LECTURE 8

DATE: 15 NOVEMBER 2021
WEEK 8

8. Computation of multiple integrals

Theorem 1 (FUBINI): $f: A\times B \to \mathbb{R}$ integrable

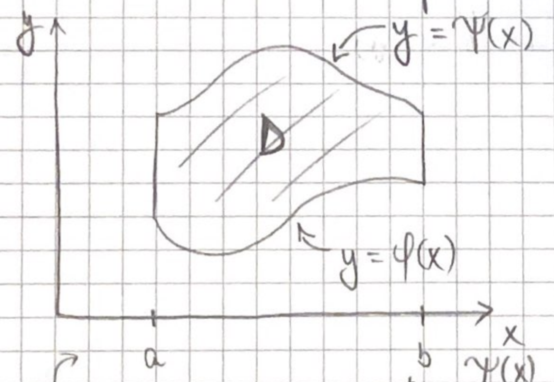
A, B bounded and Jordan measurable

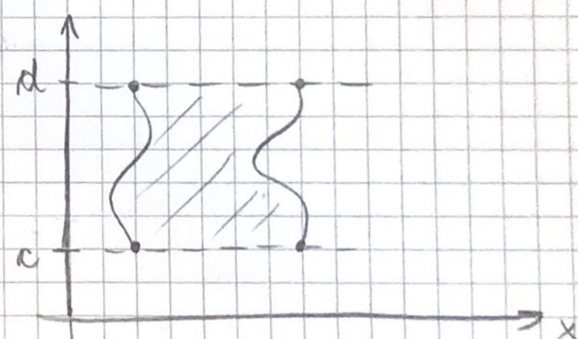
Then
$$\iint_{A\times B} f(x,y)\,dx\,dy = \int_A \left(\int_B f(x,y)\,dy\right) dx =$$
$$= \int_B \left(\int_A f(x,y)\,dx\right) dy$$

IDEA: reduce the multiple integral to the component of several simpler integrals

§ 8.1. Double integrals

We've discussed simple domains.





$$\iint_{D} f(x,y)\,dx\,dy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\right) dx$$

Notation & why we write dx dy

Whenever there is no ambiguity : simplified notation

$\int_D f(x)\,dx$ ← !

This implies it's a multiple integral, that $f = f(x_1, \dots, x_d)$ and $dx = dx_1 dx_2 \dots dx_d$

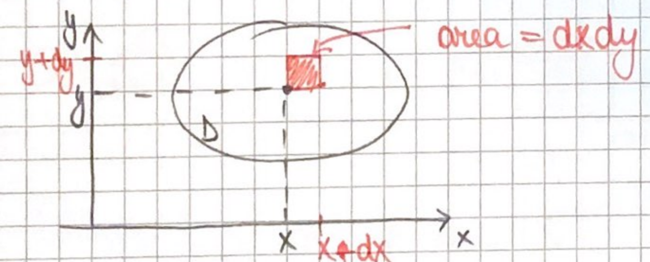
Whenever more details are helpful you write more details

e.g. $\iint_D f(x,y)\,dx\,dy$ this is a double integral

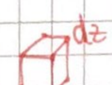
or $\int_0^1 \dots \int_0^1 e^{x_1+x_2+\dots+x_n}\,dx_1 \dots dx_n$ n integrals

Why do we write 'dx dy' (or $dx_1 dx_2 \dots dx_d$)

two reasons :
- it indicates the number (and names) of variables integrate w.r.t.
- geometric interpretation : surface element or volume element



dx, dy "infinitesimals"

dim = 3 

volume = dx dy dz

§8.2. Change of variables in the multiple integral

We need to take a step back to differential calculus and have a look at

$$f : \mathbb{R}^d \to \mathbb{R}^m \quad (\text{not just } \underbrace{f : \mathbb{R}^d \to \mathbb{R}}_{m=1})$$

$f = (f_1, f_2, \dots, f_m)$ with $f_1, \dots, f_m : \mathbb{R}^d \to \mathbb{R}$

components of f

f is called vector field or vector valued function

The question is: what's the appropriate notion of derivative for f?

(for $f : \mathbb{R}^d \to \mathbb{R}$ it was ∇f)

: Yacobi matrix

$$J_f(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \dots & \frac{\partial f_1}{\partial x_d}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \dots & \frac{\partial f_2}{\partial x_d}(x) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x) & \dots & \frac{\partial f_m}{\partial x_d}(x) \end{bmatrix}$$

$$= \begin{bmatrix} [\nabla f_1(x)] \\ [\nabla f_2(x)] \\ \dots \\ [\nabla f_m(x)] \end{bmatrix}$$

Remark: The Yacobi matrix turns out to be the Frechet differential of f

We are interested in changes of coordinates

$$u : \mathbb{R}^d \to \mathbb{R}^d \quad \left| \begin{array}{l} u = (u_1, \dots, u_d) \\ u_i \text{ has cont partial derivative} \end{array}\right.$$

Def: u is called regular
if $\det(J_u(x)) \neq 0$

Theorem 2: (change of vars in the multi integral)

Let $\Delta, D \subset \mathbb{R}^d$ bounded, closed, measurable
$M \subset \Delta$ of measure zero ($\mu(M) = 0$) and
$u : \Delta \to D$ with cont. diffable components and

(i) u is injective on $\Delta \setminus M$
(ii) u is regular on $\Delta \setminus M$

Then, if $f : D \to \mathbb{R}$ is integrable over D we have

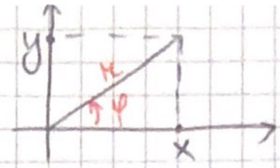
$$\int_D f(x)\,dx = \int_\Delta f(u(z)) \underbrace{|\det J_f(z)|\,dz}_{\text{the "new" infinitezimal surface / volume element}}$$

Standard changes of coordinates

2D : polar coordinates

3D : sferical coordinates, cylindrical etc.

