# Surgery theory: s-cobordism theorem

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# Overview

# Problem to solve

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We want to be able to determine when two $n-$manifolds $M$ and $N$ are homeomorphic or even better, diffeomorphic.

For this, $s-$cobordism theorem. We will use it to prove Poincaré conjecture for $n \geq 6$.

# h-cobordism def

**Definition**

A cobordism $(W; M_0, f_0, M_1, f_1)$ is called an $h-$cobordism if the inclusions

$$\partial_i W \hookrightarrow W$$

are homotopy equivalences for $i = 0, 1$.

**Definition**

An $h-$cobordism over $M_0$ is *trivial* if it is diffeomorphic relative $M_0$ to $(M_0 \times [0,1]; M_0 \times \{0\}, M_0 \times \{1\})$.

# $h-$Cobordism theorem



**Theorem ($h-$cobordism th)**

*Every $h-$cobordism over a simply connected closed manifold $M_0$ with $\dim(M_0) \geq 5$ is trivial.*

**Theorem (s-cobordism th)**

*Let $M_0$ be a closed connected smooth manifold of dimension $n \geq 6$ with fundamental group $\pi = \pi_1(M_0)$. Then*

- *$\tau(W, M_0)$ is an obstruction for $W$ being trivial;*
- *$\mathrm{Diff}(W, M_0) \leftrightarrow Wh(\pi)$.*

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# Handles

## Definition

We refer to $D^q \times D^{n-q}$ as a $q-$*handle*.
A $q-$handle has a *core* $D^q \times \{0\}$ with boundary $S^{q-1} \times \{0\}$ and a *cocore* $\{0\} \times D^{n-q}$ whose boundary $\{0\} \times S^{n-q-1}$ is called *transverse sphere* of the handle.

Given $M$ an $n-$dimensional manifold with boundary $\partial M$ and an embedding $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial M$ define a new $n-$dimensional manifold $M + (\phi^q)$ with a handle of index $q$ attached by

$$M \cup_{\phi^q} (D^q \times D^{n-q}).$$

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# Handle decomposition



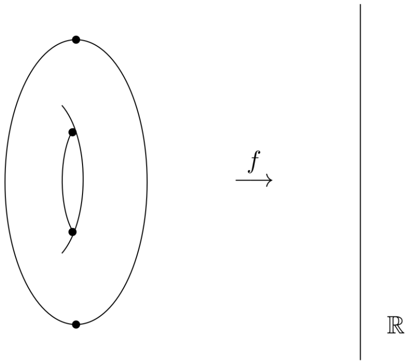
## Definition

Given a compact $n-$manifold $W$ with boundary $\partial_0 W \coprod \partial_1 W$, a *handle decomposition* of $W$ is

$$W = \partial_0 W \times [0,1] + (\phi_1^{q_1}) + (\phi_2^{q_2}) + \cdots + (\phi_r^{q_r})$$

This will be a basic tool, but how do we get a handle decomposition of our cobordisms?

# Morse function induces handle decomposition




We now want to reduce the handle decomposition given by the morse function in a way that allows us to prove our theorems.

# First lemmas

**Lemma (Isotopy Lemma)**

*Let $W$ be an $n-$dimensional compact manifold, with $\partial W = \partial_0 W \coprod \partial_1 W$. If $\phi^q, \psi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$ are isotopic embeddings, then there is a diffeomorphism $W + (\phi^q) \to W + (\psi^q)$ relative $\partial_0 W$.*

**Lemma (Diffeomorphism Lemma)**

*Given two $n-$manifolds $W, W'$ with $\partial W = \partial_0 W \coprod \partial_1 W$ and $\partial W' = \partial_0 W' \coprod \partial_1 W'$. Let $F : W \to W'$ be a diffeomorphism, inducing a diffeo $f_0 : \partial_0 W \to \partial_0 W'$.*
*Given an embedding $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$, there is an embedding $\bar{\phi}^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W'$ and a diffeomorphism $F' : W + (\phi^q) \to W' + (\bar{\phi}^q)$ which induces $f_0$ on $\partial_0 W$.*

