Lecture 14

30.05.2022

Exam 2021

2. (1p) Using the integrating factor method find (an integral representation of) the solution of the IVP $\begin{cases} x' + tx = -1 \\ x(0) = 0 \end{cases}$

3. (2.75p) Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1-function s.t. $f(-5) < 0$, $f(0) = 1$, $f(5) < 0$

a) (0.5p) Justify that the dynamical system $\dot{x} = f(x)$ does not have a global repeller equilibrium point.

b) (0.25p) Give a simple example of such function f.

c) (1.25p) Consider f from b). Represent the phase portrait in $\mathbb{R}^2 \setminus \{(0,0)\}$ of the system given in polar coordinates by $\dot{\rho} = f(\rho)$, $\dot{\theta} = 3$

d) Transform in cartesian coordinates the system from c).

4. (2p) Find the values of $h > 0$ such that the attractor equilibrium point of $\dot{x} = -3x^2 + 6x + 24$, is also an attractor fixed point of the discrete dynamical system associated to the Euler's numerical formula with stepsize $h > 0$ for the given differential equation.

4.

$$\dot{x} = -3x^2 + 6x + 24$$

$$f(x) = -3x^2 + 6x + 24$$

$$-3x^2 + 6x + 24 = 0 \quad |:(-3)$$

$$x^2 - 2x - 8 = 0$$

$$x_1 = -2$$

$$x_2 = 4$$

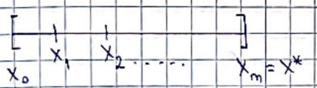
$$f'(x) = -6x + 6$$

$f'(-2) = 12 + 6 = 18 > 0 \Rightarrow$ -2 is a repeller

$f'(4) = -24 + 6 = -18 < 0 \Rightarrow$ 4 is an attractor

Euler's formula

$$y'(x) = f(x, y(x)), \quad x \in [x_0, x^*]$$



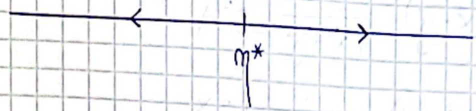
$$x_{k+1} = x_k + h$$

$$x_k = x_0 + kh, \quad k = \overline{0, n}$$

$$y_{k+1} = y_k + h f(x_k, y_k)$$

$$x_{k+1} = x_k + h(-3x_k^2 + 6x_k + 24)$$

3. a)



$\exists\ \eta_1^*$ and η_2^* equilibrium points

$\eta_1^* \in (-5;0)$

$\eta_2^* \in (0,5)$

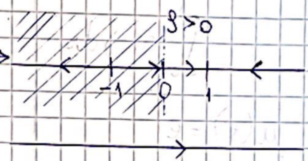
$\lim\limits_{t\to-\infty} \varphi(t,\eta) = \eta_1^*,\ \forall \eta \in \mathbb{R}$

$\lim\limits_{t\to-\infty} \underbrace{\varphi(t,\eta_2^*)} = \eta_1^* \Rightarrow \eta_2^* = \eta_1^*$ - contradiction

b) $f(x) = 1 - x^2$

c) $\dot{\rho} = 1 - \rho^2$

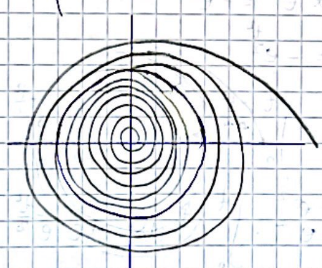
$\dot{\theta} = 3$



phase portraits

$\dot{\theta} > 0 \Rightarrow \forall$ orbit rotates anticlockwise around the origin

$\rho = 1 \Rightarrow$ correspond to an unit circle (radius $= 1$; center in origin)



d) $\begin{cases} x = \rho\cdot\cos\theta \\ y = \rho\cdot\sin\theta \end{cases} \Rightarrow \begin{cases} \dot{x} = \dot{\rho}\cos\theta - \rho\cdot\sin\theta\cdot\dot{\theta} \\ \dot{y} = \dot{\rho}\cos\theta + \rho\cdot\cos\theta\cdot\dot{\theta} \end{cases}$; $\rho = x^2+y^2$

