Chapter 1:

Review of Derivatives

Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the ”how” of differentiation. We will review all the derivative rules learned last year in Precalculus. In the next two chapters, we will review the ”why.” As a quick reference, here are those rules:

The Power Rule: $\frac{d}{dx}\left[u^n\right] = nu^{n-1}\frac{du}{dx}$

The Product Rule: $\frac{d}{dx}\left[u \cdot v\right] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

The Chain Rule: $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Here is a quick review from last year:

Identities: While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x; \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin 2x = 2 \sin x \cos x; \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverses: Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ -\textit{calculator} \ \pm 2\pi n \end{array} \right\} \qquad \sin^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi - \textit{calculator} \ \pm 2\pi n \end{array} \right\}$$

$$\tan^{-1} x = \left\{ \begin{array}{l} \textit{calculator} \ \pm 2\pi n \\ \pi + \textit{calculator} \ \pm 2\pi n \end{array} \right\} = \textit{calculator} \ \pm \pi n$$

Logarithm Rules: Here are some logarithm rules which you should recall:

$$\log_a x + \log_a y = \log_a xy$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

1.1: The Power and Exponential Rules with the Chain Rule

In Precalculus we developed the idea of the derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the derivative will be explored. In this chapter, we are primarily interested in how to find the derivative and what it is used for.

Derivative → Definition: $f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

→ Means: The function that yields the slope of the tangent line.

Numerical Derivative → Definition: $f'(a) = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

→ Means: The numerical value of the slope of the tangent line at $x = a$

Symbols for the Derivative

$\dfrac{dy}{dx}$ = "d - y - d - x" $f'(x)$ = "f prime of x" y' = "y prime"

$\dfrac{d}{dx}$ = "d - d - x" D_x = "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to Find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from Precalculus: The derivative yields the slope of the tangent line. (But there is more to it than that).

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in Precalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule:

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^x\right] = e^x$$

$$\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln a$$

The difference between these is where the variable is. The Power Rule applies when the variable is in the *base*, while the Exponential Rules apply when the variable is in the *exponent*. The difference between the two Exponential rules is what the base is. $e = 2.718281828459\ldots$, while a is any positive number other than 1.

Ex 1.1.1: Find a) $\dfrac{d}{dx}\left[x^5\right]$ and b) $\dfrac{d}{dx}\left[5^x\right]$.

Sol 1.1.1: The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\dfrac{d}{dx}\left[x^5\right] = \boxed{5x^4}$ b) $\dfrac{d}{dx}\left[5^x\right] = \boxed{5^x \ln 5}$

There are a few other basic rules that we need to remember.

$$\frac{d}{dx}[\text{constant}] = 0$$

$$\frac{d}{dx}\left[cx^n\right] = (cn)x^{n-1}$$

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

These rules allow us to easily differentiate a polynomial term by term.

Ex 1.1.2: $y = 3x^2 + 5x + 1$; find $\dfrac{dy}{dx}$.

Sol 1.1.2:

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[3x^2+5x+1\right]\\ &= (3\cdot 2)x^{2-1}+(5\cdot 1)x^{1-1}+0\\ &= \boxed{6x+5}\end{aligned}$$

Ex 1.1.3: $f(x)=x^2+4x-3+e^x$; find $f'(x)$.

Sol 1.1.3:

$$\begin{aligned}f'(x) &= \frac{d}{dx}\left[x^2+4x-3+e^x\right]\\ &= (1\cdot 2)x^{2-1}+(4\cdot 1)x^{1-1}-0+e^x\\ &= \boxed{2x+4+e^x}\end{aligned}$$

Ex 1.1.4: $y=\sqrt{x^3}+\dfrac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4$; find $\dfrac{dy}{dx}$.

Sol 1.1.4:

$$\begin{aligned}y &= \sqrt{x^3}+\frac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4\\ &= x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4\\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^{\frac{3}{2}}+4x^{-\frac{1}{2}}-x^{\frac{3}{4}}+e^4\right]\\ &= \left(1\cdot\frac{3}{2}\right)x^{\frac{3}{2}-1}+\left(4\cdot-\frac{1}{2}\right)x^{-\frac{1}{2}-1}-\left(1\cdot\frac{3}{4}\right)x^{\frac{3}{4}-1}+0\\ &= \boxed{\frac{3}{2}x^{\frac{1}{2}}-2x^{-\frac{3}{2}}-\frac{3}{4}x^{-\frac{1}{4}}}\end{aligned}$$

Note in Ex 1.1.4 that e^4 is a constant. Therefore, its derivative is 0.

As we have seen, when the variable was in the exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right] = \frac{d}{dx}\left[e^{x^2 \ln(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Ex 1.1.5: If $y = (x^2+1)(x^3-4x)$, find $\frac{dy}{dx}$.

Sol 1.1.5:

$$\begin{aligned} y &= (x^2+1)(x^3-4x) \\ &= x^5 - 4x^3 + x^3 - 4x \\ &= x^5 - 3x^3 - 4x \\ \frac{dy}{dx} &= \frac{d}{dx}\left[x^5 - 3x^3 - 4x\right] \\ &= \boxed{5x^4 - 9x^2 - 4} \end{aligned}$$

Ex 1.1.6: If $y = \frac{x^2-4x+6}{\sqrt[3]{x}}$, find $\frac{dy}{dx}$.

Sol 1.1.6:

$$\begin{aligned} y &= \frac{x^2-4x+6}{\sqrt[3]{x}} \\ &= \frac{x^2-4x+6}{x^{\frac{1}{3}}} \\ &= x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}} \end{aligned}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^{\frac{5}{3}} - 4x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}\right]$$

$$= \boxed{\frac{5}{3}x^{\frac{2}{3}} - \frac{8}{3}x^{-\frac{1}{3}} - 2x^{-\frac{4}{3}}}$$

The Chain Rule

Composite Function → Definition: A function made of two other functions, one within the other.

→ For example, $y = \sqrt{16x - x^3}$, $y = \sin\left(x^3\right)$. $y = \cos^3(x)$, and $y = \left(x^2 + 2x - 5\right)^3$. The general symbol is $f(g(x))$.

Ex 1.1.7: Given $f(x) = \cos^{-1}(x)$, $g(x) = x^2 - 1$, and $h(x) = \sqrt{1 + x^2}$, find a) $f\left(g\left(\sqrt{2}\right)\right)$, b) $h(g(1))$, and c) $f(h(g(1)))$.

Sol 1.1.7:

(a) $g\left(\sqrt{2}\right) = \left(\sqrt{2}\right)^2 - 1 = 1,$ so $f\left(g\left(\sqrt{2}\right)\right) = f(1) = \cos^{-1}(1) = \boxed{0}$.

(b) $g(1) = 0,$ so $h(g(1)) = h(0) = \sqrt{1 + 0^2} = \boxed{1}$

(c) $g(1) = 0$ and $h(g(1)) = h(0) = \sqrt{1 + 0^2} = 1,$ so $f(h(g(1))) = \cos^{-1}(1) = \boxed{0}$

So. How do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. The result is two open dolls adjacent to each other.

$$\text{The Chain Rule: } \frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

If you think of the inside function (the $g(x)$) as equaling u, we could write the Chain Rule like this:

$$\frac{d}{dx}[f(u)] = \frac{df}{du} \cdot \frac{du}{dx}$$

This is the way that most derivatives are written with the Chain Rule.

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other rules, as seen in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:

$$\frac{d}{dx}\left[u^n\right] = nu^{n-1} \cdot \frac{du}{dx}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^u\right] = e^u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = \left(a^u \cdot \ln a\right) \cdot \frac{du}{dx}$$

(where u is a function of x)

Ex 1.1.8: $\frac{d}{dx}\left[\left(4x^2 - 2x - 1\right)^{10}\right]$

Sol 1.1.8:

$$u = 4x^2 - 2x - 1 \text{ and } f(u) = u^{10}$$

$$\begin{aligned} \frac{d}{dx}\left[f(u)\right] &= f'(u) \cdot \frac{du}{dx} \\ &= 10u^9 \cdot (8x - 2) \\ &= \boxed{10\left(4x^2 - 2x - 1\right)^9 (8x - 2)} \end{aligned}$$

Ex 1.1.9: $\dfrac{d}{dx}\left[e^{4x^2}\right]$

Sol 1.1.9:

$$u = 4x^2$$

$$\begin{aligned}\frac{d}{dx}\left[e^u\right] &= e^u \cdot \frac{du}{dx} \\ &= e^{4x^2} \cdot 8x \\ &= \boxed{8xe^{4x^2}}\end{aligned}$$

Ex 1.1.10: If $y = \sqrt{16 - x^3}$, find $\dfrac{dy}{dx}$.

