8.04.2022

The dynamical system associated to an autonomous differential equation in $\mathbb{R}^n$

(1) $\dot{x} = f(x)$, $f:\mathbb{R}^n \to \mathbb{R}^n$, $n \in \mathbb{N}^*$ $\boxed{f \in C^1}$

The unknown: $t \in \mathbb{R} \longmapsto x(t) \in \mathbb{R}^n$

$\dot{x} \longrightarrow x'$, the notation used by Isaac Newton for the derivative w.r.t. time.

Ex: $\dot{x} = tx$ is non-autonomous (linear)

$\dot{x} = x$ is autonomous (linear)

$\dot{x} = 1-x^2$ autonomous (nonlinear)

(T1) The existence and uniqueness theorem for the IVP

We assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. Let $\eta \in \mathbb{R}^n$ and consider the IVP.

(2) $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$ We have that the IVP (2) has a unique sol., denoted by $\varphi(\cdot, \eta)$ whose maximal interval of def. is denoted by $I_\eta = (\alpha_\eta, \beta_\eta) \subset \mathbb{R}$.

When $\varphi(\cdot, \eta)$ is bounded in the future (i.e. on $[0, \beta_\eta)$) we have $\beta_\eta = +\infty$.

When $\varphi(\cdot, \eta)$ is bounded in the past (i.e. on $(\alpha_\eta, 0]$) we have $\alpha_\eta = -\infty$.

When $\varphi(\cdot, \eta)$ is bounded on I_η then $I_\eta = \mathbb{R}$.

~ Terminology ~

$\eta \in \mathbb{R}^n$ the <u>initial state</u> of the system

$\varphi(t,\eta) \in \mathbb{R}^n$ the <u>state</u> of the system.

<u>Def:</u> $\eta^* \in \mathbb{R}^n$ is said to be an <u>equilibrium</u> (stationary) <u>state</u> (point) of (1) when $\varphi(t,\eta^*) = \eta^*$, $\forall t \in \mathbb{R}$.

! η^* is an equilibrium point of (1) $\Leftrightarrow$ the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta^* \end{cases}$ has the unique sol., a constant function, $\varphi(t,\eta^*) = \eta^*$, $\forall t \in \mathbb{R}$

$\Leftrightarrow f(\eta^*) = 0$.

(T2) Assume that $\exists \lim_{t\to\infty} \varphi(t,\eta) = \eta^* \in \mathbb{R}^n$. Then η^* is an equilibrium point of (1).

<u>Proof:</u> $\boxed{n=1}$

$$\left.\begin{array}{l} \dot{\varphi}(t,\eta) = f(\varphi(t,\eta)) \\ \text{Hyp.}, f \text{ is cont.} \Rightarrow \exists \lim_{t\to\infty} f(\varphi(t,\eta)) = f(\eta^*) \end{array}\right\} \Rightarrow$$

$\Rightarrow \exists \lim_{t\to\infty} \dot{\varphi}(t,\eta) = f(\eta^*)$

<u>Lemma:</u> $\psi \in C^1$, $\exists \lim_{t\to\infty} \psi(t) = \eta^* \Rightarrow \lim_{t\to\infty} \dot{\psi}(t) = 0$

$\exists \lim_{t\to\infty} \dot{\psi}(t) = 0$, then

We apply the mean value th. on $[n, n+1]$, $n \in \mathbb{N}^*$.

$\exists\, a_n \in (n, n+1)$ s.t. $\psi(n+1) - \psi(n) = \psi'(a_n)$

$$\left.\begin{array}{l} \lim_{n\to\infty} \psi(n) = \eta^* \\ \lim_{n\to\infty} \psi(n+1) = \eta^* \end{array}\right\} \Rightarrow \left.\begin{array}{l} \lim_{n\to\infty} \dot{\psi}(a_n) = 0 \\ \exists \lim_{t\to\infty} \dot{\psi}(t) \end{array}\right\} \Rightarrow \lim_{t\to\infty} \dot{\psi}(t) = 0$$

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Def: Let $\eta^* \in \mathbb{R}^n$ be an equilibrium point of (1). We say that η^* is an attractor of (1) if $\exists$ a neighbourhood V_* of η^* s.t., $\lim_{t\to\infty} \varphi(t,\eta) = \eta^*$, $\forall\ \eta \in V_*$.

