The Pontryagin-Thom isomorphism

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Overview

Vector bundles

Definition

A *real vector bundle of rank* n is a map $p : V \to X$ such that

- for every $X \in X$ the set $V_x := p^{-1}(x)$ is endowed with a real vector space structure
- p verifies a local triviality condition, ie there is a *local trivialization* $\{U_\alpha, h_\alpha\}$ that is, a cover of X by the open sets U_α and homeomorphisms $h_\alpha : p^{-1}(U_\alpha) \xrightarrow{\simeq} U_\alpha \times \mathbb{R}^n$ taking V_x to $\{x\} \times \mathbb{R}^n$ by a vector space isomorphism.

Pullback bundle

A map $f : Y \to X$ induces a morphism $f^* : \mathrm{Vect}^n(X) \to \mathrm{Vect}^n(Y)$

$$\begin{tikzcd} f^*V \arrow[r, dashed, "\bar{f}"] \arrow[d, dashed] & V \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

This is called *pullback bundle*. It is actually true that every (complex or real) vectore bundle can be written as a pullback bundle.

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Classification of vector bundles

Consider the following pullbacks where ν is the tautological bundle

$$\begin{tikzcd} V \arrow[r, dashed] \arrow[d, dashed] & \nu \arrow[d] \\ X \arrow[r, "v"'] & BO \end{tikzcd} \qquad \begin{tikzcd} W \arrow[r, dashed] \arrow[d, dashed] & \nu \arrow[d] \\ Y \arrow[r, "w"'] & BU \end{tikzcd}$$

Theorem

Any real vectore bundle $V \to X$ can be written as a pullback like the one on the left. The analogous statement applies for complex vector bundles $W \to Y$ for the diagram on the right.

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Motivation

There's properties that are preserved when we suspend indefenitely a space. Recall the Freudenthal suspension theorem

Theorem

If X is a $n-1$ connected CW complex the map

$$\pi_i X \longrightarrow \pi_{i+1}\Sigma X$$

is an iso for $i < 2n-1$ and surjective for $i = 2n-1$.

Definition

The *stable homotopy group* is defined as

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Or, equivalently

$Sp \coloneqq \varprojlim \left(\mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \mathcal{S}_* \xleftarrow{\Omega} \dots \right)$ We are also interested in defining a functor that sends spaces to spectra

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The functor Σ^∞

Consider

$$Q := \operatorname*{colim}_{m \in \mathbb{Z}} \Omega^m \Sigma^m X$$

as Ω commutes with filtered colimits we have that

$$QX \xrightarrow{\simeq} \Omega Q \Sigma X$$

which will allow us to define $\Sigma^\infty X$ as the spectrum having

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Mod_R

There exists a functor $- \otimes R$ given by

$$\begin{array}{ccccc} \mathcal{S}_* & \xrightarrow{\Sigma^\infty} & \mathrm{Sp} & \xrightarrow{(-)\otimes R} & \mathrm{Mod}_R \\ (X, x) & \longmapsto & \Sigma^\infty X & \longmapsto & \tilde{C}_*(X, x; R) \end{array}$$

which is colimit preserving.

Mod_R

The inclusion $\mathrm{Mod}_R^{\mathrm{dis}}[n] \hookrightarrow \mathrm{Mod}_R$:

Given an R–module M we can make it sit in the derived category of R–modules as a chain complex with M on degree 0 and zeros elsewhere.

$$\cdots \to M \to 0 \to 0 \to \ldots$$

Additionally, we can shift this chain complex so that M is on the nth position

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Definitions

Definition (Thom space 1)

Given a vector bundle over X, $V \to X$, we one point compactify every fiber $V_x \cup \{\infty\} = S^{V_x}$ thus getting a fiber bundle $S^V \to X$. Now we collapse every infinity point of the fibers in one to get the *Thom space* of V, $\mathrm{Th}(V)$.

