AP Calculus BC

Saint Ignatius College Preparatory

Textbook Companion

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For the Greater Glory of God.

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Chapter 1:

Review of Derivatives

Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the ”how” of differentiation. We will review all the derivative rules learned last year in Precalculus. In the next two chapters, we will review the ”why.” As a quick reference, here are those rules:

The Power Rule: $\frac{d}{dx}\left[u^n\right] = nu^{n-1}\frac{du}{dx}$

The Product Rule: $\frac{d}{dx}\left[u \cdot v\right] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

The Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Here is a quick review from last year:

Identities: While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x; \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin 2x = 2 \sin x \cos x; \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverses: Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ -\textit{calculator} \ \pm 2\pi n \end{array} \right\} \qquad \sin^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi - \textit{calculator} \ \pm 2\pi n \end{array} \right\}$$

$$\tan^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi + \textit{calculator} \ \pm 2\pi n \end{array} \right\} = \textit{calculator} \ \pm \pi n$$

Logarithm Rules: Here are some logarithm rules which you should recall:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

1.1: The Power and Exponential Rules with the Chain Rule

In Precalculus we developed the idea of the derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the derivative will be explored. In this chapter, we are primarily interested in how to find the derivative and what it is used for.

Derivative $\rightarrow$ Definition: $f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$\rightarrow$ Means: The function that yields the slope of the tangent line.

Numerical Derivative $\rightarrow$ Definition: $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

$\rightarrow$ Means: The numerical value of the slope of the tangent line at $x = a$

Symbols for the Derivative

$\dfrac{dy}{dx}$ = "d - y - d - x" $f'(x)$ = "f prime of x" y' = "y prime"

$\dfrac{d}{dx}$ = "d - d - x" D_x = "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to Find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from Precalculus: The derivative yields the slope of the tangent line. (But there is more to it than that).

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in Precalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln a$$

The difference between these is where the variable is. The Power Rule applies when the variable is in the *base*, while the Exponential Rules apply when the variable is in the *exponent*. The difference between the two Exponential rules is what the base is. $e = 2.718281828459\ldots$, while a is any positive number other than 1.

Ex 1.1.1: Find a) $\dfrac{d}{dx}\left[x^5\right]$ and b) $\dfrac{d}{dx}\left[5^x\right]$.

Sol 1.1.1: The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\dfrac{d}{dx}\left[x^5\right] = \boxed{5x^4}$ b) $\dfrac{d}{dx}\left[5^x\right] = \boxed{5^x \ln 5}$

There are a few other basic rules that we need to remember.

$$\frac{d}{dx}[\text{constant}] = 0$$

$$\frac{d}{dx}\left[cx^n\right] = (cn)x^{n-1}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

These rules allow us to easily differentiate a polynomial term by term.

Ex 1.1.2: $y = 3x^2 + 5x + 1$; find $\dfrac{dy}{dx}$.

Sol 1.1.2:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[3x^2+5x+1\right]\\ &= (3\cdot 2)x^{2-1}+(5\cdot 1)x^{1-1}+0\\ &= \boxed{6x+5}\end{aligned}$$

Ex 1.1.3: $f(x)=x^2+4x-3+e^x$; find $f'(x)$.

Sol 1.1.3:

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[x^2+4x-3+e^x\right]\\ &= (1\cdot 2)x^{2-1}+(4\cdot 1)x^{1-1}-0+e^x\\ &= \boxed{2x+4+e^x}\end{aligned}$$

Ex 1.1.4: $y=\sqrt{x^3}+\dfrac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4$; find $\dfrac{dy}{dx}$.

Sol 1.1.4:

$$\begin{aligned}y &= \sqrt{x^3}+\frac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4\\ &= x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4\\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4\right]\\ &= \left(1\cdot\frac{3}{2}\right)x^{\frac{3}{2}-1}+\left(4\cdot-\frac{1}{2}\right)x^{-\frac{1}{2}-1}-\left(1\cdot\frac{3}{4}\right)x^{\frac{3}{4}-1}+0\\ &= \boxed{\frac{3}{2}x^{\frac{1}{2}}-2x^{-\frac{3}{2}}-\frac{3}{4}x^{-\frac{1}{4}}}\end{aligned}$$

Note in Ex 1.1.4 that e^4 is a constant. Therefore, its derivative is 0.

As we have seen, when the variable was in the exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right] = \frac{d}{dx}\left[e^{x^2\ln(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Ex 1.1.5: If $y = (x^2+1)(x^3-4x)$, find $\frac{dy}{dx}$.

Sol 1.1.5:

$$\begin{aligned} y &= (x^2+1)(x^3-4x) \\ &= x^5 - 4x^3 + x^3 - 4x \\ &= x^5 - 3x^3 - 4x \\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^5 - 3x^3 - 4x\right] \\ &= \boxed{5x^4 - 9x^2 - 4} \end{aligned}$$

Ex 1.1.6: If $y = \frac{x^2-4x+6}{\sqrt[3]{x}}$, find $\frac{dy}{dx}$.

Sol 1.1.6:

$$\begin{aligned} y &= \frac{x^2-4x+6}{\sqrt[3]{x}} \\ &= \frac{x^2-4x+6}{x^{\frac{1}{3}}} \\ &= x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}\right]$$

$$= \boxed{\frac{5}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{-\frac{1}{3}} - 2x^{-\frac{4}{3}}}$$

The Chain Rule

Composite Function → Definition: A function made of two other functions, one within the other.

→ For example, $y = \sqrt{16x - x^3}$, $y = \sin\left(x^3\right)$. $y = \cos^3(x)$, and $y = \left(x^2 + 2x - 5\right)^3$. The general symbol is $f(g(x))$.

Ex 1.1.7: Given $f(x) = \cos^{-1}(x)$, $g(x) = x^2 - 1$, and $h(x) = \sqrt{1 + x^2}$, find a) $f\left(g\left(\sqrt{2}\right)\right)$, b) $h(g(1))$, and c) $f(h(g(1)))$.

Sol 1.1.7:

(a) $g\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 1 = 1,$ so $f\left(g\left(\sqrt{2}\right)\right) = f(1) = \cos^{-1}(1) = \boxed{0}$.

(b) $g(1) = 0,$ so $h(g(1)) = h(0) = \sqrt{1 + 0^2} = \boxed{1}$

(c) $g(1) = 0$ and $h(g(1)) = h(0) = \sqrt{1 + 0^2} = 1,$ so $f(h(g(1))) = \cos^{-1}(1) = \boxed{0}$

So. How do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. The result is two open dolls adjacent to each other.

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If you think of the inside function (the $g(x)$) as equaling u, we could write the Chain Rule like this:

$$\frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$

This is the way that most derivatives are written with the Chain Rule.

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other rules, as seen in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot \frac{du}{dx}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln a\right) \cdot \frac{du}{dx}$$

(where u is a function of x)

Ex 1.1.8: $\frac{d}{dx}\left[(4x^2 - 2x - 1)^{10}\right]$

Sol 1.1.8:

$$u = 4x^2 - 2x - 1 \text{ and } f(u) = u^{10}$$

$$\begin{aligned}
\frac{d}{dx}\left[f(u)\right] &= f'(u) \cdot \frac{du}{dx} \\
&= 10u^9 \cdot (8x - 2) \\
&= \boxed{10\left(4x^2 - 2x - 1\right)^9 (8x - 2)}
\end{aligned}$$

Ex 1.1.9: $\dfrac{d}{dx}\left[e^{4x^2}\right]$

Sol 1.1.9:

$$\begin{aligned} u &= 4x^2 \\ \frac{d}{dx}\left[e^u\right] &= e^u \cdot \frac{du}{dx} \\ &= e^{4x^2} \cdot 8x \\ &= \boxed{8xe^{4x^2}} \end{aligned}$$

Ex 1.1.10: If $y = \sqrt{16 - x^3}$, find $\dfrac{dy}{dx}$.

Sol 1.1.10:

$$\begin{aligned} &u = 16 - x^3 \text{ and } f(u) = \sqrt{u} \\ &\frac{dy}{dx} = \frac{d}{dx}\left[f(u)\right] = f'(u) \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot -3x^2 \\ &= \boxed{\frac{-3x^2}{2\sqrt{16 - x^3}}} \end{aligned}$$

Ex 1.1.11: $\dfrac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5 + 7}\right]$

Sol 1.1.11:

$$u = x^2 + 1,\ g(u) = u^5 + 7,\ \text{and } f(g(u)) = \sqrt{g(u)}$$

$$
\begin{aligned}
\frac{d}{dx}\left[f(g(u))\right] &= f'(g(u)) \cdot g'(u) \cdot \frac{du}{dx} \\
&= \frac{1}{2\sqrt{g\left(u\right)}} \cdot 5u^4 \cdot 2x \\
&= \frac{5\left(x^2+1\right)\left(2x\right)}{2\sqrt{\left(x^2+1\right)^5+7}} = \boxed{\frac{5x\left(x^2+1\right)}{\sqrt{\left(x^2+1\right)^5+7}}}
\end{aligned}
$$

1.1 Free Response Homework

Find the derivatives of the given functions. Simplify where possible.

1. $f(x) = x^2 + 3x - 4$
2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$
3. $y = x^{-\frac{2}{3}}$
4. $y = 5e^x + 3$
5. $v(r) = \frac{4}{3}\pi r^3$
6. $g(x) = x^2 + \frac{1}{x^2}$
7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$
8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$
9. $z = \frac{A}{y^{10}} + Be^y$
10. $y = e^{x+1} + 1$

Complete the following.

11. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$
12. $\frac{d}{dx}\left[x^6 - 3\sqrt[6]{x^7} + 5^x - \frac{1}{\sqrt[3]{x^5}} + \frac{1}{8x}\right]$
13. $\frac{d}{dx}\left[x^4 - 14\sqrt[7]{x^9}\right] + 8^x - \frac{1}{\sqrt[3]{x^7}} + \frac{1}{8x}$
14. $\frac{d}{dx}\left[(x-1)\sqrt{x}\right]$
15. $\frac{d}{dz}\left[\left(z^2 - 4\right)\sqrt{z^3}\right]$
16. $\frac{d}{dx}\left[\left(x^2 - 4x + 3\right)\sqrt{x^5}\right]$
17. $\frac{d}{dt}\left[\left(4t^2 + 1\right)\left(3t^3 + 7\right)\right]$
18. $\frac{d}{dx}\left[\left(x^3 + 4x - \pi\right)^{-7}\right]$
19. $\frac{d}{dx}\left[\sqrt{3x^2 - 4x + 9}\right]$
20. $\frac{d}{dx}\left[\sqrt[7]{x^3 - 2x}\right]$
21. $\frac{d}{dy}\left[\frac{4y^3 - 2y^2 - 5y}{\sqrt{y}}\right]$
22. $\frac{d}{dv}\left[\frac{v^2 - 4v + 7}{2\sqrt{v}}\right]$
23. $\frac{d}{dw}\left[\frac{7w^2 - 4w + 1}{5w^3}\right]$
24. $\frac{d}{dw}\left[\frac{5w^2 - 3w - 4}{7w^2}\right]$
25. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$
26. $f(x) = \sqrt[5]{\left(\frac{1}{x} + 2x + e^x\right)^3}$, find $f'(x)$
27. $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$
28. $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$
29. $g(2) = 3,\ g'(2) = -4,\ f(x) = e^{g(x)}$, find $f'(2)$
30. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$

31. $f(x)=\sqrt{4-\frac{4}{9}x^2}$, find $f'\left(\sqrt{5}\right)$

32. $f(x)=e^{\sqrt{9-x^2}}$, find $f'(x)$

33. $v(t)=\sqrt{\left(\frac{E(t)}{3}+3t\right)^{\frac{3}{7}}-4}$, find $v'(t)$

34. $v(t)=\sqrt[3]{\left(\frac{C(t)}{7}+4t^2\right)^{\frac{5}{7}}-1}$, find $v'(t)$

1.1 Multiple Choice Homework

1. If $f(x)=x^{\frac{3}{2}}$, then $f'(4)=$

a) -6 b) -3 c) 3 d) 6 e) 8

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2. The derivative of $\sqrt{x}-\frac{1}{x\sqrt[3]{x}}$

a) $\frac{1}{2}x^{-\frac{1}{2}}-x^{-\frac{4}{3}}$ b) $\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{7}{3}}$ c) $\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

d) $-\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{7}{3}}$ e) $-\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

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3. Given $f(x)=\frac{1}{2x}+\frac{1}{x^2}$, find $f'(x)$

a) $-\frac{1}{2x^2}-\frac{2}{x^3}$ b) $-\frac{2}{x^2}-\frac{2}{x^3}$ c) $\frac{2}{x^2}-\frac{2}{x^3}$

d) $-\frac{1}{2x^2}+\frac{2}{x^3}$ e) $\frac{1}{2x^2}-\frac{2}{x^3}$

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4. If $f(x)=e^{5x^2}+x^4$, then $f'(1)=$

a) e^5+1 b) $5e^4+4$ c) $5e^5+1$ d) $10e+4$ e) $10e^5+4$

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5. If h is the function defined by $h(x)=e^{5x}+x+3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

---

6. If $y = \left(x^4+4\right)^2$, then $\dfrac{dy}{dx} =$

a) $2\left(x^4+4\right)$

b) $\left(4x^3\right)^2$

c) $2\left(4x^3+4\right)$

d) $4x^3\left(x^4+4\right)$

e) $8x^3\left(x^4+4\right)$

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7. If $h(x) = [f(x)]^2\, g(x)$ and $g(x) = 3$, then $h'(x) =$

a) $2f'(x)g'(x)$

b) $6f'(x)f(x)$

c) $g'(x)\,[f(x)]^2 + 2f(x)f'(x)g(x)$

d) $2f'(x)g(x) + g'(x)\,[f(x)]^2$

e) 0

---

8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[\sqrt{e^x+3}\right] = \dfrac{e^x}{2\sqrt{e^x+3}}$

II. $\dfrac{d}{dx}\left[5^{3x^2}\right] = 6x\ln(5)\left(5^{3x^2}\right)$

III. $\dfrac{d}{dx}\left[6x^3 - \pi + \sqrt[3]{x^8} - \dfrac{2}{x^3}\right] = 18x^2 + \dfrac{8}{3}\sqrt[3]{x^5} + \dfrac{6}{x^4}$

a) I only

b) II only

c) I and III only

d) I and III only

e) I, II, and III

1.2: Trig, Trig Inverse, and Log Rules

Trigonometric → Definition: A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure.

→ Means: An equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic → Definition: The inverse of an exponential function.

→ Means: An equation with log or ln in it.

