LECTURE 1

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1) Continuous dynamical systems: $t \in \mathbb{R}$ (time evolves cont)

2) Discrete dynamical systems: $t \in \mathbb{Z}$ (time evolves discrete)

1) Mathematical model of a cont. dyn. system is a differential equation: $F(t, \underbrace{x(t)}_{\text{unknown}}, x'(t), \dots, x^{(n)}(t)) = 0$

Linear differential equations (LDE)

(1) $x^{(n)}(t) + a_1(t) \cdot x^{(n-1)}(t) + a_2(t) \cdot x^{(n-2)}(t) + \dots + a_n(t) x(t) = f(t)$

where $n \in \mathbb{N}^*$ - order of the LDE (1)

$a_1, \dots, a_n \in C(\mathcal{I})$, continuous fct. on $\mathcal{I}$, $\mathcal{I} \subset \mathbb{R}$ nonempty open interval - coefficients

$f \in C(\mathcal{I})$ - the force / non-homogeneous part

Def: A solution of the LDE (1) is a function $\varphi \in C^n(\mathcal{I})$ that satisfies (1) for all $t \in \mathcal{I}$.

Why (1) is linear?

For any $x \in C^n(\mathcal{I})$ we define $\mathcal{L}(x)$ (function):

$$\mathcal{L}(x)(t) = x^{(n)}(t) + \dots + a_n(t)\, x(t), \forall t \in \mathcal{I}$$

$\Rightarrow \mathcal{L}(x) \in C(\mathcal{I})$

So, we have the map $\mathcal{L}: C^n(\mathcal{I}) \to C(\mathcal{I})$ and that $C^n(\mathcal{I})$ and $C(\mathcal{I})$ are linear/vector spaces with the usual addition and multiplication with reals.

So eq (1) $\Leftrightarrow \mathcal{L}(x) = f$

Proposition 1: The map $\mathcal{L}$ is linear, i.e.,

$$\mathcal{L}(\alpha \cdot x + \beta y) = \alpha \cdot \mathcal{L}(x) + \beta \cdot \mathcal{L}(y), \forall x, y \in C^n(\mathcal{I})$$
$$\forall \alpha, \beta \in \mathbb{R}$$

Pf:
$$\mathcal{L}(\alpha x + \beta y) = (\alpha x + \beta y)^{(n)} + a_1(t) \cdot (\alpha x + \beta y)^{(n-1)} + \dots +$$
$$+ a_n(t) \cdot (\alpha x + \beta y) =$$
$$= \alpha x^{(n)} + \beta \cdot y^{(n)} + a_1(t) \alpha \cdot x^{(n-1)} + a_1(t) \beta \cdot y^{(n-1)} + \dots$$
$$+ a_n(t) \alpha x + a_n(t) \beta y =$$
$$= \alpha (x^{(n)} + a_1(t) \cdot x^{(n-1)} + \dots + a_n(t) x) +$$
$$+ \beta (y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y) =$$
$$= \alpha \cdot \mathcal{L}(x) + \beta \cdot \mathcal{L}(y).$$

Def: The equation $\mathcal{L}(x) = 0$ (i.e. (1) with $f = 0$) is called linear homogeneous differential equation (LHDE), while when $f \neq 0$, the equation (1) or $\mathcal{L}(x) = f$ is called linear non-homogeneous differential equation (LN-HDE).

linear non-homogeneous differential equation (LN-HDE).

Consequences of the linearity of $\mathcal{L}$:

1) If $x_1, \ldots, x_m$ are solutions of a LHDE $\mathcal{L}(x) = 0$ then $\alpha_1 x_1 + \ldots + \alpha_m x_m$ is also a solution $\forall \alpha_1, \ldots, \alpha_m \in \mathbb{R}$

2) If $x_1, \ldots, x_m$ satisfy
$$\mathcal{L}(x_1) = f_1$$
$$\mathcal{L}(x_2) = f_2$$
$$\ldots\ldots$$
$$\mathcal{L}(x_m) = f_m$$
then $\alpha_1 x_1 + \ldots + \alpha_m x_m$ satisfies $\mathcal{L}(x) = \alpha_1 f_1 + \ldots + \alpha_m f_m$

Examples of differential equations

1) $x^2 + 1 = 0$, $x \in \mathbb{C}$ - algebraic equation, not a DE.

