Lecture 4

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Scalar first order LDE's

(1) $x' + a(t)x = f(t)$, $t \in I$

where $a, f \in C(I)$.

For $t_0 \in I$ fixed, consider $A(t) = -\int_{t_0}^{t} a(s)\,ds$. Thus $A'(t) = -a(t)$, $A(t_0) = 0$.

I. The integrating factor method to find the general sol of (1)

Theorem 1 The function $\mu : I \to \mathbb{R}$, $\mu(t) = e^{A(t)}$ is an integrating factor for (1).
The general sol. for (1) is

$$(2)\quad x = c \cdot e^{A(t)} + e^{A(t)} \cdot \int_{t_0}^{t} f(s)\, e^{-A(s)}\, ds, \quad c \in \mathbb{R} \text{ arbitrary.}$$

Proof.

$$x' + a(t)x = f(t) \;\Big|\cdot e^{-A(t)} \iff \underbrace{x' e^{-A(t)} + x \cdot a(t)\, e^{-A(t)}}_{\left(x \cdot e^{-A(t)}\right)'} = f(t) e^{-A(t)}$$

$$\iff \left(x \cdot e^{-A(t)}\right)' = f(t) \cdot e^{-A(t)}, \quad \forall t \in I$$

integrate

$$x \cdot e^{-A(t)} = \int_{t_0}^{t} f(s) \cdot e^{-A(s)}\, ds + c, \;\forall t \in I \;\Big|\cdot e^{A(t)} \iff x = c \cdot e^{A(t)} + e^{A(t)} \cdot \int_{t_0}^{t} f(s)\, e^{-A(s)}\, ds$$

$\forall t \in I$

where $c \in \mathbb{R}$ arbitrary.

II. The separation of variables method to find the general sol. of $x' + a(t)x = 0$

Def. A DE of the form $x' = b(t) \cdot g(x)$ is said to be a separable eq.

$\left(x' \to \dfrac{dx}{dt} \qquad \dfrac{dx}{dt} = b(t) \cdot g(x) \qquad \dfrac{dx}{g(x)} = b(t)\,dt \quad \text{we separated the variables} \right)$

Notice that $x' + a(t)x = 0$ can be written as $x' = -a(t)x$, thus is a separable equation.

Remark. We already know that $x = c \cdot e^{A(t)}$, $c \in \mathbb{R}$ is the general sol.
From here we deduce that $x = 0$ is a (constant) solution and any other sol.

Remark: we already know that $x = c\, e^{A(t)}$, $c \in \mathbb{R}$ is the general sol.
From here we deduce that $x = 0$ is a (constant) solution and any other sol. satisfies $x(t) \neq 0$ $\forall t \in I$.

The sep. of var. method: $x'(t) = -a(t)x(t)$ $\qquad x = 0$ is a sol.

Now we look for sol. $x \neq 0$ $\qquad \dfrac{x'(t)}{x(t)} = -a(t) \iff \big(\ln|x(t)|\big)' = -a(t)$

$$\overset{\text{integrate}}{\iff} \ln|x(t)| = A(t) + c, \quad c \in \mathbb{R} \iff |x(t)| = e^{A(t)+c} \iff$$

$$\iff x(t) = \pm e^{c} \cdot e^{A(t)}, \quad c \in \mathbb{R} \text{ arbitrary}$$

Recall $x = 0$ sol.

Note that $\{e^c, -e^c, 0 : c \in \mathbb{R}\} = \mathbb{R}$

$$\iff x(t) = c_1 \cdot e^{A(t)}, \quad c_1 \in \mathbb{R} \text{ arbitrary}$$

Remark. we can use the sep. of var. method to find just a solution $x_1 \neq 0$. How? when we integrate we take $c = 0$, that is

$$\ln|x(t)| = A(t) \Rightarrow x_1(t) = e^{A(t)}$$ this is a sol. of a first order LHDE

we apply the fundamental th. for LHDE and obtain that the general sol. is $x = c \cdot e^{A(t)}$, $c \in \mathbb{R}$.

III The Lagrange method (the variation of constant method) to find a particular solution of (1) $x' + a(t)x = f(t)$ (linear nonhom.)

The idea of Lagrange was to look at the general sol. of the LHDE $x' + a(t)x = 0$, which is $x_h = c \cdot e^{A(t)}$, $c \in \mathbb{R}$ arbitrary and, then, to look for a partic. sol. of (1) of the form

$$x_p = \varphi(t) \cdot e^{A(t)}, \qquad \varphi = ?$$

x_p is a sol. of (1) $\iff \varphi'(t) \cdot e^{A(t)} + \cancel{\varphi(t) \cdot e^{A(t)}(-a(t)) + a(t) \cdot \varphi(t) \cdot e^{A(t)}} = f(t)$

$$\iff \varphi'(t) = f(t) \cdot e^{-A(t)} \iff \varphi(t) = \int_{t_0}^{t} f(s) e^{-A(s)}\, ds, \quad \forall t \in I.$$

So $x_p = e^{A(t)} \int^{t} \dots e^{-A(s)} \dots$

So, $\boxed{x_p = e^{A(t)} \cdot \int_{t_0}^{t} f(s)\, e^{-A(s)}\, ds}$

Remark. Thus, the general sol. of (1) is $x = x_h + x_p$, that is we found again (2).

