Chapter 1

Linear Differential Equations. Problems

1.1 Introduction

1.1.1 Show that the function $\varphi : \mathbb{R} \to \mathbb{R}$, given by the expression $\varphi(t) = 2e^{3t}$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x' = 3x, \quad x(0) = 2.$
Represent the corresponding integral curve* and describe its long term behavior**.

**A graphical representation of a solution of some differential equation is called an integral curve or a solution curve of this equation.*

***To describe the long term behavior of some function means to decide whether it is: periodic, oscillatory around some fixed value η^* (i.e. the values of the function changes many many times from values below η^* to values above η^*), bounded, increasing, and to describe how it behaves at $\pm\infty$.*

In the following 5 problems we consider a simple gravity pendulum (idealized) that moves along a vertical circle whose radius is equal to the length of the rod. The movement initiates at the moment $t = 0$. You are required to describe the movement of a pendulum if the given function $\varphi(t)$ is the angle (measured in radians in the trigonometric sense) between the rod and the vertical.

1.1.2 Let $\eta \in \mathbb{R}^*$ be fixed. Show that the function $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \eta \sin t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x''+x = 0, \quad x(0) = 0, \quad x'(0) = \eta.$
Represent the corresponding integral curve and describe its long term behavior.
For each $\eta \in \{\frac{\pi}{18}, -\frac{\pi}{18}, \frac{\pi}{2}, \frac{\pi}{3}, 1, 2\}$ describe the movement of a pendulum.

1.1.3 Show that the function $\varphi(t) = e^{-2t}\cos t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = -2.$
Represent this integral curve and describe its long term behavior.
Describe the movement of a pendulum.

1.1.4 Show that the function $\varphi(t) = \sin 2.2t - \sin 2t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + 4.84x = -0.84\sin 2t, \quad x(0) = 0, \quad x'(0) = 0.2.$
Represent this integral curve and describe its long term behavior.
Describe the movement of a pendulum.

1.1.5 Show that the function $\varphi(t) = \sin\sqrt{6}t - \sin 2t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + 6x = -2\sin 2t, \quad x(0) = 0, \quad x'(0) = \sqrt{6} - 2.$
Represent this integral curve and describe its long term behavior.
Describe the movement of a pendulum.

1.1.6 Show that the function $\varphi(t) = t\sin t$ for all $t \in \mathbb{R}$, is a solution of the Initial Value Problem $\quad x'' + x = 2\cos t, \quad x(0) = 0, \quad x'(0) = 0.$
Represent this integral curve and describe its long term behavior.
Describe the movement of a pendulum.

1.1.7 Decide whether $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \cos t$ for all $t \in \mathbb{R}$, is a solution of the differential equations $x' + x = 0$ or $x'' - x = 0$ or $x''' + x' = 0$ or $x^{(4)} + x'' = 0$.

1.1.8 Find all constant solutions of the differential equations: a) $x' = x - x^3$; b) $x' = \sin x$; c) $x' = \dfrac{x+1}{2x^2+5}$; d) $x' = x^2 + x + 1$; e) $x' = x + 4x^3$; f) $x' = -1 + x + 4x^3$.

1.1.9 i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be defined by $x_1(1) = 1$, $x_2(t) = t$ and $x_3(t) = t^2$ for all $t \in \mathbb{R}$. Prove that they are linearly independent in the linear space $C(\mathbb{R})$ (over the filed $\mathbb{R}$ and with the usual operations).

(ii) Find all $\alpha, \beta, \gamma \in \mathbb{R}$ such that $x(t) = \alpha t^2 + \beta t + \gamma$ is a solution of a) $x' - 5x = 2t^2 + 3$ or b) $x'' = 0$ or c) $x''' = 0$. Write the solutions (of the differential equation) that you found.

1.1.10 (i) Let the functions $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$ be such that $x_1(1) = \cos t$, $x_2(t) = \sin t$ and $x_3(t) = e^t$ for all $t \in \mathbb{R}$. Prove that they are linearly independent in the linear space $C(\mathbb{R})$.

(ii) Find all $\alpha, \beta, \gamma \in \mathbb{R}$ such that $x(t) = \alpha \sin t + \beta \cos t + \gamma\, e^t$ is a solution of a) $x' + x = -3\sin t + 2\,e^t$ or b) $x'' + 4x = -3\sin t$ or c) $x'' + x = -3\sin t$ or d) $x'' + x = 0$ or e) $x''' - x'' + x' - x = 0$. Write the solutions you found.

