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Numerical Linear Algebra

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Well posed problem, ill conditioned problem, condition Number

- Recap of Previous Lecture
- Ill conditioned linear system and matrix properties
- Condition number of a matrix
- Properties of $Cond(A)$
- Perturbations in right hand side
- Perturbations in coefficients
- Perturbations in right hand side and coefficients
- Q & A

Recap of Previous Lecture

- Computational project 1
- Convergence of matrix sequences
- Matrix Series
- Triangular systems of linear equations
- Condition number

Triangular systems of linear equations 3

Example 4.1

$$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} =$$

$$\begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & \dots & .. & -1 & | & -1 \\ 0 & 1 & \dots & .. & -1 & | & -1 \\ .. & .. & \dots & .. & .. & | & .. \\ 0 & 0 & \dots & 1 & -1 & | & -1 \\ 0 & 0 & \dots & 0 & 1 & | & 1 \end{pmatrix}$$

Triangular systems of linear equations 3

Example 4.1

$$\begin{pmatrix} 1 & -1 & \ldots & .. & -1 \\ 0 & 1 & \ldots & .. & -1 \\ .. & .. & \ldots & .. & .. \\ 0 & 0 & \ldots & 1 & -1 \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, \left(\begin{array}{ccccc|c} 1 & -1 & \ldots & .. & -1 & -1 \\ 0 & 1 & \ldots & .. & -1 & -1 \\ .. & .. & \ldots & .. & .. & .. \\ 0 & 0 & \ldots & 1 & -1 & -1 \\ 0 & 0 & \ldots & 0 & 1 & 1 \end{array}\right)$$

Back substitution:

$$x_n = 1$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 0$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 0$$

$$x_{n-3} - x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-3} = x_{n-2} + x_{n-1} + x_n - 1 = 2x_{n-2} = 0$$

$$\ldots$$

Triangular systems of linear equations 4

Example 4.2

Back substitution

$$x_n = 1$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 0$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 0$$

$$x_{n-3} - x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-3} = x_{n-2} + x_{n-1} + x_n - 1 = 2x_{n-2} = 0$$

$$\dots$$

$$x_{n-k} - x_{n-k+1} - \sum_{i=n-k+2}^{n} x_i = -1 \Rightarrow$$

$$\Rightarrow x_{n-k} = x_{n-k+1} + \sum_{i=n-k+2}^{n} x_i - 1 = 2x_{n-k+1} = 0$$

$$x_{n-k-1} - x_{n-k} - \sum_{i=n-k+1}^{n} x_i = -1 \Rightarrow$$

$$\Rightarrow x_{n-k-1} = x_{n-k} + \sum_{i=n-k+1}^{n} x_i - 1 = 2x_{n-k} = 0$$

$x_n = 1, x_{n-1} = 0, x_{n-2} = 0, \dots, x_1 = 0$

Triangular systems of linear equations 5

Example 4.3

What if we have small error in data, for example $b_n = 1 + \epsilon$

$$\begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1+\epsilon \end{pmatrix},$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & ... & .. & -1 & -1 \\ 0 & 1 & ... & .. & -1 & -1 \\ .. & .. & ... & .. & .. & .. \\ 0 & 0 & ... & 1 & -1 & -1 \\ 0 & 0 & ... & 0 & 1 & 1+\epsilon \end{array}\right)$$

Triangular systems of linear equations 6

Example 4.4

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$

$$x_n = 1 + \epsilon$$

$$x_{n-1} - x_n = -1 \Rightarrow x_{n-1} = x_n - 1 = 1 + \epsilon - 1 = \epsilon$$

$$x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-2} = x_{n-1} + x_n - 1 = 2x_{n-1} = 2\epsilon$$

$$x_{n-3} - x_{n-2} - x_{n-1} - x_n = -1 \Rightarrow x_{n-3} = x_{n-2} + x_{n-1} + x_n - 1 = 2x_{n-2} = 2^2\epsilon$$

$$\dots$$

$$x_{n-k} = x_{n-k+1} + \sum_{i=n-k+2}^{n} x_i - 1 = 2x_{n-k+1} = 2^{k-1}\epsilon$$

$$x_{n-k-1} - x_{n-k} - \sum_{i=n-k+1}^{n} x_i = -1 \Rightarrow$$

$$\Rightarrow x_{n-k-1} = x_{n-k} + \sum_{i=n-k+1}^{n} x_i - 1 = 2x_{n-k} = 2^k\epsilon$$

$$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, \dots, x_1 = 2^{n-1}\epsilon$$

