Introduction to Decision Theory

Brief summary

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Contents

Chapter 1

The decision problem model

1.1 Introduction

Definition 1.1. ***Decision problem***
A decision problem is a tuple (A, Θ, L) where A is a set of actions that the decider can take, Θ is the set of states of nature, and $L : \Theta \times A \to \mathbb{R}$ is the loss function.

The sets A and Θ can be either finite or infinite. While a positive $L(\theta, a)$ indicates a loss, a negative value indicates a benefit. Naturally, the problem can also be modeled using a gain function G and the problem is equivalent to maximize $G = -L$.

If A and Θ are finite, it is useful to represent the problem as a table

$$
\begin{array}{c|cccc}
 & \theta_1 & \theta_2 & \dots & \theta_k \\
\hline
a_1 & L(\theta_1, a_1) & L(\theta_2, a_1) & \dots & L(\theta_k, a_1) \\
a_2 & L(\theta_1, a_2) & L(\theta_2, a_2) & \dots & L(\theta_k, a_2) \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
a_n & L(\theta_1, a_n) & L(\theta_2, a_n) & \dots & L(\theta_k, a_n)
\end{array}
$$

Then, we need to study how to evaluate the rows and choose the best one. Trivially, if we have actions a_i and a_j such that $L(\theta_n, a_i) \leq L(\theta_n, a_j)$ $\forall n = 1, \dots, k$, then the action a_i has less loss regardless of the state of nature, so we can discard a_j. In this case, we say that a_i dominates a_j.

Notice that this is identical to a two-person zero-sum game in game theory, with the states Θ being interpreted as the actions of the second player. The main difference between these two mathematical branches is that in game theory, we consider the players to be intelligent, meaning they choose actions that provide them with the most benefits. In decision theory, this is not the case; we do not assume that the states of nature occur with any personal interest.

1.2 Randomized actions

We can suppose that the set of actions A is endowed with a σ-algebra $\mathscr{A}$ such that $\{a\} \in \mathscr{A}$ $\forall a \in A$, and $L(\theta, \cdot)$ are measurable functions. Then, the randomized actions α are distributions functions in $(A, \mathscr{A})$.

and the loss function can be extended as

$$L(\theta, \alpha) = \int_A L(\theta, a)\alpha(a)d\alpha.$$

Naturally, $A \subset A^*$, since every $a \in A$ corresponds to the randomized action which assigns probability 1 to that action and probability 0 to any other. Therefore, the decision problem can be easily extended to (A^*, Θ, L).

Chapter 2

Decisions under risk

2.1 Introduction

Decision under risk occur when we do not know in what state of nature we are in, but we suppose (or know) the probability distribution $\pi(\theta)$ of being in the state of nature θ.
Formally, we model Θ as a probability space $(\Theta, \mathscr{A}, \pi)$, therefore the losses $L(\theta, a_1), L(\theta, a_2), \dots$ are now random variables which we will denote as $L(a_i)$ with distributions

$$F_a(x) = \pi(L(a) \leq x) \quad x \in \mathbb{R}.$$

Definition 2.1. ***Stochastic dominance***
Let A and B be random variables, then A dominates B stochastically if and only if

$$F_A(x) \leq F_B(x) \quad \forall x \in \mathbb{R}.$$

If we have $a_1, a_2 \in A$ such that $F_{a_2}(x) \leq F_{a_1}(x)$ for all $x \in \mathbb{R}$ (therefore, a_1 dominates a_2 stochastically), then a_1 is considered a better action because we have a higher probability to obtain a loss less than x with a_1 than with a_2 for all $x \in \mathbb{R}$. However, in practice, the distributions $F_{a_i}(x)$ typically intersect on some points, rendering this criterion less useful since not all actions can be directly compared.. Thus, we need to develop a functional $C(a)$ for all a, and select action a_k as the best choice when

$$C(a_k) < C(a_i) \quad \forall a_i \neq a_k \in A.$$

The following are different ways to define the functions $C(a)$. Once a criterion is established for a specific problem, the objective is to minimize $C(a)$ and select the action a that yields the minimum value.

