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Mathematical Analysis Seminar

04. 01. 2022

Sequences

$f: \mathbb{N}^k \to \mathbb{R}$

$a_1, a_2, a_3, \dots, a_i \in \mathbb{R}$, $S_k = a_1 + \dots + a_k = \sum_{n=1}^{k} a_n$

$$\sum_{n=1}^{\infty} a_n = \sum_{n\geq 1} a_n = \lim_{k\to\infty} \sum_{n=1}^{k} a_n = \lim_{k\to\infty} S_k$$

58. a) $\sum_{n=1}^{\infty} \left(-\frac{\pi}{4}\right)^n$

$$1 + q + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q} \qquad |q| < 1$$

$$\left.\begin{aligned} S_k &= -\frac{\pi}{4} + \left(-\frac{\pi}{4}\right)^2 + \dots + \left(-\frac{\pi}{4}\right)^k \\ \left|-\frac{\pi}{4}\right| &< 1 \end{aligned}\right| \Rightarrow$$

$$\Rightarrow \left(-\frac{\pi}{4}\right)\left(1 + \left(-\frac{\pi}{4}\right) + \dots + \left(-\frac{\pi}{4}\right)^{k-1}\right) = \left(-\frac{\pi}{4}\right) \cdot \frac{1-\left(-\frac{\pi}{4}\right)^k}{1+\frac{\pi}{4}}$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \left(-\frac{\pi}{4}\right) \frac{1-\left(-\frac{\pi}{4}\right)^k}{1+\frac{\pi}{4}} = -\frac{\pi}{4} \cdot \frac{1}{1+\frac{\pi}{4}} = -\frac{\pi}{\cancel{4}} \cdot \frac{\cancel{4}}{4+\pi} = \frac{-\pi}{4+\pi}$$

f) $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right) = \sum_{n\geq 2} \ln \frac{(n-1)(n+1)}{n^2} = \sum_{n\geq 2}\left(\ln \frac{n-1}{n^2} + \right.$

$\left. + \ln \frac{n+1}{n^2}\right) = \sum_{n\geq 2}^{k} \ln \frac{n-1}{n} + \sum_{n\geq 2}^{k} \ln \frac{n+1}{n} =$

$= \ln \frac{1}{2} + \ln \frac{2}{3} + \dots + \ln \frac{k-1}{k} + \ln \frac{3}{2} + \ln \frac{4}{3} + \dots + \ln \frac{k+1}{k}$

$= \ln \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \dots \cdot \frac{\cancel{k-1}}{k} + \ln \frac{\cancel{3}}{2} \cdot \frac{4}{\cancel{3}} + \dots + \ln \frac{k+1}{\cancel{k}} =$

$= \ln \frac{1}{k} + \ln \frac{k+1}{2} = \ln \frac{k+1}{2k}$

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$$\lim_{k\to\infty} \ln\frac{k+1}{k} = \ln\frac{1}{2} = \sum_{n\geq 2} \ln\left(1-\frac{1}{n^2}\right)$$

$\sum_{n\geq 1} a_n$ convergent $\Leftrightarrow \sum_{n\geq 1} a_n$ finite

divergent when it's not convergent

$\sum_{n\geq 1} a_n$ convergent $\Rightarrow \lim_{n\to\infty} a_n = 0$

$\not\Leftarrow$

$p \to q \Leftrightarrow \neg p \to \neg q$

$\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n\geq 1} a_n$ divergent

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\dots >$$

$$> 1+\frac{1}{2}+\underbrace{\frac{1}{4}+\frac{1}{4}}_{\frac{1}{2}}+\underbrace{\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}}_{\frac{1}{2}}+\underbrace{\frac{1}{16}+\dots}_{\frac{1}{2}} =$$

$$= \sum_{k=1}^{\infty} \frac{1}{2} = \infty \Rightarrow \boxed{\sum_{n\geq 1} \frac{1}{n} \text{ divergent}}$$

$$\boxed{\sum_{n\geq 1} \frac{1}{n^p} \text{ conv} \Leftrightarrow p>1}$$

56. b)

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\underbrace{\frac{1}{n^2}}_{a_n} = \frac{1}{n\cdot n} < \underbrace{\frac{1}{n(n-1)}}_{b_n} = \frac{n-(n-1)}{n(n-1)} = \frac{n}{n(n-1)} - \frac{n-1}{n(n-1)} =$$

$$= \underbrace{\frac{1}{n-1} - \frac{1}{n}}_{b_n}$$

$$\sum_{n\geq 2}\left(\frac{1}{n-1}-\frac{1}{n}\right) = \frac{1}{1} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \dots + \cancel{\frac{1}{n-1}} - \frac{1}{n} \dots \Rightarrow$$