1.1.11 Find $r \in \mathbb{R}$ such that $x(t) = e^{rt}$ is a solution of a) $x'' - 5x' + 6x = 0$ or b) $x''' - 5x'' + 6x' = 0$ or c) $x^{(4)} - 5x''' + 6x'' = 0$ or d) $x'' + 9x = 0$ or e) $x'' + x' + x = 0$.

1.1.12 Find $r \in \mathbb{R}$ such that $x(t) = t^r$ is a solution on the interval $(0, \infty)$ of a) $t^2x'' - 4tx' + 6x = 0$ or b) $t^2x'' + tx' - x = 0$ or c) $t^2x'' - x = 0$ or d) $t^2x'' + x = 0$ or e) $t^2x'' - tx' + x = 0$.

1.1.13 Find as many functions $x \in C^1(\mathbb{R})$ as you can such that: a) $x' = x$; b) $x' = 2x$; c) $x' = -x$; d) $x' = ax$, with $a \neq 0$ a real parameter.

1.1.14 Integrate the following differential equations.
a) $x' = 0$; b) $x' = 2t$; c) $x' = \sin t$; d) $x' = 2t + \sin t$; e) $x' = e^{2t}\cos t$;
f) $x' = (t^2 - 5t + 7)\sin t$; g) $x' = e^{t^2}$; h) $x'' = -3$; i) $x''' = 0$.
j) $tx' + x = 0$; k) $tx' + x = 1$; l) $2xx' = -2t$; m) $x'e^t + xe^t = 0$; n) $x'e^{2t} + 2xe^{2t} = 0$.

1.1.15 Find an integrating factor and then integrate the following differential equations. a) $x' + x = 0$; b) $x' + x = 1 + t$; c) $x' + 2x = \sin t$; d) $x' - 2x = 0$;
e) $tx' + 2x = 1$; f) $tx' + 3x = \frac{1}{t^2}$.

1.2 Understanding the fundamental theorems

1.2.1 How many solutions have each of the following problems:
a) $x'' + t^2x = 0, \quad x(0) = 0,$
b) $x'' + t^2x = 0, \quad x(0) = 0, \quad x'(0) = 0$
c) $x'' + t^2x = 0, \quad x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 1$?

1.2.2 Find the general solution of each of the following equations, looking first for some solutions of the form $x = t^r$, with $r \in \mathbb{R}$.
a) $t^2x'' - 8tx' + 20x = 0,\ t \in (0, \infty)$; b) $t^2x'' - 6x = 0,\ t \in (0, \infty)$;
c) $t^2x'' + tx' + x = 0,\ t \in (0, \infty)$;

1.2.3 Is $x = c_1e^t + c_2e^{-t}$, $c_1, c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$? Is $x = c_1\cosh t + c_2\sinh t$, $c_1, c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$? Recall that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

1.2.4 a) Verify that $y_1 = x$ and $y_2 = e^{-2x}$ are solutions of $(2x+1)y'' + 4xy' - 4y = 0$.

b) Find the solution of the Initial Value Problem:

$(2x+1)y'' + 4xy' - 4y = 0\,, \quad y(0) = 1\,, \quad y'(0) = 0.$

1.2.5 a) Find a particular solution of the form $x_p = ae^t$ (with $a \in \mathbb{R}$) for the equation $x' - 2x = e^t$.

b) Find a particular solution of the form $x_p = be^{-t}$ (with $b \in \mathbb{R}$) for the equation $x' - 2x = e^{-t}$.

c) Using the Superposition Principle, and a) and b), find a particular solution for the equation $x' - 2x = 5e^t - 3e^{-t}$.

d) Find the general solution of $x' - 2x = 5e^t - 3e^{-t}$.

1.3 First order linear differential equations

1.3.1 Find the general solution defined on the interval $I = (0, \infty)$ of the following first order linear homogeneous equations.

a) $x' + \frac{1}{t}x = 0$; b) $x' - \frac{1}{t}x = 0$; c) $x' - \frac{3}{t}x = 0$.

