Numerical Linear Algebra

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Vectors and matrices

- Vectors
- Operations on vectors
- Matrices
- Operations on matrices
- Determinants

- Vectors

Definition 1.1

A **vector** is an object that has both a magnitude and a direction.

Example 1.2

Examples of vectors:

- Displacement
- Acceleration
- Velocity
- Weight
- Force

Definition 1.3

Ordered n-tuple of objects is called a vector

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$$

- The elements $v_i, i = 1, \dots, n$ are called ***components*** of the vector
- The number n of components is called ***dimension*** of the vector v

- Operation on vectors

Definition 1.4

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} \; v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$$

- **Sum** of $u, v \in R^n$ is defined by

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \dots \\ u_n + v_n \end{bmatrix}$$

- Multiplication by scalar λ

$$\boldsymbol{\lambda u} = \lambda \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \dots \\ \lambda u_n \end{bmatrix}$$

Definition 1.5

- Two vectors u and v are **equal**, which we denote $u = v$, if they have the same size, and each of the corresponding components are the same, i.e. $u_1 = v_1, u_2 = v_2, ..., u_n = v_n$
- A **zero (null) vector** is a vector with all elements equal to zero
- A **(standard) unit vecto**r is a vector with all elements equal to zero, except one element which is equal to one.
  The $i-$th unit vector (of size n) is the unit vector with $i-$th element one, and denoted e_i.
  For example, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are unit vectors in R^3

Definition 1.6

For any two vectors $u = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix}$, the ***dot product*** of u and v is denoted by (u, v), and is defined as

$$(u, v) = u_1 v_1 + u_2 v_2 + \dots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

Example 1.7

Let $u = (0, 1, -1, 5)$ and $v = (2, 0, -1, 0)$. Then
$(u, v) = 0 \cdot 2 + 1 \cdot 0 + (-1) \cdot (-1) + 5 \cdot 0 = 1.$

Definition 1.8

The **length of vector** u in R^n is defined as $\sqrt{(u,u)}$ and denoted by $|u|$.
If $u = (u_1, u_2, u_3)$ then $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$

Definition 1.9

The dot product of two Euclidean vectors u and v is defined by

$$u \cdot v = |u||v|cos\theta$$

where θ is the angle between u and v.

Example 1.10

Determine the angle between the vector:

$$v = (1, 2, 3, 4), w = (0, -1, 3, 2)$$

Solution:

On the one hand, dot product $(v, w) = 1 \cdot 0 + 2 \cdot (-1) + 3 \cdot 3 + 4 \cdot 2 = 15$
From the last definition we have: $\cos\theta = \frac{(u,v)}{|u||v|}$

$$|u| = \sqrt{1 + 4 + 9 + 16} = \sqrt{30}$$

$$|v| = \sqrt{0 + 1 + 9 + 4} = \sqrt{14}$$

Thus,

$$\cos\theta = \frac{15}{\sqrt{30}\sqrt{14}} = \frac{15}{420} = \frac{1}{28}$$

$$\theta = \arccos\frac{1}{28}$$

Definition 1.11

The **vector (cross) product** of $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in R^3 is denoted by $u \times v$ and is calculated

$$u \times v = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Cross product results in a vector which is perpendicular to both the vectors.

Example 1.12

$$\text{Let's } u = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, v = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}.$$

Then, $u \times v = \begin{bmatrix} 2 \cdot (-5) - 1 \cdot 3 \\ 1 \cdot 0 - (-1) \cdot (-5) \\ (-1) \cdot 3 - 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} -13 \\ -5 \\ -3 \end{bmatrix}$

Example 1.13

Compute a vector that is orthogonal to the vectors

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$$

Solution:

Cross product of u and v is orthogonal to each vector u and v.

$$u \times v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -3 \end{pmatrix}$$

Definition 1.14

Let the vectors $v_1, v_2, ..., v_n$ be vectors in R^n and $c_1, c_2, ..., c_n$ be scalars. Then the sum

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

is called **linear combination** of the vectors $v_1, v_2, ..., v_n$.

Definition 1.15

A set of vectors $v_1, v_2, ..., v_n$ is called **linearly independent**, if

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n = 0$$

holds if and only if $c_1 = c_2 = ... = c_n = 0$.