2.2 Expected value criterion

$$C(a) = E[L(a)] = \int_\Theta L(\theta, a)\pi(\theta)d\theta.$$

This criterion is one of the most used in the practice due to its simplicity.

2.3 Mean-dispersion criterion

$$C(a) = E[L(a)] + \lambda\sigma(L(a)),$$

where $\sigma(L(a))$ is the dispersion of the random variable $L(a)$.

The idea is to penalize an action with large dispersion. This way, we may prefer an action such that $L(a)$ has higher expected value but lower dispersion. The constant λ quantifies the importance we attribute to this penalty. Specifically, $\lambda > 0$ if we consider loss functions, and $\lambda < 0$ if we consider gain functions.

2.4 Fixed risk criterion

$$C(a) = F_a^{-1}(\lambda),$$

where λ is the percentile of F_a, that is

$$P(L(a) > C(a)) = 1 - \lambda. \quad \lambda \in (0, 1).$$

In this criterion, λ needs to be sufficiently close to 1 to "ensure" that the loss does not exceed $C(a)$.

2.5 Maximum probability criterion

$$C(a) = 1 - F_a(K),$$

that is

$$C(a) = P(L(a) > K).$$

We aim to minimize the probability that the loss function exceeds some fixed constant K.

2.6 Expected value criterion with a safety clause

$$C(a) = \begin{cases} E[L(a)] & \text{if } 1 - F_a(K) < \varepsilon, \\ +\infty & \text{if } 1 - F_a(K) \geq \varepsilon. \end{cases}$$

In this case, we aim to discard all the actions that exceed a loss greater than K with a small probability ε. Therefore, we define $C(a) = +\infty$ if $1 - F_a(K) = P(L(a) > K) \geq \varepsilon$.

2.7 Exercises

1. Let (A, θ, L) be a decision problem, where $\Theta = (0, \infty), A = \mathbb{N}$ and $L(\theta, a) = (\theta - a)^2$. If θ has exponential distribution with mean $\mu = 7.3$, compute the best decision with the expected value criterion, mean-dispersion criterion with $\lambda = 2.5$ and the maximum probability criterion with $K = 10$.

1. Expected value criterion.

$$C(a) = E[L(a)] = \int_0^\infty (\theta - a)^2 \frac{1}{\mu} e^{-\frac{1}{\mu}\theta} d\theta = (a - \mu)^2 + \mu^2.$$

which has a minimum on $a = \mu$. However, a must be a natural value. Since $C(7) = 53.38$ and $C(8) = 53.78$, then $a = 7$ is the best action using this criterion.

2. Mean-dispersion criterion.

$$Var(L(a)) = E[L(a)^2] - E[L(a)]^2 = 4a^2\mu^2 - 16a\mu^3 + 20\mu^4.$$

Therefore,

$$C(a) = \mu^2 + (a-\mu)^2 + 5\mu\sqrt{a^2 - 4a\mu + 5\mu^2}.$$

Which has a minimum on $a = 1.7057\mu = 12.452$. Comparing with $a = 12$ and $a = 13$, we obtain that the best action is $a = 12$ with $C(a) = 358.226$.

3. Maximum probability criterion.
We need to minimize

$$C(a) = P((\theta - a)^2 > 10),$$

or equivalently, maximize

$$P(a - \sqrt{10} \leq \theta \leq a + \sqrt{10}) = \begin{cases} 1 - e^{-(a+\sqrt{10})/\mu} & \text{if } a < \sqrt{10}, \\ e^{-(a-\sqrt{10})/\mu} - e^{-(a+\sqrt{10})/\mu} & \text{if } a \geq \sqrt{10}. \end{cases}$$

Which has a maximum in $\sqrt{10}$, therefore, the minimum restricted to $\mathbb{N}$ occurs on $a = 3$.

