Lecture 10

- Trees
  - K-ary trees
- Binary tree
  - Properties
  - ADT
  - Traversals: recursive, non-recursive
  - Iterator

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2021 - 2022

Some slides borrowed from: Lect. PhD. Onet-Marian Zsuzsanna

Trees

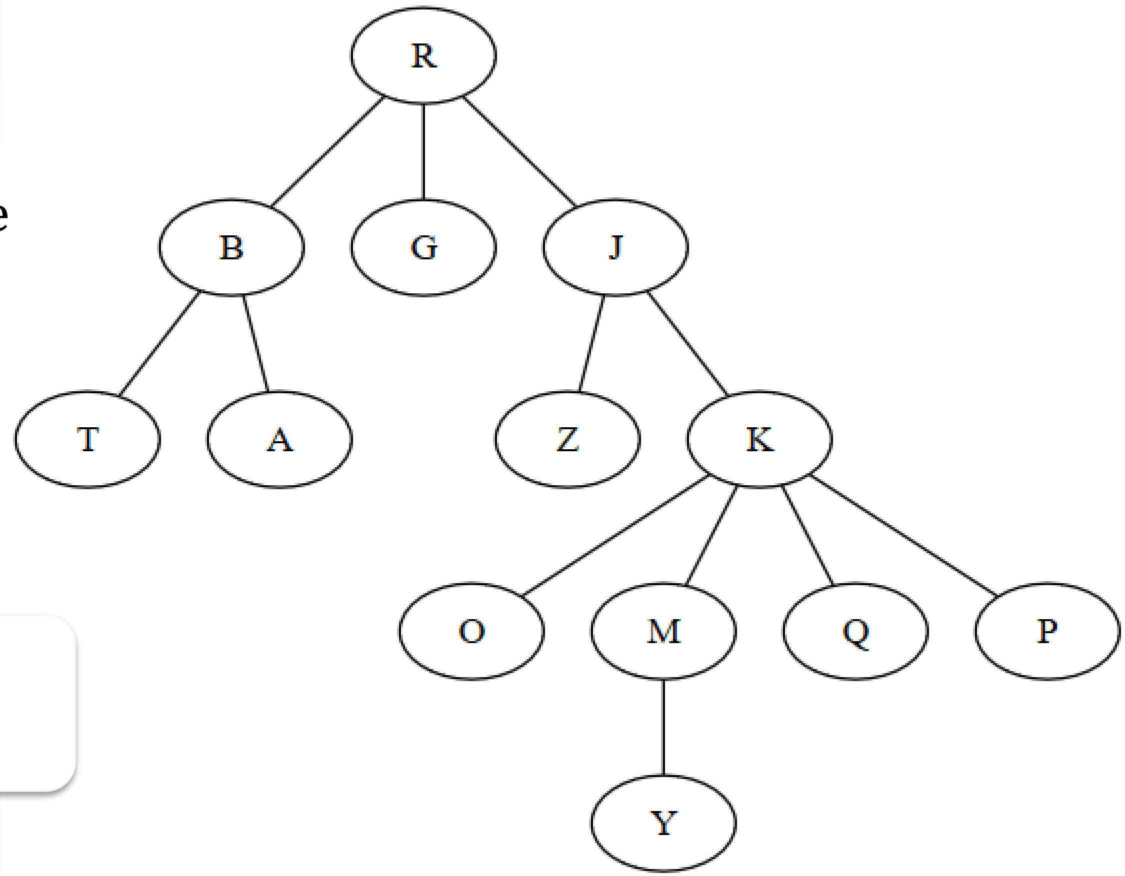
A tree is a finite set T of 0 or more nodes, with the following properties:

- If T is empty, then the tree is empty
- If T is not empty then:
  - There is a special node, R, called the root of the tree
  - The rest of the nodes are divided into k ($k \geq 0$) disjunct trees, T1, T2, ..., Tk The trees T1, T2, ..., Tk are called the subtrees (children) of R, and R is called the parent of the subtrees.

