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Chapter 4
The dynamical system generated by a differential equation[1]

We consider differential equations of the form

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given C^1 function, the unknown is a function x of variable t (from time), $\dot{x}$ is the Newton's notation for the derivative with respect to time. Equation (1) is said to be *autonomous* because the function f does not depend on t. In this lecture we define important concepts that, all together, define what is called *the dynamical system generated by* (1), such as: *the state space, the flow, the orbits, the phase portrait.*

A very important result is the following *existence and uniqueness theorem.*

Theorem 1 *Let $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $\eta \in \mathbb{R}^n$. Then the Initial Value Problem*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= \eta \end{aligned} \tag{2}$$

has a unique solution defined on an open (maximal) interval $I_\eta = (\alpha_\eta, \omega_\eta) \subset \mathbb{R}$, which, of course, is such that $0 \in I_\eta$. Denote this solution by $\varphi(\cdot, \eta)$.

If $\varphi(\cdot, \eta)$ is bounded then $I_\eta = \mathbb{R}$.

If $\varphi(\cdot, \eta)$ is bounded to the right then $\omega_\eta = \infty$.

If $\varphi(\cdot, \eta)$ is bounded to the left then $\alpha_\eta = -\infty$.

The map φ of two variables t and η defined in the previous theorem is called *the flow of the dynamical system generated by equation* (1). Some important properties of this map are

(i) $\varphi(0, \eta) = \eta$;

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(ii) $\varphi(t+s,\eta) = \varphi(t,\varphi(s,\eta))$ for each t and s when the map on either side is defined;

(iii) φ is continuous with respect to η.

It is easy to see that (i) holds true. In order to prove (ii) let us consider s and η be fixed and x_1, x_2 be two functions given by

$$x_1(t) = \varphi(t+s,\eta) \quad \text{and} \quad x_2(t) = \varphi(t,\varphi(s,\eta)).$$

By the definition of the flow, x_2 is a solution of the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= \varphi(s,\eta). \end{aligned} \tag{3}$$

In the same time we have that $x_1(0) = \varphi(s,\eta)$ and $\dot{x}_1(t) = \dfrac{d}{dt}(\varphi(t+s,\eta)) = \dot{\varphi}(t+s,\eta) = f(\varphi(t+s,\eta)) = f(x_1(t))$. Hence, x_1 is also a solution of the IVP (3). As a consequence of Theorem 1, this IVP has a unique solution, thus the two functions x_1 and x_2 must be equal. The proof of (iii) is beyond the aim of these lectures.

When working with the flow, η it is said to be *the initial state* of the dynamical system generated by equation (1), while $\varphi(t,\eta)$ is said to be the *state at time* t. According to these, the space $\mathbb{R}^n$ to which belong the states it is called *the state space* of the dynamical system generated by (1). It is also called *the phase space*.

We say that $\eta^* \in \mathbb{R}^n$ is an *equilibrium state*/point (or critical point, or stationary point or steady-state solution) of the dynamical system generated by (1) when

$$\varphi(t,\eta^*) = \eta^* \quad \text{for any} \quad t \in \mathbb{R}.$$

It is important to notice that the equilibria of (1) can be found solving in $\mathbb{R}^n$ the equation

$$f(x) = 0.$$

The *orbit* of the initial state η is

$$\gamma(\eta) = \{\ \varphi(t,\eta)\ :\ t \in I_\eta\ \}.$$

The *positive orbit* of the initial state η is

$$\gamma^+(\eta) = \{\ \varphi(t, \eta)\ :\ t \in I_\eta,\ t > 0\ \}.$$

The *negative orbit* of the initial state η is

$$\gamma^-(\eta) = \{\ \varphi(t, \eta)\ :\ t \in I_\eta,\ t < 0\ \}.$$

Note that an orbit is a curve in the state space $\mathbb{R}^n$ parameterized by the time t. Also, note that the orbit of an equilibrium point is formed only by this point, that is, $\gamma(\eta^*) = \{\ \eta^*\ \}$ when η^* is an equilibrium.

Let $\eta^* \in \mathbb{R}^n$ be an equilibrium point. We say that η^* is an *attractor* if there exists $r > 0$ such that for all η that satisfies $||\eta - \eta^*|| < r$ we have $\lim\limits_{t\to\infty} ||\varphi(t, \eta) - \eta^*|| = 0$. Here $||\cdot||$ is a norm in $\mathbb{R}^n$.

