Homework 1

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1. step by step solution:

$$u \cdot v = 0, \|u\| = 15$$
$$\implies \begin{cases} 3 \cdot x + 4 \cdot y = 0 \\ \sqrt{x^2 + y^2} = 15 \end{cases}$$
$$\implies \begin{cases} 3 \cdot x = -4 \cdot y \\ x^2 + y^2 = 15^2 \end{cases}$$
$$\implies \begin{cases} x = -\frac{4}{3} \cdot y \\ x^2 + y^2 = 15^2 \end{cases}$$
$$\implies \begin{cases} x = -\frac{4}{3} \cdot y \\ \left(-\frac{4}{3} \cdot y\right)^2 + y^2 = 15^2 \end{cases}$$
$$\implies \begin{cases} x = -\frac{4}{3} \cdot y \\ y^2 \left(\frac{16}{9} + 1\right) = 15^2 \end{cases}$$
$$\implies \begin{cases} y = \sqrt{15^2 \cdot \frac{9}{25}} = 9 \\ x = -12 \end{cases}$$
$$\implies u = (-12, 9)$$

now it's clear that

$$-12 \cdot 3 + 9 \cdot 4 = 0$$

and

$$\sqrt{(-12)^2 + 9^2} = 15$$

2. let's precalculate the cross product

$$\vec{u} \times \vec{v} = (1 \cdot 0 - (-2) \cdot (-1), (-1) \cdot 0 - (-2) \cdot 2, (-1) \cdot (-1) - 2 \cdot 1) = (-2, 4, -1)$$

and the dot product

$$\vec{u} \cdot \vec{v} = (-1) \cdot 2 + 1 \cdot (-1) + (-2) \cdot 0 = -3$$

and the magnitudes

$$\|\vec{u}\| = \sqrt{(-1)^2 + 1^2 + (-2)^2} = \sqrt{6}$$
$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$$
$$\|\vec{u} \times \vec{v}\| = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{21}$$

so now

$$\|\vec{u} \times \vec{v}\|^2 = \|\vec{u}\|^2 \|\vec{v}\|^2 - (\vec{u} \cdot \vec{v})^2$$
$$\implies \quad 21 = 6 \cdot 5 - (-3)^2$$
$$\implies \quad 21 = 30 - 9$$
$$\implies \quad 21 = 21$$

3. (a)

$$\begin{aligned}
\vec{AB} &= B - A = (0-2, 1-(-3), 2-4) = (-2,4,-2) \\
\vec{AC} &= C - A = ((-1)-2, 2-(-3), 0-4) = (-3,5,-4) \\
\vec{AB} \times \vec{AC} &= (4\cdot(-4) - 5\cdot(-2), (-2)\cdot(-4) - (-3)\cdot(-2), (-3)\cdot 4 - (-2)\cdot 5) \\
&= (-6, 2, -2) \\
\|\vec{AB} \times \vec{AC}\| &= \sqrt{(-6)^2 + 2^2 + (-2)^2} = 2\sqrt{11}
\end{aligned}$$

so the area of parallelogram $ABCD$ with adjacent sides $\vec{AB}$ and $\vec{AC}$ is $2\sqrt{11}$.

(b) the area of the triangle ABC would be half of that of the parallelogram $ABCD$ which is $\sqrt{11}$.

4. First, let's find the vectors

$$\begin{aligned}
\vec{AB} &= (6-4, 5-1, -2-0) = (2,4,-2) \\
\vec{AC} &= (5-4, 3-1, -1-0) = (1,2,-1)
\end{aligned}$$

now it's clear that $2\vec{AC} = \vec{AB}$

5. the angle between two vectors $\vec{u}$ and $\vec{v}$ is equal to

$$\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}\right)$$

so the angle between $\vec{OP}$ and $\vec{OQ}$ will be

$$\begin{aligned}
&\cos^{-1}\left(\frac{\vec{OP}\cdot\vec{OQ}}{\|\vec{OP}\|\cdot\|\vec{OQ}\|}\right) \\
=&\cos^{-1}\left(\frac{3\cdot 1 + 7\cdot 1 + (-2)\cdot(-3)}{\sqrt{3^2+7^2+(-2)^2}\cdot\sqrt{1^2+1^2+(-3)^2}}\right) \\
=&\cos^{-1}\left(\frac{3+7+6}{\sqrt{9+49+4}\cdot\sqrt{1+1+9}}\right) \\
=&\cos^{-1}\left(\frac{16}{\sqrt{62}\cdot\sqrt{11}}\right)
\end{aligned}$$

6. step by step solution:

$$\begin{aligned}
&\begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix} \\
=&\begin{pmatrix} 3\cdot 3 + 0\cdot 0 & 3\cdot(-1) + 0\cdot 2 \\ (-1)\cdot 3 + 2\cdot 0 & (-1)\cdot(-1) + 2\cdot 2 \end{pmatrix} - \begin{pmatrix} 2\cdot 3 & 2\cdot(-1) \\ 2\cdot 0 & 2\cdot 2 \end{pmatrix} \\
=&\begin{pmatrix} 9 & -3 \\ -3 & 5 \end{pmatrix} - \begin{pmatrix} 6 & -2 \\ 0 & 4 \end{pmatrix} \\
=&\begin{pmatrix} 9-6 & -3-(-2) \\ -3-0 & 5-4 \end{pmatrix} \\
=&\begin{pmatrix} 3 & -1 \\ -3 & 1 \end{pmatrix}
\end{aligned}$$

7. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $XA = B$ can be writen as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 6 & 4 \end{pmatrix}$$

which, by definition, is the following system

$$\begin{cases} 2 \cdot a + (-4) \cdot b = 2 \\ 1 \cdot a + (-3) \cdot b = 2 \\ 2 \cdot c + (-4) \cdot d = 6 \\ 1 \cdot c + (-3) \cdot d = 4 \end{cases} \implies \begin{cases} a - 2b = 1 \\ a - 3b = 2 \\ c - 2d = 3 \\ c - 3d = 4 \end{cases} \implies \begin{cases} a = -1 \\ b = -1 \\ c = 1 \\ d = -1 \end{cases}$$

so $X = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$

8. The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is the minor of the matrix.

   (a) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{pmatrix}$.