Triangular systems of linear equations 6

Example 4.5

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$
$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, ..., x_1 = 2^{n-1}\epsilon$

Triangular systems of linear equations 6

Example 4.5

Suppose right hand side contains some error ϵ and $b_n = 1 + \epsilon$

$x_n = 1 + \epsilon, x_{n-1} = \epsilon, x_{n-2} = 2^1\epsilon, x_{n-3} = 2^2\epsilon, ..., x_1 = 2^{n-1}\epsilon$

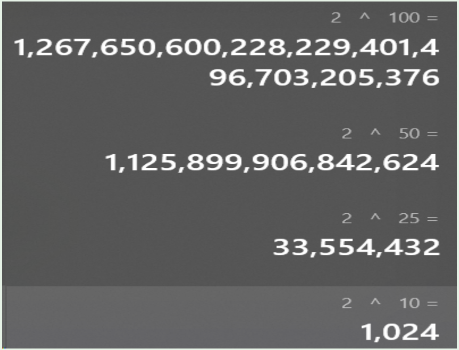

Figure: Error amplification factors 2^{n-1}

Triangular systems of linear equations 7

- $$\begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}$$

Triangular systems of linear equations 7

- $$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ .. \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1 \end{pmatrix}, x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}$$

- $$\begin{pmatrix} 1 & -1 & \dots & .. & -1 \\ 0 & 1 & \dots & .. & -1 \\ .. & .. & \dots & .. & .. \\ 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ .. \\ \tilde{x}_{n-1} \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ .. \\ -1 \\ 1+\epsilon \end{pmatrix}, \tilde{x} = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ 1+\epsilon \end{pmatrix}$$

Triangular systems of linear equations 8, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

Triangular systems of linear equations 8, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

Definition 4.6

Let $x \in \mathbb{R}$ and let $\tilde{x} \in \mathbb{R}$ denote an approximation of it.

- Absolute error: $|x - \tilde{x}|$
- Relative error: $\frac{|x-\tilde{x}|}{|x|}$

Triangular systems of linear equations 8, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

Definition 4.6

Let $x \in \mathbb{R}$ and let $\tilde{x} \in \mathbb{R}$ denote an approximation of it.

- Absolute error: $|x - \tilde{x}|$
- Relative error: $\frac{|x-\tilde{x}|}{|x|}$

Definition 4.7

Let $x \in \mathbb{R}^n$ and let $\tilde{x} \in \mathbb{R}^n$ denote an approximation of it.

- Absolute error: $\|x - \tilde{x}\|$
- Relative error: $\frac{\|x-\tilde{x}\|}{\|x\|}$

Triangular systems of linear equations 9, errors

- Can we compare two vectors?
- How do we compute errors in the solution of linear systems?

$$x = \begin{pmatrix} 0 \\ 0 \\ .. \\ 0 \\ 1 \end{pmatrix}, \tilde{x} = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ 1+\epsilon \end{pmatrix}, \quad \tilde{x} - x = \begin{pmatrix} 2^{n-1}\epsilon \\ 2^{n-2}\epsilon \\ .. \\ \epsilon \\ \epsilon \end{pmatrix}, \|\tilde{x} - x\| = 2^{n-1}|\epsilon|$$

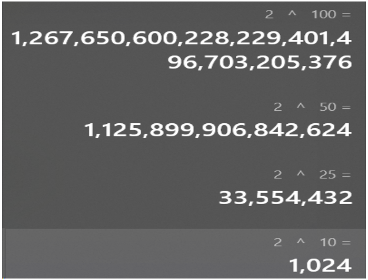

Figure: Error amplification factors 2^{n-1}

Triangular systems of linear equations 10, ill conditioned matrix

Example 4.8

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem
- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.
- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

Triangular systems of linear equations 10, ill conditioned matrix

Example 4.8

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
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- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.
- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

Theorem 4.9

The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Matrix properties and ill conditioning

Theorem 4.10

The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Matrix properties and ill conditioning

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The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Which matrix properties are important for conditioning?

