CHAPTER 11

Hyperquadrics

Contents

All equations are given with respect to an orthonormal coordinate system $\mathcal{K} = (O, \mathbf{e}_1, \dots, \mathbf{e}_n)$.

11.1 Hyperquadrics

Definition. A *hyperquadric* $\mathcal{Q}$ in $\mathbb{E}^n$ is a the set of points whose coordinates satisfy a quadratic equation, i.e.

$$\mathcal{Q} : \sum_{i,j=1}^{n} a_{ij} x_i x_j + \sum_{i}^{n} b_i x_i + c = 0, \tag{11.1}$$

with respect to some coordinate system.

- Hyperquadrics in $\mathbb{E}^2$ are called *conic sections* (see previous chapter).
- The following lectures are dedicated to hyperquadrics in $\mathbb{E}^3$. In dimension 3 we generally refer to them as *quadrics*.

- Notice that Equation (11.1) is equivalent to

$$\mathcal{Q}: \sum_{i,j=1}^{n} q_{ij}x_ix_j + \sum_{i}^{n} b_ix_i + c = 0, \tag{11.2}$$

  where $q_{ii} = a_{ii}$ and $q_{ij} = q_{ji} = \frac{a_{ij}+a_{ji}}{2}$. The matrix $Q = (q_{ij})$ is symmetric and we call it the *symmetric matrix associated to Equation* (11.2) *of the quadric* $\mathcal{Q}$.

- The matrix Q defines a homogeneous polynomial of degree 2 in the above equation.

- Notice that Equation (11.2) can be rearranged in matrix form as follows

$$\mathcal{Q}: \mathbf{x}^t \cdot Q \cdot \mathbf{x} + \mathbf{b}^t \cdot \mathbf{x} + c = 0 \tag{11.3}$$

  where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$.

11.2 Reducing to the canonical form

- Let $\mathcal{Q}$ be a hyperquadric described by Equation (11.2) with respect to the coordinate system $\mathcal{K} = (O, \boldsymbol{e})$ where $\boldsymbol{e} = (\mathbf{e}_1, \dots, \mathbf{e}_n)$.

- Let Q be the matrix associated to Equation (11.2) of $\mathcal{Q}$.

- [Step 1 - Rotation] By the Spectral Theorem, there is an orthonormal basis $\boldsymbol{e}'$ of eigenvectors for Q which diagonalizes Q. Changing the coordinate system from $\mathcal{K} = (O, \boldsymbol{e})$ to $\mathcal{K}(O, \boldsymbol{e}')$, the equation of the hyperquadric becomes

$$\mathcal{Q}: \mathbf{y}^t \cdot D \cdot \mathbf{y} + \mathbf{v}^t \cdot \mathbf{y} + c = 0 \quad \text{where} \quad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \tag{11.4}$$

  and where $\mathbf{y} = (y_1, \dots, y_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$. This change of coordinates consists in replacing $\mathbf{x}$ by $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}\mathbf{y}$ since $\mathbf{x} = \mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}\mathbf{y}$.

- Since the two bases $\boldsymbol{e}$ and $\boldsymbol{e}'$ are orthonormal, the matrix $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}$ is an orthogonal matrix, i.e. $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} \in O(n)$.

- Since Q is symmetric, the eigenvalues $\lambda_1, \dots, \lambda_n$ are real. Hence, eventually after permuting the basis vectors in $\boldsymbol{e}'$ we may assume that $\lambda_1, \dots, \lambda_p > 0$, $\lambda_{p+1}, \dots, \lambda_r < 0$ and $\lambda_{r+1}, \dots, \lambda_n = 0$ where r is the rank of Q. This permutation corresponds to changing $\boldsymbol{e}' = (\mathbf{e}'_1, \dots, \mathbf{e}'_n)$ to $\boldsymbol{e}'' = (\mathbf{e}_{\pi(1)}, \dots, \mathbf{e}_{\pi(n)})$ for some permutation π of $\{1, \dots, n\}$. The base change matrix $\mathrm{M}_{\boldsymbol{e}',\boldsymbol{e}''}$ is again orthogonal.

- So far, we have a change of coordinates from $\mathcal{K} = (O, \boldsymbol{e})$ to $\mathcal{K} = (O, \boldsymbol{e}'')$ given by the base change matrix $M_{\boldsymbol{e},\boldsymbol{e}''} = \mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}\mathrm{M}_{\boldsymbol{e}',\boldsymbol{e}''} \in O(n)$.

- Notice that $\det(\mathrm{M}_{e,e''}) = 1$ if $\mathrm{M}_{e,e''} \in SO(n)$ and $\det(\mathrm{M}_{e,e''}) = -1$ if $\mathrm{M}_{e,e''}$ is not special orthogonal. In the latter case, the matrix $\mathrm{M}_{e,e''}$ changes the orientation of the basis e''. So, if we replace one vector in e'' by minus that vector, for example $\mathbf{e}''_1 \leftrightarrow -\mathbf{e}''_1$, then e'' has the same orientation as e and $\mathrm{M}_{e,e'} \in SO(n)$ (See Examples below). We conclude that we may choose e'' such that $\mathrm{M}_{e,e''} \in SO(n)$.

