# Seminar Nr. 5, Continuous Random Variables and Continuous Random Vectors

**Theory Review**

$X : S \to \mathbb{R}$ continuous random variable with pdf $f : \mathbb{R} \to \mathbb{R}$ and cdf $F : \mathbb{R} \to \mathbb{R}$. Properties:

1. $F(x) = P(X \le x) = \int\limits_{-\infty}^{x} f(t)dt$

2. $f(x) \ge 0, \forall x \in \mathbb{R}, \int\limits_{\mathbb{R}} f(x) = 1$

3. $P(X = x) = 0, \forall x \in \mathbb{R}, P(a < X < b) = P(a \le X \le b) = \int\limits_{a}^{b} f(t)dt$

4. $F(-\infty) = 0, F(\infty) = 1$

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$(X, Y) : S \to \mathbb{R}^2$ continuous random vector with pdf $f = f_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R}$ and

cdf $F = F_{(X,Y)} : \mathbb{R}^2 \to \mathbb{R},\ F(x, y) = P(X \le x, Y \le y) = \int\limits_{-\infty}^{x} \int\limits_{-\infty}^{y} f(u, v)\ dv\ du,\ \forall (x, y) \in \mathbb{R}^2$. Properties:

1. $P(a_1 < X \le b_1, a_2 < Y \le b_2) = F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2)$
2. $F(\infty, \infty) = 1, F(-\infty, y) = F(x, -\infty) = 0,\ \forall x, y \in \mathbb{R}$
3. $F_X(x) = F(x, \infty),\ F_Y(y) = F(\infty, y),\ \forall x, y \in \mathbb{R}$ (marginal cdf's)
4. $P((X, Y) \in D) = \iint\limits_{D} f(x, y)\ dy\ dx$
5. $f_X(x) = \int\limits_{\mathbb{R}} f(x, y)dy,\ \forall x \in \mathbb{R},\ f_Y(y) = \int\limits_{\mathbb{R}} f(x, y)dx,\ \forall y \in \mathbb{R}$ (marginal densities)
6. $X$ and $Y$ are independent $<=>$ $f_{(X,Y)}(x, y) = f_X(x) f_Y(y),\ \forall (x, y) \in \mathbb{R}^2$.

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**Function** $Y = g(X)$: $X$ r.v., $g : \mathbb{R} \to \mathbb{R}$ differentiable with $g' \ne 0$, strictly monotone

$f_Y(y) = \dfrac{f_X\left(g^{-1}(y)\right)}{|g'\left(g^{-1}(y)\right)|},\ y \in g(\mathbb{R})$

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**Uniform distribution** $U(a, b),\ -\infty < a < b < \infty$ : pdf $f(x) = \dfrac{1}{b - a}, x \in [a, b]$.

**Normal distribution** $N(\mu, \sigma), \mu \in \mathbb{R}, \sigma > 0$ : pdf $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}$.

**Gamma distribution** $Gamma(a, b),\ a, b > 0$ : pdf $f(x) = \dfrac{1}{\Gamma(a) b^a} x^{a-1} e^{-\frac{x}{b}}\ ,\ x > 0$.

**Exponential distribution** $Exp(\lambda) = Gamma(1, 1/\lambda),\ \lambda > 0$ : pdf $f(x) = \lambda e^{-\lambda x}, x > 0$.
- Exponential distribution models *time*: waiting time, interarrival time, failure time, time between rare events, etc; the parameter $\lambda$ represents the frequency of rare events, measured in $\mathrm{time}^{-1}$.
- Gamma distribution models the *total* time of a multistage scheme.
- For $\alpha \in \mathbb{N}$, a $Gamma(\alpha, 1/\lambda)$ variable is the sum of $\alpha$ independent $Exp(\lambda)$ variables.

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**1.** The lifetime, in years, of some electronic component is a random variable with density

$$f(x) = \begin{cases} \dfrac{k}{x^4}, & \text{for} \quad x \ge 1 \\ 0, & \text{for} \quad x < 1. \end{cases}$$

Find
a) the constant $k$;

b) the corresponding cdf $F$;
c) the probability for the lifetime of the component to exceed 2 years.

**2.** (The Uniform property) Let $X \in U(a, b)$. For any $h > 0$ and $t, s \in [a, b - h]$,

$$P(s < X < s + h) = P(t < X < t + h).$$

The probability is only determined by the length of the interval, but not by its location.
Example: A certain flight can arrive at any time between 4:50 and 5:10 pm. Let $X$ denote the arrival time of the flight.
a) What distribution does $X$ have?
b) When is the flight more likely to arrive: between 4:50 and 4:55 or between 5 and 5:05; before 5 or after 5?

**3.** On the average, a computer experiences breakdowns every 5 months. The time until the first breakdown and the times between any two consecutive breakdowns are independent Exponential random variables. After the third breakdown, a computer requires a special maintenance.
a) Find the probability that a special maintenance is required within the next 9 months;
b) Given that a special maintenance was not required during the first 12 months, what is the probability that it will not be required within the next 4 months?

**4.** The joint density for $(X, Y)$ is $f_{(X,Y)}(x, y) = \frac{1}{16}x^3y^3$, $x, y \in [0, 2]$.
a) Find the marginal densities $f_X$, $f_Y$.
b) Are $X$ and $Y$ independent?
c) Find $P(X \leq 1)$.

**5.** Let $X$ be a random variable with density $f_X(x) = \frac{1}{4}xe^{-\frac{x}{2}}$, $x \geq 0$ and let $Y = \frac{1}{2}X + 2$. Find $f_Y$.

**6.** Let $X \in N(0, 1)$. Find the probability density function of $Y = |X|$.

**Bonus Problems**:

**7.** Let $X$ denote the velocity of a random gas molecule. According to the Maxwell-Boltzmann law, the pdf of $X$ is given by

$$f_X(x) = cx^2e^{-\beta x^2},\ x > 0,$$

where the constants $c$ and $\beta$ depend on the gas involved, its mass and its temperature. The kinetic energy of the molecule is given by $Y = \frac{1}{2}mX^2$, where $m > 0$. For a gas molecule with $c = 2, \beta = 1$ and $m = 1$, find the pdf of the kinetic energy of the molecule.

**8.** A gamer shoots at a virtual shooting board centered at the origin of the Cartesian coordinate system such that the coordinates of the hit are two independent random variables that follow the $N(0, 1)$ distribution. Find the probability that the shooter hits the upper half-plane of the shooting board at a distance between 1 and 2 from the origin.