Elementary Probability Theory

probability space

Warnings:

- probability theory is mean as hell
- it is extremely easy to fool yourself
- laying precise mathematical foundations took very long
    - when is a sequence of bits ‚random‘?
    - can we define this today?
    - more in ‚theoretical computer science‘

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the good news:

- today we know about *‚probability spaces‘*
- in the simplest form topic of *this* lecture
- kind of a safety belt
- rule in my seminars in Saarbrucken
  - *if* you say ‚probability‘
  - and you cannot define the probability space you are using
  - *then* your talk is over for the day and you can repeat it at the next session

def.

discrete probability spaces

$W = (S, p)$ *probability space*, describing a random experiment

S set, finite or countable, *sample space*

$s \in S$ sample, possible outcome of the experiment

$p: S \to [0,1]$ *probability function*

$\Sigma_{s \in S} p(s) = 1$ p(s): probability that the outcome of the experiment is s

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$S = \{0,1\}$ $p(0) = p(1) = 1/2$

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example 2: throwing a dice (with a fair dice)

$S = \{1,2,3,4,5,6\}$ $\forall s: p(s) = 1/6$

Events

def.

$W = (S, p)$ — *probability space*, describing a random experiment

S — set, finite or countable, *sample space*

$s \in S$ — sample, possible outcome of the experiment

$p: S \to [0,1]$ — *probability function*

$\Sigma_{s \in S} p(s) = 1$ — p(s): probability that the outcome of the experiment is s

$A \subseteq S$ — *event*

$p(A) = \Sigma_{a \in A} p(s)$ — *probability* of A

$a \in S$ — *elementary event*

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example for dice

$A = \{1,3,5\}$ $B = \{2,4,6\}$

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conditional probability

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example for dice

$A = \{1,3,5\}$ $B = \{2,4,6\}$

$p(A) = p(B) = 1/2$

$A, B \subseteq S, p(A) \neq 0$

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$ *probability of B given A*

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$$p(\{1\}|\{1,3,5\}) = \frac{1/6}{1/2} = 1/3$$

$$p(\{2,4,6\}|\{1,3,5\}) = \frac{0}{1/2} = 0$$

independent events

$A, B \subseteq S$ independent iff $p(A \cap B) = p(A) \cdot p(B)$

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single coin flip or single throw of dice:
too simple for example

product of probability spaces

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set of outcomes for the pair of events

$$S = S_1 \times S_2 = \{(a, b) | a \in S_1, b \in S_2\}$$

$$p(a, b) = p_1(a) \cdot p_2(b)$$

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examples

coin twice

$$S = \{(0,0), (0,1), (1,0), (1,1)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/2 \cdot 1/2 = 1/4$$

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dice twice

$$S = \{(1,1), \dots, (1,6), \dots, (6,1), \dots, (6,6)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/6 \cdot 1/6 = 36$$

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coin and dice

$$S = \{(0,1), \dots, (0,6), (1,1), \dots, (1,6)\}$$

$$p(a, b) = p_1(a) \cdot p_2(b) = 1/2 \cdot 1/6 = 1/12$$

product of probability spaces

$A, B \subseteq S$ independent iff $p(A \cap B) = p(A) \cdot p(B)$

Lemma 1. *$W_1 \times W_2$ is a probability space*

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$$p(a,b) = p_1(a) \cdot p_2(b)$$

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Proof.

$$\begin{aligned} \sum_{(a,b)\in S} p(a,b) &= \sum_{(a,b)\in S_1\times S_2} p_1(a)\cdot p_2(b) \\ &= \sum_{a\in S_1}\sum_{b\in S_2} p_1(a)\cdot p_2(b) \\ &= \sum_{a\in S_1} p_1(a)\cdot (\sum_{b\in S_2} p_2(b)) \\ &= \sum_{a\in S_1} p_1(a)\cdot 1 \\ &= 1 \end{aligned}$$

embedding events of single spaces in the product space

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$A \subseteq S_1, B \subseteq S_2$ events of single experiments

$e_1(A) = A \times S_2, e_2(B) = S_1 \times B$ embedding into S

outcome of the other event does not matter

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example: dice twice

$A = \{1\}$ $\quad en_1(A) = \{(1,1), \dots (1,6)\}$

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Lemma 2. *Embedded events have the probability of the original events in the original space.*

$$p(e_1(A)) = p_1(A) \quad , \quad p(e_2(B)) = p_2(B)$$

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$$p(e_1(A)) = p_1(A) \quad , \quad p(e_2(B)) = p_2(B)$$

proof:

$$\begin{aligned}
p(e_1(A)) &= \sum_{(a,b) \in A \times S_2} p_1(a) \cdot p_2(b) \\
&= \sum_{a \in A} \sum_{b \in S_2} p_1(a) \cdot p_2(b) \\
&= \sum_{a \in A} p_1(a) \cdot (\sum_{b \in S_2} p_2(b)) \\
&= \sum_{a \in A} p_1(a) \cdot 1 \\
&= p_1(A)
\end{aligned}$$

other case: similar

embedding events of single spaces in the product space

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$e_1(A) = A \times S_2, e_2(B) = S_1 \times B$ embedding into S