- Determinant of a matrix?

Matrix properties and ill conditioning

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The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Which matrix properties are important for conditioning?

- Determinant of a matrix?
- Eigenvalue of a matrix?

Matrix properties and ill conditioning

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The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Which matrix properties are important for conditioning?

- Determinant of a matrix?
- Eigenvalue of a matrix?
- Big or small entries of a matrix?

Matrix properties and ill conditioning

Theorem 4.10

The problem $Ax = b$ is ill posed, if $A = \begin{pmatrix} 1 & -1 & ... & .. & -1 \\ 0 & 1 & ... & .. & -1 \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 1 & -1 \\ 0 & 0 & ... & 0 & 1 \end{pmatrix}$

Which matrix properties are important for conditioning?

- Determinant of a matrix?
- Eigenvalue of a matrix?
- Big or small entries of a matrix?
- Big or small norm of a matrix?

Condition number of a matrix, 1

Definition 4.11

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix, 1

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Example 4.12

- Ill-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4.001 & 2.002 \\ 1 & 2.002 & 2.004 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 8.0021 \\ 5.006 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Condition number of a matrix, 1

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Example 4.12

- Ill-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4.001 & 2.002 \\ 1 & 2.002 & 2.004 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 8.0021 \\ 5.006 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\tilde{b} = \begin{pmatrix} 1 \\ 8.0020 \\ 5.0061 \end{pmatrix}, \tilde{x} = \begin{pmatrix} 3.0850 \\ -0.0436 \\ 1.0022 \end{pmatrix}$$

Condition number of a matrix, 1

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Example 4.12

- Ill-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4.001 & 2.002 \\ 1 & 2.002 & 2.004 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 8.0021 \\ 5.006 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\tilde{b} = \begin{pmatrix} 1 \\ 8.002\color{red}{0} \\ 5.006\color{red}{1} \end{pmatrix}, \tilde{x} = \begin{pmatrix} \color{red}{3.0850} \\ \color{red}{-0.0436} \\ 1.\color{red}{0022} \end{pmatrix}$$

Small relative perturbation: $\frac{\|b-\tilde{b}\|}{\|b\|} = 1.3975 \cdot 10^{-5}$

Large relative error: $\frac{\|x-\tilde{x}\|}{\|x\|} = 1.3461$

Condition number of a matrix, 1

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Example 4.14

- Ill-conditioned system of linear equations $Ax = b$

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```
A= [[1.    2.    1.   ]
 [2.    4.001 2.002]
 [1.    2.002 2.004]]
cond2(A)= 31062.1661044696
condFro(A)= 31326.00296885506
condInf(A)= 48170.05699998381
cond1(A)= 48170.05699998381
```

Condition number of a matrix, 1

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Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Example 4.14

- Ill-conditioned system of linear equations $Ax = b$

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A= [[1.     2.     1.    ]
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cond2(A)= 31062.1661044696
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```

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Large relative error: $\frac{\|x-\tilde{x}\|}{\|x\|} = 1.3461$

Condition number of a matrix, 2

Definition 4.15

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Example 4.16

Condition number of a matrix, 2

Definition 4.15

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Example 4.16

- Well-conditioned system of linear equations $Ax = b$

Condition number of a matrix, 2

Definition 4.15

Condition number of a matrix: $cond(A) = \|A\| \|A^{-1}\|$

Example 4.16

- Well-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Condition number of a matrix, 2

Definition 4.15

Condition number of a matrix: $cond(A) = \|A\| \|A^{-1}\|$

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- Well-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\tilde{b} = \begin{pmatrix} 3.0001 \\ 7.0001 \end{pmatrix}, \tilde{x} = \begin{pmatrix} 0.9999 \\ 1.0001 \end{pmatrix}$$

Condition number of a matrix, 2

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Example 4.16

- Well-conditioned system of linear equations $Ax = b$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, b = \begin{pmatrix} 3 \\ 7 \end{pmatrix}, x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\tilde{b} = \begin{pmatrix} 3.0001 \\ 7.0001 \end{pmatrix}, \tilde{x} = \begin{pmatrix} 0.9999 \\ 1.0001 \end{pmatrix}$$

Small relative perturbation: $\frac{\|b-\tilde{b}\|}{\|b\|} = 1.875 \cdot 10^{-5}$

Condition number of a matrix, 2

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Example 4.16

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Small relative perturbation: $\frac{\|b - \tilde{b}\|}{\|b\|} = 1.875 \cdot 10^{-5}$

Small relative error: $\frac{\|x - \tilde{x}\|}{\|x\|} = 10^{-5}$

small condition number: $cond(A) = 14.9930$

Condition number of a matrix, 2

Example 4.17

Condition number of a matrix, 2

Example 4.17

- Well-conditioned system of linear equations $Ax = b$

Condition number of a matrix, 2

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Condition number of a matrix, 2

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- Well-conditioned system of linear equations $Ax = b$

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Small relative perturbation: $\frac{\|b - \tilde{b}\|}{\|b\|} = 1.875 \cdot 10^{-5}$

Small relative error: $\frac{\|x - \tilde{x}\|}{\|x\|} = 10^{-5}$

small condition number:

