x - 2}{5x^2 + 4x + 1} < 0.7$$

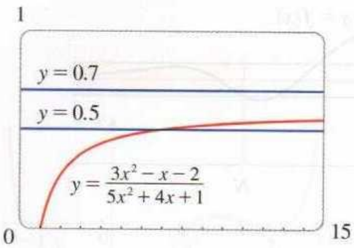


FIGURE 17

We need to determine the values of x for which the given curve lies between the horizontal lines $y = 0.5$ and $y = 0.7$. So we graph the curve and these lines in Figure 17. Then we use the graph to estimate that the curve crosses the line $y = 0.5$ when $x \approx 6.7$. To the right of this number it seems that the curve stays between the lines $y = 0.5$ and $y = 0.7$. Rounding up to be safe, we can say that

$$\text{if} \quad x > 7 \quad \text{then} \quad \left| \frac{3x^2 - x - 2}{5x^2 + 4x + 1} - 0.6 \right| < 0.1$$

In other words, for $\varepsilon = 0.1$ we can choose $N = 7$ (or any larger number) in Definition 7. ■

EXAMPLE 14 Use Definition 7 to prove that $\lim_{x \to \infty} \frac{1}{x} = 0$.

SOLUTION Given $\varepsilon > 0$, we want to find N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| < \varepsilon$$

In computing the limit we may assume that $x > 0$. Then $1/x < \varepsilon \iff x > 1/\varepsilon$. Let's choose $N = 1/\varepsilon$. So

$$\text{if} \quad x > N = \frac{1}{\varepsilon} \quad \text{then} \quad \left| \frac{1}{x} - 0 \right| = \frac{1}{x} < \varepsilon$$

Therefore, by Definition 7,

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Figure 18 illustrates the proof by showing some values of ε and the corresponding values of N.

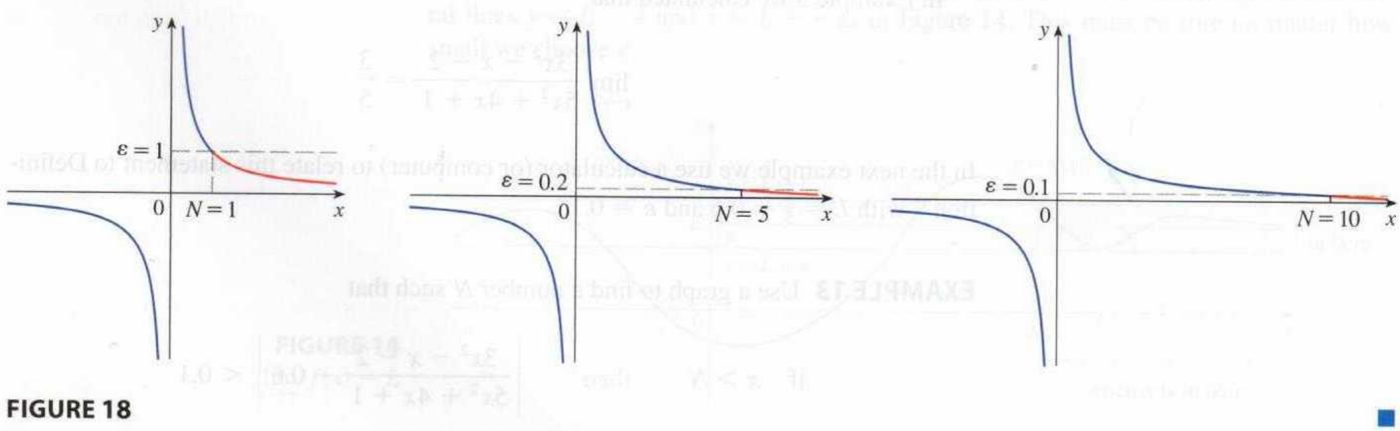


FIGURE 18 ■

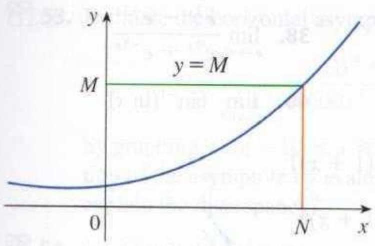


FIGURE 19
$\lim_{x \to \infty} f(x) = \infty$

Finally we note that an infinite limit at infinity can be defined as follows. The geometric illustration is given in Figure 19.

