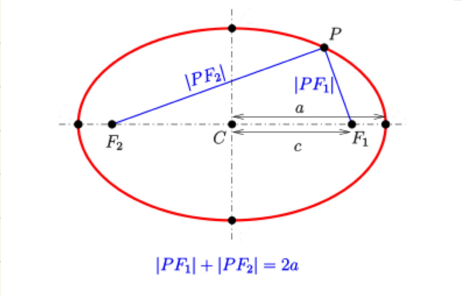
Proposition 7.1. Let F_1 and F_2 be two points in $\mathbb{E}^2$ and let a be a positive real scalar. Choose the coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$ such that F_1 and F_2 are on the Ox axis, such that $\overrightarrow{F_2F_1}$ has the same direction as $\mathbf{i}$ and such that O is the midpoint of $[F_1F_2]$. With these choices, the ellipse with focal points F_1 and F_2 for which the sum of distances from the focal points is $2a$ has an equation of the form

$$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.1}$$

for some positive scalar $b \in \mathbb{R}$. We denote this ellipse by $\mathcal{E}_{a,b}$.



$F_1(c,0)$, $F_2(-c,0)$, $M(x,y)$

So $d(M,F_1) + d(M,F_2) = 2a$

is equivalent to

$$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$$

$$\Leftrightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x-c)^2+y^2} \quad \Big| ()^2$$

$$\Rightarrow (x+c)^2+y^2 = 4a^2 - 4a\sqrt{(x-c)^2+y^2} + (x-c)^2+y^2$$

$$\Rightarrow \cancel{4}a\sqrt{(x-c)^2+y^2} = \cancel{4}a^2 - \cancel{4}xc \quad \Big| ()^2$$

$$\Rightarrow a^2(x^2 - \cancel{2xc} + c^2 + y^2) = a^4 - \cancel{2a^2xc} + x^2c^2$$

$$\Rightarrow (a^2-c^2)x^2 + y^2 = a^2(a^2-c^2) \quad \Big| : a^2(a^2-c^2)$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{\underbrace{a^2-c^2}_{=:\, b^2}} = 1$$

$=: b^2$ if $b = \sqrt{a^2-c^2}$ $\qquad a^2 - c^2 > 0$ since $a > c > 0 \Leftarrow \begin{cases} 2a \text{ is distance to focal points} \\ 2c \text{ is dist. between focal points.} \end{cases}$

this shows that the points on an ellipse satisfy eq. (7.1)

We also need to show that a point which satisfies (7.1) is on the ellipse if (x,y) is a solution to (7.1) then $y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$

then $d(M,F_2) = \sqrt{(x+c)^2 + y^2}$

$$= \sqrt{x^2 + 2cx + c^2 + b^2 - \frac{b^2}{a^2}x^2}$$

$$= \sqrt{\frac{c^2}{a^2}x^2 + 2cx + a^2}$$

$$= \sqrt{\left(\frac{c}{a}x + a\right)^2} = \left|a + \frac{c}{a}x\right|$$

Now $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{a^2} \leq 1 \Rightarrow x^2 \leq a^2 \Rightarrow |x| \leq a$

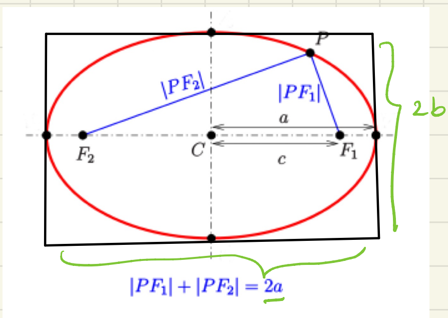
and $a > c$ $\quad (b^2 = a^2 - c^2)$

$\Rightarrow a + \frac{c}{a}x \geq a + \frac{c}{a}(-a) = a - c > 0 \quad \Rightarrow d(M,F_1) = a + \frac{c}{a}x$

similarly one shows that $d(M,F_2) = a - \frac{c}{a}x$

$\Rightarrow d(M,F_1) + d(M,F_2) = 2a$

Rem notice that $|x| \leq a$ and $|y| \leq b$ so the ellipse $\mathcal{E}_{ab}$ lies inside the rectangle with sides $2a$ and $2b$



