LECTURE 6 - DYNAMICAL SYSTEMS

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Linear diff systems with cc (cont.)

Diagonalizable matrix

Def: We say that a matrix $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable over $\mathbb{R}$ when $\exists$ a diagonal matrix $B \in \mathcal{M}_n(\mathbb{R})$ s.t. A is similar to B.

A matrix that is not diagonalizable is said to be defective.

Property: Let $A \in \mathcal{M}_n(\mathbb{R})$. We have that A is diagonalizable over $\mathbb{R}$ if and only if any eigenvalue of A is real and there exist n linearly independent eigenvectors of A.

Recall: $\lambda \in \mathbb{C}$ is an eigenvalue of $A \overset{def}{\Leftrightarrow} \exists u \in \mathbb{R}^n, u \neq 0$ s.t. $Au = \lambda u$ (u is an eigenvector corresp to the eigenvalue λ of A) $\Leftrightarrow \det(A - \lambda I_n) = 0$ (this is a polynomial eq in λ of degree n).

Assume that A is diagonalizable.

Let $\underset{u_1}{\lambda_1}, \underset{u_2}{\lambda_2}, \dots, \underset{u_n}{\lambda_n} \in \mathbb{R}$ be the eigenvalues of A, not necessarily distinct, and $u_1, u_2, \dots, u_n \in \mathbb{R}^n$ be n linearly indep. eigenvectors of A.

Then $D = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$

Property: Let $P \in \mathcal{M}_n(\mathbb{R})$ s.t. its k^{th} column is u_k. Then $A = PDP^{-1}$ and P is invertible.

Remark: $A^k = PD^kP^{-1} = P\, diag(\lambda_1^k, \lambda_2^k, \dots, \lambda_n^k)P^{-1}$

$$e^{tA} = P\,diag(e^{t\lambda_1}, e^{t\lambda_2}, \dots, e^{t\lambda_n})P^{-1}$$

Procedure to find the general sol of $X' = AX$ when $A \in \mathcal{M}_n(\mathbb{R})$ is diagonalizable.

Step 1: Find the eigenvalues and the eigenvectors of A. Decide whether A is diagonalizable or not. If it is, continue.

Step 2: Find $D = diag(\lambda_1, \dots, \lambda_n)$, $P = (u_1\ u_2\ \dots\ u_n)$

Step 3: Find $e^{tA} = P \cdot diag(e^{\lambda_1 t}, \dots, e^{\lambda_n t})P^{-1}$

Step 4: Write the gen. sol. $X = e^{tA} \cdot C$, $C \in \mathbb{R}^n$

Property: Let $A \in \mathcal{M}_n(\mathbb{R})$, λ - an eigenvalue of A and u a corresp. eigenvector. We have that $\varphi : \mathbb{R} \to \mathbb{R}^n$, $\varphi(t) = e^{\lambda t}u$, $\forall t \in \mathbb{R}$ is a sol. of $X' = AX$.

Pf: φ is a sol of $X' = AX \Leftrightarrow \varphi'(t) = A \cdot \varphi(t), \forall t \in \mathbb{R} \Leftrightarrow \lambda e^{\lambda t}u = A \cdot e^{\lambda t}u, \forall t \in \mathbb{R} \Leftrightarrow$

$\Leftrightarrow \lambda u = Au, \forall t \in \mathbb{R}$, true $\Leftrightarrow$ λ - eigenvalue, u - eigenvector of A $\square$

Consequence: Assume that A is diagonalizable. Then $e^{\lambda_1 t}u_1, e^{\lambda_2 t}u_2, \dots, e^{\lambda_n t}u_n$ are n lin. indep. sol. of $X' = AX$.

Pf: HW.

The characteristic eq. method to find the gen sol of $X' = AX$ when A is diagonalizable.

Step 1: Find the eigenvalues and the eigenvectors of A. Decide whether A is diagonalizable or not. If it is continue.

Step 2: Write the sol, $e^{\lambda_1 t}u_1, \dots, e^{\lambda_n t}u_n$

Step 3: Write the gen sol of $X' = AX$

$$X = c_1 e^{\lambda_1 t}u_1 + \dots + c_n e^{\lambda_n t}u_n, \quad c_1, \dots, c_n \in \mathbb{R}$$

Example:

1) Prove that $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$ is diagonalizable over $\mathbb{R}$.

2) Using the charact. eq method find the gen sol of $X' = AX$

1) Prove that $A=\begin{pmatrix}1&3\\1&-1\end{pmatrix}$ is diagonalizable over $\mathbb{R}$.

2) Using the charact. eq. method find the gen sol of $X'=AX$.