When η^* is an attractor, we define its basin of attraction $A_{\eta^*} = \{\eta \in \mathbb{R}^n : \lim_{t\to\infty} \varphi(t,\eta) = \eta^*\}$. When $A_{\eta^*} = \mathbb{R}^n$ we say that η^* is a global attractor.

When we replace "$t \to \infty$" to "$t \to -\infty$" we can give the similar notions replacing "attraction" with "repeller".

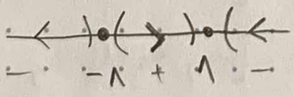
Def: The function $(t,\eta) \mapsto \varphi(t,\eta)$ is said to be the <u>FLOW</u> of (1)

Def: Let $\eta \in \mathbb{R}^n$. The <u>ORBIT</u> (trajectory) of η is
$\gamma_\eta = \{\varphi(t,\eta) : t \in \mathcal{I}_\eta\}$

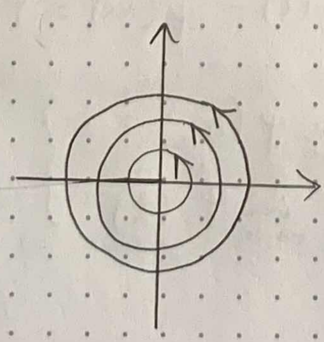
! γ_η is the image of $\varphi(t,\eta)$. The phase portrait of (1) is the representation of some "significant" orbits, together with an arrow on each orbit that indicates the future.

~ Examples ~

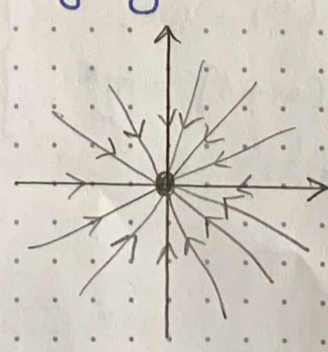
- $\dot{x} = 1 - x^2$



- $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$



- $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$



~~General procedure to~~

Lemma: Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1 function. Any non-const. sol. of $\dot{x} = f(x)$ is a strictly monoton fct.

General procedure to represent the phase portrait of (1) when $\boxed{n=1}$ (scalar dyn. syst.):

STEP 1: We find the equilibrium points, i.e. we solve the eq. $f(x) = 0$.

STEP 2: We find the sign of f on the intervals delimited by the equilibrium points.

STEP 3: We use: "The orbits of $\dot{x} = f(x)$ are the ones corresponding to the equilibrium points and the open intervals delimited by them".

We represent on $\mathbb{R}$ the orbits and ~~an~~ arrow on each orbit according to the rules:

- if $f > 0$ on the orbit, then the arrow points to the right
- if $f < 0$ —"— —"— —"— left.

~ Example ~

The orbits of $\dot{x} = 1 - x^2$ are $(-\infty, -1)$, $\{-1\}$, $(-1, 1)$, $\{1\}$, $(1, \infty)$

How do we read the phase portrait?

$\dot{x} = 1 - x^2$

Deduce that $\varphi(\cdot, 0)$ is bounded, strictly increasing, defined on $\mathbb{R}$, $\lim_{t \to \infty} \varphi(t, 0) = 1$, $\lim_{t \to -\infty} \varphi(t, 0) = -1$.