Trig Derivative Rules

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

Log Derivative Rules

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx} \qquad \frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

Note that all these rules are expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives Involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1.2.1: $\dfrac{d}{dx}\left[\sin^3(x)\right]$

Sol 1.2.1:

$$\frac{d}{dx}\left[\sin^3(x)\right] = \boxed{3\sin^2(x)\cos(x)}$$

Ex 1.2.2: $\dfrac{d}{dx}\left[\sin\left(x^3\right)\right]$

Sol 1.2.2:

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \boxed{3x^2\cos\left(x^3\right)}$$

Ex 1.2.3: $\dfrac{d}{dx}\left[\ln\left(4x^3\right)\right]$

Sol 1.2.3:

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(4x^3\right)\right] &= \frac{1}{4x^3}\cdot 12x^2 \\ &= \boxed{\frac{3}{x}}\end{aligned}$$

We could have also simplified algebraically before taking the derivative:

$$\begin{aligned}\ln\left(4x^3\right) &= \ln 4 + \ln x^3 \\ &= \ln 4 + 3\ln x \\ \frac{d}{dx}\left[\ln 4 + 3\ln x\right] &= 0 + 3\cdot\frac{1}{x} \\ &= \boxed{\frac{3}{x}}\end{aligned}$$

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 1.2.4: $\dfrac{d}{dx}\left[\sec^5\left(3x^4\right)\right]$

Sol 1.2.4:

$$\begin{aligned}\frac{d}{dx}\left[\sec^5\left(3x^4\right)\right] &= 5\sec^4\left(3x^4\right)\cdot\sec\left(3x^4\right)\tan\left(3x^4\right)\cdot\left(12x^3\right)\\ &= \boxed{60x^3\sec^5\left(3x^4\right)\tan\left(3x^4\right)}\end{aligned}$$

Ex 1.2.5: $\dfrac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right)$

Sol 1.2.5:

$$\begin{aligned}\frac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right) &= \frac{1}{\cos\left(\sqrt{x}\right)}\cdot\left(-\sin\left(\sqrt{x}\right)\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= -\tan\left(\sqrt{x}\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= \boxed{\frac{-\tan\left(\sqrt{x}\right)}{2\sqrt{x}}}\end{aligned}$$

General inverses are not all that interesting. We are more interested in particular *transcendental* inverse functions, like the natural log. Another particular kind of inverse function that bears more study is the trig inverse function. Interestingly, as with the log functions, the derivatives of these transcendental functions become algebraic functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1}u\right]=\left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1}u\right]=\left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}u\right]=\left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1}u\right]=\left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right]=\left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1}u\right]=\left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Ex 1.2.6: $\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right]$

Sol 1.2.6:

$$\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right] = \frac{1}{(3x^4)^2+1}\cdot\left(12x^3\right)$$

$$= \boxed{\frac{12x^3}{9x^8+1}}$$

Ex 1.2.7: $\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right]$

Sol 1.2.7:

$$\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right] = \frac{1}{|x^2|\sqrt{(x^2)^2-1}}\cdot 2x$$

$$= \frac{2x}{(x^2)\sqrt{(x^2)^2-1}}$$

$$= \boxed{\frac{2}{x\sqrt{x^4-1}}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] - \frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 1.2.8: If $f(x) = x^2+2x+3,\ g(x) = f^{-1}(x),$ and $g(1) = 2;$ find $g'(1)$.

Sol 1.2.8:

$$f'(x) = 2x+2 \ \therefore\ f'\left(g(x)\right) = 2\left(g(x)\right)+2$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{f'\left(g(x)\right)}$$

$$g'(1) = \frac{1}{f'\left(g(1)\right)} = 2\left(g(1)\right) + 2 = \boxed{6}$$

1.2 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin(4x)$
2. $y = 4\sec\left(x^5\right)$
3. $f(t) = \sqrt[3]{1 + \tan t}$
4. $f(\theta) = \ln(\cos(\theta))$
5. $y = a^3 + \cos^3(x)$
6. $y = \cos\left(a^3 + x^3\right)$
7. $f(x) = \cos(\ln x)$
8. $f(x) = \sqrt[5]{\ln x}$
9. $f(x) = \log_{10}(2 + \sin(x))$
10. $f(x) = \log_2(1 - 3x)$
11. $y = \sin^{-1}(e^x)$
12. $y = \tan^{-1}(\sqrt{x})$

Complete the following.

13. $\dfrac{d}{dx}\left[\sin^{-1}\left(e^{3x}\right)\right]$
14. $\dfrac{d}{dx}\left[\cot^{-1}\left(e^{2x}\right)\right]$
15. $\dfrac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right]$
16. $\dfrac{d}{dx}\left[\cot^{-1}\left(\dfrac{1}{x}\right) - \tan^{-1}(x)\right]$
17. $\dfrac{d}{dx}\left[3e^{x^2+2x}\right]$
18. $\dfrac{d}{dx}\left[3\cos\left(x^2 + 2x\right)\right]$
19. $\dfrac{d}{dx}\left[\sqrt[3]{16 + x^3}\right]$
20. $\dfrac{d}{dx}\left[\sec^{-1}\left(2x^2\right)\right]$
21. $\dfrac{d}{dx}\left[5e^{\tan(7x)}\right]$
22. $\dfrac{d}{dx}\left[\sqrt{\cos\left(1 - x^2\right)}\right]$
23. $\dfrac{d}{dx}\left[\ln^3\left(x^2 + 1\right)\right]$
24. $\dfrac{d}{dx}\left[\ln\sin\left(x^3\right)\right]$
25. $\dfrac{d}{dx}\left[\ln(\sec(x))\right]$
26. $\dfrac{d}{dx}\left[\cos\left(x^2\right)\right]$
27. $f(x) = \ln\left(x^2 + 3\right)$, find $f'(x)$
28. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g'(x)$
29. $h(x) = \sqrt{x^2 + 5}$, find $h'(x)$
30. $F(x) = \sqrt[3]{3x^2 - 6x + 1}$, find $F'(x)$
31. $y = \sin^{-1}(\cos(x))$, find y'
32. $y = \sin\left(\cos^{-1}(x)\right)$, find y'
33. $y = \tan^2(3\theta)$, find y'
34. $y = \cot^7(\sin(\theta))$, find y'

35. $y = \sin^{-1}\left(\sqrt{2}(x)\right)$, find y'

36. $y = \sin^{-1}(2x+1)$, find y'

1.2 Free Response Homework Set B

Find the derivatives of the given functions. Simplify where possible.

1. $y = \cos^{-1}\left(e^{3z}\right)$
2. $y = \tan^{-1}\sqrt{x^2-1}$
3. $y = \sec^{-1}(4x) + \csc^{-1}(4x)$
4. $f(x) = \ln\left(\tan^{-1}(5x)\right)$
5. $g(w) = \sin^{-1}(5w) + \cos^{-1}(5w)$
6. $f(t) = \sec^{-1}\sqrt{9+t^2}$

Complete the following.

7. $\dfrac{d}{d\theta}\left[e^{\csc(\theta)} + \ln\left(\cot\left(\theta^2\right)\right) - \sec(\theta)\right]$
8. $\dfrac{d}{dx}\left[\ln\left(\sec\left(x^3+5\ln x+7\right)^3\right)\right]$
9. $\dfrac{d}{dx}\left[\ln\left(\tan\left(x^2+5e^x+7\right)^3\right)\right]$
10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$
11. $\dfrac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$
12. $\dfrac{d}{dt}\left[\sin^5\left(\ln\left(7t+3\right)\right)\right]$
13. $\dfrac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$
14. $\dfrac{d}{dx}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$
15. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sqrt{9x-27x^2+\dfrac{5}{x^3}}\right]\right]$
16. $\dfrac{d}{dx}\left[\sec(5x) + \cot\left(e^x\right) - 10\ln x\right]$
17. $z = \ln(\cos(t)) + \sec\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$
18. $z = \ln(\tan(t)) + \sin\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$
19. $z = \ln(\cot(\theta)) + \sec(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$
20. $z = \ln(\cos(\theta)) + \sin(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$
21. If $g(3) = \dfrac{\pi}{2}$, $g'(3) = \dfrac{\pi}{4}$, and $f(x) = x^3 g(x) + g\left(-3\cos\left(\dfrac{\pi}{3}x\right)\right) - e^{\sin(g(x))}$, find $f'(3)$

1.2 Multiple Choice Homework

1. If $y = \sin^{-1}\left(e^{3\theta}\right)$, then $\dfrac{dy}{d\theta} =$

a) $\dfrac{1}{\sqrt{1-e^{3\theta}}}$ b) $\dfrac{1}{\sqrt{1-e^{6\theta}}}$ c) $\dfrac{1}{\sqrt{1-e^{9\theta^2}}}$

d) $-3e^{3\theta}\cos^{-1}\left(e^{3\theta}\right)$ e) $\dfrac{3e^{3\theta}}{\sqrt{1-e^{6\theta}}}$

---

2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

a) $-\csc(x)\sec^{-2}(\cos(x))$ b) $-\sin(x)\sec^{-2}(\cos(x))$ c) $-\cos(x)\csc^{-2}(\cos(x))$

d) $\dfrac{-\cos(x)}{1-\sin^2(x)}$ e) $\dfrac{-\sin(x)}{\cos^2(x)+1}$

---

3. If $h(x) = \ln\left(x^2\right)\tan^{-1}(x)$, then $h'(1) =$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{4}+1$ c) $\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2}+1$ e) $\dfrac{\pi}{2}+2$

---

4. If $f(t) = t\sqrt{1-t^2}+\cos^{-1}(t)$, then $f'(t) =$

a) $\dfrac{t-2}{2\sqrt{t^2-1}}$ b) $\dfrac{-2t^2}{\sqrt{1-t^2}}$ c) $\dfrac{-2t^2+2}{\sqrt{1-t^2}}$ d) $\dfrac{-1-t^2}{\sqrt{1-t^2}}$ e) $\dfrac{1-t^2}{\sqrt{1-t^2}}$

---

5. If h is the function defined by $h(x) = e^{5x}+x+3$, then $h'(0) =$

a) 2 b) 4 c) 5 d) 6 e) 8

---

6. Given that $f(x) = 8\sin^2(5x)$, find $f''\left(\dfrac{\pi}{30}\right)$

a) $40\sqrt{3}$ b) $40\sqrt{2}$ c) 40 d) 200 e) 0

---

7. If $g(x) = \cos^2(2x)$, then $g'(x)$ is

a) $2\cos(2x)\sin(2x)$ b) $-4\cos(2x)\sin(2x)$ c) $2\cos(2x)$

d) $-2\cos(2x)$ e) $4\cos(2x)$

---

8. If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

9. If $f(x) = \cos^2(3 - x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

10. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

1.3: Trig, Trig Inverse, and Log Rules

Remember:

$$\text{The Product Rule: } f'(x) = U \cdot \frac{dV}{dx} + V \cdot \frac{dU}{dx}$$

$$\text{The Quotient Rule: } f'(x) = \frac{V \cdot \frac{dU}{dx} - U \cdot \frac{dV}{dx}}{V^2}$$

OBJECTIVES

Find the Derivative of a Product or Quotient of Two Functions.

The Product Rule

Ex 1.3.1: $\frac{d}{dx}\left[x^2 \sin(x)\right]$

Sol 1.3.1:

$$\begin{aligned}\frac{d}{dx}\left[x^2 \sin(x)\right] &= x^2 \cdot \cos(x) + \sin(x) \cdot (2x) \\ &= \boxed{x^2 \cos(x) + 2(x)\sin(x)}\end{aligned}$$

Ex 1.3.2: $\dfrac{d}{dx}\left[5^x \cos(x)\right]$

Sol 1.3.2:

$$\frac{d}{dx}\left[5^x \cos(x)\right] = 5^x \cdot (-\sin(x)) + \cos(x) \cdot (5^x \ln 5)$$

$$= \boxed{5^x \left(\ln(5)\cos(x) + \sin(x)\right)}$$

The product rule is pretty straightforward. The tricky part is simplifying the algebra.

Ex 1.3.3: If $f(x) = x^2 e^{-\frac{x}{2}}$, find $f'(x)$

Sol 1.3.3:

$$U = x^2,\ \frac{dU}{dx} = 2x$$

$$V = e^{-\frac{x}{2}},\ \frac{dV}{dx} = e^{-\frac{x}{2}} \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$$

$$f'(x) = x^2 \cdot \left(-\frac{1}{2}e^{-\frac{x}{2}}\right) + e^{-\frac{x}{2}} \cdot 2x$$

$$= \boxed{xe^{-\frac{x}{2}}\left(-\frac{1}{2}x + 2\right)}$$

Ex 1.3.4: $\dfrac{d}{dx}\left[x\sqrt{1-x^2}\right]$

Sol 1.3.4:

$$U = x,\ \frac{dU}{dx} = 1$$

$$V = \sqrt{1-x^2} = \left(1-x^2\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = x\cdot\left(-\frac{x}{\sqrt{1-x^2}}\right) + \sqrt{1-x^2}\cdot 1$$

$$= \frac{-x^2+(1-x^2)}{\sqrt{1-x^2}}$$

$$= \boxed{\frac{1-2x^2}{\sqrt{1-x^2}}}$$

Ex 1.3.5: $\dfrac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]$

Sol 1.3.5:

$$U = (2x-3)^8,\ \frac{dU}{dx} = 8(2x-3)^7\cdot 2 = 16(2x-3)^7$$

$$V = \left(3x^2-1\right)^7,\ \frac{dV}{dx} = 7\left(3x^2-1\right)^6\cdot 6x = 42x\left(3x^2-1\right)^6$$

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right] = (2x-3)^8\cdot 42x\left(3x^2-1\right)^6 + \left(3x^2-1\right)^7\cdot 16(2x-3)^7$$

This, then, is factorable.

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right] = 42x(2x-3)^8\left(3x^2-1\right)^6 + 16\left(3x^2-1\right)^7 16(2x-3)^7$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left(21x(2x-3)+8\left(3x^2-1\right)\right)$$

$$= 2(2x-3)^7\left(3x^2-1\right)^6\left(42x^2-63x+24x^2-8\right)$$

$$= \boxed{2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2-63x-8\right)}$$

Remember that in Section 1.1 we said that we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the exponent. We can now address that situation.

Ex 1.3.6: $\dfrac{d}{dx}\left[(\cos(x))^{x^2}\right]$

Sol 1.3.6:

$$\begin{aligned}\frac{d}{dx}\left[(\cos(x))^{x^2}\right] &= \frac{d}{dx}\left[e^{x^2\ln(\cos(x))}\right] \\ &= e^{x^2\ln(\cos(x))}\cdot\left(x^2\cdot\frac{1}{\cos(x)}\cdot -(\sin(x))+\ln(\cos(x))\cdot 2x\right) \\ &= \boxed{(\cos(x))^{x^2}\left(2x\ln(\cos(x))-x^2\tan(x)\right)}\end{aligned}$$

The Quotient Rule

Ex 1.3.7: $\frac{d}{dx}\left[\frac{6x}{x^2+4}\right]$

Sol 1.3.7:

$$U = 6x,\ \frac{dU}{dx} = 6$$

$$V = x^2+4,\ \frac{dV}{dx} = 2x$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{6x}{x^2+4}\right] &= \frac{(x^2+4)\cdot 6 - 6x\cdot 2x}{(x^2+4)^2} \\ &= \frac{6x^2+24-12x^2}{(x^2+4)} \\ &= \boxed{\frac{24-6x^2}{(x^2+4)}}\end{aligned}$$

Ex 1.3.8: $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

Sol 1.3.8:

$$U = x^2+2x-3,\ \frac{dU}{dx} = 2x+2$$

$$V = x-4,\ \frac{dV}{dx} = 1$$

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2+2x-3}{x-4}\right] &= \frac{(x-4)\cdot(2x+2)-(x^2+2x-3)\cdot 1}{(x-4)^2} \\
&= \frac{2x^2-6x-8-x^2-2x+3}{(x-4)^2} \\
&= \boxed{\frac{x^2-8x-5}{(x-4)^2}}
\end{aligned}$$

Ex 1.3.9: $\dfrac{d}{dx}\left[\dfrac{x^2-4x+3}{2x^2-5x-3}\right]$

Sol 1.3.9: Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\begin{aligned}
\frac{d}{dx}\left[\frac{x^2-4x+3}{2x^2-5x-3}\right] &= \frac{d}{dx}\left[\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right] \\
&= \frac{d}{dx}\left[\frac{x-1}{2x+1}\right]
\end{aligned}$$

$$U = x-1,\ \frac{dU}{dx} = 1$$

$$V = 2x+1,\ \frac{dV}{dx} = 2$$

$$\frac{d}{dx}\left[\frac{x-1}{2x+1}\right] = \frac{(2x+1)\cdot 1-(x-1)\cdot 2}{(2x+1)^2}$$

$$= \boxed{\frac{3}{(2x+1)^2}}$$

Ex 1.3.10: $\dfrac{d}{dx}\left[\dfrac{\cot(3x)}{x^2+1}\right]$

Sol 1.3.10:

$$U = \cot(3x),\ \frac{dU}{dx} = -\csc^2(3x)\cdot 3 = -3\csc^2(3x)$$

$$V = x^2+1,\ \frac{dV}{dx} = 2x$$

$$\frac{d}{dx}\left[\frac{\cot(3x)}{x^2+1}\right] = \frac{(x^2+1)\cdot\left(-3\csc^2(3x)\right)-\cot(3x)\cdot 2x}{\left(x^2+1\right)^2}$$

$$= \frac{-3x^2\csc^2(3x)-3\csc^2(3x)-2x\cot(3x)}{\left(x^2+1\right)^2}$$

$$= \boxed{-\frac{\csc^2(3x)\left(3x^2+3\right)+2x\cot(3x)}{\left(x^2+1\right)^2}}$$

As with the Product Rule, the difficulty with the Quotient Rule arises from the algebra needed to simplify our answer.

