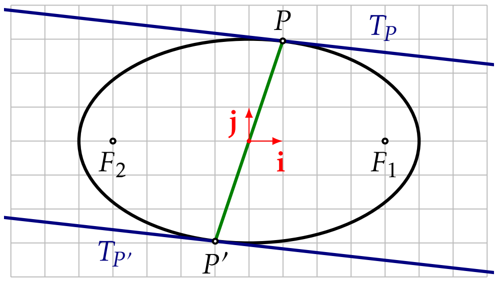
1. Determine the foci (focal points) of the Ellipse $9x^2 + 25y^2 - 225 = 0$

2. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

3. Determine the position of the line $\ell : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

4. Determine an equation of a line which is orthogonal to $\ell : 2x - 2y - 13 = 0$ and tangent to the ellipse $\mathcal{E} : x^2 + 4y^2 - 20 = 0$.

5. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.



6. Using the gradient, prove the reflective properties of an ellipse.

7. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

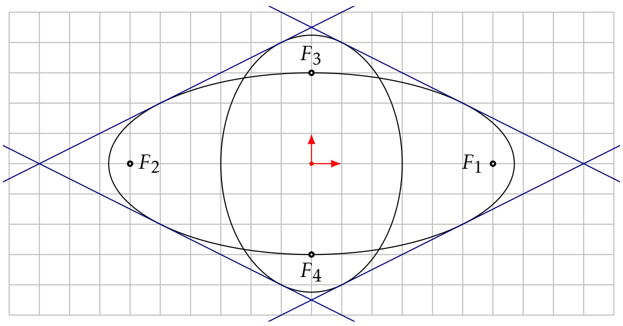


Figure 0.1: Gemeinsame Tangenten.

8. Consider the ellipse $\mathcal{E} : \frac{x^2}{4} + y^2 - 1$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse for which

1. the angle $\measuredangle F_1MF_2$ is right;
2. the angle $\measuredangle F_1MF_2$ is θ;
3. the angle $\measuredangle F_1MF_2$ is maximal.

9. Consider the ellipse $\mathcal{E} : x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as midpoint.

10. Consider the ellipse $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $\ell : x + 2y = 1$.

11. Find the equation of the circle:

1. passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$;
2. passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$;
3. tangent to both $\ell_1 : 2x+y-5 = 0$ and $\ell_2 : 2x+y+15 = 0$ if the tangency point with ℓ_1 is $M(3,-1)$.

1. Determine the foci (focal points) of the Ellipse $9x^2 + 25y^2 - 225 = 0$ $(*)$

$(*) \Leftrightarrow \dfrac{x^2}{\frac{15^2}{3^2}} + \dfrac{y^2}{\frac{15^2}{5^2}} = 1 \Leftrightarrow \dfrac{x^2}{5^2} + \dfrac{y^2}{3^2} = 1$ so $a = 5$ and $b = 3$

$b^2 = a^2 - c^2$ or $c^2 = a^2 - b^2$ so $c^2 = 16$

$\Rightarrow$ the focal points are $F_1(4,0)$ and $F_2(-4,0)$

2. Determine the intersection of the line $\ell : x + 2y - 7 = 0$ and the ellipse $\mathcal{E} : x^2 + 3y^2 - 25 = 0$.

$$\ell \cap \mathcal{E} : \left\{ \begin{array}{l} x^2 + 3y^2 - 25 = 0 \\ x + 2y - 7 = 0 \Leftrightarrow x = 7 - 2y \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} (7-2y)^2 + 3y^2 - 25 = 0 \quad (*) \\ x = 7 - 2y \end{array} \right.$$

$(*) \Leftrightarrow 49 - 28y + 4y^2 + 3y^2 - 25 = 0$

$7y^2 - 28y + 24 = 0$

$\Delta = 4^2 \cdot 7^2 - 4 \cdot 7 \cdot 24 = 4^2 \cdot 7(7-6) \Rightarrow y_{1,2} = \dfrac{4 \cdot 7 \pm 4 \cdot \sqrt{7}}{2 \cdot 7} = 2 \pm \dfrac{2}{\sqrt{7}}$

$\Rightarrow$ the two intersection points are $A(7-2y_1, y_1)$ and $B(7-2y_2, y_2)$

So $A\left(3 - \dfrac{4}{\sqrt{7}}, 2 + \dfrac{2}{\sqrt{7}}\right)$ and $B\left(3 + \dfrac{4}{\sqrt{7}}, 2 - \dfrac{2}{\sqrt{7}}\right)$

