University of Moratuwa
Faculty of Engineering
Department of Electronic & Telecommunication Engineering

Modelling and Analysis of Physiological Systems

Branched Cylinders
Dendritic Tree Approximations

Bandara D.M.D.V.
Undergraduate (Biomedical engineering)
Department Electronic and Telecommunications
Faculty of Engineering
University of Moratuwa

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Contents

Question 1

From the given equations,

$$\begin{aligned} V_1(X) &= A_1 e^{-X} + B_1 e^{X} \\ V_{21}(X) &= A_{21} e^{-X} + B_{21} e^{X} \\ V_{22}(X) &= A_{22} e^{-X} + B_{22} e^{X} \end{aligned} \tag{2}$$

$$\left.\frac{dV_1}{dX}\right|_{X=0} = -(r_i\lambda_c)_1 I_{app} \tag{3}$$

Terminal ends of daughter branches held at rest.

$$V_{21}(L_{21}) = V_{22}(L_{22}) = 0 \tag{4}$$

The membrane potential must be continuous at the nodes,

$$V_1(L1) = V_{21}(L_1) = V_{22}(L_1) = 0 \tag{5}$$

The current is conserved at the nodes,

$$\frac{-1}{(r_i\lambda_c)_1}\left.\frac{dV_1}{dX}\right|_{X=L_1} = \frac{-1}{(r_i\lambda_c)_{21}}\left.\frac{dV_{21}}{dX}\right|_{X=L_1} + \frac{-1}{(r_i\lambda_c)_1}\left.\frac{dV_{22}}{dX}\right|_{X=L_1} \tag{6}$$

From these 5 equations we can approach the required equation. First differentiating equation 2 w.r.t X,

$$\frac{dV_1}{dX} = -A_1 e^{-X} + B_1 e^{X}$$

At 0,

$$\left.\frac{dV_1}{dX}\right|_{X=0} = -A_1 + B_1$$

From equation 3;

$$-(r_i\lambda_c)_1 I_{app} = -A_1 + B_1$$

$$A_1 - B_1 = (r_i\lambda_c)_1 I_{app} \tag{I}$$

From the remaining Boundary conditions given by equation 4

$$V_{21}(L_{21}) = 0$$

$$A_{21} e^{-L_{21}} + B_{21} e^{L_{21}} = 0 \tag{II}$$

Similarly,

$$A_{22} e^{-L_{22}} + B_{22} e^{L_{22}} = 0 \tag{III}$$

Then at nodes, membrane potential must be same as given in equation 5. i.e,

$$\begin{aligned} V_1(L_1) &= V_{21}(L_1) \\ A_1 e^{-L_1} + B_1 e^{L_1} &= A_{21} e^{-L_1} + B_{21} e^{L_1} \end{aligned}$$

$$A_1 e^{-L_1} + B_1 e^{L_1} - A_{21} e^{-L_1} - B_{21} e^{L_1} = 0 \tag{IV}$$

Similarly,

$$V_{22}(L_1) = V_{21}(L_1)$$

$$A_{22}e^{-L_1} + B_{22}e^{L_1} - A_{21}e^{-L_1} - B_{21}e^{L_1} = 0 \tag{V}$$

Now by current conservation given by equation 6;

$$\frac{-1}{(r_i\lambda_c)_1}(-A_1e^{-L_1}+B_1e^{L_1}) = \frac{-1}{(r_i\lambda_c)_{21}}(-A_{21}e^{-L_1}+B_{21}e^{L_1})+\frac{-1}{(r_i\lambda_c)_1}(-A_{22}e^{-L_1}+B_{22}e^{L_1})$$

Now by substituting results from equations II, III, IV , V and simplifying we get the following equation 7.

$$\frac{A_1e^{-L_1}}{(r_i\lambda_c)_1} + \frac{B_1e^{-L_1}}{(r_i\lambda_c)_1} + \frac{A_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} - \frac{B_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} + \frac{A_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} - \frac{B_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} = 0 \tag{7}$$

Hence, the equation 7 is now verified.

Question 2

Defining the matrices as given,

$$x = \begin{pmatrix} A_1 \\ B_1 \\ A_21 \\ B_21 \\ A_22 \\ B_22 \end{pmatrix} \tag{8}$$

Then equation 7 can be written as,

$$Ax = b \tag{9}$$

Where,

$$b = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{10}$$

And,

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & e^{-L_{21}} & e^{L_{21}} & 0 & 0 & \\ 0 & 0 & 0 & 0 & & e^{-L_{22}} & e^{L_{22}} \\ e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} & 0 & 0 & \\ 0 & 0 & e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} & \\ \frac{-e^{-L_1}}{(r_i\lambda_c)_1} & \frac{e^{L_1}}{(r_i\lambda_c)_1} & \frac{e^{-L_1}}{(r_i\lambda_c)_{21}} & \frac{-e^{L_1}}{(r_i\lambda_c)_{21}} & \frac{e^{-L_1}}{(r_i\lambda_c)_{22}} & \frac{-e^{L_1}}{(r_i\lambda_c)_{22}} & \end{pmatrix} \tag{11}$$

