Chapter 1:

Review of Derivatives

Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the "how" of differentiation. We will review all the derivative rules learned last year in Precalculus. In the next two chapters, we will review the "why." As a quick reference, here are those rules:

$$\text{The Power Rule: } \frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$$

$$\text{The Product Rule: } \frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\text{The Quotient Rule: } \frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v \cdot \dfrac{du}{dx} - u \cdot \dfrac{dv}{dx}}{v^2}$$

$$\text{The Chain Rule: } \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad\qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad\qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = (e^u)\frac{du}{dx} \qquad\qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}[a^u] = (a^u \cdot \ln u)\frac{du}{dx} \qquad\qquad \frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Here is a quick review from last year:

Identities: While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x; \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin 2x = 2 \sin x \cos x; \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverses: Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1} x = \left\{ \begin{array}{l} calculator \ \pm 2\pi n \\ -calculator \ \pm 2\pi n \end{array} \right\} \qquad \sin^{-1} x = \left\{ \begin{array}{l} calculator \ \pm 2\pi n \\ \pi - calculator \ \pm 2\pi n \end{array} \right\}$$

$$\tan^{-1} x = \left\{ \begin{array}{l} calculator \ \pm 2\pi n \\ \pi + calculator \ \pm 2\pi n \end{array} \right\} = calculator \ \pm \pi n$$

Logarithm Rules: Here are some logarithm rules which you should recall:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

1.1: The Power and Exponential Rules with the Chain Rule

In Precalculus we developed the idea of the derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the derivative will be explored. In this chapter, we are primarily interested in how to find the derivative and what it is used for.

Derivative: $\rightarrow$ Definition: $f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\qquad\qquad \rightarrow$ Means: The function that yields the slope of the tangent line.

Numerical Derivative: $\rightarrow$ Definition: $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$\qquad\qquad\qquad \rightarrow$ Means: The numerical value of the slope of the tangent line at $x = a$

Symbols for the Derivative

$\dfrac{dy}{dx} =$ "d - y - d - x" $\qquad\qquad f'(x) =$ "f prime of x" $\qquad\qquad y' =$ "y prime"

$\qquad\quad \dfrac{d}{dx} =$ "d - d - x" $\qquad\qquad\qquad\qquad D_x =$ "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to Find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from Precalculus: The derivative yields the slope of the tangent line. (But there is more to it than that).

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in Precalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule: $\qquad\qquad\qquad$ The Exponential Rules:

$\dfrac{d}{dx}[x^n] = nx^{n-1} \qquad\qquad\qquad \dfrac{d}{dx}[e^x] = e^x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dfrac{d}{dx}[a^x] = a^x \cdot \ln a$

The difference between these is where the variable is. The Power Rule applies when the variable is in the *base*, while the Exponential Rules apply when the variable is in the *exponent*. The difference between the two Exponential rules is what the base is. $e = 2.718281828459\ldots$, while a is any positive number other than 1.

Ex 1.1.1: Find a) $\dfrac{d}{dx}\left[x^5\right]$ and b) $\dfrac{d}{dx}[5^x]$

Sol 1.1.1: The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

$$\text{a) } \frac{d}{dx}\left[x^5\right] = \boxed{5x^4} \qquad\qquad \text{b) } \frac{d}{dx}[5^x] = \boxed{5^x \ln 5}$$

There are a few other basic rules that we need to remember.

$$\frac{d}{dx}[\text{constant}] = 0$$

$$\frac{d}{dx}[cx^n] = (cn)x^{n-1}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

These rules allow us to easily differentiate a polynomial term by term.

Ex 1.1.2: $y = 3x^2 + 5x + 1$; find $\dfrac{dy}{dx}$.

Sol 1.1.2:

$$\frac{dy}{dx} = \frac{d}{dx}[3x^2 + 5x + 1]$$

$$= (3 \cdot 2)x^{2-1} + (5 \cdot 1)x^{1-1} + 0$$

$$= \boxed{6x + 5}$$

Ex 1.1.3: $f(x) = x^2 + 4x - 3 + e^x$; find $f'(x)$.