Trees - - terminology

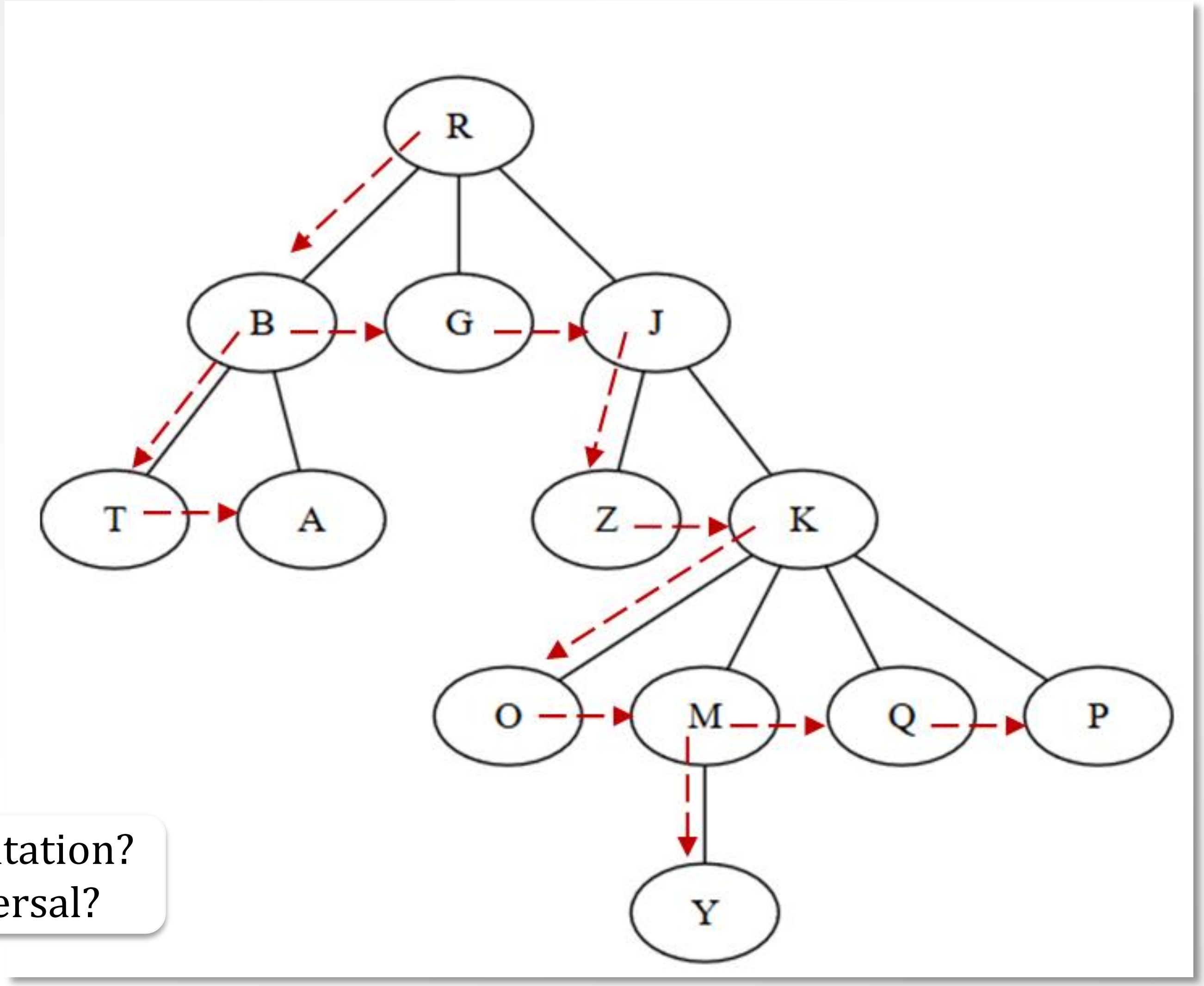
- Rooted tree, ordered tree

- Root, child, parent;
- leaf nodes, internal node;
- The degree of a node
- The depth or level of a node
- The height of a node, the height of the tree



- Depth of the root of the tree ?
- Height of a leaf?

K-ary trees



Representation?
Tree traversal?

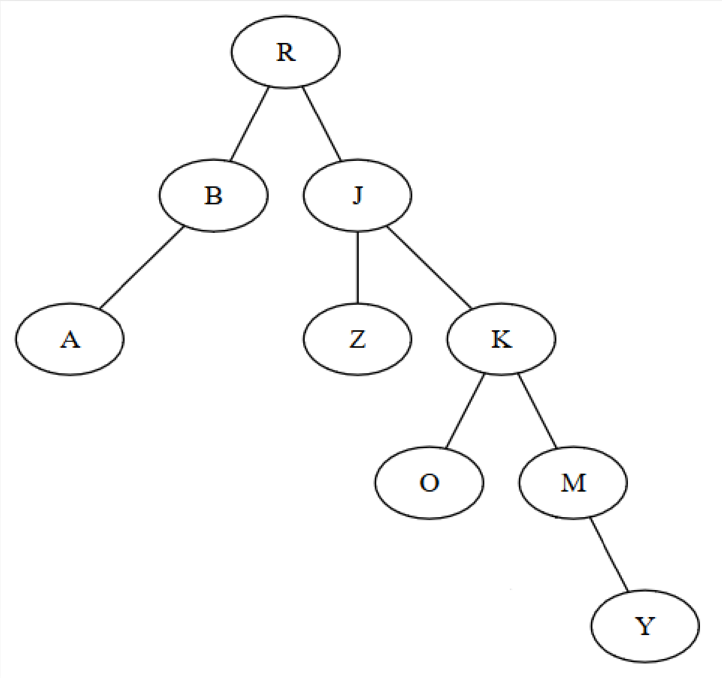
Binary trees

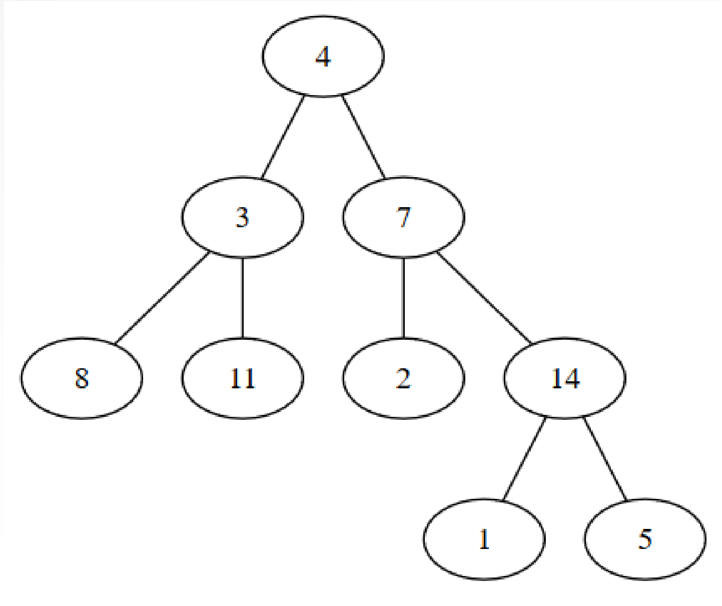
Binary tree: a tree in which each node has at most two children.

