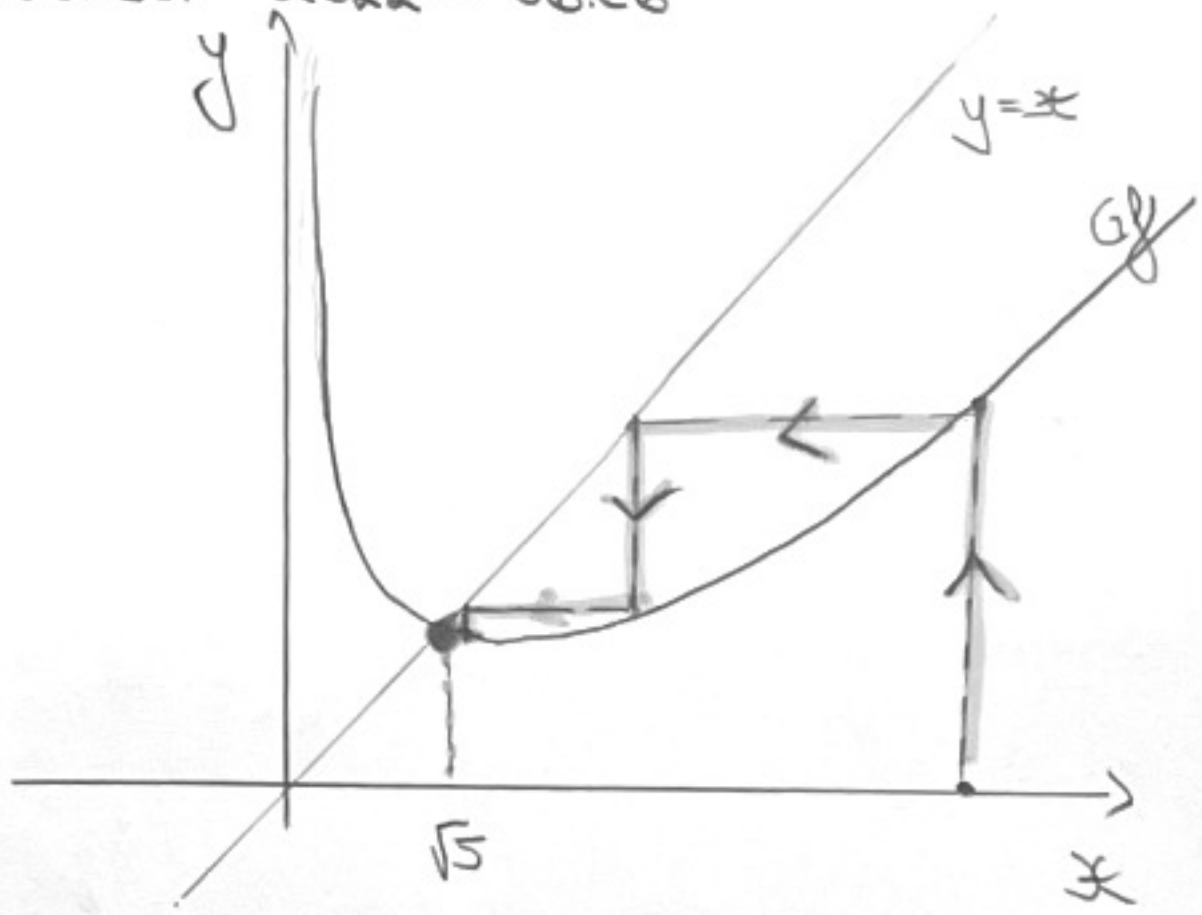
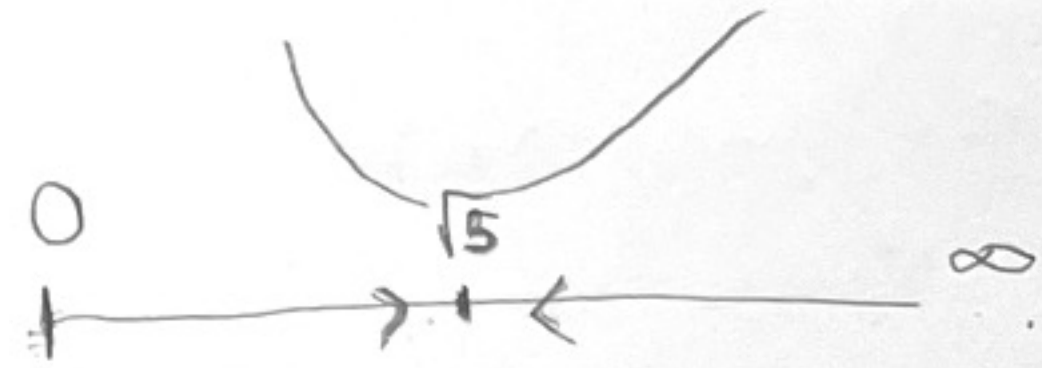
SUBIECT 2022 - 06.06

