LECTURE 7 – DYNAMICAL SYSTEMS

Monday, 5 April 2021 10:01

The dynamical system associated to an autonomous eq. in $\mathbb{R}^n$

Consider (1) $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

Keywords: state of the system, initial state, equilibrium state (stationary state), orbit (trajectory), attractor, repeller...

$\mathbb{R}^n$ is the state space of the dynamical system (1).

Let $\eta \in \mathbb{R}^n$ be fixed and the IVP: $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$ (2)

Theorem (the existence and uniqueness theorem for the IVP)

If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $\eta \in \mathbb{R}^n$ then the IVP (2) has a unique solution denoted $\varphi(t, \eta)$ such that $\varphi(\cdot, \eta): I_\eta \to \mathbb{R}^n$ where $I_\eta \subset \mathbb{R}$ is an open interval, $I_\eta = (\alpha_\eta, \beta_\eta)$ (that might depend on η).

If $\varphi(\cdot, \eta)$ is bounded on the interval $[0, \beta_\eta)$, then $\beta_\eta = \infty$

—— " —— " —— $(\alpha_\eta, 0]$, then $\alpha_\eta = -\infty$

—— " —— " —— I_η, then $I_\eta = \mathbb{R}$

Def: The function $(t, \eta) \in \mathbb{R} \times \mathbb{R}^n \mapsto \varphi(t, \eta) \in \mathbb{R}^n$ is called the flow of (2). η is an initial state and $\varphi(t, \eta)$ is the state at time t when starting from η.

Def: Let $\eta^* \in \mathbb{R}^n$. We say that η^* is an equilibrium (stationary) state (point) of (2) if $\eta(t, \eta^*) = \eta^*$, $\forall t \in \mathbb{R}$

Remark: η^* is an equilibrium $\Leftrightarrow$ the unique sol of the IVP $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta^* \end{cases}$ is the const. fct $\eta^* \Leftrightarrow f(\eta^*) = 0$

Def: Let η^* be an equilibrium point. We say that η^* is an attractor equilibrium when $\exists$ a neighborhood V_{η^*} of η^* s.t. $\forall \eta \in V_{\eta^*}$ we have $\lim_{t \to \infty} \varphi(t, \eta) = \eta^*$

Let η^* be an attractor eq. of (2). We define its basin of attraction

$$A_{\eta^*} = \{\eta \in \mathbb{R}^n : \lim_{t \to \infty} \varphi(t, \eta) = \eta^*\}.$$

When $A_{\eta^*} = \mathbb{R}^n$, we say that η^* is a global attractor.

If $+\infty$ is replaced by $-\infty$ we say that η^* is a repeller.

Def: Let $\eta \in \mathbb{R}^n$ be an initial state, its orbit is the set:

$$\gamma_\eta = \{\varphi(t, \eta) : t \in I_\eta\} \subset \mathbb{R}^n$$

Remark: γ_η is the image of $\varphi(\cdot, \eta)$. If η^* is an equilibrium, then $\gamma_{\eta^*} = \{\eta^*\}$

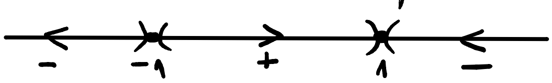
$\varphi(t, \eta)$ when $t > 0$ is said to be a future state.

when $t < 0$ is said to be a past state.

Def: The phase portrait of (2) is a representation of some significant orbits, on which we insert some arrows to indicate the future.

Without any explanation, we draw some phase portraits.

