Date 28.09.21

Mathematical Analysis. Seminar

1. Sketch the graph of the following function:

$$f : [0, \infty) \to \mathbb{R}, \quad f(x) = x^2 e^{-x}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}} \lim_{x\to\infty} \frac{2x}{e^x} = 0$$

$f(0) = 0$

$$f'(x) = (x^2 e^{-x})' = 2x e^{-x} - x^2 e^{-x} = e^{-x}(2x - x^2) =$$

$$= x\underbrace{e^{-x}}_{\neq 0}(2-x) = 0 \Rightarrow x \in \{0, 2\} \to \text{critical points of } f$$

$$f''(x) = (f'(x))' = -e^{-x}(2x - x^2) + e^{-x}(2 - 2x) =$$

$$= e^{-x}(2 - 2x - 2x + x^2) = \underbrace{e^{-x}}_{\neq 0}(x^2 - 4x + 2) \Rightarrow$$

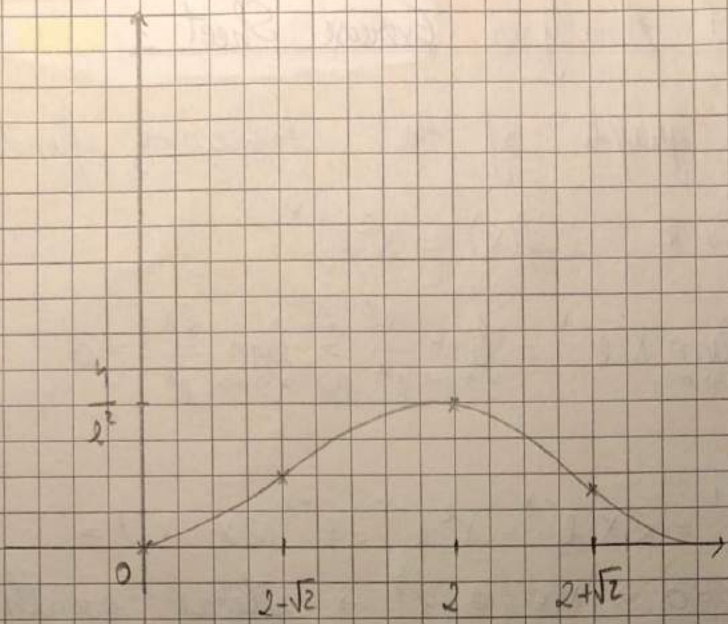
$$\Rightarrow x^2 - 4x + 2 = 0 \qquad x_{1,2} = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2} \Rightarrow$$

$$\Delta = 16 - 8 = 8$$

$$\Rightarrow x_{1,2} \in \{2 + \sqrt{2},\ 2 - \sqrt{2}\} \qquad f(2) = \frac{4}{e^2}$$

x	0		$2-\sqrt{2}$		2		$2+\sqrt{2}$		∞
$f(x)$	0	$\nearrow$		$\nearrow$	0	$\searrow$		$\searrow$	
$f'(x)$	0	$+ + +$	$+$	$+$	0	$- - -$	$-$	$- - -$	
$f''(x)$	$+$	$+ +$	0	$- - -$	$-$	$- -$	0	$+ + +$	
$f(x)$	0		0				0		

Subject : Date :......./......./.......



2. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{x^2}{1+x^2}$

$$\lim_{x\to\infty} \frac{x^2}{1+x^2} \underset{l'H}{\overset{\frac{\infty}{\infty}}{=}} 1 \qquad \lim_{x\to-\infty} \frac{x^2}{1+x^2} \underset{l'H}{\overset{\frac{\infty}{\infty}}{=}} 1$$

$$f'(x) = \left(\frac{x^2}{1+x^2}\right)' = \frac{2x(1+x^2) - x^2\, 2x}{(1+x^2)^2} = \frac{2x(1+\cancel{x^2}-\cancel{x^2})}{(1+x^2)^2} =$$

$$= \frac{2x}{(1+x^2)^2} \Rightarrow f'(x) = 0 \Leftrightarrow \frac{2x}{(1+x^2)^2} = 0 \Leftrightarrow 2x = 0 \Rightarrow x = 0$$

$$f''(x) = (f'(x))' = \left(\frac{2x}{(1+x^2)^2}\right)' = \frac{2(1+x^2)^2 - 2x \cdot 2 \cdot 2x(1+x^2)}{(1+x^2)^4} =$$

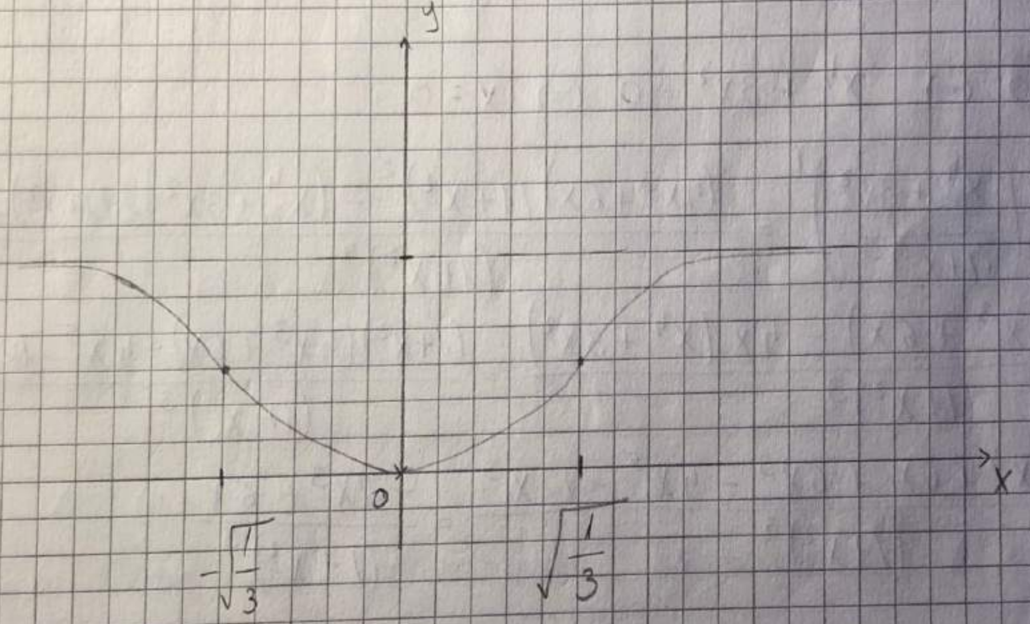
$$= \frac{(1+x^2)(2(1+x^2) - 8x^2)}{(1+x^2)^4} = \frac{2+2x^2-8x^2}{(1+x^2)^3} = \frac{2-6x^2}{(1+x^2)^3}$$

$$f''(x) = 0 \Rightarrow \frac{2-6x^2}{(1+x^2)^3} = 0 \Leftrightarrow 2-6x^2 = 0 \Rightarrow 6x^2 = 2 \Leftrightarrow$$

Subject : Date :....../....../......

$\Leftrightarrow 3x^2 = 1 \Leftrightarrow x^2 = \frac{1}{3} \Rightarrow x \in \left\{-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right\}$

x	$-\infty$		$-\sqrt{\frac{1}{3}}$		0		$\sqrt{\frac{1}{3}}$		∞
$f(x)$	1	↘		↘	0	↗		↗	1
$f'(x)$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$
$f''(x)$	$-$	$-$	0	$+$	$+$	$+$	0	$-$	$-$
$f(x)$		⌒	0		⌣		0	⌒	



3. $f : [0, \infty) \to \mathbb{R}, \quad f(x) = \frac{x^3}{1+x^2}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{1+x^2} = \infty$

$y = mx + n$ – oblique / slant asimptot

$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad n = \lim_{x\to\infty} (f(x) - mx)$

$m = \lim_{x\to\infty} \left(\frac{x^3}{1+x^2} \cdot \frac{1}{x}\right) = \lim_{x\to\infty} \frac{x^2}{1+x^2} = 1$

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$$n = \lim_{x\to\infty}\left(\frac{x^3}{1+x^2} - x\right) = \lim_{x\to\infty}\frac{\cancel{x^3} - x - \cancel{x^3}}{1+x^2} = 0$$

$y = 1$ the tg of graph

$$f'(x) = \left(\frac{x^3}{1+x^2}\right)' = \frac{3x^2(1+x^2) - x^3 \cdot 2x}{(1+x^2)^2} = \frac{3x^2 + 3x^4 - 2x^4}{(1+x^2)^2} =$$

$$= \frac{x^4 + 3x^2}{(1+x^2)^2} \geq 0$$

$$f'(x) = 0 \iff x^4 + 3x^2 = 0 \iff x = 0$$

$$f''(x) = \left(\frac{x^4+3x^2}{(1+x^2)^2}\right)' = \frac{(4x^3+6x)(1+x^2)^2 - (x^4+3x^2)2(1+x^2)2x}{(1+x^2)^4} =$$

$$= \frac{(1+x^2)(4x^3+6x) - 4x(x^4+3x^2)}{(1+x^2)^3} = \frac{(1+x^2)(4x^3+6x) - 4x^5 - 12x^3}{(1+x^2)^3} =$$

$$= \frac{4x^3 + 4x^5 + 6x + 6x^3 - 4x^5 - 12x^3}{(1+x^2)^3} = \frac{-2x^3 + 6x}{(1+x^2)^3}$$

$$f''(x) = 0 \Rightarrow -2x^3 + 6x = 0 \iff x(-2x^2+6) = 0 \Rightarrow$$

$$\Rightarrow x \in \{0, \pm\sqrt{3}\}$$

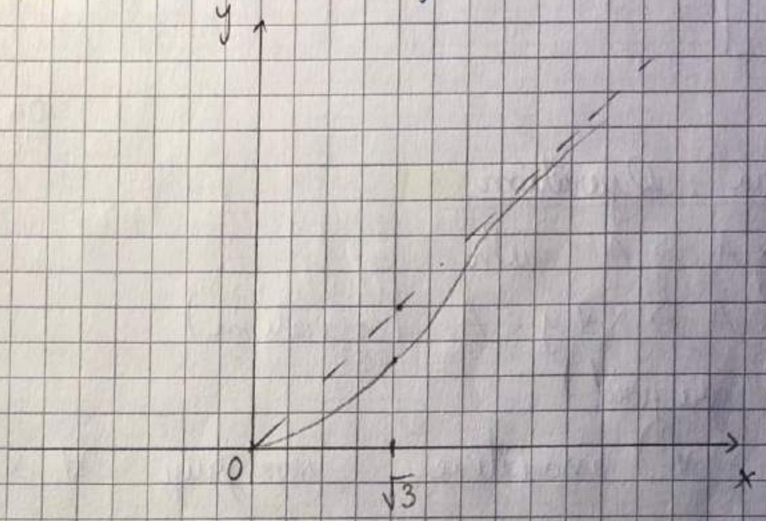
x	0				$\sqrt{3}$				∞
$f(x)$	0								
$f'(x)$	0	+	+	+	+	+	+	+	+
$f''(x)$	+	+	+	+	0	−	−	−	−
$f(x)$	0				0				

Subject : Date : ....../....../.......

$$f(\sqrt{3}) = \frac{(\sqrt{3})^3}{1+(\sqrt{3})^2} = \frac{3\sqrt{3}}{1+3} = \frac{3\sqrt{3}}{4}$$

f diff at $x_0 \overset{def}{\Longleftrightarrow} \lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$

exists and it is finite



H.W. 4.

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \dfrac{x^4}{1+x^2}$

Find a polynomial function $P(x)$ (of minimal degree) such that:

$$|f(x) - P(x)| \to 0 \quad \text{when } x \to \infty$$

$P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0$, $a \in \mathbb{R}$

$$\left| \frac{x^4}{1+x^2} - \underbrace{\overset{1+x^2}{a_0}}_{P_0(x)} \right| = \left| \frac{x^4 - a_0 - a_0 x^2}{1+x^2} \right| \xrightarrow{x \to \infty} \infty$$

$$\left| \frac{x^4}{1+x^2} - \overset{1+x^2}{\underbrace{(a_2 x^2 + a_1 x + a_0)}_{P_2(x)}} \right| = \left| \frac{x^4 - a_2 x^2 - a_1 x - a_0 -}{1+x^2} \right.$$

Subject :

Date :......./......./.......

$$\left| \frac{-a_0 x^4 - a_1 x^3 - a_0 x^2}{} \right| = \left| \frac{x^4(1-a_2) - a_1 x^3 - x^2(a_2+a_0) - a_1 x - a_0}{1+x^2} \right|$$

$$\left.\begin{aligned} a_2 &= 1 \\ a_1 &= 0 \\ a_0 &= -1 \end{aligned}\right| \Rightarrow \boxed{P_2(x) = x^2 - 1}$$

04.10.2021

Algebra . Seminar

$* : A \times A \to A$ with

$x, y \in A \Rightarrow x * y \in A$ (operation)

$(A, *)$ (grupoid)

$(A, *)$, $*$ - associative semigroup: $\forall\ x, y, z \in A$:

$(x * y) * z = x * (y * z)$

Subject: Date: 05.10.21

Mathematical Analysis. Seminar

11. Show that the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sqrt[3]{x}$, is not differentiable at 0 although its derivative at 0 exists.

$f:\mathbb{R}\to\mathbb{R}$ has a derivative at $x_0 \Leftrightarrow \lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$ exists

If this limit is finite, than f is differentiable at x_0.

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{\sqrt[3]{x}}{x}=\lim_{x\to 0}\sqrt[3]{\frac{x}{x^3}}=\lim_{x\to 0}\frac{1}{\sqrt[3]{x^2}}=\infty \text{ exists}$$

$\Rightarrow f$ has a derivative at 0

Because the limit is not finite the function f is not differentiable at 0.