Sol 1.1.3:

$$f'(x) = \frac{d}{dx}\left[x^2 + 4x - 3 + e^x\right]$$

$$= (1 \cdot 2)x^{2-1} + (4 \cdot 1)x^{1-1} - 0 + e^x$$

$$= \boxed{2x + 4 + e^x}$$

Ex 1.1.4: $y = \sqrt{x^3} + \dfrac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$; find $\dfrac{dy}{dx}$.

Sol 1.1.4:

$$y = \sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$$

$$= x^{\frac{3}{2}} + 4x^{-\frac{1}{2}} - x^{\frac{3}{4}} + e^4$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^{\frac{3}{2}} + 4x^{-\frac{1}{2}} - x^{\frac{3}{4}} + e^4 \right]$$

$$= \left(1 \cdot \frac{3}{2}\right) x^{\frac{3}{2}-1} + \left(4 \cdot -\frac{1}{2}\right) x^{-\frac{1}{2}-1} - \left(1 \cdot \frac{3}{4}\right) x^{\frac{3}{4}-1} + 0$$

$$= \boxed{\frac{3}{2} x^{\frac{1}{2}} - 2x^{-\frac{3}{2}} - \frac{3}{4} x^{-\frac{1}{4}}}$$

Note in Ex 1.1.4 that e^4 is a constant. Therefore, its derivative is 0.

As we have seen, when the variable was in the exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\dfrac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right] = \frac{d}{dx}\left[e^{x^2 \ln(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Ex 1.1.5: If $y = (x^2 + 1)(x^3 - 4x)$, find $\dfrac{dy}{dx}$.

Sol 1.1.5:

$$y = (x^2 + 1)(x^3 - 4x)$$

$$= x^5 - 4x^3 + x^3 - 4x$$

$$= x^5 - 3x^3 - 4x$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^5 - 3x^3 - 4x\right]$$

$$= \boxed{5x^4 - 9x^2 - 4}$$

Ex 1.1.6: If $y = \dfrac{x^2 - 4x + 6}{\sqrt[3]{x}}$, find $\dfrac{dy}{dx}$.

Sol 1.1.6:

$$y = \frac{x^2 - 4x + 6}{\sqrt[3]{x}}$$

$$= \frac{x^2 - 4x + 6}{x^{\frac{1}{3}}}$$

$$= x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}\right]$$

$$= \boxed{\frac{5}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{-\frac{1}{3}} - 2x^{-\frac{4}{3}}}$$

The Chain Rule

Composite Function → A function made of two other functions, one within the other. For example, $y = \sqrt{16x - x^3}$, $y = \sin x^3$, $y = \cos^3 x$, and $y = \left(x^2 + 2x - 5\right)^3$. The general symbol is $f(g(x))$.

Ex 1.1.7: Given $f(x) = \cos^{-1} x$, $g(x) = x^2 - 1$, and $h(x) = \sqrt{1 + x^2}$, find a) $f\left(g\left(\sqrt{2}\right)\right)$, b) $h(g(1))$, and c) $f(h(g(1)))$.

Sol 1.1.7:

(a) $g\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 1 = 1$, so $f\left(g\left(\sqrt{2}\right)\right) = f(1) = \cos^{-1}(1) = \boxed{0}$.

(b) $g(1) = 0$, so $h(g(1)) = h(0) = \sqrt{1 + 0^2} = \boxed{1}$

(c) $g(1) = 0$ and $h(g(1)) = h(0) = \sqrt{1 + 0^2} = 1$, so $f(h(g(1))) = \cos^{-1}(1) = \boxed{0}$

So. How do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. The result is two open dolls adjacent to each other.

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If you think of the inside function (the $g(x)$) as equaling u, we could write the Chain Rule like this:

$$\frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$

This is the way that most derivatives are written with the Chain Rule.

