Exam june 26, 2020.

complete solutions of selected problems.

$\boxed{1.}$ Denote by $\mathbb{R}^\infty$ the linear space of all sequences of real numbers $x = (x_k)_{k\geq 0}$, ~~the~~ with the natural operations. Let $S \subset \mathbb{R}^\infty$ be the set of solutions of the difference equation $x_{k+2} = k x_{k+1} + x_k$, $k \geq 0$. Let $T: S \to \mathbb{R}^2$ be defined by $T(x) = (x_0, x_1)$ for all $x \in S$. Justify that $S \neq \emptyset$ and that T is bijective. Is S a linear space of finite dimension? Justify.

<u>Solution</u> Note that $x = 0_\infty \in \mathbb{R}^\infty$ defined by $x_k = 0$ $\forall k \geq 0$ is a solution of the given DE. Thus $S \neq \emptyset$.

Also, it is easy to note that $\forall (x_0, x_1) \in \mathbb{R}^2$ there exists a unique sol. of the D.E with x_0 and x_1 as initial values. Thus T is bijective.

Now we want to check if S is a linear space.

Let $x, y \in S$ and $\alpha, \beta \in \mathbb{R}$. ~~Then~~ Denote by $z = \alpha x + \beta y$.

$$x \in S \iff x_{k+2} = k x_{k+1} + x_k, \quad \forall k \geq 0 \qquad (1)$$

$$y \in S \iff y_{k+2} = k y_{k+1} + y_k, \quad \forall k \geq 0 \qquad (2).$$

$$z \in S \iff z_{k+2} = k z_{k+1} + z_k, \quad \forall k \geq 0 \qquad (3).$$

$$z = \alpha x + \beta y \Rightarrow \quad z_{k+2} = \alpha x_{k+2} + \beta y_{k+2} \qquad (4)$$

$$k z_{k+1} + z_k = k(\alpha x_{k+1} + \beta y_{k+1}) + \alpha x_k + \beta y_k =$$

$$= \alpha(k x_{k+1} + x_k) + \beta(k y_{k+1} + y_k) \overset{(1),(2)}{=}$$

$$= \alpha x_{k+2} + \beta y_{k+2} \qquad (5)$$

$$\overset{(4),(5)}{\Longrightarrow} z_{k+2} = k z_{k+1} + z_k, \quad \forall k \geq 0 \Rightarrow z \in S.$$

Thus, S is a linear space.

It is easy to notice that T is a linear map. We know T is bijective. Then T is an izomorphism of linear spaces, which assures that $\dim S = \dim \mathbb{R}^2 = 2$.
Hence, indeed, S is a linear space of finite dimension.

<u>Remark.</u> The same problem, but for the difference equation

$x_{k+2} + k x_k = \sin k, \quad k \geq 0$

has another answer to the last question.
We prove now that S is <u>not a linear space</u> in this case.
Let $x, y \in S$ and $\alpha, \beta \in \mathbb{R}$. Denote by $z = \alpha x + \beta y$

$x \in S \Leftrightarrow \quad x_{k+2} + k x_k = \sin k, \quad k \geq 0$

$y \in S \Leftrightarrow \quad y_{k+2} + k y_k = \sin k, \quad k \geq 0$

$$z \in S \Leftrightarrow \quad z_{k+2} + k z_k = \sin k, \ k \geq 0 \Leftrightarrow$$

$$\Leftrightarrow \alpha x_{k+2} + \beta y_{k+2} + k\alpha x_k + k\beta y_k = \sin k, \ k \geq 0$$

$$\Leftrightarrow \alpha(x_{k+2} + k x_k) + \beta(y_{k+2} + k y_k) = \sin k, \ k \geq 0$$

$$\overset{(1),(2)}{\Leftrightarrow} \alpha \sin k + \beta \sin k = \sin k, \quad k \geq 0 \Leftrightarrow \alpha + \beta = 1.$$

This, of course, it is not valid $\forall\, \alpha, \beta \in \mathbb{R}$.