By simplifying equation 9 by substituting equations 8, 10, and 11.

$$Ax = b$$

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-L_{21}} & e^{L_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-L_{22}} & e^{L_{22}} \\ e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} & 0 & 0 \\ 0 & 0 & e^{-L_1} & e^{L_1} & -e^{-L_1} & -e^{L_1} \\ \frac{-e^{-L_1}}{(r_i\lambda_c)_1} & \frac{e^{L_1}}{(r_i\lambda_c)_1} & \frac{e^{-L_1}}{(r_i\lambda_c)_{21}} & \frac{-e^{L_1}}{(r_i\lambda_c)_{21}} & \frac{e^{-L_1}}{(r_i\lambda_c)_{22}} & \frac{-e^{L_1}}{(r_i\lambda_c)_{22}} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_21 \\ B_21 \\ A_22 \\ B_22 \end{pmatrix} = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & -B_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{21}e^{-L_{21}} & B_{21}e^{L_{21}} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{22}e^{-L_{22}} & B_{22}e^{L_{22}} \\ A_1e^{-L_1} & B_1e^{L_1} & -A_{21}e^{-L_1} & -B_{21}e^{L_1} & 0 & 0 \\ 0 & 0 & A_{21}e^{-L_1} & B_{21}e^{L_1} & -A_{22}e^{-L_1} & -B_{22}e^{L_1} \\ \frac{-A_1e^{-L_1}}{(r_i\lambda_c)_1} & \frac{B_1e^{L_1}}{(r_i\lambda_c)_1} & \frac{A_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} & \frac{-B_{21}e^{L_1}}{(r_i\lambda_c)_{21}} & \frac{A_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} & \frac{-B_{22}e^{L_1}}{(r_i\lambda_c)_{22}} \end{pmatrix} = \begin{pmatrix} (r_i\lambda_c)_1 I_{app} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now by equating both sides of each row,

$$A_1 - B_1 = (r_i\lambda_c)_1 I_{app} \tag{I}$$
$$A_{21}e^{-L_{21}} + B_{21}e^{L_{21}} = 0 \tag{II}$$
$$A_{22}e^{-L_{22}} + B_{22}e^{L_{22}} = 0 \tag{III}$$
$$A_1e^{-L_1} + B_1e^{L_1} - A_{21}e^{-L_1} - B_{21}e^{L_1} = 0 \tag{IV}$$
$$A_{22}e^{-L_1} + B_{22}e^{L_1} - A_{21}e^{-L_1} - B_{21}e^{L_1} = 0 \tag{V}$$

$$\frac{A_1e^{-L_1}}{(r_i\lambda_c)_1} + \frac{B_1e^{-L_1}}{(r_i\lambda_c)_1} + \frac{A_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} - \frac{B_{21}e^{-L_1}}{(r_i\lambda_c)_{21}} + \frac{A_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} - \frac{B_{22}e^{-L_1}}{(r_i\lambda_c)_{22}} = 0 \tag{7}$$

Hence, the equation 7 is obtained from the matrices.

Question 3

Matlab Code:

```
% electrical constants and derived quantities for typical
% mammalian dendrite

% Dimensions of compartments

d1 = 75e-4;                     % cm
d21 = 30e-4;                    % cm
d22 = 15e-4;                    % cm
%d21 = 47.2470e-4;        % E9 cm
%d22 = d21;                % E9 cm

l1 = 1.5;                       % dimensionless
l21 = 3.0;                      % dimensionless
l22 = 3.0;                      % dimensionless

% Electrical properties of compartments

Rm = 6e3;                       % Ohms cm^2
Rc = 90;                        % Ohms cm
Rs = 1e6;                       % Ohms

c1 = 2*(Rc*Rm)^(1/2)/pi;

rl1 = c1*d1^(-3/2);             % Ohms
rl21 = c1*d21^(-3/2);           % Ohms
rl22 = c1*d22^(-3/2);           % Ohms


% Applied current

iapp = 1e-9;    % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21
        rl1*exp(-l1)/rl22 -rl1*exp(-l1)/rl22];

b = [rl1*iapp 0 0 0 0 0]';

%% Question 03%%
x=A\b;
display(x)
```

Results are.

