23.05.2022.

Lecture 13

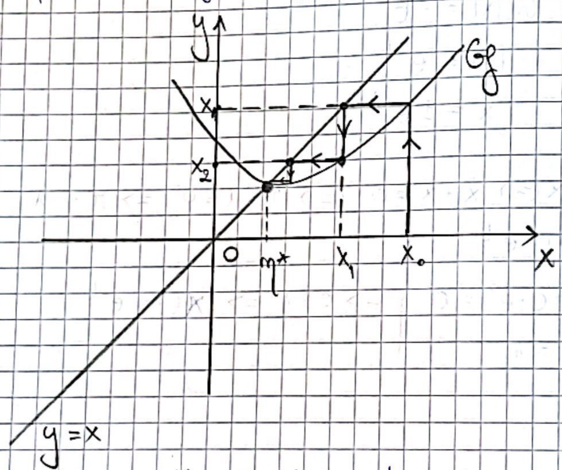
Scalar discrete dynamical systems

(1) $x_{k+1} = f(x_k)$, $k \geq 0$, $k \in \mathbb{N}$, where $f \in C^1(\mathbb{R}, \mathbb{R})$.

<u>Theorem</u> (L12) Let η^* be a fixed point of f.
If $|f'(\eta^*)| < 1$ then η^* is an attractor

The cobweb or stair-step diagram to "construct" geometrically a sequence $(x_k)_{k \geq 0}$ that satisfies (1).

- We represent the graph of f and the line $y = x$



- $G_f \cap \{y = x\}$ gives the fixed points of f

Def: If $f(\eta^*) = \eta^*$ then we say that η^* is a fixed point of f

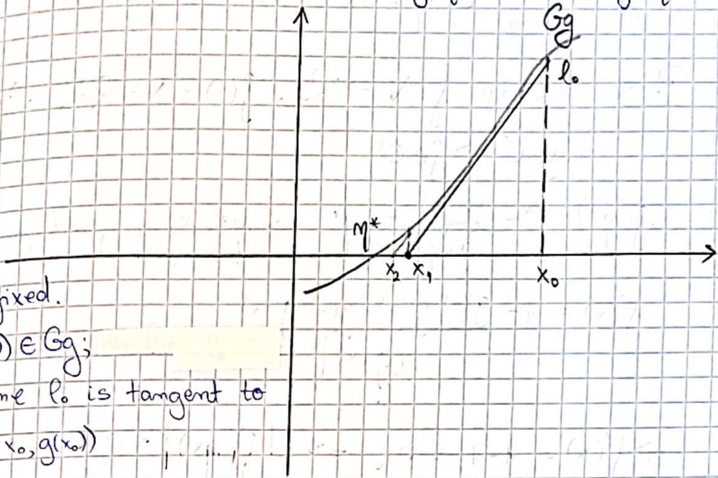
$x_0 \in \mathbb{R}$ fix

$x_1 = f(x_0)$

$x_2 = f(x_1)$

The Newton-Raphson method to find approximations for a zero of a function

Take $g: I \to \mathbb{R}$, is C^2 on the open, nonempty interval I.
We know that $\exists \eta^* \in I$ s.t. $g(\eta^*) = 0$ and $g'(\eta^*) \neq 0$



$x_0 \in I$ fixed.
$(x_0, g(x_0)) \in G_g$;
the line l_0 is tangent to G_g in $(x_0, g(x_0))$

Our aim is to find good approximations of η^*, or, more precisely, to find $(x_k)_{k \geq 0}$ such that $\lim\limits_{k \to \infty} x_k = \eta^*$

$l_0: \; y - g(x_0) = g'(x_0)(x - x_0)$

$l_0 \cap Ox: \; y = 0$ in the eq. of l_0

$\Rightarrow -g(x_0) = g'(x_0)(x_1 - x_0)$

$\Rightarrow \boxed{x_1 = x_0 - \dfrac{g(x_0)}{g'(x_0)}}$

In general $\begin{cases} x_{k+1} = x_k - \dfrac{g(x_k)}{g'(x_k)}, & k \geq 0 \\ x_0 \text{ given} \end{cases}$

Theorem 2: Let $V \subset I$ open, nonempty interval be such that $\eta^* \in V$ and $g'(\eta) \neq 0$, $\forall \eta \in V$.

