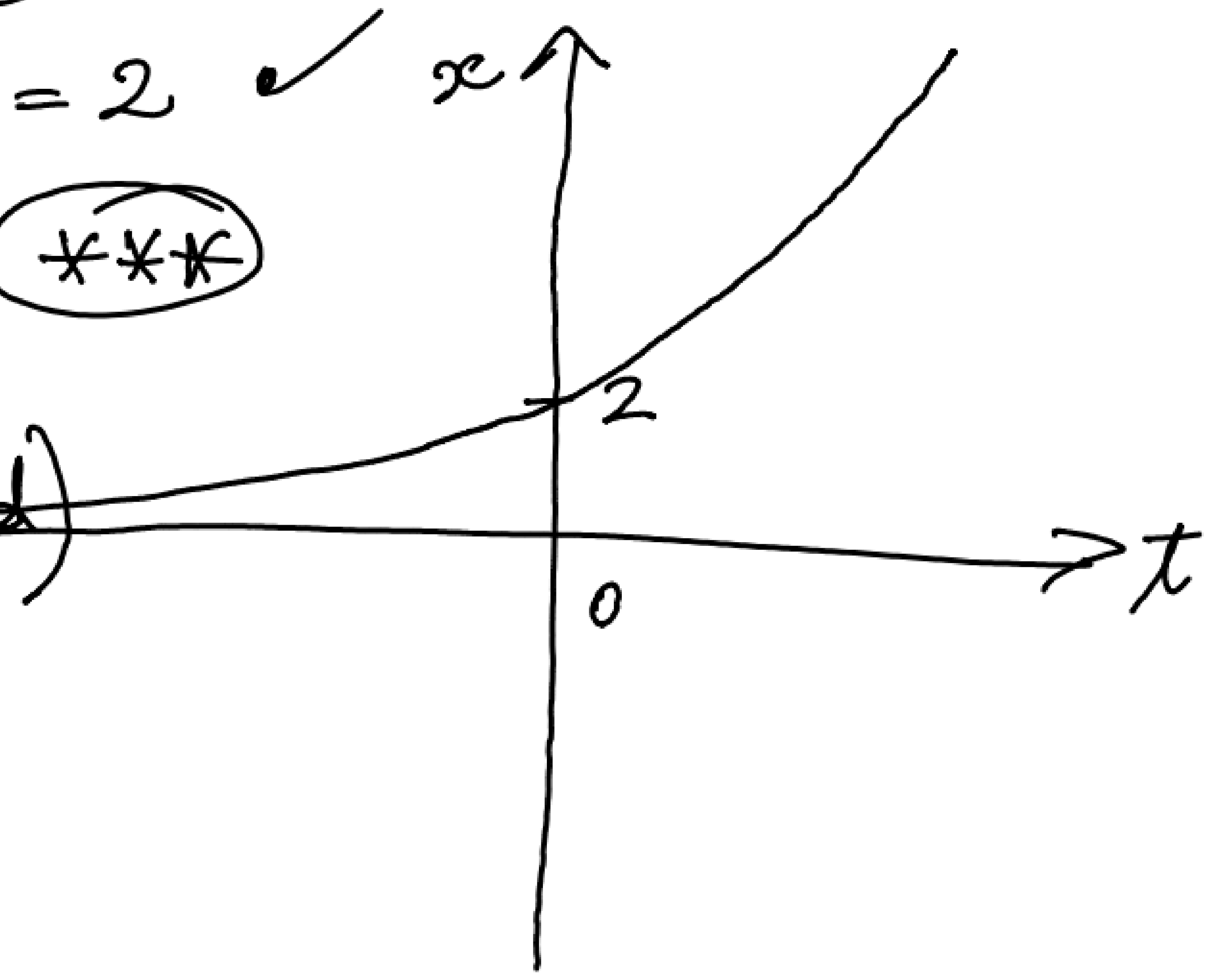
Seminar 1

$\boxed{1.1.1.}$ $\varphi(t) = 2e^{3t}$ sol of $\begin{cases} x' = 3x \\ x(0) = 2 \end{cases}$



- exponentially increasing (***)
- positive values
  (<=) 0 is a lower bound
- $\lim\limits_{t \to \infty} \varphi(t) = +\infty$
- $\lim\limits_{t \to -\infty} \varphi(t) = 0$ } (*)
- unbounded, concave (**)

Is the function :

- periodic ? : $x(t) \neq x(t+T)$ → NO
- oscillatory ? : not → NO

(**) - bounded : $\varphi(t) \geq 0$

(***) - increasing : → YES

(*) - behaviour to $\pm\infty$

Next : pendulum :