Ex 1.3.11: If $y = \dfrac{4x}{\sqrt{x^2+4}}$, find $\dfrac{dy}{dx}$

Sol 1.3.11:

$$U = 4x,\ \frac{dU}{dx} = 4$$

$$V = \sqrt{x^2+4} = \left(x^2+4\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}\cdot 2x = \frac{2x}{2\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+4}\cdot 4 - 4x\cdot\dfrac{2x}{2\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{\dfrac{4\left(x^2+4\right)}{\sqrt{x^2+4}} - \dfrac{4x^2}{\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{4x^2+16-4x^2}{(x^2+4)^{\frac{3}{2}}}$$

$$= \boxed{\frac{16}{(x^2+4)^{\frac{3}{2}}}}$$

Ex 1.3.12: Find the equation of the tangent line to $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ at $x = -\sqrt{7}$.

Sol 1.3.12: As we recall, for the equation of a line, we need a point and a slope.

$$\text{The point: } f\left(-\sqrt{7}\right) = \frac{-\sqrt{7}}{\sqrt{\left(-\sqrt{7}\right)^2+9}}$$

$$= -\frac{\sqrt{7}}{4} \to \left(-\sqrt{7}, -\frac{\sqrt{7}}{4}\right)$$

The slope is the derivative at the given x-value:

$$U = x,\ \frac{dU}{dx} = 1$$

$$V = \sqrt{x^2+9} = \left(x^2+9\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(x^2+9\right)^{-\frac{1}{2}}\cdot 2x = \frac{2x}{2\sqrt{x^2+9}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+9}\cdot 1 - x\cdot\dfrac{2x}{2\sqrt{x^2+9}}}{x^2+9}$$

Rather than simplify the algebra, we can find the slope by substituting $x = -\sqrt{7}$:

$$\left.\frac{dy}{dx}\right|_{x=-\sqrt{7}} = \frac{\sqrt{\left(-\sqrt{7}\right)^2+9}\cdot 1 - \left(-\sqrt{7}\right)\cdot\dfrac{2\left(-\sqrt{7}\right)}{2\sqrt{\left(-\sqrt{7}\right)^2+9}}}{\left(-\sqrt{7}\right)^2+9}$$

$$= \frac{4 - \frac{7}{4}}{16}$$

$$= \frac{9}{64}$$

The tangent line equation is therefore:

$$\boxed{y + \frac{\sqrt{7}}{4} = \frac{9}{64}\left(x + \sqrt{7}\right)}$$

1.3 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = t^3 \cos(t)$

2. $y = (2x-5)^4 \left(8x^2-5\right)^{-3}$

3. $y = \dfrac{\tan(x)-1}{\sec(x)}$

4. $y = \dfrac{\sin(x)}{x^2}$

5. $y = xe^{-x^2}$

6. $y = \dfrac{r}{\sqrt{r^2+1}}$

7. $y = e^{x\cos(x)}$

8. $y = e^{-5x}\cos(3x)$

9. $y = x\sin\left(\dfrac{1}{x}\right)$

10. $y = \ln\left(e^{-x} + xe^{-x}\right)$

11. $y = \dfrac{\sec^{-1}(x)}{x}$

12. $y = \dfrac{\sin(x)}{x^2}$

13. $y = \left(1+x^2\right)\tan^{-1}(x)$

14. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

15. $f(x) = x\sqrt{\ln x}$

16. $g(x) = (1+4x)^5\left(3+x-x^2\right)^8$

17. $f(x) = x\cos^{-1}(x) - \sqrt{1-x^2}$

18. $g(x) = \cos^{-1}(x) + x\sqrt{1-x^2}$

Complete the following.

19. $\dfrac{d}{dx}\left[\dfrac{3x^2+4x-3}{x^2-9}\right]$

20. $\dfrac{d}{dx}\left[\dfrac{x^3-2x^2-5x+6}{x+2}\right]$

21. $\dfrac{d}{dx}\left[\dfrac{x^5-12x^3-19x}{3x^3}\right]$

22. $\dfrac{d}{dx}\left[\dfrac{3x+3}{x^3+1}\right]$

23. $\dfrac{d}{dx}\left[\dfrac{x-4}{x^2-9x+20}\right]$

24. $\dfrac{d}{dx}\left[\dfrac{\tan(x)+5}{\sin(x)}\right]$

25. $\dfrac{d}{dx}\left[\dfrac{\sin(x)}{1-\cos(x)}\right]$

26. $\dfrac{d}{dx}\left[\dfrac{x^2}{\cos(x)}\right]$

27. $y = \dfrac{x^2-3}{x^2-4}$, find $\dfrac{dy}{dx}$

28. $f(x) = \dfrac{x^2+2x-8}{x^2-x-3}$, find $f'(x)$

29. $y = \dfrac{x^2+2x-3}{x-4}$, find y'

30. $f(x) = \dfrac{x}{\ln x}$, find $f'(x)$

31. $h(t) = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$, find $h'(t)$

32. $y = \dfrac{\tan(x)}{\cos(x)-3}$, find $\dfrac{dy}{dx}$

33. $f(x) = \left(x\sin(2x) + \tan^4\left(x^7\right)\right)^5$, find $f'(x)$

34. $f(x) = e^x - x^2 \arctan x$, find $f'(x)$

35. $f(x) = \dfrac{\tan(x)}{\tan(x)+1}$, find $f'\left(\dfrac{\pi}{4}\right)$

36. $y = x^2\sqrt{5-x^2}$, find $y'(1)$

1.3 Free Response Homework Set B

Complete the following.

1. Find the first derivative for the following function: $x(t) = e^{t^2}\sin\left(t^2 - 5t^4\right)$

2. Find the first derivative for the following function: $x(t) = e^{5t}\tan\left(3t^4\right)$

3. Find the first derivative for the following function: $y = \dfrac{x^2+2x-15}{x-3}$

4. Find the first derivative for the following function: $x(t) = e^t\left(t^2 - 5t^4\right)$

5. $\dfrac{d}{dx}\left[\dfrac{e^x + 7x^2 + 5}{\sin\left(x^3\right)}\right]$

6. $\dfrac{d}{dx}\left[e^{\sin(x)}\ln\left(\cot\left(e^x\right)\right)\right]$

7. $\dfrac{d}{dx}\left[x^2\sin\left(x^2\right) + \dfrac{x+1}{\ln x}\right]$

8. $\dfrac{d}{dx}\left[x^2\cos\left(x^2\right) + \dfrac{e^x}{x}\right]$

9. $\dfrac{d}{dx}\left[x^5\ln(5x+4) + \dfrac{x}{\ln x}\right]$

10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(x^2-3\right)}{e^{-5x}}\right]$

11. $\dfrac{d}{dx}\left[e^{x^2}\cos(x)\right]$

12. $\dfrac{d}{dx}\left[\dfrac{1+\tan(x)}{\ln(4x)}\right]$

13. $\dfrac{d}{dx}\left[\sin(t)\tan(t)\right]$

14. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{\csc(x)}\right]$

15. $\dfrac{d}{dx}\left[e^{5x^4}\ln\left(\sin(x)\right)\right]$

16. $\dfrac{d}{dx}\left[5x\sin(x) + e^{2x} - \ln\left(3x^2+1\right) + \dfrac{x}{x^2+1}\right]$

17. $\dfrac{d}{dx}\tan\left(e^x\right)\left(x^4 - 5x^3 + x\right)$

18. $\dfrac{d}{dx}\left[\dfrac{5x+2}{\ln(3x+7)}\right]$

19. $\dfrac{d}{dx}\left[\dfrac{x^5-12c^3-19c}{3c^3}\right]$

20. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sin^2(4x+2)\right]\right]$

21. $g(z)=\left(\dfrac{e^{5z}}{1+\ln z}\right)^{118}$, find $g'(z)$

22. $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$, find $g'(t)$

23. $y=\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$, find y'

24. $f(x)=x^2\arccos(x)$, find $f'(x)$

25. $y=\ln\left(u^2+1\right)-u\cot^{-1}(u)$, find $\dfrac{dy}{du}$

26. $y=\cos^{-1}\left(\dfrac{x-1}{x+1}\right)$

27. $f(t)=c\sin^{-1}\left(\dfrac{t}{c}\right)-\sqrt{c^2-t^2}$, find $f'(t)$

28. $y=4\sin^{-1}\left(\dfrac{1}{2}x\right)+x\sqrt{4-x^2}$

29. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=(h(x))^4+x\ln(h(x))$

1.3 Multiple Choice Homework

1. If $y=x^2\cos(2x)$, then $\dfrac{dy}{dx}=$

a) $-2x\sin(2x)$ b) $-4x\sin(2x)$ c) $2x(\cos(2x)-\sin(2x))$ d) $2x(\cos(2x)-x\sin(2x))$ e) $2x(\cos(2x)+\sin(2x))$

---

2. If $x(t)=2t\cos\left(t^2\right)$, find $x'(t)$.

a) $x'(t)=\sin\left(t^2\right)+3$ b) $x'(t)=-\sin\left(t^2\right)+4$ c) $x'(t)=\sin\left(t^2\right)+2$ d) $x'(t)=-4t^2\sin\left(t^2\right)$ e) $x'(t)=-4t^2\sin\left(t^2\right)+2\cos\left(t^2\right)$

---

3. If $f(x)=x\tan(x)$, then $f'\left(\dfrac{\pi}{4}\right)=$

a) $1-\dfrac{\pi}{2}$ b) $1+\dfrac{\pi}{2}$ c) $1+\dfrac{\pi}{4}$ d) $1-\dfrac{\pi}{4}$ e) $\dfrac{\pi}{2}-1$

---

4. If f is a function that is differentiable throughout its domain and is defined by $f(x)=$

$\dfrac{1+e^x}{\sin(x^2)}$, then the value of $f'(0) =$

a) -1 b) 0 c) 1 d) e e) nonexistent

---

5. If $y = \dfrac{5x-4}{4x-5}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{9}{(4x-5)^2}$ b) $\dfrac{9}{(4x-5)^2}$ c) $\dfrac{40x-41}{(4x-5)^2}$ d) $\dfrac{40x+41}{(4x-5)^2}$ e) $\dfrac{5}{4}$

---

6. If $y = \dfrac{3-2x}{3x+2}$, then $\dfrac{dy}{dx} =$

a) $\dfrac{12x+2}{(3x+2)^2}$ b) $\dfrac{12x-2}{(3x+2)^2}$ c) $\dfrac{13}{(3x+2)^2}$ d) $-\dfrac{13}{(3x+2)^2}$ e) $-\dfrac{2}{3}$

---

7. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{6x}{(4+x^2)^2}$ b) $\dfrac{3x}{(4+x^2)^2}$ c) $\dfrac{6x}{(4+x^2)^2}$ d) $-\dfrac{3x}{(4+x^2)^2}$ e) $\dfrac{3}{2x}$

---

8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[x\tan(x)\right] = x\tan(x) + x\sec^2(x)$

II. $\dfrac{d}{dx}\left[\dfrac{3}{4+x^2}\right] = \dfrac{-6x}{(4+x^2)^2}$

III. $\dfrac{d}{dx}\left[\sqrt{1-x}\right] = \dfrac{1}{2\sqrt{1-x}}$

a) I only b) II only c) III only

d) I and II only e) I, II, and III

1.4: Higher Order Derivatives

What we've been calling the derivative is actually the first derivative. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. In this section, we will explore the process of finding the higher-order derivatives.

Second Derivative → Definition: The derivative of the derivative.

Just as with the first derivative, there are several symbols for the second derivative.

Higher Order Derivative Symbols

$$\frac{d^2y}{dx^2} = \text{“d squared y, d - x squared”} \rightarrow \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4} \dots \frac{d^ny}{dx^n}$$

$$\frac{d^2}{dx^2} = \text{“d squared, d - x squared”} \rightarrow \frac{d^3}{dx^3}, \frac{d^4}{dx^4} \dots \frac{d^n}{dx^n}$$

$$f''(x) = \text{“f double prime of x”} \rightarrow f'''(x),\ f^{IV}(x) \dots f^n(x)$$

$$y'' = \text{“y double prime”}$$

OBJECTIVES

Find Higher Order Derivatives.

Ex 1.4.1: $\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]$

Sol 1.4.1:

$$\begin{aligned}\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right] &= \frac{d}{dx}\left[\frac{d}{dx}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]\right] \\ &= \frac{d}{dx}\left[4x^3 - 21x^2 - 6x + 2\right] \\ &= \boxed{12x^2 - 42x - 6}\end{aligned}$$

Ex 1.4.2: Find $\frac{d^3y}{dx^3}$ if $y = \sin(3x)$.

Sol 1.4.2:

$$y = \sin(3x)$$

$$\frac{dy}{dx} = \cos(3x) \cdot 3 = 3\cos(3x)$$

$$\frac{d^2y}{dx^2} = 3(-\sin(3x)) \cdot 3 = -9\sin(3x)$$

$$\frac{d^3y}{dx^3} = -9\cos(3x) \cdot 3 = \boxed{-27\cos(3x)}$$

More complicated functions, in particular composite functions, have a more complicated process. When the Chain Rule is applied, the result often includes a product or a quotient. Therefore, the second derivative will require the Product or Quotient Rules, as well as possibly the Chain Rule.

Ex 1.4.3: $y = e^{3x^2}$, find $\frac{d^y}{dx^2}$.

Sol 1.4.3:

$$\frac{dy}{dx} = e^{3x^2} \cdot 6x = 6xe^{3x^2}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= 6x\left(e^{3x^2} \cdot 6x\right) + e^{3x^2} \cdot 6 \\ &= 36x^2e^{3x^2} + 6e^{3x^2} \\ &= \boxed{6e^{3x^2}\left(6x^2+1\right)}\end{aligned}$$

Ex 1.4.4: $y = \sin^3(x)$, find y''.

Sol 1.4.4:

$$y' = 3\sin^2(x) \cdot \cos(x)$$

$$y'' = 3\sin^2(x)(-\sin(x)) + \cos(x)(6\sin(x)\cdot\cos(x))$$

$$= \boxed{3\sin(x)\left(2\cos^2(x) - \sin^2(x)\right)}$$

Ex 1.4.5: $f(x) = \ln\left(x^2 + 3x - 1\right)$, find $f''(x)$.

