UNIVERSITY OF COPENHAGEN

Master Thesis

Classification of $n-$connective $(2n-2)-$truncated local systems

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May 2023

KØBENHAVNS UNIVERSITET

Contents

Abstract

We prove an equivalence between the $(\infty-)$category of $n-$connective $(2n-2)-$truncated homotopy types, also known as spaces or anima, and a category of copoints in spectra. As a corrollary we obtain a classification of local systems valued on those spaces which generalizes the known gerbe classification. We use stable methods in addition to Beck-Chevalley techniques, a characterization of under-over stable categories and information from the pointed case. On the last chapter we use our theorem to re-examine the case of gerbes and the new case of local systems valued on $[n, n+1]-$spaces.

Acknowledgments

I would like to thank my supervisors Shachar Carmeli and Maxime Ramzi for all their help and guidance during the last few months. I'm grateful for the weekly meetings, question sessions and all time that they dedicated to me. They shared with me their wealth of knowledge and through their passion demostrated how enjoyable a career in math can be. It was a pleasure working with them and I see now how their influence will permeate into my future research.

Introduction

The problem of understanding local systems valued on generalized classifying spaces was motivated by a seminar about the Inmersion Conjecture [Inm] held at Copenhagen. We restricted our scope to generalized classifying spaces with non-zero homotopy groups only on the $[n,\ 2n-2]$ interval to be able to use stable techniques, as indicated by the Freudenthal suspension theorem. The paper is structured as follows.

In Chapter 1, we revert to the case of local systems where the fibers are genuine classifying spaces, that is $k-$gerbes, which we use as motivation. Using modern methods we reprove the Gerbe Classification thm1.8 which states that gerbes are determined by functors $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Ab})$ together with a cohomology class with coefficients in $\mathcal{A}$. To accomplish this we present both types of local systems as maps in An, relate them using H_k homology, and show that cohomology classes appear as the fiber of H_k.

On Chapter 2 we prove our main theorem. First we show the equivalence $\mathrm{An}_*^{[k,2k-2]} \simeq \mathrm{Sp}^{[k,2k-2]}$ in the pointed case, then we introcude some Beck-Chevalley and under-over-category tools and define the central concept of copoints. On Theorem 2.24 we show that $\mathrm{An}^{[k,2k-2]}$ is equivalent to the category of copoints, which we deduce with these tools from the pointed case. As a corollary, we get the equivalence between local systems valued in this subcategory of anima and those valued in copoints.

Lastly, on Chapter 3 we use our newly proved result to revisit $k-$gerbes and get a slightly better result that may or may not already exist on the literature. Additionally, we interrogate the new case of $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$. After applying our classification result we use postnikov towers and DG categories to present all its information in a more tangible way.

As possible further developments, we expect results analogous to our main equivalence to be true for general $\infty-$topoi. Additionally, we believe our result could help progress on the topic of de-thomification, ie finding obstructions to a space being a thom space of a bundle and preimages if possible.

Conventions

We use the language of $\infty-$categories, more information about the topic can be read at Markus Land's [IntroCat]. We consider the following conventions:

Definition. A *local system* on a space X with values in an $\infty-$category $\mathscr{C}$ is simply an object of $\mathrm{Fun}(X, \mathscr{C})$.

An	The $\infty-$category of anima / spaces / homotopy types.
$n-$connective	Refers to an object with trivial homotopy groups for $i < n$.
$m-$truncated	Refers to an object with trivial homotopy groups for $i > m$.
$\mathrm{An}^{[k,l]} \subseteq \mathrm{An}$	The subcategory of $k-$connective $l-$truncated anima.
$\mathrm{Sp}^{[k,l]} \subseteq \mathrm{Sp}$	The subcategory of $k-$connective $l-$truncated spectra.

Chapter 1

Gerbe classification

The classification of $1-$gerbes is a classical result that first appeared in [Giraud] and was later generalized to $k-$gerbes. We will focus on the latter in this chapter, providing a modern proof which will set the base for the analogous, more general theorem from Chapter 2 which constitutes the most important contribution of this paper. We start by showing a classification result for $\mathrm{An}_*^{[k,k]}$ that will provide the appropiate lens through which we should look at $k-$gerbes.

1.1 Quick pointed case

We want to look at $\mathrm{Fun}(X, \mathrm{An}_*^{[k,k]})$ for a given anima X. Let us introduce two key functors, Ω and B, which will participate in an adjunction that makes our result almost a corollary.

Loopspace functor $\Omega : \mathrm{An}_* \to \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An})$ assigns to any pointed anima $(X, x) \in \mathrm{An}_*$ the subanima of $\mathrm{map}_*(|\Delta^1|, X)$ of maps that send all vertices to the basepoint.

$$\begin{aligned} \mathrm{An}_* &\xrightarrow{\quad\Omega\quad} \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An}) \\ (X, x) &\longmapsto \mathrm{map}_*\Big((|\Delta^1|, |\partial\Delta^1|), (X, x)\Big). \end{aligned}$$

The group structure on $\Omega(X, x)$ is given as follows: we consider the simplicial set $\Delta^\bullet_{/i\sim j}$ obtained by collapsing all vertices of $\Delta^\bullet$ into one. This is a cogroup due to the degeneracy maps and we have the functor

$$\begin{aligned} \Delta^{\mathrm{op}} &\xrightarrow{\quad\omega X\quad} \mathrm{An}_* \\ \Delta^\bullet &\longmapsto \mathrm{map}_*\,(|\Delta^\bullet_{/i\sim j}|, X). \end{aligned}$$

Because of the reversed degeneracy maps the image is a grouplike monoid, with ΩX as its $1-$simplices.

Then the group operation is explicitly given by

$$(\Omega X)^{\times 2} \underset{(d_2,d_0)}{\simeq} (\omega X)_2 \xrightarrow{d_1} \Omega X,$$

$$\begin{tikzcd} & x \arrow[dr, dash, "d_0"] & \\ x \arrow[ur, dash, "d_2"] \arrow[rr, dash, "d_1"'] & & x \end{tikzcd}$$

where higher coherences are recorded by ωX.

Additionally, its clear that we can selfcompose Ω however many times we want, each time increasing the commutativity coherences and landing in the next level $\mathrm{Grp}_{\mathbb{E}_{i+1}}(\mathrm{An})$.

Classifying space functor $B : \mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An}) \to \mathrm{An}_*$ is given by the bar construction. For this we consider a simplical diagram like the following

$$\begin{tikzcd} 1 \arrow[r] & G \arrow[r, shift left=2] \arrow[r, shift right=2] & G \times G \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & G \times G \times G \arrow[l, shift left=2] \arrow[l, shift right=2] \arrow[r, shift left=6] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6] & \cdots \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \end{tikzcd}$$

where the maps upwards fill the extra coordinate with the unit and the maps downwards are multiplications of a pair of coordinates, which we geometrically realize.

Now, when can we compose B with itself? If we have $G \in \mathrm{Grp}_{\mathbb{E}_2}(\mathrm{An})$, ie a group object in $\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An})$ by Dunn additivity, the image will be again a group. This is due to the fact that B preserves products as it is given by a colimit over Δ^{op} (which is sifted) of products of the forgetful functor, which preserve products; together with the fact that the monoidal structure in An_* is cartesian and group structures are determined by monoidal information.

Thus we need an $\mathbb{E}_k-$group as input in order to compose B with itself k times.

With this ingredients we can prove the following result:

Theorem 1.1. *For $k \geq 2$ there is an equivalence between pointed $k-$connective $k-$truncated spaces and abelian groups*

$$\mathrm{An}_*^{[k,k]} \simeq \mathrm{Ab}\,.$$

Proof. The recognition principle [May], [Heb, II.21. Corollary] provides us with an adjunction

$$\begin{tikzcd} \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{An}) \arrow[rr, bend left, "B^k"] & & (\mathrm{An}_*)_{\geq k} \arrow[ll, bend left, "\Omega^k"] \end{tikzcd}$$

which restricts to an equivalence $\mathrm{Grp}_{\mathbb{E}_k}(\mathrm{An}) \simeq (\mathrm{An}_*)_{\geq k}$ when considering grouplike monoids.

Now, as we can see on the table on the right that relates groups in anima with their discrete objects, we have that $\mathrm{Grp}_{\mathbb{E}_k}(\mathrm{Set}) \simeq \mathrm{Ab}$ for $k \geq 2$. This together with the fact that $\mathrm{An}_*^{[k,k]}$ are the discrete objects of $(\mathrm{An}_*)_{\geq k}$ grants the result by restring to discrete objects in the equivalence.

Set	An_*
Grps	$\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An}_*)$
Ab	$\mathrm{Grp}_{\mathbb{E}_2}(\mathrm{An}_*)$
Ab	$\mathrm{Grp}_{\mathbb{E}_3}(\mathrm{An}_*)$
$\vdots$	$\vdots$

□

Corollary 1.2. *For $k \geq 2$ and any $X \in \mathrm{An}$, we have an equivalence*

$$\mathrm{Fun}(X, \mathrm{An}_*^{[k,k]}) \simeq \mathrm{Fun}(X, \mathrm{Ab}).$$

Remark 1.3. Now we would like a functor from $\mathrm{An}^{[k,k]}$ into Ab. Notice that we have that $\pi_k \simeq \Omega^k$ as functors $\mathrm{An}_*^{[k,k]} \to \mathrm{Ab}$, since for a space $G \in \mathrm{An}_*^{[k,k]}$ we have

$$\pi_k G \simeq \pi_0 \Omega^k G \simeq \Omega^k G$$

where the second isomorphism comes from the fact that G is $k-$truncated. Moreover, we have $H_k \simeq \pi_k$ as a consequence of the Hurewicz theorem. For this reasons, in the next section we will use H_k as it doesn't require a basepoint.

Let us introduce some auxiliary results that we will need for gerbe classification:

Proposition 1.4. *The classifying space functor B preserves fiber sequences.*

Proof. Given a fiber sequence in $\mathrm{Grp}_{\mathbb{E}_1}(\mathrm{An})$,

$$F \xrightarrow{\alpha} G \xrightarrow{\beta} H$$

we can construct a simplicial map between the bar constructions of G and H by considering

$$\beta \times \overset{k}{\cdots} \times \beta$$

on the k level, clearly making the diagram commute.

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & F \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & F \times F \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[d] & F \times F \times F \arrow[d] \\
1 \arrow[r] \arrow[d] & G \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] \arrow[d] & G \times G \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] \arrow[d] & G \times G \times G \arrow[d] \\
1 \arrow[r] & H \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & H \times H \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & H \times H \times H
\end{tikzcd}$$

Now, we want to determine the fiber of this map in sSet and for that we find the fiber at each level. As products and fibers are both types of limits, they commute and we have that the fiber at level k is the $k-$fold product of the fiber F. As the product structure of F is given by that of G, we have that the resulting simplicial object is in fact the bar construction of F. To conclude, we note that the geometric realization functor preserves finite limits [GZ, 3.1 Theorem], obtaining the result. □

1.2 General $k-$gerbes

Definition 1.5. A *$k-$gerbe* is an object of $\mathrm{Fun}(X, \mathrm{An}^{[k,k]})$.

Remark 1.6. This codifies the classical definition of a $k-$gerbe: given $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{An}^{[k,k]})$, as X is a space/$\infty-$groupoid we only hit isomorphisms on the target. We are thus associating a certain $B^k G \in \mathrm{An}^{[k,k]}$ for every point of X, an isomorphism between the classifying spaces for every path in X and isomorphisms between isomorphisms for higher homotopies on X.

On the remaining of this chapter we will restrict our attention to gerbes with connected base, ie $\pi_0 X = *$, as the general case reduces to this one by considering separatedly all connected components. In order to state the classification result we present cohomology with coefficients on a local system:

Definition 1.7. Given a space X and a local system $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Ab})$, we define its *cohomology with coefficients in $\mathcal{A}$* as

$$H^\bullet(X; \mathcal{A}) := \pi_0 \lim_X B^\bullet \mathcal{A}.$$

Theorem 1.8 (Gerbe classification)**.** *Let $k \geq 2$, there is an equivalence between $k-$gerbes up to isomorphism and local systems on abelian groups together with a cohomology class*

$$\pi_0 \mathrm{Fun}(X, \mathrm{An}^{[k,k]})^{\simeq} \simeq \{\mathcal{A} \in \pi_0 \mathrm{Fun}(X, \mathrm{Ab})^{\simeq},\ \alpha \in H^{k+1}(X, \mathcal{A})\}.$$

Remark 1.9. In order to prove the theorem, we are interested in putting both types of local systems in the same footing. Notice that, as X is connected, any local system on X with values in an $\infty-$categroy $\mathscr{C}$ hits essentially one object, which we refer to as Img. Then we can rewrite any local system as

$$X \longrightarrow B\,\mathrm{Aut}(\,\mathrm{Img})$$

where $B\,\mathrm{Aut}(\,\mathrm{Img}) \subset \mathscr{C}^{\simeq}$ is the full subgrupoid of objects isomorphic to Img in the grupoid core. Thus any local system on X is a map in the category of anima, and also $\mathrm{map}(X, B\,\mathrm{Aut}(\,\mathrm{Img}))^{\simeq} \simeq \mathrm{map}(X, B\,\mathrm{Aut}(\,\mathrm{Img}))$.

Proof of Th 1.8. The key for this theorem will be the functor $H_k : \mathrm{An}^{[k,k]} \to \mathrm{Ab}$. Postcompostion with it pointwise grants us a functor $H_k \circ - : \mathrm{Fun}(X, \mathrm{An}^{[k,k]}) \to \mathrm{Fun}(X, \mathrm{Ab})$. However, we will switch to the framework from Rmk 1.9, to more easily conclude the cohomology part. In this case, we postcompose with the induced $B\,\mathrm{Aut}(H_k)$, so if $F \in \mathrm{An}^{[k,k]}$, $H_k F = A$ and $X \to B\,\mathrm{Aut}(F)$ is a local system, we obtain the following local system valued in Ab:

$$X \longrightarrow B\,\mathrm{Aut}(F) \xrightarrow{B\,\mathrm{Aut}(H_k)} B\,\mathrm{Aut}(A)$$

We **claim** that for any $\mathcal{A} \in \mathrm{map}(X, B\,\mathrm{Aut}(A))$, the following is a fiber sequence in An

$$\lim_X B^{k+1}\mathcal{A} \longrightarrow \mathrm{map}(X, B\,\mathrm{Aut}(F)) \xrightarrow{B\,\mathrm{Aut}(H_k)\circ -} \mathrm{map}(X, B\,\mathrm{Aut}(A))$$

granting the result after applying π_0 as by definition $\pi_0 \lim_X B^{k+1}\mathcal{A} = H^{k+1}(X, \mathcal{A})$.