$\boxed{2.}$ Find the general solution of the differential equation

$x'' + 2t x' = 0.$

<u>Solution.</u> $y = x' \Rightarrow y' + 2ty = 0 \ | \cdot e^{t^2} \Rightarrow$

$$\Rightarrow y' e^{t^2} + 2ty e^{t^2} = 0 \Rightarrow (y e^{t^2})' = 0 \Rightarrow y e^{t^2} = c \Rightarrow$$

$$\Rightarrow y = c e^{-t^2}, \ c \in \mathbb{R} \Rightarrow x' = c e^{-t^2} \Rightarrow$$

$$\Rightarrow x = c_1 \int_0^t e^{-s^2} ds + c_2, \quad c_1, c_2 \in \mathbb{R}.$$

3. Consider the planar system $\dot{x} = y(y+3)$, $\dot{y} = -x(y+3)$.

(a) Represent the phase portrait.

(b) Reading the phase portrait, find $\lim_{t\to\infty} \varphi(t,4,0)$ and $\lim_{t\to\infty} \varphi(t,1,0)$ (if they exist).

(c) $\varphi(t,4,0)$ and $\varphi(t,1,0)$ are periodic functions?

<u>Solution.</u> (a) First we look for <u>the equilibria</u>.

$$\begin{cases} y(y+3) = 0 \\ -x(y+3) = 0 \end{cases} \Rightarrow (0,0) \text{ and } (a,-3) \ \forall a \in \mathbb{R}$$

are the eq.

Now we try to find a first integral.

$$\frac{dy}{dx} = \frac{-x(y+3)}{y(y+3)} \qquad y\,dy = -x\,dx \qquad y^2 = -x^2 + c$$

$\Rightarrow y^2 + x^2 = c$, $c \in \mathbb{R}$. Define $H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$.

the p.d.e. of a f.i is $y(y+3)\dfrac{\partial H}{\partial x} - x(y+3)\dfrac{\partial H}{\partial y} = 0$.

we have for $H = x^2 + y^2$: $y(y+3)\cdot 2x - x(y+3)\cdot 2y = 0$

which is valid $\forall (x,y) \in \mathbb{R}^2$. Thus, H is a global f.i.

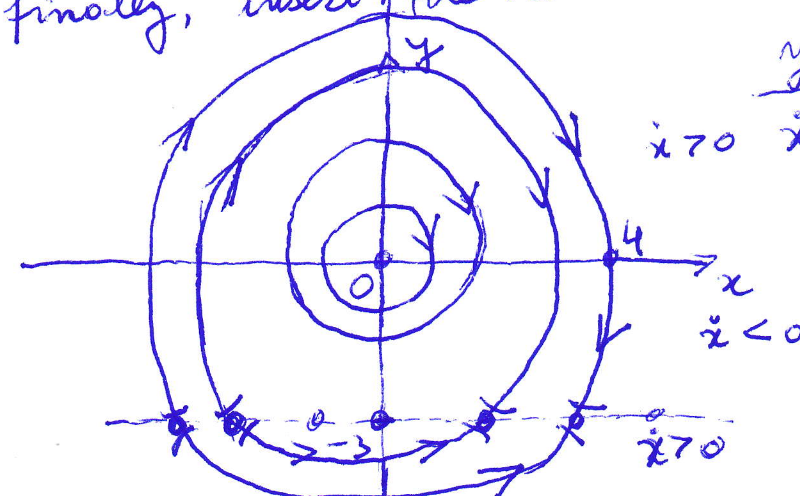
Its level curves are concentric circles centered in O.

To represent the phase portrait we follow 3 steps:

represent the level curves of H, represent the equilibria, represent the arrows on each orbit.

and finally, insert the arrows.

y	$-\infty$		-3			0		$+\infty$
$\dot{x}$	$+$	$+$	0	$-$	$-$	0	$+$	$+$



(b) $(4,0)$ is on the circle of radius 4, which intersects the line of equilibria $y=-3$ in ~~the~~ 2 points of coordinates $(a,-3)$. In order to find a, we have to solve the equation $a^2+(-3)^2=4^2$

$\Rightarrow a^2 = 7 \Rightarrow a = \pm\sqrt{7}$.

