PART III : SEQUENCES AND SERIES

LECTURE 11 : SERIES OF REALS

1. SEQUENCES OF REALS :

DEFINITION : A sequence is a map from the discrete set $\mathbb{N}^*$ to $\mathbb{R}$ (or some other space). $(a_n)_{n\in\mathbb{N}^*}$ $\quad n\in\mathbb{N}^* \longmapsto a_n \in \mathbb{R}$

DEFINITION : $(a_n)_{n\in\mathbb{N}^*}$ converges to $l\in\mathbb{R}$ if :

$\forall \varepsilon>0 \ \exists N(\varepsilon)\in\mathbb{N}^*$ such that $\forall n \in N(\varepsilon)$ we have $|a_n - l| < \varepsilon$

DEFINITION : $(a_n)_{n\in\mathbb{N}^*}$ is called fundamental (or Cauchy) sequence if :

$\forall \varepsilon>0 \ \exists N(\varepsilon)\in\mathbb{N}^*$ such that $\forall m,n > N(\varepsilon)$ we have $|a_m - a_n| < \varepsilon$

THEOREM : In $\mathbb{R}$ every Cauchy sequence converges.

REMARK : Any convergent sequence is Cauchy. However, the converse does not always hold.

2. SERIES OF REALS :

Let $(a_n)_{n\in\mathbb{N}^*}$ a sequence of reals ($a_n\in\mathbb{R}$) and define $(s_n)_{n\in\mathbb{N}^*}$ a sequence of partial sums by $s_n := a_1 + a_2 + \dots + a_n$.

DEFINITION : A series is a pair $((a_n)_{n\in\mathbb{N}^*}, (s_n)_{n\in\mathbb{N}^*})$ and is denoted by $\sum_{n=1}^{\infty} a_n$.

DEFINITION : $\sum_{n=1}^{\infty} a_n$ is convergent if $(s_n)_{n\in\mathbb{N}^*}$ is convergent.

If $s_n \xrightarrow{n\to\infty} S \in\mathbb{R}$ we write $\sum_{n=1}^{\infty} a_n = S$ and we say that S is the sum of the series.

DEFINITION : $\sum_{n=1}^{\infty} a_n$ is divergent if $s_n \longrightarrow \pm\infty$ or the limit does not exist.

THEOREM (Cauchy's general convergence crit) :

If $(s_n)_{n\in\mathbb{N}^*}$ convergent $\Rightarrow (s_n)_{n\in\mathbb{N}^*}$ is Cauchy

i.e. $\forall\varepsilon>0 \ \exists N(\varepsilon)\in\mathbb{N}^*$ such that $\forall p\in\mathbb{N}$ $|a_{N(\varepsilon)+1} + a_{N(\varepsilon)+2} + \dots + a_{N(\varepsilon)+p}| < \varepsilon$

REMARK : if $\sum_{n=1}^{\infty} a_n$ convergent $\underset{\not\Leftarrow}{\Rightarrow} a_n \xrightarrow{n\to\infty} 0$

if $a_n \not\longrightarrow 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ divergent.

3. SERIES OF POSITIVE REALS:

THEOREM (Cauchy's Integral crit):

$f:[1,\infty) \to \mathbb{R}_+$ decreasing

Define $(f_n)_{n\in\mathbb{N}^*}$ by $f_n := f(n)$

1) $\sum_{n=1}^{\infty} f_n$ convergent $\Leftrightarrow$ $\int_1^{\infty} f(x)\,dx$ convergent

2) $\sum_{n=1}^{\infty} f_n$ divergent $\Leftrightarrow$ $\int_1^{\infty} f(x)\,dx$ divergent

4. APPLICATIONS OF CAUCHY'S INTEGRAL CRIT:

a) The harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent (because $\int_1^{\infty} \frac{1}{x}\,dx$ is divergent)

b) The generalised harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (because $\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent)

c) The geometric series: $\sum_{n=1}^{\infty} q^n$ is $\begin{cases} \text{convergent if } |q|<1 \\ \text{divergent if } |q|\geq 1 \end{cases}$

5. CONVERGENCE TESTS FOR (NUMERICAL) SERIES:

A. SERIES OF POSITIVE NUMBERS:

Standing assumption $a_n > 0$ $\forall n \in \mathbb{N}^*$

COMPARISON I THEOREM: $a_n \leq b_n$ $\forall n\in\mathbb{N}^*$

$\sum_{n=1}^{\infty} a_n$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ divergent

$\sum_{n=1}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent

COMPARISON II THEOREM: $\frac{a_n}{b_n} \xrightarrow{n\to\infty} L < \infty$

if $L \neq 0$ both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ behave the same way (both convergent or divergent)

if $L = 0$ $\sum_{n=1}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent

$\sum_{n=1}^{\infty} a_n$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ divergent

CAUCHY's ROOT CRIT. THEOREM:

1) if $\sqrt[n]{a_n} \leq q \in (0,1)$, $\forall n > N_0 \in \mathbb{N}^*$ then $\sum_{n=1}^{\infty} a_n$ convergent

2) if $\sqrt[n_k]{a_{n_k}} > 1$ for some subsequence $(a_{n_k})_{k\in\mathbb{N}^*} \subset (a_n)_{n\in\mathbb{N}^*}$ then $\sum_{n=1}^{\infty} a_n$ divergent.

D'ALEMBERT RATIO TEST THEOREM:

1) if $\frac{a_{n+1}}{a_n} \leq q \in (0,1)$ $\forall n \geq N_0 \in \mathbb{N}^*$ then $\sum_{n=1}^{\infty} a_n$ convergent.