Proving the claim. We want to show that the fiber of $(B\,\mathrm{Aut}(H_k) \circ -)$ at $\mathcal{A}$ is precisely $\lim_X B^{k+1}\mathcal{A}$. For this first notice that we have a fiber sequence in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}^{[k,k]})$ of the form

$$B^k A \longrightarrow \mathrm{Aut}(B^k A) \xrightarrow{B\,\mathrm{Aut}(H_k)} \mathrm{Aut}_*(B^k A).$$

Hence we know that the fiber at $A \in \mathrm{Ab}$ is $B^k A$ and the functor that takes the fiber through $B\,\mathrm{Aut}(H_k)$ of a certain abelian group is precisely $B^{k+1} : \mathrm{Ab} \to \mathrm{An}^{[k+1,k+1]}$.

Secondly, we wish to modify our perspective on $\mathrm{map}(X, Y)$ to produce a rewriting that uses limits. Recall that limits and colimits are right and left adjoints to the constant local system functor

$$\underline{(-)} : \mathrm{An} \rightleftarrows \mathrm{An}^X : \lim_X \qquad\qquad \operatorname*{colim}_X : \mathrm{An}^X \rightleftarrows \mathrm{An} : \underline{(-)}$$

Additionally, we have that $\operatorname*{colim}_X \underline{pt} \simeq X$. We can then use both adjunctions to get

$$\mathrm{map}_{\mathrm{An}}(X, Y) \simeq \mathrm{map}_{\mathrm{An}}(\operatorname*{colim}_X \underline{\mathrm{pt}}, Y) \simeq \mathrm{map}_{\mathrm{An}^X}(\underline{\mathrm{pt}}, \underline{Y}) \simeq \mathrm{map}(*, \lim_X \underline{Y})$$

Now we can rewrite our fiber sequence (left) as the diagram on the right, as in particular we have
$* \simeq \mathrm{map}(X, *) \simeq \mathrm{map}_{\mathrm{An}^X}(\underline{\mathrm{pt}}, \underline{\mathrm{pt}}) \simeq \mathrm{map}(*, \lim_X \underline{\mathrm{pt}})$

$$\begin{tikzcd}
\mathrm{Fib} \arrow[r, dashed] \arrow[d, dashed] & \mathrm{map}(X, B\,\mathrm{Aut}(F)) \arrow[d] \\
* \arrow[r, "\{\mathcal{A}\}"] & \mathrm{map}(X, B\,\mathrm{Aut}(A)),
\end{tikzcd}
\qquad
\begin{tikzcd}
\lim_X \underline{\mathrm{pt}} \times_{\lim_X \underline{B\,\mathrm{Aut}(A)}} \lim_X \underline{B\,\mathrm{Aut}(F)} \arrow[r, dashed] \arrow[d, dashed] & \lim_X \underline{B\,\mathrm{Aut}(F)} \arrow[d] \\
\lim_X \underline{\mathrm{pt}} \arrow[r, "\mathcal{A}"] & \lim_X \underline{B\,\mathrm{Aut}(A)}.
\end{tikzcd}$$

As the limit functor preserves pullbacks, we deduce an isomorphism

$$\mathrm{Fib} \simeq \lim_X \underline{\mathrm{pt}} \times_{\lim_X \underline{B\,\mathrm{Aut}(A)}} \lim_X \underline{B\,\mathrm{Aut}(F)} \simeq \lim_X \Big(\underline{\mathrm{pt}} \times_{\underline{B\,\mathrm{Aut}(A)}} \underline{B\,\mathrm{Aut}(F)}\Big).$$

If we are able to prove that the object inside the limit is $B^{k+1}\mathcal{A}$, we would be done. This is clear as we are lifting $\mathcal{A}$ though $B\,\mathrm{Aut}(H_k)$, that is postcomposing with the functor that takes fiber through $B\,\mathrm{Aut}(H_k)$, which we determined was exactly B^{k+1}.

$$
\begin{tikzcd}
 & \mathrm{An} \arrow[d] \\
X \arrow[ur, "\underline{B\,\mathrm{Aut}(F)}"] \arrow[r, "\underline{B\,\mathrm{Aut}(A)}"] \arrow[dr, "\underline{\mathrm{pt}}"'] & \mathrm{An} \\
 & \mathrm{An} \arrow[u, "\mathcal{A}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & B\,\mathrm{Aut}(F) \arrow[d, "B\,\mathrm{Aut}(H_k(-))"] & \\
X \arrow[r, "\mathcal{A}"] & B\,\mathrm{Aut}(A) \arrow[r, "B^{k+1}"] & \mathrm{An}
\end{tikzcd}
$$

□

Remark 1.10. Notice that $B\,\mathrm{Aut}(B^k)$ provides a section of $B\,\mathrm{Aut}(H_k)$, as $B^k H_k F \simeq F$. This indicates some sort of product, but we stayed clear of formulating the theorem like that as the cohomology group depends on the element $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Ab})$ on the first coordinate. Hence $(\mathcal{A}, \gamma) \in \mathrm{Fun}(X, \mathrm{Ab}) \times H^{k+1}(X, \mathcal{A})$ does not completely make sense.

Chapter 2

Classification of $[k, 2k-2]$ local systems

We now wish to extend our range of allowed homotopy groups and get a generalization of the result in the previous chapter. Earlier we had an equivalence $H_k : \mathrm{An}_*^{[k,k]} \simeq \mathrm{Ab} : B$, however Ab is not enough if we want to consider spaces $\mathrm{An}^{[k,l]},\ k < l$, as it only accounts for one homotopy group - we will solve this problem by looking at spectra instead, which satisfies $\mathrm{Sp}^{\heartsuit} \simeq \mathrm{Ab}$.

In order to send a space in $\mathrm{An}^{[k,l]}$ to spectra we will be interested in the infinity suspension functor Σ^∞. In favor of deciding which is the longest interval $[k,l]$ possible we consult the Freudenthal suspension theorem

Theorem 2.1. *[Rezk, Th 2.1] If X is an $n-$connected pointed space then $X \xrightarrow{\sigma} \Omega\Sigma X$ is $(2n)-$connected.*

which suggests that $[k, 2(k-1)]$ is the longest interval for which Σ^∞ preserves the homotopy groups of the space, as being in $[k, \infty]$ is the same as being $(k-1)-$connected. With this in mind, on the following section we will work to get an equivalence

$$\mathrm{Fun}(X, \mathrm{An}_*^{[k,2k-2]}) \simeq \mathrm{Fun}(X, \mathrm{Sp}^{[k,2k-2]}).$$

2.1 Pointed version: $\mathrm{Fun}(X, \mathrm{An}_*^{[k,2k-2]})$

There exists an general adjunction $\Sigma^\infty : \mathrm{An}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty$ that we would like to adapt to our categories with homotopy groups concentrated on the interval $[k, 2k-2]$. Let us examine how this functors treat them:

- If we apply Ω^∞ to an object $E \in \mathrm{Sp}^{[k,2k-2]}$ we directly land on $\mathrm{An}_*^{[k,2k-2]}$ as $\pi_n E := [\Sigma^n \mathbb{S}, E] \simeq \pi_{n+m}\Omega^\infty\Sigma^m E$ for $n + m \geq 0$.
- In contrast, applying Σ^∞ to an element $X \in \mathrm{An}_*^{[k,2k-2]}$ can land outside the desired interval in spectra, as our Freudenthal suspension map is only $(2k-2)-$connected and the image might have non-trivial homotopy groups in degrees higher than $2(k-1)$.

To solve this problem we introduce the truncation on spectra $\tau_{\leq m}$ and then prove that there is an adjunction

$$\tau_{\leq 2(k-1)}\Sigma^\infty : \mathrm{An}_*^{[k,2k-2]} \rightleftarrows \mathrm{Sp}^{[k,2k-2]} : \Omega^\infty.$$

Construction 1 (Truncation)**.** To kill a cyclic subgroup of the $m-$th homotopy we pick a generator of the subgroup $\alpha \in \pi_m E = [\Sigma^m \mathbb{S}, E]$ and we cofiber it off - which we will notate as $E/\alpha := \mathrm{cofib}(\alpha)$. This results in a fiber sequence

$$\Sigma^m \mathbb{S} \xrightarrow{\alpha} E \to E/\alpha$$

which as $\pi_m \Sigma^m \mathbb{S} \simeq \mathbb{Z}$ and $\pi_i \Sigma^m \mathbb{S} \simeq 0$ for $i \leq m-1$ induces the LES

$$\dots \longrightarrow \mathbb{Z} \longrightarrow \pi_m E \longrightarrow \pi_m E/\alpha \longrightarrow 0 \longrightarrow \pi_{m-1} E \xrightarrow{\simeq} \pi_{m-1} E/\alpha \longrightarrow 0 \longrightarrow \dots$$

As a consequence, we have that $\pi_i E/\alpha \simeq \pi_i E$ for $i \leq m-1$, whereas on degree m, we have:

As the map $\pi_m E \to \pi_m E/\alpha$ is surjective with kernel generated by the image of 1 through the map in the right, we get that $\pi_m E/\alpha$ is precisely the quotient of $\pi_m E$ by the subgroup generated by alpha.

$$\begin{aligned} \mathbb{Z} &\longrightarrow \pi_m E \\ 1 &\longmapsto \alpha. \end{aligned}$$

Then to kill $\pi_m E$ in one step we just need to pick generators $\alpha_1, \alpha_2, \ldots$ for each subgroup of $\pi_m E$ and cofiber them off with a possibly infinite direct sum of spheres

$$\oplus \Sigma^m \mathbb{S} \xrightarrow{\alpha=(\alpha_1,\alpha_2,\ldots)} E \longrightarrow E/\alpha.$$

This procedure might create new classes in degrees above, which we must continue killing indefinitely upwards in addition to the already existing ones. Taking the filtered colimit of all the steps we obtain an $(m-1)-$truncated spectrum, which we denote by $\tau_{\leq m-1}E$ defining this way $\tau_{\leq m-1}$.

Theorem 2.2. *The following is an equivalence of $\infty-$categories:*

$$\begin{tikzcd} \mathrm{An}_*^{[k,2k-2]} \arrow[rr, bend left, "\tau_{\leq 2k-2}\Sigma^\infty"] \arrow[rr, phantom, "\perp"] && \mathrm{Sp}^{[k,2k-2]} \arrow[ll, bend left, "\Omega^\infty"] \end{tikzcd}.$$

Notation 2.3. To avoid cluttering, in the rest of this section we will denote

$$\boxed{\tau := \tau_{\leq 2k-2}}$$

Before diving into the proof let us introduce a general method:

Definition 2.4. A functor $F : \mathscr{C} \to \mathscr{D}$ is *conservative* if for every morphism f in $\mathscr{C}$ we have that

$$Ff \text{ is an isomorphism} \Rightarrow f \text{ is an isomorphism}.$$

Lemma 2.5. *Given an adjuction $F \dashv G$, if the unit $\eta_X : X \to GFX$ is an isomorphism and G is conservative then F and G are equivalences.*

Proof. In the mentioned conditions, if the counit $\varepsilon_Y : FGY \to Y$ is an isomorphism as well we can conclude that the functors are equivalences. Let us then look at the adjunction's triangle

$$\begin{tikzcd} GY \arrow[r, "\eta_{GY}"] \arrow[dr, "\mathrm{id}"'] & GFGY \arrow[d, "G\varepsilon_Y"] \\ & GY. \end{tikzcd}$$

This diagram presents $G\varepsilon_Y$ as homotopic to the composition of two isomorphisms, as the identity is an iso and the unit too, so it can be reversed. Then conservativity of G implies that ε_Y is an isomorphism. □

Proof of 2.2. Let us first show that it is an adjunction: given $X \in \mathrm{An}_*^{[k,2k-2]}$ and $Y \in \mathrm{Sp}^{[k,2k-2]}$ we have

$$\mathrm{map}_{\mathrm{An}_*^{[k,2k-2]}}(X, \Omega^\infty Y) \simeq \mathrm{map}_{\mathrm{An}_*}(X, \Omega^\infty Y) \simeq \mathrm{map}_{\mathrm{Sp}}(\Sigma^\infty X, Y) \simeq \mathrm{map}_{\mathrm{Sp}^{[k,2k-2]}}(\tau\Sigma^\infty X, Y),$$

where the first isomorphism is due to the fact that, as explained earlier, Ω^∞ automatically lands in $\mathrm{An}_*^{[k,2k-2]}$ on this conditions and the second is the original adjunction on Sp and An_* without further restrictions. As for the third, we use the truncation fiber sequence that exists for every object and every degree

$$\tau_{>2(k-1)}\Sigma^\infty X \longrightarrow \Sigma^\infty X \longrightarrow \tau_{\leq 2(k-1)}\Sigma^\infty X.$$

We apply $\mathrm{map}(-, Y)$ to it, wich turns fiber sequences into cofiber sequences

$$\mathrm{map}(\tau_{\leq 2(k-1)}\Sigma^\infty X, Y) \longrightarrow \mathrm{map}(\Sigma^\infty X, Y) \longrightarrow \mathrm{map}(\tau_{>2(k-1)}\Sigma^\infty X, Y).$$

and get an isomorphism as $\mathrm{map}(\tau_{>2(k-1)}\Sigma^\infty X, Y) \simeq *$ due to Y being $2(k-1)-$truncated.

Proving equivalence Let us look at the unit of our adjunction

$$\eta_X : X \longrightarrow \Omega^\infty\tau\Sigma^\infty X,$$

which we wish to show is an isomorphism. It would be enough if we find that it induces isomorphisms on degrees $[k, 2k-2]$, for that we look at the triangle:

$$\begin{tikzcd} X \arrow[d] \arrow[dr, bend left, "\eta_X"] & \\ \Omega^\infty\Sigma^\infty X \arrow[r, "\Omega^\infty\tau"] & \Omega^\infty\tau\Sigma^\infty X. \end{tikzcd}$$

Where the vertical arrow is an isomorphism on $[k, 2k-2]$ by the Freudenthal suspension thm and the horizontal one is an iso on $[k, 2k-2]$ by definition of $\tau_{\leq 2k-2}$, thus η_X is an isomorphism.

