LECTURE 2 - DYNAMICAL SYSTEMS

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Scalar linear differential equations

(1) $x^{(n)}+a_1(t)\cdot x^{(n-1)}+\dots+a_{n-1}(t)\cdot x'+a_n(t)x=f(t), t\in J\subset\mathbb{R},$

J open interval, $a_1,\dots,a_n,f\in C(J)$

The unknown $x: J\to\mathbb{R}$ - the set of scalars

$\mathcal{L}: C^n(J)\to C(J)$

$\mathcal{L}(x)(t)=x^{(n)}+a_1(t)\cdot x^{(n-1)}+\dots+a_n(t)\cdot x \quad, \forall t\in J, \forall x\in C^n(J)$

We proved that $\mathcal{L}$ is a linear map.

(1) $\Leftrightarrow \mathcal{L}(x)=f$

$\mathcal{L}(x)=0$ is a LHDE

$\ker\mathcal{L}$ - the set of all solutions of the LHDE;

- vector space.

Th 1 (The fundamental th. for LHDEs)

$\ker\mathcal{L}$ has dimension n.

Pf: $\mathbb{R}^n$ is a vector space of dim n. Fix $t_0\in J$

Let $T:\ker\mathcal{L}\to\mathbb{R}^n$ be defined by $T(\varphi)=\begin{pmatrix}\varphi(t_0)\\ \varphi'(t_0)\\ \vdots\\ \varphi^{(n-1)}(t_0)\end{pmatrix}, \forall \varphi\in\ker\mathcal{L}$

T bijective $\Leftrightarrow \forall \eta\in\mathbb{R}^n\ \exists!\ \varphi\in\ker\mathcal{L}$ s.t. $T(\varphi)=\eta \Leftrightarrow$

$\Leftrightarrow \forall \eta=\begin{pmatrix}\eta_1\\ \vdots\\ \eta_n\end{pmatrix}\in\mathbb{R}^n\ \exists!\ \varphi\in C^n(J)$ s.t. $\begin{cases}\mathcal{L}(\varphi)=0\\ \varphi(t_0)=\eta_1\\ \varphi'(t_0)=\eta_2\\ \vdots\\ \varphi^{(n-1)}(t_0)=\eta_n\end{cases}$

This is true by the $\exists!$ Theorem from Lecture 1.

T linear $\Leftrightarrow T(\alpha_1\varphi_1+\alpha_2\varphi_2)=\alpha_1T(\varphi_1)+\alpha_2T(\varphi_2), \forall \alpha_1,\alpha_2\in\mathbb{R}$
$\forall \varphi_1,\varphi_2\in\ker\mathcal{L}$

$T(\alpha_1\varphi_1+\alpha_2\varphi_2)=\begin{pmatrix}(\alpha_1\varphi_1+\alpha_2\varphi_2)(t_0)\\ (\alpha_1\varphi_1+\alpha_2\varphi_2)'(t_0)\\ \vdots\\ (\alpha_1\varphi_1+\alpha_2\varphi_2)^{(n-1)}(t_0)\end{pmatrix}=\alpha_1T(\varphi_1)+\alpha_2T(\varphi_2).\ \mathcal{Q}^{e.d}$

T bijective and T linear $\Rightarrow T$ is an isomorphism between $\ker\mathcal{L}$ and $\mathbb{R}^n\Rightarrow \dim(\ker\mathcal{L})=n$

The fundamental th. for Ln-HDE

Let $f \not\equiv 0$. The set of sol of $\mathcal{L}(x)=f$ is $\ker \mathcal{L} + \{x_p\}$, where x_p - particular sol of $\mathcal{L}(x)=f$.

In other words, the gen. sol. of eq. (1) is:

$$x = x_h + x_p, \quad x_h \text{ - general sol of } \mathcal{L}(x), \text{ which is called the LHDE assoc. to } \mathcal{L}(x)=f$$

Example:

Find the general sol. of $x'' + x = -3$.

