LECTURE 14 - DYNAMICAL SYSTEMS

Monday, 31 May 2021 10:01

Discrete dyn. systems (cont).

(1) $x_{k+1} = f(x_k),\ k \geq 0$, where $f: I \to I$ is C^1, $I \subset \mathbb{R}$, open, nonempty interval

Th: Let η^* be a fixed point of f. If $|f'(\eta^*)| < 1$ then η^* is an attractor.
If $|f'(\eta^*)| > 1$ then η^* is unstable.

Recall: η^* is, by def, a 2-periodic point of f, then if η^* is a fixed point of $f^2 = f \circ f$ and it is not a f.p of f. We say that $\{\eta^*, f(\eta^*)\}$ is a 2-periodic cycle/orbit. Note that $f(\eta^*)$ is also a 2-periodic point of f.

The 2-periodic point η^* is an attractor for f, when, by def, η^* is an attracting fixed point of f^2.

Prop: Let $\{\eta_1, \eta_2\}$ be a 2-cycle of f. If $|f'(\eta_1) \cdot f'(\eta_2)| < 1$ then this 2-cycle is an attractor of f.
If $|f'(\eta_1) \cdot f'(\eta_2)| > 1$ then this 2-cycle is unstable.

Pf: $\{\eta_1, \eta_2\}$ is a 2-cycle $\Rightarrow f(\eta_1) = \eta_2$ and $f(\eta_2) = \eta_1$ and both η_1 and η_2 are fixed points of f^2.

$(f^2)'(\eta_1) = (f \circ f)'(\eta_1) = f'(f(\eta_1)) \cdot f'(\eta_1) = f'(\eta_2) \cdot f'(\eta_1)$

The conclusion follows from the linearization method applied to η_1, as fixed point of f^2. ☐

1) We consider the map $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{50}x(100-x)$.

a) Find its fixed points and study their stability.

b) Estimate the basin of attraction of the attractor fixed point.

c) If $(x_k)_{k \geq 0}$ represent the density of fish in a lake at month k and $x_{k+1} = \frac{1}{50}x_k(100-x_k)$, $x_0 = \eta$, try to predict the fate of the fish in the cases $\eta = 80$ and resp. $\eta = 10$

a) fixed points: $f(x) = x \Leftrightarrow \frac{1}{50}x(100-x) = x \Leftrightarrow 100x - x^2 = 50x \Leftrightarrow x^2 - 50x = 0 \Leftrightarrow x(x-50) = 0$

There are 2 fixed points $\eta_1^* = 0$ and $\eta_2^* = 50$

b) $f'(x) = \frac{1}{50}(100 - 2x)$

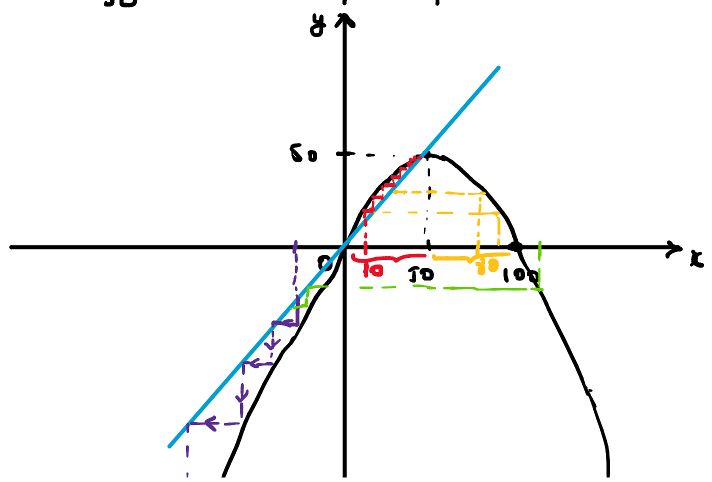
$f'(0) = 2 \Rightarrow |f'(0)| > 1 \overset{LM}{\Longrightarrow} 0$ is an unstable fixed point

$f'(50) = 0 \Rightarrow |f'(50)| < 1 \overset{LM}{\Longrightarrow} 50$ is an attractor fixed point

In order to estimate A_{50} we use the stair-step/cobweb diagram.