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other rules, as seen in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

$$\boxed{\begin{array}{cc}
\text{The Power Rule:} & \text{The Exponential Rules:} \\[2em]
\dfrac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot \dfrac{du}{dx} & \dfrac{d}{dx}\left[e^u\right] = e^u \cdot \dfrac{du}{dx} \\[2em]
& \dfrac{d}{dx}\left[a^u\right] = (a^u \cdot \ln a) \cdot \dfrac{du}{dx} \\[2em]
\multicolumn{2}{c}{(\text{where } u \text{ is a function of } x)}
\end{array}}$$

Ex 1.1.8: $\dfrac{d}{dx}\left[(4x^2 - 2x - 1)^{10}\right]$

Sol 1.1.8:

$$u = 4x^2 - 2x - 1 \text{ and } f(u) = u^{10}$$

$$\frac{d}{dx}\left[f(u)\right] = f'(u) \cdot \frac{du}{dx}$$

$$= 10u^9 \cdot (8x - 2)$$

$$= \boxed{10\left(4x^2 - 2x - 1\right)^9 (8x - 2)}$$

Ex 1.1.9: $\dfrac{d}{dx}\left[e^{4x^2}\right]$

Sol 1.1.9:

$$u = 4x^2$$

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}$$

$$= e^{4x^2} \cdot 8x$$

$$= \boxed{8xe^{4x^2}}$$

Ex 1.1.10: If $y = \sqrt{16 - x^3}$, find $\dfrac{dy}{dx}$.

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Sol 1.1.10:

$$u = 16 - x^3 \text{ and } f(u) = \sqrt{u}$$

$$\frac{dy}{dx} = \frac{d}{dx}[f(u)] = f'(u) \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}} \cdot -3x^2$$

$$= \boxed{\frac{-3x^2}{2\sqrt{16 - x^3}}}$$

Ex 1.1.11: $\dfrac{d}{dx}\left[\sqrt{(x^2+1)^5 + 7}\right]$

Sol 1.1.11:

$$u = x^2 + 1,\ g(u) = u^5 + 7,\ \text{and } f(g(u)) = \sqrt{g(u)}$$

$$\frac{d}{dx}[f(g(u))] = f'(g(u)) \cdot g'(u) \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{g(u)}} \cdot 5u^4 \cdot 2x$$

$$= \frac{5(x^2+1)(2x)}{2\sqrt{(x^2+1)^5 + 7}} = \boxed{\frac{5x(x^2+1)}{\sqrt{(x^2+1)^5 + 7}}}$$

1.1 Free Response Homework

Find the derivatives of the given functions. Simplify where possible.

1. $f(x) = x^2 + 3x - 4$

2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$

3. $y = x^{-\frac{2}{3}}$

4. $y = 5e^x + 3$

5. $v(r) = \frac{4}{3}\pi r^3$

6. $g(x) = x^2 + \frac{1}{x^2}$

7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

9. $z = \frac{A}{y^{10}} + Be^y$

10. $y = e^{x+1} + 1$

Complete the following.

11. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$

12. $\frac{d}{dx}\left[x^6 - 3\sqrt[6]{x^7} + 5^x - \frac{1}{\sqrt[3]{x^5}} + \frac{1}{8x}\right]$

13. $\frac{d}{dx}\left[x^4 - 14\sqrt[7]{x^9}\right] + 8^x - \frac{1}{\sqrt[3]{x^7}} + \frac{1}{8x}$