Sol 1.1.10:

$$u = 16 - x^3 \text{ and } f(u) = \sqrt{u}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left[f(u)\right] = f'(u) \cdot \frac{du}{dx} \\ &= \frac{1}{2\sqrt{u}} \cdot -3x^2 \\ &= \boxed{\frac{-3x^2}{2\sqrt{16 - x^3}}}\end{aligned}$$

Ex 1.1.11: $\dfrac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5 + 7}\right]$

Sol 1.1.11:

$$u = x^2 + 1,\ g(u) = u^5 + 7,\ \text{and } f(g(u)) = \sqrt{g(u)}$$

$$
\begin{aligned}
\frac{d}{dx}\left[f(g(u))\right] &= f'(g(u)) \cdot g'(u) \cdot \frac{du}{dx} \\
&= \frac{1}{2\sqrt{g\left(u\right)}} \cdot 5u^4 \cdot 2x \\
&= \frac{5\left(x^2+1\right)\left(2x\right)}{2\sqrt{\left(x^2+1\right)^5+7}} = \boxed{\frac{5x\left(x^2+1\right)}{\sqrt{\left(x^2+1\right)^5+7}}}
\end{aligned}
$$

1.1 Free Response Homework

Find the derivatives of the given functions. Simplify where possible.

1. $f(x) = x^2 + 3x - 4$
2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$
3. $y = x^{-\frac{2}{3}}$
4. $y = 5e^x + 3$
5. $v(r) = \frac{4}{3}\pi r^3$
6. $g(x) = x^2 + \frac{1}{x^2}$
7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$
8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$
9. $z = \frac{A}{y^{10}} + Be^y$
10. $y = e^{x+1} + 1$

Complete the following.

11. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$
12. $\frac{d}{dx}\left[x^6 - 3\sqrt[6]{x^7} + 5^x - \frac{1}{\sqrt[3]{x^5}} + \frac{1}{8x}\right]$
13. $\frac{d}{dx}\left[x^4 - 14\sqrt[7]{x^9}\right] + 8^x - \frac{1}{\sqrt[3]{x^7}} + \frac{1}{8x}$
14. $\frac{d}{dx}\left[(x-1)\sqrt{x}\right]$
15. $\frac{d}{dz}\left[\left(z^2 - 4\right)\sqrt{z^3}\right]$
16. $\frac{d}{dx}\left[\left(x^2 - 4x + 3\right)\sqrt{x^5}\right]$
17. $\frac{d}{dt}\left[\left(4t^2 + 1\right)\left(3t^3 + 7\right)\right]$
18. $\frac{d}{dx}\left[\left(x^3 + 4x - \pi\right)^{-7}\right]$
19. $\frac{d}{dx}\left[\sqrt{3x^2 - 4x + 9}\right]$
20. $\frac{d}{dx}\left[\sqrt[7]{x^3 - 2x}\right]$
21. $\frac{d}{dy}\left[\frac{4y^3 - 2y^2 - 5y}{\sqrt{y}}\right]$
22. $\frac{d}{dv}\left[\frac{v^2 - 4v + 7}{2\sqrt{v}}\right]$
23. $\frac{d}{dw}\left[\frac{7w^2 - 4w + 1}{5w^3}\right]$
24. $\frac{d}{dw}\left[\frac{5w^2 - 3w - 4}{7w^2}\right]$
25. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$
26. $f(x) = \sqrt[5]{\left(\frac{1}{x} + 2x + e^x\right)^3}$, find $f'(x)$
27. $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$
28. $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$
29. $g(2) = 3,\ g'(2) = -4,\ f(x) = e^{g(x)}$, find $f'(2)$
30. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$

31. $f(x)=\sqrt{4-\frac{4}{9}x^2}$, find $f'\left(\sqrt{5}\right)$

32. $f(x)=e^{\sqrt{9-x^2}}$, find $f'(x)$

33. $v(t)=\sqrt{\left(\frac{E(t)}{3}+3t\right)^{\frac{3}{7}}-4}$, find $v'(t)$

34. $v(t)=\sqrt[3]{\left(\frac{C(t)}{7}+4t^2\right)^{\frac{5}{7}}-1}$, find $v'(t)$

1.1 Multiple Choice Homework

1. If $f(x)=x^{\frac{3}{2}}$, then $f'(4)=$

a) -6 b) -3 c) 3 d) 6 e) 8

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2. The derivative of $\sqrt{x}-\frac{1}{x\sqrt[3]{x}}$

a) $\frac{1}{2}x^{-\frac{1}{2}}-x^{-\frac{4}{3}}$ b) $\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{7}{3}}$ c) $\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

d) $-\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{7}{3}}$ e) $-\frac{1}{2}x^{-\frac{1}{2}}-\frac{4}{3}x^{-\frac{1}{3}}$

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3. Given $f(x)=\frac{1}{2x}+\frac{1}{x^2}$, find $f'(x)$

a) $-\frac{1}{2x^2}-\frac{2}{x^3}$ b) $-\frac{2}{x^2}-\frac{2}{x^3}$ c) $\frac{2}{x^2}-\frac{2}{x^3}$

d) $-\frac{1}{2x^2}+\frac{2}{x^3}$ e) $\frac{1}{2x^2}-\frac{2}{x^3}$

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4. If $f(x)=e^{5x^2}+x^4$, then $f'(1)=$

a) e^5+1 b) $5e^4+4$ c) $5e^5+1$ d) $10e+4$ e) $10e^5+4$

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5. If h is the function defined by $h(x)=e^{5x}+x+3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

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6. If $y = \left(x^4 + 4\right)^2$, then $\dfrac{dy}{dx} =$

a) $2\left(x^4 + 4\right)$ b) $\left(4x^3\right)^2$ c) $2\left(4x^3 + 4\right)$

d) $4x^3\left(x^4 + 4\right)$ e) $8x^3\left(x^4 + 4\right)$

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7. If $h(x) = [f(x)]^2\, g(x)$ and $g(x) = 3$, then $h'(x) =$

a) $2f'(x)g'(x)$ b) $6f'(x)f(x)$ c) $g'(x)\,[f(x)]^2 + 2f(x)f'(x)g(x)$

d) $2f'(x)g(x) + g'(x)\,[f(x)]^2$ e) 0

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8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[\sqrt{e^x + 3}\right] = \dfrac{e^x}{2\sqrt{e^x + 3}}$

II. $\dfrac{d}{dx}\left[5^{3x^2}\right] = 6x\ln(5)\left(5^{3x^2}\right)$

III. $\dfrac{d}{dx}\left[6x^3 - \pi + \sqrt[3]{x^8} - \dfrac{2}{x^3}\right] = 18x^2 + \dfrac{8}{3}\sqrt[3]{x^5} + \dfrac{6}{x^4}$

a) I only b) II only c) I and III only

d) I and III only e) I, II, and III

1.2: Trig, Trig Inverse, and Log Rules

Trigonometric → Definition: A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure.

→ Means: An equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic → Definition: The inverse of an exponential function.

→ Means: An equation with log or ln in it.

Trig Derivative Rules

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

Log Derivative Rules

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx} \qquad \frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

Note that all these rules are expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives Involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1.2.1: $\frac{d}{dx}\left[\sin^3(x)\right]$

Sol 1.2.1:

$$\frac{d}{dx}\left[\sin^3(x)\right] = \boxed{3\sin^2(x)\cos(x)}$$

Ex 1.2.2: $\dfrac{d}{dx}\left[\sin\left(x^3\right)\right]$

Sol 1.2.2:

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \boxed{3x^2\cos\left(x^3\right)}$$

Ex 1.2.3: $\dfrac{d}{dx}\left[\ln\left(4x^3\right)\right]$

Sol 1.2.3:

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(4x^3\right)\right] &= \frac{1}{4x^3}\cdot 12x^2\\ &= \boxed{\frac{3}{x}}\end{aligned}$$

We could have also simplified algebraically before taking the derivative:

$$\begin{aligned}\ln\left(4x^3\right) &= \ln 4 + \ln x^3\\ &= \ln 4 + 3\ln x\\ \frac{d}{dx}\left[\ln 4 + 3\ln x\right] &= 0 + 3\cdot\frac{1}{x}\\ &= \boxed{\frac{3}{x}}\end{aligned}$$

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 1.2.4: $\dfrac{d}{dx}\left[\sec^5\left(3x^4\right)\right]$