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$a_n, b_n \geq 0$

$\Rightarrow S_k = 1 - \frac{1}{k} \Rightarrow \lim\limits_{k \to \infty} S_k = 1 \Rightarrow \sum\limits_{n \geq 2} b_n$ convergent $\Big\}$ $\Rightarrow$

(1)

$\Rightarrow \sum\limits_{n \geq 2} a_n$ convergent $\Rightarrow \sum\limits_{n=1}^{\infty} a_n = 1 + \sum\limits_{n \geq 2} a_n$ convergent

57. a) $\sum\limits_{n \geq 1} \underbrace{\frac{e^n}{n + 3^n}}_{a}$

$$\frac{e^n}{n+3^n} < \frac{e^n}{3^n} = b_n$$

$$\sum_{n \geq 1} \frac{e^n}{3^n} = \sum_{n \geq 1} \left(\frac{e}{3}\right)^n = \lim_{n \to \infty} \frac{e}{3} \cdot \frac{1}{1 - \frac{e}{3}} = \frac{\frac{e}{3}}{1 - \frac{e}{3}} \Rightarrow \sum_{n \geq 1} b_n \text{ conv}$$

$a_n < b_n$ $\Big| \Rightarrow$

$\xrightarrow[\text{test} <333]{\text{com}} \sum\limits_{n \geq 1} a_n$ - conv.

Comparison Test II

$a_n, b_n > 0$

$b_n > n^2$

$\lim\limits_{n \to \infty} \frac{a_n}{b_n} = L < \infty$

$a_n = n$

1. $L \neq 0 \Rightarrow (\sum a_n \text{ conv} \Leftrightarrow \sum b_n \text{ conv})$
2. $L = 0 \Rightarrow (\sum b_n \text{ conv} \Rightarrow \sum a_n \text{ conv})$

b) $\sum\limits_{n \geq 1} \frac{1}{\underbrace{n^2 - \ln n + \sin n}_{a_n}}$

$b_n = \frac{1}{n^2}$

$\frac{\ln x}{x^2}$

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$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\cancel{n^2}}{\cancel{n^2}\left(1-\underbrace{\frac{\ln n}{n^2}}_{\to 0}+\underbrace{\frac{\sin n}{n^2}}_{\to 0}\right)}=1$$

$\Rightarrow$

$$\sum_{n\geq 1}\frac{1}{n^2} \text{ conv. } (2>1)$$

Comp. Test $\underset{II}{\Longrightarrow}$ $\sum_{n\geq 1} a_n$ convergent

$$\to (n+1)^{\frac{1}{2}} \qquad n^{-\frac{3}{2}}=\frac{1}{n^{\frac{3}{2}}}$$

e) $$\sum_{n\geq 1}\frac{\sqrt{n+1}}{1+2+\ldots+n}=\sum_{n\geq 1}\underbrace{\frac{\sqrt{n+1}}{\frac{n(n+1)}{2}}}_{a_n}$$

$$b_n=\frac{1}{n^{\frac{3}{2}}} \qquad \lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\frac{\sqrt{n+1}}{\frac{n^2+n}{2}}}{\frac{1}{n^{\frac{3}{2}}}}=$$

$$=\lim_{n\to\infty} n^{\frac{3}{2}}\frac{\sqrt{n+1}}{n^2\left(\frac{1+\frac{1}{n}}{2}\right)}=2$$

b_n - conv $\left(\frac{3}{2}>1\right)$

$\Rightarrow \sum_{n\geq 1} a_n$ conv

d'Alembert Rabo Test

$a_n>0$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\Delta$$

$\Delta<1 \Rightarrow \sum a_n$ conv

$\Delta\ 1 \Rightarrow \sum a_n$ div

Raabe - Duhamel

$a_n>0$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right)=R$$

$R<1 \Rightarrow \sum a_n$ div

$R>1 \Rightarrow \sum a_n$ conv

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Cauchy Root Test

$a_n \geq 0$

$$\lim_{n\to\infty} \sqrt[n]{a_n} = C$$

$C < 1 \Rightarrow \sum a_n$ conv

$C > 1 \Rightarrow \sum a_n$ div

d) $\displaystyle\sum_{n\geq 1} \frac{2^n \cdot n!}{n^n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{2^n \cdot n!} = \lim_{n\to\infty} \frac{2(n+1)\cdot n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!}$$

$$= 2\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = 2\cdot \underbrace{\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n}_{\frac{1}{e}} = \frac{2}{e} < 1 \stackrel{\text{dAld}}{\Longrightarrow}$$

$\Rightarrow \sum_{n\geq 1} a_n$ conv.

