Introduction to Optimization Homework (2)

Dimitri Tabatadze · Monday 11-03-2024

order	$f(x_0+kh) = f(x_0) + kh\frac{1}{1!}f'(x_0) + \frac{(kh)^2}{2!}f''(x_0) + \frac{(kh)^3}{3!}f'''(x_0) + \frac{(kh)^4}{4!}f^{(\text{iv})}(x_0) + \dots + \frac{(kh)^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$
I	$f'(x_0) = \frac{f(x_0+kh) - f(x_0)}{kh} - \frac{kh}{2!}f''(x_0) - \frac{(kh)^2}{3!}f'''(x_0) - \frac{(kh)^3}{4!}f^{(\text{iv})}(x_0) - O(h^4)$
II	$f''(x_0) = 2!\frac{f(x_0+kh) - f(x_0)}{(kh)^2} - 2!\frac{1}{kh}f'(x_0) - 2!\frac{kh}{3!}f'''(x_0) - 2!\frac{(kh)^2}{4!}f^{(\text{iv})}(x_0) - O(h^3)$
III	$f'''(x_0) = 3!\frac{f(x_0+kh) - f(x_0)}{(kh)^3} - 3!\frac{1}{(kh)^2}f'(x_0) - 3!\frac{1}{2!\cdot kh}f''(x_0) - 3!\frac{kh}{4!}f^{(\text{iv})}(x_0) - O(h^2)$
IV	$f^{(\text{iv})}(x_0) = 4!\frac{f(x_0+kh) - f(x_0)}{(kh)^4} - 4!\frac{1}{(kh)^3}f'(x_0) - 4!\frac{1}{2!(kh)^2}f''(x_0) - 4!\frac{1}{3!\cdot kh}f'''(x_0) - O(h)$

1. The error term and the order of
$$f'(x_0) \approx \frac{4f(x_0+h) - 3f(x_0) - f(x_0-2h)}{6h}$$
would be calculate by Taylor's theorem
$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \dots + \frac{h^n}{n!}f^{(n)}(x_0) + O(h^{n+1})$$
$$f(x_0-2h) = f(x_0) - 2hf'(x_0) + 4h^2\frac{1}{2}f''(x_0) - \dots + (-2h)^n\frac{1}{n!}f^{(n)}(x_0) + O(h^{n+1}).$$
$$\begin{aligned} f'(x_0) &= \frac{f(x_0+h) - f(x_0) - \frac{h^2}{2}f''(x_0) - O(h^3)}{h} \\ &= \frac{f(x_0+h) - f(x_0)}{h} - \frac{h}{2}f''(x_0) - O(h^2) \\ f'(x_0) &= \frac{f(x_0) - f(x_0-2h) + 4h^2\frac{1}{2}f''(x_0) + O(h^3)}{2h} \\ &= \frac{f(x_0) - f(x_0-2h)}{2h} + hf''(x_0) + O(h^2) \end{aligned}$$
If we combine both of these, we get
$$\begin{aligned} 3f'(x_0) &= \frac{4f(x_0+h) - 3f(x_0) - f(x_0-2h)}{2h} + O(h^2) \\ f'(x_0) &= \frac{4f(x_0+h) - 3f(x_0) - f(x_0-2h)}{6h} + O(h^2). \end{aligned}$$
Now we can see that the error term for the given approx_0imation is $O(h^2)$ and the order therefore is 2.

2. As in the previous problem, we use taylors formula

$$
\begin{aligned}
f(x_0 - h) &= f(x_0) - h\frac{1}{1!}f'(x_0) + h^2\frac{1}{2!}f''(x_0) - \dots + (-h)^n\frac{1}{n!}f^{(n)}(x_0) + O(h^{n+1}) \\
f'(x_0) &= \frac{f(x_0) - f(x_0 - h) + h^2\frac{1}{2}f''(x_0) + O(h^3)}{h} \\
&= \frac{f(x_0) - f(x_0 - h)}{h} + h\frac{1}{2}f''(x_0) + O(h^2) \\
f(x_0 + 3h) &= f(x_0) + 3h\frac{1}{1!}f'(x_0) + (3h)^2\frac{1}{2!}f''(x_0) + \dots + (3h)^n\frac{1}{n!}f^{(n)}(x_0) + O(h^{n+1}) \\
f'(x_0) &= \frac{f(x_0 + 3h) - f(x_0) - h^2\frac{9}{2}f''(x_0) - O(h^3)}{3h} \\
&= \frac{f(x_0 + 3h) - f(x_0)}{3h} - h\frac{9}{2}f''(x_0) - O(h^2)
\end{aligned}
$$

then we combine these to get

$$
\begin{aligned}
20f'(x_0) &= 18\frac{f(x_0) - f(x_0 - h)}{h} + 9hf''(x_0) + 2\frac{f(x_0 + 3h) - f(x_0)}{3h} - 9hf''(x_0) + O(h^2) \\
&= 18\frac{f(x_0) - f(x_0 - h)}{h} + 2\frac{f(x_0 + 3h) - f(x_0)}{3h} + O(h^2) \\
&= \frac{53f(x_0) - 54f(x_0 - h) + 2f(x_0 + 3h)}{3h} + O(h^2) \\
f'(x) &= \frac{53f(x_0) - 54f(x_0 - h) + 2f(x_0 + 3h)}{60h} + O(h^2)
\end{aligned}
$$

a second order approximation for $f'(x_0)$

3. Similarly combining these gives

$$
\begin{aligned}
4f''(x_0) &= 2\frac{3f(x_0 + 3h) + 9f(x_0 - h) - 10f(x_0)}{9h^2} - O(h) \\
f''(x_0) &= \frac{3f(x_0 + 3h) + 9f(x_0 - h) - 10f(x_0)}{18h^2} - O(h)
\end{aligned}
$$

4. If you sum up $s = -f(x_0 - 2h) + 2f(x_0 - h) - 2f(x_0 + h) + f(x_0 + 2h)$ you will get

$$
\begin{aligned}
s = {} & (-1 + 2 - 2 + 1)f(x_0) + \\
& (2 - 2 - 2 + 2)hf'(x_0) + \\
& (-2 + 2 - 2 + 2)h^2 f''(x_0) + \\
& (8 - 2 - 2 + 8)\frac{h^3}{6}f'''(x_0) + \\
& (-16 + 2 - 2 + 16)\frac{h^4}{24}f^{(\mathrm{iv})}(x_0) + \\
& O(h^5) \\
= {} & 2h^3 f'''(x_0)
\end{aligned}
$$

$$
f'''(x_0) = \frac{-f(x_0 - 2h) + 2f(x_0 - h) - 2f(x_0 + h) + f(x_0 + 2h)}{2h^3} + O(h^2)
$$

5. გთხოვ ნუ დამაწერინებ ამ ყაზრობებს, ორი დღე ვეჭექი ამ დავალებას და ამდენის დაწერა ძლივს დავაძალე თავს, მოვკვდები კიდე რო დავწერო.