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Linear homogeneous differential systems with constant coefficients

form: matrix $A \in \mathcal{M}_n(\mathbb{R})$, $X' = AX$, where the unknown

$$X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \dots \\ x_n(t) \end{pmatrix}$$

$e^{tA} \longrightarrow$ each column is a sol. of $X' = AX$

Recall $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$, $\forall\, t \in \mathbb{R}$ (1)

ex: 1) for $A = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ we have $e^{tA} = \operatorname{diag}(e^{t\lambda_1}, e^{t\lambda_2}, \dots e^{t\lambda_n})$.

$\boxed{M_1}$ 2) we found, using the initial def., $e^{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$, $\forall t \in \mathbb{R}$

$\boxed{M_2}$ using the equivalent of def. (1)

$$\cos t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \dots$$

$$\sin t = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \dots$$

$$A \overset{\text{not}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I_2$$

$A^3 = -A$

$A^4 = (-A)\cdot A = -A^2 = I_2$

$$\Rightarrow \quad A^k = \begin{cases} (-1)^p I_2 , & k = 2p \\ (-1)^p A , & k = 2p+1 \end{cases}$$

$$e^{tA} = \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p}}{(2p)!} \cdot I_2 + \sum_{p=0}^{\infty} \frac{(-1)^p t^{2p+1}}{(2p+1)!} \cdot A = (\cos t) \cdot I_2 + (\sin t)\cdot A$$

$A \in \mathcal{M}_n(\mathbb{R}) \qquad X' = AX$

$\boxed{T1}$ For any $\eta \in \mathbb{R}^n$, the unique sol. of the IVP $\begin{cases} X' = AX \\ X(0) = \eta \end{cases}$ is

$\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{tA}\eta$.

! The general sol. of $X' = AX$ can be written as $X = e^{At} \cdot C$, $C \in \mathbb{R}^n$ arbitrary.

! Let $P \in \mathcal{M}_n(\mathbb{R})$ be invertible. Then the general solution of $X' = AX$ can also be written as $X = e^{At} P \cdot C$, $C \in \mathbb{R}^n$ arbitrary.

$(C' = PC \Leftrightarrow C = P^{-1}C')$

Similar matrices

Def. 1: Let $A, B \in \mathcal{M}_n(\mathbb{C})$. We say that A is similar to B when $\exists\, P \in \mathcal{M}_n(\mathbb{C})$, $\exists\, P^{-1}$ s.t. $A = PBP^{-1}$.

Prop. 1: If $\lambda_1 \in \mathbb{C}$ is an eigenvalue of A and $u_1 \in \mathbb{C}^n$ is an eigenvector of A, then λ_1 is an eigenvalue of B and $P^{-1}u_1$ is the corresponding eigenv. of B, when A and B are similar.

Proof: $Au_1 = \lambda_1 u_1$, $u_1 \neq 0$

$PBP^{-1}u_1 = \lambda_1 u_1 \mid \cdot P^{-1} \Rightarrow B\underbrace{P^{-1}u_1}_{\neq 0} = \lambda_1 \underbrace{P^{-1}u_1}$

Prop. 2: Assume that A and B are similar.

1) $A^k = PB^kP^{-1}$, $\forall k \in \mathbb{N}$

2) $e^{tA} = Pe^{tB}P^{-1}$, $\forall t \in \mathbb{R}$

Proof: 1) Prove by induction

$k=0$ $\quad I_n = P \cdot I_n \cdot P^{-1}$ ✓

$k=1$ $\quad A = PBP^{-1}$ ✓

$A^{k+1} = A^k \cdot A = (PB^kP^{-1}) = PB^{k+1}P^{-1}$

Def. 2: We say that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ when $\exists$ a diagonal matrix $B = \mathcal{M}_n(\mathbb{R})$ s.t. A and B are similar.

We say that $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{C}$ when $\exists$ a diagonal matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that A and B are similar.

Prop. 3: $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{C}$ if and only if $\exists$ n linearly independent eigenvectors of A in $\mathbb{C}^n$.

$A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ if and only if any eigenvalue of A in $\mathbb{R}$ and $\exists$ n lin. indep. eigenvectors of A in $\mathbb{R}^n$.

Ex: a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\qquad$ b) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\qquad$ c) $A = \begin{pmatrix} 1 & 2 \\ -2 & 3 \end{pmatrix}$

Sol: a) $\det(A - \lambda I_2) = 0$

$A - \lambda I_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ -1 & -\lambda \end{pmatrix}$

$\det(A - \lambda I_2) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow +\lambda^2 + 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i$

$\lambda_1 = i, \quad \lambda_2 = -i$

Prop. 4: If the eigenvalues of A are not repeated, (there are n distinct eigenv.) then $\exists$ n lin. indep. eigenvectors.

b) $\det(B - \lambda I_2) = 0$

$B - \lambda I_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{pmatrix}$

$\det(B - \lambda I_2) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 = 0 \quad (=)$

$(=) \quad 1 - \lambda = 0 \quad (=) \quad \lambda_1 = \lambda_2 = 1$

$u = \begin{pmatrix} a \\ b \end{pmatrix} \quad Au = \lambda u$ $\quad$ u an eigenvect. corresp. to the eigenvalue λ.