1.3.2 Find the general solution of the following first order linear nonhomogeneous equations.

a) $x' + \frac{1}{t^2}x = 1 + \frac{1}{t},\ t \in (0, \infty)$; b) $x' + tx = e^{-t^2 - t}$; c) $x' + \frac{2t}{1+t^2}x = 3$;
d) $x' - \frac{2}{t}x = t^2 \sin(2t) - 4\,t^3,\ t \in (0, \infty)$; e) $x' + \frac{1}{\sqrt{t}}x = \frac{1}{2\sqrt{t}},\ t \in (0, \infty)$;
f) $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}$, $t \in (0, \infty)$.

1.3.3 Find the general solution of $x' + \frac{1}{t}x = \frac{1}{t}e^{-2t+1}$ for $t \in (0, \infty)$. Justify the result in two ways.

1.3.4 Find the general solution of $x' - x = e^{t-1}$. Justify the result in two ways.

1.3.5 Find the general solution of the following equations by reducing their order.
a) $x''' - x'' = 0$; b) $x'' = \frac{2}{t}x'$.

1.3.6 Let $f : [0, \infty) \to \mathbb{R}$ be a C^1-function that fulfills

$$f'(t) \leq 5f(t), \quad t \in [0, \infty), \text{ and } \ f(0) = 2.$$

Prove that $f(t) \leq 2e^{5t}$, $t \in [0, \infty)$.

1.4 Linear homogeneous differential equations with constant coefficients

1.4.1 Find the general solution of each of the following differential equations.

a) $x' + 6x = 0$; b) $x'' + 4x' + 4x = 0$; c) $x'' = 0$; d) $x''' = 0$; e) $x^{(n)} = 0$;
f) $x'' + x' + x = 0$; g) $x''' - 6x'' + 11x' - 6x = 0$; h) $x^{(4)} - x = 0$; i) $x' = kx$.

1.4.2 Find the linear homogeneous differential equation with constant coefficients and of minimal order that has as solutions the following functions. Write the general solution of the differential equation that you found.

a) e^{-3t} and e^{5t}; b) $5e^{-3t}$ and $-3e^{5t}$; c) $5e^{-3t} - 3e^{5t}$; d) $5te^{-3t}$ and $-3e^{5t}$;
e) $5e^{-3t}$ and $-3te^{5t}$; f) $(5 - 3t)e^{-3t}$; g) $(5 - 3t + 2t^2)e^{-3t}$; h) $\sin 3t$;
i) $t - \sin 3t$; j) $-t \sin 3t$; k) $e^{5t} \sin 3t$; l) $e^{-3t} \sin 3t$; m) $t^7 + 1$; n) $5t - 3e^{5t}$;
o) $(t - 1)^2$; p) $2\cos^2 t$; r) $\sin^2 t$; s) $(e^t)^2$.

1.4.3 Decide whether the following statements are true or false.

a) "There exists a linear homogeneous differential equation with constant coefficients of order 7 that has as solutions $(t^3 + 2t^4)\cos 2t$ and te^{-t}."

b) "There exists a linear homogeneous differential equation with constant coefficients that has as solution $1/t$."

c) "There exists a linear homogeneous differential equation with constant coefficients that has as solution e^{t^2}."

d) "There exists a linear homogeneous differential equation with constant coefficients that has as solution $t/(1 + t^2)$."

1.4.4 Find the solution for each of the following IVPs. Here $\eta, \lambda \in \mathbb{R}$ are fixed parameters.

a) $x'' + \pi^2 x = 0,\ x(0) = 0,\ x'(0) = \eta$; b) $x'' + \lambda x = 0,\ x(0) = 0,\ x'(0) = \eta$.

1.4.5 Find all solutions for each of the following BVPs (boundary value problems).

a) $x'' + x = 0, \quad x(0) = x(\pi) = 0$; b) $x'' + x = 0, \quad x(0) = x(1) = 0$;
c) $x'' + \pi^2 x = 0, \quad x(0) = x(1) = 0$; d) $x'' + \pi^2 x = 0, \quad x(0) = x(2) = 0$.

1.4.6 Find $\lambda \in \mathbb{R}$ with the property that there exist nonnull 2π–periodic solutions of $x'' + \lambda x = 0$.

1.4.7 Find $\mu \in \mathbb{R}$ and $\omega > 0$ with the property that there exist nonnull periodic solutions of $x'' + \mu x' + \omega^2 x = 0$. In this case write the minimal period.

