LECTURE 13 - DYNAMICAL SYSTEMS

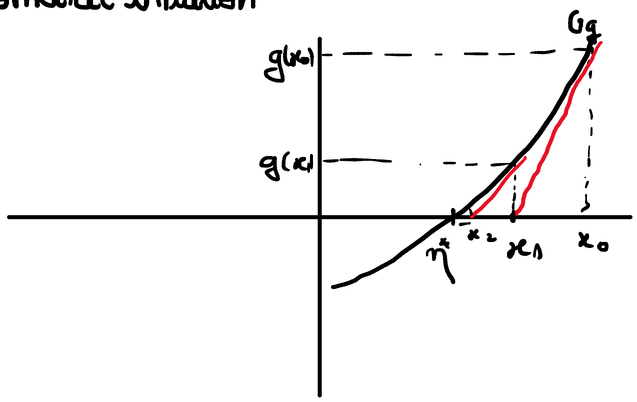
Monday, 24 May 2021 10:00

→ Discrete dynamical systems (cont)

I. The Newton - Raphson method to approximate the zeros of a scalar map

Let $g: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ nonempty, open interval. We assume that $\exists\, \eta^* \in I$ s.t. $g(\eta^*) = 0$ and $g'(\eta^*) \neq 0$.
We can't find the exact value of η^*, thus we need to approx. the values of η^*. More precisely, we need at least a seq: $(x_k)_{k\geq 0}$ s.t. $\lim_{k\to\infty} x_k = \eta^*$.

Geometric intuition



x_0

$(x_0, g(x_0)) \in G_g$

the tangent to G_g in $(x_0, g_0(x_0))$:

$$l_0: y - g(x_0) = g'(x_0)(x - x_0)$$

$l_0 \cap Ox$: $y = 0$

$$-g(x_0) = g'(x_0)(x_1 - x_0) \Rightarrow$$

$$\Rightarrow x_1 = \frac{-g(x_0)}{g'(x_0)} + x_0$$

We repeat:

$$(1)\begin{cases} x_{k+1} = x_k - \dfrac{g(x_k)}{g'(x_k)}, k \geq 0 \\ x_0 \text{ is given} \end{cases}$$

Theorem (Newton - Raphson method):

$g: V \to \mathbb{R}$, $g \in C^2(V)$, $V \subset \mathbb{R}$ open interval, $g'(x) \neq 0, \forall x \in V$. $\exists\, \eta^* \in V$ s.t. $g(\eta^*) = 0$.
There exists $\rho > 0$ s.t whenever $|x_0 - \eta^*| < \rho$ we have $\lim_{k\to\infty} x_k = \eta^*$, where $(x_k)_{k\geq 0}$ is given by (1).

Pf: Define $f: V \to \mathbb{R}$, $f(x) = x - \dfrac{g(x)}{g'(x)}$. Note that $f \in C^1(V)$

(1) $\Leftrightarrow$ $x_{k+1} = f(x_k)$, x_0 given

$$f(\eta^*) = \eta^* - \underbrace{\frac{g(\eta^*)}{g'(\eta^*)}}_{g(\eta^*)=0} = \eta^* \Rightarrow \eta^* \text{ is a fixed point of } f$$

The conclusion follows from the fact that η^* is an attracting fixed point of f. So, it remained to prove that η^* is an attractor. We use the Lin. Method:

$$f'(x) = 1 - \frac{g'(x)\cdot g'(x) - g(x)\cdot g''(x)}{(g'(x))^2}$$

$$f'(\eta^*) = 1 - (1 - 0) = 0$$

$\Rightarrow |f'(\eta^*)| < 1 \overset{LM}{\Longrightarrow} \eta^*$ is an attracting fixed point of f □

Example: (a particular case) $g: (0, \infty) \to \mathbb{R}$, $g(x) = x^2 - 3$. $\sqrt{3}$ is the unique zero-point of g, $g \in C^2(0,\infty)$
$g'(x) = 2x \neq 0, \forall x \in (0, \infty)$

