UNIVERSITY OF COPENHAGEN

Topics in Algebraic Topology

Six Functor Formalism

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Contents

Introduction

A six functor formalism consists of three pairs of adjoint functors plus a bunch of relations and compatibilities. We will present a compact definition, work out the details in the following two chapters - as well as how the information is encoded in this definition - and then on the last chapter cover how to construct six functor formalisms. First of all, where can we define them?

Definition 0.1. A *geometric setup* is a pair (C, E) where C is a $(\infty-)$category and E is a class of morphisms in C satisfying

- C has all finite limits.
- E contains all isomorphisms and is stable under base change and composition.

The central idea of the text is the following definition

Definition 0.2. A *three functor formalism* is a lax symmetric monoidal functor

$$\mathcal{D} : \mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty .$$

This encodes functors $\otimes, f^*, f_!$ and all their relations.

Definition 0.3. A *six functor formalism* is a three functor formalism for which the three functors admit right adjoints. All coherence conditions on the right adjoints are derived from the left adjoints so no extra conditions are required.

On the next chapter we will define $\mathrm{Corr}(C, E)$ and prove that it is an $\infty-$category. Afterwards in chapter two we will look at the symmetric monoidal structure on $\mathrm{Corr}(C, E)$. With this we will have an understanding of the notion of six functor formalisms as the symmetric monoidal structure on Cat_∞ is just the cartesian one given by finite products.

Six functor formalisms allows us to cleanly prove duality statements, pass from local to global and eliminate the need for arbitrary choices in our theories like looking at right or left modules, in cases where both are valid. With this as some motivation, let us begin our journey.

Chapter 1

$\infty-$category of correspondences

Definition 1.1. For $n \geq 0$ let

$$\mathsf{C}(\Delta^n) \subset (\Delta^n)^{\mathrm{op}} \times \Delta^n$$

be the subset spanned by those simplices $(i, j) \in \{0, 1, \ldots, n\}^2$ with $i \geq j$.

An edge of $\mathsf{C}(\Delta^n)$ is called *vertical* if its projection to the second factor is degenerate. A square of $\mathsf{C}(\Delta^n)$ is called *exact* if it is both cartesian and cocartesian.

Extend the above construction to a colimit preserving functor $\mathsf{C}(-) : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$. It can be proven that $\mathsf{C}(-)$ preserves finite products. Denote the right adjoint functor by $\mathrm{Corr}(-)$, then for a simplicial set K we have that $\mathrm{Corr}(K)$ is a simplicial set: $\mathrm{Corr}(K)_n = Hom(\mathsf{C}(\Delta^n), K)$.

Let us look now at the $\infty-$*category of correspondences* of the geometric setup (C, E), which is a subset of $\mathrm{Corr}(C)$:

Definition 1.2. $\mathrm{Corr}(C, E)$ is the simplicial set whose $n-$simplices are maps $\mathsf{C}(\Delta^n) \to C$ satisfying:

1. All vertical edges are to morphisms in E.
2. All exact squares are sent to cartesian squares.

So on the lower levels we have $\mathrm{Ob}(C)$ as $0-$simpices, whereas $1-$simplices are diagrams

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g\in E"] & \\ X & & Y \end{tikzcd}$$

with $W \in C$. Composition of two arrows between X & Y and Y & Z would look like

$$\begin{tikzcd} & & W'' \arrow[dl, dashed] \arrow[dr, dashed, "g'"] & & \\ & W \arrow[dl, "f"'] \arrow[dr, "g"] & & W' \arrow[dl] \arrow[dr, "h"] & \\ X & & Y & & Z \end{tikzcd}$$

which always exists as C has all finite limits and is in fact an arrow between X and Z as $g \in E \Rightarrow g' \in E$ and $h \circ g' \in E$ as E is stable under composition.

We want to prove that $\mathrm{Corr}(C, E)$ is a quasi-category. In order to do that we will present [LZ, Lemma 6.1.2.], which uses [HTT, Proposition 4.3.2.15.] (incorrectly cited in [LZ] as 4.2.3.15). We begin with some definitions from [HTT] necessary for understanding:

Definition 1.3. [HTT, Definition 2.0.0.3] A morphism $f : X \to S$ of simplicial sets is an *inner fibration* if f has the right lifting property with respect to all horn inclusions $\Lambda^n_i \subseteq \Delta^n, 0 \leq i < n$, i.e. for a diagram like the following there exists a diagonal arrow as indicated and homotopies making the triangles commute for $0 \leq i < n$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Additionally, a morphism $g : X \to S$ of simplicial sets which has the right lifting property with respect to every inclusion $\partial\Delta^n \subseteq \Delta^n$ is called a *trivial fibration*.

Definition 1.4. [Nardin, Def 1.67] Let S be a simplicial set. Then its *right cone* $S^{\triangleright}$ is the simplicial set

$$[n] \mapsto \{(f, \sigma) | f : \Delta^n \to \Delta^1, \sigma : f^{-1}(0) \to S\}$$

This should be thought of as the 1-categorical cone. I chose this definition over [HTT, Notation 1.2.8.4.] to avoid the discussion of joins of ∞−categories.

Definition 1.5. [HTT, Def 4.3.1.1] Let $f : C \to \mathcal{D}$ be an inner fibration of simplicial sets, let $\bar{p} : K^{\triangleright} \to C$ be a diagram, and let $p = \bar{p}|K$. We will say that $\bar{p}$ is an *f−colimit of p* if the map

$$C_{\bar{p}/} \to C_{p/} \underset{\mathcal{D}_{f\circ p/}}{\times} \mathcal{D}_{f\circ\bar{p}/}$$

is a trivial fibration of simplicial sets. In this case, we will also say that $\bar{p}$ is an *f−colimit diagram*.

