Numerical Linear Algebra

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QR factorization, Least squares

- Invertibility of KKT matrix
- QR factorization for constrained least squares problem
- Hausholder transformations and QR factorization
- Numerical eigenvalue problem
- Q & A

Recap of Previous Lecture

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- QR and reduced QR factorization
- Linear least squares
- Constrained least squares

Constrained Least Squares

Problem 14.1

Constrained Least Squares Problem

Constrained Least Squares

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Constrained Least Squares Problem

- $A \in \mathbb{R}^{n \times m}, n > m$

Constrained Least Squares

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- $A \in \mathbb{R}^{n \times m}, n > m$
- $C \in \mathbb{R}^{p \times m}$

Constrained Least Squares

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Constrained Least Squares Problem

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- $x \in \mathbb{R}^m, b \in \mathbb{R}^n, d \in \mathbb{R}^p$

Constrained Least Squares

Problem 14.1

Constrained Least Squares Problem

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- $x \in \mathbb{R}^m, b \in \mathbb{R}^n, d \in \mathbb{R}^p$
- $Ax = b$ - over determined system

Constrained Least Squares

Problem 14.1

Constrained Least Squares Problem

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- $f(x) = \|Ax - b\|_2$

Constrained Least Squares

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Constrained Least Squares

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KKT equation

$$\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$$

Constrained Least Squares

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KKT equation (William Karush, Harold Kuhn and Albert Tucker)

Constrained Least Squares, KKT matrix

KKT matrix

$$\begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix}$$

Constrained Least Squares, KKT matrix

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Theorem 14.2

KKT matrix is invertible iff

Constrained Least Squares, KKT matrix

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a. *C has linearly independent rows*

Constrained Least Squares, KKT matrix

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b. $\begin{pmatrix} A \\ C \end{pmatrix}$ *has linearly independent columns*

Constrained Least Squares, KKT matrix

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- Necessary condition:

Constrained Least Squares, KKT matrix

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- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied

Constrained Least Squares, KKT matrix

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- Sufficient condition:

Constrained Least Squares, KKT matrix

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  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible

Constrained Least Squares, KKT matrix

Proof of necessary condition, part a

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied

Constrained Least Squares, KKT matrix

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- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 are NOT satisfied

Constrained Least Squares, KKT matrix

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$$\Downarrow$$

- Rows of C are linearly dependent

Constrained Least Squares, KKT matrix

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- Rows of C are linearly dependent $\Rightarrow \exists \tilde{\lambda} \neq 0, C^T \tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

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- $$\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0$$

Constrained Least Squares, KKT matrix

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- Necessary condition:
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- $$\begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0$$

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- Contradiction: KKT matrix is not invertible

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 is NOT satisfied

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 is NOT satisfied

$$\Downarrow$$

- Rows of C are linearly dependent

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 is NOT satisfied

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- Rows of C are linearly dependent $\Rightarrow \exists\tilde{\lambda} \neq 0, C^T\tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 is NOT satisfied

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- Rows of C are linearly dependent $\Rightarrow \exists \tilde{\lambda} \neq 0, C^T \tilde{\lambda} = 0$

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- $$\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\lambda} \end{pmatrix} = 0$$

Constrained Least Squares, KKT matrix

Proof of necessary condition, part b

- Necessary condition:
  KKT matrix is invertible $\Rightarrow$ a. & b. of the theorem 14.2 are satisfied
- ASSUME: KKT matrix is invertible AND a. of the theorem 14.2 is NOT satisfied

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- Rows of C are linearly dependent $\Rightarrow \exists \tilde{\lambda} \neq 0, C^T \tilde{\lambda} = 0$

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- $$\begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \tilde{\lambda} \end{pmatrix} = 0$$

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- Contradiction: KKT matrix is not invertible

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible

Constrained Least Squares, KKT matrix

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Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0$

Constrained Least Squares, KKT matrix

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Constrained Least Squares, KKT matrix

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- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0$

Constrained Least Squares, KKT matrix

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Constrained Least Squares, KKT matrix

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- $x^T(2A^T A\tilde{x} + C^T \tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^T C^T \tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^T C^T = 0$

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
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- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^TA\tilde{x} + C^T\tilde{\lambda} = 0 \Downarrow$
- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^TC^T\tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^TC^T = 0 \Rightarrow \tilde{x}^TC^T\tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

