UNIVERSITY OF COPENHAGEN

Project outside course scope

The Pontryagin-Thom isomorphism

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Project info

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Title: The Pontryagin-Thom isomorphism
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Brief description
In the interest of learning infinity categories through a geometric example, we will work towards proving the Pontryagin-Thom isomorphism. In doing so we will acquire knowledge of stable homotopy theory, thom spaces, MO & MU cohomology theories and more.
Observations

Contents

Introduction

Chapter 1

Vector bundles

The purpose of this chapter is to briefly introduce vector bundles, and shape our perspective on them. We start with a classical viewpoint which we will shed gradually throught the section in favor of an $\infty-$categorical stable outlook that will provide a greater understandment on later considerations. Our main source for the classical theory is [Hatch]. Let us begin with Hatcher's defintion:

Definition 1.1. A *real vector bundle of rank k* is a map $p : V \to X$ such that

- for every $X \in X$ the set $V_x := p^{-1}(x)$ is endowed with a real vector space structure.
- p verifies a local triviality condition, ie we have $\{U_\alpha, h_\alpha\}$ comprising of a cover of $U_\alpha \subset X$ by open sets and homeomorphisms over the cover

$$\begin{tikzcd}
h_\alpha : p^{-1}(U_\alpha) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U_\alpha \times \mathbb{R}^k \arrow[dl, "\mathrm{proj}_1"] \\
& U_\alpha &
\end{tikzcd}$$

  that send V_x to $\{x\} \times \mathbb{R}^n$ by a vector space isomorphism.

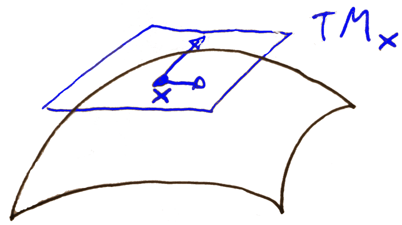
Remark 1.2. Analogously, we could define a *complex vector bundle* simply by replacing $\mathbb{R}$ by $\mathbb{C}$ above.

Example 1.3. To get acquainted with the concept, here we have some examples

1. The trivial bundle $\mathrm{proj}_1 : X \times \mathbb{R}^k \to X$.
2. Tautological bundle
3. The Möbius bundle
4. Vector bundles arise naturally, for instance, in the context of manifolds. For a manifold M the tangent bundle TM or the normal bundle $\mathcal{N}M$ are vector bundles.

Let's come back to Def 1.1. The local triviality condition establishes a way of 'gluing' all fibers to get a bundle. There is an alternative way of phrasing this condition: given $\{U_\alpha, h_\alpha\}$ for the intersection of two open sets $U_\alpha \cap U_\beta$ we can compose the respective homeomorphisms

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^k \xleftarrow{h_\beta} p^{-1}(U_\alpha \cap U_\beta) \xrightarrow{h_\alpha} (U_\alpha \cap U_\beta) \times \mathbb{R}^k.$$



To each $x \in U_\alpha \cap U_\beta$ we can then associate a vector space isomorphism by fixing the first coordinate

$$(h_\alpha \circ h_\beta^{-1})(x, -) : \mathbb{R}^k \longrightarrow \mathbb{R}^k.$$

We denote that association by $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}, k) \simeq O(k)$. Additionally, we can easily check that the collection $\{g_{\alpha\beta}\}_{\alpha\beta}$ verifies the following conditions:

- $g_{\alpha\alpha}(x) = \mathrm{id}_{\mathbb{R}^k}$.
- $g_{\alpha\beta}^{-1}(x) = g_{\beta\alpha}(x)$.
- $g_{\alpha\gamma}(x)g_{\gamma\beta}(x)g_{\beta\alpha}(x) = \mathrm{id}$, for any $x \in U_\alpha \cap U_\gamma \cap U_\beta$.

It turns out these are also valid instructions for gluing the fibers of a bundle:

Construction 1. For a fixed space X, given a cover $\{U_\alpha\}$ and functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to O(k)$ verifying the above conditions we can define a rank k vector bundle as follows: we consider the disjoint union of the product of the opens with $\mathbb{R}^k$ and we glue along $g_{\alpha\beta}$ the vector spaces in two different intersecting opens $U_\alpha \cap U_\beta$, resulting in a bundle

$$E = \coprod_\alpha U_\alpha \times \mathbb{R}^k \Big/ (x, v) \backsim (x, g_{\alpha\beta}(x) \cdot v)$$

We will now show a classification result for vector bundles that can be interpreted as assembling the maps $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{R}, k) \simeq O(k)$ into a map $X \longrightarrow BO(k)$ which defines the same vector bundle. This perspective serves as a segway into the $\infty-$categorical ways of thinking that we wish to integrate into our discussion and it will naturally introduce the K-theoretical concept of stable vector bundles.

Theorem 1.4. *Given a vector bundle $p : V \to X$ and a map $f : Y \to X$ there exist a vector bundle $V' = f^*V \to Y$ and a map $\bar{f} : V' \to V$ that sends the fibers*

$$\bar{f} : V'_y \xrightarrow{\cong} V_{f(y)}$$

for $\forall y \in Y$. We refer this bundle as a **pullback bundle**, *additionally we have that it is unique up to isomorphism.*

Proof. We consider the pullback

$$\begin{tikzcd} f^*V \arrow[r, dashed, "\bar{f}"] \arrow[d, dashed] & V \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

We will prove that $f^*V \to Y$ is a vector bundle. The pullback is defined as

$$f^*V := \left\{(v, y) \in V \times Y \,\middle|\, p(v) = f(y)\right\}.$$

Then the preimage of any $y \in Y$ is exactly $p^{-1}(f(y))$, so the vector bundle structure on $f^*V \to Y$ is provided by that of p, and it's clear how the induced map $\bar{f}$ sends isomorphically the fibers to the fibers of p.

In order to prove uniqueness up to isomorphism, consider a vector bundle $p_W : W \to Y$ verifying the conditions of the theorem. The induced map to the pullback gives us a map $W \to Y$ defined as $w \in W \mapsto (p_W(w), \bar{f}(w))$ which is an isomorphism as it clearly is an isomorphism on the fibers. □

Theorem 1.5 (Classification)**.**

Proof. □

Classification Stable equivalences, specifically Normal bundle inside a sphere and -TN?