Notation 1.6. Let C be an ∞−category, and let C^0 be a full subcategory. If $p : K \to C$ is a diagram, then we let $C^0_{/p}$ denote the fiber product $C_{/p} \underset{C}{\times} C^0$. In particular, if c is an object in C, then $C^0_{/c}$ denotes the full subcategory of $C_{/c}$ spanned by the morphisms $c' \to c$ where $c' \in C^0$.

Definition 1.7. [HTT, Def 4.3.2.2] Suppose given a commutative diagram of ∞−categories

$$\begin{tikzcd} C^0 \arrow[r, "F_0"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ C \arrow[r] \arrow[ur, "F"] & \mathcal{D}' \end{tikzcd}$$

where p is an inner fibration and the left vertical map is the inclusion of a full subcategory $C^0 \subseteq C$. We will say that F is a *p−left Kan extension of F_0 at $c \in C$* if the induced diagram

$$\begin{tikzcd} (C^0_{/c}) \arrow[r, "F_c"] \arrow[d, hook] & \mathcal{D} \arrow[d, "p"] \\ (C_{/c})^{\triangleright} \arrow[r] \arrow[ur] & \mathcal{D}' \end{tikzcd}$$

exhibits $F(c)$ as a p−colimit of F_c.

We will say that F is a *p−left Kan extension of F_0* if it is a p−left Kan extension of F_0 at c for every object $c \in C$.

Proposition 1.8. *[HTT, Proposition 4.3.2.15.] Suppose given a diagram of ∞−categories*

$$C \to \mathcal{D}' \xleftarrow{p} \mathcal{D}$$

where p is an inner fibration. Let C^0 be a full subcategory of C.

- *Let $\mathcal{K} \subseteq \mathrm{Map}_{\mathcal{D}'}(C, \mathcal{D})$ be the full subcategory spanned by those functors $F : C \to \mathcal{D}$ which are p−left Kan extensions of $F|C^0$.*
- *Let $\mathcal{K}' \subseteq \mathrm{Map}_{\mathcal{D}'}(C^0, \mathcal{D})$ be the full subcategory spanned by those functors $F^0 : C^0 \to \mathcal{D}$ with the property that, for each object $C \in C$, the induced diagram $C^0_{/C} \to \mathcal{D}$ has a p−colimit.*

Then the restriction functor $\mathcal{K} \to \mathcal{K}'$ is a trivial fibration of simplicial sets.

We are now in condition to prove that $\mathrm{Corr}(C, E)$ is an ∞−category

Theorem 1.9. *([LZ, Lemma 6.1.2], adapted) The previously defined* $\mathrm{Corr}(C, E)$ *is an ∞−category.*

Proof. We will check that $\mathrm{Corr}(C, E) \to *$ has the right lifting property with respect to $(\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) \subseteq \Delta^m \times \Delta^2$ which is equivalent to the usual inner horn inclusions by [HTT, 2.3.2.1]. Now, as $\mathsf{C}(-)$ preserves colimits, giving a map

$$f : (\Delta^m \times \Lambda^2_1) \coprod_{\partial\Delta^m \times \Lambda^2_1} (\partial\Delta^m \times \Delta^2) \to \mathrm{Corr}(C)$$

is equivalent to giving a map

$$f^\sharp : (\mathsf{C}(\Delta^m) \times \mathsf{C}(\Lambda^2_1)) \coprod_{\mathsf{C}(\partial\Delta^m) \times \mathsf{C}(\Lambda^2_1)} (\mathsf{C}(\partial\Delta^m) \times \mathsf{C}(\Delta^2)) \to C.$$

Now let $\mathcal{K}$ and $\mathcal{K}'$ be defined as in the in the dual version of the previous proposition, with $C = \mathsf{C}(\Delta^2)$, $C^0 = \mathsf{C}(\Lambda^2_1)$, and $\mathcal{D} = C$. If f factors through $\mathrm{Corr}(C, E)$, then $f^\sharp$ induces a commutative square

$$\begin{tikzcd}
\mathsf{C}(\partial\Delta^m) \arrow[r] \arrow[d, hook] & \mathcal{K} \subseteq \mathrm{Map}_{\mathcal{D}'}(\mathsf{C}(\Delta^2), C) \arrow[d] \\
\mathsf{C}(\Delta^m) \arrow[r] & \mathcal{K}' \subseteq \mathrm{Map}_{\mathcal{D}'}(\mathsf{C}(\Lambda^2_1), C)
\end{tikzcd}$$

as finite limits exist and E is closed under base change. Since the restriction map $\mathcal{K} \to \mathcal{K}'$ is a trivial fibration by the previous proposition, there exists a diagonal arrow $g^\sharp : \mathsf{C}(\Delta^m) \to \mathcal{K}$.

We regard $g^\sharp$ as a map $\mathsf{C}(\Delta^m \times \Delta^2) \simeq \mathsf{C}(\Delta^m) \times \mathsf{C}(\Delta^2) \to C$, thus induces a map $g : \Delta^m \times \Delta^2 \to \mathrm{Corr}(C)$.

Since all exact squares of $\mathsf{C}(\Delta^m \times \Delta^2)$ can be obtained by composition from exact squares either contained in the source of $f^\sharp$ or being constant under the projection to $\mathsf{C}(\Delta^m)$, the three assumptions ensure that if f factors throught $\mathrm{Corr}(C, E)$, then so does g. We found the desired lifts. □

Remark 1.10. If you check [LZ, Lemma 6.1.2], the adaptation needed is to plug $\mathcal{E}_1 = E$ and all morphisms in C as $\mathcal{E}_2$.