By the prev. th. we know that $\exists\, \rho > 0$ s.t., whenever $|x_0 - \sqrt{3}| < \rho$ we have $\lim_{k\to\infty} x_k = \sqrt{3}$, where $(x_k)_{k\geq 1}$ is given by

$$(2)\ x_{k+1} = \frac{1}{2} x_k + \frac{3}{2}\cdot\frac{1}{x_k},\ x_0$$

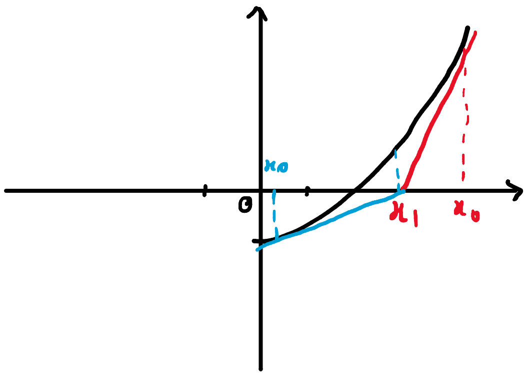
$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2-3}{2x} = x - \frac{1}{2}x + \frac{3}{2}x^{-1} = \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{x}$

Our aim now is to "find" the basin of attraction of $\sqrt{3}$.

I. We use the geom. interp. of the N-R diagram.

II. We use the stair-step diagram for f

I.



$g(x) = x^2 - 3$, $g:(0,\infty)\to\mathbb{R}$

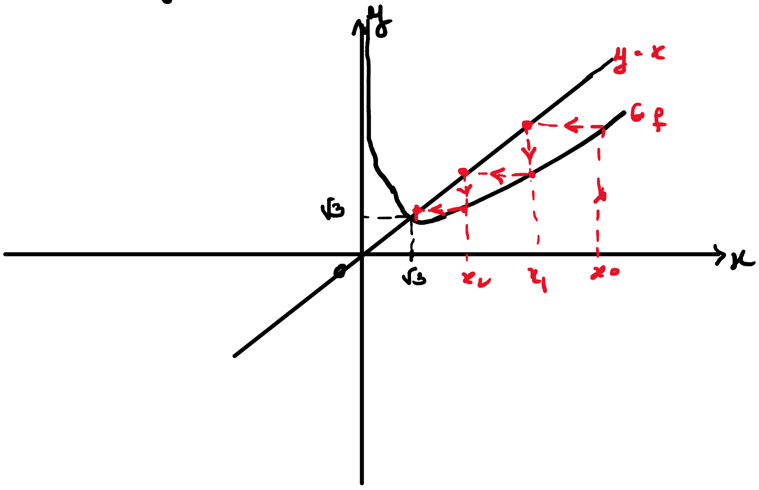
it seems that the seq. $(x_k)_{k\geq 1}$ converges to $\sqrt{3}$ for any $x_0\in(0,\infty)$

II. $f(x) = \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{x}$. We need G_f: $f(x) = x \Leftrightarrow \frac{1}{2}x + \frac{3}{2}x^{-1} = x \Leftrightarrow \frac{3}{2}\frac{1}{x} = \frac{1}{2}x \Leftrightarrow x^3 = 3 \Leftrightarrow \sqrt{3}$ is the only f.p. $x\in(0,\infty)$

$f(x) = \frac{x^2+3}{2x} > 0, \forall x\in(0,\infty)$; $f'(x) = \frac{1}{2} - \frac{3}{2}\cdot\frac{1}{x^2} = \frac{x^2-3}{2x^2}$

x	0		$\sqrt{3}$		∞
f'		$- - - - - -$	0	$+ + + + + + +$	
f	$+\infty$	$\searrow$	$\sqrt{3}$	$\nearrow$	$+\infty$

