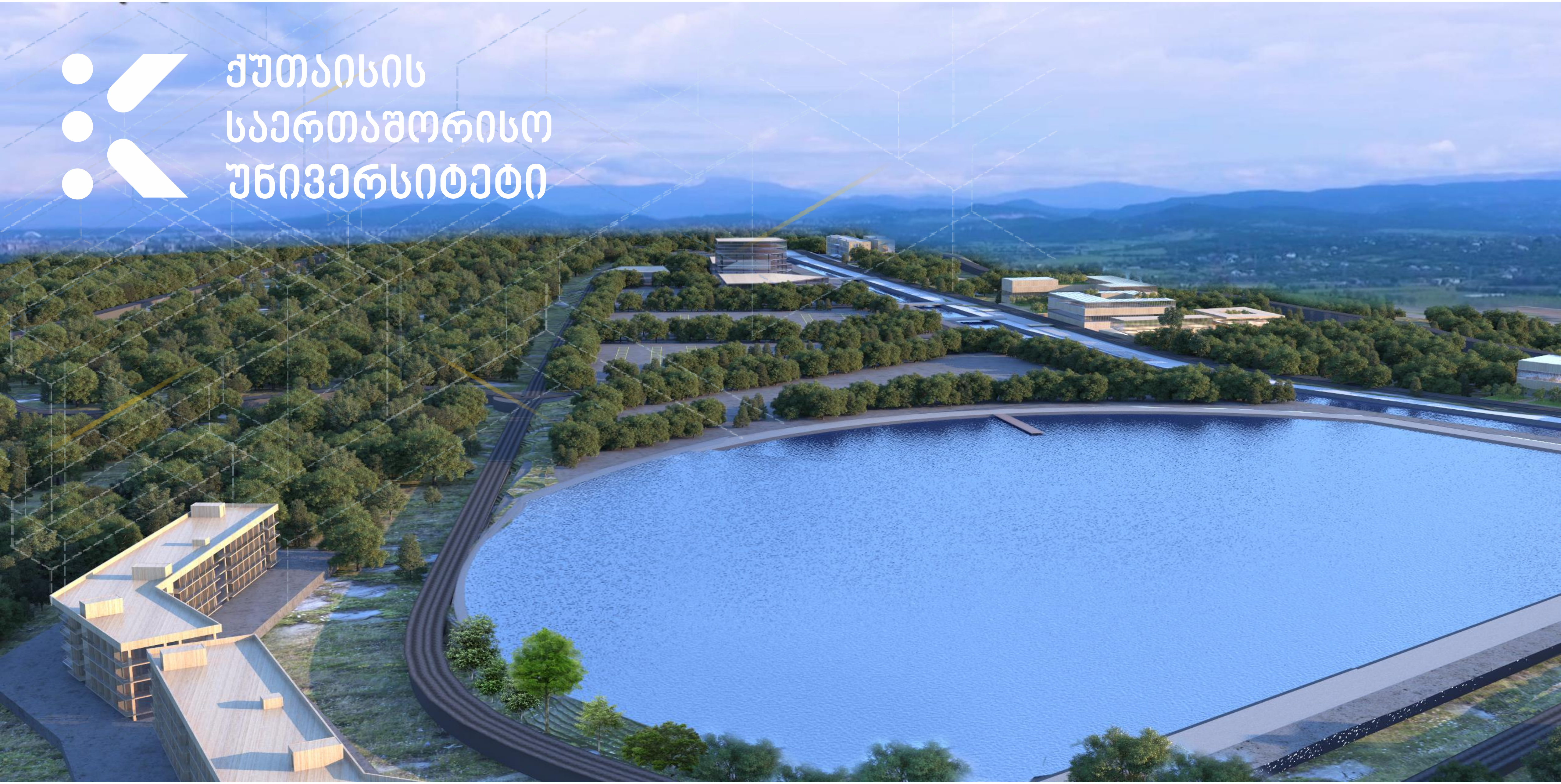
ქუთაისის საერთაშორისო უნივერსიტეტი



Properties of Limits

2.3 Calculating of Limits Using the Limit Laws

Properties of Limits (1 of 5)

In this section we use the following properties of limits, called the *Limit Laws*, to calculate limits.

