Universitatea Babeş-Bolyai, Facultatea de Matematică şi Informatică

Secţia: Informatică engleză

Curs: Dynamical Systems

Primăvara 2019

## Continuous Dynamical Systems[1]

**1.** Represent the phase portrait of the scalar dynamical system
$\dot{x} = 1 - x^2$. Find $\varphi(t, 1)$ and justify. Specify the properties of the functions $\varphi(t, 2)$ and, respectively, $\varphi(t, 0.5)$.

**2.** Let $0 < c < 1$ be a parameter and consider the scalar dynamical system
$\dot{x} = x(1 - x) - cx$.

a) Find its equilibria and study their stability using the linearization method.

b) Represent its phase portrait.

c) When $x(t) > 0$ is considered to be the density of fish in a lake, and $0 < c < 1$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a) and b). ⋄

**3.** Let $c > 1/4$ be a parameter and consider the scalar dynamical system
$\dot{x} = x(1 - x) - c$.

a) Represent its phase portrait.

b) When $x(t) \geq 0$ is considered to be the density of fish in a lake, and $c > 1/4$ to be the rate of fishing, try to predict the fate of the fish from the lake interpreting the theoretical result obtained at a). ⋄

**4.** Represent the phase portrait of the scalar dynamical system $\dot{x} = 2 - x^2$. Study the stability of the equilibrium points using the linearization method. Find $\varphi(t, \sqrt{2})$ and specify the properties of the functions $\varphi(t, -1.5)$, $\varphi(t, 0)$ and $\varphi(t, 2)$. ⋄

**5.** Represent the phase portrait of the scalar dynamical system $\dot{x} = 2x - x^2$. Study the stability of the equilibrium points using the linearization method. Find $\varphi(t, 2)$, $\varphi(t, 0)$ and study the properties of the functions $\varphi(t, -2)$, $\varphi(t, 1)$ and $\varphi(t, 3)$.

If there is an attractor, specify its basin of attraction. $\diamond$

**6.** Represent the phase portrait, specify the orbits and emphasize the properties of the solution $\varphi(\cdot, \eta)$ for any $\eta \in \mathbb{R}$ for the following equations. Specify the stability of the equilibria.

(a) $\dot{x} = x - x^2$, (b) $\dot{x} = x - x^3 + 1$, (c) $\dot{x} = x - x^3 + 0.2$, (d) $\dot{x} = -x - x^3 + 1$, (e) $\dot{x} = 2\sin x$, (f) $\dot{x} = 1 - 2\sin x$, (g) $\dot{x} = 1 - \sin x$, (h) $\dot{x} = 2 - \sin x$, (i) $\dot{x} = \tanh x$. $\diamond$

**7.** Represent the phase portrait of $\dot{x} = \lambda - x^2$. Discuss with respect to the parameter $\lambda \in \mathbb{R}$. $\diamond$

**8.** We consider the following linear planar system

(a) $\dot{x} = -6x, \quad \dot{y} = -3y$, (b) $\dot{x} = -6y, \quad \dot{y} = -3x$, (c) $\dot{x} = 6y, \quad \dot{y} = -3x$, (d) $\dot{x} = x, \quad \dot{y} = -3y$, (e) $\dot{x} = 2y, \quad \dot{y} = -3x$, (f) $\dot{x} = -2y, \quad \dot{y} = 2x$.

i) Find its flow.

ii) Specify the type and stability of this linear system.

iii) Find a first integral. There is a global first integral?

iv) Represent its phase portrait. $\diamond$

**9.** Specify the type and stability of the linear systems

(a) $\dot{x} = 4x - 5y, \quad \dot{y} = x - 2y$, (b) $\dot{x} = x + y, \quad \dot{y} = -2x + 4y$, (c) $x' = x + y, \quad y' = x - 4y$. $\diamond$

**10.** (i) For what values of the real parameter $a$, the system $\dot{x} = ax - 5y$, $\dot{y} = x - 2y$ is a center?. In that cases find the general solution of the system.

(ii) For what values of the real parameter $a$, the system from (i) has a line filled with equilibrium points? $\diamond$

**11.** There are uncoupled linear systems which are centers? $\diamond$

**12.** Find all the equilibrium points of the nonlinear planar system $\dot{x} = x(1-x)$, $\dot{y} = (y+1)(y-2)$. $\diamond$

**13*.** Represent the phase portrait of the following uncoupled (product) systems

(a) $\dot{x} = x(1-x)$, $\dot{y} = 0$ , (b) $\dot{x} = x(1-x)$, $\dot{y} = y$ , (c) $\dot{x} = x(1-x)$, $\dot{y} = (y+1)(y-2)$ . $\diamond$

**14.** Find the equilibrium points and decide whether they are or not hyperbolic, for the nonlinear planar system $\ddot{\theta} + \dot{\theta} + \theta^3 = 0$. $\diamond$

**15.** We consider the following nonlinear planar systems
$\dot{x} = -x + xy, \quad \dot{y} = -2y + 3y^2$.
a) Find its equilibrium points and study their stability using the linearization method.
b) Find $\varphi(t, 0, 2/3)$, $\varphi(t, 4, 0)$ and $\varphi(t, 1, 2/3)$.
c) Represent in the phase plane the orbits corresponding to the initial values $(0, 2/3)$, $(4, 0)$ and $(1, 2/3)$. $\diamond$