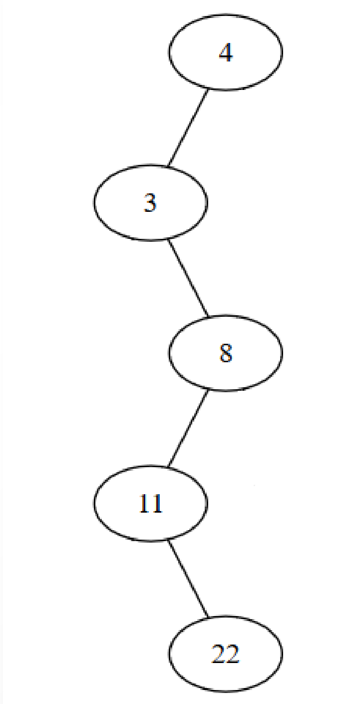
- In a binary tree we call the children of a node the left child and right child.
- Even if a node has only one child, we still have to know whether that is the left or the right one.

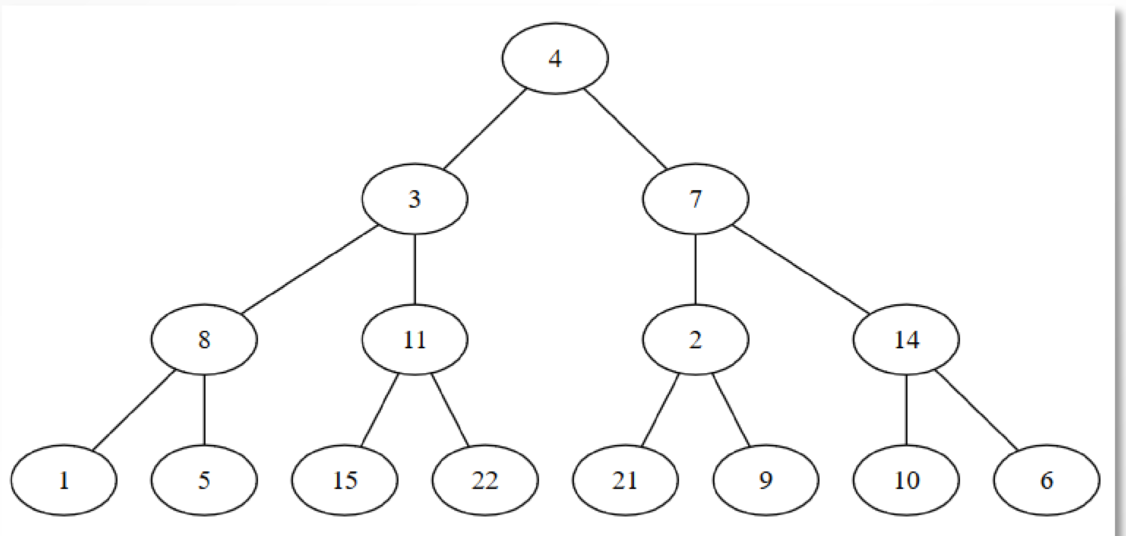
- A binary tree is called **full** if every internal node has exactly two children.
- A binary tree is called **complete** if all leaves are one the same level and all internal nodes have exactly 2 children.
- A binary tree is called **almost complete** if it is a complete binary tree except for the last level, where nodes are completed from left to right (binary heap - structure).
- A binary tree is called **balanced** if the difference between the height of the left and right subtrees of a node is at most 1.

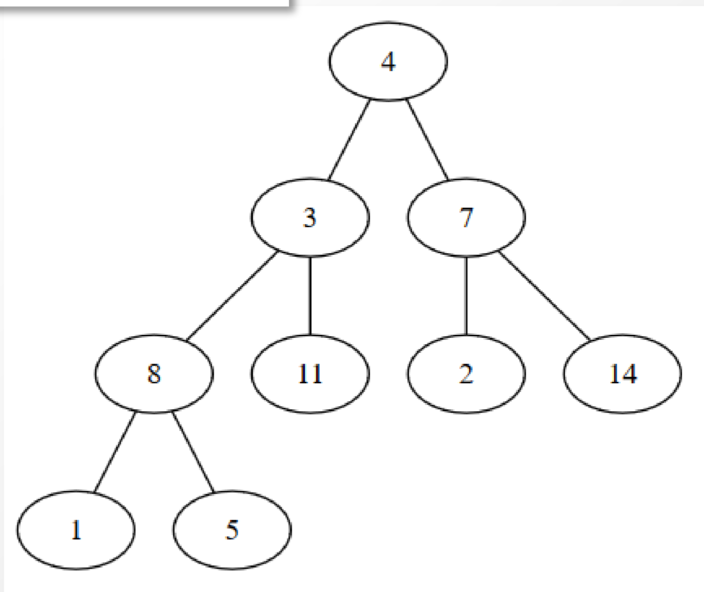
Binary trees











Binary tree - properties

- A binary tree with N nodes has exactly N - 1 edges
  (this is true for every tree, not just binary trees)

- The number of nodes in a complete binary tree of height $\boldsymbol{h}$ is $2^{\boldsymbol{h}+1} - 1$
  (it is $1 + 2 + 4 + 8 + \ldots + 2^{\boldsymbol{h}}$)
- The maximum number of nodes in a binary tree of height $\boldsymbol{h}$ is $2^{\boldsymbol{h}+1} - 1$,
  if the tree is complete.
- The minimum number of nodes in a binary tree of height $\boldsymbol{h}$ is $\boldsymbol{h} + 1$,
  if the tree is degenerate.
- A binary tree with N nodes has a height between $\log_2 N$ and N - 1.