Polar coordinates:



$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2} \\ \varphi = \operatorname{arctg}\frac{y}{x} \end{cases}$$

$r \geq 0$

$\varphi \in [0, 2\pi)$

$\mu : \Delta \to D \qquad \mu(r, \varphi) = (r\cos\varphi, r\sin\varphi)$

change of coords $\qquad r, \varphi \to x, y$

$$J_\mu(r, \varphi) = \begin{bmatrix} \frac{\partial}{\partial r}(r\cos\varphi) & \frac{\partial}{\partial \varphi}(r\cos\varphi) \\ \frac{\partial}{\partial r}(r\sin\varphi) & \frac{\partial}{\partial \varphi}(r\sin\varphi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{bmatrix}$$

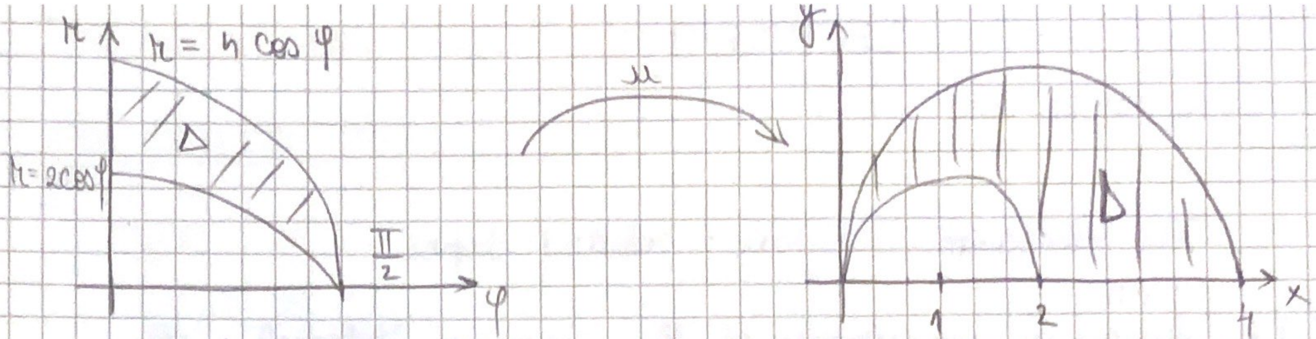
$$\det J_\mu(r, \varphi) = r(\cos\varphi)^2 + r(\sin\varphi)^2 = r \neq 0$$

(for $r \neq 0$)

ex: $I = \iint_\Delta \sqrt{x^2+y^2}\, dx\,dy \qquad D = \{(x,y) \in \mathbb{R}^2 : 2x \leq x^2+y^2 \leq 4x, \ y \geq 0\}$

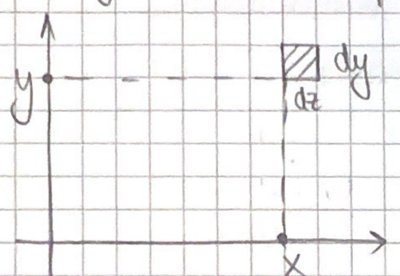
IDEA: $(x-x_0)^2 + (y-y_0)^2 = R^2$ eq. of a circle of radius R centre at (x_0, y_0)

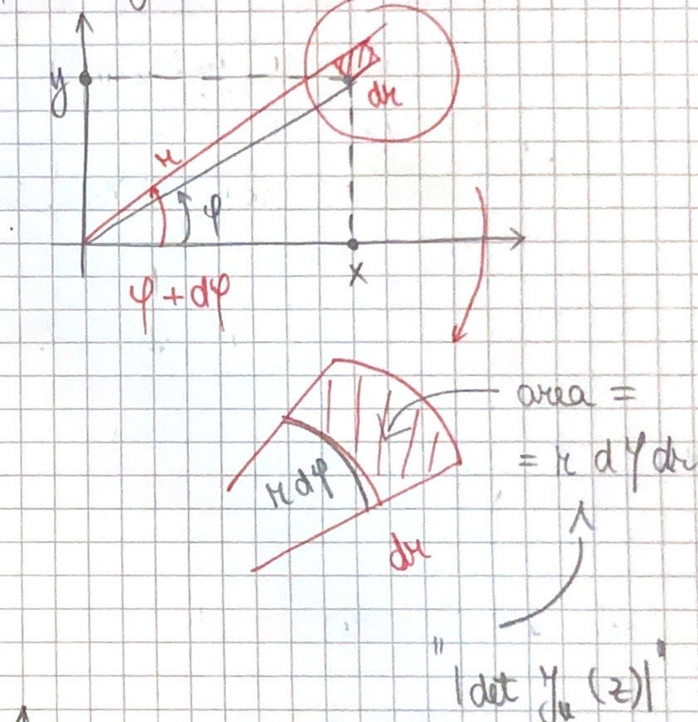
interior / exterior of circle $\leq$ / $\geq$



$$\mathcal{J} = \iint_{\Delta} r \underbrace{r}_{\det \mathcal{J}_u} \, dr \, d\varphi = \int_0^{\frac{\pi}{2}} \left(\int_{2\cos\varphi}^{4\cos\varphi} r^2 \, dr \right) d\varphi = \cdots \frac{112}{9}$$

The geometric interpretation of $|\det \mathcal{J}_u(z)| \, dz$





$d\varphi$ is "angle"

$r\,d\varphi$ is "length"

Spherical coords

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$

$|\det \mathcal{J}_u(r, \varphi, \theta)| = r^2 \sin\theta$