Sol 1.2.4:

$$\begin{aligned}\frac{d}{dx}\left[\sec^5\left(3x^4\right)\right] &= 5\sec^4\left(3x^4\right)\cdot\sec\left(3x^4\right)\tan\left(3x^4\right)\cdot\left(12x^3\right)\\ &= \boxed{60x^3\sec^5\left(3x^4\right)\tan\left(3x^4\right)}\end{aligned}$$

Ex 1.2.5: $\frac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right)$

Sol 1.2.5:

$$\begin{aligned}\frac{d}{dx}\ln\left(\cos\left(\sqrt{x}\right)\right) &= \frac{1}{\cos\left(\sqrt{x}\right)}\cdot\left(-\sin\left(\sqrt{x}\right)\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= -\tan\left(\sqrt{x}\right)\cdot\frac{1}{2\left(\sqrt{x}\right)}\\ &= \boxed{\frac{-\tan\left(\sqrt{x}\right)}{2\sqrt{x}}}\end{aligned}$$

General inverses are not all that interesting. We are more interested in particular *transcendental* inverse functions, like the natural log. Another particular kind of inverse function that bears more study is the trig inverse function. Interestingly, as with the log functions, the derivatives of these transcendental functions become algebraic functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1}u\right] = \left(\frac{1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc^{-1}u\right] = \left(\frac{-1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos^{-1}u\right] = \left(\frac{-1}{\sqrt{1-u^2}}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec^{-1}u\right] = \left(\frac{1}{|u|\sqrt{u^2-1}}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \left(\frac{1}{u^2+1}\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot^{-1}u\right] = \left(\frac{-1}{u^2+1}\right)\frac{du}{dx}$$

Ex 1.2.6: $\dfrac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right]$

Sol 1.2.6:

$$\frac{d}{dx}\left[\tan^{-1}\left(3x^4\right)\right] = \frac{1}{(3x^4)^2+1}\cdot\left(12x^3\right)$$

$$= \boxed{\frac{12x^3}{9x^8+1}}$$

Ex 1.2.7: $\dfrac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right]$

Sol 1.2.7:

$$\frac{d}{dx}\left[\sec^{-1}\left(x^2\right)\right] = \frac{1}{|x^2|\sqrt{(x^2)^2-1}}\cdot 2x$$

$$= \frac{2x}{(x^2)\sqrt{(x^2)^2-1}}$$

$$= \boxed{\frac{2}{x\sqrt{x^4-1}}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] - \frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 1.2.8: If $f(x) = x^2 + 2x + 3,\ g(x) = f^{-1}(x),$ and $g(1) = 2;$ find $g'(1)$.

Sol 1.2.8:

$$f'(x) = 2x + 2 \ \therefore\ f'\left(g(x)\right) = 2\left(g(x)\right) + 2$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{f'\left(g(x)\right)}$$

$$g'(1) = \frac{1}{f'\left(g(1)\right)} = 2\left(g(1)\right) + 2 = \boxed{6}$$

1.2 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin(4x)$
2. $y = 4\sec\left(x^5\right)$
3. $f(t) = \sqrt[3]{1 + \tan t}$
4. $f(\theta) = \ln(\cos(\theta))$
5. $y = a^3 + \cos^3(x)$
6. $y = \cos\left(a^3 + x^3\right)$
7. $f(x) = \cos(\ln x)$
8. $f(x) = \sqrt[5]{\ln x}$
9. $f(x) = \log_{10}(2 + \sin(x))$
10. $f(x) = \log_2(1 - 3x)$
11. $y = \sin^{-1}(e^x)$
12. $y = \tan^{-1}(\sqrt{x})$

Complete the following.

13. $\dfrac{d}{dx}\left[\sin^{-1}\left(e^{3x}\right)\right]$
14. $\dfrac{d}{dx}\left[\cot^{-1}\left(e^{2x}\right)\right]$
15. $\dfrac{d}{dx}\left[\tan^{-1}\left(x^2\right)\right]$
16. $\dfrac{d}{dx}\left[\cot^{-1}\left(\dfrac{1}{x}\right) - \tan^{-1}(x)\right]$
17. $\dfrac{d}{dx}\left[3e^{x^2+2x}\right]$
18. $\dfrac{d}{dx}\left[3\cos\left(x^2 + 2x\right)\right]$
19. $\dfrac{d}{dx}\left[\sqrt[3]{16 + x^3}\right]$
20. $\dfrac{d}{dx}\left[\sec^{-1}\left(2x^2\right)\right]$
21. $\dfrac{d}{dx}\left[5e^{\tan(7x)}\right]$
22. $\dfrac{d}{dx}\left[\sqrt{\cos(1 - x^2)}\right]$
23. $\dfrac{d}{dx}\left[\ln^3\left(x^2 + 1\right)\right]$
24. $\dfrac{d}{dx}\left[\ln\sin\left(x^3\right)\right]$
25. $\dfrac{d}{dx}[\ln(\sec(x))]$
26. $\dfrac{d}{dx}\left[\cos\left(x^2\right)\right]$
27. $f(x) = \ln\left(x^2 + 3\right)$, find $f'(x)$
28. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g'(x)$
29. $h(x) = \sqrt{x^2 + 5}$, find $h'(x)$
30. $F(x) = \sqrt[3]{3x^2 - 6x + 1}$, find $F'(x)$
31. $y = \sin^{-1}(\cos(x))$, find y'
32. $y = \sin\left(\cos^{-1}(x)\right)$, find y'
33. $y = \tan^2(3\theta)$, find y'
34. $y = \cot^7(\sin(\theta))$, find y'

35. $y = \sin^{-1}\left(\sqrt{2}(x)\right)$, find y'

36. $y = \sin^{-1}(2x+1)$, find y'

1.2 Free Response Homework Set B

Find the derivatives of the given functions. Simplify where possible.

1. $y = \cos^{-1}\left(e^{3z}\right)$
2. $y = \tan^{-1}\sqrt{x^2-1}$
3. $y = \sec^{-1}(4x) + \csc^{-1}(4x)$
4. $f(x) = \ln\left(\tan^{-1}(5x)\right)$
5. $g(w) = \sin^{-1}(5w) + \cos^{-1}(5w)$
6. $f(t) = \sec^{-1}\sqrt{9+t^2}$

Complete the following.

7. $\dfrac{d}{d\theta}\left[e^{\csc(\theta)} + \ln\left(\cot\left(\theta^2\right)\right) - \sec(\theta)\right]$
8. $\dfrac{d}{dx}\left[\ln\left(\sec\left(x^3+5\ln x+7\right)^3\right)\right]$
9. $\dfrac{d}{dx}\left[\ln\left(\tan\left(x^2+5e^x+7\right)^3\right)\right]$
10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$
11. $\dfrac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$
12. $\dfrac{d}{dt}\left[\sin^5\left(\ln(7t+3)\right)\right]$
13. $\dfrac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$
14. $\dfrac{d}{dx}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$
15. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sqrt{9x-27x^2+\dfrac{5}{x^3}}\right]\right]$
16. $\dfrac{d}{dx}\left[\sec(5x) + \cot\left(e^x\right) - 10\ln x\right]$
17. $z = \ln(\cos(t)) + \sec\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$
18. $z = \ln(\tan(t)) + \sin\left(e^t\right) + 7\pi^2$, find $\dfrac{dz}{dt}$
19. $z = \ln(\cot(\theta)) + \sec(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$
20. $z = \ln(\cos(\theta)) + \sin(\ln\theta) + 7\pi^2$, find $\dfrac{dz}{d\theta}$
21. If $g(3) = \dfrac{\pi}{2}$, $g'(3) = \dfrac{\pi}{4}$, and $f(x) = x^3 g(x) + g\left(-3\cos\left(\dfrac{\pi}{3}x\right)\right) - e^{\sin(g(x))}$, find $f'(3)$

1.2 Multiple Choice Homework

1. If $y = \sin^{-1}\left(e^{3\theta}\right)$, then $\dfrac{dy}{d\theta} =$

a) $\dfrac{1}{\sqrt{1-e^{3\theta}}}$ b) $\dfrac{1}{\sqrt{1-e^{6\theta}}}$ c) $\dfrac{1}{\sqrt{1-e^{9\theta^2}}}$

d) $-3e^{3\theta}\cos^{-1}\left(e^{3\theta}\right)$ e) $\dfrac{3e^{3\theta}}{\sqrt{1-e^{6\theta}}}$

---

2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

a) $-\csc(x)\sec^{-2}(\cos(x))$ b) $-\sin(x)\sec^{-2}(\cos(x))$ c) $-\cos(x)\csc^{-2}(\cos(x))$

d) $\dfrac{-\cos(x)}{1-\sin^2(x)}$ e) $\dfrac{-\sin(x)}{\cos^2(x)+1}$

---

3. If $h(x) = \ln\left(x^2\right)\tan^{-1}(x)$, then $h'(1) =$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{4}+1$ c) $\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2}+1$ e) $\dfrac{\pi}{2}+2$

