CHAPTER 7

Quadratic curves (conics)

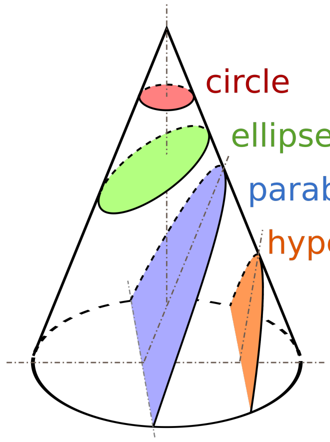
Contents

Here we consider objects in $\mathbb{E}^2$ with respect to an orthonormal coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$.

Definition. A *quadratic curve* (or *conic*) in $\mathbb{E}^2$ is a curve described by a quadratic equations

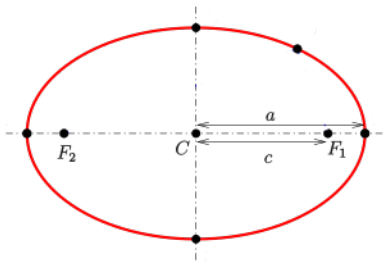
$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

for some $a, b, c, d, e, f \in \mathbb{R}$.



7.1 Ellipse

7.1.1 Geometric description



Definition. An *ellipse* is the geometric locus of points in $\mathbb{E}^2$ for which the sum of the distances from two given points, the *focal points*, is constant.

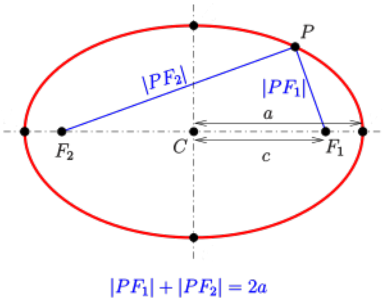
7.1.2 Canonical equation - global description

In general we can describe a conic with a so-called canonical equation. Such an equation is with respect to a well chosen coordinate system.

Proposition 7.1. Let F_1 and F_2 be two points in $\mathbb{E}^2$ and let a be a positive real scalar. Choose the coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$ such that F_1 and F_2 are on the Ox axis, such that $\overrightarrow{F_2F_1}$ has the same direction as $\mathbf{i}$ and such that O is the midpoint of $[F_1F_2]$. With these choices, the ellipse with focal points F_1 and F_2 for which the sum of distances from the focal points is $2a$ has an equation of the form

$$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.1}$$

for some positive scalar $b \in \mathbb{R}$. We denote this ellipse by $\mathcal{E}_{a,b}$.



- The equation (7.1) is called the *canonical equation of the ellipse* $\mathcal{E}_{a,b}$. Clearly, with respect to some other coordinate system, the same ellipse will have a different equation.
- If $2c$ denotes the distance between F_1 and F_2 then $b^2 = a^2 - c^2$.
- The intersections of $\mathcal{E}_{a,b}$ with the coordinate axes are the points $(\pm a, 0)$ and $(0, \pm b)$.
- The numerical quantity
$$\varepsilon = \frac{c}{a} = \sqrt{1 - \frac{b^2}{a^2}} \in [0, 1)$$
is called the *eccentricity* of the ellipse $\mathcal{E}_{a,b}$. It measures how flat or how round the ellipse is.
- The canonical equation shows that $M(x_M, y_M) \in \mathcal{E}_{a,b}$ if and only if $(\pm x_M, \pm y_M) \in \mathcal{E}_{a,b}$.

7.1.3 Parametric equations - local description

Parametric equations are never unique. Depending on what your intentions are you may prefer one over the other.

The equation (7.1) allows us to express y in terms of x:

$$y(x) = \pm \frac{b}{a}\sqrt{a^2 - x^2}.$$

This gives a partial parametrization of $\mathcal{E}_{a,b}$. For the 'northern part' we have the parametrization

$$\phi : [-a, a] \to \mathbb{E}^2 \quad \text{given by} \quad \phi(x) = \big(x, y(x)\big) = \Big(x, \frac{b}{a}\sqrt{a^2 - x^2}\Big).$$

This is the graph of the function

$$y(x) = \pm\frac{b}{a}\sqrt{a^2 - x^2} \quad \text{for which} \quad y'(x) = \frac{-bx}{a\sqrt{a^2 - x^2}} \quad \text{and} \quad y''(x) = \frac{ab}{(x-a)(x+a)\sqrt{a^2 - x^2}}.$$

