18.04.2022

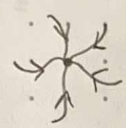
Stability of equilibria of planar systems

(1) $\dot{x} = f(x)$, where $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$

Let $\eta^* \in \mathbb{R}^2$ be an equilibrium point of (1) (i.e. $f(\eta^*) = 0_2$)

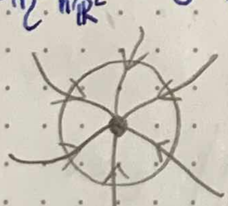
($\forall\, \eta \in \mathbb{R}^2$ $\quad t \mapsto \varphi(t, \eta)$ the unique sol. of the IVP : $\begin{cases} \dot{x} = f(x) \\ x(0) = \eta \end{cases}$)

Def: 1) η^* is an attractor of (1) when $\exists\, V \in \mathcal{V}(\eta^*)$ s.t. $\forall\, \eta \in V$ $\lim\limits_{t \to \infty} \varphi(t, \eta) = \eta^*$



2) η^* is a repeller when $\exists\, V \in \mathcal{V}(\eta^*)$ s.t. $\forall\, \eta \in V$, $\lim\limits_{t \to -\infty} \varphi(t, \eta) = \eta^*$

3) η^* is stable when $\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t. $\forall\, \eta \in \mathbb{R}^2$ with $\|\eta - \eta^*\|_{\mathbb{R}^2} < \delta$ we have $\| \varphi(t, \eta) - \underbrace{\eta^*}_{\varphi(t, \eta^*)} \|_{\mathbb{R}^2} < \varepsilon$, $\forall\, t \in [0, \infty)$



$\delta = \varepsilon$

4) η^* is unstable when it's not stable.

§1. Stability of linear planar systems.

§2. The linearization method to study the stability of an equilibrium point of a nonlinear systems.

§3. Examples

---

§1. (2) $X = AX$, where $A \in \mathcal{M}_2(\mathbb{R})$ $\det A \neq 0$.

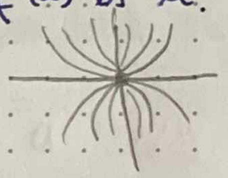
Let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of A.

! 1) $\det A = \lambda_1 \cdot \lambda_2$

2) $\det A \neq 0 \iff \eta^* = 0_2 \in \mathbb{R}^2$ is the unique equil. point of (2)

! 3) $\det A \neq 0 \iff \lambda_1 \neq 0$ and $\lambda_2 \neq 0$

Def: (The type of a linear system). We say that 0_2 of (2) is a:

→ NODE when either $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$ 

→ SADDLE when $\lambda_1 < 0 < \lambda_2$ 

→ CENTER when $\lambda_{1,2} = \pm i\beta$, $\beta \in \mathbb{R}^*$ 

→ FOCUS when $\lambda_{1,2} = \alpha \pm i\beta$, $\alpha, \beta \in \mathbb{R}^*$, $\alpha \neq 0$. 

$\boxed{T_1}$ 1) If $Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$ then $\eta^* = 0_2$ of (2) is a global attractor.

2) If $Re(\lambda_1) > 0$ and $Re(\lambda_2) > 0$ then $\eta^* = 0_2$ of (2) is a global repeller.

3) A center is stable. All the sol. of (2) with a center is a periodic function.

4) A saddle is unstable.

§2. $\dot{x} = f(x)$

$P(x) = f(\eta^*) + \underbrace{f'(\eta^*)}_{?\ \text{Jacobian matrix}}(x - \eta^*)$

for a function $f \in C^1(\mathbb{R}^2, \mathbb{R}^2)$, its Jacobian matrix in $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

$$Jf(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{pmatrix}$$

$f(x) \longrightarrow \underline{Jf(\eta^*)}(x - \eta^*)$

(3) $\dot{x} = Jf(\eta^*)\, X$

Def: 1) The linear system (3) is called the linearization of (1) around the equil. point η^*.

2) Assume that $\det Jf(\eta^*) \neq 0$. When the eq. point O_2 of (3) is a NODE / SADDLE / CENTER / FOCUS, we say that the eq. point ~~of~~ η^* of (1) is a LINEAR NODE / LINEAR SADDLE / LINEAR CENTER / LINEAR FOCUS.

3) Let $\lambda_1, \lambda_1 \in \mathbb{C}$ be the eigenvalues of $Jf(\eta^*)$. We say that η^* is a hyperbolic equil. point of (1) when:

$$Re(\lambda_1) \neq 0 \text{ and } Re(\lambda_2) \neq 0.$$

! 1) If $\det Jf(\eta^*) = 0$ then η^* is not hyperbolic.

2) Assume that $\det Jf(\eta^*) \neq 0$.

If η^* is a <u>linear center</u> then η^* is not hyperbolic.