Example 1.16

Is the following set of vectors linearly independent? If it is linearly dependent, find a linear dependence relation.

$$\{a_1, a_2\}, a_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}, a_2 = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$$

Solution:

Let's consider linear combination of these vectors

$$c_1 \begin{bmatrix} -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -c_1 + 2c_2 \\ 4c_1 - 8c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} -c_1 + 2c_2 = 0 \\ 4c_1 - 8c_2 = 0 \end{cases}$$

The system has infinitely many solutions, i.e. $\{a_1, a_2\}$ is linearly dependent. As $-c_1 + 2c_2 = 0$, it means that $c_1 = 2c_2$. So, the dependence relation is $2a_1 + a_2 = 0$

- Matrices

Definition 1.17

A collection of n vectors in R^n arranged in a rectangular array of m rows and n columns is called a **matrix**.
A matrix A, therefore, has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- It is denoted by $A = (a_{ij})_{m \times n}$ or simply by $A = (a_{ij})$, where $i = 1, ..., m$ and $j = 1, ..., n$.
- A is said to be of order (dimension) $m \times n$.
- The set of all $m \times n$ real matrices is denoted by $R^{m \times n}$

Definition 1.18

- Two matrices are **equal** if and only if
  - the order of the matrices are the same
  - the corresponding elements of the matrices are the same
- A **zero (null) matrix** is a matrix all of whose elements are zero. For example,

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  is a zero matrix with dimension 2×3
- An **identity matrix** of size n is the $n \times n$ square matrix with ones on the main diagonal and zeroes elsewhere.

$$I_1 = (1), I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots$$

- Operations on matrices

Definition 1.19

The **sum** of two m-by-n matrices A and B is calculated elementwise:

$$(A+B)_{ij} = A_{ij} + B_{ij}$$

where $i = 1, ..., m, j = 1, ..., n$

Example 1.20

$$\begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 0 & -3 \\ 1 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 1+2 & 3+0 & 0+(-3) \\ (-1)+1 & 2+3 & 5+7 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 3 & -3 \\ 0 & 5 & 12 \end{pmatrix}$$

Definition 1.21

Scalar multiplication: The product cA of a scalar c and a matrix A is computed by multiplying every entry of A by c:

$$(cA)_{ij} = (c \cdot A_{ij})$$

Example 1.22

$$3 \cdot \begin{pmatrix} 1 & 3 & 0 \\ -1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 3 \cdot 1 & 3 \cdot 3 & 3 \cdot 0 \\ 3 \cdot (-1) & 3 \cdot 2 & 3 \cdot 5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 9 & 0 \\ -3 & 6 & 15 \end{pmatrix}$$

Definition 1.23

The **transpose** of an m-by-n matrix A is the n-by-m matrix A^T formed by turning rows into columns and vice versa:

$$(A^T)_{ij} = (A_{ji})$$

Example 1.24

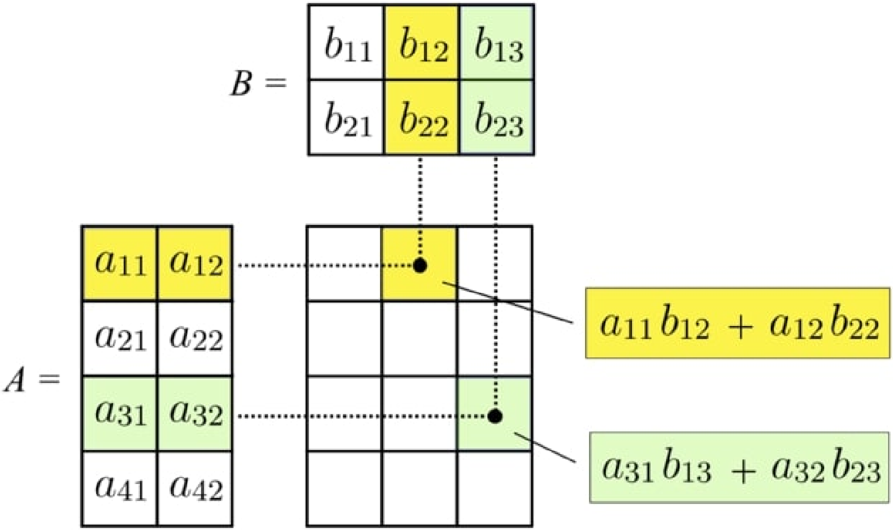
$$\begin{pmatrix} 2 & 0 & -3 \\ 1 & 3 & 7 \end{pmatrix}^T = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ -3 & 7 \end{pmatrix}$$

Definition 1.25

Matrix multiplication:
If A is an m-by-n matrix and B is an n-by-k matrix, then their **matrix product** AB is the m-by-k matrix whose entries are given by dot product of the corresponding row of A and the corresponding column of B:

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj} = \sum_{l=1}^{n} a_{il}b_{lj}$$

where $i = 1, .., m, j = 1, ..., k$


$B =$
b_{11} b_{12} b_{13}
b_{21} b_{22} b_{23}
$A =$
a_{11} a_{12}
a_{21} a_{22}
a_{31} a_{32}
a_{41} a_{42}
$a_{11}b_{12} + a_{12}b_{22}$
$a_{31}b_{13} + a_{32}b_{23}$

Example 1.26

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 2 & 3 \\ -2 & 4 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 2 & 3 \\ -2 & 4 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot 0 + 2 \cdot 2 + 3 \cdot (-2) & 1 \cdot (-1) + 2 \cdot 3 + 3 \cdot 4 \\ 0 \cdot 0 + 4 \cdot 2 + 5 \cdot (-2) & 0 \cdot (-1) + 4 \cdot 3 + 5 \cdot 4 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 17 \\ -2 & 32 \end{pmatrix}$$

Example 1.27

Show that generally matrices are non-commutative $AB \neq BA$.

