LECTURE 8 - DYNAMICAL SYSTEMS
Monday, 12 April 2021 10:00

(1) $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

$\exists!$ th, equilibrium point, flow, orbit, attractor, repellor

Def: (i) Let $U \subset \mathbb{R}^2$ open. We say that U is an <u>invariant set</u> for (1) when $\forall \eta \in U$ we have $\gamma_\eta \subset U$.

(ii) Let $\eta \in U$. We say that γ_η is a <u>periodic orbit</u> (or closed orbit) of (1) when the corresponding solution of (1) $\varphi(\cdot, \eta)$ is a periodic function.

(iii) Let $U \subset \mathbb{R}^2$ open and consider $H: U \to \mathbb{R}$ a C^1 function. We say that H is <u>a first integral in U</u> of (1) when H is not locally constant and

$$H(\varphi(t,\eta)) = H(\eta), \forall \eta \in U, \forall t \text{ s.t. } \varphi(t,\eta) \in U$$

($\Leftrightarrow$ $H/_{\gamma_\eta \cap U}$ is constant)

Def.: Let $U \subset \mathbb{R}^2$, $H: U \to \mathbb{R}$ cont. function. Let $c \in \mathbb{R}$. The c-level curve of H is $\Gamma_c = \{x \in U : H(x) = c\}$.

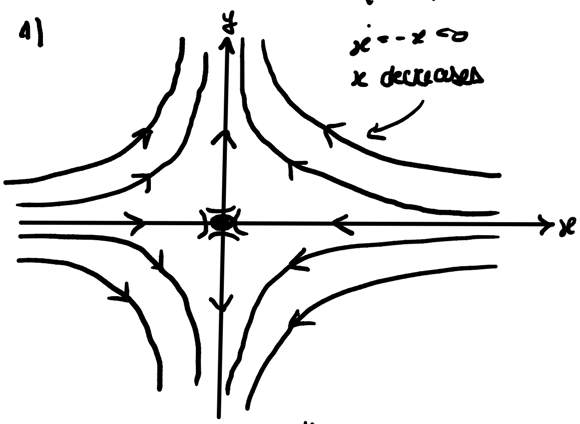
Remark. Let H be a first integral in U of (1) and U be an invariant set. Then $\forall \eta \in U$ we have $\gamma_\eta \subset \Gamma_{H(\eta)}$.
(the orbits are contained in the level curves of a first integral).

Phase portraits of linear planar systems

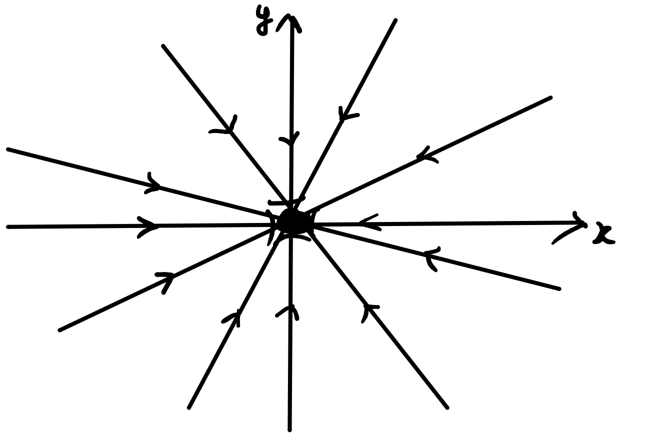
Let $A \in \mathcal{M}_2(\mathbb{R})$. (2) $\dot{X} = AX$

Remark: We have that $\eta^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \mathbb{R}^2$ is the unique equilibrium point of (2) iff $\det A \neq 0$

1)



2)