1) the charact. eq is $\det(A-\lambda I_2)=0 \Leftrightarrow \begin{vmatrix}1-\lambda & 3\\ 1 & -1-\lambda\end{vmatrix}=0 \Leftrightarrow -(1-\lambda)(1+\lambda)-3=0 \Leftrightarrow$

$\Leftrightarrow -1+\lambda^2-3=0 \Leftrightarrow \lambda^2-4=0 \Leftrightarrow$

$\Leftrightarrow \lambda_1=-2,\ \lambda_2=2$

$u_1=?$ an eigenvector corresp to $\lambda_1=-2 \Leftrightarrow u_1\in\mathbb{R}^2, u_1\neq\begin{pmatrix}0\\0\end{pmatrix}, Au_1=-2u_1, u_1=\begin{pmatrix}a\\b\end{pmatrix}$

$Au_1=-2u_1 \Leftrightarrow \begin{pmatrix}1&3\\1&-1\end{pmatrix}\cdot\begin{pmatrix}a\\b\end{pmatrix}=-2\begin{pmatrix}a\\b\end{pmatrix} \Leftrightarrow \begin{pmatrix}a+3b\\a-b\end{pmatrix}=\begin{pmatrix}-2a\\-2b\end{pmatrix} \Leftrightarrow \begin{cases}a+3b=-2a\\a-b=-2b\end{cases} \Leftrightarrow \begin{cases}3a+3b=0\\a+b=0\end{cases} \Leftrightarrow$

$\Leftrightarrow a+b=0$. Choose $a=1, b=-1 \Rightarrow u_1=\begin{pmatrix}1\\-1\end{pmatrix}$

$u_2=?$ an eigenvector corresp to $\lambda_2=2 \Leftrightarrow u_2\in\mathbb{R}^2, u_2\neq\begin{pmatrix}0\\0\end{pmatrix}, u_2=\begin{pmatrix}a\\b\end{pmatrix}$

$Au_2=2u_2 \Leftrightarrow (A-2I_2)u_2=0 \Leftrightarrow \begin{pmatrix}-1&3\\1&-3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=0 \Leftrightarrow \begin{cases}-a+3b=0\\a-3b=0\end{cases}$

Choose $a=3, b=1 \Rightarrow u_2=\begin{pmatrix}3\\1\end{pmatrix}$

So A has the eigenvalues $\lambda_1=-2, \lambda_2=2$, both real, $n=2$ and we found 2 eigenvectors $u_1=\begin{pmatrix}1\\-1\end{pmatrix}$ and $u_2=\begin{pmatrix}3\\1\end{pmatrix}$

Since $\begin{vmatrix}1&3\\-1&1\end{vmatrix}=4\neq0 \to u_1$ and u_2 are lin. indep

Thus, A is diagonalizable.

2) We have that $e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}$ and $e^{2t}\begin{pmatrix}3\\1\end{pmatrix}$ are linearly indep sol of $X'=AX \Rightarrow$

$\Rightarrow$ gen sol: $X=c_1e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix}+c_2e^{2t}\begin{pmatrix}3\\1\end{pmatrix}, c_1,c_2\in\mathbb{R}$

Remark: Write by components the system $X'=AX$ and its gen sol, using the not. $X=\begin{pmatrix}x_1\\x_2\end{pmatrix}$ or $X=\begin{pmatrix}x\\y\end{pmatrix}$.

$X'=AX \Leftrightarrow \begin{cases}x_1'=x_1+3x_2\\x_2'=x_1-x_2\end{cases}$ and its gen sol is $\begin{cases}x_1=c_1e^{-2t}+3c_2e^{2t}\\x_2=-c_1e^{-2t}+c_2e^{2t}\end{cases}, c_1,c_2\in\mathbb{R}$

Ex. 3) Find e^{tA}

$D=\begin{pmatrix}-2&0\\0&2\end{pmatrix}, P=\begin{pmatrix}1&3\\-1&1\end{pmatrix}, P^{-1}=?$ $\det P=4 \Rightarrow P^{-1}=\begin{pmatrix}\frac14&-\frac34\\\frac14&\frac14\end{pmatrix} \Rightarrow e^{tA}=\begin{pmatrix}1&3\\-1&1\end{pmatrix}\begin{pmatrix}e^{-2t}&0\\0&e^{2t}\end{pmatrix}\begin{pmatrix}\frac14&-\frac34\\\frac14&\frac14\end{pmatrix}=$

$=\begin{pmatrix}1&3\\-1&1\end{pmatrix}\begin{pmatrix}\frac14e^{-2t}&-\frac34e^{-2t}\\\frac14e^{2t}&\frac14e^{2t}\end{pmatrix}=\frac14\begin{pmatrix}e^{-2t}+3e^{2t}&-3e^{-2t}+3e^{2t}\\-e^{-2t}+e^{2t}&3e^{-2t}+e^{2t}\end{pmatrix}$. It was quite easy to compute in this way since we already computed the eigenvalues and the eigenvectors

Another method to find the gen sol. of LHDS with CC in dimension $n=2$.