Solution: Let's take matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 5 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 1 \end{bmatrix}$$

$$AB \neq BA$$

Example 1.28

Matrix multiplication is associative $(AB)C = A(BC)$. Show this for

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ 3 & -1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Solution:

$$(AB)C = \begin{pmatrix} 1 & 3 \\ 11 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 2 & 1 & -1 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

$$(AB)C = A(BC)$$

Example 1.29

Find the inverse of matrix

$$A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$$

Solution:

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then inverse matrix $A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

So, we have $A^{-1} = \frac{1}{1 \cdot 3 - 5 \cdot (-2)} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix} = \frac{1}{13} \begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$

- Determinants

Definition 1.30

The **determinant** is a scalar value that is a function of the entries of a square matrix.
The determinant of a matrix A is denoted $det(A), detA, or|A|$

Definition 1.31

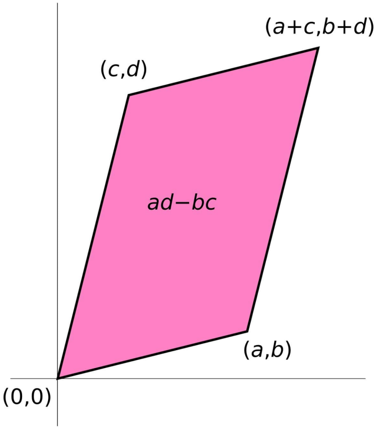
In case of 2×2 matrix the determinant can be defined as

$$detA = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Geometrically, absolute value of

$$detA = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

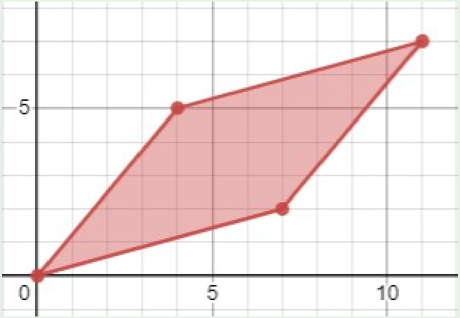
represents the area of the paralleogram which is formed with (a, b) and (c, d) vectors.



Example 1.32

Compute the area of the parallelogram with vertices $(0,0),(4,5),(7,2),(11,7)$.

Solution:



This parallelogram is spanned by two vectors $(4,5)$ and (7,2).

So, we have $Area = \left| det \begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} \right| = |4 \cdot 2 - 5 \cdot 7| = 27.$

Definition 1.33

In case of 3×3 matrix the determinant can be defined as

$$detA = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + dhc - ceg - afh - idb$$

Example 1.34

Find the determinant of the matrix:

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 3 & 0 \\ -5 & 1 & 1 \end{pmatrix}$$

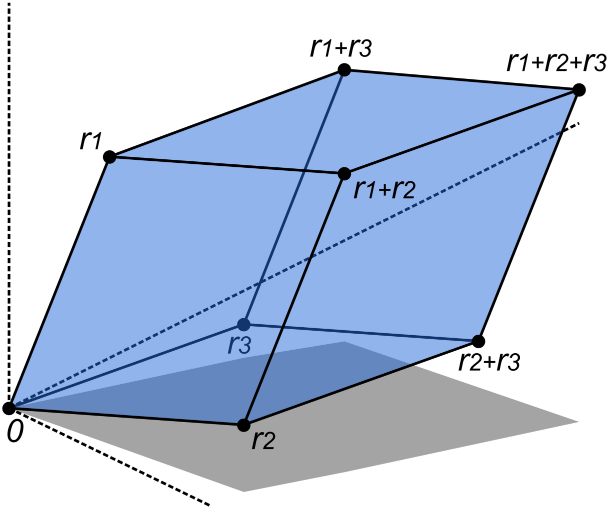
Solution:

$$|A| = \begin{vmatrix} 0 & -1 & 2 \\ 4 & 3 & 0 \\ -5 & 1 & 1 \end{vmatrix} =$$

$$0 \cdot 3 \cdot 1 + (-5) \cdot (-1) \cdot 0 + 2 \cdot 4 \cdot 1 - 2 \cdot 3 \cdot (-5) - 1 \cdot 4 \cdot (-1) - 0 \cdot 1 \cdot 0$$

$$= 0 + 0 + 8 + 30 + 4 = 42$$

Geometrically, the absolute value of the determinant of the matrix formed by the columns constructed from the vectors $r_1, r_2,$ and r_3 represents the volume of this parallelepiped.