Chapter 3

Decisions under uncertainty

3.1 Introduction

Decision under uncertainty occurs when we do not know the probability distribution of the states of nature. Similar to decisions under risk, we can define some functional $S : A \to \mathbb{R}$ and choose the action a_k such that

$$S(a_k) < S(a_i) \quad \forall a_i \neq a_k.$$

If there are multiple actions $a_1, \ldots, a_k$ with identical $S(a)$ but $S(a) < S(a_i)$ $\forall i \neq 1, \ldots, n$ then these actions are considered equivalent, and the decider can choose any of them.

3.2 Wald criterion

$$S(a) = \max_{\theta \in \Theta} L(\theta, a).$$

This criterion is also know as the minimax criterion since we choose the action that gives us

$$\min_{a \in A} \max_{\theta \in \Theta} L(\theta, a)$$

3.3 Hurwicz criterion

$$S(a) = \lambda \max_{\theta \in \Theta} L(\theta, a) + (1 - \lambda) \min_{\theta \in \Theta} L(\theta, a).$$

The parameter $\lambda \in [0, 1]$ quantifies the degree of pessimism that the decider has. When $\lambda = 1$ the decider considers the worst-case scenario for each θ and the criterion is equivalent to the Wald criterion.

3.4 Laplace criterion

$$S(a) = \frac{1}{n} \sum_{i=1}^{n} L(\theta_i, a).$$

Naturally, one can get rid of the term $1/n$ and get an equivalently criterion.

3.5 Savage criterion

$$S(a) = \max_{\theta\in\Theta}[L(\theta, a) - \min_{a\in A} L(\theta, a)].$$

The term $D(\theta, a) = L(\theta, a) - \min_{a\in A} L(\theta, a)$ is called regret function. Then, this criterion is just applying the Wald criterion to the regret function.

3.6 Exercises

1. Let the decision problem in its tabular form be

	θ_1	θ_2	θ_3	θ_4
a_1	5	-2	-1	0
a_2	-3	2	5	-2
a_3	-2	4	0	-1

Compute the best decision applying the Wald, Hurwitz, Laplace and Savage criteria.

1. Wald criterion
We compute the max for each row and take the minimum, then

$$S(a_1) = 5,\ S(a_2) = 5,\ S(a_3) = 4.$$

Therefore, $\min_a S(a) = 4$ for the action a_3.

2. Hurwitz criterion
Computing the maximum and minimum of each row, we obtain

$$S(a_1) = 5\lambda - 2(1-\lambda),\ S(a_2) = 5\lambda - 3(1-\lambda),\ S(a_3) = 4\lambda - 2(1-\lambda).$$

One can graph these lines on λ and see which of these lines represents the optimal according to λ. In that case, if $\lambda \leq 1/2$ the best action is a_2 while if $\lambda \geq 1/2$ the best one is a_3.

3. Laplace criterion Just adding all the entries of each row we obtain

$$S(a_1) = 2,\ S(a_2) = 2,\ S(a_3) = 1.$$

Therefore, the best action is a_3.

4. Savage criteria

We will construct a new table with entries $D(\theta, a)$ that is, with the regret function entries, and then compute the minimax.

	θ_1	θ_2	θ_3	θ_4
a_1	8	0	0	2
a_2	0	4	6	0
a_3	1	6	1	1

Then

$$S(a_1) = 8,\ S(a_2) = 6,\ S(a_3) = 6.$$

Therefore, the best actions here are both a_2 and a_3.

Chapter 4

Bayes and minimax decisions

4.1 Bayes decisions

Definition 4.1. ***Bayes risk***
Let (A^*, Θ, L) be a decision problem and let π be a probability distribution on Θ. Then the Bayes risk of the randomized action α with respect to π is defined as

$$r(\pi, \alpha) = \int_{\Theta} L(\theta, \alpha)\pi(\theta)d\theta.$$

The main goal now is to find the randomized action α that has a closest Bayes risk with respect to π to

$$\hat{r}(\pi) = \inf_{a \in A} r(\pi, \alpha) \qquad \textit{or} \qquad \hat{r}^*(\pi) = \inf_{\alpha \in A^*} r(\pi, \alpha)$$

depending if we consider actions in A or randomized actions in A^*.