Chapter 2

Spectra

2.1 Relation with stable phenomena

2.2 Definition and properties

$\mathrm{Sp} := \varprojlim \left(\mathrm{An}_* \xleftarrow{\Omega} \mathrm{An}_* \xleftarrow{\Omega} \mathrm{An}_* \xleftarrow{\Omega} \dots \right)$

Σ^∞

2.3 $\mathrm{Mod}_R(\mathrm{Sp})$

The functor $- \otimes R$

$$\begin{array}{ccccc} \mathrm{An}_* & \xrightarrow{\Sigma^\infty} & \mathrm{Sp} & \xrightarrow{(-)\otimes R} & Mod_R \\ (X, x) & \longmapsto & \Sigma^\infty X & \longmapsto & \tilde{C}_*(X, x; R) \end{array}$$

(Mod_R here is the derived category of $R-$modules: chain complexes of $R-$modules with quasi-isos inverted $Ch_R \left[quasi^{-1} \right]$. It can be thought of as $R-$modules in spectra.)

The inclusion $Mod_R^{dis} [n] \hookrightarrow Mod_R$

Chapter 3

Thom spaces and Thom spectra

3.1 Definition and several perspectives

There are multiple ways of constructing the Thom space, one of the most important ideas in this project. We present a couple equivalent ways as some will illustrate better the idea and others will be more useful when working with the concept.

Definition 3.1 (Thom space 1)**.** Given a vector bundle over X, $V \to X$, we one point compactify every fiber $V_x \cup \{\infty\} = S^{V_x}$ thus getting a fiber bundle $S^V \to X$. Now we collapse the infinity point of every fiber into one to get $\mathrm{Th}(V)$, the *Thom space* of V.

Definition 3.2 (Thom space 2)**.** A fiber bundle inherits a structure of metric space from $\mathbb{R}^n$ (as $O(n)$ preserves the metric, and rank n vector bundles come from a pullback of the tautological bundle in $BO(n)$). The *Thom space* of V is the quotient

$$D(V)/S(V)$$

where $D(V)$ is the disk bundle of vectors with distance ≤ 1 to the origin on each V_x and $S(V)$ is the sphere bundle of vectors with distance 1 to the origin on each V_x. Note $S(V) \subseteq D(V)$.

Definition 3.3 (Thom space 3)**.** From a $n-$rank vector bundle $V \to X$ we get the fiberwise one point compactified bundle $S^V \to X$. Now as the fibers are $n-$spheres the straightened fiber bundle looks like

$$X \longrightarrow BAut_*(S^n) \subseteq \mathrm{An}_*$$

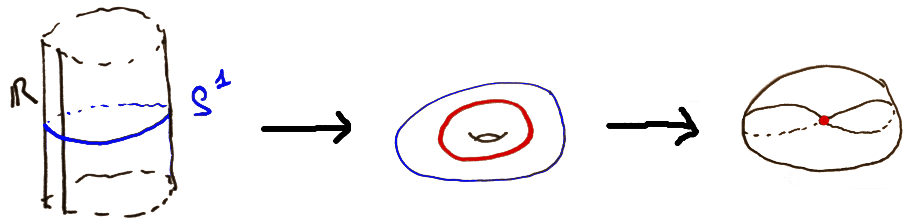
The *Thom space* of V is the pointed colimit (in An_*)

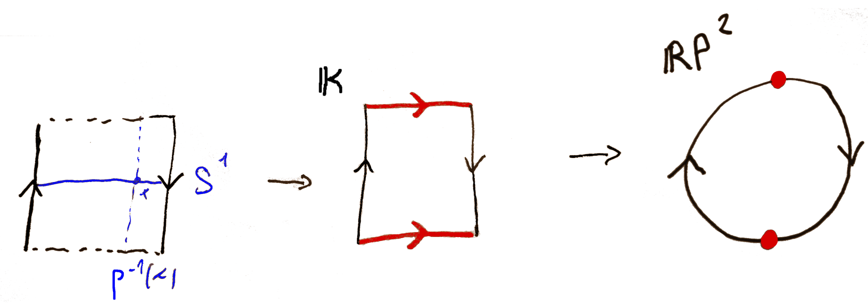
$$\underset{X}{colim_*}\, S^{V_x} \cong \mathrm{cofib}\left(\underset{X}{colim}\, \{\infty\} \longrightarrow \underset{X}{colim}\, S^{V_x}\right)$$

Remark 3.4. Definition 1 and 2 are equivalent as the open unit disk is isomorphic to the original vector space. When we then collapse the sphere bundle $S(V)$ we are adding one shared infinity point to all the fibers, ie collapsing $S(V)$ is the same as one point compactifying all fibers and identifying the infinity points all at once. This information is also encoded in definition 3 in the form of a pointed colimit, which will be a nice perspective for certain proofs.

Example 3.5. Consider the trivial rank 1 bundle on the circle S^1, which we could interpret as the cilinder [...]

Example 3.6. Consider the Möbius bundle [...]





Proposition 3.7 (Prop 3.4 of [nLab])**.** *Let $V \to X$ be a vector bundle then*

$$\mathrm{Th}(V \oplus \mathbb{R}^n) \simeq S^n \wedge Th(V) = \Sigma^n Th(V)$$

The reference contains a proof of this fact. Besides being generally useful, this provides us with a description of $(\Sigma^\infty Th(V))_n$.

3.2 J-homomorphism & Thom spectrum

The J-homomorphism can be explained in several equivalent ways, we choose one that will be useful for us but also maintains a degree of intuition present. It encodes the information of one point compactifying a vector bundle but it lands on spectra and will allow us to define Thom spectra.

