PART II : INTEGRAL CALCULUS

LECTURE 6 : ANTIDERIVATIVES & RIEMANN INTEGRAL

1. ANTIDERIVATIVES DEFINITIONS + THEOREM :

DEFINITION: $f : Y \subset \mathbb{R} \to \mathbb{R}$. F is called an antiderivative of f if $F' = f$.

The set of all antiderivatives of f is called integral of f and is denoted by: $\int f(x)\,dx = F(x) + \mathcal{C}$

THEOREM : If f is continuous on Y then f has antiderivatives.

2. RIEMANN INTEGRAL DEFINITION + THEOREMS :

DEFINITION : $f : [a,b] \to \mathbb{R}$ is called Riemann integrable if $\exists Y \in \mathbb{R}$ such that $\forall \varepsilon > 0$ $\exists \delta(\varepsilon) > 0$ with the property that for any division Δ of $[a,b]$ with $\max|x_k - x_{k-1}| < \delta$ and any ξ we have :

$$\left| Y - \sum_{k=1}^{m} f(\xi)(x_k - x_{k-1}) \right| < \varepsilon$$

THEOREM : If f is continuous then f is Riemann integrable.

LEIBNIZ - NEWTON THEOREM : If f integrable and admits antiderivatives then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

THEOREM : If f continuous then $F(x) = \int_a^x f(t)\,dt$.

PROPERTIES :

a) f, g integrable $\Rightarrow \alpha f + \beta g$ integrable : $\int(\alpha f + \beta g) = \alpha \int f + \beta \int g$

b) $f(x) \le g(x), \forall x \in [a,b] \Rightarrow \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

c) f integrable $\Rightarrow |f|$ integrable : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$

d) f continuous, g integrable $\Rightarrow \exists c$ such that $\int_a^b f(x) g(x)\,dx = f(c) \int_a^b g(x)\,dx$

e) integration by parts : $\int f(x) g'(x)\,dx = f(x)g(x) - \int f'(x) g(x)\,dx$

f) change of variables : $\int_a^b f(u(t))\,u'(t)\,dt = \int_{u(a)}^{u(b)} f(x)\,dx$

LECTURE 7: MEASURABLE SETS AND THE MULTIPLE INTEGRAL

1. JORDAN MEASURABILITY:

$B = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_d, b_d]$ → "box" in dimension d

$\text{int } B = (a_1, b_1) \times (a_2, b_2) \times \dots \times (a_d, b_d)$ → interior of B

$v(B) = (b_1 - a_1)(b_2 - a_2) \dots (b_d - a_d)$ → "volume" of the "box"

DEFINITION: We call a set $A \in \mathbb{R}^d$ elementary if it is a finite version of nonoverlapping boxes: $A = \bigcup_{i=1}^{N} B_i$, B_i = box, $\text{int } B_i \cap \text{int } B_j = \emptyset$ $\forall i \neq j$. For such A, we can define $v(A) = \sum_{i=1}^{N} v(B_i)$.

Now we consider a bounded set $D \subseteq \mathbb{R}^n$:

$$m_i(D) = \sup \{ v(A) : A \text{ elementary and } A \subseteq D \}$$
$$m_e(D) = \inf \{ v(A) : A \text{ elementary and } D \subseteq A \}$$

DEFINITION: A bounded set $D \subseteq \mathbb{R}^d$ is Jordan measurable if $m_i(D) = m_e(D)$ (inner and outer approximation coincide). The common value will be denoted by $m(D)$ and is called the Jordan measure of D.

LENGTH – AREA – VOLUME – MEASURE
$d=1$ $\quad d=2$ $\quad d=3$ $\quad d$ - arbitrary

2. THE MULTIPLE INTEGRAL IN THE SENSE OF RIEMANN:

DEF: $D \subset \mathbb{R}^d$ is bounded and Jordan measurable.

$\Delta = \{D_1, \dots, D_m\}$ is a partition of D if:

a) $\text{int } D_i \cap \text{int } D_j = \emptyset$, $i \neq j$

b) $D = D_1 \cup D_2 \cup \dots \cup D_m$

$$\|\Delta\| = \max_{1 \le i \le m} \underbrace{\delta(D_i)}_{\text{diameter of } D_i} = \max_{1 \le i \le m} \left(\sup \{ \|x-y\| : x, y \in D_i \} \right)$$

DEFINITION: $f : D \to \mathbb{R}$, $\sigma(f, \Delta, \xi) = \sum_{i=1}^{m} f(\xi_i) \, m(D_i)$ is called the Riemann sum (Jordan measure)

DEFINITION: f is Riemann integrable if the Riemann sum converges to some value $Y \in \mathbb{R}$ (as $m \to \infty$ and $\max \delta(D_i) \to 0$).