$\overset{P.P}{\Longrightarrow} \lim\limits_{t\to\infty} \varphi(t,4,0) = (\sqrt{7},-3)$.

$(1,0)$ is on the circle of radius 1, which do not intersect the line $y=-3$.

$\overset{P.P}{\Longrightarrow}$ $\gamma_{(1,0)}$ is a closed curve $\Rightarrow \varphi(t,4,0)$ is a periodic function, thus it has no limit at ∞.

(c) Since $\varphi(t,4,0)$ has limit as $t\to\infty$, we deduce that $\varphi(t,4,0)$ is not periodic. But $\varphi(t,1,0)$ is.

$\boxed{4}$ Let $x(t)$ be the temperature of a coffee at time t (measured in minutes). We have that $\dot{x}=k(20-x)$.
Find the flow associated to this differential equation.
Find $k\in\mathbb{R}$ knowing that the coffee is cooled down from $100°$ to $80°$ in 5 minutes.

Solution: Let $\eta\in\mathbb{R}$. We have to find the sol of the IVP

$$\begin{cases}\dot{x}=k(20-x)\\ x(0)=\eta\end{cases}$$

This d.e. is linear nonhom. Note $x_p=20$. The lin. hom. eq. associated is $\dot{x}+kx=0$, whose general sol. is $x_h = c\cdot e^{-kt}$, $c\in\mathbb{R}$. $\Rightarrow x = ce^{-kt}+20$

$x(0)=c+20=\eta \Rightarrow c=\eta-20$. Thus, $\varphi(t,\eta)=(\eta-20)e^{-kt}+20$.

We have $\varphi(5,100)=80 \Rightarrow (100-20)e^{-k\cdot 5}+20=80 \Rightarrow$

$\Rightarrow e^{-5k}=3/4 \Rightarrow -5k=\ln 3/4 \Rightarrow k=\frac{1}{5}\ln\frac{4}{3}$.

$\boxed{5.}$ $\boxed{(a)}$ Find the solution of the IVP $\begin{cases} x_{k+1} = -x_k + 3y_k \\ y_{k+1} = -3x_k - y_k \\ x_0 = 1/3, \quad y_0 = 0, \end{cases}$ reducing the system to a second order difference equation.

<u>Solution.</u> $\quad 3y_k = x_{k+1} + x_k$

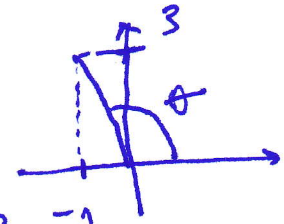
$$x_{k+2} = -x_{k+1} + 3y_{k+1} = -x_{k+1} + 3(-3x_k - y_k) =$$

$$= -x_{k+1} - 9x_k - (x_{k+1} + x_k) = -2x_{k+1} - 10x_k$$

$$x_{k+2} + 2x_{k+1} + 10x_k = 0 \qquad \text{LHDE}$$

$$r^2 + 2r + 10 = 0 \qquad r_{1,2} = -1 \pm \sqrt{-9} = -1 \pm 3i$$

$$\longmapsto \quad \operatorname{Re}(-1+3i)^k, \quad \operatorname{Im}(-1+3i)^k.$$



$$z = -1 + 3i \quad \Rightarrow \quad |z| = \sqrt{1+9} = \sqrt{10}$$

Let $\theta \in (0, \pi)$ s.t. $\cos\theta = -\dfrac{1}{\sqrt{10}}$, $\sin\theta = \dfrac{3}{\sqrt{10}}$

$$\Rightarrow z = \sqrt{10}(\cos\theta + i\sin\theta) \Rightarrow z^k = (\sqrt{10})^k(\cos k\theta + i\sin k\theta)$$

$$\Rightarrow \operatorname{Re}(z^k) = (\sqrt{10})^k \cos k\theta, \quad \operatorname{Im}(z^k) = (\sqrt{10})^k \sin k\theta$$

$$\rho \overset{not}{=} \sqrt{10} \quad \Rightarrow \quad x_k = c_1 \rho^k \cos(k\theta) + c_2 \rho^k \sin(k\theta)$$

