Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2019

## Linear difference equations. Discrete Dynamical Systems[1]

**1.** a) Find all the sequences $(x_k)_{k\geq 0}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \geq 0$.
b) Find all the sequences $(x_k)_{k\in\mathbb{Z}}$ that satisfy $x_{k+3} = k3^k + 5k - 2$, $k \in \mathbb{Z}$.

**2.** Let $\eta \in \mathbb{R}$ be a fixed parameter. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = 2x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to $\eta$.

**3.** Let $\lambda \in \mathbb{R}^*$ and $\eta \in \mathbb{R}$ be fixed parameters. Find the solution $(x_k)_{k\geq 0}$ of the initial value problem

$$x_{k+1} = \lambda x_k, \quad x_0 = \eta.$$

Check the solution you obtained. What is the long term behavior of this sequence? Discuss with respect to $\lambda$ and $\eta$.

**4.** Find the solution $(x_k)_{k\in\mathbb{Z}}$ of the initial value problem

$$x_{k+2} + x_{k+1} + x_k = 0, \quad x_0 = 0, \quad x_1 = 1.$$

Check the solution you obtained. What is the long term behavior of this sequence?

**5.** Find solutions of the form $x_k = a\,3^k$ of the difference equation $x_{k+1} = 2x_k + 3^k$, $k \geq 0$. Here we look for $a \in \mathbb{R}$.

**6.** Find solutions of the form $x_k = ak + b$ of the difference equation $x_{k+1} = 2x_k - k$, $k \geq 0$. Here we look for $a, b \in \mathbb{R}$.

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For each of the following difference equations find a particular solution of the indicated form. If no form is indicated (and the equation is nonhomogeneous), try a constant solution. Find its general solution using the fundamental theorems for linear difference equations and the characteristic equation method.

**7.** $x_{k+1} = \frac{1}{3}\, x_k$; **8.** $x_{k+1} = \frac{1}{2}\, x_k + 2$; **9.** $x_{k+1} = 2\, x_k + \frac{1}{2}$; **10.** $x_{k+1} = -3x_k$;

**11.** $x_{k+1} = 4\, x_k + 3^{k+1}$, $x_k^p = a \cdot 3^k$; **12.** $x_{k+1} = 1/3\, x_k + 2^k$, $x_k^p = a \cdot 2^k$;

**13.** $x_{k+2} - 6x_{k+1} + 9x_k = 0$; **14.** $x_{k+2} + x_{k+1} + x_k = 0$.

**15.** Find the general solution of the linear planar system

$x_{k+1} = 1/2\, x_k + y_k, \quad y_{k+1} = -1/5\, y_k$

in two ways. One way must be by reducing it to a second order difference equation in $x$.

**16.** Find the general solution of the linear planar system

$x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k$

by reducing it to a second order difference equation.

**17.** Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = -120x,\ x(0) = 1$ and solve the difference equation you obtained in two cases: $h = 0.1$ and $h = 0.001$. Discuss on the long term behavior of the solution of the differential equation and, also, of the solution of the difference equation.

**18.** Write the Euler numerical formula with stepsize $h > 0$ for the IVP $x' = 2x(1-x),\ x(0) = 1$.

**19.** Find the fixed points of the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and study their stability.

**20*.** Represent the stair-step diagram of the function from the previous exercise and try to discuss the long term behavior of the solution of $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$, $k \geq 0$ with respect to an arbitrary $x_0 \in \mathbb{R}$.

**21*.** Prove that the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = m + \varepsilon \sin x$, where $m > 0$ and $0 < \varepsilon < 1$, has a unique fixed point which is an attractor. The equation $x = m + \varepsilon \sin x$ is known as *Kepler equation* and arises in the study of planetary motion.

**22.** Study the stability of the fixed point $(0,0)$ of the following planar linear difference system:

a) $x_{k+1} = 3/5\, x_k + 1/5\, y_k, \quad y_{k+1} = 1/5\, x_k + 3/5\, y_k$;

b) $x_{k+1} = 1/2\, x_k + y_k, \quad y_{k+1} = -1/5\, y_k$;

c) $x_{k+1} = -x_k - y_k, \quad y_{k+1} = -x_k + y_k$.