Definition 1.35

Let A be $n \times n$ matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ . & & & \\ . & & & \\ . & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- The i, j minor of the matrix, denoted by M_{ij} is the determinant that results from deleting the$i$-th row and the j-th column of the matrix.
- The i, j cofactor of the matrix is defined by:

  $$C_{ij} = (-1)^{i+j} \cdot M_{ij}$$

  where M_{ij} is the i, j minor of the matrix.
- $det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij}$

Example 1.36

Let

$$A = \begin{pmatrix} 0 & -1 & 2 \\ 4 & 3 & 0 \\ -5 & 1 & 1 \end{pmatrix}$$

First of all, we must choose a column or a row of the determinant. In this case, we choose the first row, since it has a zero and therefore it will be easier to compute. So, we have:

$$\det(A) = \begin{vmatrix} 0 & -1 & 2 \\ 4 & 3 & 0 \\ -5 & 1 & 1 \end{vmatrix} = 0 \cdot C_{11} + (-1) \cdot C_{12} + 2 \cdot C_{13}$$

Example 1.37

Let's calculate cofactors.

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = 3$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 0 \\ -5 & 1 \end{vmatrix} = -4$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 3 \\ -5 & 1 \end{vmatrix} = 19$$

Thus,

$$\det(A) = \begin{vmatrix} 0 & -1 & 2 \\ 4 & 3 & 0 \\ -5 & 1 & 1 \end{vmatrix} = 0 \cdot 3 + (-1) \cdot (-4) + 2 \cdot 19 = 42$$

Definition 1.38

- The determinant of a matrix is equivalent to the determinant of its transpose.

$$det(A^T) = det(A)$$

- Any determinant that has two equal or multiple rows (or columns) is equal to zero.
- If two rows or two columns of a determinant are interchanged, the determinant gives the same result but changed sign.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix}$$

Definition 1.39

- If all elements of a row (or column) of a determinant are multiplied by some scalar number k, the value of the new determinant is k times of the given determinant.

$$\begin{vmatrix} k \cdot a & k \cdot b & k \cdot c \\ d & e & f \\ g & h & i \end{vmatrix} = k \cdot \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

- The determinant of the product of two matrices is equal to the product of the determinant of each matrix.

$$\det(AB) = \det(A) \cdot \det(B)$$

Example 1.40

Solve the equation:

$$\det\begin{pmatrix} -2x & 1 & 0 \\ 0 & 3 & 2 \\ -x & 1 & 5 \end{pmatrix} = 4$$

Solution:

$$\det\begin{pmatrix} -2x & 1 & 0 \\ 0 & 3 & 2 \\ -x & 1 & 5 \end{pmatrix} = -30x - 2x + 0 - 0 + 4x - 0 = -28x = 4$$

$$x = -\frac{1}{7}$$

Example 1.41

Solve the system: $\begin{cases} 3x + 4y = -14 \\ -2x - 3y = 11 \end{cases}$ using Cramer's rule.

Solution:

Using Cramer's rule we can write the solution as the ratio of two determinants.

$$x = \frac{\begin{vmatrix} -14 & 4 \\ 11 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{-2}{-1} = 2$$

$$y = \frac{\begin{vmatrix} 3 & -14 \\ -2 & 11 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ -2 & -3 \end{vmatrix}} = \frac{5}{-1} = -5$$

Therefore, $x = 2, y = -5$

Example 1.42

Solve the system: $\begin{cases} x + 2y + 3z = 17 \\ 3x + 2y + z = 11 \\ x - 5y + z = -5 \end{cases}$ using Cramer's rule.

Solution:

Using Cramer's rule we can write the solution as the ratio of two determinants.

$$x = \frac{\begin{vmatrix} 17 & 2 & 3 \\ 11 & 2 & 1 \\ -5 & -5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}} = \frac{-48}{-48} = 1 \qquad y = \frac{\begin{vmatrix} 1 & 17 & 3 \\ 3 & 11 & 1 \\ 1 & -5 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}} = \frac{-96}{-48} = 2$$

Example 1.42 (continued)

Example 1.43

$$z = \frac{\begin{vmatrix} 1 & 2 & 17 \\ 3 & 2 & 11 \\ 1 & -5 & -5 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -5 & 1 \end{vmatrix}} = \frac{-192}{-48} = 4$$

Therefore, $x = 1, y = 2, z = 4$