$\Rightarrow \begin{cases} \dot{x} = (1-\rho^2)\cos\theta - 3\rho\cdot\sin\theta \\ \dot{y} = (1-\rho^2)\sin\theta + 3\rho\cdot\cos\theta \end{cases}$

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$$\begin{cases} \dot{x} = (1-x^2-y^2)\cdot \dfrac{x}{\sqrt{x^2+y^2}} - 3y \\ \dot{y} = (1-x^2-y^2)\cdot \dfrac{y}{\sqrt{x^2+y^2}} + 3x \end{cases}$$

2. $[x(t)\cdot \mu(t)]' = x'(t)\cdot \mu(t) + x(t)\cdot \mu'(t)$

$x'\cdot e^{t^2} + x\cdot t\cdot e^{t^2} = (x\cdot e^{t^2})'$ False

$x'\cdot e^{\frac{t^2}{2}} + x\cdot t\cdot e^{\frac{t^2}{2}} = (x\cdot e^{\frac{t^2}{2}})'$ True

$\mu(t) = e^{\frac{t^2}{2}}$ is an integrating factor

Theory

$x' + a(t)x$

$\int_0^s a(s)\,ds$

$\mu(t) = e^0$

$x' + tx = -1 \mid \cdot e^{\frac{t^2}{2}}$ first order lin. nonhom. d.e. with nonconstant coeff.

$x'\cdot e^{\frac{t^2}{2}} + t\cdot x\cdot e^{\frac{t^2}{2}} = -e^{\frac{t^2}{2}}$

$(x\cdot e^{\frac{t^2}{2}})' = -e^{\frac{t^2}{2}}$

$x\cdot e^{\frac{t^2}{2}} = -\int_0^t e^{\frac{s^2}{2}}\,ds + c \mid \cdot e^{-\frac{t^2}{2}}, c\in\mathbb{R}$

$x = -e^{-\frac{t^2}{2}}\cdot \int_0^t e^{\frac{s^2}{2}}\,ds + c\cdot e^{-\frac{t^2}{2}}$

$\left.\begin{array}{l} x(0) = c \\ x(0) = 0 \end{array}\right\} \Rightarrow c = 0$

$\varphi(t) = -e^{-\frac{t^2}{2}}\cdot \int_0^t e^{\frac{s^2}{2}}\,ds$

$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

a) Find the principal matrix sol. of $\dot{X} = AX$.

b) Compute e^{At}

c) Find $a, b \in \mathbb{R}$ s.t. $H(x,y) = x^2 + ay^2 + bxy$ is a global first integral of $\dot{X} = AX$.

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad \dot{X} = AX \iff \begin{cases} \dot{x} = 2x - 5y \\ \dot{y} = x - 2y \end{cases}$

solutions

matrix sol. $\begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$

(IVP1) $\begin{cases} \dot{X} = AX \\ x(0) = 1 \\ y(0) = 0 \end{cases}$ $\qquad$ (IVP2) $\begin{cases} \dot{X} = AX \\ x(0) = 0 \\ y(0) = 1 \end{cases}$

$H : \mathbb{R}^2 \to \mathbb{R}$, C^1-function

H is f.i. of $\dot{X} = AX$ iff $\frac{\partial H}{\partial x} \cdot (2x - 5y) + \frac{\partial H}{\partial y}(x - 2y) = 0, \ \forall (x,y) \in \mathbb{R}^2$

$(2x + by)(2x - 5y) + (2ay + bx)(x - 2y) = 0$

$\underline{4x^2} + 2bxy - 10xy - 5by^2 + 2axy + \underline{bx^2} - 4ay^2 - 2bxy = 0$

$\begin{cases} 4 + b = 0 \\ \cancel{2b} - 10 + 2a - \cancel{2b} = 0 \\ -5b - 4a = 0 \end{cases}$

$b = -4$, $a = 5$