**23.** Find the fixed points and the 2-periodic points of the maps $f, g : \mathbb{R} \to \mathbb{R}$, $f(x) = -x^3$ and $g(x) = x^2 - 1$.

**24.** In order to study the stability of the fixed point 0 of the maps $f, g, h : \mathbb{R} \to \mathbb{R}$, $f(x) = x + x^3$, $g(x) = x - x^3$, $h(x) = x + x^2$, note that the linearization method can not be applied. Study the stability of the fixed point 0 using the stair-step diagram.

**25.** Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 1$.

(a) Find the fixed points and the 2-periodic points of $f$ and study their stability using the linearization method.

(b) Represent the graph of $f$ and find geometrically the fixed points of $f$. Also, depict the 2-periodic orbit using the stair-step diagram.

(c) Find directly $f^k(0)$ for any $k \geq 0$. Depict this orbit using the stair-step diagram.

d) Let $\eta = 2$, and, respectively, $\eta = -1/4$. Using the stair-step diagram describe the long-term behavior of the orbit that starts at $\eta$ (in other notation, of the sequence defined by $x_{k+1} = x_k^2 - 1,\ x_0 = \eta$).

**26.** Using the stair-step diagram, estimate the basin of attraction for each of the fixed points (if there is any which is an attractor) of the map

$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^2 + 5}{2x}\ .$$

**27.** Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x(1 - x)$.

a) Find its fixed points and study their stability.

b) Let $I_1 = (-\infty, 0)$, $I_2 = (0, 1)$ and $I_3 = (1, \infty)$. Find $f(I_1)$, $f(I_2)$ and $f(I_3)$.

c) Find the orbits corresponding to the initial states $\eta = 0$ and, respectively, $\eta = 1$.

d) Using the stair-step diagram, describe the long-term behavior of the orbits corresponding to the initial states: $\eta = 1/8$, $\eta = 7/8$, $\eta = -1/8$ and, respectively, $\eta = 9/8$.

e) Estimate the basin of attraction of the attractor fixed point of $f$.

**28.** Find the expression of the Fibonacci sequence

$$x_{k+2} = x_{k+1} + x_k, \; x_0 = 0, \; x_1 = 1.$$

**29*.** Find the solution of

$$x_{k+2} - 6x_{k+1} + 9x_k = 12k, \quad x_0 = 0, \quad x_1 = 0.$$

*Hint*: look for $a, b \in \mathbb{R}$ such that $(x_k)_p = ak + b$ is a particular solution of the difference equation.

**30.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (-x^2 + y/5, \, x)$.

(a) Find the fixed points of $f$ and study their stability.

(b) In case that you found an attracting fixed point, write the consequence of this fact for the sequence $(f^k(\eta))_{k\geq 0}$ where $\eta \in \mathbb{R}^2$ is properly chosen. As usual, $f^k$ denotes the $k$-th iterate of $f$.

**31.** We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \frac{1}{50}x(100 - x).$$

(a) Find its fixed points and study their stability.

(b) Using the stair-step diagram estimate the basin of attraction of the attractor fixed point.

(c) If $(x_k)_{k\geq 0}$ represent the number of fish in some lake at month $k$ and

$$x_{k+1} = \frac{1}{50}x_k(100 - x_k), \quad x_0 = \eta$$

try to predict the fate of the fish in the case $\eta = 80$ and also in the case $\eta = 10$.

**32.** We consider the IVP $\quad y' = -200y, \quad y(0) = 1,$
where the unknown is the function $y(t)$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) For $h = 0.001$, and, respectively, $h = 0.01$ find the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) and decide if it satisfies $\lim\limits_{k\to\infty} y_k = 0$.

d) Find a range of values for the step-size $h$ such that the solution $(y_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} y_k = 0$.

**33.** (a) Write the Euler's numerical formula with stepsize $h = 0.01$ to approximate the solution of the IVP $y' = y, \quad y(0) = 1$.

(b) Using (a) find a rational approximation of the Euler's constant $e$.