$\dot{x} = 1 - x^2$. $\mathbb{R}$ $(n = 1)$, the equilibrium points are -1 and 1

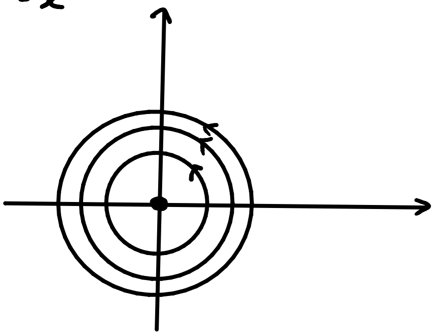


$\gamma_{-1} = \{-1\}$, $\gamma_1 = \{1\}$. The other orbits are: $(-\infty, -1)$, $(-1, 1)$, $(1, \infty)$

$\{\dot{x} = -y$

$-$ $\quad -1 \quad$ $+$ $\quad 1 \quad$ $-$ $\qquad (-\infty,-1),(-1,1),(1,\infty)$

$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$. $\mathbb{R}^2 (n=2)$, The unique equilibrium point is $(0,0)$



General method to represent the phase portrait of a scalar dynamical system

(3) $\dot{x} = f(x)$ $\quad f \in C^1(\mathbb{R})$ $\quad f : \mathbb{R} \to \mathbb{R}$

Step 1: We find all the equilibrium points, i.e. $f(x) = 0$

Step 2: We find the sign of f on each interval delimited by the equilibrium points.

Step 3: Represent on the real line the equilibrium points and the intervals delimited by them. These are the orbits of the system and we write them separately. On each orbit we insert an arrow following the rules: – $f > 0$: the arrow must indicate to the right

– $f < 0$: ——— " ——— " ——— left

Reading the phase portrait, we can deduce properties of the solutions of (3), $\varphi(\cdot,\eta)$, following the rules:

example : $\dot{x} = 1 - x^2$ $\quad$ ←)(→)(←

$\quad - \quad -1 \quad + \quad 1 \quad -$

for $\eta \in (-\infty;-1)$ we deduce that $\gamma_\eta = (-\infty,-1)$, so, by def. the image of $\varphi(\cdot,\eta)$ is $(-\infty;-1)$.

Since $f < 0$ on γ_η and $\varphi(\cdot,\eta)$ is the sol of the IVP: $\begin{cases} \dot{x} = f(x) < 0 \\ x(0) = \eta \end{cases}$ we deduce that $\varphi(\cdot,\eta)$ is strictly decreasing

Recall that $\varphi(\cdot,\eta)$ is defined on $I_\eta = (\alpha_\eta, \beta_\eta)$. Since $\varphi(\cdot,\eta)$ is strictly decreasing we deduce that and its image is $(-\infty,-1)$ → $\varphi(\cdot,\eta)$ is bounded in the past, i.e. on $(\alpha_\eta, 0]$. Then by the $\exists!$ th. we have $\alpha_\eta = -\infty$. Then $\lim_{t \to -\infty} \varphi(t,\eta) = -1$ and $\lim_{t \to \beta_\eta} \varphi(t,\eta) = -\infty$.

For $\eta = -1$ we have $\varphi(t,-1) = -1$, $\forall t \in \mathbb{R}$, that is the unique sol. of the IVP: $\begin{cases} \dot{x} = 1 - x^2 \\ x(0) = -1 \end{cases}$ is -1.

For $\eta \in (-1,1)$ we have (without proof) that $\gamma_\eta = (-1,1)$. Then, by def, the image of $\varphi(\cdot,\eta)$ is $(-1,1) \Rightarrow$ $\Rightarrow \varphi(\cdot,\eta)$ is bounded $\Rightarrow I_\eta = \mathbb{R}$, $\forall \eta \in (-1,1)$

Since $f > 0$ on $(-1,1)$ we have that $\varphi(\cdot,\eta)$ is strictly increasing. Also $\lim_{t \to \infty} \varphi(t,\eta) = 1$ and $\lim_{t \to -\infty} \varphi(t,\eta) = -1$

The following statement is true:

If $J \subset \mathbb{R}$ is an open interval delimited by the zeros of a function $f \in C^1(\mathbb{R})$ then $\forall \eta \in J$ we have that the orbit γ_η of $\dot{x} = f(x)$ is $\gamma_\eta = J$.