$f(x) = \frac{1}{50}x(100-x) \Rightarrow G_f$ is a parabola

$f(50) = \frac{1}{50} \cdot 50 \cdot 50 = 50$



$A_{50} = (0, 100)$

0 is a fixed point $\Rightarrow (x_k)_{k\geq 0}$ is identically 0 : $f^k(0) = 0, \forall k \geq 0$

$f^k(100) = 0, \forall k \geq 1$

c) $\eta = 80$, $\eta = 10$

Note that $80, 10 \in A_{50} \Rightarrow \lim_{k\to\infty} f^k(80) = 50$

$$\lim_{k\to\infty} f^k(10) = 50$$

For $\eta = 10$ we have after 1 month the density $f(10) = \frac{1}{50} \cdot \cancel{10} \cdot 90 = 18$ and then the density will (slowly) increase. The long term behaviour is that the density will be around 50.

For $\eta = 80$, since $f(80) = \frac{1}{50} \cdot 80 \cdot 20 = 32$, after 1 month the density will decrease (dramatically) and after it will slowly increase s.t. the long-term behaviour is that the density will be around 50.

Recall: The Newton's method

$g: I \to \mathbb{R}$, $g(\eta^*) = 0$ $\quad f(x) = x - \frac{g(x)}{g'(x)}$

$g'(\eta^*) \neq 0$ $\quad f(\eta^*) = \eta^*$, $f'(\eta^*) = 0$

η^* is an attractor f.p. of f

$g(x) = x^2 - 3$
$\eta^* = \sqrt{3}$ $\quad f(x) = x - \frac{x^2 - 3}{2x} = x - \frac{1}{2}x + \frac{3}{2x} = \frac{1}{2}x + \frac{3}{2x}$

$A_{\sqrt{3}} = (0, \infty)$

$z_0 = x_0 + iy_0$, $x_0 \in (0, \infty)$, $y_0 \in \mathbb{R}$

$g(x) = x^4 - 1 = (x^2 - 1)(x^2 + 1)$, $\pm 1, \pm i$ $\quad f(x) = x - \frac{x^4 - 1}{4x^3} = x - \frac{1}{4}x + \frac{1}{4x^3} = \frac{3}{4}x + \frac{1}{4x^3}$

"Complex" fixed points: $\pm 1, \pm i \in \mathbb{C}$

$g(\eta^*) = 0$, $f(x) = x - \frac{g(x)}{g'(x)}$, $f(\eta^*) = \eta^*$

Recall: $f'(x) = 1 - \frac{g'(x) \cdot g'(x) - g(x) \cdot g''(x)}{[g'(x)]^2} \Rightarrow f'(\eta^*) = 0 \Rightarrow \eta^*$ is an attractor of f

Ex: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x - \frac{1}{4}(x^2 - 2)$

a) Find the fixed points and study their stability.

b) Write the conseq. of a) for the ivp: $x_{k+1} = x_k - \frac{1}{4}(x_k^2 - 2)$, $x_0 = \eta$

c) Using the cobweb diagram study the behaviour of the seq. given at b) for any $\eta \in \mathbb{R}$.

a) $f(x) = x \Leftrightarrow \cancel{x} - \frac{1}{4}(x^2 - 2) = \cancel{x} \Leftrightarrow x = \pm\sqrt{2} \Rightarrow$ the fixed points are $-\sqrt{2}$ and $\sqrt{2}$.

$f'(x) = 1 - \frac{x}{2} \Rightarrow f'(-\sqrt{2}) = 1 + \frac{\sqrt{2}}{2} \Rightarrow |f'(-\sqrt{2})| > 1 \Rightarrow -\sqrt{2}$ is an unstable fixed point.

$f'(\sqrt{2}) = 1 - \frac{\sqrt{2}}{2} \Rightarrow |f'(\sqrt{2})| < 1 \Rightarrow \sqrt{2}$ is an attractor fixed point

$f'(x) = 1 - \frac{x}{2} \Rightarrow f'(-\sqrt{2}) = 1 + \frac{\sqrt{2}}{2} \Rightarrow |f'(-\sqrt{2})| > 1 \Rightarrow -\sqrt{2}$ is an unstable fixed point.

