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The direction field associated to a differential equation

(1) $y'(x) = f(x, y(x))$

(2) $\begin{cases} \dot{x}(t) = g_1(x(t), y(t)) \\ \dot{y}(t) = g_2(x(t), y(t)) \end{cases}$

ASSUMPTIONS: $f \in C^1(\mathbb{R}^2)$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

RECALL: We have existence and uniqueness for any ~~t~~ IVP.

The graph of a sol. of (1) is a plane curve (in $\mathbb{R}^2$), an orbit of (2) is a plane curve.

Def: The direction field associated to eq. (1) is a collection of vectors. To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope $\boxed{m = f(x_0, y_0)}$

The direction field associated to syst. (2) is a collection of vectors. To a point $(x_0, y_0) \in \mathbb{R}^2$ we associate a vector of slope

$$\boxed{m = \frac{g_2(x_0, y_0)}{g_1(x_0, y_0)}}$$

Ex: (3) $y' = 1 - \frac{x}{y^2}$ (4) $\begin{cases} \dot{x} = y^2 \\ \dot{y} = -x + y^2 \end{cases}$

Note that (3) and (4) have the same direction field. Draw the vectors corresp. to the points $(1,1)$, $(1,0)$, $(0,1)$.

$$\frac{g_2(x,y)}{g_1(x,y)} = \frac{-x+y^2}{y^2} = 1 - \frac{x}{y^2} = f(x,y) \implies$$ (3) and (4) have the same dir. field.

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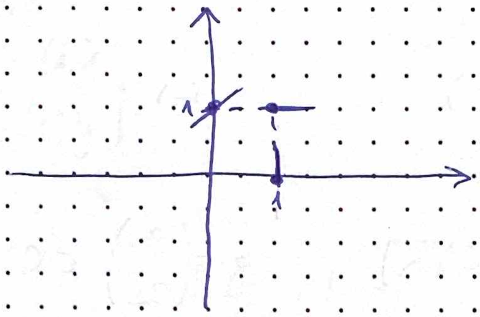
$(1,1) \rightarrow m_1 = f(1,1) = 0$

$(1,0) \rightarrow m_2 = f(1,0) = \infty$

$(0,1) \rightarrow m_3 = f(0,1) = 1$



$g(x,y) = (g_1(x,y), g_2(x,y))$ this vector has slope $\dfrac{g_2(x,y)}{g_1(x,y)}$

Property: Let $(x_0, y_0) \in \mathbb{R}^2$.

- denote by $\varphi : I \to \mathbb{R}$ the solution of the IVP $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

Then $\varphi'(x_0) = f(x_0, y_0)$, i.e. the vector of the d.f. associated to (x_0, y_0) is tangent to the solution curve that passes through this point.

- denote by $\varphi : I \to \mathbb{R}^2$ the sol. of the IVP $\begin{cases} \dot{x} = g_1(x,y) \\ \dot{y} = g_2(x,y) \\ x(0) = x_0 \\ y(0) = y_0 \end{cases}$

Then the tangent vector to the orbit of φ in (x_0, y_0) is $\dot{\varphi}(0) = \begin{pmatrix} g_1(x_0,y_0) \\ g_2(x_0,y_0) \end{pmatrix}$ whose slope is $\dfrac{g_2(x_0,y_0)}{g_1(x_0,y_0)}$

In other words, the d.f. is tangent to the orbits of (2).

Def: Let $m \in \mathbb{R} \cup \{\infty\}$. The m-izocline of (1) is $\{(x,y) \in \mathbb{R}^2 : m = f(x,y)\}$

The m-izocline of (2) is $\{(x,y) \in \mathbb{R}^2 : m = \frac{g_2(x,y)}{g_1(x,y)}\}$

Ex: (5) $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ (6) $y' = -\frac{x}{y}$

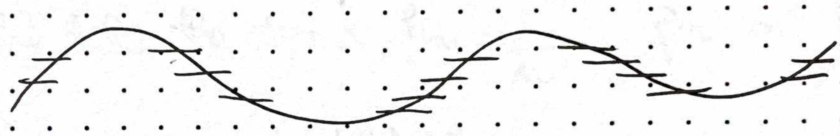
a) Note that they have the same direction field.

We have $f(x,y) = -\frac{x}{y}$, $\begin{matrix} g_1(x,y) = -y \\ g_2(x,y) = x \end{matrix} \Rightarrow \frac{g_2(x,y)}{g_1(x,y)} = f(x,y) \Rightarrow$

$\Rightarrow$ the d.f. is the same

b) Find and represent the izoclines.