3. Determine the position of the line $\ell : 2x + y - 10 = 0$ relative to the ellipse $\mathcal{E} : \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0$.

$$\ell \cap \mathcal{E} : \left\{ \begin{array}{l} 2x + y - 10 = 0 \Rightarrow y = 10 - 2x \\ \frac{x^2}{9} + \frac{y^2}{4} - 1 = 0 \Rightarrow 4x^2 + 9y^2 - 36 = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} y = 10 - 2x \\ 4x^2 + 9(10-2x)^2 - 36 = 0 \quad (*) \end{array} \right.$$

$(*) \Leftrightarrow 4x^2 + 9(100 - 40x + 4x^2) - 36 = 0$

$4x^2 + 900 - 360x + 36x^2 - 36 = 0$

$40x^2 - 360x + 864 = 0 \Leftrightarrow 5x^2 - 45x + 2^2 \cdot 3^3 = 0$

$$\begin{array}{r|l} 360 & 4 \cdot 9 \\ 10 & 2 \cdot 5 \end{array}$$

$$\Delta = 3^4 5^2 - 2^2 \cdot 5 \cdot 2^2 \cdot 3^3 = 3^3 \cdot 5\left(3 \cdot 5 - 2^4\right) < 0$$

$\Rightarrow$ l does not intersect $\mathcal{E}$

5. A *diameter* of an ellipse is the line segment determined by the intersection points of the ellipse with a line passing through the center of the ellipse. Show that the tangent lines to an ellipse at the endpoints of a diameter are parallel.

The endpoints of a diameter are symmetric relative to the origin

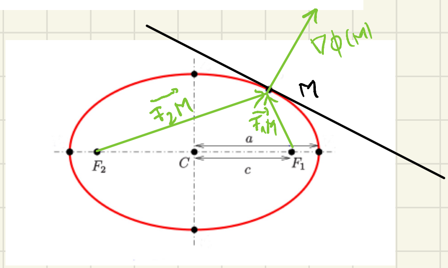
If $M(x_0, y_0)$ is one endpoint then $M(-x_0, -y_0)$ is the other endpoint.

now $T_M \mathcal{E}_{a,b} : \dfrac{x x_0}{a^2} + \dfrac{y y_0}{b^2} = 1$

and $T_M \mathcal{E}_{a,b} : \dfrac{x(-x_0)}{a^2} + \dfrac{y(-y_0)}{b^2} = 1 \iff \dfrac{x x_0}{a^2} + \dfrac{y y_0}{b^2} = -1$

both lines admit $\left(\dfrac{x_0}{a^2}, \dfrac{y_0}{b^2}\right)$ as normal vectors so they are parallel.

6. Using the gradient, prove the reflective properties of an ellipse.



$$\mathcal{E}_{a,b} : d(M, F_1) + d(M, F_2) - 2a = 0$$

$$\iff \underbrace{\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} - 2a}_{\phi(x,y)} = 0$$

$$\nabla\phi(x,y) = \left(\frac{\partial\phi}{\partial x}(x,y), \frac{\partial\phi}{\partial y}(x,y)\right)$$

$$\frac{\partial\phi}{\partial x}(x,y) = \frac{\partial}{\partial x}\left(\sqrt{(x-c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} - 2a\right) = \frac{x-c}{\sqrt{(x-c)^2 + y^2}} + \frac{x+c}{\sqrt{(x+c)^2 + y^2}}$$

$$\frac{\partial\phi}{\partial x}(x,y) = \frac{y}{\sqrt{(x-c)^2 + y^2}} + \frac{y}{\sqrt{(x+c)^2 + y^2}} =$$

$$\Rightarrow \nabla\phi(M) = \left(\frac{x_M - c}{\|\vec{F_1M}\|} + \frac{x+c}{\|\vec{F_2M}\|}, \frac{y}{\|\vec{F_1M}\|} + \frac{y}{\|\vec{F_2M}\|}\right)$$

$$= \frac{1}{\|\overrightarrow{F_1M}\|}(x_M - c, y) + \frac{1}{\|\overrightarrow{F_2M}\|}(x+c, y)$$

$$= \frac{\overrightarrow{F_1M}}{\|\overrightarrow{F_1M}\|} + \frac{\overrightarrow{F_2M}}{\|\overrightarrow{F_2M}\|}$$

$\Rightarrow$ $\nabla\phi(M)$ is a direction vector for the angle $\widehat{F_1MF_2}$ $(*)$

