Elementary Probability Theory

Warnings:

- probability theory is mean as hell
- it is extremely easy to fool yourself
- laying precise mathematical foundations took very long
 - when is a sequence of bits ,random'?
 - can we define this today?
 - more in ,theoretical computer science'

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the good news:

- today we know about ,probability spaces'
- in the simplest form topic of this lecture
- kind of a safety belt
- rule in my seminars in Saarbrucken
 - if you say ,probability'
 - and you cannot define the probability space you are using
 - then your talk is over for the day and you can repeat it at the next session

W = (S, p)

probability space, describing a random experiment

S

set, finite or countable, sample space

 $s \in S$

sample, possible outcome of the experiment

 $p: S \rightarrow [0,1]$

probability function

 $\Sigma_{s \in S} p(s) = 1$

p(s): probability that the outcome of the experiment is s

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example 1: coin flip (with unbiased coin)

$$S = \{0,1\}$$
 $p(0) = p(1) = 1/2$

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example 2: throwing a dice (with a fair dice)

$$S = \{1,2,3,4,5,6\}$$
 $\forall s: p(s) = 1/6$

def. Events

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$$A \subseteq S$$
 event $p(A) = \sum_{a \in A} p(s)$ probability of A $p(A) = \sum_{a \in A} p(s)$ elementary event

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example for dice

$$A = \{1,3,5\}$$
 $B = \{2,4,6\}$ $p(A) = p(B) = 1/2$

def.

conditional probability

$$W = (S, p)$$

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event

$$p(A) = \Sigma_{a \in A} p(s)$$

probability of A

$$a \in S$$

elementary event

example for dice

$$A = \{1,3,5\}$$
 $B = \{2,4,6\}$

$$S = \{2,4,6\}$$

$$p(A) = p(B) = 1/2$$

$$A, B \subseteq S, p(A) \neq 0$$

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$

probability of B given A

def.

conditional probability

$$W = (S, p)$$

probability space, describing a random experiment

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example 2: throwing a dice (with a fair dice)

$$S = \{1,2,3,4,5,6\}$$

$$\forall s : p(s) = 1/6$$

$$A \subseteq S$$

event

$$p(A) = \Sigma_{\alpha \in A} p(s)$$

probability of A

$$a \in S$$

elementary event

example for dice

$$A = \{1,3,5\}$$
 $B = \{2,4,6\}$

$$3 = \{2,4,6\}$$

$$p(A) = p(B) = 1/2$$

$$A, B \subseteq S, p(A) \neq 0$$

$$p(B|A) = \frac{p(B \cap A)}{p(A)}$$

probability of B given A

$$p(\{1\}|\{1,3,5\}) = \frac{1/6}{1/2} = 1/3$$

$$p(\{2,4,6\}|\{1,3,5\}) = \frac{0}{1/2} = 0$$

independent events

$$A, B \subseteq S$$
 independent iff $p(A \cap B) = p(A) \cdot p(B)$

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single coin flip or single throw of dice: too simple for example

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let's consider two experiments

$$W_1 = (S_1, p_1), W_2 = (S_2, p_2)$$

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single coin flip or single throw of dice: too simple for example

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$$W_1 = (S_1, p_1), W_2 = (S_2, p_2)$$

set of outcomes for the pair of events

$$S = S_1 \times S_2 = \{(a, b) | a \in S_1, b \in S_2\}$$

$$p(a,b) = p_1(a) \cdot p_2(b)$$

$$W = W_1 \times W_2 = (S, p)$$

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examples

coin twice

$$S = \{0,0\}, (0,1), (1,0), (1,1)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/2 \cdot 1/2 = 1/4$$

$$A, B \subseteq S$$
 independent iff

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examples

coin twice

$$S = \{(0,0), (0,1), (1,0), (1,1)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/2 \cdot 1/2 = 1/4$$

dice twice

$$S = \{(1,1), \dots, (1,6), \dots, (6,1), \dots, (6,6)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/6 \cdot 1/6 = 36$$

$$A, B \subseteq S$$
 independent iff

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examples

coin twice

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$$S = \{(1,1), \dots, (1,6), \dots, (6,1), \dots, (6,6)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/6 \cdot 1/6 = 1/36$$

coin and dice

$$S = \{(0,1), \dots, (0,6), (1,1), \dots, (1,6)\}$$

$$p(a,b) = p_1(a) \cdot p_2(b) = 1/2 \cdot 1/6 = 1/12$$

$$A, B \subseteq S$$
 independent iff

$$p(A \cap B) = p(A) \cdot p(B)$$

Lemma 1. $W_1 \times W_2$ is a probability space

single coin flip or single throw of dice: too simple for example

let's consider two experiments

$$W_1 = (S_1, p_1), W_2 = (S_2, p_2)$$

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$$p(a,b) = p_1(a) \cdot p_2(b)$$

$$W = W_1 \times W_2 = (S, p)$$

Proof.