---

4. If $f(t) = t\sqrt{1-t^2}+\cos^{-1}(t)$, then $f'(t) =$

a) $\dfrac{t-2}{2\sqrt{t^2-1}}$ b) $\dfrac{-2t^2}{\sqrt{1-t^2}}$ c) $\dfrac{-2t^2+2}{\sqrt{1-t^2}}$ d) $\dfrac{-1-t^2}{\sqrt{1-t^2}}$ e) $\dfrac{1-t^2}{\sqrt{1-t^2}}$

---

5. If h is the function defined by $h(x) = e^{5x}+x+3$, then $h'(0) =$

a) 2 b) 4 c) 5 d) 6 e) 8

---

6. Given that $f(x) = 8\sin^2(5x)$, find $f''\left(\dfrac{\pi}{30}\right)$

a) $40\sqrt{3}$ b) $40\sqrt{2}$ c) 40 d) 200 e) 0

---

7. If $g(x) = \cos^2(2x)$, then $g'(x)$ is

a) $2\cos(2x)\sin(2x)$ b) $-4\cos(2x)\sin(2x)$ c) $2\cos(2x)$

d) $-2\cos(2x)$ e) $4\cos(2x)$

---

8. If $f(x) = \sin^2(3-x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

9. If $f(x) = \cos^2(3-x)$, then $f'(0) =$

a) $-2\cos(3)$ b) $-2\sin(3)\cos(3)$ c) $6\cos(3)$

d) $2\sin(3)\cos(3)$ e) $6\sin(3)\cos(3)$

---

10. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

1.3: Trig, Trig Inverse, and Log Rules

Remember:

$$\text{The Product Rule: } f'(x) = U \cdot \frac{dV}{dx} + V \cdot \frac{dU}{dx}$$

$$\text{The Quotient Rule: } f'(x) = \frac{V \cdot \frac{dU}{dx} - U \cdot \frac{dV}{dx}}{V^2}$$

OBJECTIVES

Find the Derivative of a Product or Quotient of Two Functions.

The Product Rule

Ex 1.3.1: $\frac{d}{dx}\left[x^2 \sin(x)\right]$

Sol 1.3.1:

$$\frac{d}{dx}\left[x^2 \sin(x)\right] = x^2 \cdot \cos(x) + \sin(x) \cdot (2x)$$

$$= \boxed{x^2 \cos(x) + 2(x)\sin(x)}$$

Ex 1.3.2: $\dfrac{d}{dx}\left[5^x\cos(x)\right]$

Sol 1.3.2:

$$\begin{aligned}\frac{d}{dx}\left[5^x\cos(x)\right] &= 5^x\cdot(-\sin(x)) + \cos(x)\cdot(5^x\ln 5)\\ &= \boxed{5^x\left(\ln(5)\cos(x)+\sin(x)\right)}\end{aligned}$$

The product rule is pretty straightforward. The tricky part is simplifying the algebra.

Ex 1.3.3: If $f(x) = x^2 e^{-\frac{x}{2}}$, find $f'(x)$

Sol 1.3.3:

$$U = x^2,\ \frac{dU}{dx} = 2x$$

$$V = e^{-\frac{x}{2}},\ \frac{dV}{dx} = e^{-\frac{x}{2}}\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-\frac{x}{2}}$$

$$\begin{aligned}f'(x) &= x^2\cdot\left(-\frac{1}{2}e^{-\frac{x}{2}}\right) + e^{-\frac{x}{2}}\cdot 2x\\ &= \boxed{xe^{-\frac{x}{2}}\left(-\frac{1}{2}x+2\right)}\end{aligned}$$

Ex 1.3.4: $\dfrac{d}{dx}\left[x\sqrt{1-x^2}\right]$

Sol 1.3.4:

$$U = x,\ \frac{dU}{dx} = 1$$

$$V = \sqrt{1-x^2} = \left(1-x^2\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(1-x^2\right)^{-\frac{1}{2}}\cdot(-2x) = -\frac{x}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = x\cdot\left(-\frac{x}{\sqrt{1-x^2}}\right)+\sqrt{1-x^2}\cdot 1$$

$$= \frac{-x^2+(1-x^2)}{\sqrt{1-x^2}}$$

$$= \boxed{\frac{1-2x^2}{\sqrt{1-x^2}}}$$

Ex 1.3.5: $\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]$

Sol 1.3.5:

$$U=(2x-3)^8,\ \frac{dU}{dx}=8(2x-3)^7\cdot 2=16(2x-3)^7$$

$$V=\left(3x^2-1\right)^7,\ \frac{dV}{dx}=7\left(3x^2-1\right)^6\cdot 6x=42x\left(3x^2-1\right)^6$$

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]=(2x-3)^8\cdot 42x\left(3x^2-1\right)^6+\left(3x^2-1\right)^7\cdot 16(2x-3)^7$$

This, then, is factorable.

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]=42x(2x-3)^8\left(3x^2-1\right)^6+16\left(3x^2-1\right)^7 16(2x-3)^7$$

$$=2(2x-3)^7\left(3x^2-1\right)^6\left(21x(2x-3)+8\left(3x^2-1\right)\right)$$

$$=2(2x-3)^7\left(3x^2-1\right)^6\left(42x^2-63x+24x^2-8\right)$$

$$=\boxed{2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2-63x-8\right)}$$

Remember that in Section 1.1 we said that we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the exponent. We can now address that situation.

Ex 1.3.6: $\frac{d}{dx}\left[(\cos(x))^{x^2}\right]$

Sol 1.3.6:

$$\frac{d}{dx}\left[(\cos(x))^{x^2}\right] = \frac{d}{dx}\left[e^{x^2\ln(\cos(x))}\right]$$

$$= e^{x^2\ln(\cos(x))} \cdot \left(x^2 \cdot \frac{1}{\cos(x)} \cdot -(\sin(x)) + \ln(\cos(x)) \cdot 2x\right)$$

$$= \boxed{(\cos(x))^{x^2}\left(2x\ln(\cos(x)) - x^2\tan(x)\right)}$$

The Quotient Rule

Ex 1.3.7: $\frac{d}{dx}\left[\frac{6x}{x^2+4}\right]$

Sol 1.3.7:

$$U = 6x,\ \frac{dU}{dx} = 6$$

$$V = x^2+4,\ \frac{dV}{dx} = 2x$$

$$\frac{d}{dx}\left[\frac{6x}{x^2+4}\right] = \frac{\left(x^2+4\right)\cdot 6 - 6x\cdot 2x}{\left(x^2+4\right)^2}$$

$$= \frac{6x^2+24-12x^2}{\left(x^2+4\right)}$$

$$= \boxed{\frac{24-6x^2}{\left(x^2+4\right)}}$$

Ex 1.3.8: $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

Sol 1.3.8:

$$U = x^2+2x-3,\ \frac{dU}{dx} = 2x+2$$

$$V = x-4,\ \frac{dV}{dx} = 1$$

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2+2x-3}{x-4}\right] &= \frac{(x-4)\cdot(2x+2)-(x^2+2x-3)\cdot 1}{(x-4)^2}\\ &= \frac{2x^2-6x-8-x^2-2x+3}{(x-4)^2}\\ &= \boxed{\frac{x^2-8x-5}{(x-4)^2}}\end{aligned}$$

Ex 1.3.9: $\dfrac{d}{dx}\left[\dfrac{x^2-4x+3}{2x^2-5x-3}\right]$

Sol 1.3.9: Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left[\frac{x^2-4x+3}{2x^2-5x-3}\right] &= \frac{d}{dx}\left[\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right]\\ &= \frac{d}{dx}\left[\frac{x-1}{2x+1}\right]\end{aligned}$$

$$U = x-1,\ \frac{dU}{dx} = 1$$

$$V = 2x+1,\ \frac{dV}{dx} = 2$$

$$\frac{d}{dx}\left[\frac{x-1}{2x+1}\right] = \frac{(2x+1)\cdot 1-(x-1)\cdot 2}{(2x+1)^2}$$

$$= \boxed{\frac{3}{(2x+1)^2}}$$

Ex 1.3.10: $\dfrac{d}{dx}\left[\dfrac{\cot(3x)}{x^2+1}\right]$

Sol 1.3.10:

$$U=\cot(3x),\ \frac{dU}{dx}=-\csc^2(3x)\cdot 3=-3\csc^2(3x)$$

$$V=x^2+1,\ \frac{dV}{dx}=2x$$

$$\frac{d}{dx}\left[\frac{\cot(3x)}{x^2+1}\right] = \frac{(x^2+1)\cdot\left(-3\csc^2(3x)\right)-\cot(3x)\cdot 2x}{\left(x^2+1\right)^2}$$

$$= \frac{-3x^2\csc^2(3x)-3\csc^2(3x)-2x\cot(3x)}{\left(x^2+1\right)^2}$$

$$= \boxed{-\frac{\csc^2(3x)\left(3x^2+3\right)+2x\cot(3x)}{\left(x^2+1\right)^2}}$$

As with the Product Rule, the difficulty with the Quotient Rule arises from the algebra needed to simplify our answer.

