LECTURE 12 - DYNAMICAL SYSTEMS

Monday, 17 May 2021 10:00

Discrete dynamical systems

Ex: We consider the IVP: $\begin{cases} y' = y \\ y(0) = 1 \end{cases}$. We know that the unique sol. of the IVP is $\varphi : \mathbb{R} \to \mathbb{R}, \varphi(x) = e^x$

(i) Write the Euler's numerical formula with stepsize $h > 0$.

(ii) We prove that, for this IVP, the Euler's numerical method is convergent.

$h_n = \frac{1}{n}, n \in \mathbb{N}^*$ (thus, $h_n \xrightarrow[n\to\infty]{} 0$, (h_n) is decreasing)

We take $x^* = 1$ ($\Leftrightarrow$ we need n steps to reach $x_n = x^* = 1$)

We compute $y_n \simeq \varphi(x_n) = \varphi(1) = e$.

Prove that $y_n \xrightarrow[n\to\infty]{} e$.

(i) $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ $h > 0$ $\begin{cases} x_{k+1} = x_k + h, k \geq 0 \\ y_{k+1} = y_k + h f(x_k, y_k), k \geq 0 \end{cases}$

$\begin{cases} y' = y \\ y(0) = 1 \end{cases}$ $h > 0$ $\begin{cases} x_{k+1} = x_k + h & x_0 = 0 \\ y_{k+1} = y_k + h \cdot y_k & y_0 = 1 \end{cases}$ $\Leftrightarrow$ $\begin{cases} x_k = k \cdot h, k \geq 0 \\ y_{k+1} = (1+h) y_k, k \geq 0 \\ y_0 = 1 \end{cases}$ $\Leftrightarrow$ $\begin{cases} x_k = k \cdot h, k \geq 0 \\ y_k = (1+h)^k, k \geq 0 \end{cases}$

(ii) $h = \frac{1}{n}$

$\begin{cases} x_k = \frac{k}{n}, k = \overline{0,n} \\ y_k = (1 + \frac{1}{n})^k, k = \overline{0,n} \end{cases}$ $k = n \Rightarrow \begin{cases} x_n = 1 \\ y_n = (1 + \frac{1}{n})^n \approx \varphi(1) = e \end{cases}$, $\lim_{n\to\infty} y_n = e$

Discrete dynamical systems

(1) $x_{k+1} = f(x_k), k \geq 0$ $f : \mathbb{R}^n \to \mathbb{R}^n, n \geq 1$, continuous

Remark: Start with $x_0 = \eta \in \mathbb{R}^n$ then $x_1 = f(\eta), x_2 = f(x_1) = f(f(\eta)) = \underbrace{(f \circ f)}_{f^2}(\eta) = f^2(\eta) \ldots \; x_k = \overbrace{f^k(\eta)}^{\text{the } k^{th} \text{ iterate of } f}, \forall k \geq 0$

Any IVP has a unique sol. in the future.

Def: The positive orbit of the initial state $\eta \in \mathbb{R}^n$ is $\gamma_\eta^+ = \{\eta, f(\eta), f^2(\eta), \ldots, f^k(\eta), \ldots\}$

The orbit can be computed only when f is invertible: $\gamma_\eta = \{\ldots f^{-k}(\eta), \ldots, f^{-2}(\eta), f(\eta), f^2(\eta), \ldots\}$.

Let $\eta^* \in \mathbb{R}^n$. We say η^* is a fixed point of f when $f(\eta^*) = \eta^*$ or equivalently $\gamma_{\eta^*}^+ = \{\eta^*\}$.

Remark: If η^* - fixed point of $f \Rightarrow \eta^*$ is a fixed point of $f^k, \forall k \geq 1$

Proof: $f(\eta^*) = \eta^*$. We want to prove that $f^k(\eta^*) = \eta^*$.

$f(\eta^*) = \eta^* \Rightarrow f(f(\eta^*)) = f(\eta^*) \Rightarrow f^2(\eta^*) = \eta^*$ ... the conclusion follow by induction

Def: Let η^* be a fixed point of f. We say that η^* is a stable f.p. of f when, $\forall \varepsilon > 0, \exists \delta > 0$ s.t. whenever $\|\eta - \eta^*\| < \delta$ we have that $\|f^k(\eta) - \eta^*\| < \varepsilon, \forall k \geq 0$.

We say that η^* is an attractor of f when $\exists \rho > 0$ s.t. whenever $\|\eta - \eta^*\| < \rho$ we have $\lim_{k\to\infty} f^k(\eta) = \eta^*$

Basin of attraction: $A_{\eta^*} = \{\eta \in \mathbb{R}^n : \lim_{k\to\infty} f^k(\eta) = \eta^*\}$

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Property: If $\lim_{k\to\infty} f^k(\eta) = \tilde{\eta}$ then $\tilde{\eta}$ is a fixed point of f.