14. $\frac{d}{dx}\left[(x-1)\sqrt{x}\right]$

15. $\frac{d}{dz}\left[\left(z^2 - 4\right)\sqrt{z^3}\right]$

16. $\frac{d}{dx}\left[\left(x^2 - 4x + 3\right)\sqrt{x^5}\right]$

17. $\frac{d}{dt}\left[\left(4t^2 + 1\right)\left(3t^3 + 7\right)\right]$

18. $\frac{d}{dx}\left[\left(x^3 + 4x - \pi\right)^{-7}\right]$

19. $\frac{d}{dx}\left[\sqrt{3x^2 - 4x + 9}\right]$

20. $\frac{d}{dx}\left[\sqrt[7]{x^3 - 2x}\right]$

21. $\frac{d}{dy}\left[\frac{4y^3 - 2y^2 - 5y}{\sqrt{y}}\right]$

22. $\frac{d}{dv}\left[\frac{v^2 - 4v + 7}{2\sqrt{v}}\right]$

23. $\frac{d}{dw}\left[\frac{7w^2 - 4w + 1}{5w^3}\right]$

24. $\frac{d}{dw}\left[\frac{5w^2 - 3w - 4}{7w^2}\right]$

25. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$

26. $f(x) = \sqrt[5]{\left(\frac{1}{x} + 2x + e^x\right)^3}$, find $f'(x)$

27. $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$

28. $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$

29. $g(2) = 3$, $g'(2) = -4$, $f(x) = e^{g(x)}$, find $f'(2)$

30. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$

31. $f(x) = \sqrt{4 - \dfrac{4}{9}x^2}$, find $f'\left(\sqrt{5}\right)$

32. $f(x) = e^{\sqrt{9-x^2}}$, find $f'(x)$

33. $v(t) = \sqrt{\left(\dfrac{E(t)}{3} + 3t\right)^{\frac{3}{7}} - 4,}$, find $v'(t)$

34. $v(t) = \sqrt[3]{\left(\dfrac{C(t)}{7} + 4t^2\right)^{\frac{5}{7}} - 1}$, find $v'(t)$

1.1 Multiple Choice Homework

1. If $f(x) = x^{\frac{3}{2}}$, then $f'(4) =$

   a) -6 b) -3 c) 3 d) 6 e) 8

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2. The derivative of $\sqrt{x} - \dfrac{1}{x\sqrt[3]{x}}$

   a) $\dfrac{1}{2}x^{-\frac{1}{2}} - x^{-\frac{4}{3}}$
        b) $\dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{4}{3}x^{-\frac{7}{3}}$
        c) $\dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{4}{3}x^{-\frac{1}{3}}$

   d) $-\dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{4}{3}x^{-\frac{7}{3}}$
        e) $-\dfrac{1}{2}x^{-\frac{1}{2}} - \dfrac{4}{3}x^{-\frac{1}{3}}$

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3. Given $f(x) = \dfrac{1}{2x} + \dfrac{1}{x^2}$, find $f'(x)$

   a) $-\dfrac{1}{2x^2} - \dfrac{2}{x^3}$
        b) $-\dfrac{2}{x^2} - \dfrac{2}{x^3}$
        c) $\dfrac{2}{x^2} - \dfrac{2}{x^3}$

   d) $-\dfrac{1}{2x^2} + \dfrac{2}{x^3}$
        e) $\dfrac{1}{2x^2} - \dfrac{2}{x^3}$

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4. If $f(x) = e^{5x^2} + x^4$, then $f'(1) =$

   a) $e^5 + 1$ b) $5e^4 + 4$ c) $5e^5 + 1$ d) $10e + 4$ e) $10e^5 + 4$

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5. If h is the function defined by $h(x) = e^{5x} + x + 3$, then $h'(0)$ is

   a) 2 b) 4 c) 5 d) 6 e) 8

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6. If $y = \left(x^4 + 4\right)^2$, then $\dfrac{dy}{dx} =$

   a) $2\left(x^4 + 4\right)$ b) $\left(4x^3\right)^2$ c) $2\left(4x^3 + 4\right)$

            d) $4x^3\left(x^4 + 4\right)$ e) $8x^3\left(x^4 + 4\right)$

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7. If $h(x) = [f(x)]^2\, g(x)$ and $g(x) = 3$, then $h'(x) =$

   a) $2f'(x)g'(x)$ b) $6f'(x)f(x)$ c) $g'(x)\,[f(x)]^2 + 2f(x)f'(x)g(x)$

           d) $2f'(x)g(x) + g'(x)\,[f(x)]^2$ e) 0

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8. Which of the following statements must be true?

$$\text{I.} \quad \frac{d}{dx}\left[\sqrt{e^x + 3}\right] = \frac{e^x}{2\sqrt{e^x + 3}}$$

$$\text{II.} \quad \frac{d}{dx}\left[5^{3x^2}\right] = 6x \ln(5)\left(5^{3x^2}\right)$$

$$\text{III.} \quad \frac{d}{dx}\left[6x^3 - \pi + \sqrt[3]{x^8} - \frac{2}{x^3}\right] = 18x^2 + \frac{8}{3}\sqrt[3]{x^5} + \frac{6}{x^4}$$

   a) I only   b) II only   c) I and III only

           d) I and III only   e) I, II, and III

1.2: Trig, Trig Inverse, and Log Rules

Trigonometric: → Definition: A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure.