Definition (Thom space 2)

The *Thom space* of V is the quotient

$$D(V)/S(V)$$

where $D(V)$ is the disk bundle and $S(V) \subseteq D(V)$ is the sphere bundle.

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Colimit definition

With the perspective of definition 1, but written as a colimit we have

Definition (Thom space 3)

From a $n-$rank vector bundle $V \to X$ we get the fiberwise one point compactified bundle $S^V \to X$. Now as the fibers are $n-$spheres the straightened fiber bundle looks like

$$X \longrightarrow BAut_*(S^n) \subseteq \mathcal{S}_*$$

The *Thom space* of V is the pointed colimit (in $\mathcal{S}_*$)

$$\operatorname{colim}_{*X} S^{V_x} \simeq \operatorname{cofib}\left(\operatorname{colim}_X \{\infty\} \longrightarrow \operatorname{colim}_X S^{V_x}\right)$$

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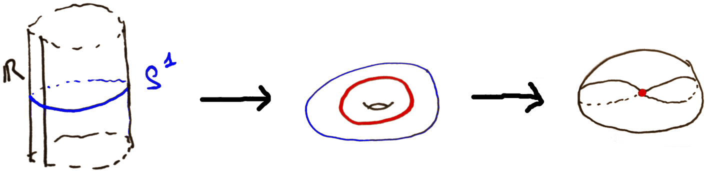
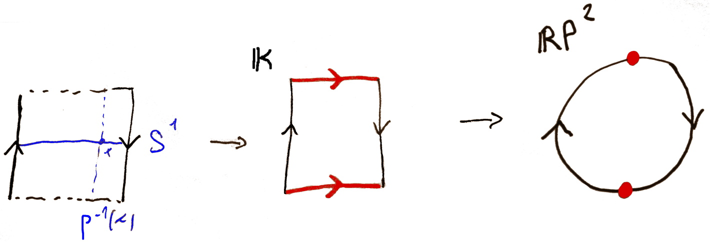
Example: cilinder bundle



Its Thom space is homotopy equivalent to S^1

Example: Möbius bundle



After collapsing the infinity points we obtain the projective plane.

Suspension of a Thom space

Property

Let $V \to X$ be a vector bundle then

$$\Sigma^n \mathrm{Th}(V) = S^n \wedge \mathrm{Th}(V) \simeq \mathrm{Th}(V \oplus \mathbb{R}^n)$$

We will use this property several times, in addition, it gives us a nice description of $(\Sigma^\infty \mathrm{Th}(V))_n$.

Definition

Given a vector bundle $V \to X$ its *Thom spectrum* is the suspension spectrum of its Thom space

$$\mathrm{Th}_{\mathrm{Sp}}(V) = \Sigma^\infty \mathrm{Th}_{\mathcal{S}_*}(V)$$

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J homomorphism

The J-homomorphism is a functor which encodes the information of one point compactifying every fiber. For our purposes we will consider the definition which lands on spectra:

Definition

The *J-homomorphism* is the following functor

$$\begin{aligned} \mathrm{BO} &\longrightarrow \mathrm{Sp} \\ V &\longmapsto \Sigma^{-\dim V}\Sigma^{\infty}S^{V} \end{aligned}$$

Thom Spectrum considerations

Consider the diagram

$$\begin{tikzcd}
 & B \arrow[d, "\xi"] \arrow[dr] & \\
M \arrow[ur, "\varphi"] \arrow[r, "-TM"'] & BO \arrow[r, "J"'] & Sp
\end{tikzcd}$$

We have $\mathrm{Th}_{Sp}(-TM) \simeq \operatorname{colim}_M J \circ (-TM) = \operatorname{colim}_M J \circ \xi \circ \varphi$

Additionally, $M\xi \simeq \operatorname{colim}_B J \circ \xi$, $MO \simeq \operatorname{colim}_{BO} J$

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Towards the Thom iso: Orientability

We want to prove the Thom isomorphism, which states that for a vector bundle $V \to X$ the homology of X is isomorphic to the homology of $\mathrm{Th}(V)$ shifted by $\dim(V)$.