1.4.8 Find $\mu \in \mathbb{R}$ and $\omega > 0$ with the property that all solutions of $x''+\mu x'+\omega^2 x = 0$ goes to 0 as $t \to \infty$.

1.5 Linear nonhomogeneous differential equations with constant coefficients

1.5.1 Decide whether the following statements are true or false.

a) "All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim\limits_{t\to\infty} x(t) = 1$."

b) "The solution of the IVP $x'' + 4x = 1, \quad x(0) = 5/4, \quad x'(0) = 0$ satisfies $x(\pi) = 5/4$."

c) "The equation $x' = 3x + t^3$ admits a polynomial solution. (*Hint.* Look for a polynomial solution of degree 3.)"

1.5.2 Let $\lambda \in \mathbb{R}$ be a parameter. Find the general solution of $x'' - x = e^{\lambda t}$ knowing that, depending on λ, it has a particular solution either of the form $ae^{\lambda t}$ or of the form $ate^{\lambda t}$.

1.5.3 Let $\omega > 0$ be a parameter and denote $\varphi(\cdot, \omega)$ the solution of the IVP

$$x'' + x = \cos(\omega t), \quad x(0) = x'(0) = 0.$$

(i) When $\omega \neq 1$ find a solution of the form $x_p(t) = a\cos(\omega t) + b\sin(\omega t)$ for $x'' + x = \cos(\omega t)$. (Here you have to determine the real coefficients a and b.)

(ii) Find a solution of the form $x_p(t) = t(a\cos t + b\sin t)$ for $x'' + x = \cos t$.

(iii) Find $\varphi(\cdot, \omega)$ for any $\omega > 0$.

(iv) Prove that $\lim_{\omega\to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

1.5.4 Let $\alpha > 0$ and $\varphi(\cdot, \alpha)$ be the solution of the IVP

$$x'' - 4x = e^{\alpha t}, \quad x(0) = x'(0) = 0.$$

(i) When $\alpha \neq 2$ find a solution of the form $x_p(t) = ae^{\alpha t}$ for $x'' - 4x = e^{\alpha t}$. (Here you have to determine the real coefficient a.)

(ii) Find a solution of the form $x_p(t) = ate^{2t}$ for $x'' - 4x = e^{2t}$.

(iii) Find $\varphi(\cdot, \alpha)$ for any $\alpha > 0$.

(iv) Prove that $\lim_{\alpha\to 2} \varphi(t, \alpha) = \varphi(t, 2)$ for each $t \in \mathbb{R}$.

1.6 Linear homogeneous planar systems with constant coefficients

1.6.1 *Let $A \in \mathcal{M}_2(\mathbb{R})$. Using both methods that we learned, the characteristic equation method and reduction to second order equation, find the general solution of the system $X' = AX$ in each of the following situations. Also, find a fundamental matrix solution and, finally, find e^{tA}, the principal matrix solution.*

a) $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$; b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; c) $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$;

d) $A = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix}$; e) $A = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}$; f) $A = \begin{pmatrix} 5 & -3 \\ 8 & -6 \end{pmatrix}$;

g) $A = \begin{pmatrix} -3 & 4 \\ -1 & 1 \end{pmatrix}$; h) $A = \begin{pmatrix} 1 & -4 \\ 2 & 5 \end{pmatrix}$; i) $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$;

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$; k) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$; l) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

m) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$; n) $A = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where $a \in \mathbb{R}$ is a parameter.

1.7 Exams problems

1.7.1 Let $f \in C(\mathbb{R})$ and $\eta \in \mathbb{R}$. Find an integral representation of the solution of the IVP $x' + x = f(t), \quad x(0) = \eta$.

1.7.2 Write the general solution of the following equations.
a) $\quad x' - 3t^2 x = t^3$; b) $\quad x' - 3t^2 x = f(t)$ where $f \in C(\mathbb{R})$.

1.7.3 Find the solution of the IVP
a) $x' + 2tx = t,\ x(0) = 0$; b) $x' + tx = 1,\ x(0) = 0$;
c) $x'' + 4x = 1,\ x(0) = 1,\ x'(0) = 0$.

1.7.4 Find the general solution of the scalar differential equation $x' - ax = at - 1$, where the unknown is the function x of variable t and $a \in \mathbb{R}^*$ is a fixed parameter.

