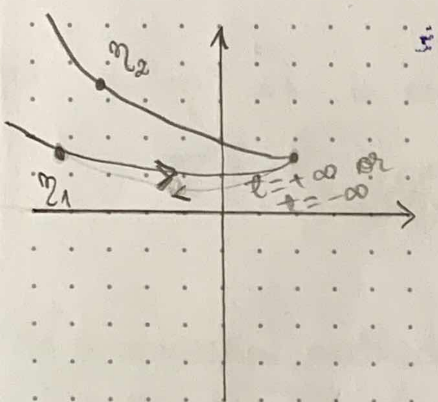
11. 04. 2022

# Planar dynamical systems

(1) $\dot{x} = f(x)$ where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

the unknown $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \in \mathbb{R}^2$ is the STATE SPACE

Lecture 7: state space, equilibrium point, orbit, phase portrait

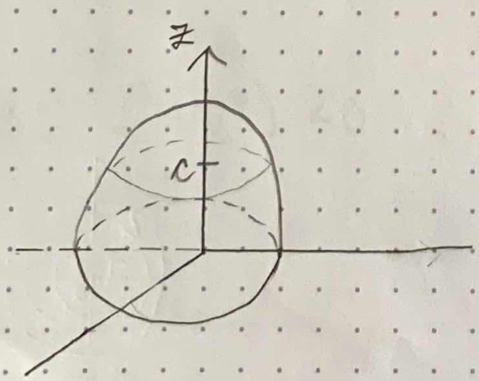



! 1) for any $\eta \in \mathbb{R}^2$ $\exists$ an orbit $\gamma_\eta$ s.t. $\eta \in \gamma_\eta$

2) for $\eta_1, \eta_2 \in \mathbb{R}^2$, $\eta_1 \neq \eta_2$ we have that either $\gamma_{\eta_1} \cap \gamma_{\eta_2} = \emptyset$ or $\gamma_{\eta_1} = \gamma_{\eta_2}$.

3) an orbit $\gamma_\eta$ ends only at infinity or near an attractor/repeller.

**Def:** Let $U \subset \mathbb{R}^2$ and $H : U \to \mathbb{R}$ continuous, $c \in \mathbb{R}$. The c-level curve of $H$ is $\Gamma_c = \{x \in U : H(x) = c\}$




$H : \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$

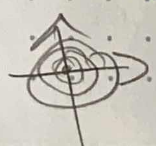
Represent in $\mathbb{R}^2$ the level curves of $H$.

$c \in \mathbb{R}$ $\quad x^2 + y^2 = c$

$c > 0$ circle centered in the origin with radius $\sqrt{c}$

$c = 0$ $\quad (0,0)$

$c < 0$ $\quad \emptyset$



**Def:** Let $U \subset \mathbb{R}^2$ be open and connected and $H: U \to \mathbb{R}$ $C^1$ function. We say that $H$ is a <u>first integral of (1) in $U$</u> if:

(i) $H$ isn't locally constant

(ii) $H(\underbrace{\varphi(t,\eta)}_{\text{sol of the sistem}}) = H(\eta)$ (is constant), $\forall \eta \in U$, $\forall t$ s.t. $\varphi(t,\eta) \in U$

**Def:** Let $U \subset \mathbb{R}^2$ be open and connected. We say that $U$ is an invariant set for (1) if $\forall \eta \in U$, we have $\underbrace{\gamma_\eta}_{\text{orbit}} \subseteq U$.

! (ii) $\Leftrightarrow$ $H\big|_{\gamma_\eta \cap U}$ is constant

Assume, in addition, that $U$ is an invariant set of (1) $\Rightarrow$ $\Rightarrow \forall \eta \in U$, $\gamma_\eta \subseteq U$.

(ii) $\Leftrightarrow$ $\gamma_\eta \subseteq \Gamma_{H(\eta)}$ (the orbits of (1) are contained in the level curves of a first integral)

**Ex:** $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$

Check that $(0,0)$ is the only equilibrium point, which is neither an attractor nor a repeller.

Check using the def. that $H: \mathbb{R}^2 \to \mathbb{R}$, $H(x,y) = x^2 + y^2$ is a global first integral. Represent the pp.

**Def:** A first integral $\mathbb{R}^2$ is said to be a global first integral.

$f(x,y) = \begin{pmatrix} -y \\ x \end{pmatrix}$

~~indeed~~ eq. points, $f(x,y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = y = 0$.

indeed, the only equil. pt. is $(0,0)$

we need the expression of the flow.

Let $\eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix} \in \mathbb{R}^2$, the ivp $\begin{cases}\dot x = -y\\ \dot y = x\\ x(0)=\eta_1\\ y(0)=\eta_2\end{cases}$

We use the reduction to a second order d.e. in $\underline{\underline{x}}$.

$$\frac{y=-\dot x}{\left.\begin{matrix}\dot x = -y\\ \dot y = x\end{matrix}\right\}} \Rightarrow \begin{array}{l}\ddot x + x = 0\\ r^2+1=0\\ r_{1,2} = \pm i \quad \longmapsto \begin{array}{l}\cos t\\ \sin t\end{array}\end{array}$$

$$\begin{cases} x = c_1\cos t + c_2 \sin t\\ y = c_1 \sin t - c_2\cos t\end{cases}, \quad c_1, c_2 \in \mathbb{R}$$

$$\left.\begin{array}{l} x(0) = c_1 = \eta_1 \Rightarrow c_1 = \eta_1\\ y(0) = -c_2 = \eta_2 \Rightarrow c_2 = -\eta_2\end{array}\right\} \Rightarrow \varphi(t,\eta_1,\eta_2) = \begin{pmatrix}\eta_1\cos t - \eta_2 \sin t\\ \eta_1\sin t + \eta_2\cos t\end{pmatrix}$$