We will need one property on the vector bundle for this to work, tho, which is:

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A vector bundle $V \longrightarrow X$ is called *orientable* if the monodromy action $\gamma_* \curvearrowright V_x$ has positive determinant $\forall \gamma \in \Pi_1(X, x)$.

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The Pontryagin-Thom isomorphism

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Theorem

Given a n-rank orientable vector bundle $V \to X$ then there exists an isomorphism

$$H^i(X;R) \simeq \widetilde{H}^{i+n}(\mathrm{Th}_{\mathcal{S}_*}(V);R)$$

where R is a coefficient ring.

For the proof recall definition 3: $\mathrm{Th}_{\mathcal{S}_*}(V) := \operatorname{colim}^*_X S^{V_x}$

Additionally, remember that for the constant map

$$x \in X \longmapsto * \in \mathcal{S}_*$$

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Proof of Thom iso

This perspective pairs perfectly with the functor

$$\begin{array}{ccc} \mathcal{S}_* & \xrightarrow{(-)\otimes R} & \mathrm{Mod}_R \\ (X, x) & \longmapsto & \tilde{C}_*(X, x; R) \end{array}$$

sending X to the chain complex from which we calculate its homology being colimit preserving.

Proof of Thom iso

We want to prove $\mathrm{Th}_{\mathcal{S}_*}(V) \otimes R \simeq (X \otimes R)[n]$ where $[n]$ shifts n positions to the right. If we're able to prove it naturally for all points we have it. Given

$$\begin{aligned} X &\xrightarrow{\quad \mathrm{th}(V) \quad} \mathcal{S}_* \\ x \in X &\longmapsto S_x = S(V_x) \end{aligned}$$

we want to prove $\mathrm{th}(V) \otimes R \simeq R[n]$ + naturality.

Proof of Thom iso

Naturality in 1−categories vs ∞−categories

$$\begin{tikzcd}
th(V)_x \arrow[r, "\cong"] \arrow[d, "\gamma"] & R[n] \arrow[d, "id_{R[n]}"] \\
th(V)_y \arrow[r, "\cong"] & R[n]
\end{tikzcd}$$

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th(V)_z \arrow[r] \arrow[dr, "\cong"'] & th(V)_y \arrow[dr, "\cong"] & \\
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Proof of Thom iso

Our situation is better

$$\begin{tikzcd}
 & \Pi_1 X \arrow[rr] & & \mathrm{Mod}_R^{\mathrm{dis}} \arrow[dl, hook'] \\
X \arrow[ur] \arrow[r, "\mathrm{th}(V)"] & \mathcal{S}_* \arrow[r, "-\otimes R"] & \mathrm{Mod}_R \arrow[ur, bend right, "H_n"'] &
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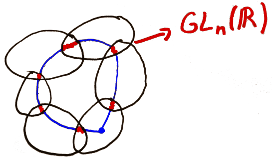
Proof of Thom iso

A loop induces an iso $\gamma_* : V_x \xrightarrow{\cong} V_x$



This will induce a map on homology

$$H_n(S^{V_x}; R) \simeq \mathbb{Z} \otimes R \longrightarrow H_n(S^{V_x}; R) \simeq \mathbb{Z} \otimes R$$

which on $\mathbb{Z}$ will be given by

$$\frac{\det(M)}{|\det(M)|} \in \{-1, 1\}$$

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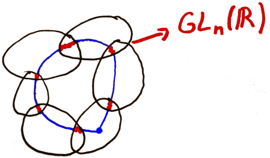

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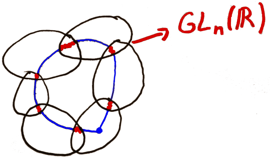

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Finishing proof and application

But as we asked for the vector bundle to be orientable this map is trivially constant one and we get the result!

