Proposition 4.1. Every plane in $\mathbb{E}^3$ can be described with a linear equation in three variables

$$ax + by + cz + d = 0 \tag{4.5}$$

relative to a fixed coordinate system and any linear equation in two variables describes a plane relative to a fixed coordinate system.

We showed that a plane has a linear eq. (see symmetric eq)

We need to show that a linear equation corresponds to a plane

In (4.5) at least one of the constants a, b, c is non-zero.

Without loss of generality we may assume $a \neq 0$

Then
$$\begin{cases} x = -\frac{b}{a}y - \frac{c}{a}z - \frac{d}{a} \\ y = y \\ z = z \end{cases} \iff \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{d}{a} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix} \quad (*)$$

(x_P, y_P, z_P) is a solution to (4.5) $\iff$ $\overrightarrow{AP}$ is a linear combination of v and w

where $A = \begin{bmatrix} -\frac{d}{a} \\ 0 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} -b/a \\ 1 \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} -c/a \\ 0 \\ 1 \end{bmatrix}$

So the solutions of (4.5) is the set

$$S = \{ P \in \mathbb{E}^3 : \overrightarrow{AP} = sv + tw \text{ for some } s, t \in \mathbb{R} \}$$

A belongs to this set. Moreover v and w are linearly independant

So the set $\{ \overrightarrow{AP} : P \in S \}$ is a vector subspace of $\mathbb{V}^3$ of dimension 2

Proposition 4.2. Suppose you have a plane $\pi : ax+by+cz+d=0$ and a point $P(x_P, y_P, z_P)$ in $\mathbb{E}^3$. The distance from P to π is

$$d(P,\pi) = \frac{|ax_p+by_p+cz_p+d|}{\sqrt{a^2+b^2+c^2}}. \tag{4.9}$$

- $\pi : ax+by+cz+d=0$ (1)
- fix a point $A \in \pi$ and let v and w be two non-collinear vectors parallel to π

then $\pi : \begin{vmatrix} x-x_A & y-y_A & z-z_A \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = 0$

$$\Leftrightarrow \pi : \underbrace{\begin{vmatrix} v_y & v_z \\ w_y & w_z \end{vmatrix}}_{= n_x} x + \underbrace{\begin{vmatrix} v_z & v_x \\ w_z & w_x \end{vmatrix}}_{= n_y} y + \underbrace{\begin{vmatrix} v_x & v_y \\ w_x & w_y \end{vmatrix}}_{= n_z} z + d' = 0 \quad (2)$$

Notice that $n(n_x, n_y, n_z) = v \times w$ is a normal vector for π

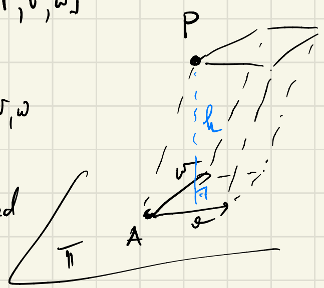
The equations (1) and (2) describe the same plane so they are proportional

$\Rightarrow \exists \lambda : \quad a = \lambda n_x \quad b = \lambda n_y \quad c = \lambda n_z \quad d = \lambda d'$

So $$\frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}} = \frac{|\lambda n_x x + \lambda n_y y + \lambda n_z z + \lambda d'|}{\sqrt{(\lambda n_x)^2 + (\lambda n_y)^2 + (\lambda n_z)^2}} = \frac{|n_x x + n_y y + n_z z + d'|}{\|n\|}$$

Now for an arbitrary point $P(x_p, y_p, z_p)$

$$|n_x x_p + n_y y_p + n_z z_p + d'| = \left| \begin{vmatrix} x_p - x_A & y_p - y_A & z_p - z_A \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \right| = [\vec{AP}, v, w]$$



and $\|n\| = \|v \times w\| =$ area of parallelogram spanned by v, w

$$\Rightarrow \frac{|ax+by+cz+d|}{\sqrt{a^2+b^2+c^2}} = h = \text{height from } P \text{ in the parallelepiped} = d(P,\pi)$$

Proposition 4.4. Every line in $\mathbb{E}^3$ can be described with two linear equations in three variables

$$\begin{cases} a_1x + b_1y + c_1z + d_1 &= 0 \\ a_2x + b_2y + c_2z + d_2 &= 0 \end{cases} \tag{4.13}$$

relative to a fixed coordinate system and any compatible system of two linear equations of rank 2 in three variable describes a line relative to a fixed coordinate system.