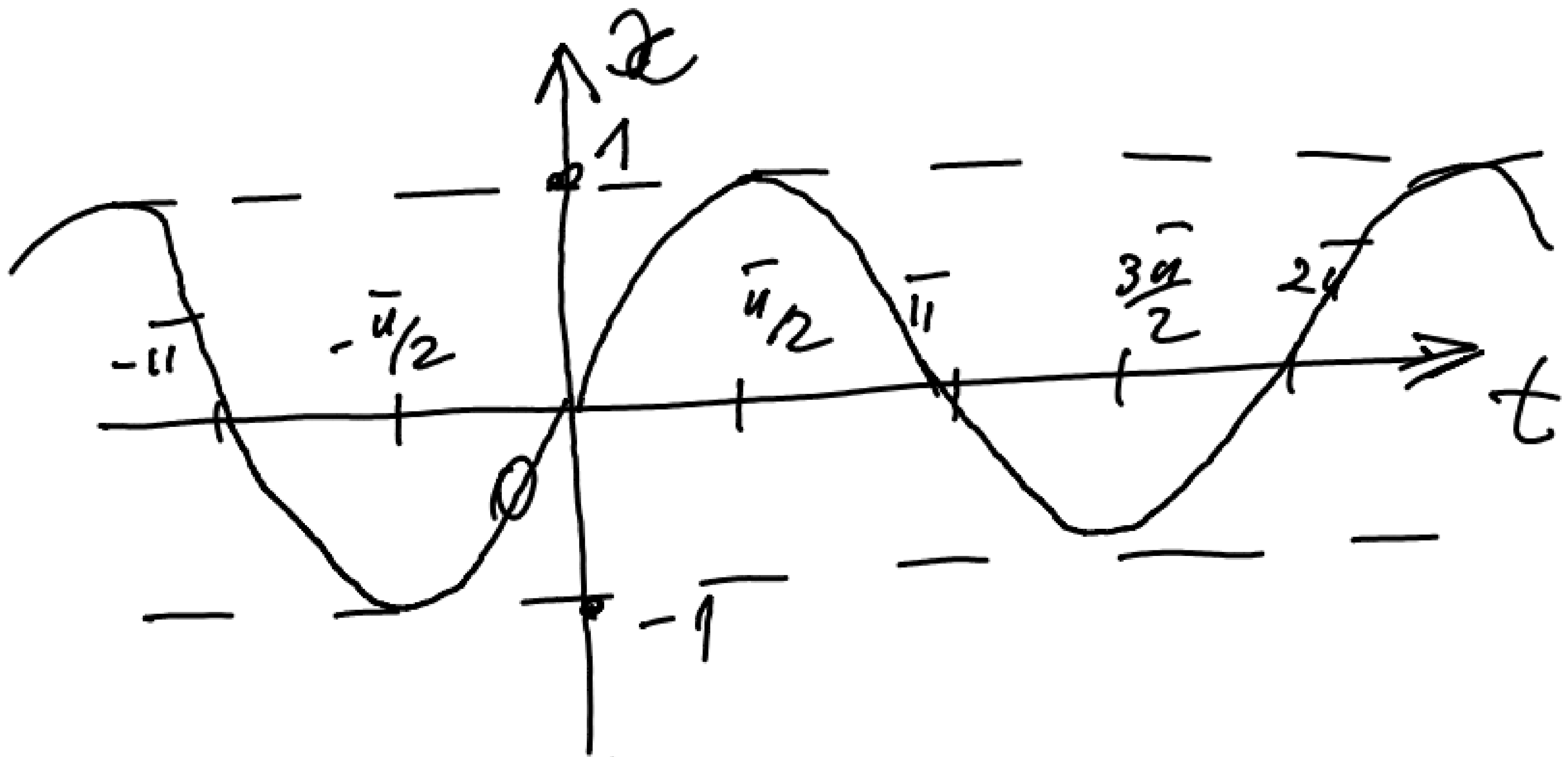
1.1.2 $\varphi(t) = \eta \cdot \sin t$ is a solution of the IVP: $\begin{cases} x'' + x = 0 \\ x(0) = 0 \\ x'(0) = \eta \end{cases}$