ADT Binary Tree

$\mathcal{BT}$ = {*bt* | *bt* binary tree with nodes containing information of type TElem}

- init(bt)
- destroy(bt)

- isEmpty(bt)

- iterator (bt, traversal, i)

- root(bt)
  - **descr:** returns the information from the root of a binary tree
  - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
  - **post:** $root \leftarrow e$, $e \in TElem$, e is the information from the root of bt
  - **throws:** an exception if bt is empty

- initLeaf(bt, e)
  - **descr:** creates a new binary tree, having only the root with a given value
  - **pre:** $e \in TElem$
  - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with only one node (its root) which contains the value e

- left(bt)
- right(bt)
  - **descr:** returns the right subtree of a binary tree
  - **pre:** $bt \in \mathcal{BT}$, $bt \neq \Phi$
  - **post:** $right \leftarrow r$, $r \in \mathcal{BT}$, r is the right subtree of bt
  - **throws:** an exception if bt is empty

- initTree(bt, left, e, right)
  - **descr:** creates a new binary tree, having a given information in the root and two given binary trees as children
  - **pre:** $left, right \in \mathcal{BT}$, $e \in TElem$
  - **post:** $bt \in \mathcal{BT}$, bt is a binary tree with left child equal to $left$, right child equal to $right$ and the information from the root is e

- insertLeftSubtree(bt, left)
- insertRightSubtree(bt, right)
  - **descr:** sets the right subtree of a binary tree to a given value (if the tree had a right subtree, it will be changed)
  - **pre:** $bt, right \in \mathcal{BT}$
  - **post:** $bt' \in \mathcal{BT}$, the right subtree of bt' is equal to $right$

Binary tree - representation

- Representation using an array, similar to a binary heap

  Disadvantage:

  depending on the form of the tree, we might waste a lot of space**.**

- Linked representation
  - with dynamic allocation
  - on an array

<u>BTNode</u>:
    info: TElem
    left: ↑ BTNode
    right: ↑ BTNode

BinaryTree:
    root: ↑ BTNode

Binary tree - traversals

Preorder traversal:

- Visit the root of the tree
- Traverse the left subtree - if exists
- Traverse the right subtree - if exists

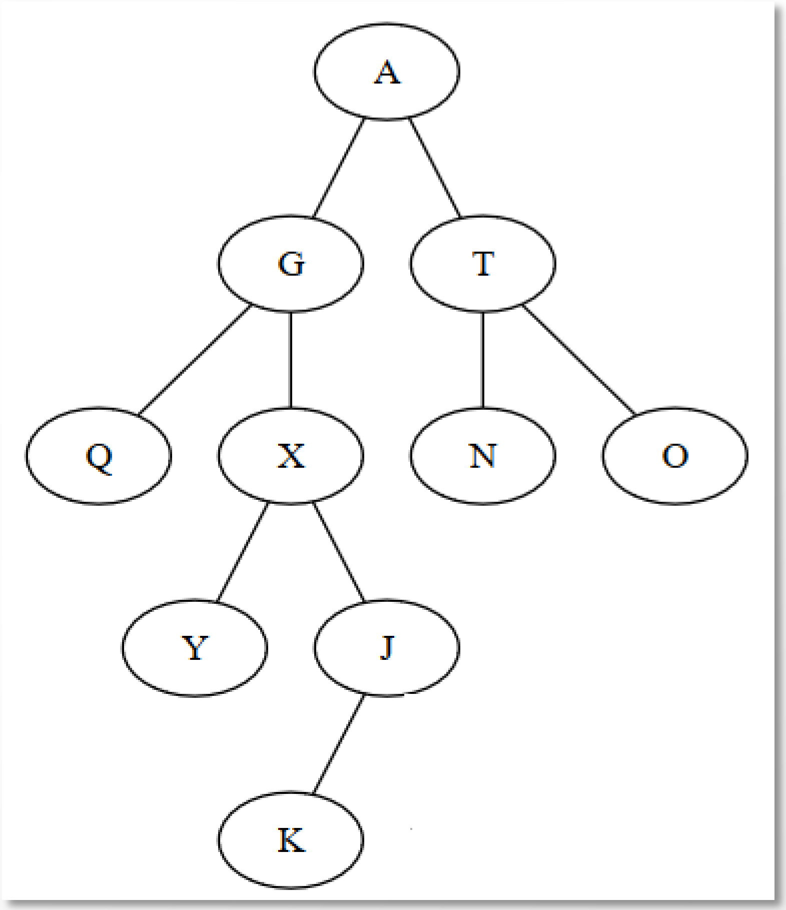
Inorder traversal:

- Traverse the left subtree - if exists
- Visit the root of the tree
- Traverse the right subtree - if exists

Postorder traversal:

- Traverse the left subtree - if exists
- Traverse the right subtree - if exists
- Visit the root of the tree

Traversal on levels

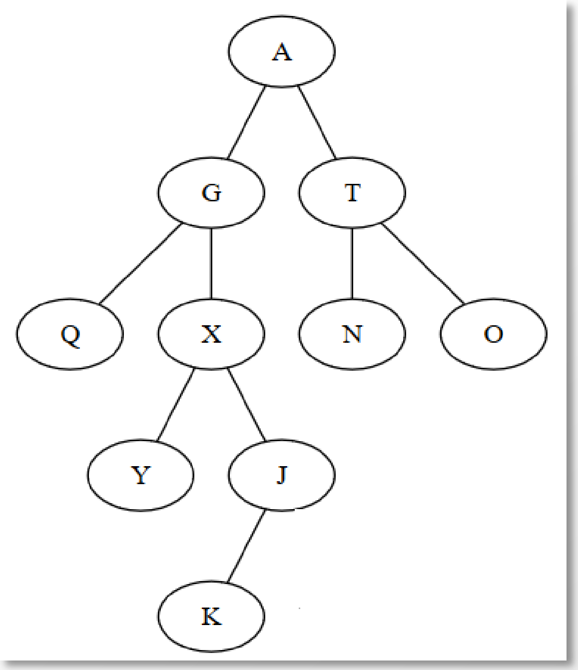


Preorder traversal - recursive implementation

subalgorithm preorderRec(tree) is:
  preorder_recursive(tree.root)
end-subalgorithm

subalgorithm preorder_recursive(node) is:
  if node ≠ NIL then
    @ visit [node].info
    preorder_recursive([node].left)
    preorder_recursive([node].right)
  end-if
end-subalgorithm

The traversal takes Θ(n) time for a tree with n nodes.



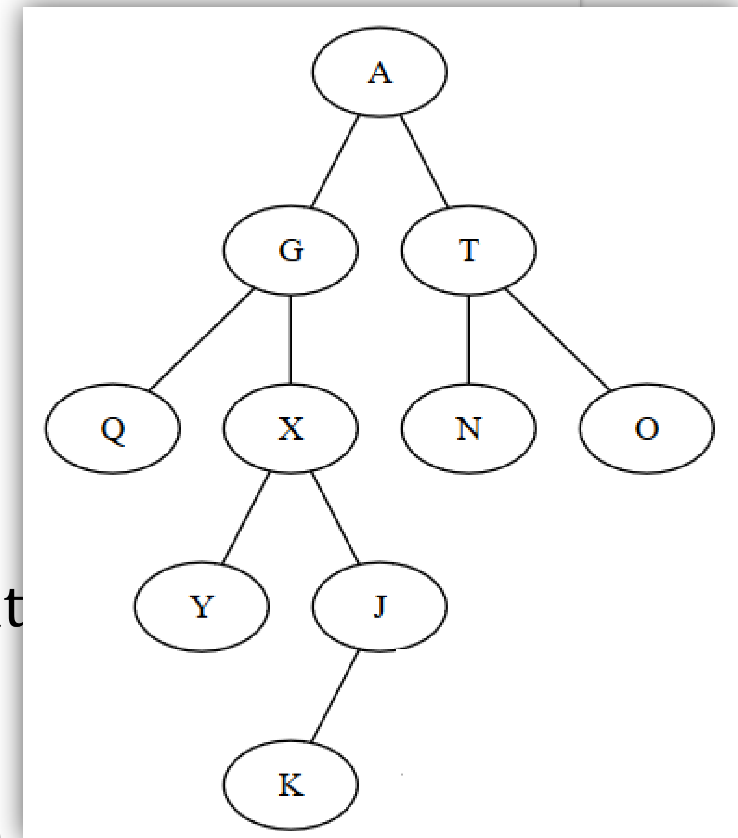
How can we implement inoder and postorder traversals?

On-level traversal

```
subalgorithm onLevel(tree) is:
  init(q)          // q is an auxiliary queue
  if tree.root ≠ NIL then
        push(q, tree.root)
  end-if
  while not isEmpty(q) execute
        currentNode ← pop(q)
        @visit currentNode
        if [currentNode].left ≠ NIL then
              push(q, [currentNode].left)
        end-if
        if [currentNode].right ≠ NIL then
              push(q, [currentNode].right)
        end-if
  end-while
end-subalgorithm
```