The limit $Y = \int_D f(x)\,dx$ or $\iiint \dots \int f(x_1, x_2, \dots, x_d)\, dx_1 \dots dx_d$

LECTURE 8 : COMPUTATION OF MULTIPLE INTEGRAL.

1. FUBINI THEOREM (only for exercises) :

$f: A\times B \to \mathbb{R}$ integrable, A, B bounded and Jordan measurable.

Then : $$\iint_{A\times B} f(x,y)\,dx\,dy = \int_A \Big(\int_B f(x,y)\,dy\Big)dx = \int_B\Big(\int_A f(x,y)\,dx\Big)dy$$

EXAMPLE : $D = [0,1]\times[1,2]$

$$\iint_D xy\,dx\,dy = \int_0^1\Big(\int_1^2 xy\,dy\Big)dx = \Big(\int_0^1 x\,dx\Big)\Big(\int_1^2 y\,dy\Big) = \frac{x^2}{2}\Big|_0^1 \cdot \frac{y^2}{2}\Big|_1^2 = \frac{3}{4}$$

2. DOUBLE INTEGRALS (only for exercises) :

① $$\iint_D f(x,y)\,dx\,dy = \int_a^b\Big(\int_{\varphi(x)}^{\gamma(x)} f(x,y)\,dy\Big)dx$$

EXAMPLE : D : $y = x$, $y = x^2$, $\varphi(x) = x^2$, $\gamma(x) = x$

$$\mathcal{Y} = \iint_D (x+y)\,dx\,dy = \int_0^1\Big(\int_{x^2}^{x}(x+y)\,dy\Big)dx = \int_0^1\Big(xy + \frac{y^2}{2}\Big)\Big|_{y=x^2}^{y=x} dx = \dots$$

② $$\iint_D f(x,y)\,dx\,dy = \int_c^d\Big(\int_{\lambda(y)}^{\mu(y)} f(x,y)\,dx\Big)dy$$

EXAMPLE : D : $x = y$, $x = y^2$, $\lambda(y) = y^2$, $\mu(y) = y$

$$\mathcal{Y} = \iint_D f(x,y)\,dx\,dy = \int_0^1\Big(\int_{y^2}^{y} f(x,y)\,dx\Big)dy = \dots$$

3. JACOBI MATRIX :

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) & \dots & \frac{\partial f_1}{\partial x_d}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) & \dots & \frac{\partial f_2}{\partial x_d}(x) \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) & \dots & \frac{\partial f_m}{\partial x_d}(x) \end{bmatrix} = \begin{bmatrix} [\nabla f_1(x)] \\ [\nabla f_2(x)] \\ \vdots \\ [\nabla f_d(x)] \end{bmatrix}$$

THEOREM: (change of variables in the multi integral)

Let $\Delta, D \subset \mathbb{R}^d$ bounded, closed, measurable

$M \subset \Delta$ of measure zero ($m(M) = 0$) and $u : \Delta \to D$ with continuous differentiable components and:

1) u is injective on $\Delta \setminus M$
2) u is regular on $\Delta \setminus M$

Then, if $f : D \to \mathbb{R}$ is integrable over D we have:

$$\int_D f(x)\,dx = \int_\Delta f(u(z)) \left| \det J_f(z) \right| dz$$

4. POLAR COORDINATES:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \Rightarrow \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \operatorname{arctg}\frac{y}{x} \end{cases} \qquad r \geq 0,\ \varphi \in [0, 2\pi)$$

$u : \Delta \to D \quad , \quad u(r, \varphi) = (r\cos\varphi, r\sin\varphi)$

$$J_u(r,\varphi) = \begin{pmatrix} \frac{\partial}{\partial r}(r\cos\varphi) & \frac{\partial}{\partial \varphi}(r\cos\varphi) \\ \frac{\partial}{\partial r}(r\sin\varphi) & \frac{\partial}{\partial \varphi}(r\sin\varphi) \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$\det J_u(r,\varphi) = r\cos^2\varphi + r\sin^2\varphi = r \neq 0$$

$$\Rightarrow \iint_D f(x,y)\,dx\,dy = \iint_\Delta f(r\cos t, r\sin t)\cdot r\,dr\,dt$$

