1. Let $A_0, \dots, A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1} + \overrightarrow{A_1A_2} + \cdots + \overrightarrow{A_{n-1}A_n} + \overrightarrow{A_nA_0}$.

2. In each of the following cases, decide if the indicated vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ can be represented with the vertices of a triangle:

1. $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$.
2. $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.
3. $\|\mathbf{u}\| = 7$, $\|\mathbf{v}\| = 3$, $\|\mathbf{w}\| = 11$.
4. $\mathbf{u}(1,0,1)$, $\mathbf{v}(0,1,0)$, $\mathbf{w}(2,2,2)$.

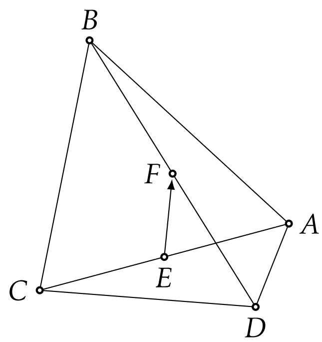
3. Let $ABCDEF$ be a regular hexagon centered at O.

1. Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.
2. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

4. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that $\overrightarrow{MN} + \overrightarrow{PQ} = 0$. Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

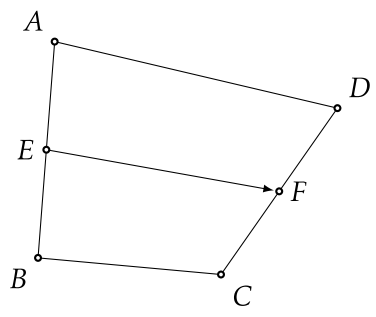
$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB}).$$



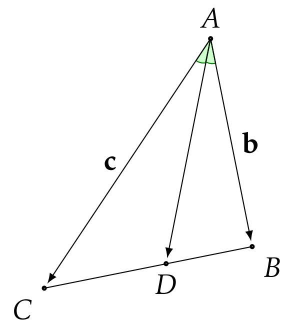
6. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC}).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.

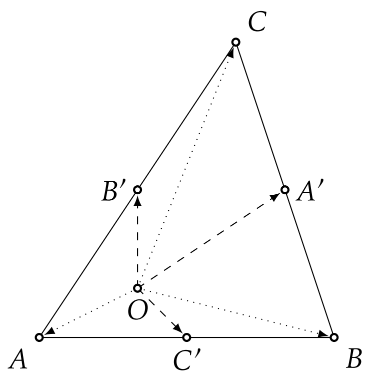


7. Let ABC be a triangle and let $D \in [BC]$ be such that AD is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b} = \overrightarrow{AB}$ und $\mathbf{c} = \overrightarrow{AC}$.

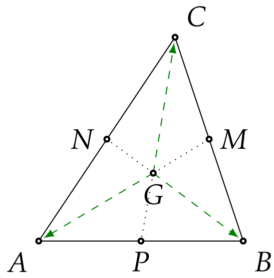


8. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$



9. Show that the medians in a triangle intersect in one point.



10. Let $ABCD$ be a tetrahedron. Determine the sums

1. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$,
2. $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$,
3. $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$.

11. Let $ABCD$ be a tetrahedron. Show that $\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}$.

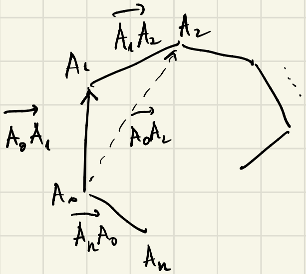
12. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

1. Let $A_0,\dots,A_n$ be the vertices of a polygon. Determine $\overrightarrow{A_0A_1}+\overrightarrow{A_1A_2}+\dots+\overrightarrow{A_{n-1}A_n}+\overrightarrow{A_nA_0}$.



$$\underbrace{\underbrace{\underbrace{\underbrace{\overrightarrow{A_0A_1}+\overrightarrow{A_1A_2}}_{\overrightarrow{A_0A_2}}+\overrightarrow{A_2A_3}}_{\overrightarrow{A_0A_3}}+\dots+\overrightarrow{A_{n-1}A_n}}_{\overrightarrow{A_0A_n}}+\overrightarrow{A_nA_0}}_{\overrightarrow{A_0A_0}=0}=$$