Sol 1.4.5:

$$f'(x) = \frac{1}{x^2+3x-1}(2x+3) = \frac{2x+3}{x^2+3x-1}$$

$$f''(x) = \frac{\left(x^2+3x-1\right)(2) - (2x+3)(2x+3)}{\left(x^2+3x-1\right)^2}$$

$$= \frac{\left(2x^2+6x-2\right) - \left(4x^2+12x+9\right)}{\left(x^2+3x-1\right)^2}$$

$$= \boxed{-\frac{2x^2-6x-11}{\left(x^2+3x-1\right)^2}}$$

Ex 1.4.6: $g(x) = \sqrt{4x^2+1}$, find $g''(x)$

Sol 1.4.6:

$$g'(x) = \frac{1}{2}\left(4x^2+1\right)^{-\frac{1}{2}}(8x) = \frac{4x}{(4x^2+1)^{\frac{1}{2}}}$$

$$
\begin{aligned}
g''(x) &= \frac{\left(4x^2+1\right)^{\frac{1}{2}}(4) - (4x)\left[\frac{1}{2}\left(4x^2+1\right)^{-\frac{1}{2}}(8x)\right]}{\left[\left(4x^2+1\right)^{\frac{1}{2}}\right]^2} \\
&= \frac{\left(4x^2+1\right)^{\frac{1}{2}}(4) - \dfrac{16x^2}{\left(4x^2+1\right)^{\frac{1}{2}}}}{\left(4x^2+1\right)} \\
&= \frac{\left(4x^2+1\right)(4) - 16x^2}{\left(4x^2+1\right)^{\frac{3}{2}}} \\
&= \boxed{\frac{4}{\left(4x^2+1\right)^{\frac{3}{2}}}}
\end{aligned}
$$

1.4 Free Response Homework

Find the second derivatives of the given functions. Simplify where possible.

1. $f(x) = x^5 + 6x^2 - 7x$
2. $h(x) = 5x^4 + 9x^3 - 4x^2 + x - 8$
3. $y = \left(x^3 + 1\right)^{\frac{2}{3}}$
4. $H(t) = \tan(3t)$
5. $g(t) = t^3 e^{5t}$
6. $y = e^{3x^2}$
7. $y = \sin^4(x)$
8. $f(t) = t\cos(t)$
9. $y = -\dfrac{4x}{x^2 + 4}$
10. $y = \dfrac{x^2 - 1}{x^2 - 4}$
11. $f(x) = x\sqrt{8 - x^2}$
12. $y = \dfrac{1}{2}x + \sin(x)$
13. $g(t) = te^{-t}$
14. $y = e^{-x^2}$
15. $y = \dfrac{x}{x^2 - 9}$
16. $B(x) = 2x - x^{\frac{2}{3}}$
17. $y = x^3 + x^2 - 7x + 15$
19. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

Complete the following:

21. $y = \cos\left(x^2\right)$, find y''
22. $y = \tan^2(x)$, find y''
23. $y = \sec(3x)$, find $\dfrac{d^2y}{dx^2}$
24. $y = xe^{2x}$, find $\dfrac{d^2y}{dx^2}$
25. $f(x) = \ln\left(x^2 + 3\right)$, find $f''(x)$
25. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g''(x)$
27. $h(x) = \sqrt{x^2 + 5}$, find $h''(x)$
28. $F(x) = \sqrt{3x^2 - 2x + 1}$, find $F''(x)$
29. $y = \dfrac{x^2 - 3}{x^2 - 10}$, find $\dfrac{d^2y}{dx^2}$
30. $y = \dfrac{3x + 3}{x^3 + 1}$, find $\dfrac{d^2y}{dx^2}$

1.4 Multiple Choice Homework

1. If f and g are twice differentiable and if $h(x) = g\left(f(x)\right)$, then $h''(x) =$

a) $g''\left(f(x)\right)$ b) $g''\left(f(x)\right)f''(x)$ c) $g''\left(f(x)\right)\left[f'(x)\right]^2$

d) $g'(f(x))\left[f'(x)\right]^2 + f'(x)\left(f''(x)\right)$ e) $g'(f(x))\, f''(x) + \left[f'(x)\right]^2 g''(f(x))$

---

2. Find $\dfrac{d^2y}{dx^2}$ if $y = \dfrac{x+2}{x-3}$

a) $-\dfrac{2}{(x-3)^2}$ b) 0 c) $\dfrac{10}{(x-3)^3}$ d) $\dfrac{2}{(x-3)^2}$ e) None of these

---

3. If $y = \ln(\cos(x))$ and $0 \le x \le \pi$, then $\dfrac{d^2y}{dx^2}$ is

a) $-\tan(x)$ b) $-\sec^2(x)$ c) $\tan(x)$ d) $\sec^2(x)$ e) $\sec(x)\tan(x)$

---

4. If $y = \ln\left(x^2+4\right)$, then $\dfrac{d^2y}{dx^2}$ is

a) $\dfrac{1}{x^2+4}$ b) $\dfrac{2x}{x^2+4}$ c) $\dfrac{-2x^2+8}{x^2+4}$ d) $\dfrac{2x}{\left(x^2+4\right)^2}$ e) $\dfrac{-2x^2+8}{\left(x^2+4\right)^2}$

---

5. If $y = e^{x^2}$, then $\dfrac{d^2y}{dx^2} =$

a) e^{x^2} b) $2e^{x^2}\left(2x^2+1\right)$ c) $2xe^{x^2}$ d) $4x^2e^{x^2}$ e) $2e^{x^2}\left(2x^2-1\right)$

---

6. If $h(t) = \ln\left(t^2+1\right)$, then $h''(-1) =$

a) $\ln 2$ b) 0 c) -1 d) -2 e) DNE

---

7. If $y = \sin\left(e^x\right)$, then $\dfrac{d^2y}{dx^2} =$

a) $\cos\left(e^x\right)$ b) $e^x\cos\left(e^x\right)$ c) $e^x\sin\left(e^x\right) + e^x\cos\left(e^x\right)$

d) $-e^x\sin\left(e^x\right) + e^x\cos\left(e^x\right)$ e) $-e^x\left(e^x\sin\left(e^x\right) - \cos\left(e^x\right)\right)$

1.5: Implicit Differentiation and Second Derivative Applications

Implicit differentiation is a technique that is considered here because of its direct impact on related rates in section 8 of this chapter. Implicit differentiation is an application of the Chain Rule where the y-function is not easily defined explicitly.

One of the most useful aspects of the Chain Rule is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of y with respect to x — that is, $\frac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra. When y cannot easily be isolated, we can treat y like we treat $g(x)$. In other words:

$$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot \left[g'(x)\right] \text{ is the same as } \frac{d}{dx}\left[f(y)\right] = f'(y) \cdot \left(\frac{dy}{dx}\right)$$

OBJECTIVES

Take Derivatives of Relations Implicitly.
Use Implicit Differentiation to Find Higher Order Derivatives.
Use the Second Derivative Test to Determine Whether a Point is at a Maximum, Minimum, or Neither.

Ex 1.5.1: Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$

Sol 1.5.1:

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right]$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}[25]$$

$$\to 2x + 2y\frac{dy}{dx} = 0$$

We can now isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \boxed{-\frac{x}{y}}$$

With this function, notice that y could have been isolated and $\frac{dy}{dx}$ could've been found **explicitly**.

$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \sqrt{25 - x^2}$$

$$\frac{dy}{dx} = -\frac{x}{\sqrt{25 - x^2}}$$

Notice that this is the same answer as we found with implicit differentiation. You could substitute y for $\sqrt{25 - x^2}$ in the denominator and come up with the same derivative, $\frac{dy}{dx} = -\frac{x}{y}$.

Ex 1.5.2: Find the derivative of $x^2 - 3y^2 + 4x - 12y - 2 = 0$ implicitly.

Sol 1.5.2:

$$\frac{d}{dx}\left[x^2 - 3y^2 + 4x - 12y - 2 = 0\right]$$

$$2x - 6y\frac{dy}{dx} + 4 - 12\frac{dy}{dx} = 0$$

$$(-6y - 12)\frac{dy}{dx} = -2x - 4$$

$$\frac{dy}{dx} = \boxed{\frac{-2x - 4}{-6y - 12}}$$

When considering functions, implicit differentiation may not seem to be a particularly powerful tool, because it is often simple to isolate y. But consider a non-function, like this circle, ellipse, or hyperbola, where y is not so easily isolated.

Ex 1.5.3: Find $\frac{dy}{dx}$ for the hyperbola $x^2 - 3xy + 3y^2 = 2$

Sol 1.5.3: It would be very difficult to solve for y here, so implicit differentiation is really our only option.

$$\frac{d}{dx}\left[x^2 - 3xy + 3y^2 = 2\right]$$

Note that $-3xy$ is a product. It will require the Product Rule.

$$2x - 3x\frac{dy}{dx} - 3y + 6y\frac{dy}{dx} = 0$$

$$(-3x + 6y)\frac{dy}{dx} = -2x + 3y$$

$$\frac{dy}{dx} = \boxed{\frac{-2x + 3y}{-3x + 6y}}$$

Ex 1.5.4: Find the equation of the line tangent to $x^3 - y^2 + 6y = -3$ at $y = 1$

a) $3x^2 - 2y = 6$ b) $3x - y = -7$ c) $3x + y = -5$

d) $x + 3y = 1$ e) $x - 3y = -5$

Sol 1.5.4: First, let's find the point of tangency.

$$x^3 - y^2 + 6y = -3 \rightarrow x^3 - (1)^2 + 6(1) = -3$$

$$x^3 = -8 \therefore x = -2$$

$$\frac{d}{dx}\left[x^3 - y^2 + 6y = -3\right]$$

$$3x^2 - 2y\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-3x^2}{-2y + 6}$$

Now, let's plug in our point that we found, $(-2, 1)$

$$\frac{dy}{dx} = \frac{-3(-2)^2}{-2(1) + 6}$$

$$\frac{dy}{dx} = -3$$

$$y - 1 = -3(x + 2)$$

$$y - 1 = -3x - 6$$

$$\rightarrow \boxed{\text{c) } 3x + y = -5}$$

Of course, if we want to find a second derivative, we can use implicit differentiation a second time.

Ex 1.5.5: Given $\dfrac{dy}{dx} = \dfrac{x+2}{3y+6}$, find $\dfrac{d^2y}{dx^2}$

Sol 1.5.5:

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x+2}{3y+6}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(3y+6) - (x+2)\left(3\dfrac{dy}{dx}\right)}{(3y+6)^2}$$

Since we already know $\dfrac{dy}{dx}$, we can substitute

$$\frac{d^2y}{dx^2} = \frac{(3y+6) - (x+2)(3)\left(\dfrac{x+2}{3y+6}\right)}{(3y+6)^2}$$

$$= \frac{(3y+6) - (x+2)(3)\left(\dfrac{x+2}{3y+6}\right)}{(3y+6)^2} \cdot \frac{3y+6}{3y+6}$$

$$= \boxed{\frac{(3y+6)^2 - 3(x+2)^2}{(3y+6)^3}}$$

Ex 1.5.6: Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for $\sin(y) = 2\cos(3x)$.

Sol 1.5.6:

$$\frac{d}{dx}\left[\sin(y) = 2\cos(3x)\right]$$

$$\cos(y)\frac{dy}{dx} = -6\sin(3x)$$

$$\frac{dy}{dx} = \boxed{-\frac{6\sin(3x)}{\cos(y)}}$$

$$\frac{d^2y}{dx^2} = \frac{-18\cos(y)\cos(3x) - 6\sin(3x)\sin(y)\frac{dy}{dx}}{\cos^2(y)}$$

$$= \frac{-18\cos(y)\cos(3x) - 6\sin(3x)\sin(y)\left(-\frac{6\sin(3x)}{\cos(y)}\right)}{\cos^2(y)}$$

$$= \boxed{\frac{-18\cos(y)\cos(3x) + 36\sin^2(3x)\sin(y)}{\cos^3(y)}}$$

AP-Style Implicit Differentiation Problems

Common Sub-topics:

- Demonstrating implicit differentiation
- Finding the equation of a tangent line
- Finding points where the tangent line is horizontal and/or vertical
- Finding points on a curve with a particular slope
- Finding the second derivative and apply the Second Derivative Test
- Finding the particular solution (this will have to wait for the next chapter)

Remember:

The Second Derivative Test

For a function f:

1) If $f'(c) = 0$ and $f''(c) > 0$, then f has a relative minimum at c

2) If $f'(c) = 0$ and $f''(c) < 0$, then f has a relative maximum at c

This is necessary because one cannot create a sign pattern without an **explicitly** stated function, so the First Derivative Test will not work on problems which require implicit differentiation to find the derivative.

OBJECTIVES

Take Derivatives of Relations Implicitly.
Use Implicit Differentiation to Find Higher Order Derivatives.
Use Separation of Variables to Find the Particular Solution to a Differential Equation.

Ex 1.5.7: Consider the curve given by $x^2 + 4xy + y^2 = -12$.

(a) Show that $\dfrac{dy}{dx} = -\dfrac{x+2y}{2x+y}$

(b) Find the point(s) where the equation of the tangent line(s) is/are horizontal.

(c) Find the value(s) of $\dfrac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

Sol 1.5.7:

(a) Because we have a $4xy$ term, we need to use the product rule

$$\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right]$$

$$2x + 4x\frac{dy}{dx} + 4y(1) + 2y\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 4y$$

$$(4x + 2y)\frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x-4y}{4x+2y} = \boxed{-\frac{x+2y}{2x+y}}$$

(b) Horizontal lines have a slope of 0, so we need to find when $\dfrac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 0$$

$$-\frac{x+2y}{2x+y} = 0$$

$$x + 2y = 0$$

$$x = -2y$$

To be on the curve, $x = -2y$ must satisfy the original equation.

$$(-2y)^2 + 4(-2y)y + y^2 = -12$$

$$4y^2 - 8y^2 + y^2 = -12$$

$$-3y^2 = -12$$

$$y^2 = 4 \therefore y = \pm 2$$

$$x = -2y \rightarrow \boxed{(-4, 2) \text{ and } (4, -2)}$$

(c) What is really asked here is to apply the Second Derivative Test, because we cannot create a sign pattern for non-functions. The y is not isolated in the equation.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx} = -\frac{x+2y}{2x+y}\right]$$

$$= \frac{(2x+y)\left(1+2\frac{dy}{dx}\right) - (x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-4,2)} = \frac{(2(-4)+2)\,(1+2(0)) - (-4+2(2))\,(2+0)}{(2(-4)+2)^2}$$

$$= \frac{6}{(-6)^2} > 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,-2)} = \frac{(2(4)-2)\,(1+2(0)) - (4+2(-2))\,(2+0)}{(2(4)+2)^2}$$

$$= \frac{6}{(-6)^2} < 0$$

$\boxed{(-4, 2) \text{ will be a minimum}}$ because the second derivative is positive.

$\boxed{(4, -2) \text{ will be a maximum}}$ because the second derivative is negative.

Be Careful!! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don't make careless mistakes.

1.5 Free Response Homework

Find $\dfrac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for y and differentiating. Show that $\dfrac{dy}{dx}$ is the same in both cases.

1. $x^2 + y^2 = 1$
2. $x^3 + 4y^2 = 16$
3. $\dfrac{1}{x} + \dfrac{1}{y} = 1$
4. $\sqrt{x} + \sqrt{y} = 4$

Find $\dfrac{dy}{dx}$ for each of these equations by implicit differentiation.

