LECTURE 1

Part I : Differential Calculus

1. Differential Calculus for functions of a single variable. Taylor's Formula.

Problem: Digital computing machines do not store values of nonlinear functions such as $\sin x$ but rather compute them (whenever needed) using only basic arithmetic operations: $+, -, \cdot, \div$.
How is this done in practice?

Math. Insight: Polynomials
$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ are constructed via a finite number of arithmetic operations.
(So, that's what we need!)

§ 1.1. Limits and Continuity

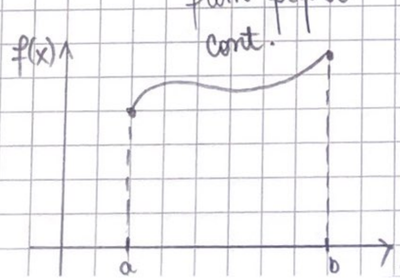
(CAUCHY 1821) „$\varepsilon - \delta$ defs"

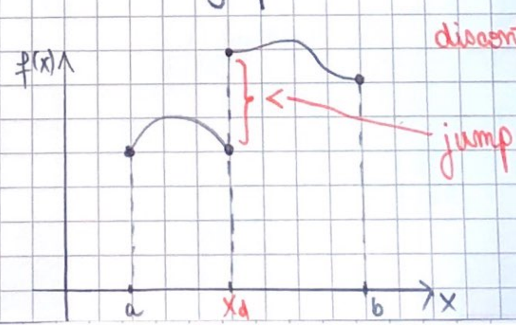
Def: (limit of f at a point x^*)
$f : (a, b) \to \mathbb{R}$
$\lim_{x \to x^*} f(x) = l$ if

$\forall \varepsilon > 0 \ \exists \delta > 0$ so that (s.t.) $|f(x) - l| < \varepsilon$ for all $x \in (a, b)$ with $|x - x^*| < \delta$

Def: $f : (a, b) \to \mathbb{R}$ is cont at $x^* \in (a, b)$ if $\lim_{x \to x^*} f(x) = f(x^*)$

Intuitively: „f cont : you can draw its graph without removing pencil from paper"





THEOREM (WEIERSTRASS ~ 1860)

$f : [a,b] \to \mathbb{R}$ cont

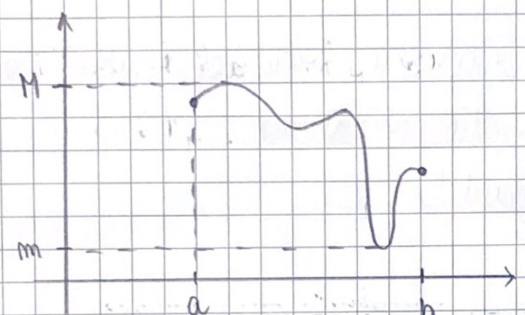
Then f is bounded and attains its minimum (m) and maximum (M) in the sense that $f([a,b]) = [m, M]$

notation for the image of $[a,b]$ through f

$$f([a,b]) = \{ y \in \mathbb{R} : \exists x \in [a,b] \text{ s.t. } y = f(x) \}$$

Remark: $\forall x \in [a,b] \; \exists y \in [m, M]$ s.t. $y = f(x)$

"you pass through all points between m and M"



§ 1.2. Differential Calculus and Mean-value Theorems

"Derivative" IDEA Newton (Velocity)

Def: (differentiability) $f : (a,b) \to \mathbb{R}$ is differentiable at $x^* \in (a,b)$ if the limit

$$\lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h}$$ exists and is finite ($< \infty$)

Remark: If the limit exists but is infinite, then we say that f has derivative at x^* (but f is not differentiable at x^*).

Why derivatives? OPTIMIZATION

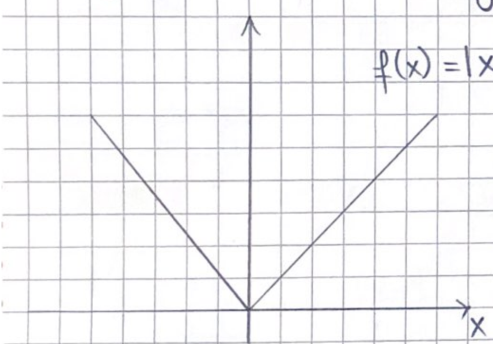
THEOREM (FERMAT)

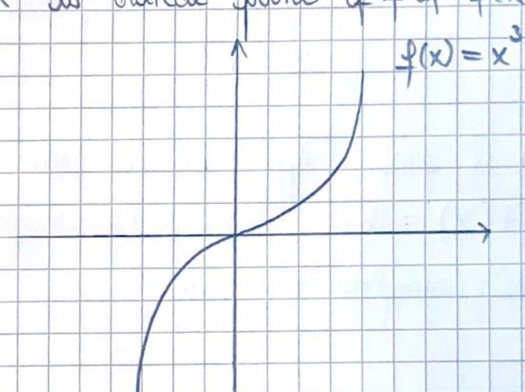
$f:(a,b)\to\mathbb{R}$ has a local minimum/maximum at $x^*\in(a,b)$.
If f is differentiable at x^* then $f'(x^*)=0$.