The proof is similar to that of Proposition 4.1:

One can show that the solutions S to (4.13) have the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} + t \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \quad t \in \mathbb{R}$$

where $A(x_A, y_A, z_A)$ is a particular solution to (4.13)

$\Rightarrow$ $\{ \overrightarrow{AP} : P \in S \}$ is a 1-dimensional vector subspace of V^3

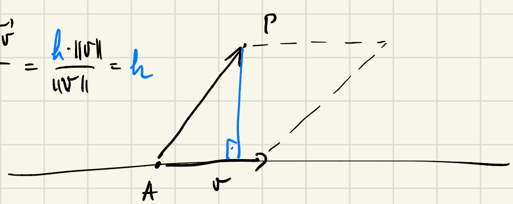
Proposition 4.5. Suppose you have a line

$$\ell_1 : \begin{cases} x = x_A + tv_x \\ y = y_A + tv_y \\ z = z_A + tv_z \end{cases} \quad \text{and a point} \quad P(x_P, y_P, z_P) \in \mathbb{E}^3.$$

The distance from P to ℓ is

$$d(P,\ell) = \frac{||\overrightarrow{PA} \times \mathbf{v}||}{||\mathbf{v}||}.$$

$$\frac{||\overrightarrow{PA} \times \vec{v}||}{||\vec{v}||} = \frac{\text{area of parallelogram spanned by } \overrightarrow{PA} \text{ and } \vec{v}}{\text{base} = ||\vec{v}||} = \frac{h \cdot ||\vec{v}||}{||\vec{v}||} = h$$



[Common perpendicular line of two skew lines] With the notation in the previous paragraph consider the line

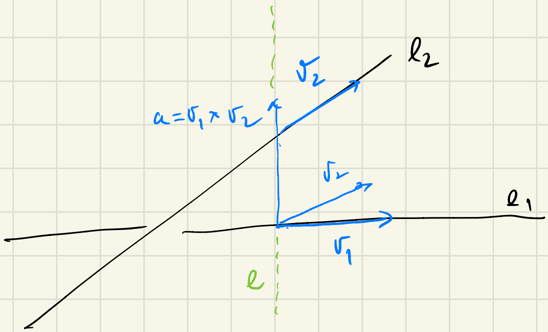
$$\ell : \begin{cases} \pi_1' : \begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ v_x & v_y & v_z \\ a_x & a_y & a_z \end{vmatrix} = 0, \\ \pi_2' : \begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ u_x & u_y & u_z \\ a_x & a_y & a_z \end{vmatrix} = 0. \end{cases}$$

$\ell = \pi_1' \cap \pi_2'$

$\Rightarrow \ell \parallel \pi_1'$ and $\ell \parallel \pi_2'$

$\Rightarrow \ell \parallel a(a_x, a_y, a_z) = v \times w$

$\Rightarrow \ell \perp \ell_1$ and $\ell \perp \ell_2$ (1)



$\left.\begin{array}{l} \ell, \ell_1 \subseteq \pi_1' \\ \ell \nparallel \ell_1 \end{array}\right\} \Rightarrow \ell$ and ℓ_1 intersect (2)

$\left.\begin{array}{l} \ell, \ell_2 \subseteq \pi_2' \\ \ell \nparallel \ell_2 \end{array}\right\} \Rightarrow \ell$ and ℓ_2 intersect (3)

The claim follows from (1) (2) and (3)