Chapter 2

Symmetric monoidal structure on $\mathrm{Corr}(C, E)$

Consider the category Fin_* of finite pointed sets. As finite sets are isomorphic if they have the same cardinality, we will actually consider the following skeleton, which we will also denote Fin_* in an abuse of notation

- Its objects are sets $\langle n \rangle = \{*, 1, 2, \ldots, n\}$
- Its morphisms are maps $\alpha : \langle n \rangle \to \langle m \rangle$ such that $\alpha(*) = *$

Morphisms are referred to as *inert* if

$$\forall i \in \{1, 2, \ldots, m\} {=:} \langle m \rangle^\circ \text{ we have that } |\alpha^{-1}(i)| \leq 1$$

i.e. points don't get mushed together. In contrast, a morphism that satisfies $\alpha^{-1}(\{*\}) = \{*\}$ is called *active*.

A particular family of inert morphisms that we will be interested in is, for $1 \leq i \leq n$, the maps $\rho^i : \langle n \rangle \to \langle 1 \rangle$ defined as

$$\rho^i(j) = \begin{cases} 1 & \text{if } j = i \\ * & \text{otherwise} \end{cases} \tag{2.1}$$

With this we can now define

Definition 2.1. A *commutative monoid* in a $\infty-$category $\mathscr{C}$ is a functor $X : \mathrm{Fin}_* \longrightarrow \mathscr{C}$ such for all $\langle n \rangle$ the map induced by ρ^i for $1 \leq i \leq n$

$$X(\langle n \rangle) \xrightarrow{(\rho^i)_i} \prod_{1 \leq i \leq n} X(\langle 1 \rangle)$$

is an equivalence.

Definition 2.2. A *symmetric monoidal $\infty-$category* $(\mathscr{C}, \otimes)$ is a commutative monoid in Cat_∞.

Thus, giving a symmetric monoidal $\infty-$category requires us to write down functors

$$\mathrm{Fin}_* \to \mathrm{Cat}_\infty$$

These type of functors are difficult to define as all higher coherences need to be specified. In order to solve this problem, we have the following theorem, due to Lurie

Theorem 2.3. *(Straightening/Unstraightening) For an $\infty-$category D, there is a canonical equivalence between the $\infty-$category of functors $Fun(D, \mathrm{Cat}_\infty)$ and the $\infty-$category of coCartesian fibrations over D, $CoCart(D)$.*

$$\begin{tikzcd} \mathrm{El}(F) \arrow[d] & & \\ \mathrm{Fin}_* & \longleftrightarrow & \left(\mathrm{Fin}_* \xrightarrow{F} \mathrm{Cat}_\infty \right) \end{tikzcd}$$

We need now to understand what a coCartesian fibration is. Following the model-free definitions from [Gee], we begin by:

Definition 2.4. Fix a functor of $\infty-$categories $p : \mathcal{E} \to D$.

A morphism $\varphi \in Hom(e_1, e_2)$ in $\mathcal{E}$ is *$p-$cocartesian* - or a *$p-$coCartesian lift* of $p(\varphi)$ relative to e_1, depending on the perspective - if the following diagram in Cat_∞ is cartesian:

$$\begin{tikzcd} \mathcal{E}_{e_2/} \arrow[r, "- \circ \varphi"] \arrow[d, "p"] & \mathcal{E}_{e_1/} \arrow[d, "p"] \\ D_{p(e_2)/} \arrow[r, "- \circ p(\varphi)"] & D_{p(e_1)/} \end{tikzcd}$$

The functor p is said to be a *coCartesian fibration* if every object of the pullback

$$\begin{tikzcd} Fun([1], D) \times_D \mathcal{E} \arrow[r, dashed] \arrow[d, dashed] & \mathcal{E} \arrow[d, "p"] \\ Fun([1], D) \arrow[r, "source"'] & D \end{tikzcd}$$

admits a $p-$coCartesian lift.

Alternative viewpoint Now looking at the coCartesian definition from the perspective used on the exercise sessions, which maybe showcases the Grothendieck construction more clearly, we consider the diagram

$$\begin{tikzcd} & & \psi_! e_1 \arrow[dl, dashed] \arrow[ddl, mapsto, "p"] \\ e_1 \arrow[urr] \arrow[r] \arrow[d, mapsto, "p"] & e_2 \arrow[d, mapsto, "p"] & \\ d_1 \arrow[r, "\psi"] & d_2 & \end{tikzcd}$$

here, we are given a morphism $d_1 \xrightarrow{\psi} d_2$ in D and an element in the fiber of the source $e_1 \in p^{-1}(d_1)$. The $p-$coCartesian lift of ψ with respect to e_1, which we denote $e_1 \to \psi_! e_1$, is the intial object in the category of lifts of ψ with source e_1.

This is why we will have that dotted arrow to any other lift of ψ. This info is encoded in the first diagram on the definition, which says that $\forall e \in \mathcal{E}$ the following is a cartesian diagram in An:

$$\begin{tikzcd}
hom_{\mathcal{E}}(e_2, e) \arrow[r] \arrow[d] & hom_{\mathcal{E}}(e_1, e) \arrow[d] \\
hom_D(p(e_2), f(e)) \arrow[r] & hom_D(p(e_1), f(e))
\end{tikzcd}$$

Additionally, the $Fun([1], D) \underset{D}{\times} \mathcal{E}$ part of the definition asserts that p is a coCartesian fibration if we have $p-$coCartesian lifts of any morphism in D with respect to any starting point in the fiber of the source of the morphism.