```
A= [[1 2]
 [3 4]]
cond2(A)= 14.933034373659265
condFro(A)= 14.999999999999998
condInf(A)= 21.0
cond1(A)= 20.999999999999996
```

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. $cond(A) \geq 1$ *for any induced norm*

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\| \|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. $cond(A) \geq 1$ *for any induced norm (Solution:* $A \cdot A^{-1} = I$*)*

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. $cond(A) \geq 1$ *for any induced norm (Solution:* $A \cdot A^{-1} = I$*)*
2. $cond(\alpha A) = cond(A)$ *for any norm*

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. *$cond(A) \geq 1$ for any induced norm (Solution: $A \cdot A^{-1} = I$)*
2. *$cond(\alpha A) = cond(A)$ for any norm (Solution: $\alpha A \cdot \alpha^{-1} A^{-1} = I$)*

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. *$cond(A) \geq 1$ for any induced norm (Solution: $A \cdot A^{-1} = I$)*
2. *$cond(\alpha A) = cond(A)$ for any norm (Solution: $\alpha A \cdot \alpha^{-1} A^{-1} = I$)*
3. *$cond(AB) \leq cond(A) cond(B)$ for any sub-multiplicative norm*

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. $cond(A) \geq 1$ *for any induced norm (Solution:* $A \cdot A^{-1} = I$*)*
2. $cond(\alpha A) = cond(A)$ *for any norm (Solution:* $\alpha A \cdot \alpha^{-1} A^{-1} = I$*)*
3. $cond(AB) \leq cond(A) cond(B)$ *for any sub-multiplicative norm*
4. $cond_1(A) = cond_\infty(A^T)$

Condition number of a matrix, properties, 1

Definition 4.18

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number of a matrix may be different in different norms

Theorem 4.19

1. $cond(A) \geq 1$ *for any induced norm (Solution:* $A \cdot A^{-1} = I$*)*
2. $cond(\alpha A) = cond(A)$ *for any norm (Solution:* $\alpha A \cdot \alpha^{-1} A^{-1} = I$*)*
3. $cond(AB) \leq cond(A) cond(B)$ *for any sub-multiplicative norm*
4. $cond_1(A) = cond_\infty(A^T)$
5. $cond(A) = cond(A^{-1})$

Condition number of a matrix, properties, 2

Definition 4.20

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number computed in one norm and condition number computed in another may be different

Theorem 4.21

1. $cond_2(A) = 1$ *iff* $A^T A = \alpha I$, $\alpha \neq 0$

Condition number of a matrix, properties, 2

Definition 4.20

Condition number of a matrix: $cond(A) = \|A\| \|A^{-1}\|$

Condition number computed in one norm and condition number computed in another may be different

Theorem 4.21

1. $cond_2(A) = 1$ *iff* $A^T A = \alpha I$, $\alpha \neq 0$
2. $cond_2(A) = cond_2(A^T)$

Condition number of a matrix, properties, 2

Definition 4.20

Condition number of a matrix: $cond(A) = \|A\| \|A^{-1}\|$

Condition number computed in one norm and condition number computed in another may be different

Theorem 4.21

1. $cond_2(A) = 1$ *iff* $A^T A = \alpha I$, $\alpha \neq 0$
2. $cond_2(A) = cond_2(A^T)$
3. $cond_2(A^T A) = (cond_2(A))^2$

Condition number of a matrix, properties, 2

Definition 4.20

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number computed in one norm and condition number computed in another may be different

Theorem 4.21

1. $cond_2(A) = 1$ *iff* $A^T A = \alpha I$, $\alpha \neq 0$
2. $cond_2(A) = cond_2(A^T)$
3. $cond_2(A^T A) = (cond_2(A))^2$
4. $cond_2(A) = \frac{\lambda_{max}}{\lambda_{min}}$, A - *symmetric positive definite*, $\lambda_{max}, \lambda_{min}$ - *eigenvalues of* A

Condition number of a matrix, properties, 2

Definition 4.20

Condition number of a matrix: $cond(A) = \|A\|\|A^{-1}\|$

Condition number computed in one norm and condition number computed in another may be different

Theorem 4.21

1. $cond_2(A) = 1$ *iff* $A^TA = \alpha I$, $\alpha \neq 0$
2. $cond_2(A) = cond_2(A^T)$
3. $cond_2(A^TA) = (cond_2(A))^2$
4. $cond_2(A) = \frac{\lambda_{max}}{\lambda_{min}}$, A - *symmetric positive definite*, $\lambda_{max}, \lambda_{min}$ - *eigenvalues of* A
5. $cond_2(A) = cond_2(AU) = cond_2(VA)$, *if* U, V *unitary matrices*