$$\sum_{(a,b)\in S} p(a,b) = \sum_{(a,b)\in S_1\times S_2} p_1(a) \cdot p_2(b)
= \sum_{a\in S_1} \sum_{b\in S_2} p_1(a) \cdot p_2(b)
= \sum_{a\in S_1} p_1(a) \cdot (\sum_{b\in S_2} p_2(b))
= \sum_{a\in S_1} p_1(a) \cdot 1
= 1$$

$$A, B \subseteq S$$
 independent iff

$$p(A \cap B) = p(A) \cdot p(B)$$

single coin flip or single throw of dice: too simple for example

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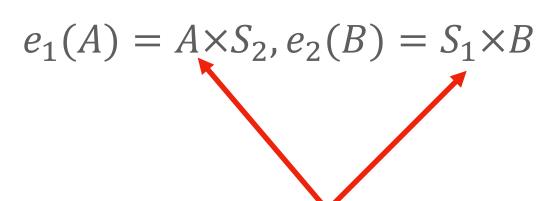
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$$p(a,b) = p_1(a) \cdot p_2(b)$$

$$W = W_1 \times W_2 = (S, p)$$

 $A \subseteq S_1, B \subseteq S_2$

events of single experiments



embedding into S

outcome of the other event does not matter

$$A, B \subseteq S$$
 independent iff

$$p(A \cap B) = p(A) \cdot p(B)$$

single coin flip or single throw of dice: too simple for example

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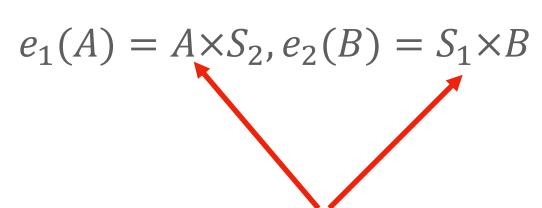
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events of single experiments



embedding into S

outcome of the other event does not matter

example: dice twice

$$A = \{1\}$$
 $en_1(A) = \{(1,1), ... (1,6)\}$

$$A, B \subseteq S$$
 independent iff

$$p(A \cap B) = p(A) \cdot p(B)$$

single coin flip or single throw of dice: too simple for example

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$$A \subseteq S_1, B \subseteq S_2$$

events of single experiments

$$e_1(A) = A \times S_2, e_2(B) = S_1 \times B$$

embedding into S

Lemma 2. Embedded events have the probability of the original events in the original space.

$$p(e_1(A)) = p_1(A)$$
 , $p(e_2(B)) = p_2(B)$

$$A, B \subseteq S$$
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$$p(e_1(A)) = p_1(A)$$
 , $p(e_2(B)) = p_2(B)$

proof:

$$p(e_{1}(A)) = \sum_{(a,b)\in A\times S_{2}} p_{1}(a) \cdot p_{2}(b)$$

$$= \sum_{a\in A} \sum_{b\in S_{2}} p_{1}(a) \cdot p_{2}(b)$$

$$= \sum_{a\in A} p_{1}(a) \cdot (\sum_{b\in S_{2}} p_{2}(b))$$

$$= \sum_{a\in A} p_{1}(a) \cdot 1$$

$$= p_{1}(A)$$

other case: similar

$$A, B \subseteq S$$
 independent iff

$$p(A \cap B) = p(A) \cdot p(B)$$

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$$A \subseteq S_1, B \subseteq S_2$$

events of single experiments

$$e_1(A) = A \times S_2, e_2(B) = S_1 \times B$$

embedding into S

Lemma 3. Embedded events from different probability spaces are independent.