10. $f:\mathbb{R}\to\mathbb{R}$ $\qquad f(x)=\begin{cases}x^2, & x\geq 0\\ -x^2, & x<0\end{cases}$

How many times is f differentiable?

f is differentiable on $\mathbb{R}\setminus\{0\}$ and we have to check differentiability at 0.

$$f'_l(0)=\lim_{x\nearrow 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\nearrow 0}\frac{-x^2-0}{x}=\lim_{x\nearrow 0}(-x)=0$$

$$f'_r(0)=\lim_{x\searrow 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\searrow 0}\frac{x^2}{x}=\lim_{x\searrow 0}x=0$$

$f'_l(0)=f'_r(0)\Rightarrow f$ is differentiable at 0 at least once

$$f'(x)=\begin{cases}2x, & x>0\\ -2x, & x<0\\ 0, & x=0\end{cases}$$

Subject : Date : ....../....../......

$$f''_l(0) = \lim_{x \nearrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \nearrow 0} \frac{-2x - 0}{x} = -2$$

$$f''_r(0) = \lim_{x \searrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \searrow 0} \frac{2x}{x} = 2$$

$f''_r(0) \neq f''_l(0) \Rightarrow f'$ is not differentiable at 0

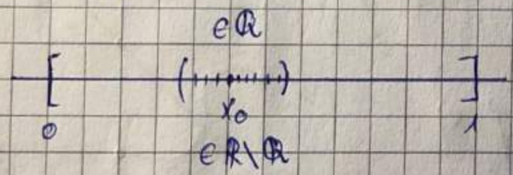
$\Rightarrow f$ is only once differentiable

12. $f, g : [0, 1] \to \mathbb{R}$ continuous functions such that $f(x) = g(x) \quad \forall\, x \in [0, 1] \cap \mathbb{Q}$ (1)

Prove that $f(x) = g(x) \quad \forall\, x \in [0, 1]$

We have to prove that $f(x) = g(x) \quad \forall\, x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})$

Let $x_0 \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})$, we have to prove that $f(x_0) = g(x_0)$



$(x_n)_{n \in \mathbb{N}^*} \quad x_n = \frac{1}{n}, \quad 1, \frac{1}{2}, \frac{1}{3}, \dots \quad \lim_{n \to \infty} x_n = 0$

$x_0 \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \Rightarrow \exists\, (x_n) \subseteq \mathbb{Q} : \boxed{\lim_{n \to \infty} x_n = x_0}$

$\sqrt{2} \simeq 1,4142\ldots$

$\mathbb{Q}$ are dense in $\mathbb{R}$

$x_1 = 1 \quad x_2 = 1.4 \quad x_3 = 1.41 \quad x_4 = 1.414\ldots \quad \overset{\in \mathbb{Q}}{(x_n)} \to \sqrt{2}$

$$\underbrace{\left(1 + \frac{1}{n}\right)^n}_{\in \mathbb{Q}} \to e \ \in \mathbb{R} \setminus \mathbb{Q}$$

f is continuous at $x_0 \Leftrightarrow \lim_{x \to x_0} f(x) = f(x_0) \Leftrightarrow$

$\Leftrightarrow \lim_{n \to \infty} x_n = x_0$ then $\lim_{n \to \infty} f(x_n) = f(x_0)$

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Subject :

Date :......./......./.......

$$\lim_{n\to\infty} x_n = x_0 \xrightarrow{g, f\ \text{cont}} \lim_{n\to\infty} f(x_n) = f(x_0),\ \lim_{n\to\infty} g(x_n) = g(x_0)$$

$$x_n \in \mathbb{Q} \Rightarrow f(x_n) = g(x_n)$$

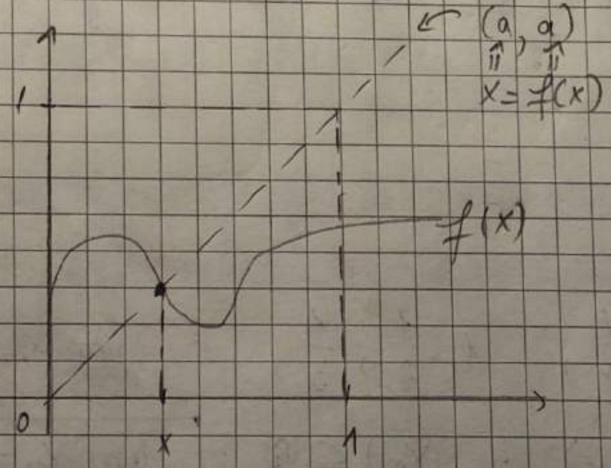
$$\Rightarrow \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = \boxed{f(x_0) = g(x_0)} \Rightarrow f(x) = g(x)$$

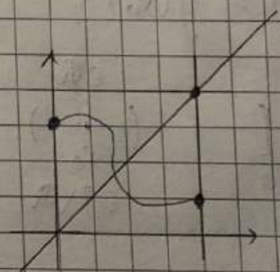
$$\forall x \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q}) \quad (2)$$

$$(1), (2) \Rightarrow f(x) = g(x) \quad \forall x \in [0, 1]$$

8. $f : [0, 1] \to [0, 1]$ continuous, prove that f has at least one fixed point.

$x_0 \in [0, 1]$, that is $f(x_0) = x_0$

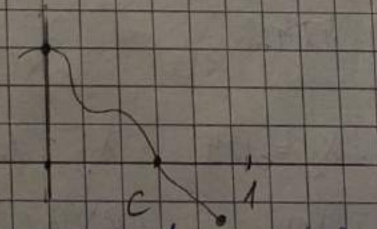




$g(x) = f(x) - x$

$g(0) = f(0) \geq 0$

$g(1) = f(1) - 1 \leq 0$



If $g(0) = 0 \to 0$ is a fixed point of f

If $g(1) = 0 \Rightarrow 1$ is a fixed point of f

We assume that $g(0) \neq 0$ and $g(1) \neq 0 \Rightarrow g(0) > 0$ and $g(1) < 0$; g continuous $\Rightarrow$ Bolzano

Subject :   Date : ....../....../......

Intermediate Value Theorem

$\Rightarrow \exists\, c \in (0,1);\ g(c) = 0 \Rightarrow g(c) = f(c) - c = 0 \Rightarrow$

$\Rightarrow$ c is a fixed point of f

7. Find a function $f: \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point but $|f|$ is continuous on $\mathbb{R}$

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

Dirichlet function

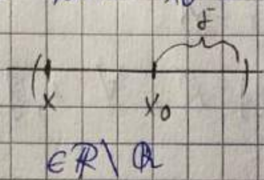
f is continuous at $x_0 \Leftrightarrow \forall\, \varepsilon > 0\ \exists\, \delta > 0:$

$$|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

$|f| = 1$ continuous at every point

Let $\varepsilon = 1$ and $x_0 \in \mathbb{Q}$ then $f(x_0) = 1$

$\forall\, \delta > 0$



$\exists\, x \in \mathbb{R} \setminus \mathbb{Q}: |x - x_0| < \delta \Rightarrow$

$\Rightarrow |f(x) - f(x_0)| = |-1 - 1| = 2 \not< 1$

$\Rightarrow$ f is not continuous at x_0

Similarly where x_0 is irrational

9. $f: D \to \mathbb{R}$, $S \subseteq D$

f is Lipschitzian on S if $\exists\, L \geq 0$:

$$\boxed{|f(x) - f(y)| \leq L|x - y|} \quad (1) \quad \forall\, x, y \in S$$

a) If f is Lipschitzian then it is also continuous

f is continuous at $y \Leftrightarrow \forall\, \overset{\text{arbitrary}}{\varepsilon} > 0\ \exists\, \delta > 0:$

$$\boxed{|x - y| < \delta} \overset{(2)}{\Rightarrow} \boxed{|f(x) - f(y)| < \varepsilon}$$

Let $\delta = \dfrac{\varepsilon}{L}$

Subject : Date :......./......./.......

$(1),(2) \Rightarrow |f(x) - f(y)| < \boxed{L \cdot \delta}$ $\Rightarrow$

$\delta = \frac{\varepsilon}{L}$; $L \cdot \delta < \varepsilon$

$\Rightarrow |f(x) - f(y)| < \cancel{L} \cdot \frac{\varepsilon}{\cancel{L}} = \varepsilon \Rightarrow f$ is continuous at y

11.10.2021

Algebra. Seminar

Homogeneous relation : $\varphi : M \to M$ [φ - tan]

Graph of the solution : φ : $\{(x, y) / x \varphi y\}$

Equivalance solution : R : $x \varphi x$

T : $\left.\begin{matrix} x \varphi y \\ y \varphi z \end{matrix}\right\} \Rightarrow x \varphi z$ related

S : $x \varphi y \Rightarrow y \varphi x$

T in def

def $d^{k+1} f(x)(\gamma) = \dfrac{d}{dt}\left(d^{k} f(x+t\gamma)(\gamma)\right)_{t=0}$

Mathematical Analysis. Seminar

12.10.2021.

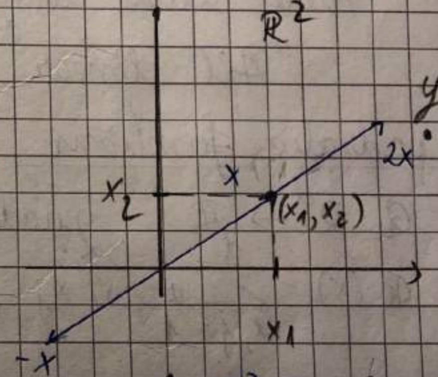
$\mathbb{R}^n = \underbrace{\mathbb{R}\times\mathbb{R}\times\dots\times\mathbb{R}}_{n \text{ times}} = \{(x_1, x_2, \dots, x_n) / x_i \in \mathbb{R}\}$

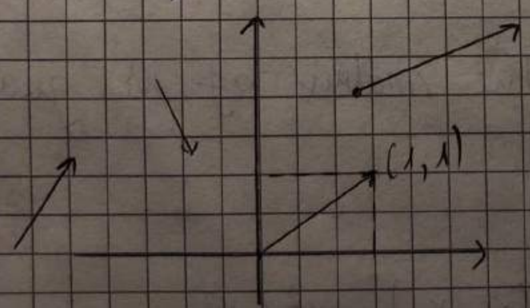
$x \in \mathbb{R}^n$, $x = (x_1, x_2, \dots, x_n)$

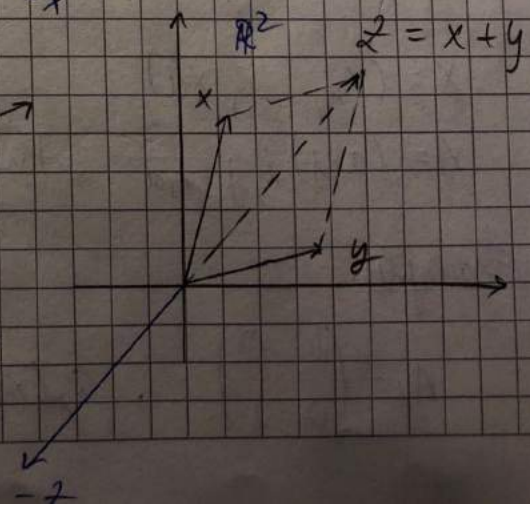
$y \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n)$

$x + y = (x_1 + y_1, \dots, x_n + y_n)$

$\underset{\text{scalar}}{\alpha} \cdot x = (\alpha \cdot x_1, \dots, \alpha \cdot x_n)$







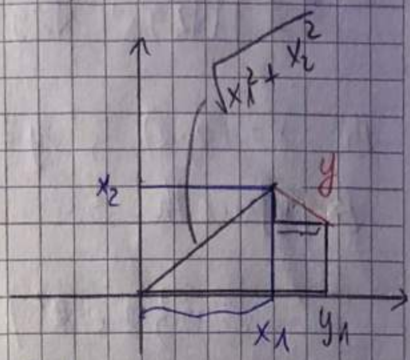
$$\langle x, y\rangle = x\cdot y = \sum_{i=1}^{m} x_i\, y_i = x_1 y_1 + \ldots + x_m y_m$$ - scalar /

$$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$\langle \alpha\cdot x, y\rangle = \alpha\cdot\langle x, y\rangle$$

$$\langle x, y\rangle = \langle y, x\rangle$$

$$\langle x, x\rangle > 0,\ \forall x \in \mathbb{R}^m \setminus \{0_m\}$$ origin of $\mathbb{R}^m$



$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$$ - Euclidean norm of x

$$\text{dist}(x, y) = d(x, y) = \sqrt{(x_1-y_1)^2 + \ldots + (x_m - y_m)^2} = \|x-y\|$$ - Euclidean distance

$$x \perp y \iff \langle x, y\rangle = 0$$

13. $x = (1, 2, -1)$, $y = (2, 0, 1) \in \mathbb{R}^3$

a) $x + y = (1+2, 2+0, -1+1) = (3, 2, 0)$

$\|y\| = \sqrt{2^2 + 0^2 + 1^2} = \sqrt{5}$

$\langle x, y\rangle = 1\cdot 2 + 2\cdot 0 + (-1)\cdot 1 = 1$

b) $z \perp x$, $z \perp y$, $\|z\| = 1$. Find $z \in \mathbb{R}^3$

$z \perp x \Rightarrow \langle (z_1, z_2, z_3), (1, 2, -1)\rangle = z_1 + 2z_2 - z_3 = 0$

$z\ y \Rightarrow \langle (z_1, z_2, z_3), (2, 0, 1)\rangle = 2z_1 + z_3 = 0$

$\|z\| = 1 \Rightarrow \sqrt{z_1^2 + z_2^2 + z_3^2} = 1$ $\qquad p \to q$

$$\begin{cases} z_1 + 2z_2 - z_3 = 0 \Rightarrow z_2 = \dfrac{z_3 - z_1}{2} = \dfrac{-3z_1}{2} \\ 2z_1 + z_3 = 0 \Rightarrow z_3 = -2z_1 \\ \sqrt{z_1^2 + z_2^2 + z_3^2} = 1 \quad |^2 \iff z_1^2 + \left(-\dfrac{3z_1}{2}\right)^2 + (-2z_1)^2 = 1 \iff \end{cases}$$

$$\iff z_1^2 + \frac{9}{4} z_1^2 + 4z_1^2 = 1 \ \Big|\cdot 4 \iff 29 z_1^2 = 4 \Rightarrow z_1 = \pm\frac{2}{\sqrt{29}}$$

$$z_1 = \pm\frac{2}{\sqrt{29}};\ z_2 = \mp\frac{3}{\sqrt{29}};\ z_3 = \mp\frac{4}{\sqrt{29}} \Rightarrow$$

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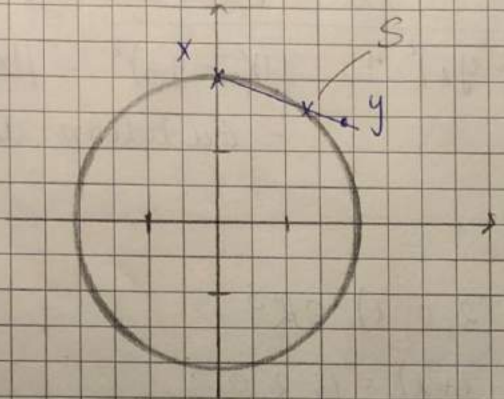
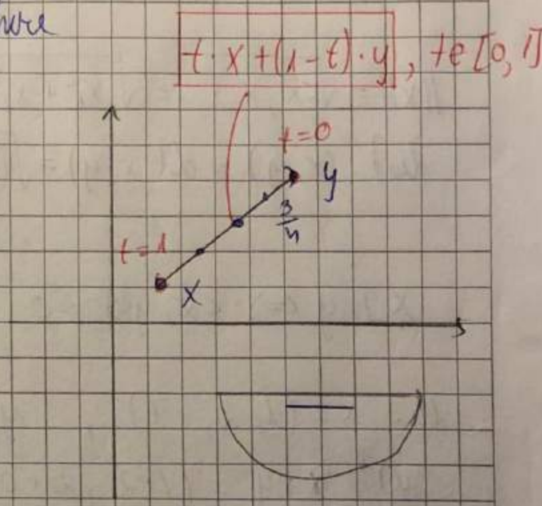
Subject : Date :....../....../......