Additionally, the functor Ω^∞ is conservative on connective spectra [HA, 1.4.3.6], so using lemma 2.5 we get that the counit is an isomorphism and thus the result.

□

Corollary 2.6. *Applying the previous equivalence pointwise on* $\mathrm{Fun}(X, -)$ *we get*

$$\mathrm{Fun}(X, \mathrm{An}_*^{[k,2k-2]}) \simeq \mathrm{Fun}(X, \mathrm{Sp}^{[k,2k-2]}).$$

2.2 Beck-Chevalley and under-over-category tricks

In this section we will introduce some general machinery that we will use later on our unpointed discussion:

Construction 2 (Beck-Chevalley)**.** Given the square in the right, where we have right adjoints $L_0 \dashv R_0$ and $L_1 \dashv R_1$, and a homotopy

$$\begin{tikzcd} A \arrow[r, bend left, "L_0"] \arrow[r, phantom, "\perp"] \arrow[d, "f"'] & B \arrow[l, bend left, dashed, "R_0"] \arrow[d, "g"] \\ C \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, dashed, "R_1"] \end{tikzcd}$$

$$g \circ L_0 \overset{H}{\simeq} L_1 \circ f$$

filling it, then we can construct a natural transformation $f \circ R_0 \Rightarrow R_1 \circ g$ as follows:

$$f \circ R_0 \xrightarrow{\eta_1(f \circ R_0)} R_1 \circ L_1 \circ f \circ R_0 \overset{R_1(H^{-1})R_0}{\simeq} R_1 \circ g \circ L_0 \circ R_0 \xrightarrow{(R_1 \circ g)\varepsilon_0} R_1 \circ g.$$

Definition 2.7. [HA, 4.7.4.13] On this conditions, we say that the square is *left adjointable* if the induced natural transformation $f \circ R_0 \Rightarrow R_1 \circ g$ is an isomorphism (it is a homotopy).

Let us explore what properties follow from left adjointability, and use this relation between the two homotopies on the square to prove an useful theorem. This theorem will be used in two different ways in the proof of the classification of local systems.

Lemma 2.8. *Given a left adjointable square with homotopy H and induced homotopy K, the following squares are filled by a homotopy:*

$$\begin{tikzcd} L_1fR_0 \arrow[r, "H^{-1}R_0"] \arrow[d, "L_1K"'] & gL_0R_0 \arrow[d, "g\varepsilon_0"] \\ L_1R_1g \arrow[r, "\varepsilon_1 g"'] & g \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, "\eta_1 f"] \arrow[d, "f\eta_0"'] & R_1L_1f \arrow[d, "R_1H^{-1}"] \\ fR_0L_0 \arrow[r, "KL_0"'] & R_1gL_0 \end{tikzcd}$$

Proof. We will prove the result for the square on the left, the one on the right is analogous. We unravel the definition of the *Beck-Chevalley-induced transformation K* to get the diagram

$$\begin{tikzcd} L_1fR_0 \arrow[r, "H^{-1}R_0"] \arrow[d, "L_1\eta_{1(fR_0)}"'] & gL_0R_0 \arrow[r, "g\varepsilon_0"] & g \\ L_1R_1L_1fR_0 \arrow[r, "L_1R_1(H^{-1})R_0"'] & L_1R_1gL_0R_0 \arrow[r, "L_1R_1g\varepsilon_0"'] \arrow[u, dashed, "\varepsilon_{1(gL_0R_0)}"'] & L_1R_1g \arrow[u, "\varepsilon_1 g"'] \end{tikzcd}$$

Which can be divided in two by writing $\varepsilon_{1(gL_0R_0)}$ on the dotted arrow. The right square can then be filled by naturality of the counit, whereas the left square is filled due to the adjunction's triangle diagram and naturality of the homotopy (or counit), as seen in the following factorization:

$$\begin{tikzcd} L_1fR_0 \arrow[r, "\mathrm{id}"] \arrow[d, "L_1\eta_{1(fR_0)}"'] & L_1fR_0 \arrow[r, "H^{-1}R_0"] & gL_0R_0 \\ L_1R_1L_1fR_0 \arrow[ur, "\varepsilon_{1(L_1fR_0)}"'] \arrow[rr, "L_1R_1(H^{-1})R_0"'] & & L_1R_1gL_0R_0 \arrow[u, "\varepsilon_{1(gL_0R_0)}"'] \end{tikzcd}$$

□

Let us now recall a possible definition for adjunction that uses the unit but avoids talking about triangle diagrams (where coherences need to be accounted for), by using the isomorphism between mapping spaces. The explicit formula for this isomorphism is what we will need:

Definition 2.9. We say that two functors $F : \mathscr{C} \rightleftarrows \mathscr{D} : G$ between $\infty-$categories are *adjoint* if there exists a natural transformation $\eta : \mathrm{id}_{\mathscr{C}} \Rightarrow GF$ such that the composite

$$\mathrm{Map}_{\mathscr{D}}(Fc, d) \xrightarrow{G} \mathrm{Map}_{\mathscr{C}}(GFc, Gd) \xrightarrow{\eta^*} \mathrm{Map}_{\mathscr{C}}(c, Gd)$$

is an isomorphism $\forall c \in \mathscr{C}, d \in \mathscr{D}$.

Remark 2.10. We could analogously use the counit $\varepsilon : FG \Rightarrow id_{\mathscr{D}}$ and ask for the composition

$$\mathrm{Map}_{\mathscr{D}}(c, Gd) \xrightarrow{F} \mathrm{Map}_{\mathscr{D}}(Fc, FGd) \xrightarrow{\varepsilon_*} \mathrm{Map}_{\mathscr{D}}(Fc, d)$$

to be an isomorphism $\forall c \in \mathscr{C}, d \in \mathscr{D}$.

Theorem 2.11. *Let the square depicted in the left be a left adjointable square with homotopy H and Beck-Chevalley-induced homotopy K. Then the right diagram has a homotopy filling it for any $a \in A, b \in B$.*

$$\begin{tikzcd} A \arrow[r, bend left, "L_0"] \arrow[d, "f"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, dashed, "R_0"] \arrow[d, "g"] \\ C \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & D \arrow[l, bend left, dashed, "R_1"] \end{tikzcd}
\qquad
\begin{tikzcd} \mathrm{Map}_A(a, R_0 b) \arrow[r, "adj"] \arrow[d, "f"'] & \mathrm{Map}_B(L_0 a, b) \arrow[d, "g"] \\ \mathrm{Map}_C(fa, fR_0 b) \arrow[d, "Kb"'] & \mathrm{Map}_D(gL_0 a, gb) \arrow[d, "H^{-1}a"] \\ \mathrm{Map}_C(fa, R_1 gb) \arrow[r, "adj"'] & \mathrm{Map}_D(L_1 fa, gb) \end{tikzcd}$$

Proof. Consider a map $a \xrightarrow{h} R_0 b$, we will run it through both composition paths in parallel:

In the left path we use f and Kb to get

$$fa \xrightarrow{fh} fR_0 b \overset{Kb}{\simeq} R_1 gb.$$

Then, we apply L_1 and postcompose with the counit for the adjunction isomorphism:

$$L_1 fa \xrightarrow{L_1 fh} L_1 fR_0 b \overset{L_1 Kb}{\simeq} L_1 R_1 gb \xrightarrow{\varepsilon_{1(gb)}} gb.$$

In the right, we apply L_0 and postcompose with the counit for the adjunction isomorphism:

$$L_0 a \xrightarrow{L_0 h} L_0 R_0 b \xrightarrow{\varepsilon_{0(b)}} b.$$

And then, we just use g and $H^{-1}a$ to get

$$L_1 fa \overset{H^{-1}a}{\simeq} gL_0 a \xrightarrow{gL_0 h} gL_0 R_0 b \xrightarrow{g\varepsilon_{0(gb)}} gb.$$

As isomorphism of functors is checked pointwise, it is enough to check that these two maps are homotopic for an arbitray map $a \xrightarrow{h} R_0 b$. Looking at the diagram formed by the two maps, we notice that we can write $H^{-1}R_0 b$ as a diagonal:

$$\begin{tikzcd} L_1 fa \arrow[r, "H^{-1}a"] \arrow[d, "L_1 fh"'] & gL_0 a \arrow[r, "gL_0 h"] & gL_0 R_0 b \arrow[d, "g\varepsilon_{0(b)}"] \\ L_1 fR_0 b \arrow[r, "L_1 Kb"'] \arrow[urr, "H^{-1}R_0 b"'] & L_1 R_1 gb \arrow[r, "\varepsilon_{1(gb)}"'] & gb \end{tikzcd}$$

Hence the square is filled by a homotopy, as the upper 'triangle' is filled by naturality of H^{-1} and the lower 'triangle' is filled by lemma 2.8. ◻

Proposition 2.12. *Let the following be a left adjointable square:*

$$\begin{tikzcd} A \arrow[r, bend left, "L_0"] \arrow[d, "f"'] \arrow[r, phantom, "\perp"] & B \arrow[l, bend left, dashed, "R_0"] \arrow[d, "g"] \\ C \arrow[r, bend left, "L_1"] \arrow[r, phantom, "\perp"] & D. \arrow[l, bend left, dashed, "R_1"] \end{tikzcd}$$

If L_0 and L_1 are essentially surjective then f being an equivalence implies that g is one as well.

Proof. To prove that g is an equivalence we check fully faithfulness and essential surjectivity.

Fully faithfulness: Given two objects $b, \tilde{b} \in B$, we want an iso

$$g : \mathrm{map}_B(b, \tilde{b}) \xrightarrow{\simeq} \mathrm{map}_D(gb, g\tilde{b}).$$

As L_0 is essentially surjective there exists an object $a \in A$ such that $b \simeq L_0 a$, thus $\mathrm{map}_B(b, \tilde{b}) \simeq \mathrm{map}_B(L_0 a, \tilde{b})$. We can then leverage our adjunctions following the previous theorem and get the chain of isomorphisms

$$\mathrm{map}_B(L_0 a, \tilde{b}) \overset{\mathrm{adj}}{\simeq} \mathrm{map}_A(a, R_0\tilde{b}) \overset{f\ \mathrm{iso}}{\simeq} \mathrm{map}_C(fa, fR_0\tilde{b}) \overset{K\tilde{b}}{\simeq} \mathrm{map}_C(fa, R_1 g\tilde{b}) \overset{\mathrm{adj}}{\simeq} \mathrm{map}_D(L_1 fa, g\tilde{b}) \overset{H^{-1}a}{\simeq} \mathrm{map}_D(gL_0 a, g\tilde{b}).$$

The given composition of isomorphisms is precisely the map induced by g due to what we proved in Th 2.11. Now, as $b \simeq L_0 a$, we have $\mathrm{map}_D(gL_0 a, g\tilde{b}) \simeq \mathrm{map}_D(gb, g\tilde{b})$.

Essential surjectivity: As L_1 is essentially surjective, for all $d \in D$ there exists $c \in C$ such that $L_1 c \simeq d$. Moreover, there exists an object $a \in A$ such that $fa \simeq c$, as f is an isomorphism. Then we have that $L_0 a$ is the preimage as

$$gL_0 a \simeq L_1 fa \simeq d$$

where the iso $L_1 f \simeq gL_0$ is a consequence of the square being left adjointable. $\square$

We have now finished laying down the Beck-Chevalley machinery and we proceed to use it to examine under-over-categories:

Lemma 2.13. *For any presentable $\infty-$category $\mathscr{C}$ and an object $x \in \mathscr{C}$ we can consider the undercategory $\mathscr{C}_{x/}$. In this conditions there is an adjunction*

$$- \amalg x : \mathscr{C} \rightleftarrows \mathscr{C}_{x/} : fgt$$

which, if $\mathscr{C}$ is stable, can be upgraded to an adjunction $-\oplus x : \mathscr{C} \rightleftarrows \mathscr{C}_{x/\ /x} : fgt$. Dually, there's an adjunction

$$fgt : \mathscr{C}_{/x} \rightleftarrows \mathscr{C} : - \times x.$$

Proposition 2.14. *Given a stable $\infty-$category $\mathscr{C}$ and an object $x \in \mathscr{C}$ we have an equivalence*

$$\mathscr{C} \simeq \mathscr{C}_{x/\ /x}.$$

Proof. We want to show that $-\oplus x : \mathscr{C} \to \mathscr{C}_{x/\ /x}$ is an equivalence. For that, as the identity is an equivalence we only need to check the conditions of the previous proposition on the square on the right.

$$\begin{tikzcd}
\mathscr{C}_{/x} \arrow[rr, bend left, "\mathrm{fgt}"] \arrow[rr, phantom, "\perp"] \arrow[d, "\mathrm{id}"'] & & \mathscr{C} \arrow[ll, bend left, "-\times x"] \arrow[d, "-\oplus x"] \\
\mathscr{C}_{/x} \arrow[rr, bend left, "-\amalg x"] \arrow[rr, phantom, "\perp"] & & \mathscr{C}_{x/\ /x} \arrow[ll, bend left, "\mathrm{fgt}"]
\end{tikzcd}$$

Let us begin with **essential surjectivity of the left adjoints**. We have that the forgetful functor meets the condition as in a stable, in particular additive, category we can always choose the zero map $0 : y \to x$ as the preimage of y.

Now for the functor $- \amalg x : \mathscr{C}_{/x} \to \mathscr{C}_{x/\ /x}$, given $y \in \mathscr{C}_{x/\ /x}$ we can consider the unit map $y \to \mathrm{fgt}(y) \amalg x$. This is an isomorphism with inverse the map which is the identity on $\mathrm{fgt}(y)$ and the original under-map on x, so $\mathrm{fgt}(y)$ is the essential preimage $\forall y \in \mathscr{C}_{x/\ /x}$.