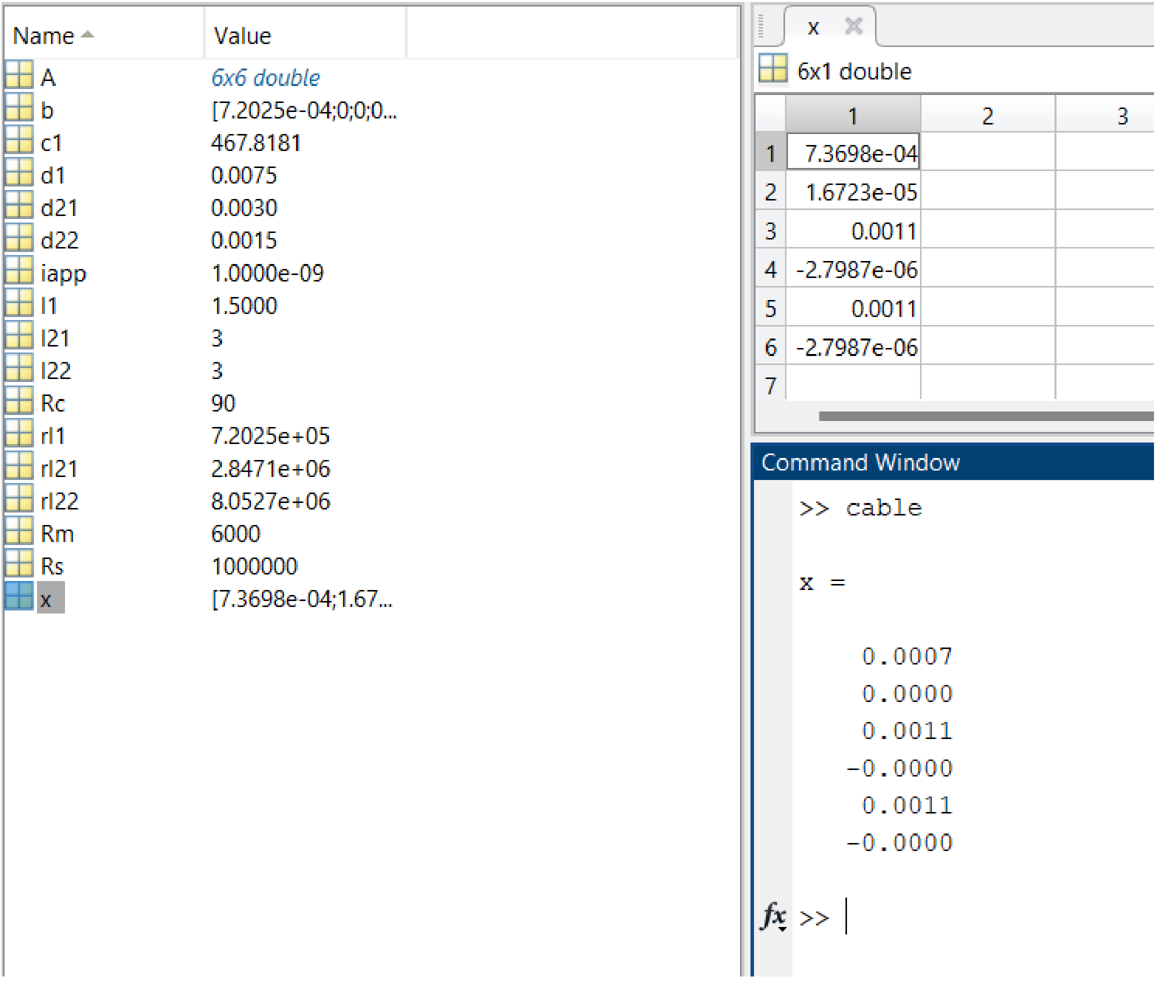


Figure 1: 03

Question 4

After copying the following code we get the plot below that.

```
%% Question 04%%%
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X (dimensionless)');
ylabel('V (Volts)');
title('Steady-state voltage - E5');
```

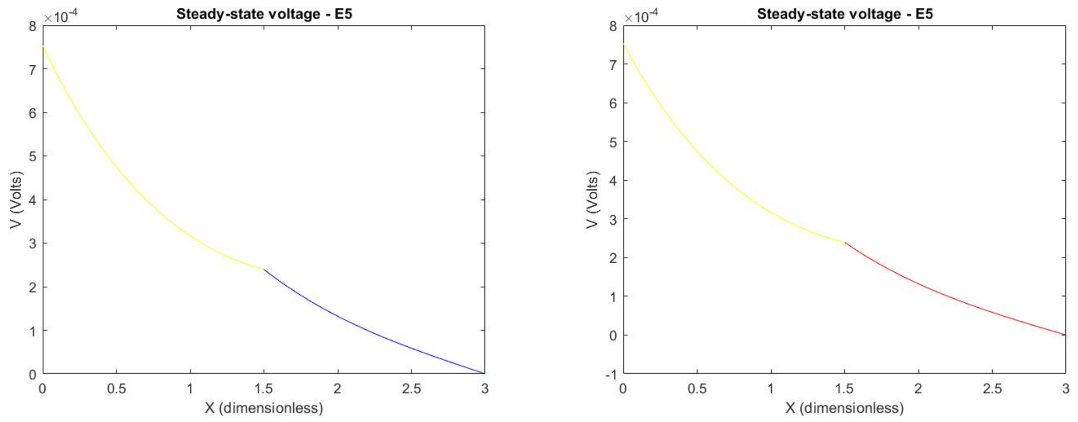
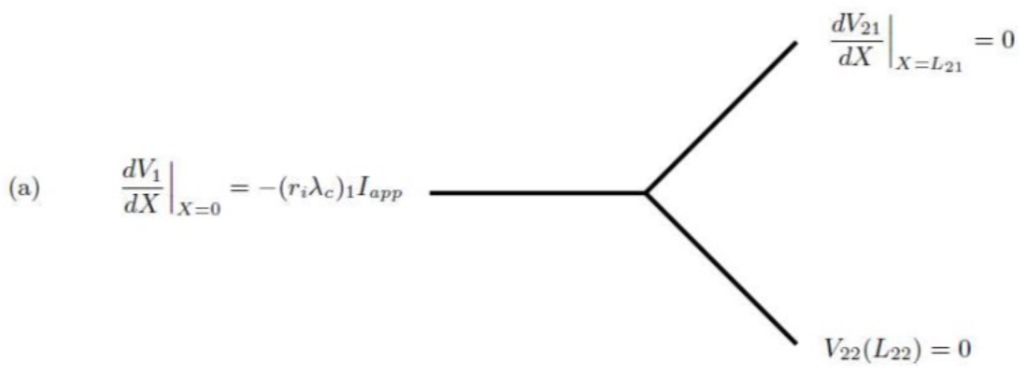