2) if $\frac{a_{n+1}}{a_n} \geq 1$ $\forall n \geq n_0 \in \mathbb{N}$ then $\sum_{n=1}^{\infty} a_n$ divergent

RAABE-DUHAMEL THEOREM:

1) if $\exists q > 1$ and $N_0 \in \mathbb{N}^*$ such that $n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq q$, $\forall n \geq N_0 \in \mathbb{N}^*$ then $\sum_{n=1}^{\infty} a_n$ convergent

2) if $\exists q > 1$ and $N_0 \in \mathbb{N}^*$ such that $n\left(\frac{a_n}{a_{n+1}} - 1\right) < q$, $\forall n \geq N_0 \in \mathbb{N}^*$ then $\sum_{n=1}^{\infty} a_n$ divergent

B. ALTERNATE SERIES:

Now a_n can also be <0.

ABEL-DIRICHLET THEOREM:

$(a_n)_{n\in\mathbb{N}^*}$ decreasing and with $a_n \xrightarrow{n\to\infty} 0$

$(b_n)_{n\in\mathbb{N}^*}$, $T_n = b_1 + \ldots + b_n$ and T_n is bounded ($|T_n| < M$, $\forall n \in \mathbb{N}^*$)

then $\sum_{n=1}^{\infty} a_n b_n$ convergent

THEOREM:

$(a_n)_{n\in\mathbb{N}^*}$ decreasing and $a_n \xrightarrow{n\to\infty} 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ convergent

LECTURE 12 : SEQUENCES AND SERIES OF FUNCTIONS

1. SEQUENCES OF FUNCTIONS :

POINTWISE CONVERGENCE DEFINITION : $f_n \xrightarrow[n\to\infty]{p.w.} f$

For any fixed $x \in [a,b]$ we have $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon, x) \in \mathbb{N}^*$ such that $\forall n > N(\varepsilon, x)$ we have $|f_n(x) - f(x)| < \varepsilon$

UNIFORM CONVERGENCE DEFINITION : $f_n \xrightarrow[n\to\infty]{u} f$ if

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ such that $\forall n > N(\varepsilon)$ (uniform w.r.t. x) we have

$|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [a,b]$

CONTINUITY THEOREM :

$f_n : [a,b] \to \mathbb{R}$ all continuous and $f_n \xrightarrow[n\to a]{u} f$, then f is continuous.

INTEGRABILITY THEOREM :

f_n all continuous and $f_n \xrightarrow{u} f$, then f integrable and $\lim\limits_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

DIFFERENTIABILITY THEOREM :

f_n all differentiable and :

$$\left.\begin{array}{l} 1)\ f_n \xrightarrow{p.w.} f \\ 2)\ f_n' \xrightarrow{u} g \end{array}\right\} \text{then } f \text{ is differentiable and } f' = g.$$

2. POWER SERIES :

$\sum\limits_{n=1}^{\infty} f_n \to$ series of functions $\quad (f_n)_{n\in\mathbb{N}^*} \to$ sequence of functions

$s_n(x) = f_1(x) + \dots + f_n(x)$, $(s_n)_{n\in\mathbb{N}^*} \to$ sequence of partial sums

DEFINITION :

$$\sum_{n=1}^{\infty} f_n \left\langle \begin{array}{l} \text{pointwise convergent} \Leftrightarrow s_n \xrightarrow{p.w.} f \\ \text{uniform convergent} \Leftrightarrow s_n \xrightarrow{u} f \end{array} \right.$$

WEIERSTRASS THEOREM :

$(f_n)_{n\in\mathbb{N}^*}$ sequence of functions, $f_n : [a,b] \to \mathbb{R}$, $(a_n)_{n\in\mathbb{N}^*}$ sequence of positive reals

and
$$\left.\begin{array}{l} 1)\ \sum\limits_{n=1}^{\infty} a_n \text{ convergent} \\ 2)\ |f_n(x)| \le a_n,\ \forall n \ge n_0,\ \forall x \in [a,b] \end{array}\right\} \text{then } \sum_{n=1}^{\infty} f_n \text{ convergent}$$

LECTURE 13 : FOURIER SERIES

1. OSCILLATIONS :

$u(t) = A \cdot \cos(\omega t + \varphi)$ A – amplitude

ω – frequency

$u(t) = a \cos \omega t + b \sin \omega t$ φ – initial phase

2. TRIGONOMETRIC SERIES :

$\frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos mx + b_m \sin mx$ is the Fourier series associated to

$f : [-\pi, \pi] \rightarrow \mathbb{R}$ integrable if : $\begin{cases} a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \\ b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \end{cases}$

f even $\Rightarrow b_m = 0$

f odd $\Rightarrow a_m = 0$

3. CONVERGENCE OF FOURIER SERIES :

$$f_m(x) = \frac{1}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} \cdot \frac{\sin \frac{2m+1}{2} t}{\sin \frac{t}{2}} \, dt$$

4. APPLICATION OF THE FOURIER SERIES :

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = ?$$

$a_m = 0$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} \sin mx \, dx = \frac{(-1)^{m+1}}{m}$$

$$\frac{a_0^2}{2} + \sum_{m=1}^{\infty} (a_m^2 + b_m^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx$$

$$\Leftrightarrow \sum_{m=1}^{\infty} \frac{1}{m^2} = \frac{1}{\pi} \cdot 2 \int_0^{\pi} \frac{x^2}{4} \, dx = \frac{1}{\pi} \cdot \frac{x^3}{6} \Big|_0^{\pi} = \frac{\pi^2}{6}$$

5. GIBBS PHENOMENON :

There is a price to pay for discontinuity (in the signal); namely the F-approx will oscilate close to the discontinuity.