Pf: $x_k \overset{not}{=} f^k(\eta) \Rightarrow x_{k+1} = f(x_k)$, $\lim_{k\to\infty} f^k(\eta) = \tilde{\eta} \in \mathbb{R}^n$, f is continuous $\Rightarrow$

$\Rightarrow \lim_{k\to\infty} x_{k+1} = \lim_{k\to\infty} f(x_k) = f(\lim_{k\to\infty} x_k) \Leftrightarrow \tilde{\eta} = f(\tilde{\eta})$ is a f.p. of f

Def: Let $\eta^* \in \mathbb{R}^n$ and $p \in \mathbb{N}, p \geq 2$. We say that η^* is p-periodic point of f when $f^p(\eta^*) = \eta^*$ and η^* is not a fixed point of $f, f^2, \dots, f^{p-1}$.

Remark: Let η^* be a p-periodic point of f. Then the unique sol. of the ivp $\begin{cases} x_{k+1} = f(x_k) \\ x_0 = \eta^* \end{cases}$ is $\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*), \eta^*, f(\eta^*), f^2(\eta^*), \dots, f^{p-1}(\eta^*), \eta^* \dots$ and the positive orbit is $\gamma^+_{\eta^*} = \{\eta^*, f(\eta^*), \dots, f^{p-1}(\eta^*)\}$ it is called a p-cycle.

Ex: Find the fixed points and the 2-periodic points of $f: \mathbb{R} \to \mathbb{R}, f(x) = 1 - 2x^2$.

Find the fixed points: $f(x) = x \Leftrightarrow 1 - 2x^2 = x \Leftrightarrow 2x^2 + x - 1 = 0$

$\Delta = 1 + 8 = 9 \Rightarrow x_{1,2} = \frac{-1 \pm 3}{4} \Rightarrow x_1 = -1,\ x_2 = \frac{1}{2}$

2 fixed points: $\eta_1^* = \frac{1}{2}, \eta_2^* = -1$. Find the 2-periodic points: $f^2(x) = x$

$f^2(x) = (f \circ f)(x) = f(f(x)) = 1 - 2(f(x))^2 = 1 - 2(1 - 2x^2)^2 = 1 - 2(1 - 4x^2 + 4x^4) = 1 - 2 + 8x^2 - 8x^4 =$
$= -8x^4 + 8x^2 - 1, \forall x \in \mathbb{R}$

$f^2(x) = x \Leftrightarrow -8x^4 + 8x^2 - 1 = x \Leftrightarrow 8x^4 - 8x^2 + x + 1 = 0$

We use that the fixed points of f are f.p of $f^2 \Rightarrow -1$ and $\frac{1}{2}$ are roots of the eq $8x^4 - 8x^2 + x + 1 = 0$.

$8x^4 - 8x^2 + x + 1 = (2x^2 + x - 1) \cdot g$

$\Leftrightarrow 8x^4 - 8x^2 + x + 1 = 4x^2(2x^2 + x - 1) - 4x^3 + 4x^2 - 8x^2 + x + 1 =$
$= 4x^2(2x^2 + x - 1) - 2x(2x^2 + x - 1) + 2x^2 - 2x - 4x^2 + x + 1 =$
$= 4x^2(2x^2 + x - 1) - 2x(2x^2 + x - 1) + 1 - 2x^2 - x =$
$= (2x^2 + x - 1)(4x^2 - 2x - 1)$

We have to find the roots of $4x^2 - 2x - 1 = 0$, which are $\frac{1-\sqrt{5}}{4}$ and $\frac{1+\sqrt{5}}{4}$

Check: $f\left(\frac{1-\sqrt{5}}{4}\right) = 1 - 2 \cdot \left(\frac{1-\sqrt{5}}{4}\right)^2 = 1 - 2 \cdot \frac{1 - 2\sqrt{5} + 5}{16} = 1 - \frac{6 - 2\sqrt{5}}{8} = 1 - \frac{3 - \sqrt{5}}{4} = \frac{4 - 3 + \sqrt{5}}{4} = \frac{1 + \sqrt{5}}{4}$

$f\left(\frac{1+\sqrt{5}}{4}\right) = 1 - 2\left(\frac{1+\sqrt{5}}{4}\right)^2 = 1 - 2 \cdot \frac{1 + 2\sqrt{5} + 5}{16} = 1 - \frac{6 + 2\sqrt{5}}{8} = 1 - \frac{3 + \sqrt{5}}{4} = \frac{4 - 3 - \sqrt{5}}{4} = \frac{1 - \sqrt{5}}{4}$

$\gamma^+_{\frac{1-\sqrt{5}}{4}} = \gamma^+_{\frac{1+\sqrt{5}}{4}} = \left\{\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right\}$ the 2-cycle.

Def: Let η^* be a p-periodic point of f. We say that the corresponding p-cycle γ^+_{η} is stable/unstable/attractor for f when η^* is a stable/unstable/attractor fixed point of f^p.