The Thom isomorphism can be used to prove

$$H^*(BU(k)) \cong \mathbb{Z}[c_1, c_2, \dots, c_k] \text{ with } |c_i| = 2i$$

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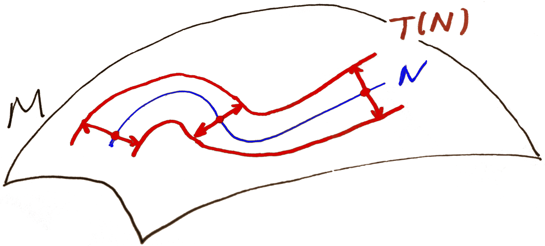
Thom collapse map

If we have an embedding $N \longrightarrow M$ we can consider the normal bundle

$$\mathcal{N}_{N/M} \coloneqq (T_n N)^{\perp} \subseteq T_n M$$

Now choose a tubular neighborhood $T(N)$ of N.



Thom collapse map

The Thom collapse map is given by collapsing the complement of the tubular neighborhood

$$M \longrightarrow M/(M \backslash T(N)) \simeq \mathrm{Th}_{\mathcal{S}_*}(N; \mathcal{N}_{N/M})$$

This mirrors the $D(V)/S(V)$ construction of the Thom space of the normal bundle. Tubular neighborhood theorem + let's check drawing.
Specially interesting the embedding

$$N \hookrightarrow \mathbb{R}^k \hookrightarrow S^k$$

and the resulting homotopy class we get. After applying Σ^∞ it looks like

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ξ—structure

Definition

Given a map $\xi : B \longrightarrow BO$ a *ξ-structure* on a virtual bundle $V : X \longrightarrow BO$ of rank 0 is a lift like the following

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ X \arrow[ur, dashed] \arrow[r, "V"'] & BO \end{tikzcd}$$

Additionally, for a vector bundle V of nonzero rank a ξ-structure is a ξ-structure in $V - \mathbb{R}^{\mathrm{rank} V}$, and a ξ-structure in a manifold M is a ξ-structure on its stable normal bundle $-TM$.

Cobordism ring

Definition

The *cobordism ring*

$$\Omega_*^\xi = \bigoplus_{n \in \mathbb{N}} \Omega_n^\xi$$

is the graded ring with cobordism classes of $n-$manifolds with a ξ-structure on degree n, where cobordisms are required to have a ξ-structure compatible with that of the bordant manifolds.

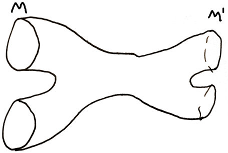

Figure: A cobordism between two manifolds.

The theorem!

Theorem (Pontryagin-Thom iso)

There is an isomorphism

$$\Omega^{\xi}_{\bullet} \cong \pi_{\bullet} M\xi$$

From manifold to homotopy class We have a n−manifold with a ξ−structure

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\underline{\mathbb{R}}^n - TM"'] & B \arrow[d, "\xi"] \\ & BO \end{tikzcd}$$

We will construct a map $\Sigma^n \mathrm{Th}_{Sp}(M; -TM) \longrightarrow \mathrm{Th}_{Sp}(B; \xi) = M\xi$ and a homotopy class

$$\alpha_M \in \pi_0 \mathrm{Th}(M; -TM)$$

such that, when sent trhough the map, we obtain an appropiate class in $\pi_n M\xi$

From manifold to homotopy class We have a $n-$manifold with a $\xi-$structure

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\underline{\mathbb{R}}^n - TM"'] & B \arrow[d, "\xi"] \\ & BO \end{tikzcd}$$

We will construct a map $\Sigma^n \mathrm{Th}_{Sp}(M; -TM) \longrightarrow \mathrm{Th}_{Sp}(B;\xi) = M\xi$ and a homotopy class

$$\alpha_M \in \pi_0 \mathrm{Th}(M; -TM)$$

such that, when sent trhough the map, we obtain an appropiate class in $\pi_n M\xi$