$\Rightarrow z \in \left\{ \left(\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-4}{\sqrt{29}} \right), \frac{1}{\sqrt{29}}(-2, 3, 4) \right\}$

14. $x = (0, 2)$, $y = (2, 1) \in \mathbb{R}^2$. Find the intersection of $[x, y]$ with S.

$S = \{ z \in \mathbb{R}^2 \mid \|z\| = 2 \}$ - sphere





$[x, y] = \underbrace{t\cdot x + (1-t)\cdot y}_{\text{convex combination of } x \text{ and } y}, \; t \in [0, 1]$

Let $w \in [x, y] \cap S \Rightarrow w = t\cdot x + (1-t)y$ and $\|w\| = 2 \Rightarrow$

$\Rightarrow w = t(0, 2) + (1-t)\cdot(2, 1)$ and $\|w\| = 2 \Rightarrow$

$\Rightarrow w = (2-2t, t+1)$ and $\sqrt{(2-2t)^2 + (t+1)^2} = 2$

$(2-2t)^2 + (t+1)^2 = 4 \Leftrightarrow \cancel{4} - 8t + 4t^2 + t^2 + 2t + 1 = \cancel{4} \Leftrightarrow$

$\Leftrightarrow 5t^2 - 6t + 1 = 0 \Leftrightarrow (5t-1)(t-1) = 0$

$\Rightarrow t \in \left\{ \frac{1}{5}, 1 \right\} \Rightarrow w \in \left\{ \left(\frac{8}{5}, \frac{6}{5} \right), (0, 2) \right\}$

15. $x, y \in \mathbb{R}^d$

$\|x+y\|^2 - \|x-y\|^2 = 4 \langle x, y \rangle$

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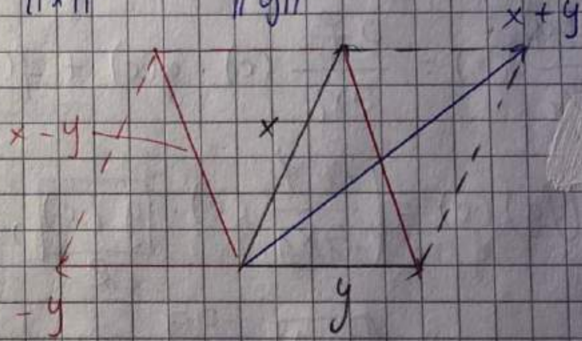
$\|x+y\|^2 = \langle x+y, x+y \rangle$

$$\begin{aligned}
\|x+y\|^2 - \|x-y\|^2 &= \langle x+y, x+y\rangle - \langle x-y, x-y\rangle = \\
&= \langle x, x+y\rangle + \langle y, x+y\rangle - (\langle x, x-y\rangle - \\
&\quad - \langle y, x-y\rangle) = \\
&= \cancel{\langle x, x\rangle} + \langle x, y\rangle + \langle y, x\rangle + \cancel{\langle y, y\rangle} - \\
&\quad - (\cancel{\langle x, x\rangle} - \langle x, y\rangle - \langle y, x\rangle + \cancel{\langle y, y\rangle}) = \\
&= 4\langle x, y\rangle
\end{aligned}$$

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= \langle x, x\rangle + \cancel{\langle x, y\rangle} + \cancel{\langle y, x\rangle} + \langle y, y\rangle + \\
&\quad + \langle x, x\rangle - \cancel{\langle x, y\rangle} - \cancel{\langle y, x\rangle} + \langle y, y\rangle = \\
&= 2\underbrace{\langle x, x\rangle}_{\|x\|^2} + 2\underbrace{\langle y, y\rangle}_{\|y\|^2} = 2(\|x\|^2 + \|y\|^2)
\end{aligned}$$

$\|x\| + \|y\| = \|x-y\|$

$x \perp y \iff \langle x, y\rangle = 0$



16. Find the first order partial derivatives of the

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

x - variable , y - constant

$$\frac{\partial f}{\partial x}(x,y) = -\sin x \cdot (\cos y) - \cos x \cdot \sin y$$

x - const , y - var.

$$\frac{\partial f}{\partial y}(x,y) = \cos x \cdot (-\sin y) - \sin x \cdot \cos y$$

d) $f: \mathbb{R} \times \mathbb{R} \times \mathbb{R}^* \to \mathbb{R}$, $f(x,y,z) = \dfrac{x \cdot e^y}{z}$

$\dfrac{\partial f}{\partial x}(x,y,z) = \dfrac{e^y}{z}$, x - var, y, z - cons

Subject : Date : ....../....../......

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{x}{z}\cdot e^{y}, \quad y - \text{var}, \quad x, z - \text{cons}$$

$$\frac{\partial f}{\partial z}(x,y,z) = (\underbrace{x\cdot e^{y}}_{\text{const}})\frac{1}{z} = \frac{-x e^{y}}{z^2}, \quad z - \text{var}$$

HW 18. $f, g : \mathbb{R}^m \to \mathbb{R}$ differentiable

$$\nabla(fg)(c) = f(c)\cdot\nabla g(c) + g(c)\cdot\nabla f(c)$$

$$\nabla f(c) = \left(\frac{\partial f}{\partial x_1}(c), \frac{\partial f}{\partial x_2}(c), \dots, \frac{\partial f}{\partial x_m}(c)\right)$$

$$\nabla(fg)(c) = \left(\frac{\partial(fg)}{\partial x_1}(c), \dots, \frac{\partial(fg)}{\partial x_m}(c)\right) \quad (fg)' = f'g + fg'$$

$$\frac{\partial(fg)}{\partial x_i}(c) = \frac{\partial f}{\partial x_i}(c)\cdot g(c) + f(c)\cdot\frac{\partial g}{\partial x_i}(c), \ \forall c \in \{1, \dots, m\}$$

$$\nabla(fg)(c) = \left(\frac{\partial f}{\partial x_1}(c)\cdot\underbrace{\boxed{g(c)}}_{\in\mathbb{R}}, \dots, \frac{\partial f}{\partial x_m}(c)\underbrace{\boxed{g(c)}}_{\in\mathbb{R}}\right) +$$

$$+ \left(\underbrace{\boxed{f(c)}}_{\in\mathbb{R}}\cdot\frac{\partial g}{\partial x_1}(c), \dots, \underbrace{\boxed{f(c)}}_{\in\mathbb{R}}\cdot\frac{\partial g}{\partial x_m}(c)\right) =$$

$$= g(c)\cdot\nabla f(c) + f(c)\cdot\nabla g(c)$$

18. 10. 2021

Algebra Seminar

$(G, *)$ group:
- $\forall$ elem. has an inv
- assoc
- id. elem.

$(R, +, \cdot)$ ring:
- $(R, +)$ ab group
- $(R^*, \cdot)$ assoc.
- distrib

Subject : Date : ....../....../......

connected neurons

$\varnothing\left(\sum w_{ij}x_j + b_i\right)$ = output of neuron "i"

input from neurons on previous layer

entire NN = function connecting IN to OUT

19.10.2021

Mathematical Analysis. Seminar

19. Cauchy - Schwarz Inequality.

$x, y \in \mathbb{R}^m$

$\langle x, y\rangle = \sum_{i=1}^{m} x_i \cdot y_i = x_1 y_1 + \dots + x_m \cdot y_m$ produs scalar

produs a 2 vec scalari

$\|x\| = \sqrt{\langle x, x\rangle}$

Prove that $|\langle x, y\rangle| \le \|x\| \cdot \|y\| \quad \forall\, x, y \in \mathbb{R}^m$

If $x = 0_m$ then the inequality is true $(0 \le 0)$

We asume that $x \neq 0_m \Rightarrow \langle x, x\rangle > 0$ $\quad \langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$

$0 \le \left\langle \frac{\langle x,y\rangle}{\langle x,x\rangle} x - y, \underbrace{\frac{\langle x,y\rangle}{\langle x,x\rangle} x - y}_{z} \right\rangle = \left\langle \frac{\langle x,y\rangle}{\langle x,x\rangle} \cdot x, \frac{\langle x,y\rangle}{\langle x,x\rangle} x - y \right\rangle -$

$- \left\langle y, \frac{\langle x,y\rangle}{\langle x,x\rangle} x - y \right\rangle = \left\langle \underbrace{\frac{x \cdot y}{x, x}}_{\in\mathbb{R}} x, \underbrace{\frac{\langle x,y\rangle}{\langle x,x\rangle}}_{\in\mathbb{R}} x \right\rangle - \left\langle \underbrace{\frac{\langle x,y\rangle}{\langle x,x\rangle}}_{\in\mathbb{R}} x, y \right\rangle -$

$- \left\langle y, \frac{\langle x,y\rangle}{\langle x,x\rangle} x \right\rangle + \langle y, y\rangle = \frac{\langle x,y\rangle}{\cancel{\langle x,x\rangle}} \cdot \frac{\langle x,y\rangle}{\cancel{\langle x,x\rangle}} \cdot \cancel{\langle x,x\rangle} -$

$- \frac{\langle x,y\rangle}{\cancel{\langle x,x\rangle}} \cancel{\langle x,y\rangle} - \frac{\langle x,y\rangle}{\langle x,x\rangle} \cdot \langle y,x\rangle + \langle y,y\rangle =$

$= \frac{-\langle x,y\rangle \cdot \langle x,y\rangle}{\langle x,x\rangle} + \langle y,y\rangle \Rightarrow \frac{\langle x,y\rangle^2}{\|x\|^2} \le \|y\|^2 \;\Big|\cdot \underbrace{\|x\|^2}_{>0} \Rightarrow$

$\Rightarrow \langle x,y\rangle^2 \le \|x\|^2 \cdot \|y\|^2 \;\Big|\sqrt{\;} \Rightarrow |\langle x,y\rangle| \le \|x\| \cdot \|y\|$

Subject :

Date :....../....../.......

20. $T: \mathbb{R}^n \to \mathbb{R}$ linear $\Leftrightarrow \begin{cases} T(x+y) = T(x) + T(y) & \forall\, x, y \in \mathbb{R}^n \\ T(\alpha \cdot x) = \alpha \cdot T(x) & \alpha \in \mathbb{R} \end{cases}$

Prove that $\exists\, \alpha_T \in \mathbb{R}^n : T(x) = \langle \alpha_T, x \rangle = \alpha_1 x_1 + \ldots + \alpha_n x_n$

$\alpha_T = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{R}^n$ $\qquad (1,2) = (1,0) + (0,2) = (1,0) + 2(0,1)$

$T(x) = T(\underline{x_1, x_2, \ldots, x_n}) = T((x_1, 0, \ldots, 0) + (0, x_2, \ldots, 0) +$
$+ \ldots + (0, \ldots, 0, x_n)) \overset{(1)}{=} T(\underbrace{x_1}_{\in \mathbb{R}}, 0, \ldots, 0) + T(0, \underbrace{x_2}_{\in \mathbb{R}}, 0, \ldots 0) + \ldots +$
$+ T(0, \ldots, 0, \underbrace{x_n}_{\in \mathbb{R}}) = T(x_1(1, 0, \ldots, 0)) + T(x_2(0, 1, 0, \ldots, 0)) +$
$+ T(x_n(0, \ldots, 0, 1)) \overset{(2)}{=} x_1 \cdot \underbrace{T(1, 0, \ldots, 0)}_{\alpha_1} + x_2 \cdot \underbrace{T(0, 1, 0, \ldots, 0)}_{\alpha_2} +$
$+ \ldots + x_n \cdot \underbrace{T(0, 0, \ldots, 0, 1)}_{\alpha_n} = x_1 \alpha_1 + x_2 \alpha_2 + \ldots + x_n \alpha_n =$
$= \langle x_1, \alpha_T \rangle = \langle \alpha_T, x \rangle$

$T: \mathbb{R}^3 \to \mathbb{R} \qquad T(x, y, z) = x + 2y + 3z$

$\left.\begin{array}{l} T(1,0,0) = 1 \\ T(0,1,0) = 2 \\ T(0,0,1) = 3 \end{array}\right\} \Rightarrow \alpha_T = (1, 2, 3)$

$T(2, 3, 0) = 2 + 2 \cdot 3 + 0 =$
$= \langle \underbrace{(1,2,3)}_{\alpha}, \underbrace{(2,3,0)}_{x} \rangle$

21 $T: \mathbb{R}^n \to \mathbb{R}$, $T(x) = \langle a, x \rangle = a_1 x_1 + \ldots + a_n x_n$

Compute the gradient and the Hessian matrix.

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 y$

$\frac{\partial f}{\partial x}(x, y) = 2xy$; $\frac{\partial f}{\partial y} = (x, y) = x^2$

Subject : Date :......./......./.......

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial x}(x^2) = 2x$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial y}(2xy) = 2x$$

$$\frac{\partial^2 f}{\partial x \partial x}(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = \frac{\partial}{\partial x}(2xy) = 2y$$

$$\frac{\partial^2 f}{\partial y \partial y}(x,y) = \frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(x,y)\right) = \frac{\partial}{\partial y}(x^2) = 0$$

$$H_f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_m}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_m \partial x_m} & \cdots & \frac{\partial^2 f}{\partial x_m \partial x_m}(x) \end{pmatrix} \qquad H_f(x,y) = \begin{pmatrix} 2y & 2x \\ 2x & 0 \end{pmatrix}$$

$$T(x) = \langle a, x \rangle = a_1 x_1 + \ldots + a_m x_m$$

$$\nabla T(x) = \left(\frac{\partial T}{\partial x_1}(x), \ldots, \frac{\partial T}{\partial x_m}(x)\right) = (a_1, a_2, \ldots, a_m) = a$$

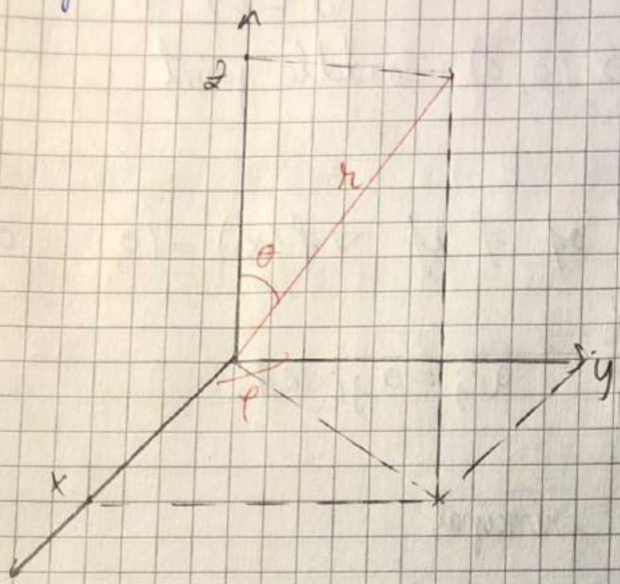
x_1 - var

$$\frac{\partial T}{\partial x_i}(x) = a_i$$

$$\frac{\partial^2 T}{\partial x_j \cdot \partial x_i}(x) = \frac{\partial}{\partial x_j}\left(\frac{\partial T}{\partial x_i}(x)\right) = \frac{\partial}{\partial x_j}(a_i) = 0$$

$$\Rightarrow HQ(x) = \nabla^2 Q(x) = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = 0_{m \times m}$$

Spherical coords



$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

$|\det J_u(r, \varphi, \theta)| =$
$= r^2 \sin\theta$

30.10.2021

Mathematical Analysis. Seminar

23. $f_1, f_2 : \mathbb{R}^2 \to \mathbb{R}$

$f_1(x, y) = x^2 + y^2 \quad ; \quad f_2(x, y) = x^2 - y^2$

a) $f_1(x, y) = \underline{1} \cdot x^2 + \underline{0} \cdot xy + \underline{0} \cdot y + \underline{1} \cdot y^2 \to f_1$ is quadratic

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$f_2(x, y) = (x, y) \cdot A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = x^2 - y^2$$

b) $\nabla f_1(x, y) = (2x, 2y) \qquad \nabla f_1(0, 0) = (0, 0)$

$\nabla f_2(x, y) = (2x, -2y) \qquad \nabla f_2(0, 0) = (0, 0)$

c) $H f_1(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad H f_1(0, 0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$|H f_1(0, 0)| = 4 > 0 \to (0, 0)$ not a saddle point

$2 > 0 \Rightarrow (0, 0)$ local min

$x^2 + y^2 \geq 0$

Subject :   Date :......./......./.......

