LECTURE 2

DATE : 4 OCTOBER 2021

2. Calculus for functions of several variables I. The Geometry $\mathbb{R}^d$, partial derivatives, the Gradient.

Why functions of several variables?

Reality is complicated...

- Volume of a cone  $V = \frac{\pi r \times h}{3}$

$V = V(r, h)$

- You buy a new BMW

Price = Price (Options) ← many : motor
leather seats
⋮

Problem : (How) can we ~~construct~~ describe all the geometric objects that we need (in $\mathbb{R}^d$) starting with a single simple concept?

Math Insight : „Yes we can!"

based on the so called inner (a dot) product.

§ 2.1. The Geometry $\mathbb{R}^d$

I will try to avoid the word „vector"

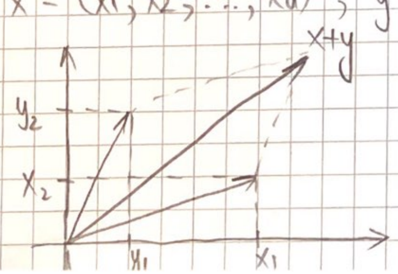
$(x_1, x_2, \dots, x_d) \in \mathbb{R}^d$ element of $\mathbb{R}^d$

cartesian prod. $\mathbb{R} \times \mathbb{R} \times \dots$

Two op. on $\mathbb{R}^d$

- „+" addition $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_d + y_d)$

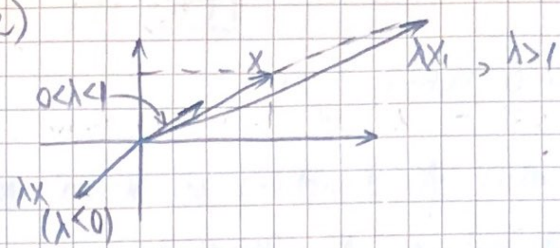
$x = (x_1, x_2, \dots, x_d)$, $y = (y_1, y_2, \dots, y_d)$



• multiplication by a scalar ($\lambda \in \mathbb{R}$)

$\lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_d)$

geom: „scaling"



You know all these things!
These are just operations with matrices

$$\lambda(x+y) = \lambda x + \lambda y, \quad (\lambda+\mu)x = \lambda x + \mu x \quad \text{etc.}$$

Careful you have $0_{\mathbb{R}}$ (scalar) and $0_{\mathbb{R}^d} = (0, \dots, 0)$

$$0_{\mathbb{R}}\, x = 0_{\mathbb{R}^d} \ \forall x \in \mathbb{R}^d \quad \text{and} \quad \lambda \cdot 0_{\mathbb{R}^d} = 0_{\mathbb{R}^d} \quad \forall \lambda \in \mathbb{R}$$

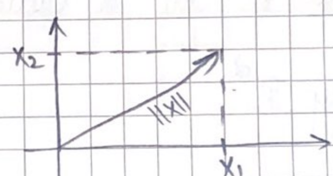
Straight line through the origin $= \{\lambda x : \lambda \in \mathbb{R}\}$ pointing in the direction of $x \neq 0_{\mathbb{R}^d}$

• the inner (dot) product (alternative notation $\langle x, y \rangle$)

$$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_d y_d$$

• norm of x (length)

$$\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + x_2^2 + \dots + x_d^2}$$



• distance $\quad \mathrm{dist}(x, y) = \|x - y\|$

the properties (see below) can be generalized axiomatically

(N1) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0_{\mathbb{R}^d}$

(N2) $\|\lambda x\| = |\lambda|\,\|x\|$, $\forall \lambda \in \mathbb{R}$, $x \in \mathbb{R}^d$

(N3) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^d$

HW: „axioms" of dist

- generalize "$[x-\varepsilon, x+\varepsilon] \subset \mathbb{R}$" $= \{z \in \mathbb{R} : |x-z| \leq \varepsilon\}$

$B_r(x) = \{y \in \mathbb{R}^d : \|x-y\| < r\}$, $r > 0$

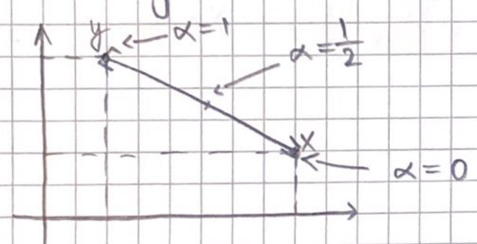
open ball centered at x and of radius r

- orthogonality in $\mathbb{R}^d$

$x \perp y \iff x \cdot y = 0$

- segments in $\mathbb{R}^d$

$$\underbrace{[x, y]}_{\text{segment}} = \{ \underbrace{(1-\alpha)x + \alpha y}_{\text{convex comb (or average)}} : \alpha \in \overbrace{[0,1]}^{\subset \mathbb{R}} \} \subset \mathbb{R}^d$$



- C convex set if $\forall x, y \in C$ we have $[x, y] \subset C$

Remark: the Ball is a convex set

Remark: $\mathbb{R}^d$ can not be ordered (completely) !!!

$\exists x, y \in \mathbb{R}^d \quad x \nleq y$

§ 2.2. Partial derivatives and the Gradient

Def: $f : \mathbb{R}^d \to \mathbb{R}$ f has partial derivative with respect to x_k at a point $x = (x_1, \dots, x_d)$ if

$$\lim_{h \to 0} \frac{f(x_1, x_2, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots) - f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots)}{h}$$

if the limit exists. Notation $\frac{\partial f}{\partial x_k}$

"keep all other variables constant and take derivative with respect to x_k"

Partial derivative → only partial information ("in direction of x_k)

That's why we need : The Gradient (full info)

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_d}(x) \right), x \in \mathbb{R}^d$$