5. $x^2 + xy = 10$
6. $x^3 + 10x^2y + 7y^2 = 60$
7. $x^2 + xy - 4y - 1 = 0$
8. $xy + 2x + 3x^2 = 4$
9. $x^2 + 4xy - 5y^2 = 4$
10. $3x^2 + xy - 4y^2 = 5$
11. $x^2 = \dfrac{x-y}{x+y}$
12. $x^2 + y^2 = \dfrac{x}{y}$
13. $y^2 = \dfrac{x-y}{x+y}$
14. $y^2 = \dfrac{x^2-1}{x+2}$
15. $x^2y^2 + x\sin(y) = 4$
16. $4\cos(x)\sin(y) = 1$
17. $e^{x^2y} = x + y$
18. $\tan(x-y) = \dfrac{y}{1+x^2}$
19. Find the equation of the line tangent to $x^2 - y^2 - 6y - 3 = 0$ at $\left(\sqrt{3}, 0\right)$.
20. Find the equation of the line tangent to $9x^2 + 4y^2 + 36x - 8y - 32 = 0$ at $(0, 2)$.
21. Find the equation of the line tangent to $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $(-1, -3)$.
22. Find the equation of the line tangent to $x^3 + \dfrac{y}{x} + y^2 = 7$ at $(1, 2)$.
23. Find the equation of the lines tangent and normal to $y - \dfrac{4}{\pi^2}x^2 = 2e^{y\sin(x)} + y^3 - 3$ through the point $\left(\dfrac{\pi}{2}, 0\right)$.

24. Find the equation of the lines tangent and normal to $x^2 + 3xy + y^2 = 11$ through the point $(1, 2)$.

25. Find $\dfrac{d^2y}{dx^2}$ if $xy + y^2 = 1$.

26. Find $\dfrac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$.

27. Find $\dfrac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$

28. Find $\dfrac{d^2y}{dx^2}$ if $x^3 + 4y^2 = 16$.

29. Consider the curve given by $3x^2 - 4xy + 5y^2 = 25$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{3x - 2y}{2x - 5y}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 2.

(c) Find the equation(s) of the line(s) tangent to $3x^2 - 4xy + 5y^2 = 25$ at the point(s) P found in part (b).

(d) Find the point(s) on $3x^2 - 4xy + 5y^2 = 25$ where the tangent line is horizontal.

30. Consider the curve given by $x^2 - xy + y^2 = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y - 2x}{2y - x}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 2.

(c) Find the equation(s) of the line(s) tangent to $x^2 - xy + y^2 = 4$ at the point(s) P found in part (b).

(d) Find the point(s) on $x^2 - xy + y^2 = 4$ where the tangent line is vertical.

31. Consider the curve given by $2x^2 - xy + y^2 = 44$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4x - y}{x - 2y}$.

(b) Determine point(s) P on the curve for which the x-coordinate is equal to 5.

(c) Find the equation(s) of the line(s) tangent to $2x^2 - xy + y^2 = 44$ at the point(s) P found in part (b).

(d) Find the point(s) on $x^2 - xy + y^2 = 4$ where the tangent line is vertical.

32. Consider the curve given by $x^2 + xy + y^2 = 12$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-y-2x}{2y+x}$.

(b) Find the point(s) P on $x^2 + xy + y^2 = 12$ where the tangent line is horizontal.

(c) Find the value(s) of $\dfrac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

33. Consider the curve given by $xy + y^3 = 4x$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{4-y}{3y^2 = x}$.

(b) Show that there are no points on the curve where the tangent line is horizontal.

(c) Find the point(s) on $xy + y^3 = 4x$ where the tangent line is vertical.

34. Consider the curve given by $x^2 + xy + y^2 = 4$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-2x-y}{x+2y}$.

(b) Find the point(s) on $x^2 + xy + y^2 = 4$ where the tangent line is horizontal.

(c) Find the y-coordinates of the point(s) where the tangent line is vertical.

1.5 Multiple Choice Homework

1. Use implicit differentiation to find the points on $x^3 - y^2 + x^2 = 0$ where the tangent line is vertical.

a) $(0, 0)$ only　　b) $(-1, 0)$ only　　c) $\left(1, \sqrt{2}\right)$ only

d) $(-1, 0)$ and $(0, 0)$　　e) No such points exist

---

2. If $x^2 + xy = 10$, then when $x = 2$, $\dfrac{dy}{dx} =$

a) $-\dfrac{7}{2}$　　b) -2　　c) $\dfrac{2}{7}$　　d) $\dfrac{3}{2}$　　e) $\dfrac{7}{2}$

---

3. What is the slope of the line tangent to the curve $y^2 + x = -2xy - 5$ at the point $(2, 1)$.

a) $-\frac{4}{3}$ b) $-\frac{3}{4}$ c) $-\frac{1}{2}$ d) $-\frac{1}{4}$ e) 0

---

4. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x.

a) $\frac{6x - 4y}{4x + 4}$ b) $\frac{9x^2 - 4}{4x + 8y}$ c) $\frac{9x^2 - 4}{4 + 8y}$ d) $\frac{9x^2 - 4y}{4x + 8y}$ e) $\frac{9x^2 - 4y}{4 + 8y}$

---

5. Given $x + xy + 2y^2 = 6$, then $\left.\frac{dy}{dx}\right|_{(2,1)} =$

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) $-\frac{1}{5}$ e) $-\frac{3}{4}$

---

6. Consider the closed curve in the xy-plane given by $2x^2 + 5x + y^2 + y = 8$. Which of the following is correct?

a) $\frac{dy}{dx} = -\frac{4x + 5}{8x + 2y + 1}$ b) $\frac{dy}{dx} = \frac{4x + 5}{2y + 1}$ c) $\frac{dy}{dx} = -\frac{4x + 5}{8x + 2y}$ d) $\frac{dy}{dx} = \frac{4x + 5}{8x + 2y}$ e) $\frac{dy}{dx} = \frac{4x + 5}{2y + 1}$

---

7. The slope of the line tangent to $xy - y^3 + 6 = 0$ at $(1, 2)$ is

a) 0 b) $-\frac{1}{12}$ c) $\frac{2}{11}$ d) $\frac{1}{6}$ e) $\frac{1}{4}$

---

8. Find the equation of the line tangent to the curve $\sec\left(x^2\right) + xy^3 = 2 - y$ at $x = 0$.

a) $y = -x$ b) $y - 1 = -x$ c) $y - 2 = -x$ d) $y - 1 = x$ e) $y - 2 = x$

---

9. If $\sin^{-1}(x) = \ln y$, then $\frac{dy}{dx} =$

a) $\dfrac{y}{\sqrt{1-x^2}}$ b) $\dfrac{xy}{\sqrt{1-x^2}}$ c) $\dfrac{y}{1+x^2}$ d) $e^{\sin^{-1}(x)}$ e) $\dfrac{e^{\sin^{-1}(x)}}{1+x^2}$

---

10. If $x^2y + yx^2 = 6$, then at $(1,3)$, $\dfrac{d^2y}{dx^2} =$

a) -18 b) -6 c) 6 d) 12 e) 18

---

11. If $y = x + \sin(xy)$, then $\dfrac{dy}{dx} =$

a) $1+\cos(xy)$ b) $1+y\cos(xy)$ c) $\dfrac{1}{1-\cos(xy)}$

d) $\dfrac{1}{1-x\cos(xy)}$ e) $\dfrac{1+y\cos(xy)}{1-\cos(xy)}$

---

12. If $\sin(xy) = x^2$, then $\dfrac{dy}{dx} =$

a) $2x\sec(xy)$ b) $\dfrac{\sec(xy)}{x^2}$ c) $2x\sec(xy)-y$

d) $\dfrac{2x\sec(xy)}{y}$ e) $\dfrac{2x\sec(xy)-y}{x}$

---

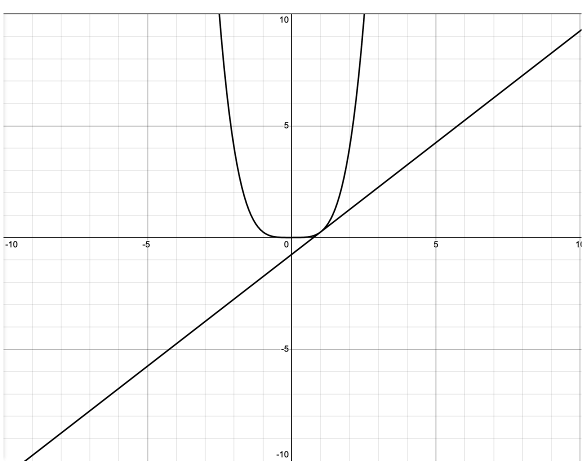
13. Given $y = \ln\left(x^2+y^2\right)$, find $\dfrac{dy}{dx}$ at the point $(1,0)$.

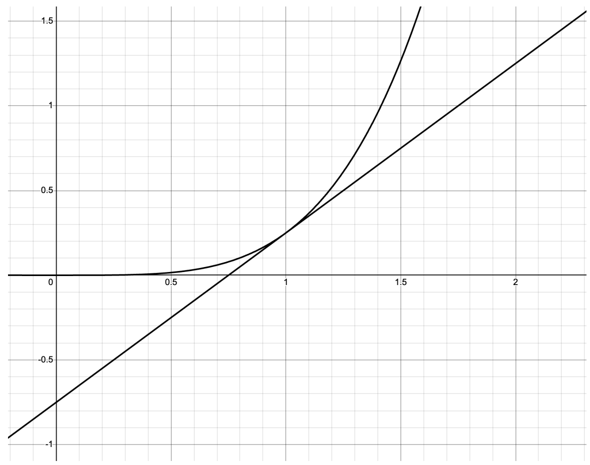
a) 0 b) 0.5 c) 1 d) 2 e) undefined

1.6: Local Linearity, Euler's Method, and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, y values on the line are very close to y values on the function. This idea is called local linearity—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency.

Consider the graphs of $y = \dfrac{1}{4}x^4$ and its tangent line at $x = 1$, given by the equation $y = x - \dfrac{3}{4}$:





The closer you zoom in, the more the line and the curve become one. The y values on the line are good approximations of the y values on the curve. For a good animation of this concept, see the following:

tangent line approximation animation

Since it's easier to find the y value of a line arithmetically than for other functions — especially transcendental functions — the tangent line approximation is useful if you have no calculator.

OBJECTIVES

Use the Equation of a Tangent Line to Approximate Function Values.

Ex 1.6.1: Find the equations of the line tangent to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.6.1: The slope of the tangent line will be $f'(-1)$

$$f'(x) = 4x^3 - 3x^2 - 4x$$

$$f'(-1) = 4(-1)^3 - 3(-1)^2 - 4(-1) = -3$$

(Note that we could've gotten this more easily with the nDeriv function on our calculator.)

$$f(-1) = 1, \therefore \boxed{y - 1 = -3(x + 1)} \text{ or } \boxed{y = -3x - 2}$$

One of the many uses of the tangent line is based on the idea of local linearity. This means that in small areas, algebraic curves act like lines — namely their tangent lines. Therefore, one can get an approximate y value for points near the point of tangency by plugging x values into the equation of the tangent line.

Ex 1.6.2: Use the tangent line equation found in **Ex 1.6.1** to get an approximate value of $f(-0.9)$.

Sol 1.6.2: While we can find the exact value of $f(-0.9)$ with a calculator, we can get a quick approximation from the tangent line.

If $x = -0.9$ on the tangent line, then:

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = \boxed{0.7}$$

This last example was somewhat trite in that we could've just plugged -0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but it is doable. Consider the next example, though.

Ex 1.6.3: Find the tangent line equation to $f(x) = e^{2x}$ at $x = 0$ and use it to approximate the value of $e^{0.2}$.

Sol 1.6.3: Without a calculator, we could not find the exact value of $e^{0.2}$. In fact, even the calculator gives us an approximate value.

$$f'(x) = 2e^{2x} \text{ and } f'(0) = 2e^{2(0)} = 2$$

$$f(0) = e^0 = 1$$

So the tangent line equation is $\boxed{y - 1 = 2(x - 0)}$ or $y = \boxed{2x + 1}$

$$e^{0.2} \approx 2(0.2) + 1 = \boxed{1.2}$$

Note that the value you get from a calculator of $e^{0.2}$ is $1.221403\ldots$. Our approximation of 1.2 seems very reasonable.

Though not as useful as practically useful (in 2 dimensions) as the tangent line, another context for the derivative is finding the equation of the normal line.

Normal Line $\rightarrow$ Definition: The line perpendicular to a curve.

Ex 1.6.4: Find the equation of the line normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.6.4: In **Ex 1.6.1**, we saw that the slope of the tangent line was $f'(-1) = -3$. The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope of $\frac{1}{3}$. This gives us

$$\boxed{y - 1 = \frac{1}{3}(x+1)} \text{ or } \boxed{y = \frac{1}{3}x - \frac{4}{3}}$$

for the equation of the normal line.

Euler's Method

OBJECTIVES

Use Euler's Method to Approximate a Numerical Solution to a Differential Equation at a Given Point.

In the previous section, we learned a little regarding approximations with tangent lines. Euler's Method is just a better approximation method. It uses more than one tangent line to get the job done.

The process is similar to approximating with tangent lines. We use $\frac{dy}{dx}$ to find a tangent line, then use that tangent line to find an approximate value for y. We then use that y value and another x value to create another "tangent line." Of course, it isn't actually a tangent line because our y value wasn't actually on the curve. We then repeat the process until we get to the value we want to approximate.

Steps to Euler's Method

1. Identify your starting point and step size.
2. Use $\dfrac{dy}{dx}$ to find the slope and make it a tangent line.
3. Find an approximate y value by plugging in $x + (\textit{1 step size})$ to the tangent line.
4. Use the approximate y value and the next x step over to make a new tangent line.
5. Repeat steps 3 and 4 until you reach your final x value – the one you actually want an approximation for.

Ex 1.6.5: Use Euler's Method with a step size of $\frac{1}{2}$ to estimate $f(3)$, where $f(x) = \ln(x)$.