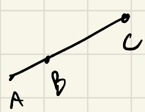
2. In each of the following cases, decide if the indicated vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ can be represented with the vertices of a triangle:

1. $\mathbf{u}(7,3)$, $\mathbf{v}(-2,-8)$, $\mathbf{w}(-5,5)$.
2. $\mathbf{u}(7,3)$, $\mathbf{v}(2,8)$, $\mathbf{w}(-5,5)$.
3. $\|\mathbf{u}\|=7$, $\|\mathbf{v}\|=3$, $\|\mathbf{w}\|=11$.
4. $\mathbf{u}(1,0,1)$, $\mathbf{v}(0,1,0)$, $\mathbf{w}(2,2,2)$.

1. $u+v+w = \begin{bmatrix}7\\3\end{bmatrix}+\begin{bmatrix}-2\\-8\end{bmatrix}+\begin{bmatrix}-5\\5\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\Rightarrow$ u, v, w can be represented on the sides of a polygon (with 3 sides, ie. Δ)

• is the triangle degenerate?



$\Updownarrow$

A, B, C collinear?

$\Updownarrow$

u, v linearly dependent which is not the case

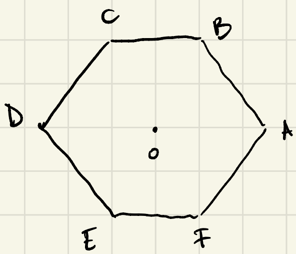
2. We have the same vectors as in 1 except v which is in the opposite direction so, the answer is yes.

3. $\|u\| + \|v\| = 10 < \|w\| = 11 \Rightarrow$ no, a triangle cannot have one side longer than the sum of the other sides

4. $\|u\| = \sqrt{2}$ $\quad \|v\| = 1$ $\quad \|w\| = 2\sqrt{3} \Rightarrow$ no, $\sqrt{2}+1 < 2\sqrt{3}$

3. Let $ABCDEF$ be a regular hexagon centered at O.

1. Express the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$ in terms of $\overrightarrow{OE}$ and $\overrightarrow{OF}$.
2. Show that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.



1.) $\overrightarrow{OA} = \overrightarrow{EF} = \overrightarrow{EO} + \overrightarrow{OF} = \overrightarrow{OF} - \overrightarrow{OE}$

$\overrightarrow{OB} = -\overrightarrow{OE}$

$\overrightarrow{OC} = -\overrightarrow{OF}$

$\overrightarrow{OD} = \overrightarrow{FE} = \overrightarrow{OE} - \overrightarrow{OF}$

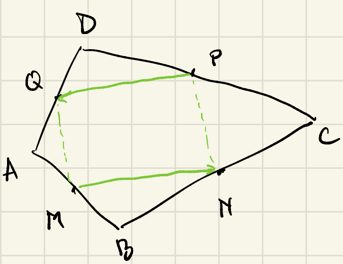
$\overrightarrow{FE} = -\overrightarrow{EF} = -\overrightarrow{OA}$

2.) $$\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = (\overrightarrow{AB} + \overrightarrow{AF}) + \overbrace{(\overrightarrow{AB} + \underset{\overrightarrow{AO}}{\underset{\parallel}{\overrightarrow{BC}}})}^{\overrightarrow{AC}} + \overbrace{(\overrightarrow{AF} + \underset{\overrightarrow{AO}}{\underset{\parallel}{\overrightarrow{FE}}})}^{\overrightarrow{AE}} + \underset{2\overrightarrow{AO}}{\underset{\parallel}{\overrightarrow{AD}}}$$

$$= 2\underbrace{(\overrightarrow{AB} + \overrightarrow{AF})}_{\overrightarrow{AO}} + 4\overrightarrow{AO}$$

$$= 6\overrightarrow{AO} = 3\overrightarrow{AD}$$

4. Let $ABCD$ be a quadrilateral. Let M, N, P, Q be the midpoints of $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively. Show that $\overrightarrow{MN} + \overrightarrow{PQ} = 0$. Deduce that the midpoints of the sides of an arbitrary quadrilateral form a parallelogram.