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# More lemmas

## Lemma (right order on handles)

*Let $W$ be an $n-$dim compact manifold with $\partial W = \partial_0 W \coprod \partial_1 W$. Suppose that $V = W + (\psi^r) + (\phi^q)$ for $q \leq r$. Then $V$ is a diffeo relative $\partial_0 W$ to*

$$V' = W + (\bar{\phi}^q) + (\psi^r)$$

*for an appropiate $\bar{\phi}^q$.*

## Lemma (Cancellation lemma)

*Now let $\phi^q : S^{q-1} \times D^{n-q} \hookrightarrow \partial_1 W$ and $\psi^{q+1} : S^q \times D^{n-1-q} \hookrightarrow \partial_1(W + (\phi^q))$ be embeddings. If $\psi^{q+1}(S^q \times \{0\})$ is transverse to the transverse sphere of $(\phi^q)$ and intersects it in exactly one point, then there is a diffeo relative $\partial_0 W$ form $W$ to $W + (\phi^q) + (\psi^{q+1})$.*

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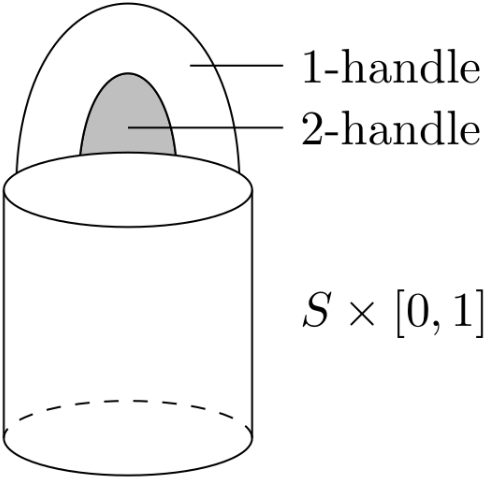
# Handle cancellation



# Notation

For $-1 \leq q \leq n$ denote

$$W_q := \partial_0 W \times [0,1] + \sum_{i=1}^{p_0}(\phi_i^0) + \sum_{i=1}^{p_1}(\phi_i^1) + \cdots + \sum_{i=1}^{p_q}(\phi_i^q);$$

$$\partial_1 W_q := \partial W_q - \partial_0 W \times \{0\};$$

$$\partial_1^\circ W_q := \partial_1 W_q - \coprod_{i=1}^{p_{q+1}} \phi^{q+1}(S^q \times \mathrm{int}(D^{n-1-q})).$$

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# Elimination lemma



## Lemma

*Suppose that for a fixed* $1 \leq q \leq n-3$ *we have that* $W$ *looks like*

$$W = \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1}) + \cdots + \sum_{i=1}^{p_n} (\phi_i^n)$$

*Now for an index* $1 \leq i_0 \leq p_q$ *suppose there is an embedding* $\psi^{q+1} : S^q \times D^{n-1-q} \hookrightarrow \partial_1^\circ W_q$ *satisfying:*

i) $\psi^{q+1}|_{S^q \times \{0\}}$ *is isotopic in* $\partial_1 W_q$ *to an embedding* $\psi_1^{q+1} : S^q \times \{0\} \hookrightarrow \partial_1 W_q$ *which meets the transverse sphere of* $(\phi_{i_0}^q)$ *transversally in exactly one point and is disjoint from transverse sphere of other handles.*

# Elimination lemma

**Lemma**

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# Elimination lemma part 2

## Continues

ii) $\psi^{q+1}|_{S^q\times\{0\}}$ is isotopic in $\partial_1 W_{q+1}$ to a trivial embedding $\psi_2^{q+1} : S^q \times \{0\} \hookrightarrow \partial_1^\circ W_{q+1}$.
Then $W$ is diffeo relative $\partial_0 W$ to

$$\partial_0 W \times [0,1] + \sum_{i=1,\dots,p_q, i\neq i_0} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\bar{\phi}_i^{q+1}) + (\psi^{q+2}) \cdots + \sum_{i=1}^{p_n} (\bar{\phi}_i^n).$$

# $W$ as a pushout

We want to construct a CW complex $(X, \partial_0 W)$ and a homotopy equivalence

$$(f, \mathrm{id}) : (W, \partial_0 W) \xrightarrow{\simeq} (X, \partial_0 W)$$

begin writing the diagram

$$\begin{tikzcd}
\coprod\limits_{1=1,\dots,p_q} S^{q-1} \times D^{n-q} \arrow[r, "\coprod\limits_{1=1,\dots,p_q} \phi_i^q"] \arrow[d] & W_{q-1} \arrow[d, dashed] \\
\coprod\limits_{1=1,\dots,p_q} D^{q} \times D^{n-q} \arrow[r, dashed] & W_q
\end{tikzcd}$$