Sol 1.6.5:

$$f(1) = \ln 1 = 0 \qquad \text{We start with 1 because we know } \ln 1 = 0.$$

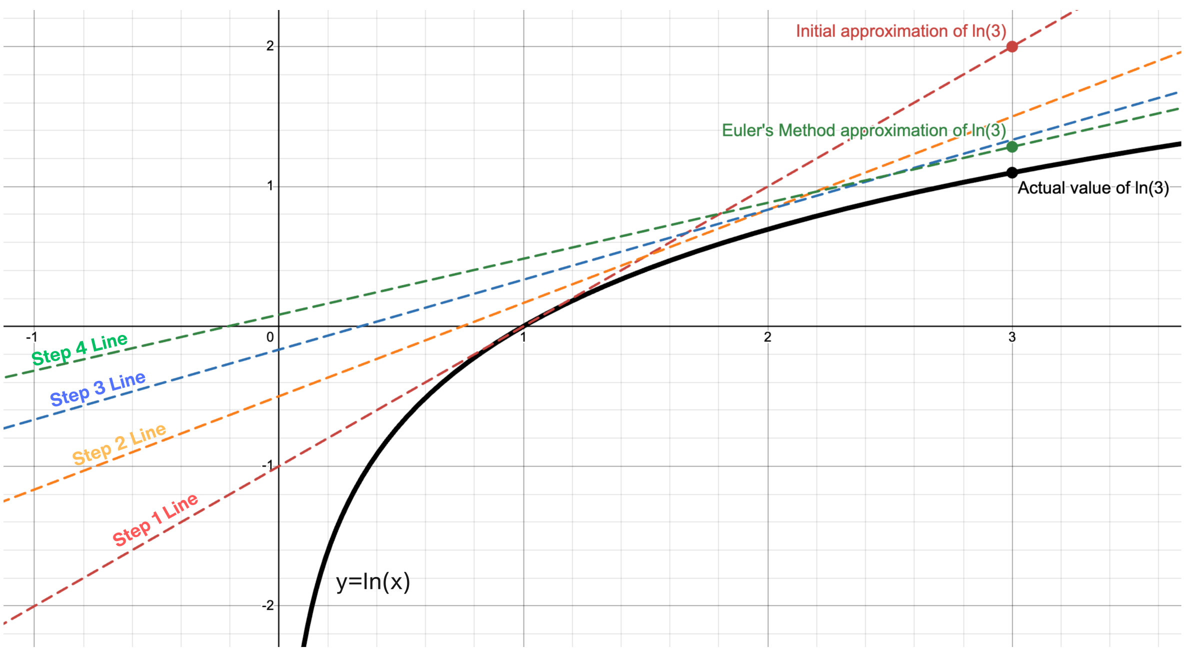
$$f'(x) = \frac{1}{x} \qquad \text{We start by taking the derivative.}$$

Note that in our chart below, we are getting our "New y" from our tangent line (step 3 above). Our "New x" comes from the "$x + (\textit{1 step size})$" step. Our new slope comes from plugging in the "New x" into $f'(x)$. For instance, for our first step below, the "New x" is equal to 1 plus the step size of $\frac{1}{2}$ given in the problem, our "New y" comes from plugging in $\frac{3}{2}$ for x and solving for y in $y - 0 = 1(x - 1)$, and our slope (for the next step) is a result of $f'\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

Step	Point	$f'(x)$ (Slope)	Tangent Line Equation	New x	New y
1	$(1, 0)$	1	$y - 0 = 1(x - 1)$	$\frac{3}{2}$	$\frac{1}{2}$
2	$\left(\frac{3}{2}, \frac{1}{2}\right)$	$\frac{2}{3}$	$y - \frac{1}{2} = \frac{2}{3}\left(x - \frac{3}{2}\right)$	2	$\frac{5}{6}$
3	$\left(2, \frac{5}{6}\right)$	$\frac{1}{2}$	$y - \frac{5}{6} = \frac{1}{2}(x - 2)$	$\frac{5}{2}$	$\frac{13}{12}$
4	$\left(\frac{5}{2}, \frac{13}{12}\right)$	$\frac{2}{5}$	$y - \frac{13}{12} = \frac{2}{5}\left(x - \frac{5}{2}\right)$	3	$\frac{77}{60}$

So, $f(3) \approx y(3) = \boxed{\frac{77}{60}}$ or $\boxed{1.28\overline{3}}$. By way of comparison, $\ln 3 \approx 1.099$.

If we had just used the initial tangent line (the tangent line from step one) to get an approximation, we would've gotten $f(3) \approx 2$. Euler's method got us a much closer approximation.



We could also use this process to approximate a value for a curve when we only know its derivative and an initial value on the curve.

Ex 1.6.6: Use Euler's method with a step size of $\frac{1}{2}$ to estimate $f\left(\frac{5}{2}\right)$ for the function whose derivative is given by $\frac{dy}{dx} = 2x + y$ with an initial value of $f(1) = 4$.

Sol 1.6.6: For this problem, to find the slope for each step, we simply we need to plug in our point into the given differential equation.

Step	**Point**	$\frac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(1, 4)$	6	$y - 4 = 6(x - 1)$	$\frac{3}{2}$	7
2	$\left(\frac{3}{2}, 7\right)$	10	$y - 7 = 10\left(x - \frac{3}{2}\right)$	2	12
3	$(2, 12)$	16	$y - 12 = 16(x - 2)$	$\frac{5}{2}$	20

$f\left(\frac{5}{2}\right) \approx y\left(\frac{5}{2}\right) = \boxed{20}$. We cannot get an exact value for this function, because we have not learned techniques regarding how to solve this differential equation yet.

Ex 1.6.7: Is the approximation in **Ex 1.6.6** an overestimate or underestimate? Why?

Sol 1.6.7: To determine this, we need to look at the concavity of the curve – this requires the second derivative.

$$\frac{dy}{dx} = 2x + y$$

$$\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} \qquad \text{Don't forget implicit differentiation!}$$

$$\frac{d^2y}{dx^2} = 2 + (2x + y)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,4)} = 2 + 2(1) + 4 = 8 \qquad \text{Plug in our initial value}$$

Since the second derivative is positive, our curve is concave up at this point, which means our tangent line lies under the curve. This is indicative of an $\boxed{\text{underestimate}}$.

Generally:

> → Your approximation will be an **overestimate** if the curve is **concave down** (since your "tangent lines" will be above the curve).
>
> → Your approximation will be an **underestimate** if the curve is **concave up** (since your "tangent lines" will be below the curve).

Ex 1.6.8: Use Euler's method with four equal step sizes to approximate $f(2)$ for $\frac{dy}{dx} = 3y - x$, given $f(0) = 1$. Is this an overestimate or an underestimate?

Sol 1.6.8: First, let's figure out our step sizes. We know that we start with $x = 0$, and we need to end up at $x = 2$. Therefore, four equal step sizes means that each step must be $+\frac{1}{2}$.

Next, let's make our table:

Step	**Point**	$\frac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(0,1)$	3	$y-1=3(x-0)$	$\frac{1}{2}$	$\frac{5}{2}$
2	$\left(\frac{1}{2},\frac{5}{2}\right)$	7	$y-\frac{5}{2}=7\left(x-\frac{1}{2}\right)$	1	6
3	$(1,6)$	17	$y-6=17(x-1)$	$\frac{3}{2}$	$\frac{29}{2}$
4	$\left(\frac{3}{2},\frac{29}{2}\right)$	42	$y-\frac{29}{2}=42\left(x-\frac{3}{2}\right)$	2	$\frac{71}{2}$

We can see that $f(2) \approx y(2) = \boxed{\frac{71}{2}}$. Now, let's solve for the second derivative to find out if this is an overestimate or underestimate.

$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 1 = 9y - 3x - 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 9(1) - 3(0) - 1 = 8$$

Because the positive second derivative indicates that y is concave up at $(0,1)$, our Euler's Method result is going to be an $\boxed{\text{underestimate}}$.

1.6 Free Response Homework

Complete the following:

1. Find the equation of the line tangent to $y = x^4 + 2e^x$ at the point $(0, 2)$.

2. Find the equation of the line tangent to $y = x + \cos(x)$ at the point $(0, 1)$.

3. Find the equation of the line tangent to $y = \sec(x) - 2\cos(x)$ at the point $\left(\frac{\pi}{3}, 0\right)$.

4. Find the equation of the line tangent to $y = x^2 e^{-x}$ at the point $\left(1, \frac{1}{e}\right)$.

5. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

6. Find the equation of the line tangent to $y = \frac{x^2 - 3}{x^2 - 4}$ when $x = -1$.

7. Find the equation of the line tangent to $f(x) = x\sqrt[4]{7 + x^2}$ when $x = 3$.

8. Find the equation of the line tangent to $y = e^{x\sin(4x)} + 2$ when $x = 0$.

9. Find the equation of the line tangent to $f(x) = x^5 - 5x + 1$ when $x = -2$ and use it to get an approximate value of $f(-1.9)$.

10. Find the equation of the line tangent to $f(x) = x\sqrt[3]{1 - x^2}$ when $x = 3$ and use it to get an approximate value of $f(3.1)$.

11. Find all points on the graph of $y = 2\sin(x) + \sin^2(x)$ where the tangent line is horizontal.

12. Find all points on the graph of $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

13. Find the equation of the lines tangent and normal to $y = -\frac{2x}{x^2 + 16}$ at $x = -1$.

14. Find the equation of the lines tangent and normal to $y = -\frac{3x}{x^2 + 1}$ at $x = 1$.

15. Find the equation of the lines tangent and normal to $y = \frac{x^2 - 4x + 3}{2x^2 - 5x - 3}$ at $x = 2$.

16. Find the equation of the lines tangent and normal to $y = x \sin\left(\frac{\pi}{2}\ln x\right)$ when $x = e$.

17. Find the equation of the lines tangent and normal to $y = x \sin\left(\frac{1}{x}\right)$ when $x = \frac{4}{\pi}$.

18. Use Euler's Method with 2 equal step sizes to find an approximation for $f(0)$, given that $f(-1) = 2$ and $\frac{dy}{dx} = 6x^2 - x^2 y$.

19. Use Euler's Method with 4 equal step sizes to find an approximation for $f(1.4)$, given that $f(1) = 0$ and $f(x) = \ln(2x - 1)$.

20. Use Euler's Method with 3 equal step sizes to find an approximation for $f(2.6)$, given that $f(2) = -2$ and $\frac{dy}{dx} = 2x + y$.

1.6 Multiple Choice Homework

1. Let f be the function given by $f(x) = 2e^{4x^2}$. For what value of x is the slope of the line tangent to the graph of f at $(x, f(x))$ equal to 3?

a) 0.168 b) 0.274 c) 0.318 d) 0.342 e) 0.551

---

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

a) $-3x - 2$ b) $-3x + 4$ c) $-x + 0.905$

d) $-x + 0.271$ e) $-x - 0.271$

---

3. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ b) $\left(\frac{1}{2}, -\frac{1}{8}\right)$ c) $\left(\frac{1}{2}, -\frac{1}{4}\right)$ d) $\left(1, -\frac{1}{2}\right)$ e) $(2, 2)$

---

4. A normal line to the graph of a function f at the point $(x, f(x))$ is defined to be the line

perpendicular to the tangent line at that point. An equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

a) $y + 12x = 38$ b) $y - 4x = 10$ c) $y + 2x = 4$

d) $y + 2x = 8$ e) $y - 2x = -4$

---

5. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{4x}{y}$ with the initial condition $f(0) = 1$. What is the best approximation for $f(1)$ using Euler's Method, starting at $x = 0$ with a step size of 0.5?

a) 1 b) 2 c) $\sqrt{5}$ d) 2.5 e) 3

---

6. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = x - y^2$ with the initial condition $f(0) = 1$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 0$ with a step size of 1?

a) -1 b) 0 c) 1 d) 2 e) 3

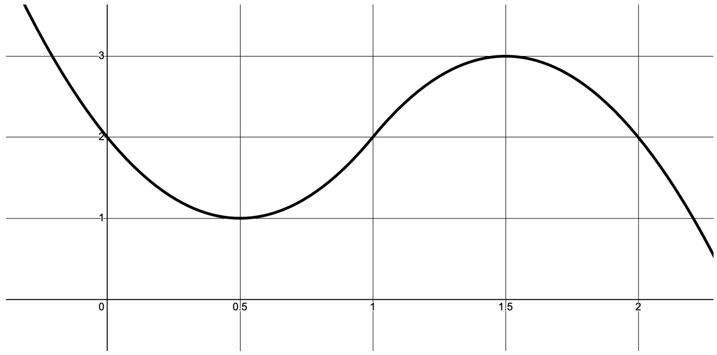
---

7. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = y - x$ with the initial condition $f(1) = 2$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 1$ with a step size of 0.5?

a) 1 b) 2 c) 3 d) 4.5 e) 6

---

8. The graph of $y = f'(x)$ is given below. Use this information and the fact that $f(0) = 3$ to find an approximate value of $f(1)$ using Euler's method with 2 equal step sizes.



a) 2.5 b) 3.5 c) 4 d) 4.5 e) 5

---

9. The table below gives selected values for the derivative of a function g on the interval $-1 \le x \le 2$. If $g(-1) = -2$ and Euler's Method with a step size of 1.5 is used to approximate $g(2)$, what is the resulting approximation?

x	-1.0	-0.5	0	0.5	1.0	1.5	2.0
$f'(x)$	2	4	3	1	0	-3	-6

a) -6.5 b) -1.5 c) 1.5 d) 2.5 e) 3

---

10. The equation of the line **normal** to the graph of $y = \dfrac{3x+4}{4x-3}$ at $(1, 7)$ is

a) $25x + y = 32$ b) $25x - y = 18$ c) $7x - y = 0$

d) $x - 25y = -174$ e) $x + 25y = 176$

---

11. The equation of the line **normal** to the graph of $y = 3x\sqrt{x^2+6} - 3$ at $(0, -3)$ is

a) $3\sqrt{6}x + y = -3$ b) $3\sqrt{6}x - y = -3$ c) $x + 3\sqrt{6}y = -3$

d) $x - 3\sqrt{6}y = 9\sqrt{6}$ e) $x + 3\sqrt{6}y = -9\sqrt{6}$

1.7: Intro to AP: Basic Derivatives Numerically and Graphically

Traditionally, calculus was an algebraically heavy subject. One of the philosophical changes that the CollegeBoard made in the 1990s was to emphasize that calculus should be understood in a variety of modes. As they state in their enduring understanding:

> *"Students should be able to work with functions represented in a veriety of ways: graphical, numerical, analytical or verbal. They should understand the connections among these representations."*

Later, they added that students should be able to verbalize their understanding and be able to communicate that understanding through proper writing. We will consider this later as we consider more context-oriented problems.

OBJECTIVES

Determine Derivative Values from Numerical or Graphical Data.

Ex 1.7.1: Assume $h(x) = f(x)g(x)$. Given the table of values below, find $h'(2)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

Sol 1.7.1:

$$h(x) = f(x)g(x)$$

$$h'(x) = f(x)g'(x) + g(x)f'(x)$$

$$h'(2) = f(2)g'(2) + g(2)f'(2)$$

$$= 1 \cdot 7 + 8 \cdot 5$$

$$= \boxed{47}$$

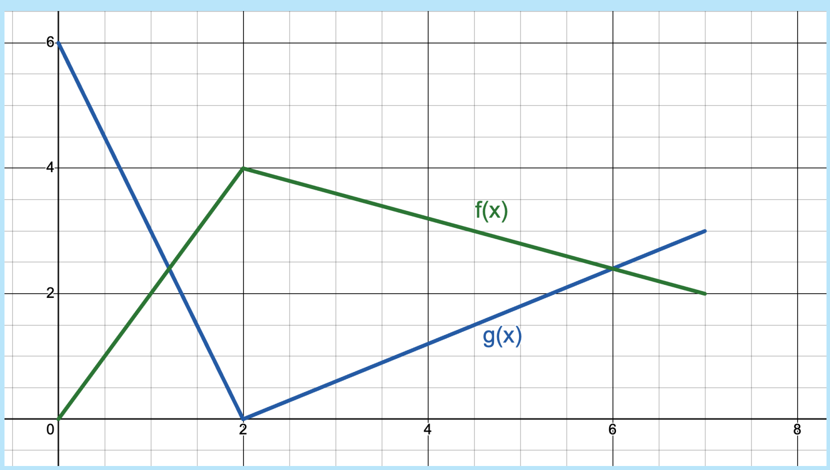
Ex 1.7.2: Using the table of values in **Ex 1.7.1:**, find $\frac{d}{dx}\left[f(g(x))\right]$ and $\frac{d}{dx}\left[g(f(x))\right]$ at $x = 1$.