From manifold to homotopy class

As $\varphi^*\xi = -TM$ we get a map between classification pullbacks

$$
\begin{tikzcd}
 & \xi \arrow[dr, dashed] \arrow[dd, dashed] & \\
\underline{\mathbb{R}}^n - TM \arrow[ur, "\overline{\varphi}"] \arrow[rr, dashed] \arrow[ddd, dashed] & & \nu \arrow[ddd] \\
 & B \arrow[ddr, "\xi"'] & \\
 & & \\
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$$

Which induces a map

$$\mathrm{Th}_{\mathbb{S}_*}(M; \underline{\mathbb{R}}^n - TM) \longrightarrow \mathrm{Th}_{\mathbb{S}_*}(B; \xi)$$

Now apply Σ^∞ and we get the desired map

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From manifold to homotopy class

Apply $\pi_n(-)$ to our map to get

$$\pi_n \Sigma^n \mathrm{Th}_{\mathcal{S}p}(M; -TM) \simeq \pi_0 \mathrm{Th}_{\mathcal{S}p}(M; -TM) \longrightarrow \pi_n M\xi$$

Embed M in a sufficiently big sphere $M \hookrightarrow S^N$. Consider the Thom collapse map

$$\Sigma^N \mathbb{S} \simeq \Sigma^\infty S^N \longrightarrow \mathrm{Th}_{\mathcal{S}p}(M; \mathcal{N}_{M/S^N})$$

As $\mathcal{N}_{M/S^N}$ is stably equivalent to $-TM$ we get the desired

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Cobordant manifolds induce homotopic maps:

Enough to consider nulcobordisms: (W, M, M', f, g) can be seen as $(W, M \sqcup M', \varnothing, f \sqcup g, \iota)$.

Given a $(W, M, \varnothing)$, follow the previous construction with the modification that we will a disk to embed W in a way that the boundary of W goes to the boundary of the disk D^{n+1}

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & S^N \arrow[d] \\ W \arrow[r, hook] & D^{N+1} \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, hook] \arrow[dr, "\varphi_W"'] & & W \arrow[dl, "\varphi_M"] \\ & B & \end{tikzcd}$$

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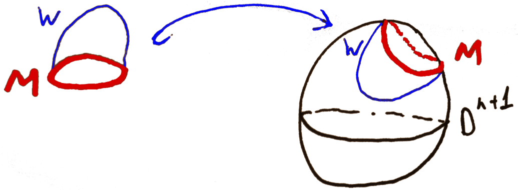

Then after the Thom collapse construction we get a diagram

$$\begin{tikzcd}
\Sigma^N \mathrm{Th}_{\mathrm{Sp}}(M;-TM) \arrow[r] & \Sigma^{N+1} \mathrm{Th}_{\mathrm{Sp}}(W;-TW) \arrow[r] & M\xi \\
\Sigma^\infty \mathbb{S}^N \arrow[u] \arrow[r] & \Sigma^\infty \mathbb{D}^{N+1} \simeq 0 \arrow[u] &
\end{tikzcd}$$

which is the desired nulhomotopy.

Map on the other direction For clearness, we will prove it for MO first and then adapt the proof for $M\xi$ We start with a class

$$\Sigma^n \mathbb{S} \longrightarrow MO$$

we will use compactness of the sphere to take a few steps, better on the blackboard

First recall $MO = \varinjlim MO_i$

$MO_k = Th(BO(k); \nu_k - \underline{\mathbb{R}}^k)$

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From homotopy class to manifold

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We have now a map

$$S^{n+N} \longrightarrow \mathrm{Th}(\mathrm{Gr}_{\mathbb{R}}(k,L); \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N)$$

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From homotopy class to manifold

Definition

Given a differentiable map between manifolds $f : M \longrightarrow N$ and a submanifold inclusion $Z \subseteq N$ we say that M and Z are *transversal* if

$$T_m M \oplus T_{f(m)} Z = T_{f(m)} N \quad \forall m \in M \text{ such that } f(m) \in Z$$

we write $M \pitchfork Z$.