Figure 2: 04

Though I have shown two plots, both blue and red overlap with each other. The yellow line indicates the steady-state voltage profile of the parent branch. Since red and blue overlap, both daughter branches have the same steady voltage profile.

Question 5

For the each given conditions,




```
%% Question 05%%
%% part (a)%%
Aa = A;
Aa(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
x= Aa\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X (dimensionless)');
ylabel('V (Volts)');
title('Steady-state voltage - E5');
```

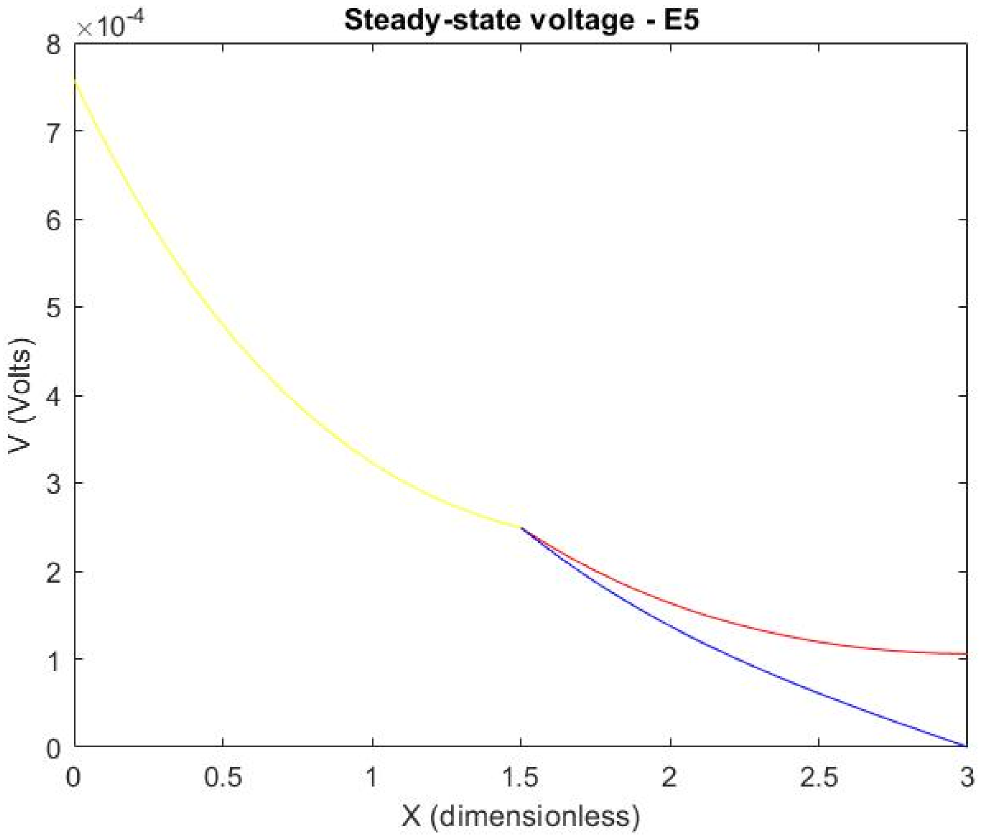
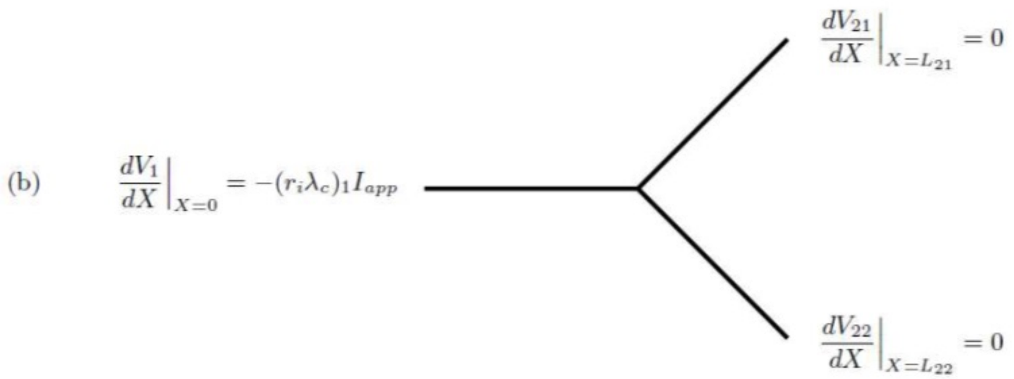


