LECTURE 11

DATE: 13 DECEMBER 2021
WEEK 12

Part III : Sequences & Series

11. Series of reals

Theme : Infinite sums and sums of infinite

$1 + (-1) + 1 + (-1) + \dots$

2 groupings : $\underbrace{(1-1)}_{=0} + \underbrace{(1-1)}_{=0} + \dots = 0$

$1 + \underbrace{(1-1)}_{=0} + \underbrace{(1-1)}_{=0} + \dots = 1$

§ 11.1. Sequences of reals :

What is a sequence ?

A sequence is a map from the discrete set $\mathbb{N}^*$ to $\mathbb{R}$ (or some other space)

$(a_n)_{n \in \mathbb{N}^*}$ $\qquad \mathbb{N}^* \ni n \longmapsto a_n \in \mathbb{R}$

Def : (a_n) converges to $l \in \mathbb{R}$ if
$\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}^*$ such that $\forall n \in N(\varepsilon)$ we have $|a_n - l| < \varepsilon$

Motivation : APPROXIMATION

standard example : $1, 1.4, 1.42 \dots \longrightarrow \sqrt{2}$

Def: $(a_n)_{n\in\mathbb{N}^*}$ is called fundamental (or Cauchy) sequence if $\forall \varepsilon>0\ \exists N(\varepsilon)\in\mathbb{N}^*$ such that $\forall m,n > N(\varepsilon)$ we have $|a_m - a_n| < \varepsilon$

Theorem 1: In $\mathbb{R}$ every Cauchy sequence converges.

Remark: Any convergent sequence is Cauchy. However, the converse does not always hold.

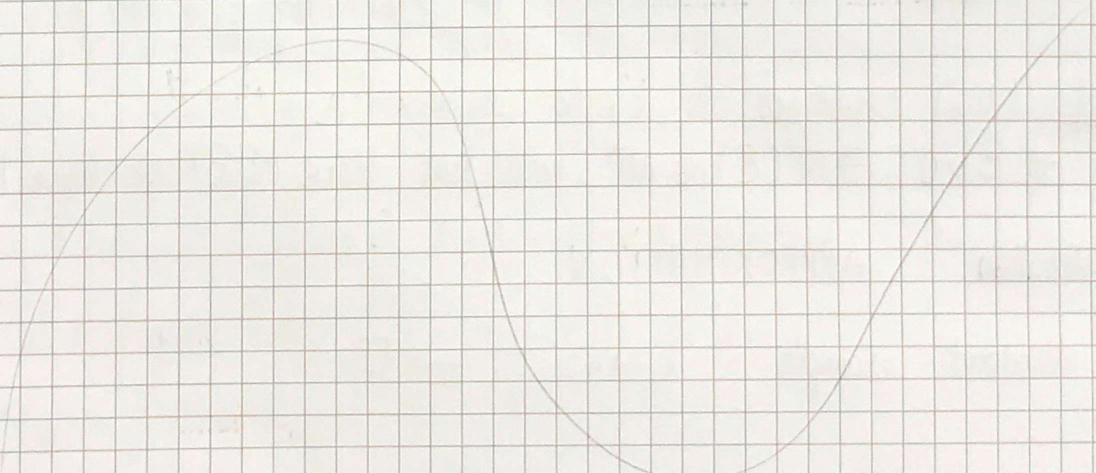
Counterexample: $\mathbb{Q}$

$a_1 = 1$, $a_2 = 1{,}4$, $a_3 = 1{,}41$, $a_4 = 1{,}414\ldots$

(better and better approximation of $\sqrt{2}$)

$(a_n)_{n\in\mathbb{N}^*} \in \mathbb{Q}$ is Cauchy but it does not converge to a limit in $\mathbb{Q}$.

In this case, we say that $\mathbb{Q}$ is not complete (by contrast to $\mathbb{R}$ which is complete = contains all limits of Cauchy sequences)



Important INSIGHTS :

1. Regard sequences as infinite length versions of $(a_1, \dots, a_n) \in \mathbb{R}^n$

2. All the above definitions work also in $\mathbb{R}^d$ (not $\mathbb{R}$) or some other, more general / abstract set X provided one can measure dist. in X.

§11.2. Series of reals :

Let $(a_n)_{n\in\mathbb{N}^*}$ sequence of reals $(a_n \in \mathbb{R})$ and define $(s_n)_{n\in\mathbb{N}^*}$ sequence of partial sums by $s_n := a_1 + a_2 + \dots + a_n$

Def : A series is a pair $((a_n)_{n\in\mathbb{N}^*}, (s_n)_{n\in\mathbb{N}^*})$ and is denoted as $\sum_{n=1}^{\infty} a_n$.