→ Means: An equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic: → Definition: The inverse of an exponential function.

→ Means: An equation with log or ln in it.

Trig Derivative Rules

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

Log Derivative Rules

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx} \qquad \frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

Note that all these rules are expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives Involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1.2.1: $\dfrac{d}{dx}\left[\sin^3(x)\right]$

Sol 1.2.1:

$$\frac{d}{dx}\left[\sin^3(x)\right] = \boxed{3\sin^2(x)\cos(x)}$$

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Ex 1.2.2: $\dfrac{d}{dx}\left[\sin\left(x^3\right)\right]$

Sol 1.2.2:

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \boxed{3x^2\cos\left(x^3\right)}$$

Ex 1.2.3: $\dfrac{d}{dx}\left[\ln\left(4x^3\right)\right]$

Sol 1.2.3:

$$\frac{d}{dx}\left[\ln\left(4x^3\right)\right] = \frac{1}{4x^3}\cdot 12x^2$$

$$= \boxed{\frac{3}{x}}$$

We could have also simplified algebraically before taking the derivative:

$$\ln\left(4x^3\right) = \ln 4 + \ln x^3$$

$$= \ln 4 + 3\ln x$$

$$\frac{d}{dx}\left[\ln 4 + 3\ln x\right] = 0 + 3\cdot\frac{1}{x}$$

$$= \boxed{\frac{3}{x}}$$

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 1.2.4: $\dfrac{d}{dx}\left[\sec^5\left(3x^4\right)\right]$

Sol 1.2.4

$$\frac{d}{dx}\left[\sec^5\left(3x^4\right)\right] = 5\sec^4\left(3x^4\right)\cdot\sec\left(3x^4\right)\tan\left(3x^4\right)\cdot\left(12x^3\right)$$

$$= \boxed{60x^3\sec^5\left(3x^4\right)\tan\left(3x^4\right)}$$

Ex 1.2.5: $\dfrac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right)$

Sol 1.2.5:

$$\frac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right) = \frac{1}{\cos\left(\sqrt{x}\right)}\cdot\left(-\sin\left(\sqrt{x}\right)\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}$$

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$$= -\tan\left(\sqrt{x}\right) \cdot \frac{1}{2\left(\sqrt{x}\right)}$$

$$= \boxed{\frac{-\tan\left(\sqrt{x}\right)}{2\sqrt{x}}}$$

General inverses are not all that interesting. We are more interested in particular *transcendental* inverse functions, like the natural log. Another particular kind of inverse function that bears more study is the trig inverse function. Interestingly, as with the log functions, the derivatives of these transcendental functions become algebraic functions.

<div align="center">

Inverse Trig Derivative Rules

</div>

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad\qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Ex 1.2.6: $\dfrac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right]$

Sol 1.2.6:

$$\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right] = \frac{1}{\left(3x^4\right)^2+1}\cdot\left(12x^3\right)$$

$$= \boxed{\frac{12x^3}{9x^8+1}}$$

Ex 1.2.7: $\dfrac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right]$

Sol 1.2.7:

$$\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right] = \frac{1}{|x^2|\sqrt{\left(x^2\right)^2-1}}\cdot 2x$$

$$= \frac{2x}{\left(x^2\right)\sqrt{\left(x^2\right)^2-1}}$$

$$= \boxed{\frac{2}{x\sqrt{x^4 - 1}}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] - \frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 1.2.8: If $f(x) = x^2 + 2x + 3$, $g(x) = f^{-1}(x)$, and $g(1) = 2$; find $g'(1)$.

