Example 11.1

$$A = \begin{pmatrix} 5 & 3 \\ 1 & 4 \end{pmatrix}, \ P = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$$

For the linear system $Ax = b$ write Richardson's method in componentwise and matrix forms using pre-conditioner matrix

Solution 11.1

$$P\frac{x^{(k+1)}-x^{(k)}}{\tau}+Ax^{(k)}=b$$

$$x^{(k+1)}=(I-\tau P^{-1}A)x^{(k)}+\tau P^{-1}b$$

$$A=\begin{pmatrix}5 & 3\\ 1 & 4\end{pmatrix},\ P=\begin{pmatrix}5 & 3\\ 3 & 5\end{pmatrix}$$

$$P^{-1}=\frac{1}{16}\begin{pmatrix}5 & -3\\ -3 & 5\end{pmatrix}$$

$$P^{-1}A=\frac{1}{16}\begin{pmatrix}5 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}5 & 3\\ 1 & 4\end{pmatrix}=\frac{1}{16}\begin{pmatrix}22 & 3\\ -10 & 11\end{pmatrix}$$

$$I-P^{-1}A=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\tau\begin{pmatrix}\frac{22}{16} & \frac{3}{16}\\ \frac{-10}{16} & \frac{11}{16}\end{pmatrix}=\begin{pmatrix}1-\frac{22}{16}\tau & \frac{3}{16}\tau\\ \frac{10}{16}\tau & 1-\frac{11}{16}\tau\end{pmatrix}$$

$$P^{-1}b=\frac{1}{16}\begin{pmatrix}5 & -3\\ -3 & 5\end{pmatrix}\begin{pmatrix}b_1\\ b_2\end{pmatrix}=\begin{pmatrix}\frac{5}{16}b_1-\frac{3}{16}b_2\\ \frac{-3}{16}b_1+\frac{5}{16}b_2\end{pmatrix}$$

Solution 11.1 - cont

$$x^{(k+1)} = (I - \tau P^{-1}A)x^{(k)} + \tau P^{-1}b$$

$$I - P^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \tau \begin{pmatrix} \frac{22}{16} & \frac{3}{16} \\ \frac{-10}{16} & \frac{11}{16} \end{pmatrix} = \begin{pmatrix} 1 - \frac{22}{16}\tau & \frac{3}{16}\tau \\ \frac{10}{16}\tau & 1 - \frac{11}{16}\tau \end{pmatrix}$$

$$P^{-1}b = \frac{1}{16}\begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} \frac{5}{16}b_1 - \frac{3}{16}b_2 \\ \frac{-3}{16}b_1 + \frac{5}{16}b_2 \end{pmatrix}$$

$$x^{(k+1)} = \begin{pmatrix} 1 - \frac{22}{16}\tau & \frac{3}{16}\tau \\ \frac{10}{16}\tau & 1 - \frac{11}{16}\tau \end{pmatrix}\begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \end{pmatrix} + \begin{pmatrix} (\frac{5}{16}b_1 - \frac{3}{16}b_2)\tau \\ (\frac{-3}{16}b_1 + \frac{5}{16}b_2)\tau \end{pmatrix} =$$

$$= \begin{pmatrix} (1 - \frac{22}{16}\tau)x_1^{(k)} + \frac{3}{16}\tau x_2^{(k)} + (\frac{5}{16}b_1 - \frac{3}{16}b_2)\tau \\ \frac{10}{16}\tau x_1^{(k)} + (1 - \frac{11}{16}\tau)x_2^{(k)} + (\frac{-3}{16}b_1 + \frac{5}{16}b_2)\tau \end{pmatrix}$$

Example 11.2

Find optimal step(parameter) for the previous example.

