Seminar 2 - LHDE with const coeff.

- $x' + cx = 0$
- $x''' = 0$
- $x^{(4)} - x = 0$
- $x' = kx$, $k \in \mathbb{R}$ fixed

To remember: Lecture 3. (§2) $\left\{\begin{array}{l} \text{- the character. eq.} \\ \text{- find the roots } \begin{array}{l}\Delta>0 \\ \Delta=0 \\ \Delta<0\end{array} \\ \text{- gen. solution} \end{array}\right.$

1.4.2

a) e^{-3t}, e^{5t}

$$\left.\begin{array}{l} e^{-3t} \to r=-3 \text{ root} \\ e^{5t} \to r=5 \text{ root} \end{array}\right\} \Rightarrow (r+3)(r-5)=0$$

$\Updownarrow$ the characteristic eq.

$$r^2 - 2r - 15 = 0$$

$$\Rightarrow x'' - 2x' - 15x = 0.$$

the diff. eq

The solution:

$$\boxed{x = c_1 e^{-3t} + c_2 e^{5t}}, \quad c_1, c_2 \in \mathbb{R}$$

(b) $5e^{-3t}$ and $-3e^{5t}$

$$\left.\begin{array}{l} 5e^{-3t} = \text{sol} \iff e^{-3t} = \text{sol} \\ -3e^{5t} = \text{sol} \iff e^{5t} = \text{sol} \end{array}\right\} \Rightarrow (b) \equiv (a)$$

c) $5e^{-3t} - 3e^{5t}$ sol of a LDHE with const coef

$\Leftrightarrow e^{-3t}$ and e^{5t} solutions $\Rightarrow$ (c) $\equiv$ (a)

d) $5te^{-3t}$ and $-3e^{5t}$ sol

$$\underbrace{te^{-3t} = \text{sol} \;\Rightarrow\; e^{-3t},\ te^{-3t}}_{r_1=-3 \text{ double root}} \qquad \underbrace{e^{5t} = \text{sol}}_{r_2 = 5 \text{ simple root}}$$

$r_1 = -3$ double root ; $r_2 = 5$ simple root

(of the characteristic equation)

$\Rightarrow$ The characteristic eq : $(r+3)^2(r-5) = 0$.

$\Leftrightarrow (r^2 + 6r + 9)(r-5) = 0$

$\Leftrightarrow r^3 + 6r^2 + 9r - 5r^2 - 30r - 45 = 0$

$\Leftrightarrow r^3 + r^2 - 21r - 45 = 0$

$\Rightarrow$ The LDHE : $x''' + x'' - 21x' - 45x = 0$.

The solution: $\underline{x = c_1 e^{-3t} + c_2 te^{-3t} + c_3 e^{5t}}$, $c_1, c_2, c_3 \in \mathbb{R}$.

f) $(5-3t)e^{-3t} = 5e^{-3t} - 3te^{-3t}$ = solution

$\Leftrightarrow e^{-3t}$ and te^{-3t} solutions $\Rightarrow$ $\underline{r = -3 \text{ double root}}$

$\Rightarrow$ The charact. eq : $(r+3)^2 = 0$

$r^2 + 6r + 9 = 0$

$\Rightarrow$ The diff. eq : $x'' + 6x' + 9x = 0$.

The gen. sol : $x(t) = \underline{c_1 e^{-3t} + c_2 te^{-3t}}$, $c_1, c_2 \in \mathbb{R}$

k) $\sin 3t$ = sol $\Leftrightarrow$ $r_{1,2} = \pm 3i$ roots of the charact eq

The characteristic eq: $(r-3i)(r+3i)=0$

$$r^2-(3i)^2=0$$

$$r^2+9=0$$

the diff. eq: $x''+9x=0$

The general solution: $\underline{x=c_1\cos 3t+c_2\sin 3t}$, $c_1, c_2 \in \mathbb{R}$

o) $(t-1)^2$ solution $\Leftrightarrow$ t^2-2t+1 = solution

$\Leftrightarrow$ $1, t, t^2$ are solutions $\Leftrightarrow$ $r=0$ is a triple root

$\Rightarrow$ The characteristic eq: $r^3=0$.

$\Rightarrow$ the diff eq: $x'''=0$.

The general solution: $\underline{x=c_1+c_2t+c_3t^2}$

$c_1, c_2, c_3 \in \mathbb{R}$.