Let $\eta^* \in \mathbb{R}^n$ be an equilibrium point which is an attractor. We define its *basin of attraction* as

$$A_{\eta^*} = \{\eta \in \mathbb{R}^n\ :\ \lim_{t\to\infty} ||\varphi(t, \eta) - \eta^*|| = 0\}.$$

When $A_{\eta^*} = \mathbb{R}^n$ we say that η^* is a *global attractor.*

Let $\Omega \subset \mathbb{R}^n$ be nonempty. We say that Ω is an invariant set if, for all $\eta \in \Omega$ we have that $\gamma_\eta \subset \Omega$. Note that, in particular, a single orbit is an invariant set.

Let Ω be an invariant set. We say that Ω is an *attractor* if, for all $\tilde{\eta} \in \Omega$ there exists $r > 0$ such that for all $\eta \in \mathbb{R}^n$ satisfying $||\eta - \tilde{\eta}|| < r$ we have $\lim\limits_{t\to\infty} \mathrm{dist}(\varphi(t, \eta), \Omega) = 0$.

The phase portrait of the dynamical system (1) is the representation in the state space $\mathbb{R}^n$ of some representative orbits, together with some arrows that indicate the evolution in time.

The particular case $n = 1$. In this case the state space is $\mathbb{R}$ and we say that equation (1) is scalar. We start with an example.

Example 1. Study the dynamical system generated by the scalar equation

$$\dot{x} = -x.$$

The state space is $\mathbb{R}$. The flow is

$$\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \varphi(t, \eta) = \eta e^{-t}.$$

There is a unique equilibrium point, $\eta^* = 0$ whose orbit is

$$\gamma(0) = \{0\}.$$

For any $\eta > 0$ we have $\quad \gamma(\eta) = \{\, \eta e^{-t} \ : \ t \in \mathbb{R}\,\} = (0, \infty)$,
$\gamma^+(\eta) = \{\, \eta e^{-t} \ : \ t > 0\,\} = (0, \eta), \quad \gamma^-(\eta) = \{\, \eta e^{-t} \ : \ t < 0\,\} = (\eta, \infty)$.
For any $\eta < 0$ we have $\quad \gamma(\eta) = (-\infty, 0), \quad \gamma^+(\eta) = (\eta, 0), \quad \gamma^-(\eta) = (-\infty, \eta)$.
Hence, the orbits are: $(-\infty, 0), \quad \{0\}, \quad (0, \infty)$. The arrows must indicate that the states of the dynamical system evolves toward the equilibrium point 0. ⋄

We have the following result.

Lemma 1 *Let $f \in C^1(\mathbb{R})$ and φ be the flow of $\dot{x} = f(x)$. Let $\eta, \xi \in \mathbb{R}$ be fixed. Then*

(i) $\gamma(\eta) \subset \mathbb{R}$ is an open interval and $\varphi(\cdot, \eta)$ is a strictly monotone function for each η which is not an equilibrium;

(ii) $\varphi(t, \eta) < \varphi(t, \xi)$ for all t, if $\eta < \xi$;

(iii) if $\gamma^+(\eta)$ is bounded, then $\lim\limits_{t\to\infty} \varphi(t, \eta) = \eta^*$, where η^* is an equilibrium point;

(iv) if $\gamma^-(\eta)$ is bounded, then $\lim\limits_{t\to-\infty} \varphi(t, \eta) = \eta^*$, where η^* is an equilibrium point.

Proof. (i) We prove first the second statement. Since η is not an equilibrium point we have that either $f(\eta) > 0$ or $f(\eta) < 0$. We consider the case when $f(\eta) > 0$ (the other case is similar).
Then we have that $\dfrac{d}{dt}\varphi(0, \eta) = f(\eta) > 0$. We assume by contradiction that there exists t_1 such that $\dfrac{d}{dt}\varphi(t_1, \eta) \leq 0$. We denote $\eta_1 = \varphi(t_1, \eta)$. Since $f(\eta) > 0$ and $f(\eta_1) \leq 0$, it follows that there exists η^* between η and η_1, such that $f(\eta^*) = 0$. But the function $\varphi(\cdot, \eta)$ is continuous on the open interval $(0, t_1)$, or $(t_1, 0)$. Hence,

it takes all the values between η and η_1. This means that there exists t_2 such that $\varphi(t_2, \eta) = \eta^*$. We consider now the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_2) &= \eta^* \end{aligned}$$

and see that it has two solutions: $\varphi(\cdot, \eta)$ and the constant function η^*. This fact contradicts the unicity property.