2) $y' = x$, $y = y(x)$, the unknown is a function $\Big] \Rightarrow$ we have y'
$\Rightarrow$ this is a first order d.e. lin, non-hom

$$y(x) = \frac{x^2}{2} + c, \; c \in \mathbb{R}$$

3) $x'(t) - x(t) = 0 \iff x' - x = 0$
first order LHDE with constant coefficient

Remark: $x_1 = e^t$ is a solution $(e^t)' - e^t = 0, \forall t \in \mathbb{R}$

$x' - x = 0 \,/ \cdot e^{-t}$

$x' \cdot e^{-t} - x \cdot e^{-t} = 0$

$x' \cdot e^{-x} - x \cdot e^{-x} = 0$

$(x \cdot e^{-t})' = 0 \Rightarrow x \cdot e^{-t} = c, c \in \mathbb{R} \Rightarrow x = c \cdot e^{t}, c \in \mathbb{R}$

4) $v \cdot t + \frac{1}{2} \cdot v' \cdot t^2 = 0 \,/ \cdot \frac{2}{t^2}$, $v = v(t)$ first order LHDE non constant coefficients

$\boxed{v' + \frac{2}{t} v = 0} \longleftrightarrow a_1(t) = \frac{2}{t}$, I is either $(-\infty, 0)$ or $(0, \infty)$

$v' \cdot t^2 + v 2t = 0 \Leftrightarrow (v \cdot t^2)' = 0 \Leftrightarrow v \cdot t^2 = c, c \in \mathbb{R} \Rightarrow$

$\Rightarrow v = \frac{c}{t^2}, c \in \mathbb{R}$

5) $x'' - x = 0$ second order LHDE with const coeff.

$x'' - x' + x' - x = 0$, $y = x' - x$

$y' + y = 0 \,/ \cdot e^t \Rightarrow y' e^t + y \cdot e^t = 0 \Rightarrow (y \cdot e^t)' = 0 \Rightarrow$

$\Rightarrow y \cdot e^t = c, c \in \mathbb{R} \Rightarrow y = c \cdot e^{-t}, c \in \mathbb{R}$

$\Rightarrow x' - x = c \cdot e^{-t} \,/ \cdot e^{-t}$ first order LN-HDE

$\Rightarrow x' e^{-t} - x \cdot e^{-t} = c \cdot e^{-2t} \Rightarrow (x \cdot e^{-t})' = c \cdot e^{-2t} \Rightarrow$

$\Rightarrow x \cdot e^{-t} = \underbrace{c \cdot \frac{1}{-2}}_{c_1} \cdot e^{-2t} + c_2 \,/ \cdot e^t, \; c, c_2 \in \mathbb{R}$

$\Rightarrow \boxed{x = c_1 \cdot e^{-t} + c_2, \; c_1, c_2 \in \mathbb{R}}$

Examples of nonlinear eq: $x' \cdot x = 1 \Big/ \cdot 2 \Rightarrow$

$\Rightarrow 2x \cdot x' = 2 \Rightarrow (x^2)' = 2 \Rightarrow \boxed{x^2 = 2t + c, \ c \in \mathbb{R}}$ general sol in implicit form

The fundamental theorems for LDEs

Theorem 1 (The existence and uniqueness theorem for the Initial Value Problem)

Let $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{R}^n$, $t_0 \in I$. The IVP for eq. (1) is the following:

$$(2) \begin{cases} L(x) = f \\ x(t_0) = \eta_1 \\ x'(t_0) = \eta_2 \\ \dots \\ x^{(n-1)}(t_0) = \eta_n \end{cases}$$

In the above hypotheses, the IVP (2) has a unique solution $\varphi \in C^n(I)$.

Theorem 2 (The fundamental theorem for LHDEs)

The set of solutions of an n^{th} order LHDE $L(x) = 0$ is a linear space of dimension n.

Hence, if $x_1, \dots, x_n$ are linearly independent solutions of $L(x) = 0$ then, the general sol. of $L(x) = 0$ is $x = c_1 x_1 + \dots + c_n x_n$, $c_1, \dots, c_n \in \mathbb{R}$

Example: Is $x = c_1 \cdot ch(t) + c_2\, sh(t)$, $c_1 c_2 \in \mathbb{R}$ the general solution of $x'' - x = 0$? Yes.

$ch(t) = \dfrac{e^t + e^{-t}}{2}$, $sh(t) = \dfrac{e^t - e^{-t}}{2}$

Method 1: $x = c_1 \dfrac{e^t + e^{-t}}{2} + c_2 \cdot \dfrac{e^t - e^{-t}}{2} \Leftrightarrow$

$$x = \underbrace{\frac{c_1 + c_2}{2}}_{d_1} \cdot e^t + \underbrace{\frac{c_1 - c_2}{2}}_{d_2} \cdot e^{-t}, \quad e^t, e^{-t} \text{ are solutions}$$

$\Rightarrow x = d_1 e^t + d_2 e^{-t}$ solution

, $\Leftarrow$ " $(c_1, c_2) \in \mathbb{R}^2 \longmapsto \left(\dfrac{c_1 + c_2}{2}, \dfrac{c_1 - c_2}{2}\right) \in \mathbb{R}^2$" is bijective

$\Leftrightarrow \begin{cases} d_1 = \dfrac{c_1 + c_2}{2} \\ d_2 = \dfrac{c_1 - c_2}{2} \end{cases}$ has a unique sol (c_1, c_2)

given (d_1, d_2)

Method 2: Prove that $ch(t)$ and $sh(t)$ are linearly independent solutions using Th. 2.