Definition 3.8. The *J-homomorphism* is the following functor

$$\begin{aligned} BO &\longrightarrow \mathrm{Sp} \\ V &\longmapsto \Sigma^{-dimV}\Sigma^\infty S^V \end{aligned}$$

Definition 3.9. Given a vector bundle $V \to X$ its *Thom spectrum* is the suspension spectrum of its Thom space

$$\mathrm{Th}_{\mathrm{Sp}}(V) = \Sigma^\infty \, \mathrm{Th}_{\mathrm{An}_*}(V)$$

Remark 3.10. Consider the following diagram, where M is a manifold and TM its tangent bundle

$$\begin{tikzcd} & B \arrow[d, "\xi"] \arrow[dr] & \\ M \arrow[ur, "\varphi"] \arrow[r, "-TM"'] & BO \arrow[r, "J"'] & \mathrm{Sp} \end{tikzcd}$$

the Thom spectrum of the stable normal bundle of M is equivalent to

$$\mathrm{Th}_{\mathrm{Sp}}(-TM) \simeq \underset{M}{colim}\ J \circ (-TM) = \underset{M}{colim}\ J \circ \xi \circ \varphi$$

Additionally, for the map $\xi : B \longrightarrow BO$ we have the Thom spectrum

$$M\xi \simeq \underset{B}{colim}\ J \circ \xi$$

and in the BO case we consider $MO \simeq \underset{BO}{colim}\ J$.

Remark 3.11. Every $M\xi$, as a spectrum, defines a homology theory which can be written as

$$M\xi_*(X) = \pi_*(M\xi \otimes \Sigma^\infty_+ X)$$

3.3 Thom isomorphism

We will prove now the Thom isomorphism, a useful result that states that the homology of the Thom space of (some) vector bundle coincides with that of the base space with a shift of indices. This shift will be determined by the rank of the vector bundle. We will first define a condition necesary for the result to be true

Definition 3.12. A vector bundle $V \longrightarrow X$ is called *orientable* if the monodromy action $\gamma_* \curvearrowright V_x$ has positive determinant $\forall \gamma \in \Pi_1(X, x)$.

Theorem 3.13 (Thom isomorphism)**.** *Given a n-rank orientable vector bundle $V \to X$ then there exists an isomorphism*

$$H^i(X; R) \simeq \widetilde{H}^{i+n}(\mathrm{Th}_{\mathrm{An}_*}(V); R)$$

where R is a coefficient ring.

Proof. Recall definition 3 of the Thom space: first we one point compactify all the fibers of the vector bundle. Straightening the resulting fiber bundle will grant us

$$\begin{aligned} X &\xrightarrow{th(V)} \mathrm{An}_* \\ x \in X &\longmapsto S_x = S(V_x) \end{aligned}$$

we then identify all the infinity points, ie $\underrightarrow{colim}_*\ th(V) = \mathrm{Th}_{\mathrm{An}_*}(V)$. We're taking the pointed colimit of the functor $th(V)$ which lands in pointed spaces as the $\{\infty\}$ point serves as a basepoint for every fiber.

Additionally, remember that for the constant map

$$\begin{aligned} X &\xrightarrow{const_*} \mathrm{An}_* \\ x &\longmapsto * \end{aligned}$$

we have $\underset{X}{colim} * \cong X$.

This perspective will prove useful when we consider the functor

$$\begin{aligned} \mathrm{An}_* &\xrightarrow{(-)\otimes R} Mod_R \\ (X, x) &\longmapsto \tilde{C}_*(X, x; R) \end{aligned}$$

sending a space to the reduced chain complex from which we can calculate its homology, as it turns out that this is colimit preserving [ref nedeed].

What we want to prove ultimately is that there's a quasi iso $\mathrm{Th}_{\mathrm{An}_*}(V) \otimes R \simeq (X \otimes R)[n]$ where $[n]$ denotes shifting the chain complex n positions to the right. Hence if we prove the claim for every point - that is $th(V) \otimes R \simeq R[n]$ - plus naturality the iso would extend to the colimits. In a regular 1−category it would be enough to prove that the diagram on the left was commutative

$$\begin{tikzcd}
th(V)_x \arrow[r, "\cong"] \arrow[d, "\gamma"] & R[n] \arrow[d, "id_{R[n]}"] \\
th(V)_y \arrow[r, "\cong"] & R[n]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & th(V)_x \arrow[ddll] \arrow[dd, "\gamma"] \arrow[drr, "\cong"] & & \\
 & & & & R[n] \arrow[dd, "id_{R[n]}"] \arrow[ddll, dashed, "id_{R[n]}"'] \\
th(V)_z \arrow[rr] \arrow[drr, "\cong"'] & & th(V)_y \arrow[drr, "\cong"] & & \\
 & & R[n] \arrow[rr, "id_{R[n]}"] & & R[n]
\end{tikzcd}$$

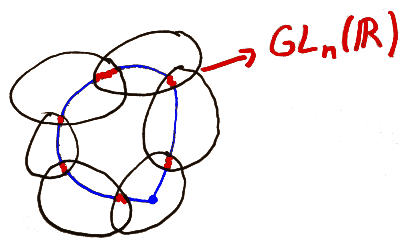
whereas on the context of ∞−categories that we're working in we have to worry about higher simplices as well. For instance, it could very well be that a 2−simplex on the fibers induced a non-trivial homotopy filling a 2−simplex with the identity of $R[n]$ on all sides: see the diagram on the right. It could be hard to check that in fact this is the case, but our situation will be simpler.

In fact $th(V) \otimes R \simeq R[n]$ as the image of $th(V)$ is a sphere $\forall x \in X$, whose formal chain complex with its reduced cohomology is precisely $R[n]$. Now luckily, as as $R[n]$ belongs to the 1−category Mod_R^{dis}, we can reduce our problem to checking commutativity of a diagram like that on the left. Additionally, precisely because Mod_R^{dis} is a 1−category, $th(V) \otimes R$ will factor through the fundamental grupoid like shown in the diagram

$$\begin{tikzcd}
 & \Pi_1 X \arrow[rr] & & Mod_R^{dis} \arrow[dl] \\
X \arrow[ur] \arrow[r, "th(V)"] & \mathrm{An}_* \arrow[r, "-\otimes R"] & Mod_R \arrow[ur, bend right, "H_n"'] &
\end{tikzcd}$$

we would like to check that $\Pi_1 X \longrightarrow Mod_R^{dis}$ is a constant map. Being constant on $\Pi_1 X$ means that any loop acts trivially.

A loop induces an iso $\gamma_* : V_x \xrightarrow{\cong} V_x$ which is characterized by the matrix obtained by multiplying all the matrices which characterize the transition functions on each of the points of the intersection between two trivializing opens that the loop γ crosses [check figure].