$$y_k = \frac{1}{3}x_{k+1} + \frac{1}{3}x_k = \frac{1}{3}c_1 \rho^{k+1}\cos(k\theta + \theta) + \frac{1}{3}c_2 \rho^{k+1}\sin(k\theta+\theta)$$

$$+ \frac{1}{3}c_1 \rho^k \cos(k\theta) + \frac{1}{3}c_2 \rho^k \sin(k\theta)$$

$$\left.\begin{array}{l} y_0 = \frac{1}{3}c_1 \underbrace{\rho\cos\theta}_{-1} + \frac{1}{3}c_2 \underbrace{\rho\sin\theta}_{3} + \frac{1}{3}c_1 = 0 \\ x_0 = c_1 = \frac{1}{3} \end{array}\right\} \Rightarrow \left| \begin{array}{l} -\frac{1}{3} + 3c_2 + \frac{1}{3} = 0 \\ c_1 = \frac{1}{3} \end{array}\right.$$

$$\Rightarrow c_1 = \frac{1}{3}, \quad c_2 = 0 \Rightarrow \begin{cases} x_k = \frac{1}{3}\rho^k \cos k\theta \quad \text{and} \\ y_k = \frac{1}{9}\rho^{k+1}\cos(k\theta+\theta) + \frac{1}{9}\rho^k\cos(k\theta). \end{cases}$$

$$\cancel{y_k = \frac{1}{9}\rho^{k+1}\cos}$$

$\boxed{5.}\boxed{(b)}$ Find $\cancel{\lim_{k\to\infty} A^{k}}\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$, where A is the matrix $\lim_{n\to\infty} A^{-n}\begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$.

of the system from (a).

Solution. ~~Let~~ we have $A = \begin{pmatrix} -1 & 3 \\ -3 & -1 \end{pmatrix}$.

Denote $X_k = \begin{pmatrix} x_k \\ y_k \end{pmatrix}$. The IVP from (a) can be written in the form $X_{k+1} = AX_k$, $X_0 = \begin{pmatrix} 1/3 \\ 0 \end{pmatrix}$

and its solution can be written in the form

$X_k = A^k X_0$. From (a) we also have

$$X_k = \begin{pmatrix} \frac{1}{3}\rho^k \cos(k\theta) \\ \frac{1}{9}\rho^{k+1}\cos(k\theta+\theta) + \frac{1}{9}\rho^k\cos(k\theta) \end{pmatrix}.$$

Take $k=-n \Rightarrow X_{-n} = A^{-n}X_0 = \begin{pmatrix} \frac{1}{3}\rho^{-n}\cos(n\theta) \\ \frac{1}{9}\rho^{-n+1}\cos(n\theta+\theta) + \\ +\frac{1}{9}\rho^{-n}\cos(n\theta) \end{pmatrix}$

$\Rightarrow \lim_{n\to\infty} A^{-n}\begin{pmatrix} 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

since $\lim_{n\to\infty} \rho^{-n}\cos(n\theta) = 0$ (we have $\rho = \sqrt{10} > 1$ and $|\cos n\theta| \le 1$).

6. Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1 map such that $f(1/3)=-1$ and $f(-1)=1/3$. Justify that $\{-1, 1/3\}$ is a 2-cycle for f. Prove that, if $|f'(-1)f'(1/3)|<1$ then the 2-cycle $\{-1, 1/3\}$ is an attractor.

<u>Solution.</u> Since $f^2(-1) = f(f(-1)) = f(1/3) = -1$, we have that, by definition, $\{-1, 1/3\}$ is a 2-cycle.

~~We have that~~

In order to prove that the 2-cycle $\{-1, 1/3\}$ is an attractor, we have to prove that (by definition) -1 is an attracting fixed point of f^2.

Since $(f^2)'(-1) = (f\circ f)'(-1) = f'(f(-1))\cdot f'(-1) = f'(\frac{1}{3})\cdot f'(-1)$ we have that, by hypothesis, $|(f^2)'(-1)| < 1$. Thus, -1 is, indeed, an attracting fixed point of f^2.