- Writing out the equation of $\mathcal{Q}$ in $\mathcal{K}''$ we have:

$$\mathcal{Q} : \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_r y_r^2 + v_1 y_1 + v_2 y_2 + \cdots + v_n y_n + c = 0. \tag{11.5}$$

- [Step 2 - translation] Notice that $r > 0$, i.e. there is at least one eigenvalue distinct from 0 otherwise $Q = 0_n$ and (11.2) is not a quadratic polynomial.

- If $v_1 \neq 0$ then

$$\lambda_1 y_1^2 + v_1 y_1 = \lambda_1\Big(y_1^2 + 2\frac{v_1}{2\lambda_1} y_1\Big) = \lambda_1\Big(y_1 + \frac{v_1}{2\lambda_1}\Big)^2 - \frac{v_1^2}{4\lambda_1^2}.$$

- Thus, if for $i \in \{1, \ldots, r\}$ we let $z_i = y_i + \frac{v_i}{2\lambda_i}$ and $z_i = y_i$ for $i > r$ then Equation (11.5) becomes

$$\mathcal{Q} : \lambda_1 z_1^2 + \lambda_2 z_2^2 + \cdots + \lambda_r z_r^2 + v_{r+1} z_{r+1} + \cdots + v_n z_n = k \tag{11.6}$$

  for some $k \in \mathbb{R}$. This change of variables corresponds to a change of coordinates from $\mathcal{K}'' = (O, e'')$ to $\mathcal{K}''' = (O', e'')$ given by the translation

$$\begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_{r+1} \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_r \\ y_{r+1} \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} \frac{v_1}{2\lambda_1} \\ \vdots \\ \frac{v_r}{2\lambda_r} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

- Notice that so far we changed the coordinates from $\mathcal{K}$ to $\mathcal{K}''$ with an orthogonal transformation in $SO(n)$ and from $\mathcal{K}''$ to $\mathcal{K}'''$ with a translation. These are isometries, so the composition of these two transformations is an isometry.

- It is possible to customize and simplify the equation (11.6) of $\mathcal{Q}$ further, by making other changes of coordinates. However these will in general not correspond to isometries (See the discussion in Section 11.3.5).

- We *reduced* Equation (11.2) to Equation (11.6). Equation (11.6) is not yet the canonical form but it is an important step towards the canonical form. We may refer to Equation (11.6) as the *intermediate canonical form*.

- In what follows we look at what more can be done if we restrict to conics and quadrics, i.e. if we look at hyperquadrics in $\mathbb{E}^2$ and $\mathbb{E}^3$.

11.3 Classification of conics - dimension 2

11.3.1 Isometric classification

- A hyperquadric in $\mathbb{E}^2$ is a curve given by an equation of the form

$$\mathcal{C}: q_{11}x^2 + 2q_{12}xy + q_{22}y^2 + b_1x + b_2y + c = 0. \tag{11.7}$$

- From the above discussion, we may apply a rotation and a translation to change the coordinate system such that Equation (11.4) becomes

$$\mathcal{C}: \lambda_1 x^2 + \lambda_2 y^2 = k \quad \text{or} \quad \mathcal{C}: \lambda_1 x^2 + v_2 y = k. \tag{11.8}$$

  where $\lambda_1 > 0$ (if $\lambda_1, \lambda_2 < 0$, multiply the whole equation by -1).

- If $\lambda_2 > 0$ and $k = 0$, then the equation has only $(0,0)$ as solution, in this case the curve is degenerate to a point, the origin.

- If $\lambda_2 > 0$ and $k < 0$, then there are no real solutions to the equation.

- If $\lambda_2 > 0$ and $k > 0$, after dividing by k the equation becomes

$$\frac{x^2}{\frac{k}{\lambda_1}} + \frac{y^2}{\frac{k}{\lambda_2}} = 1$$

  which is the equation of an ellipse. The ellipse is in canonical form if $\frac{k}{\lambda_1} > \frac{k}{\lambda_2}$. If this is not the case, we need to do one additional change of coordinates by interchanging x with y. This is again on orthogonal transformation (see Examples below).

- If $\lambda_2 < 0$ and $k = 0$, then the equation becomes

$$(\sqrt{\lambda_1}x - \sqrt{-\lambda_2}y)(\sqrt{\lambda_1}x + \sqrt{-\lambda_2}y) = 0.$$

  This is the union of two lines.

- If $\lambda_2 < 0$ and $k < 0$, we may multiply the whole equation by -1 and interchange the role of λ_1 and λ_2. Doing so, we need to do one additional change of coordinates by interchanging x with y. This is again on orthogonal transformation (see Examples below).

- If $\lambda_2 < 0$ and $k > 0$, after dividing by k the equation becomes

$$\frac{x^2}{\frac{k}{\lambda_1}} - \frac{y^2}{\frac{k}{-\lambda_2}} = 1$$

  which is a the equation of a hyperbola.

- If $\lambda_2 = 0$ we have the equation

$$\mathcal{C}: \lambda_1 x^2 + v_2 y = k.$$

  This is the case if and only if $\operatorname{rank}(Q) = 1$.