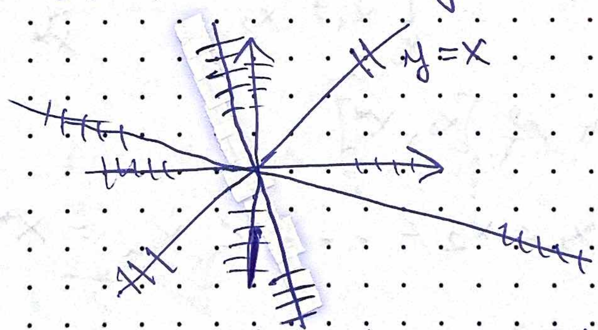
0-izocline



$m \in \mathbb{R} \cup \{\infty\}$, the m-izocline has the eq. $m = -\frac{x}{y}$ $\Leftrightarrow$ $\boxed{y = -\frac{1}{m} x}$ this is a line through the origin.

$\downarrow$

m-izocline



$m = -1 \Rightarrow y = x$ this is the -1-izocline

This means that the d.f. is orthogonal to any izocline of this example, thus to any line that passes through the origin.

So, any orbit is orthogonal to any line that passes ~~to the~~ through the origin.

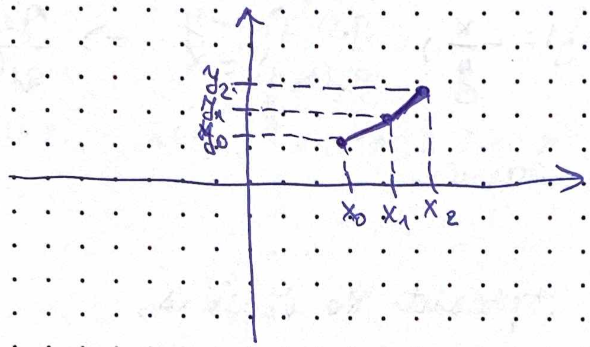
 Thus, any orbit is a circle centered in the origin.

~~# REZIST~~

Short introduction to numerical methods

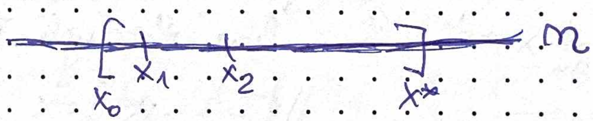
$$(1)\ \begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases} \qquad f \in C^1(\mathbb{R}^2)$$

has a unique solution $\varphi : I \to \mathbb{R}$.



~ Euler's numerical method ~

$[x, x^*] \subset I$



Take a partition of $[x_0, x^*]$

$$x_0 < x_1 < x_2 < \dots < x_n = x^*, \qquad n \in \mathbb{N},\ n \geq 1$$

↑ # of steps

$y - y_0 = m_0 (x - x_0)$

$y_1 = y_0 + (x_1 - x_0) \cdot f(x_0, y_0)$

$$y_{k+1} = y_k + (x_{k+1} - x_k) f(x_k, y_k) \quad , \quad k = \overline{0, m-1}$$

y_i — "approx." of the exact value $\varphi(x_i)$.

↑ the result of a numerical method

Usually, $x_{k+1} - x_k = h$, $\forall k = \overline{0, m-1}$ $h > 0$ = stepsize

$$\Rightarrow \begin{cases} x_{k+1} = x_k + h \\ y_{k+1} = y_k + h \cdot f(x_k, y_k) \end{cases} , \quad k = \overline{0, m-1}$$

Ex: $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ We know that the unique sol. is $\varphi(x) = e^x$.

We fix the interval $[0, 1]$.

We take the stepsize $h_m = \frac{1}{m}$ for $m \geq 1$ fixed

We compute y_m, an approx. of $\varphi(1) = e$.

We'll prove that $\lim_{m \to \infty} y_m = e$.

$$0 \left[\; \frac{1}{10} \quad \frac{2}{10} \quad \frac{3}{10} \quad \cdots \; \right] 1$$

$m = 10$

Note that $h_m = \frac{1}{m}$ the nr. of steps to cover the interval $[0, 1]$ is m. $f(x, y) = y$.

Write the Euler's numerical formula: $\begin{cases} x_{k+1} = x_k + \frac{1}{m} \\ y_{k+1} = y_k + \frac{1}{m} y_k \end{cases} , \; k = \overline{0, m-1}$

$x_0 = 0$, $y_0 = 1$

$$\Rightarrow \begin{cases} x_k = \frac{k}{m} \\ y_{k+1} = \left(1 + \frac{1}{m}\right) y_k \end{cases} , \quad k = \overline{0, m-1} , \quad y_0 = 1$$

$x_0 = 0$

BRUNNEN

$$\begin{cases} x_k = \frac{k}{n} \\ y_k = \left(1+\frac{1}{n}\right)^k \end{cases}, \quad k = \overline{0, n-1}$$

$$y_n = \left(1+\frac{1}{n}\right)^n \approx e, \ \forall n \in \mathbb{N}^*$$

$\Rightarrow \lim\limits_{n\to\infty} y_n = e.$