LECTURE 9 : THE RIEMANN INTEGRAL : RELATED TOPICS

1. TRAPEZOIDAL RULE :

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left(f(a)+f(b)\right)$$

$$\text{error} = \left| \int_a^b f(x)\,dx - \frac{b-a}{2}\left(f(a)+f(b)\right)\right| = \frac{(b-a)^3}{12}\left|f''(\xi)\right| \quad \text{for some } \xi \in (a,b)$$

- if $(b-a) \gg 1 \Rightarrow$ approximation is bad
- if $(b-a) \ll 1 \Rightarrow$ approximation is good

2. THE COMPOSITE TRAPEZOIDAL RULE :

idea : - Subdivide $[a,b]$ into m equal subintervals $x_i - x_{i-1} = \frac{b-a}{m}$

- Use trapezoidal rule on each $[x_{i-1}, x_i]$

$$\int_a^b f(x)\,dx = \sum_{i=1}^{m} \int_{x_{i-1}}^{x_i} f(x)\,dx \approx \sum_{i=1}^{m} \frac{b-a}{m}\cdot\frac{1}{2}\left(f(x_{i-1})+f(x_i)\right) = \frac{b-a}{m}\left(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{m-1} f(x_i)\right)$$

$$\text{error} = \left|\int_a^b f(x)\,dx - \sum_{i=1}^{m}\frac{b-a}{m}\cdot\frac{1}{2}\left(f(x_{i-1})+f(x_i)\right)\right| = \frac{b-a}{12}\cdot\frac{(b-a)^2}{m^2}\,f''(\xi)$$

LECTURE 10: EXTENSIONS OF THE RIEMANN INTEGRAL

1. IMPROPER INTEGRALS:

DEFINITION: $f:[a,b)\to\mathbb{R}$ ($b=\infty$ admitted), f integrable (Riemann) on any $[a,b]$, $b<b$ (locally integrable).

We call the improper $\int_a^b f(x)\,dx$ convergent (CONV) if:

$$\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx \ \exists<\infty$$

We call the improper $\int_a^b f(x)\,dx$ divergent (DIV) if:

$$\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx$$ does not exist or is equal to ∞

CAUCHY THEOREM: $f:[a,b)\to\mathbb{R}$ locally integrable.

$\int_a^b f(x)\,dx$ (CONV) $\Leftrightarrow \forall\, \varepsilon>0\ \exists\, b_\varepsilon<b$ such that $\forall\, t\in(b_\varepsilon,b): \left|\int_{b_\varepsilon}^t f(x)\,dx\right|<\varepsilon$

Remark: $\int_a^b |f(x)|\,dx$ (CONV) $\Rightarrow \int_a^b f(x)\,dx$ (CONV)

2. TESTING THE CONVERGENCE OF IMPROPER INTEGRALS:

COMPARISON I THEOREM:

$f,g:[a,b)\to\mathbb{R}$ locally integrable and $0\le f(x)\le g(x)\ \forall x\in[a,b)$

Then:

1) $\int_a^b g(x)\,dx$ (CONV) $\Rightarrow \int_a^b f(x)\,dx$ (CONV)

2) $\int_a^b f(x)\,dx$ (DIV) $\Rightarrow \int_a^b g(x)\,dx$ (DIV)

COMPARISON II THEOREM:

$g(x)>0$ on $[a,b)$ and $\lim_{x\to b}\frac{f(x)}{g(x)}=L\in\mathbb{R}\ (<\infty)$.

Then:

1) if $L\neq 0$ then $\int f$ and $\int g$ are both (DIV) or both (CONV)

2) if $L=0$ then $\int g$ (CONV) $\Rightarrow \int|f|$ (CONV)
absolute (CONV)

Remark: Usually, you compare with x^α:

$\int_1^\infty x^\alpha\,dx$: (CONV) for $\alpha<-1$; (DIV) for $\alpha\geq -1$

$\int_0^1 x^\beta\,dx$: (CONV) for $\beta>-1$; (DIV) for $\beta\leq -1$

3. IMPROPER INTEGRALS WITH PARAMETER :

$f:[a,b)\times[c,d]\to\mathbb{R}$

improper

$F:[c,d]\to\mathbb{R}$

with parameter

$$F(y)=\int_a^b f(x,y)\,dx$$

DEFINITION: The improper integral with parameter $\int_a^b f(x,y)\,dx$ converges uniformly to F (w.r.t. y) if:

$\varepsilon>0\ \exists b_\varepsilon<b : \forall t\in(b_\varepsilon,b)$ and $\forall y\in[c,d]$ we have

$\left|\int_{b_\varepsilon}^{t} f(x,y)\,dx - F(y)\right|<\varepsilon$ and $b_\varepsilon \neq b(y)$

independent of y