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- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^TC^T\tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^TC^T = 0 \Rightarrow \tilde{x}^TC^T\tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0$

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
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- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^TC^T\tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^TC^T = 0 \Rightarrow \tilde{x}^TC^T\tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0 \Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^TA\tilde{x} + C^T\tilde{\lambda} = 0 \Downarrow$
- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^TC^T\tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^TC^T = 0 \Rightarrow \tilde{x}^TC^T\tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0 \Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$ &+ condition b.(linearly independent columns of $\begin{pmatrix} A \\ C \end{pmatrix}$)

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0 \Downarrow$
- $x^T(2A^T A\tilde{x} + C^T \tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^T C^T \tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^T C^T = 0 \Rightarrow \tilde{x}^T C^T \tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0 \Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$ &+ condition b.(linearly independent columns of $\begin{pmatrix} A \\ C \end{pmatrix}$) $\Downarrow$
- Contradiction: $\tilde{x} = 0$
- $\tilde{x} = 0, 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0$ $\Downarrow$
- $x^T(2A^T A\tilde{x} + C^T \tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^T C^T \tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^T C^T = 0 \Rightarrow \tilde{x}^T C^T \tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0$ $\Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$ &+ condition b.(linearly independent columns of $\begin{pmatrix} A \\ C \end{pmatrix}$) $\Downarrow$
- Contradiction: $\tilde{x} = 0$
- $\tilde{x} = 0, 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0 \Rightarrow C^T \tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^TA\tilde{x} + C^T\tilde{\lambda} = 0 \Downarrow$
- $x^T(2A^TA\tilde{x} + C^T\tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^TC^T\tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^TC^T = 0 \Rightarrow \tilde{x}^TC^T\tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0 \Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$ &+ condition b.(linearly independent columns of $\begin{pmatrix} A \\ C \end{pmatrix}$) $\Downarrow$
- Contradiction: $\tilde{x} = 0$
- $\tilde{x} = 0, 2A^TA\tilde{x} + C^T\tilde{\lambda} = 0 \Rightarrow C^T\tilde{\lambda} = 0$ &+ condition a.(linearly independent rows of C)

Constrained Least Squares, KKT matrix

Proof of sufficient condition

- Sufficient condition:
  a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- ASSUME: a.& b. of the theorem 14.2 is satisfied AND KKT matrix is NOT invertible $\Downarrow$
- $\exists \tilde{x} \neq 0, \exists \tilde{\lambda} \neq 0, \begin{pmatrix} 2A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\lambda} \end{pmatrix} = 0 \Rightarrow 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0 \Downarrow$
- $x^T(2A^T A\tilde{x} + C^T \tilde{\lambda}) = 0 \Rightarrow 2\|A\tilde{x}\|_2^2 + \tilde{x}^T C^T \tilde{\lambda} = 0$
- $C\tilde{x} = 0 \equiv \tilde{x}^T C^T = 0 \Rightarrow \tilde{x}^T C^T \tilde{\lambda} = 0 \Rightarrow \|A\tilde{x}\|_2^2 = 0 \Rightarrow A\tilde{x} = 0 \Downarrow$
- $\begin{pmatrix} A \\ C \end{pmatrix} \tilde{x} = 0$ &+ condition b.(linearly independent columns of $\begin{pmatrix} A \\ C \end{pmatrix}$) $\Downarrow$
- Contradiction: $\tilde{x} = 0$
- $\tilde{x} = 0, 2A^T A\tilde{x} + C^T \tilde{\lambda} = 0 \Rightarrow C^T \tilde{\lambda} = 0$ &+ condition a.(linearly independent rows of C) $\Downarrow$
- Contradiction: $\tilde{\lambda} = 0$

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$
- a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$
- a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- $\begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$
- a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- $\begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$

Algorithm 14.3

- Compute $H = 2A^TA$ and $f = 2A^Tb$

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$
- a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- $\begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$

Algorithm 14.3

- Compute $H = 2A^TA$ and $f = 2A^Tb$
- Solve $\begin{pmatrix} H & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ d \end{pmatrix}$

Constrained Least Squares, KKT matrix

Solution of KKT equation

- KKT equation: $\begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$
- a. & b. of the theorem 14.2 are satisfied $\Rightarrow$ KKT matrix is invertible
- $\begin{pmatrix} \hat{x} \\ \hat{\lambda} \end{pmatrix} = \begin{pmatrix} 2A^TA & C^T \\ C & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2A^Tb \\ d \end{pmatrix}$