The first statement follows by the fact that $\gamma(\eta)$ is the image of the continuous and strictly monotone function $\varphi(\cdot, \eta)$ which, by Theorem 1, is defined on an open interval.

(ii) In these hypotheses we have that $\varphi(0, \eta) - \varphi(0, \xi) < 0$. Assume by contradiction that there exists t_1 such that $\varphi(t_1, \eta) - \varphi(t_1, \xi) \geq 0$. From here, using the continuity of the function $\varphi(t, \eta) - \varphi(t, \xi)$, we deduce that there exists t_2 such that $\varphi(t_2, \eta) - \varphi(t_2, \xi) = 0$. We consider now the IVP

$$\begin{aligned} \dot{x} &= f(x) \\ x(t_2) &= \varphi(t_2, \eta) \end{aligned}$$

and see that it has two different solutions: $\varphi(t, \eta)$ and $\varphi(t, \xi)$. This fact contradicts the unicity property.

(iii) The function $\varphi(t, \eta)$ is a solution of $\dot{x} = f(x)$, hence

$$\frac{d\varphi}{dt}(t, \eta) = f(\varphi(t, \eta)). \tag{4}$$

Since, in addition, the C^1 function $\varphi(t, \eta)$ is monotone and bounded as t goes to ∞, we deduce that there exists some $\eta^* \in \mathbb{R}$ such that

$$\lim_{t \to \infty} \varphi(t, \eta) = \eta^* \quad \text{and} \quad \lim_{t \to \infty} \frac{d\varphi}{dt}(t, \eta) = 0. \tag{5}$$

Passing to the limit as $t \to \infty$ in (4) and taking into account equations (5), we obtain that

$$0 = f(\eta^*),$$

which means that η^* must be an equilibrium point. The proof of (iv) is similar. $\square$

As a consequence of the above result we give the following procedure useful to represent the phase portrait of any scalar dynamical system $\dot{x} = f(x)$.

Step 1. Find all the equilibria, i.e. solve $f(x) = 0$.

Step 2. Represent the equilibria on the state space, $\mathbb{R}$. The orbits are the ones corresponding to the equilibria and the open intervals of $\mathbb{R}$ delimited by the equilibria.

Step 3. Determine the sign of f on each orbit. According to this sign, insert an arrow on each orbit. If the sign is $+$, the arrow must indicate that x increases, while if the sign is $-$, the arrow must indicate that x decreases.

Example 2. Consider the differential equation $\dot{x} = x - x^3$.

The state space is $\mathbb{R}$. The equilibria are -1, 0, 1.

The orbits are $(-\infty, -1)$, $\{-1\}$, $(-1, 0)$, $\{0\}$, $(0, 1)$, $\{1\}$, $(1, \infty)$.

The function $f(x) = x - x^3$ is positive on $(-\infty, -1)$, negative on $(-1, 0)$, positive on (0,1) and negative on $(1, \infty)$. $\diamond$

Example 3. How to read a phase portrait? Assume that we see a phase portrait of some scalar differential equation $\dot{x} = f(x)$ and note that, for example, the open bounded interval (a, b) is an orbit such that the arrow on it indicates to the right. Only with this information we can deduce some important properties of the flow of the differential equation having this phase portrait.

First we deduce that a and b must be *equilibria*, thus $\varphi(t, a) = a$ and $\varphi(t, b) = b$ for all $t \in \mathbb{R}$. Let $\eta \in (a, b)$ be a fixed initial state. Then $\gamma(\eta) = (a, b)$, which means that *the image* of the function $\varphi(\cdot, \eta)$ *is the open bounded interval* (a, b). The fact that the arrow indicates to the right provides the information that the function $\varphi(\cdot, \eta)$ *is strictly increasing.* By Theorem 1, since $\varphi(\cdot, \eta)$ is bounded, we must have that its interval of definition is $\mathbb{R}$. Moreover, we have $\lim\limits_{t \to -\infty} \varphi(t, \eta) = a$ and $\lim\limits_{t \to \infty} \varphi(t, \eta) = b$.