$\varphi'(t) = \eta \cdot \cos t$,

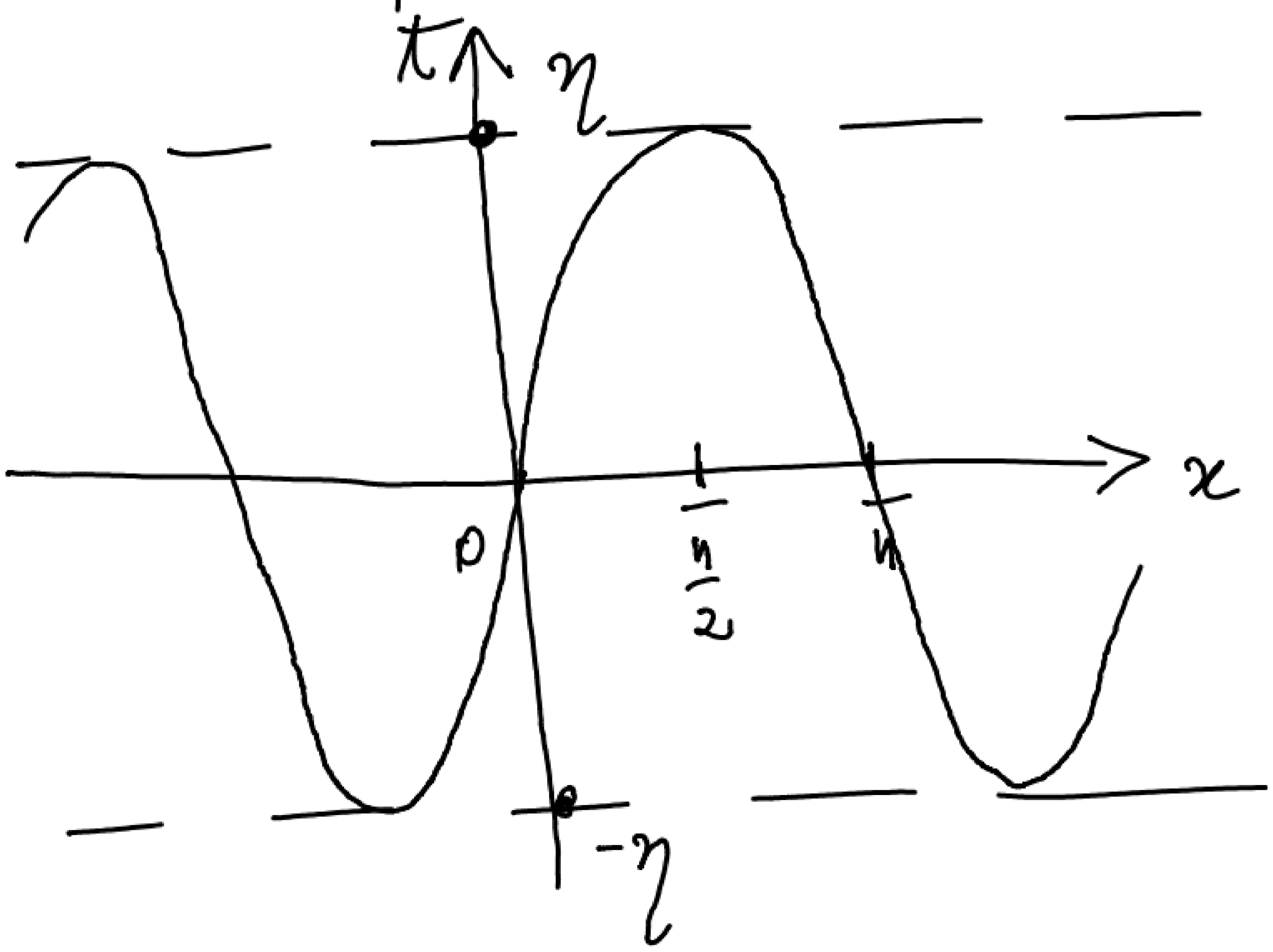
$\varphi''(t) = -\eta \cdot \sin t$

$\Rightarrow \begin{cases} \varphi''(t) + \varphi(t) = 0 & , \forall t \in \mathbb{R} \\ \varphi(0) = \eta \cdot \sin 0 = 0 \\ \varphi'(0) = \eta \cdot \cos 0 = \eta \end{cases}$

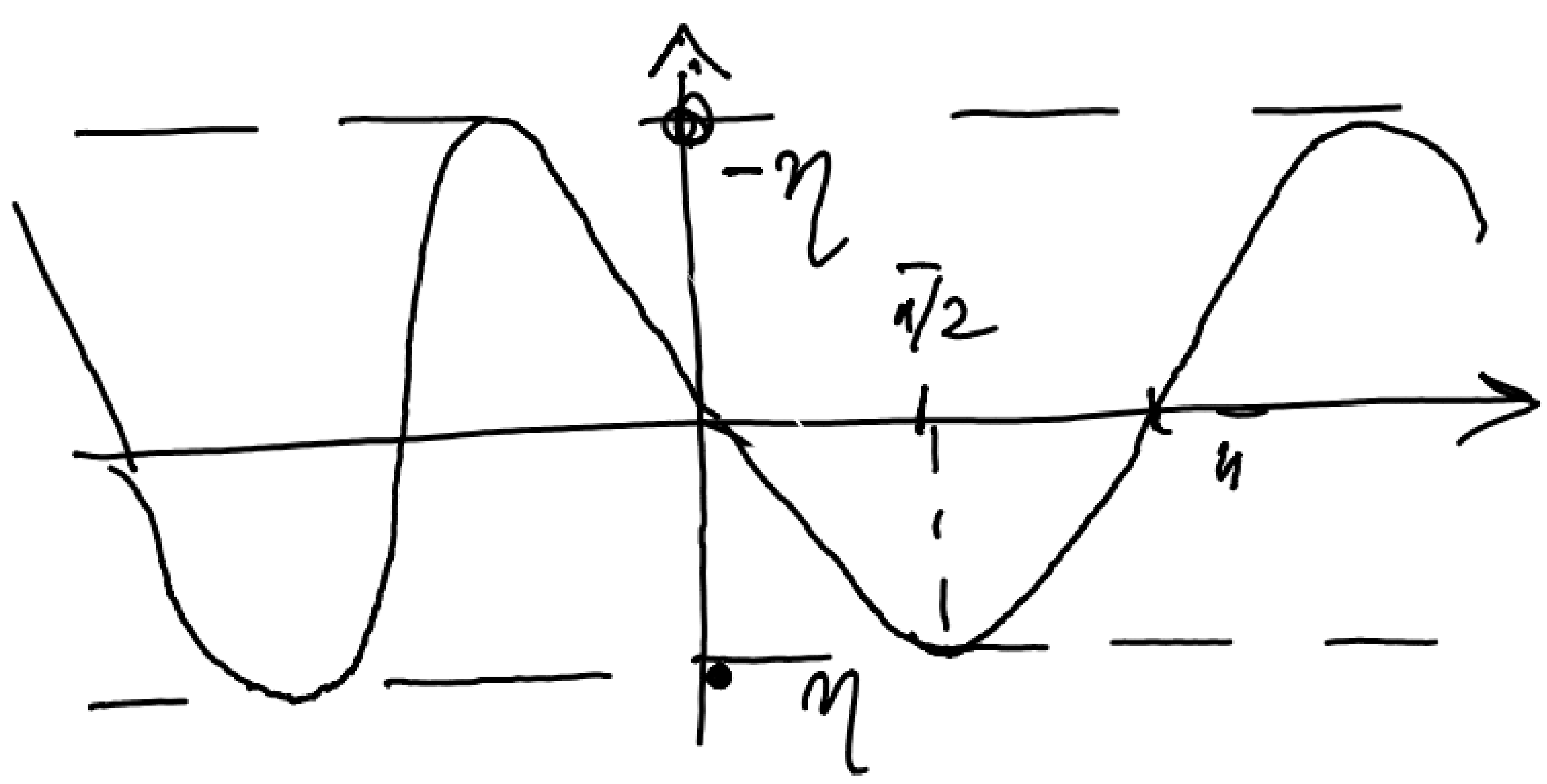
$\Rightarrow \varphi$ is a solution of the given IVP.