We can check this for some particular cases

$\dot{x} = -x$ $\qquad f(x) = -x$

x	$-\infty$		0		$+\infty$
f		$+ + + +$	0	$- - - -$	

$\longrightarrow \;)(\; \longleftarrow$

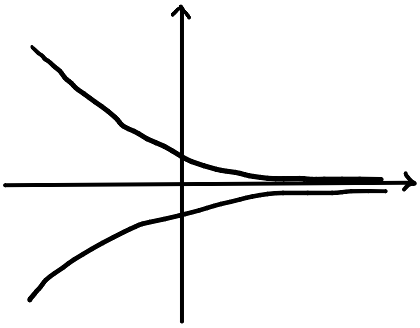
$+ \quad 0 \quad -$

our aim: find the flow. Check that the image of $\varphi(\cdot,\eta)$ is $(-\infty,0)$ when $\eta\in(-\infty;0)$, and respectively, that the image is $(0,\infty)$, when $\eta\in(0,\infty)$. Check also that $\lim\limits_{t\to\infty}\varphi(t,\eta)=0, \forall\eta\in\mathbb{R}$.

$\begin{cases}\dot x=-x\\ x(0)=\eta\end{cases}$ $\quad x=c\,e^{-t}, c\in\mathbb{R}$ $\quad$ So, the unique sol: $\varphi(t,\eta)=\eta\cdot e^{-t}, \forall\eta\in\mathbb{R}, \forall t\in\mathbb{R}$

$x(0)=c=\eta$

So we obtained the flow $\varphi:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Take $\eta\in(-\infty;0)$. Then the image of $\varphi(t,\eta)=\eta e^{-t}$ is $(-\infty,0)$.



is strictly increasing: $\dot\varphi=\underbrace{-\eta}_{>0}e^{-t}>0$

$\lim\limits_{t\to\infty}\varphi(t,\eta)=\eta\cdot e^{-\infty}=0$ $\quad \lim\limits_{t\to-\infty}\varphi(t,\eta)=\eta\, e^{+\infty}=-\infty$

Take $\eta\in(0,\infty)$. $\varphi(t,0)=0\cdot e^{-t}=0, \forall t\in\mathbb{R} \Rightarrow \eta^*=0$ is an equilibrium point

For $\eta\in\mathbb{R}$ $\lim\limits_{t\to\infty}\varphi(t,\eta)=\lim\limits_{t\to\infty}\eta\cdot e^{-t}=0$. So $\eta^*=0$ is a global attractor eq. of $\dot x=-x$

We come back to $\dot x=1-x^2$ $\quad \longleftarrow \;)(\; \longrightarrow \;)(\; \longleftarrow$

$- \quad -1 \quad + \quad 1 \quad -$

The equilibrium -1 is a repeller and $\lim\limits_{t\to-\infty}\varphi(t,\eta)=-1, \forall\eta\in(-\infty;1)$.

The equilibrium 1 is an attractor with $A_1=(-1,\infty)$, $\lim\limits_{t\to\infty}\varphi(t,\eta)=1, \forall\eta(-1,\infty)$.

$\dot x=1-x^2$ is a nonlinear eq.

The eq, as any scalar eq $\dot x=f(x)$, is a separable eq., i.e this eq can be integrated by separating the variables.

$\dot x=1-x^2$, -1 and 1 are constant solutions

$\frac{dx}{dt}=1-x^2$, $\frac{dx}{1-x^2}=dt \Rightarrow \underbrace{\int\frac{dx}{1-x^2}}=\int dt \Rightarrow$

$-\frac{\ln|x-1|-\ln|x+1|}{2}=t+c \Rightarrow \ln\left|\frac{x-1}{x+1}\right|=-2t+c \Rightarrow$

$\Rightarrow \frac{x-1}{x+1}=\pm e^{-2t}\cdot e^{c}$

$\frac{1}{1-x^2}=-\frac{1}{2}\,\frac{2}{x^2-1}=-\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)$