Limit Laws Suppose that c is a constant and the limits $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ exist. Then

1. $\lim\limits_{x \to a}\left[f(x)+g(x)\right]=\lim\limits_{x \to a} f(x)+\lim\limits_{x \to a} g(x)$

2. $\lim\limits_{x \to a}\left[f(x)-g(x)\right]=\lim\limits_{x \to a} f(x)-\lim\limits_{x \to a} g(x)$

3. $\lim\limits_{x \to a}\left[cf(x)\right]=c\lim\limits_{x \to a} f(x)$

4. $\lim\limits_{x \to a}\left[f(x)g(x)\right]=\lim\limits_{x \to a} f(x)\cdot\lim\limits_{x \to a} g(x)$

5. $\lim\limits_{x \to a}\dfrac{f(x)}{g(x)}=\dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ if $\lim\limits_{x \to a} g(x)\neq 0$

Properties of Limits (2 of 5)

These five laws can be stated verbally as follows:

Sum Law

1. The limit of a sum is the sum of the limits.

Difference Law

2. The limit of a difference is the difference of the limits.

Constant Multiple Law

3. The limit of a constant times a function is the constant times the limit of the function.

Properties of Limits (3 of 5)

Product Law **4.** The limit of a product is the product of the limits.

Quotient Law **5.** The limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0).

For instance, if $f(x)$ is close to L and $g(x)$ is close to M, it is reasonable to conclude that $f(x) + g(x)$ is close to $L + M$.

Example 1

Use the Limit Laws and the graphs of *f* and *g* in Figure 1 to evaluate the following limits, if they exist.

(a) $\lim_{x \to -2} \left[f(x) + 5g(x) \right]$ (b) $\lim_{x \to 1} \left[f(x)g(x) \right]$ (c) $\lim_{x \to 2} \dfrac{f(x)}{g(x)}$

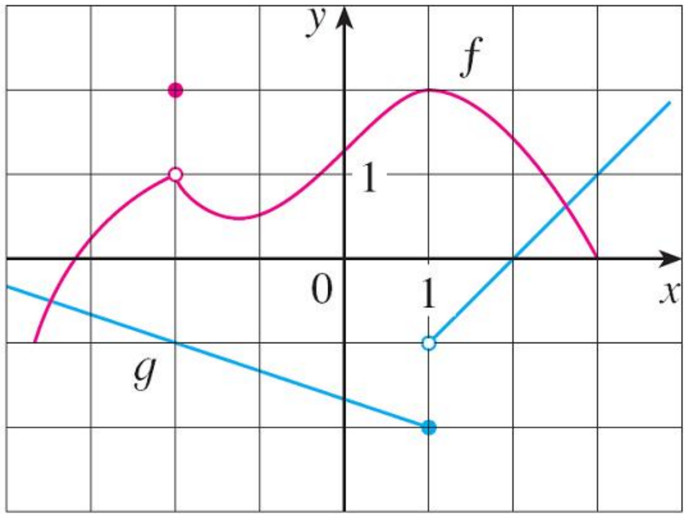


Figure 1

Example 1(a) – Solution

From the graphs of *f* and *g* we see that

$$\lim_{x\to -2} f(x) = 1 \quad \text{and} \quad \lim_{x\to -2} g(x) = -1$$

Therefore we have

$$\lim_{x\to -2}\left[f(x)+5g(x)\right] = \lim_{x\to -2} f(x) + \lim_{x\to -2}\left[5g(x)\right] \qquad \text{(by Limit Law 1)}$$

$$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x) \qquad \text{(by Limit Law 3)}$$

$$= 1 + 5(-1)$$

$$= -4$$

Example 1(b) – Solution

We see that $\lim_{x\to 1} f(x) = 2$. But $\lim_{x\to 1} g(x)$ does not exist because the left and right limits are different:

$$\lim_{x\to 1^-} g(x) = -2 \qquad \lim_{x\to 1^+} g(x) = -1$$

So we can't use Law 4 for the desired limit. But we *can* use Law 4 for the one-sided limits:

$$\lim_{x\to 1^-} \left[f(x)g(x)\right] = \lim_{x\to 1^-} f(x) \cdot \lim_{x\to 1^-} g(x) = 2\cdot(-2) = -4$$

$$\lim_{x\to 1^+} \left[f(x)g(x)\right] = \lim_{x\to 1^+} f(x) \cdot \lim_{x\to 1^+} g(x) = 2\cdot(-1) = -2$$

The left and right limits aren't equal, so $\lim_{x\to 1}\left[f(x)g(x)\right]$ does not exist.

