Universitatea Babeş–Bolyai
Facultatea de Matematică şi Informatică

Exam on Dynamical Systems, June 26, 2020

1. (1p=0.2+0.4+0.4) Denote by $\mathbb{R}^\infty$ the linear space of all sequences of real numbers $x = (x_k)_{k\geq 0}$, with the natural operations. Let $S \subset \mathbb{R}^\infty$ be the set of solutions of the difference equation $x_{k+2} - (\sin k)x_k = k$, $k \geq 0$. Let $T : S \to \mathbb{R}^2$ be defined by $T(x) = (x_0, x_1)$ for all $x \in S$. Justify that $S \neq \emptyset$ and that T is bijective. Is S a linear space of finite dimension? Justify.

2. (1p) Find the general solution of the differential equation $x'' + 2tx' = 0$.

3. Consider the planar system $\dot{x} = -y(y + x)$, $\dot{y} = x(y + x)$.
(a) (1.75p=0.25+0.5+0.25+0.75) Represent its phase portrait.
(b) (0.75p=0.5+0.25) Reading the phase portrait, find $\lim_{t\to\infty} \varphi(t, 3, 0)$, and $\lim_{t\to\infty} \varphi(t, 1, 0)$ (if they exist). Here, $\varphi(t, \eta_1, \eta_2)$ denotes, as usual, the flow of the system.
(c) (0.25p) Specify whether $\varphi(t, 3, 0)$ and $\varphi(t, 1, 0)$ are periodic functions.

4. (0.75p=0.25+0.25+0.25) Let $x(t)$ be the concentration of a radioactive substance at time t. We have that $\dot{x} = -0.05(x - 10)$. Find the flow associated to this differential equation. Find the time $T > 0$ in which the coffee cool down from 80° to 40°.

5. (a) (2p) Find the solution of the IVP $x_{k+1} = -x_k + 3y_k$, $\quad y_{k+1} = -3x_k - y_k$, $x_0 = 0, y_0 = 2$, reducing the system to a second order difference equation.
(b) (0.5p) Find $\lim_{n\to\infty} A^{-n} \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, where A is the matrix of the system from (a).

6. (1p=0.25+0.75) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 map such that $f(2) = -5$ and $f(-5) = 2$. Justify that $\{-5,\ 2\}$ is a 2-cycle for f. Prove that, if $|f'(-5)f'(2)| < 1$ then the 2-cycle $\{-5,\ 2\}$ is an attractor.