$$p(e_1(A)) \cap e_2(B)) = p(e_1(A)) \cdot p(e_2(B))$$

proof:

$$e_{1}(A) \times e_{2}(B) = (A \times S_{2}) \cap (S_{1} \times B)$$

$$= A \times B$$

$$p(e_{1}(A)) \cap e_{2}(B)) = \sum_{(a,b) \in A \times B} p_{1}(a) \cdot p_{2}(b)$$

$$= \sum_{a \in A} \sum_{b \in B} p_{1}(a) \cdot p_{2}(b)$$

$$= \sum_{a \in A} p_{1}(a) \cdot (\sum_{b \in B} p_{2}(b))$$

$$= \sum_{a \in A} p_{1}(a) \cdot p_{2}(B)$$

$$= p_{1}(A) \cdot p_{2}(B)$$

$$= p(e_{1}(A)) \cdot p(e_{2}(B)) \text{ (lemma 2)}$$

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$$W = W_1 \times \cdots \times W_n = (S, p)$$

$$S = S_1 \times \dots \times S_n \qquad p(a_1, \dots, a_n) = p_1(a_1) \cdot \dots, p_n(a_n)$$

$$W_i = (S_i, p_i) \qquad i \in \{1, ..., n\}$$

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$$S = S_1 \times \dots \times S_n \qquad p(a_1, ..., a_n) = p_1(a_1) \cdot ..., p_n(a_n)$$

Lemma 4. $S_1 \times ... \times S_n$ is a probability space.

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embedding events for single experiments

$$A_i \subseteq S_i \qquad i \in \{1, \dots, n\}$$

$$e_i(A_i) = S_1 \times \cdots S_{i-1} \times A_i \times S_{i+1} \times \cdots \times S_n \subseteq S$$

$$W_i = (S_i, p_i) \qquad i \in \{1, ..., n\}$$

$$W = W_1 \times \cdots \times W_n = (S, p)$$

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Lemma 5. Embedded events $e_i(A_i)$ have the probability of the original events A_i in the original space.

$$p(e_i(A_i)) = p_i(A_i)$$

$$W_i = (S_i, p_i) \qquad i \in \{1, \dots, n\}$$

$$W = W_1 \times \cdots \times W_n = (S, p)$$

$$S = S_1 \times \dots \times S_n \qquad p(a_1, \dots, a_n) = p_1(a_1) \cdot \dots, p_n(a_n)$$

Lemma 4. $S_1 \times ... \times S_n$ is a probability space.

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Lemma 5. Embedded events $e_i(A_i)$ have the probability of the original events A_i in the original space.

$$p(e_i(A_i)) = p_i(A_i)$$

$$B_1, \dots, B_S \subseteq S$$
 events

are mutually independent iff

 $\forall K \subseteq \{1, ..., s\}$ for all subsets of set of indices, resp. B's

$$p(\bigcap_{i\in K}B_i)=\Pi_{i\in K}p(B_i)$$

$$W_i = (S_i, p_i) \qquad i \in \{1, \dots, n\}$$

$$i \in \{1, \dots, n\}$$

$$W = W_1 \times \cdots \times W_n = (S, p)$$

$$S = S_1 \times \dots \times S_n \qquad p(a_1, \dots, a_n) = p_1(a_1) \cdot \dots, p_n(a_n)$$

Lemma 4. $S_1 \times ... \times S_n$ is a probability space.

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Lemma 6. The embedded events

$$e_1(A_1),\ldots,e_n(A_n)$$

are mutually independent.

$$W = (S, p)$$
 pro

probability space

 $X: S \to \mathbb{R}$

random variable

the real numbers

simplest case:

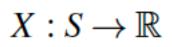
- elementary events are numbers
- X is identity

$$S \subset \mathbb{R}$$

$$X(a) = a$$

$$W = (S, p)$$

probability space



random variable

the real numbers

simplest case:

- elementary events are numbers
- X is identity
- or X is constant

 $S \subset \mathbb{R}$

$$X(a) = a$$
$$X(a) = c$$

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

$$W = (S, p)$$

probability space

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expected value of random variable X

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examples

$$X(a) = c$$

X is constant

$$E(X) = \Sigma_{a \in S} c \cdot p(a) = c \cdot \Sigma_{a \in S} p(a) = c$$

$$W = (S, p)$$

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examples

$$X(a) = c$$

X is constant

$$E(X) = \Sigma_{a \in S} c \cdot p(a) = c \cdot \Sigma_{a \in S} p(a) = c$$

$$X(a) = a$$

$$E(X) = \sum_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

$$W = (S, p)$$

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$$E(X) = \Sigma_{a \in S} c \cdot p(a) = c \cdot \Sigma_{a \in S} p(a) = c$$

coin

$$X(a) = a$$

$$E(X) = \sum_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2$$

dice

$$X(a) = a$$

$$E(X) = \sum_{a \in \{1,\dots,6\}} a \cdot 1/6 = 3 \cdot (1+6)/6 = 7/2$$

$$W = (S, p)$$

probability space

 $X:S\to\mathbb{R}$

random variable

the real numbers

simplest case:

- elementary events are numbers
- X is identity
- or X is constant

 $S \subset \mathbb{R}$

$$X(a) = a$$

 $X(a) = c$

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

examples

$$X(a) = c$$

X is constant

$$E(X) = \Sigma_{a \in S} c \cdot p(a) = c \cdot \Sigma_{a \in S} p(a) = c$$

coin

$$X(a) = a$$

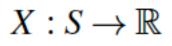
$$E(X) = \sum_{a \in \{0,1\}} a \cdot 1/2 = 0 \cdot 1/2 + 1 \cdot 1/2 = 1/2 \notin S$$

dice

$$X(a) = a$$

$$E(X) = \sum_{a \in \{1,\dots,6\}} a \cdot 1/6 = 3 \cdot (1+6)/6 = 7/2$$
 $\notin S$

$$W = (S, p)$$



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linearity

Lemma 7. If $X,Y:S\to\mathbb{R}$ are random variables and $c\in\mathbb{R}$ is a constant, then

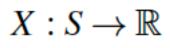
•

$$E(c \cdot X) = c \cdot E(X)$$

•

$$E(X+Y) = E(X) + E(Y)$$

$$W = (S, p)$$



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Proof.

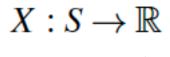
$$E(c \cdot X) = \sum_{a \in S} c \cdot X(a) \cdot p(a)$$
$$= c \cdot (\sum_{a \in S} X(a) \cdot p(a))$$
$$= c \cdot E(X)$$

$$E(X+Y) = \sum_{a \in S} (X(a) + Y(a)) \cdot p(a)$$

$$= \sum_{a \in S} X(a) \cdot p(a) + \sum_{a \in S} X(a) \cdot p(a)$$

$$= E(X) + E(Y)$$

$$W = (S, p)$$



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•

$$E(X+Y) = E(X) + E(Y)$$

by induction:

Lemma 8. Let $X_i: S \to \mathbb{R}$, $i \in \{1, ..., n\}$ be random variables. Then

$$E(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} E(X_i)$$

$$W = (S, p)$$
 probability space

$$X: S \to \mathbb{R}$$
 random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

$$W = (S, p)$$
 probability space

 $X: S \to \mathbb{R}$ random variable

expected value of random variable X

$$E(X) = \Sigma_{a \in S} X(a) \cdot p(a)$$

For $i \in \{1,2\}$ let

$$W_i = (S_i, p_i)$$

be probability spaces and let

$$X_i:S_i\to\mathbb{R}$$

be random variables in these spaces.

$$X: S_1 \times S_2 \to \mathbb{R}$$
 , $X(a,b) = X_1(a) + X_2(b)$

$$W = (S, p)$$
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 probability space

Lemma 9.

$$E(X) = E(X_1) + E(X_2)$$

 $X: S \to \mathbb{R}$

random variable

expected value of random variable X

$$E(X) = \sum_{a \in S} X(a) \cdot p(a)$$

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Lemma 9.

$$E(X) = E(X_1) + E(X_2)$$

$$\begin{split} E(X) &= \sum_{(a,b) \in S_1 \times S_2} p(a,b) \cdot X(a,b) \\ &= \sum_{a \in S_1} \sum_{b \in S_2} p_1(a) \cdot p_2(b) \cdot (X_1(a) + X_2(b)) \\ &= \sum_{a \in S_1} p_1(a) \cdot (\sum_{b \in S_2} p_2(b) \cdot (X_1(a) + X_2(b)) \\ &= \sum_{a \in S_1} p_1(a) \cdot X_1(a) \cdot (\sum_{b \in S_2} p_2(b)) + \sum_{a \in S_1} p_1(a) \cdot (\sum_{b \in S_2} p_2(b) \cdot X_2(b)) \\ &= \sum_{a \in S_1} p_1(a) \cdot X_1(a) \cdot 1 + \sum_{a \in S_1} p_1(a) \cdot E(X_2) \\ &= E(X_1) + E(X_2) \end{split}$$