Exam July 10. Solutions of selected problems

3. Let $g:\mathbb{R}^2\to\mathbb{R}$ be a C^1-function. Consider the planar differential system
$$\begin{cases}\dot{x}=-y+x\,g(x,y)\\ \dot{y}=x+y\,g(x,y).\end{cases}$$

a) (0.25p) Check that $(0,0)$ is an equilibrium point. There are other equilibrium points?

b) (1.5p) Using the linearization method, study the stability of the equilibrium point $(0,0)$. Discuss with respect to the values of $g(0,0)$.

c) (0,75p) Prove that any orbit (that does not correspond to an equilibrium point) rotates around $(0,0)$.

d) (1p) In the case that g takes the value 0 in any point of the unit circle, check that
$$\varphi\left(t,\tfrac{1}{\sqrt{2}},\tfrac{1}{\sqrt{2}}\right)=\left(\cos\left(t+\tfrac{\pi}{4}\right),\ \sin\left(t+\tfrac{\pi}{4}\right)\right),\quad \forall\, t\in\mathbb{R}.$$

<u>Solution</u> a)
$$\begin{cases}-y+x\,g(x,y)=0\\ x+y\,g(x,y)=0\end{cases}\iff\begin{cases}y=x\,g(x,y)\\ x+x[g(x,y)]^2=0\end{cases}\iff$$
$$\iff\begin{cases}y=x\,g(x,y)\\ x\,[\underbrace{1+[g(x,y)]^2}_{>0}]=0\end{cases}\iff x=y=0.$$

So, $\{(0,0)\}$ is the unique equilibrium point.

b) Let $f:\mathbb{R}^2\to\mathbb{R}^2$, $\quad f(x,y)=\begin{pmatrix}-y+x\,g(x,y)\\ x+y\,g(x,y)\end{pmatrix}$

g is C^1 $\Rightarrow$ f is C^1.

$$Jf(x,y) = \begin{pmatrix} g(x,y) + x\dfrac{\partial g}{\partial x}(x,y) & -1 + x\dfrac{\partial g}{\partial y}(x,y) \\ 1 + y\dfrac{\partial g}{\partial x}(x,y) & g(x,y) + y\dfrac{\partial g}{\partial y}(x,y) \end{pmatrix}$$

$$A := Jf(0,0) = \begin{pmatrix} g(0,0) & -1 \\ 1 & g(0,0) \end{pmatrix}$$ 0.25p

To determine the eigenvalues of A $\begin{vmatrix} g(0,0)-\lambda & -1 \\ 1 & g(0,0)-\lambda \end{vmatrix} = 0$

$\Leftrightarrow \quad [g(0,0)-\lambda]^2 + 1 = 0 \quad \Leftrightarrow \quad [g(0,0)-\lambda]^2 = -1 \quad \Leftrightarrow$

$\Leftrightarrow \quad g(0,0) - \lambda = \pm i \quad \Leftrightarrow \quad \lambda_{1,2} = g(0,0) \pm i$ 0.25p

0,25 First note that the eq. p. $(0,0)$ is hyperbolic iff $g(0,0) \neq 0$.

0,25 Thus, if $g(0,0) = 0$ the linearization method fails, and we can not sey anything about the stability of $(0,0)$.

0,25 If $g(0,0) < 0$ then, the ~~the~~ linearization method only assures that $(0,0)$ is an attractor for the given nonlinear system.

0,25 If $g(0,0) > 0$ then —"— repeller —"— .
(we applied the <u>second theorem</u> in Lecture 9)

Remark: If you wrote "global attractor" instead of "attractor" it is not correct.

c) It is sufficient if we prove that the angle θ of polar coordinates is strictly monotone (as function of time) along any orbit (except the orbit corresp. to the e.g.p $(0,0)$).

0,25

$$\tan\theta = \frac{y}{x} \Rightarrow \left.\begin{array}{l} \dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot y x - y\dot x}{x^2} \\ x = \rho\cos\theta \end{array}\right\} \Rightarrow$$

$$\Rightarrow \dot\theta = \frac{1}{\rho^2}\left[\dot y x - y\dot x\right] = \frac{1}{\rho^2}\left[x^2 + \cancel{xy\,g(x,y)} + y^2 - \cancel{xy\,g(x,y)}\right]$$

$$= \frac{\rho^2}{\rho^2} = 1 . \quad \text{So, } \dot\theta = 1 > 0 \Rightarrow$$

0,5 the angle $\theta(t)$ is strictly increasing along any orbit

d) We have ~~to~~ check that the given function satisfies

0,25

$$(*)\begin{cases} \dot x = -y + x\,g(x,y) \\ \dot y = x + y\,g(x,y) \\ x(0) = \frac{1}{\sqrt2} \\ y(0) = \frac{1}{\sqrt2} \end{cases}.$$

Denote $\tilde x(t) = \cos\left(t + \frac{\pi}{4}\right)$, $\tilde y(t) = \sin\left(t+\frac{\pi}{4}\right)$

we have $\tilde x(0) = \cos\frac{\pi}{4} = \frac{1}{\sqrt2}$

$\tilde y(0) = \sin\frac{\pi}{4} = \frac{1}{\sqrt2}$ 0,25

Also, we have that

$\tilde x(t)^2 + \tilde y(t)^2 = 1, \quad \forall t\in\mathbb{R} \Rightarrow$

$\Rightarrow (\tilde x(t), \tilde y(t)) \in$ the unit circle $\Rightarrow g(\tilde x(t), \tilde y(t)) = 0$, 0,25 $\forall t\in\mathbb{R}$.

In addition, $\tilde x'(t) = -\tilde y(t)$ and $\tilde y'(t) = \tilde x(t)$ $\forall t\in\mathbb{R}$. 0,25

Thus, $(\tilde x(t), \tilde y(t))$ indeed satisfies all the relations in $(*)$.

4. Let $a, b \in \mathbb{R}$ and $f(x) = ax^2 + bx + 1$ be such that $f(1) = 2$ and $f(2) = 1$. Study whether the discrete scalar dynamical system $x_{k+1} = f(x_k)$, $k \in \mathbb{N}$ has an attracting 2-periodic orbit.

Solution. First note that, since $f(1) = 2$ and $f(2) = 1$ we have the orbit that starts at $\eta = 1$ is $\{1, 2\}$, a 2-periodic orbit (also called 2-cycle). 0,5

Now we find a and b.

$$f(1) = 2 \iff a + b + 1 = 2 \iff \begin{cases} a + b = 1 \\ 2a + b = 0 \end{cases} \iff \begin{matrix} a = -1 \\ b = 2 \end{matrix} \quad 0{,}25$$

$$f(2) = 1 \iff 4a + 2b + 1 = 1$$

So, $f(x) = -x^2 + 2x + 1 \Rightarrow f'(x) = -2x + 2 \Rightarrow$

$f'(1) = 0$ and $f'(2) = -2 \Rightarrow |f'(1) \cdot f'(2)| = 0 < 1$

$\Rightarrow$ (~~[illegible]~~ using the Property in Lecture 14)
the 2-cycle $\{1, 2\}$ is an attractor. 0,5