Now, as a consequence of the straightening theorem we have the following equivalent definition of sym monoidal $\infty-$category:

Definition 2.5. A *symmetric monoidal $\infty-$category* is a coCartesian fibration $p : \mathscr{C}^{\otimes} \to \mathrm{Fin}_*$ such that for all finite sets $\langle n \rangle$ the functor

$$\mathscr{C}^{\otimes}_{\langle n \rangle} \xrightarrow{(\rho^i)_!} \prod_{1 \leq i \leq n} \mathscr{C}^{\otimes}_{<1>}$$

induced on the fibers $\mathscr{C}^{\otimes}_{\langle j \rangle} := p^{-1}(\langle j \rangle)$ by the inert maps $\rho^i, i \in \langle n \rangle^{\circ}$, is an equivalence.

We will not go into the details of the straightening and unstraightening functors and the exact description of the sources $\mathscr{C}^{\otimes}$ of our coCartesian fibrations. A technical discussion of this can be found at [Straight]. Nevertheless, we will look at $\mathscr{C}^{\otimes}$ in the 1-categorical case, which will give us a partial description in the $\infty-$categorical case (higher coherences missing) but will serve as an non-rigorous idea.

Sources of coCart fibrations: 1-categorical case Objects of $\mathscr{C}^{\otimes}$ are simply pairs $(\langle m \rangle, (X_i)_{1 \leq i \leq m})$ where $X_i \in \mathscr{C}$.

Morphisms $(\langle m \rangle, (X_i)_{1 \leq i \leq m}) \to (\langle n \rangle, (Y_j)_{1 \leq j \leq n})$ in $C^{\otimes}$ are given by a map $f \in \mathrm{Map}_{\mathrm{Fin}_*}(\langle m \rangle, \langle n \rangle)$ together with maps $\bigotimes_{i \in f^{-1}(j)} X_i \to Y_j$, for $1 \leq j \leq n$. This morphism gets sent to $f \in \mathrm{Map}_{\mathrm{Fin}_*}(\langle m \rangle, \langle n \rangle)$ by the coCartesian fibration.

Example 2.6. The map $f : \langle 5 \rangle \to \langle 3 \rangle$ on the left is part of the associated morphism in $C^{\otimes}$ on the right.

$$\begin{tikzcd}
\langle 5 \rangle = \Big\{ & * \arrow[d] & 1, \arrow[d] & 2, \arrow[dl] & 3, \arrow[d] & 4, \arrow[dlll] & 5 \arrow[dll] & \Big\} \\
\langle 3 \rangle = \Big\{ & *, & 1, & 2, & 3 & & & \Big\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Big(& \langle 5 \rangle, \arrow[d] & X_1 \otimes X_2 \otimes X_4 \arrow[d] & X_3 \otimes X_5 \arrow[dr] & & \Big) \\
\Big(& \langle 3 \rangle, & Y_1, & Y_2, & Y_3 & \Big)
\end{tikzcd}$$

Composition of two morphisms $(\langle m \rangle, (X_i)_{1 \leq i \leq m}) \xrightarrow{f} (\langle n \rangle, (Y_j)_{1 \leq j \leq n})$ and $(\langle n \rangle, (Y_j)_{1 \leq j \leq n}) \xrightarrow{g} (\langle l \rangle, (Z_k)_{1 \leq k \leq l})$ in $C^{\otimes}$ is given by the map $\langle m \rangle \xrightarrow{g \circ f} \langle l \rangle$ and the maps

$$\bigotimes_{i \in (g \circ f)^{-1}(k)} X_i \simeq \bigotimes_{j \in g^{-1}(k)} \bigotimes_{i \in f^{-1}(j)} X_i \to \bigotimes_{i \in (g \circ f)^{-1}(k)} Y_j \to Z_k$$

for $1 \leq k \leq l$.

We are now ready to look at how Corr(C, E) promotes to a symmetric monoidal $\infty-$category, for that we need to examine $C^{\times'}$ and $E^{\times}$:

As C has all finite products, we have the cartesian symmetric monoidal structure available. We will actually look at the symmetric monoidal structure that the coproduct $\coprod$ provides on C^{op}, define $C^{\times'} := ((C^{\mathrm{op}})^{\amalg})^{\mathrm{op}}$.

Objects of $C^{\times'}$ are pairs $(\langle m\rangle, (X_i)_{1\leq i\leq m})$ where $X_i \in C$ and morphims $(\langle m\rangle, (X_i)_{1\leq i\leq m}) \to (\langle n\rangle, (Y_j)_{1\leq j\leq n})$ are the datum of a morphism $f \in \mathrm{Map}_{\mathrm{Fin}_*}(\langle n\rangle, \langle m\rangle)$ and maps

$$X_i \to \prod_{j\in f^{-1}(i)} Y_j, \text{ with } 1 \leq i \leq m.$$

Now $E^{\times}$ is the subset of morphisms of $C^{\times'}$ that lie over identity morphisms in Fin_* and whose maps belong to E.

Proposition 2.7. *([LZ, Prop 6.1.3], adapted) Let* $\mathrm{Corr}(C, E)^{\otimes} := \mathrm{Corr}(C^{\times'}, E^{\times})$*. Then*

$$p : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Fin}_*$$

is a coCartesian fibration, with $\mathrm{Corr}(C, E)^{\otimes}_{\langle 1\rangle} \simeq \mathrm{Corr}(C, E)$.