07.01.2022

Mathematical Analysis Course

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$H \cdot [u] = [sym] \quad 2 \quad sym = 2$

$(n - k - dig)$

Mathematical Analysis. Seminar

11.01.2022.

$f: \mathbb{R} \to \mathbb{R}$ $(n+1)$ times differentiable, $x_0 \in \mathbb{R}$

$$f(x) = f(x_0) + \frac{f'(x)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots +$$

$$+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$ c between x_0 and x

$c \in (\min\{x_0, x\}, \max\{x_0, x\})$

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$$f(x) = T_n(x) + R_n(x)$$

Taylor pol. of degree n Reminder

1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \cos x$

Find the Taylor polinomial $T_2(x)$ at 0, and the remainder $R_2(x)$, deduce that $1 - \frac{x^2}{2} \leq \cos x$

$$f'(x) = (\cos x)' = -\sin x$$

$$f''(x) = (-\sin x)' = -\cos x$$

$$f'''(x) = \sin x$$

$$f(x) = f(0) + \frac{f'(0)}{1}(x-0) + \frac{f''(0)}{2}(x-0)^2 + R_2(x) =$$

$$= 1 + \frac{0}{1} \cdot x + \frac{-1}{2}(x-0)^2 + R_2(x) =$$

$$= 1 - \frac{1}{2}x^2 + R_2(x) = \cos x$$

$$R_2(x) = \frac{f'''(c)}{3!}(x-0)^3 = \frac{\sin c}{6} x^3$$

if $x \in [2, \infty)$ then $1 - \frac{x^2}{2} \leq -1 \leq \cos x$

if $x \in (-\infty, 2]$ then $1 - \frac{x^2}{2} \leq -1 \leq \cos x$

if $x \in (-2, 2)$ $1 - \frac{x^2}{2}$

$x \in (-2, 0)$; $c \in (-2, 0) \Rightarrow c \in (-\pi, 0) \Rightarrow \sin c < 0$ $\Big| \Rightarrow$

$x \in (-2, 0) \Rightarrow x^3 < 0$

$\Rightarrow x^3 \sin c > 0$

$x \in (0, 2)$: $c \in [0, 2) \Rightarrow c \in (0, \pi) \Rightarrow \sin c > 0$ $\Big| \Rightarrow x^3 \sin c > 0$

$x \in [0, 2]$: $\Rightarrow x^3 > 0$

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$\Rightarrow 1 - \frac{x^2}{2} \le \cos x$

3. $f: \mathbb{R}^2 \to \mathbb{R}, \quad f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Continuity of f at $(0,0)$

$f_x(0,0) \qquad f_y(0,0)$

I. $x = 0 \Rightarrow f(0,y) = \frac{0 \cdot y}{0 + y^2} = 0 \Rightarrow \lim_{y \to 0} f(0,y) = 0$

II. $x = y \Rightarrow f(x,x) = \frac{x \cdot x}{x^2 + x^2} = \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$

$\Rightarrow \lim_{x\to 0} f(x,x) \ne \lim_{y \to 0} f(0,y) \Rightarrow f$ not cont at $(0,0)$

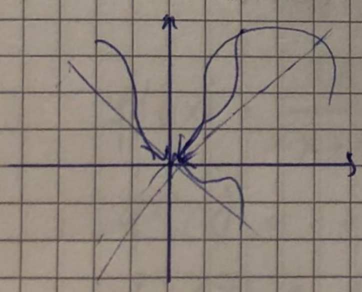
$$\left.\begin{aligned} \frac{\partial f}{\partial x}(0,0) &= \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{0}{x} = 0 \in \mathbb{R} \\ \frac{\partial f}{\partial y}(0,0) &= \lim_{y\to 0} \frac{f(y,0) - f(0,0)}{y} = \lim_{y \to 0} \frac{0}{y} = 0 \in \mathbb{R} \end{aligned}\right| \Rightarrow$$

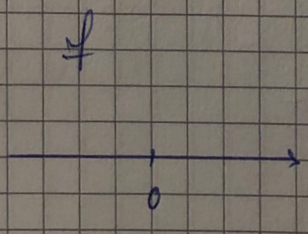
$\Rightarrow f$ is partial differentiable at $(0,0)$