~~Then $\forall x_0 \in V$ given~~ the sequence $(x_k)_{k\geq 0}$ given by (3) satisfies $\lim_{k\to\infty} x_k = \eta^*$.

Proof: Define $f : V \to \mathbb{R}$, $f(x) = x - \frac{g(x)}{g'(x)}$, $\forall x \in V$

$g \in C^2 \Rightarrow f \in C^1$

$$f(\eta^*) = \eta^* - \frac{g(\eta^*)}{g'(\eta^*)} = \eta^* \Rightarrow \eta^* \text{ is a fixed point of } f$$

$$f'(x) = 1 - \frac{g'(x)\cdot g'(x) - g(x)\cdot g''(x)}{[g'(x)]^2}$$

$$\Rightarrow f'(x) = 1 - 1 = 0 \Rightarrow |f'(\eta^*)| = 0 < 1$$

$\xrightarrow{\text{Lin method}}$ the f.p. of f, η^* is an attractor

Then $\exists \rho > 0$ s.t. $|x_0 - \eta^*| < \rho$, the seq. $(x_k)_{k\geq 0}$ given by (3) satisfies $\lim_{k\to\infty} x_k = \eta^*$.

Example: $g(x) = x^2 - 3$, $g : (0, \infty) \to \mathbb{R}$

Try to estimate the basin of attraction of the attractor as in the proof of Th. 2 for this g, using the stair-step diagram.

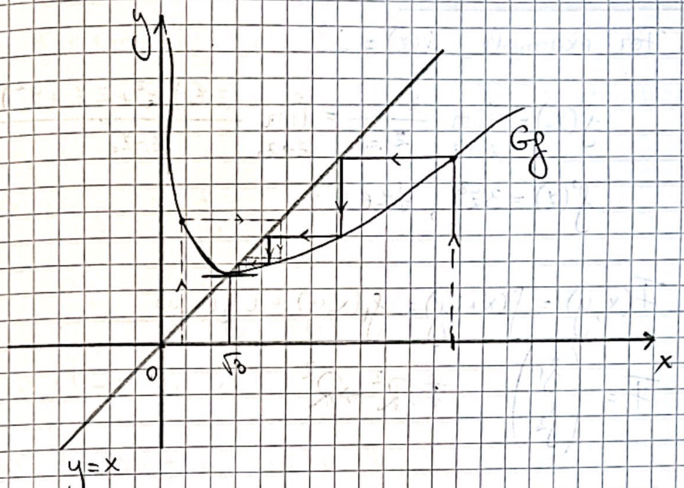
$\eta^* = \sqrt{3}$

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - 3}{2x} = \frac{2x^2 - x^2 + 3}{2x} = \frac{x^2+3}{2x} = \frac{1}{2}x + \frac{3}{2}\cdot\frac{1}{x}$$

$$\begin{cases} x_{k+1} = \frac{1}{2}x_k + \frac{3}{2}\cdot\frac{1}{x_k} \\ x_0 \text{ given} \end{cases}$$

From the proof of Th2 we know that $\sqrt{3}$ is an attracting fixed point of f, thus (as we wrote in the conclusion of Th 2) $\exists \delta > 0$ s.t. for x_0 with $|x_0 - \sqrt{3}| < \delta$ we have $\lim_{k \to \infty} x_k = \sqrt{3}$

So, we want to estimate the basin of attraction of $\sqrt{3}$, denoted $A_{\sqrt{3}}$.



$f: (0, \infty) \to \mathbb{R}$

$f'(x) = \frac{1}{2} - \frac{3}{2} \cdot \frac{1}{x^2} = \frac{x^2-3}{2x^2}$

x	0		$\sqrt{3}$		$+\infty$
f'		$- \; - \; - \; - \; -$	0	$+ \; + \; + \; + \; +$	
f	$+\infty$	$\searrow$	$\sqrt{3}$	$\nearrow$	$+\infty$

$f'(\eta^*) = 0$

$| \; | < 1$

It seems that $A_{\sqrt{3}} = (0, \infty)$

Complex maps

$f : \mathbb{C} \to \mathbb{C}$

Def: The derivative in $z_0 \in \mathbb{C}$ is (if exists) the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