Algorithm 14.3

- Compute $H = 2A^TA$ and $f = 2A^Tb$
- Solve $\begin{pmatrix} H & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ d \end{pmatrix}$

Remark 14.4

Computing A^TA is not practical: expensive, may introduce errors

QR factorization for constrained least squares

Different Lagrangian $\Rightarrow$ different KKT equation

$$L(x, \lambda) = (Ax - b, Ax - b) + 2(Cx - d, \lambda)$$

QR factorization for constrained least squares

Different Lagrangian ⇒ different KKT equation

$$L(x, \lambda) = (Ax - b, Ax - b) + 2(Cx - d, \lambda)$$

$$\begin{aligned} (A^T Ax - A^T b) + C^T \lambda &= 0 \\ Cx &= d \end{aligned}$$

QR factorization for constrained least squares

Different Lagrangian $\Rightarrow$ different KKT equation

$$L(x, \lambda) = (Ax - b, Ax - b) + 2(Cx - d, \lambda)$$

$$\begin{aligned} (A^T Ax - A^T b) + C^T \lambda &= 0 \\ Cx &= d \end{aligned}$$

$$\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$

QR factorization for constrained least squares

Different Lagrangian ⇒ different KKT equation

$$L(x, \lambda) = (Ax - b, Ax - b) + 2(Cx - d, \lambda)$$

$$\begin{aligned} (A^T Ax - A^T b) + C^T \lambda &= 0 \\ Cx &= d \end{aligned}$$

$$\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$

$$L(x, \lambda) = \frac{1}{2}(Ax - b, Ax - b) + (Cx - d, \lambda), KKT =?$$

QR factorization for constrained least squares

Different Lagrangian ⇒ different KKT equation

$$L(x, \lambda) = (Ax - b, Ax - b) + 2(Cx - d, \lambda)$$

$$\begin{aligned} (A^T Ax - A^T b) + C^T \lambda &= 0 \\ Cx &= d \end{aligned}$$

$$\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$

$$L(x, \lambda) = \frac{1}{2}(Ax - b, Ax - b) + (Cx - d, \lambda), KKT = ?$$

$$L(x, \lambda) = \alpha(Ax - b, Ax - b) + \beta(Cx - d, \lambda), KKT = ?$$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T A x - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T A x - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

- $\lambda = \xi + d$,

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T Ax - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

- $\lambda = \xi + d$, $Cx = d$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T A x - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

- $\lambda = \xi + d$, $Cx = d \Rightarrow \lambda = \xi + Cx$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T A x - A^T b) + C^T \lambda &= 0 \\
C x &= d
\end{aligned}
$$

- $\lambda = \xi + d$, $Cx = d \Rightarrow \lambda = \xi + Cx$
- $(A^T A x - A^T b) + C^T(\xi + d) = (A^T A x - A^T b) + C^T(\xi + Cx)$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T Ax - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

- $\lambda = \xi + d$, $Cx = d \Rightarrow \lambda = \xi + Cx$
- $(A^T Ax - A^T b) + C^T(\xi + d) = (A^T Ax - A^T b) + C^T(\xi + Cx)$
- $(A^T Ax - A^T b) + C^T(\xi + Cx) = (A^T A + C^T C)x - A^T b + C^T \xi$

QR factorization for constrained least squares

Change of variables, equivalent KKT equation

$$
\begin{aligned}
(A^T A x - A^T b) + C^T \lambda &= 0 \\
Cx &= d
\end{aligned}
$$

- $\lambda = \xi + d$, $Cx = d \Rightarrow \lambda = \xi + Cx$
- $(A^T A x - A^T b) + C^T(\xi + d) = (A^T A x - A^T b) + C^T(\xi + Cx)$
- $(A^T A x - A^T b) + C^T(\xi + Cx) = (A^T A + C^T C)x - A^T b + C^T \xi$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b.