$Hf_2(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ $Hf_2(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

$|Hf_2(0,0)| = -4 < 0 \to (0,0)$ saddle point

$f: \mathbb{R}^m \to \mathbb{R}$

x_0 is critical point of f if $\nabla f(x_0) = (0, \dots, 0)$

Sylvester's Criterion

$A = (a_{ij})_{i,j \in \{1,\dots,m\}}$ $a_{ij} = a_{ji} \in \mathbb{R}$

$\Delta_k = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$ principal

1) A is positive definite $\Leftrightarrow \Delta_k > 0 \ \forall k \in \{1, \dots, m\}$

2) A is negative definite $\Leftrightarrow (-1)^k \cdot \Delta_k > 0, \forall k \in \{1, \dots, m\}$

3) A is indefinite if the first Δ_k that breaks the sequence has wrong

4) If are then is 0 then it is inconclusive

x_0 is a local min. point of $f \Leftrightarrow Hf(x_0)$ positive definite

x_0 is a local max. point of $f \Leftrightarrow Hf(x_0)$ negative definite

If $Hf(x_0)$ is indefinite then x_0 is a saddle point.

$Q(x_1, \dots, x_m) = (x_1, \dots, x_m) \cdot A \cdot (x_1, \dots, x_m)^t$ - quadratic form associated to A

Q positive define $\Leftrightarrow Q(x_1, \dots, x_m) > 0, \forall (x_1, \dots, x_m) \in \mathbb{R}^m \setminus \{0_m\}$

Q neg. def. $\Leftrightarrow Q(x_1, \dots, x_m) < 0, \forall (x_1, \dots, x_m) \in \mathbb{R}^m \setminus \{0_m\}$

Subject : Date : ....../....../......

Q indefinite $\Leftrightarrow$ $Q(x_1, \dots, x_m) < 0$ and $Q(y_1, \dots, y_m) > 0$
for some $(x_1, \dots, x_m) \in \mathbb{R}^m$
$(y_1, \dots, y_m) \in \mathbb{R}^m$

24) a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 - 3x + y^2$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}(x,y) = 3x^2 - 3 \\ \frac{\partial f}{\partial y}(x,y) = 2y \end{array}\right| \Rightarrow \nabla f(x,y) = (3x^2 - 3, 2y)$$

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y)$$

$$\nabla f(x,y) = (0,0) \Rightarrow \left\{\begin{array}{l} 3x^2 - 3 = 0 \Rightarrow x \in \{1, -1\} \\ 2y = 0 \Rightarrow y \in \{0\} \end{array}\right\} \Rightarrow$$

$\Rightarrow S = \{(1,0), (-1,0)\}$ - ~~critical~~ stationary points of f

$$H_f(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix} \qquad H_f(1,0) = \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix}, \ \Delta_1 = 6 > 0, \ \Delta_2 = 12 > 0 \Rightarrow$$

$\Rightarrow H_f(1,0)$ pos. def.

$\Rightarrow (1,0)$ loc. min. point

$$H_f(-1,0) = \begin{pmatrix} -6 & 0 \\ 0 & 2 \end{pmatrix}, \ \Delta_1 = -6 < 0, \ \Delta_2 = -12 < 0 \Rightarrow$$

$\Rightarrow H_f(-1,0)$ indef.

$\Rightarrow (-1,0)$ saddle point

b) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3xy$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x}(x,y) = 3x^2 - 3y \\ \frac{\partial f}{\partial y}(x,y) = 3y^2 - 3x \end{array}\right| \Rightarrow \nabla f(x,y) = \underline{(3x^2 - 3y, 3y^2 - 3x)}$$

$$\nabla f(x,y) = (0,0) \Rightarrow \left\{\begin{array}{l} x^2 - y = 0 \Rightarrow y = x^2 \\ y^2 - x = 0 \end{array}\right| \Rightarrow$$

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Subject :   Date :......./......./.......

$\Rightarrow x^4 - x = x(x^3 - 1) = 0 \Rightarrow x \in \{0, 1\}$

$S = \{(0,0), (1,1)\}$ - critical points

$Hf(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ $Hf(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$, $\Delta_2 = -9 \Rightarrow$

$\Rightarrow Hf(0,0)$ indef. $(0,0)$ saddle point

$Hf(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$, $\left.\begin{array}{l} \Delta_1 = 6 > 0 \\ \Delta_2 = 27 > 0 \end{array}\right\} \Rightarrow$ $Hf(1,1)$ pos. def
$(1,1)$ loc. min. point

c) $f : (0, \infty) \times \mathbb{R} \to \mathbb{R}$

$f(x,y) = x(y^2 + \ln^2 x)$

$\nabla f(x,y) = (\underline{y^2 + \ln^2 x + 2\ln x}, 2xy)$

$\nabla f(x,y) = (0,0) \Rightarrow \begin{cases} y^2 + \ln^2 x + 2\ln x = 0 \\ 2\underset{x_0}{x}y = 0 \Rightarrow y = 0 \end{cases} \Rightarrow x \in \{1, e^{-2}\}$

$S = \{(1,0), (e^{-2}, 0)\}$ - critical points

$Hf(x,y) = \begin{pmatrix} \frac{2\ln x + 2}{x} & 2y \\ 2y & 2x \end{pmatrix}$

$Hf(1,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\left.\begin{array}{l} \Delta_1 = 2 > 0 \\ \Delta_2 = 4 > 0 \end{array}\right| \Rightarrow$ $Hf(1,0)$ pos. def.
$(1,0)$ loc. min. point

$Hf(e^{-2}, 0) = \begin{pmatrix} -2e^{+2} & 0 \\ 0 & 2e^{-2} \end{pmatrix}$ $\Delta_2 = -4 < 0 \Rightarrow Hf(e^{-2}, 0)$ ind.

$(e^{-2}, 0)$ saddle point

Subject : Date : ..../......./.......

d) $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^4 + y^4 - 4(x-y)^2$$

$$\nabla f(x, y) = (4x^2 - 8(x-y),\ 4y^3 + 8(x-y))$$

$$\nabla f(x, y) = (0, 0) \Rightarrow \begin{cases} x^3 - 2(x-y) = 0 \ (1) \\ y^3 + 2(x-y) = 0 \end{cases} +$$

$$x^3 + y^3 = 0 = (x+y)(x^2 - xy + y^2)$$

$$\Downarrow$$

$$\boxed{x = -y}$$

$$x^2 - xy + y^2 = \left(x - \frac{y}{2}\right)^2 + \frac{3}{4}y^2 = 0 \Rightarrow \begin{cases} y = 0 \\ x = \frac{y}{2} \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow$$

$$\Rightarrow (0, 0) \in S$$

$$\left.\begin{matrix} (1) \\ x = -y \end{matrix}\right\} \Rightarrow x^3 - 4x = 0 = x(x^2 - 4) = x(x-2)\cdot(x+2) \Rightarrow$$

$$\Rightarrow x \in \{0, 2, -2\}$$

$S = \{(0, 0), (2, -2), (-2, 2)\}$ critical points

$$Hf(x, y) = 4\begin{pmatrix} 3x^2 - 2 & 2 \\ 2 & 3y^2 - 2 \end{pmatrix}$$

$$Hf(0, 0) = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \quad \Delta_1 = -8 < 0,\ \Delta_2 = 0$$

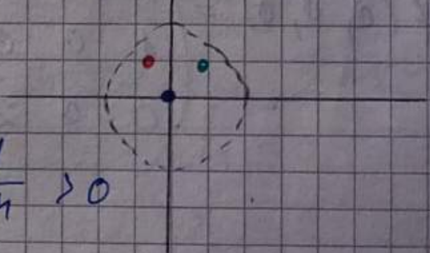
$Q(x, y) = -8(x^2 - 2xy + y^2) = -8(x-y)^2 \leq 0$ negative semi definite

$$f(0, 0) = 0$$

$$f(x, y) = x^4 + y^4 - 4(x-y)^2$$

$$a_m = \left(\frac{1}{m}, \frac{1}{m}\right) \xrightarrow{m \to \infty} (0, 0)$$

$$f(a_m) = f\left(\frac{1}{m}, \frac{1}{m}\right) = \frac{1}{m^4} + \frac{1}{m^4} > 0$$



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Subject : Date : ..../......./.......

$b_n = \left(\frac{1}{n}, 0\right) \xrightarrow{n\to\infty} (0,0)$ $\frac{1}{n}, \frac{1}{n}$ $\frac{2}{n^4} - 4\frac{2}{n^2} = \frac{2-8n^2}{n^4}$

$f(b_n) = \frac{1}{n^4} - \frac{4}{n^2} = \frac{1-4n^2}{n^4} < 0$

$\lim\left(\frac{1}{n}\right) = 0$

$\Rightarrow (0,0)$ is a saddle point

$Hf(2,-2) = Hf(-2,2) = \begin{pmatrix} 40 & 8 \\ 8 & 40 \end{pmatrix}$ $\left.\begin{array}{l} \Delta_1 = 40 > 0 \\ \Delta_2 = 40^2 - 8^2 > 0 \end{array}\right| \Rightarrow$

$\Rightarrow$ $Hf(2,-2)$, $Hf(-2,2)$ pos. definite

$(2,-2)$, $(-2,2)$ loc. min. points

25. a) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = z^2(1+xy) + xy$

$\nabla f(x,y,z) = (yz^2 + y,\ xz^2 + x,\ 2z + 2xyz)$

$\nabla f(x,y,z) = (0,0,0) \Rightarrow$

$\Rightarrow \begin{cases} yz^2 + y = 0 = y(z^2+1) \Rightarrow y = 0 \\ xz^2 + x = 0 = x(z^2+1) \Rightarrow x = 0 \\ 2z + 2xyz = 0 = 2z(1+xy) \Rightarrow z = 0 \end{cases}$

$\Rightarrow S = \{(0,0,0)\}$

$Hf(x,y,z) = \begin{pmatrix} 0 & z^2+1 & 2yz \\ z^2+1 & 0 & 2xz \\ 2yz & 2xz & 2+2xy \end{pmatrix}$

$Hf(0,0,0) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = -1 \\ \Delta_3 = -2 \end{array}$ inconclusive

Subject : Date :....../....../......

$Q(x,y,z) = 1 \cdot xy + 1 \cdot yx + 2 \cdot z^2 = 2xy + 2z^2$

$\left.\begin{array}{l} Q(1,1,1) = 4 > 0 \\ Q(-1,1,0) = -2 < 0 \end{array}\right| \Rightarrow Q$ indefinite $\Rightarrow (0,0,0)$ saddle point

2.11.2021.

Mathematical Analysis Seminar

27. $a = (1,2) \in \mathbb{R}^2$

$S = \{ (x_1, x_2) \in \mathbb{R}^2 \ / \ \|(x_1, x_2)\| = 1 \}$ unit circle

find the closest point from S to a

Let $x = (x_1, x_2) \in S \Rightarrow \|x\| = \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2} = 1 \Rightarrow$

$\Rightarrow \underline{x_1^2 + x_2^2} = 1$

$d(a,x) = \|a - x\| = \sqrt{(1-x_1)^2 + (2-x_2)^2} = \sqrt{(1 - 2x_1 + \underline{x_1^2}) +}$

$\overline{+ (4 - 4x_2 + \underline{x_2^2})} = \sqrt{6 - 2x_1 - 4x_2}$

Let $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = 6 - 2x_1 - 4x_2$ be a function

Let $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x_1, x_2) = x_1^2 + x_2^2 - 1$ be a function

$\begin{cases} f(x_1, x_2) \to \min \\ g(x_1, x_2) = 0 \text{ - constraint} \end{cases}$

Lagrange Multiplier Method

Let $L: \mathbb{R}^3 \to \mathbb{R}$, $L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda g(x_1, x_2)$

Lagrange multi

$L(x_1, x_2, \lambda) = 6 - 2x_1 - 4x_2 - \lambda \underbrace{(x_1^2 + x_2^2 - 1)}_{\alpha}$

$\nabla L(x_1, x_2, \lambda) = (-2 - 2\lambda x_1, -4 - 2\lambda x_2, 1 - x_1^2 - x_2^2)$

$\nabla L(x_1, x_2, \lambda) = (0,0,0) \Rightarrow \begin{cases} -2 - 2\lambda x_1 = 0 \\ -4 - 2\lambda x_2 = 0 \quad (\Leftrightarrow) \\ 1 - x_1^2 - x_2^2 = 0 \end{cases}$

Subject : Date :....../....../......

$$\Leftrightarrow \begin{cases} -1 = \lambda x_1 \\ -2 = \lambda x_2 \\ x_1^2 + x_2^2 = 1 \end{cases} \Rightarrow \begin{matrix} x_1 = \frac{-1}{\lambda}, \ \lambda \neq 0 \\ x_2 = \frac{-2}{\lambda}, \ \lambda \neq 0 \\ \Rightarrow \end{matrix}$$

$$\Rightarrow \left(\frac{-1}{\lambda}\right)^2 + \left(\frac{-2}{\lambda}\right)^2 = \frac{5}{\lambda^2} = 1 \Rightarrow \lambda \in \{-\sqrt{5}, \sqrt{5}\}$$

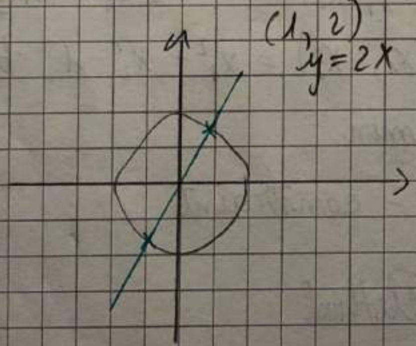
$$S = \left\{\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, -\sqrt{5}\right), \left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}, \sqrt{5}\right)\right\} -$$

- critical points of L

$$d\left((1,2), \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)\right) < d\left((1,2), \left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)\right)$$

$$(a_1 - x_1)^2 + (a_2 - x_2)^2$$

The point we were looking for is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$



28. $a, b, c > 0$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc} \quad \text{equality holds} \Leftrightarrow a = b = c$$

$$\begin{rcases} x = \sqrt[3]{a} \\ y = \sqrt[3]{b} \\ z = \sqrt[3]{c} \end{rcases} \Rightarrow \frac{x^3 + y^3 + z^3}{3} = \frac{a+b+c}{3} \overset{?}{\geq} \sqrt[3]{abc} = xyz$$

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Subject :

Date : ....../....../......