Proposition 4.1.
If the loss function is lower bounded, then

$$\hat{r}^*(\pi) = \hat{r}(\pi).$$

Geometrical interpretation

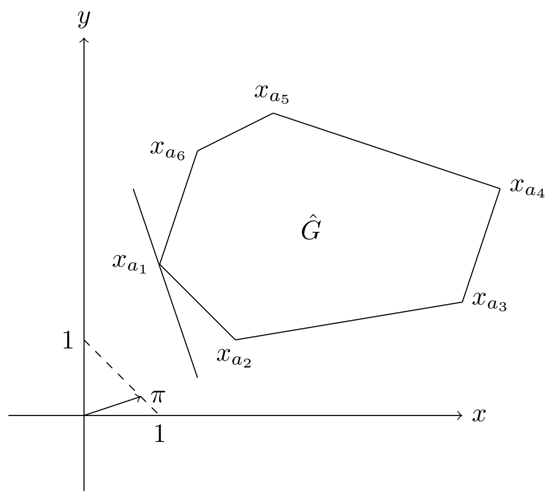
If $\Theta = \{\theta_1, \ldots, \theta_n\}$ is finite, we define the sets

$$G = \{x_a \in \mathbb{R}^n : x_a = (L(\theta_1, a), \ldots, L(\theta_n, a) : a \in A\},$$

$$G^* = \{x_\alpha \in \mathbb{R}^n : x_a = (L(\theta_1, \alpha), \ldots, L(\theta_n, \alpha) : \alpha \in A^*\}.$$

Then, each decision $a \in A$ can be represented as a point in $\mathbb{R}^n$ as $x_a = (L(\theta_1, a), \ldots, L(\theta_n, a))$, (interpreting G and G^* as the set of all possible decisions). Notice that G^* is the convex hull of G.

Let's see an example in two dimensions.



In the figure, $\Theta = \{\theta_1, \theta_2\}$ and $A = \{a_1, \ldots, a_6\}$. Then, each $\pi \in \Theta^*$ is a vector (π_1, π_2) such that $\pi_1 + \pi_2 = 1$. By definition, the perpendicular lines with π, satisfy $\pi_1 x_1 + \pi_2 x_2 = c$ and are the set of points where the Bayes risk is, precisely, c, since

$$c = \pi_1 x_1 + \pi_2 x_2 = \pi_1 L(\theta_1, \alpha) + \pi_2 L(\theta_2, \alpha) = r(\pi, \alpha).$$

Then the minimum Bayes risk is obtained with the decision that intersect the nearest line to the origin, in the figure case, a_1.

Notice that the Bayes decision may not be unique, for example, if the points x_{a_1} and x_{a_2} form a perpendicular line with π, then all randomized actions of the form $x_\alpha = \alpha x_{a_1} + (1-\alpha)x_{a_2}$ are Bayes decisions.

Naturally, this geometrical interpretation can be extended to any dimension n.

4.2 Important results

Definition 4.2. ***Lower closed set***
We define the sets

$$Q(x) = \{y \in \mathbb{R}^n : y_j \leq x_j,\ \forall j \leq n\},$$

$$\hat{G}_I = \{x \in \mathbb{R}^n : Q(x) \cap \bar{G} = \{x\}\}.$$

If $\hat{G}_I \subset \hat{G}$, then $\hat{G}$ is lower closed.

Proposition 4.2.
If Θ is finite, the loss function is lower bounded and $\hat{G}$ is lower closed, then $\forall \pi \in \Theta^*$ such that $\pi_j > 0$ $j = 1, \ldots, n$ there do exists a Bayes action.

Definition 4.3.
Let α and α' be randomized actions in A^*, then α dominates α' if and only if $L(\theta, \alpha) \leq L(\theta, \alpha')$, $\forall \theta \in \Theta$.

Definition 4.4. ***Admissible action***
$\alpha \in A^*$ is an admissible action if there do not exists any other $\alpha' \in A^*$ such that α' dominates α. The set of all admissible actions is denoted A^*_{ad}.