Fix a point $O \in E^2$

$\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM} = \frac{1}{2}(\cancel{\overrightarrow{OB}} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \cancel{\overrightarrow{OB}})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\overrightarrow{OA} + \cancel{\overrightarrow{OD}}) - \frac{1}{2}(\cancel{\overrightarrow{OD}} + \overrightarrow{OC})$

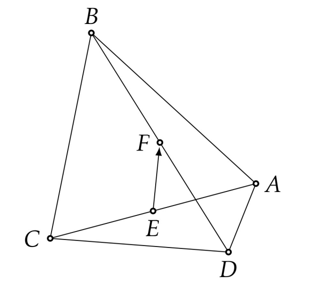
$+$

$\overrightarrow{MN} + \overrightarrow{PQ} = \ldots = 0$

$\Rightarrow \overrightarrow{MN} = \overrightarrow{QP} \Leftrightarrow MNPQ$ parallelogram

5. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AC]$ and let F be the midpoint of $[BD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD}) = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB}).$$



Let O be any point in the plane E^2

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OB}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$$

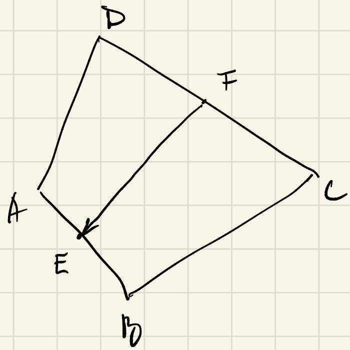
$$= \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OB} - \overrightarrow{OA} - \overrightarrow{OC}) = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{CD})$$

$$= \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{CB})$$

6. Let $ABCD$ be a quadrilateral. Let E be the midpoint of $[AB]$ and let F be the midpoint of $[CD]$. Show that

$$\overrightarrow{EF} = \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC}).$$

Deduce that the length of the midsegment in a trapezoid is the arithmetic mean of the lengths of the bases.



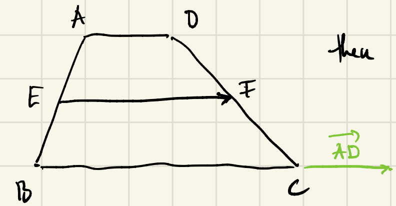
Let O be any point in E^2

$$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC}) - \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

$$= \frac{1}{2}(\overrightarrow{OD} + \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{OB})$$

$$= \frac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})$$

If ABCD is a trapezoid

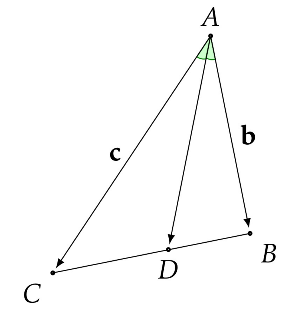


then $\overrightarrow{EF}, \overrightarrow{AD}, \overrightarrow{BC}$ are collinear.

$$|EF| = \|\overrightarrow{EF}\| = \|\tfrac{1}{2}(\overrightarrow{AD} + \overrightarrow{BC})\| = \frac{1}{2}\|\overrightarrow{AD} + \overrightarrow{BC}\| = \frac{1}{2}(\|\overrightarrow{AD}\| + \|\overrightarrow{BC}\|) = \frac{|AD| + |BC|}{2}$$

since $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are collinear and have the same orientation

7. Let ABC be a triangle and let $D \in [BC]$ be such that AD is an angle bisector. Express $\overrightarrow{AD}$ in terms of $\mathbf{b} = \overrightarrow{AB}$ und $\mathbf{c} = \overrightarrow{AC}$.