> **9 Precise Definition of an Infinite Limit at Infinity** Let f be a function defined on some interval (a, ∞). Then
>
> $$\lim_{x \to \infty} f(x) = \infty$$
>
> means that for every positive number M there is a corresponding positive number N such that
>
> $$\text{if} \quad x > N \quad \text{then} \quad f(x) > M$$

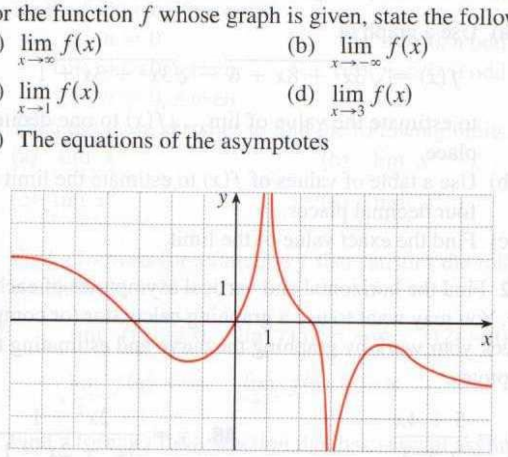
Similar definitions apply when the symbol ∞ is replaced by $-\infty$. (See Exercise 80.)

2.6 Exercises

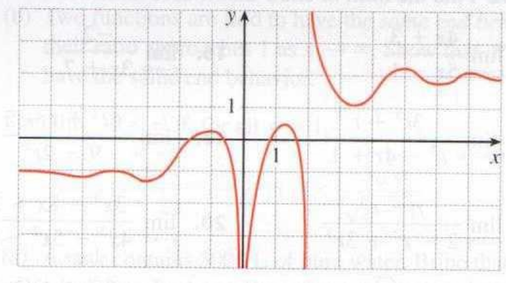
1. Explain in your own words the meaning of each of the following.
(a) $\lim_{x \to \infty} f(x) = 5$ (b) $\lim_{x \to -\infty} f(x) = 3$

2. (a) Can the graph of $y = f(x)$ intersect a vertical asymptote? Can it intersect a horizontal asymptote? Illustrate by sketching graphs.
(b) How many horizontal asymptotes can the graph of $y = f(x)$ have? Sketch graphs to illustrate the possibilities.

3. For the function f whose graph is given, state the following.
(a) $\lim_{x \to \infty} f(x)$ (b) $\lim_{x \to -\infty} f(x)$
(c) $\lim_{x \to 1} f(x)$ (d) $\lim_{x \to 3} f(x)$
(e) The equations of the asymptotes



4. For the function g whose graph is given, state the following.
(a) $\lim_{x \to \infty} g(x)$ (b) $\lim_{x \to -\infty} g(x)$
(c) $\lim_{x \to 0} g(x)$ (d) $\lim_{x \to 2^-} g(x)$
(e) $\lim_{x \to 2^+} g(x)$ (f) The equations of the asymptotes



5–10 Sketch the graph of an example of a function f that satisfies all of the given conditions.

5. $f(2) = 4$, $f(-2) = -4$, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = 2$

6. $f(0) = 0$, $\lim_{x \to 1^-} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = -2$

7. $\lim_{x \to 0} f(x) = \infty$, $\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = \infty$, $\lim_{x \to -\infty} f(x) = 1$, $\lim_{x \to \infty} f(x) = -1$

8. $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to -2^-} f(x) = \infty$, $\lim_{x \to -2^+} f(x) = -\infty$, $\lim_{x \to 2} f(x) = \infty$, $\lim_{x \to \infty} f(x) = \infty$

9. $f(0) = 0$, $\lim_{x \to 1} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = -\infty$, f is odd

10. $\lim_{x \to -\infty} f(x) = -1$, $\lim_{x \to 0^-} f(x) = \infty$, $\lim_{x \to 0^+} f(x) = -\infty$, $\lim_{x \to 3^-} f(x) = 1$, $f(3) = 4$, $\lim_{x \to 3^+} f(x) = 4$, $\lim_{x \to \infty} f(x) = 1$

11. Guess the value of the limit

$$\lim_{x \to \infty} \frac{x^2}{2^x}$$

by evaluating the function $f(x) = x^2/2^x$ for $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 20, 50$, and 100. Then use a graph of f to support your guess.