Definition 4.5. ***Complete class***
The set $A^*_{cc} \subset A^*$ is a complete class of actions if for all $\alpha \notin A^*_{cc}$ there do exists an action $\alpha' \in A^*_{cc}$ such that α' dominates α.
A complete class A^*_{cc} is said to be a minimal complete class if it does not contain any other complete class.

Proposition 4.3.
If there do exist a minimal complete class, then it is the admissible action set A^*_{ad}.

Proposition 4.4.
If Θ is finite, the loss function is lower bounded and $\hat{G}$ is lower closed, then A^*_{ad} is a minimal complete class.

4.3 Minimax decisions

Definition 4.6. ***Minimax value and decision***
Let (A^*, θ, L) a decision problem, then the minimax value is defined as

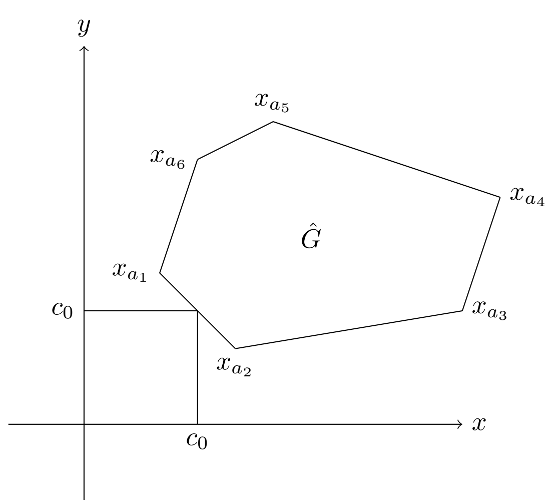
$$\bar{V} = \inf_{\alpha \in A^*} \sup_{\theta \in \Theta} L(\theta, \alpha).$$

Therefore, the minimax decision is the decision α such that

$$\sup_{\theta \in \Theta} L(\theta, \alpha) = \bar{V}$$

Geometrical interpretation

Geometrically, according to the minimax criterion, all decisions that have c fixed as their maximum coordinate are equivalent. Then the problem lies on finding the minimum c_0 such that $(c_0, \ldots, c_0)$ intersects with $\hat{G}$.



Definition 4.7. ***Least favorable distribution***
Let Θ^* be the set of all probability distributions π on Θ. Then, the least favorable distribution is such

π_0 *that satisfies*

$$\hat{r}(\pi_0) = \sup_{\pi \in \Theta^*} (\pi) = \sup_{\pi \in \Theta^*} \inf_{a \in A} r(\pi, a)$$

As the name indicates, one can think of this concept as if nature evaluates all its possible actions $\pi \in \Theta^*$ and chooses its best π according to the maximin criterion.

Proposition 4.5.
If Θ is finite and $\hat{G}$ is lower bounded, then

$$\sup_{\pi \in \Theta^*} \inf_{\alpha \in A^*} L(\pi, \alpha) = \inf_{\alpha \in A^*} \sup_{\theta \in \Theta} L(\pi, \alpha)$$

and there exist a least favorable distribution π_0 such that $\inf_{\alpha \in A^*} L(\pi_0, \alpha)$ is equal to this common value of the equality.
Further, when $\hat{G}$ is lower closed, there do exists a randomized action α_0 that is admissible, minimax and the Bayes decision with respect to π_0.

4.4 Exercises

1. Let (Θ, A, L) be a decision problem where $\Theta = A = (0, 1)$ and $L(\theta, a) = (\theta - a)^2/(\theta(1-\theta))$. Find the Bayes decision and its risk if $\pi(\theta) = \theta^{p-1}(1-\theta)^{q-1}/B(p, q)$ where $p, q > 1$.

$$r(\pi, a) = \int_0^1 \frac{(\theta - a)^2}{\theta(1-\theta)} \frac{\theta^{p-1}(1-\theta)^{q-1}}{B(p,q)} d\theta = \frac{p}{q-1} - 2a\frac{(p+q-1)}{q-1} + a^2\frac{(p+q-1)(p+q-2)}{(p-1)(q-1)},$$