(they don't have to be equal, only finit)

$f: \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(x_0, y_0) \in \mathbb{R} \Leftrightarrow$

$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0, y_0)$





$y = mx$

$(1,1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{1}{3}\right), \dots \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$

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$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x+y} =$$

if $x=0 \Rightarrow \lim_{y\searrow 0} \frac{y^2}{y} = 0$

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \quad \lim_{r\to 0} \frac{r^2\cos^2\varphi + r^2\sin^2\varphi}{r\cos\varphi + r\sin\varphi} = \lim_{r\to 0} \frac{r^2}{r\underbrace{(\cos\varphi + \sin\varphi)}_{\in[-2,2]}} \text{ } \delta$$

1. Let $\sum x_n$ be a convergent series with positive terms ($x_n \geq 0$). Are the following convergent?

a) $\sum_{n\geq 1} \frac{x_n}{1+x_n}$ $\quad 1 - \frac{1}{1+x_n}$

$$\left.\begin{array}{l} \frac{x_n}{1+x_n} \leq x_n \\ \sum x_n \text{ conv} \end{array}\right\} \xRightarrow{CT1} \sum_{n\geq 1} \frac{x_n}{1+x_n} \text{ conv}$$

b) $\sum_{n\geq 1} x_n^2$

$x_n \in [0,1) \Rightarrow x_n^2 \leq x_n$

$\sum x_n$ conv $\Rightarrow \lim_{n\to\infty} x_n = 0 \Rightarrow \exists\, N \in \mathbb{N};\ x_n < 1$

$\forall\, n > N \Rightarrow x_n^2 \leq x_n,\ \forall\, n > N$ $\Big|\ \xRightarrow{CT1}$

$\sum x_n$ conv

$\Rightarrow \sum x_n^2$ conv.

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9. $D=[0,1]\times[0,1]$, $I=\iint_D (1-x)\,y\,dx\,dy$

$I=\int_0^1\int_0^1 (1-x)y\,dx\,dy=\int_0^1 (1-x)\left(\frac{y^2}{2}\Big|_0^1\right)dx=\int_0^1 (1-x)\frac{1}{2}\,dx=$

$=\frac{1}{2}\int_0^1 dx-\frac{1}{2}\int_0^1 x\,dx=\frac{1}{2}x\Big|_0^1-\frac{1}{2}\frac{x^2}{2}\Big|_0^1=\frac{1}{2}-\frac{1}{4}=\frac{1}{4}$

10. According to the theorem of Fermat for functions of several variables $f:\mathbb{R}^n\to\mathbb{R}$ is differentiable $a\in\mathbb{R}^n$ and a is a local min for f, then $\nabla f(a)=0_{\mathbb{R}^n}$

11. $(a_n)_{n\in\mathbb{N}^*}$, $a_n>0$ such that $\sum_{n=1}^{\infty} a_n=\infty$ div

$\sum_{n=1}^{\infty} a_n^2=$ finite conv

$\sum_{n=1}^{\infty}\frac{1}{n}$ div ; $\sum_{n=1}^{\infty}\frac{1}{n^2}$ conv

12. a) $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^2+(y-1)^2$
subject to $x-y=0$

$(y-1)^2=y^2-2y+1$

Let $g:\mathbb{R}^2\to\mathbb{R}$, $g(x,y)=x-y$
$L:\mathbb{R}^2\to\mathbb{R}$, $L(x,y,\lambda)=x^2+(y-1)^2-\lambda(x-y)$

$\nabla L(x,y,\lambda)=\left(\frac{\partial L}{\partial x},\frac{\partial L}{\partial y},\frac{\partial L}{\partial \lambda}\right)=(0,0,0)$

$\frac{\partial L}{\partial x}=2x-\lambda$; $\frac{\partial L}{\partial y}=2y-2+\lambda$; $\frac{\partial L}{\partial \lambda}=-(x-y)=y-x$

$\nabla L(x,y,\lambda)=(2x-\lambda,\,2y-2+\lambda,\,y-x)=(0,0,0)\Rightarrow$

$\Rightarrow\begin{cases}2x-\lambda=0\\2y-2+\lambda=0\\y-x=0\iff y=x\end{cases}\quad\begin{cases}2x-\lambda=0\\\Rightarrow 2x+\lambda=2\ (+)\end{cases}$