② $\begin{cases} x'' + \pi^2 x = 0 \\ x(0) = 0 \\ x'(0) = \eta \end{cases}$, $\eta \in \mathbb{R}$ fixed parameter

$x'' + \pi^2 x = 0 \Rightarrow x = c_1 \cos(\pi t) + c_2 \sin(\pi t)$, $c_1, c_2 \in \mathbb{R}$

$\left.\begin{array}{l} x(0) = c_1 \\ \xRightarrow{ic} x(0) = 0 \end{array}\right\} \Rightarrow c_1 = 0. \Rightarrow x = c_2 \sin(\pi t)$

$x' = c_2 \pi \cos(\pi t) \Rightarrow \left.\begin{array}{l} x'(0) = c_2 \pi \\ \xRightarrow{ic} x'(0) = \eta \end{array}\right\} \Rightarrow c_2 \pi = \eta \Rightarrow c_2 = \frac{\eta}{\pi}$

$\Rightarrow$ The unique solution of the IVP is:

$$\boxed{x = \frac{\eta}{\pi} \cdot \sin(\pi t).}$$

1.4.5:

a) $\begin{cases} x'' + x = 0 \\ x(0) = x(\pi) = 0 \end{cases}$ $\Rightarrow x(t) = c_1 \sin t + c_2 \cos t$, $c_1, c_2 \in \mathbb{R}$

$\left.\begin{array}{l} x(0) = c_1 \sin 0 + c_2 \cos 0 = c_2 \\ \xrightarrow{ic} x(0) = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow c_2 = 0 \Rightarrow x = c_1 \sin t$

But $x(\pi) = 0 \Rightarrow \left.\begin{array}{l} x(\pi) = c_1 \sin \pi \\ x(\pi) = 0 \end{array}\right\} \Rightarrow c_1 \sin \pi = 0$

true, $\forall c_1 \in \mathbb{R}$

$\Rightarrow$ Gen. sol: $\underline{x = c \cdot \sin t}$, $c \in \mathbb{R}$.

b) $\begin{cases} x''+x=0 \\ x(0)=x(1)=0 \end{cases}$ $\Rightarrow x(t)=c_1\sin t+c_2\cos t$

$\hookrightarrow x(0)=c_1\sin 0+c_2\cos 0=0 \Rightarrow c_2=0$

$x(1)=c_1\underbrace{\sin 1}_{\neq 0}=0 \Rightarrow c_1=0.$ $\Bigg\} \Rightarrow$

$\Rightarrow c_1=c_2=0.$ $\Rightarrow$ $x=0$ the unique solution of the BVP.

1.4.6 $\lambda\in\mathbb{R}$, $\lambda=?$ such that the equation:
$x''+\lambda x=0$ has non-null 2π-periodic solutions.

Solution:

$x''+\lambda x=0$

- the characteristic eq: $r^2+\lambda=0 \Leftrightarrow r^2=-\lambda$

Case 1: $\lambda>0$, $r_{1,2}=\pm i\sqrt{\lambda}$ $\to \cos(\sqrt{\lambda}t)$, $\sin(\sqrt{\lambda}t)$

$\Rightarrow$ The general solution: $x=c_1\cdot\cos(\sqrt{\lambda}t)+c_2\sin(\sqrt{\lambda}t)$, $c_1,c_2\in\mathbb{R}$

The main period of $\cos(\sqrt{\lambda}t)$ is $\frac{2\pi}{\sqrt{\lambda}}$,

Thus, any period has the form: $\boxed{\frac{2\pi}{\sqrt{\lambda}}\cdot n,\ n\in\mathbb{N}^*}$

From hyphothesis: $T=2\pi$ = period

$\Rightarrow \frac{2\pi}{\sqrt{\lambda}}\cdot n=2\pi \Rightarrow \sqrt{\lambda}=n \Rightarrow \lambda_n=n^2$, $n\in\mathbb{N}$

Case 2: $\lambda = 0$ $\Rightarrow x'' = 0 \Rightarrow r^2 = 0 \Rightarrow r_{1,2} = 0.$
double root

$\Rightarrow \left.\begin{array}{l} e^{0t} = 1 \\ e^{0t} \cdot t = t \end{array}\right\}$ solution $\Rightarrow \underbrace{x = c_1 + c_2 t}_{\text{general sol}}$, $c_1, c_2 \in \mathbb{R}$

The solution $x = c_1 + c_2 t \neq$ periodic $(T = 2\pi)$
$(x = \text{non null})$ $\Rightarrow (\nexists)$ sol.

