Prof. Dr. Dr. h.c. Florian Rupp
Akaki Matcharashvili

Optimality Conditions for Unrestricted Optimization & Convexity

This exercise sheet consists of three parts: at first problems for the Additional/ Central Exercise Problems class are given. Their solution will be provided and can serve you as further blueprints when solving similar tasks, e.g. for the homework assignment. Then, the actual Homework Assignments are stated that will be discussed during the TTF in the following week. Please, hand-in your results of these assignments through MSTeams at the date and time specified in MSTeams. Finally, the third part consists of Graded Homework Assignments that will be corrected and contribute to the continuous assessment of our course. Please, hand-in your results of these assignments as well through MSTeams at the date and time specified in MSTeams.

Additional/ Central Exercise Problems:

Exercise 2.1: An Existence Theorem for Stationary Points

a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show: If $\overline{x}$ is a local but not a global minimum of f, then f possesses at least one additional stationary point $\hat{x}$ besides $\overline{x}$, i.e. at least one point $\hat{x} \neq \overline{x}$ with $f'(\hat{x}) = 0$.

b) Compute the stationary points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ with

$$f(x, y) \ := \ \mathrm{e}^{3y} - 3x\mathrm{e}^{y} + x^3 \,,$$

and determine, whether the computed points are local minima, local maxima, global minima, global maxima or saddle points. What do these results imply for the statement of a)?

Exercise 2.2: Sufficient Condition for a Saddle Point Recall, a stationary point is called a saddle point, if it is neither a local minimum nor a local maximum point.

Let $f : U \to \mathbb{R}$ be a function defined in an open set $U \subseteq \mathbb{R}^n$. Suppose that f is twice continuously differentiable over U and that x^* is a stationary point. Show that if $H_f(x^*)$ is an indefinite matrix, then x^* is a saddle point of f over U.

Exercise 2.3: Examples Regarding Convexity

a) Justify that the sets $\mathbb{R}^n$ and $\{x \in \mathbb{R}^n \, : \, \|x\| \leq 1\}$ are convex. Then argue that affine-linear functions $f : \mathbb{R}^n \to \mathbb{R}$, $x \mapsto a^T x + b$, with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$ are convex. (This especially implies: constant functions are convex)

b) In the lecture you have seen that each local minimum of a convex function is a global minimum Does every strictly convex function have such a (local = global) minimum? State a short proof of your answer.

Homework Assignment:

Problem 2.1: Stationary Points and their Classification — For each of the following functions on $\mathbb{R}^2$ or $\mathbb{R}^3$, respectively, find all the stationary points and classify them according to whether they are saddle points, strict/ non-strict local/ global minimum/ maximum points:

a) $f(x_1, x_2) = (4x_1^2 - x_2)^2$.

b) $f(x_1, x_2, x_3) = x_1^4 - 2x_1^2 + x_2^2 + 2x_2x_3 + 2x_3^2$.

c) $f(x_1, x_2) = 2x_2^3 - 6x_2^2 + 3x_1^2 x_2$.

d) $f(x_1, x_2) = x_1^4 + 2x_1^2 x_2 + x_2^2 - 4x_1^2 - 8x_1 - 8x_2$.

Problem 2.2: Level Sets of Convex Functions — Show that for convex functions $f : \mathbb{R}^n \to \mathbb{R}$ the level sets are convex as well. Does the converse also hold true?

Problem 2.3: Composition of Convex Functions — Let $g : X \to \mathbb{R}$ be convex function on a convex set $X \subset \mathbb{R}^n$. Let $I \subset \mathbb{R}$ be an interval with $g(X) \subset I$ and $f : I \to \mathbb{R}$ be a convex and monotonously increasing function.

a) Show that the function $f \circ g : x \in X \mapsto f(g(x)) \in \mathbb{R}$ is convex.

b) Give an example that illustrates that, in general, one cannot waive the condition on the monotonous increase of f.

Problem 2.4: Set of Optimal Solutions of Convex Problems — Let $f : K \to \mathbb{R}$ be a convex function on the non-empty convex set $K \subset \mathbb{R}^n$ and consider the convex optimization problem

$$\min_{x \in X} f(x) \,.$$

Show that the set

$$\{\, x \in K \,:\, f(x) \,\leq\, f(y) \quad \text{for all } y \in K \,\}$$

of optimal solutions of this convex optimization problem is a convex set.

Graded Homework Assignment:

Graded Problem II.1: Stationary Points — Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x_1, x_2) = x_1^2 - 5x_1 x_2^2 + 5x_2^4$.

a) Determine all stationary points of f.

b) Show that $\overline{x}_1 = 0$ is a strict global minimum of the restricted function $x_1 \to f(x_1, 0)$ and that $\overline{x}_2 = 0$ is a strict global minimum of the restricted function $x_2 \to f(0, x_2)$

c) Is $\overline{x} = 0$ a local minimum of f? If not, is $\overline{x} = 0$ a saddle point of f?

Graded Problem II.2: Properties of Quadratic Functions — Let $f : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = x^T Ax + 2b^T x + c$, where $A \in \mathbb{R}^{n \times n}$ is symmetric, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Suppose that A is positive semi-definite. Show that f is bounded below over $\mathbb{R}^n$ if and only if $b \in \{Ay : y \in \mathbb{R}^n\}$, i.e. b is in the range of A.

Note: A function f is bounded below over a set C if there exists a constant α such that $f(x) \geq \alpha$ for all $x \in C$.

Graded Problem II.3: Convex Sets — Determine whether the following sets are convex or not. Explain your answer.

1. $A = \{x \in \mathbb{R}^n : \|x\|^2 = 1\}$
2. $B = \{x \in \mathbb{R}^n : \max_{i=1,2,\ldots,n} x_i \leq 1\}$
3. $C = \{x \in \mathbb{R}^n : \min_{i=1,2,\ldots,n} x_i \leq 1\}$

Graded Problem II.4: Convex Functions — Let $f : \mathbb{R}^n \to \mathbb{R}$. Prove that f is convex if and only if for any $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, $d \neq 0$, the one-dimensional function $g_{x,d}(t) = f(x + td)$ is convex.