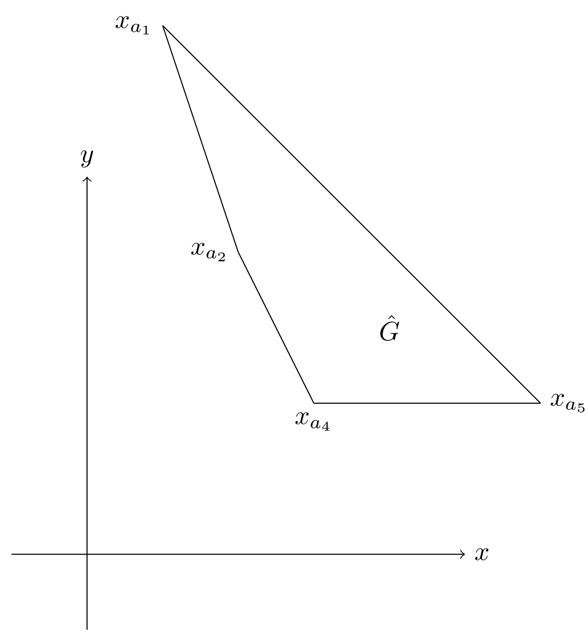
which is just a parabola with respect of am therefore we get the minimum with $(p-1)/(p+q-2)$ with risk $\hat{r} = 1/(p+q-2)$.

2. Let the decision problem be

	θ_1	θ_2
a_1	1	7
a_2	2	4
a_3	4	3
a_4	3	2
a_5	6	2

1. Represent $\hat{G}$.
2. Find the Bayes decision as a function of π.
3. Find the minimax randomized action and the least favorable distribution.

1.
The vertices of $\hat{G}$ are the points $x_{a_i} = (L(\theta_1, a_i), L(\theta_2, a_i))$.
$x_{a_1} = (1, 7), x_{a_2} = (2, 4), x_{a_3} = (4, 3), x_{a_4} = (3, 2)$ and $x_{a_5} = (6, 2)$.



2.
By applying the definition of the Bayes risk we have

$$\begin{aligned}
r(\pi, a_1) &= \pi + 7(1-\pi) = 7 - 6\pi, \\
r(\pi, a_2) &= 2\pi + 4(1-\pi) = 4 - 2\pi, \\
r(\pi, a_3) &= 4\pi + 3(1-\pi) = 3 + \pi, \\
r(\pi, a_4) &= 3\pi + 2(1-\pi) = 2 + \pi, \\
r(\pi, a_5) &= 6\pi + 2(1-\pi) = 2 + 4\pi.
\end{aligned}$$

Minimizing, we obtain that the Bayes decision is a_1 if $\pi \geq 3/4$, a_2 if $2/3 \leq \pi \leq 3/4$ and a_4 if $\pi \leq 2/3$.

3.
We have to find the least value c such that (c, c) intersects with $\hat{G}$. If we look at the graph $\hat{G}$, that intersection will occur on the line between x_{a_2} and x_{a_4} which can be easy computed as $(8/3, 8/3)$ which corresponds to the randomized action $\alpha_0 = (0, 1/3, 0, 2/3, 0)$. The least favorable distribution is then $\pi_0 = (2/3, 1/3)$.

Chapter 5

Decisions with experimentation

5.1 Introduction

When the decider can observe a random variable X before making a decision, where its distribution $P_\theta(x)$ is a function that depends on θ., we deal with decisions involving experimentation.