Assume now that the open unbounded interval (b, ∞) is an orbit such that the arrow on it indicates to the left. Let $\eta \in (b, \infty)$ be a fixed initial state. Then $\gamma(\eta) = (b, \infty)$, which means that *the image* of the function $\varphi(\cdot, \eta)$ *is the open unbounded interval* (b, ∞). The fact that the arrow indicates to the left provides the information that the function $\varphi(\cdot, \eta)$ *is strictly decreasing.* Thus, the positive orbit is the bounded interval (b, η). By Theorem 1, we deduce that $\beta_\eta = \infty$. Moreover, we have $\lim\limits_{t \to \infty} \varphi(t, \eta) = b$.

Stability of the equilibria of dynamical systems

The notion of stability is of considerable theoretical and practical importance. Roughly speaking, an equilibrium point η^* is stable if all solutions starting near η^* stay nearby. If, in addition, nearby solutions tend to η^* as $t \to \infty$, then η^* is asymptotically stable. Precise definitions were given by the Russian mathematician Aleksandr Lyapunov in 1892.

We remind that we study differential equations of the form

$$\dot{x} = f(x) \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a given C^1 function.

Definition 1 *An equilibrium point η^* of equation (1) is said to be stable if, for any given $\varepsilon > 0$, there is a $\delta > 0$ such that, for every η for which $||\eta - \eta^*|| < \delta$ we have that $||\varphi(t, \eta) - \eta^*|| < \varepsilon$ for all $t \geq 0$.*

The equilibrium point η^* is said to be unstable if it is not stable.

An equilibrium point η^* is said to be asymptotically stable if it is both stable and attractor.

Stability of linear dynamical systems. We consider

$$\dot{x} = Ax \tag{6}$$

where the matrix $A \in \mathcal{M}_n(\mathbb{R})$ is called *the matrix of the coefficients* of the linear system (6). We assume that

$$\det A \neq 0$$

such that the only equilibrium point of (6) is $\eta^* = 0$ (here 0 denotes the null vector from $\mathbb{R}^n$).

Definition 2 *We say that the linear system* (6) *is stable / asymptotically stable / unstable when its equilibrium point at the origin has this quality.*

We have the following important result. Its proof is beyond the aim of these lectures. It is given in terms of the *eigenvalues* of the matrix A. Remember that the eigenvalues of A have the property that they are the roots of the algebraic equation

$$\det(A - \lambda I_n) = 0.$$

Denote by $\sigma(A)$ the set of all eigenvalues $\lambda \in \mathbb{C}$ of the matrix A.

The notation $\Re(\lambda)$ for $\lambda \in \mathbb{C}$ means the real part of λ.

Theorem 2 *If $\Re(\lambda) < 0$ for any $\lambda \in \sigma(A)$ then the linear system $\dot{x} = Ax$ is asymptotically stable.*

If there exists some $\lambda \in \sigma(A)$ such that $\Re(\lambda) > 0$ then the linear system $\dot{x} = Ax$ is unstable.

The linearization method to study the stability of an equilibrium point of a nonlinear system. An equilibrium point η^* of (1) is said to be *hyperbolic* when $\Re(\lambda) \neq 0$ for any eigenvalue λ of the Jacobian matrix $Jf(\eta^*)$.

Theorem 3 *Let η^* be a hyperbolic equilibrium point of (1). We have that η^* is asymptotically stable / unstable if and only if the linear system*

$$\dot{x} = Jf(\eta^*)x$$

has the same quality.

Corollary 1 *Let $n = 1$ and η^* be an equilibrium point of $\dot{x} = f(x)$.*

If $f'(\eta^*) < 0$ then η^* is asymptotically stable.

If $f'(\eta^*) > 0$ then η^* is unstable.

Exercise 1. Study the stability of the equilibria of the damped pendulum equation

$$\ddot{\theta} + \frac{\nu}{m}\dot{\theta} + \frac{g}{L}\sin\theta = 0,$$

where $\nu > 0$ is the damping coefficient, m is the mass of the bob, $L > 0$ is the length of the rod and $g > 0$ is the gravity constant. What happen when $\nu = 0$?