For example, $f(z) = z^3$

$$f'(z_0) = \lim_{z \to z_0} \frac{z^3 - z_0^3}{z - z_0} = \lim_{z \to z_0} \frac{\cancel{(z - z_0)}(z^2 + z \cdot z_0 + z_0^2)}{\cancel{z - z_0}} = 3z_0^2$$

$$f'(z) = 3z^2, \ \forall z \in \mathbb{C}$$

$$F(x,y) = f(x+iy) = f_1(x,y) + i f_2(x,y)$$

$$F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \qquad F : \mathbb{R}^2 \to \mathbb{R}^2$$

Def: $\eta^* \in \mathbb{C}$ is a fixed point of $f : \mathbb{C} \to \mathbb{C}$ when $f(\eta^*) = \eta^*$

Define $(z_k)_{k \geq 0}$ a sequence in $\mathbb{C}$ by $z_k = f^k(z_0)$, or

$$\begin{cases} z_{k+1} = f(z_k) \\ z_0 \in \mathbb{C} \text{ given} \end{cases}$$

We define a similar notion of attractor.

Th: If $\eta^* \in \mathbb{C}$ is a fixed point of $f : \mathbb{C} \to \mathbb{C}$ (of class C^1) such that $|f'(\eta^*)| < 1$ then η^* is an attractor.

Example: $g : \mathbb{C} \to \mathbb{C}$, $g(z) = z^4 - 1$

Take $f(z) = z - \dfrac{g(z)}{g'(z)}$

$g(z) = 0 \Leftrightarrow z^4 - 1 = 0 \Leftrightarrow (z^2-1)(z^2+1) = 0 \Leftrightarrow z_1 = -1,\ z_2 = 1$
$z_3 = i,\ z_4 = -i$

each of them is an attractor of f

A_{-1}, A_1, A_i, A_{-i} the corresponding basins of attraction, as subsets of $\mathbb{C}$

Their shape is of a fractal

The tent map

$T : [0,1] \to \mathbb{R}$

$T(x) = 1 - |2x - 1|,\ \forall x \in [0,1]$

$T^k(x_0),\ x_0 \in [0,1]$

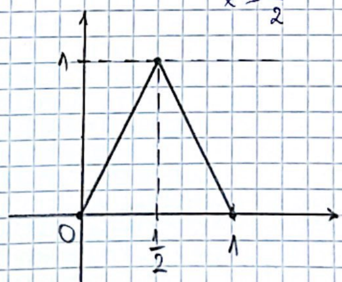
→ these sequences can have very complicated behaviour. "the tent map is chaotic"

- represent the graph of T
- note a similarity with $f(x) = 4x(1-x)$
- $T^k(\eta)$ for $\eta = \frac{3}{2^n}$, $n \geq 2$, $n \in \mathbb{N}$
- graphs of T^2 and T^3 (HW)

$2x - 1 > 0$

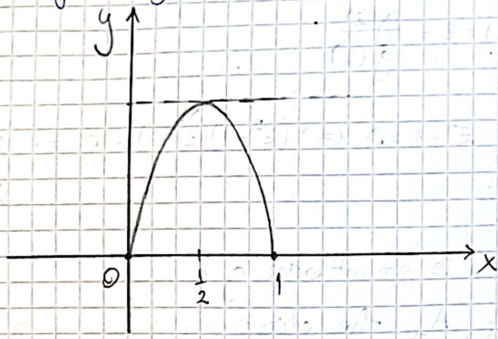
$x > \frac{1}{2}$

$$T(x) = \begin{cases} 2(1-x), & x \in \left[\frac{1}{2}, 1\right] \\ 2x, & x \in \left[0, \frac{1}{2}\right) \end{cases}$$



$T(0) = T(1) = 0$, $T\left(\frac{1}{2}\right) = 1$

The graph of $f(x) = 4x(1-x)$, $x \in [0,1]$



$f(\frac{1}{2}) = 4 \cdot \frac{1}{2} \cdot \frac{1}{2} = 1$

$x_{k+1} = 4 * x_k * (1 - x_k)$

$x_0 = 0.67$

$y_{k+1} = 4 * y_k - 4 * y_k * y_k$

$y_0 = 0.67$

$x_k \in [0,1]$ $\qquad$ $x_{30} = 0.01\ldots$

$y_k \in [0,1]$ $\qquad$ $y_{30} = 0.91\ldots$