We have to prove that $\underbrace{x^3+y^3+z^3-3xyz}_{f(x,y,z)} \geq 0$

$f: \mathbb{R}^3_+ \to \mathbb{R}$

$$f(x,y,z) = (x+y+z)(x^2+y^2+z^2-xy-xz-yz) =$$
$$= \frac{1}{2}\underbrace{(x+y+z)}_{\substack{\geq 0 \\ \neq 0}}\underbrace{\left((x-y)^2+(x-z)^2+(y-z)^2\right)}_{\geq 0} \geq 0$$

$$f(x,y,z) = 0 \Leftrightarrow (x-y)^2+(x-z)^2+(y-z)^2 = 0 \Leftrightarrow$$
$$\Leftrightarrow x-y = x-z = y-z = 0 \Leftrightarrow x = y = z$$

29. $f: \mathbb{R}^3_+ \to \mathbb{R}$, $f(x,y,z) = 2(xy+yz+xz)$ subject to $xyz = 1$. Find the minim of f.

$L: \mathbb{R}^4 \to \mathbb{R}$, $L(x,y,z,\lambda) - 2(xy+yz+xz) - \lambda(xyz-1)$

$g: \mathbb{R}^3_+ \to \mathbb{R}$, $g(x,y,z) = xyz-1$

$$\nabla L(x,y,z,\lambda) = (2y+2z-\lambda yz,\ 2x+2z-\lambda xz,\ 2x+2y-\lambda xy,\ 1-xyz)$$

$$\nabla L(x,y,z,\lambda) = (0,0,0,0) \Rightarrow \begin{cases} 2y+2z = \lambda yz & \frac{1}{x} \ \big|\cdot x \\ 2x+2z = \lambda xz & \frac{1}{y} \ \big|\cdot y \\ 2x+2y = \lambda xy & \frac{1}{z} \ \big|\cdot z \\ xyz = 1 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 2xy+2xz = \lambda \\ 2xy+2zy = \lambda \\ 2xz+2yz = \lambda \\ xyz = 1 \end{cases} \Rightarrow \left.\begin{array}{l} \Rightarrow z(x-y) = 0 \Rightarrow x = y \\ \Rightarrow x(y-z) = 0 \Rightarrow y = z \\ xyz = 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow x = y = z = 1$$

Subject : Date : ..../......./.......

$S = \{(1, 1, 1, 4)\}$ - critical point of L

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$f(x, y, z) = 2(xy + yz + zx)$$

$$\left.\begin{array}{l} a = xy \\ b = yz \\ c = zx \end{array}\right| \Rightarrow \frac{xy+yz+zx}{3} \geq \sqrt[3]{(xyz)^2} = 1 \Rightarrow$$

$$\Rightarrow xy + yz + zx \geq 3$$

$$\Rightarrow \left.\begin{array}{l} f(x, y, z) \geq 2 \cdot 3 = 6 \\ f(1, 1, 1) = 2(1+1+1) = 6 \end{array}\right\} \Rightarrow (1, 1, 1) \text{ global min. point of } f$$

30. $f: \mathbb{R}_+^3 \to \mathbb{R}$, $f(x, y, z) = xyz$ subject to $x+y+z=1$

$g: \mathbb{R}_+^3 \to \mathbb{R}$, $g(x, y, z) = x + y + z - 1$

$L: \mathbb{R}^4 \to \mathbb{R}$, $L(x, y, z, \lambda) = xyz - \lambda(x + y + z - 1)$

$\nabla L(x, y, z, \lambda) = (yz - \lambda, xz - \lambda, xy - \lambda, 1 - x - y - z)$

$$\nabla L(x, y, z, \lambda) = (0, 0, 0, 0) \Rightarrow \begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x + y + z = 1 \end{cases} \Rightarrow \underset{\neq 0}{z}(x - y) = 0 \Rightarrow y = x, \quad x(z - y) = 0 \Rightarrow z = y$$

$\Rightarrow S = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{9}\right)\right\}$ - critical point of L

$$H_L(x, y, z, \lambda) = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & z & y \\ -1 & z & 0 & y \\ -1 & y & x & 0 \end{pmatrix}$$ - bordered Hessian matrix

Subject : Date :....../....../......

$$H_L(x,\lambda) = \begin{pmatrix} 0 & \frac{\partial g}{\partial x_1}(x,\lambda) & \cdots & \frac{\partial g}{\partial x_n}(x,\lambda) \\ \frac{\partial g}{\partial x_1}(x,\lambda) & \frac{\partial^2 L}{\partial x_1^2}(x,\lambda) & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial g}{\partial x_n}(x,\lambda) & \frac{\partial^2 L}{\partial x_n \partial x_1}(x,\lambda) & & \frac{\partial^2 L}{\partial x_n^2}(x,\lambda) \end{pmatrix}$$

$n = \#$ of var. of $f = 3$

$m = \#$ of. $= 1$

We first calculate Δ_{2m+1} then the others up to Δ_{m+n}

If of them have sign $(-1)^m$ then we have a loc. min. point

If the first one has sign $(-1)^{m+1}$ and the others alternate then - loc. max. point

$$H_L\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{9}\right) = \begin{pmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{3} & 0 & \frac{1}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

$$\Delta_3 = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & \frac{1}{3} \\ -1 & \frac{1}{3} & 0 \end{vmatrix} = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} > 0 \quad (1)$$

$$\Delta_4 = \begin{vmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & \frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{3} & 0 & \frac{1}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & -1 \\ -1 & -\frac{1}{3} & 0 & \frac{1}{3} \\ -1 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} & 0 \end{vmatrix} =$$

Subject :

Date : ...../...../......

$$= (-1)(-1)^5 \begin{vmatrix} -1 & -\frac{1}{3} & 0 \\ -1 & 0 & -\frac{1}{3} \\ -1 & \frac{1}{3} & \frac{1}{3} \end{vmatrix} = -1 \cdot \frac{1}{9} + \frac{1}{3}\left(-\frac{1}{3} + \frac{1}{3}\right) = -\frac{1}{9} < 0 \quad (2)$$

$(1), (2) \Rightarrow \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ loc. max. point

9. 11. 2021.

Mathematical Analysis. Seminar

32. a) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx =$, $\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}}$ $u(x) = \cos x$, $u'(x) = \frac{du}{dx} = -\sin x$

$$= \int \frac{-du}{u} = -\int \frac{du}{u} = -\ln|u| + c, \ c \in \mathbb{R}$$

b) $\int \ln x \, dx = \int \underbrace{(x)'}_{f'} \cdot \underbrace{\ln x}_{g} dx = x \ln x - \int x (\ln x)' dx =$

$(fg)' = f'g + fg' \,|\int$

$fg = \int f'g + \int fg' \Rightarrow$

$\Rightarrow \int f'g = fg - \int fg'$

$$= x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + c, \ c \in \mathbb{R}$$

c) $\int \frac{x+2}{x^2+1} dx = \int \frac{x}{x^2+1} dx + 2\int \frac{1}{x^2+1} dx =$

$$= \frac{1}{2} \ln(x^2+1) \cdot 2 \arctan x + c, \ c \in \mathbb{R}$$

33. b) $\int \frac{dx}{(x^2+1)(x-2)} = \frac{ax+b}{x^2+1}^{\,x-2/} + \frac{c}{x-2}^{\,x^2+1/}$

$$= \frac{1}{5}\int \frac{(x^2+1)-(x^2-4)}{(x^2+1)(x-2)} dx = \frac{1}{5}\left(\int \frac{1}{x-2} dx - \int \frac{(x+2)\cancel{(x-2)}}{(x^2+1)\cancel{(x-2)}} dx\right) =$$

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Subject :

Date :......./......./.......

$$= \frac{1}{5}\left(\ln|x-2| - \int \frac{x}{x^2+1}\,dx - \int \frac{2}{x^2+1}\,dx \right) =$$

$$= \frac{1}{5}\left(\ln|x-2| - \frac{1}{2}\ln(x^2+1) - 2\arctan x \right) + c, \quad c \in \mathbb{R}$$

$x = 2u \Rightarrow dx = 2du$

33. a) $$\int \frac{dx}{(x^2+4)^2} = \int \frac{du}{(4u^2+4)^2} = \frac{1}{8}\int \frac{du}{(u^2+1)^2} =$$

$u = \tan v$

$du = \frac{dv}{\cos^2 v}$

$$= \frac{1}{8}\int \frac{dv}{\cos^2 v(\tan^2 v + 1)^2} = \frac{1}{8}\int \frac{dv}{\cos^2 v\left(\frac{\sin^2 v + \cos^2 v}{\cos^2 v}\right)^2} =$$

$$= \frac{1}{8}\int \cos^2 v\,dv = \frac{1}{8}\int \frac{1+\cos v}{2}\,dv =$$

$\cos 2x = 2\cos^2 x - 1$

$$= \frac{1}{16}\int\left(v + \frac{1}{2}\sin 2v\right) + c = \frac{1}{16}\left(\arctan\frac{x}{2} + \frac{1}{2}\sin\left(2\arctan\frac{x}{2}\right)\right) + c,$$

$c \in \mathbb{R}$

34. $$\underbrace{\int (\sin x)^4\,dx}_{I} = \int \sin^3 x \cdot \sin x\,dx = \sin^3 x \cdot (-\cos x) -$$

$$- \int 3\sin^2 x \cdot \cos x(-\cos x)\,dx = -\sin^3 x\cos x + 3\int \sin^3 x(1 -$$

$$- \sin^2 x)\,dx = -\sin^3 x\cos x + 3\int \sin^2 x\,dx - 3\underbrace{\int \sin^4 x\,dx}_{I}$$

$$\Rightarrow I = \frac{1}{4}\left(-\sin^3 x\cos x + 3\int \sin^2 x\,dx\right) =$$

$\cos 2x = \cos^2 x - \sin^2 x$

$1 - 2\sin^2 x = \cos 2x$

$$= -\frac{1}{4}\sin^3 x \cdot \cos x - \frac{3}{4}\int \frac{1-\cos 2x}{2}\,dx =$$

$$= -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\left(x - \frac{1}{2}\sin 2x\right) + c, \quad c \in \mathbb{R}$$

b) $$\int (\cos x)^5\,dx = \int (1 - \sin^2 x)^2 \cdot \cos x\,dx = \int (1-u)^2\,du =$$

$u = \sin x$

$du = \cos x\,dx$

$$= \int (1 - 2u^2 + u^4)\,du = u - \frac{2u^3}{3} + \frac{u^5}{5} + c =$$

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c, \quad c \in \mathbb{R}$$

Subject :

Date :......./......./.......

35. $\int \frac{dx}{\sqrt[4]{1+x^4}} = \int \frac{dx}{(1+x^4)^{\frac{1}{4}}} = \int \frac{dx}{x\underbrace{\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}}_{u}} =$

$u = \left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} \Big|^4$

$= \int \frac{x^4 dx}{x^5\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}}} = -\int \frac{u^{\cancel{3}\,2} du}{(u^4-1)\cancel{u}} =$

$u^4 = 1 + \frac{1}{x^4} \Rightarrow$

$\Rightarrow x^4 = \frac{1}{u^4-1}$

$\cancel{4}u^3 du = -\cancel{4}\frac{1}{x^5} dx$

$= -\int \frac{u^2 du}{(u^2+1)(u^2-1)} = -\frac{1}{2}\int \frac{(u^2+1)+(u^2-1)}{(u^2+1)(u^2-1)} du =$

$= -\frac{1}{2}\int \frac{1}{u^2-1} du - \frac{1}{2}\int \frac{1}{u^2+1} du = -\frac{1}{4} \ln\left|\frac{u-1}{u+1}\right| -$

$-\frac{1}{2} \arctan u + c = -\frac{1}{4} \ln\left|\frac{\sqrt[4]{1+\frac{1}{x}}+1}{\sqrt[4]{1+\frac{1}{x}}+1}\right| - \frac{1}{2} \arctan \sqrt[4]{1+\frac{1}{x^4}} + c,$

$c \in \mathbb{R}$

36. $\int \frac{dx}{5+4\sin x} =$

$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$= \int \frac{dx}{5+4\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}} = \int \frac{\overbrace{1+\tan^2\frac{x}{2}}^{=\frac{1}{\cos^2\frac{x}{2}}}}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}} dx =$

$\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} dx = dt$

$= \int \frac{2dt}{5t^2+8t+5} = \int \frac{2dt}{5\left(t^2+\frac{8}{5}t+\frac{16}{25}\underbrace{-\frac{16}{25}+1}\right)} =$

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \arctan \frac{x}{a} + c$

$\frac{9}{25} = \left(\frac{3}{5}\right)^2$

$= 2\int \frac{dt}{5\left(t+\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{2}{\cancel{5}} \cdot \frac{\cancel{5}}{3} \arctan \frac{5t+\frac{4}{\cancel{5}}}{\frac{3}{\cancel{5}}} + c =$

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Subject : Date : ....../....../......

$$= \frac{2}{3} \arctan \frac{5 \cdot \tan\frac{x}{2} + 4}{3} + c, \quad c \in \mathbb{R}$$

16.11.2021.