# CW construction

Construct by induction, $X_{-1} = \partial_0 W$ and $f_{-1} : \partial_0 W \times [0,1] \to \partial_0 W$. Now,

$$
\begin{tikzcd}
\coprod_{1=1,\dots,p_q} S^{q-1} \arrow[rrr, "\coprod_{1=1,\dots,p_q} f_{q-1} \circ \phi_i^q|_{S^{q-1} \times \{0\}}"] \arrow[d] & & & X_{q-1} \arrow[d, dashed] \\
\coprod_{1=1,\dots,p_q} D^q \arrow[rrr, dashed] & & & X_q
\end{tikzcd}
$$

# Auxiliar spaces

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\coprod_{1=1,\dots,p_q} S^{q-1} \arrow[r, "\coprod_{1=1,\dots,p_q} \phi_i^q|_{S^{q-1}\times\{0\}}"] \arrow[d] & W_{q-1} \arrow[d, dashed] \\
\coprod_{1=1,\dots,p_q} D^q \arrow[r, dashed] & Y_q
\end{tikzcd}
$$

$$(g_q, f_{q-1}) : (Y_q, W_{q-1}) \to (X_q, X_{q-1})$$

$$(h_q, id) : (Y_q, W_{q-1}) \to (W_q, W_{q-1})$$

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# Set up for basis

$$\begin{tikzcd} S^{q-1} \times D^{n-q} \arrow[r, "\phi^q"] \arrow[d] & W_{q-1} \arrow[d, dashed] \\ D^q \times D^{n-q} \arrow[r, dashed, "\Phi^q"] & W_q \end{tikzcd}$$

Also, consider $\widetilde{W} \xrightarrow{p} W$ universal cover of $W$ and $\widetilde{W}_q = p^{-1}(W_q)$

# Basis for $C_q(\tilde{W}, \partial_0\tilde{W})$

$$\begin{tikzcd}
 & (\widetilde{W}_q, \widetilde{W}_{q-1}) \arrow[d] \\
(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \arrow[r, "(\Phi^q,\phi^q)"] \arrow[ur, dashed, "(\tilde{\Phi}^q,\tilde{\phi}^q)"] & (W_q, W_{q-1})
\end{tikzcd}$$

The image of the generator $H_q(D^q \times D^{n-q}, S^{q-1} \times D^{n-q}) \cong \mathbb{Z}$, well-defined up to multiplication by $\gamma \in \pi$, determines a $\mathbb{Z}\pi$–basis $\{[\phi_i^q] \in C_q(\widetilde{W}, \widetilde{\partial_0 W}) \mid 1 \leq i \leq p_q\}$.

# Normal Form Lemma



## Lemma

*Normal form lemma Let* $(W; \partial_0 W, \partial_1 W)$ *be a compact* $h$*−cobordism of dim* $n \geq 6$*. Let* $2 \leq q \leq n-3$ *then there is a diffeo relative* $\partial_0 W$

$$W \cong \partial_0 W \times [0,1] + \sum_{i=1}^{p_q} (\bar{\phi}_i^q) + \sum_{i=1}^{p_{q+1}} (\bar{\phi}_i^{q+1}).$$

# Matrix of the only differential



As $W$ is an $h-$cobordism, the $\mathbb{Z}\pi-$chain cx $C_*(\widetilde{W}, \widetilde{\partial_0 W})$ is acyclic. Hence

$$d_{q+1} : H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \to H_q(\widetilde{W_q}, \widetilde{W_{q-1}})$$

is bijective.

Now, consider the matrix $A$ representing $d_{q+1}$ on the $\mathbb{Z}\pi-$bases

$$([\phi_i^{q+1}])_{1 \leq i \leq p_q} \text{ of } H_{q+1}(\widetilde{W_{q+1}}, \widetilde{W_q}) \text{ and}$$
$$([\phi_i^{q}])_{1 \leq i \leq p_q} \text{ of } H_q(\widetilde{W_q}, \widetilde{W_{q-1}}).$$

# Matrix definition of $Wh(\pi)$

Set of equivalence classes of matrices of arbitrary size with entries in $\mathbb{Z}\pi$ where two matrices are equivalent if they can be transformed into one another through:

i) Adding the $k-$th row multiplied by $x \in \mathbb{Z}\pi$ from the left to the $l-$th row ($k \neq l$);

ii)

$$B = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

iii) Inverse of $ii$): eliminate column and row $i$ if they have zeros excepto for position $(i, i)$;

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# More equivalence operations

iv) Multiplying the $i-$th row from the left with a trivial unit, i.e. $\pm\gamma$ for $\gamma \in \pi$;

v) Interchange two rows or two columns.

