LECTURE 13

DATE : 3 JANUARY 2022
WEEK 13

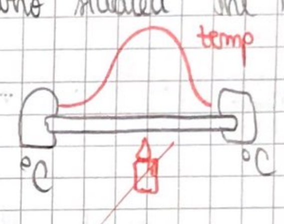
13. Fourier Series

What & Why ?

What are the most important ideas of univ level Math ?
Some very bright guy said : Optimization & Fourier analysis
signal processing, telecom, control theory
Partial Differential Equations

It all started with J. Fourier 1807, 1822 who studied the HEAT equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial t}(t,x)$$



§ 13.1. Small recap : From Power Series to Fourier Series

What is a (numerical) series?
(= "infinite sum that makes sense")

$(a_n)_{n \in \mathbb{N}^*}$ sequence of reals , $s_n = a_1 + a_2 + \dots + a_n$
sequence of partial sums

$$\sum_{n=1}^{\infty} a_n \text{ (CONV)} \iff s_n \xrightarrow{n \to \infty} S < \infty$$

the sum of the series is a number

Series of functions $f_n, f : [a,b] \longrightarrow \mathbb{R}$

$$\sum_{n=1}^{\infty} f_n \qquad \begin{array}{ll} (p\ \text{CONV}) & \text{"pointwise" w.r.t. } x \\ (u\ \text{CONV}) & \text{uniform w.r.t. } x \in [a,b] \end{array}$$

The sum of the series is a function

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

Why series of functions? (Approximation)

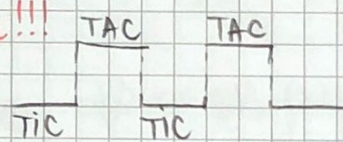
Taylor approx : $f(x) \simeq T_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

want to let $n \longrightarrow \infty$ (perfect approx)

Power series $(x_0 = 0)$ $\quad f(x) = \sum_{n=1}^{\infty} a_n x^n$

$a_n \in \mathbb{R}$ $\quad f_n(x) = x^n$

FOURIER : We can do better!!!



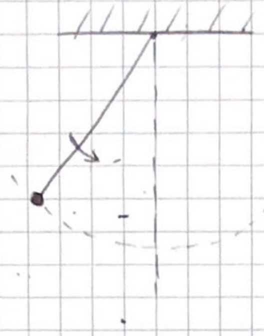
- ask for less regularity
  (Taylor : $n+1$ cont. derivatives, $n \longrightarrow \infty$)
- global approx (not just x_0-dependent)

The Price : give up Polynomials, replace them by "Trigonometric" polynomials

replace Power Series by Trigonometric Series

§ 13. 2. Fourier Series

Oscillations : Galileo Galilei (first Pend. Clock!)



$$u(t) = \underset{\text{amplitude}}{A} \cos(\underset{\text{frequency}}{\omega} t + \underset{\text{initial phase}}{\varphi})$$

$$\overset{\text{easy trigon}}{=} a \cdot \cos \omega t + b \sin \omega t$$

The Trigonometric series :

$$(1) \quad \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

is the Fourier series associated to $f : [-\pi, \pi] \to \mathbb{R}$ integrable if

$$(2) \begin{cases} a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \, dx \end{cases}$$

Remark : f even then $b_n = 0$
f odd then $a_n = 0$

Remark (Counterexample) : NOT all Trig. series are Fourier series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \cos nx$$

NOT (u CONV)
$\nexists f$ s.t. (2) holds

Fourier series are trigonometric series with assoc. "signal"

Theorem 1 (BESSEL Inequality)

Let $f : [-\pi, \pi] \longrightarrow \mathbb{R}$ square integrable

then $$s_m = \frac{a_0^2}{2} + \sum_{k=1}^{m} (a_k^2 + b_k)^2 \leq \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2 \, dx \quad (3)$$

where a_m, b_m are Fourier coefficients given in (2)

Idea of Proof:

Define $f_m(x) = \frac{a_0}{2} + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)$

Compute $$0 \leq \int_{-\pi}^{\pi} (f(x) - f_m(x))^2 \, dx = \underbrace{\int_{-\pi}^{\pi} f(x)^2 \, dx}_{T_1} - 2 \underbrace{\int_{-\pi}^{\pi} f(x) f_m(x) \, dx}_{T_2} + \underbrace{\int_{-\pi}^{\pi} f_m(x)^2 \, dx}_{T_3}$$

using the "magic" properties of sin & cos

Property 2 : $$\int_{-\pi}^{\pi} \cos kx \cos jx \, dx = 0 \quad \forall k, j \in \mathbb{N}^*, \ k \neq j$$

$$\int_{-\pi}^{\pi} \sin kx \sin jx \, dx = 0 \quad k \neq j$$

$$\int_{-\pi}^{\pi} \sin kx \cos jx \, dx = 0$$

$$\int_{-\pi}^{\pi} (\sin kx)^2 \, dx = \int_{-\pi}^{\pi} (\cos kx)^2 \, dx = \pi$$

Theorem 3 (PARSEVAL's Equality / Identity)

If the Fourier series associated to f (μ CONV) then (3) holds with equality as $n \to \infty$

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 \, dx \quad (5)$$

What's the Deep Insight behind this Fourier thing?

Square integrable functions form "space" with good geometry (just like $\mathbb{R}^n$ but with $n \to \infty$) called Hilbert space

The operations are:

"$+$" sum $\quad (f+g)(x) = f(x) + g(x)$

"αf" scaling $\quad (\alpha f)(x) = \alpha f(x)$, $\alpha \in \mathbb{R}$

"$\langle f, g \rangle$" $\quad \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx \in \mathbb{R}$

inner product / scalar product (the $\mathbb{R}^d$ notation was $x \cdot y$)

$\sin kx$, $\cos kx$ form an orthog. base

(just like $(1, 0, \dots, 0)$
$(0, 1, \dots, 0)$
$\vdots$
$(0, 0, \dots, 1, 0)$
$(0, 0, \dots, 0, 1)$ did in $\mathbb{R}^d$)

§13.3. The convergence of Fourier series

Important open question in the 19th century:
When does a Fourier series (CONV)?

Property 4 (DIRICHLET's Formula):

$$f_n(x) = \frac{1}{\pi} \int_0^{\pi} \frac{f(x+t) + f(x-t)}{2} \cdot \frac{\sin \frac{2n+1}{2} t}{\sin \frac{t}{2}} \, dt$$

in (4)

Theorem 5 (DIRICHLET):

$f : [-\pi, \pi] \to \mathbb{R}$ pointwise-diffable then Fourier series converges at any x and its sum is given by

($\exists x_i$, $i = 0, N$, $x_0 = -\pi$, $x_N = \pi$ s.t. f diff on each (x_i, x_{i+1}))

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = \frac{f(x+0) + f(x-0)}{2}$$

left-right limits

§13.4. Concluding Remarks and a Remarkable Application

Remark: Gibbs Phenomenon: there is a price to pay for discont (in the signal); namely the F-approx will oscilate close to the discont.

Application of Fourier series:

Compute sum of $\sum_{n=1}^{\infty} \frac{1}{n^2} = ? = \frac{\pi^2}{6}$

Idea: apply Parseval Theorem to $f(x) = \frac{x}{2}$ use (2) to see that

$a_n = 0$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{x}{2} \sin nx \, dx = \frac{(-1)^{n+1}}{n}$

$$(5) \quad \underbrace{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)}_{\text{reduces to } \sum_{n=1}^{\infty} \frac{1}{n^2}} = \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx}_{\frac{1}{\pi} \cdot 2 \int_0^{\pi} \frac{x^2}{4} dx = \frac{1}{\pi} \frac{x^3}{6} \Big|_0^{\pi} = \frac{\pi^2}{6}}$$