1.7.5 Let $t \in \mathbb{R}$. Using the Euler's formula compute e^{it}, $e^{i\pi}$, $e^{i\pi/2}$, $e^{(-1+i)t}$.

1.7.6 Find the linear homogeneous differential equation with constant coefficients of minimal order that has the general solution $c_1e^{-t}\cos t + c_2e^{-t}\sin t$, $c_1, c_2 \in \mathbb{R}$.

1.7.7 Find the linear homogenous differential equation of minimal order that has as solution the function $(2te^t)^2$.

1.7.8 Find the linear homogenous differential equation of minimal order that has as solution the function $1 + t(1 + e^{-t})$.

1.7.9 Let $k, \eta \in \mathbb{R}$ be fixed parameters. Find the solution of the IVP
$x' = k(21 - x), \quad x(0) = \eta.$

1.7.10 We consider the equation $x'' - x = te^{-2t}$.
a) Find a particular solution of the form $x_p(t) = (at + b)e^{-2t}$, where $a, b \in \mathbb{R}$.
b) Find the general solution.
c) Find the solution that satisfies the initial conditions $x(0) = 0$, $x'(0) = 0$.

1.7.11 Write the statement of The Superposition Principle for second order linear nonhomogeneous differential equations. Give an example.

1.7.12 Let $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined by $\mathcal{L}x = x'' - 2x' + x$ for any $x \in C^2(\mathbb{R})$.
(a) Prove that $\mathcal{L}$ is a linear map. What is the dimension of its kernel?
(b) Find the general solution of the equation $x'' - 2x' + x = \cos 2t$ knowing that it has a particular solution of the form $a\cos 2t + b\sin 2t$, for some $a, b \in \mathbb{R}$.
(c) Let $f_1(t) = e^{2t}$ and $f_2(t) = e^{-2t}$ for all $t \in \mathbb{R}$. Find a particular solution of the equation $\mathcal{L}x = 3f_1 + 5f_2$.

1.7.13 Let $a_1, a_2 \in C(\mathbb{R})$ and $\mathcal{L} : C^2(\mathbb{R}) \to C(\mathbb{R})$ be defined by $\mathcal{L}x(t) = x''(t) + a_1(t)x'(t) + a_2(t)x(t)$ for any $t \in \mathbb{R}$ and $x \in C^2(\mathbb{R})$. Prove that $\mathcal{L}$ is a linear map.

Let $\Phi : \ker\mathcal{L} \to \mathbb{R}^2$ be defined by $\Phi(x) = (x(0), x'(0))$. Prove that Φ is an isomorphism of linear spaces. What is the dimension of $\ker\mathcal{L}$?

1.7.14 a) Write the statement of the Fundamental Theorem for linear homogeneous second order differential equations.
b) The following proposition is true or false? Justify.
"The general solution of the differential equation $x'' - 9x = 0$ is
$x(t) = c_1 \cosh 3t + c_2 \sinh 3t$, where c_1, c_2 are arbitrary real constants."

1.7.15 Find the general solution of the following differential equation $x'' - x = e^{at}$. Discuss with respect to the real parameter a.

1.7.16 Write the general solution of $x'' - a^2 x = e^{bt}$, where $a > 0$ and $b \in \mathbb{R}$ are parameters.

1.7.17 Find the general solution of the following differential equations

$$x' + ax = -at + 1, \quad x'' - ax' + (a-1)x = 0,$$

where $a \in \mathbb{R} \setminus \{0, 1\}$ is a real parameter. Here the unknown is the function denoted x of independent variable t.

1.7.18 Find the solution of the IVP

$$\theta'' + 9\theta = 0, \quad \theta(0) = \frac{\pi}{2}, \quad \theta'(0) = 0.$$

Here $\theta = \theta(t)$, and t is the time variable measured in minutes. Describe the motion of a pendulum governed by this IVP. After how much time the pendulum will return to the initial position?

1.7.19 We consider the differential equation

$$x'' + 4x = \cos 2t.$$

a) Find a solution of the form $x_p = t(a \cos 2t + b \sin 2t)$, with $a, b \in \mathbb{R}$.
b) Find its general solution.
c) Describe the motion of a spring-mass system governed by this equation.

1.7.20 Describe the motion of a spring-mass system whose equation is
$x'' + k/m\ x = 0$, where $k, m > 0$.

1.7.21 Find the solution of the IVP

$$x'' + 4x' + 5x = 0, \quad x(0) = 1, \quad x'(0) = -2.$$

Represent the corresponding integral curve and describe its long-term behavior.