$4x=2\Rightarrow x=\frac{1}{2}\Rightarrow y=\frac{1}{2}$

$\Rightarrow\lambda=2\cdot\frac{1}{2}=1\to S:\left\{\left(\frac{1}{2},\frac{1}{2},1\right)\right\}$

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4. $x = (1, 0, 1), \quad y = (0, 1, 0)$

$xy = 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0 = 0 + 0 + 0 = 0$

$\|x+y\| \leq \|x\| + \|y\| \Leftrightarrow \|x+y\| \leq \sqrt{1^2 + 0 + 1^2} + \sqrt{0^2 + 1^2 + 0^2} \Leftrightarrow$

$\Leftrightarrow \|x+y\| \leq \sqrt{1+1} + \sqrt{1} \Leftrightarrow \|x+y\| \leq \sqrt{2} + \sqrt{1}$

5. $f : \mathbb{R}^3 \to \mathbb{R}, \quad f(x_1, x_2, x_3) = x_1 x_2 + x_3^2 - 3x_2$

$\nabla f(x_1, x_2, x_3) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$

$\frac{\partial f}{\partial x_1} = x_2 \ ; \quad \frac{\partial f}{\partial x_2} = x_1 - 3 \ ; \quad \frac{\partial f}{\partial x_3} = 2x_3$

$\nabla f(x_1, x_2, x_3) = (x_2, x_1 - 3, 2x_3)$

6. $f : [a, b] \to \mathbb{R}$ Riemann integrable

$\Delta = \{ a = x_0, x_1, \dots, x_m = b \}$

a division of $[a, b]$ and $\xi = \{\xi_1, \dots, \xi_m\}$

$\xi_i \in [x_{i-1}, x_i]$

The Riemann sum associated to f, Δ, ξ is

$$\sigma(f, \Delta, \xi) = \sum_{k=1}^{m} f(\xi_k)(x_k - x_{k-1})$$

7. $$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{\substack{t \to 0 \\ t > 0}} \left(2 \int_t^1 \frac{x'}{2\sqrt{x}} \right) = \lim_{t \searrow 0} \left(2 \sqrt{x} \Big|_t^1 \right) =$$

$$= \lim_{t \searrow 0} \left(2 (\sqrt{1} - \sqrt{t}) \right) = 2 \lim_{t \searrow 0} (1 - t) = 2(1 - 0) = 2 \Rightarrow \text{conv}$$

8. $$I(t) = \int_t^{2t} t^2 x \, dx$$

$$I'(t) = \left(\int_t^{2t} t^2 x \, dx \right)' = \left(t^2 \int_t^{2t} x \, dx \right)' = \left(t^2 \frac{x^2}{2} \Big|_t^{2t} \right)' =$$

$$= \left(t^2 \left(\frac{(2t)^2}{2} - \frac{t^2}{2} \right) \right)' = \left(t^2 \frac{4t^2 - t^2}{2} \right)' = \left(t^2 \frac{3t^2}{2} \right)' = \left(\frac{3t^4}{2} \right)' =$$

$$= \frac{3}{2} \cdot 4t^3 = 6t^3$$

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b) $\frac{1}{1+x^2} = 1 - x^2 + x^4 - \ldots = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ Taylor expansion

$\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \ldots = \frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$ (A)

sum of geom series is $\frac{1}{1-q}$

On the other hand for $f(x) = \frac{1}{1+x^2}$, $f(0) = 1$

$f'(x) = \left((1+x^2)^{-1}\right)' = -(1+x^2)^{-2} \cdot 2x = -\frac{2x}{(1+x^2)^2}$,

$f''(x) = \left(-\frac{2x}{(1+x^2)^2}\right)' = -\frac{2}{(1+x^2)^2} = \underbrace{2x\left(\left(1+x^2\right)^{-2}\right)'}_{\text{This will vanish in } x_0=0}$, $f''(0) = -2$

....

