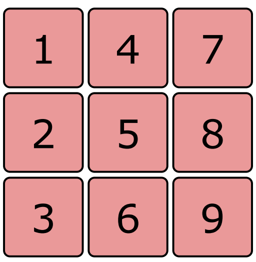
Challenge: The Matrix Chain Multiplication

In this lesson, you will solve a problem to find the minimum number of primitive multiplications required for a matrix chain multiplication.

We'll cover the following

Problem statement

Remember how matrix multiplication works. Given two matrices A and B of dimensions $(n \times m)$ and $(m \times l)$, the resulting matrix we get is AB whose dimensions are $(n \times l)$. We've shown how matrix multiplication occurs in the example below.



(3 x 3)

Matrix A

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2)

Matrix B

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 3) (3 x 2)

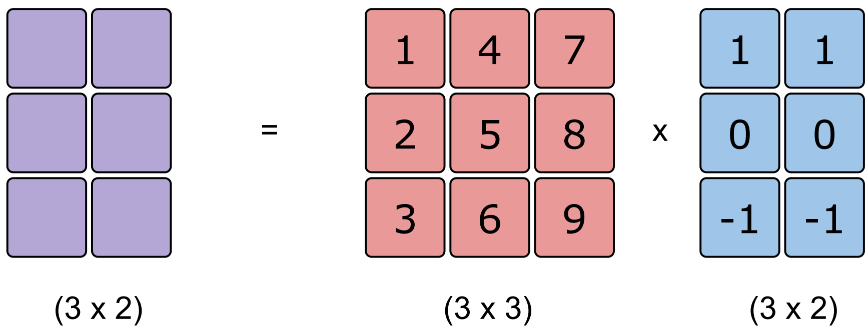
Since number of columns of A = number of rows of B, AB is fit for matrix multiplication

$$\underset{(3 \times 3)}{\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}} \quad \underset{(3 \times 2)}{\begin{bmatrix}1 & 1\\ 0 & 0\\ -1 & -1\end{bmatrix}}$$

The dimensions of AB is equal to number of rows of A x number of columns of B

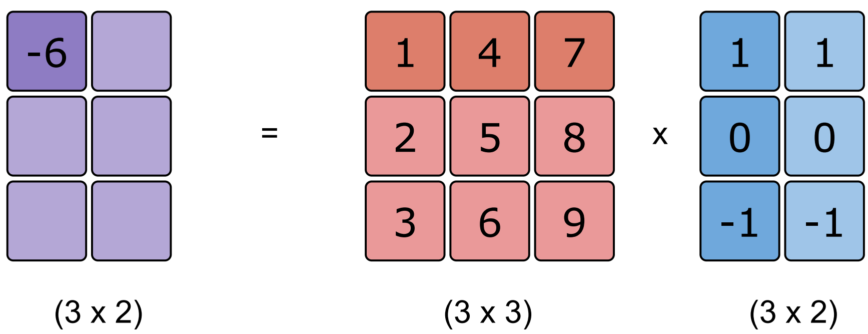
$$\underset{(3 \times 2)}{\begin{bmatrix} & \\ & \\ & \end{bmatrix}} = \underset{(3 \times 3)}{\begin{bmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{bmatrix}} \times \underset{(3 \times 2)}{\begin{bmatrix}1 & 1\\ 0 & 0\\ -1 & -1\end{bmatrix}}$$

The dimensions of AB is equal to number of rows of A x number of columns of B



Number of individual multiplications =

Now let's evaluate value for each entry in AB matrix, while we do this let's also calculate number of individual multiplications we do



Number of individual multiplications = 3

Evaluating (1,1) of AB

$$\begin{bmatrix} -6 & -6 \\ & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3

Evaluating (1,2) of AB

$$\begin{bmatrix} -6 & -6 \\ -6 & \\ & \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3 + 3

Evaluating (2,1) of AB

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \\ & \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3 + 3 + 3

Evaluating (2,2) of AB

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \\ -6 & \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3 + 3 + 3 + 3

Evaluating (3,1) of AB

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \\ -6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3 + 3 + 3 + 3 + 3

Evaluating (3,2) of AB

$$\begin{bmatrix} -6 & -6 \\ -6 & -6 \\ -6 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \end{bmatrix}$$

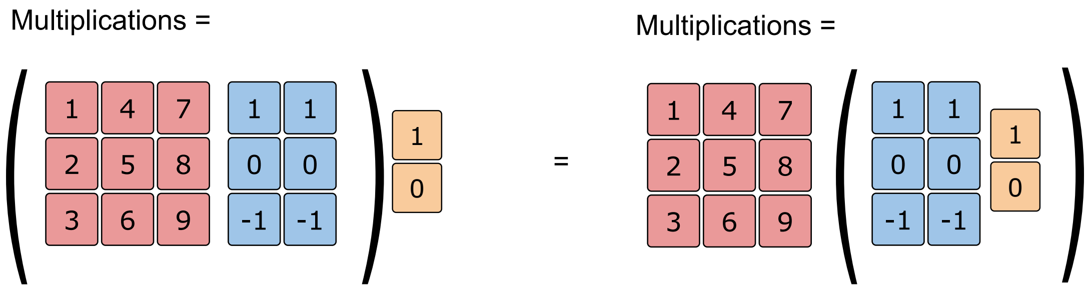
(3 x 2) (3 x 3) (3 x 2)

Number of individual multiplications = 3 + 3 + 3 + 3 + 3 + 3 = 18

Notice how total number of individual multiplications is equal to 3 x 3 x 2 or (number of rows of A x number of columns of A x number of columns of B)

You can see in the above visualization the number of primitive multiplications that take place during the evaluation of AB. To multiply two matrices of dimensions $(n \times m)$ and $(m \times l)$, the total number of primitive multiplications required are $n \times m \times l$.

