Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză, Curs: Dynamical Systems, Primăvara 2018

Seminars 2 and 3. Linear Differential Equations

Find the general solution of each of the following differential equations.

1. $x' + 6x = 0$; 2. $x'' + 4x' + 4x = 0$; 3. $x'' = 0$; 4. $x''' = 0$; 5. $x^{(n)} = 0$;
6. $x'' + x' + x = 0$; 7. $x''' - 6x'' + 11x' - 6x = 0$; 8. $x^{(4)} - x = 0$.

Find the linear homogeneous differential equation with constant coefficients and of minimal order that has as solutions the following functions. Write the general solution of the differential equation that you found.

9. e^{-3t} and e^{5t}; 10. $5e^{-3t}$ and $-3e^{5t}$; 11. $5e^{-3t} - 3e^{5t}$; 12. $5te^{-3t}$ and $-3e^{5t}$;
13. $5e^{-3t}$ and $-3te^{5t}$; 14. $(5-3t)e^{-3t}$; 15. $(5-3t+2t^2)e^{-3t}$; 16. $\sin 3t$;
17. $t - \sin 3t$; 18. $-t\sin 3t$; 19. $e^{5t}\sin 3t$; 20. $e^{-3t}\sin 3t$; 21. t^7+1; 22. $5t - 3e^{5t}$;
23. $(t-1)^2$; 24. $2\cos^2 t$; 25. $\sin^2 t$; 26. $(e^t)^2$.

Decide whether the following statements are true or false.

27. "There exists a linear homogeneous differential equation with constant coefficients of order 7 that has as solutions $(t^3 + 2t^4)\cos 2t$ and te^{-t}."

28. "There exists a linear homogeneous differential equation with constant coefficients that has as solution $1/t$."

29. "There exists a linear homogeneous differential equation with constant coefficients that has as solution e^{t^2}."

30. "There exists a linear homogeneous differential equation with constant coefficients that has as solution $t/(1+t^2)$."

Find the solution for each of the following IVPs. Here $\eta, \lambda \in \mathbb{R}$ are fixed parameters.

31. $x'' + \pi^2 x = 0, \quad x(0) = 0, \quad x'(0) = \eta$; 32. $x'' + \lambda x = 0, \quad x(0) = 0, \quad x'(0) = \eta$.

Find all solutions for each of the following BVPs (boundary value problems).

33. $x'' + x = 0, \quad x(0) = x(\pi) = 0$; 34. $x'' + x = 0, \quad x(0) = x(1) = 0$;
35. $x'' + \pi^2 x = 0, \quad x(0) = x(1) = 0$; 36. $x'' + \pi^2 x = 0, \quad x(0) = x(2) = 0$.

37. Find $\lambda \in \mathbb{R}$ with the property that there exist nonnull 2π–periodic solutions of $x'' + \lambda x = 0$.

38. Find $\mu \in \mathbb{R}$ and $\omega > 0$ with the property that there exist nonnull periodic solutions of $x'' + \mu x' + \omega^2 x = 0$. In this case write the minimal period.

39. Find $\mu \in \mathbb{R}$ and $\omega > 0$ with the property that all solutions of $x'' + \mu x' + \omega^2 x = 0$ goes to 0 as $t \to \infty$.

Decide whether the following statements are true or false.

40. "All the solutions of $x'' + 3x' + x = 1$ satisfy $\lim\limits_{t\to\infty} x(t) = 1$."

41. "The solution of the IVP $x'' + 4x = 1, \quad x(0) = 5/4, \quad x'(0) = 0$ satisfies $x(\pi) = 5/4$."

42. "The equation $x' = 3x + t^3$ admits a polynomial solution. (*Hint.* Look for a polynomial solution of degree 3.)"

43. Let $\omega > 0$ be a parameter and denote $\varphi(\cdot, \omega)$ the solution of the IVP

$$x'' + x = \cos(\omega t), \quad x(0) = x'(0) = 0.$$

(i) When $\omega \neq 1$ find a solution of the form $x_p(t) = a\cos(\omega t) + b\sin(\omega t)$ for $x'' + x = \cos(\omega t)$. (Here you have to determine the real coefficients a and b.)
(ii) Find a solution of the form $x_p(t) = t(a\cos t + b\sin t)$ for $x'' + x = \cos t$.
(iii) Find $\varphi(\cdot, \omega)$ for any $\omega > 0$.
(iv) Prove that $\lim_{\omega\to 1} \varphi(t, \omega) = \varphi(t, 1)$ for each $t \in \mathbb{R}$.

44. Let $\alpha > 0$ and $\varphi(\cdot, \alpha)$ be the solution of the IVP

$$x'' - 4x = e^{\alpha t}, \quad x(0) = x'(0) = 0.$$

(i) When $\alpha \neq 2$ find a solution of the form $x_p(t) = ae^{\alpha t}$ for $x'' - 4x = e^{\alpha t}$. (Here you have to determine the real coefficient a.)
(ii) Find a solution of the form $x_p(t) = ate^{2t}$ for $x'' - 4x = e^{2t}$.
(iii) Find $\varphi(\cdot, \alpha)$ for any $\alpha > 0$.
(iv) Prove that $\lim_{\alpha\to 2} \varphi(t, \alpha) = \varphi(t, 2)$ for each $t \in \mathbb{R}$.

Find the general solution of
45. $x' + \frac{1}{t}x = 0$; 46. $x' - \frac{1}{t}x = 0$; 47. $x' - \frac{3}{t}x = 0$;
48. $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}$; 49. $x' + tx = e^{-t^2-t}$; 50. $x' + \frac{2t}{1+t^2}x = 3$;
51. $x' - \frac{2}{t}x = t^2\sin(2t) - 4t^3$; 52. $x' + \frac{1}{\sqrt{t}}x = \frac{1}{2\sqrt{t}}$; 53. $x' + \frac{1}{t^2}x = 1 + \frac{1}{t}$.

Find the general solution of the following equation, looking first for some solutions of the form $x = t^r$, with $r \in \mathbb{R}$.
54. $t^2x'' - 8tx' + 20x = 0,\ t \in (0, \infty)$; 55. $t^2x'' - 6x = 0,\ t \in (0, \infty)$;
56. $t^2x'' + tx' + x = 0,\ t \in (0, \infty)$;

57. Is $x = c_1 \cosh t + c_2 \sinh t$, $c_1, c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$? Recall that $\cosh t = (e^t + e^{-t})/2$ and $\sinh t = (e^t - e^{-t})/2$.

58. a) Find a particular solution of the form $x_p = ae^t$ (with $a \in \mathbb{R}$) for the equation $x' - 2x = e^t$.

b) Find a particular solution of the form $x_p = be^{-t}$ (with $b \in \mathbb{R}$) for the equation $x' - 2x = e^{-t}$.

c) Find a particular solution for the equation $x' - 2x = 5e^t - 3e^{-t}$.

d) Find the general solution of $x' - 2x = 5e^t - 3e^{-t}$.