Intersection line vs. ellipse

$$\begin{cases} \dfrac{x^2}{a^2} + \dfrac{(kx+m)^2}{b^2} = 1 \\ y = kx + m \end{cases} \longrightarrow \frac{x^2}{a^2} + \frac{k^2x^2}{b^2} + \frac{2kxm}{b^2} + \frac{m^2}{b^2} = 1 \quad \Big| \cdot a^2b^2$$

$$(b^2 + a^2k^2)x^2 + 2kma^2x + a^2m^2 - a^2b^2 = 0$$

$$\Delta = 4k^2m^2a^4 - 4a^2(b^2 + a^2k^2)(m^2 - b^2)$$

$$b^2m^2 - b^4 + a^2k^2m^2 - a^2k^2b^2$$

$$= 4a^2(\cancel{k^2m^2a^2} - b^2m^2 + b^4 - \cancel{a^2k^2m^2} - a^2k^2b^2)$$

$$= 4a^2b^2(-m^2 + b^2 - a^2k^2)$$

Tangency & gradients

- here we combine the local and global description of a curve

let $\phi : I \to \mathbb{E}^2$ $\phi(t) = (x(t), y(t))$ be a parametrization of the curve C

and $E(x,y) : \mathbb{E}^2 \to \mathbb{R}$ be a function giving an equation for C

$C : E(x,y) = c$ for some constant $c \in \mathbb{R}$

then $E(\phi(t)) = c \quad \forall t$

so $\partial_t E(\phi(t)) = 0$

$$\Rightarrow \forall t_0 \quad \left.\frac{\partial E(x,y)}{\partial x}\right|_{\phi(t_0)} \left.\frac{\partial x}{\partial t}\right|_{t_0} + \left.\frac{\partial E(x,y)}{\partial y}\right|_{\phi(t_0)} \cdot \left.\frac{\partial y}{\partial t}\right|_{t_0} = 0$$

$$\Leftrightarrow \underbrace{\begin{bmatrix} \dfrac{\partial E}{\partial x} & \dfrac{\partial E}{\partial y} \end{bmatrix}_{\phi(t_0)}}_{(\nabla E)(\phi(t_0))} \cdot \underbrace{\left.\begin{bmatrix} \dfrac{\partial x}{\partial t} \\ \dfrac{\partial y}{\partial t} \end{bmatrix}\right|_{t_0}}_{\text{a tangent vector at } t_0} = 0$$

$\Rightarrow (\nabla E)(\phi(t_0))$ is orthogonal to a tangent vector at t_0

$\Rightarrow (\nabla E)(\phi(t_0))$ is a normal vector to the tangent line at t_0

Proposition 7.2. Let F_1 and F_2 be two points in $\mathbb{E}^2$ and let a be a positive real scalar. Choose the coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$ such that F_1 and F_2 are on the Ox axis, such that $\overrightarrow{F_2F_1}$ has the same direction as $\mathbf{i}$ and such that O is the midpoint of $[F_1F_2]$. With these choices, the hyperbola with focal points F_1 and F_2 for which the absolute value of the difference of distances from the focal points is $2a$ has an equation of the form

$$\mathcal{H}_{a,b} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{7.3}$$

for some positive scalar $b \in \mathbb{R}$. We denote this hyperbola by $\mathcal{H}_{a,b}$.

$$\left| d(M,F_1) - d(M,F_2) \right| = 2a$$

it is by taking absolute value that one actually obtains an equation

$$\Leftrightarrow \sqrt{(x-c)^2+y^2} - \sqrt{(x+c)^2+y^2} = \pm 2a \quad \Big|\,(\;)^2$$

$\Rightarrow$ ... calculation similar to the case of the ellipse

$\Leftarrow$ ... $d(M,F_1) = \frac{c}{a}x + a$, $d(M,F_2) = \frac{c}{a}x - a$

Rem there is a rectangle of sides $2a$ and $2b$ the diagonals of which are the asymptotes of the hyperbola