LECTURE 10

DATE: 6 DECEMBER 2021
WEEK 11

10. Extensions of the Riemann Integral: Improper integrals (w. a. parameter)

Famous model: ideal gas

MAXWELL 1860, BOLTZMANN 1872, 1877

example: $\int_0^\infty x^2 e^{-x^2}\,dx = ?$

§10.1. Improper integrals:

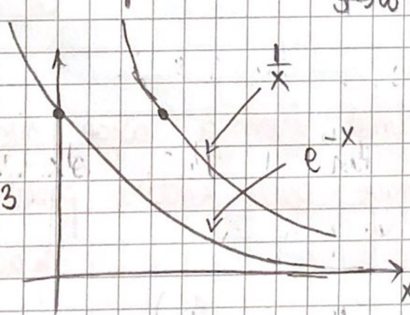
"Improper integrals are limits"

Ex 1: $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx = 1$

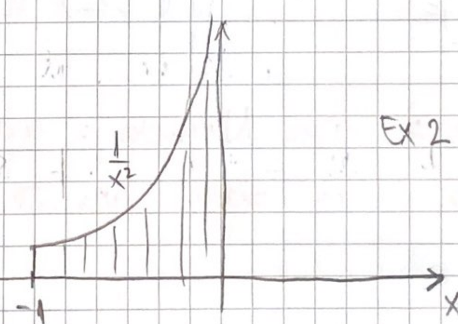
Ex 2: $\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{\substack{t\to 0\\ t<0}} \int_{-1}^t \frac{1}{x^2}\,dx = \lim_{\substack{t\to 0\\ t<0}} \left(-\frac{1}{x}\Big|_{-1}^t\right) = \infty$

Ex 3: $\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \ln x\Big|_1^t = \infty$

Ex 1 & 3



Ex 2



Def: $f : [a, b) \to \mathbb{R}$ ($b = \infty$ admitted), f integrable (Riemann) on any $[a, b]$, $b < b$ (locally integrable)

We call the improper $\int_a^b f(x)\,dx$ convergent (CONV) if $\lim_{\substack{t\to b\\ t<b}} \int_a^t f(x)\,dx \;\exists < \infty$

otherwise (DIV)

Theorem 1 (CAUCHY): $f:[a,b)\to\mathbb{R}$ locally integrable

$\int_a^b f(x)\,dx$ (CONV) $\Leftrightarrow \forall \varepsilon>0 \ \exists b_\varepsilon<b$

such that $\forall t\in(b_\varepsilon,b)$: $\left|\int_{b_\varepsilon}^{t} f(x)\,dx\right|<\varepsilon$

Meaning: (CONV) $\Leftrightarrow \int_{b_\varepsilon}^{t}$ can be made arbitrary small

§ 10.2. Testing the convergence of improper integrals:

Theorem 2 (comparison I)

$f,g:[a,b)\to\mathbb{R}$ locally integrable

and $0\le f(x)\le g(x)$ $\forall x\in[a,b)$

Then (i) $\int_a^b g$ (CONV) $\Rightarrow \int_a^b f$ (CONV)

(ii) $\int_a^b f$ (DIV) $\Rightarrow \int_a^b g$ (DIV)

Theorem 3 (comparison II)

$g(x)>0$ on $[a,b)$ and $\lim_{x\to b}\frac{f(x)}{g(x)}=L\in\mathbb{R}$ ($<\infty$)

Then (i) if $L\neq 0$ $\int f$ & $\int g$ are both (DIV) or both (CONV)

(ii) if $L=0$ $\int g$ CONV $\Rightarrow \int|f|$ (CONV)

absolute (CONV)

Remark: Usually, you compare with x^{α}

$$\int_1^{\infty} x^{\alpha}\, dx \begin{cases} \text{(CONV) for } \alpha < -1 \\ \text{(DIV) for } \alpha \geq -1 \end{cases}$$

$$\int_0^{1} x^{\beta}\, dx \begin{cases} \text{(CONV) for } \beta > -1 \\ \text{(DIV) for } \beta \leq -1 \end{cases}$$

Ex 4: $\int_1^{\infty} \frac{\sin x}{x^2}\, dx$ (CONV) because $\left|\frac{\sin x}{x^2}\right| \leq x^{-2}$ and $\int_1^{\infty} x^{-2}$ (CONV)

Ex 5: $\int_0^{\infty} e^{-x} (\sin x)^3\, dx$ (CONV) because $|e^{-x}(\sin x)^3| \leq e^{-x}$ and $\int_0^{\infty} e^{-x}\, dx = 1$ (CONV)

Remark: (should be placed after Theorem Cauchy)

$$\int_a^b |f(x)|\, dx \text{ (CONV)} \Rightarrow \int_a^b f(x)\, dx \text{ (CONV)}$$

(absolute CONV $\Rightarrow$ CONV)

Remark: $(a, b]$, (a, b) work the same way

Remark: Int by parts & change of variable both work if the improper integral is (CONV).

§10.3. Improper integrals with parameter:

$f : [a,b) \times [c,d] \to \mathbb{R}$

$F : [c,d] \to \mathbb{R}$

improper — with parameter

$$F(y) = \int_a^b f(x,y)\, dx$$

So: "Improper $\int$ with parameter are functions (of y)"

<u>Def</u>: The improper integral with parameter

$\int_a^b f(x,y)\, dx$ <u>converges uniformely</u> (w.r.t. y) to F if $\varepsilon > 0$

$\exists\, b_\varepsilon < b : \forall t \in (b_\varepsilon, b)$ and $\forall y \in [c,d]$ we have

$\left| \int_{b_\varepsilon}^{t} f(x,y)\, dx - F(y) \right| < \varepsilon$ and $b_\varepsilon \neq b(y)$

independent of y

Notation: (u CONV)

Ex 6: (Euler Gamma function)

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x}\, dx$$

$n \in \mathbb{N}$

$\Gamma(n+1) = n!$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

<u>Theorem 4</u> (Continuity): $f : [a,b) \times [c,d] \to \mathbb{R}$

If f cont. (as function of two variables) and if $\int_a^b f(x,y)\, dx$ (u CONV) then $F(y) = \int_a^b f(x,y)\, dx$ cont (in y)

Theorem 5 (Differentiability): f cont, $\frac{\partial f}{\partial y}$ cont and

$\int_a^b f(x,y)\,dx)$ (u CONV), $\int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$ (u CONV)

Then $F(y) = \int_a^b f(x,y)\,dx$ is differentiable and

$$F'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\,dx$$

Theorem 6 (Integrability):

f cont, $\int_a^b f(x,y)\,dx$ (u CONV) then

$F(y) = \int_a^b f(x,y)\,dx$ is y-integrable and

$$\int_c^d F(y)\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$$

Ex 7 : $f(x) = \begin{cases} \frac{e^x - 1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

f diffable? $f^{(n)}(0) = ?$

trick $f(x) = \int_0^1 e^{xy}\,dy$

$$f'(x) = \int_0^1 \frac{\partial}{\partial x}(e^{xy})\,dy$$

$$= \int_0^1 y\,e^{xy}\,dy = \ldots$$

$$f''(x) = \int_0^1 y^2 e^{xy}\,dy = \ldots$$

$$f'(0) = \int_0^1 y\,\cancel{e^{0y}}^{\,1}\,dy = \frac{1}{2}$$

$$f''(0) = \int_0^1 y^2 e^{0\cdot y}\,dy = \frac{1}{3}$$

$$f^{(n)}(0) = \int_0^1 y^n e^{0\cdot y}\,dy = \frac{1}{n+1}$$