Proposition 3.1. Every line in $\mathbb{E}^2$ can be described with a linear equation in two variables

$$ax + by + c = 0 \tag{3.4}$$

relative to a fixed coordinate system and any linear equation in two variables describes a line relative to a fixed coordinate system.

- We showed that a line satisfies a linear eq. (see for example symmetric eq)
- Consider the set S of points $P(x,y)$ such that $ax+by+c=0$ for some $a,b,c \in \mathbb{R}$ with a,b not both 0
- a,b not both zero $\Rightarrow$ without loss of generality we may assume $a \neq 0$

then $\begin{cases} x = -\frac{b}{a}y - \frac{c}{a} \\ y = y \end{cases}$ so $S = \{P(x,y) \in \mathbb{E}^2 : \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c/a \\ 0 \end{bmatrix} + y \begin{bmatrix} -b/a \\ 1 \end{bmatrix}\}$ for any $y \in \mathbb{R}$

the point $A = \begin{bmatrix} -c/a \\ 0 \end{bmatrix} \in S$

$\Rightarrow S = \{P(x,y) \in \mathbb{E}^2 : \underbrace{\begin{bmatrix} x - \frac{c}{a} \\ y \end{bmatrix}}_{= \overrightarrow{AP}} = t \begin{bmatrix} -b/a \\ 1 \end{bmatrix} \ \forall t \in \mathbb{R}\}$

Denote $\begin{bmatrix} -b/a \\ 1 \end{bmatrix}$ by v,

$\Rightarrow S = \{P(x,y) \in \mathbb{E}^2 : \overrightarrow{AP} = t v\}$

$\Rightarrow \phi_A(S)$ is the 1-dimensional vector subspace of $\mathbb{V}^2$ generated by v

$\Rightarrow S$ is a line

Proposition 3.2. Suppose you have a line $\ell : ax + by + c = 0$ and a point $P(x_P, y_P)$ in $\mathbb{E}^2$. The distance from P to ℓ is

$$d(P,\ell) = \frac{|ax_p + by_p + c|}{\sqrt{a^2+b^2}}.$$

- Identify $\mathbb{E}^2$ with the Oxy-plane of $\mathbb{E}^3$ relative to a right oriented orthonormal coordinate system of $\mathbb{E}^3$
- Let $A(x_A, y_A)$ and $B(x_B, y_B)$ be two points in ℓ
- For any point $Q(x_Q, y_Q) \in \mathbb{E}^2 = Oxy$ we have

$$Q \in \ell \iff A, B, Q \text{ collinear}$$

$$\iff \begin{vmatrix} x_Q & y_Q & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} = 0 \qquad (\text{see chapter } 2)$$

$$\iff x_Q\underbrace{(y_A - y_B)}_{= a'} + y_Q\underbrace{(-x_A + x_B)}_{= b'} + \underbrace{(x_A y_B - y_A x_B)}_{= c'} = 0 \quad \text{a linear eq in } x_Q \text{ and } y_Q$$

So $\ell : a'x + b'y + c' = 0$

but we also have $\ell : ax + by + c = 0$

$$\Rightarrow \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \lambda$$

then
$$\frac{|ax_p + by_p + c|}{\sqrt{a^2+b^2}} = \frac{|\lambda a' x_p + \lambda b' y_p + \lambda c'|}{\sqrt{(\lambda a')^2 + (\lambda b')^2}} = \frac{|a' x_p + b' y_p + c'|}{\sqrt{(a')^2 + (b')^2}}$$

Moreover
$$\frac{|a' x_p + b' y_p + c'|}{\sqrt{(a')^2 + (b')^2}} = \frac{\left| \begin{vmatrix} x_P & y_P & 1 \\ x_A & y_A & 1 \\ x_B & y_B & 1 \end{vmatrix} \right|}{\sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}}$$

$$= \frac{\text{area of } \Delta ABP}{\|\overrightarrow{BA}\|} = h = d(P, \ell)$$