Let us now verify that the square is **left adjointable**. First we show existence of a homotopy $-\oplus x \circ \mathrm{fgt} \simeq - \amalg x \circ \mathrm{id}$, which as an iso in $\mathrm{Fun}(\mathscr{C}_{/x}, \mathscr{C}_{x/\ /x})$ can be checked pointwise. Given $z \in \mathscr{C}_{/x}$ we have that $\mathrm{fgt}(z) \oplus x \simeq z \amalg x$ in $\mathscr{C}_{x/\ /x}$ as the following map witnesses:

Consider the map f determined by the matrix

$$f = \begin{pmatrix} \mathrm{id}_z & 0 \\ x_z & \mathrm{id}_x \end{pmatrix}$$

This the inverse of the map determined by the upper triangular matrix with identities in the diagonal and $\omega_z : z \to x$ on position $(0, 1)$.

$$\begin{tikzcd} \mathrm{fgt}(z) \oplus x \arrow[rr, "f"] \arrow[dr] & & z \amalg x \arrow[dl] \\ & x \arrow[ul, bend left] \arrow[ur, bend right, "x_z"'] & \end{tikzcd}$$

Now, checking that the induced Beck-Chevalley natural transformation $\mathrm{fgt} \circ (- \oplus x) \Rightarrow (- \times x) \circ \mathrm{id}$ is an isomorphism in $\mathrm{Fun}(\mathscr{C}, \mathscr{C}_{/x})$ is inmediate by definition of $- \oplus x$.

□

Remark 2.15. We want to show that the inverse of $- \oplus x : \mathscr{C} \to \mathscr{C}_{x/\,/x}$ is the functor $\mathrm{fib} : \mathscr{C}_{x/\,/x} \to \mathscr{C}$ that takes the fiber of the map to x. We first observe that we have

$$\mathrm{fib} \circ (- \oplus x) \simeq \mathrm{id}\,.$$

This is the case as given $y \in \mathscr{C}$, the map $y \oplus x \to x$ is the projection on the x coordinate and thus zero on y and only on y, so it will be the fiber. Now, we just need to precompose the expression with the inverse of $- \oplus x$:

$$\mathrm{fib} \simeq \mathrm{fib} \circ (- \oplus x) \circ (- \oplus x)^{-1} \simeq \mathrm{id} \circ (- \oplus x)^{-1} \simeq (- \oplus x)^{-1}.$$

2.3 Classifying $\mathrm{Fun}(X, \mathrm{An}^{[k,2k-2]})$: copoints

We now intend to remove the basepoint in the target of our local systems, a process which we have learned to be non-trivial during our $k-$gerbe discussion. Removal of the point should be compensated with extra information, in this case we will prove that we need to consider the images in spectra plus a map to the truncated sphere called copoint.

Recall that given a space X, a basepoint $x \in X$ is a choice of a map $* \to X$, and additionally, we always have a unique map $X \to *$ for any anima. Through this lens we can interpet the category of pointed spaces as the under-over-category $\mathrm{An}_{*/\,/*}$ and unpointed spaces as the over-category $\mathrm{An}_{/*}$. Note that we can apply lemma 2.13 to An with $x = *$ to get the following adjunction, where $(-)_+ := - \amalg *$

$$(-)_+ : \mathrm{An} \rightleftarrows \mathrm{An}_* : \mathrm{fgt}. \tag{2.1}$$

With this left adjoint we can define a functor from unpointed anima into spaces:

Definition 2.16. The *unreduced infinity suspension* functor $\Sigma^\infty_+(-)$ is characterized by the triangle

$$\begin{tikzcd} \mathrm{An} \arrow[rr, "\Sigma^\infty_+"] \arrow[dr, "(-)_+"'] & & \mathrm{Sp} \\ & \mathrm{An}_* \arrow[ur, "\Sigma^\infty"'] & \end{tikzcd}$$

or equivalently, by sending the point to the sphere spectrum and preserving colimits. We will be interested in the functor $\tau\Sigma^\infty_+(-) : \mathrm{An}^{[k,2k-2]} \to \mathrm{Sp}^{[0,2k-2]}$, which allows us to define copoints.

Definition 2.17. Given any anima $X \in \mathrm{An}$, its *copoint* in spectra is the image through $\tau\Sigma^\infty_+$ of the unique map $X \to *$. We denote by ω_X the copoint in spectra associated to an anima X:

$$\tau\Sigma^\infty_+ X \xrightarrow{\omega_X} \tau\mathbb{S}.$$

Now if remembering this map accounts for the information lost when removing the basepoint, we should find that the copoint adds nothing in the pointed space case.

Remark 2.18. Given $X \in \mathrm{An}_* \simeq \mathrm{An}_{*/\,/*}$ we look at what happens when we apply $\tau\Sigma^\infty_+$ and we remember the copoint. First, $(-)_+$ adds a second basepoint. Notice that the fiber of $X_+ \to S^0$ at $\{0\}$ is X, and that the under-over maps make the fiber sequence split. This is also a split cofiber (consider the pushout $* \leftarrow X \to X_+$):

$$\begin{tikzcd}
 & X \arrow[d] & \tau\Sigma^\infty X \arrow[d] \\
X \arrow[d, bend left] \arrow[r, mapsto, "(-)_+"] & X_+ \arrow[d, bend left] \arrow[r, mapsto, "\tau\Sigma^\infty"] & \tau\Sigma^\infty_+ X \arrow[d, bend left, "\omega_X"] \\
* \arrow[u, bend left, "x"] & S^0 \arrow[u, bend left] & \tau\mathbb{S} \arrow[u, bend left]
\end{tikzcd} \tag{2.2}$$

Then we apply $\tau\Sigma^\infty$, which preserves colimits as both components are left adjoints (in particular τ is left adjoint to the inclusion of truncated spectra in spectra). Therefore, we know that the result is a split (co)fiber sequence, granting an isomorphism

$$\tau\Sigma^\infty_+ X \simeq \tau\Sigma^\infty X \oplus \tau\mathbb{S}$$

and exhibiting the copoint as just a projection. This is intuitively why copoints are not extra information on pointed anima, an idea that we will formalize on proposition 2.23.

Remark 2.19. Notice that for unpointed anima we don't have an splitting as the upward maps are missing, but it is still verified that $\tau\Sigma^\infty X$ is the fiber of the copoint. As this fiber is $k-$connective and both $\tau\Sigma^\infty_+ X$ and $\tau\mathbb{S}$ are in $\mathrm{Sp}^{[0,2k-2]}$, we have that copoints belong to the following category

Definition 2.20. We define $(\mathrm{Sp}^{[0,2k-2]})^{\text{k-conn}}_{/\tau\mathbb{S}}$ to be the full subcategory of objects of $(\mathrm{Sp}^{[0,2k-2]})_{/\tau\mathbb{S}}$ whose fiber is $k-$connective, i.e. the following pullback, where the right vertical map is taking the fiber of the copoint

$$\begin{tikzcd}
(\mathrm{Sp}^{[0,2k-2]})^{\text{k-conn}}_{/\tau\mathbb{S}} \arrow[r, dashed] \arrow[d, dashed] & (\mathrm{Sp}^{[0,2k-2]})_{/\tau\mathbb{S}} \arrow[d, "\mathrm{fib}"] \\
\mathrm{Sp}^{[k,2k-2]} \arrow[r, hook] & \mathrm{Sp}^{[-1,2k-2]} \, .
\end{tikzcd}$$

Not only we find that the functor lands in this category, but it also completely fills it:

Proposition 2.21. *The functor* $\tau\Sigma^\infty_+ : \mathrm{An}^{[k,2k-2]} \to (\mathrm{Sp}^{[0,2k-2]})^{k-conn}_{/\tau\mathbb{S}}$ *is essentially surjective.*

Proof. We wish to find an essential preimage for every object $\{c : E \to \tau\mathbb{S}\}$ in the target. It is enough to find a way of providing a section for these maps, as it would reveal a splitting

$$E \simeq \mathrm{Fib}(c) \oplus \tau\mathbb{S}$$

that would cast fgt $\Omega^\infty \mathrm{Fib}(c) \in \mathrm{An}^{[k,2k-2]}_*$ as the desired preimage. This is a consequence of $\mathrm{Fib}(c)$ belonging to $\mathrm{Sp}^{[k,2k-2]}$, theorem 2.2 and the discussion in Rmk 2.18.

A section is a map $s : \tau\mathbb{S} \to E$ for which the diagram in the right can be filled by a homotopy, that is, a lift of $\mathrm{id}_{\tau\mathbb{S}}$ through c.

Notice that as $\tau\mathbb{S}$ is $(2k-2)-$truncated, $[\mathbb{S}, \tau\mathbb{S}] \simeq [\tau\mathbb{S}, \tau\mathbb{S}]$ and therefore $\mathrm{id}_{\tau\mathbb{S}} \in \pi_0\tau\mathbb{S}$.

$$\begin{tikzcd} E \arrow[r, "c"] & \tau\mathbb{S} \\ & \tau\mathbb{S} \arrow[u, "\mathrm{id}"'] \arrow[ul, dashed, "s"] \end{tikzcd}$$

It only remains to inspect the LES for the fiber sequence

$$\dots \longrightarrow \pi_0\mathrm{Fib} \longrightarrow \pi_0 E \longrightarrow \underset{\substack{\cup\!\!\!\shortmid \\ \mathrm{id}_{\tau\mathbb{S}}}}{\pi_0\tau\mathbb{S}} \longrightarrow \pi_{-1}\mathrm{Fib} \longrightarrow \dots$$

where as $\pi_{-1}\mathrm{Fib} = 0$ (fiber is $k \geq 2$ connective), the map $\pi_0 E \to \pi_0\tau\mathbb{S}$ is surjective. It is clear then that we can find a preimage for the identity. □

Notation 2.22. Motivated by the previous proposition, we rename the target category to *category of copoints*, with the intention of making more explicit what is the category that we are talking about

$$\boxed{\mathrm{coPt}_k^{2k-2} := (\mathrm{Sp}^{[0,2k-2]})_{\tau\mathbb{S}}^{k-\mathrm{conn}}.}$$

We present the last ingredient needed in order to tackle the classification result, the most important theorem in this work.

Proposition 2.23. *For $k \geq 2$, there is a equivalence between pointed $k-$connective $(2k-2)-$truncated spectra and the following under-over-category*

$$(\mathrm{Sp}^{[0,2k-2]})_{\tau\mathbb{S}/\ /\tau\mathbb{S}}^{k\text{-}conn} \simeq \mathrm{Sp}^{[k,2k-2]}\,.$$

Proof. By Prop 2.14 and remark 2.15 we have an equivalence

$$(\mathrm{Sp}^{[0,2k-2]})_{\tau\mathbb{S}/\ /\tau\mathbb{S}} \simeq \mathrm{Sp}^{[0,2k-2]}$$

given by $(- \oplus \tau\mathbb{S})$ and $\mathrm{fib}(-)$, the functor that takes the fiber of the copoint. Just by restricting to $k-$connective objects in this equivalence we get $\mathrm{Sp}^{[k,2k-2]}$ and under-over-objects with $k-$connective fiber, as we wanted. □

Theorem 2.24. *For $k \geq 2$, the following is an equivalence between $k-$connective $(2k-2)-$truncated spaces and copoints in spectra*

$$\tau\Sigma_+^\infty : \mathrm{An}^{[k,2k-2]} \to \mathrm{coPt}_k^{2k-2}\,.$$

Proof. It is enough to prove fully faithfulness of $\tau\Sigma_+^\infty$ as we showed essential surjectivity in Prop 2.21. As we embark to show

$$\mathrm{map}_{\mathrm{An}^{[k,2k-2]}}(X, Y) \simeq \mathrm{map}_{\mathrm{coPt}_k^{2k-2}}(\tau\Sigma_+^\infty X, \tau\Sigma_+^\infty Y)$$

we recall that the latter is defined as the following pullback, where are picking the maps f that admit a homotopy filling the diagram in the right

$$\begin{tikzcd} \mathrm{map}_{\mathrm{coPt}_k^{2k-2}}(\tau\Sigma_+^\infty X, \tau\Sigma_+^\infty Y) \arrow[r, dashed] \arrow[d, dashed] & \mathrm{map}(\tau\Sigma_+^\infty X, \tau\Sigma_+^\infty Y) \arrow[d, "\omega_Y \circ -"] \\ \{\omega_X\} \arrow[r, hook] & \mathrm{map}(\tau\Sigma_+^\infty X, \tau\mathbb{S}) \end{tikzcd} \qquad \begin{tikzcd} \tau\Sigma_+^\infty X \arrow[rr, "f"] \arrow[dr, bend right, "\omega_X"'] & & \tau\Sigma_+^\infty Y \arrow[dl, bend left, "\omega_Y"] \\ & \tau\mathbb{S} & \end{tikzcd}$$

We want to use all the information we have on the pointed case in our advantage. In order to facilitate that, let us look at the diagram defining pointed mapping spaces in $\mathrm{An}^{[k,2k-2]}$: we choose basepoints $x \in X$ and $y \in Y$, and we pick the elements of the mapping space such that $x \mapsto y$, or equivalently, that make the black triangle on the right commute

$$\begin{tikzcd}
\mathrm{map}_*(X,Y) \arrow[r] \arrow[d] & \mathrm{map}(X,Y) \arrow[d, "ev_x"] \\
\{y\} \arrow[r, hook] & \mathrm{map}(*,Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr] \arrow[dr, bend right] & & Y \arrow[dl, bend left] \\
 & * \arrow[ul, "x"] \arrow[ur, "y"'] \arrow[d] & \\
 & * &
\end{tikzcd}$$

Remembering the always existing unique maps to the point (drawn above in grey), we proceed to apply $\tau\Sigma^\infty_+$. We run into the slight problem that there doesn't appear to be a reason why the result should still be a fiber sequence. Let us examine what we obtain

$$\begin{tikzcd}
Fib \arrow[r, dashed] \arrow[d, dashed] & \mathrm{map}_{(\mathrm{Sp}^{[0,2k-2]})^{k-\mathrm{conn}}_{/\tau\mathbb{S}}}(\tau\Sigma^\infty_+ X, \tau\Sigma^\infty_+ Y) \arrow[d, "- \circ \tau\Sigma^\infty_+ x"] \\
\{\tau\Sigma^\infty_+ y\} \arrow[r, hook] & \mathrm{map}_{(\mathrm{Sp}^{[0,2k-2]})^{k-\mathrm{conn}}_{/\tau\mathbb{S}}}(\tau\mathbb{S}, \tau\Sigma^\infty_+ Y).
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau\Sigma^\infty_+ X \arrow[rr] \arrow[dr, bend right, "\omega_X"'] & & \tau\Sigma^\infty_+ Y \arrow[dl, bend left, "\omega_Y"] \\
 & \tau\mathbb{S} \arrow[ul] \arrow[ur] \arrow[d, "\mathrm{id}"] & \\
 & \tau\mathbb{S} &
\end{tikzcd}$$

On the left we are choosing pointings $\tau\mathbb{S} \to \tau\Sigma^\infty_+ X$ and $\tau\mathbb{S} \to \tau\Sigma^\infty_+ Y$ over the sphere and we select the maps that preserve pointings, as the vertical arrow is precomposition with the pointing on X and the horizontal arrow chooses the pointing on Y.