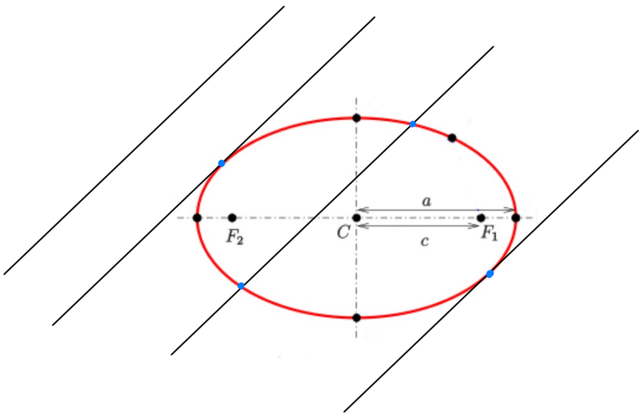
Thus, we can use the known methods to verify the monotony and the convexity of $y(x)$ which describes this part of the ellipse.

A second way of parametrizing the ellipse $\mathcal{E}_{a,b}$ is with

$$\phi : \mathbb{R} \to \mathbb{E}^2 \quad \text{defined by} \quad \phi(t) = (a\cos(t), b\sin(t)).$$

It is easy to check using equation (7.1) that this is a parametrization.

7.1.4 Relative position of a line



Consider the canonical equation of the ellipse $\mathcal{E}_{a,b}$. Let ℓ be a line with equation $y = kx + m$ (relative to the same coordinate system). The intersection of the two objects is the set of points with coordinates solutions to the system

$$\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \\ y = kx + m \end{cases} \Leftrightarrow \begin{cases} \frac{x^2}{a^2} + \frac{(kx+m)^2}{b^2} = 1 \\ y = kx + m \end{cases}.$$

The solutions to this system are $(x, y) = (x, kx + m)$ where x is a solution to the first equation. So let us discus that equation:

$$(b^2 + a^2k^2)x^2 + 2kma^2x + a^2(m^2 - b^2) = 0.$$

This is a quadratic equation in x since a, b, k, m are fixed. The discriminant of this equation is

$$\Delta = 4k^2m^2a^4 - 4a^2(m^2 - b^2)(b^2 + a^2k^2) = 4a^2b^2(a^2k^2 + b^2 - m^2).$$

So, the number of solutions is controlled by $a^2k^2 + b^2 - m^2$:

- $-\sqrt{a^2k^2+b^2} < m < \sqrt{a^2k^2+b^2}$ in which case ℓ intersects $\mathcal{E}_{a,b}$ in two distinct points.
- $m = \pm\sqrt{a^2k^2+b^2}$ in which case ℓ intersects $\mathcal{E}_{a,b}$ in a unique point. Such a point is a *double intersection point* because it is obtained as a double solution to the algebraic equation. For these two values of m, the line $\ell : y = kx + m$ is tangent to the ellipse. Therefore, if a slope k is given, there are two tangent lines to the ellipse:
$$y = kx \pm \sqrt{a^2k^2+b^2}.$$
- $m < \sqrt{a^2k^2+b^2}$ or $m > \sqrt{a^2k^2+b^2}$ in which case there is no intersection point between ℓ and $\mathcal{E}_{a,b}$.

7.1.5 Tangent line in a given point - algebraic

Consider an ellipse and a line:

$$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \ell : \begin{cases} x = x_0 + tv_x \\ y = y_0 + tv_y \end{cases}.$$

which have the point (x_0, y_0) in common. When is ℓ tangent to the ellipse? If the intersection $\mathcal{E}_{a,b} \cap \ell$ has a unique point. In order to determine when this is the case, we check which points on ℓ satisfy the equaion of the ellipse:

$$\frac{(x_0 + v_xt)^2}{a^2} + \frac{(y_0 + v_yt)^2}{b^2} = 1.$$

The parameters t satisfying the above equations corrspond to points on ℓ which lie on $\mathcal{E}_{a,b}$. The equation is equivalent to

$$\left(\frac{v_x^2}{a^2} + \frac{v_y^2}{b^2}\right)t^2 + 2\left(\frac{x_0v_x}{a^2} + \frac{y_0v_y}{b^2}\right)t + \underbrace{\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} - 1}_{=0} = 0$$