Remark: Let η^* be a p-periodic point of f s.t. its corresp. p-cycle is an attractor. Let $\eta \in A_{\eta^*, f^p}$

Then: $\lim_{k\to\infty} (f^p)^k(\eta) = \eta^*$ $\quad (f^p)^k = f^{pk}$

$$\gamma^+_\eta = \{\eta, f(\eta), f^2(\eta), \dots, f^{p-1}(\eta),$$
$$f^p(\eta), f(f^p(\eta)), f^2(f^p(\eta)), \dots, f^{p-1}(f^p(\eta)),$$
$$f^{2p}(\eta), f(f^{2p}(\eta)), \dots\}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$
$$\eta^* \quad f(\eta^*) \quad f^2(\eta^*) \quad \dots \quad f^{p-1}(\eta^*)$$

The linearization method to study the stability of fixed points of scalar maps:

Let η^* be a fixed point of the scalar map $f:\mathbb{R}\to\mathbb{R}$, which is C^1.

If $|f'(\eta^*)|<1$ then η^* is an attractor.

If $|f'(\eta^*)|>1$ then η^* is unstable.

Remark: If $|f'(\eta^*)|=1$, we say that η^* is not hyperbolic.

Ex: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=1-2x^2$ (scalar map)

Study the stab. of its fixed points $\eta_1^* = \frac{1}{2}$, $\eta_2^* = -1$ and its 2-cycle $\left\{\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right\}$

$f'(x) = -4x$: $f'(\eta_1^*) = f'(\frac{1}{2}) = -4\cdot\frac{1}{2} = -2 \Rightarrow |f'(\frac{1}{2})| = 2>1 \overset{LM}{\Rightarrow}$ the f.p. $\frac{1}{2}$ is unstable

$f'(\eta_2^*) = f'(-1) = -4\cdot(-1) = 4 \Rightarrow |f'(-1)| = 4>1 \overset{LM}{\Rightarrow}$ the f.p. -1 is unstable

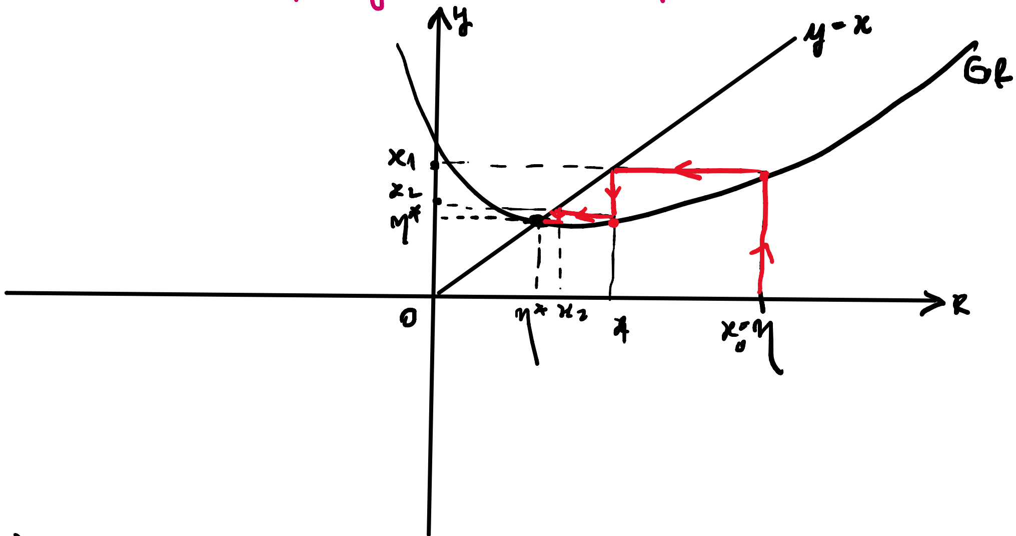
Recall: $f^2(x) = -8x^4+8x^2-1 \Rightarrow (f^2)'(x) = -32x^3+16x$

$$(f^2)'\left(\frac{1-\sqrt{5}}{4}\right) = -32\cdot\left(\frac{1-\sqrt{5}}{4}\right)^3 + 16\cdot\left(\frac{1-\sqrt{5}}{4}\right) = -32\,\frac{(1-\sqrt{5})^3}{64} + 4(1-\sqrt{5}) =$$

$$= \frac{-(1-\sqrt{5})^2(1-\sqrt{5})}{2} + 4(1-\sqrt{5}) = (1-\sqrt{5})\left(-\frac{6-2\sqrt{5}}{2}+4\right) = (1-\sqrt{5})(-3+\sqrt{5}+4) =$$
$$= (1-\sqrt{5})(1+\sqrt{5}) = 1-5 = -4$$

$\Rightarrow \left|(f^2)'\left(\frac{1-\sqrt{5}}{4}\right)\right| = 4>1 \Rightarrow$ the 2-cycle $\left\{\frac{1-\sqrt{5}}{4}, \frac{1+\sqrt{5}}{4}\right\}$ is unstable.

The cobweb/stair-step diagram for scalar maps



$f(x) = x$

$y = x$

$x_0 = \eta$ $x_1 = f(\eta)$ $x_2 = f(x_1)$

$y = x$

$x_0 = \eta \quad x_1 = f(\eta) \quad x_2 = f(x_1)$

HW: $f(x) = \frac{1}{2}x + \frac{3}{2} \cdot \frac{1}{x}$, $f : (0, \infty) \to \mathbb{R}$