Lecture 10

Planar dynamical systems

I. Phase portraits for linear planar systems

(1) $\dot{X} = AX$, $A \in \mathcal{M}_2(\mathbb{R})$, $\det A \neq 0$, $X = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ $\lambda_1, \lambda_2 \in \mathbb{C}$ - eigenval. of A

! Recall from lect. 9 :

- $\det A \neq 0 \iff$ the only equil. pt. of (1) is $\eta^* = 0 \in \mathbb{R}^2$
- $\det A \neq 0 \iff \lambda_1 \neq 0$ and $\lambda_2 \neq 0$

DEF. + Th. from L9:

If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 \leq \lambda_2 < 0$ then $\eta^* = 0_2$ is a NODE, global attractor, $\nexists$ global first integral.

If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $0 < \lambda_1 \leq \lambda_2$, then $\eta^* = 0_2$ is a NODE, global repeller, $\nexists$ global f. i.

If $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < 0 < \lambda_2$, then $\eta^* = 0_2$ is a SADDLE, unstable, $\exists$ global f. i.

If $\lambda_{1,2} = \pm i\beta$, $\beta \in \mathbb{R}^*$, then $\eta^* = 0_2$ is a CENTER stable, $\exists$ global f. i.

If $\lambda_{1,2} = \alpha \pm i\beta$, with $\beta \in \mathbb{R}^*$ then $\eta^* = 0_2$ is a FOCUS $\nexists$ global f. i. and
- global attractor when $\alpha < 0$
- global repeller when $\alpha > 0$

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Typical phase portrait of a NODE

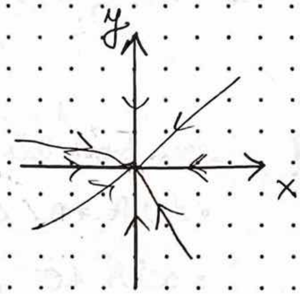
$$\begin{cases} \dot{x} = -x \\ \dot{y} = -y \end{cases} \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \text{coef. of the system}$$

$\uparrow x \quad \uparrow y$

$\rightarrow$ diag. matrice $\Rightarrow \lambda_1 = -1, \lambda_2 = -1$

$\Rightarrow$ the equil. $\eta^* = 0_2$ is a node, global attractor.

From L_S we have the phase portrait



Typical phase portrait of a SADDLE

$$\begin{cases} \dot{x} = -x \\ \dot{y} = y \end{cases} \qquad A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \lambda_1 = -1, \lambda_2 = 1 \Rightarrow$$

$\Rightarrow \eta^* = 0_2$ is a SADDLE, unstable

Find the flow: $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^{-t} \\ \eta_2 e^{t} \end{pmatrix}, \forall t \in \mathbb{R}$

$\forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$

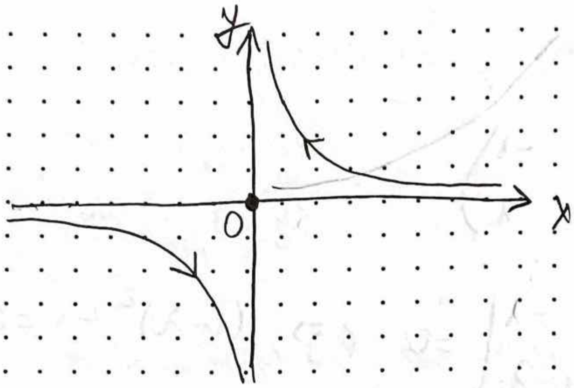
A global f.i.: $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x \cdot y$.

The phase portrait: the level curves of H are: $xy = c$, $c \in \mathbb{R}$

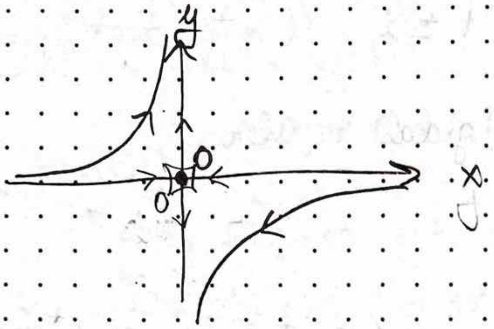
$\Rightarrow y = \frac{1}{x}$ (our curve is the graph of the function $\frac{1}{x}$)

$f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2} < 0$

x	$-\infty$		0		$+\infty$
$\frac{1}{x}$	0	$\searrow$	$-\infty$ | $+\infty$	$\searrow$	0