12. (a) Use a graph of

$$f(x) = \left(1 - \frac{2}{x}\right)^x$$

to estimate the value of $\lim_{x \to \infty} f(x)$ correct to two decimal places.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

13–14 Evaluate the limit and justify each step by indicating the appropriate properties of limits.

13. $\lim_{x \to \infty} \dfrac{2x^2 - 7}{5x^2 + x - 3}$ **14.** $\lim_{x \to \infty} \sqrt{\dfrac{9x^3 + 8x - 4}{3 - 5x + x^3}}$

15–42 Find the limit or show that it does not exist.

15. $\lim_{x \to \infty} \dfrac{4x + 3}{5x - 1}$ **16.** $\lim_{x \to \infty} \dfrac{-2}{3x + 7}$

17. $\lim_{t \to -\infty} \dfrac{3t^2 + t}{t^3 - 4t + 1}$ **18.** $\lim_{t \to -\infty} \dfrac{6t^2 + t - 5}{9 - 2t^2}$

19. $\lim_{r \to \infty} \dfrac{r - r^3}{2 - r^2 + 3r^3}$ **20.** $\lim_{x \to \infty} \dfrac{3x^3 - 8x + 2}{4x^3 - 5x^2 - 2}$

21. $\lim_{x \to \infty} \dfrac{4 - \sqrt{x}}{2 + \sqrt{x}}$ **22.** $\lim_{u \to -\infty} \dfrac{(u^2 + 1)(2u^2 - 1)}{(u^2 + 2)^2}$

23. $\lim_{x \to \infty} \dfrac{\sqrt{x + 3x^2}}{4x - 1}$ **24.** $\lim_{t \to \infty} \dfrac{t + 3}{\sqrt{2t^2 - 1}}$

25. $\lim_{x \to \infty} \dfrac{\sqrt{1 + 4x^6}}{2 - x^3}$ **26.** $\lim_{x \to -\infty} \dfrac{\sqrt{1 + 4x^6}}{2 - x^3}$

27. $\lim_{x \to -\infty} \dfrac{2x^5 - x}{x^4 + 3}$ **28.** $\lim_{q \to \infty} \dfrac{q^3 + 6q - 4}{4q^2 - 3q + 3}$

29. $\lim_{t \to \infty} \left(\sqrt{25t^2 + 2} - 5t\right)$ **30.** $\lim_{x \to -\infty} \left(\sqrt{4x^2 + 3x} + 2x\right)$

31. $\lim_{x \to \infty} \left(\sqrt{x^2 + ax} - \sqrt{x^2 + bx}\right)$

32. $\lim_{x \to \infty} \left(x - \sqrt{x}\right)$

33. $\lim_{x \to -\infty} (x^2 + 2x^7)$ **34.** $\lim_{x \to \infty} (e^{-x} + 2 \cos 3x)$

35. $\lim_{x \to \infty} (e^{-2x} \cos x)$ **36.** $\lim_{x \to \infty} \dfrac{\sin^2 x}{x^2 + 1}$

37. $\lim_{x \to \infty} \dfrac{1 - e^x}{1 + 2e^x}$ **38.** $\lim_{x \to \infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

39. $\lim_{x \to (\pi/2)^+} e^{\sec x}$ **40.** $\lim_{x \to 0^+} \tan^{-1}(\ln x)$

41. $\lim_{x \to \infty} [\ln(1 + x^2) - \ln(1 + x)]$

42. $\lim_{x \to \infty} [\ln(2 + x) - \ln(1 + x)]$

43. (a) For $f(x) = \dfrac{x}{\ln x}$ find each of the following limits.

(i) $\lim_{x \to 0^+} f(x)$ (ii) $\lim_{x \to 1^-} f(x)$ (iii) $\lim_{x \to 1^+} f(x)$

(b) Use a table of values to estimate $\lim_{x \to \infty} f(x)$.

(c) Use the information from parts (a) and (b) to make a rough sketch of the graph of f.

44. (a) For $f(x) = \dfrac{2}{x} - \dfrac{1}{\ln x}$ find each of the following limits.