Remark: The converse statement doesn't hold
($f'(x^*)=0 \not\Rightarrow x^*$ local extremum)

Def: (local extrema) x^* is a local min/max of f
$f:(a,b)\to\mathbb{R}$ if $f(x^*)\lesseqgtr f(x)$
$\forall x\in[x^*-\varepsilon, x^*+\varepsilon]$, $\varepsilon>0$

Def: (critical or stationary points) x^* is critical point of f if $f'(x^*)=0$.





0 is a minimum but
f is not diffable at 0
(so 0 is not a crit)

$f'(0)=0$ but 0 is not extr

Remark: local extrema and critical points do not coincide!

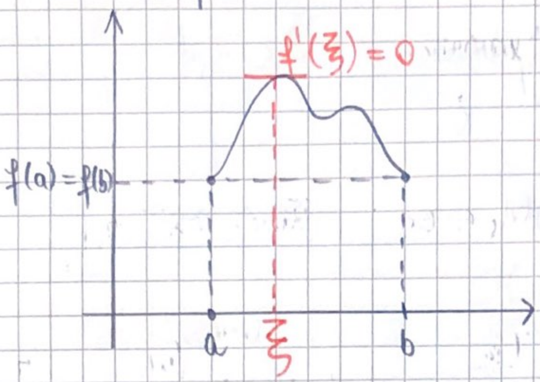
THEOREM (ROLLE)

$f:[a,b]\to\mathbb{R}$

If:
- f cont on $[a,b]$
- f diffable on (a,b)
- $f(a)=f(b)$

Then $\exists\ \xi\in(a,b)$ such that $f'(\xi)=0$.

Idea of proof : use Theorem Fermat + Theorem Weierstrass



! This means that f has a local min / max !

THEOREM (LAGRANGE)

$f : [a,b] \to \mathbb{R}$

If : • f cont on $[a,b]$

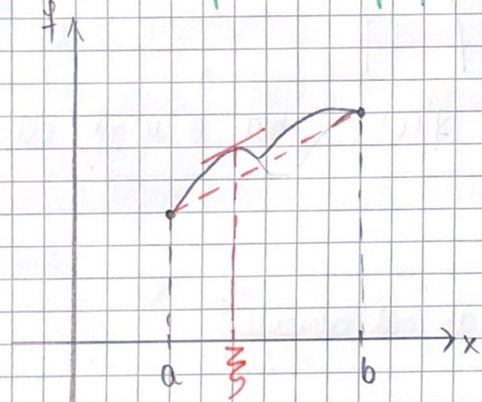
• f diffable on (a,b)

Then $\exists\ \xi \in (a,b)$ s.t. $f'(\xi) = \dfrac{f(b)-f(a)}{b-a}$

Idea of proof : Apply Theorem Rolle to :

$F(x) = (b-a)\,f(x) - x\,(f(b)-f(a))$

HW : complete the proof



§ 1.3. Taylor's Formula

IDEA: approximate (nonlinear) function by polynomial

THEOREM (TAYLOR)

$f:(a,b)\to\mathbb{R}$ diffable $n+1$ times, $x_0\in(a,b)$. Then, $\forall x\in(a,b)$
$\exists\, \xi$ between x and x_0 such that

$$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}_{T_n(x)} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}}_{R_n(x)}$$

$T_n(x)$ Taylor polyn. $\qquad R_n(x)$ Remainder

$f(x) = T_n(x) + R_n(x)$

$R_n(x) \sim \frac{1}{(n+1)!} \xrightarrow{n\to\infty} 0$

Examples: usually $x_0 = 0$ (Mac Laurin)

$$e^x \approx e^0 + \frac{e^0}{1!}(x-0) + \frac{e^0}{2!}(x-0)^2 + \dots$$
$$\approx 1 + x + \frac{1}{2}x^2 + \dots + \frac{1}{n!}x^n$$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Idea of the proof:

Step 1: Prove Taylor formula w. "integral remainder"

use integration by parts

$$(*)\ f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \int_{x_0}^{x} f^{(n+1)}(r)\cdot\frac{(x-r)^n}{n!}\,dr$$

$$f(x) = f(x_0) + \int_{x_0}^{x} 1\cdot f'(r)\,dr$$

derivatives with respect to r

$$= f(x_0) + \int_{x_0}^{x} (-(x-r))'\, f'(r)\,dr$$

integr. by parts

$$= f(x_0) - 0 + (x-x_0)f'(x_0) + \int_{x_0}^{x} \overbrace{(x-r)}^{=\frac{1}{2}(-(x-r)^2)'} f''(r)\,dr$$

and repeat integr by parts to get (*)

Step 2: Prove that integral reminder is equivalent to Lagrange reminder

$$\int_{x_0}^{x} f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt \quad \text{vs} \quad \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1}$$

HW: Find ξ s.t. $=$ holds

Hint: use Theorem Weierstrass