Explanation : $0 \in (-1,1)$, which is an orbit $\Rightarrow \underbrace{\gamma_0}_{\text{orbit corresp. to } 0} = (-1,1) \Rightarrow$

$\Rightarrow$ the image of $\varphi(\cdot,0)$ is $(-1,1)$ $\Rightarrow$ $\varphi(\cdot,0)$ is bounded $\Rightarrow$
$\Rightarrow$ the time runs from $-\infty$ to $+\infty$ $\Rightarrow$ $I_0 = \mathbb{R}$

PP $\Rightarrow$ ~~~~ $\varphi(\cdot,0)$ is strictly increasing.

phase portrait

$$\lim_{t\to\infty} \varphi(t,0) = 1$$
$$\lim_{t\to-\infty} \varphi(t,0) = -1$$

PP $\Rightarrow$ $\eta^* = 1$ is an attractor and $A_1 = (-1,\infty)$
$\eta^* = -1$ is a repeller and $B_{-1} = (-\infty, 1)$.

The linearization method.

$f:\mathbb{R}\to\mathbb{R}$, C^1 function. Let η^* be an equilibrium point of $\dot{x} = f(x)$.

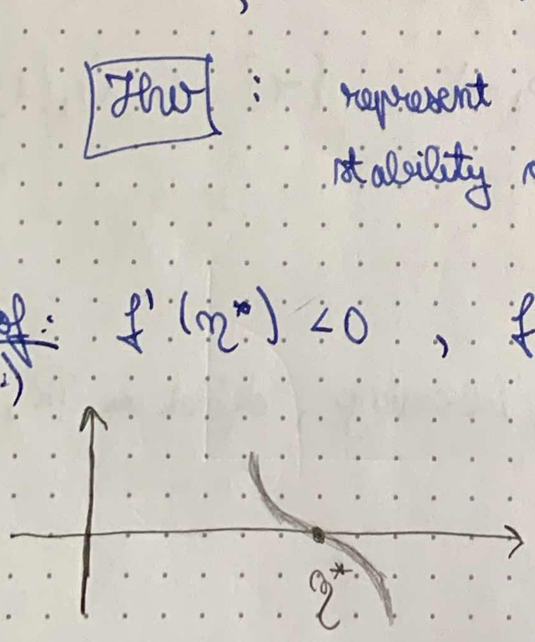
If $f'(\eta^*) < 0$, then η^* is an attractor.
If $f'(\eta^*) > 0$, then η^* is a repeller.

! when $f'(\eta^*) = 0$, the linearization method fails.

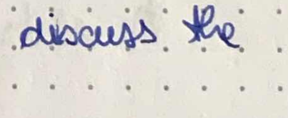
$\dot{x} = -x^3$, $\dot{x} = x^3$, $\dot{x} = x^2$, $\dot{x} = 0$.

$\boxed{\text{Hw}}$: represent the phase portrait and discuss the stability of $\eta^* = 0$.

Proof (geom.): $f'(\eta^*) < 0$, $f(\eta^*) = 0$



x	η^*
f	$+\ 0\ -$



– problem 24 (from the list)

$k > 0$ parameter ; $\dot{x} = -k(x - 21)$ is the model of Newton for cooling processes , here $x(t)$ being the temperature of a cup of tea at "t"

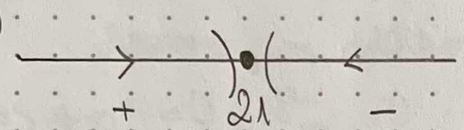
k depends on the environment.

a) Find the flow and represent the phase portrait

b) initial temp. of $45°C \rightarrow 37°C$ after 10 minutes

Find ? $\quad$? $\xrightarrow[20']{} 37°C$

a)



! find the flow

b) $\underbrace{\varphi(t, \eta)}_{\text{flow}}$, $\underset{\uparrow \text{ init. temp.}}{\eta} = ?$ s.t. $\varphi(20, \eta) = 37$

$\varphi(10, 45) = 37$ (an eq. in "k")

$\eta \in \mathbb{R}$ $\begin{cases} \dot{x} = -k(x - 21) \\ x(0) = \eta \end{cases}$

– find gen. sol.

– $\varphi(t, \eta)$ the sol. of IVP