In order for ℓ to be tangent to $\mathcal{E}_{a,b}$, there needs to be a unique solution t to the above equation. Since $t = 0$ is obviously a solution, this needs to be the *only* solution. In other words, $t = 0$ should be a double solution. For this to happen we must have

$$\frac{x_0v_x}{a^2} + \frac{y_0v_y}{b^2} = 0 \quad \Leftrightarrow \quad \mathbf{n} \cdot \mathbf{v} = 0$$

where $\mathbf{n} = \mathbf{n}(\frac{x_0}{a^2}, \frac{y_0}{b^2})$. Thus, ℓ is tangent to the ellipse if and only if the the vector $\mathbf{n}$ is orthogonal to ℓ, i.e. if and only if $\mathbf{n}$ is a normal vector for ℓ. It follows that ℓ is tangent to $\mathcal{E}_{a,b}$ in the point $(x_0, y_0) \in \mathcal{E}_{a,b}$ if and only if it satisfies the Cartesian equation:

$$\ell : \frac{x_0}{a^2}(x - x_0) + \frac{y_0}{b^2}(y - y_0) = 0.$$

We call this line the *tangent line to $\mathcal{E}_{a,b}$ at the point $(x_0, y_0) \in \mathcal{E}_{a,b}$* and denote it by $T_{(x_0,y_0)}\mathcal{E}_{a,b}$. Rearranging the above equation we see that:

$$T_{(x_0,y_0)}\mathcal{E}_{a,b} : \frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1. \tag{7.2}$$

7.1.6 Tangent line in a given point - via gradients

It is possible to describe the tangent line $T_{(x_0,y_0)}\mathcal{E}_{a,b}$ to $\mathcal{E}_{a,b}$ at the point $(x_0,y_0)\in\mathcal{E}_{a,b}$ using gradients. For this consider the map

$$\psi:\mathbb{E}^2\to\mathbb{R}\quad\text{defined by}\quad \psi(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

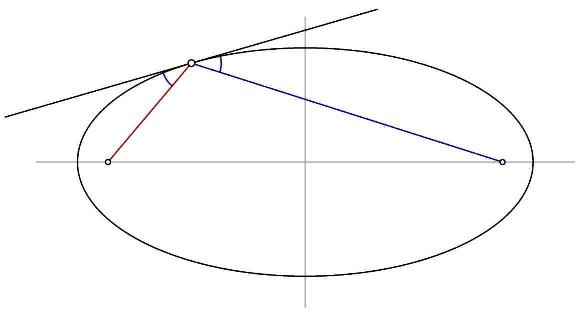
and notice that $\mathcal{E}_{a,b}=\psi^{-1}(1)$. The gradient in a point $(x_0,y_0)\in\mathcal{E}_{a,b}$ is

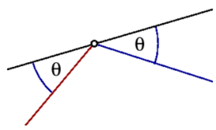
$$\nabla_{(x_0,y_0)}(\psi)=\left(2\frac{x}{a^2},2\frac{y}{b^2}\right)_{(x_0,y_0)}=2\left(\frac{x_0}{a^2},\frac{y_0}{b^2}\right).$$

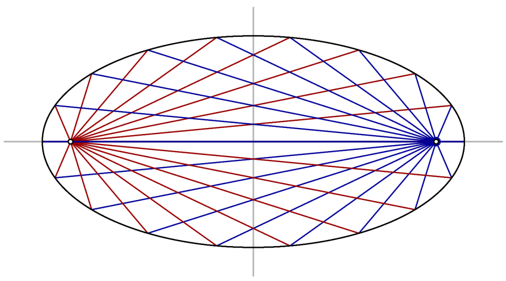
Using a parametrization $\phi:I\to\mathbb{E}^2$ of $\mathcal{E}_{a,b}$ and calculating $\partial_t\psi(\phi(t))$ with the chain rule, one shows that $\nabla_{(x_0,y_0)}(\psi)$ is orthogonal to the tangent vectors at the point (x_0,y_0). In other words, $\nabla_{(x_0,y_0)}(\psi)$ is a normal vector at the point $(x_0,y_0)\in\mathcal{E}_{a,b}$. This gives a different way of obtaining the equation (7.2).

