Finite Element Methods for Partial Differential Equations

Final Project :

1. (1) Please explain why you need to study the numerical methods of partial differential equations.

   (2) Please illustrate the advantage and shortcoming of Galerkin finite elements.

   (3) Please compare Glaerkin finite element methods and finite difference methods.

2. Following the format on page 81-85 of the lecture slides of Chapter 3 for Neumann boundary conditions $\sigma(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{p}$, derive the weak formulation, corresponding Galerkin formulation and the matrix formula.

   Consider the linear elasticity problem:

   $$\begin{cases} -\nabla \cdot \sigma = \mathbf{f} & in \quad \Omega, \\ \mathbf{u} = \mathbf{g} & on \quad \Gamma. \end{cases}$$

   where

   $$\mathbf{u} = (u_1, \, u_2)^t, \qquad \mathbf{g} = (g_1, \, g_2)^t, \qquad \mathbf{f} = (f_1, \, f_2)^t,$$

   the stress tensor

   $$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}, \qquad \sigma_{ij} = \lambda \, (\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u}),$$

   the strain tensor

   $$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}, \qquad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u_i}}{\partial \mathbf{x_j}} + \frac{\partial \mathbf{u_j}}{\partial \mathbf{x_i}}\right),$$

   and

   $$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

   Here λ and μ are Lamé parameters.

3. Consider the following 2D elliptic equation:

   $$-\nabla \cdot (c(x, y)\nabla u(x, y)) = f(x, y) \text{ in [left,right]} \times \text{[bottom,top]},$$

   $$u(x, y) = g(x, y) \text{ on } \partial\Omega,$$

   where

   $$c(x, y) = \begin{pmatrix} c_{11}(x, y) & c_{12}(x, y) \\ c_{21}(x, y) & c_{22}(x, y) \end{pmatrix}.$$

$f(x, y)$ and $g(x, y)$ are chosen such that the analytic solution is $u(x, y) = \exp^{x+y}$ and $c(x, y) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

(1) Following the format on pages 81-85 of the lecture slides of Chapter 3 for Neumann boundary conditions, derive the weak formulation, corresponding Galerkin formulation and the matrix formula.

(2) Solve the above equation on the domain $[0, 2] \times [0, 1]$ and provide the maximum/L2/H1 errors for mesh size $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$. Please plot your mesh information for $h = 1/4$.

4. Consider the following 2D parabolic equation:

$$u_t(x, y, t) - \nabla \cdot (c(x, y)\nabla u(x, y, t)) = f(x, y, t) \quad \text{in } \Omega \times [0, T],$$

$$u(x, y, t) = g(x, y, t) \quad \text{on } \partial\Omega \times [0, T],$$

$$u(x, y, 0) = u_0(x, y) \quad \text{on } \Omega,$$

where $\Omega = [left, right] \times [bottom, top]$.

(1) Deriving the matrix formulation and using the θ-scheme (following Page 38 in Chapter 4) for temporal discretization.

(2) Solve the above equation on the domain $\Omega = [0, 2] \times [0, 1]$ and $T = 1$. $f(x, y, t)$ and $g(x, y, t)$ are chosen such that the analytic solution is $u(x, y, t) = \exp^{x+y+t}$ and $c(x, y) = 2$. The mesh size $h = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$.

* For the linear finite element, choose $\Delta t = h$ when $\theta = 1/2$ and $\Delta t = 4h^2$ when $\theta = 1$.
* For the quadratic finite element, choose $\Delta t^2 \approx h^3$ when $\theta = 1/2$ and $\Delta t = 8h^3$ when $\theta = 1$.

Provide the maximum/L2/H1 errors for $T = 1$ only.

Group Work

1. Consider the 2D linear elasticity equation:

$$\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \quad \text{on} \quad \Omega$$

$$u_1 = 0,\ u_2 = 0 \quad \text{on} \quad \partial\Omega$$

where $\Omega = [0,1] \times [0,1]$, $\lambda = 1$ and $\mu = 2$. The analytic solution of this problem is

$$u_1 = \sin(\pi x)\sin(\pi y)$$

$$u_1 = x(x-1)y(y-1)$$

The right term f_1 and f_2 can be derived once we plug the analytic solutions into the elasticity equation. Solve the above equation and provide the maximum/L2/H1 errors for mesh size $h = \frac{1}{8},\ \frac{1}{16},\ \frac{1}{32},\ \frac{1}{64}$ with the linear finite element and quadratic finite element, respectively.

You may turn in your handwriting solutions in person or by email.

The .m files of your Matlab code for each part should be compressed into a single file, use your name as part of the file name, and electronically submitted to 291567422@qq.com together with a .txt file which copies all the maximum/L2/H1 errors in a table format.

All files should be submitted before 30, 2025.

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