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR$,

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR, \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR, \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR, \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR$, $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R^T R & R^T Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} R^T Q_1^T b \\ d \end{pmatrix}$$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR$, $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R^T R & R^T Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} R^T Q_1^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R & Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR, \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R^T R & R^T Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} R^T Q_1^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R & Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$$
- New variable: $y = Rx \Rightarrow$

QR factorization for constrained least squares

QR factorization of $\begin{pmatrix} A \\ C \end{pmatrix}$, equivalent KKT equation

- Assumption b. $\Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = QR, \Rightarrow \begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
- $A^T = R^T Q_1^T, C^T = R^T Q_2^T, A^T A = R^T R, C^T C = R^T R$
- $$\begin{pmatrix} A^T A + C^T C & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R^T R & R^T Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} R^T Q_1^T b \\ d \end{pmatrix}$$
- $$\begin{pmatrix} 2R & Q_2^T \\ Q_2 R & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$$
- New variable: $y = Rx \Rightarrow \begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$
- $C = Q_2 R$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$
- $C = Q_2 R \Rightarrow Q_2 = C R^{-1}$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$
- $C = Q_2 R \Rightarrow Q_2 = CR^{-1}$
- $Q_2^T \tilde{\xi} = (CR^{-1})^T \tilde{\xi} = R^{-T} C^T \tilde{\xi}$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
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- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$
- $C = Q_2 R \Rightarrow Q_2 = CR^{-1}$
- $Q_2^T \tilde{\xi} = (CR^{-1})^T \tilde{\xi} = R^{-T} C^T \tilde{\xi}$
- $\Downarrow$
- $C^T \tilde{\xi} = 0$

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists\tilde{\xi} \neq 0, Q_2^T\tilde{\xi} = 0$
- $C = Q_2 R \Rightarrow Q_2 = CR^{-1}$
- $Q_2^T\tilde{\xi} = (CR^{-1})^T\tilde{\xi} = R^{-T}C^T\tilde{\xi}$
- $\Downarrow$
- $C^T\tilde{\xi} = 0$
- C^T has linearly independent columns

QR factorization for constrained least squares

Proposition 14.5

Assumption a. is valid and Q_2 has linearly independent rows

Proof.

- Assume: Q_2 has linearly dependent rows
- $\Downarrow$
- $\exists \tilde{\xi} \neq 0, Q_2^T \tilde{\xi} = 0$
- $C = Q_2 R \Rightarrow Q_2 = C R^{-1}$
- $Q_2^T \tilde{\xi} = (C R^{-1})^T \tilde{\xi} = R^{-T} C^T \tilde{\xi}$
- $\Downarrow$
- $C^T \tilde{\xi} = 0$
- C^T has linearly independent columns
- $\Downarrow$
- Contradiction: $\xi = 0$

$\square$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
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- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$

QR factorization for constrained least squares

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- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$
- $Q_2 Q_2^T \xi = Q_2 Q_1^T b - 2d$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$
- $Q_2 Q_2^T \xi = Q_2 Q_1^T b - 2d$
- $Q_2 Q_2^T = \tilde{R}^T \tilde{R}$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$
- $Q_2 Q_2^T \xi = Q_2 Q_1^T b - 2d$
- $Q_2 Q_2^T = \tilde{R}^T \tilde{R}$
- $\tilde{R}^T \tilde{R} \xi = \tilde{R}^T \tilde{Q}^T Q_1^T b - 2d$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$
- $Q_2 Q_2^T \xi = Q_2 Q_1^T b - 2d$
- $Q_2 Q_2^T = \tilde{R}^T \tilde{R}$
- $\tilde{R}^T \tilde{R} \xi = \tilde{R}^T \tilde{Q}^T Q_1^T b - 2d$
- $$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d$$

QR factorization for constrained least squares

QR factorization of Q_2, equivalent KKT equation

- Q_2 has linearly independent rows $\Rightarrow Q_2^T = \tilde{Q}\tilde{R}$
- KKT equation $\begin{pmatrix} 2I & Q_2^T \\ Q_2 & 0 \end{pmatrix} \begin{pmatrix} y \\ \xi \end{pmatrix} = \begin{pmatrix} Q_1^T b \\ d \end{pmatrix}$
- $Q_2 y = d, 2y + Q_2^T \xi = Q_1^T b$
- $y = 0.5(Q_1^T b - Q_2^T \xi)$
- $Q_2 y = 0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi)$
- $0.5(Q_2 Q_1^T b - Q_2 Q_2^T \xi) = d$
- $Q_2 Q_2^T \xi = Q_2 Q_1^T b - 2d$
- $Q_2 Q_2^T = \tilde{R}^T \tilde{R}$
- $\tilde{R}^T \tilde{R} \xi = \tilde{R}^T \tilde{Q}^T Q_1^T b - 2d$
-
$$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d$$
- Equation for x: $Rx = y$

