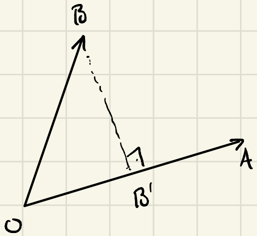
Proposition 2.1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{V}^n$ be two non-zero vectors and let $\mathbf{a}_1 = \frac{1}{\|\mathbf{a}\|}$ and $\mathbf{b}_1 = \frac{1}{\|\mathbf{b}\|}\mathbf{b}$. Then

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \cdot \mathrm{pr}^{\perp}_{\mathbf{a}_1} \mathbf{b} = \|\mathbf{b}\| \cdot \mathrm{pr}^{\perp}_{\mathbf{b}_1} \mathbf{a}.$$

Fix a point O



Let A, B be such that $a = \overrightarrow{OA}$, $b = \overrightarrow{OB}$

Let $B' \in OA$ be such that $BB' \perp OA$

Suppose $B' \neq O$. Then $\angle OB'B$ is a right angle and $\cos \angle AOB = \dfrac{\|OB'\|}{\|b\|}$

$$\Rightarrow a \cdot b = \|a\| \cdot \|b\| \cos \angle AOB = \|a\| \cdot \|\overrightarrow{OB'}\| = \|a\| \cdot \mathrm{pr}^{\perp}_{a_1} b$$

If $B' = O$ then $BO \perp OA \Rightarrow \overrightarrow{OB} \perp \overrightarrow{OA} \Rightarrow a \cdot b = 0 = \|\overrightarrow{OB'}\| = \|a\| \cdot \|\overrightarrow{OB'}\| = a \cdot \mathrm{pr}^{\perp}_{a_1} b$

The other equality is obtained by interchanging a and b.

Proposition 2.2. For a non-zero vector $\mathbf{v} \in \mathbb{V}^n$, the maps $\mathrm{Pr}^{\perp}_{\mathbf{v}} : \mathbb{V}^n \to \langle \mathbf{v} \rangle$ and $\mathrm{pr}^{\perp}_{\mathbf{v}} : \mathbb{V}^n \to \mathbb{R}$ are linear.

It is enough to show that $\mathrm{Pr}^{\perp}_{v}$ is linear since then $\forall x, y \in \mathbb{R}$ $\forall$ vectors a, b

$$\mathrm{pr}^{\perp}_{v}(xa + yb)\, v = \mathrm{Pr}^{\perp}_{v}(xa + yb) = x\, \mathrm{Pr}^{\perp}_{v}(a) + y\, \mathrm{Pr}^{\perp}_{v}(b) = \left(x\, \mathrm{pr}^{\perp}_{v}(a) + y\, \mathrm{pr}^{\perp}_{v}(b)\right) v$$

$$\Rightarrow \mathrm{pr}^{\perp}_{v}(xa + yb) = x\, \mathrm{pr}^{\perp}_{v}(a) + y\, \mathrm{pr}^{\perp}_{v}(b)$$

$$\Rightarrow \mathrm{pr}^{\perp}_{v} \text{ is linear}$$

Fix a point O and let A, B, C be such that $a = \overrightarrow{OA}$

$b = \overrightarrow{OB}$

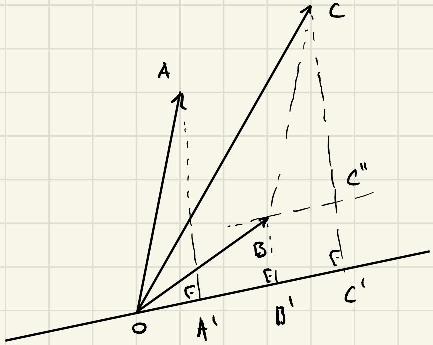
$a + b = \overrightarrow{OC}$

Further, let A', B', C' be such that

$$Pr_v^{\perp}(a) = \overrightarrow{OA'}$$

$$Pr_v^{\perp}(b) = \overrightarrow{OB'}$$

$$Pr_v^{\perp}(a+b) = \overrightarrow{OC'}$$



Take $C'' \in CC'$ such that $BC'' \parallel OC'$

Then $\Delta\, OAA'$ and $\Delta\, BCC''$ are congruent $\Rightarrow |OA| = |BC''|$

$\Rightarrow |OA| = |B'C'|$

$\Rightarrow \overrightarrow{OA'} + \overrightarrow{OB'} = \overrightarrow{OC'} \Rightarrow Pr_v^{\perp}(a) + Pr_v^{\perp}(b) = Pr_v^{\perp}(a+b)$ so $Pr_v^{\perp}$ is additive

To see that $Pr_v^{\perp}$ is homogeneous, i.e. that $Pr_v^{\perp}(x a) = x\, Pr_v^{\perp}(a)$ use Thales

Proposition 2.3. For any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}^n$ and any $\lambda \in \mathbb{R}$ we have

1. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$,
2. $(\lambda \mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$,
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$,
4. $\mathbf{a} \cdot \mathbf{a} \geq 0$,
5. $\mathbf{a} \cdot \mathbf{a} = 0 \Leftrightarrow \mathbf{a} = 0$.

1, 4, 5 follow directly from the definition

2, 3 are another way of writing the linearity of $pr_v^{\perp}$:

$$a \cdot (xb + yc) \overset{\text{Prop 2.1}}{=} \|a\| \cdot pr^{\perp}_{\frac{a}{\|a\|}}(xb+yc) \overset{\text{Prop 2.2}}{=} \|a\| \cdot \left(x\, pr^{\perp}_{\frac{a}{\|a\|}}(b) + y\, pr^{\perp}_{\frac{a}{\|a\|}}(c)\right)$$

$$= x \|a\| \cdot pr^{\perp}_{\frac{a}{\|a\|}}(b) + y \|a\| \cdot pr^{\perp}_{\frac{a}{\|a\|}}(c)$$

$$= x\, a \cdot b + y\, a \cdot c.$$

Proposition 2.4. Consider two vectors $\mathbf{a}(a_1, a_2, \dots, a_n), \mathbf{b}(b_1, b_2, \dots, b_n) \in \mathbb{V}^n$ with components relative to an *orthonormal basis*. Their scalar product is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n. \tag{2.1}$$

let $e_1, \dots, e_n$ be this basis

$$\Rightarrow \quad a \cdot b = (a_1 e_1 + a_2 e_2 + \dots + a_n e_n) \cdot (b_1 e_1 + \dots + b_n e_n)$$

$$= \sum_{\substack{i=1 \\ j=1}}^{n} a_i b_j e_i e_j$$

$$= \sum_{i=1}^{n} a_i b_i e_i^2 \qquad (\text{since for } i \neq j \; e_i \perp e_j \Leftrightarrow e_i e_j = 0)$$

$$= \sum_{i=1}^{n} a_i b_i \qquad (\text{since } e_i^2 = \|e_i\|^2 = 1)$$