$\eta = 1$.



$\eta > 0$.



$\eta < 0$.

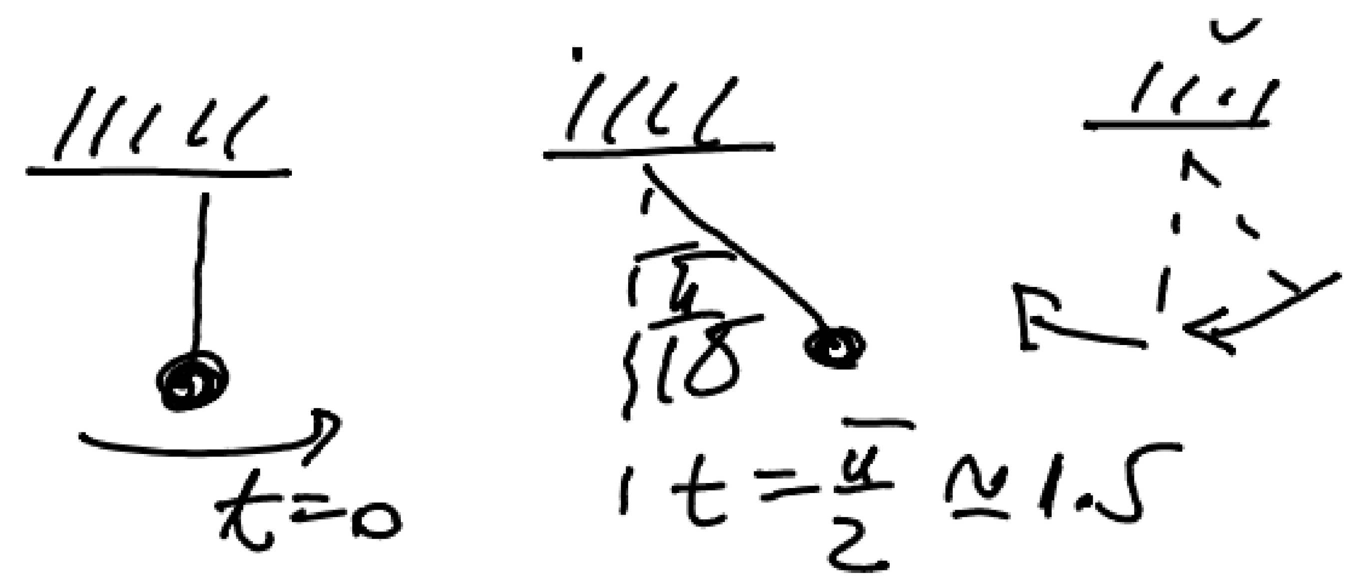
$\varphi(t)=0, \ \forall\, t\in\mathbb{R}$ $\qquad$ $\eta=0$ •