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} a + b = a \\ b = b \end{cases} \qquad \begin{pmatrix} a \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ there are no 2 lin. indep. eigen-vectors.

$\Rightarrow$ B isn't diagonalizable over $\mathbb{R}$ or $\mathbb{C}$.

c) $\lambda_1 = \lambda_2 = 1$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} a = a \\ b = b \end{cases} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are 2 lin. indep. eigenvectors of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ C is diagonalizable over $\mathbb{C}$ and $\mathbb{R}$

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The general sol. of a system $X' = AX$ in the case that A is diagonalizable matrix over $\mathbb{C}$ / over $\mathbb{R}$.

Step 1: $A \in M_n(\mathbb{R})$ compute its eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$ (they can be repeated) and the corresp. eigenvectors $u_1, u_2, \dots, u_n \in \mathbb{C}^n$. Then decide whether it is diagonalizable or not.

Step 2: Write $D = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ and $P = (u_1 \ u_2 \dots u_n)$ (its columns are the eigenvectors, where the order is important)
$A = PDP^{-1}$

Step 3: $e^{tA} = Pe^{tD}P^{-1} = P \operatorname{diag}(e^{t\lambda_1} \ e^{t\lambda_2} \ \dots \ e^{t\lambda_n}) P^{-1}$
The general solution $X = e^{tA} \cdot C$, $C \in \mathbb{R}^n$ arbitrary.

Assume that A is diagonalizable over $\mathbb{R}$, then we already know the gen. sol. of $X' = AX$.

$$X = e^{tA} \cdot P \cdot C = P \cdot \operatorname{diag}(e^{t\lambda_1} \ e^{t\lambda_2} \ \dots \ e^{t\lambda_n}) \cdot C =$$
$$= \left(e^{t\lambda_1} u_1 \quad e^{t\lambda_2} u_2 \quad \dots \quad e^{t\lambda_n} u_n \right) \cdot \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} =$$
$$= c_1 e^{t\lambda_1} u_1 + c_2 e^{t\lambda_2} u_2 + \dots + c_n e^{t\lambda_n} u_n$$
$$c_1, c_2, \dots, c_n \in \mathbb{R}$$

$\boxed{\text{Hw}}$ Let $u \in \mathbb{R}^n$ be an eigenvector of A, corresp. to the eigenvalue $\lambda \in \mathbb{R}$. Check that $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(t) = e^{t\lambda} u$ is a sol. of $X' = AX$.

Ex: $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable over $\mathbb{R}$. Find the general sol. of $X' = AX$ in two ways.

Sol: $\det(A - \lambda I_2) = 0$

$\begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = 0 \iff -(1-\lambda)(1+\lambda) - 3 = 0 \iff$

$\iff -1 + \lambda^2 - 3 = 0 \iff \lambda^2 = 4 \iff$

$\iff \lambda_1 = -2, \ \lambda_2 = 2$

So A has 2 real eigenvalues $\lambda_1 \neq \lambda_2 \Rightarrow A$ is diagonalizable over $\mathbb{R}$.

$u_1 \neq 0 \quad Au_1 = -2u_1 \qquad u_1 = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = -2 \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{cases} a + 3b = -2a \\ a - b = -2b \end{cases} \iff \begin{cases} 3(a+b) = 0 \\ a + b = 0 \end{cases}$

$\iff a + b = 0 \Rightarrow b = -a \Rightarrow u_1 = a \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad a = 1 \ \Rightarrow u_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$u_2 \neq 0 \quad Au_2 = 2u_2 \qquad u_2 = \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{cases} a + 3b = 2a \\ a - b = 2b \end{cases} \Rightarrow \begin{cases} -a + 3b = 0 \\ a - 3b = 0 \end{cases} \Rightarrow a - 3b = 0 \Rightarrow$

$\Rightarrow a = 3b \Rightarrow u_2 = b \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow b = 1 \qquad u_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\boxed{M_1}$ 1^{st} method to write the gen. sol.

$D = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \qquad P^{-1} = ?$

$e^{tA} = P \cdot \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{pmatrix} \cdot P^{-1} \qquad X = e^{tA} C, \quad C \in \mathbb{R}^2$ arbitr.

$\boxed{M_2}$ 2^{nd} method to write the gen. sol.

$X = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 e^{-2t} + 3c_2 e^{2t} \\ -c_1 e^{-2t} + c_2 e^{2t} \end{pmatrix}$

$c_1, c_2 \in \mathbb{R}$