Ex 1.3.11: If $y=\dfrac{4x}{\sqrt{x^2+4}}$, find $\dfrac{dy}{dx}$

Sol 1.3.11:

$$U=4x,\ \frac{dU}{dx}=4$$

$$V=\sqrt{x^2+4}=\left(x^2+4\right)^{\frac{1}{2}},\ \frac{dV}{dx}=\frac{1}{2}\left(x^2+4\right)^{-\frac{1}{2}}\cdot 2x=\frac{2x}{2\sqrt{x^2+4}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+4}\cdot 4 - 4x\cdot\dfrac{2x}{2\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{\dfrac{4\left(x^2+4\right)}{\sqrt{x^2+4}} - \dfrac{4x^2}{\sqrt{x^2+4}}}{x^2+4}$$

$$= \frac{4x^2+16-4x^2}{(x^2+4)^{\frac{3}{2}}}$$

$$= \boxed{\frac{16}{(x^2+4)^{\frac{3}{2}}}}$$

Ex 1.3.12: Find the equation of the tangent line to $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ at $x = -\sqrt{7}$.

Sol 1.3.12: As we recall, for the equation of a line, we need a point and a slope.

$$\text{The point: } f\left(-\sqrt{7}\right) = \frac{-\sqrt{7}}{\sqrt{\left(-\sqrt{7}\right)^2+9}}$$

$$= -\frac{\sqrt{7}}{4} \to \left(-\sqrt{7}, -\frac{\sqrt{7}}{4}\right)$$

The slope is the derivative at the given x-value:

$$U = x,\ \frac{dU}{dx} = 1$$

$$V = \sqrt{x^2+9} = \left(x^2+9\right)^{\frac{1}{2}},\ \frac{dV}{dx} = \frac{1}{2}\left(x^2+9\right)^{-\frac{1}{2}}\cdot 2x = \frac{2x}{2\sqrt{x^2+9}}$$

$$\frac{dy}{dx} = \frac{\sqrt{x^2+9}\cdot 1 - x\cdot\dfrac{2x}{2\sqrt{x^2+9}}}{x^2+9}$$

Rather than simplify the algebra, we can find the slope by substituting $x = -\sqrt{7}$:

$$\left.\frac{dy}{dx}\right|_{x=-\sqrt{7}} = \frac{\sqrt{\left(-\sqrt{7}\right)^2+9}\cdot 1 - \left(-\sqrt{7}\right)\cdot\dfrac{2\left(-\sqrt{7}\right)}{2\sqrt{\left(-\sqrt{7}\right)^2+9}}}{\left(-\sqrt{7}\right)^2+9}$$

$$= \frac{4 - \frac{7}{4}}{16}$$

$$= \frac{9}{64}$$

The tangent line equation is therefore:

$$\boxed{y + \frac{\sqrt{7}}{4} = \frac{9}{64}\left(x + \sqrt{7}\right)}$$

1.3 Free Response Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = t^3 \cos(t)$

2. $y = (2x-5)^4 \left(8x^2-5\right)^{-3}$

3. $y = \dfrac{\tan(x)-1}{\sec(x)}$

4. $y = \dfrac{\sin(x)}{x^2}$

5. $y = xe^{-x^2}$

6. $y = \dfrac{r}{\sqrt{r^2+1}}$

7. $y = e^{x\cos(x)}$

8. $y = e^{-5x}\cos(3x)$

9. $y = x\sin\left(\dfrac{1}{x}\right)$

10. $y = \ln\left(e^{-x} + xe^{-x}\right)$

11. $y = \dfrac{\sec^{-1}(x)}{x}$

12. $y = \dfrac{\sin(x)}{x^2}$

13. $y = \left(1+x^2\right)\tan^{-1}(x)$

14. $y = \ln\left(x^2+4\right) - x\tan^{-1}\left(\dfrac{x}{2}\right)$

15. $f(x) = x\sqrt{\ln x}$

16. $g(x) = (1+4x)^5\left(3+x-x^2\right)^8$

17. $f(x) = x\cos^{-1}(x) - \sqrt{1-x^2}$

18. $g(x) = \cos^{-1}(x) + x\sqrt{1-x^2}$

Complete the following.

19. $\dfrac{d}{dx}\left[\dfrac{3x^2+4x-3}{x^2-9}\right]$

20. $\dfrac{d}{dx}\left[\dfrac{x^3-2x^2-5x+6}{x+2}\right]$

21. $\dfrac{d}{dx}\left[\dfrac{x^5-12x^3-19x}{3x^3}\right]$

22. $\dfrac{d}{dx}\left[\dfrac{3x+3}{x^3+1}\right]$

23. $\dfrac{d}{dx}\left[\dfrac{x-4}{x^2-9x+20}\right]$

24. $\dfrac{d}{dx}\left[\dfrac{\tan(x)+5}{\sin(x)}\right]$

25. $\dfrac{d}{dx}\left[\dfrac{\sin(x)}{1-\cos(x)}\right]$

26. $\dfrac{d}{dx}\left[\dfrac{x^2}{\cos(x)}\right]$

27. $y = \dfrac{x^2-3}{x^2-4}$, find $\dfrac{dy}{dx}$

28. $f(x) = \dfrac{x^2+2x-8}{x^2-x-3}$, find $f'(x)$

29. $y = \dfrac{x^2+2x-3}{x-4}$, find y'

30. $f(x) = \dfrac{x}{\ln x}$, find $f'(x)$

31. $h(t) = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$, find $h'(t)$

32. $y = \dfrac{\tan(x)}{\cos(x)-3}$, find $\dfrac{dy}{dx}$

33. $f(x) = \left(x\sin(2x) + \tan^4\left(x^7\right)\right)^5$, find $f'(x)$

34. $f(x) = e^x - x^2 \arctan x$, find $f'(x)$

35. $f(x) = \dfrac{\tan(x)}{\tan(x)+1}$, find $f'\left(\dfrac{\pi}{4}\right)$

36. $y = x^2\sqrt{5-x^2}$, find $y'(1)$

1.3 Free Response Homework Set B

Complete the following.

1. Find the first derivative for the following function: $x(t) = e^{t^2}\sin\left(t^2 - 5t^4\right)$

2. Find the first derivative for the following function: $x(t) = e^{5t}\tan\left(3t^4\right)$

3. Find the first derivative for the following function: $y = \dfrac{x^2+2x-15}{x-3}$

4. Find the first derivative for the following function: $x(t) = e^t\left(t^2 - 5t^4\right)$

5. $\dfrac{d}{dx}\left[\dfrac{e^x + 7x^2 + 5}{\sin\left(x^3\right)}\right]$

6. $\dfrac{d}{dx}\left[e^{\sin(x)}\ln\left(\cot\left(e^x\right)\right)\right]$

7. $\dfrac{d}{dx}\left[x^2\sin\left(x^2\right) + \dfrac{x+1}{\ln x}\right]$

8. $\dfrac{d}{dx}\left[x^2\cos\left(x^2\right) + \dfrac{e^x}{x}\right]$

9. $\dfrac{d}{dx}\left[x^5\ln(5x+4) + \dfrac{x}{\ln x}\right]$

10. $\dfrac{d}{dx}\left[\dfrac{\cos\left(x^2-3\right)}{e^{-5x}}\right]$

11. $\dfrac{d}{dx}\left[e^{x^2}\cos(x)\right]$

12. $\dfrac{d}{dx}\left[\dfrac{1+\tan(x)}{\ln(4x)}\right]$

13. $\dfrac{d}{dx}\left[\sin(t)\tan(t)\right]$

14. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{\csc(x)}\right]$

15. $\dfrac{d}{dx}\left[e^{5x^4}\ln\left(\sin(x)\right)\right]$

16. $\dfrac{d}{dx}\left[5x\sin(x) + e^{2x} - \ln\left(3x^2+1\right) + \dfrac{x}{x^2+1}\right]$