Inspecting the diagram in the right is clear that the pullback is the image of $\mathrm{map}_{\mathrm{An}_{*/\,/*}}(X,Y)$, which is

$$\mathrm{map}_{(\mathrm{Sp}^{[0,2k-2]})^{\text{k-conn}}_{\tau\mathbb{S}/\,/\tau\mathbb{S}}}(\tau\Sigma^\infty_+ X, \tau\Sigma^\infty_+ Y) \simeq \mathrm{map}_{\mathrm{Sp}^{[k,2k-2]}}(\tau\Sigma^\infty X, \tau\Sigma^\infty Y)$$

where the isomorphism is given by 2.23. We have found the following map of fiber sequences, where the middle map is precisely what we need to be an equivalence.

$$\begin{tikzcd}
\mathrm{map}_*(X,Y) \arrow[r, "\simeq"] \arrow[d] & \mathrm{map}_{\mathrm{Sp}^{[k,2k-2]}}(\tau\Sigma^\infty X, \tau\Sigma^\infty Y)) \arrow[d] \\
\mathrm{map}(X,Y) \arrow[r] \arrow[d, "\mathrm{ev}_x"] & \mathrm{map}_{\mathrm{coPt}_k^{2k-2}}(\tau\Sigma^\infty_+ X, \tau\Sigma^\infty_+ Y) \arrow[d] \\
\mathrm{map}(*,Y) \arrow[r, "\dagger"] & \mathrm{map}_{\mathrm{coPt}_k^{2k-2}}(\tau\mathbb{S}, \tau\Sigma^\infty_+ Y)
\end{tikzcd}$$

The upper map is an isomorphism due to the pointed result from thm 2.2. If we additionally show the bottom map † to be an isomorphism, we can use a 5−lemma argument to show that the middle map is an isomorphism as well, granting the result.

It is enough to prove that the next square on the left is left adjointable to show that the bottom map in the fiber map is an isomorphism. If this is the case, by Th 2.11 we have that the square in the right (where

$(Y, y) \in \mathrm{An}_*$) is filled by a homotopy, thus presenting † as a composition of isomorphisms. The vertical maps on the right diagram are isomorphisms for the adjunctions $(-)_+ \dashv \mathrm{fgt}$ and $\mathrm{fgt} \dashv (- \oplus \tau\mathbb{S})$, whereas the bottom map is an iso due to Th 2.2.

$$\begin{tikzcd}
\mathrm{An} \arrow[r, "\tau\Sigma^\infty_+"] \arrow[d, bend right, "(-)_*"'] & (\mathrm{Sp}^{[0,2k-2]})^{k-\mathrm{conn}}_{/\tau\mathbb{S}} \arrow[d, bend right, "\mathrm{fgt}"'] \\
\mathrm{An}_* \arrow[r, "\tau\Sigma^\infty"] \arrow[u, bend right, "\mathrm{fgt}"'] & \mathrm{Sp}^{[0,2k-2]} \arrow[u, bend right, "-\oplus\tau\mathbb{S}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{map}_{\mathrm{An}}(*, Y) \arrow[r, "\dagger"] \arrow[d, "\simeq"'] & \mathrm{map}_{(\mathrm{Sp}^{[0,2k-2]})^{k-\mathrm{conn}}_{/\tau\mathbb{S}}}(\tau\mathbb{S}, \tau\Sigma^\infty Y \oplus \tau\mathbb{S}) \arrow[d, "\simeq"] \\
\mathrm{map}_{\mathrm{An}_*}(S^0, Y) \arrow[r, "\simeq"'] & \mathrm{map}_{\mathrm{Sp}^{[0,2k-2]}}(\tau\mathbb{S}, \tau\Sigma^\infty Y)
\end{tikzcd}$$

We begin noticing that $\tau\Sigma^\infty_+ \simeq \tau\Sigma^\infty \circ (-)_+$ by definition, so we have a homotopy in the square on the left with the left adjoints. Now for left adjointness we need to prove that the Beck-Chevalley induced natural transfomation

$$\tau\Sigma^\infty_+ \circ \mathrm{fgt} \to (\tau\Sigma^\infty -) \oplus \tau\mathbb{S}$$

is an equivalence, but this is precisely what was shown in Rmk 2.18. Therefore the result is proven.

□

Remark 2.25. The discussion on Rmk 2.18 suggests that the inverse of $\tau\Sigma^\infty_+$ is precisely the composition

$$\mathrm{coPt}_k^{2k-2} \xrightarrow{\mathrm{fib}} \mathrm{Sp}^{[k,2k-2]} \xrightarrow{\Omega^\infty} \mathrm{An}_*^{[k,2k-2]} \xrightarrow{\mathrm{fgt}} \mathrm{An}^{[k,2k-2]}.$$

To prove that this is the case we show $\mathrm{fgt} \circ \Omega^\infty \circ \mathrm{fib} \circ \tau\Sigma^\infty_+ \simeq \mathrm{id}$ which can then be precomposed with $(\tau\Sigma^\infty_+)^{-1}$ to conclude. This expresion is true as $\tau\Sigma^\infty_+$ sends an anima $X \in \mathrm{An}^{[k,2k-2]}$ to the copoint

$$\tau\Sigma^\infty X \longrightarrow \tau\Sigma^\infty_+ X \longrightarrow \tau\mathbb{S}$$

which has fiber $\tau\Sigma^\infty X$, to which we can apply Ω^∞ to get X with some particular choice of basepoint. If we forget the basepoint, we're back at X.

Corollary 2.26. *For $k \geq 2$, the previous result can be applied pointwise in* $\mathrm{Fun}(X, -)$ *to get an equivalence*

$$\mathrm{Fun}(X, \mathrm{An}^{[k,2k-2]}) \simeq \mathrm{Fun}(X, \mathrm{coPt}_k^{2k-2}).$$

Equivalently, as $\mathrm{coPt}_k^{2k-2} = (\mathrm{Sp}^{[0,2k-2]})^{k\text{-}conn}_{/\tau\mathbb{S}}$*, we can regard the right hand side as a fiber sequence in local systems where the fiber of the map to the constant local system $\underline{\tau\mathbb{S}}$ is k−connective*

$$\mathrm{Fun}(X, \mathrm{An}^{[k,2k-2]}) \simeq \mathrm{Fun}(X, \mathrm{Sp}^{[0,2k-2]})^{k\text{-}conn}_{/\underline{\tau\mathbb{S}}}.$$

Chapter 3

Parametrized results

3.1 Gerbe classification

Adapting the theorem from the previous section we get another gerbe classification result

Theorem 3.1. *For $k \geq 2$, there is an equivalence*

$$\mathrm{Fun}(X, \mathrm{An}^{[k,k]}) \simeq \mathrm{Fun}(X, \mathrm{coPt}_k^k) \simeq \mathrm{Fun}(X, \mathrm{Sp}^{[0,k]})^{k-conn}_{\underline{\tau_{\leq k}\mathbb{S}}}.$$

We will now be interested on how this new result implies the original one. For a clean proof we will need to discuss some elements beforehand, including an alternative definition of a cohomology group with coefficients in a local system:

Definition 3.2. Given a space X and a local system $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Ab}) \simeq \mathrm{Fun}(X, \mathrm{Sp}^{\heartsuit})$, we define the *cohomology of X with coefficients in $\mathcal{A}$* as

$$H^{k+1}(X, \mathcal{A}) \coloneqq \pi_0 \hom_{\mathrm{Fun}(X,\mathrm{Sp})}(\underline{\mathbb{S}}, \Sigma^{k+1}\mathcal{A})$$

Remark 3.3. Notice that as Ab has a canonical $\mathbb{Z}$–action, we could interpret $\mathcal{A}$ as hitting the underlying spectrum of a module in spectra. Using the adjunction between spectra and $\mathbb{Z}$–modules in spectra

$$- \otimes \mathbb{Z} : \mathrm{Sp} \rightleftarrows \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}) : \mathrm{fgt} \tag{3.1}$$

which can be seen in [HA, Cor 4.2.4.8], we can take the mate of $\hom_{\mathrm{Sp}^X}(\underline{\mathbb{S}}, \Sigma^{k+1}\mathcal{A})$ by applying $- \otimes \mathbb{Z}$ to the source and get an equivalent definition:

$$H^{k+1}(X, \mathcal{A}) \simeq \pi_0 \hom_{\mathrm{Fun}(X,\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))}(\underline{\mathbb{Z}}, \Sigma^{k+1}\mathcal{A}).$$

Remark 3.4. In order to show that this definition is equivalent to that of chapter 1 we recall that for any symmetric monoidal presentable ∞–category $\mathscr{C}$ and an object $c \in \mathscr{C}$ there is an adjunction

$$\begin{tikzcd} \mathrm{An} \arrow[r, bend left, "-\otimes c"] \arrow[r, phantom, "\perp"] & \mathscr{C}. \arrow[l, bend left, "\mathrm{map}(c,-)"] \end{tikzcd} \tag{3.2}$$

Given any $X \in \mathrm{An}$, we can apply apply the adjunction pointwise on $\mathrm{Fun}(X, -)$ to get another adjunction

$$- \otimes c : \mathrm{An}^X \rightleftarrows \mathscr{C}^X : \mathrm{map}(c, -). \tag{3.3}$$

As $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$ is sym monodial presentable we have the tools to relate the different cohomology definitions

Lemma 3.5. *There is an isomorphism*

$$\pi_0 \underset{X}{\lim} B^\bullet \mathcal{A} \simeq \pi_0 \hom_{\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X}(\underline{\mathbb{Z}}, \Sigma^\bullet \mathcal{A}).$$

Proof. Consider $\mathscr{C} = \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$ on corollary 3.3 to obtain an adjunction between the pointwise $-\otimes\mathbb{Z}$ functor and the pointwise $\mathrm{map}(\mathbb{Z}, -)$ functor. In particular, $\mathrm{map}(\mathbb{Z}, -)$ is equivalent to the composition

$$\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X \xrightarrow{\mathrm{fgt}} \mathrm{Sp}^X \xrightarrow{\Omega^\infty} \mathrm{An}_*^X \xrightarrow{\mathrm{fgt}} \mathrm{An}^X,$$

as given a $M \in \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X$ we have the following isomorphisms, where we use $\mathbb{S}\otimes\mathbb{Z} \simeq \mathbb{Z}$ and $* \coprod * = S^0$

$$\mathrm{map}_{\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X}(\mathbb{Z}, M) \overset{(-\otimes\mathbb{Z}) \dashv \mathrm{fgt}}{\simeq} \mathrm{map}_{\mathrm{Sp}^X}(\mathbb{S}, \mathrm{fgt}(M)) \overset{\Sigma^\infty \dashv \Omega^\infty}{\simeq} \mathrm{map}_{\mathrm{An}_*^X}(S^0, \Omega^\infty \mathrm{fgt}(M)) \overset{(-)_+ \dashv \mathrm{fgt}}{\simeq} \mathrm{map}_{\mathrm{An}^X}(*, \mathrm{fgt}\, \Omega^\infty \mathrm{fgt}(M)).$$

By Yoneda this pointwise isomorphisms assemble to an isomorphism of fucntors. Good, we have established the adjunction in the left and we wish to combine it with the one in the right to conclude

$$-\otimes\mathbb{Z} : \mathrm{An}^X \rightleftarrows \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X : \Omega^\infty \qquad\qquad \underline{(-)} : \mathrm{An} \rightleftarrows \mathrm{An}^X : \underset{X}{\lim}\,.$$

For that we need two extra facts: that $\underline{\mathrm{pt}} \otimes \mathbb{Z} \simeq \underline{\mathrm{pt} \otimes \mathbb{Z}} \simeq \underline{\mathbb{Z}}$ and that $\Omega^\infty\Sigma^\bullet\mathcal{A} \simeq B^\bullet\mathcal{A}$, where this is due to the pointwise images being Eilenberg-MacLane spaces with the same homotopy groups. These allow us to access the adjunctions and present the following chain of isomorphims, proving the result

$$\hom(\underline{\mathbb{Z}}, \Sigma^\bullet\mathcal{A}) \simeq \hom(\underline{\mathrm{pt}} \otimes \mathbb{Z}, \Sigma^\bullet\mathcal{A}) \simeq \mathrm{map}_{\mathrm{An}^X}(\underline{\mathrm{pt}}, \Omega^\infty\Sigma^\bullet\mathcal{A}) \simeq \mathrm{map}_{\mathrm{An}^X}(\underline{\mathrm{pt}}, B^\bullet\mathcal{A}) \simeq \mathrm{map}_{\mathrm{An}}(*, \underset{X}{\lim} B^\bullet\mathcal{A}).$$

□

Additionally, for the gerbe classification proof we will need a precise notion of why the operation of rolling a fiber sequence is an equivalence in Sp. This fundamentally reduces to the fact that in spectra cofiber and fiber sequences are equivalent, let us define the category of cofiber sequences:

Definition 3.6. Consider the full subcategory of $\mathrm{Sp}^{\Delta^1\times\Delta^1}$ of cocartesian squares. The *category of cofiber sequences* denoted by Cof, is the subcategory of cocartesian squares like the one in the left. Equivalently, it is the pullback on the right where we pick the lower left position of the square to be zero:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathrm{Cof} = \left\{\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right\} \arrow[r, dashed] \arrow[d, dashed] & \left\{\begin{smallmatrix} X & Y \\ W & Z \end{smallmatrix}\right\} \arrow[d, "ev_{1,0}"] \\ 0 \arrow[r, hook] & \mathrm{Sp}, \end{tikzcd}$$

Lemma 3.7. *The following functors are equivalences,*

$$s_{cof} : \mathrm{Sp}^{\Delta^1} \longrightarrow Cof \qquad\qquad d_{fib} : Cof \longrightarrow \mathrm{Sp}^{\Delta^1}$$

$$(X \to Y) \longmapsto \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & cof(f) \end{tikzcd}\right) \qquad\qquad \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & cof(f) \end{tikzcd}\right) \longmapsto (Y \to cof(f))$$

Proof. The result is trivial for s_{cof}, it adds the cofiber of the map which you can forget to go back. In the case of d_{fib} we can go back by adding the fiber and noting that fiber sequences are cofiber sequences. $\square$

Remark 3.8 (Rolling)**.** The composition $s_{\mathrm{cof}} \circ d_{\mathrm{fib}} : \mathrm{Sp}^{\Delta^1} \to \mathrm{Sp}^{\Delta^1}$

$$(f : X \to Y) \longmapsto \left(\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & \mathrm{cof}(f) \end{tikzcd}\right) \longmapsto (g : Y \to \mathrm{cof}(f))$$

is an equivalence, and $\mathrm{cof}(f) \simeq \Sigma\,\mathrm{fib}(f)$ as the right diagram shows. Additionally, given $X \in \mathrm{An}$, we can apply this pointwise in $\mathrm{Fun}(X, -)$ to show that rolling is an equivalence in local systems.