Sol 1.2.8:

$$f'(x) = 2x + 2 \ \therefore \ f'(g(x)) = 2(g(x)) + 2$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{f'(g(x))}$$

$$g'(1) = \frac{1}{f'(g(1))} = 2(g(1)) + 2 = \boxed{6}$$

1.2 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin(4x)$

2. $y = 4\sec\left(x^5\right)$

3. $f(t) = \sqrt[3]{1 + \tan t}$

4. $f(\theta) = \ln\left(\cos(\theta)\right)$

5. $y = a^3 + \cos^3(x)$

6. $y = \cos\left(a^3 + x^3\right)$

7. $f(x) = \cos\left(\ln x\right)$

8. $f(x) = \sqrt[5]{\ln x}$

9. $f(x) = \log_{10}\left(2 + \sin(x)\right)$

10. $f(x) = \log_2(1 - 3x)$

11. $y = \sin^{-1}\left(e^x\right)$

12. $y = \tan^{-1}\left(\sqrt{x}\right)$

Complete the following.

13. $\dfrac{d}{dx}\left[\sin^{-1}\left(e^{3x}\right)\right]$

14. $\dfrac{d}{dx}\left[\cot^{-1}\left(e^{2x}\right)\right]$

15. $\dfrac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right]$

16. $\dfrac{d}{dx}\left[\cot^{-1}\left(\dfrac{1}{x}\right) - \tan^{-1}(x)\right]$

17. $\dfrac{d}{dx}\left[3e^{x^2 + 2x}\right]$

18. $\dfrac{d}{dx}\left[3\cos\left(x^2 + 2x\right)\right]$

19. $\dfrac{d}{dx}\left[\sqrt[3]{16 + x^3}\right]$

20. $\dfrac{d}{dx}\left[\sec^{-1}\left(2x^2\right)\right]$

21. $\dfrac{d}{dx}\left[5e^{\tan(7x)}\right]$

22. $\dfrac{d}{dx}\left[\sqrt{\cos\left(1 - x^2\right)}\right]$

23. $\dfrac{d}{dx}\left[\ln^3\left(x^2 + 1\right)\right]$

24. $\dfrac{d}{dx}\left[\ln\sin\left(x^3\right)\right]$

25. $\dfrac{d}{dx}\left[\ln\left(\sec(x)\right)\right]$

26. $\dfrac{d}{dx}\left[\cos\left(x^2\right)\right]$

27. $f(x) = \ln\left(x^2 + 3\right)$, find $f'(x)$

28. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g'(x)$

29. $h(x) = \sqrt{x^2 + 5}$, find $h'(x)$

30. $F(x) = \sqrt[3]{3x^2 - 6x + 1}$, find $F'(x)$

31. $y = \sin^{-1}\left(\cos(x)\right)$, find y'

32. $y = \sin\left(\cos^{-1}(x)\right)$, find y'

33. $y = \tan^2\left(3\theta\right)$, find y'

34. $y = \cot^7\left(\sin(\theta)\right)$, find y'

35. $y = \sin^{-1}\left(\sqrt{2}(x)\right)$, find y'

36. $y = \sin^{-1}(2x + 1)$, find y'

The following table shows some values of $g(x)$, $g'(x)$, and $h(x)$, where $h(x) = g^{-1}(x)$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
1	2	3	$\dfrac{1}{2}$	$\dfrac{1}{3}$
3	1	2	-2	$\dfrac{1}{2}$

37. Find $h'(1)$

38. Find $g'(1)$

1.2 Free Response Homework Set B

Find the derivatives of the given functions. Simplify where possible.

1. $y = \cos^{-1}\left(e^{3z}\right)$

2. $y = \tan^{-1}\sqrt{x^2 - 1}$

3. $y = \sec^{-1}(4x) + \csc^{-1}(4x)$

4. $f(x) = \ln\left(\tan^{-1}(5x)\right)$

5. $g(w) = \sin^{-1}(5w) + \cos^{-1}(5w)$

6. $f(t) = \sec^{-1}\sqrt{9 + t^2}$

Complete the following.