**34.** Let $g : I \to \mathbb{R}$ be a $C^1$ map such that $g'(x) \neq 0$ for all $x$ in the interval $I$. Assume that there exists $r \in I$ such that $g(r) = 0$. Prove that for $\eta \in I$ sufficiently close to $r$ the unique solution $(x_k)_{k\geq 0}$ of the IVP

$$x_{k+1} = x_k - \frac{g(x_k)}{g'(x_k)}, \quad x_0 = \eta$$

satisfies

$$\lim_{k\to\infty} x_k = r.$$

**35.** Let $a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{R}$ be such that $a_{12} \neq 0$. Show that the roots of the characteristic equation corresponding to the second-order difference equation obtained by reducing the linear system

$$x_{k+1} = a_{11}x_k + a_{12}y_k, \quad y_{k+1} = a_{21}x_k + a_{22}y_k$$

are the eigenvalues of the matrix associated to this system.

**36.** We consider the map

$$T : [0.1] \to \mathbb{R}, \quad T(x) = 1 - |2x - 1|.$$

a) Represent the graph of $T$. Find its fixed points.

b) Let $n \in \mathbb{N}$, $n \geq 2$. Find the orbit of the initial state $\frac{3}{2^n}$.

c) Find the 2-periodic points of $T$.

d) Represent the graphs of $T^2$ and $T^3$. How many fixed points they have?

e) The map $T$ has a 2-periodic orbit? Or a 3-periodic orbit? $T$ has a 2019-periodic orbit?

**37.** We consider the IVP $y' = 1 + xy^2,\ y(0) = 0$. Write the Euler numerical formula on the interval $[0, 1]$ with step-size $h = 0.02$. Specify the initial values and the number of steps necessary to find the approximate value of $\varphi(0.5)$ and, respectively, of $\varphi(1)$. Here with $\varphi$ is denoted the exact solution of the given IVP.

**38.** We consider the difference equation

$$x_{k+2} + x_k = \cos\frac{k\pi}{2} \ .$$

a) Find a solution of the form $(x_k)_p = ak\cos\frac{k\pi}{2}$, with $a \in \mathbb{R}$. (Hint: we recall that $\cos(t + \pi) = -\cos t$ for any $t \in \mathbb{R}$)

b) Find its general solution.

c) Find the solution with $x_0 = x_1 = 0$ and describe its long-term behavior (is it periodic? is it bounded? is it oscillatory around 0?).

**39.** We consider the linear difference system

$$x_{k+1} = \frac{3}{5}x_k + \frac{1}{5}y_k, \quad y_{k+1} = \frac{1}{5}x_k + \frac{3}{5}y_k.$$

a) Study the stability of this system.

b) Find its general solution.

**40.** Find the linear homogeneous difference equation with constant coefficients, of minimal order, which has as solutions the two sequences

$$1,\ \frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ \frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ \frac{1}{2^5}\ ,\ \dots$$

and

$$1,\ -\frac{1}{2}\ ,\ \frac{1}{2^2}\ ,\ -\frac{1}{2^3}\ ,\ \frac{1}{2^4}\ ,\ -\frac{1}{2^5}\ ,\ \dots\ .$$

**41.** We consider the scalar difference equation

$$x_{k+1} = x_k + \lambda x_k(2 - x_k),$$

whose unknown is the sequence $(x_k)_{k\geq 0}$, and where $\lambda \in (0,1)$ is a parameter. Find its constant solutions (fixed points) and study their stability. Discuss with respect to the parameter $\lambda$.

**42.** We consider the IVP $x' = -10^3 x, \quad x(0) = 1$.

a) Find the solution and its limit as $t \to \infty$.

b) Write the Euler's numerical formula with constant step-size h.

c) Find a range of values for the step-size h such that the solution $(x_k)_{k\geq 0}$ of the difference equation found at b) satisfies $\lim\limits_{k\to\infty} x_k = 0$.

**43.** We consider the map $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$ and, given $x_0 \in \mathbb{R}$, consider the sequence $(x_k)_{k\geq 0}$ satisfying the recurrence

$$x_{k+1} = f(x_k)\ .$$

a) Find the fixed points of $f$, and study their stability.

b) Find $(x_k)_{k\geq 0}$ when $x_0 = \sqrt{2}$.

c) There exists an $x_0 \in \mathbb{R} \setminus \{\sqrt{2}\}$ such that $\lim\limits_{k\to\infty} x_k = \sqrt{2}$?

d) There exists an $x_0 \in \mathbb{R}$ such that $\lim\limits_{k\to\infty} x_k = 2$?