$\boxed{x' y = 1}$



$\boxed{x' y = -1}$

Typical phase portrait of a CENTER

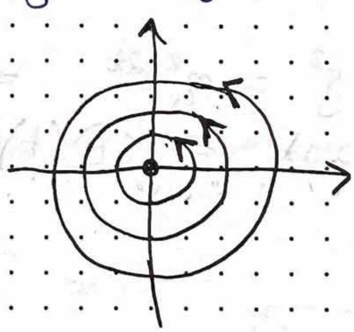
$$\begin{cases} \dot{x} = -y \\ \dot{y} = x \end{cases}, \quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\det(A - \lambda I) = 0 \Leftrightarrow \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow$

$\Leftrightarrow \lambda^2 = -1 \Rightarrow \lambda_{1,2} = \pm i$ (Note that $\mathrm{Re}(\pm i) = 0$)

Indeed $\eta^* = O_2$ is a center, stable, it is a global f.i. from L_8: $H : \mathbb{R}^2 \to \mathbb{R}$, $H(x, y) = x^2 + y^2$.

The phase portrait is:



HW: $\begin{cases} x' = -y \\ y' = 4x \end{cases} \Rightarrow$ the orbits will be ellipses

Typical phase portrait of a FOCUS

$$\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases} \qquad A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\det(A - \lambda I_2) = 0 \iff \begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \iff (1-\lambda)^2 + 1 = 0 \iff$$

$$\iff \boxed{1 - \lambda = \pm i} \iff \lambda_{1,2} = 1 \pm i \quad \text{(note that } \mathrm{Re}(\lambda_{1,2}) = 1 > 0$$

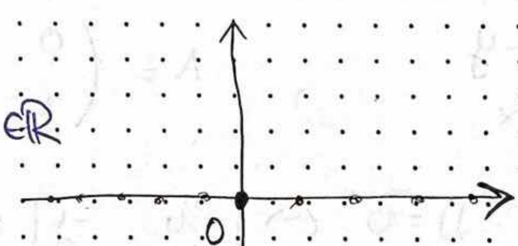
Then $\eta^* = 0_2$ is a FOCUS, global repeller.

$$\left.\frac{dy}{dx} = \frac{x+y}{x-y}\quad\right\} \text{ not useful to compute this}$$

The flow is: $\varphi(t, \eta_1, \eta_2) = \begin{pmatrix} \eta_1 e^t \cos t - \eta_2 e^t \sin t \\ \eta_1 e^t \sin t + \eta_2 e^t \cos t \end{pmatrix}, \forall t \in \mathbb{R}, \forall \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \in \mathbb{R}^2$

- particular case: $\eta_2 = 0$

$$\varphi(t, \eta_1, \boxed{0}) = \begin{pmatrix} \eta_1 e^t \cos t \\ \eta_1 e^t \sin t \end{pmatrix}, t \in \mathbb{R}$$



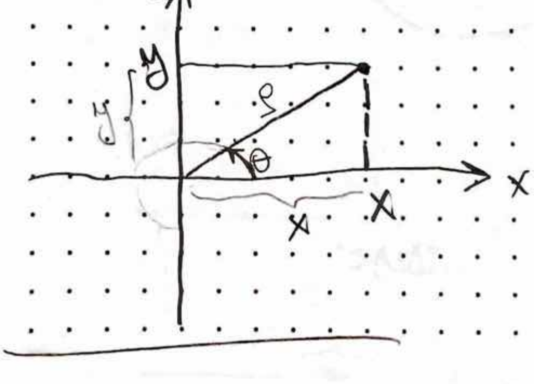
parametric eq.: $\begin{cases} x = \eta_1 e^t \cos t \\ y = \eta_1 e^t \sin t \end{cases}, t \in \mathbb{R}$

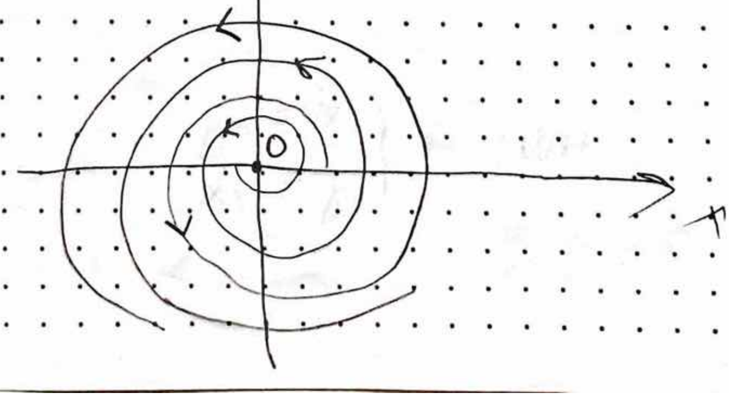
$$\boxed{\begin{aligned} \rho^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}}$$