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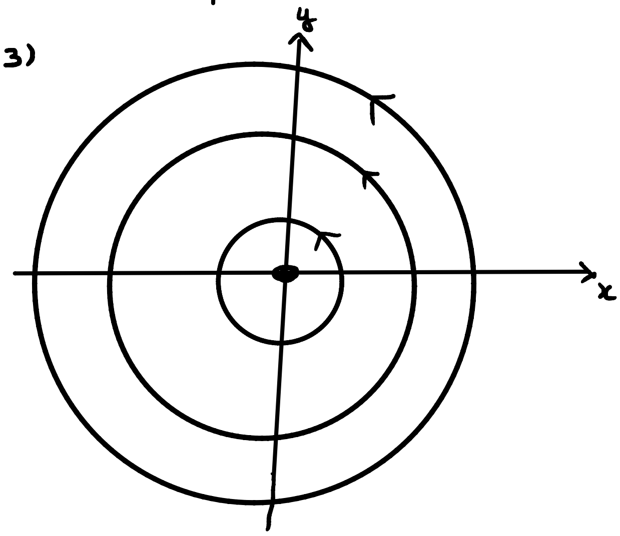

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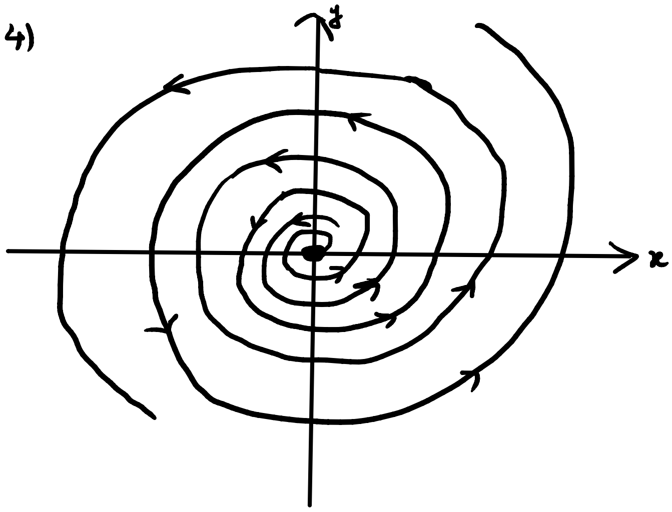


a) $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ b) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ c) $\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases}$ d) $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$

Decide if there exist: - a global attractor / a global repeller
- a global first integral

Find the flow.

a) $\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}$ the flow
check that $H: \mathbb{R}^2 \to \mathbb{R}, H(x,y) = x^2+y^2$ is a global first integral

det $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$ and ivp $\begin{cases} \dot{x} = -y \\ \dot{y} = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$ $\quad \ddot{x} = -\dot{y} = -x \quad y = -\dot{x}$

$\ddot{x} + x = 0 \Rightarrow k^2+1 = 0 \Rightarrow k_{1,2} = \pm i \Rightarrow \begin{cases} x = c_1 \cos t + c_2 \sin t \\ y = c_1 \sin t - c_2 \sin t \end{cases}$

$\begin{cases} x(0) = c_1 = \eta_1 \\ y(0) = -c_2 = \eta_2 \end{cases} \Rightarrow \begin{cases} c_1 = \eta_1 \\ c_2 = -\eta_2 \end{cases} \Rightarrow \varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 \cos t - \eta_2 \sin t \\ \eta_1 \sin t + \eta_2 \cos t \end{pmatrix}, \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2, \forall t \in \mathbb{R}$

check that $H: \mathbb{R}^2 \to \mathbb{R}, H(x,y) = x^2+y^2$ is a global first integral.
global: it is defined on $\mathbb{R}^2$ ✓ $H \in C^1(\mathbb{R}^2)$ ✓ not locally constant ✓

$H(\varphi(t,\eta)) \overset{?}{=} H(\eta), \forall \eta \in \mathbb{R}^2, \forall t \in \mathbb{R}$.