⇒ periodic with main period $T=2\pi$

⇒ bounded between $-|\eta|$ and $|\eta|$

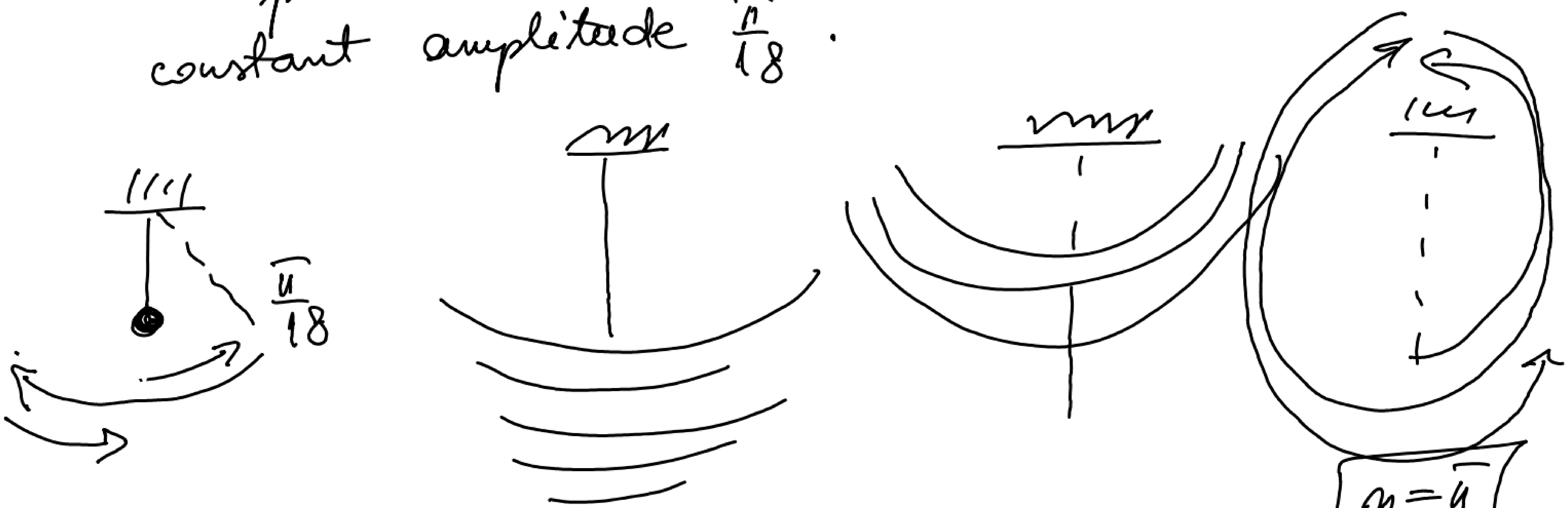
⇒ oscillatory around the value 0, amplitude $|\eta|$.

• $\eta=\frac{\pi}{18}$, $\varphi(t)=\frac{\pi}{18}\cdot\sin t$



• $t=0$, $\varphi(0)=0$, $\varphi'(0)=\frac{\pi}{18}$

pendulum oscillates around the vertical with constant amplitude $\frac{\pi}{18}$.



$\boxed{\eta=\pi}$

$\varphi(t)=\frac{3\pi}{2}\sin t$, $\varphi(t)=6\pi\sin t$ ...

...

$\boxed{1.1.3}$

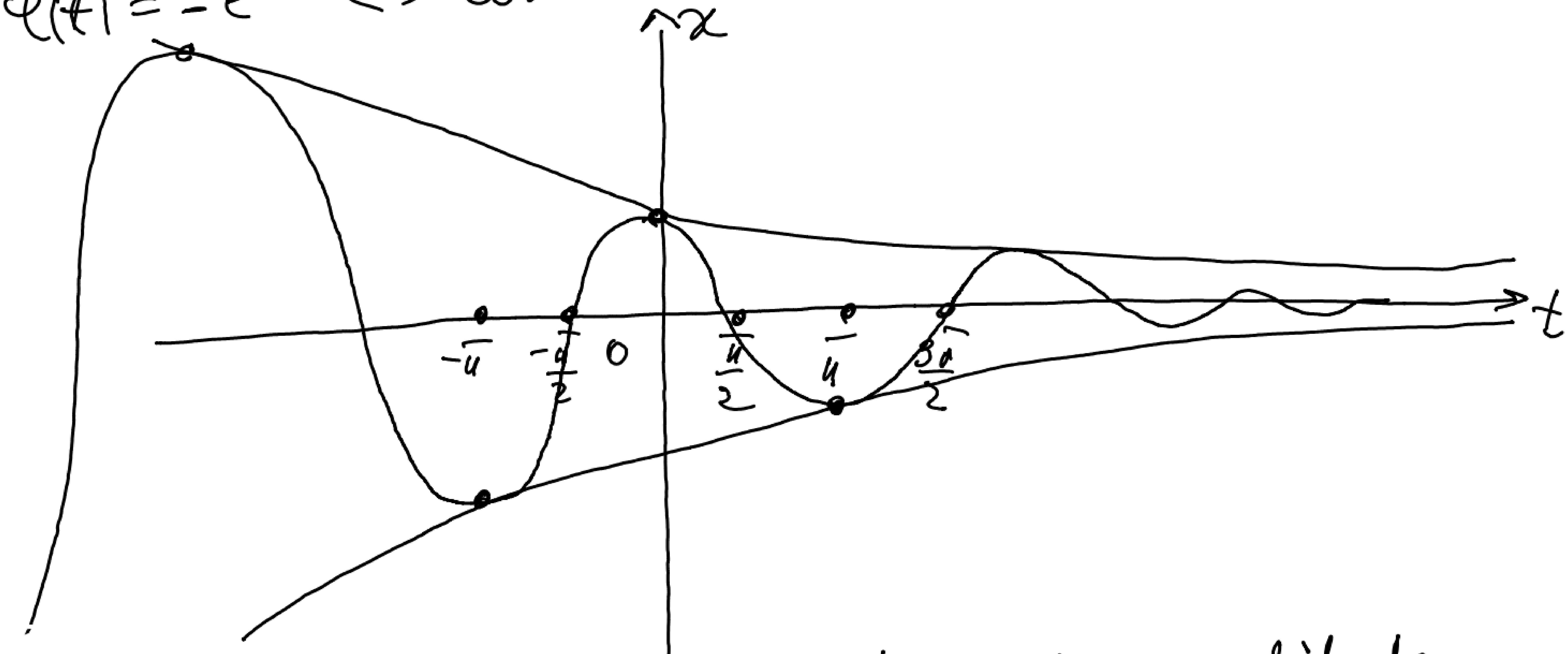
$\varphi(t) = e^{-2t} \cdot \cos t$ is a solution of: $\begin{cases} x'' + 4x' - 5x = 0. \\ x(0) = 1 \\ x'(0) = 2 \end{cases}$

