Dynamical Systems

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1 Questions from lecture 3

$$x'' + \omega_0^2 x = A\cos\omega t \tag{1}$$

$$x = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t + x_p$$

$$x_p = ?$$

Case 3.1 $\omega \neq \omega_0$

Q: Is any solution periodic?

$$\begin{aligned} x_p &= a\cos\omega t + b\sin\omega t \\ x_p' &= -a\omega\sin\omega t + b\omega\cos\omega t \\ x_p'' &= -a\omega^2\cos\omega t - b\omega^2\sin\omega t \end{aligned}$$

Substituting in (1) we get:

$$\begin{aligned} &a\omega_0^2\cos\omega t + b\omega_0^2\sin\omega t - a\omega^2\cos\omega t - b\omega^2\sin\omega t = A\cos\omega t \\ &\qquad \Rightarrow a(\omega_0^2 - \omega^2)\cos\omega t + b(\omega_0^2 - \omega^2)\sin\omega t = A\cos\omega t + 0\sin\omega t. \end{aligned}$$

But $\sin\omega t$ and $\cos\omega t$ are linearly independent, so:

$$\begin{cases} a(\omega_0^2 - \omega^2) &= A \\ b(\omega_0^2 - \omega^2) &= 0 \end{cases} \Rightarrow \begin{cases} a = \dfrac{A}{\omega_0^2 - \omega^2} \\ b = 0 \end{cases} .$$

Thus, we get:

$$x_p = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

The general solution is:

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

Let us try and see if there are periodic solutions. We will search for solutions with period $T = \frac{2\pi}{\omega_0}$. For this to happen, $x(t)$ has to equal $x(t+\frac{2\pi}{\omega_0})$.

$$x(t + \frac{2\pi}{\omega_0}) = c_1 \cos(\omega_0(t + \frac{2\pi}{\omega_0})) + c_2 \sin(\omega_0(t + \frac{2\pi}{\omega_0})) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega(t + \frac{2\pi}{\omega_0}))$$
$$\Rightarrow x(t + \frac{2\pi}{\omega_0}) = c_1 \cos(\omega_0 t + 2\pi) + c_2 \sin(\omega_0 t + 2\pi) + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t + 2\pi \frac{\omega}{\omega_0})$$
$$\Rightarrow x(t + \frac{2\pi}{\omega_0}) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos(\omega t + 2\pi \frac{\omega}{\omega_0}).$$

By forcing $x(t) = x(t + \frac{2\pi}{\omega_0})$ we get that $\cos \omega t = \cos(\omega t + 2\pi \frac{\omega}{\omega_0})$, therefore $\frac{\omega}{\omega_0}$ has to be an integer, so there are an infinite number of periodic solutions if the external force is chosen in an appropriate manner.

Case 3.2 $\omega = \omega_0$

Q: Show that any solution is unbounded.

$$\begin{aligned}
x_p &= t(a \cos \omega t + b \sin \omega t) \\
x_p' &= a \cos \omega t + b \sin \omega t + t(-a\omega \sin \omega t + b\omega \cos \omega t) \\
x_p'' &= -2a\omega \sin \omega t + 2b\omega \cos \omega t + t(-a\omega^2 \cos \omega t - b\omega^2 \sin \omega t)
\end{aligned}$$

Substituting in (1) we get:

$$-2a\omega \sin \omega t + 2b\omega \cos \omega t - \omega^2 t(a \cos \omega t + b \sin \omega t) + \omega^2 t(a \cos \omega t + b \sin \omega t) = A \cos \omega t$$

$$\Rightarrow -2a\omega \sin \omega t + 2b\omega \cos \omega t = A \cos \omega t + 0 \sin \omega t.$$

But $\sin \omega t$ and $\cos \omega t$ are linearly independent, so:

$$\begin{cases} -2a\omega & = 0 \\ 2b\omega & = A \end{cases} \Rightarrow \begin{cases} a = 0 \\ b = \dfrac{A}{2\omega} \end{cases} .$$

Thus, we get:

$$x_p = \frac{A}{2\omega} t \sin \omega t.$$

The general solution is:

$$x(t) = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t + \frac{A}{2\omega} t \sin \omega t.$$

It is easy to see that regardless of the choices for c_1 and c_2, the third term is unbounded (we can choose a sequence $a_n = 2\pi n + \frac{\pi}{2\omega} \to \infty$ as $n \to \infty$ to show this), therefore all solutions are unbounded.

2 Resonance

The following information is copied from wikipedia.

Resonance describes the phenomenon of increased amplitude that occurs when the frequency of a periodically applied force (or a Fourier component of it) is equal or close to a natural frequency of the system on which it acts. When an oscillating force is applied at a resonant frequency of a dynamic system, the system will oscillate at a higher amplitude than when the same force is applied at other, non-resonant frequencies.

A column of soldiers marching in regular step on a narrow and structurally flexible bridge can set it into dangerously large amplitude oscillations. April 12, 1831, the Broughton Suspension Bridge near Salford, England collapsed while a group of British soldiers were marching across. Since then, the British Army has had a standing order for soldiers to break stride when marching across bridges, to avoid resonance from their regular marching pattern affecting the bridge.

Structural resonance of a suspension bridge induced by winds can lead to its catastrophic collapse. Several early suspension bridges in Europe and USA were destroyed by structural resonance induced by modest winds. The collapse of the Tacoma Narrows Bridge on 7 November 1940 is characterized in physics as a classic example of resonance.

- Tacoma Bridge Collapse: `https://www.youtube.com/watch?v=XggxeuFDaDU`
- Breaking Glass with Sound: `https://www.youtube.com/watch?v=CdUoFIZSuX0`