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem
- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
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- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

Condition number of a matrix, properties, 3

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Example 4.22

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- J.Hadamard concept of well posedness:

Condition number of a matrix, properties, 3

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- J.Hadamard concept of well posedness:
  - a solution exists

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem
- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.
- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

- J.Hadamard concept of well posedness:
  - a solution exists
  - the solution is unique

Condition number of a matrix, properties, 3

- condition number characterize ill- and well-conditioned problems

Example 4.22

- Two systems of linear equations with the same matrix A, $Ax = b$ and $A\tilde{x} = \tilde{b}$
- b and $\tilde{b}$ are two different inputs for the same problem
- **Well conditioned problem:** if b and $\tilde{b}$ are close then x and $\tilde{x}$ are close.
- **Ill conditioned problem:** even if b and $\tilde{b}$ are close then x and $\tilde{x}$ differ from each other drastically.

- J.Hadamard concept of well posedness:
  - a solution exists
  - the solution is unique
  - solution changes continuously together with input

Condition number of a matrix, properties, 4

- J.Hadamard concept of well posedness:
  - solution of the problem exists

Condition number of a matrix, properties, 4

- J.Hadamard concept of well posedness:
    - solution of the problem exists
    - solution of the problem is unique

Condition number of a matrix, properties, 4

- J.Hadamard concept of well posedness:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data

Condition number of a matrix, properties, 4

- J.Hadamard concept of well posedness:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \circ \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}\,b_{11} & a_{12}\,b_{12} & a_{13}\,b_{13} \\ a_{21}\,b_{21} & a_{22}\,b_{22} & a_{23}\,b_{23} \\ a_{31}\,b_{31} & a_{32}\,b_{32} & a_{33}\,b_{33} \end{bmatrix}.$$

Figure: Jacques Hadamard, Hadamard product, source Wikipedia

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning
  - well-conditioned problem:
    small perturbations in input data cause small perturbations in solutions
  - ill-conditioned problem:

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning
  - well-conditioned problem:
    small perturbations in input data cause small perturbations in solutions
  - ill-conditioned problem:
    small perturbations in input data cause big perturbations in solutions

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning
  - well-conditioned problem:
    small perturbations in input data cause small perturbations in solutions
  - ill-conditioned problem:
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- Example of well-conditioned problem ?

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning
  - well-conditioned problem:
    small perturbations in input data cause small perturbations in solutions
  - ill-conditioned problem:
    small perturbations in input data cause big perturbations in solutions
- Example of well-conditioned problem ?
- Example of ill-conditioned problem ?

Condition number of a matrix, properties, 5

- J.Hadamard concept of well posed problem:
  - solution of the problem exists
  - solution of the problem is unique
  - solution depends continuously on input data
- Example of well-posed problem ?
- Concept of ill- and well-conditioning
  - well-conditioned problem:
    small perturbations in input data cause small perturbations in solutions
  - ill-conditioned problem:
    small perturbations in input data cause big perturbations in solutions
