# LECTURE 9

DATE : 22 NOVEMBER 2021
WEEK 9

## 9. The Riemann Integral : related topics

### § 9.1. Numerical computation of the Riemann Integral : the Trapezoidal Rule

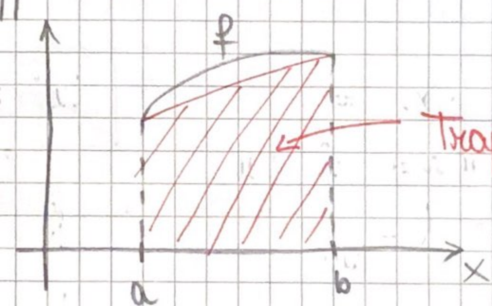
$$\int_a^b f(x)\,dx = F(b) - F(a) \quad \text{where} \quad F' = f$$

antiderivative

Most functions don't admit simple / nice antiderivatives

$\Longrightarrow$ you want to approximate the Riemann Integral

Trapezoidal rule




$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a) + f(b)\right)$$

How good is this ?

$$\text{error} = \left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left(f(a)+f(b)\right)\right| = \frac{(b-a)^3}{12}\left|f''(\xi)\right|$$

for some $\xi \in (a, b)$

roughly
- if $(b-a) \gg 1$ approx is bad
- if $(b-a) \ll 1$ approx is good

How can we exploit this?

## The Composite Trapezoidal rule

IDEA: Subdivide $[a,b]$ into $n$ equal subintervals

$$x_i - x_{i-1} = \frac{b-a}{n}$$

use Trapezoidal rule on each $[x_{i-1}, x_i]$

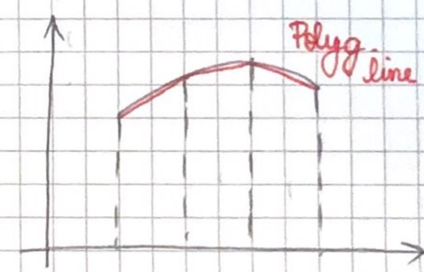
$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{n} \frac{b-a}{n}\cdot\frac{1}{2}\left(f(x_{i-1}) + f(x_i)\right)$$

$$\left(\text{this rewrites as } \frac{b-a}{n}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(x_i)\right)\right)$$

$$\text{error} = \left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} \frac{b-a}{n}\cdot\frac{1}{2}\left(f(x_{i-1})+f(x_i)\right)\right| = \frac{b-a}{12}\cdot\frac{(b-a)^2}{n^2} f''(\xi)$$

( far better than $\frac{1}{n}$ because comput. effort $\sim n$ )

Remark: What we have done is just: approximate the function $f$ by a piece-wise liniar function (polygonal line)

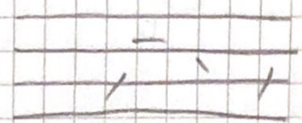



• Formula contains an average times length of interval.

# § 9.2. Integrals and Probability Theory

$$\text{Probability} = \frac{\text{Favorable outcomes}}{\text{Total nr of outcomes}}$$

The Buffon Needle Problem (1733)

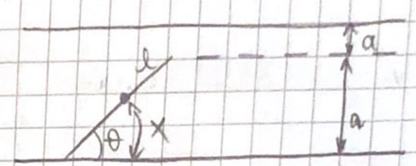


Throw Needles on the table cloth (randomly)
What is the probability (for one needle) to cross a line?

$2l$ = length of needle

$2a$ = distance between line

$(l < a)$




$x$ = distance from center of needle to closest line

$x \in [a, b]$

$\theta$ = angle needle makes with horizontal lines

$\theta \in [0, \pi]$

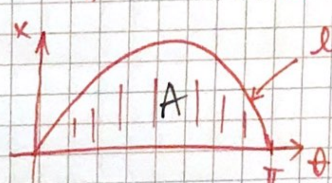
$\Omega$ = all possible outcomes
$= \{(x, \theta) \in [0, a] \times [0, \pi]\}$

$m(\Omega) = a\pi$ ($\Omega$ is a box)

$A$ = favorable outcomes
$= \{(x, \theta) : \frac{0 \le x \le l \sin\theta}{\theta \in [0, \pi]}\}$

if $l \sin\theta > x$ you don't cross (geometry)

$$m(A) = \iint_A dx\, d\theta = \int_0^{\pi} \left( \int_0^{l \sin\theta} dx \right) d\theta \dots = 2l$$




$$\text{Probab} = \frac{m(A)}{m(\Omega)} = \frac{2l}{a\pi}$$

## § 9.3. Monte Carlo integration

IDEA (Stan. ULAM): use randomness to solve a deterministic problem

Name ("Monte Carlo") comes from Ulam's uncle to used to lose family money in the MC Casinos

This idea was famously used in computations for the bombs.

How does it work (Naive...)

$$y = \int_\Omega f(x)\,dx, \quad x = (x_1, \dots, x_N) \quad N \text{ very large}$$

Riemann: subdivide $\Omega$ in $n$ subdomains of equal measure

$m(\Omega_i) = \frac{1}{n} m(\Omega)$

$$\left( m(\Omega) = \int_\Omega dx \right)$$

$$y \approx \sum_{i=1}^{n} f(\xi_i)\, \underbrace{m(\Omega_i)}_{\frac{m(\Omega)}{n}}$$

this rewrites as an average

$$= m(\Omega)\, \frac{1}{n} \sum_{i=1}^{n} f(\xi_i)$$

Ulam: choose them randomly

Law of Large Numbers: average obtained from a large nr. of samples should be ~~above~~ close to the average over entire possibility space