- If $v_2 = 0$ and $k \geq 0$ then we have two lines described by the equation $(\sqrt{\lambda_1}x - \sqrt{k})(\sqrt{\lambda_1}x + \sqrt{k}) = 0$. If $k = 0$ this is a double-line, $x^2 = 0$.

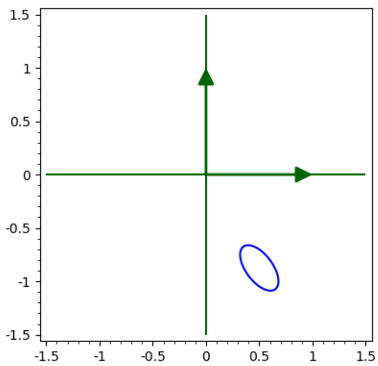
- If $v_2 = 0$ and $k < 0$ then we have no solutions.

- If $v_2 \neq 0$ then, dividing by v_2, the equation becomes
$$\frac{\lambda_1}{|v_1|}x^2 = \frac{k}{|v_1|} - y$$
and we may change the coordinates $(x, \frac{k}{|v_1|} - y) \to (x, y)$ such that the equation becomes
$$x^2 = \frac{|v_1|}{\lambda_1}y.$$
This change of coordinates corresponds to a reflection and a translation (again an isometric change of coordinates) and we recognize the equation of a parabola with parameter $p = \frac{|v_1|}{2\lambda_1}$. However, here again, we don't yet have the canonical form of a parabola. In order to obtain $\mathcal{P}_p$ we need to interchange the x-axis with the y-axis.

- [Conclusion] Starting with an equation of the form (11.7), we may use rotations, translations and reflections to recognize that we obtain either
  - degenerate cases: two lines, double lines, points, or
  - non-degenerate cases:
  $$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{H}_{a,b} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{P}_p : y^2 = 2px.$$

  So, the curves described by an equation of the form (11.7) are conic sections.

11.3.2 Example - ellipse



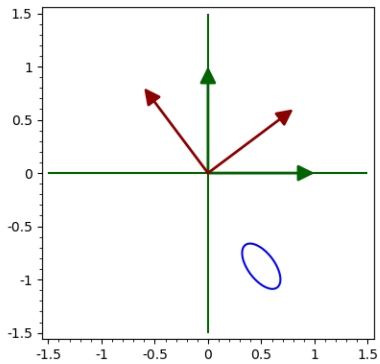
Consider the curve with equation
$$\mathcal{C} : 73x^2 + 72xy + 52y^2 - 10x + 55y + 25 = 0. \tag{11.9}$$

It is the ellipse in the above image, however, a priori it is not at all clear that $\mathcal{C}$ is an ellipse. The symmetric matrix associated to this equation is

$$Q = \begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}$$

and in matrix form Equation (11.9) becomes

$$\mathcal{C} : \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 73 & 36 \\ 36 & 52 \end{bmatrix}}_{Q} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} -10 & 55 \end{bmatrix}}_{b} \begin{bmatrix} x \\ y \end{bmatrix} + 25 = 0. \tag{11.10}$$



The eigenvalues of Q are $\lambda_1 = 100$ and $\lambda_2 = 25$. An eigenvector for the eigenvalue λ_1 is $(4,3)$ and an eigenvector for the eigenvalue λ_2 is $(3,-4)$. These two vectors form an orthogonal basis, thus, an orthonormal basis is $\boldsymbol{e}' = (\mathbf{e}'_1(4/5,3/5), \mathbf{e}'_2(3/5,-4/5))$. The base change matrix from $\boldsymbol{e}'$ to $\boldsymbol{e}$ is

$$\mathrm{M}_{e,e'} = \frac{1}{5}\begin{bmatrix} 4 & 3 \\ 3 & -4 \end{bmatrix}.$$

We know that this matrix is orthogonal, which in this example is easy to check directly. In particular $\mathrm{M}_{e,e'}^{-1} = \mathrm{M}_{e,e'}^{t}$. Moreover, the determinant is -1 which means that $\mathrm{M}_{e,e'}$ is not a direct isometrie, i.e. $\mathrm{M}_{e,e'} \in O(2) \setminus SO(2)$. If we change the direction of the second eigenvector, we still have an eigenvector for the eigenvalue λ_2 but with respect to the basis $\boldsymbol{e}' = (\mathbf{e}'_1(4/5,3/5), \mathbf{e}'_2(-3/5,4/5))$ we have

$$\mathrm{M}_{e,e'} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \in SO(2).$$