Figure 3: 05.01




```
%% part (b)%%
Ab = Aa;
Ab(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x=Ab\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

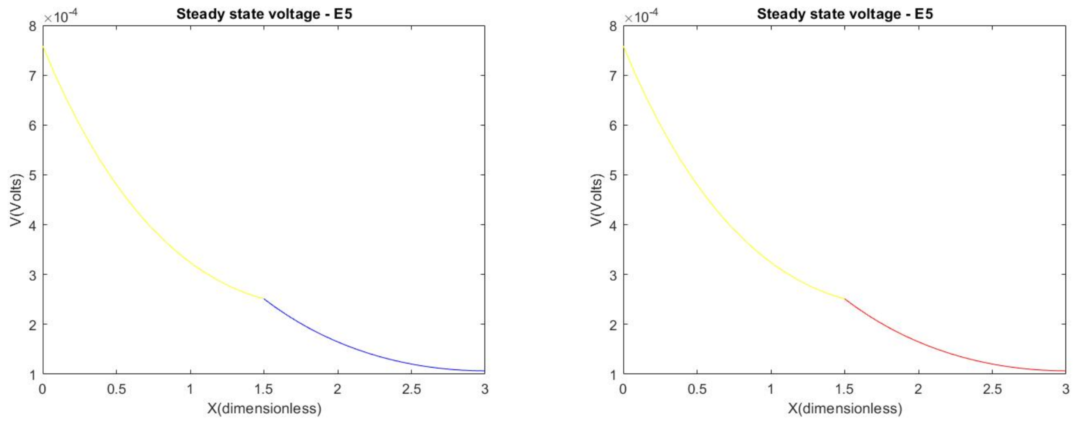
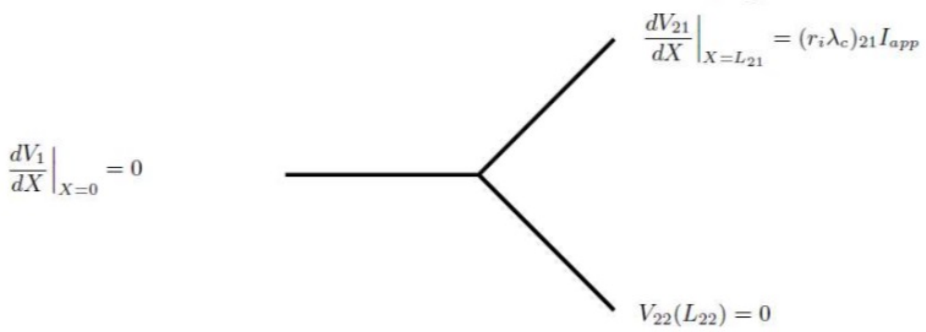


Figure 4: 05.02

(c)




```
%% part (c)%%
Ac = A;
Ac(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
bc = b;
bc(1) = 0;
bc(2) = rl21*iapp;
x=Ac\bc;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