1.c





$A_{\sqrt{5}} = (0, \infty)$

1.a $f(\eta^*) = \eta^*$

$$\frac{\eta^{*2} + 5}{2\eta^*} = \eta^* \Rightarrow \eta^{*2} + 5 = 2\eta^{*2} \Rightarrow \eta^{*2} = 5 \Rightarrow \eta^* = \pm\sqrt{5}$$

but $\eta^* \in (0,\infty)$ } $\Rightarrow \eta^* = \sqrt{5}$ the unique fixed point

1b. $f'(x) = \dfrac{x^2 - 5}{2x}$

$$f'(\sqrt{5}) = \left| \frac{(\sqrt{5})^2 - 5}{2 \cdot (\sqrt{5})^2} \right| = 0 < 1 \Rightarrow \eta^* \text{ is an attractor}$$

2. $\begin{cases} \dot{x} = -y(x^2 + y^2) \\ \dot{y} = x(x^2 + y^2) \end{cases}$

a) $f(x,y) = \begin{pmatrix} -y(x^2+y^2) \\ x(x^2+y^2) \end{pmatrix}$

Equilibria $\Leftrightarrow f(x,y) = 0_2 \Rightarrow \begin{cases} -y(x^2+y^2) = 0 \\ x(x^2+y^2) = 0 \end{cases} \Rightarrow x = y = 0 \Rightarrow$

$\Rightarrow$ the only equil point is $(0,0)$

b) $Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} (-yx^2 - \cancel{y^3})'_x & (-yx^2 - y^3)'_y \\ (x^3 + xy^2)'_x & (\cancel{x^3} + xy^2)'_y \end{pmatrix}$

$= \begin{pmatrix} -y \cdot 2x & -x^2 - 3y^2 \\ 3x^2 + y^2 & 2xy \end{pmatrix}$

$J(f(\eta^*)) = J(f(0,0)) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\det(J(f(0,0)) - \lambda I_2) = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda^2$

$\Rightarrow \lambda^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 0 \Rightarrow Re(\lambda_1) = Re(\lambda_2) = 0 \Rightarrow (0,0)$ the point is not hyperbolic

2.c) $\dfrac{\dot\theta}{\cos^2\theta} = \dfrac{\dot y \cdot x - y\dot x}{x^2} = \dfrac{x^2(x^2+y^2) + y^2(x^2+y^2)}{x^2} = \dfrac{(x^2+y^2)(x^2+y^2)}{x^2} =$

$= \dfrac{(x^2+y^2)^2}{x^2} = \dfrac{\rho^{\cancel{4}\,2}}{\cancel{\rho^2}\cancel{\cos^2\theta}}$

Obs: $x^2+y^2 = \rho^2$

$x^2 = \rho^2\cos^2\theta$

$\Rightarrow \dot\theta = \rho^2 \,/\!\int dt \Rightarrow \theta(t) = \rho^2 \cdot t$

- $\varphi(t,1,0)$
  $x = 1$, $y = 0$ $\quad \rho = \sqrt{1^2+0^2} = 1$, $\theta(t) = \rho^2 \cdot t = t$ $\Rightarrow$ $x = \rho\cos\theta = \cos t$, $y = \rho\sin\theta = \sin t$
- $\varphi(t,2,0)$
  $x = 2$, $y = 0$ $\quad \rho = \sqrt{2^2+0^2} = \sqrt{4} = 2$, $\theta(t) = \rho^2 \cdot t = 4t$ $\Rightarrow$ $x = \rho\cos\theta = 2\cos 4t$, $y = \rho\sin\theta = 2\sin 4t$
- $\varphi(t,3,0)$
  $x = 3$, $y = 0$ $\quad \rho = \sqrt{3^2+0^2} = 3$, $\theta(t) = \rho^2 t = 9t$ $\Rightarrow$ $x = 3\cos 9t$, $y = 3\sin 9t$

d) $\begin{cases} x' = -y(x^2+y^2) \\ y' = x(x^2+y^2) \end{cases} \Rightarrow \begin{cases} \dfrac{dx}{dt} = -y(x^2+y^2) \\ \dfrac{dy}{dt} = x(x^2+y^2) \end{cases}$