Reflecting in $\mathcal{E}$ at point M $\Leftrightarrow$ reflecting in $T_M\mathcal{E}$

$\Leftrightarrow$ incoming angle $\equiv$ outgoing angle $\Leftrightarrow$ $(*)$

7. Determine the common tangents to the ellipses

$$\frac{x^2}{45} + \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{9} + \frac{y^2}{18} = 1.$$

Consider the equation of tangent lines if the slope is given

$$y = kx \pm \sqrt{a^2k^2 + b^2}$$

For the first ellipse we have

$$y = kx \pm \sqrt{45k^2 + 9}$$

Such a line is tangent to the second ellipse if the intersection point with the second ellipse is a double intersection point

$\Leftrightarrow$ discriminant is zero for the equation

$$\frac{x^2}{9} + \frac{\left(kx \pm \sqrt{45k^2+9}\right)^2}{18} = 1$$

$$(2 + k^2)\, x^2 \pm 2\sqrt{45k^2+9}\, kx + 45k^2 - 9 = 0$$

$$\ldots \quad \Delta = -72(2k-1)(2k+1)$$

so the two slopes are $k = \pm\frac{1}{2}$ and the four tangent lines are

$$\pm x + 2y \pm 9 = 0$$

Method II the tangent lines for the two ellipses are

$$y = kx \pm \sqrt{45k^2+9} \quad \text{and} \quad y = kx \pm \sqrt{9k^2+18}$$

they are equal if $\sqrt{45k^2+9} = \sqrt{9k^2+18}$

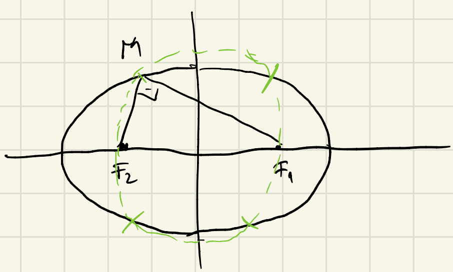
$$45k^2 + 9 = 9k^2 + 18$$

$$36k^2 = 9$$

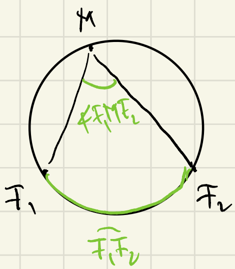
$$k^2 = \frac{1}{4} \iff k = \pm\frac{1}{2}$$

8. Consider the ellipse $\mathcal{E} : \frac{x^2}{4} + y^2 - 1 = 0$ with focal points F_1 and F_2. Determine the points M, situated on the ellipse for which

1. the angle $\measuredangle F_1MF_2$ is right;
2. the angle $\measuredangle F_1MF_2$ is θ;
3. the angle $\measuredangle F_1MF_2$ is maximal.



1. recall that for a circle passing through M, F_1, F_2 the measure of $\measuredangle F_1MF_2$ is twice the measure of the arc $\overset{\frown}{F_1F_2}$



So $\measuredangle F_1MF_2$ is a right angle if and only if F_1F_2 is a diameter

⇒ The points in 1. are the intersection of $\mathcal{E}$ with the circle centered at O and of radius c

2. $$\cos\theta = \cos \widehat{F_1 M F_2} = \frac{\overrightarrow{MF_1}\cdot\overrightarrow{MF_2}}{\|\overrightarrow{MF_1}\|\cdot\|\overrightarrow{MF_2}\|}$$

$\|\overrightarrow{MF_1}\| = a - \frac{c}{a}x_M$ $\quad$ $\|\overrightarrow{MF_2}\| = a + \frac{c}{a}x_M$ (see lecture)

$a=2 \quad c=\sqrt{4-1}=\sqrt{3}$

$\overrightarrow{MF_1} = (c - x_M, y_M)$ $\quad$ $\overrightarrow{MF_2} = (-c - x_M, y_M)$

$$\Rightarrow \overrightarrow{MF_1}\cdot\overrightarrow{MF_2} = x_M^2 - c^2 + y_M^2 = x_M^2 - 3 + 1 - \frac{x_M^2}{4} = \frac{3}{4}x_M^2 - 2$$

$$\Rightarrow \cos(\theta) = \frac{\frac{3}{4}x_M^2 - 2}{4 - \frac{3}{4}x_M^2}$$

$$\Rightarrow \theta = \arccos \frac{\frac{3}{4}x_M^2 - 2}{4 - \frac{3}{4}x_M^2}$$