Before proving the result we will take a look at objects and morphisms of $\mathrm{Corr}(C, E)^{\otimes}$ and state an auxiliary lemma.

Idea of $\mathrm{Corr}(C, E)^{\otimes}$ Objects are pairs $(\langle m\rangle, (X_i)_{1\leq i\leq m})$ with $X_i \in C$, and morphisms $(\langle n\rangle, (Y_j)_{1\leq j\leq n}) \to (\langle m\rangle, (X_i)_{1\leq i\leq m})$ are maps $f : \langle m\rangle \to \langle n\rangle$ together with correspondences

$$\begin{tikzcd} & Z_j \arrow[dl] \arrow[dr, "\in E"] & \\ Y_j & & \prod_{i\in f^{-1}(j)} X_i \end{tikzcd}$$

for $1 \leq j \leq n$.

Lemma 2.8. *([LZ, Lemma 6.1.4],adapted) Given a morphism* f *of* $\mathrm{Corr}(C, E)^{\otimes}$ *of the form*

$$\begin{tikzcd} & (Z_j)_{1\leq j\leq n} \arrow[dl] \arrow[dr] & \\ (Y_j)_{1\leq j\leq n} & & (X_i)_{1\leq i\leq m} \end{tikzcd}$$

lying over an edge $f : \langle m\rangle \to \langle n\rangle$ *in* Fin_* *through* p*, then* f *is a* $p-$*coCartesian lift if and only if*

1. *for every* $1 \leq j \leq n$*, the induced morphism* $Z_j \to Y_j$ *is an isomorphim, and*
2. *for every* $1 \leq j \leq n$*, the induced morphisms* $Z_j \to \prod_{i\in f^{-1}(j)} X_i$ *are isomorphisms.*

Proof. (of Proposition 2.7) By theorem 1.9, $\mathrm{Corr}(C, E)^{\otimes}$ is an $\infty-$category. By the previous lemma we know that p is a coCartesian fibration since C admits finite products. Moreover, we have the isomorphism $\mathrm{Corr}(C, E)^{\otimes}_{\langle n\rangle} \simeq \prod_{1\leq i\leq n} \mathrm{Corr}(C, E)^{\otimes}_{\langle 1\rangle}$ induced by $(\rho^i)_i$. We now know that this promotes to a symmetric monoidal structure and that the underlying $\infty-$category is $\mathrm{Corr}(C, E)$ from construction. □

Definition 2.9. Given two symmetric monoidal $\infty-$categories $C^{\otimes}, D^{\otimes} \to \mathrm{Fin}_*$ We say that $F^{\otimes} : \mathscr{C}^{\otimes} \to D^{\otimes}$ is a *lax symmetric monoidal functor* over Fin_* if it preserves coCartesian lifts of ρ^i. We say that it is *symmetric monoidal* if it preserves all coCartesian lifts.

It turns out that the cartesian symmetric monoidal sturcture is special when dealing with lax symmetric monoidal functors, and we get the following result.

Theorem 2.10. *[LurHA, Prop 2.4.1.7] Let $(\mathscr{C}, \otimes)$ promote to a symmetric monoidal ∞−category and D be an ∞−cat admitting finite products, endowed with the cartesian symmetric monoidal structure.*

Then anima of lax symmetric monoidal functors $\mathscr{C}^{\otimes} \to D^{\times}$ is equivalent to the anima of functors

$$F : \mathscr{C}^{\otimes} \to D$$

such that $\forall \langle m \rangle \in \mathrm{Fin}_*$ and $X_i \in \mathscr{C}$, for $1 \leq i \leq m$, the map

$$F((I, (X_i)_{1 \leq i \leq m})) \xrightarrow{(\rho^i)_i} \prod_{1 \leq i \leq m} F(X_i)$$

is an equivalence.

We arrive at the ultimate definition of a 6-functor formalism, which we will use in the next chapter.

Definition 2.11. A *three functor formalism* is a lax sym monoidal functor $\mathrm{Corr}(C, E) \to \mathrm{Cat}_\infty$. Equivalently, it is a functor $\mathcal{D} : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Cat}_\infty$ such that $\forall \langle n \rangle \in \mathrm{Fin}_*$ and $X_i \in C,\ i \in \langle n \rangle$, the functor

$$\mathcal{D}((\langle n \rangle, (X_i)_{1 \leq i \leq n})) \to \prod_{1 \leq i \leq n} \mathcal{D}(X_i)$$

is an equivalence.

2.0.1 Info encoded in the definition

We will only look at what data the three functor formalism encodes, as all coherence relations on the right adjoints come automatically from the left adjoints.

From the lax symmetric monoidal structure of the functor we get a canonical "exterior tensor product" functor $\mathcal{D}(X) \otimes \mathcal{D}(X) \to \mathcal{D}(X \times X)$ that composed with the image $(\mathcal{D}(X \times X) \to \mathcal{D}(X))$ of the first correspondence of the next diagram defines the **tensor product** $\otimes$ on $\mathcal{D}(X)$.

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\mathrm{id}"] & \\ X \times X & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

The image of the middle correspondence on the diagram defines the **pullback functor** $f^* : \mathcal{D}(Y) \to \mathcal{D}(X)$ for any map $f : X \to Y$.

On the other hand, the image of the right correspondence of the diagram defines the **lower shriek functor** $g_! : \mathcal{D}(X) \to \mathcal{D}(Y)$ for any $g : X \to Y$ in E.