$$\rho^2 = x^2 + y^2 = \eta_1^2 e^{2t} \implies \rho(t) = |\eta_1| e^t$$

grows exponentially

$$\tan\theta = \tan t \implies \theta(t) = t, \quad t \in \mathbb{R}$$





II. Phase portraits using polar coordinates

$$(1)\begin{cases} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{cases}$$

To transform (1) to polar coordinates, means to consider new unknowns as $(\rho(t), \theta(t))$ instead of $(x(t), y(t))$ related by

$$\begin{cases} \rho(t)^2 = x(t)^2 + y(t)^2 \\ \tan\theta(t) = \dfrac{y(t)}{x(t)} \end{cases} \quad (2)$$

step 1: Take the derivative w.r.t. ~~t~~ t in (2)

$$(3)\begin{cases} \rho\dot{\rho} = x\dot{x} + y\dot{y} \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\dot{y}x - y\dot{x}}{x^2} \end{cases}$$

step 2: Replace in (3): $\begin{vmatrix} \dot{x} = f_1(x,y) \\ \dot{y} = f_2(x,y) \end{vmatrix}$

step 3: Then replace: $\begin{vmatrix} x = \rho\cos\theta \\ y = \rho\sin\theta \end{vmatrix}$

Come back to $\begin{cases} \dot{x} = x - y \\ \dot{y} = x + y \end{cases}$

- We transform it to polar coordinates:

$$\begin{cases} \rho\dot{\rho} = x(x-y) + y\cdot(x+y) \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x(x+y) - y(x-y)}{x^2} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \rho\dot{\rho} = x^2+y^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{x^2+y^2}{x^2} \end{cases} \Rightarrow \begin{cases} \rho\dot{\rho} = \rho^2 \\ \dfrac{\dot{\theta}}{\cos^2\theta} = \dfrac{\rho^2}{\rho^2\cos^2\theta} \end{cases} \Rightarrow$$

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$\Rightarrow \begin{cases} \dot{\rho} = \rho \\ \dot{\theta} = 1 \end{cases} \Rightarrow \begin{cases} \rho(t) = \rho_0 e^t \\ \theta(t) = t + \theta_0 \end{cases}$ the same

$\dot{\rho} > 0 \Rightarrow \rho$ is increasing, so we go farther away from the origin (spiral around the origin)

$\dot{\rho} < 0 \Rightarrow \rho$ is decreasing, so we approach the origin (spiral)

$\dot{\rho} = 0 \Rightarrow \rho$ is constant ~~along~~ around an orbit (circle)

$\dot{\theta} > 0 \Rightarrow \theta \nearrow \Rightarrow$ counter-clockwise rotation

$\dot{\theta} < 0 \Rightarrow \theta \searrow \Rightarrow$ clockwise rotation

$\dot{\theta} = 0 \Rightarrow \theta$ is const. $\Rightarrow$ the orbit lies on a line passing through O_2

Exercise:
$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \end{cases}$$

a.) Check that $\varphi(t,1,0) = (\cos t, \sin t) \quad \forall t \in \mathbb{R}$

b.) Pass to polar coord. and represent the phase portrait, Reading the p.p., specify the stability of the equil. $(0,0)$ Is there an attractor?

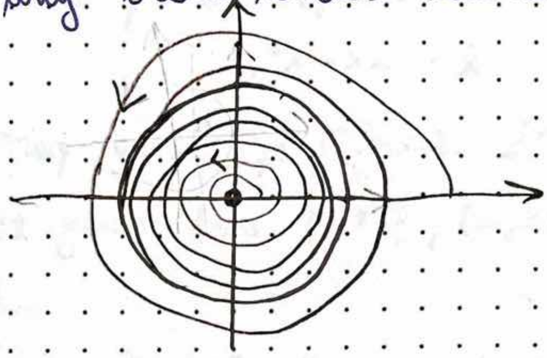
SOL: a) Recall that, by def., $\varphi(t,1,0)$ is the sol. of the IVP

$$\begin{cases} \dot{x} = -y + x(1-x^2-y^2) \\ \dot{y} = x + y(1-x^2-y^2) \quad (*) \\ x(0) = 1 \\ y(0) = 0 \end{cases}$$

We have to replace $x = \cos t$, $y = \sin t$ in (*) and show that we obtain valid relations. ✓

b) $\begin{cases} \dot{\rho} = \rho(1-\rho^2) \\ \dot{\theta} = 1 \quad > 0 \end{cases}$ $\Rightarrow$ any orbit rotates counter-clockwise

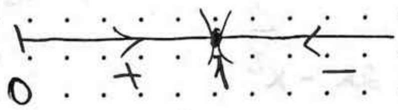
$\begin{cases} x = \cos t \\ y = \sin t \end{cases}, \; t \in \mathbb{R}$



$(0,0)$ is an eq. point

phase portrait of ρ

$\underbrace{\rho(1+\rho)}_{>0}(1-\rho)$



The circle is an attractor.

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