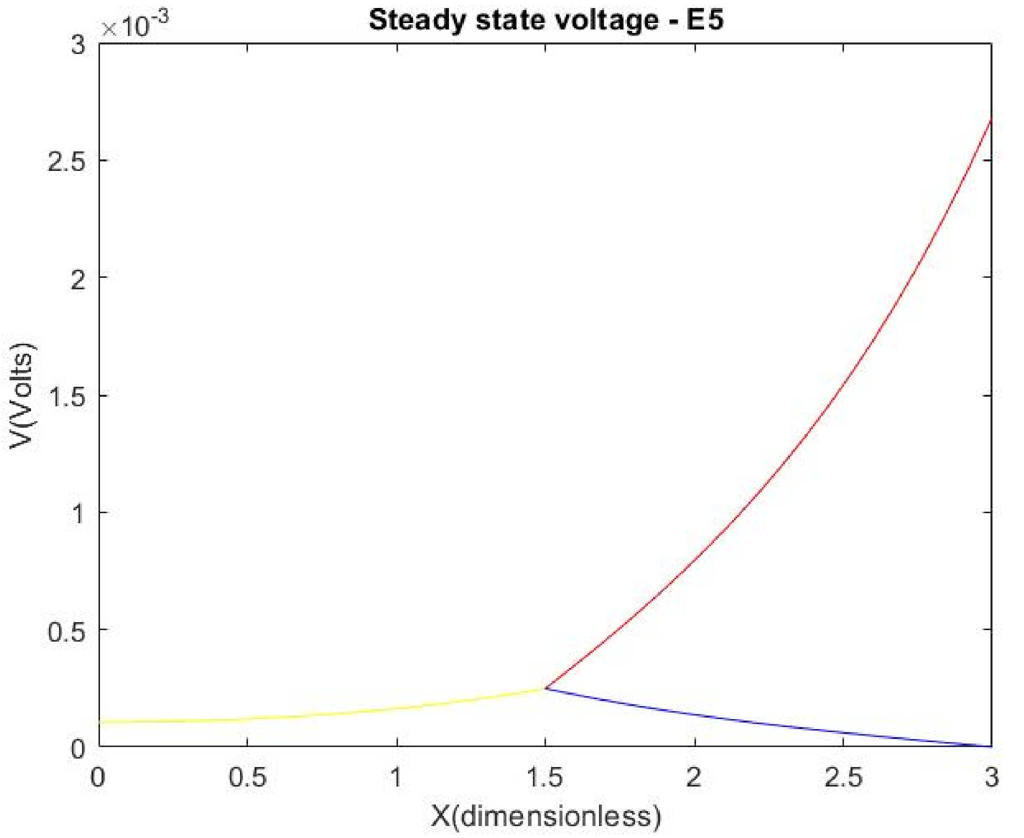
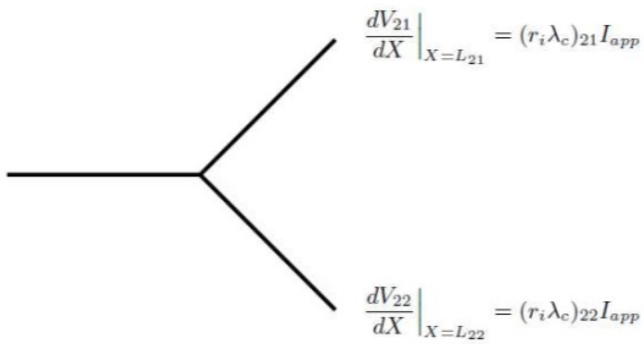


Figure 5: 05.03

(d) $\left.\frac{dV_1}{dX}\right|_{X=0} = 0$




```
%% part (d)%%
bd = bc;
bd(3) = rl22*iapp;
x=Ac\bd;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

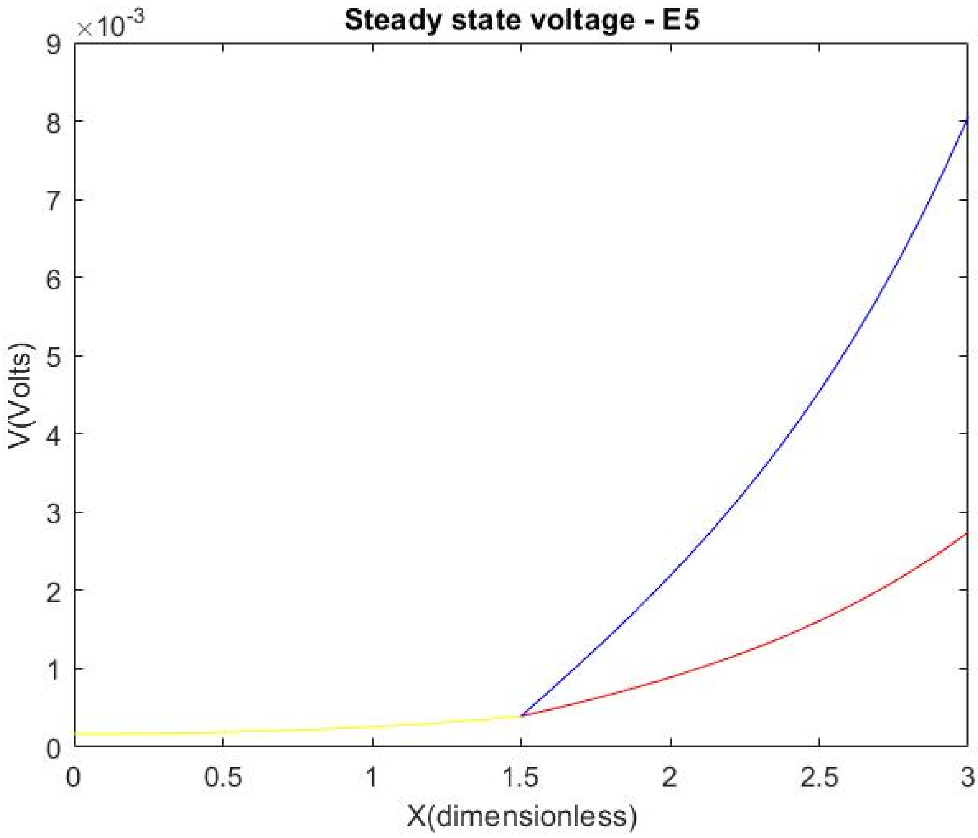
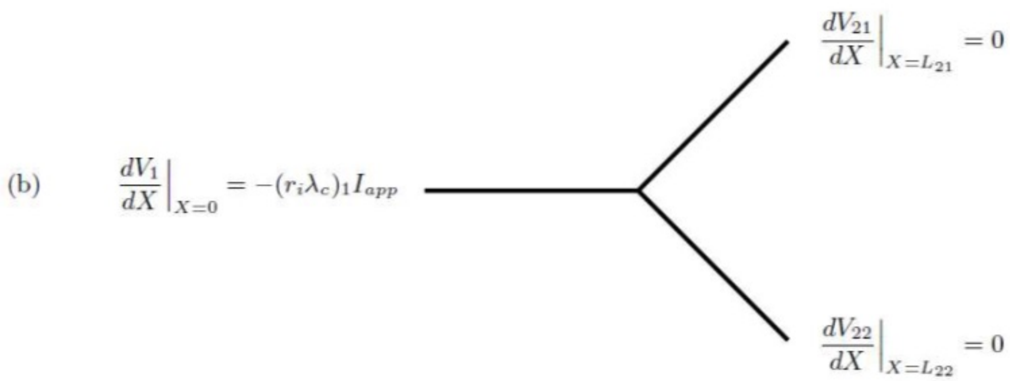


Figure 6: 05.03

Question 6