17. $\dfrac{d}{dx}\tan\left(e^x\right)\left(x^4 - 5x^3 + x\right)$

18. $\dfrac{d}{dx}\left[\dfrac{5x+2}{\ln(3x+7)}\right]$

19. $\dfrac{d}{dx}\left[\dfrac{x^5-12c^3-19c}{3c^3}\right]$

20. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sin^2(4x+2)\right]\right]$

21. $g(z)=\left(\dfrac{e^{5z}}{1+\ln z}\right)^{118}$, find $g'(z)$

22. $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$, find $g'(t)$

23. $y=\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$, find y'

24. $f(x)=x^2\arccos(x)$, find $f'(x)$

25. $y=\ln\left(u^2+1\right)-u\cot^{-1}(u)$, find $\dfrac{dy}{du}$

26. $y=\cos^{-1}\left(\dfrac{x-1}{x+1}\right)$

27. $f(t)=c\sin^{-1}\left(\dfrac{t}{c}\right)-\sqrt{c^2-t^2}$, find $f'(t)$

28. $y=4\sin^{-1}\left(\dfrac{1}{2}x\right)+x\sqrt{4-x^2}$

29. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=(h(x))^4+x\ln(h(x))$

1.3 Multiple Choice Homework

1. If $y=x^2\cos(2x)$, then $\dfrac{dy}{dx}=$

a) $-2x\sin(2x)$ b) $-4x\sin(2x)$ c) $2x\left(\cos(2x)-\sin(2x)\right)$

d) $2x\left(\cos(2x)-x\sin(2x)\right)$ e) $2x\left(\cos(2x)+\sin(2x)\right)$

---

2. If $x(t)=2t\cos\left(t^2\right)$, find $x'(t)$.

a) $x'(t)=\sin\left(t^2\right)+3$ b) $x'(t)=-\sin\left(t^2\right)+4$ c) $x'(t)=\sin\left(t^2\right)+2$

d) $x'(t)=-4t^2\sin\left(t^2\right)$ e) $x'(t)=-4t^2\sin\left(t^2\right)+2\cos\left(t^2\right)$

---

3. If $f(x)=x\tan(x)$, then $f'\left(\dfrac{\pi}{4}\right)=$

a) $1-\dfrac{\pi}{2}$ b) $1+\dfrac{\pi}{2}$ c) $1+\dfrac{\pi}{4}$ d) $1-\dfrac{\pi}{4}$ e) $\dfrac{\pi}{2}-1$

---

4. If f is a function that is differentiable throughout its domain and is defined by $f(x)=$

$\dfrac{1+e^x}{\sin(x^2)}$, then the value of $f'(0) =$

a) -1 b) 0 c) 1 d) e e) nonexistent

---

5. If $y = \dfrac{5x-4}{4x-5}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{9}{(4x-5)^2}$ b) $\dfrac{9}{(4x-5)^2}$ c) $\dfrac{40x-41}{(4x-5)^2}$ d) $\dfrac{40x+41}{(4x-5)^2}$ e) $\dfrac{5}{4}$

---

6. If $y = \dfrac{3-2x}{3x+2}$, then $\dfrac{dy}{dx} =$

a) $\dfrac{12x+2}{(3x+2)^2}$ b) $\dfrac{12x-2}{(3x+2)^2}$ c) $\dfrac{13}{(3x+2)^2}$ d) $-\dfrac{13}{(3x+2)^2}$ e) $-\dfrac{2}{3}$

---

7. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

a) $-\dfrac{6x}{(4+x^2)^2}$ b) $\dfrac{3x}{(4+x^2)^2}$ c) $\dfrac{6x}{(4+x^2)^2}$ d) $-\dfrac{3x}{(4+x^2)^2}$ e) $\dfrac{3}{2x}$

---

8. Which of the following statements must be true?

I. $\dfrac{d}{dx}\left[x\tan(x)\right] = x\tan(x) + x\sec^2(x)$

II. $\dfrac{d}{dx}\left[\dfrac{3}{4+x^2}\right] = \dfrac{-6x}{(4+x^2)^2}$

III. $\dfrac{d}{dx}\left[\sqrt{1-x}\right] = \dfrac{1}{2\sqrt{1-x}}$

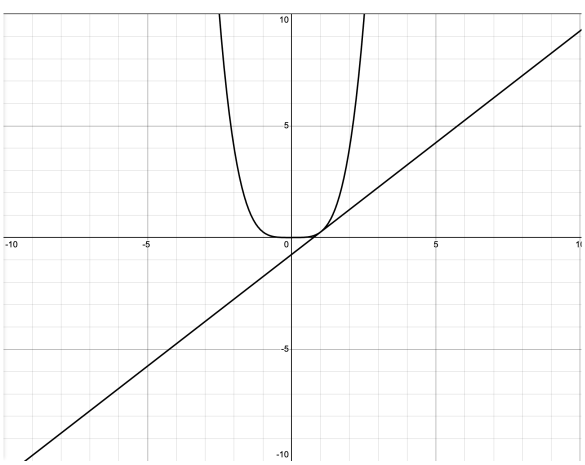
a) I only b) II only c) III only

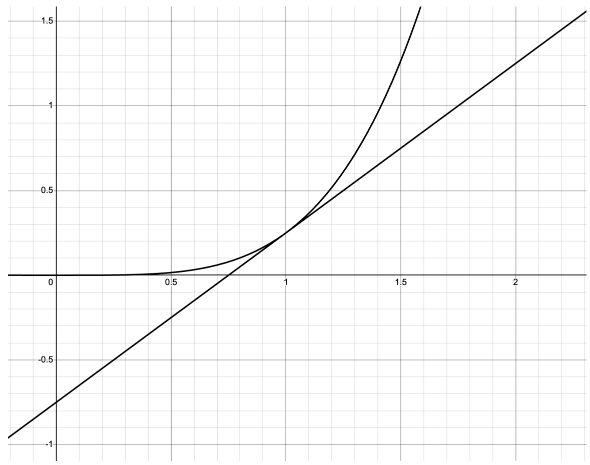
d) I and II only e) I, II, and III

1.4: Local Linearity, Euler's Method, and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, y values on the line are very close to y values on the function. This idea is called local linearity—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency.

Consider the graphs of $y = \dfrac{1}{4}x^4$ and its tangent line at $x = 1$, given by the equation $y = x - \dfrac{3}{4}$:





The closer you zoom in, the more the line and the curve become one. The y values on the line are good approximations of the y values on the curve. For a good animation of this concept, see the following:

tangent line approximation animation

Since it's easier to find the y value of a line arithmetically than for other functions — especially transcendental functions — the tangent line approximation is useful if you have no calculator.

OBJECTIVES

Use the Equation of a Tangent Line to Approximate Function Values.

Ex 1.4.1: Find the equations of the line tangent to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.4.1: The slope of the tangent line will be $f'(-1)$

$$f'(x) = 4x^3 - 3x^2 - 4x$$

$$f'(-1) = 4(-1)^3 - 3(-1)^2 - 4(-1) = -3$$

(Note that we could've gotten this more easily with the nDeriv function on our calculator.)

$$f(-1) = 1, \therefore \boxed{y - 1 = -3(x + 1)} \text{ or } \boxed{y = -3x - 2}$$

One of the many uses of the tangent line is based on the idea of local linearity. This means that in small areas, algebraic curves act like lines — namely their tangent lines. Therefore, one can get an approximate y value for points near the point of tangency by plugging x values into the equation of the tangent line.

Ex 1.4.2: Use the tangent line equation found in **Ex 1.4.1** to get an approximate value of $f(-0.9)$.

Sol 1.4.2: While we can find the exact value of $f(-0.9)$ with a calculator, we can get a quick approximation from the tangent line.

If $x = -0.9$ on the tangent line, then:

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = \boxed{0.7}$$

This last example was somewhat trite in that we could've just plugged -0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but it is doable. Consider the next example, though.

Ex 1.4.3: Find the tangent line equation to $f(x) = e^{2x}$ at $x = 0$ and use it to approximate the value of $e^{0.2}$.

Sol 1.4.3: Without a calculator, we could not find the exact value of $e^{0.2}$. In fact, even the calculator gives us an approximate value.

$$f'(x) = 2e^{2x} \text{ and } f'(0) = 2e^{2(0)} = 2$$

$$f(0) = e^0 = 1$$

So the tangent line equation is $\boxed{y - 1 = 2(x - 0)}$ or $y = \boxed{2x + 1}$

$$e^{0.2} \approx 2(0.2) + 1 = \boxed{1.2}$$

Note that the value you get from a calculator of $e^{0.2}$ is $1.221403\ldots$. Our approximation of 1.2 seems very reasonable.