By the angle bisector theorem we have

$$\frac{|AC|}{|CD|} = \frac{|AB|}{|BD|} \qquad (*)$$

Now $\overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = c + \dfrac{|CD|}{|CB|}\overrightarrow{CB}$

$$= c + \frac{|CD|}{|CB|}(\overrightarrow{AB} - \overrightarrow{AC})$$

$$= c + \frac{|CD|}{|CB|}(b - c)$$

$$= \left(1 - \frac{|CD|}{|CB|}\right)c + \frac{|CD|}{|CB|}b$$

By $(*)$ we have $\dfrac{|BD|}{|CD|} = \dfrac{|AB|}{|AC|} \overset{+1}{\Rightarrow} \dfrac{\overbrace{|BD| + |CD|}^{|BC|}}{|CD|} = \dfrac{|AB| + |AC|}{|AC|}$

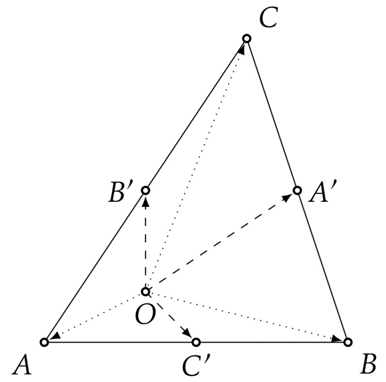
So $|CD| = \dfrac{|BC| \cdot |AC|}{|AB| + |AC|}$

$$\Rightarrow \overrightarrow{AD} = \left(1 - \frac{|AC|}{|AB| + |AC|}\right)c + \frac{|AC|}{|AB| + |AC|}b$$

$$\overrightarrow{AD} = \frac{|AB|}{|AB| + |AC|}c + \frac{|AC|}{|AB| + |AC|}b$$

8. Let A', B' and C' be midpoints of the sides of a triangle ABC. Show that for any point O we have

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$



$$\overrightarrow{OA'} = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC})$$

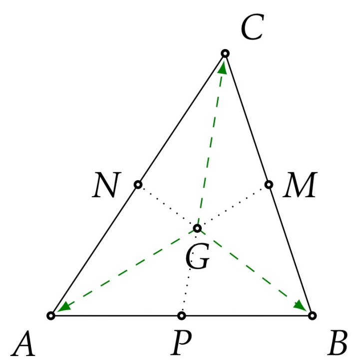
$$\overrightarrow{OB'} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC})$$

$$\overrightarrow{OC'} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$$

$+$

$$= \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

9. Show that the medians in a triangle intersect in one point.



Let P be the midpoint of AB

M —— " —— BC

N —— " —— CA

For any point $O \in \mathbb{E}^2$ we have

consider the vector

$$v = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$$

Then

$$v = (\overrightarrow{OA} + \overrightarrow{OB}) + \overrightarrow{OC} = 2\,\overrightarrow{OP} + \overrightarrow{OC} \quad (1)$$

$$v = (\overrightarrow{OA} + \overrightarrow{OC}) + \overrightarrow{OB} = 2\,\overrightarrow{ON} + \overrightarrow{OB} \quad (2)$$

$$v = (\overrightarrow{OB} + \overrightarrow{OC}) + \overrightarrow{OA} = 2\,\overrightarrow{OM} + \overrightarrow{OA} \quad (3)$$

So, if we choose O on CP it follows from (1) that $v \parallel \overrightarrow{CP}$

Similar, if we choose O on BM ——— || ——— $v \parallel \vec{BM}$

——— || ——— AM ——— || ——— $v \parallel \vec{AM}$

Now, if $O \in CP \cap BM \Rightarrow v \parallel \vec{CP}$ and $v \parallel \vec{BM}$

$\Rightarrow v = 0$ (else $\vec{CP} \parallel \vec{BM} \Leftrightarrow CP \parallel BM$ which is impossible)

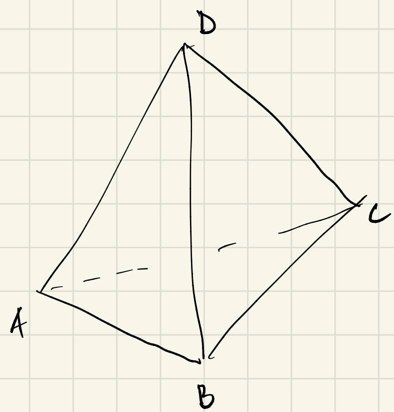
(3)
$\Rightarrow 0 = 2\vec{OM} + \vec{OA} \Rightarrow \vec{OM}, \vec{OA}$ are proportional $\Rightarrow O, M, A$ are collinear

So if $O \in CP \cap BM$ then $O \in AM$, i.e the intersection point of two medians lies on the third median

i.e all three medians intersect in one point

10. Let $ABCD$ be a tetrahedron. Determine the sums

1. $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$,
2. $\overrightarrow{AD} + \overrightarrow{BC} + \overrightarrow{DB}$,
3. $\overrightarrow{AB} + \overrightarrow{CD} + \overrightarrow{BC} + \overrightarrow{DA}$.