We can obtain the probability of being in each state of nature after observing the experimentation using the Bayes formula

$$\pi(\theta|x) = \frac{\pi(\theta)P_\theta(x)}{P(x)}$$

where $P(x)$ is the marginal distribution of x

$$P(x) = \int_\Theta P_\theta(x)\pi(\theta)d\theta.$$

Then, the decision problem (Θ, A, L) is modified into (Θ, D, R), where D is the set of all rules $d : X \to A$, in a way that for each x the decider selects the action a and the loss function is now the risk function

$$R(\theta, d) = \int_X L(\theta, d(x))P_\theta(x)dx.$$

5.2 Bayes decision rules

We can extend easily the previous concepts to Bayes decisions. Then, the Bayes risk with experimentation is

$$r(\pi, d) = \int_\Theta R(\theta, d)\pi(\theta)d\theta = \int_\Theta \int_X L(\theta, d(x))P_\theta(x)\pi(\theta)dxd\theta.$$

or its equivalent, by applying the Bayes formula

$$r(\pi, d) = \int_X \int_\Theta L(\theta, d(x))\pi(\theta|x)P(x)d\theta dx.$$

5.3 Exercises

1. Find the Bayes decision rule to the decision problem with experimentation

	θ_1	θ_2
a_1	0	10
a_2	5	-2

$$P_{\theta_1}(X=1))=3/4,\ P_{\theta_1}(X=0))=1/4,$$

$$P_{\theta_2}(X=1))=1/3,\ P_{\theta_2}(X=0))=2/3.$$

Let $\pi=(\pi,1-\pi)$ be the prior probability of the state of nature. We can compute the posterior probabilities

$$\pi(\theta_1|X=1)=\frac{\pi 3/4}{\pi 3/4+(1-\pi)1/3}=\frac{9\pi}{5\pi+4}$$

$$\pi(\theta_2|X=1)=\frac{(1-\pi)1/3}{\pi 3/4+(1-\pi)1/3}=\frac{4(1-\pi)}{5\pi+4}$$

The Bayes action with this prior probability distribution π_1 is

$$r(\pi_1,a_1)=L(\theta_1,a_1)\pi(\theta_1|X=1)+L(\theta_2,a_1)\pi(\theta_2|X=1)=\frac{40(1-\pi)}{5\pi+4},$$

$$r(\pi_1,a_2)=L(\theta_1,a_2)\pi(\theta_1|X=1)+L(\theta_2,a_2)\pi(\theta_2|X=1)=\frac{53\pi-8}{5\pi+4}.$$

Minimizing this we obtain

$$d^*_\pi(1)=\begin{cases}a_1 & \text{if } \pi\geq 16/31,\\ a_2 & \text{if } \pi<16/31.\end{cases}$$

Analogously, with $X=0$

$$\pi(\theta_1|X=0)=\frac{\pi 1/4}{\pi 1/4+(1-\pi)2/3}=\frac{3\pi}{8-5\pi}$$

$$\pi(\theta_2|X=0)=\frac{(1-\pi)2/3}{\pi 1/4+(1-\pi)2/3}=\frac{8(1-\pi)}{8-5\pi}$$

The Bayes action with this prior probability distribution π_0 is

$$r(\pi_0,a_1)=L(\theta_1,a_1)\pi(\theta_1|X=0)+L(\theta_2,a_1)\pi(\theta_2|X=0)=\frac{80(1-\pi)}{8-5\pi},$$

$$r(\pi_0,a_2)=L(\theta_1,a_2)\pi(\theta_1|X=0)+L(\theta_2,a_2)\pi(\theta_2|X=0)=\frac{31\pi-16}{8-5\pi}.$$

Minimizing this we obtain

$$d^*_\pi(0)=\begin{cases}a_1 & \text{if } \pi\geq 32/37,\\ a_2 & \text{if } \pi<32/37.\end{cases}$$

Concluding that the Bayes rule with respect of π is

$$\begin{cases} d^*_\pi(1) = a_1,\ d^*_\pi(0) = a_1 & \text{if } \pi \geq 32/37, \\ d^*_\pi(1) = a_1,\ d^*_\pi(0) = a_2 & \text{if } 16/31 \leq \pi \leq 32/37, \\ d^*_\pi(1) = a_2,\ d^*_\pi(0) = a_2 & \text{if } \pi \leq 16/31. \end{cases}$$

References

[1] Ricardo Vélez Ibarrola. *Introducción a la Teoría de la Decisión*. UNED, 2012.