   The determinant $\det(A)$ can be calculated in the following way:

$$\begin{aligned} \det(A) &= (1 \cdot (-1) \cdot 3) + (1 \cdot 1 \cdot 0) + (1 \cdot (-1) \cdot 2) - (2 \cdot (-1) \cdot 0) - (1 \cdot (-1) \cdot 1) - (1 \cdot 1 \cdot 3) \\ &= -3 + 0 - 2 - 0 + 1 - 3 \\ &= -7 \end{aligned}$$

   Now to calculate $\operatorname{adj}(A)$, we first need to calculate all 9 cofactors

$$\begin{aligned} C_{11} &= (-1)^{1+1} \cdot ((-1) \cdot 3 - 1 \cdot (-1)) && = -2 \\ C_{12} &= (-1)^{1+2} \cdot (1 \cdot 3 - 1 \cdot 0) && = -2 \\ C_{13} &= (-1)^{1+3} \cdot (1 \cdot (-1) - (-1) \cdot 0) && = -1 \\ C_{21} &= (-1)^{2+1} \cdot (1 \cdot 3 - 2 \cdot (-1)) && = -5 \\ C_{22} &= (-1)^{2+2} \cdot (1 \cdot 3 - 2 \cdot 2) && = -1 \\ C_{23} &= (-1)^{2+3} \cdot (1 \cdot (-1) - 1 \cdot 0) && = 2 \\ C_{31} &= (-1)^{3+1} \cdot (1 \cdot 1 - 2 \cdot (-1)) && = 3 \\ C_{32} &= (-1)^{3+2} \cdot (1 \cdot 1 - 2 \cdot 1) && = 1 \\ C_{33} &= (-1)^{3+3} \cdot (1 \cdot (-1) - 1 \cdot 1) && = -2 \end{aligned}$$

   and we get that

$$\operatorname{adj}(A) = \begin{pmatrix} -2 & -5 & 3 \\ -2 & -1 & 1 \\ -1 & 2 & -2 \end{pmatrix}$$

   now we can finaly find the inverse

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \frac{1}{-7} \cdot \begin{pmatrix} -2 & -5 & 3 \\ -2 & -1 & 1 \\ -1 & 2 & -2 \end{pmatrix} = \begin{pmatrix} 2/7 & 5/7 & -3/7 \\ 2/7 & 1/7 & -1/7 \\ 1/7 & -2/7 & 2/7 \end{pmatrix}$$

(b) Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

The determinant $\det(B)$ can be calculated in the following way:

$$\begin{aligned} \det(B) &= (1 \cdot 1 \cdot 1) + (2 \cdot 2 \cdot 2) + (3 \cdot 3 \cdot 3) - (3 \cdot 1 \cdot 2) - (2 \cdot 3 \cdot 1) - (1 \cdot 2 \cdot 3) \\ &= 1 + 8 + 27 - 6 - 6 - 6 \\ &= 18 \end{aligned}$$

Now to calculate $\operatorname{adj}(B)$, we first need to calculate all 9 cofactors

$$\begin{aligned} C_{11} &= (-1)^{1+1} \cdot (1 \cdot 1 - 2 \cdot 3) && = -5 \\ C_{12} &= (-1)^{1+2} \cdot (3 \cdot 1 - 2 \cdot 2) && = 1 \\ C_{13} &= (-1)^{1+3} \cdot (3 \cdot 3 - 1 \cdot 2) && = 7 \\ C_{21} &= (-1)^{2+1} \cdot (2 \cdot 1 - 3 \cdot 3) && = 7 \\ C_{22} &= (-1)^{2+2} \cdot (1 \cdot 1 - 3 \cdot 2) && = -5 \\ C_{23} &= (-1)^{2+3} \cdot (1 \cdot 3 - 2 \cdot 2) && = 1 \\ C_{31} &= (-1)^{3+1} \cdot (2 \cdot 2 - 3 \cdot 1) && = 1 \\ C_{32} &= (-1)^{3+2} \cdot (1 \cdot 2 - 3 \cdot 3) && = 7 \\ C_{33} &= (-1)^{3+3} \cdot (1 \cdot 1 - 2 \cdot 3) && = -5 \end{aligned}$$

and we get that

$$\operatorname{adj}(A) = \begin{pmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{pmatrix}$$

now we can finaly find the inverse

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A) = \frac{1}{18} \cdot \begin{pmatrix} -5 & 7 & 1 \\ 1 & -5 & 7 \\ 7 & 1 & -5 \end{pmatrix} = \begin{pmatrix} -5/18 & 7/18 & 1/18 \\ 1/18 & -5/18 & 7/18 \\ 7/17 & 1/18 & -5/18 \end{pmatrix}$$