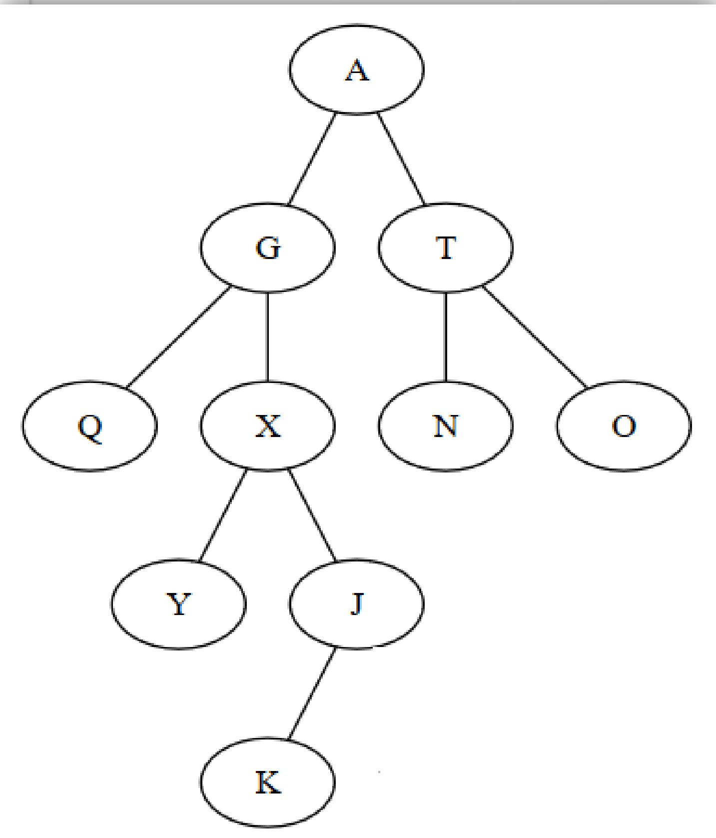
Preorder traversal - non-recursive implementation

```
subalgorithm preorder(tree) is:
  init(s)          //s is an auxiliary stack
  if tree.root ≠ NIL then
        push(s, tree.root)
  end-if
  while not isEmpty(s) execute
        currentNode ← pop(s)
        @visit currentNode
        if [currentNode].right ≠ NIL then
              push(s, [currentNode].right
        end-if
        if [currentNode].left ≠ NIL then
              push(s, [currentNode].left)
        end-if
  end-while
end-subalgorithm
```



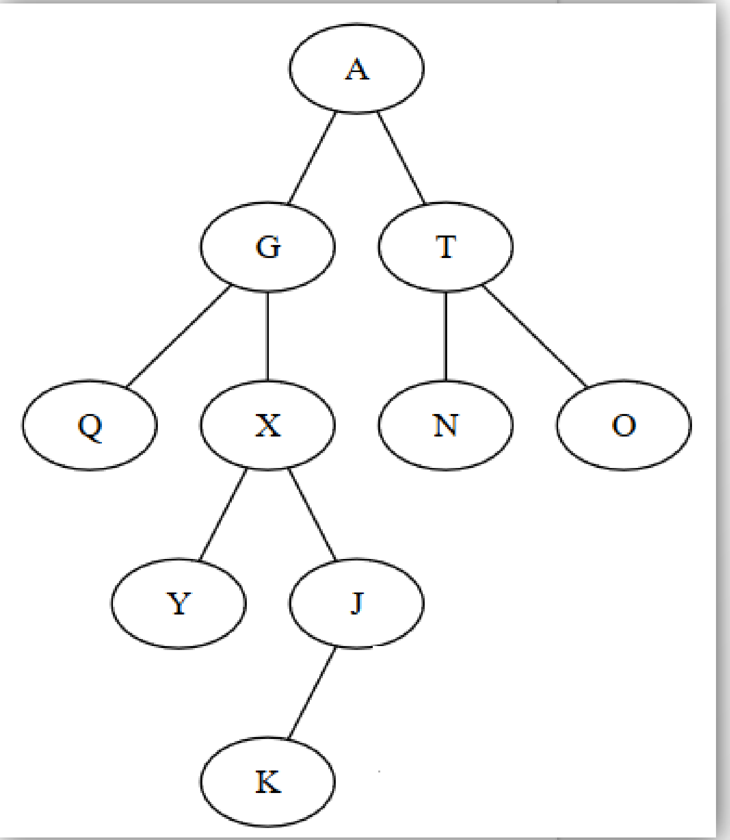
Inorder traversal - non-recursive implementation

```
subalgorithm inorder(tree) is:
    init(s)          //s is an auxiliary stack
    currentNode ← tree.root
    while currentNode ≠ NIL execute
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
    while not isEmpty(s) execute
        currentNode ← pop(s)
        @visit currentNode
        currentNode ← [currentNode].right
        while currentNode ≠ NIL execute
            push(s, currentNode)
            currentNode ← [currentNode].left
        end-while
    end-while
end-subalgorithm
```



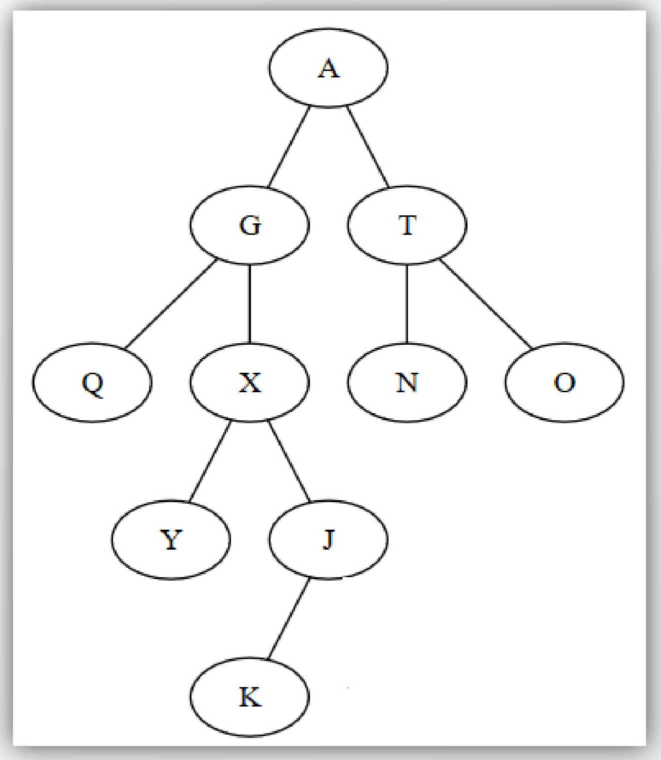
Postorder traversal - non-recursive , 2 stacks

```
subalgorithm preorder(tree) is:
  init(s1); init(s2)
  if tree.root ≠ NIL then
          push(s1, tree.root)
  end-if
  while not isEmpty(s1) execute
          currentNode ← pop(s1)
          push(s2, currentNode)
          if [currentNode].left ≠ NIL then
                  push(s1, [currentNode].left)
          end-if
          if [currentNode].right ≠ NIL then
                  push(s1, [currentNode].right)
          end-if
  end-while
  while not isEmpty(s2) execute
          currentNode ← pop(s2)
          @ visit    currentNode
  end-while
end-subalgorithm
```