$H(\varphi(t,\eta)) = (\eta_1 \cos t - \eta_2 \sin t)^2 + (\eta_1 \sin t + \eta_2 \cos t)^2 =$

$= \eta_1^2 \cos^2 t - \cancel{2\eta_1\eta_2 \cos t \sin t} + \eta_2^2 \cdot \sin^2 t + \eta_1^2 \sin^2 t + \cancel{2\eta_1\eta_2 \sin t \cos t} + \eta_2^2 \cos^2 t =$

$= \eta_1^2 + \eta_2^2 = H(\eta)$

$\Rightarrow H$ is a global first integral

$H(x,y) = x^2+y^2$. Its level curves are $x^2+y^2 = c, c \in \mathbb{R}$

$c = 0 \quad x^2+y^2 = 0$ is $(0,0)$ $\qquad x^2+y^2 = c > 0$ circle centered in the origin with $\sqrt{c}$ radius

$c < 0 \quad x^2+y^2 < 0$ is $\emptyset$

These level curves are the orbits of the system. The phase portrait is a). When $y > 0$ we have $\dot{x} = -y < 0 \Rightarrow$
$\Rightarrow x$ decreases along the orbit $\Rightarrow$ the arrow points to the left.

b) $\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases}$ $\to$ the flow
$\to$ check that O_2 is a global attractor
$\to$ find a first integral in $\mathbb{R} \times (0,\infty)$.

$\to$ the flow: $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{-t} \end{pmatrix}, \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2, \forall t \in \mathbb{R}$

$\Rightarrow \lim\limits_{t \to \infty} \varphi(t,\eta) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = O_2, \forall \eta \in \mathbb{R}^2 \overset{\text{def}}{\Rightarrow} O_2$ is a global attractor

Take $H: \mathbb{R} \times (0,\infty) \to \mathbb{R}, H(x,y) = \frac{x}{y}$

check that this is a first integral in $\mathbb{R} \times (0,\infty)$

$H \in C^1(\mathbb{R} \times (0,\infty))$, it is not locally constant ✓

$H(\varphi(t,\eta)) = \frac{\eta_1 e^{-t}}{\eta_2 e^{-t}} = \frac{\eta_1}{\eta_2} = H(\eta), \forall \eta \in \mathbb{R} \times (0,\infty), \forall t \in \mathbb{R}$ ✓

So, indeed, H is a first integral in $\mathbb{R} \times (0,\infty)$.

Notice that $\mathbb{R} \times (0,\infty)$ is an invariant set, since whenever $(\eta_1, \eta_2) \in \mathbb{R} \times (0,\infty)$ we have

So, indeed, H is a first integral in $\mathbb{R}\times(0,\infty)$.

Notice that $\mathbb{R}\times(0,\infty)$ is an invariant set, since whenever $(\eta_1,\eta_2)\in\mathbb{R}\times(0,\infty)$ we have $\varphi(t,\eta_1,\eta_2)\in\mathbb{R}\times(0,\infty), \forall t\in\mathbb{R}$

Note that the same expr., $\frac{x}{y}$ defines a f.i in $\mathbb{R}\times(-\infty,0)$.

The shape of the orbits: the shape of the level curves of H $\Leftrightarrow$

$\Leftrightarrow \frac{x}{y}=c \qquad y=\frac{1}{c}x$ these are lines that pass through $(0,0)$ $\Rightarrow$ The phase portrait is 3).

c) $\begin{cases}\dot{x}=-x\\ \dot{y}=y\end{cases}$ → the flow
→ find a global first integral
→ the shape of the orbits

$\varphi(t,\eta_1,\eta_2)=\begin{pmatrix}\eta_1 e^{-t}\\ \eta_2 e^{t}\end{pmatrix}$ $\quad H(x,y)=x\cdot y$, $H:\mathbb{R}^2\to\mathbb{R}$ $\xrightarrow{HW}$ check that this is a f.i

$\hookrightarrow$ global since it is defined on $\mathbb{R}^2$.