- Example of well-conditioned problem ?
- Example of ill-conditioned problem ?
- well-conditioned vs well-posed ?

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

- *A is invertible*

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

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Suppose

- *A is invertible*
- $Ax = b$

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

- *A is invertible*
- $Ax = b$
- *δb is perturbation of b*

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

- A *is invertible*
- $Ax = b$
- δb *is perturbation of* b
- δx *is pertubation caused by* δb

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

- *A is invertible*
- *$Ax = b$*
- *δb is perturbation of b*
- *δx is pertubation caused by δb*

Then the following holds true:

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

Perturbations in linear system $Ax = b$, case - RHS b , 1

Theorem 4.23

(Right perturbation theorem)
Suppose

- *A is invertible*
- *$Ax = b$*
- *δb is perturbation of b*
- *δx is pertubation caused by δb*

Then the following holds true:

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

$$\frac{1}{cond(A)}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq cond(A)\frac{\|\delta b\|}{\|b\|}$$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$
- $\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\|$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$
- $\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\|$
- $\|b\| = \|Ax\| \leq \|A\|\|x\|$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$
- $\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\|$
- $\|b\| = \|Ax\| \leq \|A\|\|x\|$
- $\|\delta x\|\|b\| \leq \|A^{-1}\|\|\delta b\|\|A\|\|x\|$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$
- $\|\delta x\| = \|A^{-1}\delta b\| \le \|A^{-1}\|\|\delta b\|$
- $\|b\| = \|Ax\| \le \|A\|\|x\|$
- $\|\delta x\|\|b\| \le \|A^{-1}\|\|\delta b\|\|A\|\|x\|$
- $\frac{\|\delta x\|}{\|x\|} \le \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|} = cond(A)\frac{\|\delta b\|}{\|b\|}$

Perturbations in linear system $Ax = b$, case - RHS b , 2

Proof.

(Right perturbation theorem, part 1)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $\delta x = A^{-1}\delta b$
- $\|\delta x\| = \|A^{-1}\delta b\| \leq \|A^{-1}\|\|\delta b\|$
- $\|b\| = \|Ax\| \leq \|A\|\|x\|$
- $\|\delta x\|\|b\| \leq \|A^{-1}\|\|\delta b\|\|A\|\|x\|$
- $\frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|} = cond(A)\frac{\|\delta b\|}{\|b\|}$

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

□

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1}b$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1}b$
- $\|\delta b\| = \|A\delta x\| \leq \|A\| \|\delta x\|$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1}b$
- $\|\delta b\| = \|A\delta x\| \leq \|A\|\|\delta x\|$
- $\|x\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\|$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1}b$
- $\|\delta b\| = \|A\delta x\| \leq \|A\|\|\delta x\|$
- $\|x\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\|$
- $\|\delta b\|\|x\| \leq \|A\|\|\delta x\|\|A^{-1}\|\|b\|$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1} b$
- $\|\delta b\| = \|A\delta x\| \leq \|A\| \|\delta x\|$
- $\|x\| = \|A^{-1} b\| \leq \|A^{-1}\| \|b\|$
- $\|\delta b\| \|x\| \leq \|A\| \|\delta x\| \|A^{-1}\| \|b\|$
- $\frac{\|\delta b\|}{\|b\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta x\|}{\|x\|} = cond(A) \frac{\|\delta x\|}{\|x\|}$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Proof.

(Right perturbation theorem, part 2)

- $Ax = b$
- $A(x + \delta x) = b + \delta b$
- $A\delta x = \delta b$
- $x = A^{-1} b$
- $\|\delta b\| = \|A\delta x\| \leq \|A\| \|\delta x\|$
- $\|x\| = \|A^{-1} b\| \leq \|A^{-1}\| \|b\|$
- $\|\delta b\| \|x\| \leq \|A\| \|\delta x\| \|A^{-1}\| \|b\|$