Phase portraits of planar systems

Phase portraits of linear planar systems. We consider $\dot{x} = Ax$ where $A \in \mathcal{M}_2(\mathbb{R})$ with $\det A \neq 0$. In this case the state space is $\mathbb{R}^2$ and the orbits are curves. Denote by $\lambda_1, \lambda_2 \in \mathbb{C}$ the two eigenvalues of A. In the next definition the equilibrium point at the origin is classified as *node, focus, center, saddle*, depending on the eigenvalues of A.

Definition 3 *The equilibrium point $\eta^* = 0$ of the linear planar system $\dot{x} = Ax$ is a*

(i) **node** *if $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$.*

(ii) **saddle** *if $\lambda_1 < 0 < \lambda_2$.*

(iii) **focus** *if $\lambda_{1,2} = \alpha \pm i\beta$ with $\beta \neq 0$ and $\alpha \neq 0$.*

(iv) **center** *if $\lambda_{1,2} = \pm i\beta$ with $\beta \neq 0$.*

Remark. Note that the following affirmations are valid.

A node can be either asymptotically stable (when $\lambda_1 \leq \lambda_2 < 0$) or unstable (when $0 < \lambda_1 \leq \lambda_2$).

Any saddle is unstable.

A focus can be either asymptotically stable (when $\alpha < 0$) or unstable (when $\alpha > 0$).

Any center is stable, but not asymptotically stable.

In the sequel we provide some examples of linear planar systems having the equilibrium point at the origin of each of the types presented in the above definition. For each system given as an example our purpose is to study its type and stability, then to find the flow, the orbits and represent the phase portrait.

Example 1. $\quad \dot{x} = -x, \quad \dot{y} = -2y.$

The matrix of the system is $A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, which have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Hence the equilibrium point at the origin is an *asymptotically stable node.*

In order to find the flow we have to consider the IVP

$$\dot{x} = -x, \quad \dot{y} = -2y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{-t},\ \eta_2 e^{-2t}).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1 e^{-t},\, \eta_2 e^{-2t}) \quad : \quad t \in \mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1 e^{-t},\ y = \eta_2 e^{-2t},\ t \in \mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$\eta_1^2\, y = \eta_2\, x^2,$$

which, in general, is an equation of a parabola with the vertex in the origin. In the special case $\eta_1 = 0$ this is the equation $x = 0$, that is the Oy axis, while in the special case $\eta_2 = 0$ this is the equation $y = 0$, that is the Ox axis. Note that each orbit lie on one of these planar curves, but it is not the whole parabola or the whole line. More precisely, we have

$$\begin{array}{lcl}
\gamma(\eta) & = & \{(x,y) \in \mathbb{R}^2 \quad : \quad \eta_1^2\, y = \eta_2\, x^2, \quad \eta_1 x > 0, \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1\eta_2 \neq 0,\\
\gamma(\eta) & = & \{(0,y) \in \mathbb{R}^2 \quad : \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1 = 0,\ \eta_2 \neq 0,\\
\gamma(\eta) & = & \{(x,0) \in \mathbb{R}^2 \quad : \quad \eta_1 x > 0\} \quad \text{when} \quad \eta_1 \neq 0,\ \eta_2 = 0,\\
\gamma(0) & = & \{0\}.
\end{array}$$

On each orbit the arrows must point toward the origin.

Example 2. $\quad \dot{x} = x, \quad \dot{y} = -y.$

The matrix of the system is $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which have the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence the equilibrium point at the origin is a *saddle*, which is *unstable.* In order to find the flow we have to consider the IVP

$$\dot{x} = x, \quad \dot{y} = -y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^{t},\, \eta_2 e^{-t}).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1 e^{t},\, \eta_2 e^{-t}) \quad : \quad t \in \mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1 e^t,\ y = \eta_2 e^{-t},\ t \in \mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$xy = \eta_1 \eta_2,$$

which, in general, is an equation of a hyperbola. More precisely, we have

$$\begin{aligned}
\gamma(\eta) &= \{(x,y) \in \mathbb{R}^2 \quad : \quad xy = \eta_1\eta_2, \quad \eta_1 x > 0, \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1\eta_2 \neq 0, \\
\gamma(\eta) &= \{(0,y) \in \mathbb{R}^2 \quad : \quad \eta_2 y > 0\} \quad \text{when} \quad \eta_1 = 0,\ \eta_2 \neq 0, \\
\gamma(\eta) &= \{(x,0) \in \mathbb{R}^2 \quad : \quad \eta_1 x > 0\} \quad \text{when} \quad \eta_1 \neq 0,\ \eta_2 = 0, \\
\gamma(0) &= \{0\}.
\end{aligned}$$