1.7.22 We consider the differential equation $x'' + 2ax' + 4x = 0$, where $a > 0$ is a real parameter. Write the general solution and describe the long-term behavior of the solutions (for $t \in (0, \infty)$). Discuss with respect to the parameter a.

1.7.23 (a) Find the general solution of the following differential equation

$$\varphi'' + \frac{9}{4}\varphi = 0.$$

(b) (True or False) "All the solutions of $\varphi'' + \frac{9}{4}\varphi = 0$ are periodic with a period $T = 4\pi$."

1.7.24 We say that a differential equation exhibit resonance when all its solutions are unbounded.

For what values of the mass m will $mx''+25x = 12\cos(36\pi t)$ exhibit resonance?

1.7.25 Find the general solution of $\ddot{\theta} + \dot{\theta} + \theta = 0$. Prove that $\lim_{t\to\infty} \theta(t) = 0$ for any solution θ of this differential equation.

1.7.26 Let $\alpha \in \mathbb{R}$. Describe the long-term behavior of the function $x(t) = e^{\alpha t}\cos 2t$, $t \in \mathbb{R}$. Discuss with respect to α.

1.7.27 Let $\gamma > 0$. Decide if the following statement is true.

"All the solutions of $x'' + \gamma x' + 9x = 0$ satisfy $\lim_{t\to\infty} x(t) = 0$."

1.7.28 Find $\gamma \in \mathbb{R}$ such that all the solutions of the differential equation $x''+\gamma x'+ 9x = 0$ are periodic.

1.7.29 We consider the differential equation

$$t^2x'' + 2tx' - 2x = 0, \quad t \in (0, \infty).$$

a) Find solutions of the form $x(t) = t^r$ where $r \in \mathbb{R}$ has to be determined.

b) Specify its type and find its general solution.

c) Find the solution of the IVP

$$t^2x'' + 2tx' - 2x = 0, \quad x(1) = 0, \quad x'(1) = 1.$$

1.7.30 Find the general solution of the differential equation

$$x^2u'' - 6xu' + 10u = 0,$$

whose unknown is the function u of variable x. Hint: look for solutions of the form $u = x^r$, with $r \in \mathbb{R}$.

1.7.31 Find the solution of the following Initial Value Problem
$y'' - \frac{y'}{x} = x^2 \quad y(2) = 0\,, \quad y'(2) = 4.$

1.7.32 a) Find a particular solution of the form $x_p(t) = a\,t^2\,e^t$ (where the real coefficient a has to be determined) for

$$x'' - 2x' + x = e^t\,.$$

b) Find a constant solution for

$$x'' - 2x' + x = 5\,.$$

c) Find the general solution of the differential equation:

$$x'' - 2x' + x = 10 + 5e^t\,.$$

d) Find the solution of the IVP

$$x'' - 2x' + x = 5\,, \quad x(0) = 5\,, \quad x'(0) = 0.$$

1.7.33 Decide if the following statement is true.
"The following BVP has at least a solution

$$x'' + 9x = 0, \quad x(0) = 0, \quad x(\pi) = 9."$$

1.7.34 We use the notation

$$\mathcal{L}(x) = x'' + 25x.$$

(i) Find the solution of the IVP

$$\mathcal{L}(x) = 0, \quad x(0) = 0, \quad x'(0) = 1.$$

Represent this integral curve and describe its long-term behavior.

(ii) Let $\varphi_1(t) = t\,\cos(5t)$ and $\varphi_2(t) = t\,\sin(5t)$ for all $t \in \mathbb{R}$. Compute $\mathcal{L}(5)$, $\mathcal{L}(\varphi_1)$ and $\mathcal{L}(\varphi_2)$.

(iii) Find a constant solution for $\mathcal{L}(x) = 5$.
(iv) Find the general solution of the differential equation
$\mathcal{L}(x) = 25 - 25\sin(5t)$.

1.7.35 We consider the differential equation

$$x' + \frac{1}{t^2}x = 0\,, \quad t \in (-\infty, 0).$$

a) Check that $x = e^{1/t}$ is a solution of this d.e..
b) Find the solution of the IVP $x' + \frac{1}{t^2}x = 0, \quad x(-1) = 1.$
c) Find the general solution of $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}$, $t \in (-\infty, 0)$.