(i) $\lim_{x \to \infty} f(x)$ (ii) $\lim_{x \to 0^+} f(x)$

(iii) $\lim_{x \to 1^-} f(x)$ (iv) $\lim_{x \to 1^+} f(x)$

(b) Use the information from part (a) to make a rough sketch of the graph of f.

45. (a) Estimate the value of

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + x + 1} + x\right)$$

by graphing the function $f(x) = \sqrt{x^2 + x + 1} + x$.

(b) Use a table of values of $f(x)$ to guess the value of the limit.

(c) Prove that your guess is correct.

46. (a) Use a graph of

$$f(x) = \sqrt{3x^2 + 8x + 6} - \sqrt{3x^2 + 3x + 1}$$

to estimate the value of $\lim_{x \to \infty} f(x)$ to one decimal place.

(b) Use a table of values of $f(x)$ to estimate the limit to four decimal places.

(c) Find the exact value of the limit.

47–52 Find the horizontal and vertical asymptotes of each curve. You may want to use a graphing calculator (or computer) to check your work by graphing the curve and estimating the asymptotes.

47. $y = \dfrac{5 + 4x}{x + 3}$ **48.** $y = \dfrac{2x^2 + 1}{3x^2 + 2x - 1}$

49. $y = \dfrac{2x^2 + x - 1}{x^2 + x - 2}$ **50.** $y = \dfrac{1 + x^4}{x^2 - x^4}$

51. $y = \dfrac{x^3 - x}{x^2 - 6x + 5}$ **52.** $y = \dfrac{2e^x}{e^x - 5}$

53. Estimate the horizontal asymptote of the function

$$f(x) = \frac{3x^3 + 500x^2}{x^3 + 500x^2 + 100x + 2000}$$

by graphing f for $-10 \leqslant x \leqslant 10$. Then calculate the equation of the asymptote by evaluating the limit. How do you explain the discrepancy?

54. (a) Graph the function

$$f(x) = \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

How many horizontal and vertical asymptotes do you observe? Use the graph to estimate the values of the limits

$$\lim_{x \to \infty} \frac{\sqrt{2x^2 + 1}}{3x - 5} \quad \text{and} \quad \lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{3x - 5}$$

(b) By calculating values of $f(x)$, give numerical estimates of the limits in part (a).

(c) Calculate the exact values of the limits in part (a). Did you get the same value or different values for these two limits? [In view of your answer to part (a), you might have to check your calculation for the second limit.]

55. Let P and Q be polynomials. Find

$$\lim_{x \to \infty} \frac{P(x)}{Q(x)}$$

if the degree of P is (a) less than the degree of Q and (b) greater than the degree of Q.

56. Make a rough sketch of the curve $y = x^n$ (n an integer) for the following five cases:

(i) $n = 0$ (ii) $n > 0$, n odd
(iii) $n > 0$, n even (iv) $n < 0$, n odd
(v) $n < 0$, n even

Then use these sketches to find the following limits.

(a) $\lim_{x \to 0^+} x^n$ (b) $\lim_{x \to 0^-} x^n$
(c) $\lim_{x \to \infty} x^n$ (d) $\lim_{x \to -\infty} x^n$

57. Find a formula for a function f that satisfies the following conditions:

$$\lim_{x \to \pm\infty} f(x) = 0, \quad \lim_{x \to 0} f(x) = -\infty, \quad f(2) = 0,$$
$$\lim_{x \to 3^-} f(x) = \infty, \quad \lim_{x \to 3^+} f(x) = -\infty$$

58. Find a formula for a function that has vertical asymptotes $x = 1$ and $x = 3$ and horizontal asymptote $y = 1$.

59. A function f is a ratio of quadratic functions and has a vertical asymptote $x = 4$ and just one x-intercept, $x = 1$. It is known that f has a removable discontinuity at $x = -1$ and $\lim_{x \to -1} f(x) = 2$. Evaluate

(a) $f(0)$ (b) $\lim_{x \to \infty} f(x)$

60–64 Find the limits as $x \to \infty$ and as $x \to -\infty$. Use this information, together with intercepts, to give a rough sketch of the graph as in Example 12.