Changing the coordinate system from $\mathcal{K}$ to $\mathcal{K}' = (O, \boldsymbol{e}')$, Equation (11.9) becomes

$$\mathcal{C} : \begin{bmatrix} x' & y' \end{bmatrix} M_{e,e'}^{t} Q M_{e,e'} \begin{bmatrix} x' \\ y' \end{bmatrix} + b M_{e,e'} \begin{bmatrix} x' \\ y' \end{bmatrix} + 25 = 0$$

which one calculates to be

$$\mathcal{C}: \begin{bmatrix} x' & y' \end{bmatrix} \underbrace{\begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix}}_{Q'} \begin{bmatrix} x' \\ y' \end{bmatrix} + \underbrace{\begin{bmatrix} -25 & 50 \end{bmatrix}}_{b'} \begin{bmatrix} x' \\ y' \end{bmatrix} + 25 = 0$$

and we have

$$\mathcal{C}: 100x'^2 + 25y'^2 + 25x' + 50y' + 25 = 0$$

equivalently

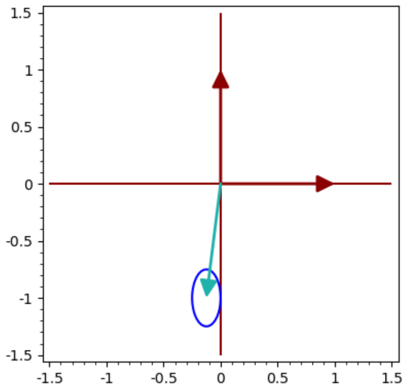
$$\mathcal{C}: 4x'^2 + y'^2 + x' + 2y' + 1 = 0$$

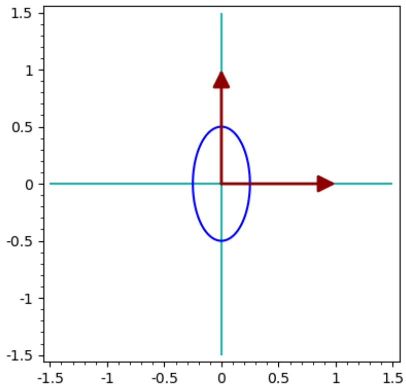
equivalently

$$\mathcal{C}: 4\Big(x'^2 + \frac{x'}{4} + \frac{1}{64}\Big) - \frac{1}{16} + \big(y'^2 + 2y' + 1\big) - 1 + 1 = 0$$

equivalently

$$\mathcal{C}: 4\Big(x' + \frac{1}{4}\Big)^2 + \big(y' + 1\big)^2 - \frac{1}{16} = 0.$$





Now, let us change the coordinate system again, using a translation of vector $(-\frac{1}{4}, -1)$. The new coordinate system is $\mathcal{K}'' = (O'', \boldsymbol{e}'')$ where $O'' = (-\frac{1}{4}, -1)$ and the basis $\boldsymbol{e}'' = \boldsymbol{e}'$ doesn't change. In $\mathcal{K}''$ the equation of $\mathcal{C}$ becomes

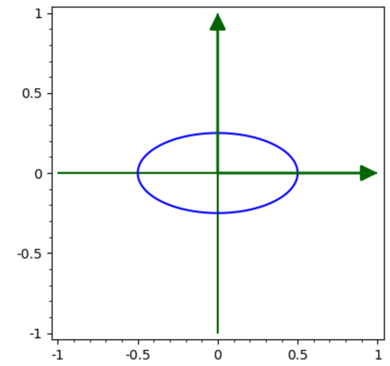
$$\mathcal{C}: 4x''^2 + y''^2 - \frac{1}{16} = 0 \quad \Leftrightarrow \quad \frac{x''^2}{\frac{1}{64}} + \frac{y''^2}{1} - 1 = 0.$$

Clearly, this is the equation of an ellipse. But it is not yet in canonical form because the focal points are on the y-axes. So, in order to obtain the canonical form we need to do a final coordinate change and permute the coordinate axes, for instance with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SO(2) \quad \text{or with} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in O(2).$$

With any of the two transformations we obtain

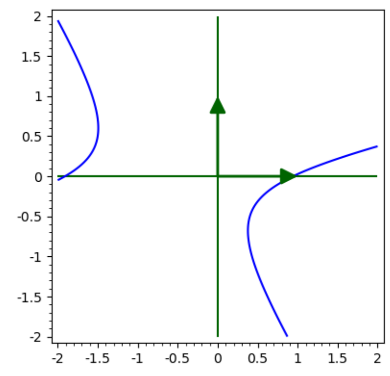
$$\mathcal{C} = \mathcal{E}_{1,\frac{1}{8}} : \frac{y'''^2}{1} + \frac{x'''^2}{\frac{1}{64}} - 1 = 0.$$



To recap:

- We changed the coordinates from $\mathcal{K}$ to $\mathcal{K}'$ with the rotation $M_{e,e'}$ of angle θ where $\cos(\theta) = \frac{4}{5}$.
- We changed the coordinates from $\mathcal{K}'$ to $\mathcal{K}''$ with a translation of vector $(-\frac{1}{4}, -1)$.
- We changed the coordinates from $\mathcal{K}''$ to $\mathcal{K}'''$ in order to interchange the variables. And we obtained $\mathcal{E}_{1,\frac{1}{8}}$.