Exercises Find the general sol. of a) $x' - 2tx = t$, $t \in \mathbb{R}$

b) $x' - 2tx = \frac{2}{\sqrt{\pi}}$, $t \in \mathbb{R}$

(HW) c) $x' + \frac{1}{t}x = \frac{1}{t} e^{-2t+1}$, $t \in (0, \infty)$

a) $x' - 2tx = t$ first order L nonhom DE

Step 1 we write the LHDE associated $x' - 2tx = 0$

apply the sep. of var. method $x' = 2tx$ $\quad \frac{dx}{dt} = 2tx$ $\quad \int \frac{dx}{x} = \int 2t\, dt$

$x = 0$ is a sol.

$\ln|x| = t^2 + c$, $\quad |x| = e^{t^2+c}$, $\quad \left.\begin{array}{l} x = \pm e^c \cdot e^{t^2} \\ x = 0 \end{array}\right\} \Rightarrow x_h = c_1 \cdot e^{t^2}$, $c_1 \in \mathbb{R}$

Step 2 $x_p = ?$ Note that $x_p = -\frac{1}{2}$ is a sol.

Step 3 the gen. sol. $x = c_1 \cdot e^{t^2} - \frac{1}{2}$, $c_1 \in \mathbb{R}$

b) $x' - 2tx = \frac{2}{\sqrt{\pi}}$

Step 1 it has the same hom. part as a). Thus $x_h = c_1 \cdot e^{t^2}$, $c_1 \in \mathbb{R}$

Step 2. $x_p = ?$ we apply the Lagrange method

$x_p = \varphi(t) \cdot e^{t^2}$

x_p is a sol. $\Leftrightarrow$ $\varphi'(t) \cdot e^{t^2} + \cancel{\varphi(t) \cdot e^{t^2} \cdot 2t} - \cancel{2t\varphi(t)e^{t^2}} = \frac{2}{\sqrt{\pi}}$ $\Leftrightarrow$

$\Leftrightarrow \varphi'(t) = \frac{2}{\sqrt{\pi}} e^{-t^2}$ $\Leftrightarrow$ $\varphi(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-s^2}\, ds$ it is known that the primitive of e^{-t^2} ...

The Gauss error function ...

$\sqrt{\pi}$ $\sqrt{\pi}\int_0$

The Gauss error function

$$\operatorname{erf}(t) \overset{not}{=} \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}\,ds, \quad \forall t\in\mathbb{R}$$

the primitive of e^{-t^2} is not a finite combination of elementary functions, thus we are not able to write its expression.

So, $\varphi(t) = \operatorname{erf}(t) \Rightarrow \underline{x_p = \operatorname{erf}(t)\cdot e^{t^2}},$

<u>Step 3</u> $\quad x = c\cdot e^{t^2} + \operatorname{erf}(t)\cdot e^{t^2}, \quad c\in\mathbb{R}$ arbitrary.

Find an integrating factor for a) and b). It should be the same since both eq. have the same homogeneous part.

$\underline{x' - 2tx} = t \;\big|\cdot\cdot \qquad a(t) = -2t \qquad A(t) = -\int_0^t a(s)\,ds = 2\int_0^t s\,ds =$

$= s^2\big|_0^t = t^2$

$\mu(t) = e^{-A(t)} = e^{-t^2}$ an integrating factor

$$x' - 2tx = t \;\big|\cdot e^{-t^2} \iff x'\cdot e^{-t^2} - 2txe^{-t^2} = t\cdot e^{-t^2} \iff$$

$$\iff \left(x\cdot e^{-t^2}\right)' = te^{-t^2} \iff x\cdot e^{-t^2} = -\frac{1}{2}e^{-t^2} + c \;\Big|\cdot e^{t^2}, \quad c\in\mathbb{R} \iff$$

$$\left(\int te^{-t^2}dt = -\frac{1}{2}\int\left(e^{-t^2}\right)'dt = -\frac{1}{2}e^{-t^2}\right)$$

$$\iff \boxed{x = c\cdot e^{t^2} - \frac{1}{2}, \quad c\in\mathbb{R}.}$$

<u>Remark. on the Gauss error function</u> $\quad \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-s^2}ds.$

Since $\int_0^\infty e^{-s^2}ds = \frac{\sqrt{\pi}}{2}$ we have $\lim_{t\to\infty}\operatorname{erf}(t) = 1.$

It is easy to prove that erf is an odd function, i.e. $\operatorname{erf}(-t) = -\operatorname{erf}(t), \forall t\in\mathbb{R}$

Thus we have $\lim_{t\to-\infty} \operatorname{erf}(t) = -1$.

Also, note that $\operatorname{erf}(0)=0$, erf is a strictly increasing function (since $\operatorname{erf}'(t) = \frac{2e^{-t^2}}{\sqrt{\pi}}$)

Thus its graph looks like