Mathematical Analysis Seminar

37. a) $\iint_A \frac{x}{x^2+y^2}\,dx\,dy \quad ; \quad A = \underbrace{[1,2]}_{x} \times \underbrace{[0,1]}_{y}$

$$I = \int_1^2 \left(\int_0^1 \frac{x}{x^2+y^2}\,dy \right) dx = \int_0^1 \int_1^2 \frac{x}{x^2+y^2}\,dx\,dy$$

$$\int_1^2 \frac{x}{x^2+y^2}\,dx = \frac{1}{2} \ln(x^2+y^2) \Big|_{x=1}^{x=2} =$$

$$= \frac{1}{2} \left(\ln(4+y^2) - \ln(1+y^2) \right)$$

$$\int f'g = fg - \int fg' \qquad \underset{f}{y} \quad \underset{g}{\ln(4+y^2)}$$

$$I = \int_0^1 \frac{1}{2} \left(\ln(4+y^2) - \ln(1+y^2) \right) dy =$$

$$= \frac{1}{2} \left(\underset{f}{y} \cdot \underbrace{\ln(4+y^2)}_{g} \Big|_0^1 - \int_0^1 y \cdot \frac{2y}{4+y^2}\,dy - y \ln(1+y^2) \Big|_0^1 + \right.$$

$$\left. + \int_0^1 y \frac{2y}{1+y^2}\,dy \right) = \frac{1}{2} \left(\ln 5 - 2 \int_0^1 \left(\frac{y^2+4}{4+y^2} - \frac{4}{4+y^2} \right) dy - \right.$$

$$\left. - \ln 2 + 2 \int_0^1 \left(\frac{y^2+1}{1+y^2} - \frac{1}{1+y^2} \right) dy \right) =$$

$$= \frac{1}{2} \left(\ln 5 - 2 \left(y\Big|_0^1 - 4 \frac{1}{2} \arctan \frac{y}{2} \Big|_0^1 \right) + 2 \left(y\Big|_0^1 - \right.\right.$$

$$\left.\left. - \arctan y \Big|_0^1 \right) \right) = \frac{1}{2} \left(\ln \frac{5}{2} - 2 \left(-2 \arctan \frac{1}{2} - 1 + \right.\right.$$

$$\left.\left. + \arctan 1 \right) \right) = \ln \sqrt{\frac{5}{2}} + 4 \arctan \frac{1}{2} - 2\frac{\pi}{4}$$

Subject : Date : ....../....../.......

37. c) $\iint_A (\sin x + \sin y)\, dx\, dy$, $A = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{4}]$

$$I = \int_0^{\pi/4}\int_0^{\pi/2} (\sin x + \sin y)\, dx\, dy \quad ; \int_0^{\pi/2} dy \int_0^{\pi/2} \sin x\, dx + \int_0^{\pi/2} dx \int_0^{\pi/4} \sin y\, dy$$

$$\int_0^{\pi/2} (\sin x + \sin y)\, dx = (-\cos x + x \sin y)\Big|_0^{x=\pi/2} =$$

$$= \pi/2 \cdot \sin y + 1$$

$$I = \int_0^{\pi/4} \left(\frac{\pi}{2} \sin y + 1\right) dy = \left[\frac{\pi}{2}(-\cos y) + y\right]\Big|_0^{\pi/4} = \frac{\pi}{2}\left(-\frac{\sqrt{2}}{2}\right) + \frac{\pi}{4} + \frac{\pi}{2} =$$

$$= \frac{3\pi - \sqrt{2} - \pi}{4} = \frac{\pi}{4}(3 - \sqrt{2})$$

d) $\int \dots \int_A e^{x_1 + \dots + x_n}\, dx_1 \dots dx_n$, $A = [0,1] \times \dots \times [0,1]$

$$I = \int_0^1 \dots \int_0^1 e^{x_1} \cdot e^{x_2} \cdot \dots \cdot e^{x_n}\, dx_1 = \int_0^1 e^{x_n} \int_0^1 e^{x_{n-1}} \dots \int_0^1 e^{x_1}\, dx_1 \dots dx_n =$$

$$= \int_0^1 e^{x_n}\, dx_n \cdot \dots \cdot \int_0^1 e^{x_1}\, dx_1 = e^{x_n}\Big|_0^1 \cdot \dots \cdot e^{x_1}\Big|_0^1 =$$

$$= \underbrace{(e-1) \cdot \dots \cdot (e-1)}_{n \text{ times}} = (e-1)^n$$

39. M is bounded by $y = 2x^2$ and $y = x^2 + 1$

a) Write M as a simple set w.r.t. the y-axis

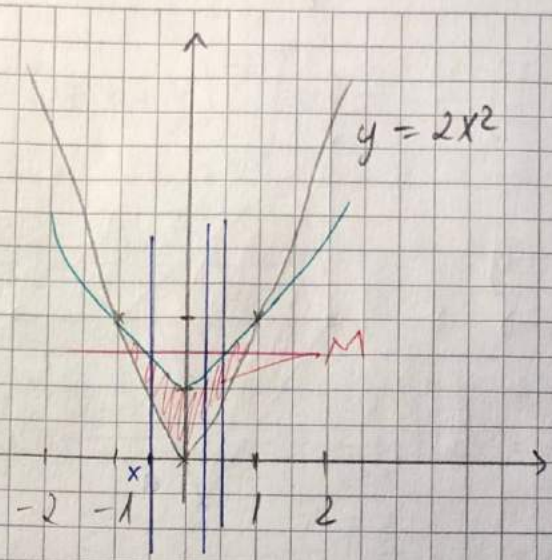
b) Is M simple w.r.t. the x-axis?

c) $\iint_M (x + 2y)\, dx\, dy = ?$

M is simple set w.r.t. the y-axis if $\exists\, a, b \in \mathbb{R} : a < b$ and $\exists\, \varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ continuous : $\varphi_1(x) \le y \le \varphi_2(x)$ and $a \le x \le b$

Subject :

Date :....../....../......



a) $M = \{(x,y) \in \mathbb{R}^2 \mid \underset{a}{-1} \le x \le \underset{b}{1}$ and $2x^2 \le y \le x^2+1\}$ -

- w. r. t the y-axis

b) $\nexists\ \psi_1, \psi_2 : \mathbb{R} \to \mathbb{R} : \psi_1(y) \le x \le \psi_2(y),\ \forall (x,y) \in M$

c) $\iint_M (x+2y)\,dx\,dy = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} (x+2y)\,dy\,dx = \int_{-1}^{1}\int_{2x^2}^{x^2+1} (x+2y)\,dy\,dx =$

$= \int_{-1}^{1} (xy + y^2)\Big|_{y=2x^2}^{y=x^2+1} dx = \int_{-1}^{1} (x^3 + x + (x^2+1)^2 - 2x^3 - 4x^4)\,dx =$

$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx = \left(\frac{-3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right)\Big|_{-1}^{1}$

$= \frac{-6}{5} + \frac{4}{3} + 2 = \frac{32}{15}$

40. $\iiint_M \frac{1}{(1+x+y+z)^3}\,dx\,dy\,dz$

M is bounded by $x+y+z=1$ and by the coordanate planes

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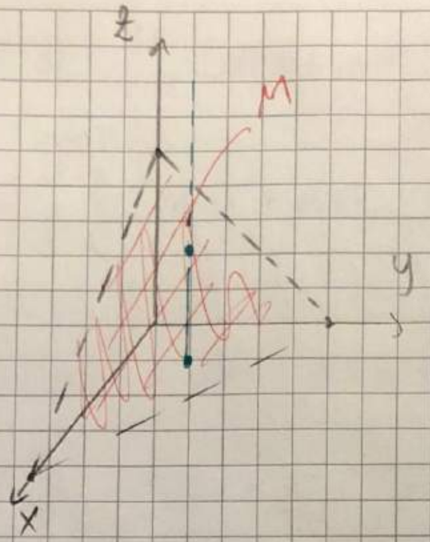
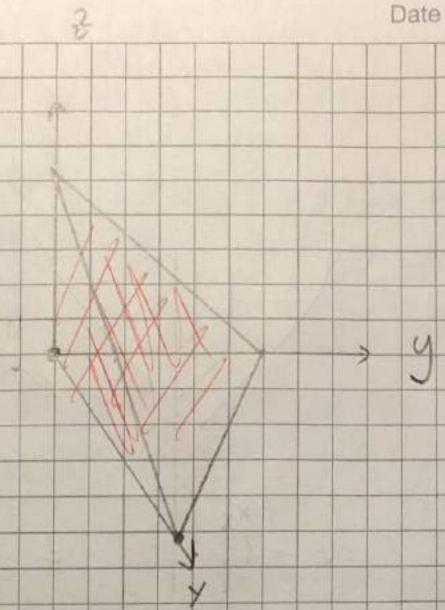
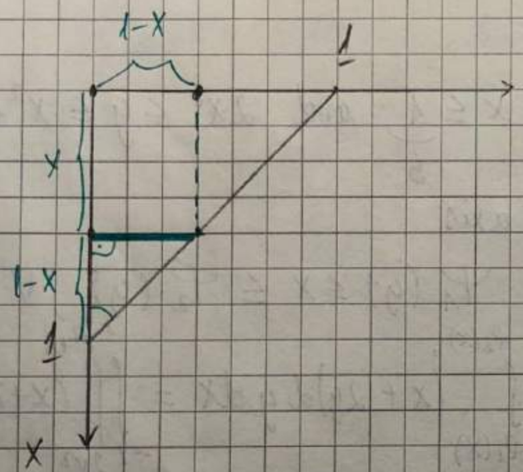
Subject : Date :......./......./.......







$$\begin{cases} 0 \le \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{cases}$$

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{1}{(1+x+y+z)^3}\, dz\, dy\, dx$$

$$\int_0^{1-x-y} (1+x+y+z)^{-3} dz = \frac{(1+x+y+z)^{-2}}{-2} \Bigg|_{z=0}^{z=1-x-y} =$$

$$= -\frac{1}{2}\left((1+\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y})^{-2} - (1+x+y)^{-2} \right) =$$

$$= -\frac{1}{2}\left(\frac{1}{4} - (1+x+y)^{2} \right)$$

Subject : Date : ....../....../......

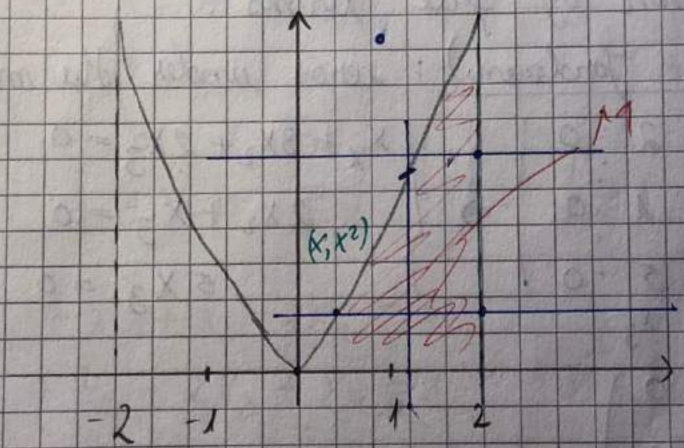
$$-\frac{1}{2}\int_0^{1-x}\left[\frac{1}{4}-(1+x+y)^{-2}\right]dy = -\frac{1}{2}\left(\frac{1}{4}y-\frac{(1+x+y)^{-1}}{+1}\right)\Bigg|_{y=0}^{y=1-x} =$$

$$= -\frac{1}{2}\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{1+x}\right)$$

$$I = \int_0^1\left(-\frac{1}{2}\left(\frac{1-x}{4}+\frac{1}{2}-\frac{1}{1+x}\right)\right)dx = -\frac{1}{2}\left(\frac{1}{4}x-\frac{x^2}{2}+\frac{1}{2}x-\right.$$

$$\left.-\ln(x+1)\right)\Big|_{x=0}^{x=1} = -\frac{1}{2}\left(\frac{1}{4}-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\ln 2\right) = \frac{\ln 2}{2}-\frac{1}{8}$$

41. M is bounded by $y=x^2$, $x=2$ and $y=0$



$M=\{(x,y)\in\mathbb{R}^2 \mid \underbrace{0}\le x\le\underbrace{2}, \quad 0\le y\le x^2\}$ - w.r.t the y-axis

$M=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 4, \quad \sqrt{y}\le x\le 2\}$ - w.r.t the x-axis

$$\int_0^2\int_0^{x^2} xy\,dy\,dx = \int_0^4\int_{\sqrt{y}}^2 xy\,dx\,dy$$

Scanned with CamScanner

1944 Oppenheimer convinces Fermi to join Sept. 1944
Fermi = assoc. Director

1945 July 16^{th} Trinity Test
August 6 & 9 Hiroshima & Nagasaki

23.11.2021.

Mathematical Analysis Seminar

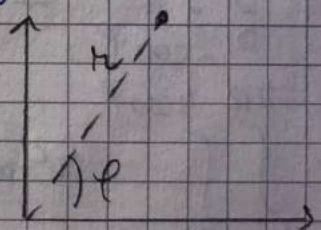
43. $I=\iint_D \sqrt{x^2+y^2}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 / 2x\le x^2+y^2\le$
$\le 4x,\ y\ge 0\}$

$$\begin{cases} x = r\cdot\cos\varphi \\ y = r\cdot\sin\varphi \end{cases}$$

$r\ge 0,\ \varphi\in[0, 2\pi)$

$x=\tan u$

$\frac{dx}{dn}$ $\qquad dx=\frac{du}{\cos^2 u}$



Subject :

Date :....../....../......

$$\det J(r, \varphi) = \begin{vmatrix} \frac{\partial x}{\partial r}(r, \varphi) & \frac{\partial x}{\partial \varphi}(r, \varphi) \\ \frac{\partial y}{\partial r}(r, \varphi) & \frac{\partial y}{\partial \varphi}(r, \varphi) \end{vmatrix} =$$

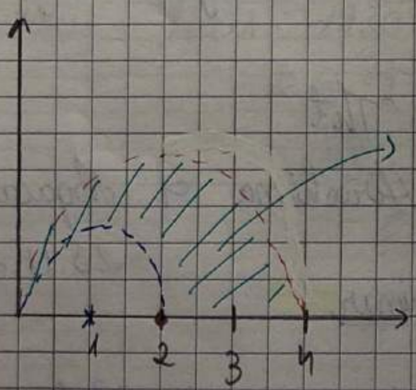
$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r(\cos^2\varphi + \sin^2\varphi) = r \geq 0$$

$$2x \leq x^2 + y^2 \leq 4x$$
$$\downarrow$$
$$2(r\cos\varphi) \leq r^2\cos^2\varphi + r^2\sin^2\varphi \leq 4(r\cos\varphi)$$
$$2r\cos\varphi \leq r^2 \leq 4r\cos\varphi \quad | : r \neq 0$$
$$2\cos\varphi \leq r \leq 4\cos\varphi \qquad x^2+y^2=R^2$$

$$\begin{cases} x^2+y^2 \geq 2x \\ x^2+y^2 \leq 4x \\ y \geq 0 \end{cases} \Rightarrow \begin{cases} (x-1)^2+y^2 \geq 1 \\ (x-2)^2+y^2 \leq 4 \\ y \geq 0 \end{cases}$$

$(x-1)^2+y^2 \geq 1$ - exterior circle radius 1, center at (1, 0)

$(x-2)^2+y^2 \leq 4$ - interior of circle with radius 2 center at (2, 0)