On each orbit the arrows must point such that x moves away from 0, while y moves toward 0.

Example 3. $\dot{x} = -y, \quad \dot{y} = x.$

The matrix of the system is $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, which have the eigenvalues $\lambda_{1,2} = \pm i$. Hence the equilibrium point at the origin is a *center*, which is *stable* but not asymptotically stable.

In order to find the flow we have to consider the IVP

$$\dot{x} = -y, \quad \dot{y} = x, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 \cos t - \eta_2 \sin t,\ \eta_1 \sin t + \eta_2 \cos t).$$

The orbit corresponding to a fixed initial state $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$ is

$$\gamma(\eta) = \{(\eta_1 \cos t - \eta_2 \sin t,\ \eta_1 \sin t + \eta_2 \cos t) \quad : \quad t \in \mathbb{R}\}.$$

In other words, the orbit is the curve in the plane xOy of parametric equations

$$x = \eta_1 \cos t - \eta_2 \sin t,\ y = \eta_1 \sin t + \eta_2 \cos t,\ t \in \mathbb{R}.$$

Note that the parameter t can be eliminated and thus obtain the cartesian equation

$$x^2 + y^2 = \eta_1^2 + \eta_2^2,$$

which, in general, is an equation of a circle with the center at the origin and radius $\sqrt{\eta_1^2 + \eta_2^2}$. More precisely, we have

$$\begin{aligned}
\gamma(\eta) &= \{(x,y) \in \mathbb{R}^2 \quad : \quad x^2 + y^2 = \eta_1^2 + \eta_2^2\} \quad \text{when} \quad \eta_1^2 + \eta_2^2 \neq 0, \\
\gamma(0) &= \{0\}.
\end{aligned}$$

On each orbit the arrows must point in the trigonometric sense.

Example 4. $\quad \dot{x} = x - y, \quad \dot{y} = x + y.$
The matrix of the system is $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, which have the eigenvalues $\lambda_{1,2} = 1 \pm i$.
Hence the equilibrium point at the origin is an *unstable focus*.
In order to find the flow we have to consider the IVP

$$\dot{x} = x - y, \quad \dot{y} = x + y, \quad x(0) = \eta_1, \quad y(0) = \eta_2$$

for each fixed $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. Calculations yields that the flow $\varphi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ has the expression

$$\varphi(t, \eta_1, \eta_2) = (\eta_1 e^t \cos t - \eta_2 e^t \sin t,\ \eta_2 e^t \cos t + \eta_1 e^t \sin t).$$

In order to find the shape of the orbits it is more convenient to pass to polar coordinates, that is, instead of the unknowns $x(t)$ and $y(t)$, to consider new unknowns $\rho(t)$ and $\theta(t)$ related by

$$x(t) = \rho(t) \cos \theta(t), \quad y(t) = \rho(t) \sin \theta(t), \tag{7}$$

where

$$\rho(t) > 0 \quad \text{for any} \quad t \in \mathbb{R}.$$

We can write equivalently

$$\rho(t)^2 = x(t)^2 + y(t)^2, \quad \tan \theta(t) = \frac{y(t)}{x(t)}\,. \tag{8}$$

Our aim is to find a system satisfied by the new unknowns ρ and θ. In this system their derivatives will be involved. We will show two ways to find this new system.