7. $\dfrac{d}{d\theta}\left[e^{\csc(\theta)} + \ln\left(\cot\left(\theta^2\right)\right) - \sec(\theta)\right]$

8. $\dfrac{d}{dx}\left[\ln\left(\sec\left(x^3 + 5\ln x + 7\right)^3\right)\right]$

9. $\dfrac{d}{dx}\left[\ln\left(\tan\left(x^2 + 5e^x + 7\right)^3\right)\right]$

10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$

11. $\dfrac{d}{dx}\left[\ln\left(\sqrt{x^2 + 4x - 5}\right)\right]$

12. $\dfrac{d}{dt}\left[\sin^5\left(\ln\left(7t + 3\right)\right)\right]$

13. $\dfrac{d}{dx}\left[\csc\left(\ln\left(7x^2 + x\right)\right)\right]$

14. $\dfrac{d}{dx}\left[\ln\left(\sqrt{e^{4t^2 + 6}}\right)\right]$

15. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sqrt{9x - 27x^2 + \dfrac{5}{x^3}}\right]\right]$

16. $\dfrac{d}{dx}\left[\sec(5x) + \cot\left(e^x\right) - 10\ln x\right]$

17. $z = \ln\left(\cos(t)\right) + \sec\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$

18. $z = \ln\left(\tan(t)\right) + \sin\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$

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19. $z = \ln\left(\cot(\theta)\right) + \sec\left(\ln\theta\right) + 7\pi^2$, find $\dfrac{dz}{d\theta}$

20. $z = \ln\left(\cos(\theta)\right) + \sin\left(\ln\theta\right) + 7\pi^2$, find $\dfrac{dz}{d\theta}$

21. If $g(3) = \dfrac{\pi}{2}$, $g'(3) = \dfrac{\pi}{4}$, and $f(x) = x^3 g(x) + g\left(-3\cos\left(\dfrac{\pi}{3}x\right)\right) - e^{\sin(g(x))}$, find $f'(3)$

1.2 Multiple Choice Homework

1. If $y = \sin^{-1}\left(e^{3\theta}\right)$, then $\dfrac{dy}{d\theta} =$

   a) $\dfrac{1}{\sqrt{1 - e^{3\theta}}}$

   b) $\dfrac{1}{\sqrt{1 - e^{6\theta}}}$

   c) $\dfrac{1}{\sqrt{1 - e^{9\theta^2}}}$

   d) $-3e^{3\theta}\cos^{-1}\left(e^{3\theta}\right)$

   e) $\dfrac{3e^{3\theta}}{\sqrt{1 - e^{6\theta}}}$

---

2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

   a) $-\csc(x)\sec^{-2}(\cos(x))$

   b) $-\sin(x)\sec^{-2}(\cos(x))$

   c) $-\cos(x)\csc^{-2}(\cos(x))$

   d) $\dfrac{-\cos(x)}{1 - \sin^2(x)}$

   e) $\dfrac{-\sin(x)}{\cos^2(x) + 1}$

---

3. If $h(x) = \ln\left(x^2\right)\tan^{-1}(x)$, then $h'(1) =$

   a) $\dfrac{\pi}{4}$

   b) $\dfrac{\pi}{4} + 1$

   c) $\dfrac{\pi}{2}$

   d) $\dfrac{\pi}{2} + 1$

   e) $\dfrac{\pi}{2} + 2$

---

4. If $f(t) = t\sqrt{1 - t^2} + \cos^{-1}(t)$, then $f'(t) =$

   a) $\dfrac{t - 2}{2\sqrt{t^2 - 1}}$

   b) $\dfrac{-2t^2}{\sqrt{1 - t^2}}$

   c) $\dfrac{-2t^2 + 2}{\sqrt{1 - t^2}}$

   d) $\dfrac{-1 - t^2}{\sqrt{1 - t^2}}$

   e) $\dfrac{1 - t^2}{\sqrt{1 - t^2}}$

---

5. If h is the function defined by $h(x) = e^{5x} + x + 3$, then $h'(0) =$

   a) 2

   b) 4

   c) 5

   d) 6

   e) 8

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6. Given that $f(x) = 8\sin^2(5x)$, find $f''\left(\dfrac{\pi}{30}\right)$

   a) $40\sqrt{3}$ b) $40\sqrt{2}$ c) 40 d) 200 e) 0

7. If $g(x) = \cos^2(2x)$, then $g'(x)$ is

   a) $2\cos(2x)\sin(2x)$ b) $-4\cos(2x)\sin(2x)$ c) $2\cos(2x)$

   d) $-2\cos(2x)$ e) $4\cos(2x)$

8. If $f(x) = \sin^2(3-x)$, then $f'(0) =$

   a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

   d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

9. If $f(x) = \cos^2(3-x)$, then $f'(0) =$

   a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

   d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

10. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

   a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

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1.3: Trig, Trig Inverse, and Log Rules