In particular, a correspondence

$$\begin{tikzcd} & W \arrow[dl, "k"'] \arrow[dr, "h"] & \\ X & & Y \end{tikzcd}$$

gets sent to the functor $h_! k^* : \mathcal{D}(X) \to \mathcal{D}(Y)$.

Base change. Knowing this, from compatibility with composition and the following correspondence

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & X' \arrow[dl, dashed, "f'"'] \arrow[dr, dashed, "g'"] & & \\ & Y' \arrow[dl, "\mathrm{id}"'] \arrow[dr, "g"] & & X \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}"] & \\ Y' & & Y & & X \end{tikzcd}$$

we get the base change formula $g^* f_! \simeq f'_! g'^*$ for any cartesian diagram like the one on the left.
Projection formula. given $f : X \to Y$ in E, $A \in \mathcal{D}(X)$ and $B \in \mathcal{D}(Y)$,

$$f_!(A \otimes f^*(B)) \simeq f_!(A) \otimes B.$$

We present two ways of checking this, one more related with the way that we showed how the three functor formalism encoded the rest of the information and the other using ideas related to modules that may shine more light on what's happening underneath.

Strategy 2.12. *Given $f : X \to Y$ in E, the projection formula on* $\mathrm{Corr}(C, E)^{\otimes}$ *is equivalent to saying that $f_!$ promotes to a homomorphism of $\mathcal{D}(Y)-$modules.*

How can we then look at this situation on the context of modules? First a couple definitions:

Definition 2.13. A *coalgebra A* in the symmetric monoidal $\infty-$category C is a algebra object in the opposite category C^{op}.

In the 1-category case we would have an object A of C together with the comultiplication $\Delta : A \to A \otimes A$ and counit $e : A \to *$ maps verifying coassociativity and counitality constraints.

We now want to look at comodules over a coalgebra, however, in order to not go through a long detour which is not the point of these notes, we will look at the 1-categorical case exclusively. The appropiate definition of comodule for $\infty-$categories can be found in [LurHA, Def 3.3.3.8.].

Definition 2.14. Given a coalgebra A in C (symmetric monodial 1-category) and an object $M \in C$, a *left $A-$coaction* is a morphism $\rho : M \to A \otimes M$ satisfying coassociativity and counitality. An *$A-$comodule* is an object M together with a left $A-$coaction.

The way that we will use this concepts is noticing that every object in C is a coalgebra: as C has all finite products we can endow it with the cartesian symmetric monoidal structure. Hence, the usual diagonal map works as comultiplication and the map to the terminal object (empty product) works as counit map. Coassociativity and counitality are clear.

Now if we have a morphism $a \xrightarrow{f} b$ with $a, b \in C$, then it is a map of coalgebras by

$$\begin{tikzcd} a \arrow[r, "\Delta"] \arrow[d, "f"] & a \times a \arrow[d, "f\times f"] \\ b \arrow[r, "\Delta"] & b \times b \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ b \arrow[r] & *. \end{tikzcd}$$

Moreover, we have that a is a $b-$comodule as we have the commutative coaction diagram:

$$\begin{tikzcd} a \arrow[r, "(f{,}\mathrm{id})"] \arrow[d, "(f{,}\mathrm{id})"] & b \times a \arrow[d, "\Delta\times\mathrm{id}"] \\ b \times a \arrow[r, "\mathrm{id}\times(f{,}\mathrm{id})"] & b \times b \times a \end{tikzcd}$$

In addition, b is trivially a $b-$module and f can be viewed as a map of modules.

Consider now the map $F : (C, \times) \to \mathrm{Corr}(C, E)^{\otimes}$ given by sending $a \mapsto (\langle 1\rangle, a)$ and a morphism $a \xrightarrow{f} b$ to

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "="] & \\ b & & a \end{tikzcd}$$

which is a morphism from b to a. This functor reverses morphisms but leaves them otherwise untouched. For this reason the structure is preserved and we have that for an object $a \in Ob(C)$, the object $F(a)$ is an algebra, with the maps

$$\begin{tikzcd} & a \arrow[dl, "f"'] \arrow[dr, "="] & \\ a \times a & & a \end{tikzcd} \qquad \begin{tikzcd} & a \arrow[dl, "e"'] \arrow[dr, "="] & \\ * & & a \end{tikzcd}$$

verifying the desired properties as consequence of the properties in C. By the same reasons as before we have that $F(f)$ is a map of algebras and induces a $F(b)-$module structure on $F(a)$.

Finally, consider the functor $C_E \to \mathrm{Corr}(C, E)^{\otimes}$ sending $a \mapsto F(a)$ and a morphism $\epsilon : a \to b$ in E to the correspondence $\tilde{\epsilon}$:

$$\begin{tikzcd} & a \arrow[dl, "="'] \arrow[dr, "\epsilon"] & \\ a & & b \end{tikzcd}$$

Then $\tilde{\epsilon}$ is a map of $F(b)-$modules as the diagram

$$\begin{tikzcd} F(a) \times F(b) \arrow[r, "\tilde{\epsilon}\times\mathrm{id}"] \arrow[d, "\rho^{\mathrm{op}}"] & F(b) \times F(b) \arrow[d, "\rho^{\mathrm{op}}"] \\ F(a) \arrow[r, "\tilde{\epsilon}"] & F(b) \end{tikzcd}$$

commutes because $F(\epsilon)$ is precisely the morphism that endows $F(a)$ with a $F(b)-$module structure. Now recall that $\tilde{\epsilon}$ is $\epsilon_!$ and being a map of $F(b)-$modules gives us the desired isomorphism, we can pull $\epsilon_!$ out of the tensor product.