QR factorization for constrained least squares

Equivalent KKT equation

QR factorization for constrained least squares

Equivalent KKT equation

- KKT equation $\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$

QR factorization for constrained least squares

Equivalent KKT equation

- KKT equation $\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$
- Equivalent equaition

$$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d,$$
$$y = 0.5(Q_1^T b - Q_2^T \xi),$$
$$R\hat{x} = y$$

Algorithm 14.6

QR factorization for constrained least squares

Equivalent KKT equation

- KKT equation $\begin{pmatrix} A^TA & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^Tb \\ d \end{pmatrix}$
- Equivalent equaition

$$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d,$$

$$y = 0.5(Q_1^T b - Q_2^T \xi),$$

$$R\hat{x} = y$$

Algorithm 14.6

1. Compute QR factorization: $\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$

QR factorization for constrained least squares

Equivalent KKT equation

- KKT equation $\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$
- Equivalent equaition

$$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d,$$
$$y = 0.5(Q_1^T b - Q_2^T \xi),$$
$$R\hat{x} = y$$

Algorithm 14.6

1. Compute QR factorization: $\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
2. Compute QR factorization: $Q_2^T = \tilde{Q}\tilde{R}$

QR factorization for constrained least squares

Equivalent KKT equation

- KKT equation $\begin{pmatrix} A^T A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \lambda \end{pmatrix} = \begin{pmatrix} A^T b \\ d \end{pmatrix}$
- Equivalent equaition

$$\tilde{R}\xi = \tilde{Q}^T Q_1^T b - 2\tilde{R}^{-T} d,$$

$$y = 0.5(Q_1^T b - Q_2^T \xi),$$

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Algorithm 14.6

1. Compute QR factorization: $\begin{pmatrix} A \\ C \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} R$
2. Compute QR factorization: $Q_2^T = \tilde{Q}\tilde{R}$
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QR factorization for constrained least squares

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7. Find solution of constrained least squares problem: solve $R\hat{x} = y$

Hausholder transformation, full QR factorization

Definition 14.7

$$H = I - \frac{2vv^T}{v^T v}$$

Hausholder transformation, full QR factorization

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H - Hausholder matrix

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Hausholder transformation, full QR factorization

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Alston Hausholder (1904-1993)

American mathematician, celebrated numerical analyst, former director of Mathematics and Computer Science Division of Oak Ridge National Laboratory.

Hausholder transformation, full QR factorization

Hausholder transformation

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- v - unit vector, $\|v\|_2 = 1$, normal to hyperplane

Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

Hausholder transformation

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- Geometric interpetation

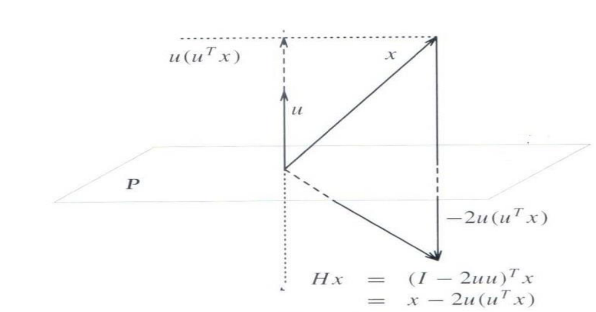
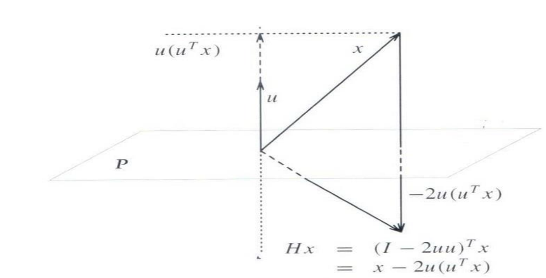


Figure: Source: Biswa Nath Datta

Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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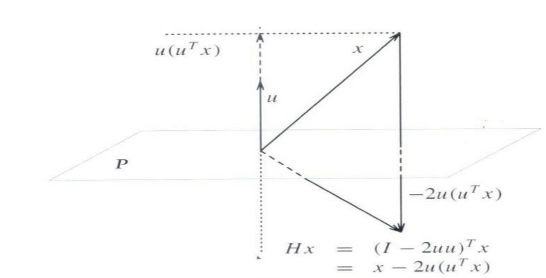