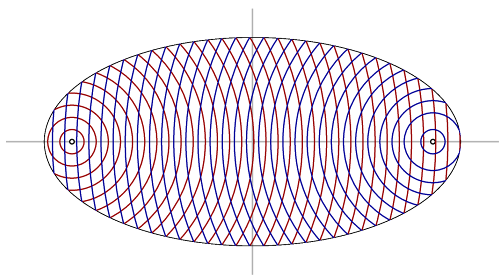
7.1.7 Applications

An ellipse has reflective properties:



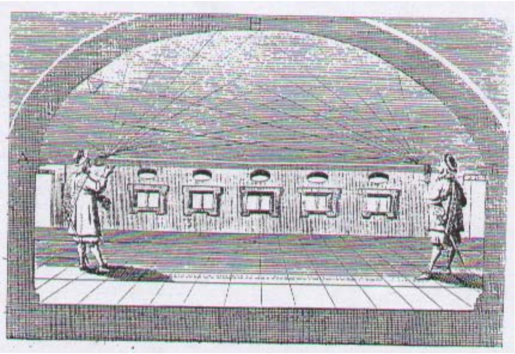






These properties are exploited in praxis.

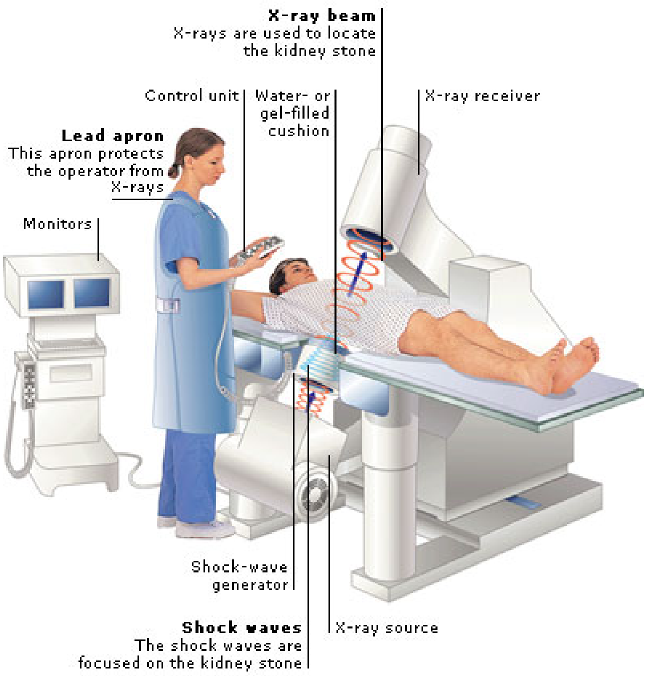
- [Elliptical rooms]



(a) Elliptical room.

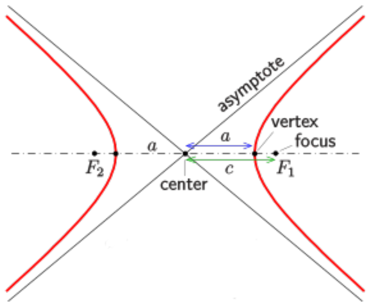
(b) The National Statuary Hall in Washington, D.C.

- [Lithotripsy]



7.2 Hyperbola

7.2.1 Geometric description



Definition. A *hyperbola* is the geometric locus of points in $\mathbb{E}^2$ for which the difference of the distances from two given points, the *focal points*, is constant.

7.2.2 Canonical equation - global description

Proposition 7.2. Let F_1 and F_2 be two points in $\mathbb{E}^2$ and let a be a positive real scalar. Choose the coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$ such that F_1 and F_2 are on the Ox axis, such that $\overrightarrow{F_2F_1}$ has the same direction as $\mathbf{i}$ and such that O is the midpoint of $[F_1F_2]$. With these choices, the hyperbola with focal points F_1 and F_2 for which the absolute value of the difference of distances from the focal points is $2a$ has an equation of the form

$$\mathcal{H}_{a,b} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{7.3}$$

for some positive scalar $b \in \mathbb{R}$. We denote this hyperbola by $\mathcal{H}_{a,b}$.

- The equation (7.3) is called the *canonical equation of the hyperbola* $\mathcal{H}_{a,b}$. Clearly, with respect to some other coordinate system, the same hyperbola will have a different equation.
- If $2c$ denotes the distance between F_1 and F_2 then $b^2 = c^2 - a^2$.
- The intersections of $\mathcal{H}_{a,b}$ with the coordinate axes are the points $(\pm a, 0)$.
- The numerical quantity
$$\varepsilon = \frac{c}{a} = \sqrt{1 + \frac{b^2}{a^2}} \in (1, \infty)$$
is called the *eccentricity* of the hyperbola $\mathcal{H}_{a,b}$. It measures how open or how closed the two branches of the hyperbola are.
- The canonical equation shows that $M(x_M, y_M) \in \mathcal{H}_{a,b}$ if and only if $(\pm x_M, \pm y_M) \in \mathcal{H}_{a,b}$.