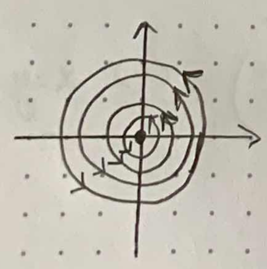
$$\forall t\in\mathbb{R},\ \forall\, \eta = \begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\in\mathbb{R}^2$$

Assume by contradiction that $(0,0)$ is an attractor. Then for $\eta$ close to $0 = \begin{pmatrix}0\\0\end{pmatrix}$

$\lim_{t\to\infty} (t,\eta) = 0$. False.

We check that the fct. is a global first integral.
$H \in C^1(\mathbb{R}^2)$ is not locally const.

$$H(\varphi(t,\eta)) = (\eta_1\cos t - \eta_2\sin t)^2 + (\eta_1\sin t + \eta_2\cos t)^2 =$$
$$= \eta_1^2\cos^2 t - \cancel{2\eta_1\eta_2\cos t\sin t} + \eta_2^2\sin^2 t +$$
$$+ \eta_1^2\sin^2 t + \cancel{2\eta_1\eta_2\sin t\cos t} + \eta_2^2\cos^2 t =$$
$$= \eta_1^2(\cos^2 t + \sin^2 t) + \eta_2^2(\sin^2 t + \cos^2 t) =$$
$$= \eta_1^2 + \eta_2^2 = H(\eta), \quad \forall t\in\mathbb{R}$$



Ex: $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ Check that $(0,0)$ is the only equil. pt., which is a global attractor.
Check that $H: \mathbb{R}\times(0,\infty) \to \mathbb{R}$, $H(x,y) = \frac{x}{y}$ is a first integral. Represent the pp.

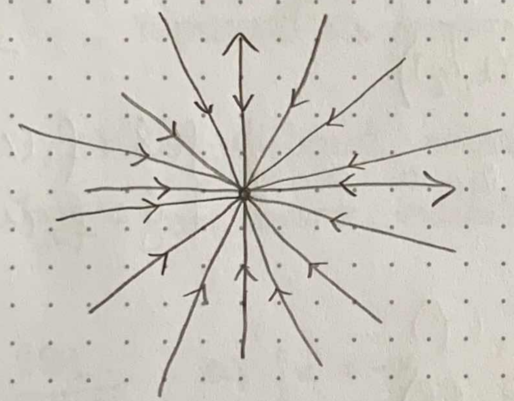
Compute the flow.

$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

$$\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{-t} \end{pmatrix}, \forall t \in \mathbb{R}$$

$\lim\limits_{t\to\infty} \varphi(t,\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Rightarrow$ the equilibrium point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a global attractor

$\forall \eta \in \mathbb{R}^2$



Note that $H_2: \mathbb{R}\times(-\infty,0) \to \mathbb{R}$
$H_2(x,y) = \frac{x}{y}$ is also a f.i.

Let $\eta = \begin{pmatrix} \eta_1 \\ 0 \end{pmatrix} \in \mathbb{R}^2$.

$$\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ 0 \end{pmatrix}$$

$$\gamma_\eta = \left\{ \begin{pmatrix} \eta_1 e^{-t} \\ 0 \end{pmatrix} : t \in \mathbb{R} \right\}$$

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~ Another method to check that a given function is a first integral:

Let $U \subset \mathbb{R}^2$ open and connected, $H : U \to \mathbb{R}$, $C^1$ and is not locally constant.

$H$ is a first integral of $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$ if and only if

$$f_1(x,y) \cdot \frac{\partial H}{\partial x}(x,y) + f_2(x,y) \cdot \frac{\partial H}{\partial y}(x,y) = 0, \quad \forall (x,y) \in U$$

<u>Proof</u>: $H$ is a f.i $\Leftrightarrow$ $H(\varphi(t,\eta)) = H(\eta)$, $\forall \eta, \forall t \ldots \Leftrightarrow$

$\Leftrightarrow \frac{d}{dt} H(\varphi(t,\eta)) = 0, \ \forall \eta, \forall t$

$\Leftrightarrow \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}; \quad \frac{\partial H}{\partial x}(\varphi(t,\eta)) \cdot \underbrace{\dot{\varphi}_1(t,\eta)}_{=f_1(\varphi(t,\eta))} + \frac{\partial H}{\partial y}(\varphi(t,\eta)) \cdot \overbrace{\dot{\varphi}_2(t,\eta)}^{=f_2(\varphi(t,\eta))} = 0 \quad \forall t$

by def.: $t \mapsto \varphi(t,\eta) = \begin{pmatrix} \varphi_1(t,\eta) \\ \varphi_2(t,\eta) \end{pmatrix}$ is a sol. of $\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$

$\Rightarrow \begin{cases} \dot{\varphi}_1 = f_1(\varphi_1, \varphi_2) \\ \dot{\varphi}_2 = f_2(\varphi_1, \varphi_2) \end{cases} \Rightarrow \begin{cases} \dot{\varphi}_1 = f_1(\varphi) \\ \dot{\varphi}_2 = f_2(\varphi) \end{cases}$

$\varphi(t,\eta)$ can be an arbitrary point in $U$