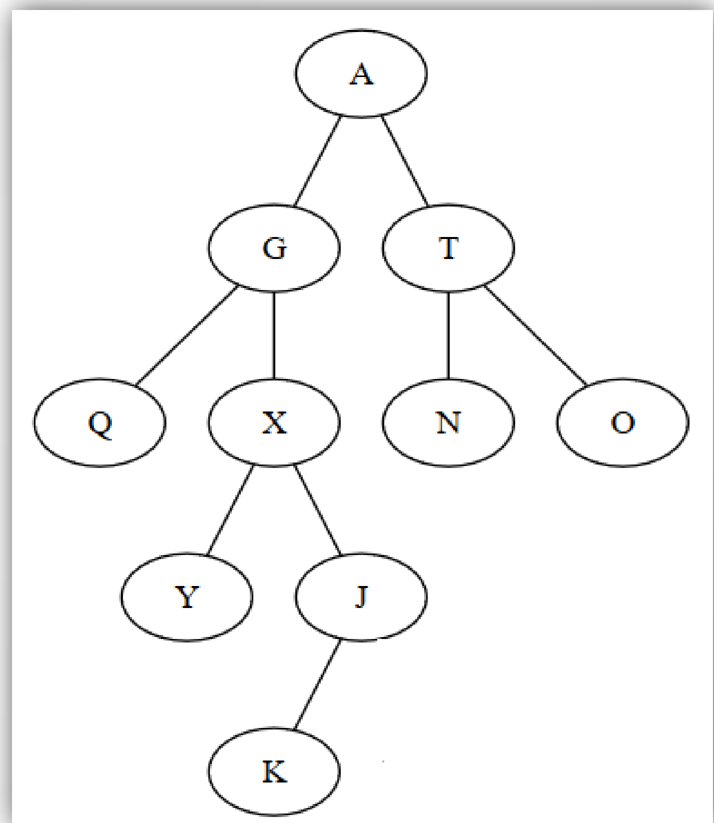
Postorder traversal with one stack

```
subalgorithm postorder(tree) is:
    init(s)                              //s is an auxiliary stack
    node ← tree.root
    while node ≠ NIL execute
        if [node].right ≠ NIL then
                push(s, [node].right)
        end-if
        push(s, node)
        node ← [node].left
    end-while
    while not isEmpty(s) execute
        node ← pop(s)
        if [node].right ≠ NIL and (not isEmpty(s)) and
                        [node].right = top(s) then
                pop(s)
                push(s, node)
                node ← [node].right
        else
                @visit node
                node ← NIL
        end-if
        while node ≠ NIL execute
                if [node].right ≠ NIL then
                        push(s, [node].right)
                end-if
                push(s, node)
                node ← [node].left
        end-while
    end-while
end-subalgorithm
```



Postorder traversal

- Node: A (Stack:)
- Node: NIL (Stack: T A X G Q)
- Visit Q, Node NIL (Stack: T A X G)
- Node: X (Stack: T A G)
- Node: NIL (Stack: T A G J X Y)
- Visit Y, Node: NIL (Stack: T A G J X)
- Node: J (Stack: T A G X)
- Node: NIL (Stack: T A G X J K)
- Visit K, Node: NIL (Stack: T A G X J)
- Visit J, Node: NIL (Stack: T A G X)
- Visit X, Node: NIL (Stack: T A G)
- Visit G, Node: NIL (Stack: T A)
- Node: T (Stack: A)
- Node: NIL (Stack: A O T N)



Traversal without a stack

Preorder, postorder and inorder traversals can be implemented without an auxiliary stack if:

- we use a representation for a node, where we keep a pointer to the parent node
- and an information to show "the state" of a node

Use flag *visited*:

- When we start the traversal we assume that all nodes have the visited flag set to false.
- During the traversal we set the flags to true, but when traversal is over, we have to make sure that they are set to false again (otherwise a second traversal is not possible).