$$P\frac{x^{(k+1)} - x^{(k)}}{\alpha} + Ax^{(k)} = b$$

$$\lambda_1(P^{-1}A) > \lambda_2(P^{-1}A) > ... > \lambda_n(P^{-1}A) > 0$$

$$\alpha_{opt} = \frac{2}{\lambda_{max} + \lambda_{min}}$$

Solution 11.2

$$P^{-1}A = \frac{1}{16}\begin{pmatrix} 22 & 3 \\ -10 & 11 \end{pmatrix}$$

$$\lambda_{max} = \frac{17}{16},\ \lambda_{min} = 1$$

$$\alpha_{opt} = \frac{2}{\frac{17}{16}+1} = \frac{2}{33/16} = \frac{32}{33}$$

Example 11.3

The linear system $Ax = b$ given by

$$\begin{cases} 4x_1 + 3x_2 = 24 \\ 3x_1 + 4x_2 - x_3 = 30 \\ -x_2 + 4x_3 = -24 \end{cases}$$

has the solution $(3, 4, -5)^t$. Compare the iterations from the Gauss-Seidel method and the SOR method with $\omega = 1.25$ using $x^{(0)} = (1, 1, 1)^t$ for both methods.

Solution

Gaus-Seidel method:

$$x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}), i = 1, 2, ..., n, k = 0, 1, ...$$

$$x_1^{(k)} = -0.75x_2^{(k-1)} + 6$$

$$x_2^{(k)} = -0.75x_1^{(k)} + 0.25x_3^{(k-1)} + 7.5$$

$$x_3^{(k)} = 0.25x_2^{k} - 6$$

Solution - cont.

SOR method:

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}) + (1-\omega)x_i^{(k)}, i = 1, 2, ..., n$$

$$x_1^{(k)} = -0.25x_1^{(k-1)} - 0.9375x_2^{(k-1)} + 7.5$$
$$x_2^{(k)} = -0.9375x_1^{(k)} - 0.25x_2^{(k-1)} + 0.3125x_3^{(k-1)} + 9.375$$
$$x_3^{(k)} = 0.3125x_2^{k} - 0.25x_3^{(k-1)} - 7.5$$

Solution - cont.

Seven iterations:

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	5.250000	3.1406250	3.0878906	3.0549316	3.0343323	3.0214577	3.0134110
$x_2^{(k)}$	1	3.812500	3.8828125	3.9267578	3.9542236	3.9713898	3.9821186	3.9888241
$x_3^{(k)}$	1	−5.046875	−5.0292969	−5.0183105	−5.0114441	−5.0071526	−5.0044703	−5.0027940

k	0	1	2	3	4	5	6	7
$x_1^{(k)}$	1	6.312500	2.6223145	3.1333027	2.9570512	3.0037211	2.9963276	3.0000498
$x_2^{(k)}$	1	3.5195313	3.9585266	4.0102646	4.0074838	4.0029250	4.0009262	4.0002586
$x_3^{(k)}$	1	−6.6501465	−4.6004238	−5.0966863	−4.9734897	−5.0057135	−4.9982822	−5.0003486

For the iterates to be accurate to seven decimal places, the Gauss-Seidel method requires 34 iterations, as opposed to 14 iterations for the SOR method with $\omega = 1.25$

Convergence theorems

1. If $a_{ii} \neq 0$, for each $i = 1, 2, ..., n$, then $\rho(T_\omega) \geq |\omega - 1|$. This implies that the SOR method can converge only if $0 < \omega < 2$.
2. If A is a positive definite matrix and $0 < \omega < 2$, then the SOR method converges for any choice of initial approximate vector $x^{(0)}$.
3. If A is positive definite and tridiagonal, then $\rho(T_g) = [\rho(T_j)]^2 < 1$, and the optimal choice of ω for the SOR method is

$$\omega = \frac{2}{1 + \sqrt{1 - [\rho(T_j)]^2}}$$

Example 11.4

Find the optimal choice of ω for the SOR method for the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

Solution

$$T_j = D^{-1}(L+U) = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & -3 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -0.75 & 0 \\ -0.75 & 0 & 0.25 \\ 0 & 0.25 & 0 \end{pmatrix}$$

Find eigenvalues.
Spectral radius: $\rho(T_j) = \sqrt{0.625}$.
Thus,

$$\omega = \frac{2}{1+\sqrt{1-[\rho(T_j)]^2}} = \frac{2}{1+\sqrt{1-0.625}} \approx 1.24$$