$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Y \arrow[r] & \mathrm{cof}(f) \end{tikzcd}$$

We're now in a position to tackle the main theorem of this section, where we will use the rolling equivalence to showcase $k-$gerbes as a certain type of map which we will send to modules using the adjunction from Rmk 3.3, where we can check the cohomology definition quite explicitly.

Theorem 3.9 (Gerbe classification)**.** *Let $k \geq 2$, there is an equivalence between $k-$gerbes up to isomorphism and local systems on groups together with a cohomology class*

$$\pi_0\,\mathrm{Fun}(X, \mathrm{An}^{[k,k]})^{\simeq} \simeq \{\mathcal{A} \in \pi_0\,\mathrm{Fun}(X, \mathrm{Ab})^{\simeq},\ \alpha \in H^{k+1}(X, \mathcal{A})\}.$$

Proof. Th 3.1 grants us that $k-$gerbes are equivalent to copoints, ie maps in $\mathrm{Fun}(X, \mathrm{Sp}^{[0,k]})$ of the form

$$E \longrightarrow \underline{\tau_{\leq k}\mathbb{S}}$$

whose fiber belongs to $\mathrm{Fun}(X, \mathrm{Sp}^{[k,k]})$. We choose to regard the fiber as being suspended from a functor $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Sp}^{[0,0]})$, and thus the fiber sequence reads $\Sigma^k\mathcal{A} \to E \to \underline{\tau_{\leq k}\mathbb{S}}$.

Rolling the copoint results in $\underline{\tau_{\leq k}\mathbb{S}} \to \Sigma^{k+1}\mathcal{A}$, which will be a more convenient form for the proof. We want to use Rmk 3.8 to show that the category of these maps is equivalent to $k-$gerbes. For that we define two auxiliary subcategories of $\mathrm{Fun}(X, \mathrm{Cof})$ as pullbacks of the diagrams

$$\begin{tikzcd} \left\{\begin{smallmatrix} \Sigma^k\mathcal{A} & E \\ 0 & B \end{smallmatrix}\right\} \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Cof}^X = \left\{\begin{smallmatrix} F & E \\ 0 & B \end{smallmatrix}\right\} \arrow[d, "ev_{0,0}"] \\ \mathrm{Fun}(X, \mathrm{Sp}^{[k,k]}) \arrow[r, hook] & \mathrm{Fun}(X, \mathrm{Sp}) \end{tikzcd} \qquad \begin{tikzcd} \left\{\begin{smallmatrix} E & B \\ 0 & \Sigma^{k+1}\mathcal{A} \end{smallmatrix}\right\} \arrow[r, dashed] \arrow[d, dashed] & \mathrm{Cof}^X = \left\{\begin{smallmatrix} E & B \\ 0 & CF \end{smallmatrix}\right\} \arrow[d, "ev_{1,1}"] \\ \mathrm{Fun}(X, \mathrm{Sp}^{[k+1,k+1]}) \arrow[r, hook] & \mathrm{Fun}(X, \mathrm{Sp}) \end{tikzcd}$$

The map induced by $d_{\mathrm{fib}} \circ s_{\mathrm{cof}}$ produces an equivalence between the two subcategories:

$$\left\{\begin{tikzcd} \Sigma^k\mathcal{A} \arrow[r, "f"] \arrow[d] & E \arrow[d] \\ 0 \arrow[r] & B \end{tikzcd}\right\} \longrightarrow \{E \to B\} \longrightarrow \left\{\begin{tikzcd} E \arrow[r, "g"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & \Sigma^{k+1}\mathcal{A} \end{tikzcd}\right\}.$$

Now, if we force the local system B to be $\underline{\tau_{\leq k}\mathbb{S}}$ we obtain an equivalence between $k-$gerbes and rolled $k-$gerbes. We accomplish this by taking pullbacks along the evaluations $ev_{1,1}$ and $ev_{0,1}$ of the map that picks the constant truncated sphere local system. As the evaluations are in particular fibrations, our equivalence lifts to an equivalence of the pullbacks, which we draw in blue:

$$\begin{tikzcd}
 & \left\{\begin{smallmatrix} E & \underline{\tau_{\leq k}\mathbb{S}} \\ 0 & \Sigma^{k+1}\mathcal{A} \end{smallmatrix}\right\} \arrow[rr, dashed] \arrow[dd, dashed] & & \left\{\begin{smallmatrix} E & B \\ 0 & \Sigma^{k+1}\mathcal{A} \end{smallmatrix}\right\} \arrow[dd, "ev_{0,1}"] \\
\left\{\begin{smallmatrix} \Sigma^k\mathcal{A} & E \\ 0 & \underline{\tau_{\leq k}\mathbb{S}} \end{smallmatrix}\right\} \arrow[ur, "\simeq"] \arrow[rr, dashed] \arrow[dr, dashed] & & \left\{\begin{smallmatrix} \Sigma^k\mathcal{A} & E \\ 0 & B \end{smallmatrix}\right\} \arrow[ur, "\simeq"] \arrow[dr, "ev_{1,1}"'] & \\
 & * \arrow[rr, "\{\underline{\tau_{\leq k}\mathbb{S}}\}"] & & \mathrm{Fun}(X, \mathrm{Sp}^{[0,k]})
\end{tikzcd}$$

Thus we obtain an equivalence that characterizes $k-$gerbes:

$$\left\{E \to \underline{\tau_{\leq k}\mathbb{S}}\right\} \xrightarrow{\simeq} \left\{\underline{\tau_{\leq k}\mathbb{S}} \to \Sigma^{k+1}\mathcal{A}\right\}.$$

We wish to show how the maps in the right encode local systems on abelian groups together with a cohomology class. First, recall that $\mathrm{Sp}^{\heartsuit} \simeq \mathrm{Ab}$ so if we $(k+1)-$desuspend the target we get a functor $\mathcal{A} \in \mathrm{Fun}(X, \mathrm{Ab})$ as we wanted.

Now for the cohomology class, start by noticing that $\Sigma^{k+1}\mathcal{A}$ can be interpreted as a module in spectra due to it being discrete. Additionally, it is $(k+1)-$truncated, so we take the mate of $\underline{\tau_{\leq k}\mathbb{S}} \to \Sigma^{k+1}\mathcal{A}$ through the following adjunctions to work on the category of truncated modules:

$$- \otimes \mathbb{Z} : \mathrm{Sp} \rightleftarrows \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}) : \mathrm{fgt} \qquad\qquad \tau_{\leq k+1} : \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}) \rightleftarrows \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})_{\leq k+1} : \iota$$

Which results in $\tau_{\leq k+1}(\underline{\tau_{\leq k}\mathbb{S}} \otimes \mathbb{Z}) \to \Sigma^{k+1}\mathcal{A}$. Notice that the $\mathbb{Z}-$linearization of a constant functor is constant and the truncation of a constant functor is constant, too. Then we have

$$\tau_{\leq k+1}(\underline{\tau_{\leq k}\mathbb{S}} \otimes \mathbb{Z}) \simeq \tau_{\leq k+1}(\underline{\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z}}) \simeq \underline{\tau_{\leq k+1}(\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z})}.$$

Therefore, the map that we obtained earlier is exactly a map

$$\underline{\tau_{\leq k+1}(\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z})} \longrightarrow \Sigma^{k+1}\mathcal{A}.$$

This is a cohomology class. Let us prove $\underline{\tau_{\leq k+1}(\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z})} \simeq \underline{\mathbb{Z}}$. Every spectrum fits into a (co)fiber sequence with its upper and lower truncations for any positive integer, in particular we're interested in degree k

$$\underline{\tau_{>k}\mathbb{S}} \longrightarrow \underline{\mathbb{S}} \longrightarrow \underline{\tau_{\leq k}\mathbb{S}}$$

as the object on the right appears inside our expresion. Now we roll the sequence to get $\underline{\mathbb{S}} \to \underline{\tau_{\leq k}\mathbb{S}} \to \underline{\Sigma\tau_{>k}\mathbb{S}}$ and using that $- \otimes \mathbb{Z}$ preserves cofibers as a left adjoint, together with the fact that $\mathbb{S}$ is the unit of the tensor product of spectra, we obtain a (co)fiber sequence in $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$

$$\underline{\mathbb{Z}} \longrightarrow \underline{\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z}} \longrightarrow \underline{(\Sigma\tau_{>k}\mathbb{S}) \otimes \mathbb{Z}}.$$

The tensor product of an $m-$connective and an $n-$connective object is $(m+n)-$connective [HA, 2.2.1.3, 2.2.1.4, 7.1.1.7]. Thus $\underline{(\Sigma\tau_{>k}\mathbb{S}) \otimes \mathbb{Z}}$ is $(k+2)-$connective as $\mathbb{Z}$ is connective and $\Sigma\tau_{>k}\mathbb{S}$ is $(k+2)-$connective.

Now given a map $f : B \to C$ we have that $\tau_{\leq k+1} f : \tau_{\leq k+1} B \to \tau_{\leq k+1} C$ is an isomorphism iff $\pi_i(f)$ is an iso $\forall i \leq k+1$. As $\underline{(\Sigma\tau_{>k}\mathbb{S}) \otimes \mathbb{Z}}$ is $(k+2)-$connective and $\tau_{\leq k+1}\mathbb{Z} \simeq \mathbb{Z}$, we conclude $\underline{\tau_{\leq k+1}(\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z})} \simeq \underline{\mathbb{Z}}$. Then, as a element of

$$\hom_{\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})^X}(\underline{\mathbb{Z}}, \Sigma^{k+1}\mathcal{A})$$

the previous map is a representative of a cohomology class. $\square$

3.2 Presenting $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$ in cohomological terms

Having classified $k-$gerbes we focus our attention on interrogating the natural next step, functors belonging to $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$, which are equivalent to a fiber sequence in $\mathrm{Fun}(X, \mathrm{Sp}^{[0,k+1]})$ of the form

$$\mathscr{B} \longrightarrow E \longrightarrow \underline{\tau_{\leq k+1}\mathbb{S}}$$

where $\mathscr{B} \in \mathrm{Fun}(X, \mathrm{Sp}^{[k,k+1]})$, analogously to what we saw in Cor 2.26.

Theorem 3.10. *For $k \geq 2$, we have an equivalence between $\pi_0 \mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})^{\simeq}$ and isomorphism classes of squares on $\mathrm{Fun}(X, \mathrm{Sp}^{[0,k+3]})$ of the following form, where $(-)[n] := \Sigma^n(-)$*

$$\begin{tikzcd}
\underline{\tau_{\leq k}\mathbb{S}} \arrow[r] \arrow[d] & \pi_k\mathscr{B}[k+1] \arrow[d] \\
\underline{\pi_{k+1}\mathbb{S}[k+2]} \arrow[r] & \pi_{k+1}\mathscr{B}[k+3].
\end{tikzcd}$$

Proof. Given a local system in $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$, we want to divide the equivalent fiber sequence valued on spectra into $1-$type information. For this we use postnikov towers, where we will find only one non-trivial step passing from k to $k+1$, highlighted on the left. Additionally, we can check pointwise that the $k-$truncated sequence is a fiber sequence as well, so rolling is allowed.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathscr{B} \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \underline{\tau_{\leq k+1}\mathbb{S}} \arrow[d] \\
\mathscr{B} \arrow[r] \arrow[d] & E \arrow[r, "f"] \arrow[d] & \underline{\tau_{\leq k+1}\mathbb{S}} \arrow[d] \\
\tau_{\leq k}\mathscr{B} \arrow[r] \arrow[d] & \tau_{\leq k}E \arrow[r, "\tau_{\leq k}f"] \arrow[d] & \underline{\tau_{\leq k}\mathbb{S}} \arrow[d] \\
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & * \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \vdots \arrow[d] & \\
\pi_{k+1}E[k+1] \arrow[r] & \tau_{\leq k+1}E \arrow[r] \arrow[d] & \pi_{k+2}E[k+3] \\
\pi_kE[k] \arrow[r] & \tau_{\leq k}E \arrow[r, "k-\mathrm{inv}"] \arrow[d] & \pi_{k+1}E[k+2] \\
\pi_{k-1}E[k-1] \arrow[r] & \tau_{\leq k-1}E \arrow[r] \arrow[d] & \pi_kE[k+1] \\
 & \vdots &
\end{tikzcd}$$

Now as cocartesian squares are cartesian and viceversa in Sp, we have that $k-$invariants (depicted on the right) completely classify extensions to the next level of the postnikov tower. As a consequence, from the next diagram where we represent the non-trivial step on the postnikov tower and its $k-$invariants, we can conclude that functors of the form $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$ are classified by the bottom squares.

$$\begin{tikzcd}
\mathscr{B} \arrow[r] \arrow[d] & E \arrow[r, "f"] \arrow[d] & \underline{\tau_{\leq k+1}\mathbb{S}} \arrow[d] & \\
\pi_k\mathscr{B}[k] \arrow[r] \arrow[d, "k-\mathrm{inv}"'] & \tau_{\leq k}E \arrow[r, "\tau_{\leq k}f"] \arrow[d, "k-\mathrm{inv}"'] & \underline{\tau_{\leq k}\mathbb{S}} \arrow[r] \arrow[d, "k-\mathrm{inv}"'] & \pi_k\mathscr{B}[k+1] \arrow[d] \\
\pi_{k+1}\mathscr{B}[k+2] \arrow[r] & \pi_{k+1}E[k+2] \arrow[r] & \underline{\pi_{k+1}\mathbb{S}[k+2]} \arrow[r] & \pi_{k+1}\mathscr{B}[k+3]
\end{tikzcd}$$