One thing you might remember from matrices is that $AB \neq BA$, meaning that the matrix multiplication is non-commutative. But matrix multiplication satisfies associative law, i.e., $(AB)C = A(BC)$. This means that the order in which chain multiplication of more than two matrices takes place does not affect the correctness of the result. One thing it does affect, however, is the number of primitive multiplications that we have to do.



Associativity of matrices, and number of multiplications

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Multiplications = 18

$$\left(\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1&1\\0&0\\-1&-1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}-6&-6\\-6&-6\\-6&-6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

=

Multiplications = 6

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\left(\begin{bmatrix}1&1\\0&0\\-1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Associativity of matrices, and number of multiplications

Multiplications = 18 + 6

$$\left(\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1&1\\0&0\\-1&-1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}-6&-6\\-6&-6\\-6&-6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix}-6\\-6\\-6\end{bmatrix}$$

=

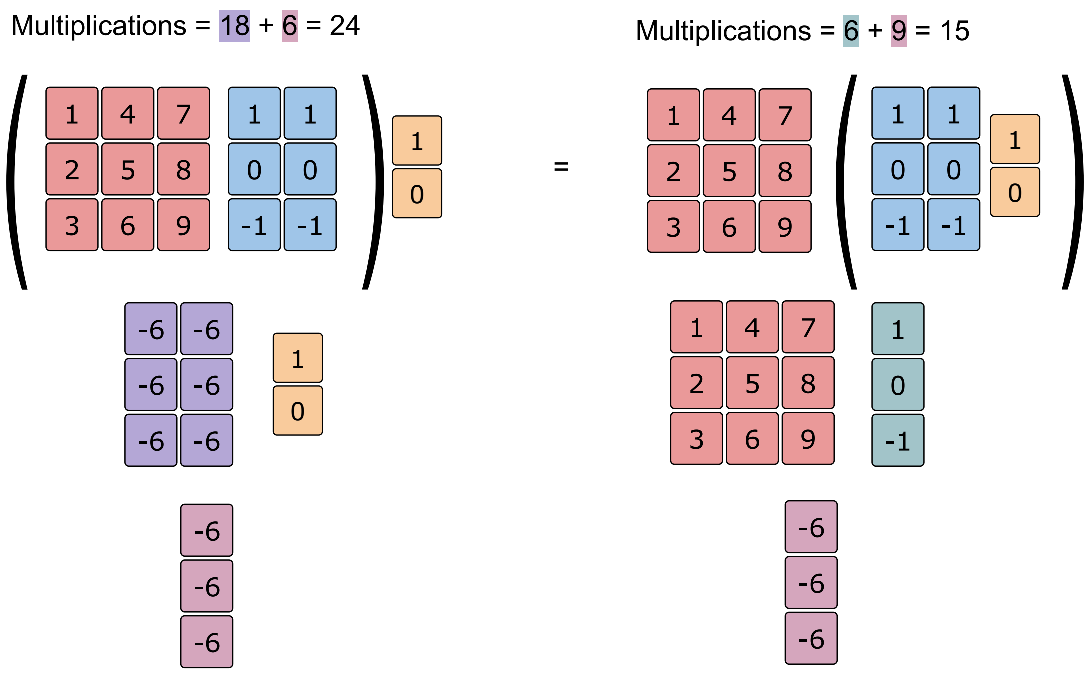
Multiplications = 6 + 9

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\left(\begin{bmatrix}1&1\\0&0\\-1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)$$

$$\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\begin{bmatrix}-6\\-6\\-6\end{bmatrix}$$

Associativity of matrices, and number of multiplications



Same result, different number of multiplications

Now that you have grasped the basics of matrix multiplications, let's look at the problem. You are given a chain of matrices to be multiplied. You have to find the least number of primitive multiplications possible to evaluate the result.

Input

Your algorithm will take as input `dims`, a list denoting the dimensions of the matrices to be multiplied given. Since columns of the first matrix and rows of the second matrix are the same, we have not repeated this information in the list `dims`. So now the dimensions of the first matrix would be $(dims[0] \times dims[1])$ and the second matrix would have dimensions of $(dims[1] \times dims[2])$. Similarly, the dimensions of the third matrix would be $(dims[2] \times dims[3])$.

```
dims = [3,3,2,1]
```

Output

Your algorithm would return the least number of primitive multiplications required to multiply the chain of matrices whose dimensions are given to you in `dims`.

```
MinMultplications([3,3,2,1]) = 15
```

Coding challenge

First, write down a few examples on a piece of paper and try to solve them manually. Once you have done that, write a brute force algorithm and then think about how you can make it more efficient. Best of luck!

```
import numpy as np

# to get maximum value of int, you can use np.inf

def minMultiplications(dims):
    # write your code here
    return 0

stressTesting = True
```

Hint 1 of 1

Think in context of placing two pairs of parentheses at each step e.g. $(M_1M_2)(M_3M_4)$. Find the most optimal placement at each step by exploring all the options.

In the next lesson, we will go over some solutions to this problem.