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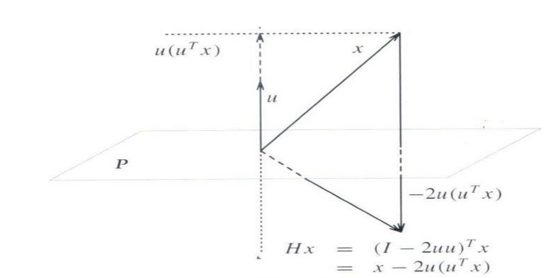
- $\|Hx\|_2 = \|x\|_2$

Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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Hausholder transformation, full QR factorization

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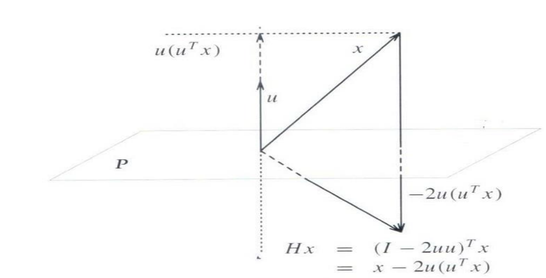

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- $\|Hx\|_2 = \|x\|_2 \Downarrow$
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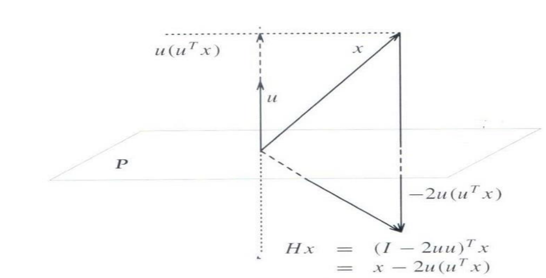
Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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Hausholder transformation, full QR factorization

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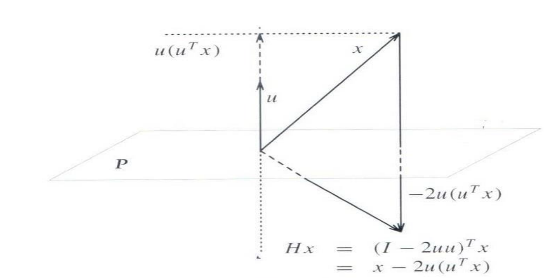

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Hausholder transformation, full QR factorization

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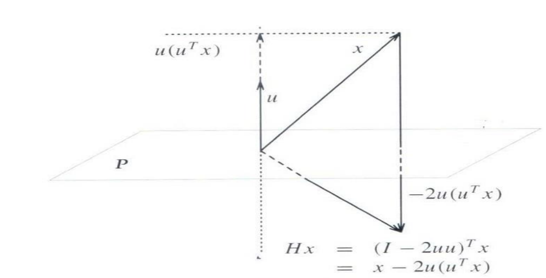

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Hausholder transformation, full QR factorization

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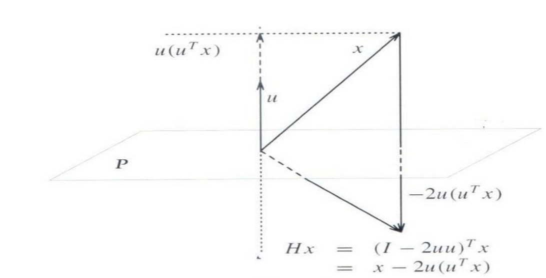

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Hausholder transformation, full QR factorization

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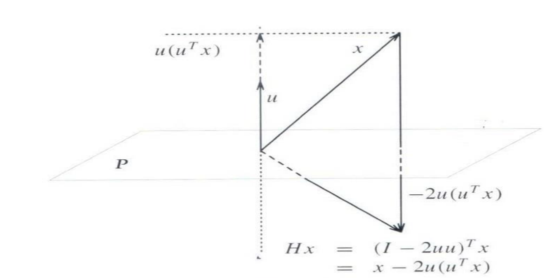

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Hausholder transformation, full QR factorization