$(1)\ \begin{cases}x'=a_{11}x+a_{12}y\\y'=a_{21}x+a_{22}y\end{cases}$ where $A=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}\in\mathcal{M}_2(\mathbb{R})$

Case 1: Uncoupled systems (A is diagonal or $a_{12}=a_{21}=0$)

$\begin{cases}x'=a_{11}x\\y'=a_{22}y\end{cases} \Leftrightarrow \begin{cases}x=c_1e^{a_{11}t}\\y=c_2e^{a_{22}t}\end{cases}, c_1,c_2\in\mathbb{R}$

Case 2: Coupled systems (either $a_{12}\neq0$ or $a_{21}\neq0$)

$(1)\begin{cases}x'=a_{11}x+a_{12}y\ (2)\\y'=a_{21}x+a_{22}y\ (3)\end{cases}$ Consider the case $a_{12}\neq0$

Reduction to a second order diff. eq.

$a_{12}\neq0 \Rightarrow$ the second order DE will be in x. Thus, we want to obtain from (1) a rel. between x', x''

and x, without y.

$$y \overset{(2)}{=} \frac{1}{a_{12}}(x' - a_{11}x)$$

$$x'' \underset{(2)}{=} a_{11}x' + a_{12}y' \underset{(3)}{=} a_{11}x' + a_{12}(a_{21}x + a_{22}y) =$$
$$= a_{11}x' + a_{12}a_{21}x + \cancel{a_{12}}\cdot a_{22}\cdot \frac{1}{\cancel{a_{12}}}(x' - a_{11}x) =$$
$$= a_{11}x' + a_{12}a_{21}x + a_{22}x' - a_{11}a_{22}x \Rightarrow$$

$\Rightarrow x'' - (a_{11}+a_{22})x' + (a_{11}a_{22} - a_{12}a_{21})x = 0$ (4) second order LHDE with CC.

Remark: The charact. polynomial of eq. (4) is $p(\kappa) = \kappa^2 - (a_{11}+a_{22})\kappa + (a_{11}a_{22} - a_{12}a_{21}) =$
$= \kappa^2 - tr(A)\kappa + \det(A)$, which also is the charact. polynomial of A

Exercise: Using the above method, find the gen. sol of $X' = AX$ with $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$. Then find e^{tA} using the gen sol. that we found and that $E(t) = e^{tA}$ satisfies $\begin{cases} X' = AX \\ X(0) = I_2 \end{cases}$.

$\begin{cases} x' = x + 3y \\ y' = x - y \end{cases}$, this is a coupled system, here it is possible to find the second order eq. both in x or in y.

$y = \frac{1}{3}(x' - x)$ $\quad x'' = x' + 3y' = x' + 3(x-y) = x' + 3x - \cancel{3}\cdot\frac{1}{\cancel{3}}(x'-x) = \cancel{x'} + 3x - \cancel{x'} + x = 4x$

$x'' - 4x = 0$ $\quad \kappa_1 = -2 \mapsto e^{-2t}$ $\quad x = c_1 e^{-2t} + c_2 e^{2t}$

$\kappa^2 - 4 = 0$ $\quad \kappa_2 = 2 \mapsto e^{2t}$ $\quad y = \frac{1}{3}(-2c_1 e^{-2t} + 2c_2 e^{2t} - c_1 e^{-2t} - c_2 e^{2t}) \Leftrightarrow$

$\Rightarrow \begin{cases} x = c_1 e^{-2t} + c_2 e^{2t} \\ y = -c_1 e^{-2t} + \frac{1}{3} c_2 e^{2t} \end{cases}$, $c_1, c_2 \in \mathbb{R}$

$e^{tA} = ?$ the first column of e^{tA} is the sol. of the ivp $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$ (5)

the second column of e^{tA} is the sol of the ivp $\begin{cases} X' = AX \\ X(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$ (6)

(5): $\begin{cases} x(0) = c_1 + c_2 = 1 \\ y(0) = -c_1 + \frac{1}{3} c_2 = 0 \end{cases} \Leftrightarrow \frac{4}{3} c_2 = 1 \Rightarrow c_2 = \frac{3}{4} \Rightarrow c_1 = \frac{c_2}{3} = \frac{1}{4} \Rightarrow$ the sol of (5) is $X_1 = \begin{pmatrix} \frac{1}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ -\frac{1}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$

(6): $\begin{cases} x(0) = c_1 + c_2 = 0 \\ y(0) = -c_1 + \frac{1}{3} c_2 = 1 \end{cases} \Leftrightarrow c_2 = \frac{3}{4} \Rightarrow c_1 = -c_2 = -\frac{3}{4} \Rightarrow$ the sol of (6) is $X_2 = \begin{pmatrix} -\frac{3}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ \frac{3}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$

So, $e^{tA} = (X_1 \;\; X_2) = \begin{pmatrix} \frac{1}{4} e^{-2t} + \frac{3}{4} e^{2t} & -\frac{3}{4} e^{-2t} + \frac{3}{4} e^{2t} \\ -\frac{1}{4} e^{-2t} + \frac{1}{4} e^{2t} & \frac{3}{4} e^{-2t} + \frac{1}{4} e^{2t} \end{pmatrix}$.