Moreover, by lemma 3.7 and Rmk 3.8, all information on the bottom squares is captured in the rolled square on the right of the diagram. □

We will spend the rest of this section examining this square further to give a more usable description. Notice that $\pi_{k+1}\mathscr{B}[k+3] \in \mathrm{Fun}(X, \mathrm{Sp}^{[k+3,k+3]})$ can be interpreted as belonging to $\mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$, so we might as well $\mathbb{Z}-$linearize the whole square, ie take the mates of all maps with respect to the adjunction

$$- \otimes \mathbb{Z} : \mathrm{Fun}(X, \mathrm{Sp}) \rightleftarrows \mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})) : \mathrm{fgt}.$$

Additionally, $\pi_{k+1}\mathscr{B}[k+3]$ is $(k+3)-$truncated, so we take the mate through the adjunction between inclusion of $(k+3)-$truncated local systems on modules and $\tau_{\leq k+3}$, obtaining a square of the form

$$\begin{tikzcd}
\underline{\tau_{\leq k+3}(\tau_{\leq k}\mathbb{S} \otimes \mathbb{Z})} \arrow[r] \arrow[d] & \tau_{\leq k+3}(\pi_k\mathscr{B}[k+1] \otimes \mathbb{Z}) \arrow[d] \\
\underline{\tau_{\leq k+3}(\pi_{k+1}\mathbb{S}[k+2] \otimes \mathbb{Z})} \arrow[r] & \pi_{k+1}\mathscr{B}[k+3].
\end{tikzcd}$$

Determining the modified terms We wish to unravel what $\mathbb{Z}-$linearization plus truncation have done. It is verified that for any spectrum A which is the underlying spectrum of a $\mathbb{Z}-$module we have

$$A \otimes_{\mathbb{S}} \mathbb{Z} \simeq A \otimes_{\mathbb{Z}} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z})$$

and tensor product between $\mathbb{Z}$'s is precisely the integral steenrod algebra $\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathbb{Z} \oplus {}^{\mathbb{Z}}\!/\!_{2}[2] \oplus \dots$. This allow us to calculate the objects on the diagonal, as

$$\pi_k\mathscr{B}[k+1] \otimes \mathbb{Z} \simeq \pi_k\mathscr{B}[k+1] \oplus {}^{\pi_k\mathscr{B}}\!/\!_{2}[k+3] \oplus \dots$$
$$\underline{\pi_{k+1}\mathbb{S}[k+2]} \otimes \mathbb{Z} \simeq \underline{\pi_{k+1}\mathbb{S}[k+2]} \oplus \underline{{}^{\pi_{k+1}\mathbb{S}}\!/\!_{2}[k+4]} \oplus \dots$$

which after $(k+3)-$truncating will result in

- $\tau_{\leq k+3}(\pi_k\mathscr{B}[k+1] \otimes \mathbb{Z}) \simeq \pi_k\mathscr{B}[k+1] \oplus {}^{\pi_k\mathscr{B}}\!/\!_{2}[k+3]$
- $\underline{\tau_{\leq k+3}(\pi_{k+1}\mathbb{S}[k+2] \otimes \mathbb{Z})} \simeq \underline{\pi_{k+1}\mathbb{S}[k+2]}$.

For the $\mathbb{Z}-$linearization of the truncated sphere we will need a little more work:

Calculating $\underline{\tau_{\leq k+3}(\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z})}$. We start by applying $-\otimes\mathbb{Z}$ to the usual truncation co/fiber sequence and get

$$\tau_{>k}\mathbb{S}\otimes\mathbb{Z}\longrightarrow\mathbb{Z}\longrightarrow\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z}.$$

Earlier, when we used a similar argument for the cohomology class we took advantage of the fact that we were $(k+1)-$truncating and the fiber was $(k+1)-$connective, per contra the present case require us to $(k+3)-$truncate, which results in some homotopy groups of the fiber bleeding into the right term $\tau_{k+3}(\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z})$. For now, let us roll the sequence to get the isomorphic sequence

$$\mathbb{Z}\longrightarrow\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z}\longrightarrow\Sigma(\tau_{>k}\mathbb{S}\otimes\mathbb{Z}).$$

Here the last term verifies $\Sigma(\tau_{>k}\mathbb{S}\otimes\mathbb{Z})\simeq(\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z}$ as smash product of spectra preserves colimits in each variable (in fact this plus $\mathbb{S}$ being the unit characterizes the monoidal structure [HA, 4.8.2.9, 4.8.2.18]).

We have that $(\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z}$ is $(k+2)-$connective as its terms are $(k+2)$ and $0-$connective, respectively. Hence only the homotopy groups on positions $k+2$ and $k+3$ of $\tau_{\leq k+3}((\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z})$ could be different from zero. Moreover, as $k\geq 2$ and $\mathbb{Z}$ is $1-$truncated, the LES associated to the sequence degenerates on a series of isomorphisms, thus we could determine $\tau_{\leq k+3}(\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z})$ by calculating π_{k+2} and π_{k+3} of $\tau_{\leq k+3}((\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z})$.

Lemma 3.11. *Given an $n-$connective spectrum X and $\eta\in\pi_1\mathbb{S}\simeq\mathbb{Z}/2$ a generator, we have*

$$\tau_{\leq n+1}(X\otimes\mathbb{Z})\simeq\tau_{\leq n+1}(X\otimes\mathbb{S}/\eta)\simeq\tau_{\leq n+1}\operatorname{cofib}(\Sigma X\xrightarrow{\eta}X).$$

Proof. For the first isomorphism look at the diagram on the right: consider the map $\mathbb{S}\to\mathbb{Z}$ picking the generator of $\pi_0\mathbb{Z}=\mathbb{Z}$. Precomposition with η is the zero map as $\pi_1\mathbb{Z}=0$, thus there is a map from the cofiber $\mathbb{S}/\eta$. Conisder the fiber of this map

$$\begin{tikzcd}
\Sigma\mathbb{S} \arrow[r, "\eta"] \arrow[rr, bend left, "0"] & \mathbb{S} \arrow[r, "1"] \arrow[d] & \mathbb{Z} \\
& \mathbb{S}/\eta \arrow[ur, dashed] &
\end{tikzcd}$$

$$\mathrm{Fib}\to\mathbb{S}/\eta\to\mathbb{Z}.$$

It is verified that Fib is $2-$connective.

This is due to the map being an isomorphism on π_0 (we picked the generator of $\pi_0\mathbb{Z}$ earlier) and $\pi_1\mathbb{S}/\eta\simeq 0$. Now, as $-\otimes X$ is a left adjoint (Rmk 3.4) and fibers are cofibers in spectra, we conclude that the following is a cofiber sequence

$$X\otimes\mathrm{Fib}\longrightarrow X\otimes\mathbb{S}/\eta\longrightarrow X\otimes\mathbb{Z}$$

Furthermore, as X is $n-$connective, $X\otimes\mathrm{Fib}$ is $(n+2)-$connective, hence the map on the right will induce the first isomorphism after applying $\tau_{\leq n+1}$. On the other hand, the sphere spectrum is the unit of the tensor product and $\mathbb{S}/\eta:=\operatorname{cofib}(\Sigma\mathbb{S}\xrightarrow{\eta}\mathbb{S})$ therefore $\tau_{\leq n+1}(X\otimes\mathbb{S}/\eta)\simeq\tau_{\leq n+1}\operatorname{cofib}(\Sigma X\xrightarrow{\eta}X)$. □

Considering $n=k+2$ and $X=\Sigma\tau_{>k}\mathbb{S}$, we can use the result to find the desired homotopy groups of $\tau_{\leq k+3}((\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z})$. Denote

$$C:=\tau_{\leq n+1}\operatorname{cofib}(\Sigma X\xrightarrow{\eta}X)$$

and consider the following cofiber LES, where zero groups are *red*:

$$\begin{tikzcd}
{\color{red}\pi_{n+2}C} \arrow[r] & \pi_{n+1}\Sigma X \arrow[r, "\eta"] \arrow[d, phantom, "\wr"] & \pi_{n+1}X \arrow[r] & \pi_{n+1}C \arrow[r, "0"] & \pi_n\Sigma X \arrow[r] \arrow[d, phantom, "\wr"] & \pi_nX \arrow[r, "\simeq"] & \pi_nC \arrow[r] & {\color{red}\pi_{n-1}\Sigma X} \\
& \pi_nX & & & {\color{red}\pi_{n-1}X} & & &
\end{tikzcd}$$

From this we deduce that $\pi_nC\simeq\pi_nX$ and $\pi_{n+1}C\simeq{}^{\pi_{n+1}X}\!/_{\eta\cdot\pi_nX}$, therefore

- $\pi_{k+2}((\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z}) \simeq \pi_{k+2}\Sigma\tau_{>k}\mathbb{S} \simeq \pi_{k+1}\mathbb{S}$
- $\pi_{k+3}((\Sigma\tau_{>k}\mathbb{S})\otimes\mathbb{Z}) \simeq {}^{\pi_{k+3}\Sigma\tau_{>k}\mathbb{S}}\!/_{\eta\cdot\pi_{k+2}\Sigma\tau_{>k}\mathbb{S}} \simeq {}^{\pi_{k+2}\mathbb{S}}\!/_{\eta\cdot\pi_{k+1}\mathbb{S}}.$

Remark 3.12. Any local system $Y \in \mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$ which is $k-$connective for some integer k splits as the direct sum of its homotopy groups. The idea for the proof would be to use induction on $(k+i)-$truncatedness.

For $i = 0$ it is verified trivially that the local system splits as the direct sum of its homotopy groups, as there is only one. Assume that the statement holds for n, and consider the postnikov tower of the group. We want step n to produce an isomorphism

$$\tau_{\leq n+1}Y \simeq \tau_{\leq n}Y \oplus \pi_{n+1}Y[n+1]$$

but $\tau_{\leq n+1}Y$ is the cofiber of the $n-$invariant, a map that goes from degree n to $n+2$, thus has to be zero. This fact is ultimately a consequence of $\mathbb{Z}$ having projective dimension 1. Hence, we conclude

- $\underline{\tau_{\leq k+3}(\tau_{\leq k}\mathbb{S}\otimes\mathbb{Z})} \simeq \underline{\mathbb{Z}\oplus\pi_{k+1}\mathbb{S}[k+2]\oplus {}^{\pi_{k+2}\mathbb{S}}\!/_{\eta\cdot\pi_{k+1}\mathbb{S}}[k+3]}.$

With this we have determined the four corners of the square, we now wish to classify the maps and the homotopy filling the square. For that we will introduce a new tool.

3.2.1 DG category origin of $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$

DG categories will allow us to give more concrete explicit representatives of isomorphim classes of the squares that classify $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$. Let us start by their definition:

Definition 3.13 (Def 1.3.1.1, [HA]). A *differential graded category $\mathscr{C}$* over a commutative ring R is a $1-$category that verifies

- Enrichment on $Ch(R)$: $\hom(X,Y)$ is a chain complex of $R-$modules for all $X, Y \in \mathscr{C}$

  $$\cdots \longrightarrow \hom(X,Y)_2 \longrightarrow \hom(X,Y)_1 \longrightarrow \hom(X,Y)_0 \longrightarrow \hom(X,Y)_{-1} \longrightarrow \cdots$$

- $R-$linearity of composition: for all $X, Y, Z \in \mathscr{C}$ the composition $\hom(Y,Z) \underset{R}{\otimes} \hom(X,Y) \to \hom(X,Z)$ can be identified with a collection of $R-$bilinear maps

  $$\circ : \hom(Y,Z)_p \times \hom(X,Y)_q \longrightarrow \hom(X,Z)_{p+q}$$

  satisfying the **Leibniz rule** $d(g\circ f) = dg\circ f + (-1)^p g\circ df$. Composition of this $R-$bilinear maps should be strictly associative.

- identity morphisms: for all $X \in \mathscr{C}$, we have $\mathrm{id}_X \in \hom(X,X)_0$ such that

  $$g\circ \mathrm{id}_X = g \qquad \mathrm{id}_X \circ f = f$$

  for all $f \in \hom(Y,X)_p$ and $g \in \hom(X,Y)_q$.

Remark 3.14. When $R = \mathbb{Z}$ we simply talk of *diferential graded categories*. Additionally, note that $\forall X \in \mathscr{C}$ it is verified that $d\,\mathrm{id}_X = 0$, as a consequence of

$$d\,\mathrm{id}_X = d(\mathrm{id}_x\circ\mathrm{id}_X) = (d\,\mathrm{id}_x)\circ\mathrm{id}_X + \mathrm{id}_X\circ(d\,\mathrm{id}_X) = 2d\,\mathrm{id}_X\,.$$

Example 3.15. [HA, 1.3.2.1] We will be interested in particular in $\mathrm{Ch}(\mathbb{Z})_{dg}$, the *diferential graded category of chain complexes of* $\mathbb{Z}$*−modules*. Given two objects $A, B \in \mathrm{Ch}(\mathbb{Z})_{dg}$ the complex of morphisms between A and B is an internal object which is defined by

$$\hom(A, B)_n := \prod_{i \in \mathbb{Z}} \hom(A_i, B_{i+n})$$

endowed with a differential

$$\begin{aligned} d : \hom(A, B)_n &\longrightarrow \hom(A, B)_{n-1} \\ f &\longmapsto \partial_B \circ f - (-1)^n f \circ \partial_A. \end{aligned}$$

Hence, elements of $\hom(A, B)_0$ are usual chain maps, on $\hom(A, B)_1$ we find chain homotopies and as we keep increasing the degree we see higher homotopies. In this context and as remarked in [HA, 1.3.2.3] we can identify $\pi_n \hom(A, B)$ with the group of chain-homotopy classes of maps from $A_\bullet$ to the shifted complex $B_{\bullet+n}$, that is

$$H_n(\hom(A, B)) = {}^{\hom(A, B)_n}\!/\!_{d \hom(A, B)_{n+1}}.$$

Composition of morphisms automatically induces $\mathbb{Z}$−bilinear composition maps

$$\hom(A, B)_n \times \hom(B, C)_m \longrightarrow \hom(A, C)_{n+m}.$$

Lastly, the chain map in $\hom(A, A)_0$ that is the identity at every level functions as the identity morphism for every $A \in \mathrm{Ch}(\mathbb{Z})_{dg}$. Fom now on we refer to $\mathrm{Ch}(\mathbb{Z})_{dg}$ simply as $\mathrm{Ch}(\mathbb{Z})$.