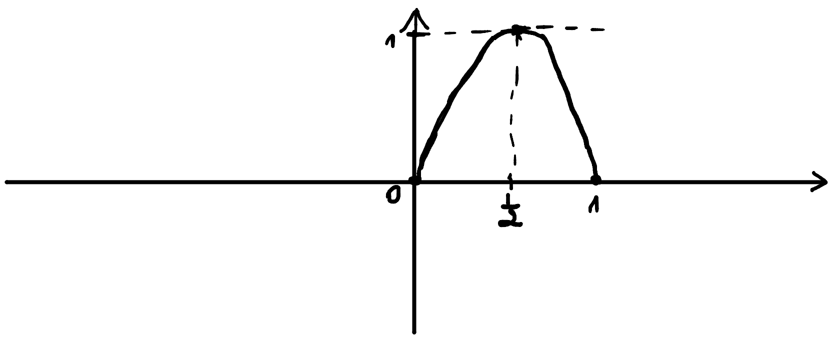
$y = \frac{1}{2}x$ is an asymptote at $+\infty$



$x_{k+1} = f(x_k)$
$x_1 = f(x_0)$
$x_2 = f(x_1)$

It seems that $A_{\sqrt{3}} = (0,\infty)$

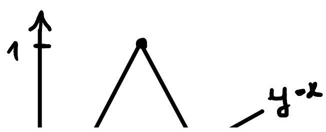
The tent map: in the law $f(x) = 4x(1-x)$ is "chaotic"



$f(\frac{1}{2}) = 4\cdot\frac{1}{2}\cdot\frac{1}{2} = 1$

$x\in[0,1] \Rightarrow f(x)\in[0,1]$

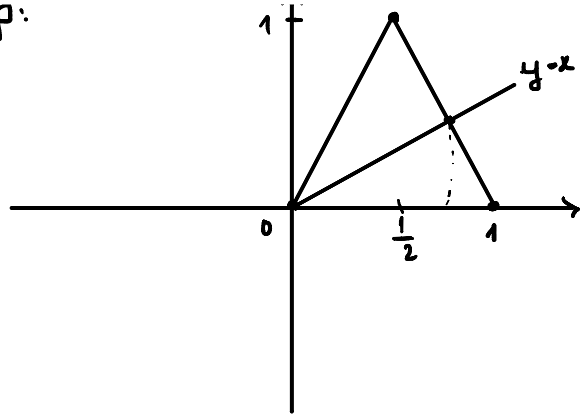
The tent map:



$T:[0,1]\to\mathbb{R}$

$T(x) = 1 - |2x-1| = \begin{cases} 2x, & x\in[0,\frac{1}{2}] \\ 2(1-x), & x\in[\frac{1}{2},1] \end{cases}$, $\forall x\in[0,1]$

The tent map:



$T:[0,1]\to\mathbb{R}$

$T(x)=1-|2x-1|=\begin{cases}2x, & x\in[0,\frac{1}{2}]\\ 2(1-x), & x\in[\frac{1}{2},1]\end{cases}, \forall x\in[0,1]$

a) Rep. G_T. Find the fixed points.

b) Comp. the orbits corresp. to: $\eta=\frac{3}{2^n}, n\geq 2$

c) Solve: $T(x)=0, T(x)=1, T(x)=\frac{1}{2}, T^2(x)=0, T^2(x)=1, T^2(x)=\frac{1}{2}$

d) Rep. G_{T^2} and G_{T^3}. How many fixed points they have?

e) T has a 2-periodic orbit? T has a 3-periodic orbit?

There is a result: if a scalar map has a 3-periodic orbit, then it has a p-per. orbit for any $p\in\mathbb{N}^*$.

a) $2(1-x)=x \Leftrightarrow 2-2x=x \Leftrightarrow x=\frac{2}{3}$

T has 2 f.p: $\eta_1^*=0, \eta_2^*=\frac{2}{3}$

b) $\eta=\frac{3}{2^n}, n\geq 2$

$n=2, \eta=\frac{3}{4}$ $T(\eta)=T(\frac{3}{4})=2(1-\frac{3}{4})=2\cdot\frac{1}{4}=\frac{1}{2}$