Case 3: $\lambda < 0$, $x'' + \lambda x = 0$

The characteristic eq: $r^2 + \lambda = 0.$

$r^2 = -\lambda > 0$

$\Rightarrow r_{1,2} = \pm\sqrt{-\lambda} \in \mathbb{R}$ $\left.\begin{array}{l} \nearrow e^{\sqrt{-\lambda}t} \\ \searrow e^{-\sqrt{-\lambda}t} \end{array}\right\}$ solutions

Thus, the general solution is:

$x = c_1 e^{\sqrt{-\lambda}t} + c_2 e^{-\sqrt{-\lambda}t}$, $c_1, c_2 \in \mathbb{R}.$

↳ not periodic $(T = 2\pi)$
$\Rightarrow (\nexists)$ sol

$\Rightarrow$ Conclusion:

$\lambda_n = n^2$, $n \in \mathbb{N}^*$

$\Rightarrow$ the eq: $\underline{x'' + n^2 x = 0.}$

1.4.7. $\mu \in \mathbb{R}$, $\omega > 0$, $x'' + \mu x' + \omega^2 x = 0$

$\mu = ?$ such that the equation has non-null periodic solutions.

Solution:

The characteristic equation: $r^2 + \mu r + \omega^2 = 0$.

- $\Delta > 0$, $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2 \to e^{r_1 t}; e^{r_2 t}$
- $\Delta = 0$, $r_{1,2} \in \mathbb{R}$, $r_1 = r_2$ (double) $\to e^{rt}; te^{rt}$.
- $\Delta < 0$, $r_{1,2} = \alpha \pm i\beta \notin \mathbb{R} \to e^{\alpha t}\cos\beta t; e^{\alpha t}\sin\beta t$

A second order LHDE with constant coefficients has periodic solutions $\Leftrightarrow$ $\Delta < 0$ and $Re(root) = 0$.

Here, in our case $\Delta = \mu^2 - 4\omega^2 < 0$

$\Rightarrow r_{1,2} = \alpha \pm i\beta$

$r_1 + r_2 = 2\alpha = 0$, but $r_1 + r_2 = -\mu$ (Viète)

Follows that the equation has periodic sol $\Leftrightarrow$ $\mu = 0$

$\Rightarrow x'' + \omega^2 x = 0$

1.4.8 $\mu \in \mathbb{R}$, $\omega > 0$, $x'' + \mu x' + \omega^2 x = 0$.

$\mu = ?$, $\omega = ?$ such that $\lim_{t \to \infty} x(t) = 0$ for $\forall$ sol x.

Solution:

From hypothesis : $\lim_{t \to \infty} x(t) = 0$ $\forall$ sol x.

$\Longleftrightarrow$

$\Delta > 0$, $r_1 < 0$; $r_2 < 0$.

$\Delta = 0$, $r < 0$.

$\Delta < 0$, $\alpha < 0$.

Viéte $\Longrightarrow$ $\begin{cases} r_1 + r_2 = -\mu \\ r_1 r_2 = \omega > 0 \end{cases}$

Case 1: $\Delta > 0$ and $r_1 < 0$, $r_2 < 0 \Longleftrightarrow -\mu < 0 \Leftrightarrow \underline{\mu > 0}$

Case 2: $\Delta = 0$ and $r_1 = r_2 = r < 0 \Leftrightarrow -\mu < 0 \Leftrightarrow \underline{\mu > 0}$

Case 3: $\Delta < 0$, $r_{1,2} = \alpha \pm i\beta$, $\alpha < 0$

$\Longleftrightarrow \begin{cases} r_1 + r_2 = 2\alpha < 0 \\ r_1 \cdot r_2 = \alpha^2 + \beta^2 > 0 \end{cases}$ $\Leftrightarrow -\mu < 0 \Leftrightarrow \underline{\mu > 0}$

$\Rightarrow$ Conclusion: $\boxed{\text{any sol} \xrightarrow[t \to \infty]{} 0 \Leftrightarrow \mu > 0}$