Taylor $f(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \ldots$

so $\frac{1}{1+x^2} = 1 + 0 - x^2 + \ldots$ (B) (cf (A) and (B)) ✓

Second test

1. $\frac{d}{dx}(\sin \omega x) = \frac{d}{dx}(\sin \omega x) \cdot \frac{d}{dx}(\omega x) = \omega \cos \omega x$, $\omega \in \mathbb{R}$

2. $(\ln x)^2$ antiderivative $\Rightarrow F(x) = (\ln x)^2$

$F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x)$

$\int \ln^2 x\,dx = \int x' \ln^2 x\,dx = x \ln^2 x - \int x\,(\ln^2 x)'\,dx =$

$= x \ln^2 x - \int \left(\cancel{x} \cdot 2 \ln x \cdot \frac{1}{\cancel{x}}\right) dx = x \ln^2 x - 2 \int x' \ln x\,dx = =$

$= x \ln^2 x - 2x \ln x + 2 \int x \cdot \frac{1}{x}\,dx = x \ln^2 x - 2x \ln x + x + C$

3. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - x$

$f'(x) = 3x^2 - 1$

$f'(c) = 0 \Leftrightarrow 3x^2 - 1 = 0 \Leftrightarrow x^2 = \frac{1}{3} \Leftrightarrow x = \pm \frac{\sqrt{3}}{3}$

$\Rightarrow S: \left\{-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right\}$

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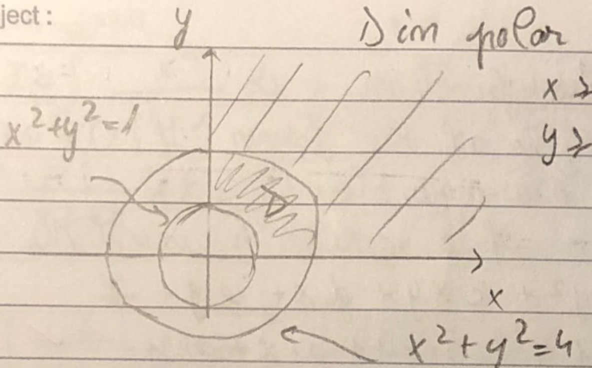

$x \geq 0$
$y \geq 0$

$$I = \int_1^2 \int_0^{\pi/2} \frac{1}{r} \, r \, dr \, d\varphi = \int_1^2 dr \int_0^{\pi/2} d\varphi = \frac{\pi}{2}$$ ✓

Jacobi det of transform / chang of coords

12. $f: \mathbb{R}^m \to \mathbb{R}$ differentiable
$a = (a_1, \dots, a_m)$, $b_m = (b_1, \dots, b_m) \in \mathbb{R}^m$
$c \in [a, b] \subset \mathbb{R}^m$ on the segment connecting a and b

$$f(b) - f(a) = \nabla f(c) \cdot (b-a)$$

Define $F: [0,1] \to \mathbb{R}$, $F(t) = f((1-t)a + tb)$
$F(0) = f(a)$
$F(1) = f(b)$

$$(*) \quad \frac{d}{dt} F(t) \overset{\text{CHAIN}}{=\!=} \nabla f((1-t)a+tb) \cdot \underbrace{\frac{d}{dt}((1-t)a+tb)}_{(b-a)}$$

F satisf. hyp of Th. Lagrange for function of one var, hence $\exists \, t_c \in (0,1)$: $F(1) - F(0) = F'(t_c)(1-0) \iff$
$\iff f(b) - f(a) = \nabla f(c) \cdot (b-a)$; $c = (1-t_c)a - t_c b$ ✓

13. a) $\sum_{m=0}^{\infty} (-1)^m x^{2m}$ converges for any $0 \leq x < 1$

$$\sum_{m=0}^{\infty} (-1)^m x^{2m} = \sum_{m=0}^{\infty} (-x^2)^m = \sum_{m=0}^{\infty} q^m \quad \text{(CONV)}$$ ✓

with $q = -x^2$ and $|q| = |x^2| < 1$

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7. $Q: \mathbb{R}^2 \to \mathbb{R}$ quadratic function

a quadratic function, is of the form $f(x) = ax^2+bx+c$
of one variable

a quadratic function of 2 variables is of the form
$$f(x,y) = ax^2 + by^2 + cxy + dx + ey + f$$
$$f(x,y) = x^2+y^2 = 1\cdot x^2 + 1\cdot y^2 + 0\cdot xy + 0\cdot x + 0\cdot y + 0$$ ✓

8. $\sum_{n=0}^{\infty} q^n, \ |q|<1$

$$s_n = 1 + q + q^2 + \dots + q^n = \frac{1-q^{n+1}}{1-q} \quad \text{geom. progression}$$

So $s_n \xrightarrow[|q|<1]{n\to\infty} \frac{1}{1-q}$ $\left[(s_n) \text{ CONV} \Rightarrow \sum_{n=0}^{\infty} q^n \text{ CONV} \right] \Rightarrow$ a) ✓

9. term of the Taylor expansion
$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \dots$$ ✓