Though not as useful as practically useful (in 2 dimensions) as the tangent line, another context for the derivative is finding the equation of the normal line.

Normal Line $\rightarrow$ Definition: The line perpendicular to a curve.

Ex 1.4.4: Find the equation of the line normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$.

Sol 1.4.4: In **Ex 1.4.1**, we saw that the slope of the tangent line was $f'(-1) = -3$. The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope of $\frac{1}{3}$. This gives us

$$\boxed{y - 1 = \frac{1}{3}(x+1)} \text{ or } \boxed{y = \frac{1}{3}x - \frac{4}{3}}$$

for the equation of the normal line.

Euler's Method

OBJECTIVES

Use Euler's Method to Approximate a Numerical Solution to a Differential Equation at a Given Point.

In the previous section, we learned a little regarding approximations with tangent lines. Euler's Method is just a better approximation method. It uses more than one tangent line to get the job done.

The process is similar to approximating with tangent lines. We use $\frac{dy}{dx}$ to find a tangent line, then use that tangent line to find an approximate value for y. We then use that y value and another x value to create another "tangent line." Of course, it isn't actually a tangent line because our y value wasn't actually on the curve. We then repeat the process until we get to the value we want to approximate.

Steps to Euler's Method

1. Identify your starting point and step size.
2. Use $\dfrac{dy}{dx}$ to find the slope and make it a tangent line.
3. Find an approximate y value by plugging in $x + (\textit{1 step size})$ to the tangent line.
4. Use the approximate y value and the next x step over to make a new tangent line.
5. Repeat steps 3 and 4 until you reach your final x value – the one you actually want an approximation for.

Ex 1.4.5: Use Euler's Method with a step size of $\frac{1}{2}$ to estimate $f(3)$, where $f(x) = \ln(x)$.

Sol 1.4.5:

$f(1) = \ln 1 = 0$ — We start with 1 because we know $\ln 1 = 0$.

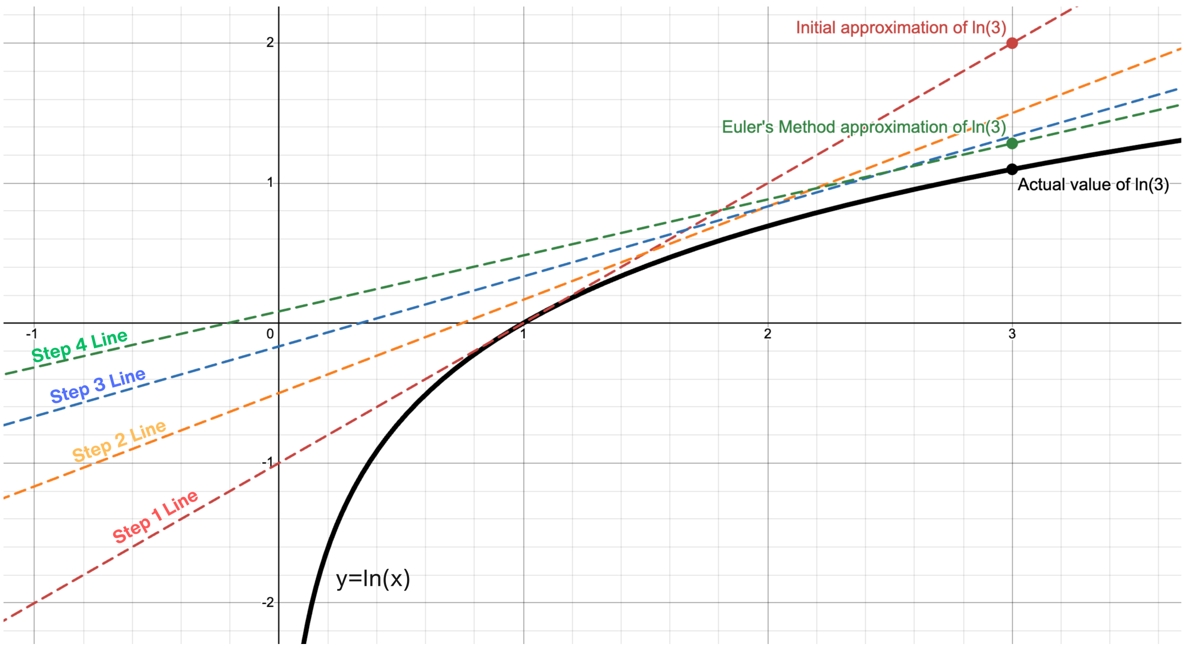
$f'(x) = \dfrac{1}{x}$ — We start by taking the derivative.

Note that in our chart below, we are getting our "New y" from our tangent line (step 3 above). Our "New x" comes from the "$x + (\textit{1 step size})$" step. Our new slope comes from plugging in the "New x" into $f'(x)$. For instance, for our first step below, the "New x" is equal to 1 plus the step size of $\frac{1}{2}$ given in the problem, our "New y" comes from plugging in $\frac{3}{2}$ for x and solving for y in $y - 0 = 1(x - 1)$, and our slope (for the next step) is a result of $f'\left(\frac{3}{2}\right) = \frac{1}{\frac{3}{2}} = \frac{2}{3}$.

Step	**Point**	$f'(x)$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(1, 0)$	1	$y - 0 = 1(x - 1)$	$\frac{3}{2}$	$\frac{1}{2}$
2	$\left(\frac{3}{2}, \frac{1}{2}\right)$	$\frac{2}{3}$	$y - \frac{1}{2} = \frac{2}{3}\left(x - \frac{3}{2}\right)$	2	$\frac{5}{6}$
3	$\left(2, \frac{5}{6}\right)$	$\frac{1}{2}$	$y - \frac{5}{6} = \frac{1}{2}(x - 2)$	$\frac{5}{2}$	$\frac{13}{12}$
4	$\left(\frac{5}{2}, \frac{13}{12}\right)$	$\frac{2}{5}$	$y - \frac{13}{12} = \frac{2}{5}\left(x - \frac{5}{2}\right)$	3	$\frac{77}{60}$

So, $f(3) \approx y(3) = \boxed{\dfrac{77}{60}}$ or $\boxed{1.28\overline{3}}$. By way of comparison, $\ln 3 \approx 1.099$.

If we had just used the initial tangent line (the tangent line from step one) to get an approximation, we would've gotten $f(3) \approx 2$. Euler's method got us a much closer approximation.



We could also use this process to approximate a value for a curve when we only know its derivative and an initial value on the curve.

Ex 1.4.6: Use Euler's method with a step size of $\dfrac{1}{2}$ to estimate $f\left(\dfrac{5}{2}\right)$ for the function whose derivative is given by $\dfrac{dy}{dx} = 2x + y$ with an initial value of $f(1) = 4$.

Sol 1.4.6: For this problem, to find the slope for each step, we simply we need to plug in our point into the given differential equation.

Step	**Point**	$\frac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(1, 4)$	6	$y - 4 = 6(x - 1)$	$\frac{3}{2}$	7
2	$\left(\frac{3}{2}, 7\right)$	10	$y - 7 = 10\left(x - \frac{3}{2}\right)$	2	12
3	$(2, 12)$	16	$y - 12 = 16(x - 2)$	$\frac{5}{2}$	20

$f\left(\frac{5}{2}\right) \approx y\left(\frac{5}{2}\right) = \boxed{20}$. We cannot get an exact value for this function, because we have not learned techniques regarding how to solve this differential equation yet.

Ex 1.4.7: Is the approximation in **Ex 1.4.6** an overestimate or underestimate? Why?

Sol 1.4.7: To determine this, we need to look at the concavity of the curve – this requires the second derivative.

$$\frac{dy}{dx} = 2x + y$$

$$\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx} \qquad \text{Don't forget implicit differentiation!}$$

$$\frac{d^2y}{dx^2} = 2 + (2x + y)$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,4)} = 2 + 2(1) + 4 = 8 \qquad \text{Plug in our initial value}$$

Since the second derivative is positive, our curve is concave up at this point, which means our tangent line lies under the curve. This is indicative of an $\boxed{\text{underestimate}}$.

Generally:

→ Your approximation will be an **overestimate** if the curve is **concave down** (since your "tangent lines" will be above the curve).

→ Your approximation will be an **underestimate** if the curve is **concave up** (since your "tangent lines" will be below the curve).

Ex 1.4.8: Use Euler's method with four equal step sizes to approximate $f(2)$ for $\dfrac{dy}{dx} = 3y - x$, given $f(0) = 1$. Is this an overestimate or an underestimate?

Sol 1.4.8: First, let's figure out our step sizes. We know that we start with $x = 0$, and we need to end up at $x = 2$. Therefore, four equal step sizes means that each step must be $+\dfrac{1}{2}$.