Strategy 2.15. *Regard the correspondence* $(\langle 2\rangle, (X, Y)) \to (\langle 1\rangle, Y)$ *associated to the projection formula as two different compositions.*

First consider the composition $(\langle 2\rangle, (X, Y)) \to (\langle 2\rangle, (Y, Y)) \to (\langle 1\rangle, Y)$ where the last map is active, which can be visualized as

$$\begin{tikzcd} & & X \times Y \times Y \arrow[dl, dashed, "\mathrm{id}\times\mathrm{proj}"'] \arrow[dr, dashed, "f"] & & \\ & X \times Y \arrow[dl, "\mathrm{id}\times\mathrm{id}"'] \arrow[dr, "f\times\mathrm{id}"] & & Y \arrow[dl, "\Delta"'] \arrow[dr, "\mathrm{id}"] & \\ X \times Y & & Y \times Y & & Y \end{tikzcd}$$

here the composite unravels to $(A, B) \mapsto (f_!(A), B) \mapsto f_!(A) \otimes B$.

On the other hand we can express the corrsepondence as the composite $(\langle 2\rangle, (X, Y)) \to (\langle 1\rangle, X) \to (\langle 1\rangle, Y)$ where the first map is active. The diagram visualizing this case would be

$$\begin{tikzcd}[column sep=small]
 & & X \times X \arrow[dl, dashed, "\mathrm{proj}"'] \arrow[dr, dashed, "\mathrm{proj}"] & & \\
 & X \arrow[dl, "\mathrm{id} \times f"'] \arrow[dr, "\mathrm{id}"] & & X \arrow[dl, "\mathrm{id}"'] \arrow[dr, "f"] & \\
X \times Y & & X & & Y
\end{tikzcd}$$

Here the composite unravels to $(A, B) \mapsto A \otimes f^*(B) \mapsto f_!(A \otimes f^*(B))$.

Checking that these two compositions are equivalent is the same as checking on the last square of the previous strategy that there is an homotopy witnessing that the square commutes.

Chapter 3

Constructing six functor formalisms: I & P factorization

Given a geometric setting (C, E), we want to construct a lax symmetric monoidal functor

$$\mathcal{D} : \mathrm{Corr}(C, E)^{\otimes} \longrightarrow \mathrm{Cat}_{\infty}^{\times}$$

or equivalently, a functor

$$\mathcal{D} : \mathrm{Corr}(C, E)^{\otimes} \longrightarrow \mathrm{Cat}_{\infty}$$

such that for any finite set I and $X_i \in C$ for $i \in I$, the functor

$$\mathcal{D}((I, (X_i)_i)) \xrightarrow{(\rho^i)_i} \prod_{i \in I} \mathcal{D}(X_i)$$

is an equivalence.

Seed functor In practice, it is easy to construct a functor

$$\mathcal{D}_0 : C^{\mathrm{op}} \longrightarrow \mathrm{CAlg}(\mathrm{Cat}_{\infty})$$

to symmetric monoidal $\infty-$categories encoding $\otimes$ and f^*. We would like to extend this functor from $C^{\mathrm{op}} \simeq \mathrm{Corr}(C, \mathrm{isom})$ to $\mathrm{Corr}(C, E)$.

The subclasses $I, P \subset E$. Additionally, "in nature", we can sometimes find two special classes of morphisms $I, P \subset E$ of 'open immersions' and 'proper' maps such that for $f \in I$, the functor $f_!$ is the left adjoint of f^*, whereas for $f \in P$, $f_!$ is the right adjoint of f^*, and such that any $f \in E$ admits a factorization $f \simeq \bar{f} j$ with $j \in I$ and $\bar{f} \in P$, i.e. a compactification.

Now we wish to define the whole functor lower shriek as $f_! := \bar{f}_! j_!$ where the latter two functors are defined as the right and left adjoint of the pullback functor respectively. This could however, depend on the compactification, to solve that we introduce:

Conditions 3.1. *For $f_!$ to be well-defined, we will need:*

1. *The classes I and P contain all isomorphism and are stable under base change and composition.*
2. *For a diagram in C of the form*

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "f"'] & & Y \arrow[dl, "f'"] \\
& Z &
\end{tikzcd}$$

if $f, f' \in I$ (resp. P) then $g \in I$ (resp. P).

3. *If $f \in I \cap P$, then f is $n-$truncated for some $n \geq -2$.*
4. *Any arrow $f \in E$ can be factored as: $f \simeq \bar{f}j$ with $j \in I$ and $\bar{f} \in P$.*
5. *For all $f \in I$, the functor f^* admits a left adjoint $f_!$ satisfying base change and the projection formula.*
6. *For all $f \in P$, the functor f^* admits a right adjoint f_* satisfying the base change and the projection formula.*
7. *For any Cartesian diagram*

$$\begin{tikzcd}
X' \arrow[r, "j'"] \arrow[d, "g'"] & X \arrow[d, "g"] \\
Y' \arrow[r, "j"] & Y
\end{tikzcd}$$

with $j \in I$ (hence $j' \in I$) and $g \in P$ (hence $g' \in P$), the canonical map $j_! g'^* \to g_* j'_!$ is an equivalence.

We will now show how this conditions imply that the lower shriek functor is well defined through the following constructions.

CONSTRUCTION 1. *For $f \in I \cap P$, there is a canonical equivalence $Nm_f : f_! \xrightarrow{\simeq} f_*$ between the left and right adjoint of f^*, given by the norm map.*

By condition 3, f must be $n-$truncated. Let us argue by induction on n.