10. $f: \mathbb{R}^2 \to \mathbb{R}$, $\nabla f(x_1^*, x_2^*) = (0,0)$, $(x_1^*, x_2^*) \in \mathbb{R}^2$
(x_1^*, x_2^*) is neither a local min nor a local max

$f(x_1, x_2) = x_1^2 - x_2^2$, $f(0,0) = 0^2 - 0^2 = 0$

$\frac{\partial f}{\partial x_1}(x_1,x_2) = 2x_1$; $\frac{\partial f}{\partial x_2}(x_1,x_2) = -2x_2$

$\frac{\partial f}{\partial x_1}(0,0) = 0$; $\frac{\partial f}{\partial x_2}(0,0) = 0$ $\Bigg| \Rightarrow \nabla f(0,0) = (0,0)$

(0,0) neither loc. min or loc max bcs.

$f(0,0) = 0 \le f(x_1, 0) = x_1^2$, $\forall x_1 \in \mathbb{R}$ while

$f(0,0) = 0 \geq f(0, x_2) = -2x_2^2$, $\forall x_2 \in \mathbb{R}$

11. $I = \iint_D \frac{1}{\sqrt{x^2+y^2}}\, dx\, dy$; $D = \{(x,y) \in \mathbb{R}^2 : 1 \le x^2+y^2 \le 4, x \ge 0, y \ge 0\}$

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $r \in [1, 2]$, $\theta \in [0, \frac{\pi}{2}]$
$r = \sqrt{x^2+y^2}$, $\theta = \operatorname{arctg} \frac{y}{x}$

Polar coords.

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4. $I=\int_{-\infty}^{\infty}\frac{x^2}{x^2+1}dx=\int_{-\infty}^{0}\frac{x^2}{x^2+1}dx+\int_{0}^{\infty}\frac{x^2}{x^2+1}dx=I'+I''$

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$I'=\int_{-\infty}^{0}\frac{x^2}{x^2+1}dx=\lim_{t\to-\infty}\int_{t}^{0}\frac{x^2}{x^2+1}dx$

$\int\frac{x^2}{x^2+1}dx=\int\frac{x^2+1-1}{x^2+1}dx=\int dx-\int\frac{1}{x^2+1}dx=x-\operatorname{arctg}x+C$

$I'=\lim_{t\to-\infty}(x-\operatorname{arctg}x)\Big|_t^0=\lim_{t\to-\infty}(0-\operatorname{arctg}0-t+\operatorname{arctg}t)=$

$=\lim_{t\to-\infty}(\operatorname{arctg}t-t)=-\frac{\pi}{2}+\infty=\infty$

$I''=\int_{0}^{\infty}\frac{x^2}{x^2+1}dx=\lim_{t\to\infty}\int_{0}^{t}\frac{x^2}{x^2+1}dx=\lim_{t\to\infty}(x-\operatorname{arctg}x)\Big|_0^t=$

$=\lim_{t\to\infty}(t-\operatorname{arctg}t)=\infty-\frac{\pi}{2}=\infty$

$I=\infty+\infty=\infty$ => divergent => b) ✓

if the result of the limit exists and is a finite number => conv
if the result of the limit doesn't exist or the limit is equal with $-\infty$ or ∞ → DIV

5. $D=[1,2]\times[0,1]$ $I=\iint_D xy\,dx\,dy$

$I=\int_1^2\int_0^1 xy\,dx\,dy=\int_1^2 x\cdot\left(\frac{y^2}{2}\Big|_0^1\right)dx=\int_1^2 x\left(\frac{1}{2}-\frac{0}{2}\right)dx=$

$=\int_1^2\frac{1}{2}x\,dx=\frac{1}{2}\cdot\frac{x^2}{2}\Big|_1^2=\frac{1}{2}\left(\frac{4}{2}-\frac{1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}=\frac{3}{4}$ ✓

6. According to the Theorem of Fermat for functions of several variables, if $f: B_r(x^*)\subset\mathbb{R}^m\to\mathbb{R}$ Fréchet differentiable in x^* and x^* is a local minimum (maximum) for f, then $\nabla f(x^*)=0_{\mathbb{R}^m}$ ✓

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Exam MA

1. $x = (1,1,0)$, $y = (0,1,1) \in \mathbb{R}^3$

$d(x,y) = \|x-y\| = \sqrt{(1-0)^2 + (1-1)^2 + (0-1)^2} = \sqrt{1+0+1} = \sqrt{2}$ ✓