$f'(\sqrt{2}) = 1 - \frac{\sqrt{2}}{2} \Rightarrow |f'(\sqrt{2})| < 1 \Rightarrow \sqrt{2}$ is an attractor fixed point

b) Note that $\forall \eta \in \mathbb{R} \Rightarrow x_k = f^k(\eta)$, i.e. $(x_k)_{k \geq 0}$ is the seq. of iterations of f starting with η.

$\sqrt{2}$ is a fixed point of $f \Rightarrow f^k(\sqrt{2}) = \sqrt{2}, \forall k \geq 0$

$-\sqrt{2}$ — " — " — $\Rightarrow f^k(-\sqrt{2}) = -\sqrt{2}, \forall k \geq 0$

$\sqrt{2}$ is an attracting f. p. of $f \overset{\text{def}}{\Leftrightarrow} \exists \rho > 0$ s.t. for any $\eta \in \mathbb{R}$ with $|\eta - \sqrt{2}| < \rho$, we have $\lim_{k \to \infty} f^k(\eta) = \sqrt{2}$.

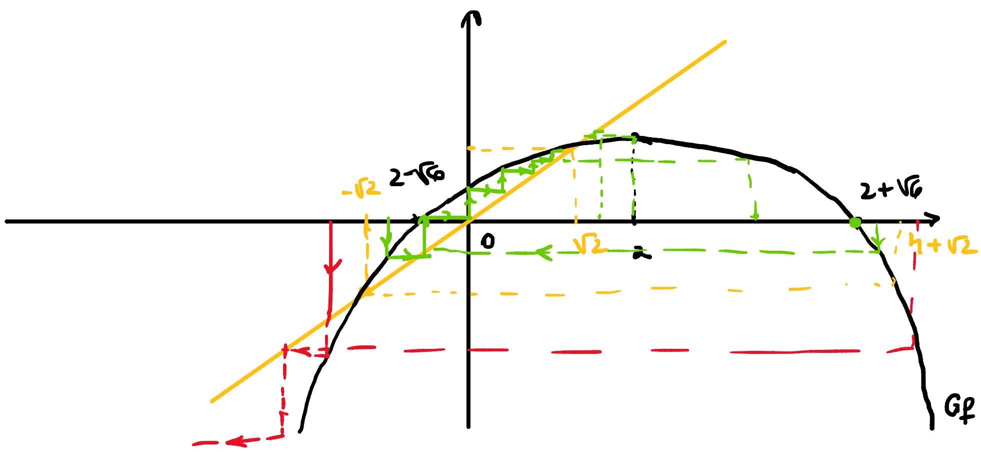
c) $f(x) = x - \frac{1}{4}(x^2 - 2)$, $f'(x) = 1 - \frac{1}{2}x = \frac{2-x}{2}$

x	$-\infty$	$-\sqrt{2}$	$2-\sqrt{6}$	$\sqrt{2}$	2	$2+\sqrt{6}$	$+\infty$
$f'(x)$	+ + + + + + + +				0	- - - - - -	
$f(x)$			0		$\frac{3}{2}$	0	

$f(x) = 0 \Leftrightarrow x - \frac{1}{4}(x^2 - 2) = 0 \Leftrightarrow 4x - x^2 + 2 = 0 \Leftrightarrow$

$\Leftrightarrow x^2 - 4x - 2 = 0 \Rightarrow x_{1,2} = 2 \pm \sqrt{6}$

$f(2) = 2 - \frac{1}{4}(4-2) = 2 - \frac{1}{2} = \frac{3}{2}$



It seems that $A_{\sqrt{2}} = (-\sqrt{2}, 4+\sqrt{2})$

$f(4+\sqrt{2}) = 4 + \sqrt{2} - \frac{1}{4}((4+\sqrt{2})^2 - 2) = 4 + \sqrt{2} - \frac{1}{4}(4+\sqrt{2}-\sqrt{2})(4+\sqrt{2}+\sqrt{2}) = -\sqrt{2}$

$f^k(4+\sqrt{2}) = -\sqrt{2}, \forall k \geq 1$

$\lim_{k \to \infty} f^k(\eta) = \sqrt{2}, \forall \eta \in (-\sqrt{2}, 4+\sqrt{2})$

$\lim_{k \to \infty} f^k(\eta) = -\infty, \forall \eta \in (-\infty; -\sqrt{2}) \cup (4+\sqrt{2}; +\infty)$

+ monotonicity