This is a second order Ln-HDE. $x = x_h + x_p$

$x_h = ?$ $x'' + x = 0$

$x = \cos(t)$, $x' = -\sin(t)$, $x'' = -\cos(t) \Rightarrow x'' + x = 0, \forall t \in \mathbb{R}$

$x = \sin(t)$ is also a sol.

We have $\{\cos t, \sin t\}$ two linearly indep. sol. of $x'' + x = 0 \Rightarrow$

$\Rightarrow x_h = c_1 \cdot \cos t + c_2 \cdot \sin t$, $c_1, c_2 \in \mathbb{R}$

$x_p = -3 \Rightarrow (-3)'' + (-3) = -3$, true

$\Rightarrow x = c_1 \cos t + c_2 \sin t - 3$, $c_1, c_2 \in \mathbb{R}$.

Scalar first order LDE

(2) $x' + a(t)x = f(t), t \in J$ where $a, f \in C(J)$. Let $t_0 \in J$

Let $A(t) = -\int_{t_0}^{t} a(s)\,ds$. Then $A'(t) = -a(t)$, $A(t_0) = 0$

I. The integrating factor method to find the gen. sol of (2).

we have that $\mu(t) = e^{-A(t)}$ is an integrating factor

we multiply (2) with $\mu(t)$ and obtain:

$$x' \cdot e^{-A(t)} + a(t) \cdot x \cdot e^{-A(t)} = f(t) e^{-A(t)} \Leftrightarrow$$

$$\Leftrightarrow \left(x \cdot e^{-A(t)}\right)' = f(t) \cdot e^{-A(t)} \Leftrightarrow \quad x \cdot e^{-A(t)} = \int_{t_0}^{t} f(s) e^{-A(s)}\,ds + c \; \Big/ e^{A(t)} \Leftrightarrow$$

(integrate "t")

$$x = c \cdot e^{A(t)} + e^{A(t)} \cdot \int_{t_0}^{t} f(s) \cdot e^{-A(s)}\,ds, \; c \in \mathbb{R}$$

II. The separation of var. method to find the gen sol. $x' + a(t)x = 0$

we already know that its gen. sol is $x = c \cdot e^{A(t)}$, $c \in \mathbb{R}$

$x' = -a(t)x$.

Remark: any eq. $x' = \underbrace{a(t) \cdot g(x)}_{t \text{ and } x \text{ are separated}}$ is said to be a separable eq.

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$x = 0$ is a solution

$x(t) \neq 0, \forall t \in J$

$$\Rightarrow \underbrace{\frac{x'(t)}{x(t)}}_{(\ln|x(t)|)'} = -a(t) \Rightarrow (\ln|x(t)|)' = -a(t) \Leftrightarrow \ln|x(t)| = -\int_{t_0}^{t} a(s)\,ds + c \Leftrightarrow$$

$$\Leftrightarrow \ln|x(t)| = A(t) + c \Leftrightarrow |x(t)| = e^{A(t)+c} \Leftrightarrow$$

$$\Leftrightarrow x(t) = \pm e^{A(t)}\cdot e^{c} \Rightarrow \begin{cases} x(t) = \pm e^{c}\cdot e^{A(t)}, c \in \mathbb{R} \\ x = 0 \end{cases}$$

Remark $\{e^{c}, -e^{c}, 0;\ c \in \mathbb{R}\} = \mathbb{R} \Leftrightarrow x = c_1\cdot e^{A(t)}, c_1 \in \mathbb{R}$

III. The Lagrange method (the variation of the constant method)

- to find a particular sol. to a first order Ln-HDE $x' + a(t)x = f$.

The gen sol of the LHDE ass. $x' + a(t)x = 0$ is $x_h = c\cdot e^{A(t)}, c \in \mathbb{R}$

The idea of Lagrange was to look for x_p sol. of (2).