11.3.3 Example - hyperbola



Consider the curve with equation

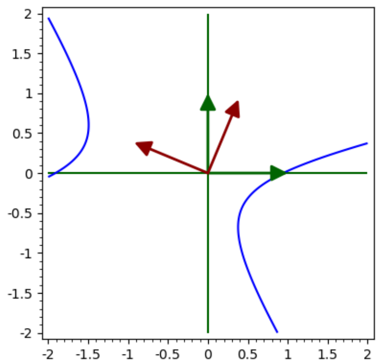
$$\mathcal{C} : -94x^2 + 360xy + 263y^2 - 91x + 221y + 169 = 0. \tag{11.11}$$

It is the hyperbola in the above image, however, a priori it is not at all clear that $\mathcal{C}$ is a hyperbola. The symmetric matrix associated to this equation is

$$Q = \begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix}$$

and in matrix form Equation (11.11) becomes

$$\mathcal{C} : \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} -94 & 180 \\ 180 & 263 \end{bmatrix}}_{Q} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} -91 & 221 \end{bmatrix}}_{b} \begin{bmatrix} x \\ y \end{bmatrix} + 25 = 0. \tag{11.12}$$



The eigenvalues of Q are $\lambda_1 = 338$ and $\lambda_2 = -169$. An eigenvector for the eigenvalue λ_1 is $(5, 12)$ and an eigenvector for the eigenvalue λ_2 is $(12, -5)$. These two vectors form an orthogonal basis, thus, an orthonormal basis is $\boldsymbol{e}' = (\mathbf{e}'_1(5/13, 12/13), \mathbf{e}'_2(12/13, -5/13))$. The base change matrix from $\boldsymbol{e}'$ to $\boldsymbol{e}$ is

$$\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} = \frac{1}{13} \begin{bmatrix} 5 & 12 \\ 12 & -5 \end{bmatrix}.$$

We know that this matrix is orthogonal, which in this example is easy to check directly. In particular $\mathrm{M}^{-1}_{\boldsymbol{e},\boldsymbol{e}'} = \mathrm{M}^{t}_{\boldsymbol{e},\boldsymbol{e}'}$. Moreover, the determinant is -1 which means that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}$ is not a direct isometrie, i.e. $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} \in O(2) \setminus SO(2)$. If we change the direction of the second eigenvector, we still have an eigenvector for the eigenvalue λ_2 but with respect to the basis $\boldsymbol{e}' = (\mathbf{e}'_1(5/13, 12/13), \mathbf{e}'_2(-12/13, 5/13))$ we have

$$\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} = \frac{1}{13} \begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix} \in SO(2).$$

Changing the coordinate system from $\mathcal{K}$ to $\mathcal{K}' = (O, \boldsymbol{e}')$, Equation (11.11) becomes

$$\mathcal{C} : \begin{bmatrix} x' & y' \end{bmatrix} M^{t}_{\boldsymbol{e},\boldsymbol{e}'} Q M_{\boldsymbol{e},\boldsymbol{e}'} \begin{bmatrix} x' \\ y' \end{bmatrix} + b M_{\boldsymbol{e},\boldsymbol{e}'} \begin{bmatrix} x' \\ y' \end{bmatrix} + 169 = 0$$

which one calculates to be

$$\mathcal{C}:\begin{bmatrix}x' & y'\end{bmatrix}\underbrace{\begin{bmatrix}338 & 0\\ 0 & -169\end{bmatrix}}_{Q'}\begin{bmatrix}x'\\ y'\end{bmatrix}+\underbrace{\begin{bmatrix}169 & 169\end{bmatrix}}_{b'}\begin{bmatrix}x'\\ y'\end{bmatrix}+169=0$$

and we have

$$\mathcal{C}: 338x'^2-169y'^2+169x'+169y'+169=0$$

equivalently

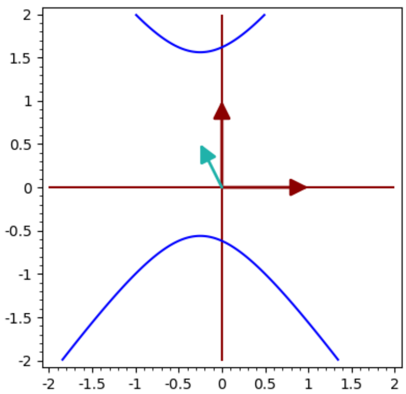
$$\mathcal{C}: 2x'^2-y'^2+x'+y'+1=0$$

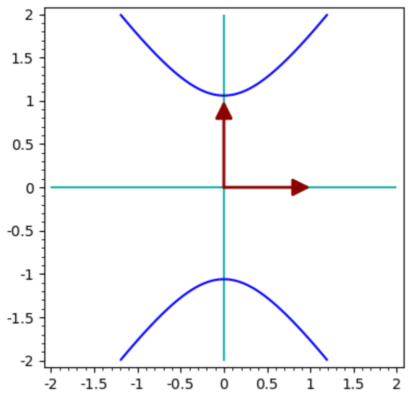
equivalently

$$\mathcal{C}: 2\Big(x'^2+\frac{x'}{2}+\frac{1}{16}\Big)-\frac{1}{8}-\Big(y'^2-y'+\frac{1}{4}\Big)+\frac{1}{4}+1=0$$

equivalently

$$\mathcal{C}: 2\Big(x'+\frac{1}{4}\Big)^2+\Big(y'-\frac{1}{2}\Big)^2+\frac{9}{8}=0.$$





Now, let us change the coordinate system again, using a translation of vector $(-\frac{1}{4},\frac{1}{2})$. The new coordinate system is $\mathcal{K}''=(O'',\boldsymbol{e}'')$ where $O''=(-\frac{1}{4},\frac{1}{2})$ and the basis $\boldsymbol{e}''=\boldsymbol{e}'$ doesn't change. In $\mathcal{K}''$ the equation of $\mathcal{C}$ becomes