Sol 1.7.2: These are two different but similar problems, so let's consider them individually.

$$\begin{aligned}\frac{d}{dx}\left[f(g(x))\right]\bigg|_{x=1} &= f'(g(1))\cdot g'(1)\\ &= f'(2)\cdot 6\\ &= 5\cdot 6\\ &= \boxed{30}\end{aligned}$$

$$\begin{aligned}\frac{d}{dx}\left[g(f(x))\right]\bigg|_{x=1} &= g'(f(1))\cdot f'(1)\\ &= g'(3)\cdot 4\\ &= 9\cdot 4\\ &= \boxed{36}\end{aligned}$$

Ex 1.7.3: Given the graph below, find (a) $w'(1)$ if $w = \dfrac{g(x)}{f(x)}$ and (b) $v'(1)$ if $v = g(f(x))$.



Sol 1.7.3:

(a)

$$w = \frac{g(x)}{f(x)} \therefore w' = \frac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}$$

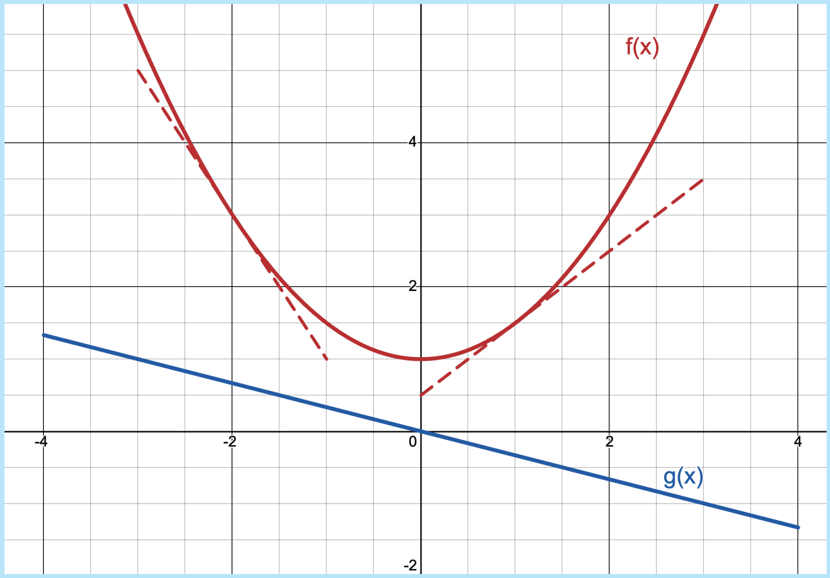
$$\begin{aligned} w'(1) &= \frac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2} \\ &= \frac{2 \cdot (-3) - 3 \cdot 2}{2^2} \\ &= \boxed{-3} \end{aligned}$$

(b)

$$\begin{aligned} v(x) &= g(f(x)) \\ v'(x) &= g'(f(x)) \cdot f'(x) \\ v'(1) &= g'(f(1)) \cdot f'(1) \\ &= g'(2) \cdot 2 \\ &= \boxed{DNE} \end{aligned}$$

(Note that $g'(2)$ does not exist. The slope cannot be determined at $x = 1$ because the slopes to the left and right of $x = 1$ are different. The is called a corner point, or a cusp point, and will be explored further in a later chapter.)

Ex 1.7.4: The figure below shows the graph of the functions f and g. The graphs of the lines tangent to the graph of f at $x = -2$ and $x = 1$ are also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $-\dfrac{5}{6}$ b) $-\dfrac{1}{2}$ c) $\dfrac{1}{6}$ d) $\dfrac{1}{3}$ e) $\dfrac{1}{2}$

Sol 1.7.4:

$$B(x) = f(x) \cdot g(x)$$

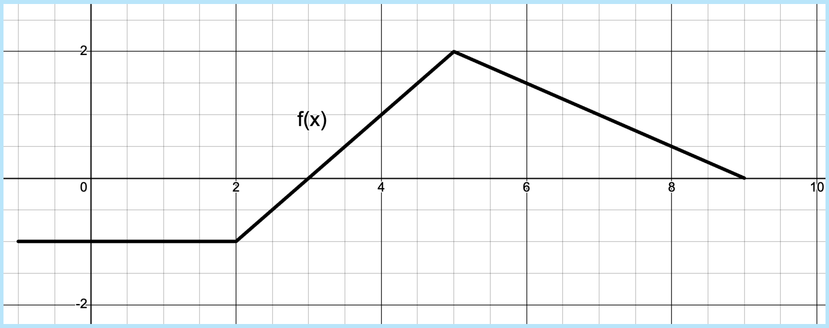
$$B'(x) = f(x)g'(x) + g(x)f'(x)$$

$$B'(1) = f(1)g'(1) + g(1)f'(1)$$

$$= \frac{3}{2} \cdot \left(-\frac{1}{3}\right) + \left(-\frac{2}{3}\right) \cdot 1$$

$$= \boxed{\text{a)} \ -\frac{5}{6}}$$

Ex 1.7.5: Let $f(x)$ be the function whose graph is given below and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below. Furthermore, let h be the function defined by $h(x) = \ln\left(x^2 + 4\right)$.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

(a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(3)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{h\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

Sol 1.7.5:

(a) $f(4) = 1$ and $f'(4) = 1$. Therefore, the tangent line equation is

$$\boxed{y - 1 = 1(x - 4)}$$

(b)

$$h(x) = \ln\left(x^2 + 4\right)$$

$$h'(x) = \frac{2x}{x^2 + 4}$$

$$K(x) = h(f(x)) \therefore K = h'(f(x)) \cdot f'(x)$$

$$\begin{aligned} K'(3) &= h'(f(3)) \cdot f'(3) \\ &= h'(0) \cdot 1 \\ &= 0 \cdot 1 \\ &= \boxed{0} \end{aligned}$$

(c)

$$M(x) = g(x) \cdot f(x)$$

$$M'(x) = g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

$$\begin{aligned} M'(6) &= g(6) \cdot f'(6) + f(6) \cdot g'(6) \\ &= 6 \cdot \frac{1}{2} + 12 \cdot \frac{3}{2} \\ &= \boxed{15} \end{aligned}$$

(d)

$$J(x) = \frac{g(x)}{h\left(\frac{1}{2}x\right)}$$

$$J'(x) = \frac{h\left(\frac{1}{2}x\right) \cdot g'(x) - g(x) \cdot h'\left(\frac{1}{2}x\right) \cdot \frac{1}{2}}{\left[h\left(\frac{1}{2}x\right)\right]^2}$$

$$\begin{aligned} J'(8) &= \frac{h(4) \cdot g'(8) - g(8) \cdot h'(4) \cdot \frac{1}{2}}{[h(4)]^2} \\ &= \boxed{\frac{8 \ln 8 - \frac{4}{5}}{\ln^2 8}} \end{aligned}$$

1.7 Free Response Homework

1. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a) $g'(2)$, where $g(x) = [f(x)]^3$

b) $h'2$, where $h(x) = f(x^3)$

2. The following table shows some values of $g(x),\ g'(x),$ and $h(x)$, where $h(x) = g^{-1}(x)$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
1	2	3	$\frac{1}{2}$	$\frac{1}{3}$
3	1	2	-2	$\frac{1}{2}$

a) Find $g'(1)$

b) Find $h'(1)$

For problems 3 – 14, refer to the values in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	2	8	4	3
8	8	-12	2	4

3. If $h(x) = f(g(x))$, find $h'(8)$

4. If $h(x) = f(g(x))$, find $h'(2)$

5. If $k(x) = g(f(x))$, find $k'(2)$

6. If $m(x) = f(f(x))$, find $m'(4)$

7. If $P_1(x) = f(x)g(x)$, find $P_1'(2)$

8. If $P_1(x) = f(x)g(x)$, find $P_1'(8)$

9. If $P_2(x) = f(2x)g(x)$, find $P_2'(2)$

10. If $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3'(4)$

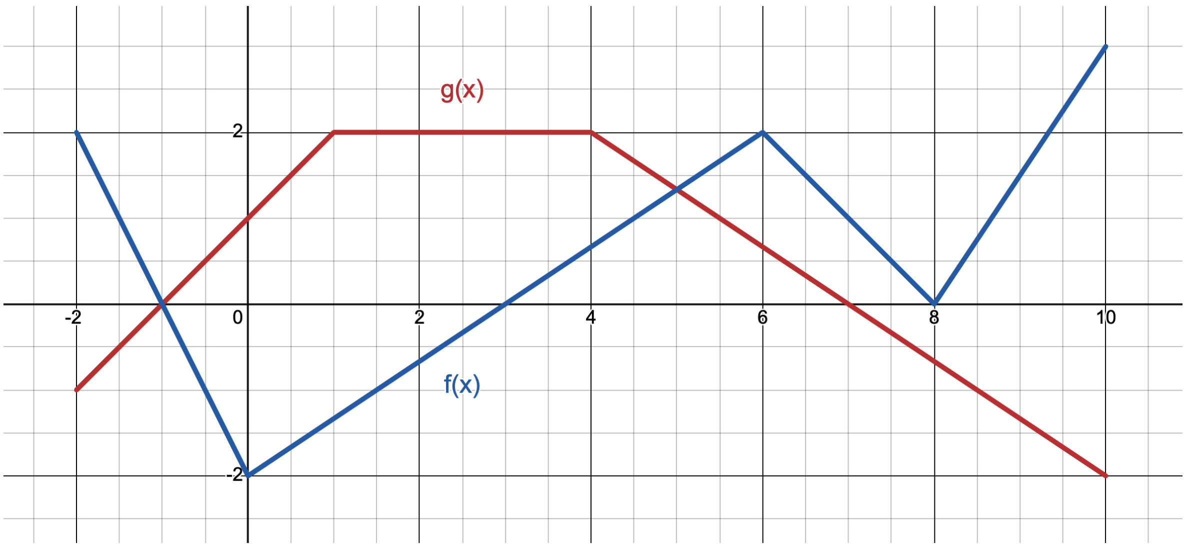
11. If $Q_1(x) = \dfrac{f(x)}{g(x)}$, find $Q_1'(2)$

12. If $Q_2(x) = \dfrac{g(x)}{f(x)}$, find $Q_2'(8)$

13. If $Q_3(x) = \dfrac{f(2x)}{g(x)}$, find $Q_3'(4)$

14. If $Q_4(x) = \dfrac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4'(4)$

For problems 15 – 26, the graphs of $f(x)$ and $g(x)$ are given below.



15. If $u = f(g(x))$, find $u'(2)$

16. $v = g(f(x))$, find $v'(4)$

17. If $w = g(g(x))$, find $w'(6)$

18. If $t = f(f(x))$, find $t'(8)$

19. If $P_1(x) = f(x)g(x)$, find $P_1'(2)$

20. If $P_1(x) = f(x)g(x)$, find $P_1'(8)$

21. If $P_2(x) = f(2x)g(x)$, find $P_2'(2)$

10. If $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3'(2)$

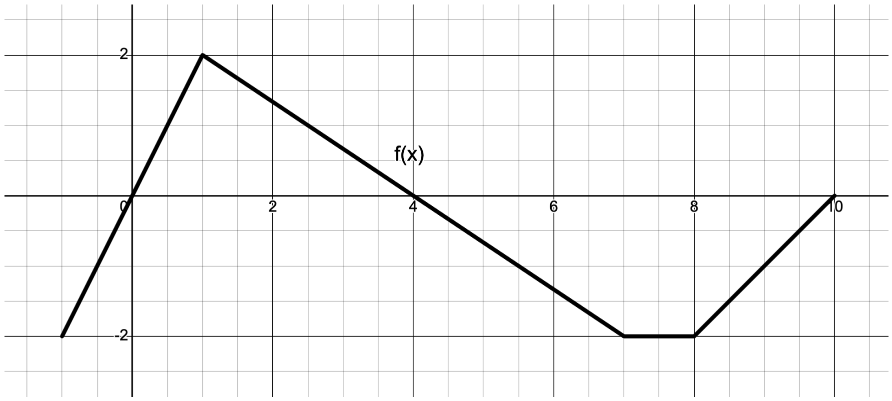
23. If $Q_1(x) = \dfrac{f(x)}{g(x)}$, find $Q_1'(2)$

24. If $Q_2(x) = \dfrac{g(x)}{f(x)}$, find $Q_2'(8)$

25. If $Q_3(x) = \dfrac{f(2x)}{g(x)}$, find $Q_3'(4)$

26. If $Q_4(x) = \dfrac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4'(4)$

27. Let $f(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

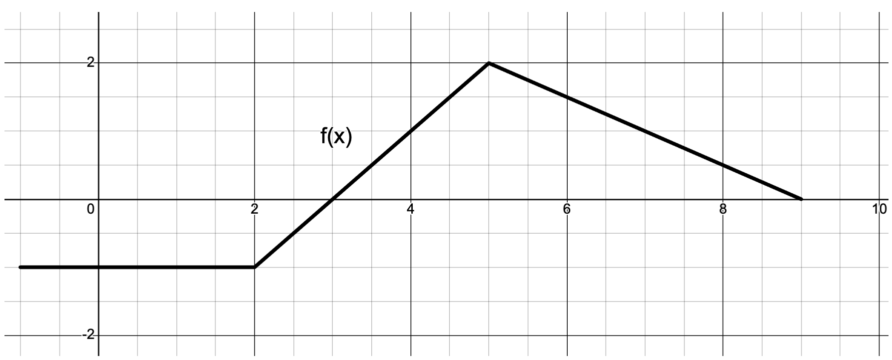
(a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = g(f(x))$. Find $K'(1)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

(d) Let J be the function defined by $J(x) = \dfrac{f(2x)}{g(x)}$. Find $J'(2)$.

28. Let $f(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

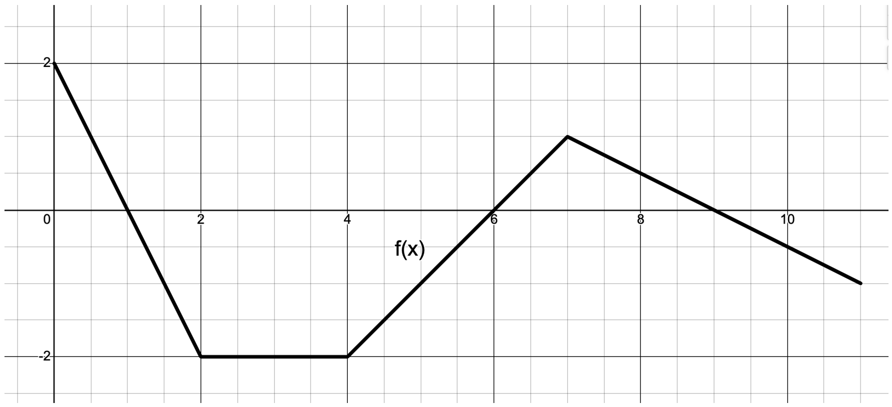
(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = g(g(x))$. Find $K'(8)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(2x)}{f(x)}$. Find $J'(1)$.

29. Let $f(x)$ be the function defined by $f(x) = 4x - x^3$, let $h(x)$ be the function whose graph is given below, and let $g(x)$ be a differentiable function with selected values of $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

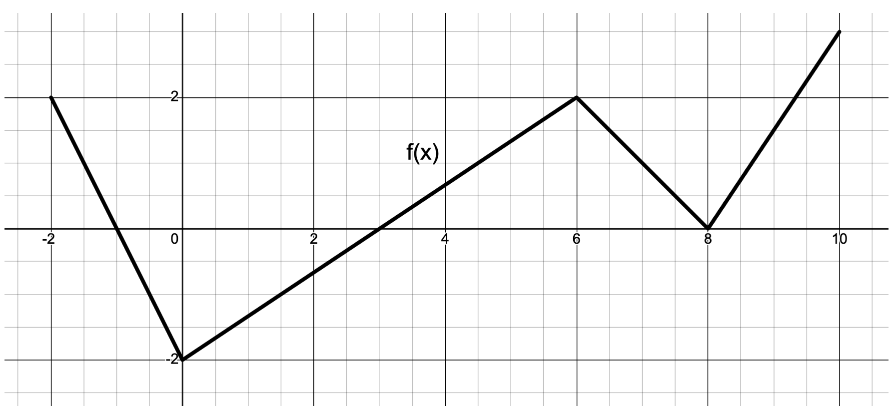
(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(1)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{f(x)}$. Find $J'(4)$.