$\varphi \in [0, \frac{\pi}{2})$

$$I = \int_0^{\pi/2}\int_{2\cos\varphi}^{4\cos\varphi} \underbrace{\sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}}_{r} \cdot \underbrace{|\det J(r, \varphi)|}_{r} \, dr\, d\varphi =$$

$$= \int_0^{\pi/2}\int_{2\cos\varphi}^{4\cos\varphi} r^2 \, dr\, d\varphi = \int_0^{\pi/2} \frac{r^3}{3}\Bigg|_{2\cos\varphi}^{4\cos\varphi} d\varphi =$$

Subject : Date :......./......./.......

$$= \frac{64-8}{3} \cdot \int_0^{\pi/2} \cos^3 \varphi \, d\varphi = \frac{56}{3} \cdot \int_0^{\pi/2} (1-\sin^2\varphi)\cdot \cos\varphi \, d\varphi =$$

$$= \frac{56}{3} \left(\sin\varphi - \frac{\sin^3\varphi}{3}\right)\Big|_0^{\pi/2} = \frac{56}{3}\left(1-\frac{1}{3}\right) = \frac{112}{9}$$

44. $I = \iiint_D \frac{dx\,dy\,dz}{x^2+y^2+z^2}$, $D = \{(x,y,z) \in \mathbb{R}^3 / 1 \le x^2 + y^2 + z^2 \le 4,\ z \ge 0\}$

$$\Rightarrow \begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad r \ge 0,\ \varphi \in [0, 2\pi),\ \theta [0, \pi]$$

$\boxed{\text{Lecture 8}}$

$$\begin{cases} 1 \le \overbrace{\underbrace{r^2\sin^2\theta}\cos^2\varphi + \underbrace{r^2\sin^2\theta}\sin^2\varphi}^{1} + r^2\cos^2\theta \le 4 \\ \underset{\ge 0}{r}\cos\theta \ge 0 \\ \varphi \in [0, 2\pi) \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 1 \le r^2 \le 4 \quad \Rightarrow \quad 1 \le r \le 2 \\ \theta \in [0, \pi/2] \\ \varphi \in [0, 2\pi) \end{cases}$$

$$\det J(r, \varphi, \theta) = \begin{vmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{vmatrix} =$$

$$= \cos\theta\,(-r^2\sin\theta\sin\varphi\cos\theta\sin\varphi - r^2\sin\theta\cos\theta\cos\varphi) +$$

$$- r\sin\theta\,(\underbrace{r\sin^2\theta}\cos^2\varphi + \underbrace{r\cdot\sin^2\theta}\sin^2\varphi) =$$

$$= \underbrace{-r^2\sin\theta\cos^2\theta}\sin^2\varphi - \underbrace{r^2\cdot\cos^2\theta\sin\theta}\cdot\underbrace{\cos^2\varphi} - r\sin\theta \cdot r\sin^2\theta =$$

$$= \underbrace{-r^2\sin\theta}\cdot\cos^2\theta \underbrace{- r^2\sin\theta}\sin^2\theta = -r^2\sin\theta$$

$$I = \int_0^{\pi/2}\int_0^{2\pi}\int_1^2 \frac{1}{r^2}\, r^2 \sin\theta \, dr\, d\varphi\, d\theta =$$

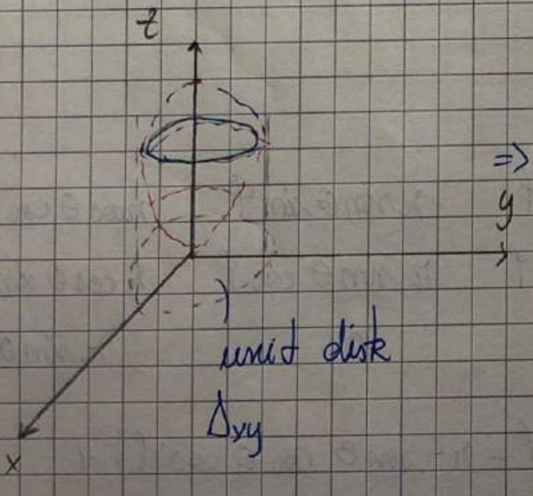
$$= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\varphi \int_1^2 dr = -\cos\theta \Big|_0^{\pi/2} \cdot 2\pi \cdot 1 = 2\pi$$

45. find the volume of $D = \{(x,y,z) \in \mathbb{R}^3 \,/\, z \geq x^2+y^2,$
$(z-2)^2 \geq x^2+y^2,\ z \leq 2\}$

$$\left.\begin{array}{l}(z-2)^2 \geq x^2+y^2 \Rightarrow |z-2| \geq \sqrt{x^2+y^2} \\ z \leq 2\end{array}\right| \Rightarrow 2-z \geq \sqrt{x^2+y^2} \Rightarrow$$

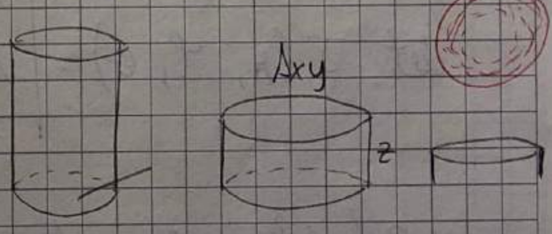
$$\Rightarrow \begin{array}{l} z \leq 2-\sqrt{x^2+y^2} \\ z \geq x^2+y^2 \end{array}$$

$$vol(\Delta) = \iiint_\Delta dx\,dy\,dz = \iint_{\Delta xy} dx\,dy \int_{x^2+y^2}^{2-\sqrt{x^2+y^2}} dz \quad = x^2+y^2 = R^2$$



$z \geq x^2+y^2$ - disk with radius $\sqrt{z}$

$\Rightarrow (z-2)^2 \geq x^2+y^2$ - disk with radius $2-z$



$x = r\cos\varphi,\ r \in [0,1]$

$y = r\sin\varphi,\ \varphi \in [0, 2\pi]$

$$= \iint_{\Delta xy} \left(2 - \sqrt{x^2+y^2} - (x^2+y^2)\right) dx\,dy =$$

$$= \int_0^1\int_0^{2\pi} (2 - r - r^2)\, r\, d\varphi\, dr = \int_0^{2\pi} d\varphi \cdot \int_0^1 (2r - r^2 - r^3)\, dr =$$

$= 2\pi \cdot \left(r^2 - \frac{r^3}{3} - \frac{r^4}{4}\right)\Big|_0^1 = 2\pi\left(1 - \frac{1}{3} - \frac{1}{4}\right) = 2\pi \frac{5}{12} = \frac{5\pi}{6}$

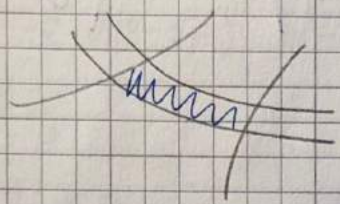
$$z \geq x^2 + y^2$$
$$(z-2)^2 \geq x^2 + y^2$$

$$\left|\begin{array}{l} z = x^2 + y^2 \\ (z-2)^2 = x^2 + y^2 \end{array}\right| -$$

$$\left.\begin{array}{r} z^2 - 5z + 4 = 0 = (z-1)(z-4) \\ z \leq 2 \end{array}\right\} \Rightarrow z = 1$$

46. $I = \iint_D (x^2 + y^2)\, dx\, dy$

$xy = 1 \ , \ xy = 2 \ , \ x^2 = y^2 = 1 \ , \ x^2 - y^2 = 4$



$$\begin{cases} u = xy \\ v = x^2 - y^2 \end{cases} \longrightarrow \det J(x,y) \overset{\det}{=} \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix} =$$

$J(r, \varphi)$

$\int f = \int \det J(r, \varphi)\, dr\, d\varphi$

$$= -2y^2 - 2x^2 = -2(x^2 + y^2)$$

$$J(x,y) \cdot J(u,v) = I_2 \Rightarrow \det J(u,v) = \frac{1}{\det J(x,y)}$$

$$\Rightarrow \det J(u,v) = \frac{1}{-2(x^2+y^2)}$$

$$I = \int_1^2 \int_1^4 \cancel{(x^2+y^2)} \cdot \frac{1}{2\cancel{(x^2+y^2)}}\, dv\, du = \frac{1}{2} \cdot 3 = \frac{3}{2}$$

$$\mathbb{R}^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \mathbb{R}^2 \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

(a, b)

T

T^{-1}

$$f^{(n)}(0) = \int_0 y^n e^{0y}\, dy = \frac{1}{n+1}$$

07.12.2021

Mathematical Analysis Seminar

47) $\int_0^\infty e^{-x^2} dx$

$$\int_0^\infty f(x)\,dx = \lim_{t\to\infty} \int_0^x f(x)\,dx = \lim_{t\to\infty} F(t) - F(0)$$

$F'(x) = f(x)$

$x = ty, \quad t>0$

$dx = t\,dy$

$$I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t^2y^2} t\,dy = t\cdot\int_0^\infty e^{-t^2y^2} dy \Big| \cdot e^{-t^2}$$

$$I e^{-t^2} = t e^{-t^2} \int_0^\infty e^{-t^2y^2} dy \Big| \int_0^\infty dt$$

$$I \cdot \underbrace{\int_0^\infty e^{-t^2} dt}_{I} = \int_0^\infty t e^{-t^2} \int_0^\infty e^{-t^2y^2} dy\, dt$$

$$I^2 = \int_0^\infty \underbrace{\int_0^\infty t e^{-t^2(1+y^2)} dt}_{J(y)}\, dy$$

Subject : Date : ....../....../......

$$J(y)=\int_0^\infty t\,\underbrace{e^{-t^2(1+y^2)}}\,dt \qquad t e^{-t^2} \qquad -\frac{1}{2}\left(e^{-t^2}\right)'$$

$$\frac{\partial}{\partial t}\left(e^{-t^2(1+y^2)}\right)=-2t(1+y^2)\cdot\underbrace{e^{-t^2(1+y^2)}}$$

$$J(y)=\frac{1}{-2(1+y^2)}\int_0^\infty\frac{\partial}{\partial t}\left(e^{-t^2(1+y^2)}\right)dt=$$

$$=\frac{1}{-2(1+y^2)}\cdot e^{-t^2(1+y^2)}\Big|_{t=0}^{t=\infty}=\underbrace{\lim_{t\to\infty}\frac{1}{-2(1+y^2)e^{t^2(1+y^2)}}}_{0}+$$

$$+\frac{1}{2(1+y^2)}$$

$$I^2=\int_0^\infty J(y)\,dy=\int_0^\infty\frac{1}{2(1+y^2)}\,dy=\frac{1}{2}\arctan y\Big|_0^\infty=$$

$$=\frac{1}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi}{4}$$

$$\left.\begin{array}{l} I>0 \\ e^{-x^2}>0\end{array}\right\} \Rightarrow I=\frac{\sqrt{\pi}}{2}$$

48) $\displaystyle\int_0^\infty\frac{\sin x}{x}\,dx$

$$I(t)=\int_0^\infty\frac{\sin x}{x}e^{-tx}\,dx\ ,\quad I(0)=\int_0^\infty\frac{\sin x}{x}\,dx$$

$$I'(t)=\int_0^\infty\frac{\partial}{\partial t}\left(\frac{\sin x}{x}e^{-tx}\right)dx=\int_0^\infty\frac{\sin x}{\cancel{x}}\left(-\cancel{x}e^{-tx}\right)dx=$$

$$=-\int_0^\infty\sin x\cdot e^{-tx}\,dx$$

$$\int\underbrace{\sin x}_{f}\cdot\underbrace{e^{-tx}}_{g'}\,dx=\frac{-e^{-tx}}{t}\sin x-\int\underbrace{\frac{-e^{-tx}}{t}}_{g}\underbrace{\cos x}_{f'}\,dx=$$

Subject : Date : ...../...../.....

$$= \frac{-e^{-tx}\sin x}{t} - \left(\frac{e^{-tx}}{t^2}\cos x - \int \frac{e^{-tx}}{t^2}\sin x\,dx\right) =$$

$$= \frac{-e^{-tx}(t\sin x + \cos x)}{t^2} - \frac{1}{t^2}\int e^{-tx}\sin x\,dx$$

$$\frac{t^2}{1} + \frac{1}{t^2} = \frac{t^2+1}{t^2}$$

$$\int e^{-tx}\sin x\,dx = \frac{-e^{-tx}(t\sin x + \cos x)}{t^2+1} + c$$

$$I'(t) = -\int_0^\infty e^{-tx}\sin x \cdot dx = \left.\frac{e^{-tx}(t\sin x + \cos x)}{t^2+1}\right|_{x=0}^{x=\infty} =$$

$$= \lim_{x\to\infty} \frac{t\overbrace{\sin x}^{\in[-1,1]} + \overbrace{\cos x}^{\in[-1,1]}}{\underbrace{e^{tx}}_{\to\infty}(t^2+1)} - \frac{1}{t^2+1} = -\frac{1}{t^2+1}$$

$$I'(t) = -\frac{1}{t^2+1} \;\Big|\; \int dt \Rightarrow I(t) = \int -\frac{1}{t^2+1}\,dt =$$

$$= -\arctan t + c$$

$$I(t) = -\arctan t + c = \int_0^\infty \frac{\sin x}{x}\cdot \underbrace{e^{-tx}}_{0}\,dx$$

$$t\to\infty \Rightarrow \lim_{t\to\infty}(-\arctan t) + c = 0 \Rightarrow c = \frac{\pi}{2}$$

49) $$\int_0^\infty \frac{\sin x}{x} e^{-x}\,dx = I(1) = -\arctan 1 + \frac{\pi}{2} = -\frac{\pi}{4} +$$

$$+ \frac{\pi}{2} = \frac{\pi}{4}$$

Subject : Date : ....../....../......

