LECTURE 6

DATE: 01 NOVEMBER 2021

Recap Part I (Differential Calculus)

Main Objective: Optimization
(Relevant to Machine Learning)

Lecture 1: Diff Calculus in dimension = 1
Main (new) Result: TAYLOR's Theorem

Lecture 2: Geometry of $\mathbb{R}^d$ ($d > 1$)
$x \cdot y = x_1 y_1 + x_2 y_2 + \dots + x_d y_d$
inner/scalar product
allows us to compute distances, angles

Lecture 3: Derivatives for $f: \mathbb{R}^d \to \mathbb{R}$
The Gradient ∇f:
Hesse Matrix $\nabla^2 f$
Chain Rule

Lecture 4: Optimization
Least Squares

Lecture 5: Constraint Optimization (Lagrange Multiplier Method)
(Planner) Curves and the Implicit Function Theorem

Part II : Integral Calculus

6. Antiderivatives & the Riemann Integral

Def: $f : I \subset \mathbb{R} \to \mathbb{R}$

F is an antiderivative of f if $F' = f$.

The set of all antiderivatives of f is called integral of f and by:

$$\int f(x)\,dx = F(x) + C$$

(two antiderivatives differ only by a const)

Theorem 1: f cont on I then f has antiderivatives

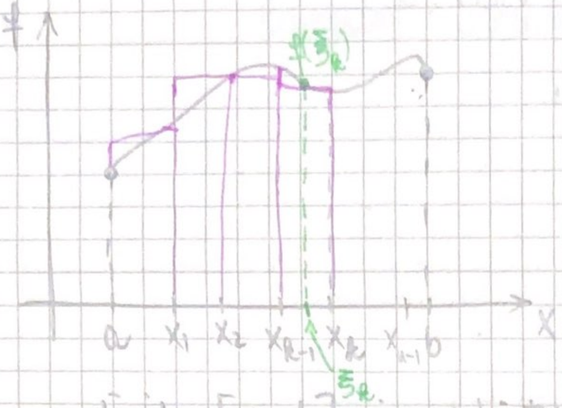
§6.1. Nonelementary functions & antiderivatives:

elementary functions = constructed via a finite nr of operations applied to x^{α}, a^{x}, $\sin x$

The Fundamental difference between differential and integral calculus is that:

- the derivative of an elementary f is elementary
- the antiderivative of an elementary f need not be ele

§6.2. The Riemann Integral



Question:

area under the graph of $f = ?$

$\Delta = \{x_0, \dots, x_n\}$

divide $[a, b]$ in subintervals $x_0 = a < x_1 < \dots < x_n = b$

in each subinterval pick one intermediate point

$\xi = \{\xi_1, \dots, \xi_n\}$, $\xi \in [x_{k-1}, x_k]$

Area under graph (one slice) $\approx f(\xi_k)(x_k - x_{k-1})$

$$\sigma(f; \Delta; \xi) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

the sum of areas of all slices

take n larger & larger $\Rightarrow$ better & better approx

Def: $f : [a, b] \to \mathbb{R}$ is called Riemann integrable if $\exists\, y \in \mathbb{R}$ such that $\forall \varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ with the property that for any Δ of $[a, b]$ with $\max_k |x_k - x_{k-1}| < \delta$ and any ξ we have

$$\left| y - \sum_{k=1}^{n} f(\xi)(x_k - x_{k-1}) \right| < \varepsilon$$

Theorem 2: f cont $\Rightarrow$ f Riemann integrable

Theorem 3 : (LEIBNIZ-NEWTON) Fundamental Theorem of Calculus

If f integrable and admits antiderivatives then

$$\underbrace{\int_a^b f(x)\,dx}_{\text{Riemann integral (notation)}} = F(b) - F(a)$$

Theorem 4 : If f cont then

$$F(x) = \int_a^x f(t)\,dt \quad \text{is antiderivative of } f$$

Properties :

- f, g integrable $\Rightarrow$ $\alpha f + \beta g$ integrable $\int(\alpha f+\beta g) = \alpha\int f + \beta\int g$
- $f(x) \leq g(x),\ \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$
- f integrable $\Rightarrow$ $|f|$ integrable $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$
- f cont, g integrable $\Rightarrow \exists\, c$ such that

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

- integration by parts
- change variables $\qquad f$ cont, u diffable

$$\int_a^b f(u(t))\,u'(t)\,dt = \int_{u(a)}^{u(b)} f(x)\,dx$$

Remark : Integrable functions need not be continuous !