The level curves of H: $xy=c$, $c\in\mathbb{R}$

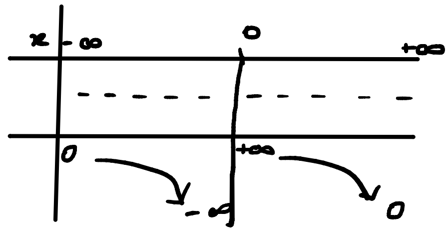
$c=0 \Rightarrow x=0$ and $y=0$ are level curves

$c=1 \Rightarrow y=\frac{1}{x}$ - hyperbola

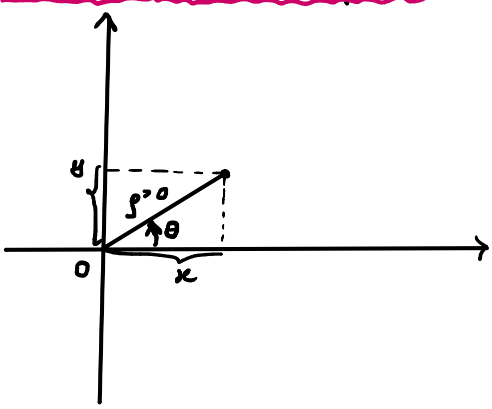
The phase portrait is 1).

$f(x)=\frac{1}{x}$

$f'(x)=-\frac{1}{x^2}$



Polar coordinates in the plane



Property: For any $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$ there exists a unique pair $(\rho,\theta)\in(0,\infty)\times[0,2\pi)$ such that

(1) $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\end{cases} \Leftrightarrow z+iy=\rho\cdot e^{i\theta}$

(ρ,θ) are called the polar coordinates of a point $A\in\mathbb{R}^2\setminus\{(0,0)\}$ of cartesian coord. (x,y).

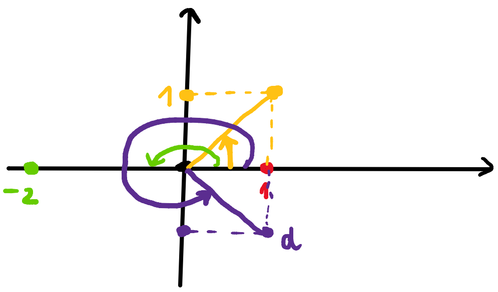
Remark: (1) $\Leftrightarrow \rho=\sqrt{x^2+y^2}$, $\cos\theta=\frac{x}{\rho}$, $\sin\theta=\frac{y}{\rho}$

Exercises: For the following points of given cartesian coordinates, find their polar coordinates.

a) $\begin{cases}x=1\\ y=0\end{cases}$ $\quad \rho=1$, $\theta=0$ $\Rightarrow z=1=1\cdot e^{i\cdot 0}$

b) $\begin{cases}x=-2\\ y=0\end{cases}$ $\quad \rho=2$, $\begin{rcases}\cos\theta=-1\\ \sin\theta=0\end{rcases}\Rightarrow$ $\Rightarrow\theta=\pi$ $\Rightarrow z=-2=2e^{i\pi}$

c) $\begin{cases}x=1\\ y=1\end{cases}$ $\Rightarrow \rho=\sqrt{2}\Rightarrow$ $\theta=\frac{\pi}{4}$ $\Rightarrow z=1+i=\sqrt{2}e^{i\frac{\pi}{4}}$



d) $\begin{cases} x = 1 \\ y = -1 \end{cases}$ $\rho = \sqrt{2}$

$\cos\theta = \frac{\sqrt{2}}{2} \to \theta = \frac{7\pi}{4}$

$\sin\theta = -\frac{\sqrt{2}}{2}$

e) $\begin{cases} x = \eta_1 \cos t - \eta_2 \sin t \\ y = \eta_1 \sin t + \eta_2 \cos t \end{cases}$, $t \in \mathbb{R}$, $\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$, $\eta \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\rho(t) = \sqrt{\eta_1^2 + \eta_2^2} = \rho_0$

denote $\rho_0 = \sqrt{\eta_1^2 + \eta_2^2}$

$\theta_0 \in [0, 2\pi)$, $\eta_1 = \rho_0 \cos\theta_0$

$\eta_2 = \rho_0 \sin\theta_0$

(ρ_0, θ_0) the polar coord of the initial point (η_1, η_2).

Show that $\theta(t) = t - \theta_0$, $\forall t \in \mathbb{R}$. <u>HW</u>