**16.** Consider the following planar system

$$\dot{x} = 2x - x^2, \quad \dot{y} = y - xy^2.$$

a) Find the equilibrium points and study their stability.
b) Find $\varphi(t, 2, 1/2)$, $\varphi(t, 2, 0)$ and $\varphi(t, 0, 2)$. $\diamond$

**17.** Find the polar coordinates of the following points of cartesian coordinates. Represent all these points in the plane.
$(0, 1,\ (1, 0),\ (2, 0),\ (0, 3),\ (-3, 0),\ (0, -2),\ (1, 1),\ (1, 1/2),\ (-2, 1),\ (-6, -3)$, $(\eta_1 \cos t - \eta_2 \sin t,\ \eta_1 \sin t + \eta_2 \cos t)$ where $t \in \mathbb{R}$ and $(\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, $(\eta_1 e^t \cos t - \eta_2 e^t \sin t,\ \eta_1 e^t \sin t + \eta_2 e^t \cos t)$ where $t \in \mathbb{R}$ and $(\eta_1, \eta_2) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.
$\diamond$

**18.** We consider the linear planar system $\dot{x} = -x + y$, $\dot{y} = -x - y$. Specify its type and stability. Pass to polar coordinates. Represent its phase portrait. $\diamond$

**19.** Consider the following planar system

$$\dot{x} = x - 2\,x\,y, \quad \dot{y} = x - y.$$

Find the equilibrium points and study their stability. $\diamond$

**20.** Consider the following planar system

$$\dot{x} = -y(x^2 + y^2), \quad \dot{y} = x(x^2 + y^2).$$

a) Does this system have other equilibria besides $(0,0)$? Justify.

b) Decide whether the equilibrium point $(0,0)$ is hyperbolic or not.

c) Verify that $\varphi(t,1,0) = (\cos t, \sin t)$, $\varphi(t,2,0) = (2\cos 4t, 2\sin 4t)$ for all $t \in \mathbb{R}$. Find $\varphi(t,3,0)$. Represent the corresponding orbits.

d) Pass to polar coordinates and represent the phase portrait. Deduce that all the solutions of the system are periodic. $\diamond$

**21.** We consider the planar system

$\dot{x} = -y + x(1 - x^2 - y^2)$, $\dot{y} = x + y(1 - x^2 - y^2)$.

a) Study the stability of the equilibrium point $(0,0)$ using the linearization method. There are other equilibrium points?

b) Check that $\varphi(t,1,0) = (\cos t, \sin t)$ for all $t \in \mathbb{R}$. Represent the corresponding orbit. $\diamond$

**22.** Consider the following planar system

$$\dot{x} = -y - x(x^2 + y^2), \quad \dot{y} = x - y(x^2 + y^2).$$

a) Does this system have other equilibria besides $(0,0)$? Justify.

b)Prove that the equilibrium point $(0,0)$ is not hyperbolic.

c) Use the system obtained by passing to polar coordinates to determine the shape of the orbits. $\diamond$

**23.** Denote by $x(t)$ the density of a trout population at time $t$. Describe its evolution in each of the following cases.

(a) $\dot{x} = 100x$.

(b) $\dot{x} = 100x - x^2$.

Assume that someone is fishing, first with a constant rate $k > 0$, that is,

(c) $\dot{x} = 100x - x^2 - k$,

and another one, in another basin, with rate proportional to the density, $kx(t)$, where $k > 0$, that is,

(d) $\dot{x} = 100x - x^2 - kx$.

Discuss with respect to $k$. $\diamond$

**24.** For each $k > 0$ we consider the differential equation

$$\dot{x} = -k\,(x - 21),$$

which is the model of Newton for cooling processes, here $x(t)$ being the temperature of a cup of tea at time $t$.

(a) Find its flow. (b) An experiment revealed the following fact. A cup of tea with initial temperature of $49°C$ has a temperature of $37°C$ after 10 minutes. Find the initial temperature of a cup of tea such that after 20 minutes the tea has $37°C$. $\diamond$

**25.** Find a global first integral of the following nonlinear planar system $\ddot{\theta} + \omega^2 \sin\theta = 0$ (where $\omega > 0$). $\diamond$

**26**. Find a first integral in $(0, \infty) \times (0, \infty)$ of the following nonlinear planar system $\dot{x} = N_1 x - xy, \quad \dot{y} = -N_2 y + xy$ (where $N_1, N_2 > 0$). $\diamond$

**27**. Find a first integral in $(0, \infty) \times (0, \infty)$ of the following nonlinear planar system $\dot{x} = x - xy, \quad \dot{y} = -0.3y + 0.3xy$. $\diamond$

**28.** a) Give an example of a coupled linear planar system which has a node at the origin.

b) Give an example of a coupled linear planar system which has a saddle at the origin. $\diamond$

**29.** Find the the flow of each of the systems. Decide wether, for every $\eta \in \mathbb{R}^2$ we have $\lim\limits_{t\to\infty} \varphi(t, \eta) = 0$ or we have that $\varphi(t, \eta)$ is bounded for $t \in (0, \infty)$.

a) $x' = -2x, \;\; y' = -3y$

b) $x' = -2x, \;\; y' = 3y$

c) $x' = -3x, \;\; y' = x - 3y$

d) $x' = y, \;\; y' = \omega^2 x$ (here $\omega > 0$ is a parameter)

e) $x' = -x - y, \;\; y' = x - y$

f) $x' = -5x - 9y, \;\; y' = 2x + y$. $\diamond$