1. $\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$

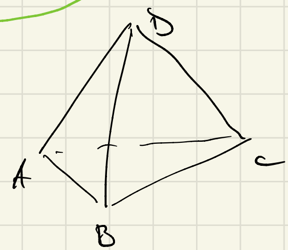
2. $\vec{AD} + \vec{BC} + \vec{DB} = \vec{AB} + \vec{BC} = \vec{AC}$

3. $\vec{AD} + \vec{CD} + \vec{BC} + \vec{DA} = \vec{AA} = 0$

11. Let $ABCD$ be a tetrahedron. Show that $\overrightarrow{AD} + \overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{AC}$.

$$\vec{AD} + \vec{BC} = (\vec{AB} + \vec{BD}) + (\vec{BA} + \vec{AC})$$

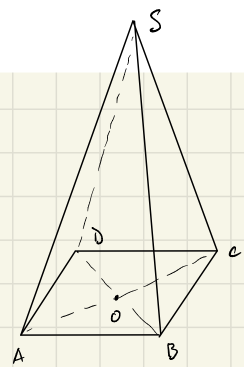
$$= \vec{BD} + \vec{AC} + \underbrace{\vec{AB} + \vec{BA}}_{=0}$$



12. Let $SABCD$ be a pyramid with apex S and base the parallelogram $ABCD$. Show that

$$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} + \overrightarrow{SD} = 4\overrightarrow{SO}$$

where O is the center of the parallelogram.



$$\underbrace{\vec{SA} + \vec{SC}}_{= 2\vec{SO}} + \underbrace{\vec{SB} + \vec{SD}}_{= 2\vec{SO}} = 4\vec{SO}$$

13. In $\mathbb{E}^3$ consider the parallelograms $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$. Show that the midpoints of the segments $[A_1B_1]$, $[A_2B_2]$, $[A_3B_3]$ and $[A_4B_4]$ are the vertices of a parallelogram.

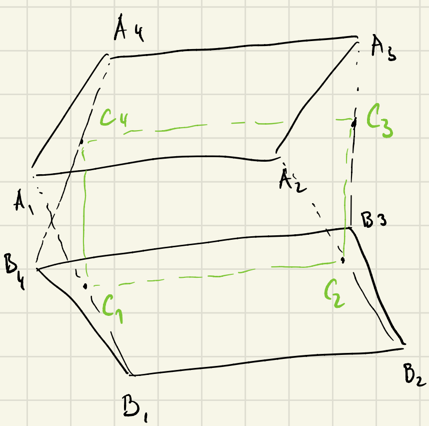
Let C_i be the midpoint of $[A_iB_i]$

$$\vec{C_1C_2} = \vec{OC_2} - \vec{OC_1} = \frac{1}{2}(\vec{OA_2} + \vec{OB_2} - \vec{OA_1} - \vec{OB_1})$$

$$= \frac{1}{2}(\vec{A_1A_2} + \vec{B_1B_2})$$

Similar $\vec{C_3C_4} = \frac{1}{2}(\vec{A_3A_4} + \vec{B_3B_4})$

(see Problem 6)



$$\Rightarrow \vec{C_1C_2} + \vec{C_3C_4} = \frac{1}{2}(\underbrace{\vec{A_1A_2} + \vec{A_3A_4}}_{=0} + \underbrace{\vec{B_1B_2} + \vec{B_3B_4}}_{=0}) = 0 \Rightarrow \vec{C_1C_2} = \vec{C_4C_3}$$

since $A_1A_2A_3A_4$ is a parallelogram

$\Rightarrow C_1C_2C_3C_4$ parallelogram.