We have at our disposal the following result from [Kosinski, Differentiable Manifolds]:

Theorem

Let Z be a compact submanifold of N, U an open neighborhood of Z in N and $f : M \longrightarrow N$ a smooth map. Then there is an isotopy h_t of N that is the identity outside of U and such that $f \pitchfork h_1(Z)$.

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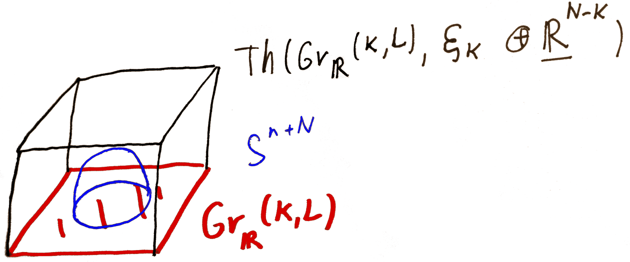
$Gr_{\mathbb{R}}(k, L)$ sits as a $0-$section inside $\nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N$ inside the Thom space. Apply the theorem to

$$\begin{tikzcd}
S^{n+N} \arrow[r, "f"] & Th(Gr_{\mathbb{R}}(k, L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\
U \arrow[u, phantom, "\cup\!\mid"] \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \arrow[u, phantom, "\cup\!\mid"] \\
 & Gr_{\mathbb{R}}(k, L) \arrow[u, phantom, "\cup\!\mid"]
\end{tikzcd}$$

From homotopy class to manifold

Visualization



We will take now M, the preimage of $Gr_{\mathbb{R}}(k, L) \cap f(S^{n+N})$ through f Thanks to the tranversality condition we know that M is a manifold by [Proposition IV.1.4] in the Kosinski book.

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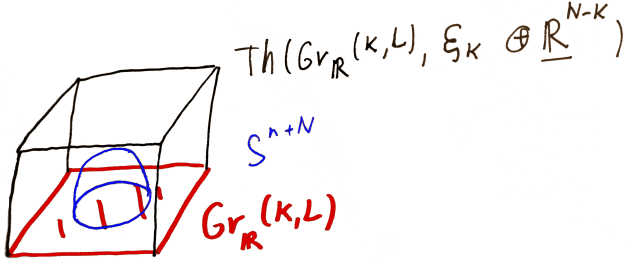

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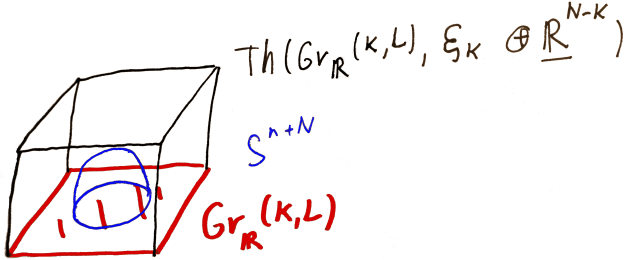

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From homotopy class to manifold

To get that it is $n-$dimensional we look at

$$\begin{tikzcd}
U \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
M \arrow[u, dashed] \arrow[r, dashed] & \mathrm{Gr}_{\mathbb{R}}(k, L) \arrow[u]
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The codimensions in one side and the other are the same. Thus we have $(n+N) - \dim(M) = N \Rightarrow \dim(M) = n$

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Second map: adjustments for $M\xi$

Firstly, let's consider the following pullbacks

$$\begin{tikzcd} B_k \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d] \\ BO(k) \arrow[r] & BO \end{tikzcd} \qquad \begin{tikzcd} B_{k,n} \arrow[r, dashed] \arrow[d, dashed] & B_k \arrow[d] \\ Gr_{\mathbb{R}}(k,n) \arrow[r] & BO(k) \end{tikzcd}$$

Additionally, we have that $B_k = \varinjlim B_{k,n}$ and $M\xi_k = Th_{Sp}(B_k;\xi_k)$.