Method 1. We take the derivatives in the equalities (7) and obtain

$$\dot{x} = \dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta, \quad \dot{y} = \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta.$$

After we replace in our system, $\quad \dot{x} = x - y, \quad \dot{y} = x + y$, we obtain

$$\dot{\rho}\cos\theta - \rho\dot{\theta}\sin\theta = \rho\cos\theta - \rho\sin\theta, \quad \dot{\rho}\sin\theta + \rho\dot{\theta}\cos\theta = \rho\cos\theta + \rho\sin\theta.$$

Calculations yields the system

$$\dot{\rho} = \rho, \quad \dot{\theta} = 1,$$

whose solution for a given initial state (ρ_0, θ_0) is given by

$$\rho(t) = \rho_0 e^t, \quad \theta(t) = \theta_0 + t,$$

which defines a logarithmic spiral in the (x, y) plane.

Since $\rho(t)$ is strictly increasing, the arrow on each orbit must point toward the infinity.

Method 2. We show that we arrive to the same system by taking the derivatives in the equalities (8). We obtain

$$2\rho\dot{\rho} = 2x\dot{x} + 2y\dot{y}, \quad \frac{\dot{\theta}}{\cos^2\theta} = \frac{\dot{y}x - y\dot{x}}{x^2}.$$

After we replace $\quad \dot{x} = x - y, \quad \dot{y} = x + y$, we obtain

$$\rho\dot{\rho} = x^2 - xy + xy + y^2, \quad \dot{\theta} = (x^2 + xy - xy + y^2)\frac{\cos^2\theta}{x^2},$$

which further can be written

$$\rho\dot{\rho} = \rho^2, \quad \dot{\theta} = \rho^2\frac{\cos^2\theta}{\rho^2\cos^2\theta}.$$

It is not difficult to see that we arrive to the same system.

When studying nonlinear systems, the linearization method gives also information about the behavior of the orbits in a neighborhood of an equilibrium point. More precisely, we have the following result for planar systems.

Theorem 4 *Let $n = 2$, $f \in C^2(\mathbb{R}^2, \mathbb{R}^2)$ and η^* be a hyperbolic equilibrium point of $\dot{x} = f(x)$. Then η^* is a node / saddle / focus if and only if for the linear system $\dot{x} = Jf(\eta^*)x$, the origin has the same type.*

First integrals. The cartesian differential equation of the orbits of a planar system. We consider the planar autonomous system

$$\begin{aligned} \dot{x} &= f_1(x, y) \\ \dot{y} &= f_2(x, y) \end{aligned} \tag{9}$$

where $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is a given C^1 function. As usual, denote its local flow by $\varphi : \{(t, \eta) \ : \ t \in I_\eta, \quad \eta \in \mathbb{R}^2\} \to \mathbb{R}^2$.

Definition 4 *Let $U \subset \mathbb{R}^2$ be an open nonempty set. We say that $H : U \to \mathbb{R}$ is a first integral in U of* (9) *if it is a non-constant C^1 function and*

$$H(\varphi(t, \eta)) = H(\eta), \quad \eta \in U, \quad t \in I_\eta.$$

A first integral in $U = \mathbb{R}^2$ is said to be a global first integral.

Remark. Let $H : U \to \mathbb{R}$ be a first integral. We have that the orbits of (9) in U lie on the level curves of H.

Example 1. We saw that the orbits of the linear system with a center at the origin $\dot{x} = -y, \ \dot{y} = x$ are the circles of cartesian equation $x^2 + y^2 = c$, for any real constant $c \geq 0$. Hence, taking into account the definition of a first integral we can say that the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + y^2$ is a first integral in $\mathbb{R}^2$ of this system. $\diamond$

Example 2. For the linear system with a saddle at the origin $\dot{x} = x, \ \dot{y} = -y$ the function $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = xy$ is a first integral in $\mathbb{R}^2$. $\diamond$

Example 3. There are systems without a first integral in $\mathbb{R}^2$. For the linear system with a node at the origin $\dot{x} - y, \ \dot{y} = x$ the function $H : \mathbb{R}^2 \setminus \{(0, y) \ : \ y \in \mathbb{R}\} \to \mathbb{R}$, $H(x, y) = \frac{y}{x^2}$ is a first integral in $\mathbb{R}^2 \setminus \{(0, y) \ : \ y \in \mathbb{R}\}$ of this system. In fact, this system does not have a first integral defined in a neighborhood of the origin. Only centers and saddles have this property. $\diamond$

In each of the previous examples a first integral was found after long calculations: first we found the flow, after the parametric equations of the orbits, and after the cartesian equation of the orbits. We used only the definitions of an orbit and, respectively, of a first integral and we had the advantage that the systems were simple enough to find explicitly their solutions. On the other hand, note that the a priori knowledge of a first integral is very helpful to draw the phase portrait.