$T^2(\eta)=T^2(\frac{3}{4})=T(\frac{1}{2})=1$

$T^3(\eta)=T^3(\frac{3}{4})=T(1)=0$

$T^4(\eta)=T(0)=0, 0\ldots$

$\gamma^+_{\frac{3}{4}}=\{\frac{3}{4},\frac{1}{2},1,0,0,\ldots\}$

$n=3, \eta=\frac{3}{8}$ $T(\frac{3}{8})=2\cdot\frac{3}{8}=\frac{3}{4}$

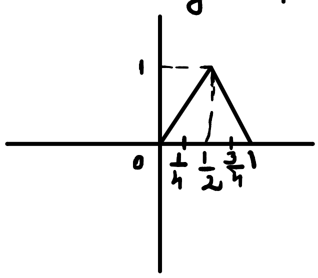
$\gamma^+_{\frac{3}{8}}=\{\frac{3}{8},\frac{3}{4},\frac{1}{2},1,0,0,\ldots\}$

$n\geq 3, T(\underbrace{\frac{3}{2^n}}_{<\frac{1}{2}})=2\cdot\frac{3}{2^n}=\frac{3}{2^{n-1}}$

$\gamma^+_{\frac{3}{2^n}}=\{\frac{3}{2^n},\frac{3}{2^{n-1}},\ldots,\frac{3}{8},\frac{3}{4},\frac{1}{2},1,0,\ldots\}$

These orbits are eventually the fixed point 0.

c) $T(x)=0$ $\Leftrightarrow x\in\{0,1\}$



$T(x)=1 \iff x=\frac{1}{2}$

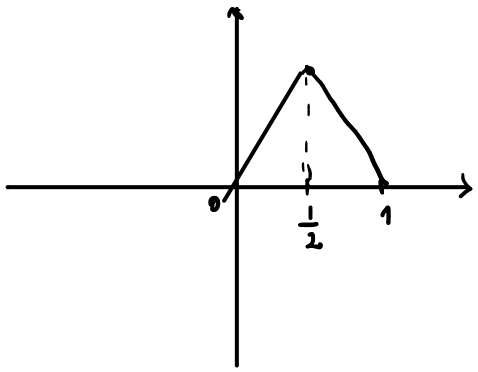
$T(x)=\frac{1}{2} \iff x\in\{\frac{1}{4},\frac{3}{4}\}$

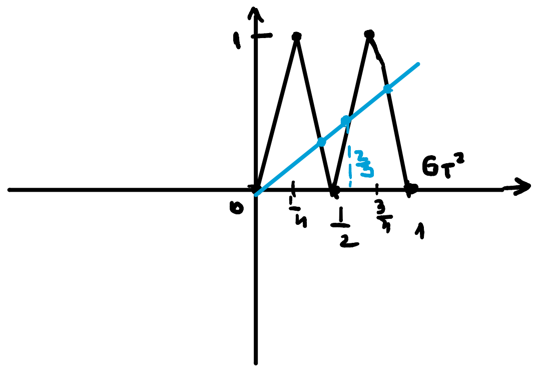
$T^2(x)=0 \iff T(T(x))=0 \iff T(x)\in\{0,1\} \Rightarrow x\in\{0,1,\frac{1}{2}\}$

$T^2(x)=1 \iff T(T(x))=1 \iff T(x)=\frac{1}{2} \Rightarrow x\in\{\frac{1}{4},\frac{3}{4}\}$

$T^2(x)=\frac{1}{2} \iff T(T(x))=\frac{1}{2} \iff T(x)\in\{\frac{1}{4},\frac{3}{4}\} \Rightarrow x\in\{\frac{1}{8},\frac{4}{8},\frac{3}{8},\frac{5}{8}\}$

d) $T^2(x)=T(T(x))=1-|2T(x)-1|=1-|2[1-|2x-1|]-1|,\ \forall x\in[0,1]$





T^2 has 4 fixed points, two of them are the fixed points of 0 and $\frac{2}{3}$

$\frac{2}{3}<\frac{3}{4}$

and the other 2 are $\eta_1\in(\frac{1}{4},\frac{1}{2}),\ \eta_2\in(\frac{3}{4},1)$

So, η_1 and η_2 are 2-periodic points of T, thus $\{\eta_1,\eta_2\}$ is a 2-cycle.

It can be proved that T^3 has 8 fixed points. Two of them are the fixed points of T and the other 6 are, in fact, two 3-cycles.