1. Compute $\frac{d}{dx}(\sin wx) = \dots$ for $w \in \mathbb{R}$.

2. An antiderivative of $(\ln x)^2$ is ...

3. The critical points of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$ are ...

4. Let $x = (1, 0, 1)$, $y = (0, 1, 0)$, $x \cdot y = \dots$, $\|x + y\| = \dots$

5. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x_1, x_2, x_3) = x_1 x_2 + x_3^2 - 3x_2$, $\nabla f(x_1, x_2, x_3) = \dots$

6. $f: [a, b] \to \mathbb{R}$ Riemann integrable. Let $\Delta = \{a = x_0, x_1, \dots, x_n = b\}$ a division of $[a, b]$ and $\xi = \{\xi_1, \dots, \xi_n\}$, $\xi_i \in [x_{i-1}, x_i]$.
The Riemann sum associated to $f, \Delta, \xi$ is ...

7. $\int_0^1 \frac{1}{\sqrt{x}}\, dx$ converges or diverges? (mark the correct answer)

8. Let $I(t) = \int_t^{2t} t^2 x \, dx$. $I'(t) = \dots$

9. $D = [0,1] \times [0,1]$, $\iint_D (1-x) y \, dx dy = \dots$

10. According to the theorem of Fermat for functions of several variables $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable $a \in \mathbb{R}^n$ and $a$ is a local min. for $f$, then $\dots = 0_{\mathbb{R}^n}$.

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11. Give an example of a sequence $(a_n)_{n \in \mathbb{N}^*}$, $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n = \infty$ (DIV) but $\sum_{n=1}^{\infty} a_n^2 =$ finite (CONV). For 1 bonus point, prove the DIV resp. CONV of the two series.

12. Choose one of
a) Find the local extrema of $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + (y-1)^2$ subject to $x - y = 0$
b) $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4,\ y \ge 0\}$, $\iint_D \frac{1}{x^2+y^2}\, dx dy = \dots$

13. Give a geometric interpretation of Lagrange's mean value theorem for functions of one variable.
(Theorem will be given, you don't need to know it.)

Second test

1. $\frac{d}{dx}(\sin \omega x) = \frac{d}{dx}(\sin \omega x) \cdot \frac{d}{dx}(\omega x) = \omega \cos \omega x$ , $\omega \in \mathbb{R}$

2. $(\ln x)^2$ antiderivative $\Rightarrow f(x) = (\ln x)^2$

$F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C$

$\int \ln^2 x\,dx = \int x' \ln^2 x\,dx = x \ln^2 x - \int x (\ln^2 x)'\,dx =$

$= x \ln^2 x - \int \left(\cancel{x} \cdot 2 \ln x \cdot \frac{1}{\cancel{x}}\right) dx = x \ln^2 x - 2 \int x' \ln x\,dx =$

$= x \ln^2 x - 2x \ln x + 2 \int x \cdot \frac{1}{x}\,dx = x \ln^2 x - 2x \ln x + 2x + C$

3. $f : \mathbb{R} \to \mathbb{R}$ , $f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

$f'(x) = 0 \Leftrightarrow 3x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{3} \Leftrightarrow x = \pm \frac{\sqrt{3}}{3}$

$\Rightarrow S : \left\{ -\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3} \right\}$

$$||x+y|| = |(1, 1, 1)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

Subject :

Date :......./......./.......

4. $x = (1, 0, 1)$, $y = (0, 1, 0)$

$xy = 1\cdot 0 + 0\cdot 1 + 1\cdot 0 = 0+0+0 = 0$

$$\left( \begin{array}{l} ||x+y|| \le ||x|| + ||y|| \Leftrightarrow ||x+y|| \le \sqrt{1^2+0+1^2} + \sqrt{0^2+1^2+0^2} \Leftrightarrow \\ \Leftrightarrow ||x+y|| \le \sqrt{1+1} + \sqrt{1} \Leftrightarrow ||x+y|| \le \sqrt{2}+\sqrt{1} \end{array} \right)$$

5. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x_1, x_2, x_3) = x_1x_2 + x_3^2 - 3x_2$

$$\nabla f(x_1, x_2, x_3) = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

$$\frac{\partial f}{\partial x_1} = x_2 \ ; \quad \frac{\partial f}{\partial x_2} = x_1 - 3 \ ; \quad \frac{\partial f}{\partial x_3} = 2x_3$$

$$\nabla f(x_1, x_2, x_3) = (x_2, x_1 - 3, 2x_3)$$

6. $f: [a, b] \to \mathbb{R}$ Riemann integrable

$\Delta = \{a_1 = x_0, x_1, \dots, x_m = b\}$

a division of $[a, b]$ and $\xi = \{\xi_1, \dots, \xi_m\}$

$\xi_i \in [x_{i-1}, x_i]$

The Riemann sum associated to $f, \Delta, \xi$ is

$$\sigma(f, \Delta, \xi) = \sum_{k=1}^{m} f(\xi_k)(x_k - x_{k-1})$$

7. $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{\substack{t\to 0\\ t>0}} \left( 2\int_t^1 \frac{dx}{2\sqrt{x}} \right) = \lim_{t \searrow 0} \left( 2\sqrt{x} \Big|_t^1 \right) =$$

$$= \lim_{t\searrow 0} \left(2(\sqrt{1} - \sqrt{t})\right) = 2 \lim_{t\searrow 0}(1-t) = 2(1-0) = 2 \Rightarrow \text{CONV}$$

8. $$I(t) = \int_t^{2t} t^2 x\,dx$$

$$I'(t) = \left( \int_t^{2t} t^2 x\,dx \right)' = \left( t^2 \int_t^{2t} x\,dx \right)' = \left( t^2 \frac{x^2}{2} \Big|_t^{2t} \right)' =$$

$$= \left( t^2 \left( \frac{(2t)^2}{2} - \frac{t^2}{2} \right) \right)' = \left( t^2 \frac{4t^2 - t^2}{2} \right)' = \left( t^2 \frac{3t^2}{2} \right)' = \left( \frac{3t^4}{2} \right)' =$$

$$= \frac{3}{2} \cdot 4t^3 = 6t^3$$

9. $D = [0,1] \times [0,1]$, $I = \iint_D (1-x)y \, dx\, dy$

$$I = \int_0^1\int_0^1 (1-x)y \, dx\, dy = \int_0^1 (1-x)\left(\frac{y^2}{2}\Big|_0^1\right) dx = \int_0^1 (1-x)\frac{1}{2}\, dx =$$

$$= \frac{1}{2}\int_0^1 dx - \frac{1}{2}\int_0^1 x\, dx = \frac{1}{2}x\Big|_0^1 - \frac{1}{2}\frac{x^2}{2}\Big|_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

10. According to the theorem of Fermat for functions of several variables $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable $a \in \mathbb{R}^n$ and a is a local min for $f$, then $\nabla f(a) = 0_{\mathbb{R}^n}$

11. $(a_n)_{n\in\mathbb{N}^*}$, $a_n > 0$ such that $\sum_{n=1}^{\infty} a_n = \infty$ DIV

$\sum_{n=1}^{\infty} a_n^2 =$ finite CONV

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIV ; $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONV

12. a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^2 + (y-1)^2$
subject to $x - y = 0$

$(y-1)^2 = y^2 - 2y + 1$

Let $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x - y$
$L: \mathbb{R}^2 \to \mathbb{R}$, $L(x,y,\lambda) = x^2 + (y-1)^2 - \lambda(x-y)$

$\nabla L(x,y,\lambda) = \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial \lambda}\right) = (0,0,0)$

$\frac{\partial L}{\partial x} = 2x - \lambda$; $\frac{\partial L}{\partial y} = 2y - 2 + \lambda$; $\frac{\partial L}{\partial \lambda} = -(x-y) = y - x$

$\nabla L(x,y,\lambda) = (2x - \lambda, 2y - 2 + \lambda, y - x) = (0,0,0) \Rightarrow$

$$\Rightarrow \begin{cases} 2x - \lambda = 0 \\ 2y - 2 + \lambda = 0 \\ y - x = 0 \iff y = x \end{cases} \Rightarrow \begin{cases} 2x - \lambda = 0 \\ 2x + \lambda = 2 \quad (+) \end{cases}$$

$4x = 2 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$

$\Rightarrow \lambda = 2 \cdot \frac{1}{2} = 1 \to S: \left\{\left(\frac{1}{2}, \frac{1}{2}, 1\right)\right\}$