Lemma 3. *Embedded events from different probability spaces are independent.*

$$p(e_1(A)) \cap e_2(B)) = p(e_1(A)) \cdot p(e_2(B))$$

proof:

$$\begin{aligned}
e_1(A) \times e_2(B) &= (A \times S_2) \cap (S_1 \times B) \\
&= A \times B \\
p(e_1(A)) \cap e_2(B)) &= \sum_{(a,b) \in A \times B} p_1(a) \cdot p_2(b) \\
&= \sum_{a \in A} \sum_{b \in B} p_1(a) \cdot p_2(b) \\
&= \sum_{a \in A} p_1(a) \cdot (\sum_{b \in B} p_2(b)) \\
&= \sum_{a \in A} p_1(a) \cdot p_2(B) \\
&= p_1(A) \cdot p_2(B) \\
&= p(e_1(A)) \cdot p(e_2(B)) \quad \text{(lemma 2)}
\end{aligned}$$

n independent experiments

$W_i = (S_i, p_i)$ $\qquad$ $i \in \{1, \dots, n\}$

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Lemma 4. *$S_1 \times \ldots \times S_n$ is a probability space.*

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embedding events for single experiments

$A_i \subseteq S_i \qquad i \in \{1, \dots, n\}$

$e_i(A_i) = S_1 \times \cdots S_{i-1} \times A_i \times S_{i+1} \times \cdots \times S_n \subseteq S$

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Lemma 5. *Embedded events $e_i(A_i)$ have the probability of the original events A_i in the original space.*

$$p(e_i(A_i)) = p_i(A_i)$$

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$B_1, \dots, B_s \subseteq S$ $\quad\quad$ events

are *mutually independent* iff

$\forall K \subseteq \{1, \dots, s\}$ for all subsets of set of indices, resp. B's

$$p(\bigcap_{i \in K} B_i) = \Pi_{i \in K} p(B_i)$$

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Lemma 6. *The embedded events*

$$e_1(A_1), \dots, e_n(A_n)$$

are mutually independent .

random variables and their expected value

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

the real numbers

simplest case:

- elementary events are numbers
- X is identity

$S \subset \mathbb{R}$

$X(a) = a$

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$X(a) = c$

expected value of random variable X

$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$

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examples

$X(a) = c$ X is constant

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coin $X(a) = a$

$$E(X) = \Sigma_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

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X is constant

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$$E(X) = \Sigma_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

dice $X(a) = a$

$$E(X) = \Sigma_{a \in \{1,\ldots,6\}} a \cdot 1/6 = 3 \cdot (1+6)/6 = 7/2$$

random variables and their expected value

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

the real numbers

simplest case:

- elementary events are numbers
- X is identity
- or X is constant

$S \subset \mathbb{R}$

$X(a) = a$
$X(a) = c$

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

examples

$X(a) = c$ X is constant

$$E(X) = \Sigma_{a \in S} c \cdot p(a) = c \cdot \Sigma_{a \in S} p(a) = c$$

coin $X(a) = a$

$$E(X) = \Sigma_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2 \quad \notin S$$

dice $X(a) = a$

$$E(X) = \Sigma_{a \in \{1,\ldots,6\}} a \cdot 1/6 = 3 \cdot (1+6)/6 = 7/2 \quad \notin S$$

linearity

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expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

Lemma 7. *If $X, Y : S \to \mathbb{R}$ are random variables and $c \in \mathbb{R}$ is a constant, then*

- $$E(c \cdot X) = c \cdot E(X)$$
- $$E(X + Y) = E(X) + E(Y)$$

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Proof.

$$\begin{aligned} E(c \cdot X) &= \Sigma_{a \in S} c \cdot X(a) \cdot p(a) \\ &= c \cdot (\Sigma_{a \in S} X(a) \cdot p(a)) \\ &= c \cdot E(X) \end{aligned}$$

$$\begin{aligned} E(X+Y) &= \Sigma_{a \in S}(X(a) + Y(a)) \cdot p(a) \\ &= \Sigma_{a \in S} X(a) \cdot p(a) + \Sigma_{a \in S} X(a) \cdot p(a) \\ &= E(X) + E(Y) \end{aligned}$$

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by induction:

Lemma 8. *Let $X_i : S \to \mathbb{R}$, $i \in \{1, \dots, n\}$ be random variables. Then*

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$$

random variables from different independent experiments

$W = (S, p)$ probability space

$X : S \to \mathbb{R}$ random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

random variables from different independent experiments

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expected value of random variable X

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For $i \in \{1,2\}$ let

$$W_i = (S_i, p_i)$$

be probability spaces and let

$$X_i : S_i \to \mathbb{R}$$

be random variables in these spaces.

$$X : S_1 \times S_2 \to \mathbb{R} \quad , \quad X(a,b) = X_1(a) + X_2(b)$$

random variables from different independent experiments

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$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

Lemma 9.

For $i \in \{1,2\}$ let

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$$E(X) = E(X_1) + E(X_2)$$

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Lemma 9.

$$E(X) = E(X_1) + E(X_2)$$

$$\begin{aligned}
E(X) &= \sum_{(a,b) \in S_1 \times S_2} p(a,b) \cdot X(a,b) \\
&= \sum_{a \in S_1} \sum_{b \in S_2} p_1(a) \cdot p_2(b) \cdot (X_1(a) + X_2(b)) \\
&= \sum_{a \in S_1} p_1(a) \cdot (\sum_{b \in S_2} p_2(b) \cdot (X_1(a) + X_2(b)) \\
&= \sum_{a \in S_1} p_1(a) \cdot X_1(a) \cdot (\sum_{b \in S_2} p_2(b)) + \sum_{a \in S_1} p_1(a) \cdot (\sum_{b \in S_2} p_2(b) \cdot X_2(b)) \\
&= \sum_{a \in S_1} p_1(a) \cdot X_1(a) \cdot 1 + \sum_{a \in S_1} p_1(a) \cdot E(X_2) \\
&= E(X_1) + E(X_2)
\end{aligned}$$