Next, let's make our table:

Step	**Point**	$\dfrac{dy}{dx}$ **(Slope)**	**Tangent Line Equation**	**New** x	**New** y
1	$(0, 1)$	3	$y - 1 = 3(x - 0)$	$\dfrac{1}{2}$	$\dfrac{5}{2}$
2	$\left(\dfrac{1}{2}, \dfrac{5}{2}\right)$	7	$y - \dfrac{5}{2} = 7\left(x - \dfrac{1}{2}\right)$	1	6
3	$(1, 6)$	17	$y - 6 = 17(x - 1)$	$\dfrac{3}{2}$	$\dfrac{29}{2}$
4	$\left(\dfrac{3}{2}, \dfrac{29}{2}\right)$	42	$y - \dfrac{29}{2} = 42\left(x - \dfrac{3}{2}\right)$	2	$\dfrac{71}{2}$

We can see that $f(2) \approx y(2) = \boxed{\dfrac{71}{2}}$. Now, let's solve for the second derivative to find out if this is an overestimate or underestimate.

$$\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 1 = 9y - 3x - 1$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 9(1) - 3(0) - 1 = 8$$

Because the positive second derivative indicates that y is concave up at $(0, 1)$, our Euler's Method result is going to be an $\boxed{\text{underestimate}}$.

1.4 Free Response Homework

Complete the following:

1. Find the equation of the line tangent to $y = x^4 + 2e^x$ at the point $(0, 2)$.

2. Find the equation of the line tangent to $y = x + \cos(x)$ at the point $(0, 1)$.

3. Find the equation of the line tangent to $y = \sec(x) - 2\cos(x)$ at the point $\left(\frac{\pi}{3}, 0\right)$.

4. Find the equation of the line tangent to $y = x^2 e^{-x}$ at the point $\left(1, \frac{1}{e}\right)$.

5. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

6. Find the equation of the line tangent to $y = \frac{x^2 - 3}{x^2 - 4}$ when $x = -1$.

7. Find the equation of the line tangent to $f(x) = x\sqrt[4]{7 + x^2}$ when $x = 3$.

8. Find the equation of the line tangent to $y = e^{x \sin(4x)} + 2$ when $x = 0$.

9. Find the equation of the line tangent to $f(x) = x^5 - 5x + 1$ when $x = -2$ and use it to get an approximate value of $f(-1.9)$.

10. Find the equation of the line tangent to $f(x) = x\sqrt[3]{1 - x^2}$ when $x = 3$ and use it to get an approximate value of $f(3.1)$.

11. Find all points on the graph of $y = 2\sin(x) + \sin^2(x)$ where the tangent line is horizontal.

12. Find all points on the graph of $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent line is horizontal.

13. Find the equation of the lines tangent and normal to $y = -\frac{2x}{x^2 + 16}$ at $x = -1$.

14. Find the equation of the lines tangent and normal to $y = -\frac{3x}{x^2 + 1}$ at $x = 1$.

15. Find the equation of the lines tangent and normal to $y = \frac{x^2 - 4x + 3}{2x^2 - 5x - 3}$ at $x = 2$.

16. Find the equation of the lines tangent and normal to $y = x \sin\left(\frac{\pi}{2} \ln x\right)$ when $x = e$.

17. Find the equation of the lines tangent and normal to $y = x \sin\left(\frac{1}{x}\right)$ when $x = \frac{4}{\pi}$.

18. Use Euler's Method with 2 equal step sizes to find an approximation for $f(0)$, given that $f(-1) = 2$ and $\frac{dy}{dx} = 6x^2 - x^2 y$.

19. Use Euler's Method with 4 equal step sizes to find an approximation for $f(1.4)$, given that $f(1) = 0$ and $f(x) = \ln(2x - 1)$.

20. Use Euler's Method with 3 equal step sizes to find an approximation for $f(2.6)$, given that $f(2) = -2$ and $\frac{dy}{dx} = 2x + y$.

1.4 Multiple Choice Homework

1. Let f be the function given by $f(x) = 2e^{4x^2}$. For what value of x is the slope of the line tangent to the graph of f at $(x, f(x))$ equal to 3?

a) 0.168 b) 0.274 c) 0.318 d) 0.342 e) 0.551

---

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

a) $-3x - 2$ b) $-3x + 4$ c) $-x + 0.905$ d) $-x + 0.271$ e) $-x - 0.271$

---

3. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent line parallel to the line $2x - 4y = 3$?

a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ b) $\left(\frac{1}{2}, -\frac{1}{8}\right)$ c) $\left(\frac{1}{2}, -\frac{1}{4}\right)$ d) $\left(1, -\frac{1}{2}\right)$ e) $(2, 2)$

---

4. A normal line to the graph of a function f at the point $(x, f(x))$ is defined to be the line

perpendicular to the tangent line at that point. An equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

a) $y + 12x = 38$ b) $y - 4x = 10$ c) $y + 2x = 4$

d) $y + 2x = 8$ e) $y - 2x = -4$

---

5. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = \dfrac{4x}{y}$ with the initial condition $f(0) = 1$. What is the best approximation for $f(1)$ using Euler's Method, starting at $x = 0$ with a step size of 0.5?

a) 1 b) 2 c) $\sqrt{5}$ d) 2.5 e) 3

---

6. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = x - y^2$ with the initial condition $f(0) = 1$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 0$ with a step size of 1?

a) -1 b) 0 c) 1 d) 2 e) 3

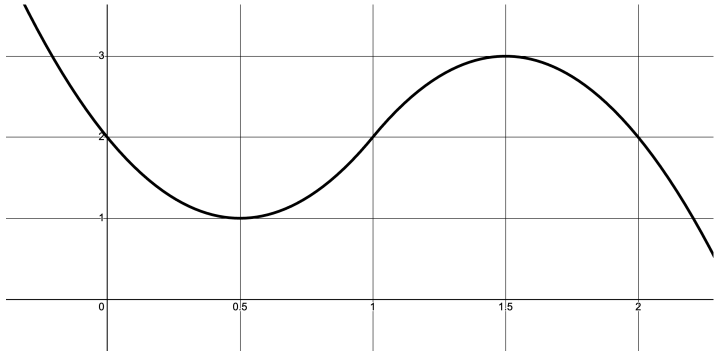
---

7. Let $y = f(x)$ be the solution to the differential equation $\dfrac{dy}{dx} = y - x$ with the initial condition $f(1) = 2$. What is the best approximation for $f(2)$ using Euler's Method, starting at $x = 1$ with a step size of 0.5?

a) 1 b) 2 c) 3 d) 4.5 e) 6

---

8. The graph of $y = f'(x)$ is given below. Use this information and the fact that $f(0) = 3$ to find an approximate value of $f(1)$ using Euler's method with 2 equal step sizes.



a) 2.5 b) 3.5 c) 4 d) 4.5 e) 5

---

9. The table below gives selected values for the derivative of a function g on the interval $-1 \le x \le 2$. If $g(-1) = -2$ and Euler's Method with a step size of 1.5 is used to approximate $g(2)$, what is the resulting approximation?

x	-1.0	-0.5	0	0.5	1.0	1.5	2.0
$f'(x)$	2	4	3	1	0	-3	-6

a) -6.5 b) -1.5 c) 1.5 d) 2.5 e) 3

---

10. The equation of the line **normal** to the graph of $y = \dfrac{3x+4}{4x-3}$ at $(1, 7)$ is

a) $25x + y = 32$ b) $25x - y = 18$ c) $7x - y = 0$ d) $x - 25y = -174$ e) $x + 25y = 176$

---

11. The equation of the line **normal** to the graph of $y = 3x\sqrt{x^2+6} - 3$ at $(0, -3)$ is

a) $3\sqrt{6}x + y = -3$ b) $3\sqrt{6}x - y = -3$ c) $x + 3\sqrt{6}y = -3$ d) $x - 3\sqrt{6}y = 9\sqrt{6}$ e) $x + 3\sqrt{6}y = -9\sqrt{6}$

1.5: Intro to AP: Basic Derivatives Numerically and Graphically

Traditionally, calculus was an algebraically heavy subject. One of the philosophical changes that the CollegeBoard made in the 1990s was to emphasize that calculus should be understood in a variety of modes. As they state in their enduring understanding:

> *"Students should be able to work with functions represented in a veriety of ways: graphical, numerical, analytical or verbal. They should understand the connections among these representations."*

Later, they added that students should be able to verbalize their understanding and be able to communicate that understanding through proper writing. We will consider this later as we consider more context-oriented problems.