This in turn will induce an iso on the compactified fibers and on its homology

$$\begin{array}{ccc} H_n(S^{V_x};R) & \longrightarrow & H_n(S^{V_x};R) \\ \wr\| & & \wr\| \\ \mathbb{Z}\otimes R & \longrightarrow & \mathbb{Z}\otimes R \end{array}$$

If M is the matrix which induced the iso on the fibers, iso on $\mathbb{Z}$ will be given by multiplying by the sign

$$\frac{det(M)}{|\,det(M)\,|} \in \{-1, 1\}$$

If $char(R) = 2$ we would have $-1 = 1$ so the action would be trivial in any case. Otherwise, we know that the sign will be always one, because of the imposed condition of the fiber bundle being orientable! So we get the desired result.

□

Remark 3.14. Recall that all complex vector bundles are orientable and thus fulfill the conditions of the Thom isomorphism.

Remark 3.15. The result is also true for non-orientable bundles if we restrict to the case $R = \mathbb{Z}/2$.

3.4 Calculating $H_*(MU)$

At first glance computing the homology of MU might seem somewhat arbitrary but it will actually allow us to compute $\pi_*(MU)$ through a spectral sequence. This, as we will see in the next chapter, gives us interesting information about cobordism classes of manifolds with some extra structure.

We know that $MU_k = \mathrm{Th}_{\mathrm{An}_*}(BU(k); \xi_k - \underline{\mathbb{C}}^k)$ thus using the Thom isomorphism 3.3, as the vector bundle is 0−dimensional, we get $H^*(BU(k)) \simeq H^*(MU_k)$. We will prove that

$$H^*(BU(k)) \cong \mathbb{Z}[c_1, c_2, \ldots, c_k] \text{ with } |c_i| = 2i \tag{3.1}$$

which is enough as we will have $H^*(BU(k)) \simeq H_*(BU(k))$, $BU = \varinjlim BU(k)$, $MU = \varinjlim MU_k$ and homology commutes with colimits, ie $H_*(BU) = \varinjlim H_*(BU(k))$.

Computing $H^*(BU(k))$ To prove 3.1 we will use induction. Assume that it is true for $n-1$, let's prove it for n. We have the following fiber sequence

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

This is due to a general fact about Lie groups, namely if we have a closed Lie subgroup $H \subseteq G$ inside a Lie group G then we have a fiber sequence

$$G/H \to BH \to BG$$

Moreover if $G \curvearrowright X$ transitively with stabilizer H then $G/H \cong X$.

Returning to our particular case, as $\pi_1 S^{2n-1}$ is trivial for $n \neq 1$, we can consider the Serre spectral sequence for cohomology which looks like

$$
\begin{tikzcd}[row sep=small, column sep=small]
2n & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2n-1 & e \arrow[dddrrrr] & eH^1 & eH^2 & \cdots & eH^{2n} & eH^{2n+1} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
1 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\
0 & 1 & H^1 & H^2 & \cdots & H^{2n} & H^{2n+1} & \cdots \\
 & 0 & 1 & 2 & \cdots & 2n & 2n+1 & \cdots
\end{tikzcd}
$$

Where $e \in H^{2n-1}(S^{2n-1})$ is a generator and H^i denotes $H^i(BU(n-1))$. Only the $2n$ differential can be different from zero for degree reasons (d_r changes degree by $(r, -r+1)$), and we have the following Gysin sequence

$$
\begin{tikzcd}
 & H^{\bullet-1}(BU(n-1)) \arrow[d] & H^{\bullet-1}(BU(n)) \arrow[l] & H^{\bullet-2n-1}(BU(n)) \arrow[l] & \cdots \arrow[l] \\
 & H^{\bullet-2n}(BU(n)) \arrow[r, "-\cup e"] & H^{\bullet}(BU(n)) \arrow[r] & H^{\bullet}(BU(n-1)) \arrow[d, "d_{2n}"] & \\
\cdots & H^{\bullet+1}(BU(n-1)) \arrow[l] & H^{\bullet+1}(BU(n)) \arrow[l] & H^{\bullet-2n+1}(BU(n)) \arrow[l] &
\end{tikzcd}
$$

We assume that $H^*(BU(n-1)) \cong \mathbb{Z}[c_1, c_2, \ldots, c_{n-1}]$ with $|c_i| = 2i$.

Look at $H^\bullet(BU(n)) \longrightarrow H^\bullet(BU(n-1))$. For $\bullet < 2n-1$ the differentials would land on negative degree, where all the elements of the page are zero; additionally, the same happens for the source of $-\cup e$ for $\bullet \leq 2n-1$. Then as $H^{2n-1}(BU(n-1)) = 0$ we can conclude that $H^i(BU(n)) \simeq H^i(BU(n-1))$ for $i \leq 2n-1$.

We now have to prove that $H^{2n}(BU(n)) \simeq \mathbb{Z}$ and that homology is zero for degrees greater than $2n$. Let's look at the following part of the gysin sequence:

$$
\begin{tikzcd}
H^{2n+k-1}(BU(n)) \arrow[r] & H^{2n+k-1}(BU(n-1)) \arrow[r] & H^k(BU(n)) \arrow[d] \\
H^{k+1}(BU(n)) & H^{2n+k}(BU(n-1)) \arrow[l] & H^{2n+k}(BU(n)) \arrow[l]
\end{tikzcd}
$$

For $k = 0$ we get $H^{2n}(BU(n)) \simeq H^0(BU(n)) \simeq \mathbb{Z}$:

$$
\begin{tikzcd}
H^{2n-1}(BU(n-1)) = 0 \arrow[r] & H^0(BU(n)) \simeq \mathbb{Z} \arrow[d] \\
H^{2n}(BU(n-1)) = 0 & H^{2n}(BU(n)) \arrow[l]
\end{tikzcd}
$$

For k odd we have $0 = H^k(BU(n)) \longrightarrow H^{2n+k}(BU(n)) \longrightarrow H^{2n+k}(BU(n-1)) = 0$ whereas for $k > 0$ and even we have

Chapter 4

Pontryagin-Thom isomorphism

4.1 Introduction

This chapter is dedicated to the Pontryagin-Thom isomorphism, which states that the cobordism ring of manifolds with a $\xi-$structure is isomorphic to the n homotopy group of the $M\xi$ spectrum.

We'll start building some tools that we will use and then define all the needed concepts to be able to attack the theorem.