1.7.36 Find the general solution of $x' + \frac{1}{t}x = -2$ for $t \in (0, \infty)$. Justify the result in two ways.

1.7.37 Consider the differential equation $x' + \frac{2t}{1+t^2}x = 3$.
(i) Find its general solution.
(ii) Find its solution that satisfies $x(0) = 1$. Is this solution bounded? What about the other solutions of the given equation? Justify.

1.7.38 We consider the system $X' = A(t)X + f(t)$, $t \in \mathbb{R}$, where

$$A(t) = \begin{pmatrix} 5 & 4e^t \\ -7e^{-t} & -7 \end{pmatrix}, \quad f(t) = \begin{pmatrix} -t \\ 1 \end{pmatrix}.$$

Also, let

$$U(t) = \begin{pmatrix} -4e^{-2t} & e^t \\ 7e^{-3t} & -1 \end{pmatrix}.$$

(i) Prove that $U(t)$ is a fundamental matrix solution of the system $X' = A(t)X$;
(ii) Find the principal matrix solution of the system $X' = A(t)X$;
(iii) Find the solution of the system $X' = A(t)X$ that satisfies $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; and the one that satisfies $X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$;
(iv) Find the matrix solution $V(t)$ of the system $X' = A(t)X$ that satisfies $V(0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$; Is V a fundamental matrix solution?
(v) Find the solution of the system $X' = A(t)X$ that satisfies $X(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$;
(vi) Find the solution of the system $X' = A(t)X + f(t)$ that satisfies $X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

1.7.39 Let $n \geq 1$ and $A \in \mathcal{M}_n(\mathbb{R})$. Assume that there exists an eigenvalue $\lambda \in \mathbb{C}$ of A and two linearly independent vectors $v_1, v_2 \in \mathbb{C}^n$ such that $(A - \lambda I_n)v_1 = 0$, $(A - \lambda I_n)v_2 = v_1$. Prove that $\varphi_1(t) = e^{\lambda t}v_1$ and $\varphi_2(t) = e^{\lambda t}(tv_1 + v_2)$ are two linearly independent solutions of the linear system $X' = AX$.

1.7.40 Show that any solution of the system $X' = AX$ satisfies $\lim_{t\to\infty} X(t) = 0$, in each of the following situations:

a) $A = \begin{pmatrix} -3 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 2 & -5 \end{pmatrix}$; b) $A = \begin{pmatrix} -5 & 3 \\ -3 & 1 \end{pmatrix}$; c) $A = \begin{pmatrix} -5 & 9 \\ -2 & 1 \end{pmatrix}$.

1.7.41 Find the general solution and than show that any solution of the system $X' = AX$ is bounded for $t \in [0, \infty)$, in each of the following situations:

a) $A = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$; b) $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -5 \end{pmatrix}$.

1.7.42 Let $A = \begin{pmatrix} -5 & 1 \\ -1 & 1 \end{pmatrix}$. Specify the type of the system $X' = AX$. Prove that any solution of the system $X' = AX$ satisfies $\lim_{t\to\infty} X(t) = 0$.

1.7.43 Let $A \in \mathcal{M}_2(\mathbb{R})$.

(a) Let $\eta \in \mathbb{R}^2$. Write a representation formula for the solution of the IVP

$$X' = AX, \quad X(0) = \eta.$$

(b) Let $t \in \mathbb{R}$. Using the definition of the matrix exponential, compute

$$e^{t\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}, \quad e^{t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \quad e^{t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}.$$

(c) Let $\lambda \in \mathbb{R}$ be an eigenvalue of A and $v_1 \in \mathbb{R}^2$ be an eigenvector of A corresponding to λ. Assume that there exists $v_2 \in \mathbb{R}^2$ such that

$$(A - \lambda I_2)v_2 = v_1.$$

Prove that $\{v_1, v_2\}$ are linearly independent. Prove that

$$\varphi_1(t) = e^{\lambda t}v_1 \quad \text{and} \quad \varphi_2(t) = e^{\lambda t}(tv_1 + v_2)$$

are solutions of the linear system $X' = AX$.

(d) Take

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Apply (c) to find two solutions of the system $X' = AX$. Write the matrix solution of this system whose columns are the two solutions found. Is this a fundamental matrix solution? Find the principal matrix solution. Find e^{tA}.