```
% electrical constants and derived quantities for typical
% mammalian dendrite

% Dimensions of compartments

d1 = 75e-4;                          % cm
%d21 = 30e-4;                        % cm
%d22 = 15e-4;                        % cm
d21 = 47.2470e-4;           % E9 cm
d22 = d21;                   % E9 cm

l1 = 1.5;                            % dimensionless
l21 = 3.0;                           % dimensionless
l22 = 3.0;                           % dimensionless

% Electrical properties of compartments

Rm = 6e3;                            % Ohms cm^2
Rc = 90;                             % Ohms cm
Rs = 1e6;                            % Ohms

c1 = 2*(Rc*Rm)^(1/2)/pi;
```

```
rl1 = c1*d1^(-3/2);                    % Ohms
rl21 = c1*d21^(-3/2);                  % Ohms
rl22 = c1*d22^(-3/2);                  % Ohms


% Applied current

iapp = 1e-9;     % Amps

% Coefficient matrices

A = [1 -1 0 0 0 0;
     0 0 exp(-l21) exp(l21) 0 0;
     0 0 0 0 exp(-l22) exp(l22);
     exp(-l1) exp(l1) -exp(-l1) -exp(l1) 0 0;
     0 0 exp(-l1) exp(l1) -exp(-l1) -exp(l1);
     -exp(-l1) exp(l1) rl1*exp(-l1)/rl21 -rl1*exp(l1)/rl21
         rl1*exp(-l1)/rl22 -rl1*exp(-l1)/rl22];

b = [rl1*iapp 0 0 0 0]';

%% Question 06%%
```




```
%% Figure - part (b)%%
Ab=A;
Ab(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
Ab(3,:) = [0 0 0 0 -exp(-l22) exp(l22)];
x=Ab\b;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

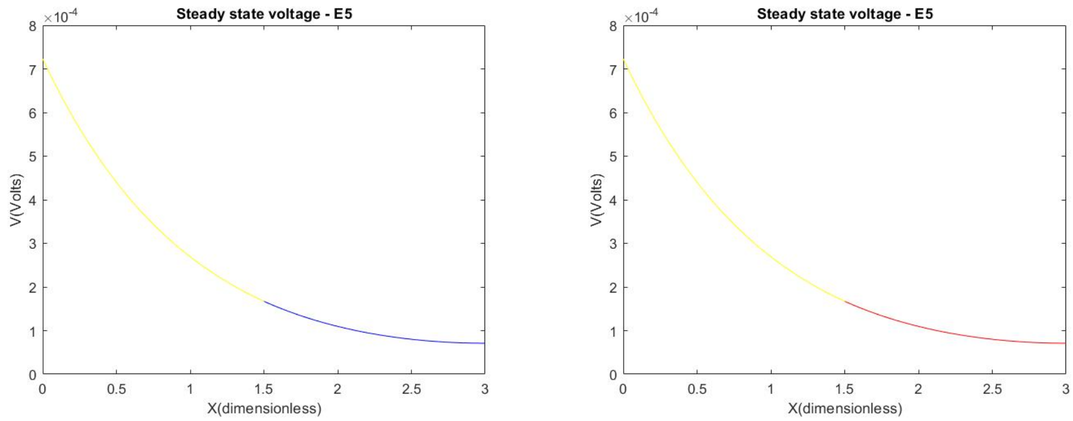
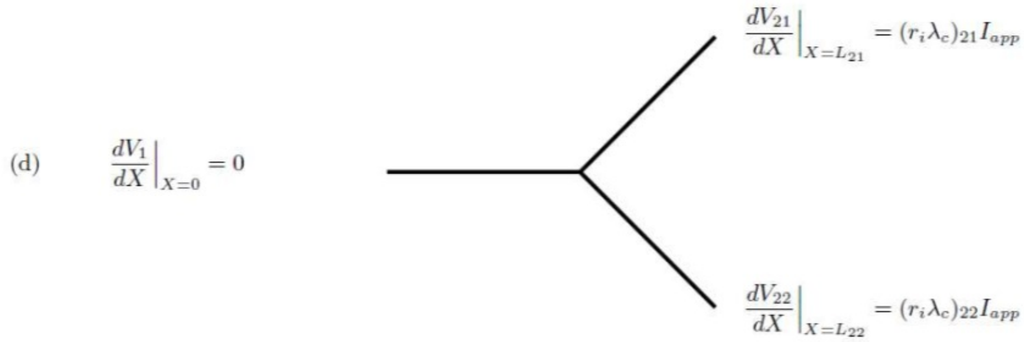