→ check – Hw

$\varphi(t) = 0 \iff \underbrace{e^{-2t}}_{\neq 0} \cdot \cos t = 0 \iff \cos t = 0 \iff t_k = \frac{\pi}{2}(2k+1),\ k \in \mathbb{Z}$

$\varphi(t) = e^{-2t} \iff \cos t = 1 \iff t_k = 2k\pi,\ k \in \mathbb{Z}$

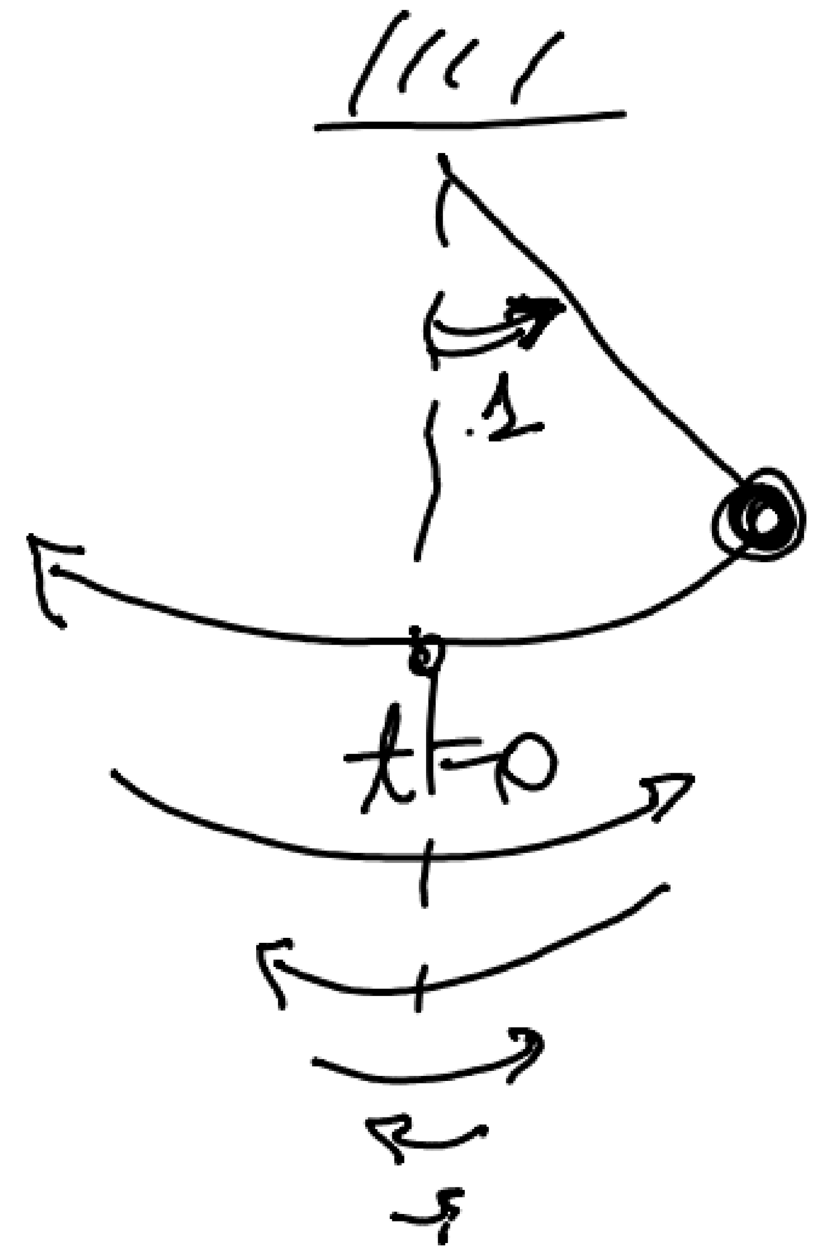
$\varphi(t) = -e^{-2t} \iff \cos t = -1 \iff t_k = (2k+1)\pi,\ k \in \mathbb{Z}$



→ oscilatory with expr decreasing amplitude.