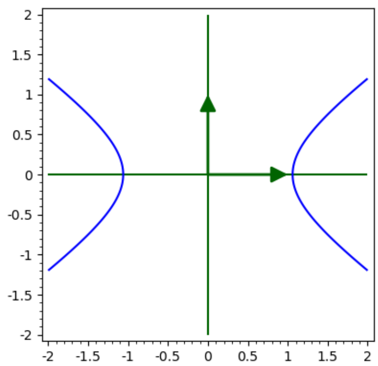
$$\mathcal{C}: 2x''^2-y''^2+\frac{9}{8}=0 \quad\Leftrightarrow\quad -\frac{x''^2}{\frac{9}{16}}+\frac{y''^2}{\frac{9}{8}}-1=0.$$

Clearly, this is the equation of a hyperbola. But it is not yet in canonical form because the focal points are on the y-axes. So, in order to obtain the canonical form we need to do a final coordinate change and permute the coordinate axes, for instance with

$$\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\in SO(2) \quad\text{or with}\quad \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\in O(2).$$

With any of the two transformations we obtain

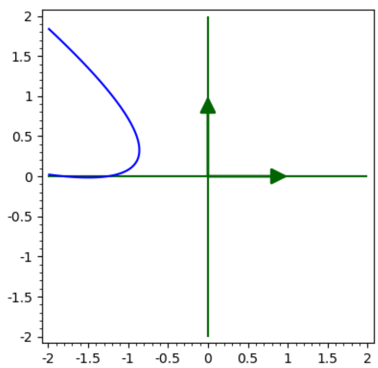
$$\mathcal{C} = \mathcal{H}_{\frac{3}{4},\frac{3}{\sqrt{2}}} : \frac{x''^2}{\frac{9}{16}} - \frac{y''^2}{\frac{9}{8}} - 1 = 0.$$



To recap:

- We changed the coordinates from $\mathcal{K}$ to $\mathcal{K}'$ with the rotation $\mathrm{M}_{e,e'}$ of angle θ where $\cos(\theta) = \frac{5}{13}$.
- We changed the coordinates from $\mathcal{K}'$ to $\mathcal{K}''$ with a translation of vector $(-\frac{1}{4}, \frac{1}{2})$.
- We changed the coordinates from $\mathcal{K}''$ to $\mathcal{K}'''$ in order to interchange the variables. And we obtained $\mathcal{H}_{\frac{3}{4},\frac{3}{\sqrt{2}}}$.

11.3.4 Example - parabola



Consider the curve with equation

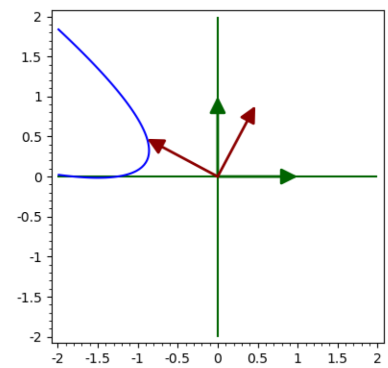
$$\mathcal{C} : 128x^2 + 480xy + 450y^2 + 391x + 119y + 289 = 0. \tag{11.13}$$

It is the parabola in the above image, however, a priori it is not at all clear that $\mathcal{C}$ is a parabola. The symmetric matrix associated to this equation is

$$Q = \begin{bmatrix} 128 & 240 \\ 240 & 225 \end{bmatrix}$$

and in matrix form Equation (11.13) becomes

$$\mathcal{C} : \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 128 & 240 \\ 240 & 225 \end{bmatrix}}_{Q} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 391 & 119 \end{bmatrix}}_{b} \begin{bmatrix} x \\ y \end{bmatrix} + 25 = 0. \tag{11.14}$$



The eigenvalues of Q are $\lambda_1 = 578$ and $\lambda_2 = 0$. An eigenvector for the eigenvalue λ_1 is $(8,15)$ and an eigenvector for the eigenvalue λ_2 is $(15,-8)$. These two vectors form an orthogonal basis, thus, an orthonormal basis is $\boldsymbol{e}' = (\mathbf{e}'_1(8/17, 15/17), \mathbf{e}'_2(15/17, -8/17))$. The base change matrix from $\boldsymbol{e}'$ to $\boldsymbol{e}$ is

$$\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} = \frac{1}{17} \begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix}.$$

We know that this matrix is orthogonal, which in this example is easy to check directly. In particular $\mathrm{M}^{-1}_{\boldsymbol{e},\boldsymbol{e}'} = \mathrm{M}^{t}_{\boldsymbol{e},\boldsymbol{e}'}$. Moreover, the determinant is -1 which means that $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'}$ is not a direct isometrie, i.e. $\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} \in O(2) \setminus SO(2)$. If we change the direction of the second eigenvector, we still have an eigenvector for the eigenvalue λ_2 but with respect to the basis $\boldsymbol{e}' = (\mathbf{e}'_1(8/17, 15/17), \mathbf{e}'_2(-15/17, 8/17))$ we have

$$\mathrm{M}_{\boldsymbol{e},\boldsymbol{e}'} = \frac{1}{17} \begin{bmatrix} 8 & -15 \\ 15 & 8 \end{bmatrix} \in SO(2).$$