$x \cdot y = 1\cdot 0 + 1\cdot 1 + 0\cdot 1 = 0+1+0 = 1$ ✓

2. $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x_1,x_2,x_3) = x_1x_2x_3 + x_3^2x_2$

$\frac{\partial f}{\partial x_1}(x_1,x_2,x_3) = x_2x_3 + \underbrace{(x_3^2x_2)'}_{\text{constant}} = x_2x_3 + 0 = x_2x_3$ ✓

$\frac{\partial f}{\partial x_2}(x_1,x_2,x_3) = x_1x_3 + x_3^2 = x_3(x_1+x_3)$ ✓

$\frac{\partial f}{\partial x_3}(x_1,x_2,x_3) = x_1x_2 + 2x_3x_2 = x_2(x_1+2x_3)$ ✓

3. $f: [a,b] \to \mathbb{R}$ Riemann integrale func.

$\Delta = \{a = x_0, x_1, \ldots, x_{m-1}, x_m = b\}$ a division

$\varepsilon = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{m-1}, \varepsilon_m\}$ with $\varepsilon_i \in [x_{i-1}, x_i]$
a system of intermediate points

The Riemann sum associated with f, Δ, ε:

$$\sigma(f, \Delta, \varepsilon) = \sum_{k=1}^{m} f(\varepsilon_k)(x_k - x_{k-1})$$ ✓

4. $f: [a,b] \to \mathbb{R}$ be a function defined on a closed interval $[a,b]$ of the real numbers, $\mathbb{R}$, and $P = \{[x_0,x_1],[x_1,x_2],\ldots,[x_{m-1},x_m]\}$, be a partition of I, where $a = x_0 < x_1 < x_2 < \cdots < x_m = b$

A Riemann sum S of f over I with partition P is defined as $S = \sum_{i=1}^{m} f(x_i^*)\Delta x_i$ where $\Delta x_i = x_i - x_{i-1}$

and $x_i^* \in [x_{i-1}, x_i]$. One might produce different Riemann sums depending on wich x_i^*'s are chosen. In the end this will not matter, if the function is Riemann integrable, when the difference or width of the summands Δx_i approaches zero

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c) $\sum_{n\geq 1} \sqrt{x_n}$

$\sum \frac{1}{n^2}$ conv but $\sum \frac{1}{n}$ div $\Rightarrow \sum \sqrt{x_n}$ not conv in general

d) $\sum_{n\geq 1} \frac{\sqrt{x_n}}{n}$ $\quad - \frac{1}{n}$

$a = x_n$

$\sqrt{\frac{x_n}{n^2}}$ $\quad b = \sqrt{x_n}$ $\quad \frac{a+b}{2} \geq \sqrt{ab}$ $\quad b = \frac{1}{n^2}$

$\frac{a+b}{2} = \frac{x_n + \frac{1}{n^2}}{2} \geq \frac{\sqrt{x_n}}{n} \geq \sqrt{ab}$

$\sum \frac{x_n + \frac{1}{n^2}}{2} = \frac{1}{2}\left(\underbrace{\sum x_n}_{\in \mathbb{R}} + \underbrace{\sum \frac{1}{n^2}}_{\in \mathbb{R}}\right) \in \mathbb{R}$ $\quad \overset{CT1}{\Longrightarrow} \sum \frac{\sqrt{x_n}}{n}$ conv

$f = T_n(x) + R_n(x)$

f can be expanded as a Taylor series $\Leftrightarrow$

$\Leftrightarrow \lim_{n\to\infty} R_n(x) = 0$

$f: (-1, \infty) \to \mathbb{R}$, $f(x) = \ln(x+1)$

Show that f can be expanded as a Taylor series around 0 on $[0, 1]$

$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$, $c \in (0, x) \subseteq (0, 1)$

$f'(x) = \frac{1}{x+1} = (x+1)^{-1}$

$f''(x) = -1(x+1)^{2}$

$f'''(x) = 2(x+1)^{-3}$

$\vdots$

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$$f^{(n)}(x) = (-1)^{n+1} (n-1)! \, (x+1)^{-n}$$

$$\Rightarrow R_n(x) = \frac{(-1)^{n+2} \cdot n! \, (c+1)^{-n-1}}{(n+1)!} x^{n+1} =$$

$$= (-1)^{n+2} \underbrace{\left(\frac{x}{c+1}\right)^{n+1}}_{\to 0} \underbrace{\frac{1}{n+1}}_{\to 0} \to 0$$

$\Rightarrow f$ can be expanded