Group structure given by

$$[A] \cdot [B] \coloneqq [(A \oplus I_m) \cdot (B \oplus I_n)]$$

then $I_i$ acts as unit for any $i \in \mathbb{N}$. Inverse of $[A]$ is $[A^{-1}]$.

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# Proof of $h-$cobordism theorem

**Proposition**

*The Whitehead group of the trivial group vanishes.*

For $s-$cobordism theorem it will be enough to prove that $[A] = \tau(W, M_0)$.

# Important lemma

## Lemma

i) *Let* $(W; \partial_0 W, \partial_1 W)$ *be an* $n-$*dim compact* $h-$*cobordism for* $n \geq 6$ *and* $A$ *be the matrix of* $d_{q+1}$. *If* $[A] = 0$ *in* $Wh(\pi)$, *then* $W$ *is trivial* $\partial_0 W$;

ii) *Consider* $u \in Wh(\pi)$, *a closed manifold* $M$ *of dim* $n - 1 \geq 5$ *with* $\pi_1(M) = \pi$ *and* $2 \leq q \leq n - 3$. *Then we can find*

$$W = M \times [0,1] + \sum_{i=1}^{p_q} (\phi_i^q) + \sum_{i=1}^{p_{q+1}} (\phi_i^{q+1})$$

*such that* $[A] = u$.

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# $K_1(R)$ for a ring $R$



Let $R$ be associative ring with unit. Denote by $GL(n, R)$ the group of invertible $(n, n)-$matrices with entries in $R$. Consider

$$\dots \subseteq GL(n, R) \subseteq GL(n+1, R) \subseteq \dots$$

given by stabilization

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

consider the colimit $GL(R)$, then consider the abelianization $K_1(R) := GL(R)/[GL(R), GL(R)]$

# $K_1(\mathbb{Z}G)$ definition $Wh(G)$

We have the canonical ring homomorphism $\mathbb{Z} \to R$ and the bijection

$$\begin{aligned} \det : K_1(\mathbb{Z}) &\longrightarrow \{\pm 1\} \\ [A] &\longmapsto \det(A) \end{aligned}$$

We define $\widetilde{K}_1(R)$ as the cokernel of $K_1(\mathbb{Z}) \to K_1(R)$.

## Definition

For a group $G$, and the map

$$\begin{aligned} G \times \{\pm 1\} &\xrightarrow{k} K_1(\mathbb{Z}G) \\ (g, \pm 1) &\longmapsto (\pm g) \in M_{(1,1)}(\mathbb{Z}G) \end{aligned}$$

then the *Whitehead group* $Wh(G) := \operatorname{coker}(k)$.

# Elementary matrices

Denote $E_n(i,j)$ with $1 \leq i,j \leq n$ be the $(n,n)-$matrix with one at $(i,j)$ and zero else where.
An elementary matrix is a matrix of the form $I_n + r \cdot E_n(i,j)$.
Denote $E(R)$ the subgroup spanned by elementary matrix. Turns out $E(R) = [GL(R), GL(R)]$.
The 2 definitions of $Wh(G)$ coincide.

# Chain contraction

Given a free finite based chain cx:

## Definition

A *chain contraction* $\gamma_*$ for an $R$−chain cx $C_*$ is a collection of homomorphism $\gamma_p : C_p \to C_{p+1}$ for $p \in \mathbb{Z}$ such that

$$\partial_{p+1} \circ \gamma_p + \gamma_{p-1} \circ \partial_p = \mathrm{id}_{C_p}.$$

Put $C_{odd} = \bigoplus_{p \in \mathbb{Z}} C_{2p+1}$ and $C_{ev} = \bigoplus_{p \in \mathbb{Z}} C_{2p}$ and let $\gamma_*$ and $\delta_*$ be chain contractions. Define $R$−homomorphisms

$$(\partial_* + \gamma_*)_{odd} : C_{odd} \to C_{ev},$$

$$(\partial_* + \delta_*)_{ev} : C_{ev} \to C_{odd}.$$

# Reidemeister torsion

Let $A$ be the matrix for $(\partial_* + \gamma_*)_{odd}$ and $B$ for $(\partial_* + \delta_*)_{ev}$, turns out $[A] = -[B]$.