$\dfrac{\frac{dy}{\cancel{dt}}}{\frac{dx}{\cancel{dt}}} = \dfrac{x\cancel{(x^2+y^2)}}{-y\cancel{(x^2+y^2)}} \Rightarrow \dfrac{dy}{dx} = \dfrac{x}{-y} \Rightarrow -y\,dy = x\,dx \;/\!\int \Rightarrow -\int y\,dy = \int x\,dx \Rightarrow$

$\Rightarrow -\dfrac{y^2}{2} = \dfrac{x^2}{2} + C \;/\cdot 2 \Rightarrow -y^2 = x^2 + C \Rightarrow C = -x^2 - y^2 = H(x,y)$ first integral

$\dfrac{\partial H}{\partial x} \cdot f_1 + \dfrac{\partial H}{\partial y} f_2 = 0 = (-x^2-y^2)'_x \cdot (-y(x^2+y^2)) + (-x^2-y^2)'_y (x(x^2+y^2)) = 0$

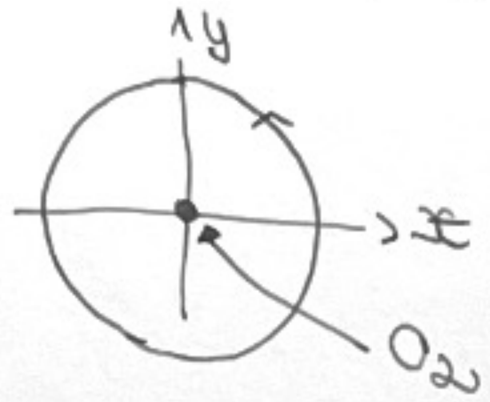
$\Rightarrow +2xy(x^2+y^2) - 2yx(x^2+y^2) = 0$ TRUE

e) $\rho\dot{\rho} = x\cdot\dot{x} + y\dot{y} = x(-y(x^2+y^2)) + y(x(x^2+y^2)) =$

$= -xy(x^2+y^2) + xy(x^2+y^2) = 0$

$\Rightarrow \dot{\rho} = 0 \Rightarrow \rho$ is constant along the orbit $\Rightarrow$ the orbit is a circle centered in O_2

$\Rightarrow \dot{\theta} = \rho^2 \Rightarrow > 0 \Rightarrow \theta \nearrow 0 \Rightarrow$ contraclockwise rotation



XAM 6 IUNIE 2022 — Teorema de existență și unicitate — $ax^2+bx+c=0$

a) $\begin{cases} x' = ax - 5y \\ y' = x - 2y \end{cases} \Rightarrow A = \begin{vmatrix} a & -5 \\ 1 & -2 \end{vmatrix}$

CENTRE $\Rightarrow \lambda_1, \lambda_2 = \pm i\beta$

$\begin{cases} \Delta < \\ b = 0 \end{cases}$

$\det(A - \lambda I) = \begin{vmatrix} a-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (a-\lambda)(-2-\lambda)+5$

$= -2a - a\lambda + 2\lambda + \lambda^2 + 5$

$= \lambda^2 + \lambda(2-a) - 2a + 5$

$\lambda^2 + \lambda(2-a) - 2a + 5$

$\Rightarrow 2 - a = 0 \Rightarrow a = 2$

vf: $\lambda^2 - 4 + 5 = 0$

$\lambda^2 + 1 = 0$

$\Delta = -4 = (2i)^2$

$\lambda_1 = \frac{2i}{2} = i$

$\lambda_2 = -\frac{2i}{2} = -i$

b) $a = 0$

$\begin{cases} x' = -5y \\ y' = x - 2y \end{cases} \quad /(\,)'_t \quad \Rightarrow y = \frac{x'}{-5}$

$x'' = -5y' = -5(x - 2y) = -5x + 10y = -5x + \cancel{10}^{2} \cdot \frac{x'}{\cancel{-5}} = -5x - 2x'$

$x'' + 2x' + 5x = 0$

$r^2 - 2r + 5 = 0$

$\Delta = 4 - 4 \cdot 5$

$= -16 = (4i)^2$

FOCUS, GLOBAL ATTRACTOR

$\Uparrow$

$r_1 = \frac{-2 + 4i}{2} = \boxed{-1 + 2i} \Rightarrow x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$

$\alpha = -1$

$\beta = 2$

$r_2 = 1 - 2i$

$x'(t) = c_1(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) + c_2(-e^{-t}\sin 2t + 2e^{-t}\cos 2t)$

the general solution

$\Rightarrow \begin{cases} x(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t \\ y(t) = \frac{c_1}{-5}(-e^{-t}\cos 2t - 2e^{-t}\sin 2t) - \frac{c_2}{5}(-e^{-t}\sin 2t + 2e^{-t}\cos 2t) \end{cases}$