- $\cap\, Ot$ → gives the no of zeros (here: ∞ zeros)
  - ↳ amplitude is osc. decr. for $t \to +\infty$
- $\nexists \lim_{t \to -\infty} \varphi(t)$ (φ – unbounded on $(-\infty, 0)$)
- $\exists \lim_{t \to \infty} \varphi(t) = 0.$
- $\varphi(0) = 1$ and $\varphi'(0) = -2 < 0$

<u>Notice</u>: the pendulum oscillate around the vertical with exponentially decreasing amplitude.



$\boxed{1.1.11.}$ $r \in \mathbb{R}$? such that

a) $x = e^{rt}$ is a solution of $x'' - 5x' + 6x = 0$.

$\Rightarrow x' = re^{rt}$, $x'' = r^2 e^{rt}$

$\Rightarrow r^2 e^{rt} - 5re^{rt} + 6e^{rt} = 0 \quad \forall t \in \mathbb{R}$

$r^2 - 5r + 6 = 0. \qquad (e^{rt} \neq 0).$

$\Rightarrow r_1 = 2, \ r_2 = 3$

Thus e^{2t}, e^{3t} are solutions of $x'' - 5x' + 6x = 0$.

$\xRightarrow[\text{Lect 1}]{\text{Th 2}} \quad \underline{x = c_1 \cdot e^{2t} + c_2 \cdot e^{3t}}, \quad c_1, c_2 \in \mathbb{R}$

general solution

d) $x = e^{rt}$ solution of : $x'' + 9x = 0$

$r^2 e^{rt} + 9e^{rt} = 0 \quad , \forall t \in \mathbb{R} \quad \Big| : e^{rt} \neq 0$

$r^2 + 9 = 0 \Rightarrow r_{1,2} = \pm 3i \in \mathbb{C} \setminus \mathbb{R}$

<u>Remark:</u> $\underline{\underline{e^{3it} = \cos 3t + i \sin 3t}}$

1.1.12. $r \in \mathbb{R}$ $r = ?$ s.t $x(t) = t^r$ sol of :

a) $t^2 x'' - 4 t x' + 6x = 0.$

$$x(t) = t^r \Rightarrow x'(t) = r t^{r-1} \Rightarrow x'' = r(r-1) t^{r-2}$$

$$\stackrel{eq}{\Longrightarrow} t^2 \cdot r (r-1) \cdot t^{r-2} - 4t \cdot r \cdot t^{r-1} + 6t^r = 0 \quad , \forall t \in (0, \infty)$$

$$t^r \left[r^2 - r - 4r + 6 \right] = 0 \quad , \forall t \in (0, \infty)$$

$$t^r \neq 0 \quad \Longrightarrow \quad r^2 - 5r + 6 = 0.$$

$$\Rightarrow r_1 = 2, \ r_2 = 3$$

Thus: t^2 and t^3 are solutions of eq

$\xrightarrow[\text{Lect 1}]{\text{Th 2}}$ the general solution is :

$$x = c_1 \cdot t^2 + c_2 \cdot t^3 \quad , c_1, c_2 \in \mathbb{R}.$$

1.1.9

(ii) Find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $x(t) = \alpha t^2 + \beta t + \gamma$ solution of:

(a) $x' - 5x = 2t^2 + 3$

Here: $x(t) = \alpha t^2 + \beta t + \gamma$

$x'(t) = 2\alpha t + \beta$

Replace in eq:

$2\alpha t + \beta - 5\alpha t^2 - 5\beta t - 5\gamma = 2t^2 + 3, \quad \forall t \in \mathbb{R}$

$(-5\alpha - 2)t^2 + (2\alpha - 5\beta)t + (\beta - 5\gamma - 3) = 0, \quad \forall t \in \mathbb{R}$

$\{1, t, t^2\}$ are linearly independent

$\underbrace{at^2 + bt + c}_{\text{here is a linear combination of } \{1, t, t^2\}} = 0 \quad \forall t \in \mathbb{R} \iff a = b = c = 0$

$$\Rightarrow \begin{cases} -5\alpha - 2 = 0 \\ 2\alpha - 5\beta = 0 \\ \beta - 5\gamma - 3 = 0 \end{cases} \Rightarrow \quad \alpha = -\frac{2}{5}, \quad \beta = -\frac{4}{25}, \quad \gamma = -\frac{79}{125}$$

Thus: $x = -\frac{2}{5}t^2 - \frac{4}{25}t - \frac{79}{125}$ solution.

(i)* Let $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$, $x_1(t) = 1$

$x_2(t) = t$

$x_3(t) = t^2$

Prove x_1, x_2, x_3 = linearly independent in lin sp $C(\mathbb{R})$.

$$c_1 \cdot x_1 + c_2 \cdot x_2 + c_3 \cdot x_3 = 0 \quad , \forall t$$

$$c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2 = 0.$$

- $t = 0$: $\quad c_1 = 0$
- $t = 1$: $\quad c_1 + c_2 + c_3 = 0$
- $t = 2$: $\quad c_1 + 2c_2 + 4c_3 = 0$

$$\Rightarrow \begin{cases} 2c_2 + 4c_3 = 0 \\ c_2 + c_3 = 0 \quad | \cdot 2 \end{cases} \quad \ominus$$

$$\underline{c_3 = 0}$$

$$\hookrightarrow \underline{c_2 = 0}.$$

$\boxed{1.1.7^*}$ decide whether: $\varphi:\mathbb{R}\to\mathbb{R}$, $\varphi(t)=\cos t$ is a solution of following diff eq:

$x'' + x = 0 \to$ YES

$x'' - x = 0 \to$ NO

$x''' + x'' = 0 \to$ NO

$x^{(4)} + x'' = 0 \to$ YES

$\boxed{1.1.8.^*}$ Find all constant solutions of the differential equations. ($\Leftrightarrow$ Find all $x = c$)

(a) $\left.\begin{array}{l} x' = x - x^3 \\ x = c \end{array}\right\} \Rightarrow \quad 0 = c - c^3 \Rightarrow c_{1,2} = \pm 1$, $c_3 = 0$

(b) $\left.\begin{array}{l} x' = \sin x \\ x = c \end{array}\right\} \Rightarrow 0 = \sin c \Rightarrow c = k\pi,\ k \in \mathbb{Z}$

(c) $\left.\begin{array}{l} x' = \dfrac{x+1}{2x^3+5} \\ x = c,\ x' = 0 \end{array}\right\} \Rightarrow \dfrac{c+1}{2c^3+5} = 0 \Rightarrow c = -1$

(d) $x' = x^2 + x + 1 \Rightarrow c^2 + c - 1 = 0.$ $\to$ No sol

$\Delta < 0$

(e) $x' = x + 4x^3 \Rightarrow 0 = c + 4c^3 \Rightarrow c = 0.$

(f) $\left.\begin{array}{l} x' = -1 + x + 4x^3 \\ c = 0 \end{array}\right\} \Rightarrow \dots = c = \frac{1}{2}$

...

$\boxed{1.1.10^*}$

(i) Let $x_1, x_2, x_3 : \mathbb{R} \to \mathbb{R}$, $x_1(t) = \cos t$

$x_2(t) = \sin t$

$x_3(t) = e^t$, $\forall t \in \mathbb{R}$

Prove that x_1, x_2, x_3 – linearly indep in $C(\mathbb{R})$.

<u>Solution</u>:

$c_1 \cdot \cos t + c_2 \cdot \sin t + c_3 \cdot e^t = 0$, $\forall t$

- $t = 0$: $c_1 + 0 + c_3 = 0$
- $t = \frac{\pi}{2}$: $0 + c_2 + c_3 \cdot e^{\pi/2} = 0$
- $t = \pi$: $-c_1 + c_3 e^{\pi} = 0$

$\Rightarrow c_3 = 0$

$\Rightarrow c_1 = 0$

$\Rightarrow c_2 = 0$

$\Downarrow$

lin. indep.

$\boxed{1.1.15^*}$ Find an integrating factor, integrate the eq:

(a)* $x' + x = 0 \quad | \cdot e^t$

$x' \cdot e^t + x \cdot e^t = 0.$

$(x \cdot e^t)' = 0. \quad \Rightarrow x \cdot e^t = C \quad \Rightarrow x = C \cdot e^{-t}, \; C \in \mathbb{R}$

(b)* $x' + x = 1 + t \quad | \cdot e^t$

$(x \cdot e^t)' = (1+t) \cdot e^t$

$x \cdot e^t = \int e^t (1+t)\, dt$ $\qquad g' = e^t \Rightarrow g = e^t$

$x \cdot e^t = e^t + \int t e^t dx$ $\qquad f = t \Rightarrow f' = 1$

$x \cdot e^t = e^t + t e^t - \int e^t dt$

$x e^t = \cancel{e^t} + t e^t - \cancel{e^t} + c \quad | : e^t$

$\underline{x = t + C \cdot e^{-t}, \quad C \in \mathbb{R}}$

(c)* $x' + 2x = \sin t \quad | \cdot e^{2t}$

$x' \cdot e^{2t} + 2x e^{2t} = \sin t \cdot e^{2t}$

$(x \cdot e^{2t})' = \sin t \cdot e^{2t} \quad | \int dt$

$x \cdot e^{2t} = \int \sin t \cdot e^{2t} dt$

$I = \int \sin t \, e^{2t} dt = \frac{1}{2} \int \sin t \cdot (e^{2t})' dt =$

$= \frac{1}{2} \sin t \cdot e^{2t} - \frac{1}{2} \int \cos t \cdot e^{2t} dt =$

$= \frac{1}{2} \sin t \cdot e^{2t} - \frac{1}{4} \cos t\, e^{2t} + \frac{-1}{4} \int \sin t \cdot e^{2t} dt$

$\Rightarrow \frac{5}{4} \cdot y = \frac{1}{2} \sin t\, e^{2t} - \frac{1}{4} \cos t\, e^{2t} + C$

$\Rightarrow x = \frac{2}{5} \sin t - \frac{1}{5} \cos t\, e^{2t} + C \cdot e^{-2t}$

(e)* $t x' + 2x = 1 \quad \Big| \cdot t$

$t^2 \cdot x' + 2tx = t$

$(t^2 \cdot x)' = t \quad \Big| \int dt$

$t^2 \cdot x = \int t\, dt$

$t^2 x = \frac{t^2}{2} + C$

$x = \frac{1}{2} + C \cdot \frac{1}{t^2} \quad , C \in \mathbb{R}$