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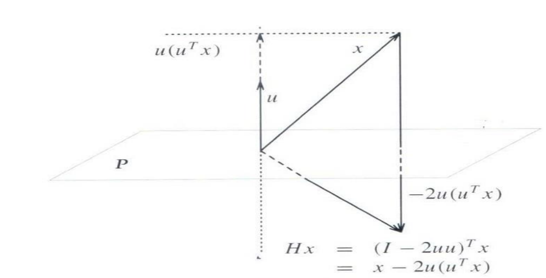

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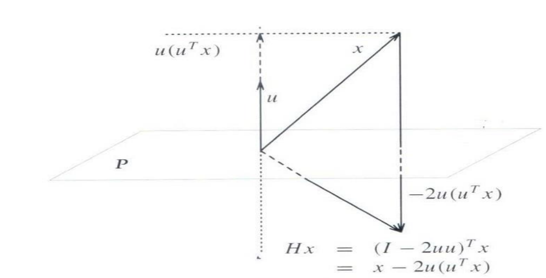
Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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Hausholder transformation, full QR factorization

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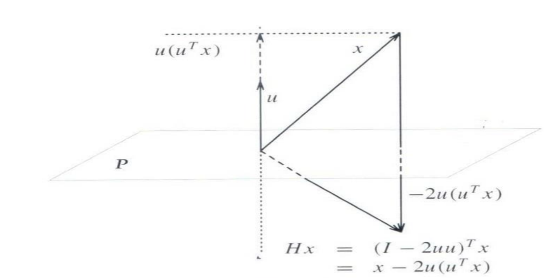

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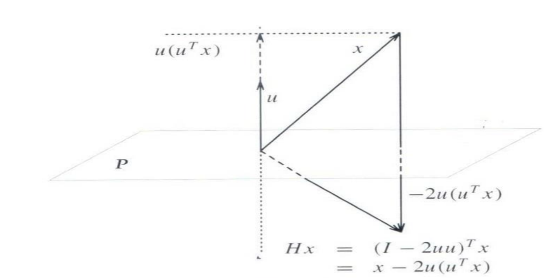
Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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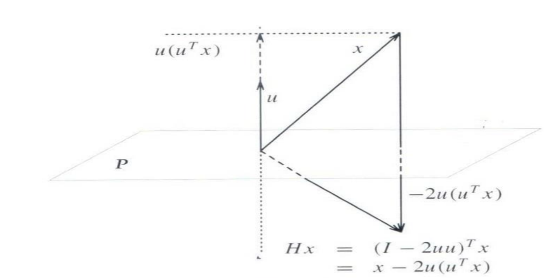

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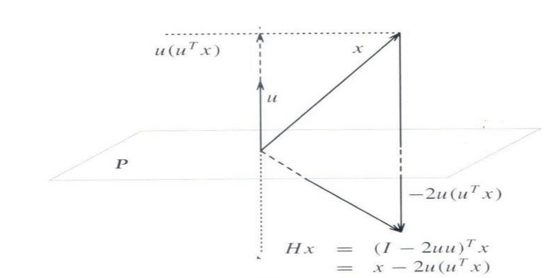

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Hausholder transformation, full QR factorization

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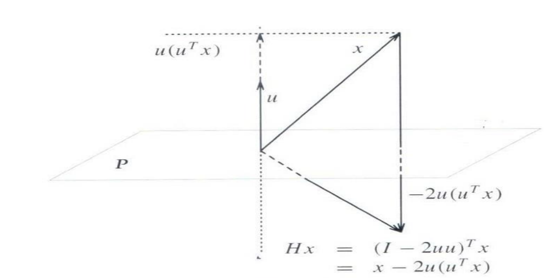

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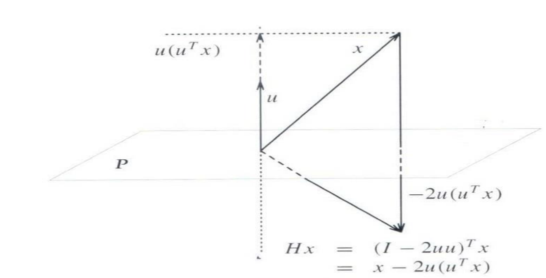
Hausholder transformation, full QR factorization

Properties of Hausholder transformation, reflection



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Hausholder transformation, full QR factorization

