LECTURE 12

DATE : 13 DECEMBER 2021
WEEK 12

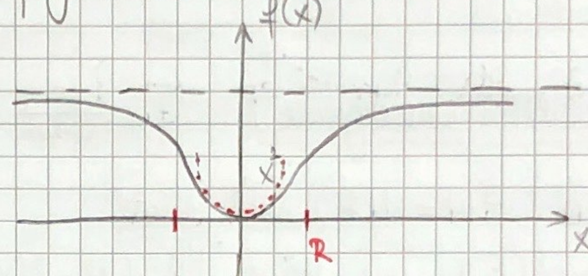
12. Sequences & series of functions

Motivation : approximation again !

Taylor $f(x) = T_n(x) + R_n(x)$

small reminder

Taylor polynomial



Ex : $f(x) = \frac{x^2}{1+x^2}$

$f(x) = x^2 + a_4 x^4 + a_6 x^6 + \dots$

Taylor around $x_0 = 0$

define a sequence of functions

$f_0(x) = 0$, $f_1(x) = 0$, $f_2(x) = x^2$, $f_3(x) = 0$, $f_4(x) = a_4 x^4$, etc

and $\sum_{n=0}^{N} f_n(x) = \sum_{n=0}^{N} a_n x^n \xrightarrow{N\to\infty} f(x)$

§ 12.1. Sequences of functions

$f, f_n : [a,b] \to \mathbb{R}$ $n = 1, 2, \dots$

Def [pointwise convergence] $f_n \xrightarrow[n\to\infty]{p.w} f$

For any fixed $x \in [a,b]$ we have $\lim_{n\to\infty} f_n(x) = f(x)$

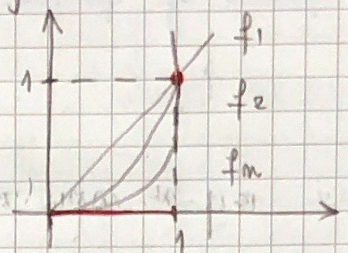
(i.e., $\forall \varepsilon > 0$ $\exists N = N(\varepsilon, x) \in \mathbb{N}^*$ such that $\forall n > N(\varepsilon, x)$ we have $|f_n(x) - f(x)| < \varepsilon$

"For every x you could have different N!")

Ex: $f, f_n : [0,1] \to \mathbb{R}$ $\quad f_n(x) = x^n$

are all cont.

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} =: f(x)$$



but notice that

$f(x)$ is discontinuous

This motivates:

Def [uniform convergence] $\quad f_n \xrightarrow[n\to\infty]{u} f$ if

$\forall \varepsilon > 0 \quad \exists N = N(\varepsilon)$ such that

$\forall n > N(\varepsilon)$ we have $|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [a,b]$

uniform with respect to x

Theorem 1 (Continuity) $\quad f_n : [a,b] \to \mathbb{R}$ all cont and

$f_n \xrightarrow[n\to a]{u} f$. Then f is continuous.

Theorem 2 (Integrability) $\quad f_n$ all continuous, $f_n \xrightarrow{u} f$

Then f integrable and $\lim\limits_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

Theorem 3 (differentiability) $\quad f_n$ all differentiable and

(i) $f_n \xrightarrow{p.w} f$

(ii) $f_n' \xrightarrow{u} g$

then f is differentiable and $f' = g$

§12.2. Power series

Series of functions $\sum_{n=1}^{\infty} f_n$ (notation)

$(f_n)_{n\in\mathbb{N}^*}$ a sequence of functions

$s_n(x) = f_1(x) + \dots + f_n(x)$, $(s_n)_{n\in\mathbb{N}^*}$

sequence of partial sums (again functions)

Def: $\sum_{n=1}^{\infty} f_n$ $<$ (p.w. CONV) $:\Leftrightarrow s_n \xrightarrow{p.w.} f$.
(u. CONV) $:\Leftrightarrow s_n \xrightarrow{u} f$

Recall : Approx (motivates notation)

Notation $\sum_{n=1}^{\infty} f_n \xrightarrow{p.w.} f$ or $\sum_{n=1}^{\infty} f_n \xrightarrow{u} f$

Theorem 4 (WEIERSTRASS)

$(f_n)_{n\in\mathbb{N}^*}$ sequence of functions , $f_n : [a,b] \to \mathbb{R}$,
$(a_n)_{n\in\mathbb{N}^*}$ sequence of positive reals and :

(i) $\sum_{n=1}^{\infty} a_n$ (CONV)

(ii) $|f_n(x)| \le a_n \quad \forall n \geq n_0 , \forall x \in [a,b]$

Then $\sum_{n=1}^{\infty} f_n$ (CONV)

Power series $\sum_{n=0}^{\infty} a_n x^n$

special interesting case of $\sum_{n=1}^{\infty} f_n$

Theorem 5 (ABEL I) $\sum_{n=0}^{\infty} a_n x^n$ power series

$\exists\, R \in [0,\infty]$ such that power series (u CONV) on $[0,R]$

radius of conv.

Proof (idea) : If $R=0$ nothing to prove

If $R>0$ such that $\sum_{n=0}^{\infty} a_n R^n$ (CONV)

if it holds, this implies $|a_n R^n| \leq M$, $\forall n \in \mathbb{N}$

For $|x|<R$ rewrite (forcing $\frac{1}{R}$)

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n R^n \left(\frac{x}{R}\right)^n$$

$$\leq \sum_{n=0}^{\infty} \underbrace{|a_n R^n|}_{\leq M} \left|\frac{x}{R}\right|^n$$

$$\leq M \sum_{n=0}^{\infty} \left|\frac{x}{R}\right|^n \quad \left(\left|\frac{x}{R}\right|<1\right)$$

and the desired result follows from (CONV) of the geometric series

So, a radius of convergence exist — How can we complete it?

Theorem 6 (J. HADAMARD) $\sum_{n=0}^{\infty} a_n x^n$ power series

If $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists, then $R = \dfrac{1}{\lim_{n\to\infty} \sqrt[n]{a_n}}$

(with $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$)

Proof based on Cauchy's root crit (which is part of the Additional material to see § 11.3)

Remark: If $\lim_{n\to\infty} \dfrac{a_n}{a_{n+1}}$ exists, then $R = \lim_{n\to\infty} \dfrac{a_n}{a_{n+1}}$

Theorem 7 (ABEL II) continuity of functions approx by power series

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ is continuous at $x = R$ if $\sum_{n=0}^{\infty} a_n R^n$ (CONV)

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generalizes the result in the 1930s

Theorem ((STONE-) WEIERSTRASS 1885)

Any cont. $f : [a, b] \to \mathbb{R}$ can be arbitrarily well approx (unif) by polynomial functions.

Weierstrass' Proof = NOT constructive

1912 S.N BERNSTEIN constructive proof based on Bernstein Polynomials

probability inspired

requires knowledge of values $f(x_i)$ at many points x_i