In the rest of the cases, it is hyperbolic.

$\boxed{T_2}$ Let η^* be a hyperbolic equil. point of the nonlinear system (1).

If η^* is a <u>LINEAR ATTRACTING NODE / FOCUS</u>, then η^* is an attractor for the nonlinear system (1).

If η^* is a <u>LINEAR REPELLING NODE / FOCUS</u>, then η^* is a repeller for the nonlinear system (1).

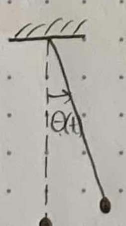
If η^* is a <u>LINEAR SADDLE</u>, then η^* is an unstable equil. point of (1).

! If we have to study the stability of the equilibria of a second order scalar diff. eq.: $\ddot{x} = f(x, \dot{x})$. We just have to ~~the~~ write the equivalent planar system with unknowns x and $y = \dot{x}$. This system is:

$$\begin{cases} \dot{x} = y \\ \dot{y} = f(x, y) \end{cases}$$

§3. Examples

1) The idealized ~~pen~~ pendulum equation.



$$\ddot{\theta} + \nu \dot{\theta} + \underbrace{\sin\theta}_{\substack{\text{second order} \\ \text{nonlinear diff. eq.}}} = 0$$

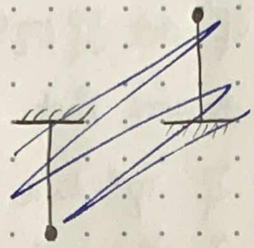
$\nu > 0$ damping constant

<u>$\nu = 0$ ideal case</u>

$\ddot{\theta} + \sin\theta = 0$ $\quad x = \theta$ $\quad y = \dot{\theta}$

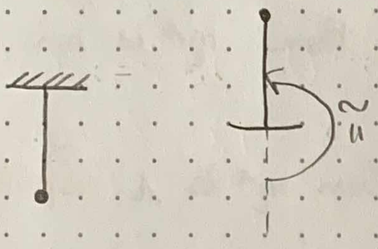
$$\begin{cases} \dot{x} = y \\ \dot{y} = -\sin x \end{cases}$$

Nonlinear planar system



Study the stability of the equilibria.

$$f(x,y) = \begin{pmatrix} y \\ -\sin x \end{pmatrix} \quad f \in C^1(\mathbb{R}^2;\mathbb{R}^2)$$



look for the equilibria: $f(x,y) = 0 \iff \begin{cases} y = 0 \\ -\sin x = 0 \end{cases} \Rightarrow$

$$\Rightarrow \begin{cases} x = k\pi \\ y = 0 \end{cases}, k \in \mathbb{Z}$$

$(k\pi, 0)$, $k \in \mathbb{Z}$

Physically we have 2 equil. points: $\eta_1^* = (0,0)$ and $\eta_2^* = (\pi, 0)$.
Since our system is nonlinear, we apply the lin. method.

$$Jf = \begin{pmatrix} 0 & 1 \\ -\cos x & 0 \end{pmatrix} \quad \boxed{\eta_2^* = (\pi, 0)}, \quad Jf(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \iff \lambda^2 - 1 = 0 \iff \lambda^2 = 1 \Rightarrow$$

$\Rightarrow \lambda_1 = 1$, $\lambda_2 = -1$

$\Rightarrow$ η_2^* is a linear saddle, thus it is hyperbolic $\overset{Thm}{\Longrightarrow}$ $\eta^* = (\pi, 0)$ is UNSTABLE.

A the eq. of the eigenvalues: $\det(A - \lambda I_n) = 0$

$$\boxed{\eta_1^* = (0,0)}, \quad Jf(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda_{1,2} = \pm i \Rightarrow$$

η_1^* is a linear center, it's not hyperbolic $\Rightarrow$
$\Rightarrow$ the LM fails

$\boxed{T_3}$ If η^* is a LINEAR CENTER and there exists a first integral of (1) well defined in a neighbourhood of η^* then η^* is a stable equil. point of (1).

$$\frac{dy}{dx} = \frac{-\sin x}{y} \iff y\,dy = -\sin x\,dx \overset{\int}{\iff} \frac{y^2}{2} = \cos x + c,\ c \in \mathbb{R}$$

$$H(x,y) = \frac{1}{2}y^2 - \cos x$$

$H : \mathbb{R}^2 \to \mathbb{R}$, C^1 check : " $\frac{\partial H}{\partial x} \cdot y + \frac{\partial H}{\partial y} \cdot (-\sin x) = 0$

$\forall (x,y) \in \mathbb{R}^2$

$$\left.\begin{array}{l} \Rightarrow H - \text{global first integral} \\ \eta^* \text{ is a linear center} \end{array}\right\} \overset{T_3}{\Longrightarrow} \underline{\eta^* \text{ is stable}}$$

BRUNNEN