7.2.3 Parametric equations - local description

Parametric equations are never unique. Depending on what your intentions are you may prefer one over the other.

The equation (7.3) allows us to express y in terms of x:

$$y(x) = \pm\frac{b}{a}\sqrt{x^2-a^2}.$$

This gives a partial parametrization of $\mathcal{H}_{a,b}$. For the 'northern part' we have the parametrization

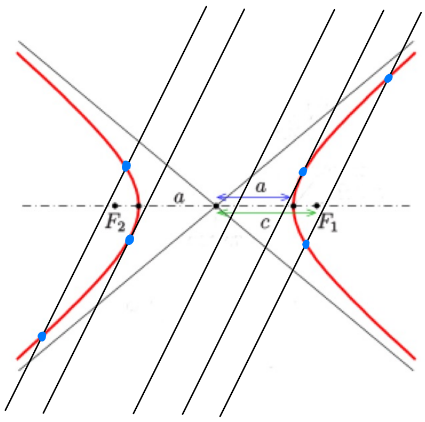
$$\phi : (-\infty,-a]\cup[a,\infty)\to\mathbb{E}^2 \quad \text{given by} \quad \phi(x) = \big(x, y(x)\big) = \Big(x, \frac{b}{a}\sqrt{x^2-a^2}\Big).$$

This is the graph of the function

$$y(x) = \pm\frac{b}{a}\sqrt{x^2-a^2} \quad \text{for which} \quad y'(x) = \frac{bx}{a\sqrt{x^2-a^2}} \quad \text{and} \quad y''(x) = \frac{ab}{(a-x)(x+a)\sqrt{x^2-a^2}}$$

Thus, we can use the known methods to verify the monotony and the convexity of $y(x)$ which describes this part of the hyperbola.

7.2.4 Relative position of a line



Consider the canonical equation of the hyperbola $\mathcal{H}_{a,b}$. Let ℓ be a line with equation $y = kx+m$ (relative to the same coordinate system). The intersection of the two objects is the set of points with coordinates solutions to the system

$$\begin{cases} \frac{x^2}{a^2}-\frac{y^2}{b^2} = 1 \\ y = kx+m \end{cases} \quad \Leftrightarrow \quad \begin{cases} \frac{x^2}{a^2}-\frac{(kx+m)^2}{b^2} = 1 \\ y = kx+m \end{cases}.$$

The solutions to this system are $(x,y) = (x, kx+m)$ where x is a solution to the first equation. So let us discus that equation:

$$(b^2-a^2k^2)x^2 - 2kma^2x - a^2(m^2+b^2) = 0 \tag{7.4}$$

This is a quadratic equation in x since a, b, k, m are fixed. The discriminant of this equation is

$$\Delta = 4k^2m^2a^4 + 4a^2(m^2 + b^2)(b^2 - a^2k^2) = 4a^2b^2(m^2 + b^2 - a^2k^2).$$

So, the number of solutions is controlled by $m^2 + b^2 - a^2k^2$ ... *if* the equation is quadratic. Suppose equation (7.4) is quadratic, i.e. $b^2 - a^2k^2 \neq 0$.

- $m < \sqrt{a^2k^2 - b^2}$ or $m > \sqrt{a^2k^2 - b^2}$ in which case ℓ intersects $\mathcal{H}_{a,b}$ in two distinct points.
- $m = \pm\sqrt{a^2k^2 - b^2}$ in which case ℓ intersects $\mathcal{H}_{a,b}$ in a unique point. Such a point is a *double intersection point* because it is obtained as a double solution to the algebraic equation. For these two values of m, the line $\ell : y = kx + m$ is tangent to the hyperbola. Therefore, if a slope k is given such that $b^2 - a^2k^2 \neq 0 \Leftrightarrow k \neq \pm\frac{b}{a}$, there are two tangent lines to the hyperbola:

$$y = kx \pm \sqrt{a^2k^2 - b^2}.$$

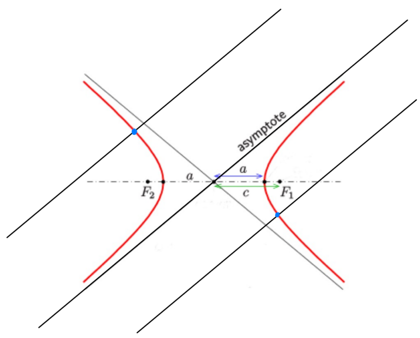
- $-\sqrt{a^2k^2 - b^2} < m < \sqrt{a^2k^2 - b^2}$ in which case there is no intersection point between ℓ and $\mathcal{H}_{a,b}$.

Suppose equation (7.4) is not quadratic, i.e. $b^2 - a^2k^2 = 0$ and the equation is

$$-2kma^2x - a^2(m^2 + b^2) = 0$$

Notice that $k \neq 0$ and $a \neq 0$ and that $k = \pm\frac{b}{a}$. We have two cases:

- $m \neq 0$, hence the unique solution $x = -\frac{m^2+b^2}{2a^2}$ which corresponds to a unique intersection point. In this cases it is a *simple intersection point*, it corresponds to a simple solution of an algebraic equation (not a double solution).
- $m = 0$ in which case there is no intersection point and ℓ is either $y = \frac{b}{a}x$ or $y = -\frac{b}{a}x$. These are the two asymptotes of the hyperbola $\mathcal{H}_{a,b}$. One can check with the parametrization in the previous section that these two lines really are asymptotes.



7.2.5 Tangent line in a given point

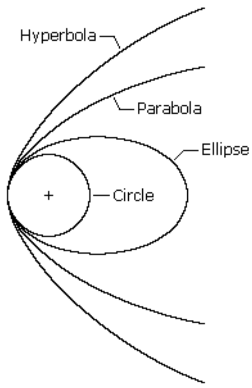
The *tangent line to $\mathcal{H}_{a,b}$ at the point* $(x_0, y_0) \in \mathcal{H}_{a,b}$ has an equation of the form

$$T_{(x_0,y_0)}\mathcal{H}_{a,b} : \frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1. \tag{7.5}$$

This can be deduced either with the algebraic method or via the gradient as in the case of the ellipse.

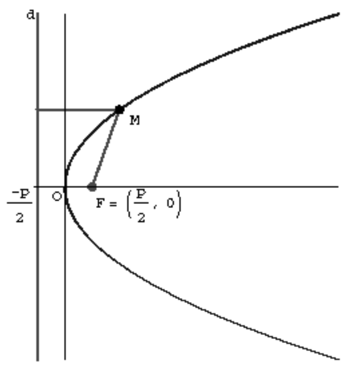
7.2.6 Applications

- [Two body problem] Newton generalized Kepler's laws to apply to any two bodies orbiting each other. The shape of an orbit is a conic section with the center of mass at one focus (first law of orbital motion).



7.3 Parabola

7.3.1 Geometric description



Definition. A *parabola* is the geometric locus of points in $\mathbb{E}^2$ for which the the distances from a given point, the *focal point*, equals the distance to a given line, the *directrix*.

7.3.2 Canonical equation - global description

Proposition 7.3. Let F be a point, let d be a line in $\mathbb{E}^2$ and let p be a positie real scalar. Choose the coordinate system $Oxy = (O, \mathbf{i}, \mathbf{j})$ such that F lies on the Ox axis, such that the Ox axis is orthogonal to d, the origin is at equal distance from d and F and the vector $\mathbf{i}$ has the same direction as $\overrightarrow{OF}$. With these choices, the parabola with focal point F and directrix d for which the $d(F,d) = p$ has an equation of the form

$$\mathcal{P}_p : y^2 = 2px \tag{7.6}$$

We denote this parabola by $\mathcal{P}_p$.

- The equation (7.6) is called the *canonical equation of the parabola* $\mathcal{P}_p$. Clearly, with respect to some other coordinate system, the same parabola will have a different equation.
- The focal point is $F(\frac{p}{2}, 0)$ and the directrix has equation $d : x = -\frac{p}{2}$.
- The intersections of $\mathcal{P}_p$ with the coordinate axes is the point $(0,0)$.
- The canonical equation shows that $M(x_M, y_M) \in \mathcal{P}_p$ if and only if $(x_M, \pm y_M) \in \mathcal{P}_p$.