Changing the coordinate system from $\mathcal{K}$ to $\mathcal{K}' = (O, \boldsymbol{e}')$, Equation (11.13) becomes

$$\mathcal{C} : \begin{bmatrix} x' & y' \end{bmatrix} M^t_{e,e'} Q M_{e,e'} \begin{bmatrix} x' \\ y' \end{bmatrix} + b M_{e,e'} \begin{bmatrix} x' \\ y' \end{bmatrix} + 289 = 0$$

which one calculates to be

$$\mathcal{C} : \begin{bmatrix} x' & y' \end{bmatrix} \underbrace{\begin{bmatrix} 578 & 0 \\ 0 & 0 \end{bmatrix}}_{Q'} \begin{bmatrix} x' \\ y' \end{bmatrix} + \underbrace{\begin{bmatrix} 289 & -289 \end{bmatrix}}_{b'} \begin{bmatrix} x' \\ y' \end{bmatrix} + 289 = 0$$

and we have

$$\mathcal{C} : 578x'^2 + 289x' - 289y' + 289 = 0$$

equivalently

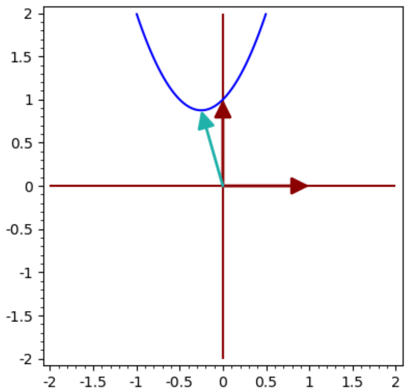
$$\mathcal{C} : 2x'^2 + x' - y' + 1 = 0$$

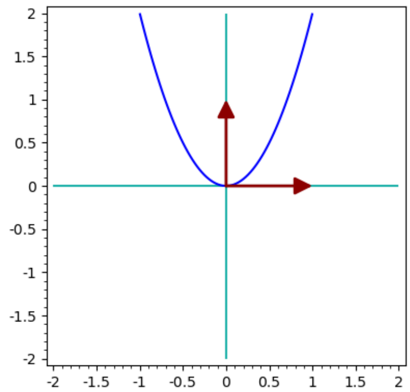
equivalently

$$\mathcal{C} : 2\Big(x'^2 + \frac{x'}{2} + \frac{1}{16}\Big) - \frac{1}{8} - y' + 1 = 0$$

equivalently

$$\mathcal{C} : 2\Big(x' + \frac{1}{4}\Big)^2 - \Big(y' - \frac{7}{8}\Big) = 0.$$





Now, let us change the coordinate system again, using a translation of vector $(-\frac{1}{4}, \frac{7}{8})$. The new coordinate system is $\mathcal{K}'' = (O'', \boldsymbol{e}'')$ where $O'' = (-\frac{1}{4}, \frac{7}{8})$ and the basis $\boldsymbol{e}'' = \boldsymbol{e}'$ doesn't change. In $\mathcal{K}''$ the equation of $\mathcal{C}$ becomes

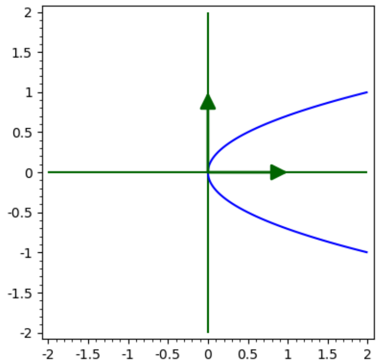
$$\mathcal{C} : 2x''^2 - y'' = 0 \quad \Leftrightarrow \quad x''^2 = \frac{1}{2}y''.$$

Clearly, this is the equation of a parabola. But it is not yet in canonical form because the focal point is on the y-axes. So, in order to obtain the canonical form we need to do a final coordinate change and permute the coordinate axes, for instance with

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SO(2) \quad \text{or with} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in O(2).$$

With any of the two transformations we obtain

$$\mathcal{C} = \mathcal{P}_{\frac{1}{4}} : y''^2 = 2\frac{1}{4}x.$$



To recap:

- We changed the coordinates from $\mathcal{K}$ to $\mathcal{K}'$ with the rotation $\mathrm{M}_{e,e'}$ of angle θ where $\cos(\theta) = \frac{8}{17}$.
- We changed the coordinates from $\mathcal{K}'$ to $\mathcal{K}''$ with a translation of vector $(-\frac{1}{4}, \frac{7}{8})$.
- We changed the coordinates from $\mathcal{K}''$ to $\mathcal{K}'''$ in order to interchange the variables. And we obtained $\mathcal{P}_{\frac{1}{4}}$.