$\frac{12}{4} - \frac{6}{4}$

3. Consider $f(x) = \dfrac{\frac{3}{4}x - 2}{4 - \frac{3}{4}x}$ $\quad$ $f'(x) = \dfrac{\frac{3}{4}x\left(4 - \frac{3}{4}x\right) + \frac{3}{4}x\left(\frac{3}{4}x - 2\right)}{\left(4 - \frac{3}{4}x\right)^2} = \dfrac{\frac{3}{2}x}{\left(4 - \frac{3}{4}x\right)^2}$

⇒ $x=0$ local minima/maxima

$$f''(x) = \frac{\frac{3}{2}\left(4-\frac{3}{4}x\right)^2 - 2\left(4-\frac{3}{4}x\right)\cdot -\frac{3}{4}\cdot\frac{3}{2}x}{\left(4-\frac{3}{4}x\right)^4} = \underbrace{\frac{3}{2}\cdot\frac{\overbrace{\left(4-\frac{3}{4}x\right)}^{\geq 0}}{\left(4-\frac{3}{4}x\right)^4}}_{>0}\Big[\underbrace{4-\frac{3}{4}x + \frac{3}{2}x}_{>0}\Big]$$

⇒ $x=0$ local minima for $\cos(\theta)$

⇒ $M(0,1)$ is the point with $\measuredangle F_1MF_2$ maximal

this is visible in the picture

9. Consider the ellipse $\mathcal{E} : x^2 + 4y^2 = 25$. Find the chords on the ellipse which have the point $A(7/2, 7/4)$ as midpoint.

consider the right hand side of the ellipse:

$$\left(x = \sqrt{25-4y^2},\ y\right)$$

The midpoint of two such points is

$$\left(\frac{1}{2}\left(\sqrt{25-4y_1^2} + \sqrt{25-4y_2^2}\right), \frac{1}{2}(y_1+y_2)\right) = \left(\frac{7}{2}, \frac{7}{4}\right)$$

so $\quad y_1 + y_2 = \frac{7}{2} \Rightarrow y_2 = \frac{7}{2} - y_1 \quad$ and

$$\sqrt{25-4y_1^2} + \sqrt{25-4\left(\frac{7}{2}-y_1\right)^2} = 7$$

$$\sqrt{25-(7-2y_1)^2} = 7 - \sqrt{25-4y_1^2} \quad \Big|\,(\ )^2$$

$$\cancel{25} - 49 + 28y_1 - \cancel{4y_1^2} = 49 - 14\sqrt{25-4y_1^2} + \cancel{25} - \cancel{4y_1^2} \quad \Big|\,:7$$

$$-7 \quad\ 4 \qquad\qquad\quad 7 \quad -2$$

$$4y_1 - 14 = -2\sqrt{25-4y_1^2}$$

$$2y_1 - 7 = \sqrt{25-4y_1^2} \quad \Big|\,(\ )^2$$

$$4y_1^2 - 28y_1 + 49 = 25 - 4y_1^2$$

$$8y_1^2 - 28y_1 + 24 = 0 \quad \Big|\,:4$$

$$2y_1^2 - 7y_1 + 6 = 0$$

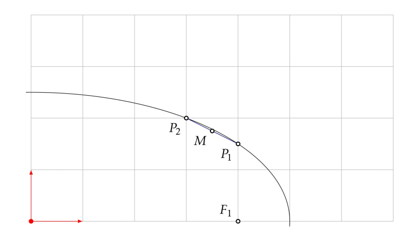
$$\Delta = 49 - 48 \qquad y_1 \begin{cases} \frac{7-1}{4} = \frac{3}{2} \\ \frac{7+1}{4} = 2 \end{cases}$$

$y \in [-b, b] = \left[-\frac{5}{2}, \frac{5}{2}\right]$

so both values are ok

$\Rightarrow$ we obtained two points $\quad x = \sqrt{25-4y^2} \begin{cases} 4 & M\left(4, \frac{3}{2}\right) \\ 3 & N(3, 2) \end{cases}$

$\Rightarrow$ the chord is the line segment $[MN]$



10. Consider the ellipse $\mathcal{E} : \frac{x^2}{25} + \frac{y^2}{9} = 1$. Determine the geometric locus of the midpoints of the chords on the ellipse which are parallel to the line $\ell : x + 2y = 1$.