4.2 Thom collapse map

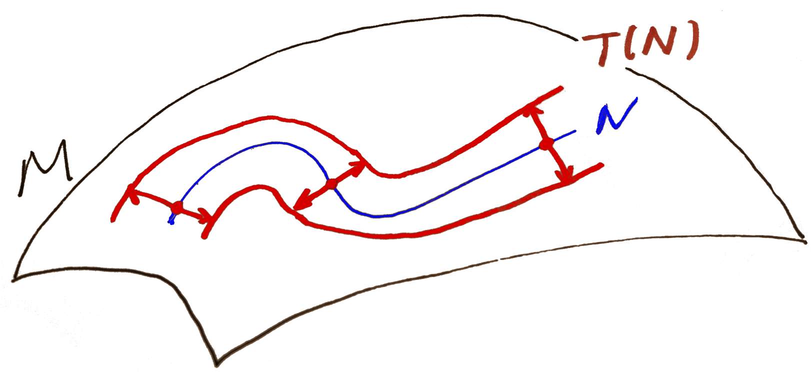
Let M be a manifold and N be a closed submanifold of M, $N \subseteq M$, we can then look at the normal bundle of N in M, defined as

$$\mathcal{N}_{N/M} \coloneqq (T_n N)^{\perp} \subseteq T_n M$$

We now choose a tubular neighborhood of N , denote it $T(N)$, sufficiently small as to not change its homotopy type. The Thom collapse map consists in collapsing the complement of $T(N)$ in M

$$M \longrightarrow M/(M \backslash T(N)) \simeq \mathrm{Th}_{\mathrm{An}_*}(N; \mathcal{N}_{N/M})$$

The result is homotopy equivalent to $\mathrm{Th}_{\mathrm{An}_*}(N; \mathcal{N}_{N/M})$ as this map mirrors the $D(V)/S(V)$ construction of the Thom space:



The tubular neighborhood $T(N)$ acts as the interior of the disk bundle $D(\mathcal{N}_{N/M})$ (the tubular neighborhood theorem is useful for establishing that this works) and the complement of $T(N)$ contributes the $S(\mathcal{N}_{N/M})$ part.

Note that collapsing the border of the disk is the same as one point compactifying the interior of the disk, the point being the collapsed complementary.

We're particularly interested in the case where M is $\mathbb{R}^n$ for some $n \in \mathbb{N}$ or better, its one point compactification S^n, we can always find a $k \in \mathbb{N}$ big enough to be able to embed N in $\mathbb{R}^k$

$$N \hookrightarrow \mathbb{R}^k \hookrightarrow S^k$$

thus getting a Thom collapse map $S^k \longrightarrow \mathrm{Th}_{\mathrm{An}_*}(N; -TN)$ as the normal bundle of N in the spere is stably equivalent to $-TN$.

Then applying the functor Σ^∞ to that map we get a morphism of spectra

$$\Sigma^k \mathbb{S} \simeq \Sigma^\infty S^k \longrightarrow \mathrm{Th}_{\mathrm{Sp}}(N; \mathcal{N}_{N/S^k})$$

This will prove really useful to find a homotopy class from a manifold in the Pontryagin-Thom isomorphism construction.

4.3 The theorem

Definition 4.1. Given a map $\xi : B \longrightarrow BO$ a *ξ-structure* on a virtual bundle $V : X \longrightarrow BO$ of rank 0 is a lift like the following

$$\begin{tikzcd} & B \arrow[d, "\xi"] \\ X \arrow[ur, dashed] \arrow[r, "V"'] & BO \end{tikzcd}$$

Additionally, for a vector bundle V of nonzero rank a ξ-structure is a ξ-structure in $V - \underline{\mathbb{R}}^{rankV}$, and a ξ-structure in a manifold M is a ξ-structure on its stable normal bundle $-TM$.

Remark 4.2. Following the previous definition, an *almost complex structure* on a manifold M is a lift

$$\begin{tikzcd} & & BU \arrow[d] \\ M \arrow[urr, dashed] \arrow[r, "\underline{\mathbb{C}}^n - TM"'] & BO(2n) \arrow[r] & BO \end{tikzcd}$$

Remark 4.3. Another particular case of the definition is the one where $B := *$ which encodes the information of a trivialization of its stable normal bundle

$$\begin{tikzcd} & * \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "-TM"'] & BO \end{tikzcd}$$

we call such a lift *framing* and a manifold together with such a lift a *framed manifold*.

Definition 4.4. The *cobordism ring*

$$\Omega_*^\xi = \bigoplus_{n \in \mathbb{N}} \Omega_n^\xi$$

is the graded ring with cobordism classes of n−manifolds with a ξ-structure on degree n, where cobordisms are required to have a ξ-structure compatible with that of the bordant manifolds.

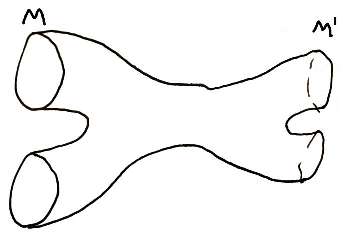


Figure 4.1: A cobordism between two manifolds.

The ring structure on Ω_*^ξ is given by disjoint union of two manifolds as the group structure (empty manifold is the identity element. As $M \times [0,1] \simeq M$ the inverse of $(M, \varphi : M \to B)$ is $M \times \{1\}$ with the induced $\xi-$structure as $(W, M \times \{0\} \sqcup M \times \{1\}, \emptyset)$ is a valid cobordism) and product of two manifolds as the non-invertible operation.

Theorem 4.5 (Pontryagin-Thom iso)**.** *There is an isomorphism*

$$\Omega_\bullet^\xi \cong \pi_\bullet M\xi$$

Proof. We will construct the maps in both directions and prove that they are each other's inverses.

From manifold to homotopy class Starting with a $n-$manifold M with a $\xi-$structure, we want to construct a homotopy class in $\pi_n M\xi$. This will be done in two steps:

1) Construct a map $\Sigma^n \operatorname{Th}_{\mathrm{Sp}}(M; -TM) \longrightarrow \operatorname{Th}_{\mathrm{Sp}}(B; \xi) = M\xi$.