**Definition**

Given a finite based free contractible $R-$chain cx $C_*$ define its *Reidemeister torsion* as

$$\rho(C_*) = [A] \ \in \tilde{K}_1(R).$$

For $f_* : C_* \to D_*$ chain homotopy equiv of finite based free $R-$chain cxs, then $cone(f_*)$ is a contractible finite based free $R-$chain cx.

# Whitehead torsion

## Definition

Given a chain homotopy equivalence of finite based free $R$–chain cx $f_* : C_* \to D_*$ define its *Whitehead torsion* by

$$\tau(f_*) := \rho(cone_*(f_*)) \in \tilde{K}_1(R)$$

## Definition

Given a homotopy equivalence $f : X \to Y$ it induces a chain homotopy equivalence $C_*(\tilde{f}) : C_*(\tilde{X}) \to C_*(\tilde{Y})$ between chain cx of their universal covers. We define its *Whitehead torsion* as

$$\tau(f) := \tau(C_*(\tilde{f})) \in Wh(\pi_1(Y, y)).$$

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$$\tau(f) \coloneqq \tau(C_*(\tilde{f})) \in Wh(\pi_1(Y, y)).$$

# Whitehead torsion of an $h$−cobordism

## Definition

The *Whitehead torsion of an* $h$*−cobordism* $(W; M_0, f_0, M_1, f_1)$ is defined as

$$\tau(W, M_0) \coloneqq (i_0 \circ f_0)^{-1}(\tau(i_0 \circ f_0 : M_0 \to W))$$

belonging to $Wh(\Pi(M_0))$, where we equip $W$ and $M_0$ with a CW structure.

# s-cobordism theorem

## Theorem (s-cobordism theorem)

*Let $M_0$ be a closed connected smooth manifold of dimension $n \geq 6$ with fundamental group $\pi = \pi_1(M_0)$. Then:*

- *Let $(W; M_0, f_0, M_1, f_1)$ be an $h-$cobordism over $M_0$. Then*

  $$W \text{ is trivial over } M_0 \Leftrightarrow \tau(W, M_0) \in Wh(\pi) \text{ vanishes;}$$

- *For any $x \in Wh(\pi)$ ther exists an $h-$cobordism $(W; M_0, f_0, M_1, f_1)$ over $M_0$ with $\tau(W, M_0) = x \in Wh(\pi)$;*

# s-cobordism theorem

**Theorem (s-cobordism th continues)**

- *The function*

$$\begin{aligned} \mathrm{Diff}(W, M_0) &\longrightarrow Wh(\pi) \\ (W; M_0, f_0, M_1, f_1) &\longmapsto \tau(W, M_0) \end{aligned}$$

*yields a bijection between diffeomorphism classes of* $h-$*cobordisms relative* $M_0$ *and* $Wh(\pi)$.

# Poincaré Conjecture

**Theorem (Poincaré Conjecture)**

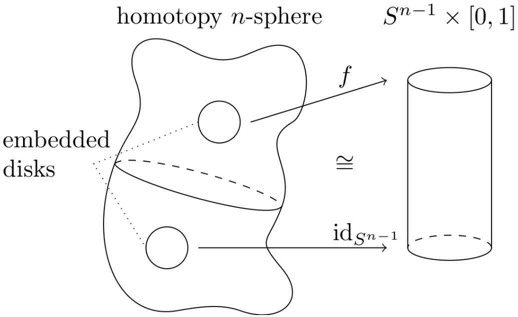
*If a closed* $n-$*dimensional manifold* $M$ *with* $\dim(M) \geq 6$ *is simply connected and its homology sphere, then* $M$ *is homeomorphic to* $S^n$.

# Proof

- Find a map: use Hurewicz $\pi_n(M) \cong H_n(M) \cong H_n(S^n) \cong \mathbb{Z}$ take the map $f : S^n \to M$ inducing 1.
- As a consequence of Hurewicz, CW and Whitehead theorem we get $f$ homotopy equivalence
- Now consider $W = M - (\mathrm{int}(D_0^n) \coprod \mathrm{int}(D_1^n))$ two embedded disjoint disks

# Proof

Use h−cobordism theorem




Fill the disks with *Alexander trick*. Note: not diffeo only homeo.

# The end

## Thanks for you attention!

Images: Remy Bohm ,*Morse theory and handle decomposition*
Ideas: Wolfgang Lück and Tibor Macko, *Surgery Theory: Foundations*