Example 4. Knowing that $H(x,y) = y^2+2x^2$ is a first integral in $\mathbb{R}^2$ of the system $\dot{x} = y,\ \dot{y} = -2x$, represent its phase portrait.

First note that the level curves of H are ellipses that encircle the origin. Hence, these are the orbits of our system. The arrows on each orbit must point in the clockwise direction. $\diamond$

New questions arise: *How to check that a given function is a first integral? How to find a first integral?* The answer to the first question is given by the following result.

Proposition 1 *A nonconstant C^1 function $H : U \to \mathbb{R}$ is a first integral in U of* (9) *if and only if it satisfies the first order linear partial differential equation*

$$f_1(x,y)\frac{\partial H}{\partial x}(x,y) + f_2(x,y)\frac{\partial H}{\partial y}(x,y) = 0,\ \textit{for any } (x,y) \in U. \tag{10}$$

Example 5. We want to check that $H(x,y) = y^2 + 2x^2$ is a first integral in $\mathbb{R}^2$ of the system $\dot{x} = y,\ \dot{y} = -2x$. In the case of this system equation (10) becomes

$$y\frac{\partial H}{\partial x}(x,y) - 2x\frac{\partial H}{\partial y}(x,y) = 0.$$

It is not difficult to check that this equation is identically satisfied in $\mathbb{R}^2$ by the function $H(x,y) = y^2 + 2x^2$. $\diamond$

Of course, Proposition 1 gives also the answer to the second question, *How to find a first integral?*, only that we do not know how to solve a first order linear partial differential equation. It is not the aim of this course to explain all these in detail, but we will give the following helpful practical result.

A first integral of the planar system (9) (or, equivalently, a solution of the linear partial differential equation (10)) can be found by integrating the equation

$$\frac{dy}{dx} = \frac{f_2(x,y)}{f_1(x,y)}, \tag{11}$$

which is called *the cartesian differential equation of the orbits of* (9). After the integration of (11) we look for a function of two variables H such that we can write the general solution of (11) as $H(x, y) = c$, $c \in \mathbb{R}$. This H is a first integral of (9).

Example 5. We come back to the system $\dot{x} = y,\ \dot{y} = -2x$. This time we want to find a first integral. The previous statement says that we need to integrate the equation

$$\frac{dy}{dx} = \frac{-2x}{y}.$$

This is separable, and it can be written as $ydy = -2xdx$. After integration we obtain $y^2/2 = -x^2 + c$, $c \in \mathbb{R}$. Hence $H(x, y) = y^2/2 + x^2$ is a first integral in $\mathbb{R}^2$. $\diamond$

With the previous example, note that the first integral is not unique. Having one first integral, we can find many more, for example by multiplying it with any non null constant.

Exercise 1. Find a first integral in $\mathbb{R}^2$ of the undamped pendulum system

$$\dot{x} = y,\ \dot{y} = -\omega^2 \sin x,$$

where $\omega > 0$ is a real parameter. Show that there exists a region U in the state space $\mathbb{R}^2$ where the orbits are closed curves that encircle the origin, thus the origin is an equilibrium point of center type and it is stable. Note that the equilibrium point at the origin is not hyperbolic, which implies that the linearization method fails.

Similarly show that there exists a region U_k in the state space $\mathbb{R}^2$ where the orbits are closed curves that encircle the equilibrium point $(2k\pi, 0)$ for any $k \in \mathbb{Z}$.

Apply the linearization method to study the behavior of the orbits around the equilibrium point $(\pi, 0)$ and similarly around $(2k\pi + \pi, 0)$ for any $k \in \mathbb{Z}$.

Represent the phase portrait in $\mathbb{R}^2$.

Exercise 2. Find a first integral in the first quadrant $(0, \infty) \times (0, \infty)$ of the Lotka-Volterra system (also called the prey-predator system)

$$\dot{x} = N_1 x - xy,\ \dot{y} = -N_2 y + xy,$$

where $N_1, N_2 > 0$ are real parameters. [2]