Step 1 only depends on the $\xi-$structure on M:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[dr, "\underline{\mathbb{R}}^n - TM"'] & B \arrow[d, "\xi"] \\ & BO \end{tikzcd}$$

As $\varphi^*\xi = -TM$, then there is an induced map $\overline{\varphi}$ between the classifying pullbacks:

$$\begin{tikzcd} & \xi \arrow[dd, dashed] \arrow[dr, dashed] & \\ \underline{\mathbb{R}}^n - TM \arrow[ur, "\overline{\varphi}"] \arrow[rr, dashed] \arrow[ddd, dashed] & & \nu \arrow[ddd] \\ & B \arrow[ddr, "\xi"] & \\ & & \\ M \arrow[uur, "\varphi"] \arrow[rr, "\underline{\mathbb{R}}^n - TM"'] & & BO \end{tikzcd} \qquad \begin{tikzcd} \underline{\mathbb{R}}^n - TM \arrow[r, "\overline{\varphi}"] \arrow[d, dashed] & \xi \arrow[d, dashed] \\ M \arrow[r, "\varphi"'] & B \end{tikzcd}$$

where ν is the tautological bundle over BO. The square diagram on the right grants us a map

$$\Sigma^n \operatorname{Th}_{\mathrm{Sp}}(M; -TM) \simeq \operatorname{Th}_{\mathrm{Sp}}(M; \underline{\mathbb{R}}^n - TM) \longrightarrow \operatorname{Th}_{\mathrm{Sp}}(B; \xi) = M\xi$$

2) Cosntruct a class $\alpha_M \in \pi_0 Th(M; -TM)$.

Applying the functor $\pi_n(-)$ to the map from step 1 gives us

$$\pi_n\Sigma^n \operatorname{Th}_{\mathrm{Sp}}(M;-TM) \simeq \pi_0 \operatorname{Th}_{\mathrm{Sp}}(M;-TM) \longrightarrow \pi_n M\xi$$

We will now construct an appropiate class in $\pi_0 \operatorname{Th}_{\mathrm{Sp}}(M;-TM)$ such that its image is the desired class in $\pi_n M\xi$. Start by embedding our manifold in a big enough sphere $M \hookrightarrow \mathbb{R}^N \hookrightarrow S^N$ then the Thom collapse construction grants us a map

$$\Sigma^N \mathbb{S} \simeq \Sigma^\infty S^N \longrightarrow \operatorname{Th}_{\mathrm{Sp}}(M;\mathcal{N}_{M/S^N})$$

but as M/S^N and $-TM \oplus \underline{\mathbb{R}}^N$ are stably equivalent

$$\operatorname{Th}_{\mathrm{Sp}}(M;\mathcal{N}_{M/S^N}) \simeq \operatorname{Th}_{\mathrm{Sp}}(M;-TM \oplus \underline{\mathbb{R}}^N) \simeq \Sigma^N \operatorname{Th}_{\mathrm{Sp}}(M;-TM)$$

Hence we have the desired class in π_0, namely: $\alpha_M : \Sigma^N \mathbb{S} \longrightarrow \Sigma^N \operatorname{Th}_{\mathrm{Sp}}(M;-TM)$. Notice that step 2 is entirely dependent on the manifold and not on the ξ−structure chosen.

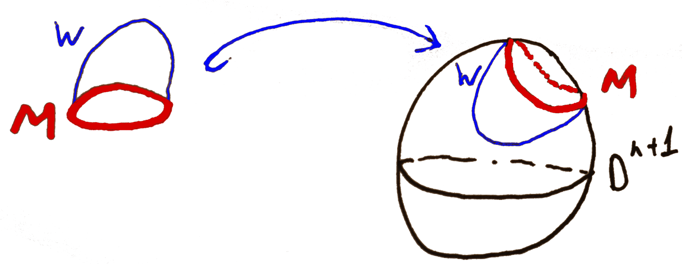
Class in $\pi_n M\xi$ is cobordism invariant It's enough to prove that it is invariant for nulcobordisms, as we can regard a cobordism (W, M, M', f, g) as nulcobordant $(W, M \sqcup M', \varnothing, f \sqcup g, \iota)$.

Consider then a cobordism $(W, M, \varnothing)$, following the construction of our homotopy class we would like to embed the whole cobordism, thom collapse it and get a nulhomotopy for the class associated to M. Now ir order to smoothly embed W, a manifold with boundary, we have to send the boundary to a boundary and we will be able to do so in a sufficiently large disk like shown in the left

$$\begin{tikzcd} M \arrow[r, hook] \arrow[d] & S^N \arrow[d] \\ W \arrow[r, hook] & D^{N+1} \end{tikzcd} \qquad \begin{tikzcd} M \arrow[rr, hook] \arrow[dr, "\varphi_W"'] & & W \arrow[dl, "\varphi_M"] \\ & B & \end{tikzcd}$$

as cobordisms are required to have ξ−structures compatible with those of the bordant manifolds we have the diagram on the right. Then when we thom collapse and compose with the map to ξ we see that the class factors through a disk, which is contractible thus the class is trivial.

$$\begin{tikzcd} \Sigma^N \operatorname{Th}_{\mathrm{Sp}}(M;-TM) \arrow[r] & \Sigma^{N+1} \operatorname{Th}_{\mathrm{Sp}}(W;-TW) \arrow[r] & M\xi \\ \Sigma^\infty \mathbb{S}^N \arrow[u] \arrow[r] & \Sigma^\infty \mathbb{D}^{N+1} \simeq 0 \arrow[u] & \end{tikzcd}$$



Map in the other direction We start with a class in $\pi_n M\xi$ ie a map $\Sigma^n\mathbb{S} \longrightarrow M\xi$ and we will assign to it an $n-$manifold with a $\xi-$structure. In order to make the argument clearer we will show it for MO and then highlight the adjustments that need to be made for the general $M\xi$.