Construction 3 (DG nerve)**.** Given a dg category $\mathscr{C}$ we can associate a simplicial set $\mathrm{N}_{dg}(\mathscr{C})$ to it in the following way:

- Vertices of $\mathrm{N}_{dg}(\mathscr{C})$ correspond to the objects of $\mathscr{C}$.
- Edges between two vertices x and y correspond to $\mathrm{Z}^0(x, y) = \{f \in \hom(x, y)_0 \ / \ df = 0\}$.
- Triangles with three vertices 0, 1 and 2 are given by three edges $f_{01} \in \mathbb{Z}^0(0, 1)$, $f_{12} \in \mathbb{Z}^0(1, 2)$ and $f_{02} \in \mathbb{Z}^0(0, 2)$ together with a homotopy $h_{012} \in \hom(0, 2)_1$ that verifies $dh_{012} = f_{02} - f_{12} \circ f_{01}$.

$$\begin{tikzcd} & 1 \arrow[dr, "f_{12}"] \arrow[d, phantom, "h_{012}"] & \\ 0 \arrow[ur, "f_{01}"] \arrow[rr, "f_{02}"'] & & 2 \end{tikzcd}$$

- n−simplices are given by pairs of collections $(\{X_i\}_{0 \leq i \leq n}, \{f_I\}_{I \subseteq [n]})$ where $X_i \in \mathscr{C}$ for $i = 0, 1, \ldots, n$ and for every subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subseteq [n]$ with $m \geq 0$, we have that $f_I \in \hom(X_{i_-}, X_{i_+})_m$ and it satisfies
$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{I - i_j} - f_{i_j < \cdots < i_1 < i_+} \circ f_{i_- < i_m < \cdots < i_j}).$$

Proposition 3.16. *[HA, 1.3.1.10, 1.3.1.12] Let* $\mathscr{C}$ *be a diferential graded category, then* $N_{dg}(\mathscr{C})$ *is an* ∞*−category. Additionally, for any* $x, y \in \mathscr{C}$ *we have*

$$H_i(\hom_{\mathscr{C}}(x, y)) \simeq \pi_{-i} \operatorname{map}_{N_{dg}(\mathscr{C})}(x, y).$$

With this, we are ready to explore how to construct $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$ and $\mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$ local systems valued on it as nerves of DG categories. In the case of modules the story is straightforward:

Lemma 3.17. *[HA, 7.1.1.16] Let* $\mathbf{A}$ *be a ring. There is an equivalence* $\mathcal{D}(\mathbf{A}) \simeq \mathrm{Mod}_{\mathbf{A}}(\mathrm{Sp})$.

Additionally, we have that injective resolution lets us access $\mathcal{D}(\boldsymbol{A})$ by just considering the dg nerve:

Lemma 3.18. *[HA, 1.3.4.5, 1.3.4.6] We have an equivalence* $N_{dg}(Ch^{inj}(\mathbf{A})) \simeq Ch(\mathbf{A})[quasi^{-1}] \simeq \mathcal{D}(\mathbf{A})$.

This is what we mean by DG origin of $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})$. Let us find this origin also on local systems. As it happens, the naïve approach of just using the same equivalence pointwise doesn't work. We have that, as $\mathrm{Ch}^{inj}(\boldsymbol{A})$ is a 1−category, functors from X actually factor through the fundamental group $\pi_1 X$ and thus

$$\mathrm{Fun}(X, \mathrm{Ch}^{inj}(\boldsymbol{A})) \simeq \mathrm{Fun}(\pi_1 X, \mathrm{Ch}^{inj}(\boldsymbol{A}))$$

whose image through N_{dg} is $\mathrm{Fun}(\pi_1 X, \mathcal{D}(\boldsymbol{A}))$. In order to circumvent this issue we consider sheaves, in which we can embed local systems as follows

Lemma 3.19. *[DAG, Th A.4.19] Given an* ∞*−category* $\mathscr{C}$*, we have that locally constant sheaves on* $X \in \mathrm{An}$ *with values in* $\mathscr{C}$ *are equivalent to local systems on X with values in* $\mathscr{C}$*:*

$$\mathrm{Sh}(X, \mathscr{C})_{loc\ const} \simeq \mathrm{Fun}(X, \mathscr{C}).$$

The idea is to access $\mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$ as locally constant sheaves on $\mathrm{Sh}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$, which we can reach as a DG nerve:

$$\mathrm{N}_{dg}(\mathrm{Sh}^{inj}(X, \mathrm{Ch}(\mathbb{Z}))) \simeq \mathrm{Sh}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp})) \tag{3.4}$$

With this realization, we can continue dissecting the classifying squares.

3.2.2 Maps and homotopy of the classifying square

We have determined the objects on the corners of the square and we're interested in the maps between them and the homotopy which fills the diagram.

$$\begin{tikzcd}
\underline{\mathbb{Z} \oplus \pi_{k+1}\mathbb{S}[k+2] \oplus {}^{\pi_{k+2}\mathbb{S}}/_{\eta}[k+3]} \arrow[r] \arrow[d] & \pi_k\mathscr{B}[k+1] \oplus {}^{\pi_k\mathscr{B}}/_{2}[k+3] \arrow[d] \\
\underline{\pi_{k+1}\mathbb{S}[k+2]} \arrow[r] & \pi_{k+1}\mathscr{B}[k+3].
\end{tikzcd}$$

Note that, because of the direct sum form, these maps will be represented by matrices of maps between the components. Following Th 3.10, we are interested in isomorphism classes of these maps, that is choices of maps

$$\mathbb{S} \to \mathrm{Map}(M[k], N[l]). \tag{3.5}$$

Here is where Ext groups become relevant. From our module perspective we have the following definition:

Definition 3.20. In $\mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))$ we define *Ext groups* as follows

$$\mathrm{Ext}^i(A, B) := \pi_{-i}\,\mathrm{Map}(A, B).$$

Remark 3.21. Recall that $\mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}) \simeq \mathcal{D}(\mathbb{Z})$. In the face of that, it is relevant to point out that this definition coincides with the classical notion of Ext group in the derived setting. We had

$$\mathrm{Ext}^i_{\mathcal{D}(\mathbb{Z})}(A, B) := \pi_0\,\mathrm{hom}_{\mathrm{Fun}(X, \mathcal{D}(\mathbb{Z}))}(A, \Sigma^i B)$$

hence, as $\mathrm{Map}(A, \Sigma^i B) \simeq \Sigma^i\,\mathrm{Map}(A, B)$, the two definitions are equivalent as Σ^{-i} is an equivalence

$$\Sigma^{-i}\mathbb{S} \to \mathrm{Map}(A, B) \qquad\qquad \mathbb{S} \to \Sigma^i\,\mathrm{Map}(A, B).$$

This is useful as it allows us to access results already proven in this setting, in particular we will be interested in the fact that $\mathrm{Ext}^i_{\mathcal{D}(\mathbb{Z})}(-, -)$ is the constant *zero* group functor if $i < 0$.

With the previous definition we can represent isomorphism classes of the components maps shown on (3.5) as classes on the $(l-k)$−Ext group, that is maps $\Sigma^{k-l}\mathbb{S} \to \mathrm{Map}(M, N)$ as we have:

$$\mathrm{Map}(M[k], N[l]) \simeq \mathrm{Map}(M[k], \Sigma^{l-k}N[k]) \simeq \Sigma^{l-k}\,\mathrm{Map}(M[k], N[k]) \simeq \Sigma^{l-k}\,\mathrm{Map}(M, N)$$

We have gathered the following information: iso classes of the maps on the square are given by matrices with entries in $\mathrm{Ext}^n(-,-)$ groups, where n is the change of degree. Additionally, by Def 3.2, maps from $\mathbb{Z}$ to another local system are in particular cohomology classes with coefficient in that local system. A explicit description of the maps is inside our reach:

$$\begin{tikzcd}
\underline{\mathbb{Z} \oplus \pi_{k+1}\mathbb{S}[k+2] \oplus {}^{\pi_{k+2}\mathbb{S}}\!/_{\eta}[k+3]} \arrow[r] \arrow[d] & \pi_k\mathscr{B}[k+1] \oplus {}^{\pi_k\mathscr{B}}\!/_{2}[k+3] \arrow[d] \\
\underline{\pi_{k+1}\mathbb{S}[k+2]} \arrow[r] & \pi_{k+1}\mathscr{B}[k+3].
\end{tikzcd}$$

- The left vertical map is a fixed map

$$\begin{pmatrix} 0 & \mathrm{id} & \mathrm{Ext}^{-1}({}^{\pi_{k+2}\mathbb{S}}\!/_{\eta}, \pi_{k+1}\mathbb{S}) \end{pmatrix},$$

  in particular it is the $\mathbb{Z}$−linearization of the k−invariant of $\mathbb{S}$. Due to this, the middle term is the identity, whereas the first term is zero as over $\mathbb{Z}$ we have resolutions of max lenght 2.

- And finally, the bottom horizontal map is given by an element of $\mathrm{Ext}^1(\pi_{k+1}\mathbb{S}, \pi_{k+1}\mathscr{B})$.

- The upper horizontal map is given by an element of

$$\begin{pmatrix} H^{k+1}(X; \pi_k\mathscr{B}) & H^{k+3}(X; {}^{\pi_k\mathscr{B}}\!/_{2}) \\ \mathrm{Ext}^{-1}(\pi_{k+1}\mathbb{S}, \pi_k\mathscr{B}) & \mathrm{Ext}^{1}(\pi_{k+1}\mathbb{S}, {}^{\pi_k\mathscr{B}}\!/_{2}) \\ \mathrm{Ext}^{-2}({}^{\pi_{k+2}\mathbb{S}}\!/_{\eta}, \pi_k\mathscr{B}) & \mathrm{Ext}^{0}({}^{\pi_{k+2}\mathbb{S}}\!/_{\eta}, {}^{\pi_k\mathscr{B}}\!/_{2}) \end{pmatrix}^{\mathrm{T}}.$$

- The right vertical map is given by an element of

$$\begin{pmatrix} \mathrm{Ext}^2(\pi_k\mathscr{B}, \pi_{k+1}\mathscr{B}) & \mathrm{Ext}^0({}^{\pi_k\mathscr{B}}\!/_{2}, \pi_{k+1}\mathscr{B}) \end{pmatrix}.$$

From our DG perspective, recall that we have $H_i(\hom_{\mathscr{C}}(x,y)) \simeq \pi_{-i}\,\mathrm{map}_{\mathrm{N}_{dg}(\mathscr{C})}(x,y)$. Hence, classes in Ext^i group are represented by cycles on the i level of the hom chain complex and moreover, homotopies between two class representatives z, w will belong to level $i+1$ of the hom chain complex and verify $\partial H = w - z$. Thus we know that for a square

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[d, "b"] \\
C \arrow[r, "g"] & D
\end{tikzcd}$$

the homotopies filling it will be represented by elements $H \in \hom(A, D)_1$ whose boundary is the difference of the two compositions. ie $\partial H = ba - gf$. Another way to put it is to say that the following is a fiber sequence

$$\hom(A, D)_1^{\partial} \to \mathrm{Fun}\left(\blacksquare, \mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))\right) \to \mathrm{Fun}\left(\square, \mathrm{Fun}(X, \mathrm{Mod}_{\mathbb{Z}}(\mathrm{Sp}))\right),$$

where the right map forgets the homotopy filling the diagram and the fiber verifies the boundary condition. If we switch to homotopy classes of these objects we have that the fiber is in particular a torsor acting on the squares.

Lastly, we show that the choice of representative for the homotopy is coherent with the choice of representative of the Ext classes, as a sort of well definedness check. Imagine the previous square was filled by a homotopy H such that $\partial(H) = ba - gf$, but we replace a by $a + \partial x$ where $x \in \hom(A, B)_{|a|+1}$. Then we can inspect the difference of the two compositions to spot the following:

$$\left.\begin{array}{l} b(a+\partial(x)) - gf = ba + b\partial(x) - gf = \partial(H) + b\partial(x) \\ \partial(bx) = \partial(b)x + (-1)^{|b|} b\partial(x) \end{array}\right\} \Rightarrow \partial(H) + b\partial x = \partial(H) + (-1)^{|b|+1}\partial(bx)$$

The last expression is equal to $\partial(H + (-1)^{|b|+1}bx)$ and thus our new homotopy will be $H + (-1)^{|b|+1}bx$. Note that the smae argument works for representatives of the Ext classes in any of the other sides of the square.

We conclude that the data of a functor in $\mathrm{Fun}(X, \mathrm{An}^{[k,k+1]})$ is given by a square in the locally constant subcategory of $\mathrm{Sh}^{inj}(X, \mathrm{Ch}(\mathbb{Z}))$ like the following, with a homotopy filling it whose boundary is the difference of the elements chosen as representatives of the arrows:

$$\begin{tikzcd}
\underline{\mathbb{Z} \oplus \pi_{k+1}\mathbb{S}[k+2] \oplus {}^{\pi_{k+2}\mathbb{S}}\!/\!_{\eta}[k+3]} \arrow[rr, "\begin{pmatrix} H^{k+1}(X;\pi_k\mathscr{B}) & 0 & 0 \\ H^{k+3}(X;{}^{\pi_k\mathscr{B}}\!/\!_2) & \mathrm{Ext}^1(\pi_{k+1}\mathbb{S},{}^{\pi_k\mathscr{B}}\!/\!_2) & \mathrm{Ext}^0({}^{\pi_{k+2}\mathbb{S}}\!/\!_{\eta},{}^{\pi_k\mathscr{B}}\!/\!_2) \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix}"'] & & \pi_k\mathscr{B}[k+1] \oplus {}^{\pi_k\mathscr{B}}\!/\!_2[k+3] \arrow[d, "\begin{pmatrix} \mathrm{Ext}^2(\pi_k\mathscr{B},\pi_{k+1}\mathscr{B}) & \mathrm{Ext}^0({}^{\pi_k\mathscr{B}}\!/\!_2,\pi_{k+1}\mathscr{B}) \end{pmatrix}"'] \\
\underline{\pi_{k+1}\mathbb{S}[k+2]} \arrow[rr, "\mathrm{Ext}^1(\pi_{k+1}\mathbb{S},\pi_{k+1}\mathscr{B})"] & & \pi_{k+1}\mathscr{B}[k+3].
\end{tikzcd}$$

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