11.3.5 Affine classification

- In the isometric classification, we allowed only changes of coordinates which are isometries. The non-degenerate curves that we obtained are:

$$\mathcal{E}_{a,b} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{H}_{a,b} : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad \mathcal{P}_p : y^2 = 2px.$$

- It is possible to change the coordinates further and rescale the basis vectors of the coordinate axes. Rescalings are clearly not isometries.
- As an example let us consider the case of $\mathcal{E}_{a,b}$, if we replace x with ax and y with by we are doing the following coordinate change

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and then

$$\mathcal{E}_{a,b} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 1 \quad \Leftrightarrow \quad \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 1.$$

So, after changing coordinates the equation of $\mathcal{E}_{a,b}$ becomes

$$x'^2 + y'^2 = 1$$

- Such affine transformations can be applied to Equation (11.6) in general for hypersurfaces.
- If we restrict attention to $\mathbb{E}^2$, after inspecting the possible cases, one shows that the solutions to a quadratic equation is one of the possibilities indicated in the following table.

Case	$r = \operatorname{rank} Q$	$(p, r-p)$	k	equation	name
(a)	2	$(0,2)$	-1	$-x^2-y^2-1=0$	imaginary ellipse (IE)
(a)	2	$(1,1)$	-1	$x^2-y^2-1=0$	**hyperbola** (H)
(a)	2	$(2,0)$	-1	$x^2+y^2-1=0$	**ellipse** (E)
(a)	2	$(0,2)$ or $(2,0)$	0	$-x^2-y^2=0$	two complex lines
(a)	2	$(1,1)$	0	$x^2-y^2=0$	**two real lines**
(a)	1	$(0,1)$ or $(1,0)$	1	$x^2+1=0$	two complex lines
(a)	1	$(1,1)$	-1	$x^2-1=0$	**two real lines**
(a)	1	$(1,1)$	0	$x^2=0$	**a real double-line**
Case	$r = \operatorname{rank} Q$	$(p, r-p)$	k'	equation	name
(b)	1	$(0,1)$ or $(1,0)$	1	$x^2-y=0$	**parabola** (P)

11.4 Classification of quadrics - dimension 3

- A hyperquadric in $\mathbb{E}^3$ is a curve given by an equation of the form

$$\mathcal{C} : q_{11}x^2 + q_{22}y^2 + q_{33}z^2 + 2q_{12}xy + 2q_{12}yz + 2q_{12}xz + b_1x + b_2y + b_3z + c = 0. \tag{11.15}$$

- From the above discussion, we may apply an orthogonal change of coordinates and a translation to change the coordinate system such that Equation (11.4) becomes

$$\mathcal{Q} : \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_r x_r^2 + v_{r+1}x_{r+1} + \cdots + v_n x_n = k. \tag{11.16}$$

- The classification in this case is similar: we work out all possible cases to see what we obtain.
- One important remark is that the base change matrix in Step 1, the matrix $\mathrm{M}_{e,e''}$ used to obtain Equation (11.16), is an element of the group $SO(3)$. So, by Euler's theorem, this is indeed a rotation around an axis.
- Furthermore, one can 'stretch' the coordinate axes with affine transformations which are not isometries in order to show that one may change the coordinate system such that Equation is one of the possibilities listed in the following table.

Case	$r = \operatorname{rank} Q$	$(p, r-p)$	k	equation	name
(a)	3	$(3,0)$	-1	$x^2+y^2+z^2-1=0$	**ellipsoid** (E)
(a)	3	$(2,1)$	-1	$x^2+y^2-z^2-1=0$	**hyperboloid of one sheet** (H1)
(a)	3	$(1,2)$	-1	$x^2-y^2-z^2-1=0$	**hyperboloid of two sheets** (H2)
(a)	3	$(0,3)$	-1	$-x^2-y^2-z^2-1=0$	imaginary ellipsoid (IE)
(a)	3	$(3,0)$	0	$x^2+y^2+z^2=0$	imaginary cone
(a)	3	$(2,1)$	0	$x^2+y^2-z^2=0$	(real, elliptic) **cone**
(a)	2	$(0,2)$	-1	$-x^2-y^2-1=0$	cylinder on imaginary ellipse
(a)	2	$(1,1)$	-1	$x^2-y^2-1=0$	**cylinder** on hyperbola
(a)	2	$(2,0)$	-1	$x^2+y^2-1=0$	**cylinder** on ellipse
(a)	2	$(0,2)$ or $(2,0)$	0	$-x^2-y^2=0$	cylinder on two complex lines
(a)	2	$(1,1)$	0	$x^2-y^2=0$	**cylinder** on two real lines
(a)	1	$(1,0)$	1	$x^2+1=0$	two complex planes
(a)	1	$(1,0)$	-1	$x^2-1=0$	**two real planes**
(a)	1	$(1,0)$	0	$x^2=0$	**a double plane**
(a)	1	$(0,1)$	-1	$x^2+1=0$	two complex planes
(a)	1	$(0,1)$	1	$x^2-1=0$	**two real planes**
(a)	1	$(0,1)$	0	$x^2=0$	**a double plane**

Case	$r = \operatorname{rank} Q$	$(p, r-p)$	k'	equation	name
(b)	2	$(2,0)$ or $(0,2)$	-1	$x^2+y^2-z=0$	**elliptic paraboloid** (EP)
(b)	2	$(1,1)$	-1	$x^2-y^2-z=0$	**hyperbolic paraboloid** (HP)
(b)	1	$(1,0)$	1	$x^2+y=0$	**cylinder** on parabola