lines parallel to ℓ have an eq. of the form

$$\ell_m : x + 2y = m \qquad \text{for } m \in \mathbb{R}$$

let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the intersection points of ℓ with $\mathcal{E}$

the midpoint of the chord P_1P_2 is

$$M_m\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) \underset{x=m-2y}{=} \left(\frac{m-2y_1+m-2y_2}{2}, \frac{y_1+y_2}{2}\right)$$

so $\quad M_m\left(m - y_1 - y_2, \frac{y_1+y_2}{2}\right)$

So the geometric locus that we look for is the set of points M_m which exist, these correspond to the values m for which ℓ_m intersects

the intersection of ℓ_m with $\mathcal{E}$ are obtained with

$$\frac{(m-2y)^2}{25} - \frac{y^2}{9} = 1$$

view this as an equation in y and impose the condition $\Delta \geq 0$ (at least 1 sol) to get the values for m

11. Find the equation of the circle:

1. passing through $A(3,1)$ and $B(-1,3)$ and having the center on the line $\ell : 3x - y - 2 = 0$;
2. passing through $A(1,1)$, $B(1,-1)$ and $C(2,0)$;
3. tangent to both $\ell_1 : 2x + y - 5 = 0$ and $\ell_2 : 2x + y + 15 = 0$ if the tangency point with ℓ_1 is $M(3,-1)$.

1.) A generic pt. on ℓ is $P(x, 3x-2)$

$$d(P,A)^2 = (3-x)^2 + (3-3x)^2 = 18 - 24x + 10x^2$$

$$d(P,B)^2 = \dots = 26 - 28x + 10x^2$$

since $r^2 = d(P,A)^2 = d(P,B)^2$

$\Rightarrow$ $18 - 24x + 10x^2 = 26 - 28x + 10x^2$

$\Rightarrow$ $x = 2$

$\Rightarrow$ center of circle is $I(2,4)$

and radius is $r = d(I,A) = \sqrt{1^2+3^2} = \sqrt{10}$

So the circle is $\mathcal{C} : (x-2)^2 + (y-4)^2 = 10$

2.) $\mathcal{C} \ni A(1,1), B(1,-1), C(2,0)$

$$\mathcal{C} : (x-a)^2 + (y-b)^2 = r^2$$

$A \in \mathcal{C} \Rightarrow (1-a)^2 + (1-b)^2 = r^2$

$1 - 2a + a^2 + 1 - 2b + b^2 = r^2$ (1)

$B \in \mathcal{C} \Rightarrow (1-a)^2 + (1+b)^2 = r^2$

$1 - 2a + a^2 + 1 + 2b + b^2 = r^2$ (2)

$C \in \mathcal{C} \Rightarrow (2-a)^2 + b^2 = r^2$

$4 - 4a + a^2 + b^2 = r^2$ (3)

so, a, b, r have to satisfy

$$(*) \begin{cases} 1-2a+a^2+1-2b+b^2=r^2 & (1) \\ 1-2a+a^2+1+2b+b^2=r^2 & (2) \\ 4-4a+a^2+b^2=r^2 & (3) \end{cases}$$

"$(1)-(2)$": $\quad -4b=0 \Rightarrow \boxed{b=0}$

$$\Rightarrow (*) \overset{b=0}{\Leftrightarrow} \begin{cases} 2-2a+a^2=r^2 \\ 4-4a+a^2=r^2 \end{cases} \quad -$$

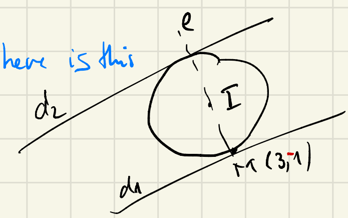
$$-2+2a=0 \Rightarrow \boxed{a=1} \Rightarrow r^2=1$$

so $\mathcal{C}: (x-1)^2+y^2=1.$

3.) the normal vectors of the two lines are the same

so in fact $d_1 \parallel d_2$

so in fact the picture here is this



a line $l \perp d_1$, $l \ni M$ is

$$l: \begin{cases} x=3+2t \\ y=-1+t \end{cases}$$

and $l \cap d_2$: $\quad 2(3+2t)+(-1+t)+15=0$

$6+4t$

$5t+20=0 \Rightarrow t=-4$

$\Rightarrow l \cap d_2 = N(-5,-5)$

$\Rightarrow$ center I of $\mathcal{C}$ is midpoint of MN, ie $I(-1,-3)$

and radius is $d(I,M)=d(I,N)=\sqrt{20}$

$\Rightarrow \mathcal{C}: (x+1)^2+(y+3)^2=20$