We know that $MO = \underrightarrow{colim}\, MO_i$ (and $MO_k = Th(BO(k); \nu_k - \underline{\mathbb{R}}^k)$) so, as the sphere is compact, the class will factor through MO_k for some big enough $k \in \mathbb{N}$.

$$\begin{tikzcd} \Sigma^n\mathbb{S} \arrow[r] \arrow[dr] & MO \\ & MO_k \arrow[u] \end{tikzcd}$$

For the same reason, this diagonal map of spectra can be realized as a map of spaces in the stable range:

$$S^{n+N} \longrightarrow \Sigma^N Th(BO(k); \nu_k - \underline{\mathbb{R}}^k) \simeq Th(BO(k); \nu_k - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N)$$

But recall that $BO(k) = \underrightarrow{colim}\, Gr_{\mathbb{R}}(k, n)$ thus for a big enough $L \in \mathbb{N}$ we have

$$\begin{tikzcd} S^{n+N} \arrow[r] \arrow[dr, "f"'] & Th(BO(k); \nu_k - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\ & Th(Gr_{\mathbb{R}}(k, L); \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \arrow[u, "\simeq"] \end{tikzcd}$$

We have saved the gap to geometry as grassmanians are manifolds! In order to continue we need to do a little detour, we start by defining a desirable property that will make our argument work:

Definition 4.6. Given a differentiable map between manifolds $f : M \longrightarrow N$ and a submanifold inclusion $Z \subseteq N$ we say that M and Z are *transversal* if

$$T_m M \oplus T_{f(m)} Z = T_{f(m)} N \quad \forall m \in M \text{ such that } f(m) \in Z$$

we write $M \pitchfork Z$.

We have at our disposal the following result,[Kos, Corolary IV.2.4]:

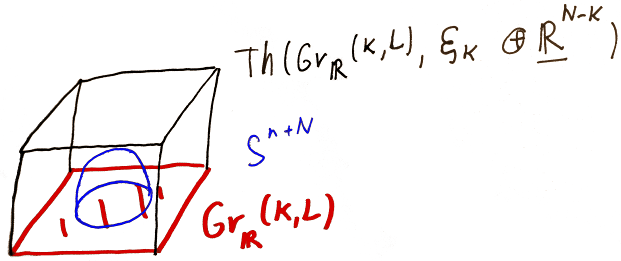
Theorem 4.7. *Let Z be a compact submanifold of N, U an open neighborhood of Z in N and $f : M \longrightarrow N$ a smooth map. Then there is an isotopy h_t of N that is the identity outside of U and such that $f \pitchfork h_1(Z)$.*

which grants us a transversal replacement of a compact submanifold. First we homotopy the map f to a smooth map and in order to apply the theorem we look at the diagram of spaces

$$\begin{tikzcd} H_n(S^{V_x}; R) \arrow[r] \arrow[d, phantom, "\wr\|"] & H_n(S^{V_x}; R) \arrow[d, phantom, "\wr\|"] \\ \mathbb{Z} \otimes R \arrow[r] & \mathbb{Z} \otimes R \end{tikzcd}$$

where the inclusion of $Gr_{\mathbb{R}}(k, L)$ is given by the $0-$section and U is the preimage of a tubular neighborhood of the grassmanian in $\nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N$. Now the transversal replacement deforms $Gr_{\mathbb{R}}(k, L)$ inside the Thom space so that it sits transversally to S^{n+N}.

Now we consider the inverse image of the modified $Gr_{\mathbb{R}}(k, L)$ through the map f from the sphere to the Thom space. We know that this is a manifold by [Kos, Proposition IV.1.4] and it will be the manifold M that we associate to the class in $\pi_n MO$. Notice that it comes with a choice of embedding $M \hookrightarrow S^{n+N}$.



$$\begin{tikzcd}
U \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
M \arrow[u, dashed] \arrow[r, dashed] & Gr_{\mathbb{R}}(k,L) \arrow[u]
\end{tikzcd}$$

M will be n−dimensional as it codimension of M in U is the same as codimension of $Gr_{\mathbb{R}}(k,L)$ in $\nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N$. As it happens the latter is precisely the rank of $\nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N$ ie $k + (N - k) = N$; this is because for any bundle over a manifold $V \to M$ it is true that $\mathcal{N}_{V\setminus M} \cong V$. Then as $U \subseteq S^{n+N}$ we have $(n + N) - dim(M) = N \Rightarrow dim(M) = n$.

- Adjustments for $M\xi$

Firstly, let's consider the following pullbacks

$$\begin{tikzcd}
B_k \arrow[r, dashed] \arrow[d, dashed] & B \arrow[d] \\
BO(k) \arrow[r] & BO
\end{tikzcd}
\qquad
\begin{tikzcd}
B_{k,n} \arrow[r, dashed] \arrow[d, dashed] & B_k \arrow[d] \\
Gr_{\mathbb{R}}(k,n) \arrow[r] & BO(k)
\end{tikzcd}$$

we have that $B_k = \underrightarrow{colim}\, B_{k,n}$ and $M\xi_k := \Omega^\infty \Sigma^k M\xi = \mathrm{Th}(B_k;\xi_k)$. Now for the first part the argument works exactly as for MO, as we only used compactness of the sphere. Additionally we can postcompose the homotopy class with $M\xi \to MO$ to see that the map to the Thom space of $BO(k)$ factors through the Thom space of B_k.

$$\begin{tikzcd}
\Sigma^n \mathbb{S} \arrow[r] \arrow[dr] & M\xi \arrow[r] & MO \\
& \mathrm{Th}_{\mathrm{Sp}}(B_k;\xi_k) \arrow[u] \arrow[r] & \mathrm{Th}_{\mathrm{Sp}}(BO(k);\nu_k - \mathbb{R}^k) \arrow[u]
\end{tikzcd}$$

Now using compactness of the sphere, a big $L \in \mathbb{N}$ for the colimits and the factorization we arrive at a situation like

$$\begin{tikzcd}
S^{n+N} \arrow[r] & \mathrm{Th}_{\mathrm{An}_*}(B_{k,L}, \xi_{k,L} \otimes \mathbb{R}^N) \arrow[r, "f"] & \mathrm{Th}_{\mathrm{An}_*}(Gr_{\mathbb{R}}(k,L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\
U \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & \xi_{k,L} \oplus \mathbb{R}^N \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \arrow[u, phantom, sloped, "\subseteq"] \\
 & B_{k,L} \arrow[u, phantom, sloped, "\subseteq"] \arrow[r] & Gr_{\mathbb{R}}(k,L) \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

where we use the transversal replacement theorem in the long map to the grassmanian and we take the exact same pullback as last time: inverse image of $Gr_{\mathbb{R}}(k,L)$ through the now transversal map. We need to rely on the grassmanian on the MO side to get a manifold at all as $B_{k,L}$ doesn't have to be a manifold.

$$\begin{tikzcd}
U \arrow[r] \arrow[rr, bend left, "smooth\&transversal"] & \xi_{k,L} \oplus \mathbb{R}^N \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
M \arrow[u, dotted] \arrow[rr, dashed, bend left] & B_{k,L} \arrow[u] \arrow[r] & Gr_{\mathbb{R}}(k,L) \arrow[u]
\end{tikzcd}$$

What we will do now is proving that the composite $M \to Gr_{\mathbb{R}}(k,L) \to BO$ is exactly $-TM$ and then prove that we have a factorization

$$\begin{tikzcd}
M \arrow[r, dashed] \arrow[rr, bend left] & B_{k,L} \arrow[r] & Gr_{\mathbb{R}}(k,L)
\end{tikzcd}$$

which, if we look at the pullback defining $B_{k,L}$, will give us a lift to B which was exactly what we wanted, to find a manifold with a ξ−structure.