Def : $\sum_{n=1}^{\infty} a_n$ is (CONV) if $(s_n)_{n\in\mathbb{N}^*}$ conv.

If $s_n \xrightarrow{n\to\infty} S \in \mathbb{R}$ we write $\sum_{n=1}^{\infty} a_n = S$ and say that S is the sum of the series.

Def : $\sum_{n=1}^{\infty} a_n$ is (DIV) if $s_n \to \pm\infty$ or $\lim \nexists$

Meaning of (CONV) : infinite sum makes sense

There are two questions :
- Does $\sum_{n=1}^{\infty} a_n$ CONV ?
- What is the sum ?

Theorem 2 : Cauchy's general convergence crit.

Based on Theorem 1 $(s_n)_{n\in\mathbb{N}^*}$ CONV $\Longleftrightarrow$ $(s_n)_{n\in\mathbb{N}^*}$ Cauchy

i.e. $\forall \varepsilon > 0 \ \exists N(\varepsilon) \in \mathbb{N}^*$ such that

$$\forall p \in \mathbb{N} \quad |a_{N(\varepsilon)+1} + a_{N(\varepsilon)+2} + \dots + a_{N(\varepsilon)+p}| < \varepsilon$$

Remark : $a_n \xrightarrow{n\to\infty} 0$ is a necessary (but not sufficient) condition for $\sum_{n=1}^{\infty} a_n$ (CONV)

i.e $\sum_{n=1}^{\infty} a_n$ (CONV) $\underset{\not\Longleftarrow}{\Longrightarrow}$ $a_n \xrightarrow{n\to\infty} 0$

Counterexample (see later) : $\sum_{n=1}^{\infty} \frac{1}{n}$ (DIV)

this is used to prove that a sequence is divergent:

if you have / prove $a_n \not\to 0$ then $\sum_{n=1}^{\infty} a_n$ (DIV)

Example : $a_n = 1 \quad \forall n = 1, 2, \dots$

const. sequence generates a (DIV) series

§ 11. 3. Series of positive reals

Standing assumption $a_n > 0 \quad \forall n \in \mathbb{N}$

Theorem 3 (CAUCHY's Integral crit.):

$f : [1, \infty) \longrightarrow \mathbb{R}_+$ decreasing

Define $(f_n)_{n \in \mathbb{N}^*}$ by $f_n := f(n)$

$$\sum_{n=1}^{\infty} f_n \quad \begin{matrix} (\text{CONV}) & \Leftrightarrow & (\text{CONV}) \\ (\text{DIV}) & \Leftrightarrow & (\text{DIV}) \end{matrix} \quad \int_1^{\infty} f(x)\, dx$$

Connection between series & improper integrals!

Proof: $\quad s_n = f_1 + \ldots + f_n \,, \quad (s_n)_{n \in \mathbb{N}^*}$

f decreasing $\Rightarrow f_{n+1} = f(n+1) \leq f(x) \leq f(n) = f_n$ for any $x \in [n, n+1]$

integrate over $[n, n+1]$ with respect to x

i.e. $\quad f_{n+1} \leq f(x) \leq f_n \quad \Big| \int_n^{n+1} dx$

$$f_{n+1} \underbrace{\int_n^{n+1} dx}_{1} \leq \int_n^{n+1} f(x)\, dx \leq f_n \underbrace{\int_n^{n+1} dx}_{1}$$

so $\quad f_{n+1} \leq \int_n^{n+1} f(x)\, dx \leq f_n \quad \Big| \sum_{n=1}^{N}$

$$s_{N+1} - f_1 \leq \int_1^{N+1} f(x)\, dx \leq s_N$$

conclusion follows from (CONV)/(DIV) of $\int$

Applications of Cauchy's Integral Criteria

1. The harmonic series : $\sum_{n=1}^{\infty} \frac{1}{n}$ is (DIV)

(because $\int_1^{\infty} \frac{1}{x}\,dx$ is (DIV))

2. The gen. harmonic series : $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (CONV)

(because $\int_1^{\infty} \frac{1}{x^2}\,dx$ is (CONV)

3. The geometric series : $\sum_{\substack{n=1 \\ \text{or } n=0}}^{\infty} q^n$

is (CONV) if $|q| < 1$ and (DIV) otherwise

Remark : Computing the sum of a series can be very tricky!

Example is $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is (CONV) but sum is not easy to compute (see Lecture 3rd Jan)

§ 11. 3. Series of positive reals (Cont'ed)

There will be additional material (uploaded in Files) which you'll have to study on your own

(CONV) criteria