30. Let h be the function defined by $h(x) = \sin(x) + e^{\cos(3x)}$, let f(x) be the function whose graph is given below, and let g(x) be a differentiable function with selected values of $g(x)$ and $g'(x)$ given in the table below.



x	$g(x)$	$g'(x)$
-4	3	2
-2	5	-1
0	7	0
2	5	-1
4	3	2

(a) Find the equation of the line tangent to $h(x)$ at $x = \dfrac{\pi}{2}$.

(b) Let K be the function defined by $K(x) = f(h(x))$. Find $K'\left(\dfrac{\pi}{2}\right)$.

(c) Let M be the function defined by $M(x) = f(x) \cdot g(x)$. Find $M'(0)$.

(d) Let J be the function defined by $J(x) = g(2x) \cdot f(x)$. Find $J'(2)$.

31. 2017 AP Calculus AB #6

1.7 Multiple Choice Homework

1. Let the function f be differentiable on the interval $[0, 2.5]$ and g be defined by $g(x) = f(f(x))$. Use the table to find $g'(1.5)$.

x	0	0.5	1	1.5	2	2.5
$f(x)$	0.5	1.5	2	2.5	1	0
$f'(x)$	0.1	0.3	0.6	1.1	2	2.2

a) 0 b) 1.24 c) 1.65 d) 2.08 e) 2.42

---

2. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given in the table below, find $h'(2)$, where $h(x) = g(x) \cdot f(x)$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

a) -12 b) -2 c) 0 d) 30 e) 64

---

3. Let $f(x)$ and $g(x)$ be differentiable functions. The table below gives the values of $f(x)$ and $g(x)$, and their derivatives, at several values of x.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	-6
2	1	8	-5	7
3	7	-2	7	9

If $h(x) = \dfrac{f(x)}{g(x)}$, what is the value of $h'(2)$?

a) -4 b) -63 c) 51 d) $-\dfrac{47}{64}$ e) $-\dfrac{33}{64}$

---

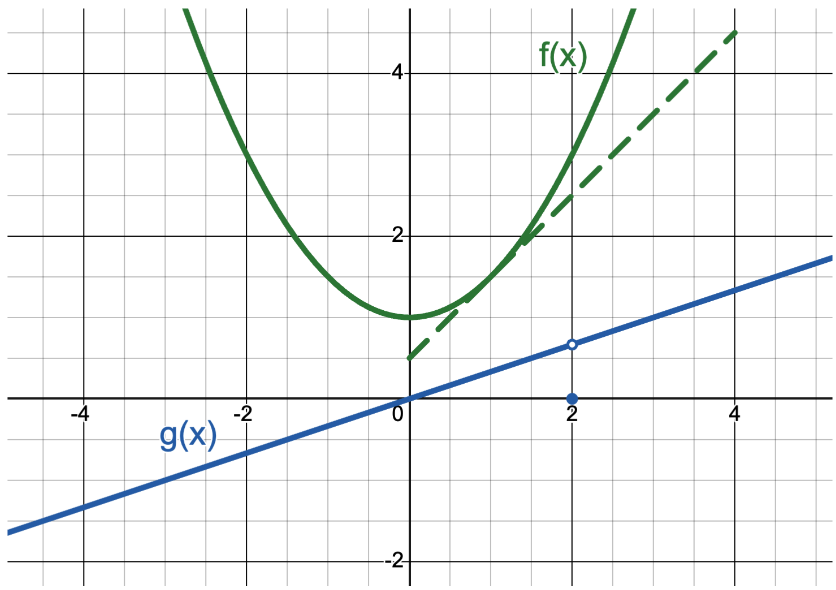
4. Let $h(x) = g(x) \cdot f\left(x^3\right)$. According to the table below, what is the value of $h'(2)$?

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	-3	0	9	10
2	4	6	-4	1
4	9	2	3	3
8	-1	1	2	5

a) -6 b) 2 c) 11 d) 24 e) 143

---

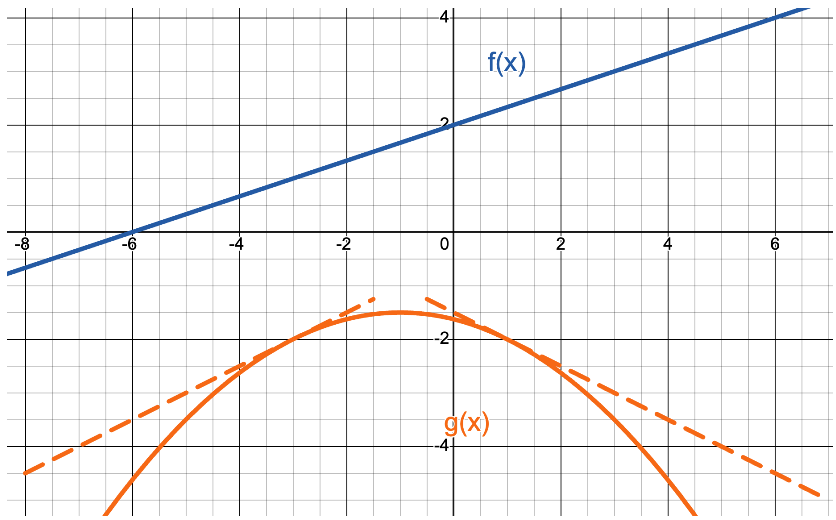
5. The figure below shows the graph of the functions f and g. The graph of the line tangent to the graph of f at $x = 1$ is also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $\dfrac{5}{6}$ b) $-\dfrac{1}{2}$ c) $-\dfrac{1}{6}$ d) $\dfrac{1}{3}$ e) $\dfrac{7}{6}$

---

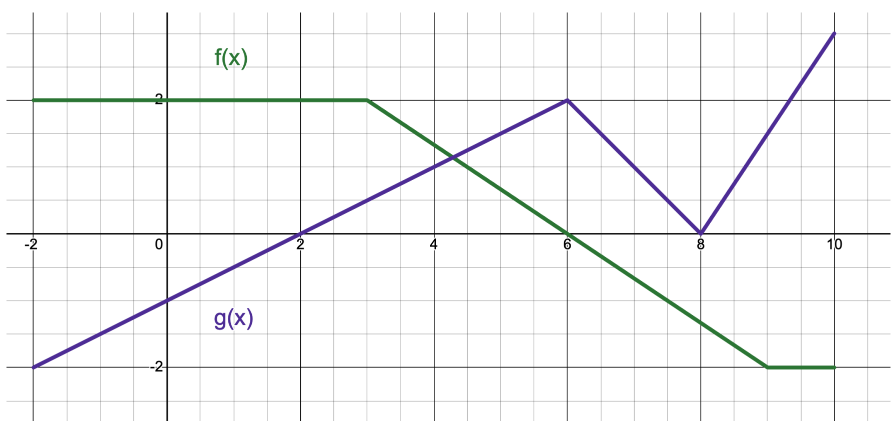
6. The figure below shows the graph of the functions f and g. The graphs of the lines tangent to the graph of g at $x = -3$ and $x = 1$ are also shown. If $B(x) = f(g(x))$, what is $B'(1)$?



a) $-\frac{1}{2}$ b) $-\frac{1}{6}$ c) $\frac{1}{6}$ d) $\frac{1}{3}$ e) $\frac{1}{2}$

---

7. Given the graphs of the two functions below and the fact that $B(x) = f(g(x))$, $B'(4) =$



a) 0 b) 1 c) $\frac{1}{2}$ d) $-\frac{1}{3}$ e) DNE

1.8: Related Rates

In this course, derivatives have primarily been interpreted as the slope of the tangent line. But, as with rectilinear motion, there are other contexts for the derivative. One overarching concept is that the derivative is a **rate of change**. The tendency is to think of rates as distance per time unit, like miles per hour or meters per second, but even slope is a rate of change—it is just that the rise and run are both measured as distances.

The idea behind related rates is two-fold. First, change is occurring in two or more measurements that are related to each other by the geometry (or algebra) of the situation. Second, an implicit Chain Rule situation exists in that the x and y-values are functions of time, which may or may not be a variable in the problem. Therefore, when taking the derivative of an x or y, an **implicit rate term** $\left(\dfrac{dx}{dx} \text{ or } \dfrac{dy}{dt}\right)$ often occurs.

OBJECTIVES

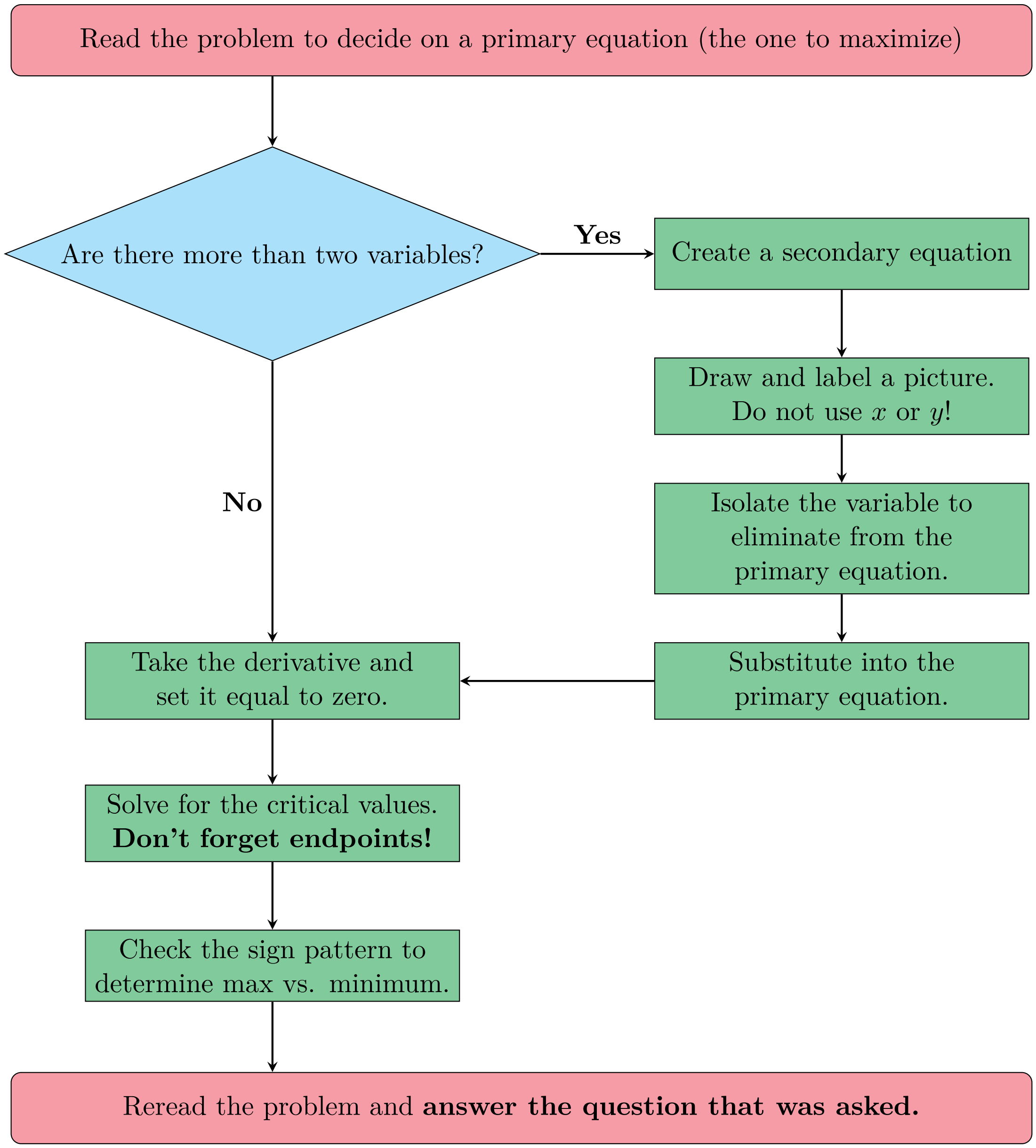
Solve Related Rates Problems.

At first glance, related rates problems might seem like optimization problems that we've seen last year. Consider the following example:

Ex 1.8.1: The volume of a cylindrical cola can is 32π in^3. What is the minimum surface area for such a can?

The word "minimum" tells us that we have an optimization problem. Recall our workflow for tackling optimization problems:

Strategy For Optimization Problems



So, let's tackle our example.

Sol 1.8.1: The problem asks to minimize surface area, which is determined by:

$$S = 2\pi r^2 + 2\pi r h$$

As there are more than two variables in this equation, either r or h needs to be eliminated in this formula before differentiating. The volume is $V = \pi r^2 h = 32\pi$, so $h = \dfrac{32}{r^2}$ and

$$S = 2\pi r^2 + 2\pi r\left(\frac{32}{r^2}\right)$$

$$= 2\pi r^2 + \frac{64\pi}{r}$$

$$S' = 4\pi r - \frac{64\pi}{r^2} = 0$$

$$\therefore 4\pi r = \frac{64\pi}{r^2}$$

$$r^3 = 16 \to r = 2.5198$$

Now, let's make a sign pattern to determine if this critical value is a minimum.

$$\begin{array}{cccc} S' & - & 0 & + \\ & \longleftarrow & \longrightarrow & \\ r & & 2.520 & \end{array}$$

Because the derivative changes from negative to positive, we know that 2.5198 is a minimum. We can plug it back into our surface area equation to find our minimum surface area.

$$S(2.5198) = 2\pi(2.5198)^2 + \frac{64\pi}{2.5198}$$

$$S(2.5198) = 119.687 \text{ in}^2$$

Therefore, the minimum surface area of the cola can is $\boxed{119.687 \text{ in}^2}$.

A related rates problem is characterized by various measurements that are changing **in relation to each other**. The variables are still related to each other through a geometric or physical relationship. The key difference from other differentiation problems is that we differentiate implicitly with respect to time, rather than a variable like x. In other words,

Optimization Problems:	Apply $\dfrac{d}{dx}$
Related Rates Problems:	Apply $\dfrac{d}{dt}$

Now, let's take a look at this different (related rates) cola problem.

Ex 1.8.2: The volume of a cylindrical cola can is 32π in^3. The height of the can is changing at $\frac{1}{4}$ in/sec. If the radius changes at the same time so as to maintain the volume, how fast is the radius shrinking when the can is 4 inches tall?

Sol 1.8.2:

$$V = \pi r^2 h = 32\pi$$

$$h = \frac{32}{r^2}$$

$$\frac{d}{dt}\left[h = \frac{32}{r^2}\right]$$

$$\frac{dh}{dt} = -\frac{64}{r^2}\left(\frac{dr}{dt}\right)$$

Now that we have a method to find what we are looking for, $\frac{dr}{dt}$, let's substitute in the values that we are given in the problem.

$$h = 4 \to \frac{32}{r^2} = 4 \to r^2 = 8$$

$$\frac{1}{4} = -\frac{64}{8}\left(\frac{dr}{dt}\right)$$

$$\frac{dr}{dt} = \boxed{-\frac{1}{32}\text{ in/sec}}$$

As we can see, many of the steps that we take in the related rates cola problem is similar to those of the optimization cola problem.