- $\frac{\|\delta b\|}{\|b\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta x\|}{\|x\|} = cond(A) \frac{\|\delta x\|}{\|x\|}$

$$\frac{1}{\|A\| \|A^{-1}\|} \frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\| \|A^{-1}\| \frac{\|\delta b\|}{\|b\|}$$

$\square$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Right perturbation theorem, questions, discussion

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

Perturbations in linear system $Ax = b$, case - RHS b , 3

Right perturbation theorem, questions, discussion

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

- Left inequality

Perturbations in linear system $Ax = b$, case - RHS b , 3

Right perturbation theorem, questions, discussion

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

- Left inequality
- Right inequality

Perturbations in linear system $Ax = b$, case - RHS b , 3

Right perturbation theorem, questions, discussion

$$\frac{1}{\|A\|\|A^{-1}\|}\frac{\|\delta b\|}{\|b\|} \leq \frac{\|\delta x\|}{\|x\|} \leq \|A\|\|A^{-1}\|\frac{\|\delta b\|}{\|b\|}$$

- Left inequality
- Right inequality
- can lower and upper bounds reached?

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*
- $Ax = b$

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*
- $Ax = b$
- *δA is perturbation of A*

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*
- *$Ax = b$*
- *δA is perturbation of A*
- *δx is pertubation caused by δA*

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*
- $Ax = b$
- *δA is perturbation of A*
- *δx is pertubation caused by δA*
- $\|\delta A\| < 1/\|A^{-1}\|$

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- *A is invertible*
- *$Ax = b$*
- *δA is perturbation of A*
- *δx is pertubation caused by δA*
- *$\|\delta A\| < 1/\|A^{-1}\|$*

Then the following holds true:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\delta A\|}{1 - \|A^{-1}\|\|\delta A\|}$$

Perturbations in linear system $Ax = b$, case - matrix A , 1

Theorem 4.24

(Left perturbation theorem)
Suppose

- A *is invertible*
- $Ax = b$
- δA *is perturbation of* A
- δx *is pertubation caused by* δA
- $\|\delta A\| < 1/\|A^{-1}\|$

Then the following holds true:

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\delta A\|}{1 - \|A^{-1}\|\|\delta A\|}$$

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{cond(A)\frac{\|\delta A\|}{\|A\|}}{1 - cond(A)\frac{\|\delta A\|}{\|A\|}}$$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$
- $Ax + A\delta x + \delta Ax + \delta A\delta x = b$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$
- $Ax + A\delta x + \delta Ax + \delta A\delta x = b$
- $A\delta x + \delta Ax + \delta A\delta x = 0$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$
- $Ax + A\delta x + \delta Ax + \delta A\delta x = b$
- $A\delta x + \delta Ax + \delta A\delta x = 0$
- $A\delta x = -\delta A(x + \delta x)$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$
- $Ax + A\delta x + \delta Ax + \delta A\delta x = b$
- $A\delta x + \delta Ax + \delta A\delta x = 0$
- $A\delta x = -\delta A(x + \delta x)$
- $\delta x = -A^{-1}\delta A(x + \delta x)$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

(Left perturbation theorem, part 1)

- $Ax = b$
- $(A + \delta A)(x + \delta x) = b$
- $Ax + A\delta x + \delta A x + \delta A \delta x = b$
- $A\delta x + \delta A x + \delta A \delta x = 0$
- $A\delta x = -\delta A(x + \delta x)$
- $\delta x = -A^{-1}\delta A(x + \delta x)$
- $\|\delta x\| \leq \|A^{-1}\|\|\delta A\|(\|x\| + \|\delta x\|)$

Perturbations in linear system $Ax = b$, case - matrix A , 2

Proof.

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- $(1 - \|A^{-1}\|\|\delta A\|)\|\delta x\| \leq \|A^{-1}\|\|\delta A\|\|x\|$

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$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\|\|\delta A\|}{1 - \|A^{-1}\|\|\delta A\|} \quad \equiv \quad \frac{\|\delta x\|}{\|x\|} \leq \frac{cond(A)\frac{\|\delta A\|}{\|A\|}}{1 - cond(A)\frac{\|\delta A\|}{\|A\|}}$$

□

Q & A