- The map $M \to Gr_{\mathbb{R}}(k,L) \to BO$ is exactly $-TM$

$$\begin{tikzcd}
 & & \xi_{k,L} \oplus \underline{\mathbb{R}}^N \arrow[dr] & \\
U \arrow[r] & \mathcal{N}_{S^{n+N} \setminus M} \arrow[ur] \arrow[rr] & & \nu_{k,L} \oplus \underline{\mathbb{R}}^{N-k} \\
 & & B_{k,L} \arrow[uu] \arrow[dr] & \\
M \arrow[uu] \arrow[r] & M \arrow[uu] \arrow[ur] \arrow[rr] & & Gr_{\mathbb{R}}(k,L) \arrow[uu]
\end{tikzcd}$$

- $M \longrightarrow Gr_{\mathbb{R}}(k,L)$ factors through $B_{k,L}$

This is the easy part, the following square diagram is a pullback as $\xi_{k,L} \oplus \mathbb{R}^N \to \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N$ is an iso on the fibers (as in fact it comes from a pullback). Then the universal property of the pullback grants us the desired factorization map

$$\begin{tikzcd}
& \xi_{k,L} \oplus \mathbb{R}^N \arrow[r] & \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N \\
& B_{k,L} \arrow[u] \arrow[r] & Gr_{\mathbb{R}}(k,L) \arrow[u] \\
M \arrow[ur, dashed] \arrow[uur, bend left] \arrow[urr, bend right] & &
\end{tikzcd}$$

getting the manifold M with a $\xi-$structure that we were looking for.

Homotopic maps induce bordant manifolds Recall that a map f in spectra is the same as maps $f_n : A_n \to B_n$ between the spaces on each degree and homotopies $\mathcal{H}_n : f_n \simeq \Omega f_{n+1}$

$$Map_{\mathrm{Sp}}(A,B) \cong \varprojlim Map_{\mathrm{An}_*}(A_n, B_n)$$

Then a homotopy between maps $f, g : \Sigma^n \mathbb{S} \longrightarrow M\xi$ is the same as homotopies $\mathcal{K}_i : f_i \simeq g_i$ where $f_i, g_i : S^{n+i} \longrightarrow M\xi_i$ equipped with higher homotopies $\mathcal{H}_i : \mathcal{K}_i \simeq \Omega\mathcal{K}_{i+1}$.

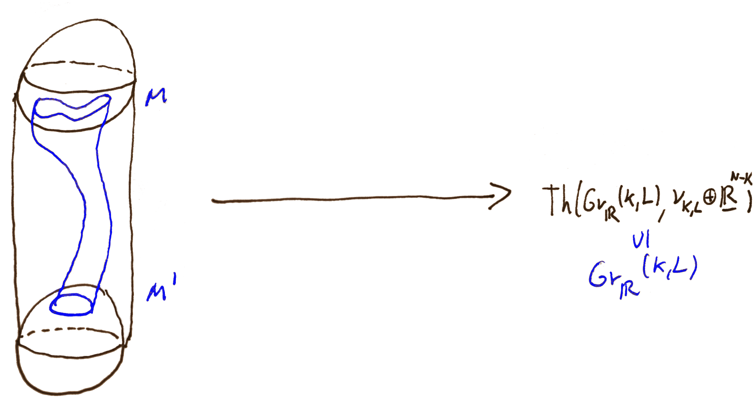
I is also compact, subsequently, the arguments that we used still work. We arrive at a homotopy

$$S^{n+N} \times I \longrightarrow Th(Gr_{\mathbb{R}}(k,L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N)$$

Now, analogously to the argument to a single map we will use the diagram

$$\begin{tikzcd}
S^{n+N} \times I \arrow[r, "f"] & Th(Gr_{\mathbb{R}}(k,L), \nu_{k,L} - \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^N) \\
& Gr_{\mathbb{R}}(k,L) \arrow[u, "\subseteq"']
\end{tikzcd}$$

to get a manifold in $S^{n+N} \times I$ as preimage of the grassmanian and the slices of the manifold in $S^{n+N} \times \{0\}$ and $S^{n+N} \times \{1\}$ will be precisely the manifolds associated to each of the two homotopic maps. That's precisely the cobordism we were looking for, and we can use the same argument as before for it having dimension $n+1$ and getting the right tangential structure.



For the same reason, the homotopies chosen to make the map from the sphere S^{n+N} smooth and the inclusion of $Gr_{\mathbb{C}}(k, L)$ transversal do not matter as we can compose homotopies from two smooth (or transversal) choices to the original map to get a homotopy between the choices and then obtain a cobordism between the two maybe different results that they provide.

Both compositions are the identity

- $\pi_k M\xi \longrightarrow \Omega_k^\xi \longrightarrow \pi_k M\xi$

  We obtained
- $\Omega_k^\xi \longrightarrow \pi_k M\xi \longrightarrow \Omega_k^\xi$

□

The special case of $\Omega_\bullet^U \cong \pi_\bullet MU$ is interesting as we are able to calculate the right side. This can be done by feeding the homology groups $H_* MU$ that we calculated in section 3.4 to the Adams spectral sequence - which is degenerate in this case - calculating $\pi_* MU$ as a result. It turns out that it is the polynomial algebra on even degrees and the generating classes look like complex projective spaces on $\Omega_\bullet^U$. We don't have time to go into details, so for a more complete explanation check [Mil]. We have then fully described Ω_*^U, the cobordism ring with a almost complex structure!

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