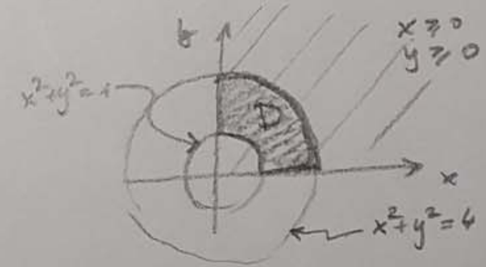
11. $I = \iint_D \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$, $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4,\ x \geq 0,\ y \geq 0\}$



Polar coords:

$x = r\cos\varphi$ D in polar $r \in [1,2]$

$y = r\sin\varphi$ $\varphi \in [0, \frac{\pi}{2}]$

$r = \sqrt{x^2+y^2}$

$$I = \iint_{1\ 0}^{2\ \pi/2} \frac{1}{r}\, r\, dr\, d\varphi = \int_1^2 dr \int_0^{\pi/2} d\varphi = \frac{\pi}{2}$$

Jacobi det of transform / change of coords

12. Define $F : [0,1] \to \mathbb{R}$, $F(t) = f((1-t)a + tb)$, $F(0) = f(a)$, $F(1) = f(b)$

$$(*)\ \frac{d}{dt}F(t) \overset{\text{CHAIN}}{=} \nabla f((1-t)a+tb) \cdot \underbrace{\frac{d}{dt}((1-t)a+tb)}_{(b-a)}$$

F satisf. hyp. of T Lagrange for functions of one var,

hence $\exists\, t_c \in (0,1)$: $F(1) - F(0) = F'(t_c)\,\cancel{(1-0)}^{=1}$

$$\underset{f(b)}{\overset{\parallel}{F(1)}} - \underset{f(a)}{\overset{\parallel}{F(0)}} = \overset{\parallel}{\nabla f(c) \cdot (b-a)}$$

$c = (1-t_c)a - t_c b$

13. a) $\sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} q^n$ (CONV)

$0 \le x < 1$ with $q = -x^2$ and $|q| = |x^2| < 1$

b) $$\sum_{n=0}^{\infty} (-1)^n x^{2n} = \underbrace{1 - x^2 + x^4 - x^6 + \ldots}_{\text{sum of geom series is } \frac{1}{1-q}} = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2} \quad (A)$$

On the other hand for $f(x) = \frac{1}{1+x^2}$, $f(0) = 1$

$f'(x) = \left((1+x^2)^{-1}\right)' = -(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$, $f'(0) = 0$

$f''(x) = \left(-\frac{2x}{(1+x^2)^2}\right)' = -\frac{2}{(1+x^2)^2} - \underbrace{2x\left((1+x^2)^{-2}\right)'}_{\text{This will vanish in } x_0 = 0}$, $f''(0) = -2$

...

Taylor $f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots$

so $\frac{1}{1+x^2} = 1 + 0 - x^2 + \ldots$ (B) (cf. (A) and (B))

Name:

Group:

Mathematical Analysis (R)[1]

1. Let $x = (1,1,0), y = (0,1,1) \in \mathbb{R}^3$. Compute the distance $d(x,y) = \ldots\ldots\ldots$ and $x \cdot y = 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 1$

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$, $\quad f(x_1, x_2, x_3) = x_1 x_2 x_3 + x_3^2 x_2$. The partial derivatives of f are
$\frac{\partial f}{\partial x_1}(x_1,x_2,x_3) = x_2 x_3$, $\frac{\partial f}{\partial x_2}(x_1,x_2,x_3) = x_1 x_3 + x_3^2$, $\frac{\partial f}{\partial x_3}(x_1,x_2,x_3) = x_1 x_2 + 2x_3 x_2$

3. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function, $\Delta = \{a = x_0, x_1, \ldots, x_{n-1}, x_n = b\}$ a division and $\xi = \{\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n\}$ with $\xi_i \in [x_{i-1}, x_i]$ a system of intermediate points. The Riemann sum associated to f, Δ, ξ is
$\ldots\ldots\ldots \sigma(f,\Delta,\xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$

4. The improper integral $\int_{-\infty}^{\infty} \frac{x^2}{x^2+1} dx$ (a) converges or (b) diverges? (mark the correct answer)

5. Let $D = [1,2] \times [0,1]$. Compute $\iint_D xy\, dxdy = \int_0^1 \left(\int_1^2 xy\, dx \right) dy = \left(\int_0^1 y\, dy \right) \cdot \left(\int_1^2 x\, dx \right) = \underbrace{\underbrace{\frac{1}{2} y^2 \Big|_0^1}_{\frac{1}{2}} \cdot \underbrace{\frac{1}{2} x^2 \Big|_1^2}_{\frac{1}{2}(4-1)}}_{=\frac{3}{4}}$

6. According to the Theorem of Fermat for functions of several variables, if $f : B_r(x^*) \subset \mathbb{R}^n \to \mathbb{R}$ Fréchet differentiable in x^* and x^* is a local minimum (maximum) for f, then $\nabla f(x^*) = 0_{\mathbb{R}^n}$

7. Give an example of a quadratic function $Q : \mathbb{R}^2 \to \mathbb{R}$.
$Q(x_1, x_2) = x_1^2 + x_2^2$

8. The series $\sum_{n=0}^{\infty} q^n$, $|q| < 1$ (a) converges or (b) diverges? (mark the correct answer)
[*Prove your claim, for +1 bonus point.*]
$s_n = 1 + q + \ldots + q^n = \frac{1-q^{n+1}}{1-q}$ geom. progression
so $s_n \xrightarrow[|q|<1]{n\to\infty} \frac{1}{1-q}$ $[(s_n)\text{ conv} \Rightarrow \sum q^n \text{ conv}]$

9. Fill in the next term of the Taylor expansion $f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \ldots \frac{f''(x_0)}{2!}(x-x_0)^2$

10. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ for which $\nabla f(x_1^*, x_2^*) = (0,0)$, at $(x_1^*, x_2^*) \in \mathbb{R}^2$, but (x_1^*, x_2^*) is neither a local minimum nor a local maximum (draw the graph of f if you prefer).
$\ldots f(x_1, x_2) = x_1^2 - x_2^2$, $f(0,0) = 0$
$\left.\begin{array}{l} \frac{\partial f}{\partial x_1}(x_1,x_2) = 2x_1, \quad \frac{\partial f}{\partial x_1}(0,0) = 0 \\ \frac{\partial f}{\partial x_2}(x_1,x_2) = -2x_2, \quad \frac{\partial f}{\partial x_2}(0,0) = 0 \end{array}\right\} \nabla f(0,0) = (0,0)$

$(0,0)$ neither loc. min nor max because
$f(0,0) = 0 \leq f(x_1,0) = x_1^2 \ \forall x_1 \in \mathbb{R}$
while
$f(0,0) = 0 \geq f(0,x_2) = -x_2^2 \ \forall x_2 \in \mathbb{R}$

Give detailed solutions to the following exercies on the next pages.

11. Compute $\iint_D \frac{1}{\sqrt{x^2+y^2}} dxdy$, where $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4, x \geq 0, y \geq 0\}$.

12. Using Lagrange's mean value Theorem and the chain rule, prove that if $f : \mathbb{R}^n \to \mathbb{R}$ differentiable and $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, then there exists $c \in [a,b] \subset \mathbb{R}^n$ on the segment connecting a and b, such that $f(b) - f(a) = \nabla f(c) \cdot (b - a)$.

13. a) Prove that the series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges for any $0 \leq x < 1$.

b) Argue that $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ is a Taylor expansion.

[1] 17.02.2020