Half of the argument only depended on the compactness of the sphere, we can get to the map of spaces

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Second map: adjustments for $M\xi$

Postcompose with $M\xi \longrightarrow MO$ to get

$$\begin{tikzcd}
\Sigma^n \mathbb{S} \arrow[r] \arrow[dr] & M\xi \arrow[r] & MO \\
& \mathrm{Th}_{\mathrm{Sp}}(B_k;\xi_k) \arrow[u] \arrow[r] & \mathrm{Th}_{\mathrm{Sp}}(BO(k);\nu_k - \mathbb{R}^k) \arrow[u]
\end{tikzcd}$$

Second map: adjustments for $M\xi$

$$\begin{array}{ccccc}
S^{n+N} & \longrightarrow & \mathrm{Th}_{\mathcal{S}_*}(B_{k,L}, \xi_{k,L} \otimes \mathbb{R}^N) & \xrightarrow{f} & \mathrm{Th}_{\mathcal{S}_*}(\mathrm{Gr}_{\mathbb{R}}(k, L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\
\cup\!| & & \cup\!| & & \cup\!| \\
U & \longrightarrow & \xi_{k,L} \oplus \mathbb{R}^N & \longrightarrow & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
 & & \cup\!| & & \cup\!| \\
 & & B_{k,L} & \longrightarrow & \mathrm{Gr}_{\mathbb{R}}(k, L)
\end{array}$$

Second map: adjustments for $M\xi$

$$
\begin{tikzcd}
U \arrow[r] \arrow[rr, bend left, "\text{smooth \& transversal}"] & \xi_{k,L} \oplus \mathbb{R}^N \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
M \arrow[u, dotted] \arrow[rr, dashed, bend left] & B_{k,L} \arrow[u] \arrow[r] & \mathrm{Gr}_{\mathbb{R}}(k, L) \arrow[u]
\end{tikzcd}
$$

Second map: adjustments for $M\xi$

$$
\begin{tikzcd}
 & & \xi_{k,L} \oplus \underline{\mathbb{R}}^N \arrow[dr] & \\
U \arrow[r] & \mathcal{N}_{S^{n+N} \setminus M} \arrow[ur] \arrow[rr] & & \nu_{k,L} \oplus \underline{\mathbb{R}}^{N-k} \\
 & & B_{k,L} \arrow[uu] \arrow[dr] & \\
M \arrow[uu] \arrow[r] & M \arrow[uu] \arrow[ur] \arrow[rr] & & Gr_{\mathbb{R}}(k, L) \arrow[uu]
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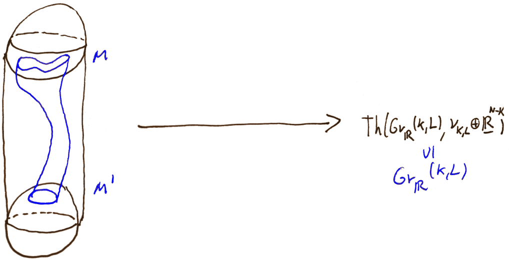
$$

Homotopic maps induce cobordant manifolds

Homotopic maps induce cobordant manifolds The homtopy is realized in the stable range by a homotopy of spaces

$$\begin{array}{ccc} S^{n+N} \times I & \xrightarrow{\quad f \quad} & \mathrm{Th}(\mathrm{Gr}_{\mathbb{R}}(k, L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\ & & \uparrow \subseteq \\ & & \mathrm{Gr}_{\mathbb{R}}(k, L) \end{array}$$

take the preimage:



Conclusions

Useful theorem that in knowable situations allows us to sort cobordism classes of manifolds with a certain $\xi-$structure.

For instance, for the almost complex structure given by a lift to BU, using the homology of MU calculated using the Thom iso + the adams spectral sequence, which is degenerated in this case we get that the cobordism ring of manifolds with an almost complex structure is a polinomial algebra with generators on even degrees. On the manifold side the generators are the complex projective spaces.

Proving something like this by hand could be much harder, if doable.

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The end

Thanks for you attention!