Example 1(c) – Solution

The graphs show that

$$\lim_{x \to 2} f(x) \approx 1.4 \text{ and } \lim_{x \to 2} g(x) = 0$$

Because the limit of the denominator is 0, we can't use Law 5.

The given limit does not exist because the denominator approaches 0 while the numerator approaches a nonzero number.

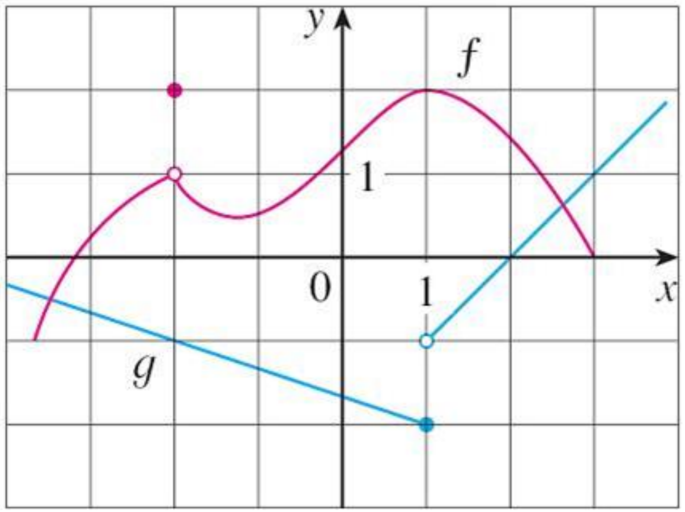


Figure 1

Properties of Limits (4 of 5)

If we use the Product Law repeatedly with $g(x) = f(x)$, we obtain the following law.

Power Law **6.** $\lim\limits_{x \to a} \left[f(x)\right]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$ where n is a positive integer

Root Law **7.** $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)}$ where n is a positive integer

[If n is even, we assume that $\lim\limits_{x \to a} f(x) > 0$.]

In applying these seven limit laws, we need to use two special limits:

8. $\lim\limits_{x \to a} c = c$ **9.** $\lim\limits_{x \to a} x = a$

Properties of Limits (5 of 5)

These limits are obvious from an intuitive point of view (state them in words or draw graphs of $y = c$ and $y = x$).

If we now put $f(x) = x$ in Law 6 and use Law 9, we get another useful special limit for power functions.

10. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

If we put $f(x) = x$ in Law 7 and use Law 9, we get a similar special limit for roots.

11. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer

(If n is even, we assume that $a > 0$.)

Evaluating Limits by Direct Substitution

Evaluating Limits by Direct Substitution (1 of 2)

Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

Functions that have the Direct Substitution Property are called *continuous at a*.

Example 3

Find

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}.$$

Solution:

Let $f(x) = (x^2 - 1) / (x - 1)$. We can’t find the limit by substituting $x = 1$ because $f(1)$ isn’t defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0.

Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$

Example 3 – Solution

The numerator and denominator have a common factor of $x - 1$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$.

Therefore we can cancel the common factor, $x - 1$, and then compute the limit by direct substitution as follows:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1}$$

$$= \lim_{x \to 1} (x+1) = 1 + 1 = 2$$

Evaluating Limits by Direct Substitution (2 of 2)

In general, we have the following useful fact.

If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.

Using One-Sided Limits

Using One-Sided Limits (1 of 1)

Some limits are best calculated by first finding the left- and right-hand limits.

The following theorem says that a two-sided limit exists if and only if both of the one-sided limits exist and are equal.

1 Theorem $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

When computing one-sided limits, we use the fact that the Limit Laws also hold for one-sided limits.

Example 7

Show that $\lim_{x \to 0} |x| = 0$.