Inorder traversal without a stack

```
subalgorithm inorderNoStack(tree) is:
    current ← tree.root
    while current ≠ NIL execute
        if [current].left ≠ NIL and [[current].left].visited = false then
                current ← [current].left
        else if [current].visited = false then
                @visit current
                [current].visited ← true
        else if [current].right ≠ NIL and [[current].right].visited = false then
                current ← [current].right
        else
                current ← [current].parent
        end-if ... end-if
    end-while
    if tree.root ≠ NIL then
        [tree.root].visited ← false
    end-if
end-subalgorithm
```

Binary tree iterator

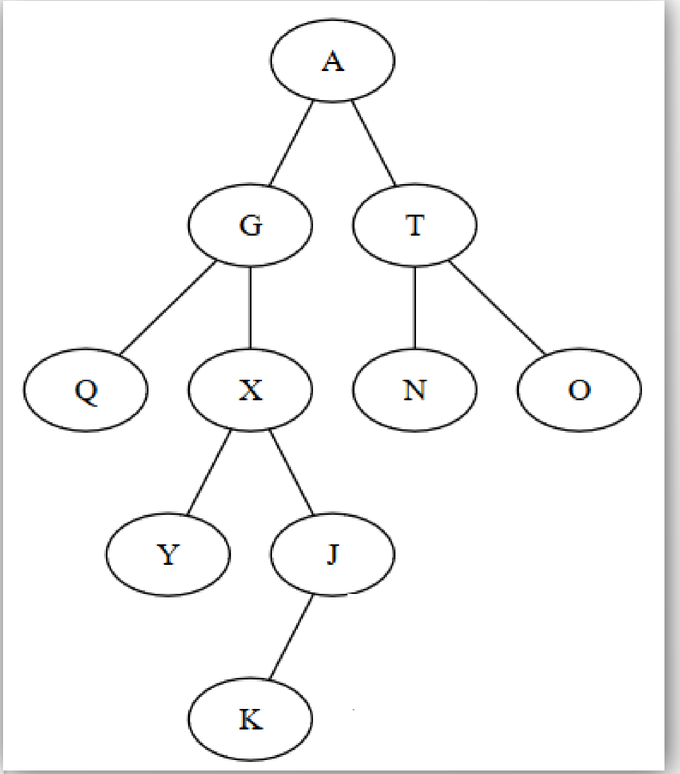
- For defining an iterator, we have to divide the traversal code into the functions of an iterator: init, getCurrent, next, valid

- We are going to work with nodes without parent node.
  - The iterator will use a stack.

<u>Iterator</u>:
    bt: BinaryTree
    s: Stack
    currentNode: ↑ BTNode

Remember: Inorder traversal - non-recursive

```
subalgorithm inorder(tree) is:
    init(s)                    // s is an auxiliary stack
    currentNode ← tree.root
    while currentNode ≠ NIL execute
        push(s, currentNode)
        currentNode ← [currentNode].left
    end-while
    while not isEmpty(s) execute
        currentNode ← pop(s)
        @visit currentNode
        currentNode ← [currentNode].right
        while currentNode ≠ NIL execute
            push(s, currentNode)
            currentNode ← [currentNode].left
        end-while
    end-while
end-subalgorithm
```



Inorder binary tree iterator

```
subalgorithm init (it, bt) is:
    it.bt ← bt
    init(it.s)
    node ← bt.root
    while node ≠ NIL execute
        push(it.s, node)
        node ← [node].left
    end-while
    if not isEmpty(it.s) then
        it.currentNode ← pop(it.s)
    else
        it.currentNode ← NIL
    end-if
end-subalgorithm
```

```
subalgorithm next(it) is:
    node ← [node].right
    while node ≠ NIL execute
        push(it.s, node)
        node ← [node].left
    end-while
    if not isEmpty(it.s) then
        it.currentNode ← pop(it.s)
    else
        it.currentNode ← NIL
    end-if
end-subalgorithm
```

```
function valid(it) is:
    valid ← ( it.currentNode ≠ NIL )
end-function
```

```
function getCurrent(it) is:
    getCurrent ← [it.currentNode].info
end-function
```

Tree traversals: problems

Think about:

- Assume you have a binary tree, you do not know how it looks like, but you have the preorder and inorder traversal of the tree. Give an algorithm for building the tree based on these two traversals.

  e.g.:

  Preorder: A B F G H E L M

  Inorder: B G F H A L E M

- Can you rebuild the tree if you have the postorder and the inorder traversal?
- Can you rebuild the tree if you have the preorder and the postorder traversal?

  e.g.: Pre: 1 2 3 4 5

  Post: 5 4 3 2 1 or: 3 4 2 5 1

What if the tree is full?

Tree traversals: problems

Rebuild the tree if you have the preorder and the postorder traversal of a tree with distinct elements. We also know that the tree is full.

- The first element in the preorder traversal is the ***root*** of the tree.
- If there are next elements , the next one is its ***left_child***.
- We will find the ***index*** of ***left_child*** in the postorder traversal.
- All the elements to the left of this ***index*** and element at this index will be in the left subtree of ***root***. And all the elements to the right of this index will be in the right subtree of the ***root***.

Now we have the preorder and postorder traversals for the 2 subtrees of the ***root***!

e.g.:

Preorder : 1, 2, 4, 5, 3, 6, 8, 9, 7

Postorder : 4, 5, 2, 8, 9, 6, 7, 3, 1

Tree: other problems

Think about:

Give the iterative and recursive algorithms for the following problems:

- Search for a given element in a binary tree.
- Determine the height of a binary tree.
- Determine the level at which a given element appears in a binary tree.
- Determine the parent of a node containing a given element .

Tree traversal

When we have a tree that is not binary, there are two possible traversals:

Level order (breadth first) - looks exactly the same as in case of a binary tree, we use an auxiliary queue to store the nodes.

Depth first - similar to breadth first, but we use an auxiliary stack to store the nodes (it is the generalizations of preorder traversal). By using a stack, the algorithms will always go as deep as possible in the tree and only once a whole subtree of a node was visited we pass to the next child.