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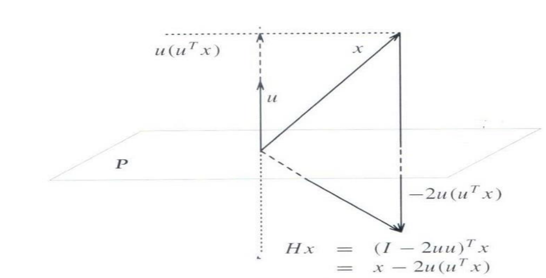

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Hausholder transformation, full QR factorization

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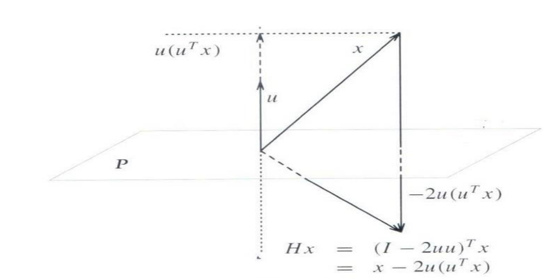

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Hausholder transformation, full QR factorization

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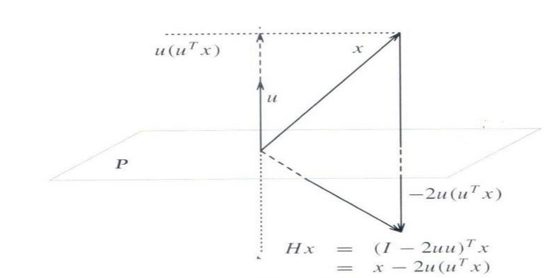

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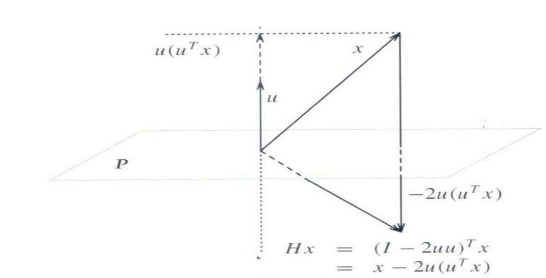

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Hausholder transformation, full QR factorization

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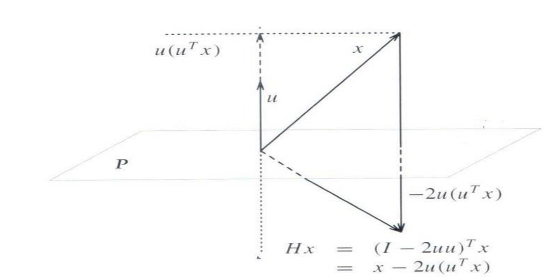

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Hausholder transformation, full QR factorization

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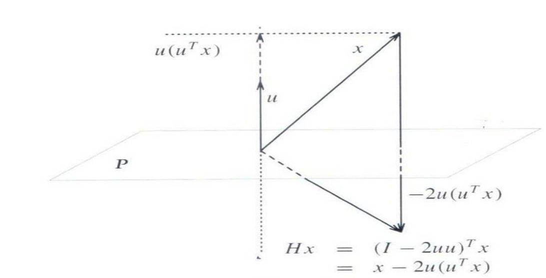

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Hausholder transformation, full QR factorization

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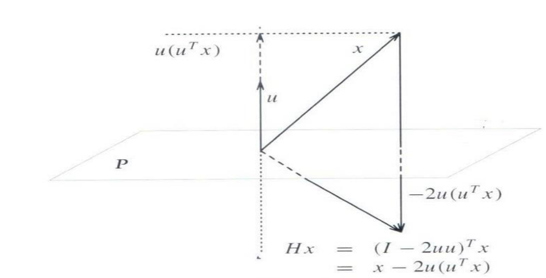

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Hausholder transformation, full QR factorization

Theorem 14.8

$v \in \mathcal{R}^n, H = I - \frac{2vv^T}{v^T v}$

Hausholder transformation, full QR factorization

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- *H is symmetric*

Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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Proof by direct verification (Exercise)

Hausholder transformation, full QR factorization

Theorem 14.9

$v \in \mathcal{R}^n, H = I - \frac{2vv^T}{v^T v}$ *(Creating zeros in a vector)*

Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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Hausholder transformation, full QR factorization

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$v \in \mathcal{R}^n, H = I - \frac{2vv^T}{v^Tv}$ *(Creating zeros in a vector)*

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Proof by direct verification (Exercise)

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Hausholder's method of QR factorization

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Q & A