7.3.3 Parametric equations - local description

Parametric equations are never unique. Depending on what your intentions are you may prefer one over the other.

The equation (7.6) allows us to express y in terms of x:

$$y(x) = \pm\sqrt{2px}.$$

This gives a partial parametrization of $\mathcal{P}_p$. For the 'northern part' we have the parametrization

$$\phi : [0, \infty) \to \mathbb{E}^2 \quad \text{given by} \quad \phi(x) = \big(x, y(x)\big) = \big(x, \sqrt{2px}\big).$$

This is the graph of the function

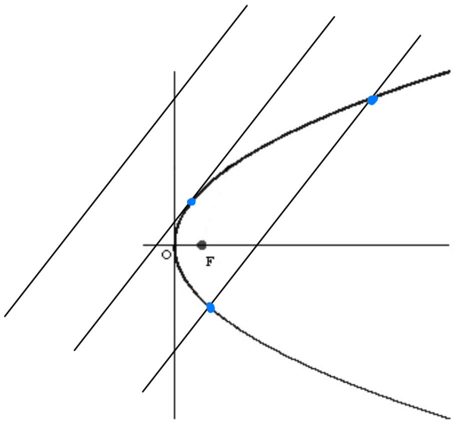
$$y(x) = \sqrt{2px} \quad \text{for which} \quad y'(x) = \frac{\sqrt{2p}}{2\sqrt{x}} \quad \text{and} \quad y''(x) = -\frac{\sqrt{2p}}{4x^{3/2}}$$

Thus, we can use the known methods to verify the monotony and the convexity of $y(x)$ which describes this part of the parabola.

We can in fact parametrize the whole parabola if we express x in terms of y, which is another way of reading equation (7.6). We then have the parametrization

$$\phi : \mathbb{R} \to \mathbb{E}^2 \quad \text{given by} \quad \phi(x) = \big(x(y), y\big) = \Big(\frac{y^2}{2p}, y\Big).$$

7.3.4 Relative position of a line



Consider the canonical equation of the parabola $\mathcal{P}_p$. Let ℓ be a line with equation $y = kx + m$ (relative to the same coordinate system). The intersection of the two objects is the set of points with coordinates solutions to the system

$$\begin{cases} y^2 = 2px \\ y = kx + m \end{cases} \Leftrightarrow \begin{cases} (kx+m)^2 = 2px \\ y = kx + m \end{cases}.$$

The solutions to this system are $(x, y) = (x, kx + m)$ where x is a solution to the first equation. So let us discus that equation:

$$k^2x^2 + 2(km - p)x + m^2 = 0 \tag{7.7}$$

This is a quadratic equation in x since p, k, m are fixed. The discriminant of this equation is

$$\Delta = 4p(p - 2km).$$

So, the number of solutions is controlled by $p - 2km$:

- $km < p/2$ in which case ℓ intersects $\mathcal{P}_p$ in two distinct points.
- $km = p/2$ in which case ℓ intersects $\mathcal{P}_p$ in a unique point. Such a point is a *double intersection point* because it is obtained as a double solution to the algebraic equation. For this value of m, the line $\ell : y = kx + m$ is tangent to the parabola. Therefore, if a slope k is given, there is one tangent line to the parabola:
$$y = kx + \frac{p}{2k}.$$
- $km > p/2$ in which case there is no intersection point between ℓ and $\mathcal{P}_p$.

7.3.5 Tangent line in a given point

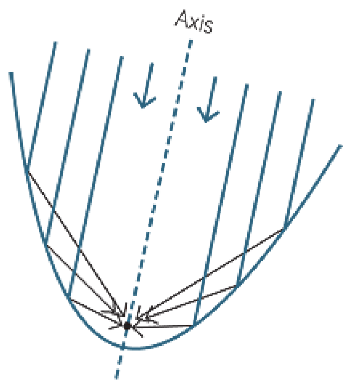
The *tangent line to* $\mathcal{P}_p$ *at the point* $(x_0, y_0) \in \mathcal{P}_p$ has an equation of the form

$$T_{(x_0,y_0)}\mathcal{P}_p : yy_0 = p(x + x_0) \tag{7.8}$$

This can be deduced either with the algebraic method or via the gradient as in the case of the ellipse.

7.3.6 Applications

A parabola has reflective properties:



These properties are used for lenses, parabolic reflectors, satellite dishes, etc.