50) $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx, \quad a,b>0$

$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\,dx, \quad a>0$

a) Convergence of $B(a,b)$ for fixed a,b

$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = \int_0^{1/2} x^{a-1}(1-x)^{b-1}\,dx + \int_{1/2}^1 x^{a-1}(1-x)^{b-1}\,dx$

$x\in[0,\frac{1}{2}] \Rightarrow x^{a-1}\cdot\underbrace{(1-x)^{b-1}}_{\frac{1}{(1-x)^{1-b}} \le c,\ c\in\mathbb{R}} \le x^{a-1}\cdot c \Rightarrow$

$\frac{1}{\left(\frac{1}{2}\right)^{1-b}} = 2^{1-b}$

$\Rightarrow \int_0^{1/2} x^{a-1}(1-x)^{b-1}\,dx \le c\int_0^{1/2} x^{a-1}\,dx = c\cdot\frac{x^a}{a}\Big|_0^{1/2} < \infty \quad (1)$

$x\in[\frac{1}{2},1] \Rightarrow \underbrace{x^{a-1}}_{\frac{1}{x^{1-a}} \le c,\ c\in\mathbb{R}}\cdot(1-x)^{b-1} \le c\cdot(1-x)^{b-1} \Rightarrow$

$\Rightarrow \int_{1/2}^1 x^{a-1}(1-x)^{b-1}\,dx \le c\cdot\int_{1/2}^1 (1-x)^{b-1}\,dx =$

$= \frac{-c(1-x)^b}{b}\Big|_{1/2}^1 < \infty \quad (2)$

$(1),(2) \Rightarrow B(a,b) < \infty \Rightarrow B(a,b)$ is convergent

b) Convergent of $\Gamma(a)$

Subject : Date :....../....../.......

$$\Gamma(a)=\int_0^\infty x^{a-1}e^{-x}\,dx=\int_0^1 x^{a-1}e^{-x}\,dx+\int_1^\infty x^{a-1}e^{-x}\,dx$$

$$x\in[0,1]\Rightarrow x^{a-1}\cdot\underbrace{e^{-x}}_{\frac{1}{e^x}}\le x^{a-1}\Rightarrow\int_0^1 x^{a-1}e^{-x}\,dx\le$$

$$\le\int_0^1 x^{a-1}\,dx=\frac{x^a}{a}\Big|_0^1=\frac{1}{a}<\infty \quad (1)$$

$$x\in[1,\infty)\qquad \lim_{x\to\infty}\frac{x^{a-1}}{e^{x/2}}=0\Rightarrow$$

$$\Rightarrow\exists N\in\mathbb{N}:\ \forall\ x\ge N\quad e^{x/2}\overset{(*)}{>}x^{a-1}$$

$$\int_1^\infty x^{a-1}\cdot e^{-x}\,dx=\underbrace{\int_1^N x^{a-1}\cdot e^{-x}\,dx}_{<\infty\ (2)}+\underbrace{\int_N^\infty x^{a-1}\cdot e^{-x}\,dx}_{<\infty\ (3)}$$

$$\int_N^\infty x^{a-1}e^{-x}\,dx\overset{(*)}{<}\int_N^\infty e^{x/2}\cdot e^{-x}\,dx=\int_N^\infty e^{-\frac{x}{2}}\,dx=$$

$$=-2e^{-\frac{x}{2}}\Big|_N^\infty=-2\left(0-\frac{1}{e^{N/2}}\right)<\infty$$

(1), (2), (3) $\Rightarrow \Gamma(a)$ is convergent

c) $\Gamma(a+1)=a\,\Gamma(a)$ $\qquad \Gamma(a)=\int_0^\infty x^{a-1}e^{-x}\,dx$

$$\Gamma(a+1)=\int_0^\infty \underbrace{x^a}\cdot e^{-x}\,dx=-e^{-x}x^a\Big|_0^\infty+\int_0^\infty a x^{a-1}\cdot e^{-x}\,dx=$$

$$=\lim_{x\to\infty}\frac{-x^a}{e^x}+0+a\cdot\underbrace{\int_0^\infty x^{a-1}\cdot e^{-x}\,dx}_{\Gamma(a)}=$$

$$=a\,\Gamma(a)$$

Subject : Date : ....../....../......

Prove that $\Gamma(n+1) = n!$, $\forall n \in \mathbb{N}$

$\Gamma(n+1) = n\,\Gamma(n) = n(n-1)\cdot\Gamma(n-1) = \dots = n(n-1)\cdot\dots\cdot\Gamma(1)$

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -\lim_{x\to\infty}\frac{1}{e^x} + 1 = 1$$

$\Rightarrow \Gamma(n+1) = n!$

Assume that $\Gamma(k+1) = k!$

$\Gamma(k+2) = \Gamma((k+1)+1) = ((k+1)\cdot\Gamma(k+1)) =$
$= (k+1)\cdot k! = (k+1)!$

d) $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2}\,dx$ $\qquad \Gamma(a) = \int_0^\infty x^{a-1}\cdot e^{-x}\,dx$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x}\,dx$$

$x = t^2$

$t = x^{1/2} \Rightarrow dt = \frac{1}{2}x^{-\frac{1}{2}}dx$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty 2e^{-t^2}\,dt = 2\int_0^\infty e^{-t^2}\,dt$$

$$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

13.12.2021

Algebra Seminar

1. $B = (v_1, v_2, v_3) = ((1,0,1), (0,1,1), (1,1,1))$
$B' = (v_1', v_2', v_3') = ((1,1,0), (-1,0,0), (0,0,1))$

$T_{BB'}$, $T_{B'B}$, $u = (2,0,-1)$ in B and in B'

Subject : Date : 14.12.21

Mathematical Analysis. Seminar

1. Comparison Test

$f, g : [a, b) \quad 0 \le f(x) \le g(x) \quad \forall x \in [a, b)$

$\int g$ convergent $\Rightarrow$ $\int f$ convergent

$\int f$ divergent $\Rightarrow$ $\int g$ divergent

2. Comparison Test

$g(x) > 0 \quad \forall x \in [a, b)$

$$\lim_{x \to b} \frac{f(x)}{g(x)} = L \in \mathbb{R}$$

if $L \neq 0$ $\quad \int f$ conv $\Leftrightarrow$ $\int g$ conv

if $L = 0$ $\quad \int g$ conv $\Rightarrow$ $\int f$ conv

51. a) $$\int_0^\infty \underbrace{\frac{\arctan x}{1+x^2}}_{f(x)} dx$$

$$\lim_{x\to\infty} \frac{\arctan x}{1+x^2} = 0 \quad , \quad \arctan x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$x \in [0, \infty) \Rightarrow \arctan x \in \left[0, \frac{\pi}{2}\right) \to 0 \le \underbrace{\frac{\arctan x}{1+x^2}}_{f(x)} \le \underbrace{\frac{\pi}{2} \cdot \frac{1}{1+x^2}}_{g(x)}$$

$$\int_0^\infty \frac{\pi}{2} \cdot \frac{1}{1+x^2}\, dx = \frac{\pi}{2} \arctan x \Big|_0^\infty = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4} < \infty \Rightarrow$$

$\Rightarrow \int g$ conv. $\Rightarrow$

1. Crit. Test $\Rightarrow$ $$\int_0^\infty \frac{\arctan x}{1+x^2} dx \text{ convergent}$$

Subject : Date :....../....../......

b) $\int_2^{\infty} \frac{x-1}{x^2+x+1}\, dx$; $g(x) = \frac{x^2+x+1}{x-1}$; $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$

$\int_2^{\infty} \frac{x^2+x+1}{x-1}\, dx$ diverg. because $\lim_{x\to\infty} \frac{x^2+x+1}{x-1} \neq 0$; $g(x) = \frac{1}{x}$

$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$ $\int_2^{\infty} \frac{1}{x}\, dx$ div. $\xRightarrow{2.C.T.}$ $\int_2^{\infty} \frac{x-1}{x^2+x+1}\, dx$ div.

$a > 0$ $\int_a^{\infty} \frac{1}{x^p}\, dx$ conv. $\Leftrightarrow$ $\boxed{p > 1}$ $\sum_{n\geq 1} \frac{1}{n^p}$ conv. $\Leftrightarrow p > 1$

c) $\int_0^1 (\ln x)^2\, dx$ $y = -t \Rightarrow dy = -dt$

$x \to 0 \Rightarrow \underbrace{\ln x}_{y} \to -\infty$ $\boxed{y = \ln x} \Rightarrow x = e^y \Rightarrow dx = e^y dy$

$\Rightarrow \int_0^1 (\ln x)^2\, dx = \int_{-\infty}^0 y^2 e^y\, dy = \int_{\infty}^0 t^2 e^{-t} (-dt) = \int_0^{\infty} \underline{t^2 e^{-t}}\, dt$ (52a))

$g(t) = \frac{1}{t^2+1}$

$\lim_{x\to\infty} \frac{f(x)}{g(t)} = \lim_{t\to\infty} \frac{t^2 \cdot e^{-t}}{\frac{1}{t^2+1}} = \lim_{t\to\infty} \frac{t^4+t^2}{e^t} \overset{l'H}{=} 0$

$\int_0^{\infty} g(t)\, dt = \int_0^{\infty} \frac{1}{t^2+1}\, dt = \arctan t \Big|_0^{\infty} = \frac{\pi}{2} < \infty$

$\Rightarrow \int g$ conv.

$\xRightarrow{2.\ Crit.\ Test.}$

$\Rightarrow \int_0^{\infty} t^2 e^{-t}\, dt$ conv. $\Rightarrow \int_0^1 (\ln x)^2\, dx$ conv.

52.b) $P(x)$ polynomial function of degree n

$P(x) = a_n x^n + \ldots + a_1 x + a_0$, $a_i \in \mathbb{R}$, $a_n \neq 0$

Prove that $\int_0^{\infty} P(x) e^{-x}\, dx = P(0) + P'(0) + \ldots + P^{(n)}(0)$

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Subject :

Date :......./......./.......

$$\int_0^\infty P(x)e^{-x}dx = P(x)(-e^{-x})\Big|_0^\infty - \int_0^\infty P'(x)(-e^{-x})\,dx =$$

$$F(a)-F(b)=\int_b^a$$

$$= P(0) + \int_0^\infty P'(x)e^{-x}\,dx =$$

$$\lim_{x\to\infty}\frac{P(x)}{e^x} \overset{l'H}{=} \lim_{x\to\infty}\frac{P'(x)}{e^x} = \dots = \lim_{x\to\infty}\frac{P^{(m)}(x)}{e^x} = \lim_{x\to\infty}\frac{a_m\cdot m!}{e^x} = 0$$

$$= P(0) + P'(x)(-e^{-x})\Big|_0^\infty + \int_0^\infty P''(x)e^{-x}\,dx =$$

$$= P(0) + P'(x) + \dots + P^{(m-1)}(x) + \int_0^\infty P^{(m)}(x)e^{-x}\,dx =$$

$$= P(0) + \dots + P^{(m-1)}(0) + P^{(m)}(0) + \int_0^\infty \underbrace{P^{(m+1)}(x)\cdot e^{-x}}_{0}\,dx =$$

$$= P(0) + \dots + P^{(m)}(0)$$

53). a) $\displaystyle\int_0^1 \frac{x^5-1}{\ln x}\,dx$; $\displaystyle I(a) = \int_0^1 \frac{x^a-1}{\ln x}\,dx$

$$I'(a) = \int_0^1 \frac{\partial}{\partial a}\left(\frac{x^a-1}{\ln x}\right)dx = \int_0^1 \frac{x^a \cancel{\ln x}}{\cancel{\ln x}}\,dx = \frac{x^{a+1}}{a+1}\Big|_0^1 =$$

$$= \frac{1}{a+1} \Rightarrow I(a) = \int \frac{1}{a+1}\,da = \ln(a+1) + c,\ c\in\mathbb{R}$$

$$I(0) = \ln 1 + c = c = \int_0^1 \frac{x^0-1}{\ln x}\,dx = 0 \Rightarrow c = 0$$

$$\Rightarrow I(a) = \ln(a+1)$$

$$\int_0^1 \frac{x^5-1}{\ln x}\,dx = I(5) = \ln(6)$$

b) $\displaystyle\int_0^\infty \frac{\arctan 2x}{x(1+x^2)}\,dx$ $\qquad$ $\displaystyle I(y) = \int_0^\infty \frac{\arctan xy}{x(1+x^2)}\,dx$

$$I'(y) = \int_0^\infty \frac{\partial}{\partial y}\left(\frac{\arctan xy}{x(1+x^2)}\right)dx = \int_0^\infty \frac{1}{1+x^2y^2}\cdot \cancel{x}\cdot\frac{1}{\cancel{x}(1+x^2)}\,dx =$$

$$= \int_0^\infty \frac{1}{(1+x^2y^2)(1+x^2)}\,dx \qquad\qquad 1-\frac{1}{y^2} = \frac{y^2-1}{y^2}$$

$$\frac{1}{y^2}\left(\int_0^\infty \frac{1}{\left(\frac{1}{y^2}+x^2\right)(1+x^2)}\right)= \qquad\qquad 1 = \left[(1+x^2) - \left(\frac{1}{y^2}+x^2\right)\right]$$

$$= \frac{1}{y^2}\cdot\frac{y^2}{y^2-1}\int_0^\infty \frac{(1+x^2)-\left(\frac{1}{y^2}+x^2\right)}{\left(\frac{1}{y^2}+x^2\right)(1+x^2)}\,dx = \frac{1}{y^2-1}\left(\int_0^\infty\frac{dx}{\frac{1}{y^2}+x^2} - \int_0^\infty \frac{dx}{1+x^2}\right)=$$

$$=\frac{1}{y^2-1}\left(y\cdot\arctan xy\Big|_0^\infty - \arctan x\Big|_0^\infty\right)=$$

$$=\frac{1}{y^2-1}\left(\lim_{x\to\infty}(y\arctan xy - \arctan x)\right)=$$

$$=\frac{1}{y^2-1}\left(y\cdot\frac{\pi}{2}-\frac{\pi}{2}\right)=\frac{\pi}{2}\cdot\frac{y-1}{y^2-1}=\frac{\pi}{2}\cdot\frac{1}{y+1}=I'(y)$$

$$\Rightarrow I(y)=\int\frac{\pi}{2}\cdot\frac{1}{y+1}\,dy=\frac{\pi}{2}\cdot\ln(y+1)+c,\quad c\in\mathbb{R}$$

$$I(0)=\frac{\pi}{2}\cdot\ln 1+c=c=\int_0^\infty\frac{\arctan 0\cdot x}{x(1+x^2)}\,dx=0\Rightarrow c=0$$

$$\Rightarrow I(y)=\frac{\pi}{2}\cdot\ln(y+1)$$

$$\int_0^\infty\frac{\arctan 2x}{x(1+x^2)}\,dx=I(2)=\frac{\pi}{2}\ln 3$$

51. a) $\displaystyle\int_0^\infty\frac{\arctan x}{x^2+1}\,dx=\frac{1}{2}(\arctan x)^2\Big|_0^\infty=\frac{\pi^2}{8}$

b) $\displaystyle\int_2^\infty\frac{x-1}{x^2+x+1}\,dx=\frac{1}{2}\int_2^\infty\left(\frac{2x+1}{x^2+x+1}-\frac{3}{x^2+x+1}\right)dx=$

Subject :

Date :......./......./.......

$$= \frac{1}{2}\left(\ln(x^2+x+1)\Big|_2^{\infty} - 3\int_2^{\infty} \frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} = \frac{1}{2}\ln(x^2+x+1)\Big|_2^{\infty} -\right.$$

$$- \underbrace{\frac{3}{\cancel{2}}\cdot\frac{\cancel{2}}{\sqrt{3}} \arctan \frac{\cancel{x}+\frac{1}{2})}{\frac{\sqrt{3}}{2}}}\Big|_2^{\infty}$$

$$\int \frac{dt}{t^2+a^2} = \frac{1}{a}\arctan\frac{t}{a}$$

$$\sqrt{3}\arctan\frac{(2x+1)}{\sqrt{3}}\Big|_2^{\infty} \in \mathbb{R}$$

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Algebra Seminar