Remember:

The Product Rule: $f'(x) = U \cdot \dfrac{dV}{dx} + V \cdot \dfrac{dU}{dx}$

The Quotient Rule: $f'(x) = \dfrac{V \cdot \dfrac{dU}{dx} - U \cdot \dfrac{dV}{dx}}{V^2}$

OBJECTIVES

Find the Derivative of a Product or Quotient of Two Functions.

The Product Rule

Ex 1.3.1: $\dfrac{d}{dx}\left[x^2\sin(x)\right]$

Sol 1.3.1:

$$\frac{d}{dx}\left[x^2\sin(x)\right] = x^2\cdot\cos(x) + \sin(x)\cdot(2x)$$

$$= \boxed{x^2\cos(x) + 2(x)\sin(x)}$$

Ex 1.3.2: $\dfrac{d}{dx}\left[5^x\cos(x)\right]$

Sol 1.3.2:

$$\frac{d}{dx}\left[5^x\cos(x)\right] = 5^x\cdot(-\sin(x)) + \cos(x)\cdot(5^x\ln 5)$$

$$= \boxed{5^x\left(\ln(5)\cos(x) + \sin(x)\right)}$$

The product rule is pretty straightforward. The tricky part is simplifying the algebra.

Ex 1.3.3: If $f(x) = x^2 e^{-\frac{x}{2}}$, find $f'(x)$

Sol 1.3.3:

$$U = x^2, \ \frac{dU}{dx} = 2x$$

$$V = e^{-\frac{x}{2}}, \ \frac{dV}{dx} = e^{-\frac{x}{2}}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$$

$$f'(x) = x^2\cdot\left(-\frac{1}{2}e^{-\frac{x}{2}}\right) + e^{-\frac{x}{2}}\cdot 2x$$

$$= \boxed{xe^{-\frac{x}{2}}\left(-\frac{1}{2}x + 2\right)}$$

Ex 1.3.4: $\dfrac{d}{dx}\left[x\sqrt{1-x^2}\right]$

Sol 1.3.4:

$$U = x, \ \frac{dU}{dx} = 1$$

$$V = \sqrt{1-x^2} = \left(1-x^2\right)^{\frac{1}{2}}, \ \frac{dV}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\cdot(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = x \cdot \left(-\frac{x}{\sqrt{1-x^2}}\right) + \sqrt{1-x^2} \cdot 1$$

$$= \frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}}$$

$$= \boxed{\frac{1-2x^2}{\sqrt{1-x^2}}}$$

Ex 1.3.5: $\dfrac{d}{dx}\left[(2x-3)^8 \left(3x^2-1\right)^7\right]$

Sol 1.3.5:

$$U = (2x-3)^8, \ \frac{dU}{dx} = 8(2x-3)^7 \cdot 2 = 16(2x-3)^7$$

$$V = \left(3x^2-1\right)^7, \ \frac{dV}{dx} = 7\left(3x^2-1\right)^6 \cdot 6x = 42x\left(3x^2-1\right)^6$$

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right] = (2x-3)^8 \cdot 42x\left(3x^2-1\right)^6 + \left(3x^2-1\right)^7 \cdot 16(2x-3)^7$$

This, then, is factorable.

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right] = 42x(2x-3)^8\left(3x^2-1\right)^6 + 16\left(3x^2-1\right)^7 16(2x-3)^7$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left[21x(2x-3) + 8\left(3x^2-1\right)\right]$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left(42x^2 - 63x + 24x^2 - 8\right)$$

$$= \boxed{2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2 - 63x - 8\right)}$$