60. $y = 2x^3 - x^4$ **61.** $y = x^4 - x^6$

62. $y = x^3(x + 2)^2(x - 1)$

63. $y = (3 - x)(1 + x)^2(1 - x)^4$

64. $y = x^2(x^2 - 1)^2(x + 2)$

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65. (a) Use the Squeeze Theorem to evaluate $\lim_{x \to \infty} \frac{\sin x}{x}$.

(b) Graph $f(x) = (\sin x)/x$. How many times does the graph cross the asymptote?

66. End Behavior of a Function By the *end behavior* of a function we mean the behavior of its values as $x \to \infty$ and as $x \to -\infty$.

(a) Describe and compare the end behavior of the functions

$$P(x) = 3x^5 - 5x^3 + 2x \qquad Q(x) = 3x^5$$

by graphing both functions in the viewing rectangles $[-2, 2]$ by $[-2, 2]$ and $[-10, 10]$ by $[-10{,}000, 10{,}000]$.

(b) Two functions are said to have the *same end behavior* if their ratio approaches 1 as $x \to \infty$. Show that P and Q have the same end behavior.

67. Find $\lim_{x \to \infty} f(x)$ if, for all $x > 1$,

$$\frac{10e^x - 21}{2e^x} < f(x) < \frac{5\sqrt{x}}{\sqrt{x - 1}}$$

68. (a) A tank contains 5000 L of pure water. Brine that contains 30 g of salt per liter of water is pumped into the tank at a rate of 25 L/min. Show that the concentration of salt after t minutes (in grams per liter) is

$$C(t) = \frac{30t}{200 + t}$$

(b) What happens to the concentration as $t \to \infty$?

69. In Chapter 9 we will be able to show, under certain assumptions, that the velocity $v(t)$ of a falling raindrop at time t is

$$v(t) = v^*(1 - e^{-gt/v^*})$$

where g is the acceleration due to gravity and v^* is the *terminal velocity* of the raindrop.

(a) Find $\lim_{t \to \infty} v(t)$.

(b) Graph $v(t)$ if $v^* = 1$ m/s and $g = 9.8$ m/s^2. How long does it take for the velocity of the raindrop to reach 99% of its terminal velocity?

70. (a) By graphing $y = e^{-x/10}$ and $y = 0.1$ on a common screen, discover how large you need to make x so that $e^{-x/10} < 0.1$.

(b) Can you solve part (a) without using a graph?

71. Use a graph to find a number N such that

$$\text{if} \quad x > N \quad \text{then} \quad \left| \frac{3x^2+1}{2x^2+x+1} - 1.5 \right| < 0.05$$

72. For the limit

$$\lim_{x\to\infty} \frac{1-3x}{\sqrt{x^2+1}} = -3$$

illustrate Definition 7 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

73. For the limit

$$\lim_{x\to-\infty} \frac{1-3x}{\sqrt{x^2+1}} = 3$$

illustrate Definition 8 by finding values of N that correspond to $\varepsilon = 0.1$ and $\varepsilon = 0.05$.

74. For the limit

$$\lim_{x\to\infty} \sqrt{x \ln x} = \infty$$

illustrate Definition 9 by finding a value of N that corresponds to $M = 100$.

75. (a) How large do we have to take x so that $1/x^2 < 0.0001$?

(b) Taking $r = 2$ in Theorem 5, we have the statement

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Prove this directly using Definition 7.

76. (a) How large do we have to take x so that $1/\sqrt{x} < 0.0001$?

(b) Taking $r = \frac{1}{2}$ in Theorem 5, we have the statement

$$\lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$$

Prove this directly using Definition 7.

77. Use Definition 8 to prove that $\lim_{x\to-\infty} \frac{1}{x} = 0$.

78. Prove, using Definition 9, that $\lim_{x\to\infty} x^3 = \infty$.

79. Use Definition 9 to prove that $\lim_{x\to\infty} e^x = \infty$.

80. Formulate a precise definition of

$$\lim_{x\to-\infty} f(x) = -\infty$$

Then use your definition to prove that

$$\lim_{x\to-\infty} (1 + x^3) = -\infty$$

81. (a) Prove that

$$\lim_{x\to\infty} f(x) = \lim_{t\to0^+} f(1/t)$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{t\to0^-} f(1/t)$$

assuming that these limits exist.

(b) Use part (a) and Exercise 65 to find

$$\lim_{x\to0^+} x \sin \frac{1}{x}$$