Solution:

We know that

$$|x| = \begin{cases} x & \text{if} \quad x \geq 0 \\ -x & \text{if} \quad x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

Example 7 – Solution

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0.$$

Example 8

Prove that the limit $lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution

Recall that $|x| = x \; if \; x > 0$; $|x| = -x \; if \; x < 0$. Then we have

$$lim_{x \to 0+} \frac{|x|}{x} = lim_{x \to 0+} \frac{x}{x} = lim_{x \to 0+} 1 = 1.$$

$$lim_{x \to 0-} \frac{|x|}{x} = lim_{x \to 0-} \frac{-x}{x} = lim_{x \to 0-} (-1) = -1.$$

The Squeeze Theorem

The Squeeze Theorem (1 of 2)

The following two theorems describe how the limits of functions are related when the values of one function are greater than (or equal to) those of another.

2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

3 The Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L$$

The Squeeze Theorem (2 of 2)

The Squeeze Theorem, which is sometimes called the Sandwich Theorem or the Pinching Theorem, is illustrated by Figure 7.

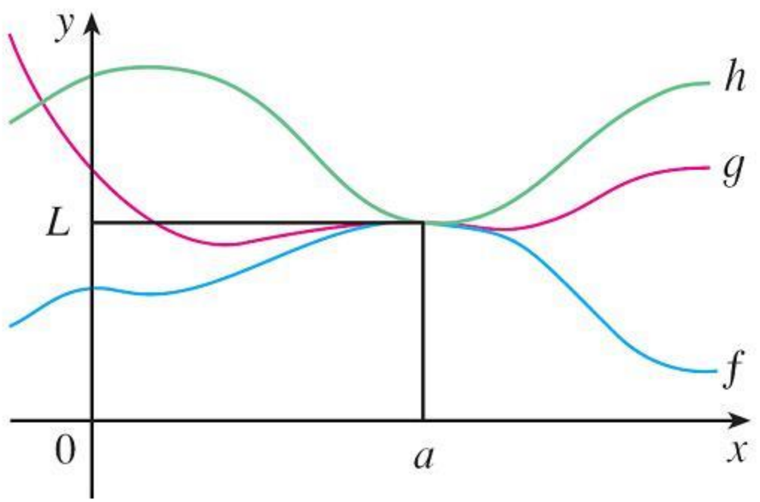


Figure 7

It says that if $g(x)$ is squeezed between $f(x)$ and $h(x)$ near a, and if f and h have the same limit L at a, then g is forced to have the same limit L at a.

Example 11

Show that

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0.$$

Solution:

First note that we **cannot** rewrite the limit as the product of the limits $\lim_{x \to 0} x^2$ and $\lim_{x \to 0} \sin(1/x)$ because $\lim_{x \to 0} \sin(1/x)$ does not exist.

We *can* find the limit by using the Squeeze Theorem.

Example 11 – Solution (1 of 3)

To apply the Squeeze Theorem we need to find a function f smaller than $g(x) = x^2 \sin(1/x)$ and a function h bigger than g such that both $f(x)$ and $h(x)$ approach 0 as $x \to 0$.

To do this we use our knowledge of the sine function. Because the sine of any number lies between −1 and 1, we can write

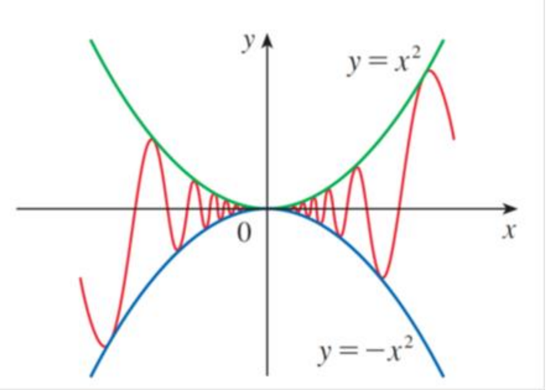
$$-1 \le \sin\frac{1}{x} \le 1 \qquad \mathbf{4}$$

Example 11 – Solution (2 of 3)

Any inequality remains true when multiplied by a positive number. We know that $x^2 \geq 0$ for all x and so, multiplying each side of the inequalities in (4) by x^2, we get

$$-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$$

as illustrated by Figure 8.



$y = x^2 \sin(1/x)$

Figure 8

Example 11 – Solution (3 of 3)

We know that

$$\lim_{x \to 0} x^2 = 0 \quad \text{and} \quad \lim_{x \to 0}(-x^2) = 0$$

Taking $f(x) = -x^2, g(x) = x^2 \sin(1/x)$, and $h(x) = x^2$ in the Squeeze Theorem, we obtain

$$\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$$