$x_p = \varphi(t)\cdot e^{A(t)}$, and to find the function $\varphi(t)$.

$$x_p \text{ sol. of (2)} \Leftrightarrow x_p' + a(t)\cdot x_p = f \Rightarrow \underbrace{\varphi'\cdot e^{A(t)} + \cancel{\varphi\cdot \underbrace{A'}_{-a(t)} e^{A(t)}}}_{x_p'} + \underbrace{\cancel{a(t)\cdot\varphi\cdot e^{A(t)}}}_{a(t)x_p} = f$$

$$\Leftrightarrow \varphi'(t)\cdot e^{A(t)} = f(t) \,/\cdot e^{-A(t)} \Leftrightarrow$$

$$\Leftrightarrow \varphi'(t) = f(t)\cdot e^{-A(t)} \Leftarrow \varphi(t) = \int_{t_0}^{t} f(s)\cdot e^{-A(s)}\,ds. \text{ Then } x_p = e^{A(t)}\cdot\int_{t_0}^{t} f(s)\cdot e^{-A(s)}\,ds$$

IV. Another method to find the gen. sol. of $x' + a(t)x = 0$

Step 1: check that $e^{A(t)}$ is a solution.

$$(e^{A(t)})' = A'(t)\cdot e^{A(t)} = -a(t)\cdot e^{A(t)} \Rightarrow$$

$$\Rightarrow (e^{A(t)})' + a(t)\cdot e^{A(t)} = 0, \forall t \in \mathbb{R}$$

Step 2: we use the fundamental th. for LHDE $\Rightarrow$

$$\Rightarrow x = c\cdot e^{A(t)}, c \in \mathbb{R}$$

Examples:

$x' = 2t\cdot x - t$ and $x' = 2t x - t^2$

Examples:

$x' - 2t\cdot x = t$ and $x' - 2t\cdot x = \frac{2}{\sqrt{\pi}}$

these are first order Ln-HDEs with the same homogeneous part

$\rightarrow$ find the gen sol. of $x' - 2tx = 0$

$x' = 2tx \qquad x' \longrightarrow \frac{dx}{dt}$

$\frac{dx}{dt} = 2tx$ separating x and $t \Rightarrow \int \frac{dx}{x} = \int 2t\,dt \Rightarrow$

$\Rightarrow \ln|x| = t^2 + c,\ c \in \mathbb{R} \Rightarrow |x| = e^c \cdot e^{t^2},\ c \in \mathbb{R} \Rightarrow$

$\Rightarrow x = \pm e^c \cdot e^{t^2},\ c \in \mathbb{R}$ $\Rightarrow \begin{cases} x = \pm e^c \cdot e^{t^2} \\ x = 0 \end{cases} \Rightarrow x_h = c_1 \cdot e^{t^2},\ c_1 \in \mathbb{R}$

$x = 0$ - solution

$x' - 2tx = t$

$x_p = ?$ we see that $x_p = -\frac{1}{2} \Rightarrow$ the gen sol is $x = c\cdot e^{t^2} - \frac{1}{2},\ c \in \mathbb{R}$

$x' - 2tx = \frac{2}{\sqrt{\pi}}$, apply Lagrange

$x_h = c\cdot e^{t^2},\ c \in \mathbb{R}$, $x_p = \varphi(t)\cdot e^{t^2}$, $\varphi = ?$

$x_p' - 2t x_p = \frac{2}{\sqrt{\pi}} \Leftrightarrow \varphi'(t)\cdot e^{t^2} + \cancel{\varphi(t)\cdot 2te^{t^2}} - \cancel{2t\,\varphi \cdot e^{t^2}} = \frac{2}{\sqrt{\pi}},\ t \in \mathbb{R}$

$\Leftrightarrow \varphi'(t)\cdot e^{t^2} = \frac{2}{\sqrt{\pi}} \Big/ \cdot e^{-t^2},\ t \in \mathbb{R} \Leftrightarrow \varphi'(t) = \frac{2}{\sqrt{\pi}} \cdot e^{-t^2} \Leftrightarrow$

$\Leftrightarrow \varphi(t) = \frac{2}{\sqrt{\pi}} \cdot \int_0^t e^{-s^2} ds = \operatorname{erf}(t)$, the error function

So, $x_p = \varphi(t)\cdot e^{t^2} \Rightarrow x_p = \operatorname{erf}(t)\cdot e^{t^2}$

$\Rightarrow$ the gen sol is $x = e^{t^2}(c + \operatorname{erf}(t)),\ c \in \mathbb{R}$