Figure 7: 06.01




```
%% Figure, part (d)%%
Ad = A;
Ad(2,:) = [0 0 -exp(-l21) exp(l21) 0 0];
bd = b;
bd(1) = 0;
bd(2) = rl21*iapp;
bd(3) = rl22*iapp;
x=Ad\bd;
y1=linspace(0,l1,20);
y21=linspace(l1,l21,20);
y22=linspace(l1,l22,20);
v1=x(1)*exp(-y1)+x(2)*exp(y1);
v21=x(3)*exp(-y21)+x(4)*exp(y21);
v22=x(5)*exp(-y22)+x(6)*exp(y22);
plot(y1,v1,'y-',y21,v21,'r-',y22,v22,'b-');
xlabel('X(dimensionless)');
ylabel('V(Volts)');
title('Steady state voltage - E5');
```

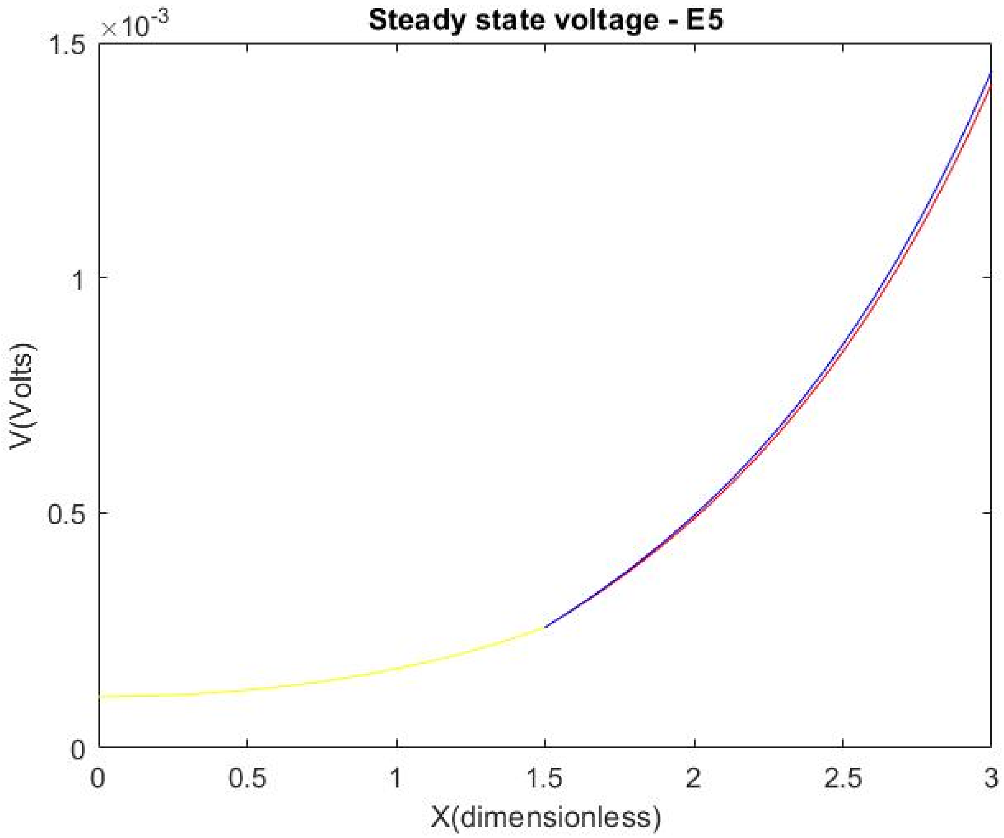


Figure 8: 06.02

When comparing both graphs after changing $d21 = d22 = 47.2470 * 10^4 cm$ both plots became continuously differentiable throughout. Also, there's a slight variation in the two daughter branches which is negligible.