- If $n = -2$ then f is an isomorphism and the claim is clear - its inverse is both left and right adjoint.
- Induction step: claim is true for $m - 1$, let us prove it in the m case. Consider the diagram:

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r, "g"] \arrow[d, "h"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

As I and P are stable under base change, $f \in I \cap P \Rightarrow g \in I \cap P$. Now applying condition 2 on the triangle at the right of Δ we get $\Delta \in I \cap P$. Moreover, by the properties of n-truncatedness, Δ is $(m-1)$-truncated.

Then by the induction step we have $\Delta_! \simeq \Delta_*$ $(*)$. Now

$$f_! \underset{g \circ \Delta \simeq \mathrm{id}}{\simeq} f_! g_* \Delta_* \underset{(*)}{\simeq} f_! g_* \Delta_! \underset{\text{cond } 7}{\simeq} f_* h_! \Delta_! \underset{h \circ \Delta \simeq \mathrm{id}}{\simeq} f_*$$

Where we are applying condition 7 ($f_! g_* \simeq f_* h_!$) on the inner square diagram which is cartesian. We have then finished construction 1.

CONSTRUCTION 2. *For any diagram*

$$\begin{tikzcd} Z \arrow[r, "h"] \arrow[d, "k"] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

with $h, g \in I$ and $k, f \in P$, there is a natural isomorphism $g_! k_* \simeq f_* h_!$.

Consider the diagram

$$\begin{tikzcd} Z \arrow[dr, dashed, "l"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"] & & \\ & X' \times_X Y \arrow[r, "g'"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ & X' \arrow[r, "g"] & X \end{tikzcd}$$

As I and P are closed under base change, $f' \in P$ and $g' \in I$. Now using this, by condition 2 applied to both triangle diagrams we get $l \in I \cap P$, which implies $l_! \simeq l_*$ $(*)$ by construction 1. Thus, we get

$$g_! k_* \underset{k \simeq f' \circ l}{\simeq} g_! f'_* l_* \underset{(*)}{\simeq} g_! f'_* l_! \underset{\text{cond } 7}{\simeq} f_* g'_! l_! \underset{g' \circ l \simeq h}{\simeq} f_* h_!$$

where again we can apply condition 7 ($g_! f'_* \simeq f_* g'_!$) to the inner square because it is cartesian. We have shown construction 2 using construction 1.

CONSTRUCTION 3. *Let $f : X \to Y$ in E be written in two ways as $f \simeq \bar{f} j$ and $\bar{f}' j'$, with $j, j' \in I$ and $\bar{f}, \bar{f}' \in P$. Then there is a canonical isomorphism*

$$\bar{f}_* j_! \simeq \bar{f}'_* j'_!.$$

So we have

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "j'"] \arrow[dr, "f"'] & \bar{X}' \arrow[d, "\bar{f}'"] \\ & Y \end{tikzcd}$$

consider now the following diagram

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drrr, bend left, "\mathrm{id}"] \arrow[dddr, bend right, "\mathrm{id}"'] & & & \\
& X \underset{Y}{\times} X \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & X \arrow[d, "j'"] \\
& & \bar{X} \underset{Y}{\times} \bar{X}' \arrow[r] \arrow[d] & \bar{X}' \arrow[d, "\bar{f}'"] \\
& X \arrow[r, "j"'] & \bar{X} \arrow[r, "\bar{f}"'] & Y
\end{tikzcd}$$

as E can be shown to have a 2-out-of-3 property similar to condition 2, we have that $X \to X \underset{Y}{\times} X \to \bar{X} \underset{Y}{\times} \bar{X}'$ is in E and thus can be factored as the triangle on the left

$$\begin{tikzcd}
X \arrow[r, "j''"] \arrow[dr] & \bar{X}'' \arrow[d] \arrow[dr, "f''"] & \\
& \bar{X} \underset{Y}{\times} \bar{X}' \arrow[r] & Y
\end{tikzcd}$$

with $j'' \in I$ and the vertical map in P. Consequently, $f'' \in P$ as P is closed under composition (and base change, so the horizontal map to Y is in P). We want to prove

$$\bar{f}_* j_! \simeq \bar{f}''_* j''_! \simeq \bar{f}'_* j'_!$$

we will restrict our attention to one side, the other will be analogous. Using any map $g : \bar{X}'' \to \bar{X}$ over Y (for example $\bar{X}'' \to \bar{X} \underset{Y}{\times} \bar{X}' \to \bar{X}$) we have the following commutative diagram

$$\begin{tikzcd}
\bar{X}'' \arrow[r, "g"] \arrow[d, "\bar{f}''"'] & \bar{X} \arrow[d, "\bar{f}"] \\
Y \arrow[r, "="] & Y
\end{tikzcd}$$

now applying construction 2 we get $j_! \simeq g_* j''_!$ which, postcomposing with $\bar{f}_*$ gives us

$$\bar{f}_* j_! \simeq \bar{f}_* g_* j''_! \underset{\bar{f} \circ g \simeq \bar{f}''}{\simeq} \bar{f}''_* j''_!.$$

We have proven construction 3 using construction 2.

This allows us to believe the following statement without proof, with which we will end.

Theorem 3.2. *[Mann, Prop A.5.10] Under the above assumptions on C, E, I and P there is a canonical extension of $\mathcal{D}_0$ to a lax symmetric monoidal functor*

$$\mathcal{D} : \mathrm{Corr}(C, E)^{\otimes} \to \mathrm{Cat}_\infty^{\times}$$

such that

- *for $f \in I$, $f_! \dashv f^*$.*
- *for $f \in P$, $f^* \dashv f_!$.*

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